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TOPICS IN INHOMOGENEOUS BERNOULLI PERCOLATION:
a study of two models

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A study of two models

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Tópicos em percolação de Bernoulli não-homogênea

Um estudo de dois modelos

Onderwerpen in inhomogene Bernoulli-percolatie

Een studie van twee modellen

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Resumo

Esta tese é uma investigação de alguns aspectos matemáticos de percolação de elos de Bernoulli não-homogênea em dois grafos $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ distintos; em cada um deles, consideramos uma decomposição $\mathbb{E}' \cup \mathbb{E}''$ do conjunto de elos $\mathbb{E}$ em questão e, dados $p, q \in [0, 1]$, atribuímos parâmetros p e q aos elos de $\mathbb{E}'$ e $\mathbb{E}''$, respectivamente. Em tal formulação, um dos conjuntos, digamos $\mathbb{E}''$, é visto como o conjunto de inomogeneidades.

O primeiro grafo $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ considerado é aquele induzido pelo produto cartesiano de um grafo infinito e conexo $G = (V, E)$ e o conjunto dos inteiros $\mathbb{Z}$. Escolhemos uma coleção infinita C de subgrafos finitos e conexos de G e trabalhamos com o modelo de percolação de elos de Bernoulli em $\mathbb{G}$ que atribui probabilidade q de estar aberto a cada elo cuja projeção em G incide sobre algum subgrafo em C, e probabilidade p para os demais elos. Aqui, focamos nossa atenção no parâmetro crítico para percolação, $p_c(q)$, definido como o supremo dos valores de p para os quais percolação com parâmetros p, q não ocorre. Mostramos que a função $q \mapsto p_c(q)$ é contínua em $(0, 1)$, no caso em que os grafos de C estão "suficientemente espaçados entre si" em G e seus conjuntos de vértices possuem cardinalidade uniformemente limitada.

O segundo grafo é a usual rede hipercúbica d-dimensional, $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, $d \geq 3$, onde definimos o modelo de percolação de Bernoulli não-homogênea em que cada elo contido no subespaço s-dimensional $\mathbb{Z}^s \times \{0\}^{d-s}$, $2 \leq s < d$, está aberto com probabilidade q, e os demais elos estão abertos com probabilidade p. Definindo $q_c(p)$ como o supremo dos valores de q para os quais percolação com parâmetros p, q não ocorre e denotando o ponto crítico para percolação homogênea em $\mathbb{L}^d$ por $p_c \in (0, 1)$, provamos dois resultados: primeiro, a unicidade do aglomerado infinito na fase supercrítica de parâmetros (p, q), para $p \neq p_c$; segundo, mostramos que, para $p < p_c$, o ponto crítico $(p, q_c(p))$ pode ser aproximado por pontos críticos de slabs, no espírito do clássico teorema de Grimmett e Marstrand para percolação homogênea.

Palavras-chave: percolação não-homogênea, unicidade, curva crítica, teorema de Grimmett–Marstrand.

Samenvatting

In dit proefschrift bestuderen we enige wiskundige aspecten van inhomogene Bernoulli-percolatie op twee verschillende grafen $\mathbb{G} = (\mathbb{V}, \mathbb{E})$; in beide gevallen beschouwen we een decompositie $\mathbb{E}' \cup \mathbb{E}''$ van de betreffende verzameling kanten $\mathbb{E}$ van de graaf, en voor gegeven $p, q \in [0, 1]$ kennen we parameters p en q toe, respectievelijk aan de kanten in $\mathbb{E}'$ en aan die in $\mathbb{E}''$. In zo'n formulering wordt een van de twee verzamelingen kanten, zeg $\mathbb{E}''$, opgevat als de verzameling inhomogeniteiten.

De eerste graaf, $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, die we beschouwen wordt verkregen door het Cartesisch product te nemen van een oneindige samenhangende graaf $G = (V, E)$ en de verzameling gehele getallen $\mathbb{Z}$. We kiezen een oneindige collectie C, bestaande uit eindige samenhangende subgrafen van G, en beschouwen het Bernoulli kantpercolatie-model op $\mathbb{G}$ dat een kans q om open te zijn toekent aan elke kant waarvan de projectie op G in een subgraaf uit C ligt, en een kans p om open te zijn aan elke andere kant in de graaf. We laten zien dat de kritieke percolatiedrempel $p_c(q)$ een continue functie op $(0, 1)$ is, onder de voorwaarde dat de grafen in C voldoende ver uit elkaar liggen en dat hun verzamelingen punten een uniform begrensde cardinaliteit hebben.

De tweede graaf die we beschouwen is het standaard d-dimensionale rooster, $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, $d \geq 3$, waarop we werken met het inhomogene Bernoulli-percolatiemodel waarin iedere kant in het s-dimensionale hypervlak $\mathbb{Z}^d \times \{0\}^{d-s}$, $2 \leq s \leq d$, open is met kans q en iedere andere kant open is met kans p.

Daarvoor bewijzen we twee resultaten: Ten eerste bewijzen we de uniciteit van de oneindige cluster in de superkritische fase, voor parameters (p, q), als $p \neq p_c$, waar $p_c \in (0, 1)$ de drempel voor homogene percolatie op $\mathbb{L}^d$ aangeeft; en ten tweede laten we zien dat het kritieke punt $(p, q_c(p))$ benaderd kan worden door de kritieke punten van "slabs"van eindige dikte, voor elke $p < p_c$.

Abstract

This thesis is an investigation of some mathematical aspects of inhomogeneous Bernoulli bond percolation in two different graphs $\mathbb{G} = (\mathbb{V}, \mathbb{E})$. In each of them, we consider a decomposition $\mathbb{E}' \cup \mathbb{E}''$ of the relevant edge set $\mathbb{E}$ and, given $p, q \in [0, 1]$, we assign parameters p and q to the edges of $\mathbb{E}'$ and $\mathbb{E}''$, respectively. In such formulation, one of the sets, say $\mathbb{E}''$, is regarded as the set of inhomogeneities.

The first graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ we consider is the one induced by the cartesian product of an infinite and connected graph $G = (V, E)$ and the set of integers $\mathbb{Z}$. We choose an infinite collection $\mathcal{C}$ of finite connected subgraphs of G and consider the Bernoulli bond percolation model on $\mathbb{G}$ which assigns probability q of being open to each edge whose projection onto G lies in some subgraph of $\mathcal{C}$ and probability p to every other edge. Here, we focus our attention on the critical percolation threshold, $p_c(q)$, defined as the supremum of the values of p for which percolation with parameters p, q does not occur. We show that the function $q \mapsto p_c(q)$ is continuous in $(0, 1)$, provided that the graphs in $\mathcal{C}$ are "suffciciently spaced from each other" on G and their vertex sets have uniformly bounded cardinality.

The second graph is the ordinary d-dimensional hypercubic lattice, $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, $d \geq 3$, where we define the inhomogeneous Bernoulli percolation model in which every edge inside the s-dimensional subspace $\mathbb{Z}^s \times \{0\}^{d-s}$, $2 \leq s < d$, is open with probability q and every other edge is open with probability p. Defining $q_c(p)$ as the supremum of the values of q for which percolation with parameters p, q does not occur and letting $p_c \in (0, 1)$ be the threshold for homogeneous percolation on $\mathbb{L}^d$, we prove two results: first, the uniqueness of the infinite cluster in the supercritical phase of parameters (p, q), whenever $p \neq p_c$; second, we show that, for any $p < p_c$, the critical point $(p, q_c(p))$ can be approximated by the critical points of slabs, in the spirit of the classical theorem of Grimmett and Marstrand for homogeneous percolation.

Keywords: inhomogeneous percolation, uniqueness, critical curve, Grimmett–Marstrand theorem.

Contents

Acknowledgements **viii**

Resumo **ix**

Samenvatting **x**

Abstract **xi**

Introduction **xiii**

The object of study of this thesis . . . xiii

Overview of this thesis . . . xiv

Basic definitions and notations . . . xv

1 Inhomogeneous Percolation on ladder graphs:
Continuity of the critical curve **1**

1.1 Overview of the chapter . . . 1

1.2 Definition of the model and main result . . . 2

1.3 Proof of Theorem 1.1 . . . 4

2 Percolation on $\mathbb{Z}^d$ with a sublattice of inhomogeneities:
The uniqueness problem **11**

2.1 Overview of the chapter . . . 11

2.2 Definition of the model and main result . . . 12

2.3 General background . . . 14

2.4 Proof of Theorem 2.1: the case $p < p_c(d)$ . . . 21

2.5 Proof of Theorem 2.1: the case $p > p_c(d)$ . . . 22

3 Percolation on $\mathbb{Z}^d$ with a sublattice of inhomogeneities:
Approximation on slabs **27**

3.1 Overview of the chapter . . . 27

3.2 Statement of the main result . . . 27

3.3 Technical lemmas . . . 28

3.4 Proof of Theorem 3.1: The renormalization process . . . 39

Bibliography **55**

Introduction

The object of study of this thesis. Percolation Theory is a well-established discipline, having its mathematical roots back in 1957, with the seminal work of Broadbent and Hammersley [6]. Initially proposed as a model for the transport of fluid through a porous medium, the theory has evolved up to the present days, reaching a wide range of topics, specially in mathematics and physics. From the physicist's standpoint, it has been applied to the study of disordered physical systems, such as random electrical networks, the modeling of epidemics and oil recovery. As for the mathematician's point of view, it has become the source of elegant and challenging problems in combinatorics, probability theory, graph theory and analysis.

This thesis is an investigation of some mathematical aspects of inhomogeneous Bernoulli bond percolation on two different graphs $\mathbb{G} = (\mathbb{V}, \mathbb{E})$; in each of them, we consider a decomposition $\mathbb{E}' \cup \mathbb{E}''$ of the relevant edge set $\mathbb{E}$ and, given $p, q \in [0, 1]$, we assign parameters p and q to the edges of $\mathbb{E}'$ and $\mathbb{E}''$, respectively. In such a formulation, one of the sets, say $\mathbb{E}''$, is regarded as the set of inhomogeneities. In our study, we analyze, in both models, some properties of the critical curve $q \mapsto p_c(q)$ (or $p \mapsto q_c(p)$), where $p_c(q)$ is the supremum of the values of p for which percolation with parameters p, q does not occur. In one of the models, we also prove the uniqueness of the infinite cluster in the supercritical phase.

The above-mentioned topics are of intrinsic interest. Perhaps one of the earliest works concerning the behavior of critical curves is due to Kesten, presented in [19]. Considering the square lattice $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}^2)$ and choosing $\mathbb{E}''$ and $\mathbb{E}'$ to be respectively the sets of vertical and horizontal edges, he proves that $p_c(q) = 1 - q$. Later on, Zhang [25] also considers the square lattice, but with the edge set $\mathbb{E}''$ being only the vertical edges within the y-axis and $\mathbb{E}' = \mathbb{E}^2 \setminus \mathbb{E}''$. He proves that, for any $q < 1$, there is no percolation at $p = 1/2$, which implies that $p_c(q)$ is constant in the interval $[0, 1)$. In the context of long-range percolation, de Lima, Rolla and Valesin [9] consider an oriented, d-regular, rooted tree $\mathbb{T}_{d,k}$, where besides the usual set of "short bonds" $\mathbb{E}'$, there is a set $\mathbb{E}''$ of "long edges" of length $k \in \mathbb{N}$, pointing from each vertex x to its d^k descendants at distance k. They show that $q \mapsto p_c(q)$ is continuous and strictly decreasing in the region where it is positive. This conclusion is also achieved by Couto, de Lima and Sanchis [8], where the authors consider the slab of thickness k induced by the vertex set $\mathbb{Z}^2 \times \{0, \ldots, k\}$, with $\mathbb{E}'$ and $\mathbb{E}''$ being respectively the sets of edges parallel and perpendicular to the xy-plane. As for the number of infinite clusters in a supercritical percolation configuration, this is one of the most basic questions studied in percolation theory. For invariant percolation on the d-dimensional lattice, major contributions in proving the uniqueness of the infinite cluster are those of Aizenman, Kesten and Newman [1] and Burton and Keane [7]. An extension of the latters' argument to more general graphs can be found in the book of Lyons and Peres [20].

Overview of this thesis. This thesis consists of three chapters, each of them containing one main result. In Chapter 1, we present an extension of the recent work of Szabó and Valesin [23], published in [10]. In [23], the authors have proved the continuity of the critical curve $q \mapsto p_c(q)$ on the interval $(0, 1)$, when $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is the graph induced by the cartesian product between an infinite and connected graph $G = (V, E)$ and the set of integers $\mathbb{Z}$, the set $\mathbb{E}''$ is obtained by selecting finite subsets $V' \subset V$, $E' \subset E$ and defining

$$\mathbb{E}'' = (\cup_{u \in V'} \{\{(u, n), (u, n + 1)\} : n \in \mathbb{Z}\}) \cup (\cup_{\{u,v\} \in E'} \{\{(u, n), (v, n)\} : n \in \mathbb{Z}\}),$$

and $\mathbb{E}' = \mathbb{E} \setminus \mathbb{E}''$. In Chapter 1 ([10]), we extend their result, in the sense that the continuity of $q \mapsto p_c(q)$ still holds if V' and E' are infinite sets, provided that the set $V' \cup (\cup_{e \in E'} e)$ do not possess arbitrarily large connected components in G, and the graph-theoretic distance between any two such components is bigger than 2. This is achieved through the construction of a coupling which allows us to understand how a small change in the parameters p and q of the model affects the percolation behavior and is largely based on the ideas of [9] and [23].

Chapters 2 and 3 are devoted to the study of inhomogeneous percolation on $\mathbb{Z}^d$, $d \geq 3$, with a sublattice of inhomogeneities. In this setting, the graph $\mathbb{G}$ is the d-dimensional lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, the set $\mathbb{E}''$ is the set of edges within the subspace $\mathbb{Z}^s \times \{0\}^{d-s}$, $2 \leq s < d$, and $\mathbb{E}' = \mathbb{E}^d \setminus \mathbb{E}''$. Some properties of this model have already been addressed by Iliev, Janse van Rensburg and Madras [18]. Besides several classical results that have been transferred from the homogeneous to the inhomogeneous percolation setting, the authors have proved that the critical function $p \mapsto q_c(p)$ is strictly decreasing in the interval $[0, p_c(d)]$, where

$p_c(d) \in (0,1)$ here denotes the critical point for percolation on $\mathbb{L}^d$ in the homogeneous case. This is particularly interesting since it shows the existence of parameters (p,q), $p < p_c(d) < q < p_c(s)$, for which there is an infinite cluster almost surely. To complement [18], we provide the following additional results:

In Chapter 2, we prove the uniqueness of the infinite cluster in the supercritical phase. As we shall discuss further, the lack of invariance of the percolation measure under a transitive group of automorphisms of $\mathbb{L}^d$ plays against a direct application of the existing techniques of [1, 7, 20]. We will then explore some other properties of our model, so that we can overcome this issue and conveniently adapt the known arguments to prove the uniqueness of the infinite cluster in our case.

In Chapter 3, we address the problem of whether for any $p \in [0, p_c(d))$, the critical point $(p, q_c(p)) \in [0,1]^2$ can be approximated by the critical point of the restriction of the inhomogeneous process to a slab $\mathbb{Z}^2 \times \{-N, \ldots, N\}^{d-2}$, for large $N \in \mathbb{N}$. Here, the classical work of Grimmett and Marstrand [13] serves as the standard reference for providing the building blocks that give an affirmative answer to this question. We shall see that, since we are dealing with a supercritical regime of parameters (p,q), where $p < p_c(d) < q < p_c(s)$, the construction of a suitable renormalization process for our case possesses some particularities that contrast with the usual approach of [13]. As a consequence, the finite-size criterion used in the construction of long-range connections should be modified accordingly, which in turn introduces some technical obstacle in the renormalization procedure that must be properly dealt with.

Basic definitions and notations. The requirements for the reader to follow this study are the knowledge of elementary probability theory, real analysis and some concepts from graph theory and ergodic theory. Fundamental definitions that are common to every chapter of this text are introduced in the following. A detailed account on percolation theory can be found in the book of Grimmett [14].

We begin with some terminology from graph theory. We say that $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is a ***graph*** with vertex set $\mathbb{V}$ and edge set $\mathbb{E}$ if $\mathbb{V}$ is a non-empty countable set and $\mathbb{E}$ is a subset of the family of subsets of $\mathbb{V}$ with two elements. For example, the d-dimensional lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ is the graph with vertex set $\mathbb{Z}^d$ and edge set $\mathbb{E}^d := \{\{x,y\} \subset \mathbb{Z}^d : \|x-y\|_1 = 1\}$, where $\|x-y\|_1 = \sum_{i=1}^d |x_i - y_i|$. If $\mathbb{G}' = (\mathbb{V}', \mathbb{E}')$ is a graph, $\mathbb{V}' \subset \mathbb{V}$ and $\mathbb{E}' \subset \mathbb{E}$, we say that $\mathbb{G}'$ is a ***subgraph*** of $\mathbb{G}$. Given $x, y \in \mathbb{V}$, a ***path*** $\pi(x,y)$ from x to y on $\mathbb{G}$ is a set of distinct vertices

$\pi(x, y) = \{x = v_0, v_1, \ldots, v_m = y\} \subset \mathbb{V}$, such that $\{v_i, v_{i+1}\} \in \mathbb{E}$ for every $i = 1, \ldots, m$. Denoting by $\mathcal{P}(x, y)$ the set of paths from x to y, the graph-theoretic ***distance*** between x and y on $\mathbb{G}$ is the number $\mathrm{dist}_{\mathbb{G}}(x, y) := \inf\{|\pi(x, y)| : \pi \in \mathcal{P}(x, y)\} - 1$. That is, $\mathrm{dist}_{\mathbb{G}}(x, y)$ is the number of edges within the shortest path from x to y. The ***degree*** of a vertex $x \in \mathbb{V}$ is the number $\deg(x) := |\{y \in \mathbb{V} : \{x, y\} \in \mathbb{E}\}|$. If $\deg(x) = 0$, we say that x is an ***isolated vertex***. The graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is said to have ***bounded degree*** if there is some $k \in \mathbb{N}$ such that $\deg(x) \leq k$ for every $x \in \mathbb{V}$.

Next, we introduce the relevant definitions for working with inhomogeneous Bernoulli bond percolation. Briefly, a percolation process on a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is defined as a probability measure on the set of subgraphs of $\mathbb{G}$. Among the many possible variants, we shall work with bond percolation models, in which every edge of $\mathbb{E}$ can be ***open (retained)*** or ***closed (removed)***, states represented by 1 and 0, respectively. Thus, a typical percolation configuration is an element of $\Omega = \{0, 1\}^{\mathbb{E}}$; this set can be regarded as the set of subgraphs of $\mathbb{G}$ induced by their open edges. That is, an element $\omega \in \Omega$ is associated with the subgraph $(\mathsf{V}(\omega), \mathsf{E}(\omega))$, where $\mathsf{E}(\omega) = \{e \in E : \omega(e) = 1\}$ and $\mathsf{V}(\omega) = \{x \in V : \exists e \in \mathsf{E}(\omega) \text{ such that } x \in e\}$, and conversely, a subgraph $(\mathbb{V}', \mathbb{E}') \subset \mathbb{G}$ with no isolated vertices induces the configuration $\omega \in \Omega$, given by $\omega(e) = 1$ if $e \in \mathbb{E}'$ and $\omega(e) = 0$ otherwise. The underlying σ-algebra $\mathcal{F}$ of the process is the one generated by the finite-dimensional cylinder sets of Ω. As for the probability measure, let $b(\alpha)$ be the Bernoulli measure with parameter $\alpha \in [0, 1]$ and let $\mathbb{E}' \cup \mathbb{E}''$ be a decomposition of the edge set $\mathbb{E}$. Given $p, q \in [0, 1]$, we define $P_{p,q} := \prod_{e \in \mathbb{E}'} b(p) \times \prod_{e \in \mathbb{E}''} b(q)$. That is, $P_{p,q}$ is the product measure on $(\Omega, \mathcal{F})$ with densities p and q on $\mathbb{E}'$ and $\mathbb{E}''$, respectively.

Given a configuration $\omega \in \Omega = \{0, 1\}^{\mathbb{E}}$, an ***open path*** in $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is a set of vertices $\{v_0, v_1, \ldots, v_m\} \subset \mathbb{V}$, such that $\omega(\{v_i, v_{i+1}\}) = 1$ for every $i = 0, \ldots, m - 1$. For $x, y \in \mathbb{V}$, we say that x is ***connected*** to y in ω if either $x = y$ or there is an open path from x to y, this event being denoted by $\{x \leftrightarrow y\}$. The ***cluster*** $C(x)$ of x in ω is the random set $C(x) := \{y \in V : x \leftrightarrow y\}$. If $|C(x)| = \infty$, we say that the vertex x ***percolates*** and write $\{x \leftrightarrow \infty\}$ for the event of such configurations.

We end this introductory section recalling an important result, extensively used in percolation theory, called the FKG Inequality, named after Fortuin, Kasteleyn and Ginibre [12]. This result is inserted in the more general context of correlation inequalities on partially ordered sets, but here we state a specific version, first proved by Harris [17], for the case of Bernoulli percolation. We refer the reader to [14] for a proof of the result.

Consider the following partial ordering of the elements of $\Omega = \{0,1\}^{\mathbb{E}}$: given $\omega, \omega' \in \Omega$, we say that $\omega \preccurlyeq \omega'$ if and only if $\omega(e) \leq \omega'(e)$ for every $e \in \mathbb{E}$. In this context, we say that an event $A \in \mathcal{F}$ is ***increasing*** if the following property holds: if $\omega \in A$, $\omega' \in \Omega$ and $\omega \preccurlyeq \omega'$, then $\omega' \in A$. The event $\{x \leftrightarrow \infty\}$ is an example of an increasing event. An event $A \in \mathcal{F}$ is ***decreasing*** if A^c is increasing.

Theorem 0.1 (FKG Inequality)**.** *If $A, B \in \mathcal{A}$ are both increasing or both decreasing events, then*

$$P_{p,q}(A \cap B) \geq P_{p,q}(A)P_{p,q}(B). \tag{1}$$

This result is fairly intuitive in the sense that, if A and B are positively correlated events, then the probability of A conditioned that B occurs must be at least the probability of A itself. For instance, if we condition on the event that there is an open path joining two vertices $x, y \in \mathbb{V}$, then it is more likely to have an open path joining x to a third vertex $z \in \mathbb{V}$.

