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Teoria de Blocos para Grupos Profinitos

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FOLHA DE APROVAÇÃO

BLOCK THEORY FOR PROFINITE GROUPS

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"*Profesor Cocoon, sin saber exactamente a que se enfrentan... ¿Diría usted que es tiempo... de estrellarse las cabezas unos contra otros y sacarse los sesos?*" (Kent Brockman- Los Simpsons)

Abstract

The objective of this work is to study the techniques developed in the theory of blocks for finite groups and then, using the machinery of profinite groups and results from the modular representation theory of profinite groups, to extend the fundamental results of the theory of blocks of finite groups to profinite groups. We are thus interested in studying the block structure of the complete group algebra $k[[G]]$ of a profinite group G, where k is a field of characteristic p.

Our approach is as follows. We extend the concepts and fundamental properties of relative projectivity and vertices from profinite $k[[G]]$-modules to pseudocompact $k[[G]]$-modules. We introduce the concept of blocks of profinite groups, characterizing a block of a profinite group G as the inverse limit of blocks of finite groups G/N, where N is a open normal subgroup of G. Then we introduce the concept of defect group for a block of a profinite group, developing the basic properties and characterizations of these groups analogous to those existing for the finite case. We demonstrate a version of Brauer's Correspondence Theorem for virtually pro-p groups. Finally, we study the structure of the blocks of a profinite group with cyclic defect group. We demonstrate that these blocks have a Brauer tree algebra structure analogous to the finite case and we demonstrate that the Brauer trees for these blocks are all star type trees when the cyclic defect group is $\mathbb{Z}_p$.

Key words: Pseudocompact algebra, profinite group, inverse limit, block, defect group, Brauer tree, Brauer tree algebra.

Resumo

O objetivo deste trabalho é estudar as técnicas desenvolvidas na teoria de blocos para grupos finitos e então, utilizando o maquinario de grupos profinitos e os resultados da teoria das representações modulares para grupos profinitos, estender os resultados fundamentais da teoria de blocos de grupos finitos para grupos profinitos. Estamos, portanto, interessados em estudar a estrutura dos blocos da álgebra de grupo completa $k[[G]]$ de um grupo profinito G, onde k é um corpo de característica p.

Nossa abordagem foi feita como segue. Estendemos os conceitos e propriedades fundamentais de relatividade projetiva e vértices de $k[[G]]$-módulos profinitios para $k[[G]]$-módulos pseudocompactos. Introduzimos o conceito de blocos de grupos profinitos, caracterizando um bloco de um grupo profinito G como o limite inverso de blocos de grupos finitos G/N, onde N é um subgrupo normal aberto de G. Posteriormente introduzimos o conceito de grupo de defeito para um bloco de um grupo profinito, desenvolvendo as propriedades básicas e caracterizações destes grupos análogas às existentes para o caso finito. Demonstramos uma versão do Teorema de Correspondência de Brauer para grupos virtualmente pro-p. Finalmente, estudamos a estrutura dos blocos de um grupo profinito com grupo de defeito cíclico. Demonstramos que estes blocos possuem uma estrutura de álgebra de árvore de Brauer análoga ao caso finito e demonstramos que as árvores de Brauer para estes blocos são todas árvores do tipo estrela quando o grupo de defeito é $\mathbb{Z}_p$.

Palavras Chave: Álgebra pseudocompacta, grupo profinito, limite inverso, bloco, grupo de defeito, árvore de Brauer, álgebra de árvore de Brauer.

Contents

Chapter 1

Introduction

The modular representation theory of finite groups is an area of algebra in which the basic problem is to describe what modules can arise over the group algebra $k[G]$, where G is a finite group and k is a field of characteristic $p > 0$ dividing the order of G. The approach of this theory is not to classify the indecomposable modules in the sense of ordinary representation theory, $|G|$ coprime to p or $p = 0$, but instead to find methods for organizing the modules over a particular group algebra.

The fact that the characteristic of k divides the order of G implies immediately that the group algebra $k[G]$ is not semisimple. With this, the majority of $k[G]$-modules are not completely reducible, therefore not every $k[G]$-module is projective. Hence, we can ask "How close to being projective is a $k[G]$-module?" in this regard, the beautifully simple concepts of relative projectivity, vertex and sources are very important.

Studying the structure of the modules defined over $k[G]$ could be a difficult task. Considering a decomposition of $k[G]$ into a direct product of indecomposable algebras, called blocks, the indecomposable $k[G]$-modules can be treated as modules for one of these blocks. Studying modules for the blocks might be an easier task. This work was started by Richard Brauer in the 1930s. He studied finite group actions on vector spaces over fields with positive characteristic. Brauer observed that each block is associated with a special subgroup, called a defect group. Furthermore, he found that if D is a p-subgroup of G then there is a correspondence between the blocks of G with defect group D and blocks of the normalizer of D in G with defect group D. This is called the Brauer correspondence (see Theorem 2.3.1).

While Brauer was studying the defect groups, he observed that for a defect group of prime order, it is possible to construct a graph, called the Brauer tree, that encodes information of the blocks [5]. Basically, this graph is a tree with a cyclic ordering between the edges. This result was later extended to blocks with cyclic defect group by E. C. Dade, [8]. It is possible to relate Brauer trees with the structure of blocks with cyclic defect group. This is the notion of a Brauer tree algebra. A finite dimensional algebra A is a *Brauer tree algebra* if there is a Brauer tree such that the edges of the tree correspond to the simple modules S in such a way that the corresponding indecomposable projective module P_S has the following description. $P_S/rad(P_S) \cong soc(P_S) \cong S$ and $rad(P_S)/soc(P_S)$ is a direct sum of two (possibly zero) uniserial modules U and V corresponding to the two vertices u and v at the end of the edge. The composition factors of U, V can be read off from the graph. Blocks with cyclic defect groups are Brauer tree algebras (see Theorem 2.4.5).

For infinite groups in general, modular representation theory cannot be trivially reproduced by changing finite groups to infinite groups. There are basic concepts and properties that use strongly the finiteness of the group G. In [22] and [21], J. MacQuarrie transferred certain foundational results from the modular representation theory of finite groups to the wider context of profinite groups. Profinite groups are a category of groups where the objects can be arbitrarily large, they are usually infinite groups, but come equipped with a strong connection with certain finite quotient groups.

If k is a finite field of characteristic p and G is a profinite group, we associate to G the complete group algebra which will be denoted by $k[[G]]$. This is a topological algebra, and, as a topological space, $k[[G]]$ is Hausdorff, compact and totally disconnected. If k is an infinite discrete field, that is, k is an infinite field with discrete topology, then $k[[G]]$ still makes sense. This algebra will not be compact, but it is pseudocompact (see for example [6]). During the development of this work we note that many results about profinite algebras and modules remain true for the case of pseudocompact objects.

Based on the work of J. MacQuarrie for modular representations of profinite groups, we begin a study of the blocks of $k[[G]]$, where G is a profinite group and k is a field of characteristic p. We define the defect group of a block of G, prove its basic properties and give several alternative characterizations (Theorem 7.3.4). We then prove a Brauer Correspondence for blocks of virtually pro-p groups (Theorem 8.0.7).

We then turn to the study of blocks with cyclic defect group. When k is algebraically closed and B is a block of $k[[G]]$ with cyclic defect group, Theorem 9.2.3 describes the structure of the block and its finitely generated indecomposable projective modules, and encodes this information in a Brauer tree. We prove, analogously to the finite case that blocks with cyclic defect group have the structure of Brauer tree algebra. Furthermore, we observe in Theorem 9.2.5 that there is only one type of Brauer tree associated to blocks of $k[[G]]$ with infinite cyclic defect group, that is, Brauer trees of star type (a Brauer tree with a central vertex with all edges emanating from this vertex).

We give here a very brief overview of each chapter, drawing attention to the main results. In Chapter 2, we begin discussing a variety of main the definitions and results from the block theory approach to the modular representation theory of finite groups. In Chapter 3 we give an introduction to profinite groups. Next, the necessary pseudocompactness machinery is introduced. In Chapter 4 we introduce the useful tool of coinvariant modules and give several properties. This structure lets us work with pseudocompact modules in a easier way. In Chapter 5 we introduce basic tools to be used from the modular representation theory of profinite groups. In Chapter 4 and Chapter 5 we do not demand that the field k be finite. So the results presented here belong to the pseudocompact world. The results of [22] were stated only for the case of k finite, but frequently the proofs of [22] pass through without change to the pseudocompact world. Where more care is required we describe the necessary modifications (for example see Lemma 4.3.1). Hence in Chapter 4 and Chapter 5, several results from [22] that were proved for profinite algebras will be proved for the more general class of pseudocompact algebras. In Chapter 5 we introduce the notion of trace as a set (Definition 5.2.7). Analogously to the finite theory, where the trace map is a central tool of block theory, the notion of trace introduced here will be important too.

In Chapter 6 we introduce our main object of study, the blocks of a profinite group. We prove several basic properties of blocks, and provide a well-behaved inverse system (Proposition 6.1.2) that will allow us to deduce information about a block of G from information about blocks of the finite quotients of G. In Chapter 7 we introduce the defect group of a block of a profinite group (Definition 7.1.1) and prove its basic properties. Furthermore, we introduce the analogous concept of Brauer homomorphism (Definition 2.2.13) and we establish several characterizations of the defect group (Theorem 7.3.4).

Chapter 8 is dedicated to the proof of a Brauer correspondence for virtually pro-p groups (Theorem 8.0.7). In this chapter several results proved apply arbitrary profinite groups, but, unfortunately, there is a technical result that it was only possible to confirm for virtually pro-p groups (Lemma 8.0.6).

Finally, in Chapter 9 we do a detailed discussion of the structure of blocks with cyclic defect group. We describe the structure of the finitely generated indecomposable projective modules of blocks with cyclic defect groups (Section 9.1). Next we introduce the concept of a Brauer tree and we prove in Theorem 9.2.3 that blocks with cyclic defect group have the structure of Brauer tree algebras in the same sense as in the finite case. Furthermore, in Theorem 9.2.5 we prove that there is only one type of Brauer tree for blocks with infinite cyclic defect group. At the end of this chapter we present some simple examples of Brauer trees and Brauer tree algebras for blocks of profinite groups with cyclic defect groups.

Introdução

A teoria das representações modulares de grupos finitos é uma área da álgebra cujo problema básico é descrever quais módulos podem surgir sobre a álgebra de grupo $k[G]$ onde, G é um grupo finito e k é um corpo de característica $p > 0$ dividindo a ordem de G. A abordagem desta teoria não é classificar os módulos indecomponíveis no sentido da teoria das representações comum, $|G|$ coprimo para p ou $p = 0$, em vez disso, o objetivo é encontrar métodos para organizar os módulos sobre uma álgebra de grupo particular.

O fato da característica de k dividir a ordem de G implica de forma imediata que a álgebra de grupo $k[G]$ não é semissimples. Com isso, a maioria dos $k[G]$-módulos não são completamente redutíveis e portanto nem todo $k[G]$-módulo é projetivo. Assim é natural perguntar "Quão perto de ser projetivo está um $k[G]$-módulo?" Com isto, os belos e simples conceitos de projetividade relativa, vértice e fontes são muito importantes.

Estudar a estrutura dos módulos definidos em $k[G]$ pode ser uma tarefa difícil. Considerando uma decomposição de $k[G]$ em um produto direto de álgebras indecomponíveis, chamados blocos, os $k[G]$-módulos indecomponíveis podem ser tratados como módulos para um desses blocos. Estudar módulos para os blocos pode ser uma tarefa mais fácil.

Este trabalho foi iniciado por Richard Brauer na década de 1930. Ele estudou ações de grupos finitos em espaços vetoriais sobre corpos com características positivas. Brauer observou que cada bloco está associado a um subgrupo especial, denominado grupo de defeito. Além disso, Brauer descobriu que se D é um p-subgrupo de G, então existe uma correspondência entre os blocos de G com o grupo de defeitos D e os blocos do normalizador de D em G com grupo de defeito D. Isso é conhecido na literatura como a correspondência de Brauer (ver Teorema 2.3.1).

Enquanto Brauer estudava os grupos de defeito, ele observou que para um grupo de defeito de ordem primo é possível construir um grafo, denominado árvore de Brauer, que codifica as informações dos blocos [5]. Basicamente este grafo é uma árvore com uma ordem cíclica entre as arestas. Este resultado foi posteriormente estendido para blocos com grupo de defeitos cíclicos por E. C. Dade, [8]. É possível relacionar as árvores Brauer e a estrutura de blocos com grupo de defeito cíclico. Esta é a noção de uma álgebra da árvore de Brauer. Uma álgebra de dimensão finita A é uma *álgebra de árvore de Brauer* se houver uma árvore de Brauer tal que as arestas da árvore correspondam aos módulos S simples de tal forma que o módulo projetivo indecomponível correspondente P_S tem a seguinte descrição: $P_S/rad(P_S) \cong soc(P_S) \cong S$ e $rad(P_S)/soc(P_S)$ é uma soma direta de dois (com um deles podendo ser nulo) módulos unisseriais U e V correspondentes aos dois vértices u e v no final da aresta. Os fatores de composição de U, V podem ser lidos no grafo. Os blocos com grupos de defeito cíclico são álgebras da árvore de Brauer (ver Teorema 2.4.5).

No caso de grupos infinitos em geral a teoria das representações modulares não pode ser trivialmente reproduzida pela troca de grupos finitos por grupos infinitos uma vez que, existem conceitos básicos e propriedades que utilizam fortemente a finitude do grupo G. Em [22] e [21], J. MacQuarrie transferiu certos resultados fundamentais da teoria das representações modulares de grupos finitos para o contexto mais amplo de grupos profinitos. Grupos profinitos são uma categoria de grupos em que os objetos podem ser arbitrariamente grandes, geralmente são infinitos, porém eles vêm equipados com uma forte conexão com certos grupos quocientes finitos.

Se k é um corpo finito de característica p e G é um grupo profinito, associamos a G a álgebra de grupo completa que será denotada por $k[[G]]$. Esta é uma álgebra topológica e como espaço topológico, $k[[G]]$ é Hausdorff, compacto e totalmente desconexo. Se k é um corpo infinito discreto, isto é, k é um corpo infinito com topologia discreta, então $k[[G]]$ ainda faz sentido. Esta álgebra não será compacta, mas é pseudocompacta (ver por exemplo [6]). Durante o desenvolvimento deste trabalho, notamos que muitos resultados sobre álgebras e módulos profinitos permanecem verdadeiros para o caso de objetos pseudocompactos.

Com base no trabalho de J.MacQuarrie para representações modulares de grupos profinitos, começamos um estudo dos blocos de $k[[G]]$, onde G é um grupo profinito e k é um corpo da característica p. Definimos o grupo de defeitos de um bloco de G, provamos suas propriedades básicas e damos várias caracterizações alternativas (Teorema 7.3.4). Em seguida, provamos uma Correspondência de Brauer para blocos de grupos virtualmente pro-p (Teorema 8.0.7).

Então, passamos ao estudo de blocos com grupo de defeitos cíclicos. Quando k é algebricamente fechado e B é um bloco de $k[[G]]$ com grupo de defeito cíclico, o Teorema 9.2.3 descreve a estrutura do bloco e seus módulos projetivos indecomponíveis finitamente gerados e codifica essas informações em uma árvore Brauer. Provamos analogamente ao caso finito, que blocos com grupo de defeito cíclico possuem a estrutura da álgebra de árvore de Brauer. Além disso, observamos no Teorema 9.2.5

que existe apenas um tipo de árvore Brauer associada a blocos de $k[[G]]$ com grupo de defeitos cíclicos infinitos, ou seja, árvores Brauer do tipo estrela (uma árvore de Brauer com um vértice central com todas as arestas emanando desse vértice).

Damos aqui uma breve descrição de cada capítulo, com ênfase nos principais resultados. No Capítulo 2, começamos discutindo uma variedade de definições e resultados principais da teoria de blocos focando na teoria da representações modulares de grupos finitos. No Capítulo 3, damos uma introdução aos grupos profinitos. Em seguida, o maquinário de pseudocompacidade necessário é introduzido. No Capítulo 4, apresentamos a ferramenta útil dos módulos coinvariantes e fornecemos várias propriedades. Essa estrutura nos permite trabalhar com módulos pseudocompactos de uma maneira mais fácil. No Capítulo 5, apresentamos as ferramentas básicas a serem usadas a partir da teoria das representações modulares de grupos profinitos. No Capítulo 4 e no Capítulo 5, não exigimos que o campo k seja finito. Portanto, os resultados apresentados aqui pertencem ao mundo pseudocompacto. Os resultados de [22] foram enunciados apenas para o caso de k finito, mas freqüentemente as provas de [22] passam sem mudança para o mundo pseudocompacto. Onde deveremos tomar mais cuidado, descrevemos as modificações necessárias (por exemplo, veja Lema 4.3.1). Conseqüentemente, nos Capítulos 4 e 5, vários resultados de [22] que foram provados para álgebras profinitas serão provados para a classe mais geral de álgebras pseudocompactas. No Capítulo 5, introduzimos a noção de traço como um conjunto (Definição 5.2.7). Analogamente à teoria finita, onde o mapa de traços é uma ferramenta central da teoria de blocos, a noção de traço introduzida aqui também será importante.

No Capítulo 6, apresentamos nosso principal objeto de estudo, os blocos de um grupo profinito. Provamos várias propriedades básicas de blocos e fornecemos um sistema inverso bem comportado (Proposição 6.1.2) que nos permitirá deduzir informações sobre um bloco de G a partir de informações sobre blocos dos quocientes finitos de G. No Capítulo 7, introduzimos o grupo de defeitos de um bloco de um grupo profinito (Definição 7.1.1) e provamos suas propriedades básicas. Além disso, introduzimos o conceito análogo de homomorfismo de Brauer (Definição 2.2.13) e estabelecemos várias caracterizações do grupo de defeitos (Teorema 7.3.4).

O Capítulo 8 é dedicado à prova de uma correspondência de Brauer para grupos virtualmente pro-p (Teorema 8.0.7). Neste capítulo vários resultados provaram ser aplicados para grupos profinitos arbitrários, mas infelizmente há um resultado técnico que só foi possível confirmar para grupos virtualmente pro-p (Lemma 8.0.6).

Finalmente, no Capítulo 9, fazemos uma discussão detalhada da estrutura de blocos com grupo de defeito cíclico. Descrevemos a estrutura dos módulos projetivos indecomponíveis finitamente gerados de blocos com grupo de defeito cíclico (Seção 9.1). A seguir introduzimos o conceito de árvore de Brauer e provamos no Teorema 9.2.3 que blocos com grupo de defeito cíclico possuem a estrutura de álgebras de árvore de Brauer no mesmo sentido que no caso finito. Além disso, no Teorema 9.2.5 provamos que existe apenas um tipo de árvore Brauer para blocos com grupo de defeito cíclico infinitos. No final deste capítulo, apresentamos alguns exemplos simples de árvores de Brauer e álgebras de árvores de Brauer para blocos de grupos profinitos com grupos de defeitos cíclicos.

Chapter 2

Block Theory for Finite Group Algebras

In this chapter, we present a summary of the part of the modular representation theory and the block theory of finite group algebras. The idea of this chapter is to give a guide to the readers of the basic tools required to study the block theory for finite groups.

The main sources to write this chapter were the books "Local Representation Theory" by J.L. Alperin [1], "Representations and Cohomology I" by D.J. Benson [3], "A Course in Finite Group Representation Theory" by Peter Webb [31] and "The Block Theory of Finite Group Algebras, Volume 1 and 2" by M. Linckelmann [19],[20]. We consider that Alperin's book is the most accessible reference for this topic, but we must at times be careful since this book assumes that the base field k is algebraically closed, where in practice ours might not be.

A basic knowledge of rings, fields, algebras and modules for these objects is assumed. Unless explicitly stated, the term "module" will refer to a *left module*.

2.1 Relative Projectivity

Consider a finite group G and k a field of characteristic $p > 0$. We want to know what are the modules over the group algebra $k[G]$. If the characteristic of k divides the order of G, many $k[G]$-modules are not projective. But we can approach in the modules that are "almost projective", in the sense explained bellow (2.1.2). We use the notation ${}_AX_B$ to denote the $A - B$-bimodule X.

Definition 2.1.1. *1. Let H be a subgroup of G and V a left $k[H]$-module. The* ***induced*** $k[G]$***-module*** *is defined by*

$$V\uparrow^G := {}_{k[G]}k[G]_{k[H]} \otimes_{k[H]} V = k[G] \otimes_H V,$$

where the multiplication from G is given by $g(x \otimes v) = gx \otimes v$, for all $g \in G$, $x \in k[G]$ and $v \in V$.

2. If U is a $k[G]$-module, the ***restricted*** $k[H]$***-module*** *$U\downarrow_H$ is the original $k[G]$-module U with action restricted to the subalgebra $k[H]$.*

If H is a subgroup of G the functor $(\_)\uparrow_H^G$ is left adjoint to $(\_)\downarrow_H^G$. The unit $\eta : 1 \to (\_)\uparrow^G\downarrow_H$ is given by $\eta_V(v) = 1 \otimes v$ and the counit $\varepsilon : (\_)\downarrow_H\uparrow^G \to 1$ by $\varepsilon_U(g \otimes u) = gu$.

Definition 2.1.2. *A $k[G]$-module U is* ***relatively*** H***-projective*** *if given any diagram of $k[G]$-modules and $k[G]$-module homomorphisms of the form*

$$\begin{array}{ccc} & & U \\ & & \big\downarrow{\scriptstyle \varphi} \\ V & \underset{\beta}{\twoheadrightarrow} & W \end{array}$$

such that there exists a $k[H]$-module homomorphism $U \longrightarrow V$ making the triangle commute, then there exists a $k[G]$-module homomorphism $U \longrightarrow V$ making the triangle commute.

Observe that a projective module is the same thing as a 1-projective module. The next result gives a useful characterization of relative projectivity. We write $U \mid V$ to mean that the module U is isomorphic to a direct summand of the module V.

Lemma 2.1.3. *Let G be a finite group and H a subgroup of G. If U is a $k[G]$-module then the following are equivalent:*

1. *U is relatively H-projective.*

2. *If a surjective homomorphism $W \twoheadrightarrow U$ of $k[G]$-modueles splits as a $k[H]$-module homomorphism, then it splits as a $k[G]$-module homomorphism.*

3. $U|\ (U\downarrow_H\uparrow^G)$.

4. $U|\ (X\uparrow^G)$ *for some $k[H]$-module X.*

An important characterization of relative projectivity is Higman's criterion. Before we establish this criterion we introduce one tool that will be involved.

Definition 2.1.4. *If $H \leqslant G$ and U_1, U_2 are $k[G]$-modules, then the **trace map** is the map $Tr_H^G : Hom_{k[H]}(U_1\downarrow_H, U_2\downarrow_H) \longrightarrow Hom_{k[G]}(U_1, U_2)$ defined by $\alpha \longmapsto \sum_{g\in G/H} g\alpha g^{-1}$, where G/H denotes a set of left coset representatives of H in G.*

The properties of this map are discussed in [3, Lemma 3.6.3]. Now, we state Higman's Criterion:

Theorem 2.1.5. *Let G be a finite group and let $H \leqslant G$. Then a $k[G]$-module U is relatively H-projective if and only if $id_U \in Tr_H^G(Hom_{k[H]}(U\downarrow_H, U\downarrow_H))$.*

Our interest is to study the indecomposable modules relatively projective to subgroups of G. Observe that modules projective relative to small subgroups may be considered as "closer" to being projective.

Definition 2.1.6. *Let U be an indecomposable $k[G]$-module. A subgroup Q of G is called a **vertex** of U if it is a minimal subgroup of G such that U relatively Q-projective.*

If Q is a vertex of U, then a **source** of U is an indecomposable $k[Q]$-module S such that $U \mid S\uparrow^G$.

If U be an indecomposable $k[G]$-module, by [3, Proposition 3.10.2], the vertices of U are p-subgroups of G and they are conjugate in G.

Proposition 2.1.7. *Let U be an indecomposable $k[G]$-module with vertex Q, and P Sylow p-subgroup of G which contains Q. Then $dim(U)$ is divisible by $|P : Q|$.*

For a proof see [12, Theorem 9, Corollary 2].

Example 2.1.8. *Let P be a Sylow $p-$subgroup of G. Proposition 2.1.7 shows that P is a vertex of the trivial $k[G]$-module, since it has dimension* 1.

2.1.1 G-algebras and the Trace map

Let G be a finite group and A be a finite dimensional associative k-algebra. Then A is a G-**algebra** over k if A can be endowed with an action $G \times A \longrightarrow A$ of G on A, written $(g, a) \longrightarrow {}^g a$ for any $a \in A$ and $g \in G$, such that the map sending $a \in A$ to ${}^g a$ is a k-algebra automorphism of A for every $g \in G$. If A is a G-algebra over k, we denote for every subgroup H of G by A^H the subalgebra of all H-fixed points in A; that is, $A^H = \{a \in A \mid {}^h a = a \; \forall \; h \in H\}$.

Observe that, if H, L are subgroups of G, and if $H \leqslant L$, then $A^L \subseteq A^H$.

Definition 2.1.9. *Let H, L be subgroups of G with $H \leqslant L$. The **trace map** is the linear map $Tr_H^L : A^H \longrightarrow A^L$ defined by $Tr_H^L(a) = \sum_{g \in L/H} {}^g a$, where L/H denotes a set of left coset representatives of H in L.*

If U, W are finite dimensional $k[G]$-modules, $Hom_k(U, W)$ is a finite dimensional G-algebra with action $({}^g\rho)(u) = g\rho(g^{-1}u)$. In particular, when $G = L$, the notations Tr_H^G used in Definitions 2.1.4 and 2.1.9 are consistent for finite groups.

Lemma 2.1.10. *Let G be a finite group, H, L subgroups of G and let A be a finite dimensional G-algebra. Then*

1. *If $H \leqslant L$, for any $a \in A^H$ and $b \in A^L$, $bTr_H^L(a) = Tr_H^L(ba)$ and $Tr_H^L(a)b = Tr_H^L(ab)$.*
2. *If $H \leqslant L$, then $Tr_L^G \circ Tr_H^L = Tr_L^G$.*
3. *[Mackey's Formula]For any $a \in A^L$, $Tr_L^G(a) = \sum\limits_{g \in H\backslash G/L} Tr_{H \cap {}^g L}^H({}^g a)$, where $H\backslash G/L$ denote the set of double coset representatives of H and L in G.*

For a proof of these properties of the trace map you can confer [19, Proposition 2.5.4, 2.5.5]. Observe that, by Part 1 of 2.1.10, $Tr_H^L(A^H)$ is an ideal of A^L.