1 Inhomogeneous Percolation on ladder graphs: Continuity of the critical curve

1.1 Overview of the chapter

In this chapter, we present an extension of the work of Szabó and Valesin [23], published in [10]. It regards the inhomogeneous Bernoulli bond percolation model on a graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, where the relevant edge set $\mathbb{E}$ can be written as a decomposition $\mathbb{E}' \cup \mathbb{E}''$, and parameters p and q, both in $[0, 1]$, are assigned to the edges of $\mathbb{E}'$ and $\mathbb{E}''$, respectively. In [23], the authors considered $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ to be the graph induced by the cartesian product between an infinite and connected graph $G = (V, E)$ and the set of integers $\mathbb{Z}$; the set $\mathbb{E}''$ was chosen by selecting finite subsets $V' \subset V$, $E' \subset E$ and defining

$$\mathbb{E}'' = (\cup_{u \in V'} \{\{(u, n), (u, n + 1)\} : n \in \mathbb{Z}\}) \cup (\cup_{\{u,v\} \in E'} \{\{(u, n), (v, n)\} : n \in \mathbb{Z}\}),$$

and $\mathbb{E}' = \mathbb{E} \setminus \mathbb{E}''$. They have proved the continuity of the critical curve $q \mapsto p_c(q)$ on the interval $(0, 1)$, where $p_c(q)$ is the supremum of the values of p for which percolation with parameters p, q does not occur. In [10], we extend this result in the sense that the continuity of $p_c(q)$ still holds if V' and E' are infinite sets, provided that the set of vertices $V' \cup (\cup_{e \in E'} e)$ do not possess arbitrarily large connected components in G, and the graph-theoretic distance between any two such components is bigger than 2. This is achieved through the construction of a coupling (a combination of Lemmas 1.5 and 1.6), which allows us to understand how a small change in the parameters p and q of the model affects the percolation behavior.

Aspects of the critical curve have been explored by several authors in different models [8, 9, 18, 19, 23, 25]; some of these results are mentioned in the **Introduction**. With respect to our model, we shall define it rigorously and state the main result in Section 1.2. In Section 1.3, we develop some technical lemmas and prove the main result.

In the next sections, we use the following notation: for a graph $G = (V, E)$, vertices $u, w \in V$ and subsets $U, W \subset V$, we denote by $\mathrm{dist}_G(u, w)$ the graph-theoretic distance between $u, w \in V$, and $\mathrm{dist}_G(U, W) := \min_{\substack{u \in U \\ w \in W}} \mathrm{dist}_G(u, w)$. We also define $\mathsf{E}_U := \{e \in E : e \subset U\}$.

1.2 Definition of the model and main result

Let $G = (V, E)$ be an infinite and connected graph with bounded degree and define $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V} := V \times \mathbb{Z}$ and

$$\mathbb{E} := \big\{\{(u, n), (v, n)\} : \{u, v\} \in E, n \in \mathbb{Z}\big\} \cup \big\{\{(u, n), (u, n + 1)\} : u \in V, n \in \mathbb{Z}\big\}.$$

Consider the following Bernoulli percolation process on $\mathbb{G}$. Every edge of $\mathbb{E}$ can be ***open*** or ***closed***, states represented by 1 and 0, respectively. Thus, a typical percolation configuration is an element of $\Omega = \{0, 1\}^{\mathbb{E}}$. As usual, the underlying σ-algebra $\mathcal{F}$ is the one generated by the finite-dimensional cylinder sets in Ω. Given $p \in [0, 1]$ and $q \in (0, 1)$, the governing probability $P_{p,q}$ of the process is the product measure on $(\Omega, \mathcal{F})$ with densities p and q on the edges of $\mathbb{E}$, specified as follows:

Fix a family of subgraphs of G, denoted by

$$\big\{G^{(r)} = \big(U^{(r)}, E^{(r)}\big)\big\}_{r \in \mathbb{N}}, \tag{1.1}$$

such that:

- $G^{(r)}$ is finite and connected for every $r \in \mathbb{N}$;
- $E^{(r)} = \mathsf{E}_{U^{(r)}}$;
- $\mathrm{dist}_G(U^{(i)}, U^{(j)}) \geq 3$, for every $i \neq j$.

For each $r \in \mathbb{N}$, let

$$\begin{aligned} \mathbb{E}^{\mathrm{in},(r)} := &\big\{\{(u, n), (v, n)\} : \{u, v\} \in E^{(r)}, n \in \mathbb{Z}\big\} \\ &\cup \big\{\{(u, n), (u, n + 1)\} : u \in U^{(r)}, n \in \mathbb{Z}\big\}. \end{aligned} \tag{1.2}$$

We assign parameter q to each edge of $\mathbb{E}^{\mathrm{in},(r)}$, for every $r \in \mathbb{N}$, and parameter p to each edge of $\mathbb{E} \setminus (\cup_{r \in \mathbb{N}} \mathbb{E}^{\mathrm{in},(r)})$.

In what follows, we shall work with the notions of open paths, connectivity between vertices and percolation of vertices. The reader is referred to the **Introduction** of this thesis for an account of these definitions. First, given $p, q \in [0,1]$, fix a vertex $v \in V$ and note that whether or not $P_{p,q}((v,0) \leftrightarrow \infty) > 0$ depends on the values of p and q. Our aim is to understand the shape of the surfaces determined by the sets of percolative and non-percolative parameters $(p,q) \in [0,1]^2$. Thus, our object of interest is the ***critical parameter function***, $p_c : [0,1] \to [0,1]$, defined by

$$p_c(q) := \sup\big\{p \in [0,1] : P_{p,q}((v,0) \leftrightarrow \infty) = 0\big\}.$$

Given $p, q \in (0,1)$ and $x, y \in \mathbb{V}$, the connectivity of $\mathbb{G}$ implies that $P_{p,q}(x \leftrightarrow y) > 0$. Thus, if $P_{p,q}(x \leftrightarrow \infty) > 0$, then the FKG Inequality (1) implies

$$P_{p,q}(y \leftrightarrow \infty) \geq P_{p,q}(y \leftrightarrow x, x \leftrightarrow \infty) \geq P_{p,q}(y \leftrightarrow x)P_{p,q}(x \leftrightarrow \infty) > 0,$$

since $\{y \leftrightarrow x\}$ and $\{x \leftrightarrow \infty\}$ are increasing events. Therefore, although the value of $P_{p,q}(x \leftrightarrow \infty)$ may depend on $x \in \mathbb{V}$, the value of $p_c(q)$ does not depend on the choice of $x \in \mathbb{V}$.

What we shall prove is a simple generalization of Theorem 1 of [23]. It states that the continuity of $p_c(q)$ still holds, provided that the cardinalities of the sets $U^{(r)}$ are uniformly bounded. In the context of Section 1.2, the model of [23] is the case where $G^{(r)} = (\emptyset, \emptyset)$ for every $r \geq 2$.

Theorem 1.1 (Continuity of the critical curve)**.** *If* $\mathrm{dist}_G(U^{(i)}, U^{(j)}) \geq 3$ *for every* $i \neq j$ *and* $\sup_{r\in\mathbb{N}}|U^{(r)}| < \infty$, *then* $p_c(q)$ *is continuous in* $(0,1)$.

Remark 1. Just as we have based our non-oriented percolation model upon the one of Szabó and Valesin [23], we can generalize the oriented model also present in [23] in an analogous manner. In this setting, the set of vertices $\mathbb{V}$ of the oriented graph is the same as the non-oriented case, and the set of oriented edges is $\vec{\mathbb{E}} = \{\langle(u,n),(v,n+1)\rangle : \{u,v\} \in E, n \in \mathbb{Z}\}$. The inhomogeneities are assigned to the set

$$\vec{\mathbb{E}}'' = \cup_{\{u,v\}\in E'}\big(\{\langle(u,n),(v,n+1)\rangle : n \in \mathbb{Z}\} \cup \{\langle(v,n),(u,n+1)\rangle : n \in \mathbb{Z}\}\big),$$

where E' is some finite subset of E. By a similar reasoning we shall present in the sequel, the continuity of the critical parameter for the oriented model also holds.

1.3 Proof of Theorem 1.1

Theorem 1.1 is a consequence of the following proposition:

Proposition 1.2. *Fix $p, q \in (0,1)$ and let $\lambda = \min(p, 1-p)$. If $\sup_{r\in\mathbb{N}}|U^{(r)}| < \infty$ and $\mathrm{dist}_G\big(U^{(i)}, U^{(j)}\big) \geq 3$ for every $i \neq j$, then for any $\varepsilon \in (0,\lambda)$, there exists $\eta = \eta(p,q,\varepsilon) > 0$ such that*

$$P_{p-\varepsilon,q+\delta}((v,0) \leftrightarrow \infty) \leq P_{p+\varepsilon,q-\delta}((v,0) \leftrightarrow \infty)$$

for every $\delta \in (0,\eta)$ and $v \in V \setminus (\cup_{r\in\mathbb{N}} U^{(r)})$.

Proof of Theorem 1.1. Since $q \mapsto p_c(q)$ is non-increasing, any discontinuity, if it exists, must be a jump. Suppose p_c is discontinuous at some point $q_0 \in (0,1)$, let $a = \lim_{q\downarrow q_0} p_c(q)$ and $b = \lim_{q\uparrow q_0} p_c(q)$. Then, for any $p \in (a,b)$, we can find an $\varepsilon > 0$ such that

$$P_{p+\varepsilon,q_0-\delta}((v,0) \leftrightarrow \infty) = 0 < P_{p-\varepsilon,q_0+\delta}((v,0) \leftrightarrow \infty)$$

for every $\delta > 0$ and $v \in V$, a contradiction to Proposition 1.2. □

The proof of Proposition 1.2 is based on the construction of a coupling which allows us to understand how a small change in the parameters p and q of the model affects the percolation behavior. This construction is done in several steps. First, we split $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ into an appropriate family of connected subgraphs $(\mathbb{V}_\alpha, \mathbb{E}_\alpha)$, $\alpha \in I$, such that $\{\mathbb{E}_\alpha\}_{\alpha\in I}$ constitutes a decomposition of $\mathbb{E}$. Second, we define coupling measures on each $\mathbb{E}_\alpha$, in such a way that the increase of parameter p compensates the decrease, by some small amount, of parameter q, in the sense of preserving the connections between boundary vertices of $\mathbb{V}_\alpha$. This property will play an important role when we consider percolation on the graph $\mathbb{G}$ as a whole. Third, we verify that we can set the same parameters for each coupling measure, provided that $\sup_{r\in\mathbb{N}}|U^{(r)}| < \infty$. Finally, we merge these couplings altogether by considering the product measure of each one. The first and second steps consist of ideas developed in [9] and [23]. The third step, which allows extending the result of [23], is the main result of [10]. To introduce them rigorously, we begin with some definitions.

For $r \in \mathbb{N}$, $n \in \mathbb{Z}$, let $L_r := |U^{(r)}|$ and

$$\begin{aligned} \mathbb{V}_n^{(r)} &:= \big\{(v,m) \in \mathbb{V} : \mathrm{dist}_G\big(v, U^{(r)}\big) \le 1, (2L_r+2)n \le m \le (2L_r+2)(n+1)\big\}; \\ \mathbb{E}_n^{(r)} &:= \mathsf{E}_{\mathbb{V}_n^{(r)}} \setminus \mathsf{E}_{V\times\{2L_r(n+1)\}}; \\ \mathbb{E}^{(r)} &:= \cup_{n\in\mathbb{Z}} \mathbb{E}_n^{(r)}. \end{aligned} \tag{1.3}$$

Based on these definitions, note that:

- Since $G=(V,E)$ has bounded degree and $|U^{(r)}| < \infty$, it follows that the graph $(\mathbb{V}_n^{(r)}, \mathbb{E}_n^{(r)})$ is finite;
- $\mathbb{E}_n^{(r)} \cap \mathbb{E}_{n'}^{(r)} = \emptyset$, for every $n \neq n'$;
- Since we are assuming $\mathrm{dist}_G(U^{(r)}, U^{(r')}) \ge 3$, for every $r \neq r'$, it follows that, given $n, n' \in \mathbb{Z}$, we have $\mathrm{dist}_{\mathbb{G}}(\mathbb{V}_n^{(r)}, \mathbb{V}_{n'}^{(r')}) \ge 1$. Therefore, $\mathbb{E}_n^{(r)} \cap \mathbb{E}_{n'}^{(r)} = \emptyset$, $\forall n, n' \in \mathbb{Z}$ and $r \neq r'$.

Next, recall the definition of $\mathbb{E}^{\mathrm{in},(r)}$ in (1.2) and let

$$\mathbb{E}_n^{\partial,(r)} := \mathbb{E}_n^{(r)} \setminus \mathbb{E}^{\mathrm{in},(r)}, \quad \mathbb{E}_n^{\mathrm{in},(r)} := \mathbb{E}_n^{(r)} \cap \mathbb{E}^{\mathrm{in},(r)}, \quad \mathbb{E}_O := \mathbb{E} \setminus \big(\cup_{r\in\mathbb{N}} \mathbb{E}^{(r)}\big).$$

One should also observe that $\mathbb{E}$ is a disjoint union of the sets above:

$$\mathbb{E} = \mathbb{E}_O \cup \big[\cup_{r\in\mathbb{N}} \mathbb{E}^{(r)}\big] = \mathbb{E}_O \cup \Big[\cup_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}} \mathbb{E}_n^{(r)}\Big] = \mathbb{E}_O \cup \Big[\cup_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}} \big(\mathbb{E}_n^{\partial,(r)} \cup \mathbb{E}_n^{\mathrm{in},(r)}\big)\Big].$$

Thus, letting

$$\Omega_O := \{0,1\}^{\mathbb{E}_O}, \quad \Omega_n^{(r)} := \{0,1\}^{\mathbb{E}_n^{(r)}}, \quad \Omega_n^{\partial,(r)} := \{0,1\}^{\mathbb{E}_n^{\partial,(r)}}, \quad \Omega_n^{\mathrm{in},(r)} := \{0,1\}^{\mathbb{E}_n^{\mathrm{in},(r)}},$$

we can write

$$\Omega = \Omega_O \times \prod_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}} \Omega_n^{(r)} = \Omega_O \times \prod_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}} \big(\Omega_n^{\partial,(r)} \times \Omega_n^{\mathrm{in},(r)}\big).$$

Finally, let

$$\begin{aligned} \partial\mathbb{V}_n^{(r)} := \big\{(v,m) \in \mathbb{V}_n^{(r)} &: \mathrm{dist}_G(v, U^{(r)}) = 1\big\} \\ &\cup \big[U^{(r)} \times \{(2L_r+2)\}\big] \cup \big[U^{(r)} \times \{(2L_r+2)\}\big], \end{aligned} \tag{1.4}$$

and, for $A \subset \partial \mathbb{V}_n^{(r)}$ and $\omega_n^{(r)} \in \Omega_n^{(r)}$, define the random set

$$C_n^{(r)}\big(A, \omega_n^{(r)}\big) \coloneqq \Big\{(v,m) \in \partial \mathbb{V}_n^{(r)} : \exists (v_0, m_0) \in A, (v,m) \xleftrightarrow{(\mathbb{V}_n^{(r)}, \mathbb{E}_n^{(r)})} (v_0, m_0) \text{ in } \omega_n^{(r)}\Big\}, \tag{1.5}$$

where $a \overset{G'}{\longleftrightarrow} b$ indicates that a and b are connected by a path entirely contained in the graph G'.

For $p, q \in [0,1]$ and $E' \subset \mathbb{E}$, let $P_{p,q}|_{E'}$ be the measure $P_{p,q}$ restricted to $\{0,1\}^{E'}$. It is clear that

$$P_{p,q} = P_{p,q}|_{\mathbb{E}_O} \times \prod_{\substack{r \in \mathbb{N} \\ n \in \mathbb{Z}}} P_{p,q}|_{\mathbb{E}_n^{(r)}}.$$

With these definitions in hand, we are ready to establish the facts necessary for the proof of Proposition 1.2.

Lemma 1.3. *Let $p, q \in (0,1)$ and $\lambda = \min(p, 1-p)$. For any $\varepsilon \in (0, \lambda)$ and $\delta \in (0,1)$ such that $(q-\delta, q+\delta) \subset [0,1]$, there exists a coupling $\mu_O = (\omega_O, \omega'_O)$ on Ω_O^2 such that*

- $\omega_O \overset{(d)}{=} P_{p-\varepsilon, q+\delta}|_{\mathbb{E}_O}$;
- $\omega'_O \overset{(d)}{=} P_{p+\varepsilon, q-\delta}|_{\mathbb{E}_O}$;
- $\omega_O \le \omega'_O$ *for every* $(\omega_O, \omega'_O) \in \Omega_O^2$.

Proof. This construction is standard. Let $Z = (Z_1, Z_2) \in \Omega_O^2$ be a pair of random elements defined in some probability space, such that the marginals Z_1 and Z_2 are independent on every edge of $\mathbb{E}_O$ and assign each edge to be open with probabilities $p - \varepsilon$ and $2\varepsilon/(1-p+\varepsilon)$, respectively. Taking $\omega_O = Z_1$ and $\omega'_O = Z_1 \vee Z_2$, define μ_O to be the distribution of (ω_O, ω'_O) and the claim readily follows. □

To properly compare percolation configurations in $(\mathbb{V}_n^{(r)}, \mathbb{E}_n^{(r)})$ at different parameter values, we make use of the following result, proved in [9] and also used in [23]. It is based on Doeblin's maximal coupling lemma (see [24], Chapter 1.4).

Lemma 1.4. *Let $\{P_\theta\}_{\theta \in (0,1)}$ be probability measures on a finite set S, such that $\theta \mapsto P_\theta(z)$ is continuous in $(0,1)$ for every $z \in S$. If $P_\tau(\bar{x}) > 0$ for some $\tau \in (0,1)$ and $\bar{x} \in S$, then, for every $\alpha, \beta \in (0,1)$ close enough to τ, there exists, on a larger probability space $(S^2, \mathbb{P})$, a coupling $X, Y \in S$, such that $X \overset{(d)}{=} P_\alpha$, $Y \overset{(d)}{=} P_\beta$ and*

$$\mathbb{P}(\{X = Y\} \cup \{X = \bar{x}\} \cup \{Y = \bar{x}\}) = 1.$$

Proof. Since $\theta \mapsto P_\theta(z)$ is continuous in $(0,1)$ for every $z \in S$, then the function

$$h(\theta,\gamma) := 1 - \sum_{z \neq \bar{x}} P_\theta(z) \vee P_\gamma(z)$$

is also continuous. By hypothesis, we have $h(\tau,\tau) = 1 - \sum_{z\neq\bar{x}} P_\tau(z) = P_\tau(\bar{x}) > 0$, so that $h(\alpha,\beta) > 0$ for every (α,β) close enough to (τ,τ).

Now, let $\mathbb{P}$ be the probability measure on S^2 defined by

$$\mathbb{P}(z_1,z_2) = \begin{cases} 1 - \sum_{z\neq\bar{x}} P_\alpha(z) \vee P_\beta(z), & \text{if } z_1 = z_2 = \bar{x}; \\ P_\alpha(z) \wedge P_\beta(z), & \text{if } z_1 = z_2 = z \neq \bar{x}; \\ [P_\alpha(z) - P_\beta(z)]^+, & \text{if } z_1 = z \neq \bar{x} \text{ and } z_2 = \bar{x}; \\ [P_\beta(z) - P_\alpha(z)]^+, & \text{if } z_1 = \bar{x} \text{ and } z_2 = z \neq \bar{x}; \\ 0 & \text{if } z_1 \neq z_2 \text{ and } z_1, z_2 \neq \bar{x}. \end{cases}$$

Thus, if $X, Y : S^2 \to S$ are defined by $X(x,y) = x$ and $Y(x,y) = y$, the result readily follows. □

The next lemma is one of the fundamental facts established in [23]. It is motivated by the observation that if a vertex $v \in \mathbb{V} \setminus \mathbb{V}_n^{(r)}$ percolates, then closing some edges within $\mathbb{V}_n^{(r)} \setminus \partial\mathbb{V}_n^{(r)}$ does not change the percolative behavior of v, as long as these closed edges do not interfere in the connectivity between the vertices of $\partial\mathbb{V}_n^{(r)}$. To make this assertion precise, we make use of the set $C_n^{(r)}(A, \omega_n^{(r)})$, defined by (1.5).

Lemma 1.5 (Coupling two configurations inside a finite cylinder)**.** *Let $r \in \mathbb{N}$, $n \in \mathbb{Z}$, $p, q \in (0,1)$ and $\lambda = \min(p, 1-p)$. For any $\varepsilon \in (0,\lambda)$, there exists $\eta^{(r)} > 0$, such that if $\delta \in (0, \eta^{(r)})$, there is a coupling $\mu_n^{(r)} = (\omega_n^{(r)}, {\omega'}_n^{(r)})$ on $\Omega_n^{(r)} \times \Omega_n^{(r)}$ with the following properties:*

- $\omega_n^{(r)} \overset{(d)}{=} P_{p-\varepsilon,q+\delta}|_{\mathbb{E}_n^{(r)}}$*;*
- ${\omega'}_n^{(r)} \overset{(d)}{=} P_{p+\varepsilon,q-\delta}|_{\mathbb{E}_n^{(r)}}$*;*
- $C_n^{(r)}(A, \omega_n^{(r)}) \subset C_n^{(r)}(A, {\omega'}_n^{(r)})$ *for every $A \in \partial\mathbb{V}_n^{(r)}$ almost surely.*

Moreover, the value of $\eta^{(r)} > 0$ depends only on the choice of q, p, ε and the graph $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$.

Proof. Let $r \in \mathbb{N}$ and $n \in \mathbb{Z}$. The measures $\mu_n^{(r)}$ will be translations of $\mu_0^{(r)}$, hence we shall construct only the latter. Our aim is to use Lemma 1.4 to properly compare percolation configurations in $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$ at different parameter values. To do so, we must first point out the relevant objects in the setting of the referred lemma.

For the finite set, we consider $S = \Omega_0^{\partial,(r)} \times \Omega_0^{\partial,(r)} \times \Omega_0^{\text{in},(r)}$.

Next, for $p \in (0,1)$ and $\lambda = \min(p, 1-p)$, fix $\varepsilon \in (0,\lambda)$ and let $\{P_{p,\varepsilon,t}\}_{t\in(0,1)}$ be the family of probability measures on $S = \Omega_0^{\partial,(r)} \times \Omega_0^{\partial,(r)} \times \Omega_0^{\text{in},(r)}$ such that, independently, each edge in the first copy of $\mathbb{E}_0^{\partial,(r)}$ is open with probability $p-\varepsilon$, each edge in the second copy of $\mathbb{E}_0^{\partial,(r)}$ is open with probability $2\varepsilon/(1-p+\varepsilon)$, and each edge in $\mathbb{E}_0^{\text{in},(r)}$ is open with probability t. For every $z \in S$, the application $t \mapsto P_{p,\varepsilon,t}(z)$ is a polynomial, therefore it is continuous on $(0,1)$.

In this context, we consider $\bar{x} = (\bar{x}^{\partial,(r),1}, \bar{x}^{\partial,(r),2}, \bar{x}^{\text{in},(r)}) \in S$, where $\bar{x}^{\partial,(r),1}(e) = 0$ for every edge e in the first copy of $\mathbb{E}_0^{\partial,(r)}$, $\bar{x}^{\partial,(r),2}(e) = 1$ for every edge e in the second copy of $\mathbb{E}_0^{\partial,(r)}$, and $\bar{x}^{\text{in},(r)}$ is defined according to the following rule:

Let $G^{(r)} = (U^{(r)}, E^{(r)})$ be the subgraph of G specified in (1.1) and recall that $L_r = |U^{(r)}|$. Define $\Delta U^{(r)} := \{v \in V : \text{dist}_G(v, U^{(r)}) = 1\}$ and assume that the vertices $w_1, \ldots, w_{L_r} \in U^{(r)}$ are enumerated so that

$$\text{dist}_G\big(w_j, \Delta U^{(r)}\big) \le \text{dist}_G\big(w_{j+1}, \Delta U^{(r)}\big) \quad \forall j = 1, \ldots, L_r - 1.$$

For a fixed $j = 1, \ldots, L_r - 1$, choose a vertex $w_j' \in \Delta U^{(r)}$ such that $\text{dist}_G(w_j, \Delta U^{(r)}) = \text{dist}_G(w_j, w_j')$, and a shortest path $\gamma_j = \{w_j = x_1, x_2, \ldots, x_k = w_j'\}$ from w_j to w_j', both of them specified according to some predefined order. Let

$$\Gamma_j := \gamma_j \cap U^{(r)} = \gamma_j \setminus \{w_j'\},$$

and, for $m, m' \in \mathbb{N}$, $m < m'$, denote

$$W_m^{m'}(j) := \{(w_j, m), (w_j, m+1), \ldots, (w_j, m')\}.$$

We set $\bar{x}^{\text{in},(r)}(e) = 1$ if and only if

$$e \subset \big(U^{(r)} \times \{L_r + 1\}\big), \tag{1.6}$$

or, for some $j \in \{1, \ldots, L_r\}$, we have

$$e \subset \big[W_0^j(j) \cup (\Gamma_j \times \{j\})\big] \cup \big[W_{2L_r+2-j}^{2L_r+2}(j) \cup (\Gamma_j \times \{2L_r + 2 - j\})\big]. \tag{1.7}$$

Thus, note that, for any $q \in (0,1)$, we have $P_{p,\varepsilon,q}(\bar{x}) > 0$. Hence, Lemma 1.4 implies the existence of $\eta^{(r)} = \eta(p, \varepsilon, q, \mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)}) > 0$, such that if $\delta \in (0, \eta^{(r)})$, then there exists a coupling

$$X = \big(X_0^{\partial,(r),1}, X_0^{\partial,(r),2}, X_0^{\text{in},(r)}\big), \quad Y = \big(Y_0^{\partial,(r),1}, Y_0^{\partial,(r),2}, Y_0^{\text{in},(r)}\big),$$

where $X, Y \in S$ possess the following properties:

- The values of $X_0^{\partial,(r),1}, X_0^{\partial,(r),2}, X_0^{\text{in},(r)}$ are independent on all edges, and the same is true for $Y_0^{\partial,(r),1}, Y_0^{\partial,(r),2}, Y_0^{\text{in},(r)}$;
- $X_0^{\partial,(r),1}$ and $Y_0^{\partial,(r),1}$ assign each edge of the first copy of $\mathbb{E}_0^{\partial,(r)}$ to be open with probability $p-\varepsilon$;
- $X_0^{\partial,(r),2}$ and $Y_0^{\partial,(r),2}$ assign each edge of the second copy of $\mathbb{E}_0^{\partial,(r)}$ to be open with probability $2\varepsilon/(1-p+\varepsilon)$;
- $X_0^{\text{in},(r)}$ and $Y_0^{\text{in},(r)}$ assign each edge of $\mathbb{E}_0^{\text{in},(r)}$ to be open with probabilities $q+\delta$ and $q-\delta$, respectively;
- $\mathbb{P}(\{X=Y\} \cup \{X=\bar{x}\} \cup \{Y=\bar{x}\}) = 1$.

Now, let $\omega_0^{(r)}, {\omega'}_0^{(r)} \in (\Omega_0^{\partial,(r)} \times \Omega_0^{\text{in},(r)}) = \Omega_0^{(r)}$ be given by

$$\omega_0^{(r)} = \big(X_0^{\partial,(r),1}, X_0^{\text{in},(r)}\big), \quad {\omega'}_0^{(r)} = \big(Y_0^{\partial,(r),1} \vee Y_0^{\partial,(r),2}, Y_0^{\text{in},(r)}\big),$$

and define $\mu_0^{(r)}$ to be the distribution of the pair $(\omega_0^{(r)}, {\omega'}_0^{(r)})$. The first four properties of X and Y listed above imply that $\omega_0^{(r)} \overset{(d)}{=} P_{p-\varepsilon,q+\delta}|_{\mathbb{E}_0^{(r)}}$ and ${\omega'}_0^{(r)} \overset{(d)}{=} P_{p+\varepsilon,q-\delta}|_{\mathbb{E}_0^{(r)}}$, so that the first two properties listed in the statement of Lemma 1.5 are satisfied. To show that $C_0^{(r)}(A,\omega_0^{(r)}) \subset C_0^{(r)}(A,{\omega'}_0^{(r)})$ for every $A \in \partial\mathbb{V}_0^{(r)}$ almost surely, it suffices to check that this property holds in the event $\{X=Y\} \cup \{X=\bar{x}\} \cup \{Y=\bar{x}\}$ of probability one. As a matter of fact,

- If $X=Y$, then $\omega_0^{(r)}(e) \le {\omega'}_0^{(r)}(e)$ for every $e \in \mathbb{E}_0^{(r)}$, so that the property immediately follows.
- If $X=\bar{x}$, then $\omega_0^{(r)} = (\mathbf{0}, \bar{x}^{\text{in},(r)}) \in (\Omega_0^{\partial,(r)} \times \Omega_0^{\text{in},(r)})$. The only open edges in this configuration are those indicated in (1.6) and (1.7), which are not capable of connecting any two vertices of $\partial\mathbb{V}_0^{(r)}$. Therefore, $C_0^{(r)}(A,\omega_0^{(r)}) = A \subset C_0^{(r)}(A,{\omega'}_0^{(r)})$ for every $A \subset \partial\mathbb{V}_0^{(r)}$.
- If $Y=\bar{x}$, then ${\omega'}_0^{(r)} = (\mathbf{1}, \bar{x}^{\text{in},(r)}) \in (\Omega_0^{\partial,(r)} \times \Omega_0^{\text{in},(r)})$. Since in this configuration every edge of $\mathbb{E}_0^{\partial,(r)}$ and every edge indicated in (1.7) is open, any vertex of $\partial\mathbb{V}_0^{(r)}$ is connected to $U^{(r)} \times \{L_r+1\}$. By (1.6), every edge inside this set is also open, so that $C_0^{(r)}(A,{\omega'}_0^{(r)}) = \partial\mathbb{V}_0^{(r)} \supset C_0^{(r)}(A,\omega_0^{(r)})$ for every non-empty subset $A \subset \partial\mathbb{V}_0^{(r)}$.

Thus, we conclude the proof of Lemma 1.5. □

The key fact that allows us to extend the results in [23] to the model defined in Section 1.2 is our main contribution to this study and the last ingredient used in the proof of Proposition 1.2.

Lemma 1.6 (Same coupling parameter for all cylinders)**.** *If* $\sup_{r\in\mathbb{N}}|U^{(r)}| < \infty$*, then for any* $\varepsilon > 0$ *fixed, the sequence* $\{\eta^{(r)}\}_{n\in\mathbb{N}}$ *in Lemma 1.5 may be chosen bounded away from* 0*.*

Proof. From Lemma 1.5, it follows that, for every $r \in \mathbb{N}$, the value of $\eta^{(r)} > 0$ depends on the choice of q, p, ε and the graph $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$. Note that while the values of q, p and ε are the same for every $r \in \mathbb{N}$, the graphs $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$ may differ. However, there are only a finite number of graphs that $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$ can assume. As a matter of fact, recalling that $\Delta U^{(r)} = \{v \in V : \mathrm{dist}_G(v, U^{(r)}) = 1\}$, one may observe by definition (1.3) that $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$ is constructed from the vertex set $U^{(r)} \cup \Delta U^{(r)}$ and from the edges with both endpoints within $U^{(r)} \cup \Delta U^{(r)}$. Since $\sup_{r\in\mathbb{N}}|U^{(r)}| < \infty$ and G has bounded degree, we have $M = \sup_{r\in\mathbb{N}}|U^{(r)} \cup \partial U^{(r)}| < \infty$. Since there are only a finite number of graphs of bounded degree with at most M vertices, the claim regarding $(\mathbb{V}_0^{(r)}, \mathbb{E}_0^{(r)})$ follows, that is, $\eta := \inf_{r\in\mathbb{N}} \eta^{(r)} > 0$. □

Proof of Proposition 1.2. Lemmas 1.5 and 1.6 imply the following result: let $p, q \in (0, 1)$ and $\lambda = \min(p, 1 - p)$. For any $\varepsilon \in (0, \lambda)$, there exists $\eta > 0$ such that if $\delta \in (0, \eta)$, there is a family of couplings $\{\mu_n^{(r)}\}_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}}$, with each $\mu_n^{(r)} = (\omega_n^{(r)}, {\omega'}_n^{(r)})$ defined on $\Omega_n^{(r)} \times \Omega_n^{(r)}$ and having the following property:

- $\omega_n^{(r)} \overset{(d)}{=} P_{p-\varepsilon,q+\delta}|_{\mathbb{E}_n^{(r)}}$;
- ${\omega'}_n^{(r)} \overset{(d)}{=} P_{p+\varepsilon,q-\delta}|_{\mathbb{E}_n^{(r)}}$;
- $C_n^{(r)}(A, \omega_n^{(r)}) \subset C_n^{(r)}(A, {\omega'}_n^{(r)})$ for every $A \in \partial\mathbb{V}_n^{(r)}$ almost surely.

Let μ_O be the coupling of Lemma 1.3 and define the coupling $\mu = (\omega, \omega')$ on Ω^2 by

$$\mu = \mu_O \times \prod_{\substack{r\in\mathbb{N}\\ n\in\mathbb{Z}}} \mu_n^{(r)}.$$

Thus, it is clear that $\omega \overset{(d)}{=} P_{p-\varepsilon,q+\delta}$, $\omega' \overset{(d)}{=} P_{p+\varepsilon,q-\delta}$ and, almost surely, for every $v \in V \setminus (\cup_{r\in\mathbb{N}} U^{(r)})$, if $(v, 0) \leftrightarrow \infty$ in ω, then $(v, 0) \leftrightarrow \infty$ in ω'. □

2 Percolation on $\mathbb{Z}^d$ with a sublattice of inhomogeneities: The uniqueness problem

2.1 Overview of the chapter

Let $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, $d \geq 3$, be the d-dimensional hypercubic lattice. In this chapter, we consider the inhomogeneous Bernoulli percolation model on $\mathbb{L}^d$ in which every edge inside the s-dimensional subspace $\mathbb{Z}^s \times \{0\}^{d-s}$, $2 \leq s < d$, is open with probability q and every other edge is open with probability p. We prove the uniqueness of the infinite cluster in the supercritical regime whenever $p \neq p_c(d)$, where $p_c(d)$ denotes the threshold for homogeneous percolation.

For homogeneous percolation on the d-dimensional lattice, major contributions to this topic are those of Aizenman, Kesten and Newman [1] and Burton and Keane [7]. An extension of the latters' argument to more general graphs can be found in the book of Lyons and Peres [20], where the authors make use of minimal spanning forests to establish the uniqueness of the infinite cluster under the conditions of amenability of the graph, insertion-tolerance of the process and invariance of the percolation measure under a transitive group of automorphisms (these terms are precisely defined in Section 2.3). In Section 2.2, we define the model rigorously, state the main result and discuss the caveats of our setting that we must overcome. When $p \neq q$, we shall see that the lack of invariance of the percolation measure under a transitive group of automorphisms of $\mathbb{L}^d$ plays against the direct application of existing techniques. Therefore, we explore some specific features of our model, which in turn allow us to properly adapt the known arguments to deal with the inhomogeneous process.

In Section 2.3, we develop the general background used throughout the chapter, introducing the notions of automorphisms, transitivity, invariance, ergodicity and insertion tolerance, as well as establishing the essential facts for dealing with the uniqueness problem. In Sections 2.4 and 2.5, we provide the proof of the uniqueness of the infinite cluster in the supercritical phase of the parameters (p, q), respectively when $p < p_c(d)$ and $p > p_c(d)$. Although the techniques used in both cases are unable to provide a proof for $p = p_c(d)$, we conjecture that uniqueness also holds in this case.

2.2 Definition of the model and main result

From now on, the percolation process we are going to consider is the one studied by Iliev, Janse van Rensburg and Madras [18], which consists of Bernoulli bond percolation on the d-dimensional lattice, with an s-dimensional sublattice of inhomogeneities. Formally speaking, given $d \geq 3$, let $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, where $\mathbb{E}^d = \{\{x, y\} \subset \mathbb{Z}^d : \|x - y\|_1 = 1\}$ and $\|x - y\|_1 = \sum_{i=1}^{d} |x_i - y_i|$. For $2 \leq s < d$, define $H \coloneqq \mathbb{Z}^s \times \{0\}^{d-s}$ and $\mathsf{E}_H \coloneqq \{e \in \mathbb{E}^d : e \subset H\}$. Let $\Omega = \{0, 1\}^{\mathbb{E}^d}$ and $\mathcal{F}$ be the σ-algebra generated by the finite-dimensional cylinder sets of Ω. For $p, q \in [0, 1]$, the governing probability measure of the process is the product measure on $(\Omega, \mathcal{F})$ given by $P_{p,q} \coloneqq \prod_{e \in \mathsf{E}_H} b(q) \times \prod_{e \in \mathbb{E}^d \setminus \mathsf{E}_H} b(p)$, where $b(\alpha)$ denotes the Bernoulli measure with parameter $\alpha \in [0, 1]$. That is, each edge of E_H is open with probability q and each edge of $\mathbb{E}^d \setminus \mathsf{E}_H$ is open with probability p, independently of any other edge.

In [18], the authors generalized several classical results of homogeneous bond percolation to this inhomogeneous setting. Besides, they presented the phase-diagram for percolation and showed that the critical curve $p \mapsto q_c(p)$ is strictly decreasing for $p \in [0, p_c(d)]$, where $p_c(d)$ is the threshold for homogeneous Bernoulli bond percolation on $\mathbb{L}^d$. This is particularly interesting since it guarantees the existence of a set of parameters $(p, q) \in [0, 1]^2$ such that $p < p_c(d) < q < p_c(s)$ and there is an infinite cluster $P_{p,q}$-almost surely. In what follows, we shall prove the uniqueness of the infinite cluster in the supercritical phase, a result that has not yet been considered for the model described above.

The definitions of open paths, connectivity between vertices and percolation of vertices are present in the Section "**Basic definitions and notations**", in the **Introduction**. Besides, we will need to work with the notion of amenability; that is, a graph $G = (V, E)$ is ***amenable*** if there exists a sequence $\{V_n\}_{n \in \mathbb{N}}$ of finite subsets of V such that $|\Delta_v V_n| / |V_n| \to 0$ as $n \to \infty$, where $\Delta_v V_n \coloneqq \{x \in V \setminus V_n : \exists y \in V_n \text{ s.t. } \{x, y\} \in E\}$. For $\omega \in \Omega$, we recall that the cluster of a vertex $x \in \mathbb{Z}^d$ is denoted by $C(x)$. Since we are interested in investigating how many infinite clusters exist in the supercritical phase for a given configuration $\omega \in \Omega$, we define the ***number of infinite components*** of ω as the random variable $N_\infty : \Omega \to \mathbb{N} \cup \{\infty\}$, given by

$$N_\infty \coloneqq |\{C(x) : x \in \mathbb{Z}^d, |C(x)| = \infty\}|.$$

Additionally, for any $d \in \mathbb{N}$, let $p_c(d) := \sup\{p : P_{p,p}(o \leftrightarrow \infty \text{ in } \mathbb{L}^d) = 0\}$. The main result of this chapter is the following:

Theorem 2.1 (Uniqueness of the infinite cluster)**.** *If $p \neq p_c(d)$, then $N_\infty \in \{0, 1\}$ $P_{p,q}$-a.s. for every $q \in [0, 1]$.*

Before we move on to the proof, let us briefly discuss the issues that appear in our model and are not covered by the existing literature regarding the determination of the number of infinite components. For a precise definition of the terminology used here, see Section 2.3. On amenable graphs such as $\mathbb{L}^d$, there is an important property used in [7] and [20] which plays a key role to determine the uniqueness of the infinite component in the supercritical phase, namely the invariance of the percolation measure under a transitive group of automorphisms of the graph. Under this condition, assuming $N_\infty = \infty$, one can find a positive lower bound for the probability of any vertex $x \in \mathbb{Z}^d$ to be a trifurcation. This fact, together with the observation that the number of trifurcations lying inside any box of $\mathbb{Z}^d$ cannot exceed the size of its boundary, implies the non-amenability of the graph, a contradiction. However, in our model, the group of automorphisms for which $P_{p,q}$ is invariant is not transitive, hence the above argument cannot be applied. As a matter of fact, if $p < p_c(d)$ and $P_{p,q}(o \leftrightarrow \infty) > 0$, the probability that a vertex x is a trifurcation decays exponentially fast with the distance between x and H, which leads us to the conclusion that the expected number of trifurcations in a box of length n is of order n^s, yielding no contradiction with the fact that the size of the boundary of the box is of order n^{d-1}. On the other hand, when $p > p_c(d)$, we must ensure that setting the parameter q to any value other than p does not cause the appearance of any new infinite cluster around the subspace H. We shall circumvent these difficulties by exploring additional properties of the percolation measure $P_{p,q}$.

The proof of Theorem 2.1 is divided in two cases, namely the case when $p < p_c(d)$ and the case when $p > p_c(d)$, because different techniques are used in each situation. To deal with the case $p < p_c(d)$, we first develop some results regarding a less restrictive bond percolation process $\mathbf{P}$ on a graph $G = (V, E)$, that comprises the process $P_{p,q}$ on the lattice $\mathbb{L}^d$ as a particular instance, and then show the impossibility of having more than one infinite cluster in the supercritical phase for the inhomogeneous percolation model. Here, our argument is an adaptation of the use of minimal spanning forests as in Chapter 7 of [20], together with the exponential decay of the probability of the one-arm event for subcritical homogeneous percolation, derived by Menshikov [21], Aizenman and Barsky [2] and Duminil-Copin and

Tassion [11]. When $p > p_c(d)$ and $q \in [0,1]$, we make use of the so-called mass transport principle as in Häggström and Peres [16], to show that, under the increasing coupling of percolation configurations induced by the family $\{U(e) : e \in \mathbb{E}^d\}$ of i.i.d. random variables having uniform distribution in $[0,1]$, each infinite cluster of the inhomogeneous process on $\mathbb{Z}^d$ with parameters (p,q) contains an infinite cluster of the homogeneous percolation process on the half-space $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$ with parameter p. As we mentioned earlier, although the techniques used in both cases are unable to provide a proof for $p = p_c(d)$, we conjecture that uniqueness also holds in this case.

2.3 General background

We begin with some definitions. A ***(vertex)-automorphism*** of a graph $G = (V, E)$ is a bijection $g : V \to V$ such that $\{g(u), g(v)\} \in E$ if and only if $\{u, v\} \in E$. We write $\mathrm{Aut}(G)$ for the group of automorphisms of G. Given a subgroup $\Gamma \subset \mathrm{Aut}(G)$, we say that Γ ***acts transitively*** on G if, for any $u, v \in V$, we have $g(u) = v$ for some $g \in \Gamma$. We say that G is ***transitive*** if $\mathrm{Aut}(G)$ itself acts transitively on G.

For any bond percolation process $(\Omega, \mathcal{F}, \mathbf{P})$ on $G = (V, E)$, note that every $g \in \mathrm{Aut}(G)$ induces a transformation $\hat{g} : \Omega \to \Omega$, given by

$$[\hat{g}(\omega)](\{u,v\}) = \omega(\{g^{-1}u, g^{-1}v\}), \quad \{u,v\} \in E.$$

We say that $\mathbf{P}$ is **Γ*-invariant*** if $\mathbf{P}(\hat{g}A) = \mathbf{P}(A)$ for every $A \in \mathcal{F}$ and $g \in \Gamma$.

Now, let $\mathcal{I}_\Gamma := \{A \in \mathcal{F} : \hat{g}A = A, \forall\, g \in \Gamma\}$. That is, $\mathcal{I}_\Gamma \subset \mathcal{F}$ is the σ-field of events of $\mathcal{F}$ that are invariant under the action of all elements of Γ. We call the measure $\mathbf{P}$ **Γ*-ergodic*** if $\mathbf{P}(A) \in \{0,1\}$ for every $A \in \mathcal{I}_\Gamma$.

Finally, given $\omega \in \Omega$ and $F \subset E$, let

$$\Pi_F\omega(e) := \begin{cases} 1, & \text{if } e \in F, \\ \omega(e), & \text{if } e \notin F. \end{cases}$$

That is, $\Pi_F\omega \in \Omega$ is the configuration obtained by opening the edges of F in ω. We also denote by $\Pi_{\neg F}\omega$ the configuration obtained by closing the edges of F in ω (the same expression as above, but with 0 in place of 1). For any event $A \in \mathcal{F}$, we define $\Pi_F A := \{\Pi_F\omega : \omega \in A\}$ and $\Pi_{\neg F}A := \{\Pi_{\neg F}\omega : \omega \in A\}$.

A bond percolation process $\mathbf{P}$ on G is ***insertion tolerant*** (resp. ***deletion tolerant***) if we have $\mathbf{P}(\Pi_F A) > 0$ (resp. $\mathbf{P}(\Pi_{\neg F} A) > 0$) for any finite subset $F \subset E$ and any event $A \in \mathcal{F}$ satisfying $\mathbf{P}(A) > 0$. If a process is both insertion and deletion tolerant, it is said to have the ***finite-energy property***.

Having defined all the relevant concepts, from now on we regard $\mathbf{P}$ as an insertion-tolerant bond percolation process on $G = (V, E)$, which is invariant and ergodic for some subgroup $\Gamma \subset \mathrm{Aut}(G)$. Moreover, for $S \subset V$, define $\mathsf{E}_S := \{e \in E : e \subset S\}$, $\Gamma|_S := \{g|_S : g \in \Gamma\}$ and $C_S(u) := C(u) \cap S$. We shall also require that there exists a set $S \subset V$, such that $\Gamma|_S$ acts transitively on the subgraph (S, E_S) and

$$\mathbf{P}(|C(u)| = \infty, |C_S(u)| < \infty) = 0 \quad \text{for every } u \in V. \tag{2.1}$$

One can note that $P_{p,q}$ is a process of the above kind: as a matter of fact, $P_{p,q}$ is invariant under the subgroup Γ of translations of $\mathbb{Z}^d$ parallel to the subspace $H = \mathbb{Z}^s \times \{0\}^{d-s}$, and insertion-tolerance comes from the fact that the states of the edges of $\mathbb{E}^d$ are independent of each other. A proof of condition (2.1) with $S = H$ is postponed to the later sections. As for the ergodicity of $P_{p,q}$ under Γ, the argument for Bernoulli percolation is canonical: let $A \in \mathcal{I}_\Gamma$ and $\varepsilon > 0$. Then, there is a cylinder event $B \in \mathcal{F}$ such that $P_{p,q}(A \triangle B) < \varepsilon$. Since $P_{p,q}$ is invariant under Γ, we have $P_{p,q}(\gamma A \triangle \gamma B) = P_{p,q}(\gamma(A \triangle B)) < \varepsilon$ for every $\gamma \in \Gamma$. If $F \subset \mathbb{E}^d$ is the finite set of edges which determines the event B, then there exists a translation $\gamma \in \Gamma$ such that F and $\gamma F = \{\{\gamma x, \gamma y\} : \{x, y\} \in F\}$ are disjoint. In turn, it follows that B and γB are independent. Since for any events C_1, C_2 and D, we have

$$|P_{p,q}(C_1 \cap D) - P_{p,q}(C_2 \cap D)| \leq P_{p,q}([C_1 \cap D] \triangle [C_2 \cap D]) \leq P_{p,q}(C_1 \triangle C_2),$$

we conclude that

$$\begin{aligned}
|P_{p,q}(A) - P_{p,q}(A)^2| &= |P_{p,q}(A \cap \gamma A) - P_{p,q}(A)^2| \\
&\leq |P_{p,q}(A \cap \gamma A) - P_{p,q}(B \cap \gamma A)| + |P_{p,q}(B \cap \gamma A) - P_{p,q}(B \cap \gamma B)| \\
&\qquad + |P_{p,q}(B \cap \gamma B) - P_{p,q}(B)^2| + |P_{p,q}(B)^2 - P_{p,q}(A)^2| \\
&\leq P_{p,q}(A \triangle B) + P_{p,q}(\gamma A \triangle \gamma B) + |P_{p,q}(B) P_{p,q}(\gamma B) - P_{p,q}(B)^2| \\
&\qquad + (P_{p,q}(A) + P_{p,q}(B))|P_{p,q}(B) - P_{p,q}(A)| \\
&< \varepsilon + \varepsilon + 0 + 2\varepsilon,
\end{aligned}$$

which implies that $P_{p,q}(A) \in \{0, 1\}$.