2.2 Blocks and Defect Groups

Sometimes studying the modules over $k[G]$ can be a hard task, but considering a decomposition of $k[G]$ into indecomposable direct algebra factors, called *blocks*, then instead of studying the structure of modules over $k[G]$ we can study the modules for the blocks of $k[G]$, which might be easier.

Definition 2.2.1. *Let A be a finite dimensional k-algebra. An element $e \in A$ is **idempotent** if $e^2 = e$. Two idempotents e, f of A are **orthogonal** if $ef = fe = 0$. A non-zero idempotent is **primitive** if it cannot be written as the sum of two non-zero orthogonal idempotents.*

*We denote by $Z(A)$ the center of A. An idempotent $e \in A$ is called **centrally primitive** if e is a primitive idempotent considered as an idempotent of $Z(A)$.*

There is a unique decomposition of 1_A as $1_A = e_1 + \cdots + e_n$, where each e_i is a centrally primitive idempotent, and that decomposition corresponds to a unique decomposition of A as $A = B_1 \times ... \times B_n$, where each $B_i = Ae_i$ ([31, Proposition 3.6.1]). Each B_i is called a **block** of A and each e_i is called **block idempotent**.

Example 2.2.2. 1. *Let G be the symmetric group S_3 of order 6. Observe that $G = C_3 \rtimes C_2 =< a, b : a^3 = b^2 = 1,\ bab^{-1} = a^{-1} >$, where $C_3 =< a >$ is the cyclic group generated by a and $C_2 =< b >$ is the cyclic group generated by b.*

If k is a field of characteristic 2, then the block idempotents of $k[G]$ are $e_1 = 1 + a + a^2$ and $e_2 = a + a^2$. Furthermore, observe that $k[G]e_1 \cong k[x]/(x^2)$ and $k[G]e_2 \cong M_2(k)$.

If k is a field of characteristic 3, since C_3 is normal in G, by [3, Proposition 6.2.2], $k[G]$ only has one block.

2. *Let G be the group $S_3 \times S_3$. Note first that if H, L are finite groups, then $k[H \times L] \cong k[H] \otimes_k k[L]$ by [19, Theorem 1.1.4]. If k is of characteristic 2, then by [29, Proposition 2.3], the block idempotents of $k[G]$ are $\{e_i \otimes e_j : 1 \leqslant i, j \leqslant 2\}$. So $k[G] = B_1 \times B_2 \times B_3 \times B_4 = k[G](e_1 \otimes e_1) \times k[G](e_1 \otimes e_2) \times k[G](e_2 \otimes e_1) \times k[G](e_2 \otimes e_2)$.*

If k is a field of characteristic 3, since $C_3 \times C_3$ is normal in G, then, by [3, Proposition 6.2.2], $k[G]$ only has one block.

3. *Let k be a field of characteristic 2 and let G be the group $\underbrace{S_3 \times S_3 \times \cdots \times S_3}_{n\text{-times}}$, for $n > 2$. In this case, the blocks of G are of the form*

$$k[G](e_1' \otimes e_2' \otimes \quad \cdots \quad \otimes e_n'),$$

where either $e_i' = e_1$ or $e_i' = e_2$, for $1 \leqslant i \leqslant n$.

Definition 2.2.3. *Let U be an A-module and B_i a block of A. Then U **lies in** B_i if $B_iU = U$ and $B_jU = 0$ for all $j \neq i$.*

If U is an A-module then U has a unique direct sum decomposition $U = U_1 \oplus ... \oplus U_n$, where each U_i lies in the block B_i (cf. [1, IV,§13, Proposition 2]).

So, suppose that $k[G]$ has a decomposition into indecomposable direct algebra factors

$$k[G] = B_1 \times \cdots \times B_n,$$

where each block B_i has the form $k[G]e_i$ for some block idempotent e_i.

The next result says that the simple modules can give interesting information about the structure of the blocks.

Proposition 2.2.4. *Let S, T be simple $k[G]$-modules. Then the following are equivalent:*

(1) S, T lie in the same block of G.

(2) There is a sequence of simple $k[G]$-modules $S = S_1, S_2, ..., S_n = T$ such that S_j, S_{j+1}, for each $j \in \{1, 2, ..., n-1\}$, are composition factors of an indecomposable projective $k[G]$-module.

(3) *There is a sequence of simple $k[G]$-modules $S = T_1, T_2, ..., T_m = T$, such that T_j, T_{j+1}, for each $j \in \{1, 2, ..., m-1\}$, are equal or there is a non-split extension of one by the other.*

For a proof see [31, Proposition 12.1.7]

2.2.1 Defect groups

The defects groups are subgroups of G that measure how long are the blocks to be semisimple algebras. To define a defect of a block of G, our main tool will be the relative trace map. In this section we will consider $k[G]$ as a G-algebra with G acting on $k[G]$ by conjugation.

Definition 2.2.5. *Let B be a block of $k[G]$ for a finite group G with block idempotent e. A* ***defect group*** *of B is a minimal subgroup D of G such that $e \in Tr_D^G(k[G]^D)$.*

Theorem 2.2.6. *Let G be a finite group and B a block of G with block idempotent e. Let D be a defect group of B.*

1. *The group D is a p-subgroup of G.*
2. *For any subgroup H of G such that $e \in Tr_H^G(k[G]^H)$ there is an element $g \in G$ such that $D \subseteq {}^gH = gHg^{-1}$.*
3. *The defect groups of B form a G-conjugacy class of p-subgroups of G.*

For a proof see [19, Theorem 5.6.5].

If the order $|D| = p^d$, then d is called the **defect** of B. The defect of a block B can be characterized through the dimension of simple modules lying in B as follows:

Theorem 2.2.7. *Let k be an algebraically closed field of characteristic p, let G be a finite group of order $p^a m$ with $(p, m) = 1$ and let B be a block of $k[G]$ with defect d. Then d is the smallest integer such that p^{a-d} divides the dimensions of all simple modules lying in B.*

For a proof see [7, §86.5].

Proposition 2.2.8. *If B is a block of G with block idempotent e, and defect group D, then any $k[G]$-module lying in B is D-projective, and there is a simple $k[G]$-module T lying in B with vertex D.*

For a proof see [16, Theorem 2.2].

Observe that if B is a block of the finite group G and the trivial module k lies in B, then, follow from Proposition 2.2.8 and example 2.1.8 that B has defect group a p-Sylow subgroup of G.

Now, considering $k[G]$ as a $k[G \times G]$-module, with the action $(g_1, g_2)x = g_1 x g_2^{-1}$, for all $g_1, g_2 \in G$ and $x \in k[G]$, we have that each block B of $k[G]$ is the same as an indecomposable summand of $k[G]$ as a $k[G \times G]$-module, with the above multiplication.

Definition 2.2.9. *Let G be a finite group. The* ***diagonal homomorphism*** *is the group homomorphism $\delta : G \longrightarrow G \times G$, defined by $\delta(g) = (g, g)$.*

Observe that if H and K are subgroups of G with $\delta(H)$ and $\delta(K)$ conjugate in $G \times G$ then H and K are conjugate in G (cf. [1, IV,§13, p. 96]).

So, we can frame defect groups in terms of relative projective modules.

Observe that $k[G] \cong k \uparrow_{\delta(G)}^{G\times G}$ as $k[G \times G]$-module (cf. [19, Proposition 5.11.6]). Thus a block B is $\delta(G)$-projective, and it has a vertex $\delta(Q)$, as a $k[G \times G]$-module, where Q is a p-subgroup of G.

Theorem 2.2.10. *Let D be a subgroup of G and let B be a block of G with block idempotent e. The following statements are equivalent.*

1. *D is a defect group of B.*
2. *$\delta(D)$ is a vertex of B as $k[G \times G]$-module.*

For a proof see [31, Theorem 12.4.5].

The following theorem, due J.A. Green, describes defect group as an intersection of two Sylow subgroups.

Theorem 2.2.11. *The defect group D of any block B of G is expressible as an intersection $P \cap gPg^{-1}$, for some $g \in G$, where P is a p-Sylow subgroup of G containing D.*

For a proof see [3, Proposition 6.1.1].

Corollary 2.2.12. *Let G be a finite group and B a block of G with defect group D. Then D is the largest normal p-subgroup of $N_G(D)$.*

For a proof see [31, Corollary 12.3.4].

2.2.2 Brauer construction and defect groups

Definition 2.2.13. 1. *Given a subgroup D of G, the* ***Brauer quotient*** *is defined as the quotient algebra*

$$k[G]^{[D]} = k[G]^D / \sum_{Q \subsetneqq D} Tr_Q^D(k[G]^Q). \tag{2.1}$$

2. *The* Brauer homomorphism *is the natural projection*

$$Br_D : k[G]^D \longrightarrow k[G]^{[D]}.$$

Observe that $Tr_Q^D(k[G]^Q)$ is an ideal of $k[G]^D$ for each subgroup Q of D. So, the sum $\sum_{Q \subsetneqq D} Tr_Q^D(k[G]^Q)$ is again an ideal.

With the help of the Brauer homomorphism is possible provide one more characterization of the defect group, useful to understand the Brauer correspondence (see 2.3.1).

Theorem 2.2.14. *Let D be a p-subgroup of a finite group G and B a block of G with block idempotent e. The following are equivalent:*

1. *B has a defect group D.*
2. *$\delta(D)$ is a vertex of B as $k[G \times G]$-module.*

3. *D is a maximal p-subgroup such that $Br_D \neq 0$.*

4. *$e \in Tr_D^G(k[G]^D)$ and $Br_D(e) \neq 0$.*

For a proof see [20, Theorem 6.2.1].

We finish this section with some examples of defect groups:

Example 2.2.15. 1. *Let k be a field of characteristic 2 and let G be the group $S_3 =< a, b : a^3 = b^2 = 1, bab^{-1} = a^{-1} >$. In Example 2.2.2 we saw that $k[G]$ has two blocks $B_1 = k[G]e_1$ and $B_2 = k[G]e_2$, where $e_1 = 1 + a + a^2$ and $e_2 = a + a^2$ are the respective block idempotents. Observe that $Tr_{C_2}^G(e_1) = e_1$ and $Tr_1^G(a) = e_2$, where $C_2 =< b >$ (the cyclic group generated by b). Additionally, since $C_G(C_2) = C_2$, then $Br_{C_2}(e_1) = 1$, $Br_{C_2}(e_2) = 0$, $Br_1(e_1) = e_1$ and $Br_1(e_2) = e_2$. Then, by Theorem 2.2.14, $< b >$ is a defect group of B_1 and 1 is a defect group of B_2.*

If k is a field of characteristic 3, then as we noted in Example 2.2.2, $k[G]$ only has one block, and by Proposition 2.2.8, C_3 is a defect group of this block.

2. *Let k be a field of characteristic 2 and let G be the group $S_3 \times S_3$. In Example 2.2.2 we noted that $k[G]$ has four blocks: $k[G](e_1 \otimes e_1)$, $k[G](e_1 \otimes e_2)$, $k[G](e_2 \otimes e_1)$ and $k[G](e_2 \otimes e_2)$. By [29, Proposition 2.6], $k[G](e_i \otimes e_j)$ has a defect group $D_i \times D_j$, where D_i is a defect group of $k[S_3]e_i$ and D_j is a defect group of $k[S_3]e_j$, for $i, j \in \{1, 2\}$.*

3. *Let k be a field of characteristic 2 and let G be the group $\underbrace{S_3 \times S_3 \times \cdots \times S_3}_{n-times}$, for $n > 2$. Then, the block idempotents are of the form $e'_1 \otimes e'_2 \otimes \cdots \otimes e'_n$, where either $e'_i = e_1$ or $e'_i = e_2$, for $1 \leqslant i \leqslant n$. The defect groups are of the form $\prod_{i=1}^n D_i$, where D_i is a defect group of $k[S_3]e'_i$, for $1 \leqslant i \leqslant n$.*

2.3 Brauer correspondence for finite groups

Let D be a p-subgroup of G. It is possible to establish a bijective correspondence between the blocks of $k[N_G(D)]$ with defect group D and the blocks of $k[G]$ with defect group D.

By [3, Lemma 6.2.1], $k[G]^D = k[C_G(D)] \oplus \sum_{Q \lneqq D} Tr_Q^D(k[G]^Q)$. So we consider the Brauer map as the surjective map $Br_D : k[G]^D \longrightarrow k[C_G(D)]$. If B is a block of G with defect group D and block idempotent e, then $Br_D(e)$ turns out to be a block idempotent of $N_G(D)$, whose corresponding block has defect group D. This map establishes the following correspondence:

Theorem 2.3.1 (Brauer's Correspondence)**.** *Let G be a finite group and D be a p-subgroup of G. The Brauer map Br_D establishes a one-one correspondence between the blocks of G with defect group D and the blocks of $N_G(D)$ with defect group D.*

For a proof see [31, Theorem 12.6.4].

2.4 Blocks with cyclic defect groups. Brauer Trees and Brauer Tree Algebras

In this section, we give a summary of the techniques related with the study of blocks of finite groups with cyclic defect groups. If G is a finite group, the structure of a block of G with cyclic defect group is determined by the properties of a certain type of graph with extra structure, called a *Brauer tree.* The information of the blocks can be encoded inside this finite graph. For more details, we recommend [20, Chapter 11], [3, Chapter 6] and [1, Chapter V].

Throughout of this section, we consider k to be an algebraically closed field of characteristic p.

Denote by $\mathcal{S}$ a set of representatives of the isomorphism classes of simple modules in B and $|\mathcal{S}|$ the number of elements of $\mathcal{S}$.

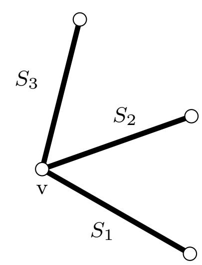
Lemma 2.4.1. *Let G be a finite group and B a block of G with non-trivial cyclic defect group D. Then $|\mathcal{S}|$ divides $p-1$.*

Proof. By [20, Theorem 11.1.3], $|\mathcal{S}|$ is equal to the order of the inertial group E (cf. [20, Definition 6.7.7]). But $|E|$ divides $p-1$ by [20, Theorem 11.1.1], so $|\mathcal{S}|$ divides $p-1$. □

2.4.1 Brauer trees and Brauer tree algebras

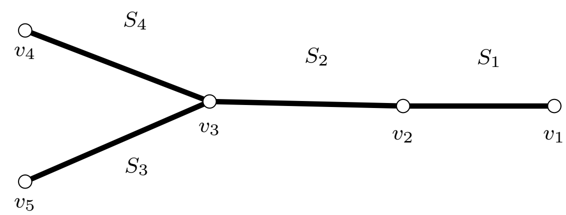
Definition 2.4.2. *A **Brauer tree** Γ is a finite, connected, undirected graph without loops or cycles (so it is a tree) together with a circular ordering of the edges emanating from each vertex. A **Brauer tree with exceptional vertex of multiplicity** m is a Brauer tree with a distinguished vertex, to which we attach a positive integer m.*

The cyclic ordering on the edges around a vertex is given by circling the vertex in an anti-clockwise direction. For example



the edge S_1 emanating from the vertex v has a "next" edge S_2, emanating from v and to the edge S_2 emanating from a vertex v has a "next" edge S_3, and the edge S_3 emanating from a vertex v has a "next" edge S_1. Call γ_v the cyclic permutation on the set of edges adjacent to v associating the "next edge" to any edge in the above ordering.

Another example, consider the following Brauer tree



then to the edges emanating from the vertex v_3, the ordering is S_4, S_3, S_2. To the vertex v_2 the ordering is S_2, S_1, and to the vertex v_1 the ordering is S_1. To the vertices v_4, v_5 the order is analogous to the case of v_1.

Following, we define the radical and the socle of a module before introducing the Brauer tree algebra's concept.

Definition 2.4.3. *Let A be finite dimensional k-algebra and let U be an A-module.*

1. *The **radical of** U, denoted by $rad(U)$, is the intersection of all maximal A-modules.*

2. *The **socle of** U, denoted by $soc(U)$ is the maximal semisimple submodule of U.*

Observe that by [31, Theorem 7.3.8] there is one-to-one correspondence between isomorphism classes of indecomposdable projective A-modules and isomorphism classes of simple A-modules given by: if P is an indecomposable projective A-module, then $P/rad(P)$ is isomorphic to a simple A-modulo S, and, on the other hand, if S is a simple A-module, there is an indecomposable projective module P_S, unique until isomorphism, such that $P_S/rad(P_S) \cong S$.

Finally, we are ready to introduce the concept of a Brauer tree algebra.

Definition 2.4.4. *Given a Brauer tree Γ, we say that a finite dimensional k-algebra A is the Brauer tree algebra associated to Γ if*

1. *There is a one-to-one correspondence between the edges of the tree and the isomorphism classes of simple A-modules,*

2. *the top $P/rad(P)$ of the indecomposable projective A-module P is isomorphic to the socle of P,*

3. *the projective cover P corresponding to the edge S is such that*

$$rad(P)/soc(P) \cong U^v(S) \oplus U^w(S)$$

for two (possibly zero) uniserial A-modules $U^v(S)$ and $U^w(S)$, where v, w are the vertices adjacent to the edge S,

4. *if v is not the exceptional vertex and if v is adjacent to the edge S then $U^v(S)$ has $s(v) - 1$ composition factors, where $s(v)$ is the number of edges adjacent to v,*

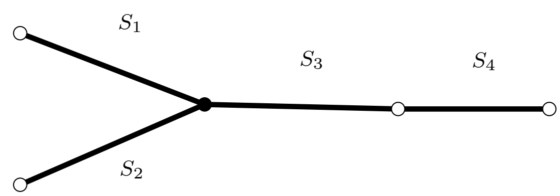
5. *if v is the exceptional vertex with multiplicity m, and if v is adjacent to S, then $U^v(S)$ has $m \cdot s(v) - 1$ composition factors,*

6. *if v is adjacent to S then the composition factors of $U^v(S)$ are described as*

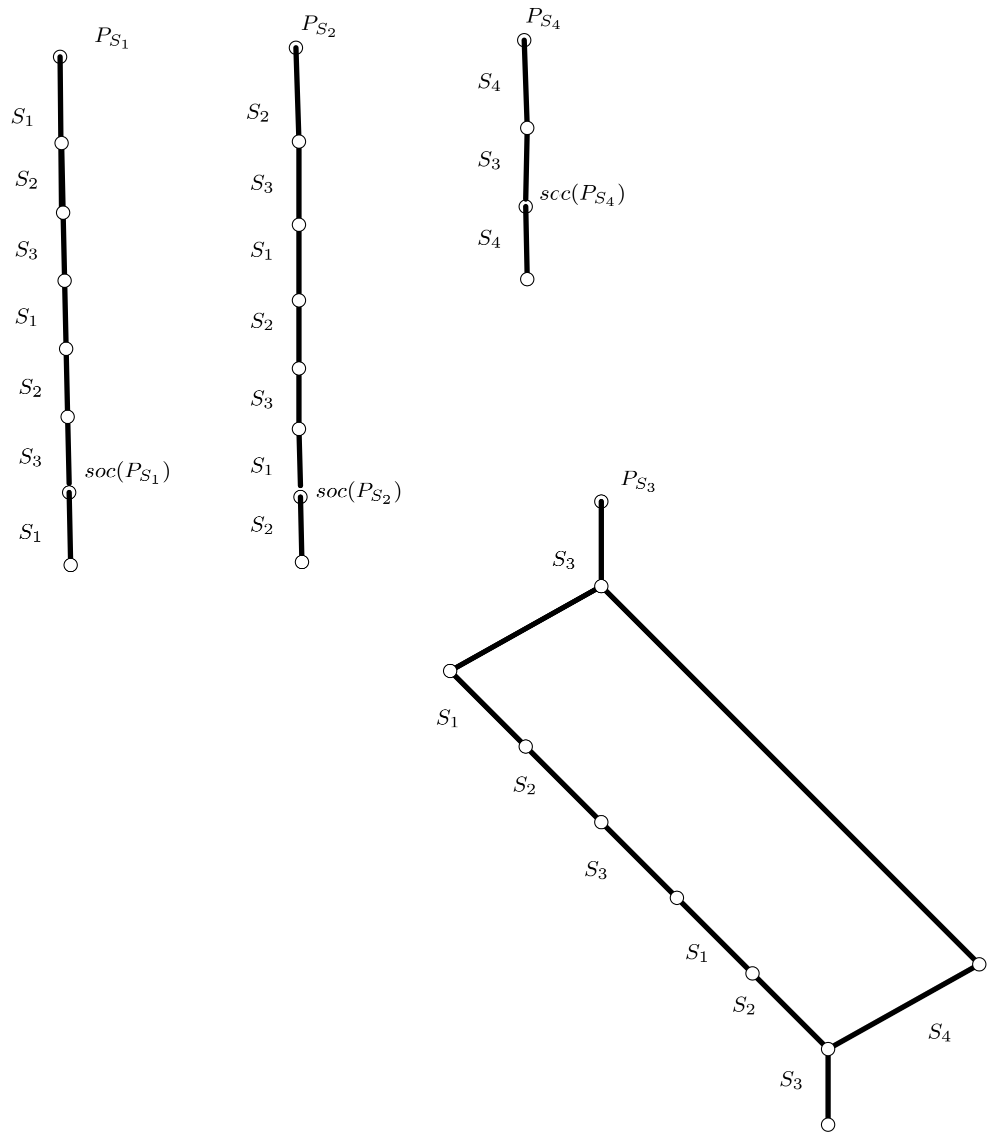
$$rad^j(U^v(S))/rad^{j+1}(U^v(S)) \cong \gamma_v^{j+1}(S),$$

for all j as long as j is smaller than the number of composition factors of $U^v(S)$.

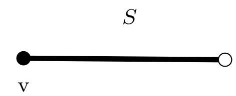
Some examples will help to better understand the idea of a Brauer tree algebra. The following examples are from [1, V, §17]. Consider the following Brauer tree, with exceptional vertex colored black with multiplicity two:



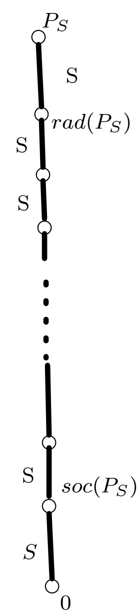
The Brauer tree algebra A has four indecomposable projective modules, and these modules can be represented by the following diagrams:


P_{S_1}
S_1
S_2
S_3
S_1
S_2
S_3
$soc(P_{S_1})$
S_1
P_{S_2}
S_2
S_3
S_1
S_2
S_3
S_1
$soc(P_{S_2})$
S_2
P_{S_4}
S_4
S_3
$scc(P_{S_4})$
S_4
P_{S_3}
S_3
S_1
S_2
S_3
S_1
S_2
S_4
S_3

Another example, Let Γ be the Brauer tree with exceptional vertex v and multiplicity m



Then the Brauer tree algebra A has exactly one simple module S, and the projective module corresponding to the Brauer tree algebra can be represented by the following diagram:



where S appears $m+1$ times in total.

It is possible to show that this Brauer tree algebra is the group algebra $k[G]$ when G is a cyclic p-group of order p^n, and in this case $m = p^n - 1$. The simple module S is the trivial $k[G]$-module k. This example shows that the group algebra of a cyclic p-group is a Brauer tree algebra.

Observe that this group algebra is indecomposable so has only one block, and this block is a Brauer tree algebra. With some conditions, we can find a relation between the blocks of group algebras and Brauer tree algebras. The idea is that blocks with cyclic defect groups, can be seen as Brauer tree algebras.

Theorem 2.4.5. *Suppose k algebraically closed and B is a block of G with non-trivial cyclic defect group D. Then B is a Brauer tree algebra for a tree with $|\mathcal{S}|$ edges and multiplicity $\frac{|D|-1}{|\mathcal{S}|}$.*

For a proof see [33, Theorem 5.10.37].

Chapter 3

Inverse Limits, Profinite Groups and Pseudocompactness

In this chapter we introduce the basic definitions and standard results required to work with pseudocompact objects. We start by introducing the notion of inverse limits of topological spaces and defining profinite groups. Next we introduce the concepts and properties of pseudocompact algebras and modules. The properties presented here are oriented to develop a block theory for profinite groups.

The main sources to write this chapter were: for the first part the books "Profinite Groups" by L. Ribes and P. Zalesskii [25] and "Profinite Groups" by J. Wilson [32]. For the second part the books "Algebraic topology" by S. Leftschetz [18] and the articles [11],[15].

Basic knowledge about topological spaces, continuity and compactness are required.

3.1 Inverse Limits and Profinite Groups

To see more details about these topics we recommend the books [25] and [32].

Definition 3.1.1. *A **directed set** is a partially ordered set I with the additional property that for all $i_1, i_2 \in I$, there exists $j \in I$ such that $j \geqslant i_1$ and $j \geqslant i_2$.*

Definition 3.1.2. *An* ***inverse system*** $\{X_i, \varphi_{ij}, I\}$ *of topological spaces* X_i *is a family of topological spaces indexed by a directed set* I*, together with a continuous map* $\varphi_{ij} : X_j \longrightarrow X_i$ *wherever* $i \leqslant j$*. The maps must satisfy that the diagram*

$$\begin{array}{ccc} X_k & \xrightarrow{\varphi_{ik}} & X_i \\ & {\scriptstyle \varphi_{jk}} \searrow \quad \nearrow {\scriptstyle \varphi_{ij}} & \\ & X_j & \end{array}$$

commutes whenever $i \leqslant j \leqslant k$ *and that* $\varphi_{ii} = id_{X_i}$*.*

Definition 3.1.3. *Given an inverse system* $\{X_i, \varphi_{ij}, I\}$ *and a topological space* Y*, a set of continuous maps* $\{\psi_i : Y \longrightarrow X_i \mid i \in I\}$ *is said to be compatible if whenever* $i \leqslant j$ *we have* $\varphi_{ij}\psi_j = \psi_i$*.*

Definition 3.1.4. *The* ***inverse limit****,* $\varprojlim_{i \in I} X_i = (X, \varphi_i)$*, of the inverse system* $\{X_i, \varphi_{ij}, I\}$ *is a topological space* X *together with a compatible set of continuous mappings* $\{\varphi_i : X \longrightarrow X_i\}$ *having the following universal property:*

Whenever Y *is a topological space and* $\{\psi_i : Y \longrightarrow X_i\}$ *is a compatible set of continuous maps, there exists a unique continuous map* $\psi : Y \longrightarrow X$ *such that* $\varphi_i\psi = \psi_i$ *for each* $i \in I$*.*

Thus, we require that there exists a unique ψ such that the following diagrams commute:

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & X \\ & {\scriptstyle \psi_i} \searrow \quad \swarrow {\scriptstyle \varphi_i} & \\ & X_i & \end{array}$$

Observe that if the inverse limit exists, then it is unique. We will always demand that a finite set have the discrete topology. If the objects in our system are finite, several topological properties pass to the limit. Discrete spaces are Hausdorff and totally disconnected, which passes to the limit. A finite space is compact, and this property passes to the cartesian product by Tychonoff's theorem. If X is an inverse limit of an inverse system consisting of finite sets, then we say that X is a **profinite space**.