For a percolation process $\mathbf{P}$ as above, Newman and Schulman [22] proved that the we cannot have $N_\infty \in \{2, 3, \dots\}$ with positive probability. This is expressed by the following result:

Theorem 2.2 (Theorem 7.5 of [20]). *Let $G = (V, E)$ be a connected graph. For any $S \subset V$ and $\Gamma \subset \mathrm{Aut}(G)$, define $\mathsf{E}_S := \{e \in E : e \subset S\}$ and $\Gamma|_S := \{g|_S : g \in \Gamma\}$. Let $\mathbf{P}$ be an insertion-tolerant bond percolation process on G, such that there exist an infinite connected set $S \subset V$ and a subgroup $\Gamma \subset \mathrm{Aut}(G)$ with the following properties:*

i. *$\mathbf{P}$ is invariant and ergodic under Γ;*

ii. *$\Gamma|_S$ acts transitively on the subgraph (S, E_S).*

Then $N_\infty \in \{0, 1, \infty\}$ $\mathbf{P}$-a.s..

Proof. Note that the action of any element of Γ on a configuration $\omega \in \Omega$ does not change the value of N_∞. Hence, N_∞ is measurable with respect to $\mathcal{I}_\Gamma$ and, by ergodicity, it is constant $\mathbf{P}$-a.s.. If $N_\infty \in \{2, 3, \dots\}$ almost surely, then, there exist $x, y \in V$ such that the event $A = \{|C(x)| = \infty\} \cap \{|C(y)| = \infty\} \cap \{C(x) \cap C(y) = \emptyset\}$ has positive probability. Let $\{x = x_1, x_2, \dots, x_n = y\}$ be a path from x to y and F the set of edges within it. By insertion tolerance, we have both $\mathbf{P}(A) > 0$ and $\mathbf{P}(\Pi_F A) > 0$. But this contradicts the fact that N_∞ is constant almost surely, since $N_\infty(\Pi_F \omega) < N_\infty(\omega)$ for every $\omega \in A$. Therefore, $N_\infty \in \{0, 1, \infty\}$ $\mathbf{P}$-a.s.. □

Thus, what comes next is intended to rule out the case $N_\infty = \infty$, using a similar approach to Theorem 7.9 of [20]. We emphasize that, unless $p = q$, this result cannot be applied directly in the present situation: if $p \neq q$, the only subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{L}^d)$ for which $P_{p,q}$ is invariant is that of the translations parallel to the subspace H, and Γ does not act transitively on $\mathbb{L}^d$, as required by the theorem.

First, we introduce some sets of vertices and edges of a graph $G = (V, E)$ that will be needed in our proof. For a subset $K \subset V$ and a subgraph $G' = (V', E') \subset G$, we define the ***exterior vertex boundary*** of K in G' and the ***exterior edge boundary*** of K in G' respectively as the sets

$$\Delta_v^{G'} K := \{y \in V' \setminus K : \exists x \in K \text{ such that } \{x, y\} \in E'\},$$

$$\Delta_e^{G'} K := \{\{x, y\} \in E' : x \in K, y \in V' \setminus K\}.$$

In particular, $\Delta_v K := \Delta_v^G K$ and $\Delta_e K := \Delta_e^G K$. For any vertex $u \in V$, we define the ***degree of the vertex*** u in K as the number $\deg_K(u) := |\Delta_v\{u\} \cap K|$. We also write $\deg(u) := \deg_V(u)$.

The relation between these sets we are going to use is expressed in the next result, which is Exercise 7.3 of [20].

Lemma 2.3. *Let $T = (V_T, E_T)$ be a tree with* $\deg(u) \geq 2$ *for all $u \in V_T$ and consider the set $B := \{u \in V_T : \deg(u) \geq 3\}$. Then, for every finite set $K \subset V_T$, we have*

$$|\Delta_v K| \geq |K \cap B| + 2. \tag{2.2}$$

Proof. We proceed by induction on $|K|$. If $|K| = 1$, then $K = \{u\} \subset V_T$. Hence, $|\Delta_v K| = \deg(u) \geq |K \cap B| + 2$ and the result follows easily. Now, suppose (2.2) holds for every $K \subset V_T$ with $|K| = n \in \mathbb{N}$.

If $|K| = n + 1$, choose $u \in K$ such that $\deg_K(u) \leq 1$ (this leaf always exists because K is finite). Hence, $|K \setminus u| = n$ and, by the induction hypothesis, we have

$$|\Delta_v (K \setminus u)| \geq |(K \setminus u) \cap B| + 2. \tag{2.3}$$

We now consider the following cases:

If $\deg_K(u) = 0$, then

- If $u \in B$, then $|\Delta_v K| \geq |\Delta_v (K \setminus u)| + 3$ and $|K \cap B| = |(K \setminus u) \cap B| + 1$.
- If $u \notin B$, then $|\Delta_v K| \geq |\Delta_v (K \setminus u)| + 2$ and $|K \cap B| = |(K \setminus u) \cap B|$.

If $\deg_K(u) = 1$, then

- If $u \in B$, then $|\Delta_v K| \geq |\Delta_v (K \setminus u)| + 1$ and $|K \cap B| = |(K \setminus u) \cap B| + 1$.
- If $u \notin B$, then $|\Delta_v K| \geq |\Delta_v (K \setminus u)|$ and $|K \cap B| = |(K \setminus u) \cap B|$.

Thus, one can easily note that the bound (2.3) implies (2.2) in all the above-mentioned cases. □

Until the end of this section, it will be useful to keep in mind the correspondence between the space $\Omega = \{0, 1\}^E$ and the set of the subgraphs of $G = (V, E)$ induced by their open edges. We shall regard the configurations of Ω in both ways, referring to the most convenient manner when necessary.

We now state a version of Lemma 7.7 of [20], specifically designed to deal with inhomogeneous percolation on $\mathbb{Z}^d$ with a sublattice of defects and similar models. The proof of this version is carried out in the same way as that of its counterpart in [20], with minor modifications. For the statement of the lemma, we need the following definition: a vertex $u \in V$ is called a ***furcation*** of a configuration $\omega \in \Omega$ if u percolates in ω and removing all edges incident to u splits $C(u)$ into at least three distinct infinite clusters. The ***set of furcations*** of a configuration ω will be denoted by $\Lambda(\omega)$. For $S \subset V$, recall that $C_S(u) := C(u) \cap S$.

Lemma 2.4. *Let $G = (V, E)$ be a connected graph and $\mathbf{P}$ be an insertion-tolerant bond percolation process on G. Suppose there exist a subgroup $\Gamma \subset \mathrm{Aut}(G)$ and a (necessarily infinite) connected set $S \subset V$ such that*

i. $\mathbf{P}$ *is invariant under* Γ*;*

ii. $\mathbf{P}(|C(u)| = \infty, |C_S(u)| < \infty) = 0$ *for every* $u \in V$.

If $\mathbf{P}(\omega : N_\infty(\omega) = \infty) > 0$*, then there exists, on a larger probability space* $(\widetilde{\Omega}, \widetilde{\mathbf{P}})$*, a coupling* $(\mathfrak{F}, \omega)$ *with the following properties:*

a. $\mathfrak{F} \subset \omega$ *and* $\mathfrak{F}$ *is a random forest;*

b. *The distribution of the pair* $(\mathfrak{F}, \omega)$ *is* Γ*-invariant;*

c. $\widetilde{\mathbf{P}}(\Lambda(\mathfrak{F}) \cap S \neq \emptyset) > 0$.

Proof. Let $\{U(e) : e \in E\}$ be independent random variables with uniform distribution on $[0, 1]$ and let P_U be the associated product measure on $[0, 1]^E$. On the product space $\widetilde{\Omega} = [0, 1]^E \times \{0, 1\}^E$, consider $\widetilde{\mathbf{P}} := P_U \times \mathbf{P}$ and let $\mathfrak{F} : \widetilde{\Omega} \to \{0, 1\}^E$ be the random element of Ω given by

$$\mathfrak{F}(U, \omega)(e) = \begin{cases} 1, & \text{if } \omega(e) = 1 \text{ and } \nexists \text{ cycle } \gamma, e \in \gamma \subset \omega \text{ s.t. } U(e) = \max_{e' \in \gamma} U(e'), \\ 0, & \text{otherwise.} \end{cases}$$

The graph associated with $\mathfrak{F}(U, \omega)$ is called the ***free minimal spanning forest*** of (U, ω). By definition, $\mathfrak{F} \subset \omega$ and $\mathfrak{F}$ has no cycles, since for every cycle in ω, the edge with the largest value of $U(e)$ is not in $\mathfrak{F}$. Moreover, the Γ-invariance of P_U and $\mathbf{P}$ implies that the distribution of the pair $(\mathfrak{F}, \omega)$ is Γ-invariant as well. We have thus proved properties ***a*** and ***b***.

In order to establish ***c***, we first show that every component (tree) of $\mathfrak{F}$ that lies in an infinite component of ω is $\widetilde{\mathbf{P}}$-a.s. infinite. As a matter of fact, if $T = (V_T, E_T)$ is a finite component of $\mathfrak{F}$ and $T \subset \eta = (V_\eta, E_\eta)$ for some infinite component $\eta \subset \omega$, then there exists an edge $f \in [\Delta_e V_T] \cap E_\eta$ such that $U(f) = \min_{e' \in [\Delta_e V_T] \cap E_\eta} U(e')$. Such an edge is $\widetilde{\mathbf{P}}$-a.s. unique, since the uniform random variables are almost surely distinct in every edge. If $\gamma \subset \eta$ is a cycle containing f, then there exists $f' \in [\Delta_e V_T] \cap E_\eta$, $f' \neq f$, such that $f' \subset \gamma$. Since $U(f') > \min_{e' \in [\Delta_e V_T] \cap E_\eta} U(e') = U(f)$, this implies $f \in E_T$, a contradiction.

Hence, if $\mathbf{P}(\omega : N_\infty(\omega) = \infty) > 0$, there exist $x, y, z \in V$ such that, with positive probability, the components $C(x)$, $C(y)$ and $C(z)$ are infinite and mutually disjoint. Since $\mathbf{P}(|C(u)| = \infty, |C_S(u)| < \infty) = 0$ for every $u \in V$, we may suppose $x, y, z \in S$. For such vertices, let

$$A := \{|C(u)| = \infty, \forall u \in \{x, y, z\}\} \cap \{C(u) \cap C(v) = \emptyset, \forall u, v \in \{x, y, z\}, u \neq v\}$$

and $T = (V_T, E_T)$ be a finite tree with $x, y, z \in V_T \subset S$ (this tree always exists because S is connected). By insertion tolerance, $\mathbf{P}(\Pi_{E_T} A) > 0$ and, on the event $\Pi_{E_T} A$, we have that $V_T \subset C(x)$ and $C(x) \setminus V_T$ has at least three infinite components. Thus, the event consisting of the configurations (U, ω) such that $U(e) < 1/2$ for every $e \in E_T$, $U(f) \geq 1/2$ for every $f \in \Delta_e V_T$ and $\omega \in \Pi_{E_T} A$ has positive probability $\widetilde{\mathbf{P}}$. On this event, $\mathfrak{F}$ contains T and there is some vertex in $V_T \subset S$ which is a furcation of $\mathfrak{F}$. □

When $\mathbf{P}$ is insertion-tolerant and invariant under $\mathrm{Aut}(G)$, the uniqueness of the infinite cluster is established for amenable graphs by proving the claim that if $\mathbf{P}(\omega : N_\infty(\omega) = \infty) > 0$, then G is non-amenable, see for example Theorems 7.6 and 7.9 of [20]. What we shall exhibit in the next result is a simple and straightforward generalization of this fact. It will help us to make a proper argument regarding the uniqueness of the infinite cluster in the inhomogeneous percolation model on $\mathbb{Z}^d$ with a sublattice of defects, which is not invariant under $\mathrm{Aut}(\mathbb{L}^d)$. Its proof is carried out in much the same way as the theorems mentioned above.

For a graph $G = (V, E)$, let $S', S \subset V$ with $|S'| < \infty$, and define

$$C(S', S) := \{u \in S' : \exists v \in S \text{ such that } v \leftrightarrow u\}.$$

Lemma 2.5 (Criterion for non-amenability). *Let $G = (V, E)$ be a connected graph and $\mathbf{P}$ be an insertion-tolerant bond percolation process on G. Suppose there exist a subgroup $\Gamma \subset \mathrm{Aut}(G)$ and a connected set $S \subset V$, such that the following conditions hold:*

i. $\mathbf{P}$ *is invariant and ergodic under* Γ*;*

ii. $\mathbf{P}(|C(u)| = \infty, |C_S(u)| < \infty) = 0$ *for every* $u \in V$*;*

iii. $\Gamma|_S$ *acts transitively on the subgraph* (S, E_S)*, where* $\mathsf{E}_S := \{e \in E : e \subset S\}$.

If $\mathbf{P}(\omega : N_\infty(\omega) = \infty) > 0$*, then there exists a constant* $c > 0$ *such that, for every finite set* $R \subset V$ *satisfying* $R \cap S \neq \emptyset$*, we have*

$$\frac{\mathbf{E}|C(\Delta_v R; S)|}{|R \cap S|} \geq c. \tag{2.4}$$

Before proving this result, note that if $\mathbf{P}$ is invariant and ergodic under Γ and Γ acts transitively on G, we can take $S = V$ and inequality (2.4) becomes the non-amenability condition for $G = (V, E)$. Besides, we would like to stress the importance of the quantity $\mathbf{E}|C(\Delta_v R; S)|$ in (2.4). If this term is replaced by a larger one, such as $|\Delta_v R|$, then it is not possible to extract any useful information from our percolation model. For instance, if $B_n = \{-n, \dots, n\}^d$ and we consider the inhomogeneous percolation process on $\mathbb{Z}^d$ defined in Section 2.2 for $d = 3$, $H = \mathbb{Z}^2 \times \{0\}$ and $p < p_c(3) < q < p_c(2)$, it follows that there is a constant $c > 0$ such that $|\Delta_v B_n| \geq c|B_n \cap H|$ for all $n \in \mathbb{N}$. Nevertheless, we shall see in the next section that inequality (2.4) does not hold for B_n on such a model, therefore $P_{p,q}(\omega : N_\infty(\omega) = \infty) = 0$.

Proof of Lemma 2.5. Let $\widetilde{\mathbf{P}}$ and $\mathfrak{F}$ be as in Lemma 2.4. Conditions ***i*** – ***iii*** imply that there is a constant $c > 0$ such that $\widetilde{\mathbf{P}}(u \in \Lambda(\mathfrak{F})) = c$ for every $u \in S$. Hence, the expected number of furcations of $\mathfrak{F}$ in $R \cap S$ is

$$\widetilde{\mathbf{E}}|\Lambda(\mathfrak{F}) \cap R \cap S| = \sum_{u \in R \cap S} \widetilde{\mathbf{P}}(u \in \Lambda(\mathfrak{F})) = c|R \cap S|. \tag{2.5}$$

Let $\mathcal{T}$ be the set of the infinite components (trees) of $\mathfrak{F}$. Also, consider the process of inductively removing the leaves of a tree. Applying this process to any $T = (V_T, E_T) \in \mathcal{T}$, we are left, at the end of the procedure, with an infinite tree $T' = (V_{T'}, E_{T'}) \subset T$ that has no leaves and $\Lambda(T') = \{u \in V_{T'} : \deg(u) \geq 3\} = \Lambda(T)$. Thus, an application of Lemma 2.3 with $K = R \cap V_{T'}$ yields

$$\begin{aligned} \left|\Delta_v^T (R \cap V_T)\right| &\geq \left|\Delta_v^{T'} (R \cap V_{T'})\right| \\ &\geq |R \cap V_{T'} \cap \Lambda(T')| = |R \cap \Lambda(T)|. \end{aligned}$$

Observing that $[\Delta_v^T (R \cap V_T)] \subset [\Delta_v R \cap V_T]$ and summing up the above inequality over all trees $T \in \mathcal{T}$, we arrive at

$$|\Delta_v R \cap V_{\mathfrak{F}_\infty}| \geq |R \cap \Lambda(\mathfrak{F}_\infty)| = |R \cap \Lambda(\mathfrak{F})|, \tag{2.6}$$

where $\mathfrak{F}_\infty := \cup_{T \in \mathcal{T}} T$.

Finally, by property **a.** of Lemma 2.4, we have $\mathfrak{F}_\infty \subset \omega_\infty$, where ω_∞ is the union of all the infinite components of ω. Since every vertex of ω_∞ is connected to S by condition ***ii*** and

$\mathbf{P}(N_\infty = \infty) = 1$ by ergodicity, it follows that $\widetilde{\mathbf{P}}$-a.s.

$$|\Delta_v R \cap V_{\mathfrak{F}_\infty}| \le |\Delta_v R \cap V_{\omega_\infty}| \le |C(\Delta_v R; S)|. \tag{2.7}$$

Combining equations (2.6) and (2.7), taking the expectation $\widetilde{\mathbf{E}}$ and using equality (2.5), we conclude that

$$\mathbf{E}|C(\Delta_v R; S)| \ge c|R \cap S|.$$

□

2.4 Proof of Theorem 2.1: the case $p < p_c(d)$

Returning to the inhomogeneous percolation process on $\mathbb{Z}^d$ defined in Section 2.2, we recall that the conditions of Lemma 2.5 are satisfied for $\mathbf{P} = P_{p,q}$ and $S = H = \mathbb{Z}^s \times \{0\}^{d-s}$. In the case $p < p_c(d)$ and $P_{p,q}(N_\infty > 0) = 1$, condition (2.1) is trivially satisfied since there is no infinite cluster on $\mathbb{Z}^d \setminus H$ almost surely. By Theorem 2.2, we then have $N_\infty \in \{0, 1, \infty\}$ $P_{p,q}$-a.s.. However, going in the opposite direction of having infinitely many infinite clusters, we have the following result:

Proposition 2.6 (Violation of non-amenability criterion). *Let $B_n = \{-n, \dots, n\}^d$, $n \in \mathbb{N}$. If $p < p_c(d)$, then*

$$\frac{E_{p,q}|C(\Delta_v B_n; H)|}{|B_n \cap H|} \xrightarrow[n\to\infty]{} 0.$$

Proof. By the exponential decay of the one-arm event in the homogeneous model with parameter $p < p_c(d)$ [2, 11, 21], there exists a positive constant $c_p > 0$ such that $P_{p,q}(u \leftrightarrow H) \le \exp\{-c_p \operatorname{dist}(u, H)\}$ for any vertex $u \in \mathbb{Z}^d$, where $\operatorname{dist}(u, H)$ denotes the graph-theoretical distance between u and H. Therefore, taking $\alpha > (d - s - 1)/c_p$ and observing that $\Delta_v B_{n-1} \subset \partial B_n := B_n \setminus B_{n-1}$, we have

$$\begin{aligned}
E_{p,q}|C(\Delta_v B_{n-1}; H)| &\le E_{p,q}|C(\partial B_n; H)| \\
&= \sum_{\substack{u\in\partial B_n \\ \operatorname{dist}(u,H)<\alpha\log n}} P_{p,q}(u \leftrightarrow H) + \sum_{\substack{u\in\partial B_n \\ \operatorname{dist}(u,H)\ge\alpha\log n}} P_{p,q}(u \leftrightarrow H) \\
&\le C\Big[n^{s-1}(\alpha \log n)^{d-s} + n^{d-1}\exp\{-c_p\alpha\log n\}\Big] \\
&\le C'|B_{n-1} \cap H| \times \left[\frac{(\alpha\log n)^{d-s}}{n} + n^{d-s-1-c_p\alpha}\right],
\end{aligned}$$

for positive constants $C = C(s, d)$ and $C' = C'(s, d)$. Observing that the last term in brackets goes to zero as $n \to \infty$, the result follows. □

As an immediate consequence of Lemma 2.5 and Proposition 2.6, we can rule out the case $N_\infty = \infty$ when $p < p_c(d)$.

Corollary 2.7. *If $p < p_c(d)$ then $N_\infty \in \{0, 1\}$ $P_{p,q}$-a.s..*

2.5 Proof of Theorem 2.1: the case $p > p_c(d)$

To work with the case $p > p_c(d)$, recall that the set of edges whose vertices both belong to the subspace $H = \mathbb{Z}^s \times \{0\}^{d-s}$ is denoted by $\mathsf{E}_H := \{e \in \mathbb{E}^d : e \subset H\}$ and let P be the probability measure associated with the family $\{U(e) : e \in \mathbb{E}^d\}$ of i.i.d. random variables having uniform distribution in $[0, 1]$. Also, consider the decomposition $\mathbb{E}^d = E^+ \cup E^- \cup \mathsf{E}_H$, where $E^+ := \{\{x, y\} \in \mathbb{E}^d : (x_d \vee y_d) > 0\}$ and $E^- := \mathbb{E}^d \setminus (E^+ \cup \mathsf{E}_H)$, and for $p, q, t \in [0, 1]$, let $\omega_{p,q,t} \in \{0, 1\}^{\mathbb{E}^d}$ be the Bernoulli bond percolation process on $\mathbb{L}^d$ given by

$$\omega_{p,q,t}(e) := \begin{cases} \mathbf{1}_{\{U(e)\le p\}} & \text{if } e \in E^+, \\ \mathbf{1}_{\{U(e)\le q\}} & \text{if } e \in \mathsf{E}_H, \\ \mathbf{1}_{\{U(e)\le t\}} & \text{if } e \in E^-. \end{cases}$$

To establish the uniqueness of the infinite cluster when $p > p_c(d)$ and therefore conclude the proof of Theorem 2.1, we make use of the above coupling and the technique used in the proof of Proposition 3.1 of [16] to derive the following result:

Proposition 2.8. *If $p > p_c(d)$ and $q \in [0, 1]$, then $N_\infty = 1$ $P_{p,q}$-a.s..*

The proof of Proposition 2.8 relies on the so-called ***mass transport principle***. As pointed out in [16], it was first used in the percolation setting by Häggström [15] and fully developed by Benjamini, Lyons, Peres and Schramm [5]. For our purposes, it suffices to state a particular version of this principle, based on Theorem 2.1 of [16]:

Theorem 2.9 (The Mass-Transport Principle). *Let $\Gamma \subset \mathrm{Aut}(\mathbb{L}^d)$ be the subgroup of translations parallel to the subspace $H = \mathbb{Z}^s \times \{0\}^{d-s}$. If $(\Omega, \mathbf{P})$ is any Γ-invariant bond percolation process on $\mathbb{L}^d$ and $m(x, y, \omega)$ is a nonnegative function of $x, y \in H$, $\omega \in \Omega$ such that $m(x, y, \omega) = m(\gamma x, \gamma y, \gamma\omega)$ for all x, y and ω and $\gamma \in \Gamma$, then*

$$\sum_{y\in H} \int_\Omega m(x, y, \omega)\, \mathrm{d}\mathbf{P}(\omega) = \sum_{y\in H} \int_\Omega m(y, x, \omega)\, \mathrm{d}\mathbf{P}(\omega) \quad \forall x \in H. \tag{2.8}$$

Proof. By definition of Γ, given $x, y \in H$, there exists a unique $\gamma \in \Gamma$ such that $y = \gamma x$. Since $m(x, y, \omega) = m(\gamma x, \gamma y, \gamma\omega)$ for all x, y and ω and $\gamma \in \Gamma$, the invariance of $\mathbf{P}$ under Γ implies

$$\begin{aligned}\sum_{y\in H}\int_\Omega m(x, y, \omega)\,\mathrm{d}\mathbf{P}(\omega) &= \sum_{\gamma\in\Gamma}\int_\Omega m(x, \gamma x, \omega)\,\mathrm{d}\mathbf{P}(\omega)\\ &= \sum_{\gamma\in\Gamma}\int_\Omega m(\gamma^{-1}x, x, \gamma^{-1}\omega)\,\mathrm{d}\mathbf{P}(\gamma^{-1}\omega)\\ &= \sum_{y\in H}\int_\Omega m(y, x, \omega)\,\mathrm{d}\mathbf{P}(\omega).\end{aligned}$$

□

This result can be viewed as the mass transport principle applied just on the subspace H. To make proper use of this technique, we must establish condition (2.1), regarding the connected component $C(v, \omega_{p,q,t})$ of $v \in \mathbb{Z}^d$ in the configuration $\omega_{p,q,t}$.

Lemma 2.10. *If $p > p_c(d)$ and $q \in [0, 1]$, then for every $v \in H$ we have*

$$P(|C(v, \omega_{p,0,0})| = \infty, |C(v, \omega_{p,0,0}) \cap H| < \infty) = 0, \tag{2.9}$$

$$P(|C(v, \omega_{p,q,p})| = \infty, |C(v, \omega_{p,q,p}) \cap H| < \infty) = 0. \tag{2.10}$$

Proof. As proved by Barsky, Grimmett and Newman [4], the critical point for homogeneous percolation on half-spaces is $p_c(d)$, hence $P(|C(o, \omega_{p,0,0})| = \infty) > 0$. Since P is Γ-invariant, ergodicity implies that there are P-a.s. infinitely many vertices in H belonging to an infinite cluster of $\omega_{p,0,0}$ when $p > p_c(d)$. As also mentioned in [4], the infinite cluster of $\omega_{p,0,0}$ is almost surely unique. Therefore, if $v \in H$ belongs to this cluster, then $|C(v, \omega_{p,0,0}) \cap H| = \infty$ almost surely and equality (2.9) holds.

To prove (2.10), suppose that for some $p > p_c(d)$ and $q \in [0, 1]$, there is a finite set $F \subset H$ such that the event

$$B = \{U \in [0, 1]^{\mathbb{E}^d} : |C(o, \omega_{p,q,p}(U))| = \infty,\ C(o, \omega_{p,q,p}(U)) \cap H = F\}$$

has positive probability. Since, for any $U \in B$, every edge within $C(o, \omega_{p,q,p}(U))$ that is incident to H is contained in $\Delta_e F \cap \Delta_e H$, if we p-close every edge in $\Delta_e F \cap \Delta_e H$, we are mapped to a configuration U' such that, for some vertex $x \in \Delta_v F \setminus H$, we have $|C(x, \omega_{p,t,p}(U'))| = \infty$ and $|C(x, \omega_{p,t,p}(U')) \cap H| < \infty$ not only for $t = q$, but for every $t \in [0, 1]$. In particular, this holds for $t = p$. Therefore, denoting by B' the event of such configurations, the finite-energy property implies that $P(B') > 0$. But this is a contradiction, since ergodicity and uniqueness of the infinite cluster of $\omega_{p,p,p}$ imply that there are almost surely infinitely many vertices in H belonging to the infinite cluster of $\omega_{p,p,p}$ when $p > p_c(d)$. □

Proof of Proposition 2.8. We shall show that every infinite cluster of $\omega_{p,q,p}$ contains an infinite cluster of $\omega_{p,0,0}$. Uniqueness for $\omega_{p,q,p}$ follows from the fact that the cluster of $\omega_{p,0,0}$ is almost surely unique. This is the same proof as that of Proposition 3.1 of [16]. We present the reasoning again to indicate the places where Lemma 2.10 should be applied.

Let $\omega = (\omega_1, \omega_2)$ be the coupling of the processes $\omega_1 = \omega_{p,0,0}$ and $\omega_2 = \omega_{p,q,p}$, with $p > p_c(d)$ and $q \in [0, 1]$, and denote by P_i the marginal distribution of ω_i, $i = 1, 2$. Let $C(u, \omega_i)$ be the connected component of $u \in \mathbb{Z}^d$ in the configuration ω_i and $C(\infty, \omega_i)$ be the union of the infinite clusters in the configuration ω_i. Since P_i is invariant only by automorphisms $\gamma \in \mathrm{Aut}(\mathbb{L}^d)$ satisfying $\gamma(H) = H$, we shall use properties (2.9) and (2.10) to restrict our analysis to the subspace H. Hence, we also consider the random sets $C_H(u, \omega_i) := C(u, \omega_i) \cap H$ and $C_H(\infty, \omega_i) := C(\infty, \omega_i) \cap H$.

For $u, v \in \mathbb{Z}^d$, recall that $\mathrm{dist}(u, v)$ denotes the graph-theoretic distance between u and v. Given $u \in H$, define

$$D_1(u) := \inf\{\mathrm{dist}(u, v) : v \in C_H(\infty, \omega_1)\};$$
$$A(u) := \{D_1(u) > 0\} \cap \left\{D_1(u) = \min_{v \in C_H(u,\omega_2)} D_1(v)\right\}.$$

That is, $A(u)$ is the event where $u \in H$ is one of the vertices of $C_H(u, \omega_2)$ that are closest to $C_H(\infty, \omega_1)$ in the configuration ω_1, this distance being positive.

By properties (2.9) and (2.10), every connected component of $C(\infty, \omega_i)$, $i = 1, 2$, intersects H at infinitely many vertices almost surely. Hence, since $\omega_1 \subset \omega_2$, if $u \in C_H(\infty, \omega_2)$, then one of the following events occurs:

- $u \in C_H(\infty, \omega_1)$;
- $u \notin C_H(\infty, \omega_1)$, $\exists v \in C(\infty, \omega_1)$ such that $u \in C_H(v, \omega_2)$;
- $u \notin C_H(\infty, \omega_1)$, $\forall v \in C(\infty, \omega_1)$, $u \notin C_H(v, \omega_2)$, $|C(u, \omega_2)| = \infty$.

For any configuration in the first two events, it follows that $C(u, \omega_2)$ contains an infinite cluster of $C(\infty, \omega_1)$. For any $\omega = (\omega_1, \omega_2)$ in the last event, there exists a vertex $x \in C_H(u, \omega_2)$ such that $D_1(x) = \min_{v \in C_H(u,\omega_2)} D_1(v) > 0$. In other words, this configuration belongs to the event $\cup_{x \in H}[\{|C(x,\omega_2)| = \infty\} \cap A(x)]$. Therefore, the proposition is proved if we show that

$$P(\{|C(u,\omega_2)| = \infty\} \cap A(u)) = 0 \quad \forall u \in H.$$

We begin by analyzing the event $\{|C(u,\omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) > 1\}$. For $u, v \in H$, let

$$A_{u,v} := \{v \in C_H(u,\omega_2)\} \cap \left\{0 < D_1(v) < \min_{\substack{w \in C_H(u,\omega_2) \\ w \neq v}} D_1(w)\right\},$$

that is, $A_{u,v}$ is the event in which $v \in C_H(u,\omega_2)$ and is the only vertex of $C_H(u,\omega_2)$ that is closest to $C_H(\infty,\omega_1)$ in the configuration ω_1.

For every $\omega = (\omega_1,\omega_2) \in \{|C(u,\omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) > 1\}$, if we open (in ω_2 only) an edge $\{u,w\} \in \mathsf{E}_H$ with $D_1(w) = D_1(u) - 1$ and close every other edge incident to w, we are mapped to a configuration in $B_{w,w} := \{|C(w,\omega_2)| = \infty\} \cap A_{w,w}$. Since P_2 has the finite-energy property, if we show that $P(B_{w,w}) = 0$, then we must have $P(\{|C(u,\omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) > 1\}) = 0$, and the first part of the proof is completed.

Define $m(u,v,\omega) := \mathbf{1}_{A_{u,v}}(\omega)$ and, as in Theorem 2.9, let $\Gamma \subset \mathrm{Aut}(\mathbb{L}^d)$ be the subgroup of translations parallel to the subspace $H = \mathbb{Z}^s \times \{0\}^{d-s}$. Since P is Γ-invariant, $m(x,y,\omega) = m(\gamma x,\gamma y,\gamma\omega)$ for all $x,y \in H$, $\omega = (\omega_1,\omega_2)$ and $\gamma \in \Gamma$, and $A_{u,v} \cap A_{u,w} = \emptyset$ if $v \neq w$, the mass-transport principle (2.8) yields

$$\begin{aligned}\int_\Omega \sum_{v \in H} m(v,u,\omega)\,\mathrm{d}P(\omega) &= \sum_{v \in H} \int_\Omega m(u,v,\omega)\,\mathrm{d}P(\omega) \\ &= \sum_{v \in H} P(A_{u,v}) = P(\cup_{v \in H} A_{u,v}) < 1.\end{aligned} \tag{2.11}$$

By property (2.10), we have $|C_H(u,\omega_2)| = \infty$ almost surely for every configuration $\omega \in B_{u,u} := \{|C(u,\omega_2)| = \infty\} \cap A_{u,u}$, and consequently $\sum_{v \in H} m(v,u,\omega) = \infty$ for every $\omega \in B_{u,u}$. This fact implies $P(B_{u,u}) = 0$ for all $u \in H$, since otherwise we would have

$$\int_\Omega \sum_{v \in H} m(v,u,\omega)\,\mathrm{d}P(\omega) \geq \int_{B_{u,u}} \sum_{v \in H} m(v,u,\omega)\,\mathrm{d}P(\omega) = \int_{B_{u,u}} \infty\,\mathrm{d}P(\omega) = \infty,$$

a contradiction with (2.11).

Now, it remains to show that $P(\{|C(u,\omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) = 1\}) = 0$. For a subset $V \subset \mathbb{Z}^d$ and $x, y \in V$, let $\mathrm{dist}_V(x,y)$ be the graph-theoretic distance between x and y in the subgraph of $\mathbb{Z}^d$ induced by V. For $w \in H$, define the random set

$$S(w) := \begin{cases} \emptyset, & \text{if } w \notin C_H(\infty, \omega_1), \\ \left\{ \begin{array}{c} v \in C_H(w, \omega_2) : \operatorname{dist}_{C(w,\omega_2)}(v, w) \\ < \operatorname{dist}_{C(w,\omega_2)}(v, x) \ \forall x \in C(\infty, \omega_1) \setminus \{w\} \end{array} \right\}, & \text{if } w \in C_H(\infty, \omega_1). \end{cases}$$

That is, $S(w)$ is the set of vertices $v \in C_H(w, \omega_2)$ such that w is the only vertex of $C(\infty, \omega_1)$ closest to v in the metric of $C(w, \omega_2)$.

Note that, for any $\omega = (\omega_1, \omega_2) \in \{|C(u, \omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) = 1\}$, if we open (in ω_2 only) an edge $\{u, w\} \in \mathsf{E}_H$ with $w \in C_H(\infty, \omega_1)$, we are mapped to a configuration in $\{|S(w)| = \infty\}$. Since P_2 is insertion-tolerant, we conclude that $P(\{|C(u, \omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) = 1\}) = 0$ if we show that $P(|S(w)| = \infty) = 0$.

Let $m(u, w, \omega) = \mathbf{1}_{\{u \in S(w)\}}$. Again by the mass-transport principle (2.8), we have

$$\begin{aligned} \int_\Omega \sum_{w \in H} m(w, u, \omega) \,\mathrm{d}P(\omega) &= \sum_{w \in H} \int_\Omega m(u, w, \omega) \,\mathrm{d}P(\omega) \\ &= \sum_{w \in H} P(u \in S(w)) = P(\cup_{w \in H} \{u \in S(w)\}) < 1. \end{aligned} \tag{2.12}$$

By property (2.10), we have $\sum_{w \in H} m(w, u, \omega) = \infty$ for any $\omega \in \{|S(u)| = \infty\}$, and this fact together with (2.12) implies $P(|S(u)| = \infty) = 0$, similarly to the previous case. Since P_2 is insertion-tolerant, we conclude that $P(\{|C(u, \omega_2)| = \infty\} \cap A(u) \cap \{D_1(u) = 1\}) = 0$. □

3 Percolation on $\mathbb{Z}^d$ with a sublattice of inhomogeneities: Approximation on slabs

3.1 Overview of the chapter

In this chapter, we continue to work with the inhomogeneous Bernoulli bond percolation process on $\mathbb{L}^d$, defined in Section 2.2. The problem we address here regards the critical curve $p \mapsto q_c(p)$ of the model, where $q_c(p)$ is the supremum of the values of q for which percolation with parameters p, q does not occur. We ask if, for any $p \in [0, p_c(d))$, the critical point $(p, q_c(p)) \in [0,1]^2$ can be approximated, in any direction, by the critical point of the restriction of the inhomogeneous process to a slab $\mathbb{Z}^2 \times \{-N, \ldots, N\}^{d-2}$, for large $N \in \mathbb{N}$. Here, the classical work of Grimmett and Marstrand [13] serves as the standard reference for providing the building blocks that give an affirmative answer to this question. We have to manage the construction of a suitable renormalization process, which possesses some particularities that arise with the introduction of inhomogeneities, contrasting with the usual approach of [13].

In Section 3.2, we state the main result of this chapter and discuss its relevance. In Section 3.3, we discuss the key ideas used in the proof and develop the technical lemmas needed in Section 3.4, where we provide the proof of the main result through a renormalization argument.

3.2 Statement of the main result

As we mentioned, the main result of this chapter is a claim about the ***critical parameter function***, $q_c : [0,1] \to [0,1]$, defined by

$$q_c(p) := \sup\{q \in [0,1] : P_{p,q}(o \leftrightarrow \infty) = 0\}.$$

In [18], the authors showed that $q_c(p)$ is strictly decreasing in the interval $[0, p_c(d)]$, which in turn implies that there exists a set of parameters $(p, q) \in [0, 1]^2$ such that $p < p_c(d) < q < p_c(s)$ and $P_{p,q}(o \leftrightarrow \infty) > 0$. Complementing this study, one can ask whether $q_c(p)$ is a continuous function in the interval $[0, p_c(d))$. Besides the intrinsic motivation of this problem, a positive answer to this question leads us to a better knowledge of our model. For instance, in Theorem 1 of [18], the authors show that if (p, q) is an interior point of $\{(p, q) : P_{p,q}(o \leftrightarrow \infty) = 0\}$, then $E_{p,q}|C(o)| < \infty$. Therefore, if $q_c(p)$ is continuous, we conclude that the phase-transition of our model is "sharp", i.e.,

$$q_c(p) = \sup\{q \in [0, 1] : E_{p,q}|C(o)| < \infty\}.$$

Thus, the main theorem of this chapter arose as an effort to prove the (left)-continuity of $q_c(p)$ in the interval $[0, p_c(d))$. Denoting by q_c^N the analogous function for the restriction of the inhomogeneous percolation process on $\mathbb{Z}^d$ with subspace of defects $H = \mathbb{Z}^s \times \{0\}^{d-s}$ to the slab $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$, we have the following result:

Theorem 3.1 (Approximation on slabs)**.** *Let $p < p_c(d)$. For every $\eta > 0$, there exists an $N \in \mathbb{N}$ such that*

$$q_c^N(p + \eta) < q_c(p) + \eta.$$

Although the idea of essential enhancements cannot be directly applied to determine the continuity of $q_c(p)$, the work of Aizenman and Grimmett [3] implies that $q_c^N(p)$ is continuous and strictly decreasing in the interval $[0, p_c(d))$, for every $N \in \mathbb{N}$. Therefore, the left-continuity of $q_c(p)$ would follow if we could replace $q_c^N(p + \eta)$ by $q_c^N(p)$ in the statement of Theorem 3.1. Since we were not able to make this improvement, the continuity of $q_c(p)$ remains an open problem.

3.3 Technical lemmas

The proof of Theorem 3.1 is accomplished using the ideas developed by Grimmett and Marstrand in [13]. It involves dividing a region of $\mathbb{L}^d$ into a family of large adjacent blocks and constructing a site percolation process using these new pieces. As we will see in Section 3.4, a block is deemed "open" if there is a certain large open path on $\mathbb{L}^d$ within it and its neighbors. This is done in such a way that an infinite open path in the "block lattice" implies an infinite open path on the slab $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$, and we shall show that this happens with positive probability.

To achieve this behavior, we must guarantee that the probability of a block to be open is sufficiently high. Since we are considering supercritical parameters $(p, q) \in [0, 1]^2$, with $p < p_c(d)$, it follows that the infinite cluster of $\mathbb{Z}^d$ intersects H at infinitely many vertices almost surely, and the probability that a vertex $x \in \mathbb{Z}^d$ percolates decays exponentially fast with $\mathrm{dist}_{\mathbb{L}^d}(x, H)$ [2, 11, 21]. This technicality may lead us to the undesirable situation in which good blocks consisting of vertices that are “distant” from H have a probability of being open that is not sufficiently high. Thus, in what follows, we adapt the techniques used in [13] to the inhomogeneous setting. We do so by modifying the finite-size criterion that determines the shape of the long-range paths inside the blocks, ensuring that these paths “begin and end in the subspace H”.

This type of long-range connections is rigorously achieved by Lemma 3.5, whose proof relies on Lemmas 3.2 and 3.3, and Claim 3.4. Following the proof, we discuss some problems regarding a direct application of Lemma 3.5, and show how to overcome them through Lemma 3.6. Finally, we find in Lemma 3.7 the condition used to show that, if the blocks in the “block lattice” are open with high probability, then we can achieve percolation on the slab $\mathbb{Z}^2 \times \{-N, \ldots, N\}^{d-2}$ with positive probability.

Every result stated in this section has an analogous counterpart in [13], and this correspondence will be indicated. We shall also highlight the relevant aspects that are particular to our case. From now on, we denote $\theta(p, q) := P_{p,q}(o \leftrightarrow \infty)$.

Recall that $H = \mathbb{Z}^s \times \{0\}^{d-s}$ and that $\Delta_v S$ and $\Delta_e S$ denote the external vertex and edge boundaries of a set $S \subset \mathbb{Z}^d$, respectively. Also, define the internal vertex boundary of S by $\partial S := \{x \in S : \exists y \in \mathbb{Z}^d \setminus S, \{x, y\} \in \mathbb{E}^d\}$. For $m \in \mathbb{N}$, let $B_m := \{-m, \ldots, m\}^d$ and $B_m^H := B_m \cap H$.

Given $\alpha, \beta > 0$ and $n \in \mathbb{N}$, let

$$S_n^{\alpha,\beta} := \{x \in H : \beta n + 1 \leq \|x\|_\infty \leq \beta n + \alpha n\},$$

and, for $m \in \mathbb{N}$ with $\beta n > m$, consider the random set

$$U_n^{\alpha,\beta} := \left\{x \in \Delta_v S_n^{\alpha,\beta} : x \xleftrightarrow{B_{\beta n+\alpha n} \setminus S_n^{\alpha,\beta}} B_m^H\right\}, \tag{3.1}$$

where $x \overset{S}{\longleftrightarrow} B$ indicates that the vertex x is connected to the set B by an open path whose vertices are entirely contained in the set S. Our first task is to show that, in the regime $p < p_c(d) < q < p_c(s)$, if the cluster of B_m^H is infinite for some $m \in \mathbb{N}$, then it is unlikely that

$U_n^{\alpha,\beta}$ consists of just a few vertices, as $n \to \infty$. We work with definition (3.1) because, unlike the homogeneous percolation process, since we are considering $\theta(p,q) > 0$ and $p < p_c(d)$, we know that the probability of a vertex $x \in B_{\alpha n+\beta n}$ to be connected to B_m^H within $B_{\beta n+\alpha n}$ decays exponentially fast with $\mathrm{dist}_{\mathbb{L}^d}(x,H)$. Therefore, when we search for such vertices, we are compelled to consider only the candidates lying near the subspace H. The following result is the equivalent of Lemma 3 of [13].