In that case, if X is a profinite space, then it is a Hausdorff, compact and totally disconnected topological space. For a proof of these assertions see [25, Chapter 1]. If Y is a subset of a topological space X, denote by $\overline{Y}$ the topological closure of Y in X.

Lemma 3.1.5. *Let X be an inverse limit of an inverse system $\{X_i, \varphi_{ij}\}$ of topological spaces and let Y be a subset of X. Then:*

1. *The $\varphi_i(Y)$ and $\overline{\varphi_i(Y)}$ form an inverse system of subsets of X_i, and $\overline{Y} = \bigcap_i \varphi_i^{-1}(\overline{\varphi_i(Y)}) = \varprojlim_i \overline{\varphi_i(Y)}$.*
2. *$\overline{Y} = \varprojlim_i \varphi_i(Y) = \varprojlim_i \overline{\varphi_i(Y)}$.*

For a proof see Corollary of [4, Propostion 9, p. 49].

Definition 3.1.6. *A subset J of the directed set I is **cofinal** if, for every $i \in I$ there is $j \in J$ with $j \geqslant i$. In this case whenever $\{X_i \ : \ i \in I\}$ is an inverse system indexed by I, then $\{X_i \ : \ i \in J\}$ is also an inverse system and $\varprojlim_{j\in I} X_i \cong \varprojlim_{i\in J} X_i$.*

If we add a group structure to the objects of our inverse system of finite spaces, and if the system maps are group homomorphisms, then the inverse limit is called a **profinite group**.

We use the notation $H \leqslant_O G$ to denote open subgroups of G, $H \leqslant_C G$ to denote closed subgroups of G, $N \trianglelefteq_O G$ to denote open normal subgroups of G and $N \trianglelefteq_C G$ to denote closed normal subgroups of G.

Example 3.1.7.

- *If G is a profinite group and I is directed set of open normal subgroups of G ordered by reverse inclusion such that $\bigcap\{N \,:\, N \in I\} = 1_G$, then $G = \varprojlim_{N\in I} G/N$.*
- *Consider the system of p-groups indexed by the natural numbers $\mathbb{N}$ and given by*

$$\begin{array}{c} \vdots \\ \big\downarrow \\ \mathbb{Z}/p^3\mathbb{Z} \\ \big\downarrow \\ \mathbb{Z}/p^2\mathbb{Z} \\ \big\downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

where the maps are just the standard projection maps. This inverse system has inverse limit the $p-$adic integers $\mathbb{Z}_p$.

Definition 3.1.8.
- *A profinite group is a **pro-**p **group** it is the inverse limit of finite p-groups*
- *A profinite group is **virtually pro-**p if it has an open pro-p subgroup.*

The $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ is an example of a pro-p group and a virtually pro-p group. The group $GL_n(\mathbb{Z}_p)$ of invertible $n \times n$ matrices with entries in $\mathbb{Z}_p$ is virtually pro-p but not pro-p.

Remark 3.1.9. *Throughout this work the directed set will usually be a set of open normal subgroups of a profinite group G, ordered by reverse inclusion. In this case, cofinal means that for each open normal subgroup M of G, there is N in the cofinal subset with $N \leqslant M$.*

3.2 Pseudocompactness

In this section we collect well-known definitions, results and constructions about pseudocompact algebras and modules.

Unless otherwise specified, throughout this section k will denote a discrete field of characteristic p. When the coefficient ring is unspecified it is assumed to be in k, so, for example, by *algebra* we understand k-algebra. Furthermore, throughout this section all modules are assumed topological left modules.

Definition 3.2.1. *A* ***pseudocompact algebra*** *is an associative, unital, Hausdorff topological k-algebra A having a basis of 0 consisting of open ideals I with cofinite dimension in A that intersect in 0 and such that $A = \varprojlim_I A/I$.*

A pseudocompact algebra can be defined equivalently as an inverse limit of discrete finite dimensional algebras in the category of topological algebras. An example of a pseudocompact algebra is the complete group algebra $k[[G]]$ of a profinite group G, that we define as the inverse limit of the finite dimensional group algebras $k[G/N]$ as N runs through the open normal subgroups of G.

Definition 3.2.2. *Let A be a psudocompact algebra. The* ***Jacobson radical*** *$J(A)$ of A is the intersection of the maximal open left ideals of A.*

Lemma 3.2.3. *Let A be a pseudocompact algebra. Then:*

1. *Write $A = \varprojlim_I \{A/I, \alpha_{II'} : A/I' \to A/I\}$ as an inverse limit of finite dimensional quotient algebras A/I. Then*

$$J(A) = \varprojlim_I J(A/I).$$

2. *Let A, B be pseudocompact algebras and let $\alpha : A \to B$ be a continuous surjective algebra homomorphism. Then $\alpha(J(A)) = J(B)$.*

For a proof of Item 1., see [15, Lemma 2.3] and for Item 2. see [15, Corollary 2.4].

Definition 3.2.4. *A pseudocompact algebra A is* local *if it has a unique maximal left ideal. If A is local, then the Jacobson radical $J(A)$ is the unique maximal left ideal.*

Lemma 3.2.5. *Let A, B be pseudocompact algebras with A local.*

1. *If e is a non-zero idempotent of A, then $e \notin J(A)$*

2. *If $f : A \longrightarrow B$ is a non zero algebra homomorphism, then $f(A)$ is a local algebra.*

Proof. 1. Write $A = \varprojlim_I \{A/I, \varphi_{II'}\}$. By Lemma 3.2.3, $J(A) = \varprojlim_I J(A/I)$. Since $\bigcap I = 0$, there exists an open ideal I_0 of A such that $e \notin I_0$. Since e is idempotent and $\varphi_{I_0} : A \to A/I_0$ is an algebra homomorphism, then $\varphi_{I_0}(e)$ is idempotent in A/I_0. Since $\varphi_{I_0}(J(A)) = J(A/I_0)$ by Lemma 3.2.3, then $\varphi(e) \notin J(A/I_0)$ by [19, Theorem 1.10.5]. Now, working in the cofinal subsystem of A/I_0 with $I \subseteq I_0$, $\varphi_I(e) \notin J(A/I)$, and hence $e \notin \varprojlim J(A/I) = J(A)$.

2. Since A is local and $f(A) \cong A/K$, with $K = ker(f)$, then $K \subseteq J(A)$. Now, using the one-one correspondence between ideals of A contained in K and ideals of A/K we have that $I/K \subseteq J(A)/K$ for all proper ideals I/K of A/K. Then $A/K \cong f(A)$ is local.

□

Definition 3.2.6. *A **pseudocompact** A**-module** is a topological A-module U possessing a basis of 0 consisting of open submodules V of finite codimension that intersect in 0 and such that $U = \varprojlim_V U/V$.*

Definition 3.2.7. *Let $\{U_i, \varphi_{ij}\}$ and $\{V_i, \psi_{ij}\}$ be two inverse systems of pseudocompact A-modules, indexed by I. A map of inverse systems $H : \{U_i, \varphi_{ij}\} \to \{V_i, \psi_{ij}\}$ is a set of continuous maps $\{h_i : U_i \to V_i : \ i \in I\}$ such that if $i \leqslant j$, then the following diagram commutes:*

$$\begin{array}{ccc} U_i & \xrightarrow{\varphi_{ij}} & U_j \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle h_j} \\ V_i & \xrightarrow[\psi_{ij}]{} & V_j \end{array}$$

*The maps h_i are called the **components** of H.*

Any map of inverse systems $H : \{U_i, \varphi_{ij}\} \to \{V_i, \psi_{ij}\}$ induces a unique continuous map $\varprojlim H : \varprojlim h_i : \varprojlim U_i \to \varprojlim V_i$ (cf. [25, Ch 1,§1.1]). If $H : \{U_i, \varphi_{ij}\} \to \{V_i, \psi_{ij}\}$ is such that each h_i is onto, then $\varprojlim h_i$ is onto and if each component h_i is 1-1, then $\varprojlim h_i$ is 1-1 (cf. [11, IV. §3, Lemma 1]).

The notion of linearly compact modules and their properties is a useful tool, that will be frequently used in this work to confirm that topological modules are pseudocompact.

Definition 3.2.8. *If V is a topological vector space, cosets of closed subspaces of V are called* ***closed linear varieties****. We say that V is* ***linearly compact*** *if for every family $\mathcal{F} = \{W_i : i \in I\}$ of closed linear varieties with the finite intersection property, we have that $\bigcap_{i \in I} W_i \neq \varnothing$.*

Lemma 3.2.9. *Let A be a pseudocompact algebra and V, U pseudocompact A-modules.*

1. *The module V is linearly compact.*
2. *If ever $\rho : V \longrightarrow U$ is a continuous homomorphism, then $\rho(V)$ is linearly compact and hence closed in U.*
3. *The submodule abstractly generated by a finite subset of V is closed.*
4. *If U is finitely generated as an A-module, then every A-module homomorphism $U \to V$ is continuous.*

For a proof of Items 1 to 3 see [15, Lemma 2.2] and for Item 4 see [28, Proposition 3.5].

Lemma 3.2.10.

1. *If V is a discrete finite dimensional vector space, then V is linearly compact space.*
2. *A product of linearly compact spaces is linearly compact.*
3. *Let $\{V_i, \varphi_{ij}, I\}$ be an inverse system of linearly compact vector spaces and let W_i be a closed linear variety in V_i for each i such that $\varphi_{ij}(W_i) \subseteq W_j$. Then $\{W_i, \varphi_{ij}, I\}$ forms an inverse system and its inverse limit is not empty.*
4. *A continuous bijective map between two linearly compact spaces is an isomorphism.*

For a proof see [18, II, §6, 27.]

Chapter 4

Complete tensor product, Coinvariants and Homomorphisms

In this chapter we introduce the concept and properties of coinvariant modules. If k is an infinite discrete field of characteristic p, then the completed algebra $k[[G]]$ will not be compact but, it is pseudocompact. Coinvariant modules are a useful tool that let us handle easier the pseudocompact $k[[G]]$-modules. This such tool was used in [22, §2] for profinite algebras. Many results about profinite algebras and modules remain true in the case of pseudocompact objects.

We start this chapter introducing basic definitions and standard results related with the notion of tensor products of pseudocompact modules. Next we introduce the concept and properties of coinvariant modules that we required for develop the block theory for profinite groups.

4.1 Complete tensor product

Several times in this section we will give results for pseudocompact algebras and pseudocompact modules but we will cite a proof for the profinite case. When we do this, the proof in the given reference passes without change to the pseudocompact case.

Let A be a pseudocompact k-algebra, U a pseudocompact right A-module, V pseudocompact left A-modules and W a k-vector space. A continuous linear map $\varphi : U \times V \to W$ is said to be **A-middle linear** if, for all $u, u_1, u_2 \in U,\ v, v_1, v_2 \in V$ and $\lambda \in A$, we have

$$\begin{aligned} \varphi(u_1 + u_2, v) &= \varphi(u_1, v) + \varphi(u_2, v), \\ \varphi(u, v_1 + v_2) &= \varphi(u, v_1) + \varphi(u, v_2) \\ \varphi(u\lambda, v) &= \varphi(u, \lambda v) \end{aligned}$$

Definition 4.1.1. *Let U be a right A-module and V a left A-module, then the* ***complete tensor product*** *of U and V over A is a pseudocompact k-vector space $U\widehat{\otimes}_A V$ and a middle linear map $\varphi : U \times V \to U\widehat{\otimes}_A V$ with the following universal property:*

Given any pseudocompact k-vector space W and continuous A-middle linear map $\psi : U \times V \to W$, there exists a unique continuous linear transformation $\varphi' : U\widehat{\otimes}_A V \to W$ such that $\varphi'\varphi = \psi$.

Observe that the universal property is analogous to the universal property of the abstract tensor product. There is an obvious and useful description of the complete tensor product as an inverse limit:

Lemma 4.1.2. *Let A be a pseudocompact k-algebra. The completed tensor product $U\widehat{\otimes}_A V$ of the pseudocompact right A-module $U = \varprojlim U_i$ and the pseudocompact left A-module $V = \varprojlim V_j$ is defined by*

$$U\widehat{\otimes}_A V = \varprojlim_{i,j} U_i \otimes_A V_j$$

The equivalence of these statement for profinite modules is given in [25, Lemma 5.5.1], but the proof works for pseudocompact modules .

Remark 4.1.3. *Observe that the completed tensor product $U\widehat{\otimes}_A V$ is a topological completion of the abstract tensor product $U \otimes_A V$. Then $U \otimes_A V$ is dense in $U\widehat{\otimes}_A V$.*

This can be interpreted as the statement that $U\widehat{\otimes}_A V$ is topologically generated by the set of elements of the form $u\widehat{\otimes}v$ with $u \in U$ and $v \in V$.

Next, we state some general properties of the completed tensor product that are in direct analogy with those of the abstract tensor product.

Proposition 4.1.4. *Lat A be a pseudocompact k-algebra. Let U, U_1, U_2 be pseudocompact A-modules and V, V_1, V_2 be pseudocompact left A-modules. Then:*

1. *$X\widehat{\otimes}_A$ __ and __ $\widehat{\otimes}_A V$ are right exact covariant functors.*
2. *$U\widehat{\otimes}_A(V_1 \oplus V_2) \cong U\widehat{\otimes}_A V_1 \oplus U\widehat{\otimes}_A V_2$*
3. *$(U_1 \oplus U_2)\widehat{\otimes}_A V \cong U_1\widehat{\otimes}_A V \oplus U_2\widehat{\otimes}_A V$.*
4. *$A\widehat{\otimes}_A V \cong V$ and $U\widehat{\otimes}_A A \cong U$.*
5. *If either U or V is finitely presented as an A-module, or if both U, V are finitely generated as A-modules, then $U \otimes_A V \cong U\widehat{\otimes}_A V$*

For a proof of Item 1 to 4 see [25, Proposition 5.5.3] and for Item 5 see [24, Proposition 2.2].

Proposition 4.1.5. *Let A, B be pseudocompact k-algebras. Let U be a pseudocompact right B-module, V be a pseudocompact $B-A$-bimodule, and W be a pseudocompact left A-module. Then*

1. *$V\widehat{\otimes}_A W$ is a left B-module with multiplication $\lambda(v\widehat{\otimes}w) = \lambda v\widehat{\otimes}w$, for $\lambda \in B$, $v \in V$ and $w \in W$.*
2. *$U\widehat{\otimes}_B V$ is a right A-module with multiplication $(u\widehat{\otimes}v)\beta = u\widehat{\otimes}v\beta$, for $\beta \in A$, $v \in V$ and $u \in U$.*
3. *$U\widehat{\otimes}_B(V\widehat{\otimes}_A W) \cong (U\widehat{\otimes}_B V)\widehat{\otimes}_A W$.*

The proof is exactly the same as for the finite case, you can see [13, Chapter IV, Theorem 5.8].

4.1.1 Induction and restriction

Let G be a profinite group and H a closed subgroup of G. If V is a pseudocompact $k[[H]]$-module, then the **induced** $k[[G]]$**-module** is defined by

$$V\uparrow^G = k[[G]]\widehat{\otimes}_{k[[H]]}V,$$

with action from G on the left factor.

If U is a $k[[G]]$-module, then the **restricted** $k[[H]]$**-module** $U\downarrow_H$ is the original module U with coefficients restricted to the subalgebra $k[[H]]$.

Remark 4.1.6. *Observe that induction is left (but not right) adjoint to restriction. For more details see [24, §2.2].*

Lemma 4.1.7. *Let G be a profinite group, $H \leqslant_C G$ and U a pseudocompact $k[[G]]$-module. Then there is an isomorphism of left $k[[G]]$-modules*

$$k[[G]]\widehat{\otimes}_{k[[H]]}U \cong k[[G/H]]\widehat{\otimes}_k U,$$

where the action of $k[[G]]$ on $k[[G]]\widehat{\otimes}_{k[[H]]}U$ is via left multiplication on $k[[G]]$, and its action on $k[[G/H]]\widehat{\otimes}_k U$ is the diagonal action.

Proof. Following, step-by-step, the proof of [25, Proposition 5.8.1], we can define continuous homomorphisms $\Phi : k[[G]]\widehat{\otimes}_{k[[H]]}U \longrightarrow k[[G/H]]\widehat{\otimes}_k U$, given by $g\widehat{\otimes}u \longmapsto gH\widehat{\otimes}gu$, and $\Psi : k[[G/H]]\widehat{\otimes}_k U \longrightarrow k[[G]]\widehat{\otimes}_{k[[H]]}U$, given by $gH\widehat{\otimes}u \longmapsto g\widehat{\otimes}g^{-1}u$. Observe that Φ and Ψ are mutually inverse:

$$\begin{aligned}\Phi(\Psi(gH\widehat{\otimes}u)) &= \Phi(g\widehat{\otimes}g^{-1}u) = gH\widehat{\otimes}gg^{-1}u = gH\widehat{\otimes}u.\\ \Psi(\Phi(g\widehat{\otimes}u)) &= \Psi(gH\widehat{\otimes}gu) = g\widehat{\otimes}g^{-1}gu = g\widehat{\otimes}u.\end{aligned}$$

□

Lemma 4.1.8. *Let G be a profinite group, $H \leqslant_C G$ and $U = \varprojlim_N U_N$ a pseudocompact $k[[G]]$-module. Then $U \downarrow_H \uparrow^G \cong \varprojlim_N U_N \downarrow_{HN} \uparrow^G$.*

Proof. This follows from [25, Lemma 5.5.2, Proposition 5.2.2, Proposition 5.8.1]. □

4.2 Coinvariants

In this section we introduce the notion of coinvariant modules. These objects will provide us with a tool to control the behaviour of the inverse systems of $k[[G]]$-modules.

Definition 4.2.1. *Let U be a pseudocompact $k[[G]]$-module and N a closed normal subgroup of G. The module of N-**coinvariant** U_N is defined as $k\widehat{\otimes}_{k[[N]]}U \cong U/I_N U$, where k is considered as the trivial $k[[H]]$-module and I_N denotes the augmentation ideal of $k[[N]]$ (that is, the kernel of the map $k[[N]] \to k$ given by $n \longmapsto 1$).*

The action of G on U_N is given by $g(\lambda\widehat{\otimes}u) = \lambda\widehat{\otimes}gu$.

Observe that N acts trivially on U_N, since an element $n \in N$ acts on a generator of U_N as follows:

$$n(\lambda\widehat{\otimes}u) = \lambda\widehat{\otimes}nu = \lambda n\widehat{\otimes}u = \lambda\widehat{\otimes}u.$$

Thus we can consider U_N to be a $k[G/N]$-module if we choose.

The module U_N with the canonical quotient map $\varphi_N : U \to U_N$, satisfies the following universal property (cf. [22, Lemma 2.5]):

Every continuous $k[[G]]$-module homomorphism ρ from U to a pseudocompact $k[[G]]$-module X on which N acts trivially factors uniquely through the canonical projection map $\varphi_N : U \longrightarrow U_N$. That is, there is a unique continuous homomorphism $\rho' : U_N \longrightarrow X$ such that $\rho = \rho'\varphi_N$.

We collect here several technical properties of coinvariant modules:

Lemma 4.2.2. *Let G be a profinite group, N, M closed normal subgroups of G with $N \leqslant M$ and H a closed subgroup of G. Let U, W be pseudocompact $k[[G]]$-modules and let V be a $k[[H]]$-module. Then*

1. *$(U_N)_M$ is naturally isomorphic to U_M.*
2. *$(U \oplus W)_N \cong U_N \oplus W_N$.*
3. *$V_{H\cap N}$ is naturally a $k[[HN/N]]$-module.*
4. *$(V \uparrow^G)_N \cong V_{H\cap N} \uparrow^{G/N}$.*
5. *$U_N \downarrow_{HN/N} \cong (U \downarrow_{HN})_N$.*

For a proof of this Lemma it is sufficient follow, step-by-step, the proof of [22, Lemma 2.6].

The properties of the completed tensor product imply that $(\_)_N$ is a right exact functor from the category of finitely generated $k[[G]]$-modules to the category of finitely generated $k[[G/N]]$-modules.

Lemma 4.2.3. *If U is a finitely generated pseudocompact $k[[G]]$-module and $N \trianglelefteq_O G$, then U_N is a finite dimensional pseudocompact module.*

Proof. If U is generated by n elements $x_1, x_2, ..., x_n$ then so is U_N by $1\widehat{\otimes}x_1, 1\widehat{\otimes}x_2, ..., 1\widehat{\otimes}x_n$. But U_N is a module for the finite dimensional group algebra $k[G/N]$. So $\dim_k(U_N) \leqslant |G/N|^n$. □

Proposition 4.2.4. *If U is a pseudocompact $k[[G]]$-module, then $\{U_N : N \trianglelefteq_O G\}$ together with the set of canonical quotient maps forms a surjective inverse system with inverse limit U.*

For a proof of this Proposition it is sufficient to follow, step-by-step, the proof of [22, Proposition 2.7].

Remark 4.2.5. *Throughout this text, whenever $N \leqslant M$ are closed normal subgroups of the profinite group G, the notation φ_{MN} will be reserved exclusively for the canonical maps $U_N \to U_M$.*

In the special case of $U = k[[G]]$ we have $k[[G]]_N = k[[G/N]]$ and the corresponding maps φ_N, φ_{MN} are in fact algebra homomorphisms.

Lemma 4.2.6. *Let U be a pseudocompact $k[[G]]$-module and V a closed submodule of U. Suppose that for each $N \trianglelefteq_O G$, U_N has a submodule W_N such that $\varphi_N(V) \subseteq W_N$, $\varphi_{MN}(W_N) \subseteq W_M$ whenever $N \leqslant M$, and $\varprojlim_N W_N = V$. Then $U/V \cong \varprojlim_N U_N/W_N$.*

Proof. For each N, let $\pi_N : U_N \longrightarrow U_N/W_N$ be the canonical projection. If $N \leqslant M$, $\pi_M\varphi_{MN}(W_N) \subseteq \pi_M(W_M) = 0$. So there is a surjective map $\overline{\varphi}_{MN} : U_N/W_N \to U_M/W_M$, given by $u + W_N \longmapsto \varphi_{MN}(u) + W_M$. Since $\{U_N, \varphi_{MN}\}$ is an inverse system, then $\{U_N/W_N, \overline{\varphi}_{MN}\}$ form an inverse system. We assert that $U/V \cong \varprojlim_N\{U_N/W_N, \overline{\varphi}_{MN}\}$.

Let $\pi : U \to U/V$ be the canonical projection map. For each $N \trianglelefteq_O G$, $\pi_N\varphi_N(V) \subseteq \pi_N(W_N) = 0$, so there is a map $\pi'_N : U/V \to U_N/W_N$, given by $u+V \longmapsto \varphi_N(u)+W_N$. Observe that if $N \leqslant M$, $\overline{\varphi}_{MN}\pi'_N = \pi'_M$. Then we have a map of inverse systems $\{\pi'_N\} : \{U/V, id\} \to \{U_N/W_N, \overline{\varphi}_{MN}\}$ with surjective components. By [11, IV. §3, Lemma 1], the induced map $\pi' : U/V \to \varprojlim_N U_N/W_N$, given by $u + V \to (\pi'_N(u))_N$, is a continuous surjective map.

But, π' is injective, since

$$\begin{aligned} \pi'(u + V)) = 0 &\Leftrightarrow \varphi_N(u) + W_N = W_N, \quad \forall\ N \\ &\Leftrightarrow \varphi_N(u) \in W_N, \quad \forall\ N \\ &\Leftrightarrow u \in \bigcap_N \varphi_N^{-1}(W_N) \\ &\Leftrightarrow u \in V. \end{aligned}$$

Hence π' is a continuous bijective map from U/V to $\varprojlim_N U_N/W_N$. By Lemma 3.2.10, $U/V \cong \varprojlim_N U_N/W_N$.

□

4.3 Homomorphisms

Let U and W be pseudocompact $k[[G]]$-modules, then $\mathrm{Hom}_{k[[G]]}(U, W)$ denotes the topological k-vector space of continuous $k[[G]]$-module homomorphisms from U to W with the compact-open topology. If $W = \varprojlim_i W_i$, then $\mathrm{Hom}_A(U, W) = \varprojlim_i Hom_{k[[G]]}(U, W_i)$, where we make $\{\mathrm{Hom}_{k[[G]]}(U, W_i)\}$ into an inverse system via the maps

$$\begin{aligned} \eta_{ij} : \mathrm{Hom}_{k[[G]]}(U, W_j) &\longrightarrow \mathrm{Hom}_{k[[G]]}(U, W_i) \\ \alpha_j &\longmapsto \varphi_{ij}\alpha_j, \end{aligned}$$

where $\varphi_{ij} : W_j \longrightarrow W_i$ is the map from the inverse system of W. Furthermore, for each i the maps $\mathrm{Hom}_{k[[G]]}(U, W) \longrightarrow \mathrm{Hom}_{k[[G]]}(U, W_i)$ are given by $\alpha \longmapsto \varphi_i\alpha$, where φ_i is the standard projection from W to W_i.