Lemma 3.2. *For any $k, m \in \mathbb{N}$, $\alpha, \beta > 0$ and $p < p_c(d) < q < p_c(s)$, we have*

$$P_{p,q}\big(|U_n^{\alpha,\beta}| \le k, B_m^H \leftrightarrow \infty\big) \underset{n}{\longrightarrow} 0.$$

Proof. Under the conditions of the lemma we have

$$P_{p,q}\big(|U_n^{\alpha,\beta}| \le k, B_m^H \leftrightarrow \infty\big) \le P_{p,q}\big(|U_n^{\alpha,\beta}| = 0, B_m^H \leftrightarrow \infty\big) + P_{p,q}\big(1 \le |U_n^{\alpha,\beta}| \le k\big).$$

Hence, the result is proved if we show that the two probabilities on the right-hand side of the inequality above go to zero as $n \to \infty$. To see this, note that since $p < p_c(d)$, the exponential decay of the radius of the open cluster [2, 11, 21] implies that there is a constant $c_p > 0$ such that

$$P_{p,q}\big(|U_n^{\alpha,\beta}| = 0, B_m^H \leftrightarrow \infty\big) \le P_{p,q}\Big(B_{\beta n} \xleftrightarrow{B_{\beta n+\alpha n}\setminus S_n^{\alpha,\beta}} \partial B_{\beta n+\alpha n}\Big) \le \big|\partial B_{\beta n}\big| e^{-c_p \alpha n}. \tag{3.2}$$

Also, since the random variable $|U_n^{\alpha,\beta}|$ does not depend on the states of the edges in $\Delta_e S_n^{\alpha,\beta}$, given $j \in \{1,\ldots,k\}$, we have

$$\begin{aligned}
P_{p,q}\big(|U_n^{\alpha,\beta}| = j\big)(1-q)^k &\le P_{p,q}\big(|U_n^{\alpha,\beta}| = j\big)(1-q)^j \\
&\le P_{p,q}\big(|U_n^{\alpha,\beta}| = j, \Delta_e U_n^{\alpha,\beta} \cap \Delta_e S_n^{\alpha,\beta} \text{ closed}\big) \\
\le P_{p,q}\Big(\Big\{B_m^H \leftrightarrow \partial B_{\beta n},\ B_m^H &\not\leftrightarrow \partial B_{\beta n+\alpha n}\Big\} \cup \Big\{B_m^H \xleftrightarrow{B_{\beta n+\alpha n}\setminus S_n^{\alpha,\beta}} \partial B_{\beta n+\alpha n}\Big\}\Big) \\
&\le P_{p,q}\big(B_m^H \leftrightarrow \partial B_{\beta n}, |C(B_m^H)| < \infty\big) + |\partial B_{\beta n}| e^{-c_p \alpha n},
\end{aligned}$$

where $C(B_m^H)$ denotes the open cluster of B_m^H. Consequently, it follows that

$$\begin{aligned}
P_{p,q}\big(1 \le |U_n^{\alpha,\beta}| \le k\big) &= (1-q)^{-k} \sum_{j=1}^{k} (1-q)^k P_{p,q}\big(|U_n^{\alpha,\beta}| = j\big) \\
&\le (1-q)^{-k} k \Big[P_{p,q}\big(B_m^H \leftrightarrow \partial B_{\beta n}, |C(B_m^H)| < \infty\big) + |\partial B_{\beta n}| e^{-c_p \alpha n}\Big].
\end{aligned} \tag{3.3}$$

Thus, the proof is completed by observing that the right-hand sides of (3.2) and (3.3) go to zero as $n \longrightarrow \infty$. □

The next result we state is the equivalent of Lemma 4 of [13]. It says that if B_m^H percolates, then for sufficiently large n, there is always a portion of $\Delta_v S_n^{\alpha,\beta}$ where we can find as many sites connected to B_m^H as we like with positive probability, which goes to one as $m \longrightarrow \infty$.

Define the sets

$$\begin{aligned}
F_n^{\alpha,\beta} &:= [\beta n + 1, \beta n + \alpha n] \times [0, \beta n + \alpha n]^{s-1} \times \{0\}^{d-s}, \\
T_n^{\alpha,\beta} &:= \Delta_v F_n^{\alpha,\beta} \cap B_{\beta n+\alpha n}, \\
V_n^{\alpha,\beta} &:= \left\{ x \in T_n^{\alpha,\beta} : x \xleftrightarrow{B_{\beta n+\alpha n} \setminus F_n^{\alpha,\beta}} B_m^H \right\}.
\end{aligned}$$

Lemma 3.3. *For any $k, m \in \mathbb{N}$, $\alpha, \beta > 0$ and $p < p_c(d) < q < p_c(s)$, we have*

$$\liminf_n P_{p,q}\big(|V_n^{\alpha,\beta}| \geq k\big) \geq 1 - P_{p,q}\big(B_m^H \nleftrightarrow \infty\big)^{1/s2^s}.$$

Proof. Since the parameters $\alpha, \beta > 0$ do not play any important role in this proof, we shall omit them when we refer to the sets $T_n^{\alpha,\beta}$, $V_n^{\alpha,\beta}$, $S_n^{\alpha,\beta}$ and $U_n^{\alpha,\beta}$ henceforth.

For $y \in \{-1, 1\}^s$ and $i = 1, \ldots, s$, let $\sigma_y : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, and $\mathcal{L}_i : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be the mappings

$$\big[\sigma_y(x)\big]_j = \begin{cases} y_j x_j, & \text{if } j \leq s, \\ x_j, & \text{otherwise,} \end{cases} \qquad [\mathcal{L}_i(x)]_j = \begin{cases} x_i, & \text{if } j = 1, \\ x_1, & \text{if } j = i, \\ x_j, & \text{otherwise,} \end{cases}$$

and consider the random set

$$V_{n,y,i} := \left\{ x \in \mathcal{L}_i(\sigma_y(T_n)) : \xleftrightarrow{B_{\beta n+\alpha n} \setminus S_n} B_m^H \right\}.$$

Note that

$$\Delta_v S_n \subset \left[\cup_{\substack{i\in\{1,\ldots,s\} \\ y\in\{-1,1\}^s}} \mathcal{L}_i(\sigma_y(T_n)) \right],$$

which implies, by the definition of U_n (3.1), that

$$\{|U_n| \leq s2^s k\} \supset \left[\cap_{\substack{i\in\{1,\ldots,s\} \\ y\in\{-1,1\}^s}} \left\{ |V_{n,y,i}| \leq k \right\} \right].$$

The events $\{|V_{n,y,i}| \leq k\}$ are decreasing and have the same probability for every $i \in \{1, \ldots, s\}$ and $y \in \{-1, 1\}^s$. Hence, observing that $V_{n,1,1} = V_n$, the FKG Inequality (1) implies

$$P_{p,q}(|U_n| \leq s2^s k) \geq P_{p,q}(|V_n| \leq k)^{s2^s},$$

whence

$$P_{p,q}(|V_n| \geq k) \geq 1 - \left[P_{p,q}\big(|U_n| \leq s2^s k, B_m^H \leftrightarrow \infty\big) + P_{p,q}\big(B_m^H \not\leftrightarrow \infty\big)\right]^{1/s2^s}.$$

Taking $\liminf_n$ on the inequality above and then using Lemma 3.2 yields the desired result. □

Now, we go one step further and show that, if the origin percolates for some $p < p_c(d) < q < p_c(s)$, then for sufficiently large n and m, it is very likely to have B_m^H connected to some translate $x + B_m^H$ which is contained in $F_n^{\alpha,\beta}$ and whose edges are all open. That is, we shall establish the equivalent of Lemma 5 of [13]. Although the proof of our result is carried out similarly as its counterpart, one of its steps uses a more general argument. This is done to avoid the verification, at a certain point of the proof, that $2m + 1$ divides both $\alpha n + 1$ and $\alpha n + \beta n + 1$, for some $\alpha, \beta > 0$ and $m, n \in \mathbb{N}$.

For $m \in \mathbb{N}$ and $x \in H$, we say that $x + B_m^H$ is an ***m*-seed** if every edge in $x + B_m^H$ is open. Thus, we define, for $\alpha n > 2m + 1$,

$$K_{m,n}^{\alpha,\beta} := \left\{x \in T_n^{\alpha,\beta} : \exists y \in F_n^{\alpha,\beta}, \{x, y\} \in \mathbb{E}^d, \omega(\{x, y\}) = 1, y \text{ is in an } m\text{-seed in } F_n^{\alpha,\beta}\right\}.$$

The strategy here is the following: provided that we can find any large number of vertices in $|V_n^{\alpha,\beta}|$ with probability as high as we need, we additionally require that some fixed number of these vertices are connected to a seed in $F_n^{\alpha,\beta}$. Using the structure of $\mathbb{Z}^d$ we can ensure that these candidates are far away from each other in such a way that all the possible seeds are mutually disjoint. Hence, if we have many such candidates, we can conclude that B_m^H is connected to $K_{m,n}^{\alpha,\beta}$ with high probability.

The following assertion describes a structural property of $\mathbb{Z}^d$ we will make use of:

Claim 3.4. *For every $M, k \in \mathbb{N}$, $M \geq 2$, there exists $T(M, k) \in \mathbb{N}$ such that if $A \subset \mathbb{Z}^d$ and $|A| > T(M, k)$, then there is a subset $\{x_1, \ldots, x_M\} \subset A$ satisfying $\|x_i - x_j\|_\infty > k$ for every $i \neq j$, where $1 \leq i, j \leq M$.*

Proof. Let $T(M,k) = \sum_{j=1}^{M-1} j(2k+1)^d$. If $M = 2$ and $|A| > (2k+1)^d$, it follows that there are at least two vertices $x, y \in A$ such that $\|x-y\|_\infty > k$, since the ball $B(x,k) = \{y \in \mathbb{Z}^d : \|y-x\|_\infty \leq k\}$ has $(2k+1)^d$ vertices for any $x \in \mathbb{Z}^d$.

Now, suppose the result holds for some $M \geq 2$. Then, if any set $A \subset \mathbb{Z}^d$ satisfies $|A| > T(M+1,k) = T(M,k) + M(2k+1)^d$, it follows by the induction hypothesis that there is a subset $\{x_1, \ldots, x_M\} \subset A$ satisfying $\|x_i - x_j\|_\infty > k$ for every $i \neq j$, where $1 \leq i, j \leq M$. Moreover, it follows that $\sum_{j=1}^{M} |B(x_j,k)| \leq M(2k+1)^d < T(M+1,k) < |A|$. This immediately implies that there exists $x_{M+1} \in A$ such that $\|x_{M+1} - x_j\|_\infty > k$ for every $1 \leq j \leq M$, and the claim is proved. □

Lemma 3.5. *If $\theta(p,q) > 0$ and $p < p_c(d) < q < p_c(s)$, then for every $\alpha, \beta, \eta \in (0,\infty)$, there exist $m, n \in \mathbb{N}$ such that*

$$P_{p,q}\Big(B_m^H \xleftrightarrow{B_{\beta n'+\alpha n'}} K_{m,n'}^{\alpha,\beta}\Big) > 1 - \eta \quad \textit{for all } n' \geq n.$$

Proof. If $\theta(p,q) > 0$, then there exists $m \in \mathbb{N}$ such that

$$P_{p,q}(B_m^H \leftrightarrow \infty) > 1 - \left(\frac{\eta}{2}\right)^{s2^s}. \tag{3.4}$$

Let $M \in \mathbb{N}$ be such that

$$pP_{p,q}(B_m^H \text{ is an } m\text{-seed}) > 1 - \left(\frac{\eta}{2}\right)^{1/M} \tag{3.5}$$

and fix $l = T(M, 2(2m+1)+2)$ as in Claim 3.4. By Lemma 3.3 and (3.4), it follows that there exists an $n \in \mathbb{N}$ such that

$$P_{p,q}\big(|V_{n'}^{\alpha,\beta}| \geq l\big) > 1 - \frac{\eta}{2} \quad \text{for all } n' \geq n. \tag{3.6}$$

Now, let $n' \geq n$ and note that Claim 3.4 ensures that, for every configuration in the event $\{|V_{n'}^{\alpha,\beta}| \geq l\}$, there is a subset $\{x_1, \ldots, x_M\} \subset V_{n'}^{\alpha,\beta}$ satisfying $\|x_i - x_j\|_\infty > 2(2m+1)+2$ for every $i \neq j$, where $1 \leq i, j \leq M$. Hence, if y_i is the unique neighbor of x_i that belongs to $F_{n'}^{\alpha,\beta}$ and $B_{m,i}^H \subset H$ is a box of side length $2m$ containing y_i, then $B_{m,i}^H \cap B_{m,j}^H = \emptyset$ for every $i \neq j$, $1 \leq i, j \leq M$. Since the event $\{|V_{n'}^{\alpha,\beta}| \geq l\}$ does not depend on the states of the edges in $S_{n'}^{\alpha,\beta}$

and of $\Delta_e S^{\alpha,\beta}_{n'}$, inequalities (3.5) and (3.6) imply

$$\begin{aligned} P_{p,q}\Big(B^H_m \xleftrightarrow{B_{\beta n'+\alpha n'}} K^{\alpha,\beta}_{m,n'}\Big) &\geq P_{p,q}\Big(\big\{\big|V^{\alpha,\beta}_{n'}\big| \geq l\big\} \cap \Big[\cup_{i=1}^{M}\big\{x_i \in K^{\alpha,\beta}_{m,n'}\big\}\Big]\Big) \\ &\geq 1-\eta. \end{aligned}$$

□

Recall that, for $S \subset \mathbb{Z}^d$, we have $\mathsf{E}_S := \{e \in \mathbb{E}^d : e \subset S\}$, and let P be the probability measure associated with the family $\{U(e) : e \in \mathbb{E}^d\}$ of i.i.d. random variables having uniform distribution in $[0,1]$. In this context, for $p \in [0,1]$, we say that $e \in \mathbb{E}^d$ is ***p*-open** if $U(e) \leq p$ and ***p*-closed** otherwise. We also say that a subset $F \subset \mathbb{E}^d$ is **(p,q)-open** if every edge of $F \cap (\mathbb{E}^d \setminus \mathsf{E}_H)$ is p-open and every edge of $F \cap \mathsf{E}_H$ is q-open.

The idea for proving Theorem 3.1 is to recursively grow the cluster of the origin of $\mathbb{Z}^d$ to more distant regions, jumping from a recently obtained seed to a farther one, and keep this process going indefinitely with positive probability. Similarly to [13], due to the geometrical nature of our connections, it is not possible to perform such an exploration independently. As a matter of fact, any attempt to reach a new open seed from a recently obtained one always involves an already explored region of $\mathbb{Z}^d$ that contains closed edges in its external boundary, creating a problem to the direct application of Lemma 3.5. Lemma 3.6, which is the equivalent of Lemma 6 of [13], solves this issue by stating that if we give these explored closed edges a small extra chance to be open, then the desired long-range connections can be attained with high probability P.

Lemma 3.6 (Finite-size criterion)**.** *Assume that $\theta(p,q) > 0$ for some $p < p_c(d) < q < p_c(s)$. Then, for every $\varepsilon, \delta > 0$ and $\alpha, \beta > 0$, there exist $m, n \in \mathbb{N}$ with the following property:*

Suppose $n' \in \mathbb{N}$ and $R \subset \mathbb{Z}^d$ satisfy $B^H_m \subset R \subset B_{\beta n'+\alpha n'}$ and $(R \cup \Delta_v R) \cap T^{\alpha,\beta}_{n'} = \emptyset$. Also, let $\gamma : \Delta_e R \cap \mathsf{E}_{B_{\beta n'+\alpha n'}} \to [0, 1-\delta]$ be any function and define the events

$$\begin{aligned} E_{n'} &:= \left\{ \begin{aligned} &\textit{there is a path joining } R \textit{ to } K^{\alpha,\beta}_{m,n'} \textit{ which is } (p,q)\textit{-open} \\ &\textit{outside } \Delta_e R \textit{ and } (\gamma(f)+\delta)\textit{-open in its only edge } f \in \Delta_e R \end{aligned} \right\}, \\ F_{n'} &:= \left\{ f \textit{ is } \gamma(f)\textit{-closed for every } f \in \Delta_e R \cap \mathsf{E}_{B_{\beta n'+\alpha n'}} \right\}. \end{aligned}$$

Then $P(E_{n'}|F_{n'}) > 1-\varepsilon$ for every $n' \geq n$.

Proof. To improve the readability of the proof, we ommit the superscripts $\alpha, \beta > 0$ from the set

$K_{m,n}^{\alpha,\beta}$.

Suppose $\theta(p,q) > 0$ for some $p < p_c(d) < q < p_c(s)$ and let $\varepsilon, \delta > 0$. Choose $t \in \mathbb{N}$ such that

$$(1-\delta)^t < \frac{\varepsilon}{2}, \tag{3.7}$$

and also $\eta > 0$ so that

$$\eta < \frac{\varepsilon}{2}(1-q)^t. \tag{3.8}$$

By Lemma 3.5, there exist $m, n \in \mathbb{N}$ such that

$$P_{p,q}\Big(B_m^H \xleftrightarrow{B_{\beta n'+\alpha n'}} K_{m,n'}\Big) > 1-\eta \quad \text{for all } n' \geq n.$$

Let $n' \geq n$ and consider the region R and the function γ as in the hypotheses of the lemma. Since $R \supset B_m^H$, it follows that

$$P_{p,q}\Big(\partial R \xleftrightarrow{B_{\beta n'+\alpha n'}} K_{m,n'}\Big) > 1-\eta. \tag{3.9}$$

Let $K \subset T_{n'}^{\alpha,\beta}$ and denote by $U_{n'}(K)$ the set of edges $\{x,y\} \subset B_{\beta n'+\alpha n'}$ such that

i. $x \in R$, $y \notin R$;

ii. There is an open path joining y to K in $B_{\beta n'+\alpha n'}$ without using any edge contained in $R \cup \Delta_v R$.

Thus, observing that for any $A \subset \Delta_e R$ the event $\{U_{n'}(K) = A\}$ does not depend on the states of the edges in $U_{n'}(K)$, which in turn are independent from one another, we have

$$\begin{aligned}
P_{p,q}\big(\partial R \nleftrightarrow K \text{ in } B_{\beta n'+\alpha n'}\big) &= P_{p,q}(U_{n'}(K) = \emptyset \text{ or } e \text{ is closed for every } e \in U_{n'}(K)) \\
&\geq P_{p,q}(|U_{n'}(K)| \leq t, U_{n'}(K) \text{ closed}) \\
&= \sum_{\substack{A \subset \Delta_e R \\ |A| \leq t}} P_{p,q}(U_{n'}(K) = A) P_{p,q}(U_{n'}(K) \text{ closed} | U_{n'}(K) = A) \\
&= \sum_{\substack{A \subset \Delta_e R \\ |A| \leq t}} (1-q)^{|A|} P_{p,q}(U_{n'}(K) = A) \\
&\geq (1-q)^t P_{p,q}(|U_{n'}(K)| \leq t).
\end{aligned} \tag{3.10}$$

Now, note that the events $\{|U_{n'}(K)| \leq t\}$ and $\{K_{m,n'} = K\}$ are independent, as well as the events $\{\partial R \nleftrightarrow K \text{ in } B_{\beta n'+\alpha n'}\}$ and $\{K_{m,n'} = K\}$. Combining these two facts together with inequalities (3.10) and (3.9), we arrive at

$$\begin{aligned} P_{p,q}(|U_{n'}(K_{m,n'})| \le t) &= \sum_{K \subset T_{n'}^{\alpha,\beta}} P_{p,q}(|U_{n'}(K)| \le t) P_{p,q}(K_{m,n'} = K) \\ &\le \sum_{K \subset T_{n'}^{\alpha,\beta}} (1-q)^{-t} P_{p,q}(\partial R \nleftrightarrow K \text{ in } B_{\beta n'+\alpha n'}) P_{p,q}(K_{m,n'} = K) \\ &\le \eta(1-q)^{-t}, \end{aligned}$$

whence (3.8) implies

$$P_{p,q}(|U_{n'}(K_{m,n'})| > t) > 1 - \frac{\varepsilon}{2}. \tag{3.11}$$

Finally, consider the coupling measure P, defined before the statement of Lemma 3.6. Since for any $A \subset \Delta_e R$ the event $\{U_{n'}(K_{m,n'}) = A\}$ is independent of the states of the edges in A and these states are independent of one another, conditioning on $F_{n'}$ gives us

$$\begin{aligned} &P(e \text{ is } (\gamma(e)+\delta)\text{-closed } \forall e \in U_{n'}(K_{m,n'}), |U_{n'}(K_{m,n'})| > t | F_{n'}) \\ &\quad = \sum_{\substack{A \subset \Delta_e R \\ |A| > t}} P(e \text{ is } (\gamma(e)+\delta)\text{-closed } \forall e \in A, U_{n'}(K_{m,n'}) = A | F_{n'}) \\ &\quad = \sum_{\substack{A \subset \Delta_e R \\ |A| > t}} (1-\delta)^{|A|} P(U_{n'}(K_{m,n'}) = A | F_{n'}) \le (1-\delta)^t \\ &\quad < \frac{\varepsilon}{2}, \end{aligned} \tag{3.12}$$

where the last inequality comes from (3.7). Combining (3.11) and (3.12), it follows that

$$\begin{aligned} &P(\exists e \in U_{n'}(K_{m,n'}) \text{ such that } e \text{ is } (\gamma(e)+\delta)\text{-open} | F_{n'}) \\ &\quad \ge P(\exists e \in U_{n'}(K_{m,n'}) \text{ such that } e \text{ is } (\gamma(e)+\delta)\text{-open}, |U_{n'}(K_{m,n'})| > t | F_{n'}) \\ &\quad > P(|U_{n'}(K_{m,n'})| > t | F_{n'}) - \frac{\varepsilon}{2} > 1 - \varepsilon, \end{aligned}$$

and the proof is complete. □

Remark 2. It is important to emphasize the condition "for every $n' \ge n$" in the statement of Lemma 3.6. Further on, we will need to choose a finite number of pairs $(\alpha_1, \beta_1), \ldots, (\alpha_l, \beta_l)$, and check that there exists $n_0 \in \mathbb{N}$ such that $P(E_{n_0}^{\alpha_i,\beta_i} | F_{n_0}^{\alpha_i,\beta_i})$ is sufficiently large, for every $i = 1, \ldots, l$. Since for each pair (α_i, β_i), there exists $n(\alpha_i, \beta_i) \in \mathbb{N}$ such that $P(E_{n'}^{\alpha_i,\beta_i} | F_{n'}^{\alpha_i,\beta_i})$ is sufficiently large for every $n' \geq n(\alpha_i, \beta_i)$, the desired result is achieved if we consider $n_0 = \max_{1\leq i\leq l} n(\alpha_i, \beta_i)$. The necessity of working with boxes of multiple sizes is particular to our setting. This technicality differs from [13], where the authors needed to use just one size of box in their renormalization process.

The last technical result we need is Lemma 1 of [13], stated in the following. It is the condition that allows us to show that, if the blocks in the “block lattice” are open with high probability, then percolation on the block lattice implies in percolation the slab $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$ with positive probability. Let $G = (V, E)$ be an infinite and connected graph. Suppose we have a collection of random variables $\{Z(x) \in \{0, 1\} : x \in V\}$ defined on some probability space $(\Omega, \mathcal{F}, \mu)$, let $f_1, f_2, \dots$ be an ordering of the edges in E and fix $x_1 \in V$. Consider the following random sequence $\mathcal{S} = \{S_t = (A_t, B_t)\}_{t\in\mathbb{N}}$ of ordered pairs of subsets of V: let

$$S_1 = \begin{cases} (\{x_1\}, \emptyset), & \text{if } Z(x_1) = 1, \\ (\emptyset, \{x_1\}), & \text{if } Z(x_1) = 0. \end{cases}$$

Having obtained $S_1, \dots, S_t$ for $t \geq 1$, we define S_{t+1} in the following manner: denote $f_i = \{u_i, v_i\}$ and let $j_{t+1} = \inf\{i : u_i \in A_t, v_i \in V \setminus (A_t \cup B_t)\}$, with the convention that $\inf \emptyset = \infty$. If $j_{t+1} < \infty$, let $x_{t+1} = v_{j_{t+1}}$ and declare

$$S_{t+1} = \begin{cases} (A_t \cup \{x_{t+1}\}, B_t), & \text{if } Z(x_{t+1}) = 1 \\ (A_t, \{x_{t+1}\} \cup B_t), & \text{if } Z(x_{t+1}) = 0. \end{cases}$$

Otherwise, declare $S_{t+1} = S_t$. We call $\mathcal{S}$ the ***cluster-growth process*** of the vertex x_1 with respect to $(Z(x))_{x\in V}$. Note that the, in the context of site percolation, the open cluster $C(x_1)$ of x_1 with respect to $(Z(x))_{x\in V}$ is the set $A_\infty = \cup_{t\geq 1} A_t$ and its external vertex boundary is the set $B_\infty = \cup_{t\geq 1} B_t$.