In particular, when U is finitely generated, $\mathrm{Hom}_{k[[G]]}(U, W)$ is a pseudocompact $k[[G]]$-module. For more details you can confer [24, §2.2].

Lemma 4.3.1. *Let U, W be pseudocompact $k[[G]]$-modules. Then $Hom_{k[[G]]}(U, W) \cong \varprojlim_{N \trianglelefteq_O G} Hom_{k[[G]]}(U_N, W_N)$.*

Proof. Write $U = \varprojlim_N U_N$ and $W = \varprojlim_N W_N$. It is sufficient to confirm that for each $N \trianglelefteq_O G$, $\mathrm{Hom}_{k[[G]]}(U_N, W_N) \cong \mathrm{Hom}_{k[[G]]}(U, W_N)$.

Consider the continuous map $\Gamma_N : \mathrm{Hom}_{k[[G]]}(U_N, W_N) \to \mathrm{Hom}_{k[[G]]}(U, W_N)$ given by $\alpha \longmapsto \alpha\varphi_N$, where φ_N is the canonical projection $U \to U_N$. Observe that Γ_N is injective since $\alpha \in \ker(\Gamma_N)$ if, and only if, $\alpha(\varphi_N(u)) = 0$ for all $u \in U$. This is equivalent to $\alpha = 0$. On the other hand, by the universal property of coinvariant modules, Γ_N is surjective. So $\mathrm{Hom}_{k[[G]]}(U_N, W_N) \cong \mathrm{Hom}_{k[[G]]}(U, W_N)$.

Now, $\mathrm{Hom}_{k[[G]]}(U, W) = \varprojlim_N \mathrm{Hom}_{k[[G]]}(U, W_N) \cong \varprojlim_N \mathrm{Hom}_{k[[G]]}(U_N, W_N)$. □

Lemma 4.3.2. *If U is a finitely generated pseudocompact A-module, then $E = End_A(U)$ is a pseudocompact A-algebra.*

For a proof see [24, Lemma 2.3].

In particular, when U is an indecomposable finitely generated $k[[G]]$-module, $\mathrm{End}_{k[[G]]}(U)$ is a local ring. To see this we use the notion of algebraic compactness and its properties.

Proposition 4.3.3. *Let G be a profinite group and let U be an indecomposable finitely generated $k[[G]]$-module. Then U has local endomorphism ring.*

Proof. Since U is pseudocompact, it is linearly compact by Item 1 from Lemma 3.2.9. But, linearly compact modules are algebraically compact (cf. [9, Proposition 4.11]). It hence follows from [9, Proposition 4.10] that the abstract endomorphism ring of U is local.

Since U is finitely generated, by Item 2 of Lemma 3.2.9, the rings of abstract and continuous endomorphisms coincide. □

Lemma 4.3.4. *Let U, W be pseudocompact $k[[G]]$-modules with U finitely generated. If $\pi : W \longrightarrow U$ is a surjective $k[[G]]$-homomorphism such that $\pi_N : W_N \longrightarrow U_N$ is split for each $N \trianglelefteq_O G$, then π is a split homomorphism.*

Proof. Write $W = \varprojlim_N \{W_N, \varphi_{MN}\}$ and $U = \varprojlim_N \{U_N, \varphi_{MN}\}$. Since $\pi_N : W_N \longrightarrow U_N$ splits for each $N \trianglelefteq_O G$, then there is $\iota_N : U_N \longrightarrow W_N$ such that $\pi_N \iota_N = id_{U_N}$.

Let $X_N \subseteq \mathrm{Hom}_{k[[G]]}(U_N, W_N)$ be the non-emtpy set of splitting maps of π_N. For $N \leqslant M$, the map $X_N \to X_M$ given by $\alpha \longmapsto \alpha_M$ makes X_N into an inverse system.

We claim that each X_N is a closed linear variety. For each $N \trianglelefteq_O G$, consider the continuous linear map $\gamma_N : \mathrm{Hom}_{k[[G]]}(U_N, W_N) \to \mathrm{End}_{k[[G]]}(U_N)$ given by $\alpha \to \pi_N \alpha$. Then $\gamma_N^{-1}(id_{U_N}) = X_N$. Moreover, a simple verification shows that the set X_N is the same as $\{\alpha + \ker(\gamma_N)\}$, for some $\alpha \in X_N$. Then $X_N = \alpha + \ker(\gamma_N)$ is a closed linear variety. Now, by Item 3 of Lemma 3.2.10, $\varprojlim_N X_N \neq \varnothing$. An element $\iota \in \varprojlim_N X_N$ is a splitting of π.

□

4.4 Radicals and socles of pseudocompact $k[[G]]$-modules and coinvariants

In this section we introduce the notion of radical and socle of a pseudocompact module, and we developed the relation with coinvariant modules. Radical and socles of pseudocompact modules will be key tools to study the structure of blocks of $k[[G]]$.

Definition 4.4.1. *Let A be a pseudocompact k-algebra and let U be a pseudocompact A-module. The* ***radical of*** *U, denoted by $rad(U)$, is the intersection of the maximal open submodules of U.*

Lemma 4.4.2. *Let U be pseudocompact $k[[G]]$-module with $U = \varprojlim_N \{U_N, \varphi_{MN}\}$. Then*

1. *$rad(U) = \varprojlim_N rad(U_N)$, and the inverse system $\{rad(U_N), \varphi_{MN}\}$ is surjective.*
2. *For any $i \geqslant 1$, $rad^i(U) = \varprojlim_N rad^i(U_N)$, where $rad^i(U) = rad(rad^{i-1}(U))$ and the inverse system is surjective.*
3. *$U/rad(U) \cong \varprojlim_N U_N/rad(U_N)$.*
4. *For any $j \in \mathbb{N}$, $rad^j(U)/rad^{j+1}(U) \cong \varprojlim_N rad^j(U_N)/rad^{j+1}(U_N)$.*

Proof. 1. Given open subgroups N, M with $N \leqslant M$, $\varphi_{MN} : U_N \longrightarrow U_M$ is a surjective homomorphism. Then, by [2, Proposition 9.15], $\varphi_{MN}(rad(U_N)) = rad(U_M)$. Thus, the restriction of the inverse system of U_N to their radicals yields an inverse system with inverse limit $\varprojlim_N rad(U_N) \subseteq U$. Furthermore, $rad(U)$ is sent to $rad(U_N)$, for each N, so that $rad(U) \subseteq \varprojlim_N rad(U_N)$. On the other hand, let $u \notin rad(U)$. Then there is a maximal open submodule W of U such that $u \notin W$. Then, there is $N' \trianglelefteq_O G$ such that $I_{N'}U \subseteq W$ and $u \notin \varphi_{N'}(W)$. But, by The Correspondence Theorem, for modules (cf. [10, §10.2, Theorem 4]) $\varphi_{N'}(W)$ is maximal in $U_{N'}$, so $rad(U_{N'}) \subseteq \varphi_{N'}(W)$ and hence $\varphi_{N'}(u) \notin rad(U_{N'})$. Then $\varphi_N(u) \notin rad(U_N)$ for all $N \leqslant N'$. Hence, $u \notin \varprojlim_{N \leqslant N'} rad(U_N)$. Then, $\varprojlim_N rad(U_N) \subseteq rad(U)$.

2. This is a particular case of Item 1.

3. The result follows by applying Lemma 4.2.6 with $V = rad(U)$ and $W_N = rad(U_N)$.

4. This is a particular case of Item 3.

□

Definition 4.4.3. *Let A be a pseudocompact algebra and U a pseudocompact A-module. We say that U is* ***semisimple*** *is for every closed submodule W of U, there is a closed submodule W' of U such that $U = W \oplus W'$.*

Lemma 4.4.4. *Let A be a pseudocompact algebra and U a pseudocompact A-module. The following are equivalent:*

1. *U is a direct product of simple modules.*

2. *U is semisimple.*

3. *Every open submodule of U has a complement.*

For a proof see [14, Lemma 3.9].

Observe that if $N \leqslant M$ are open subgroups of G, and X is a simple module of U_N, then $\varphi_{MN}(X)$ is simple or zero. Hence $\varphi_{MN}(soc(U_N)) \subseteq soc(U_M)$.

Lemma 4.4.5. *Let G be a profinite group and U a finitely generated pseudocompact $k[[G]]$-module with $U = \varprojlim_N \{U_N, \varphi_{MN}\}$. Then $\varprojlim_N \{soc(U_N), \varphi_{MN}\}$ is the maximal closed semisimple submodule of U.*

Proof. First we confirm that $L = \varprojlim_N \{soc(U_N), \varphi_{MN}\}$ is a semisimple submodule of U. Let W be an open submodule of L. Then L/W is finite dimensional. Consider the canonical projection $q : L \to L/W$. By Lemma 4.4.4, it is sufficient to confirm that q splits.

Since $(\_)_N$ is a right exact functor, for each $N \trianglelefteq_O G$, we have surjective maps $q_N : L_N \to (L/W)_N$. Since $L_N \subseteq soc(U_N)$, and $soc(U_N)$ is a semisimple submodule of U_N, then q_N splits, for each $N \trianglelefteq_O G$. So, by Lemma 4.3.4, the map q is split, as required.

To finish we confirm that every closed semisimple submodule of U is contained in L. Let V be a closed semisimple submodule of U. By Lemma 4.4.4, V is a direct product of simple modules. It is thus sufficient to confirm that every simple submodule S of V is contained in L. Let S be a simple submodule of V. Then, $\varphi_N(S) \subseteq soc(U_N)$, for each N. so $S \subseteq \varprojlim_N Soc(U_N) = L$. Hence, $V \subseteq L$. □

Remark 4.4.6. *Observe that the inverse system of* $\{soc(U_N), \varphi_{MN}\}$ *need not be surjective, and it can happen that* $soc(U) = 0$.

Definition 4.4.7. *The* socle *of* U, *denoted by* $soc(U)$, *is the maximal closed semisimple submodule of* U.

Lemma 4.4.8. *Let* G *be a profinite group and* U *a finitely generated pseudocompact* $k[[G]]$*-module with* $U = \varprojlim_N \{U_N, \varphi_{MN}\}$. *Then* $U/soc(U) \cong \varprojlim_N U_N/soc(U_N)$.

Proof. The result follows by applying Lemma 4.2.6 with $V = soc(U)$ and $W_N = soc(U_N)$.

□

Chapter 5

Tools of the Modular Representation Theory of Profinite Groups

In this chapter we introduce the basic definitions and results of the modular representation theory for profinite groups required in the next chapters. Furthermore, we introduce analogous notions of the trace map for pseudocompact algebras. In the finite theory the trace map is a central piece in the study of the block theory of finite groups, and the analogous notion introduced here will have an equivalent importance in the block theory of profinite groups.

5.1 Relative Projectivity

Between the years 2005 and 2011, some results of the modular representation theory of finite groups were extended to the category of profinite groups. In [22], [21], were extended the notions of relative projectivity and vertex of the modules defined over finite group algebras to modules over completed group algebras defined over finite field of characteristic p.

Recall that if k is an infinite discrete field of characteristic p, then $k[[G]]$ is a pseudocompact algebra. As in Chapters 3 and 4, many results presented here, about profinite algebra and modules, remain true in case of pseudocompact objects.

Definition 5.1.1. *Let H be a closed subgroup of G. A pseudocompact $k[[G]]$-module U is **relatively H-projective** if given any diagram of pseudocompact $k[[G]]$-modules and $k[[G]]$-module homomorphisms of the form*

$$\begin{array}{ccc} & & U \\ & & \downarrow{\scriptstyle \varphi} \\ V & \underset{\beta}{\twoheadrightarrow} & W \end{array}$$

such that there exists a continuous $k[[H]]$-module homomorphism $U \longrightarrow V$ making the triangle commute, then there exists a continuous $k[[G]]$-module homomorphism $U \longrightarrow V$ making the triangle commute.

Observe that a projective module is the same thing as a 1-projective module. Many familiar characterizations of relatively projective modules follow exactly as finite case:

Lemma 5.1.2. *Let G be a profinite group, H a closed subgroup of G and U be a pseudocompact $k[[G]]$-module. The following are equivalent:*

1. *U is relatively H-projective.*
2. *If a continuous $k[[G]]$-module epimorphism $V \longrightarrow U$ splits as a $k[[H]]$-module homomorphism, then it splits as a $k[[G]]$-module homomorphism.*
3. *U is isomorphic to a direct summand of $U \downarrow_H \uparrow^G$.*
4. *The natural projection $\pi : U \downarrow_H \uparrow^G \longrightarrow U$ sending $g \widehat{\otimes} u \longmapsto gu$ splits.*
5. *U is a direct summand of a module induced from some pseudocompact $k[[H]]$-module.*

Proof. 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3. $\Leftrightarrow$ 5. can be proved in a similar way to the finite case (cf. Lemma 2.1.3) and 4. $\Rightarrow$ 3. is obvious.

It remains to confirm 3. $\Rightarrow$ 4. Observe that π is component at U of the counit of the induction-restriction adjuction (see Remark 4.1.6). So, if $\alpha : U \longrightarrow U \downarrow_H \uparrow^G$ and $\beta : U \downarrow_H \uparrow^G \longrightarrow U$ are the split homomorphisms coming from 3., then, by properties of adjoint functors, $\beta = \pi\gamma \uparrow_H$, for some endomorphism $\gamma : U \downarrow_H \longrightarrow U \downarrow_H$. Then $id_U = \beta\alpha = (\pi\gamma \uparrow^G)\alpha = \pi(\gamma \uparrow_H \alpha)$. Hence π is split. $\square$

Remark 5.1.3. *Here and elsewhere, when U, V are $k[[G]]$-modules, we write $U|V$ to mean that U is isomorphic to a continuous direct summand of V.*

The fact of U being a pseudocompact $k[[G]]$-module and being able to write U as inverse limit of its coinvariant modules U_N (cf. Proposition 4.2.4) let us find new characterizations of relative projectivity of U.

Proposition 5.1.4. *Let U be a finitely generated pseudocompact $k[[G]]$-module, and $H \leqslant_C G$. Then U is relatively H-projective if and only if U is relatively HN-projective for every $N \trianglelefteq_O G$.*

Proof. Assume that U is relatively HN-projective for every $N \trianglelefteq_O G$. Then, by Lemma 5.1.2, the natural projection $\pi_N : U \downarrow_{HN}\uparrow^G \longrightarrow U$ splits. Denote by I_N the non-empty set of splittings of π_N. Observe that I_N is a closed linear variety, since I_N is the inverse image of id_U by the continuous map $\rho_N : Hom_{k[[G]]}(U, U \downarrow_{HN}\uparrow^G) \longrightarrow End_{k[[G]]}(U)$ given by $\alpha \longmapsto \pi_N\alpha$.

Now, the proof of [22, Proposition 3.3] can be followed step-by-step, since $\varprojlim_N I_N \neq \varnothing$ by Item 3 of Lemma 3.2.10. $\square$

Proposition 5.1.5. *Let U be a finitely generated pseudocompact $k[[G]]$-module, and $H \leqslant_C G$. Then U is relatively H-projective if, and only if, U_N is relatively HN-projective for every $N \trianglelefteq_O G$.*

Proof. We essentially follow the proof of [22, Proposition 3.5]. Fix $M \trianglelefteq_O G$ and consider the cofinal system of open normal subgroups N of G contained in M (see Remmark 3.1.9). The module U_N is relatively HM-projective, so by Lemma 5.1.2, the canonical projection $\pi_N : U_N \downarrow_{HM}\uparrow^G \rightarrow U_N$ splits. But $U_N \downarrow_{HM}\uparrow^G \cong (U \downarrow_{HM}\uparrow^G)_N$ by Lemma 4.2.2. Then, by Lemma 4.3.4, U is relatively HM-projective. Now, following the proof of [22, Proposition 2.13], the result follow. $\square$

Next, we define the trace map, and with this finally we state our characterization of finitely generated relatively H-projective $k[[G]]$-modules:

Definition 5.1.6. *If $H \leqslant_O G$ and U, W are $k[[G]]$-modules, then* ***the trace map***

$$Tr_H^G : Hom_{k[[H]]}(U\downarrow_H, W\downarrow_H) \longrightarrow Hom_{k[[G]]}(U, W),$$

is defined by $\alpha \longrightarrow \sum_{s \in G/H} s\alpha s^{-1}$, where G/H denotes a set of left coset representatives of H in G.

The following theorem collect all characterization for relatively projective modules.

Theorem 5.1.7. *Let G be a profinite group, let $H \leqslant_C G$, and let U be a finitely generated pseudocompact $k[[G]]$-module. Then the following are equivalent:*

1. *U is relatively H-projective.*
2. *If ever a continuous $k[[G]]$-epimorphism $V \longrightarrow U$ splits as a $k[[H]]$-module homomorphism, then it splits as a $k[[G]]$-module homomorphism.*
3. *U is isomorphic to a direct summand of $U\downarrow_H\uparrow^G$.*
4. *The natural projection $\pi : U\downarrow_H\uparrow^G \longrightarrow U$ sending $g\widehat{\otimes}u \longmapsto gu$ splits.*
5. *U is isomorphic to a direct summand of a module induced from some pseudocompact $k[[H]]$-module.*
6. *U is relatively HN-projective for every $N \trianglelefteq_O G$.*
7. *U_N is relatively HN-projective for every $N \trianglelefteq_O G$.*
8. *For every $N \trianglelefteq_O G$ there exists a continuous $k[[HN]]$-endomorphism α_N of U such that $id_U = Tr_{HN}^G(\alpha_N)$.*

The last item can be proved mimicking the finite case.

5.1.1 Vertices

Thanks to the pass from profinite modules to pseudocompact modules, mentioned in the previous section, the theory of vertices and sources developed in [22] for profinite modules can be transferred to the pseudocompact case. In this section, we collect the more relevant results about vertices.

Mostly proofs of the following results can be done by mimicking in the proofs of the profinite case, we refer the reader interested in the proofs to [22].

Definition 5.1.8. *If U a finitely generated indecomposable pseudocompact $k[[G]]$-module, a* ***vertex*** *of U is a minimal closed subgroup Q such that U is relatively projective to Q.*

The following results were proved for profinite $k[[G]]$-modules in [22, §4], however, they proofs are valid in a more general context as pseudocompact $k[[G]]$-modules.

Lemma 5.1.9. *If U is an indecomposable finitely generated pseudocompact $k[[G]]$-module, then a vertex of U exists.*

Theorem 5.1.10. *Let G be a profinite group, U an indecomposable finitely generated $k[[G]]$-module, and let Q, R be vertices of U. Then there exists $x \in G$ such that $Q = xRx^{-1}$.*

Lemma 5.1.11. *If U is a finitely generated indecomposable $k[[G]]$-module, then any vertex of U is a pro-p group.*

Example 5.1.12. *Let G be a profinite group with Sylow p-subgroup P. Then P is a vertex of the trivial $k[[G]]$-module k.*

If k had a vertex Q properly contained in P, then for some $N \trianglelefteq_O G$ we have that QN/N is properly contained in PN/N. But then the module $k_N = k$ would be relatively QN/N-projective, which contradicts the fact that any vertex of the trivial $k[G/N]$-module is a p-Sylow subgroup of G/N by [1, III, §9, p.67].

Lemma 5.1.13. *Let G be a profinite group, $N \trianglelefteq_C G$ and U a finitely generated indecomposable $k[[G]]$-module on which N acts trivially. If U has vertex Q, then U has vertex QN/N as $k[G/N]$-module.*

Proof. Since N act trivially on U, then $U \cong U_N$. So U has vertex R/N as a $k[G/N]$-module for some $N \leqslant R \leqslant QN$. Furthermore, N acts trivially on $U \downarrow_R \uparrow^G$, since for $g \in G$, $n \in N$ and $u \in U$,

$$n(g\widehat{\otimes}_R u) = ng\widehat{\otimes}_R u = gn'\widehat{\otimes}_R u = g\widehat{\otimes}_R n'u = g\widehat{\otimes}_R u.$$

So, by Lemma 4.2.2,

$$U\downarrow_R\uparrow^G \cong (U\downarrow_R\uparrow^G)_N \cong U_N\downarrow_{R/N}\uparrow^{G/N} \cong U\downarrow_{R/N}\uparrow^{G/N}.$$

Hence $U \mid U\downarrow_R\uparrow^G$ as a $k[[G]]$-module. Thus U is R-projective, so that ${}^gQ \leqslant R$, for some $g \in G$.

But, $({}^gQ)N \leqslant R$ if and only if ${}^gQN \leqslant R$, since N is normal in G. So we have that ${}^gQN \leqslant R \leqslant QN$. Hence $R = QN$. □

The next result is a version of [12, Theorem 9] for profinite groups.

Proposition 5.1.14. *Let U be an indecomposable finite dimensional $k[[G]]$-module with vertex Q. If P is a p-Sylow subgroup of G containing Q, then Q is open in P.*

Proof. Since P, Q are closed subgroups of G, to see that Q is open in P it is sufficient to confirm that the index of Q in P is finite.

Since U is finite dimensional, there is a cofinal system of open normal subgroups N of G such that $U \cong U_N$. Then, by Lemma 5.1.13, U has vertex QN/N as $k[G/N]$-module. Furthermore, by [32, Proposition 2.2.3], PN/N is a p-Sylow subgroup of G/N, for each N in the cofinal system. Since $Q \leqslant P$, then $QN/N \leqslant PN/N$.

By Proposition 2.1.7, $|PN/N : QN/N|$ divides $dim(U)$, for each N in the cofinal system. Then, for each N in the cofinal system, $|PN/N : QN/N|$ is bounded above by $dim(U)$. Thus $|P : Q| < \infty$. □

5.2 Relative traces

In the modular representation theory of finite groups, the trace map is a tool frequently used. For modular representations of profinite groups and block theory it will be a useful tool as in the finite case. In this section we introduce the trace map for profinite groups and open subgroups analogously as the finite case. Furthermore, for closed subgroups we introduce a trace from a closed subgroup as a subset. This new notion will be a key tool to be used in the next chapters.

Definition 5.2.1. *Let G be a profinite group and A a pseudocompact k-algebra. We say that A is a pseudocompact* ***G-algebra*** *if A is endowed with a continuous action $G \times A \longrightarrow A$ of G on A, written $(g, a) \longmapsto {}^g a$ for all $a \in A$ and $g \in G$, such that the continuous map sending $a \in A$ to ${}^g a$ is a k-algebra automorphism.*

Definition 5.2.2. *If A, B are pseudocompact G-algebras, a continuous k-algebra homomorphism $f : A \longrightarrow B$ is a* ***homomorphism of G-algebras*** *if $f({}^g a) = {}^g f(a)$ for all $a \in A$ and $g \in G$.*

Definition 5.2.3. *If H is a closed subgroup of G, the* ***subalgebra of fixed elements*** *of A by H is defined by $A^H = \{a : {}^h a = a, \ \forall \ h \in H\}$.*

Observe that A^H is closed in A, since, for each $h \in H$, we can consider the continuous map $\rho_h : A \longrightarrow A \times A$ given by $a \longmapsto (a, {}^h a)$, so $A^H = \bigcap_{h \in H} \rho_h^{-1}(\{(a, a) : \ a \in A\})$.

Given $g \in G$, we have that

$$A^{g^{-1}Hg} = {}^g(A^H) = \{{}^g a : a \in A^H\}.$$

Observe that if L, H are closed subgroups of G with $L \leqslant H$, then $A^H \subseteq A^L$.

Example 5.2.4. *The complete group algebra $k[[G]]$ can be considered as a G-algebra with action of G given by conjugation, that is, ${}^g x = gxg^{-1}$, for each $g \in G$ and $x \in k[[G]]$.*

Definition 5.2.5. *Let A be a pseudocompact G-algebra and H, L subgroups of G with $H \leqslant_O L$. The corresponding* ***trace map*** *is the continuous map*

$$\begin{aligned} Tr_H^L : A^H &\longrightarrow A^L \\ a &\longmapsto \sum_{g \in L/H} {}^g a, \end{aligned}$$

where L/H denote a set of left coset representatives of H in L.

The next result is a version of [19, Proposition 2.5.4, 2.5.5] for pseudocompact G-algebra.

Lemma 5.2.6. *Let G be a profinite group and let A be a pseudocompact G-algebra. Then*

1. *If H is an open subgroup of G, for any $a \in A^H$ and $b \in A^G$, $bTr_H^G(a) = Tr_H^G(ba)$ and $Tr_H^G(a)b = Tr_H^G(ab)$.*
2. *If H, L are open subgroups of G with $L \leqslant H$, then $Tr_H^G \circ Tr_L^H = Tr_L^G$.*
3. *[Mackey's Formula]If H is a closed subgroup and L an open subgroup of G. Then, for any $a \in A^L$, $Tr_L^G(a) = \sum\limits_{g \in H\backslash G/L} Tr^H_{H\cap {}^gL}({}^ga)$.*

These properties of the trace map carry through just as for finite groups.

If H is a closed subgroup and N, M are open normal subgroups of G with $N \leqslant M$, by Item 2 of Lemma 5.2.6, we have that

$$Tr_{HN}^G(A^{HN}) = Tr_{HM}^G(Tr_{HN}^{HM}(A^{HN})) \subseteq Tr_{HM}^G(A^{HM}).$$

Definition 5.2.7. *Let A be a pseudocompact G-algebra. If H is a closed subgroup of G, define the* ***trace of*** *H as $Tr_H^G(A^H) = \bigcap_{N \trianglelefteq_O G} Tr_{HN}^G(A^{HN})$.*

The following result is a useful tool that relates the set trace of H with the trace map for finite groups.

Lemma 5.2.8. *Let G be a profinite group and D be a closed pro-p subgroup of G. Considering $k[[G]]$ as G-algebra with multiplication given by conjugation, then $Tr_D^G(k[[G]]^D) \subseteq \varprojlim_N Tr_{DN/N}^{G/N}(k[G/N]^{DN/N})$.*

Proof. We first make $\{Tr_{DN/N}^{G/N}(k[G/N]^{DN/N})\}$ into an inverse system via the maps $\Phi_{MN} : Tr_{DN/N}^{G/N}(k[G/N]^{DN/N}) \longrightarrow Tr_{DM/M}^{G/M}(k[G/M]^{DM/M})$ given by the canonical projection $\varphi_{MN} : k[G/N] \longrightarrow k[G/M]$ restricted to $Tr^{G/N}_{DN/N}(k[G/N]^{DN/N})$, whenever $N \leqslant M$. Let $y \in Tr^{G/N}_{DN/N}(k[G/N]^{DN/N})$, so there is $a \in k[G/N]^{DN/N}$ such that $y = Tr^{G/N}_{DN/N}(a) = \sum_{gN \in G/DN} {}^{gN}a$. It is sufficient to show that $\varphi_{MN}(y) \in Tr^{G/M}_{DM/M}(k[G/M]^{DM/M})$. Observe that $\varphi_{MN}(a) \in k[G/M]^{DM/M}$.

Since $DN \leqslant DM \leqslant G$, we can consider a set X of left coset representatives of DN in DM. On the other hand, we can consider a set Z of left coset representatives of DM in G. So $ZX = \{zx{:}z \in Z, x \in X\}$ is a set of left coset representatives of DN in G. Now, we can write y as $y = \sum_{zx \in ZX} {}^{zx}a$. Then

$$\begin{aligned}
\Phi_{MN}(y) = \varphi_{MN}(y) &= \varphi_{MN}(Tr^{G/N}_{DN/N}(a)) \\
&= \varphi_{MN}(\sum_{zx \in ZX} {}^{zxN}a) \\
&= \sum_{z \in Z} \sum_{x \in X} \varphi_{MN}({}^{zxN}a) \\
&= \sum_{z \in Z} \sum_{x \in X} {}^{\varphi_{MN}(zN)\varphi_{MN}(xN)}\varphi_{MN}(a) \\
&= \sum_{z \in Z} \left| \frac{DM}{DN} \right| {}^{\varphi_{MN}(zN)}\varphi_{MN}(a) \\
&= \left| \frac{DM}{DN} \right| Tr^{G/M}_{DM/M}(\varphi_{MN}(a)) \\
&\in Tr^{G/M}_{DM/M}(k[G/M]^{DM/M}).
\end{aligned}$$

Thus we have $\varprojlim_N \{Tr^{G/N}_{DN/N}(k[G/N]^{DN/N}), \Phi_{MN}\}$ with compatible maps $\Phi_N : \varprojlim_N Tr^{G/N}_{DN/N}(k[G/N]^{DN/N}) \longrightarrow Tr^{G/N}_{DN/N}(k[G/N]^{DN/N})$ given by $x \longmapsto \varphi_N(x)$. Now, let $y \in Tr^G_Q(k[[G]]^Q)$. We must confirm that $y \in \varprojlim_N Tr^{G/N}_{QN/N}(k[G/N]^{QN/N}))$. But, by [4, II. §5, Proposition 9],

$$\varprojlim_N Tr^{G/N}_{QN/N}(k[G/N]^{QN/N})) = \bigcap_{N \trianglelefteq_O G} \varphi_N^{-1}(Tr^{G/N}_{QN/N}(k[G/N]^{QN/N})).$$

So we must confirm that $y \in \bigcap_{N \trianglelefteq_O G} \varphi_N^{-1}(Tr^{G/N}_{QN/N}(k[G/N]^{QN/N}))$.

By definition 5.2.7, $y \in Tr^G_{QN}(k[[G]]^{QN})$, for each $N \trianglelefteq_O G$. Fix $N \trianglelefteq_O G$. Then $y = Tr^G_{QN}(a) = \sum_{g \in G/QN} {}^g a$, for some $a \in k[[G]]^{QN}$. It is sufficient to show that $\Phi_N(y) \in Tr^{G/N}_{QN/N}(k[G/N]^{QN/N}))$. So,

$$\begin{aligned}
\Phi_N(y) = \varphi_N(y) &= \varphi_N(\sum_{g \in G/QN} {}^g a) \\
&= \sum_{g \in G/QN} \varphi_N({}^g a) \\
&= \sum_{g \in G/QN} {}^{\varphi_N(g)}\varphi_N(a) \\
&= \sum_{g \in G/QN} {}^{gN}\varphi_N(a) \in Tr^{G/N}_{QN/N}(k[G/N]^{QN/N})).
\end{aligned}$$

□

Chapter 6

Blocks of Profinite Groups and Modules

In this chapter we arrive at the core of this work, the development of a systematic block theory for completed group algebras. We will begin by addressing basic notions related to the block decomposition of a group algebra and the behavior of the modules defined on an algebra that admits such a decomposition.

Throughout this chapter, k is an infinite discrete field of characteristic p and A a pseudocompact k-algebra. We have the following definitions equal as in the finite case (cf. Definition 2.2.1).

Definition 6.0.1. *An element $e \in A$ is* ***idempotent*** *if $e^2 = e$. Two idempotents e, f of A are* ***orthogonal*** *if $ef = fe = 0$. A non-zero idempotent is* ***primitive*** *if it cannot be written as a sum of two non-zero orthogonal idempotents.*

We denote by $Z(A)$ the center of A.

Definition 6.0.2. *An idempotent $e \in A$ is called* ***centrally primitive*** *if e is a primitive idempotent of $Z(A)$.*

Lemma 6.0.3. *Let A be a pseudocompact k-algebra. There is a unique set of pairwise orthogonal centrally primitive idempotents $E = \{e_i : \; i \in I\}$ in A such that*

$$A = \prod_{i \in I} B_i = \prod_{i \in I} Ae_i. \tag{6.1}$$

Proof. By [11, IV. §3, Corollaries 1,2] applied to $Z(A)$ there is a set of pairwise orthogonal centrally primitive idempotents $E = \{e_i : i \in I\}$ in A such that

$$A = \prod_{i \in I} B_i = \prod_{i \in I} Ae_i.$$

Note if there is another set of pairwise orthogonal centrally primitive idempotents $F = \{f_i : i \in I\}$ with the same property as E, by [27, II, Theorem 6.6.64], there is an invertible element $x \in A$ such that $xe_i = f_i x$ for every $i \in I$. But since $e_i, f_i \in Z(A)$ then $xe_i = xf_i$. So $e_i = f_i$ for every $i \in I$. Hence $E = F$. □

Each B_i is called **block of** A (or A-block), each e_i is called a **block idempotent** and E is called the complete set of centrally primitive orthogonal idempotents of A. The decomposition of A like (6.1) is called The **block decomposition**.

Lemma 6.0.4. *Let A be a pseudocompact algebra and let $e \in A$ be an idempotent. If X is a not necessarily closed ideal of A, then $e\overline{X}e = \overline{eXe}$.*

Proof. First, we confirm that $e\overline{X}e$ is closed in eAe. Observe that $e\overline{X}e = \overline{X} \cap eAe$, since if $y \in \overline{X} \cap eAe$, we have that $y = eae$, for some $a \in A$ and $y \in \overline{X}$. Then $y = eae = e(eae)e = eye \in e\overline{X}e$. On the other hand, if $eye \in e\overline{X}e$, then $eye \in eAe$ and $eye \in \overline{X}$, since $\overline{X}$ is an ideal of A. Thus $e\overline{X}e$ is closed in eAe. It now follows by the definition of closure that $\overline{eXe} \subseteq e\overline{X}e$.

Let $f : A \longrightarrow eAe$ be the continuous map given by $a \longmapsto eae$. Then $f(\overline{X}) \subseteq \overline{f(X)}$. That is $e\overline{X}e \subseteq \overline{eXe}$. Hence $e\overline{X}e = \overline{eXe}$. □

Lemma 6.0.5. *Let A be a pseudocompact k-algebra and e be a primitive idempotent of A. Then*

1. *eAe is a local algebra.*

2. *(Rosenberg's Lemma) If $\mathcal{J}$ is a set of closed ideals of A and $e \in \overline{\sum_{I \in \mathcal{J}} I}$, then there is some $I \in \mathcal{J}$ such that $e \in I$.*

Proof. 1. By [30, Theorem 29.15], eAe is a linearly compact algebra, hence a pseudocompact algebra, with unity e. On the other hand, by [11, IV. §3, Corollary 1], Ae is an indecomposable projective pseudocompact A-module. Then, by Lemma 4.3.2, $End(Ae)$ is pseudocompact, and by Proposition 4.3.3, $\mathrm{End}_A(Ae)$ is a local pseudocompact algebra. Now, arguing as the finite case (cf. [3, Lemma 1.3.3]) we obtain that $eAe \cong End_A(Ae)^{op}$: Define the maps

$$\begin{aligned} \rho : \mathrm{End}_A(Ae)^{op} &\longrightarrow eAe \quad , \quad \gamma : eAe \longrightarrow \mathrm{End}_A(Ae)^{op} \\ f &\longmapsto f(e) \quad , \quad x \longmapsto \gamma(x)(ae) = aex. \end{aligned}$$

Observe that $f(e) \in eAe$ since $f(e) \in Ae$ and $f(e) = f(ee) = ef(e) \in eAe$. Furthermore, $\gamma(x) \in End_A(Ae)$ since $\gamma(x)(ae)e = aexe = aex \in Ae$, for some $ae \in Ae$.

We confirm that γ and ρ are algebra homomorphisms. Let $f_1, f_2 \in End_A(Ae)$, then $\rho(f_1 f_2) = (f_1 f_2)(e) = f_2(f_1(e)) = f_2(f_1(e)e) = f_1(e)f_2(e) = \rho(f_1)\rho(f_2)$.

Let $x, y \in eAe$ and $ae \in Ae$, then $\gamma(xy)(ae) = aexy = \gamma(y)(aex) = \gamma(y)(\gamma(x)(ae)) = \gamma(x)\gamma(y)(ae)$.

It remains to confirm that ρ and γ are mutually inverse. Let $x \in eAe$, then $\rho\gamma(x) = \gamma(x)(e) = ex = x$. Since $x \in eAe$ is arbitrary, then $\rho\gamma = id_{End(Ae)}$. On the other hand, let $f \in End_A(Ae)$ and $ae \in Ae$, then $\gamma\rho(f)(ae) = \gamma(f(e))(ae) = aef(e) = f(aee) = f(ae)$. Again, since $f \in End_A(Ae)$ and $ae \in Ae$ is arbitrary, $\gamma\rho = id_{eAe}$.

2. By Lemma 6.0.4, $e(\overline{\sum_{I \in \mathcal{J}} I})e = \overline{e(\sum_{I \in \mathcal{J}} I)e} = \overline{\sum_{I \in \mathcal{J}} eIe}$. So, if $e \in \overline{\sum_{I \in \mathcal{J}} I}$ then $e \in \overline{\sum_{I \in \mathcal{J}} eIe}$. Since the algebra eAe is local by Item 1, then each ideal eIe, with $I \in \mathcal{J}$, is either contained in the Jacobson radical of eAe or equal to eAe. Thus there is at least one $I \in \mathcal{J}$ such that $eIe = eAe$. So $e \in I$.

□

Definition 6.0.6. *Let U be a pseudocompact A-module. We say that U **lies** in a block B of A if $BU = U$ and $B'U = 0$ for all $B' \neq B$.*

If B is a pseudocompact algebra with unity e, then U lies in B_i if, and only if, $eU = U$ and $e'U = 0$ for all $e' \in E$ distinct from e.

If U is a pseudocompact A-module lying in some block B of A, and V is a closed submodule of U, then V and U/V lie in B. If U_1, U_2 are A-modules lying in B then $U_1 \bigoplus U_2$ lies in B.

The next result is a pseudocompact version of [1, IV, §13, Proposition 2].

Proposition 6.0.7. *Let $A = \prod_{i\in I} B_i$ be the block decomposition of A and let U be a pseudocompact A-module. Then U has a unique decomposition of the form $\prod\limits_{i\in I} U_i$, where U_i lies in the block B_i.*

Proof. Let e_i be the block idempotent of B_i. Define $U_i = e_iU$, a A-module lying in B_i. Observe that, by Lemma 3.2.9 and Lemma 3.2.10, $\prod\limits_{i\in I} U_i$ is a pseudocompact A-module.

Define for each $i \in I$ the continuous homomorphism $\rho_i : U \to U_i$ given by $u \longmapsto e_iu$. Now, for each finite subset K of I, define the continuous homomorphism $\rho_K : U \longrightarrow \prod\limits_{i\in K} U_i$ by $\rho_K(u) = (\rho_i(u))_{i\in K}$.

The homomorphisms ρ_K induce a continuous surjective homomorphism $\rho : U \longrightarrow \prod\limits_{i\in I} U_i$, which is an injective since $\ker(\rho) = \bigcap\limits_{i\in I} \ker(\rho_i)$ and if $u \in \ker(\rho)$, then $u = u \cdot 1 = (e_i)_{i\in I}u = (e_iu)_{i\in I} = (\rho_i(u))_{i\in I} = 0$.

So ρ is a continuous bijective homomorphism, and since U is pseudocompact, by Lemma 3.2.10, $U \cong \prod\limits_{i\in I} U_i$.

The uniqueness of the decomposition follows from the uniqueness of the complete set of centrally primitive orthogonal idempotents. □

Observe that, in particular, indecomposable pseudocompact modules lie in a unique block. Furthermore, if B is a block of A, there is a simple A-module lying in B: The left A-module B lies in B, so if V is a maximal submodule of B, then B/V is a simple module lying in B.

6.0.1 Finite dimensional modules in $k[[G]]$-blocks

Recall our convention that the symbols φ_N and φ_{MN} are reserved for the canonical projections between coinvariant modules.

Remark 6.0.8. *By [23, Proposition 6.4], the complete set of centrally primitive orthogonal idempotents E in $k[[G]]$ is a discrete set and can be obtained as the direct limit of the corresponding complete sets E_N of $k[G/N]$. We make the direct system explicit, as we will use in future:*

If $N \leqslant M$ are open normal subgroups of G, define $\psi_{MN} : E_M \longrightarrow E_N$ by sending $c \in E_M$ to the unique centrally primitive idempotent d of $k[G/N]$ such that $\varphi_{MN}(d)c \neq 0$. Since c is primitive as a central idempotent, this is equivalent to saying that $\varphi_{MN}(d)c = c$.

Lemma 6.0.9. *Let U, V be finite dimensional $k[[G]]$-modules lying in the same block of G. Then there is $N_0 \trianglelefteq_O G$ acting trivially on U, V and such that U, V lie in the same block of G/N_0.*

Proof. Since U, V are finite dimensional, it is sufficient to prove the lemma supposing that both are indecomposable $k[[G]]$-modules. Let e be the block idempotent such that $eU = U, eV = V$ and $e'U = 0 = e'V$ for all $e' \in E \smallsetminus \{e\}$.

Since U has finite dimension, the automorphism group $Aut(U)$ of U has the discrete topology. Then the kernel K of the continuous group homomorphism $\sigma : G \longrightarrow Aut(U)$ is an open normal subgroup of G acting trivially on U. Analogously for V we have an open normal subgroup K' of G acting trivially on V. Since $M = K \cap K'$ acts trivially on U and V, then U and V can be treated as $k[G/M]$-modules.

Since M acts trivially on U and V, then $x \cdot U = \varphi_M(x) \cdot U$ and $x \cdot V = \varphi_M(x) \cdot V$ for all $x \in k[[G]]$. In particular, $U = e \cdot U = \varphi_M(e) \cdot U$ and $V = e \cdot V = \varphi_M(e) \cdot V$.

Using the notation in Remark 6.0.8, if f, g are the block idempotents of the $k[G/M]$-modules U, V respectively, then, since $\varphi_M(e)f = f$ and $\varphi_M(e)g = g$, we have that $\psi_M(f) = \psi_M(g) = e$

Now, by construction of the direct limit there is $N_0 \leqslant M$ such that $\psi_{MN_0}(f) = \psi_{MN_0}(g) = e_{N_0}$. Thus U, V lie in the same $k[G/N_0]$-block with block idempotent e_{N_0}. ∎

The following results show a relation between simple modules lying in the same block of G. The statement presented is analogous to the finite case presented by Webb [31, Proposition 12.1.7].

Proposition 6.0.10. *Let S, T be simple $k[[G]]$-modules. Then the following are equivalent:*

(1) S, T lie in the same block of G.

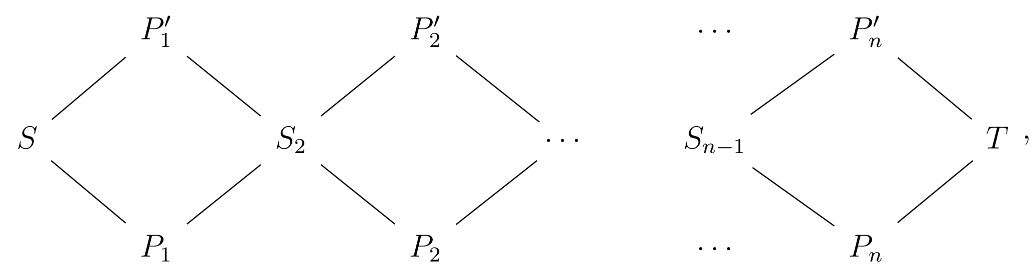
(2) There is a sequence of simple $k[[G]]$-modules $S = S_1, S_2, ..., S_n = T$ such that S_j, S_{j+1}, for each $j \in \{1, 2, ..., n-1\}$, are composition factors of an indecomposable projective pseudocompact $k[[G]]$-module.

(3) There is a sequence of simple $k[[G]]$-modules $S = T_1, T_2, ..., T_m = T$, such that T_j, T_{j+1}, for each $j \in \{1, 2, ..., m-1\}$, are equal or there is a non-split extension of one by the other.

Proof. Item (2) says that each pair of simple modules S_i, S_{i+1} are composition factors of an indecomposable projective module P_i, $1 \leqslant i < n$, where $S_1 = S$ and $S_n = T$. The following diagram represents this relation

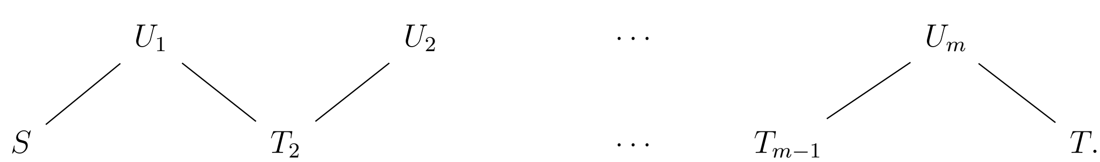
The edges point at which simple modules are composition factors of which projective modules. Since each P_i ($1 \leqslant i < n$) is indecomposable, it only lies in one $k[[G]]$-block. Hence each composition factor of P_i lies in the same block. In particular each S_i lie in the same $k[[G]]$-block as P_i and P_{i+1}. Then all the P_i lie in the same $k[[G]]-$block, hence S, T lie in the same $k[[G]]$-block. So we have proved (2) $\Rightarrow$ (1).

Assume that (1) is true, that is, S, T lie in the same block of G. Then, by Lemma 6.0.9, S, T lie in the same block of G/N, for some $N \trianglelefteq_O G$ acting trivially on S and T. By [31, Proposition 12.1.7], there is a sequence of simple $k[G/N]$-modules $S = S_1, S_2, ..., S_n = T$ such that S_j, S_{j+1}, are composition factors of an indecomposable projective $k[G/N]$-module P_i, for each $j \in \{1, 2, ...m-1\}$. By [11, IV. §3, Corollaries 1,2], $k[[G]] = \prod_{i \in I} P'_i$ a product of indecomposable projective $k[[G]]$-modules. Then $k[G/N] = \prod_{i \in I} (P'_i)_N$. In particular, P_i is isomorphic to a direct summand of the projective $k[G/N]$-module $(P'_i)_N$. Since the homomorphism $\varphi_N : k[[G]] \longrightarrow k[G/N]$, defined in Remark 4.2.5, is surjective, then S_i, S_{i+1} are composition factors of P'_i too. Hence the sequence of $k[[G]]$-modules $S = S_1, S_2, ..., S_n = T$ has the property required, proving (1) $\Rightarrow$ (2). This argument can be represented with the following diagram



Assume that (3) is true, then there is a sequence of simple $k[[G]]$-modules $S = T_1, T_2, ..., T_m = T$, such that there is a non-split extension of T_i by T_{i+1} for every $1 \leqslant i \leqslant m$. Then for each $1 \leqslant i \leqslant m$, there is an indecomposable module U_i such that the following sequence $0 \longrightarrow T_i \longrightarrow U_i \longrightarrow T_{i+1} \longrightarrow 0$ (or $0 \longrightarrow T_{i+1} \longrightarrow U_i \longrightarrow T_i \longrightarrow 0$) is exact. So, T_i, T_{i+1} are composition factors of U_i for all $i \in \{1, ..., m\}$.

We can express this relation with the following diagram:



Since each U_i, $1 \leqslant i \leqslant m$, is indecomposable, then it can be lie only in one block of G. Thus each T_i lies in one block of G and hence S, T lie in the same block of G proving (3) ⇒ (1).

Assume that (1) is true. By Lemma 6.0.9, there is $N \trianglelefteq_O G$ acting trivially on S and T such that S, T lie in the same block of G/N. By [31, Proposition 12.1.7], there is a sequence of simple $k[G/N]$-modules $S = T_1, T_2, ..., T_m = T$, with a non-split extension of T_i by T_{i+1}, for every $1 \leqslant i \leqslant m$. The non-split of sequence $k[G/N]$-modules from finite version are non-split sequence of $k[[G]]$-modules in the obvious way proving (1) ⇒ (3). □

6.1 Blocks of G as inverse limits of finite dimensional blocks.

Consider $k[[G]]$ as a module for the complete group algebra $k[[G \times G]]$ with the following continuous multiplication:

$$\begin{array}{rcl} k[[G \times G]] \times k[[G]] & \longrightarrow & k[[G]] \\ ((g_1, g_2), x) & \longrightarrow & g_1 x g_2^{-1}, \end{array}$$

where $g_1, g_2 \in G$ and $x \in k[[G]]$. The canonical projections $\varphi_{MN} : k[G/N] \to k[G/M]$ and $\varphi_N : k[[G]] \to k[G/N]$ defined in Remark 4.2.5 are $k[[G \times G]]$-module homomorphisms.

Each block of $k[[G]]$ is equal to an indecomposable summand of $k[[G]]$ as a $k[[G \times G]]$-module. Furthermore, the blocks of $k[[G]]$ are pairwise non-isomorphic as $k[[G \times G]]$-modules, since the annihilators of the blocks in $k[[G \times 1]]$ are pairwise non-equal (cf. [1, p. 96]).

Let B be a block of G, considered as $k[[G \times G]]$-module, and $N \trianglelefteq_O G$. Define B_N as the coinvariant module $B_{N \times N}$. Observe that $B_{N \times N} = B_{N \times 1}$, since given $x \in B_{N \times 1}$ and $n, m \in N$, there is $m' \in N$ such that

$$(n,m)x = nxm^{-1} = nm'x = (nm',1)x = x,$$

so that $N \times N$ acts trivially on $B_{N\times 1}$. By Proposition 4.2.4, $B = \varprojlim_N \{B_N, \varphi_{MN}\}$, where, whenever $N \leqslant M$, $\varphi_{MN} : B_N \longrightarrow B_M$ is the canonical quotient map.

Define the diagonal map $\delta : G \longrightarrow G \times G$ as $\delta(g) = (g,g)$, for each $g \in G$.

Lemma 6.1.1. *$k[[G]]$ as a $k[[G \times G]]$-module is isomorphic to the induced module $k \uparrow_{\delta(G)}^{G\times G}$.*

Proof. We essentially follow the proof of the finite case [19, Proposition 5.11.6].

Consider the continuous maps $\rho : k[[G]] \longrightarrow k \uparrow_{\delta(G)}^{G\times G}$, defined by $g \longmapsto (g,1)\widehat{\otimes}1$, for each $g \in G$. Note that $(g,1)\widehat{\otimes}1 = (1,g^{-1})\widehat{\otimes}1$, since

$$(g,1)\widehat{\otimes}1 = (1,g^{-1})(g,g)\widehat{\otimes}1 = (1,g^{-1})\widehat{\otimes}(g,g)1 = (1,g^{-1})\widehat{\otimes}g1g^{-1} = (1,g^{-1})\widehat{\otimes}1.$$

Then, if $x, y \in G$, then $(x,y)\rho(g) = (x,y)(g,1)\widehat{\otimes}1 = (xg,y)\widehat{\otimes}1 = (xgy^{-1},1)\widehat{\otimes}1 = \rho((x,y)g)$. Hence ρ is a $k[[G \times G]]$-module homomorphism.

On the other hand, let $\gamma : k \uparrow_{\delta(G)}^{G\times G} \longmapsto k[[G]]$ be defined by $(g,h)\widehat{\otimes}\lambda \longmapsto g\lambda h^{-1}$, for each $g, h \in H$ and $\lambda \in k$. Observe that γ are well defined, since if $g, h, z \in G$ and $\lambda \in k$, then

$$\gamma((gz,hz)\widehat{\otimes}\lambda) = gz\lambda z^{-1}h^{-1} = gzz^{-1}h^{-1}\lambda = gh^{-1}\lambda = g\lambda h^{-1}.$$

Furthermore, if $x, y \in G$, then $(x,y)\gamma((g,h)\widehat{\otimes}\lambda) = (x,y)g\lambda h^{-1} = xg\lambda h^{-1}y^{-1} = \gamma((x,y)(g,h)\widehat{\otimes}\lambda)$. Hence γ is a $k[[G \times G]]$-module homomorphism.

Now, we confirm that γ and ρ are mutually inverse.

$$\begin{aligned}
\rho\gamma((g,h)\widehat{\otimes}1) &= \rho(g\lambda h^{-1})) = (gh^{-1},1)\widehat{\otimes}1) = (g,h)\widehat{\otimes}1. \\
\gamma\rho(g) &= \gamma((g,1)\widehat{\otimes}1) = g.
\end{aligned}$$

□

Proposition 6.1.2. *Let B be a block of G. There is an open normal subgroup N_0 of G such that B is the inverse limit of blocks of G/N associated to the cofinal system of open normal subgroups N of G contained in N_0.*

Proof. Let E be the set of blocks idempotents in $k[[G]]$. We use the direct system $\{E_N, \psi_{MN}\}$ of sets of blocks idempotents of G/N that was described in Remark 6.0.8.

If $e \in E$ is the block idempotent of B, there is $N_0 \trianglelefteq_O G$ and $e_0 \in E_{N_0}$ such that $e = \psi_{N_0}(e_0)$. Consider the cofinal system $\mathcal{N}$ of open normal subgroups N of G with $N \leqslant N_0$.