Now, let $p_c^{\text{site}}(G) \in (0, 1)$ be the Bernoulli site percolation threshold for G and define

$$\rho(\mathcal{S}, t) := \begin{cases} \mu(Z(x_{t+1}) = 1 | S_1, \dots, S_t), & \text{if } j_{t+1} < \infty, \\ 1, & \text{otherwise.} \end{cases}$$

The next result states that the cluster of x_1 with respect to $(Z(x))_{x\in V}$ is infinite with positive probability μ, provided that, when performing the cluster-growth process of x_1, the conditional probability of augmenting the set A_t at any step $t \in \mathbb{N}$ exceeds the parameter of a supercritical Bernoulli site percolation process on G.

Lemma 3.7 (Renormalization condition). *If there exists $\lambda \in (p_c^{site}(G), 1)$ such that*

$$\rho(\mathcal{S}, t) \geq \lambda \ \ \textit{for all } t \in \mathbb{N}, \tag{3.13}$$

then $\mu(|A_\infty| = \infty) > 0$.

Proof. Let P be the probability measure associated with the family $\{U(x) : x \in V\}$ of i.i.d. random variables having uniform distribution on $[0,1]$, fix $x_1 \in V$ and consider the following random sequence $\mathcal{S}^* = \{S_t^* = (A_t^*, B_t^*)\}_{t\in\mathbb{N}}$ of ordered pairs of subsets of V: first, let

$$S_1^* = \begin{cases} (\{x_1\}, \emptyset), & \text{if } U(x_1) \leq \mu(Z(x_1) = 1), \\ (\emptyset, \{x_1\}), & \text{otherwise.} \end{cases}$$

Having obtained $S_1^*, \ldots, S_t^*$ for $t \geq 1$, let j_{t+1} and x_{t+1} be defined analogously as it was described for the sequence $\mathcal{S}$. If $j_{t+1} < \infty$, declare

$$S_{t+1}^* = \begin{cases} (A_t^* \cup \{x_{t+1}\}, B_t^*), & \text{if } U(x_{t+1}) \leq \mu(Z(x_{t+1}) = 1 | S_t = S_t^*, \ldots, S_1 = S_1^*), \\ (A_t^*, \{x_{t+1}\} \cup B_t^*), & \text{otherwise,} \end{cases}$$

and let $S_{t+1}^* = S_t^*$ if $j_{t+1} = \infty$.

Additionally, fix $\lambda \in (p_c^{\text{site}}(G), 1)$ and let $\mathcal{S}^\lambda = \{S_t^\lambda = (A_t^\lambda, B_t^\lambda)\}_{t\in\mathbb{N}}$ be the cluster-growth process of x_1 with respect to the random variables $\{\mathbf{1}_{\{U(x)\leq\lambda\}} : x \in V\}$.

Writing $A_\infty^\tau = \cup_{t\geq 1} A_t^\tau$ and $B_\infty^\tau = \cup_{t\geq 1} B_t^\tau$, $\tau \in \{*, \lambda\}$, it suffices to prove that

i. $\mathcal{S}$ and $\mathcal{S}^*$ have the same distribution;

ii. $A_\infty^* \supset A_\infty^\lambda$;

As a matter of fact, combining these assertions and observing that A_∞^λ has the same distribution as the open cluster of x_1 under the Bernoulli site-percolation measure on G with parameter $\lambda > p_c^{\text{site}}(G)$, it follows that

$$\mu(|A_\infty| = \infty) = P(|A_\infty^*| = \infty) \geq P(|A_\infty^\lambda| = \infty) = P_\lambda(|C(x_1)| = \infty) > 0.$$

To establish **i**, we show that if $\mu(S_t = \sigma_t, \dots, S_1 = \sigma_1) > 0$ for some sequence $\sigma_1, \dots, \sigma_t$, then

$$P(S_t^* = \sigma_t, \dots, S_1^* = \sigma_1) = \mu(S_t = \sigma_t, \dots, S_1 = \sigma_1). \tag{3.14}$$

We proceed by induction on $t \in \mathbb{N}$. For $t = 1$ and $\sigma_1 = (\{x_1\}, \emptyset)$,

$$P(S_1^* = \sigma_1) = P(U(x_1) \le \mu(Z(x_1) = 1)) = \mu(Z(x_1) = 1) = \mu(S_1 = \sigma_1),$$

and the same also holds for $\sigma_1 = (\emptyset, \{x_1\})$. Now, suppose (3.14) holds for some $t \ge 1$, let $\hat{\rho}(\mathcal{S}^*, t) = \mu(Z(x_{t+1}) = 1 | S_t = S_t^*, \dots, S_1 = S_1^*)$ and $\sigma_{t+1} = (A_t^* \cup \{x_{t+1}\}, B_t^*)$. Thus,

$$\begin{aligned}
P(S_{t+1}^* = \sigma_{t+1}, \dots, S_1^* = \sigma_1) &= P(U(x_{t+1}) \le \hat{\rho}(\mathcal{S}^*, t), S_t^* = \sigma_t, \dots, S_1^* = \sigma_1) \\
&= P(U(x_{t+1}) \le \hat{\rho}(\mathcal{S}^*, t) | S_t^* = \sigma_t, \dots, S_1^* = \sigma_1) P(S_t^* = \sigma_t, \dots, S_1^* = \sigma_1) \\
&= \mu(Z(x_{t+1}) = 1 | S_t = \sigma_t, \dots, S_1 = \sigma_1) \mu(S_t = \sigma_t, \dots, S_1 = \sigma_1) \\
&= \mu(S_{t+1} = \sigma_{t+1}, \dots, S_1 = \sigma_1),
\end{aligned}$$

and the same result holds for $\sigma_{t+1} = (A_t^*, B_t^* \cup \{x_{t+1}\})$.

Assertion **ii** is a consequence of the fact that condition (3.13) implies that every vertex of B_∞^* is λ-closed, therefore $A_\infty^\lambda \cap B_\infty^* = \emptyset$ and $A_\infty^\lambda \subset A_\infty^*$. □

We stress that an analogous result also holds if we introduce an orientation to the edges of G. This is particularly important in our case, since the renormalized graph we shall consider in the sequel is an oriented one.

3.4 Proof of Theorem 3.1: The renormalization process

To prove Theorem 3.1, it suffices to show that, for any $p < p_c(d)$, $\eta > 0$ and $q = q_c(p) + \eta/2$, there exists $N \in \mathbb{N}$ such that, with positive probability, the origin lies in an infinite $(p + \eta/2, q + \eta/2)$-open cluster within $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$. As already mentioned, we rely on the classical approach of Grimmett and Marstrand [13] to show that the restriction of the inhomogeneous process with parameters $p + \eta/2$ and $q + \eta/2$ to the slab $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$ stochastically dominates a supercritical percolation process on the graph $G = (V, E)$, with vertex set $V = \{x \in \mathbb{Z}^+ \times \mathbb{Z} : x_1 + x_2 \text{ is even}\}$ and edge set $E = \{\{x, x + (1, \pm 1)\} : x \in V\}$. The orientation of the edges is to be taken from x to $x + (1, \pm 1)$, for every $x \in V$. The stochastic domination occurs in the sense that if the cluster of the origin of the latter is infinite, then the cluster of the origin of the former is infinite as well.

The above idea is carried out with the aid of the following renormalization scheme: we construct a (dependent) oriented site percolation process on G, defined in terms of some special events lying in the space $([0,1]^{\mathbb{E}^d}, P)$, where P denotes the probability measure associated with the family $\{U(e) : e \in \mathbb{E}^d\}$ of i.i.d. random variables having uniform distribution in $[0,1]$. We do this by specifying a collection of random variables $\{Z(x) \in \{0,1\} : x \in V\}$, which encode information about the existence of large $(p+\eta/2, q+\eta/2)$-open paths in $\mathbb{Z}^2 \times \{-N, \ldots, N\}^{d-2}$. In particular, when considering the cluster-growth process of the origin with respect to $(Z(x))_{x\in V}$, we will require that

i. property (3.13) holds for some $\lambda \in (\vec{p}_c^{\,\text{site}}(G), 1)$, so that $|A_\infty|$ is infinite with positive probability by Lemma 3.7;

ii. if $|A_\infty| = \infty$, then the origin percolates in $\mathbb{Z}^2 \times \{-N, \ldots, N\}^{d-2}$ through a $(p+\eta/2, q+\eta/2)$-open path.

It is clear that these two conditions combined immediately imply the desired conclusion. Thus, we proceed to the construction of the process $(Z(x))_{x\in V}$.

Having fixed $p < p_c(d)$, let $\eta > 0$ be small and define

$$q = q_c(p) + \eta/2 \qquad \delta = \frac{1}{16}\eta, \qquad \varepsilon = \frac{1}{150}\big(1 - \vec{p}_c^{\,\text{site}}(G)\big). \tag{3.15}$$

Also, consider $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 1/100$ and $\beta_1 = \beta_2/2 = \beta_3/(2+\alpha+\alpha^2) = 1$. Since $\theta(p,q) > 0$, Lemma 3.6 guarantees the existence of $m, n \in \mathbb{N}$ such that $P(E_n|F_n) > 1 - \varepsilon$ for each given pair (α_i, β_i).

For a vertex $x \in V$ and a subset $A \subset V$, let $x + A := \{x + a : a \in A\}$. Also, let $\vec{u}_1, \ldots, \vec{u}_d$ be the canonical basis of $\mathbb{R}^d$ and, for $N = 6n$, let $\Lambda(N) = B_N \cup (2N\vec{u}_2 + B_N)$. The fundamental blocks of the renormalized lattice are the ***site-blocks***

$$\Lambda_x = \Lambda_x(N) := 4Nx + \Lambda(N), \quad x \in V,$$

which can be written as the union of a "lower" and an "upper" translate of B_N, namely

$$\Lambda_x^l = \Lambda_x^l(N) := 4Nx + B_N,$$
$$\Lambda_x^u = \Lambda_x^u(N) := 2N\vec{u}_2 + \Lambda_x^l(N).$$

The adjacency relation between site-blocks is the one inherited from $G = (V, E)$. That is, for $x, y \in V$, the boxes Λ_x and Λ_y are adjacent if and only if $\{x, y\} \in E$. The long-range connections in $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$ we are going to build will occur between adjacent site-blocks, using its edges and the edges within the ***passage-blocks***

$$\Pi_x = \Pi_x(N) := [\Lambda_x + 2N(\vec{u}_1 + \vec{u}_2)] \cup [\Lambda_x + 2N(\vec{u}_1 - \vec{u}_2)], \quad x \in V.$$

Having set up the renormalization structure, we are now in a position to define the random variables $Z(x)$, $x \in V$. We will specify them recursively, considering the first coordinate of each $x = (x_1, x_2) \in V$. The idea is to make $Z(x)$ encode information about connections between seeds inside the site-blocks Λ_x, $\Lambda_{x+(1,1)}$ and $\Lambda_{x+(1,-1)}$. These open paths will be contained in $\Lambda_x \cup \Pi_x \cup \Lambda^l_{x+(1,1)} \cup \Lambda^u_{x+(1,-1)}$ and possess connectivity features such that requirements **i.** and **ii.** are fulfilled for $\lambda = [1 + \vec{p}_c^{\,\text{site}}(G)]/2$.

We begin by determining the event $\{Z(o) = 1\}$. This will be achieved through the application of a sequential algorithm, which constructs an increasing sequence $E_1, E_2, \dots$ of edge-sets by making repeated use of Lemma 3.6. At each step k of the algorithm, we acquire information about the values of $U(e)$ for certain $e \in \mathbb{E}^d$, and record this information into suitable functions $\gamma_k, \zeta_k : \mathbb{E}^d \to [0, 1]$, in such a way that every $e \in \mathbb{E}^d$ is $\gamma_k(e)$-closed and $\zeta_k(e)$-open and

$$\gamma_k(e) \le \gamma_{k+1}(e), \qquad\qquad \zeta_k(e) \ge \zeta_{k+1}(e).$$

In this context, we respectively regard γ_k and ζ_k as the acquired “negative” and “positive” information about the states of the edges of $\mathbb{E}^d$ up to step k. At the end of each step, the ζ_k-open cluster of the origin within $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$ will have grown larger and closer to the site-blocks $\Lambda_{(1,1)}$ and $\Lambda_{(1,-1)}$, as we use Lemma 3.6 to reach new open seeds from the previouly open ones in a coordinated manner.

In our process, a single attempt of growing the cluster of the origin in the setting of Lemma 3.6 will be called a ***step*** of the exploration. The determination of $Z(o) = 1$ is constituted by a (finite) sequence of successful steps, specified in the sequel. To make the construction clear, we gather some particular subsequences of steps together, according to the “direction of growth” of the cluster, and call them ***phases*** of the exploration. A picture of a configuration such that $Z(o) = 1$ is illustrated in Figure 3.5. This event occurs if we succeed in each of the following phases:

Phase 1: Let $E_1 = \mathsf{E}_{B_m^H}$. This phase is successful if every edge in E_1 is q-open. In this case, we set

$$\gamma_1(e) = 0, \qquad \text{for all } e \in \mathbb{E}^d,$$
$$\zeta_1(e) = \begin{cases} q, & \text{if } e \in E_1, \\ 1, & \text{otherwise,} \end{cases}$$

so that every edge $e \in \mathbb{E}^d$ is $\gamma_1(e)$-closed and $\zeta_1(e)$-open.

Phase 2: Provided that Phase 1 is successful, we attempt to connect the seed B_m^H to another q-open m-seed lying in the passage-block Π_o by using Lemma 3.6 in the first series of steps in the same direction.

Let $\mathcal{P}$ be the collection of all paths in $\mathbb{Z}^d$ and denote the ***edge-boundary*** of a subset $E' \subset \mathbb{E}^d$ by $\Delta E' := \{f \in \mathbb{E}^d \setminus E' : \exists e \in E' \text{ such that } |f \cap e| = 1\}$. Given $V' \subset \mathbb{Z}^d$, $E' \subset \mathbb{E}^d$ with $(E' \cup \Delta E') \subset \mathsf{E}_{V'}$, and $\gamma : \mathbb{E}^d \to [0,1]$, define

$$\begin{aligned} \mathcal{P}(V', E', \gamma) :=& \big\{\pi = \{x_1, \dots, x_k\} \in \mathcal{P} : \pi \subset V', \{x_1, x_2\} \in \Delta E' \text{ and is } \gamma(\{x_1, x_2\})\text{-open}, \\ & \{x_i, x_{i+1}\} \in (E' \cup \Delta E')^c \text{ and is } (p,q)\text{-open } \forall i = 2, \dots, k-1\big\}, \\ \mathcal{V}(V', E', \gamma) :=& \bigcup_{\pi \in \mathcal{P}(V', E', \gamma)} \pi. \end{aligned}$$

Now, set $D_1 = B_{n+\alpha n}$ and let $E_2 = E_1 \cup \widetilde{E}_2$, where $\widetilde{E}_2$ is the set of all edges with both vertices in $\mathcal{V}(D_1, E_1, \gamma_1 + \delta)$. This step is successful if there exists an edge in E_2 having an endvertex in

$$\begin{aligned} K_{m,n}^{\alpha,1} = \big\{& x \in T_n^{\alpha,1} : \exists y \in F_n^{\alpha,1} \text{ such that } \{x,y\} \in E \text{ and is } (p,q)\text{-open}, \\ & y \text{ is in a } q\text{-open } m\text{-seed in } F_n^{\alpha,1}\big\}. \end{aligned}$$

Conditioned that Phase 1 is successful, Lemma 3.6 implies that this step is successful with probability at least $1 - \varepsilon$. In this case, let

$$\gamma_2(e) = \begin{cases} \gamma_1(e), & \text{if } e \notin \mathsf{E}_{D_1}, \\ \gamma_1(e) + \delta, & \text{if } e \in \Delta E_1 \setminus E_2, \\ q, & \text{if } e \in (\Delta E_2 \setminus \Delta E_1) \cap \mathsf{E}_{D_1} \cap \mathsf{E}_H, \\ p, & \text{if } e \in (\Delta E_2 \setminus \Delta E_1) \cap \mathsf{E}_{D_1} \cap \mathsf{E}_H^c, \\ 0, & \text{otherwise,} \end{cases}$$

$$\zeta_2(e) = \begin{cases} \zeta_1(e), & \text{if } e \in E_1, \\ \gamma_1(e) + \delta, & \text{if } e \in \Delta E_1 \cap E_2, \\ q, & \text{if } e \in E_2 \setminus (E_1 \cup \Delta E_1) \cap \mathsf{E}_{D_1} \cap \mathsf{E}_H, \\ p, & \text{if } e \in E_2 \setminus (E_1 \cup \Delta E_1) \cap \mathsf{E}_{D_1} \cap \mathsf{E}_H^c, \\ 1, & \text{otherwise.} \end{cases}$$

Figure 3.1 illustrates a successful realization of the first step of this phase.

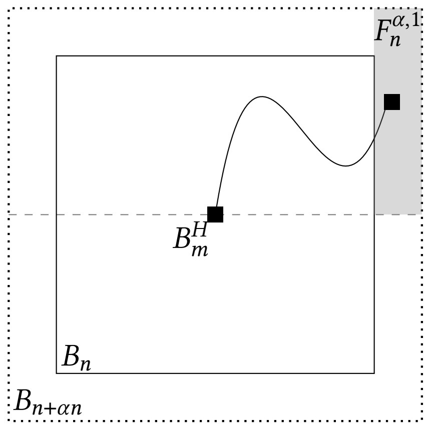

Figure 3.1: A successful realization of the first step, projected onto $\mathbb{Z}^2 \times \{0\}^{d-2}$. The black squares represent the q-open m-seeds, connected by a ζ_2-open path indicated by the black curve, whose edges are contained in the dotted box $B_{n+\alpha n}$. The gray region represents the set $F_n^{\alpha,1}$, where the new seed is found.

Having succeeded with the first step, let $b_2 \in \mathbb{Z}^s \times \{0\}^{d-s}$ be the center of the earliest seed in $F_n^{\alpha,1}$ (in some ordering of all centers) connected to B_m^H and let

$$D_2 = b_2 + B_{n+\alpha n}.$$

In this second step, we proceed to link the seed $b_2 + B_m^H$ to a new seed $b_3 + B_m^H$ inside D_2, in such a way that if we denote $b_k = (b_{k,1}, \dots, b_{k,d})$, we have

$$b_{3,1} - b_{2,1} \in [n, n + \alpha n],$$
$$|b_{3,i}| \leq n + \alpha n, \ \forall i = 2, \dots, s,$$
$$b_{3,i} = 0, \ \forall i = s + 1, \dots, d.$$

Observe that the first condition imposes a direction for the cluster of the origin to grow and the second condition constrains it to some adequate boundaries. The third condition is the requirement $b_3 + B_m^H \subset H$. They can be achieved through a steering argument analogous to the one in [13]: for a vertex $v = (v_1, \ldots, v_d) \in \mathbb{Z}^d$, let $\sigma_v : \mathbb{Z}^d \to \mathbb{Z}^d$ be the application given by

$$[\sigma_v(x)]_i = \begin{cases} -\operatorname{sgn}(v_i)x_i, & \text{if } i = 2, \ldots, s, \\ x_i, & \text{if } i = 1 \text{ or } i = s+1, \ldots, d. \end{cases} \tag{3.16}$$

We regard σ_v as the ***steering function***, which, in the present Phase, is given by (3.16). Its definition will be modified for the subsequent Phases, whenever necessary.

Let $E_3 = E_2 \cup \widetilde{E}_3$, where $\widetilde{E}_3$ is the set of all edges with both vertices in $\mathcal{V}(D_2, E_2, \gamma_2 + \delta)$. This step is successful if there exists an edge in E_3 having an endvertex in

$$\begin{aligned} b_2 + \sigma_{b_2} K_{m,n}^{\alpha,1} := \big\{ x \in b_2 + \sigma_{b_2} T_n^{\alpha,1} : \exists y \in b_2 + \sigma_{b_2} F_n^{\alpha,1} \text{ such that } \{x, y\} \in E, \\ \{x, y\} \text{ is } (p,q)\text{-open and } y \text{ is in a } q\text{-open } m\text{-seed in } b_2 + \sigma_{b_2} F_n^{\alpha,1} \big\}. \end{aligned}$$

Just as before, in case of success, we update the values of the random variables $U(e)$, $e \in \mathbb{E}^d$, recording them into the functions $\gamma_3, \zeta_3 : \mathbb{E}^d \to [0,1]$. Note that, by Lemma 3.6, conditioned that Phase 1 and the previous step are successful, this step is successful with probability greater than $1 - \varepsilon$.

The above procedure illustrates how we should proceed with the sequential algorithm in order to find our suitable seed in Π_o: from the ζ_k-open cluster of $b_k + B_m^H$ inside the box $D_k = b_k + B_{n+\alpha n}$, we give a small increase $\delta > 0$ on the parameter of the edges in its external boundary in order to open some of them. In turn, from the endpoints of these newly open edges, we try to find a (p,q)-open path to a new q-open m-seed $b_{k+1} + B_m^H$, satisfying

$$\begin{aligned} b_{k+1,1} - b_{k,1} &\in [n, n + \alpha n], \\ |b_{k+1,i}| &\leq n + \alpha n, \ \forall i = 2, \ldots, s \\ b_{k+1,i} &= 0, \ \forall i = s+1, \ldots, d. \end{aligned} \tag{3.17}$$

Given that the previous steps are successful, this happens with probability at least $1 - \varepsilon$, since in each application of Lemma 3.6, the already explored region R together with its external vertex boundary, $\Delta_v R$, never intersects $b_k + \sigma_{b_k} T_n^{\alpha,1}$. In this case, the updated values of the random variables $U(e)$, $e \in \mathbb{E}^d$, are recorded into functions $\gamma_{k+1}, \zeta_{k+1} : \mathbb{E}^d \to [0,1]$ accordingly.

The exploration process stops when we finally find a q-open m-seed $(c_2 + B_m^H) \subset \Pi_o$, such that

$$c_{2,1} \in [9n, 10n + \alpha n],$$

$$|c_{2,i}| \leq n + \alpha n, \ \forall i = 2, \ldots, s$$

$$c_{2,i} = 0, \ \forall i = s+1, \ldots, d,$$

and we say that Phase 2 is successful if such a seed is reached. Since (3.17) implies that $b_{k+1,1} \geq b_{k,1} + n$ and our initial seed is $o + B_m^H$, this is possible after the application of at most nine of the described steps. Therefore, conditioned that Phase 1 is successful, we have

$$P(\text{Phase 2 successful}|\text{Phase 1 successful}) \geq (1-\varepsilon)^9,$$

and every edge $e \in \mathbb{E}^d$ is $\gamma_{10}(e)$-closed and $\zeta_{10}(e)$-open at the end of the procedure. Figure 3.2 represents a successful connection between B_m^H and $c_2 + B_m^H$.