For each $N \leqslant N_0$, let $e_N = \psi_{N_0 N}(e_0)$ the unique centrally primitive idempotent of $k[G/N]$ such that $\varphi_{N_0 N}(e_N)e_0 = e_0$. Observe that, whenever $N \leqslant M \leqslant N_0$, $\varphi_{MN}(e_N)e_M = e_M$. So, for each $N \leqslant N_0$, the block X_N of G/N, with block idempotent e_N, is a direct summand of B_N as a $k[[G \times G]]$-module, then there are $k[[G \times G]]$-homomorphisms $\iota_N : X_N \longrightarrow B_N$ inclusion, and $\pi_N : B_N \longrightarrow X_N$ multiplication by e_N with $\pi_N \iota_N = id_{X_N}$. Now we can form a new inverse system of blocks X_N via the maps $\gamma_{MN} : X_N \longrightarrow X_M$ given by $x_N \longmapsto \pi_M \varphi_{MN} \iota_N(x_N)$, whenever $N \leqslant M$. In particular, $\gamma_{MN}(e_N) = e_M$, since

$$\gamma_{MN}(e_N) = \pi_M \varphi_{MN} \iota_N(e_N) = \varphi_{MN}(e_N)e_M = e_M,$$

where the last equality is a consequence of e_M being a primitive idempotent. Then $\gamma_{MN}(x_N) \in X_M$, for each $x_N \in X_N$. Furthermore, γ_{MN} is a surjective homomorphism, since if $ye_M \in X_M$ with $y \in k[G/M]$, then there is $x \in k[G/N]$ such that $y = \varphi_{MN}(x)$, so

$$\begin{aligned} ye_M &= \varphi_{MN}(x)e_M \\ &= \varphi_{MN}(x)\varphi_{MN}(e_N)e_M \\ &= \varphi_{MN}(xe_N)e_M \\ &= \gamma_{MN}(xe_N). \end{aligned}$$

Furthermore, if $N \leqslant M \leqslant K$, and $xe_N \in X_N$, then

$$\begin{aligned} \gamma_{KM}\gamma_{MN}(xe_N) &= (\pi_K\varphi_{KM}\iota_M)(\pi_M\varphi_{MN}\iota_N)(xe_N) \\ &= \varphi_{KM}\iota_M(\varphi_{MN}(xe_N)\cdot e_M)\cdot e_K \\ &= \varphi_{KM}\iota_M(\varphi_{MN}(x)\varphi_{MN}(e_N)\cdot e_M)\cdot e_K \\ &= \varphi_{KM}\iota_M(\varphi_{MN}(x)e_M)\cdot e_K \\ &= \varphi_{KM}(\varphi_{MN}(x)e_M)\cdot e_K \\ &= \varphi_{KM}(\varphi_{MN}(x))\varphi_{KM}(e_M)\cdot e_K \\ &= \varphi_{KN}(x)\cdot e_K \\ &= \varphi_{KN}(x)\varphi_{KN}(e_N)\cdot e_K \\ &= (\pi_K\varphi_{KN}\iota_N)(xe_N) \\ &= \gamma_{KN}(xe_N). \end{aligned}$$

Thus $\gamma_{KM}\gamma_{MN} = \gamma_{KN}$. Hence $\{X_N, \gamma_{MN}\}$ form an inverse system. Denote by X the inverse limit of the inverse system of X_N.

Now, observe that the split homomorphisms π_N are components of a map of inverse systems from $\{B_N, \varphi_{MN}\}$ to $\{X_N, \gamma_{MN}\}$. Then, by Lemma 4.3.4, the induced continuous map $\pi = \varprojlim_N \pi_N$ is a split $k[[G \times G]]$-homomorphism. Hence X is a direct summand of B. But B is indecomposable as a $k[[G \times G]]$-module, thus

$$B = X = \varprojlim_N X_N.$$

□

Remark 6.1.3. *For the rest of this chapter,given a block B of G with block idempotent e, the notation $\{X_N, \gamma_{MN}, N \in \mathcal{N}\}$ will refer to a fixed inverse system of finite dimensional blocks of G/N as constructed in Proposition 6.1.2.*

Note that this inverse system is not unique, due to the choice of N_0 and the block idempotent e_0 of G/N_0, but it will be important that once these choices are made, the maps γ_{MN} are canonical.

Example 6.1.4. *Let k be a field of characteristic 2 and let G be the profinite group $\prod_{i\in I} S_3$, the infinite direct product of copies of the group $S_3 =< a, b \mid a^3 = b_2 = 1, bab^{-1} = a^{-1} >$. Considering the directed set of finite subsets $J \subset I$, the complete group algebra $k[[G]]$ is the inverse limit of an inverse system $\{k[\prod_{i\in J} S_3], \varphi_{KJ}\}$ of finite group algebras, where if $K, J \subset I$ are finite subsets of I with $K \subseteq J$, then $\varphi_{KJ} : k[\prod_{i\in J} S_3] \to k[\prod_{i\in K} S_3]$ is the map induced from the canonical projection $\prod_{i\in J} S_3 \to \prod_{i\in K} S_3$.*

By example 2.2.2-1, we know that $e_1 = 1 + a + a^2$ and $e_2 = a + a^2$ are the block idempotents of $k[S_3]$. We assert that given a finite subset J_0 of I, the block idempotent e of $k[[G]]$ is characterized by the property that for each finite subset J of I, $\varphi_J(e) = \bigotimes_{j\in J} e'_j$, where $e'_j = e_2$ if $j \in J_0$, otherwise $e'_j = e_1$.

To prove the assertion, fix a finite subset J of I and some $i \in I - J$ for which $e'_i = e_2$, it is sufficient to confirm that in this case, $\varphi_J(e) = 0$.

Let $J' = J \cup \{i\}$. Then the projection $\varphi_{JJ'} : k[\prod_{j\in J'} S_3] \to k[\prod_{j\in J} S_3]$ sends the finite tensor product $\bigotimes_{j\in J'} e'_j$ to $2\cdot(\bigotimes_{j\in J} e'_j) = 0$. So, $\varphi_J(e) = \varphi_{JJ'}\varphi_{J'}(e) = \varphi_{JJ'}(\bigotimes_{j\in J'} e'_j) = 0$, proving the assertion.

Now, using Proposition 6.1.2, if B is a block of G with block idempotent e and $J_0 \subset I$ is the finite subset of I for which $e'_j = e_2$ for each $j \in J_0$, then $B = \varprojlim_{J_0\subseteq J} X_J$, where X_J is the block of the finite group $\prod_{j\in J} S_3$ with block idempotent $\bigotimes_{j\in J} e'_j$, where $e'_j = e_2$ for $j \in J_0$ and $e'_j = e_1$ otherwise.

Corollary 6.1.5. *Let B be a block of G with block idempotent e. Let S be a finite dimensional $k[[G]]$-module lying in B. Then there is an open normal subgroup N_0 of*

G acting trivially on S such that S lies in the block X_{N_0} of G/N_0. Furthermore, S lies in X_N for each $N \trianglelefteq_O G$ contained in N_0.

Proof. Write $B = \varprojlim_N \{X_N, \gamma_{MN}, N \in \mathcal{N}\}$, where X_N is block of G/N with block idempotent e_N as described in Remark 6.1.3. Fix $M \in \mathcal{N}$. There is a simple $k[G/M]$-module T lying in X_M. Then T lies in B. If $N \leqslant M$, then T is a $k[G/N]$-module with multiplication $xT = \varphi_{MN}(x)T$, for $x \in k[G/N]$. Furthermore T lies in X_N, since $e_N T = \varphi_{MN}(e_N)T = e_M T = T$.

Since S, T are finite dimensional $k[[G]]$-modules lying in the same block B, by Lemma 6.0.9, there is $N_0 \trianglelefteq_O G$ with $N_0 \leqslant M$, such that S, T lie in the same $k[G/N_0]$-block. But T lies in X_{N_0} hence so does S.

□

Proposition 6.1.6. *Let G be a profinite group, R a closed subgroup of G and B a block of G considered as a $k[[G \times G]]$-module. Then B is $\delta(R)$-projective if, and only if, X_N is $\delta(R)(N \times N)$-projective for each $N \trianglelefteq_O G$.*

Proof. ($\Rightarrow$) If B is $\delta(R)$-projective, then $B_N = B_{N \times N}$ is $\delta(R)(N \times N)$-projective for each $N \trianglelefteq_O G$ by Theorem 5.1.7. Since $X_N \mid B_N$, then X_N is $\delta(R)(N \times N)$-projective for each $N \trianglelefteq_O G$.

($\Leftarrow$) Assume that X_N is $\delta(R)(N \times N)$-projective for each $N \trianglelefteq_O G$.

We use the isomorphism of Lemma 4.1.7 to write

$$B_N \downarrow_{\delta(R)(N\times N)} \uparrow^{G\times G} \cong k[[G \times G/\delta(R)(N \times N)]] \widehat{\otimes}_k B_N.$$

Since $X_N \mid B_N$, for each $N \trianglelefteq_O G$, there are canonical maps $\pi_N : B_N \longrightarrow X_N$ and $\iota_N : X_N \longrightarrow B_N$ such that $\pi_N \iota_N = id_{X_N}$. Note that $\{\pi_N\}$ is a map of inverse systems, cf. the proof of Proposition 6.1.2. Then, for each $N \trianglelefteq_O G$, define the continuous $k[[G \times G]]$-homomorphisms

$$\begin{array}{rcl} q_N : k[[G \times G/\delta(R)(N \times N)]]\widehat{\otimes}_k B_N & \to & X_N \\ z\widehat{\otimes} b & \longmapsto & \pi_N(b). \end{array}$$

We assert that q_N splits as a $k[[G \times G]]$-homomorphism: q_N is the composition of the split homomorphisms $k[[G \times G/\delta(R)(N \times N)]]\widehat{\otimes}_k B_N \to k[[G \times G/\delta(R)(N \times N)]]\widehat{\otimes}_k X_N$ and the canonical projection $X_N \downarrow_{\delta(R)(N\times N)}\uparrow^{G\times G} X_N \to X_N$. But since X_N is $\delta(R)(N \times N)$-projective for each $N \trianglelefteq_O G$, by Theorem 5.1.7, the canonical projection is split, so q_N is the composition of two split maps.

Using that π_N are components of a map of inverse systems between $\{B_N\} \to \{X_N\}$, it can be checked that q_N are the components of a map of inverse systems. So, by Lemma 4.3.4, the induced map $q : k[[G \times G/\delta(R)]]\widehat{\otimes}_k B \longrightarrow \varprojlim X_N = B$ is split.

□

Chapter 7

Defect Groups

In this chapter we introduce the concept and properties of the defects groups in analogy with the theory developed for blocks of finite groups. Throughout this chapter we will observe that defect groups are central pieces in the study of blocks.

From now on we will suppose that k is algebraically closed. While this is often not necessary, it simplifies several arguments.

7.1 Definition and basic properties

Recall that we treat $k[[G]]$ as a G-algebra with action given by conjugation. Whenever we treat a subalgebra or quotient algebra of $k[[G]]$ as a G-algebra, the action from G is induced from this one.

Note that the center $Z(k[[G]])$ of $k[[G]]$ is the same as $k[[G]]^G$. So every block idempotent of $k[[G]]$ belongs to $k[[G]]^G$.

Definition 7.1.1. *Let B be a block of a profinite group G with block idempotent e. A **defect group** of B is a closed subgroup D of G such that $e \in Tr_D^G(k[[G]]^D)$ and minimal with this property.*

Theorem 7.1.2. *Let B be a block of G. A defect group of B exists.*

Proof. Let e be the block idempotent of B and consider $\mathcal{D} = \{H \leqslant_C G : e \in Tr_H^G(k[[G]]^H)\}$. Make $\mathcal{D}$ into a partial order, ordering by inclusion. Observe that $\mathcal{D} \neq \emptyset$, since $G \in \mathcal{D}$. Let $\mathcal{C}$ be a chain of elements of $\mathcal{D}$. We assert that $L = \bigcap_{H \in \mathcal{C}} H \in \mathcal{D}$.

Fix $N \trianglelefteq_O G$. Since LN is open in G, $LN = HN$ for some $H \in \mathcal{C}$. Hence $e \in Tr_{LN}^G(k[[G]]^{LN})$ for all $N \trianglelefteq_O G$. So $L \in \mathcal{D}$ and by the definition of $Tr_L^G(k[[G]]^L)$ (cf. Definition 5.2.7) it is a lower bound for $\mathcal{C}$. By Zorn's Lemma, there is a minimal element D in $\mathcal{D}$. □

Analogous to the finite case (cf. Proposition 2.2.8) we have the following result:

Proposition 7.1.3. *If B is a block of G with block idempotent e, and defect group D, then any finitely generated $k[[G]]$-module lying in B is D-projective.*

Proof. Let U be a finitely generated $k[[G]]$-module lying in B. By Theorem 5.1.7, it is sufficient to confirm that for each $N \trianglelefteq_O G$, there exists a continuous $k[[DN]]$-endomorphism α_N of U such that $id_U = Tr_{DN}^G(\alpha_N)$.

Fix $N \trianglelefteq_O G$. Since D is a defect group of B, then $e \in Tr_{DN}^G(k[[G]]^{DN})$. So there is $x_N \in k[[G]]^{DN}$ such that $e = Tr_{DN}^G(x_N)$. Now consider the continuous map $\alpha_N : U \longrightarrow U$ defined by $u \longmapsto x_N u$. Then $\alpha_N \in End_{k[[DN]]}(U)$, and $id_U = Tr_{DN}^G(\alpha_N)$, since for each $u \in U$,

$$\begin{aligned}
Tr_{DN}^G(\alpha_N)(u) &= \sum_{g \in G/DN} g\alpha_N(g^{-1}u) \\
&= \sum_{g \in G/DN} g x_N(g^{-1}u) \\
&= \sum_{g \in G/DN} (g x_N g^{-1})u \\
&= Tr_{DN}^G(x_N)u \\
&= eu \\
&= u.
\end{aligned}$$

So, by Theorem 5.1.7, U is D-projective. $\square$

The next result is a version profinite of [1, IV, §13, Proposition 3].

Proposition 7.1.4. *Let B be a block of G. Then B has a vertex $\delta(H)$, as a $k[[G \times G]]$-module, where H is a pro-p subgroup of G.*

Proof. By Lemma 6.1.1, $k[[G]] \cong k \uparrow_{\delta(G)}^{G\times G}$ as $k[[G \times G]]$-module. Then B is $\delta(G)$-projective. Now, by Lemma 5.1.11, B as $k[[G \times G]]$-module has a vertex a closed pro-p subgroup of $G \times G$ contained in $\delta(G)$, that is, a subgroup of the form $\delta(H)$, where H is a pro-p subgroup of G. $\square$

The next result is a version for profinite groups of [31, Lemma 12.4.4] and will be used to the proof of Proposition 7.1.6.

Proposition 7.1.5. *Let H be a closed subgroup of G and B a block of G with block idempotent e. Then $e \in Tr_H^G(k[[G]]^H)$ if, and only if, B is $\delta(H)$-projective as a $k[[G \times G]]$-module.*

Proof. ($\Rightarrow$) Suppose that $e \in Tr_H^G(k[[G]]^H)$. So for each $N \trianglelefteq_O G$, there is $x_N \in k[[G]]^{HN}$ such that $e = Tr_{HN}^G(x_N)$. Note that writing $k[[G]] = B \oplus B'$, $Tr_{HN}^G(B^{HN}) \subseteq B$ and $Tr_{HN}^G(B'^{HN}) \subseteq B'$. Hence we may suppose that $x_N \in B^{HN}$. We assert that $B \downarrow_{\delta(G)(N\times N)}$ is $\delta(H)(N \times N)$-projective. Consider the continuous map $\alpha_N : B \longrightarrow B$ given by $y \longmapsto x_N y$. This map is a $k[[\delta(H)(N \times N)]]$-homomorphism since

$$\begin{aligned} \alpha_N((hn_1, hn_2)y) &= x_N hn_1 y(hn_2)^{-1} \\ &= hn_1((hn_1)^{-1}x_N hn_1)y(hn_2)^{-1} \\ &= hn_1 x_N y(hn_2)^{-1} \\ &= (hn_1, hn_2)\alpha_N(y). \end{aligned}$$

Now, observe that if R is a set of left coset representatives of HN in G, then $\{(r, r) : r \in R\}$ is a set of left coset representatives of $\delta(H)(N \times N)$ in $\delta(G)(N \times N)$. So

$$
\begin{aligned}
Tr_{\delta(H)(N\times N)}^{\delta(G)(N\times N)}(\alpha_N)(e) &= \sum_{(r,r)\in\delta(G)(N\times N)/\delta(H)(N\times N)} (r,r)\alpha_N((r^{-1},r^{-1})e) \\
&= \sum_{(r,r)\in\delta(G)(N\times N)/\delta(H)(N\times N)} r\alpha_N(r^{-1}er))r^{-1} \\
&= \sum_{(r,r)\in\delta(G)(N\times N)/\delta(H)(N\times N)} rx_N r^{-1}err^{-1} \\
&= Tr_{HN}^{G}(x_N)e \\
&= ee \\
&= e
\end{aligned}
$$

Thus $Tr_{\delta(H)(N\times N)}^{\delta(G)(N\times N)}(\alpha_N)$ is the identity map on B and, by Theorem 5.1.7, $B\downarrow_{\delta(G)(N\times N)}$ is $\delta(H)(N\times N)$-projective. Let Z be a $k[[\delta(H)(N\times N)]]$-module such that

$$B\downarrow_{\delta(G)(N\times N)} \mid Z\uparrow^{\delta(G)(N\times N)} .$$

Since B is $\delta(G)(N\times N)$-projective for each $N \trianglelefteq_O G$, by Theorem 5.1.7, we have that

$$B \;\mid\; B\downarrow_{\delta(G)(N\times N)}\uparrow^{G\times G} \mid Z\uparrow^{\delta(G)(N\times N)}\uparrow^{G\times G}$$

So, for each $N \trianglelefteq_O G$, B is $\delta(H)(N\times N)$-projective. By Theorem 5.1.7, B is $\delta(H)$-projective.

($\Leftarrow$) Now, assume that B is $\delta(H)$-projective as a $k[[G\times G]]$-module. Then B is $(G\times HN)$-projective for each $N \trianglelefteq_O G$. By Theorem 5.1.7, for each $N \trianglelefteq_O G$ there is a continuous $k[[G\times HN]]$-endomorphism α_N of B such that $id_B = Tr_{G\times HN}^{G\times G}(\alpha_N)$. Now, arguing as in the finite case (cf. [20, Propodition 6.2.3]), α_N is in particular a left $k[[G]]$-endomorphism of B as a left B-module. Then α_N is given by a right multiplication, that is, for some $x \in B$, $\alpha_N(y) = yx$ for all $y \in B$.

On the other hand, $\alpha_N(yh) = \alpha_N(y)h$ for all $h \in HN$. So $yhx = yxh$ for all $y \in B$ and all $h \in HN$. Considering, $y = e$ we have $ehx = exh$. But since e acts trivially on elements of B, we have that $hx = xh$. So, $hxh^{-1} = x$ and hence $x \in B^{HN} \subseteq k[[G]]^{HN}$.

Now, if l runs over a set of left coset representatives of HN in G, then $(1,l)$ runs over a set of representatives of $G \times HN$ in $G \times G$. So, $id_B = Tr_{G\times HN}^{G\times G}(\alpha_N) = \sum_{(1,l)\in \frac{G\times G}{G\times HN}}(1,l)\alpha_N(1,l^{-1})$. Then, for all $y \in B$,

$$\begin{aligned} y = id_B(y) &= Tr_{G\times HN}^{G\times G}(\alpha_N)(y) \\ &= \sum_{(1,l)\in G\times G/G\times HN} (1,l)\alpha_N((1,l^{-1})y) \\ &= \sum_{(1,l)\in G\times G/G\times HN} 1\alpha_N(1yl)l^{-1} \\ &= \sum_{(1,l)\in G\times G/G\times HN} ylxl^{-1} \\ &= y\sum_{l\in G/HN} lxl^{-1} \\ &= yTr_{HN}^G(x). \end{aligned}$$

But the unique possibility for $y = yTr_{HN}^G(x)$ for all $y \in B$ is for $Tr_{HN}^G(x)$ be the unity of B, that is, $e = Tr_{HN}^G(x) \in Tr_{HN}^G(k[[G]]^{HN})$. So, for every $N \trianglelefteq_O G$, $e \in Tr_{HN}^G(k[[G]]^{HN})$. Then $e \in \bigcap_N Tr_{HN}^G(k[[G]]^{HN}) = Tr_H^G(k[[G]]^H)$.

□

The next result is a version of [31, Theorem 12.4.5] for profinite groups:

Proposition 7.1.6. *Let D be a closed subgroup of G and let B be a block of G with block idempotent e. The following statements are equivalent.*

1. *D is a defect group of B.*
2. *$\delta(D)$ is a vertex of B as $k[[G \times G]]$-module.*

Proof. This follows from Proposition 7.1.5. □

Proposition 7.1.7. *Let G be a profinite group and B a block of G. Then the defect groups of B are a conjugacy class of pro-p subgroups of G.*

Proof. Since vertices are pro-p subgroups of G by Proposition 7.1.4, so are defect groups by Proposition 7.1.6. It remains to check conjugacy. Let Q, D be defect groups of B. By Proposition 7.1.6, $\delta(Q), \delta(D)$ are vertices of B as a $k[[G \times G]]$-module. Additionally, by Theorem 5.1.10 and Lemma 5.1.11, there is $x, y \in G$ such that $(x, y)(d, d)(x^{-1}, y^{-1}) = (q, q) \in \delta(G)$, for $d \in D, q \in Q$. Then $xdx^{-1} = ydy^{-1}$ and so

$$\begin{aligned}\delta(q) = (x, y)(d, d)(x^{-1}, y^{-1}) &= (xdx^{-1}, ydy^{-1}) \\ &= (xdx^{-1}, xdx^{-1}) \\ &= \delta(x)\delta(d)\delta(x^{-1}).\end{aligned}$$

Then, $\delta(Q) = \delta(x)\delta(D)\delta(x^{-1})$ and so $Q = xDx^{-1}$.

□

Proposition 7.1.8. *Let B be a block of G with defect group D and let P be a p-Sylow subgroup of G containing D. Then D is open in P.*

Proof. Let S be a simple $k[[G]]$-module lying in B with vertex Q contained in D, as we may by Proposition 7.1.3. By Proposition 5.1.14, Q is open in P, and hence D is open in P. □

7.2 Blocks and vertices

In this section we show that defect groups are well-behaved with respect to the inverse system of finite dimensional blocks of Remark 6.1.3.

The following result, due to Green in the finite case [12, Theorem 12], is rarely mentioned for finite groups, but becomes incredibly useful for profinite groups because simple modules are finite dimensional:

Proposition 7.2.1. *Let G be a profinite group and B a block of G with defect group D. There is a simple module T lying in B with vertex D.*

Proof. Write $B = \varprojlim_N X_N$ as in Remark 6.1.3. Let $\mathcal{S}$ be the set of all simple modules lying in B. There is $T \in \mathcal{S}$ with the following property: $dim(T) = p^r a$, with $p \nmid a$ and r as small as possible among all simple $k[[G]]$-modules in B.

Using Corollary 6.1.5, fix a cofinal system $\mathcal{N}$ of $N \trianglelefteq_O G$ acting trivially on T such that T lies in X_N. By Proposition 7.1.3, T has a vertex Q contained in D. We want to show that $Q = D$, then it is sufficient to confirm that $D \leqslant Q$.

Let P be a Sylow subgroup of G containing Q. By [32, Proposition 2.2.3], PN/N is a p-Sylow subgroup of G/N, and $Q \leqslant P$, then $QN/N \leqslant PN/N$. So, by Proposition 2.1.7, $|PN/N : QN/N|$ divides $dim(T)$. So it follows from Theorem 2.2.7 that X_N has defect group QN/N.

Now, by Proposition 7.1.6, X_N has vertex $\delta(QN/N)$ as a $k[G/N \times G/N]$-module. But, $\delta(\frac{QN}{N}) \cong \frac{\delta(Q)(N\times N)}{(N\times N)}$ (through the isomorphism $(q,q)(N \times N) \longmapsto (qN, qN)$), so X_N is $\delta(Q)(N \times N)$-projective for each $N \in \mathcal{N}$. By Proposition 6.1.6, B is $\delta(Q)$-projective. Then $\delta(D) \leqslant \delta(Q)$, and thus $D \leqslant Q$. Hence $Q = D$. □

Corollary 7.2.2. *Let B be a block of G with defect group D. There is $N_0 \trianglelefteq_O G$ such that X_N has defect group DN/N for every open normal subgroup N of G with $N \leqslant N_0$.*

Proof. By Proposition 7.2.1, we can consider a simple module T lying in B with vertex D. By Corollary 6.1.5, there is $N_0 \trianglelefteq_O G$ such that T lies in X_{N_0}. Consider an open normal subgroup N of G contained in N_0. Then, by Lemma 5.1.13, T has vertex DN/N as a $k[G/N]$-module.

By Proposition 2.2.8 there is a defect group $\frac{D_{(N)}}{N}$ of X_N such that $\frac{DN}{N} \leqslant \frac{D_{(N)}}{N}$. But, since B is relatively $\delta(D)$-projective as a $k[[G \times G]]$-module, then X_N is $\delta(DN/N)$-projective as a $k[G/N \times G/N]$-module. So, $DN/N = D_{(N)}/N$, as required. □

Example 7.2.3. *Let k be a field of characteristic 2 and let G be the profinite group $\prod_{i\in I} S_3$, the infinite direct product of copies of the group $S_3 =< a, b \mid a^3 = b_2 = 1, bab^{-1} = a^{-1} >$. Consider the block of G with block idempotent e. As we saw in Example 6.1.4, if J_0 is the finite subset of I with $e_i' = e_2$, then $B = \varprojlim_{J_0 \subseteq J} X_J$, where* *each X_J is a block of the finite group $\prod_{j\in J} S_3$ with block idempotent $\bigotimes_{j\in J} e'_j$, where $e'_j = e_2$ if $j \in J_0$, otherwise $e'_j = e_1$. Then B has a defect group $D = \prod_{i\in I} D_i$, where $D_i = 1$ if $i \in J_0$, otherwise $D_i =< b >$.*

Furthermore, observe that D has index $2^{|J_0|}$ in the 2-Sylow subgroup $\prod_{i\in I} < b >$, where $|J_0|$ denote the size of J_0. In particular, D is open in $\prod_{i\in I} < b >$.

Now, as in the finite case (cf. Theorem 2.2.11), we can give the important property that any defect group is the intersection of two Sylow subgroups.

Theorem 7.2.4. *Let G be a profinite group and B a block of G with defect group D. Then D is expressible as an intersection $P \cap gPg^{-1}$, for some $g \in C_G(D)$, where P is a Sylow pro-p subgroup of G containing D.*

Proof. Write $B = \varprojlim_N X_N$ as in Remark 6.1.3. By Corollary 7.2.2, we can take a cofinal system of open normal subgroups N of G such that each X_N has defect group DN/N.

Let P be a Sylow pro-p subgroup of G containing D. By [32, Proposition 2.2.3], PN/N is a Sylow p-subgroup of GN, and by Theorem 2.2.11, there is $gN \in C_{G/N}(DN/N)$, such that $DN/N = PN/N \cap {}^{gN}PN/N$. Denote by C_N the set $\rho_N^{-1}(\{gN \in C_{G/N}(DN/N) : DN/N = PN/N \cap {}^{gN}PN/N\}) \subseteq \rho_N^{-1}(C_G(DN/N))$, where ρ_N is the continuous projection from G to G/N.

We thus have a collection of closed, non-empty sets $\{C_N : N \trianglelefteq_O G\}$. We wish to show that their intersection is non-empty. Since G is compact it suffices to confirm that any intersection of finitely many of them is non-empty.

Let $N_1, ..., N_n$ be open normal subgroups of G. Then $M = N_1 \cap ... \cap N_n \trianglelefteq_O G$ and so by the previous argument $C_M \neq \varnothing$. This means that there exists $x \in G$ such that $DM/M = (PM/M) \cap {}^{xM}(PM/M)$. Then, for each $i \in \{1, 2, ..., n\}$, $DN_i/N_i = (PN_i/N_i) \cap {}^{xM}(PN_i/N_i)$. So $C_M \subseteq C_{N_1} \cap ... \cap C_{N_n}$ and thus $C_{N_1} \cap ... \cap C_{N_n} \neq \varnothing$.

Then, there is $x \in \bigcap_N C_N \subseteq \bigcap_N \rho_N^{-1}(C_{G/N}(DN/N)) = C_G(D)$, such that $DN/N = PN/N \cap^{xN} PN/N$ for each $N \trianglelefteq_O G$. Then,

$$
\begin{aligned}
D &= \bigcap_N \rho_N^{-1}(DN/N) \\
&= \bigcap_N \rho_N^{-1}((PN/N) \cap {}^{xM}(PN/N)) \\
&= \bigcap_N \rho_N^{-1}(PN/N) \cap \rho_N^{-1}({}^{xN}(PN/N)) \\
&= P \cap {}^{x}P.
\end{aligned}
$$

□

For a profinite group G, we denote by $O_p(G)$ the largest normal pro-p subgroup of G. Observe that $O_p(G)$ is the intersection of the Sylow p-subgroups of G, since the Sylow p-subgroups are closed under conjugation, and $O_p(G)$ is contained in some Sylow p-subgroup P of G. Then ${}^{g}(O_p(G)) = O_p(G) \leqslant {}^{g}P$ for every $g \in G$. But, by Sylow's Theorem (cf. [32, Proposition 2.2.2]), ${}^{g}P$ is a Sylow p-subgroup, so $O_P(G)$ is contained in their intersection.

The next result is a version of [31, Corollary 12.3.4] for profinite groups:

Corollary 7.2.5. *Let G be a profinite group and B a block of G with defect group D. Then $D = O_p(N_G(D))$.*

Proof. Let P be a p-Sylow subgroup of G containing a p-Sylow subgroup Q of $N_G(D)$. Then $P \cap N_G(D) = Q$. By Theorem 7.2.4, $D = P \cap {}^{g}P$ for some $g \in C_G(D)$. In particular, $g \in N_G(D)$. Then ${}^{g}(P \cap N_G(D)) = {}^{g}P \cap N_G(D)$ is a Sylow p-subgroup of $N_G(D)$ by [32, Theorem 2.2.2]. Thus, by Theorem 7.2.4, $D = (P \cap N_G(D)) \cap ({}^{g}P \cap N_G(D)$. Then $O_p(N_G(D))) \leqslant D$. On the other hand, D is a closed normal pro-p subgroup of $N_G(D)$, so $D \leqslant O_p(N_G(D))$. Hence $D = O_p(N_G(D))$. □

7.3 Brauer homomorphism and defect group

We wish to give two characterizations of defect groups of blocks of G through the Brauer homomorphism (Definition 7.3.1).

Consider $k[[G]]$ as a G-algebra with action given by conjugation. Let D be a closed pro-p subgroup of G. Observe that, for each proper open subgroup Q of D (denoted $Q \lneqq_O D$), $Tr_Q^D(k[[G]]^Q)$ is an ideal of $k[[G]]^D$ by Item 1 of Lemma 5.2.6. Thus $\sum_{Q \lneqq_O D} Tr_Q^D(A^Q)$ is an abstract ideal of $k[[G]]^D$ and its closure is a closed ideal of $k[[G]]^D$ [30, Theorem 4.2].

The next definition is motivated by [26, §3]:

Definition 7.3.1. *1. Let G be a profinite group and let A be a pseudocompact G-algebra. For a closed pro-p subgroup D of G, the* Brauer quotient *is defined as the quotient algebra*

$$A^{[D]} = A^D / \overline{\sum_{Q \lneqq_O D} Tr_Q^D(A^Q)}. \tag{7.1}$$

2. The Brauer homomorphism *is the natural projection*

$$Br_D : A^D \longrightarrow A^{[D]}.$$

Analogous to [3, Lemma 6.2.4] we have the following characterization for the defect groups:

Theorem 7.3.2. *Let B be a block of a profinite group G with block idempotent e. Then B has defect group D if, and only if, $e \in Tr_D^G(k[[G]]^D)$ and $Br_D(e) \neq 0$.*

Proof. ($\Rightarrow$) Assume that B has defect group D. Then $e \in Tr_D^G(k[[G]]^D)$. It remains to confirm that $Br_D(e) \neq 0$. To do this, we suppose that $Br_D(e) = 0$ and argue to a contradiction.

Since $Br_D(e) = 0$, then $e \in \overline{\sum_{Q \lneqq_O D} Tr_Q^D(k[[G]]^Q)}$. So, by Lemma 6.0.5, $e \in Tr_Q^D(k[[G]]^Q)$ for some $Q \lneqq_O D$. Fix an open normal subgroup M of G such that DM/M is a defect group of X_M (by Corollary 7.2.2) and such that QM/M has the same index in DM/M as Q in D. Then $\varphi_M(e) \in Tr_{QM/M}^{DM/M}(k[G/M]^{QM/M})$ and $Br_{DM/M}(\varphi_M(e)) = 0$

since $Br_{DM/M}$ is an algebra homomorphism. This contradicts the finite version of the result (cf. [3, Lemma 6.2.4]), since X_M has defect group DM/M.

($\Leftarrow$) Now, assume that $e \in Tr_D^G(k[[G]]^D)$ and $Br_D(e) \neq 0$. Then B has a defect group R contained in D by definition of defect group. By Proposition 5.1.14, R is open in D. So there is $M \trianglelefteq_O G$ such that $RM \cap D = R$. So $RM \lneqq DM$.

Since R is a defect group of B, then $e \in Tr_{RM}^G(k[[G]]^{RM})$. So, there is $a \in k[[G]]^{RM}$ such that $e = Tr_{RM}^G(a)$. Applying Mackey's formula (Lemma 5.2.6) we have that $e = \sum_{g \in D\backslash G/RM} Tr_{D \cap {}^g RM}^D({}^g a)$.

Observe that, for each $g \in D\backslash G/RM$, $D \cap {}^g RM \lneqq_O D$, since otherwise,

$$\begin{aligned} D \cap {}^g RM = D &\Rightarrow D \leqslant {}^g RM \\ &\Rightarrow RM \leqslant DM \leqslant {}^g RM \\ &\Rightarrow g \in N_G(RM) \\ &\Rightarrow D \cap {}^g RM = D \cap RM \lneqq D, \end{aligned}$$

a contradiction.

It follows that if R is proper in D, then $e \in ker(Br_D) = \overline{\sum_{Q \lneqq_O D} Tr_Q^D(k[[G]]^Q)}$, contrary to our hypothesis. Hence $R = D$ proving the theorem. $\square$

We wish to collect together the characterizations of a defect group proved throughout this chapter, and provide one further characterization that will help with the Brauer Correspondence for virtually pro-p groups (cf. Theorem 8.0.7). First we need a version of [31, Lemma 12.5.1] for profinite groups.

Lemma 7.3.3. *Let H, D be closed pro-p subgroups of G. If, for each $N \trianglelefteq_O G$, $Br_D(Tr_{HN}^G(a)) \neq 0$, for some $a \in k[[G]]^{HN}$, then D is conjugate to a subgroup H of G.*

Proof. Fix some $N \trianglelefteq_O G$ and a such that $Br_D(Tr_{HN}^G(a)) \neq 0$. Applying Mackey's formula,

$$Tr^G_{HN}(a) = \sum_{g \in D\backslash G/HN} Tr^D_{{}^gHN \cap D}({}^g a).$$

Since $Br_D(Tr^G_{HN}(a)) \neq 0$, there is $g \in G$, such that ${}^gHN \cap D = D$. So $D \leqslant {}^gHN$.

Let C_N be the set of all $g \in G$ such that $D \leqslant {}^gHN$. Observe that C_N is a closed subset of G since each C_N is a union of set of the form $DgHN$ for some set of $g \in G$. But such sets are unions of cosets of HN, and there are only a finitely number of these cosets. So C_N is a finite union of closed sets, hence C_N is closed.

Consider the collection of closed, non-empty sets $\{C_N{:}N \trianglelefteq_O G\}$. We assert that $\bigcap_N C_N \neq \emptyset$. Since G is compact, it is enough to show that any finite intersection of objects of the collection is non-empty.

Let $N_1, ..., N_n$ be open normal subgroups of G, then $N_1 \cap \cdots \cap N_n$ is an open normal subgroup of G. So $C_{N_1 \cap \cdots \cap N_n} \neq \emptyset$, by previous argument. then there is $g \in G$ such that $DN \subseteq gH(N_1 \cap \cdots \cap N_n)g^{-1} \subseteq gHN_1g^{-1} \cap \cdots \cap gHN_ng^{-1}$. So $C_{N_1 \cap \cdots \cap N_n} \subseteq C_{N_1} \cap \cdots \cap C_{N_n}$. Hence $C_{N_1} \cap \cdots \cap C_{N_n} \neq \emptyset$.

Then since $\bigcap_N C_N \neq \emptyset$, there is $x \in G$ such that $D \subseteq {}^xHN$ for every $N \trianglelefteq_O G$. Hence

$$\begin{aligned} D^x &\subseteq HN, \ \forall N \trianglelefteq_O G \\ D^x &\subseteq \bigcap_N HN \\ D^x &\subseteq H \text{ (by [32, Proposition 0.3.3])} \\ D &\subseteq {}^xH. \end{aligned}$$

□

Now we can give our complete characterization of defect groups for a block of G:

Theorem 7.3.4. *Let G be a profinite group, B a block of G with block idempotent e. The following are equivalent for a closed subgroup D of G:*

1. *B has a defect group D.*

2. *$\delta(D)$ is a vertex of B as a $k[[G \times G]]$-module.*

3. *$e \in Tr_D^G(k[[G]]^D)$ and $Br_D(e) \neq 0$*

4. *D is a maximal pro-p subgroup of G such that $Br_D(e) \neq 0$.*

Proof. 1. ⇔ 2. follow from Proposition 7.1.6, 1. ⇔ 3. follow from Theorem 7.3.2.

3. ⇒ 4. Assume that D is a subgroup of G such that $e \in Tr_D^G(k[[G]]^D)$ and $Br_D(e) \neq 0$. Let Q be a pro-p subgroup of G for which $Br_Q(e) \neq 0$. Then, by Lemma 7.3.3, Q is a subgroup of a conjugate of D. Thus D is a maximal subgroup of G with $Br_D(e) \neq 0$.

4. ⇒ 1. Assume that D is a maximal pro-p subgroup of G such that $Br_D(e) \neq 0$. Let H be a defect group of B. Then, $e \in Tr_H^G(k[[G]]^H)$ and $Br_H(e) \neq 0$. By Lemma 7.3.3, D is conjugate to a subgroup of H. So, by Lemma 7.3.3, we may assume that H contains D. But D is by hypothesis maximal with $Br_D(e) \neq 0$ and hence $D = H$, as required.

□

Chapter 8

Brauer Correspondence for Virtually pro-p Groups

The Brauer correspondence for finite groups describes a relationship between the blocks of the finite group G with defect group D and the blocks of the normalizer in G of D with defect group D. In this chapter we demonstrate a version of Brauer correspondence for blocks of virtually pro-p groups, using the results obtained in the previous chapters. We follow the approaches given in [3, §6.2], and [31, §12.6].

Remark 8.0.1. *Observe that $k[[G]]^D$ is a $N_G(D)$-algebra, since if $g \in N_G(D)$ and $x \in k[[G]]^D$, then $d(gxg^{-1})d^{-1} = g(g^{-1}dg)x(g^{-1}d^{-1}g)g^{-1} = gxg^{-1}$, for every $d \in D$. So $gxg^{-1} \in k[[G]]^D$.*

In particular, $k[[G]]^{[D]}$ is a $N_G(D)$-algebra and Br_D is a $N_G(D)$-algebra homomorphism.

Lemma 8.0.2. *Let G be a profinite group and D a closed pro-p subgroup of G. Then $k[[G]]^{[D]} \cong k[[C_G(D)]]$.*

Proof. If we consider G as G-space with action given by conjugation, then $C_G(D) = G^D = \{g \in G : dgd^{-1} = g, \forall d \in D\}$. Then, by [26, Lemma 3.4], $k[[G]]^{[D]} \cong k[[G^D]] = k[[C_G(D)]]$. □

It will be convenient for us to consider Br_D as taking values in $k[[C_G(D)]]$. Thus we will regard Br_D as

$$Br_D : k[[G]]^D \longrightarrow k[[G]]^{[D]} \longrightarrow k[[C_G(D)]], \tag{8.1}$$

where the second arrow is the inverse of the isomorphism of Lemma 8.0.2.

The next result is a pseudocompact version of [3, Proposition 6.2.2].

Lemma 8.0.3. *Let G be a profinite group and D a closed normal pro-p subgroup of G. Then every idempotent in $Z(k[[G]])$ lies in $k[[C_G(D)]]$.*

Proof. Fix an idempotent $e \in Z(k[[G]])$. For each $N \trianglelefteq_O G$, DN/N is a normal p-subgroup of G/N and $\varphi_N(e)$ is a central idempotent of $k[G/N]$. Then, by [3, Proposition 6.2.2], $\varphi_N(e) \in k[C_{G/N}(DN/N)]$ for each $N \trianglelefteq_O G$. So $e \in \varprojlim k[C_{G/N}(DN/N)] = k[[C_G(D)]]$.

∎

Lemma 8.0.4. *Let G be a profinite group, D a closed pro-p subgroup of G and H a closed subgroup of G such that $DC_G(D) \leqslant H \leqslant N_G(D)$. Let b be a block of H with block idempotent f and defect group D. There is a unique block B of G with block idempotent e such that $f = f \cdot Br_D(e)$.*

Proof. By hyphotesis $DC_G(D) \leqslant H \leqslant N_G(D)$, so D is normal in H. By Lemma 8.0.3, $f \in k[[C_H(D)]]$. But, again by hypothesis, $C_H(D) = C_G(D)$. So $f \in k[[C_G(D)]]$.

On the other hand, $Br_D : k[[G]]^D \to k[[C_G(D)]]$ is a surjective algebra homomorphism, then $Br_D(1) = 1$ and so there is a block idempotent e such that $f = Br_D(e)f$. If e' is another block idempotent with this property, then $f = Br_D(e_i)fBr_D(e')f = Br_D(e_i)Br_D(e')f = Br_D(e_i \cdot e')f$. So $e_i = e'$ since distinct block idempotents are orthogonal. ∎

Definition 8.0.5. *Let G be a profinite group, D a closed pro-p subgroup of G and H a closed subgroup of G with $DC_G(D) \leqslant H \leqslant N_G(D)$. If b is a block of H with block idempotent f, define the Brauer correspondent b^G of b to be the unique block of G with block idempotent e such that $f = f \cdot Br_D(e)$.*

The following Lemma corresponds to Lemma [3, Lemma 6.2.5]. This is a central piece to establish a Brauer correspondence and it will proved for G being a virtually pro-p. The reason to the restriction for virtually pro-p groups is that when G is virtually pro-p, D is open in G and hence Tr_D^G is defined as in the finite case.

Lemma 8.0.6. *Let G be a virtually pro-p group and let D be an open subgroup. Then the diagram*

$$\begin{array}{ccc} k[[G]]^D & \xrightarrow{\;Br_D\;} & k[[C_G(D)]] \\ \Big\downarrow{\scriptstyle Tr_D^G} & & \Big\downarrow{\scriptstyle Tr_D^{N_G(D)}} \\ Tr_D^G(k[[G]]^D) & \xrightarrow[\;Br_D\;]{} & Tr_D^{N_G(D)}(k[[C_G(D)]]) \end{array}$$

commutes. In particular the lower map is surjective.

Proof. Since D is open in G, then Tr_D^G is a map just like it was defined for finite groups. So the proof is similar to the finite case.

Let $x \in k[[G]]^D$. Then,

$$\begin{aligned} Tr_D^G(x) &= \sum_{g \in D\backslash G/D} Tr^D_{D\cap {}^gD}({}^gx) \text{ (by Lemma 5.2.6-Mackey's formula)} \\ &= \sum_{\substack{g\in D\backslash G/D,\\ D\cap {}^gD \lneqq D}} Tr^D_{D\cap {}^gD}({}^gx) + \sum_{\substack{g\in D\backslash G/D,\\ D\cap {}^gD = D}} Tr^D_{D\cap {}^gD}({}^gx) \\ &= \sum_{\substack{g\in D\backslash G/D,\\ D\cap {}^gD \lneqq D}} Tr^D_{D\cap {}^gD}({}^gx) + \sum_{\substack{g\in D\backslash G/D,\\ D\cap {}^gD = D}} {}^gx. \end{aligned}$$

Observe that, if $g \in G$ is such that $D \cap {}^gD = D$, then $g \in N_G(D)$. Thus,

$$\begin{aligned}
Br_D(Tr_D^G(x)) &= Br_D(\sum_{\substack{g\in D\backslash G/D,\\ D\cap {}^gD\lneqq D}} Tr^D_{D\cap {}^gD}({}^gx) + \sum_{\substack{g\in D\backslash G/D,\\ D\cap {}^gD=D}} {}^gx) \\
&= Br_D(\sum_{\substack{g\in D\backslash G/D,\\ D\cap {}^gD\lneqq D}} Tr^D_{D\cap {}^gD}({}^gx)) + Br_D(\sum_{\substack{g\in D\backslash G/D,\\ D\cap {}^gD=D}} {}^gx) \\
&= Br_D(\sum_{\substack{g\in D\backslash G/D,\\ D\cap {}^gD=D}} {}^gx) \text{ (By Definition of } Br_D) \\
&= Br_D(\sum_{g\in N_G(D)/D} {}^gx) \\
&= \sum_{g\in N_G(D)/D} Br_D({}^gx) \\
&= \sum_{g\in N_G(D)/D} {}^gBr_D(x) \\
&= Tr_D^{N_G(D)}(Br_D(x)).
\end{aligned}$$

The penultimate equality follows from fact that Br_D is an $N_G(D)$-algebra homomorphism. So $Br_D \circ Tr_D^G = Tr_D^{N_G(D)} \circ Br_D$. Since the top map and right map are surjective, then the bottom map is also surjective.

□

In this point, we can establish and prove a version of the Brauer Correspondence for virtually pro-p groups:

Theorem 8.0.7. *Let G be a virtually pro-p group and D an open pro-p subgroup of G. The map Φ sending a block b of $N_G(D)$ to its Brauer correspondent b^G induces a bijection between the blocks of $N_G(D)$ with defect group D and the blocks of G with defect group D.*

Proof. Since $D \trianglelefteq_C N_G(D)$, by Lemma 8.0.3, each block idempotent in $k[[N_G(D)]]$ lies in $k[[C_{N_G(D)}(D)]]$. But, $C_{N_G(D)}(D) = C_G(D)$, so each block idempotent in $k[[N_G(D)]]$ lies in $k[[C_G(D)]]$.

Let b be a block of $N_G(D)$ with block idempotent f and defect group D, and let b^G be the Brauer correspondent of b with block idempotent e. By Theorem 7.3.4, there is a defect group R of b^G containing D. We assert that $R = D$.

By Lemma 5.2.6-1, $Tr_D^G(k[[G]]^D)$ is an ideal of $Z(k[[G]])$. So, we may consider the ideal $eTr_D^G(k[[G]]^D)$ of $eZ(k[[G]])$. By Lemma 8.0.6 and since Br_D is an algebra homomorphism, then

$$Br_D(eTr_D^G(k[[G]]^D)) = Br_D(e)Br_D(Tr_D^G(k[[G]]^D)) = Br_D(e)Tr_D^{N_G(D)}(k[[C_G(D)]]),$$

which contains f. Note that $f \notin J(k[[C_G(D)]])$, since otherwise, $\varphi_N(f)$ would be a non-zero idempotent in $J(k[[C_G(D)]]_N)$, for some N, which is impossible since the radical of a finite dimensional algebra is nilpotent. Then, $Br_D(eTr_D^G(k[[G]]^D)) \nsubseteq J(k[[C_G(D)]])$. So $eTr_D^G(k[[G]]^D) \nsubseteq J(eZ(k[[G]]))$. Now, since e is a primitive idempotent of $Z(k[[G]])$, by Lemma 6.0.5, $eZ(k[[G]])$ is a local algebra and hence $eTr_D^G(k[[G]]^D) = eZ(k[[G]])$. By Lemma 5.2.6,

$$\begin{aligned} e \in eTr_D^G(k[[G]]^D) &= eTr_R^G Tr_D^R(k[[G]]^D) \\ &= Tr_R^G(eTr_D^R(k[[G]]^D)) \\ &= Tr_R^G Tr_D^R(ek[[G]]^D) \\ &= Tr_D^G(ek[[G]]^D). \end{aligned}$$

Then $e \in Tr_D^G(k[[G]]^D)$. But, since b^G has defect group R, then R is the smallest subgroup such that $e \in Tr_R^G(k[[G]]^R)$. So $R \leqslant D$. Hence $D = R$.

The argument above shows that Φ is well-defined, so it remains to check that it is bijective.

Injectivity: Since $eTr_D^G(k[[G]]^D) = eZ(k[[G]])$ is a local algebra and Br_D is a homomorphism, then $Br_D(eTr_D^G(k[[G]]^D))$ is a local algebra. So f is the only block idempotent in $Br_D(eTr_D^G(k[[G]]^D))$.

Let b_1, b_2 are two blocks of $N_G(D)$ with defect group D and block idempotents f_1 and f_2 respectively. If $\Phi(b_1) = \Phi(b_2)$, then $b_1^G = b_2^G$ has defect group D and block idempotent e. Since f_1, f_2 lie in the local algebra $Br_D(eTr_D^G(k[[G]]^D))$, then $f_1 = f_2$. So $b_1 = b_2$ and hence Φ is injective.

Surjectivity: Let B be a block of G with defect group D and block idempotent e. Then, by Theorem 7.3.4, D is such that $Br_D(e) \neq 0$ and $e \in Tr_D^G(k[[G]]^D)$. Furthermore, by Lemma 8.0.6, $Br_D(Tr_D^G(k[[G]]^D)) = Tr_D^{N_G(D)}(k[[C_G(D)]])$. So $Br_D(e)$ is a central idempotent of $k[[C_G(D)]]$ in $Tr_D^{N_G(D)}(k[[C_G(D)]])$.

Let b be a block of $N_G(D)$ with block idempotent f such that $f = fBr_D(e)$. If b has defect group D, then $\Phi(b) = B$. It remain to confirm that b has defect group D.

We have that, $f \in Tr_D^{N_G(D)}(k[[C_G(D)]])$, since, writing $Br_D(e) = Tr_D^{N_G(D)}(a)$ for some $a \in k[[C_G(D)]]$, we have

$$\begin{aligned} f &= fBr_D(e) \\ &= fTr_D^{N_G(D)}(a) \\ &= Tr_D^{N_G(D)}(fa) \quad \text{(by Lemma 5.2.6)} \\ &\in Tr_D^{N_G(D)}(k[[C_G(D)]]). \end{aligned}$$

Since $f \in Tr_D^{N_G(D)}(k[[C_G(D)]])$, there is a defect group R of b contained in D. On the other hand, by Theorem 7.2.4, $R = Q \cap {}^gQ$ for a Sylow p-subgroup Q of $N_G(D)$ and some $g \in C_{N_G(D)}(R)$. But, by Corollary 7.2.5, $D = O_p(N_G(D))$ is the intersection of all Sylow p-subgroups of $N_G(D)$, then $D \leqslant R$. Hence $R = D$. Thus, $B = \Phi(b)$. □

Remark 8.0.8. *Note that 8.0.7 is false when we replace the word "open" in its statement with the word "closed".*

Let G be the free pro-p group of rank 2, freely generated by x and y. Let D be a closed maximal cyclic subgroup of G. By [17, Theorem 5.1], $N_G(D)$ is a closed cyclic group of G, but D is maximal cyclic subgroup of G, then $D = N_G(D)$.