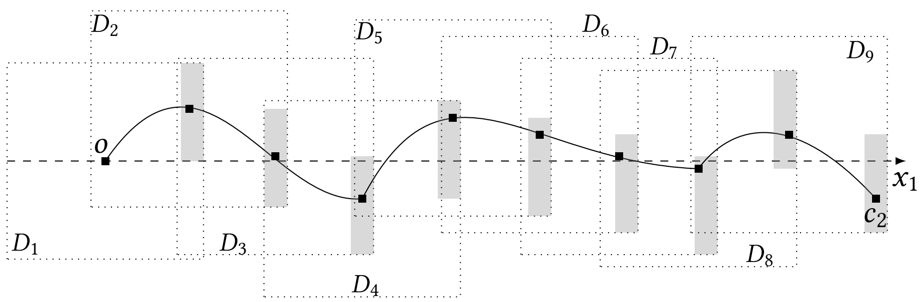

Figure 3.2: A successful realization of Phase 1, linking B_m^H to $c_2 + B_m^H$. Each black square represents the open seed obtained at the end of each step. They are linked by paths indicated by the black curves, obtained through successive applications of Lemma 3.6. Each application of the lemma considers a box $D_k = b_k + B_{n+\alpha n}$, $k = 1, \ldots, 9$, depicted by the dotted boxes. The gray regions represent the sets $b_k + \sigma_{b_k} F_n^{\alpha,1}$, where seed $b_{k+1} + B_m^H$ is found at the end of the k-th step. The dashed line is the reference by which the steering occurs, relative to the x_1-axis.

Phase 3: So far, the sequential algorithm has been applied following the restrictions imposed by (3.17), which can be interpreted as requiring the cluster of the origin to "grow along the x_1-axis in the positive direction, keeping its coordinates bounded in the other directions". Having reached seed $c_2 + B_m^H \subset \Pi_o$, we continue the exploration process in order to find a path in $\Pi_o \cup \Lambda^l_{(1,1)} \cup \Lambda^u_{(1,-1)}$ to open seeds in the site-blocks $\Lambda^l_{(1,1)}$ and $\Lambda^u_{(1,-1)}$, which means that a change of direction is necessary. As a condition for applying Lemma 3.6, this needs to be done in such a way that we do not analyze previously explored edges in the region where we intend to place the next seeds. Hence, we branch out the cluster of $c_2 + B_m^H$ into an upper and a lower component by inspecting, in two steps, the edges inside boxes of sizes $2n + \alpha n$ and $2n + 2\alpha n + \alpha^2 n$, both centered in c_2.

To put it rigorously, let $\mathcal{L} : \mathbb{R}^d \to \mathbb{R}^d$ be the linear mapping given by

$$\mathcal{L}(x_1, x_2, x_3, \dots, x_d) = (x_2, -x_1, x_3, \dots, x_d),$$

and define the steering function $\sigma_v : \mathbb{Z}^d \to \mathbb{Z}^d$, $v \in \mathbb{Z}^d$, by

$$[\sigma_v(x)]_i = \begin{cases} -\operatorname{sgn}(v_i)x_i, & \text{if } i = 3, \dots, s, \\ x_i, & \text{if } i = 1, 2 \text{ or } i = s+1, \dots, d. \end{cases}$$

The application $\mathcal{L}$ is a rotation of the x_1x_2-plane by $-\pi/2$ and introduces the change of direction of the exploration process from being parallel to the x_1-axis to being parallel to the x_2-axis. As before, σ_v will act to keep the other $d-2$ coordinates bounded. Let

$$D_{10} = c_2 + B_{2n+\alpha n}$$

and $E_{11} = E_{10} \cup \widetilde{E}_{11}$, where $\widetilde{E}_{11}$ is the set of all edges with both vertices in $\mathcal{V}(D_{10}, E_{10}, \gamma_{10} + \delta)$. This step is successful if there exists an edge in E_{11} having an endvertex in

$$\begin{aligned} c_2 + \mathcal{L}\sigma_{c_2}K^{\alpha,2}_{m,n} := \big\{ & x \in c_2 + \mathcal{L}\sigma_{c_2}T^{\alpha,2}_n : \exists y \in c_2 + \mathcal{L}\sigma_{c_2}F^{\alpha,2}_n \text{ such that } \{x,y\} \in E, \\ & \{x,y\} \text{ is } (p,q)\text{-open and } y \text{ is in a } q\text{-open } m\text{-seed in } c_2 + \mathcal{L}\sigma_{c_2}F^{\alpha,2}_n \big\}. \end{aligned}$$

After succeeding, we record the updated values of the random variables $U(e)$ into the functions $\gamma_{11}, \zeta_{11} : \mathbb{E}^d \to [0,1]$ and repeat the same step using a slightly bigger box than D_{10},

$$D_{11} = c_2 + B_{2n+2\alpha n+\alpha^2 n},$$

this time to find an edge in E_{12} having an endvertex in $c_2 - \mathcal{L}\sigma_{c_2}K^{\alpha,2+\alpha+\alpha^2}_{m,n}$. The size D_{11} is bigger to ensure that the edges incident to $c_2 - \mathcal{L}\sigma_{c_2}T^{\alpha,2+\alpha+\alpha^2}_n$ have not been explored before. If we succeed, we call the "lower" and the "upper" seeds $c_3^l + B^H_m$ and $c_3^u + B^H_m$, respectively. Thus,

$$P(\text{Phase 3 successful}|\text{Phases 1 and 2 successful}) \geq (1-\varepsilon)^2,$$

and every edge $e \in \mathbb{E}^d$ is $\gamma_{12}(e)$-closed and $\zeta_{12}(e)$-open in this case. Figure 3.3 illustrates a successful connection at Phase 3.

One can notice that Lemma 3.6 is not applicable if, instead of using the box $c_2 + B_{2n+\alpha n}$,

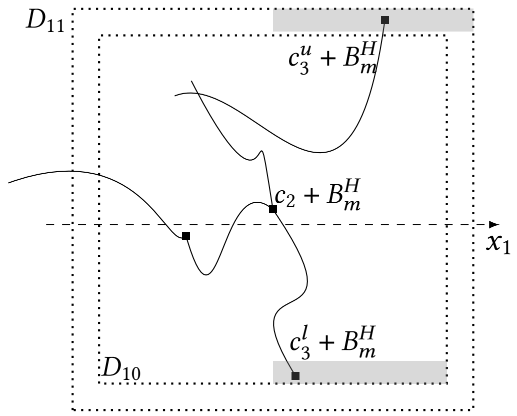

Figure 3.3: A successful connection at Phase 3, projected onto $\mathbb{Z}^2 \times \{0\}^{d-2}$. The connections between seeds occur in the same way as described in Figure 3.2.

we had considered $D_{10} = c_2 + B_{n+\alpha n}$. In this situation, as shown in Figure 3.4, we have $D_9 \cap (c_2 + \mathcal{L}\sigma_{c_2} T_n^{\alpha,1}) \neq \emptyset$, which implies that the vertices of this region may have been revealed in the previous step. Therefore, the requiremets for the subset R in the statement of Lemma 3.6 are not satisfied under this setting. This fact also explains why the renormalization scheme of Grimmett and Marstrand [13] cannot be adapted in a straightforward manner, using only one size of box, as mentioned in Remark 2.

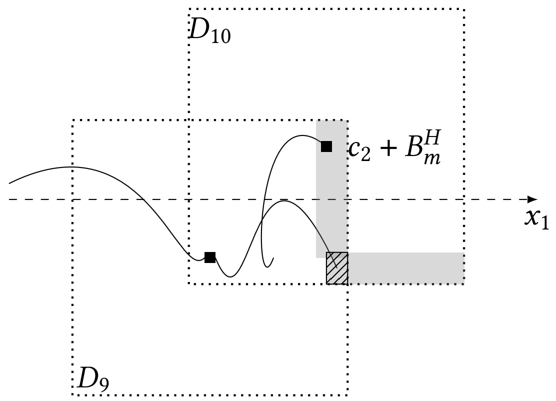

Figure 3.4: An illustration of the issue that appears if we consider $D_{10} = c_2 + B_{n+\alpha n}$. Once seed $c_2 + B_m^H$ is reached from the open paths obtained at previous steps (indicated by the black curves), we should make a change of direction, as explained in Phase 3. However, if we attempt to make such a change using D_{10} as a translate of $B_{n+\alpha n}$, then a portion of the region where seed $c_3^l + B_m^H$ (or $c_3^r + B_m^H$, depending on the position of D_9) is supposed to be found may have already been explored. As the hatched region indicates, this might be the case when we applied Lemma 3.6 using the box D_9.

Phase 4: From now on, all the subsequent phases will consist of explorations analogous to the ones in Phases 2 and 3, hence we will only give a brief explanation on how the cluster grows and mention the number of steps necessary for the accomplishment of each phase.

At Phase 4, we attempt to link $c_3^l + B_m^H$ to a q-open m-seed $(c_4 + B_m^H) \subset \Pi_o$, such that

$$
\begin{aligned}
c_{4,1} &\in [9n, 15n],\\
c_{4,2} &\in [-9n, -10n - \alpha n]\\
|c_{4,i}| &\leq 3n, \ \forall i = 3, \dots, s\\
c_{4,i} &= 0, \ \forall i = s+1, \dots, d.
\end{aligned}
$$

This phase is analogous to Phase 2, with the difference that, in the present case, we grow the cluster along the x_2-axis in the negative direction and use the plane $x_1 = 12n$ as the reference for steering the first coordinate. The steering reference for the other $s-2$ coordinates do not change. Since $c_{3,2}^l \leq n$ and $c_{4,2} \in [-9n, -10n - \alpha n]$, it takes at most 12 applications of Lemma 3.6 to reach a seed as mentioned above. Therefore, Phase 4 is successful with probability at least $(1-\varepsilon)^{12}$, conditioned on the event that we succeed at the previous phases.

Phase 5: Here we prepare another change of direction in the explored open cluster, analogous to the step used in Phase 3. We attempt to link $c_4 + B_m^H$ to a q-open m-seed $(c_5 + B_m^H) \subset \sigma_{c_4} F_n^{\alpha,2}$, where $\sigma_v : \mathbb{Z}^d \to \mathbb{Z}^d$, $v \in \mathbb{Z}^d$ is the steering function

$$
[\sigma_v(x)]_i = \begin{cases} -x_2, & \text{if } i = 2,\\ -\operatorname{sgn}(v_i) x_i, & \text{if } i = 3, \dots, s,\\ x_i, & \text{if } i = 1 \text{ or } i = s+1, \dots, d. \end{cases}
$$

This phase is successful with probability at least $1 - \varepsilon$, conditioned on the previous phases being successful as well.

Phase 6: Here we complete the exploration of the lower branch of the cluster of the origin. We attempt to link $c_5 + B_m^H$ to a seed $(\mathscr{s}_o + B_m^H) \subset \Lambda^u_{(1,-1)}$, with $\mathscr{s}_o = (\mathscr{s}_{o,1}, \dots, \mathscr{s}_{o,d}) \in \mathbb{Z}^d$ satisfying

$$
\begin{aligned}
\mathscr{s}_{o,1} &\in [24n, 25n + \alpha n],\\
\mathscr{s}_{o,2} &\in [-9n, -15n]\\
|\mathscr{s}_{o,i}| &\leq 3n, \ \forall i = 3, \dots, s\\
\mathscr{s}_{o,i} &= 0, \ \forall i = s+1, \dots, d.
\end{aligned}
$$

We perform a process similar to that of Phases 2 and 4, growing the cluster of $c_5 + B_m^H$ along the x_1-axis in the positive direction, using the plane $x_2 = -12n$ as the reference for steering the second coordinate and keeping the steering rule for the remaining coordinates the same as before. As usual, we use a translate of $B_{n+\alpha n}$ in each application of Lemma 3.6. If such a seed is reached, we declare Phase 6 successful. Since $c_{5,1} \geq 11n$ and $\mathcal{s}_{o,1} \in [24n, 25n + \alpha n]$, this is achieved within at most 13 applications of Lemma 3.6, hence the probability of success is at least $(1-\varepsilon)^{13}$.

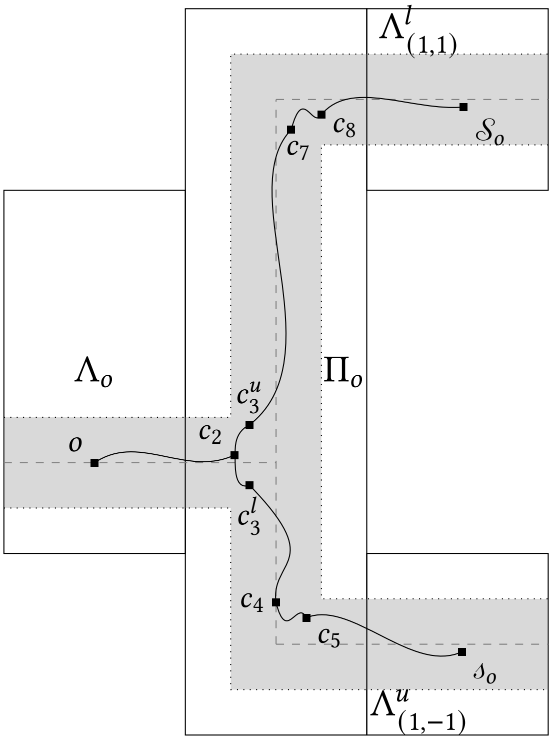

Figure 3.5: A configuration in the event $\{Z(o) = 1\}$, projected onto $\mathbb{Z}^2 \times \{0\}^{d-2}$. Each tiny black square represents the open seed obtained at the end of each phase. They are linked by paths represented by the black curves, obtained through successive applications of Lemma 3.6. The dashed lines represent the reference by which the steering occurs, relative to the x_1x_2-plane. As a consequence of adopting this reference and the parameters (α_i, β_i), $i = 1, 2, 3$, every open seed found in the exploration process lies inside the gray region, within a distance of $3n$ from the dashed lines.

Phases 7, 8 and 9: These are essentially reproductions of Phases 4, 5 and 6, respectively. This time, we apply the sequential algorithm to the "upper" branch of the cluster of the origin, attempting to link $(c_3^u + B_m^H) \subset \Pi_o$ to an open seed $(\mathcal{S}_o + B_m^H) \subset \Lambda^l_{(1,1)}$. The only relevant difference occurs at Phase 7, where Lemma 3.6 must be applied at most 24 times, instead of 12 times as in Phase 4. This is so because the box $2N\vec{u}_1 + \Lambda_o^u \subset \Pi_o$ necessarily needs to be entirely crossed during the exploration process along the x_2-axis in the positive direction.

If we succeed at all these phases, we declare $Z(o) = 1$. A configuration of this kind is illustrated in Figure 3.5. During this process, we have used Lemma 3.6 at most 75 times, therefore (3.15) implies that

$$P\Big(Z(o) = 1\Big|B_m^H \text{ is a seed}\Big) \geq (1-\varepsilon)^{75} \geq 1 - 75\varepsilon \geq \frac{1}{2}\big(1 + \vec{p}_c^{\,\text{site}}(G)\big). \tag{3.18}$$

We should also have updated the functions γ_k and ζ_k to the same extent. Thus, if $k_{\max} \in \mathbb{N}$ is the maximum number of steps used in the determination of $Z(o)$, it follows that $k_{\max} \leq 75$. Moreover, we claim that

$$\gamma_{k_{\max}}(e) \leq \zeta_{k_{\max}}(e) \leq q\mathbf{1}_{\mathsf{E}_H}(e) + p\mathbf{1}_{\mathsf{E}_H^c}(e) + 8\delta \quad \forall e \in E_{k_{\max}}, \tag{3.19}$$

which implies that every edge of $E_{k_{\max}}$ is $(p + \eta/2, q + \eta/2)$-open, since $8\delta \leq \eta/2$ by (3.15).

As a matter of fact, note that the general rule for updating the edges of $\mathbb{Z}^d$ is

$$\gamma_{k+1}(e) = \begin{cases} \gamma_k(e), & \text{if } e \notin \mathsf{E}_{D_k}, \\ \gamma_k(e) + \delta, & \text{if } e \in \Delta E_k \setminus E_{k+1}, \\ q, & \text{if } e \in (\Delta E_{k+1} \setminus \Delta E_k) \cap \mathsf{E}_{D_k} \cap \mathsf{E}_H, \\ p, & \text{if } e \in (\Delta E_{k+1} \setminus \Delta E_k) \cap \mathsf{E}_{D_k} \cap \mathsf{E}_H^c, \\ 0, & \text{otherwise,} \end{cases}$$

$$\zeta_{k+1}(e) = \begin{cases} \zeta_k(e), & \text{if } e \in E_k, \\ \gamma_k(e) + \delta, & \text{if } e \in \Delta E_k \cap E_{k+1}, \\ q, & \text{if } e \in E_{k+1} \setminus (E_k \cup \Delta E_k) \cap \mathsf{E}_{D_k} \cap \mathsf{E}_H, \\ p, & \text{if } e \in E_{k+1} \setminus (E_k \cup \Delta E_k) \cap \mathsf{E}_{D_k} \cap \mathsf{E}_H^c, \\ 1, & \text{otherwise.} \end{cases}$$

This means that any edge $e \in \mathbb{Z}^d$ such that $\zeta_{k+1}(e) = \gamma_k(e) + \delta$ or $\gamma_{k+1}(e) = \gamma_k(e) + \delta$ belongs to ΔE_k. By definition of the exploration process, this inspected edge must be contained in the box D_k. Since a box D_k, $k = 1, \ldots, k_{\max}$, intersects at most 8 other boxes (this is the case of boxes D_{10} and D_{11} used at Phase 3), such an edge is inspected at most 8 times. Therefore, $\zeta_{k_{\max}}(e) \leq q\mathbf{1}_{\mathsf{E}_H}(e) + p\mathbf{1}_{\mathsf{E}_H^c}(e) + 8\delta$ for every $e \in E_{k_{\max}}$.

If $Z(o) = 1$, we continue to apply the exploration process described, in order to determine the states of the random variables $Z(1,-1)$ and $Z(1,1)$. For each random variable, the process goes the same way as for $Z(o)$: we start with $(\mathcal{s}_o + B_m^H) \subset \Lambda_{(1,-1)}$ and $(\mathcal{S}_o + B_m^H) \subset \Lambda_{(1,1)}$ as the initial q-open m-seeds, respectively, and apply Lemma 3.6 at most 75 times, reproducing Phases 2-9 in the relevant site and passage blocks. This involves augmenting the set of explored edges $E_{k_{\max}}$ by successive applications of Lemma 3.6. By the observations made in the previous paragraph, it follows that every edge in the augmented set is $(p + \eta/2, q + \eta/2)$-open.

In general, for $x \in V \subset \mathbb{Z}^2$, we say that $Z(x) = 1$ if Phases 2-9 can be successfully performed in the region $\Lambda_x \cup \Pi_x \cup \Lambda^l_{x+(1,1)} \cup \Lambda^u_{x+(1,-1)}$, using $(\mathcal{s}_{x-(1,-1)} + B_m^H) \subset \Lambda^l_x$, if it exists, as the initial q-open m-seed, or using $(\mathcal{S}_{x-(1,1)} + B_m^H) \subset \Lambda^u_x$, if such a seed exists and the former do not. Otherwise, we say that $Z(x) = 0$.

The definition of $Z(x)$ together with the choice of $N = 6n$ imply that, for any $l \in \mathbb{N}$, given that the variables $Z((x_1, x_2))$ with $x_1 < l$ have been determined, the states of the variables $Z((x_1, x_2))$ with $x_1 = l$ are independent of each other, since the set of edges used in the exploration of the corresponding boxes are all disjoint. We use this fact to conclude the proof of Theorem 3.1 in the following manner: for $x, y \in V$, we say that $x \le y$ if $x_1 < y_1$ or $x_1 = y_1$ and $x_2 \le y_2$. This naturally defines an ordering of the sites of V. If we consider the cluster-growth of $o \in V$ with respect to $(Z(x))_{x \in V}$ according to this ordering, it follows that, at each stage, conditioned on the past exploration, the chance of augmenting the open cluster by one vertex is at least $(1 + \vec{p}_c^{\,\text{site}}(G))/2$ by (3.18), so that (3.13) is satisfied. By Lemma 3.7, it follows that there is a positive probability of the cluster of the origin on $G = (V, E)$ induced by $(Z(x))_{x \in V}$ to be infinite. On this event, there exists an infinite $(p + \eta/2, q + \eta/2)$-open path of $\mathbb{Z}^d$ within the slab $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$. □

Figure 3.6 shows a cluster-growth process with all possible types of open and closed site-blocks.

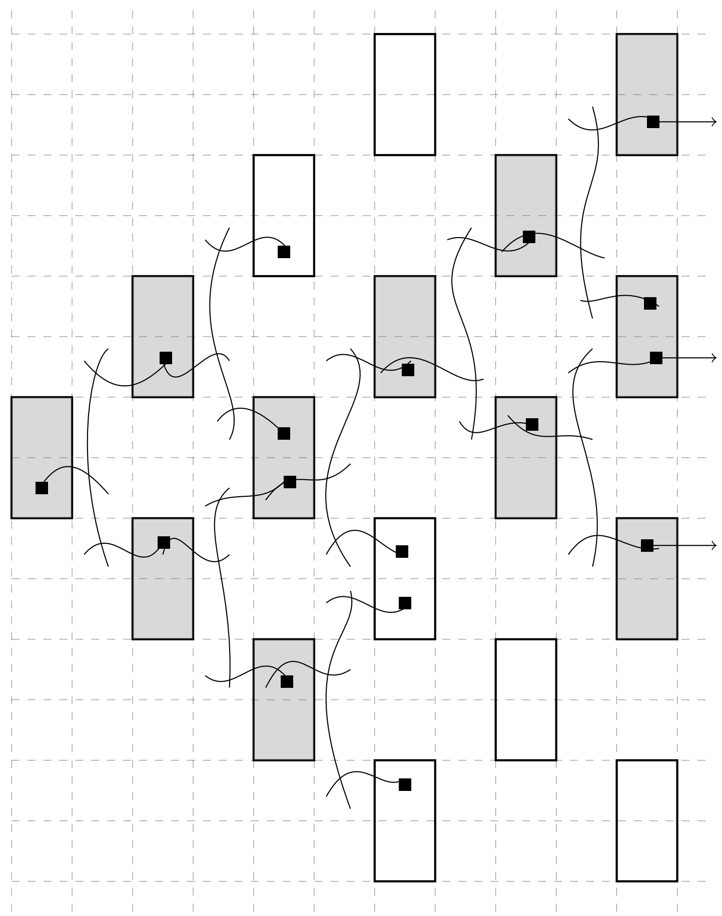

Figure 3.6: A cluster-growth process of $o \in V$ with respect to $(Z(x))_{x \in V}$. The gray site-blocks indicate $Z(x) = 1$ and the white ones indicate $Z(x) = 0$. Successful paths between adjacent site-blocks are indicated by the black curves and unsuccessful paths are omitted. Every possible combination between the placement of seeds and the value of $Z(x)$ is represented above.

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