There is of course a block of $k[[N_G(D)]]$ *with defect group* D *(namely* $k[[N_G(D)]]$ *itself), but there is not a block of* $k[[G]]$ *with defect group* D *(* $k[[G]]$ *has only one block and its defect group is* G*).*

Chapter 9

Blocks with Cyclic Defect Group - Brauer Tree Algebras

As we have seen through of all this work, inside of the block theory, the research is focused on the study of the behaviour of defect groups. In both cases, finite groups and profinite groups, the behaviour of these special subgroups gives information of the block decomposition of $k[[G]]$. In particular, for finite group algebras, when the defect groups are cyclic subgroups, it is possible to encode the information of the blocks in special graphs, called Brauer trees. In this chapter we will move from finite group algebras to complete group algebras, and we will show that it is possible realize pseudocompact blocks with cyclic defect group as Brauer tree algebras.

9.1 Blocks with cyclic defect groups

Remark 9.1.1. *Throughout our discussion in this section k will be an algebraically closed field of characteristic p, G a profinite group and B a block of G with non-trivial cyclic defect group D, that is, D is a profinite group possessing a dense cyclic abstract subgroup. We consider $\mathcal{P} = \{P_i : i \in \mathcal{I}\}$ a set of representatives of the isomorphism classes of indecomposable projective modules in B and $\mathcal{S} = \{S_i := P_i/rad(P_i) : i \in \mathcal{I}\}$ a set of representatives of the isomorphism classes of the simple modules in B.*

Write $B = \varprojlim_{N\in\mathcal{N}} X_N$, where $\mathcal{N}$ is a cofinal system of open normal subgroups of G such that X_N is a block with cyclic defect group DN/N, as in Remark 6.1.3.

Lemma 9.1.2. *Let B be a block of a profinite group G with cyclic defect group D. Then $\mathcal{S}$ and $\mathcal{P}$ are finite sets.*

Proof. Since for each $i \in \mathcal{I}$, $S_i = P_i/rad(P_i) \in \mathcal{S}$, where $P_i \in \mathcal{P}$, to prove that $\mathcal{S}$ and $\mathcal{P}$ are finite sets, it is sufficient to confirm that $\mathcal{S}$ is a finite set.

For each N, the number $|\mathcal{S}_N|$ of isomorphism classes of simple modules in X_N divides $p-1$ by Lemma 2.4.1. Hence $|\mathcal{S}_N| \leqslant p-1$, for each $N \in \mathcal{N}$.

Assume for contradiction that $|\mathcal{S}| \geqslant p$ and fix a set $\{S_1, ..., S_p\}$ of distinct simples in $\mathcal{S}$. By Corollary 6.1.5, there is $N \in \mathcal{N}$ such that $\{S_1, ..., S_p\}$ are distinct simple modules in X_N. But this contradicts the paragraph above. □

Remark 9.1.3. *Since $\mathcal{S}$ is a finite set by Lemma 9.1.2, from now on we assume in addition that for each $N \in \mathcal{N}$, N acts trivially on every simple module in B.*

Proposition 9.1.4. *Let $P_i \in \mathcal{P}$. For each $N \in \mathcal{N}$, P_{i_N} is non-zero and indecomposable.*

Proof. Let $\pi : P_i \to S_i$ be the canonical projection. Since $(\_)_N$ is right exact, then $\pi_N : P_{i_N} \to S_{i_N} \neq 0$ is surjective. Then $P_{i_N} \neq 0$ for all $N \in \mathcal{N}$. Furthermore, $I_N P_i \subseteq rad(P_i)$ for every N, and hence $P_{i_N}/rad(P_{i_N}) \cong P_i/rad(P_i) = S_i$. Hence P_{i_N} is indecomposable. □

An interesting property of blocks B with cyclic defect groups is that we can find a cofinal set of $N \trianglelefteq_O G$ such that B_N is no longer a direct product of blocks, but just a block.

Lemma 9.1.5. *Let B be a block of G with cyclic defect group D. There is $N_0 \trianglelefteq_O G$ acting trivially on each $S_i \in \mathcal{S}$ and such that B_N is a block for each $N \leqslant N_0$.*

Proof. By Lemma 9.1.2, $\mathcal{S}$ is finite. So we have the same number of simple modules in each B_N.

But by Corollary 6.1.5, there is $N_0' \in \mathcal{N}$ such that every $S_i \in \mathcal{S}$ lies in X_N, for each $N \leqslant N_0'$. Since each block of B_N has a simple module and every simple module lies in X_N, then $X_N = B_N$. $\square$

Lemma 9.1.6. *Let $P_i \in \mathcal{P}$. Then the multiplicity of P_i is equal to $dim_k(S_i)$*

Proof. Since P_{i_N} is indecomposable for each $N \in \mathcal{N}$ by Proposition 9.1.4, and P_{i_N} is the projective cover of S_i as a $k[G/N]$-module, by [31, Theorem 7.3.9], $dim_k(S_i)$ is the multiplicity of P_{i_N} for each $N \in \mathcal{N}$. Since the multiplicity of P_i in $k[[G]]$ is equal to the multiplicity of P_{i_N} in $k[G/N]$, then the multiplicity of P_i is $dim_k(S_i)$. $\square$

Lemma 9.1.7. *Let P_i be a non-simple indecomposable projective module. Then $rad(P_i)/soc(P_i) \cong \varprojlim_N rad(P_{i_N})/soc(P_{i_N})$.*

Proof. Since P_i is not simple, there is $N' \in \mathcal{N}$ such that P_{i_N} is indecomposable and not simple. For this N', $soc(P_{i_{N'}}) \subseteq rad(P_{i_{N'}})$. We work in the cofinal system of open normal subgroups N of G with $N \leqslant N'$. The result follows by applying Lemma 4.2.6 with $U = rad(P_i)$, $V = soc(P_i)$ and $W_N = soc(P_{i_N})$. $\square$

9.1.1 Indecomposable projective modules of blocks with cyclic defect group

We maintain the notation fixed in Remark 9.1.1. The objective of this section is to give a pseudocompact version of [20, Theorem 11.1.8].

Remark 9.1.8. *Throughout this section fix a cofinal system $\mathcal{M}$ of open normal subgroups N of G such that B_N is a block with cyclic defect group DN/N and with each $N \in \mathcal{M}$, acting trivially on the $S_i \in \mathcal{S}$. It follow from our conditions on N that the set $\mathcal{S}$ of representatives of the isomorphism classes of the simple modules in B can be canonically identified with a set $\mathcal{S}_N$ of representatives of the isomorphism classes of the simple modules in B_N, via the map $S_i \longmapsto S_{i_N}$.*

Definition 9.1.9. *Let A be a pseudocompact k-algebra.*

1. *A* composition series *for a finite dimensional A-module U is a finite series of submodules*

$$\{0\} = U_0 \subset U_1 \subset \ldots \subset U_n = U$$

with U_i maximal in U_{i+1} for $0 \leqslant i \leqslant n-1$. The simple factors U_{i+1}/U_i are called composition factors *of U and the integer $l(U) = n$ is called its* length*.*

2. *A finite dimensional A-module U is* uniserial *if it has a unique composition series.*

3. *A pseudocompact A-module U is* pro-uniserial *if it can be expressed as the inverse limit of finite dimensional uniserial A-modules.*

4. *The simple module S is a* composition factor *of the pseudocompact module U if it a composition factor of some finite dimensional quotient of U.*

Lemma 9.1.10. *Fix $P_i \in \mathcal{P}$. Then $rad(P_i)/rad^2(P_i) = T_i \oplus T_i'$, where T_i, T_i' are non-isomorphic simple modules or zero.*

Proof. If P_i is a simple module, the we have nothing to do, so assume that P_i is a non-simple module. Since B_N is a block with cyclic defect group DN/N, for each $N \in \mathcal{M}$ and P_{i_N} is indecomposable by Proposition 9.1.4, then, by [19, Theorem 11.1.8], there are two unique uniserial submodules X_{i_N}, Y_{i_N} of P_{i_N} such that $rad(P_{i_N}) = X_{i_N} + Y_{i_N}$, $soc(P_{i_N}) = X_{i_N} \cap Y_{i_N}$.

We will analyze two cases. In the first case, at least one of the modules X_{i_N}, Y_{i_N} is simple for every N. Assume without loss of generality that Y_{i_N} is simple for each N. Since $Y_{i_N} = soc(Y_{i_N}) = soc(P_{i_N}) = soc(X_{i_N}) \subseteq X_{i_N}$, then $Y_{i_N} \subseteq X_{i_N}$. So $rad(P_{i_N}) = X_{i_N} + Y_{i_N} = X_{i_N}$ and $rad^2(P_{i_N}) = rad(X_{i_N} + Y_{i_N}) = rad(X_{i_N})$. Then $rad(P_{i_N})/rad^2(P_{i_N}) = X_{i_N}/rad(X_{i_N}) = T_{i_N}$ for some simple module T_{i_N}. Then since $rad(P_i)/rad^2(P_i) \cong \varprojlim_N rad(P_{i_N})/rad^2(P_{i_N}) = \varprojlim_N T_{i_N}$ by Lemma 4.4.2, then these T_{i_N} are all isomorphic and $rad(P_i)/rad^2(P_i)$ is simple as required.

For the second case, assume that X_{i_N}, Y_{i_N} are non-simple submodules of P_{i_N} for some N. Then $soc(P_{i_N}) \subseteq rad(X_{i_N}),\ rad(Y_{i_N})$. Let $\pi_N : X_{i_N} \to X_{i_N}/rad(X_{i_N})$, $\pi_N' : Y_{i_N} \to$

$Y_{i_N}/rad(Y_{i_N})$ be the canonical projections, and consider the map $\gamma_N : X_{i_N} + Y_{i_N} \to X_{i_N}/rad(X_{i_N}) \oplus Y_{i_N}/rad(Y_{i_N})$ given by $x + y \longmapsto (\pi_N(x),\ \pi'_N(y))$ for $x \in X_{i_N}$ and $y \in Y_{i_N}$. The simple modules $X_{i_N}/rad(X_{i_N})$, $Y_{i_N}/rad(Y_{i_N})$ are non-isomorphic by [20, Theorem 11.1.8]. We confirm that γ_N is well-defined: Let $x + y = x' + y' \in X_{i_N} + Y_{i_N}$, then $z = x - x' = y - y' \in X_{i_N} \cap Y_{i_N} = soc(P_{i_N}) \subseteq rad(X_{i_N}),\ rad(Y_{i_N})$. Then $\gamma_N(x - x') = \gamma_N(y - y') = 0$. So

$$\begin{aligned} 0 &= \gamma_N(x - x') + \gamma_N(y - y') \\ &= \pi_N(x - x') + \pi'_N(y - y') \\ &= \pi_N(x) - \pi_N(x') + \pi'_N(y) - \pi'_N(y') \\ &\Rightarrow \pi_N(x) + \pi'_N(y) = \pi_N(x') + \pi'_N(y') \\ &\Rightarrow \gamma_N(x + y) = \gamma_N(x' + y'). \end{aligned}$$

We assert that $ker(\gamma_N) = rad(X_{i_N}) + rad(Y_{i_N})$. Now, a simple confirmation shows that $rad(U_{i_N}) + rad(V_{i_N}) \subseteq ker(\gamma_N)$. On the other hand, $x + y \in ker(\gamma_N)$ if, and only if, $x \in rad(X_{i_N})$ and $y \in rad(Y_{i_N})$. Then $x + y \in rad(X_{i_N}) + rad(Y_{i_N})$. Hence, $ker(\gamma_N) = rad(X_{i_N}) + rad(Y_{i_N})$.

It follows that $(X_{i_N} + Y_{i_N})/(rad(X_{i_N}) + rad(Y_{i_N}))$ is the direct sum of two simple modules $T_{i_N} = X_{i_N}/rad(X_{i_N})$ and $T'_{i_N} = Y_{i_N}/rad(Y_{i_N})$. Next we check that $rad(X_{i_N} + Y_{i_N}) = rad(X_{i_N}) + rad(Y_{i_N})$.

By [19, Theorem 11.1.8], every submodule of P_{i_N} is equal to $X'_N + Y'_N$ for some submodule X'_N of X_{i_N} and some submodule Y'_N of Y_{i_N}. Hence the maximal submodules of $X_{i_N} + Y_{i_N}$ have the form $X_{i_N} + Y'_N$ with Y'_N maximal in Y_{i_N} and $X'_N + Y_{i_N}$ with X'_N maximal in X_{i_N}. But, since X_{i_N}, Y_{i_N} are uniserial, they have only one maximal submodule, $rad(X_{i_N})$ for X_{i_N} and $rad(Y_{i_N})$ for Y_{i_N}. Then $rad(X_{i_N} + Y_{i_N}) = (rad(X_{i_N}) + Y_{i_N}) \cap (X_{i_N} + rad(Y_{i_N})) = rad(X_{i_N}) + rad(Y_{i_N})$.

So, $rad(P_{i_N})/rad^2(P_{i_N}) = (X_{i_N} + Y_{i_N})/rad(X_{i_N} + Y_{i_N}) = T_{i_N} \oplus T'_{i_N}$. Since $rad(P_i)/rad^2(P_i) \cong \varprojlim_N rad(P_{i_N})/rad^2(P_{i_N})$, then $rad(P_i)/rad^2(P_i)$ is also a direct sum of two simple modules. □

Proposition 9.1.11. *Fix $P_i \in \mathcal{P}$. There are unique pro-uniserial submodules X_i, Y_i of P_i satisfying the following properties:*

1. $X_i \cap Y_i = soc(P_i)$.

2. $X_i + Y_i = rad(P_i)$.

3. $\frac{rad(P_i)}{soc(P_i)} \cong \frac{X_i}{soc(P_i)} \oplus \frac{Y_i}{soc(P_i)}$, *where* $\frac{X_i}{soc(P_i)}$, $\frac{Y_i}{soc(P_i)}$ *have no common composition factors.*

Proof. By Lemma 9.1.10, $rad^2(P_i)$ is open in P_i. So, consider the cofinal system of $N \in \mathcal{M}$ such that $I_N P_i \subseteq rad^2(P_i)$. By Lemma 9.1.10, $rad(P_i)/rad^2(P_i) \cong T_i \oplus T_i'$, where T_i, T_i' are non-isomorphic simple modules or zero. If one of them is 0, let it be T_i'. In this case, set X_{i_N} to be $rad(P_{i_N})$ and Y_{i_N} to be $soc(P_{i_N})$. If T_i, T_i' are both non-zero, then for each N in the cofinal system, let X_{i_N} be the uniserial maximal submodule of $rad(P_{i_N})$ such that $X_{i_N}/rad(X_{i_N}) \cong T_i$ and let Y_{i_N} be the uniserial maximal submodule of $rad(P_{i_N})$ such that $Y_{i_N}/rad(Y_{i_N}) \cong T_i'$. By [20, Theorem 11.1.8], X_{i_N} and Y_{i_N} are the unique submodules with this property.

Since φ_{MN} sends $rad(P_{i_N})$ onto $rad(P_{i_M})$, then $\varphi_{MN}(X_{i_N}) = X_{i_M}$ and $\varphi_{MN}(Y_{i_N}) \subseteq Y_{i_M}$. So, define $X_i = \varprojlim_N \{X_{i_N}, \varphi_{MN}\}$ and $Y_i = \varprojlim_N \{Y_{i_N}, \varphi_{MN}\}$.

Next, we confirm that X_i and Y_i satisfy the properties 1,2 and 3 of the statement:

By [20, Theorem 11.1.8], $X_{i_N} \cap Y_{i_N} = soc(P_{i_N})$, $X_{i_N} + Y_{i_N} = rad(P_{i_N})$ and $\frac{rad(P_{i_N})}{soc(P_{i_N})} = \frac{X_{i_N}}{soc(P_{i_N})} \oplus \frac{Y_{i_N}}{soc(P_{i_N})}$, where $\frac{X_{i_N}}{soc(P_{i_N})}$, $\frac{Y_{i_N}}{soc(P_{i_N})}$ have no common composition factors.

By Lemma 4.4.2, $rad(P_i) = \varprojlim_N \{rad(P_{i_N}), \varphi_{MN}\} = \varprojlim_N \{X_{i_N} + Y_{i_N}, \varphi_{MN}\} = X_i + Y_i$, so property 2 follows.

Now, by Lemma 4.4.5, $soc(P_i) = \varprojlim_N \{soc(P_{i_N}), \varphi_{MN}\}$. Since $X_i = \varprojlim_N X_{i_N} = \bigcap_N \varphi_N^{-1}(\varphi_N(X_i))$ and $Y_i = \varprojlim_N Y_{i_N} = \bigcap_N \varphi_N^{-1}(\varphi_N(Y_i))$, then,

$$\begin{aligned} X_i \cap Y_i &= \left(\bigcap_N \varphi_N^{-1}(\varphi_N(X_i))\right) \cap \left(\bigcap_N \varphi_N^{-1}(\varphi_N(Y_i))\right) \\ &= \bigcap_N \varphi_N^{-1}(\varphi_N(X_i)) \cap \varphi_N^{-1}(\varphi_N(Y_i)) \\ &= \bigcap_N \varphi_N^{-1}(\varphi_N(X_i) \cap \varphi_N(Y_i)) \\ &= \bigcap_N \varphi_N^{-1}(X_{i_N} \cap Y_{i_N}) \\ &= \bigcap_N \varphi_N^{-1}(soc(P_{i_N})) \\ &= soc(P_i) \end{aligned}$$

so property 1 follows.

It remains to confirm that X_i and Y_i satisfy property 3. Applying Lemma 4.2.6 with $U = X_i$, $V = soc(P_i)$ and $W_N = soc(P_{i_N})$ we have that $\frac{X_i}{soc(P_i)} \cong \varprojlim_N \frac{X_{i_N}}{soc(P_{i_N})}$. Analogously, Applying Lemma 4.2.6 with $U = Y_i$, $V = soc(P_i)$ and $W_N = soc(P_{i_N})$ we have that $\frac{Y_i}{soc(P_i)} \cong \varprojlim_N \frac{Y_{i_N}}{soc(P_{i_N})}$. Then, $\frac{X_i}{soc(P_i)} \oplus \frac{Y_i}{soc(P_i)} \cong \varprojlim_N \frac{X_{i_N}}{soc(P_{i_N})} \oplus \frac{Y_{i_N}}{soc(P_{i_N})}$.

Now, it remains to confirm that $\frac{rad(P_i)}{soc(P_i)} \cong \frac{X_i}{soc(P_i)} \oplus \frac{Y_i}{soc(P_i)}$. For each $N \in \mathcal{M}$, $\frac{rad(P_{i_N})}{soc(P_{i_N})} = \frac{X_{i_N}}{soc(P_{i_N})} \oplus \frac{Y_{i_N}}{soc(P_{i_N})}$, and, by Lemma 9.1.7, $\frac{rad(P_i)}{soc(P_i)} \cong \varprojlim_N \frac{rad(P_{i_N})}{soc(P_{i_N})}$. Then,

$$\frac{X_i}{soc(P_i)} \oplus \frac{Y_i}{soc(P_i)} \cong \varprojlim_N \frac{X_{i_N}}{soc(P_{i_N})} \oplus \frac{Y_{i_N}}{soc(P_{i_N})} = \varprojlim_N \frac{rad(P_{i_N})}{soc(P_{i_N})} \cong \frac{rad(P_i)}{soc(P_i)}.$$

Then $\frac{X_i}{soc(P_i)} \oplus \frac{Y_i}{soc(P_i)} \cong \frac{rad(P_i)}{soc(P_i)}$. The uniqueness of X_i and Y_i, follows from the uniqueness of Y_{i_N}, X_{i_N} for each N. □

Note that Proposition 9.1.11 implies that for $n \in \mathbb{N}$, the modules $P_i/rad^n(P_i)$ are finite dimensional and hence that each $rad^n(P_i)$ is open in P_i. Note that from the above proof the maps $X_{i_N} \to X_{i_M}$ are surjective whenever $N \leqslant M$ and hence the maps $\frac{X_{i_N}}{soc(P_{i_N})} \to \frac{X_{i_M}}{soc(P_{i_M})}$ are surjective. The maps $Y_{i_N} \to Y_{i_M}$ are surjective except perhaps when $Y_{i_N} = soc(P_{i_N})$. But in this case $\frac{Y_{i_N}}{soc(P_{i_N})} = \frac{Y_{i_M}}{soc(P_{i_M})} = 0$, so the map

$\frac{Y_{i_N}}{soc(P_{i_N})} \to \frac{Y_{i_M}}{soc(P_{i_M})}$ is surjective anyway. We will use that these maps are surjective in the rest of the Chapter.

When G is finite, the block B has more structure. Namely there are permutations ρ, σ of $\mathcal{S}$ and the uniserial submodules of each P_i can be named U_i and V_i in a such way that the distinct composition factors of U_i are given by the ρ-orbit of S_i as we descend the (unique) composition series of U_i, and similarly the distinct composition factors of V_i are given by the σ-orbit of S_i. We lift this structure to profinite groups.

If ever W is a pseudocompact B-module, denote by $Fact(W) \subseteq \mathcal{S}$ the set of distinct representatives of the isomorphism classes of composition factors of W.

The next Lemma is a pseudocompact version of [20, Theorem 11.1.8]:

Lemma 9.1.12. *There are two permutations ρ, σ of $\mathcal{S}$ and the submodules X_i, Y_i of 9.1.11, can be renamed U_i, V_i in a such way that for each i,*

- *the first $|S_i^\rho|$ composition factors of U_i from the top are*

$$\frac{U_i}{rad(U_i)} \cong \rho(S_i),\ \frac{rad(U_i)}{rad^2(U_i)} \cong \rho^2(S_i), ...,\ \frac{rad^{|S_i^\rho|-1}(U_i)}{rad^{|S_i^\rho|}(U_i)} \cong \rho^{|S_i^\rho|}(S_i) = S_i,$$

- *the first $|S_i^\sigma|$ composition factors of V_i from the top are*

$$\frac{V_i}{rad(V_i)} \cong \sigma(S_i),\ \frac{rad(V_i)}{rad^2(V_i)} \cong \sigma^2(S_i), ...,\ \frac{rad^{|S_i^\sigma|-1}(V_i)}{rad^{|S_i^\sigma|}(V_i)} \cong \sigma^{|S_i^\sigma|}(S_i) = S_i,$$

where $|S_i^\rho|$, $|S_i^\sigma|$ are the sizes of the ρ-orbit S_i^ρ of S_i and the σ-orbit S_i^σ of S_i.

Proof. We maintain the notations from the proof of Proposition 9.1.11. Since $\mathcal{S}$ is finite, we can find $N_0 \in \mathcal{M}$ such that $Fact(X_i) = Fact(X_{i_{N_0}})$ and $Fact(Y_i) = Fact(Y_{i_{N_0}})$ for each i.

Since B_{N_0} is a block of G/N_0 with cyclic defect group DN_0/N_0, by the finite version of this result [20, Theorem 11.1.8], there are permutations ρ_{N_0}, σ_{N_0} of $\mathcal{S} = \mathcal{S}_{N_0}$ satisfying the conclusions of the Lemma with respect to an appropriate renaming of $X_{i_{N_0}}$ and $Y_{i_{N_0}}$ as $U_{i_{N_0}}$ and $V_{i_{N_0}}$. By how we chose X_i, Y_i in Proposition 9.1.11, $\varphi_{N_0}(X_i) = X_{i_{N_0}}$. Rename X_i by U_i if $X_{i_{N_0}} = U_{i_{N_0}}$ and by V_i if $X_{i_{N_0}} = V_{i_{N_0}}$. Rename Y_i to be the other one. This renaming is unambiguous unless $X_i = Y_i = soc(P_i)$, in which case one may rename arbitrarily.

Define $\rho := \rho_{N_0}$ and $\sigma := \sigma_{N_0}$. For each i, the condition that $Fact(U_i) = Fact(U_{i_{N_0}})$ implies that $I_{N_0}U_i \subseteq rad^{|S_i^\rho|}(U_i)$, since otherwise not every isomorphism class of composition factor of U_i would appear as a composition factor of $U_{i_{N_0}}$, and similarly with V_i. Hence, the first $|S_i^\rho|$ composition factors of U_i are

$$\frac{U_i}{rad(U_i)} \cong \rho(S_i),\ \frac{rad(U_i)}{rad^2(U_i)} \cong \rho^2(S_i), ...,\ \frac{rad^{|S_i^\rho|-1}(U_i)}{rad^{|S_i^\rho|}(U_i)} \cong \rho^{|S_i^\rho|}(S_i) = S_i,$$

and similarly with V_i, as required. □

For each $N \in \mathcal{M}$, we define the permutations $\rho_N := \rho$ and $\sigma_N := \sigma$ of $\mathcal{S}_N = \mathcal{S}$. By construction, ρ_N and σ_N satisfy [20, Theorem 11.1.8] with respect to the block B_N of G/N.

9.2 Brauer trees and Brauer tree algebras

In this section we give a pseudocompact version of the notion of a Brauer tree for a block of G with cyclic defect group [20, Theorem 11.1.9] and we give a pseudocompact version of [33, Theorem 5.10.37]. Additionally we give a description of the Brauer trees for blocks of $k[[G]]$.

We maintain the notation fixed in Remark 9.1.1. Now, fix a cofinal system $\mathcal{M}'$ of open normal subgroups N of G such that B_N is a block with cyclic defect group DN/N, each $N \in \mathcal{M}'$ acts trivially on each $S_i \in \mathcal{S}$, $Fact(U_i) = Fact(U_{i_N})$, $Fact(V_i) = Fact(V_{i_N})$ for every $i \in \mathcal{I}$ and, if D is a finite, then $|D| = |DN/N|$ for every $N \in \mathcal{M}'$.

Using the canonical identification $\mathcal{S} = \mathcal{S}_N$, $\rho_N := \rho$ and $\sigma_N := \sigma$ for each $N \in \mathcal{M}'$ as in Lemma 9.1.12, denote by $\Gamma(B_N)$ the Brauer tree of B_N, meaning:

$\Gamma(B_N)$ is a tree with vertices the ρ-orbits S_i^ρ and the σ-orbits S_i^σ of $\mathcal{S}$, the edges are the elements of $\mathcal{S}$, and the edge S_i joins the ρ-orbit of S_i and the σ-orbit of S_i. Unless $|DN/N| = p$ and $|\mathcal{S}| = p-1$, there is a unique exceptional vertex, to which we attribute the number $m_N = \frac{|DN/N|-1}{|\mathcal{S}|}$ (cf. [20, Theorem 11.1.9]).

There is a cyclic ordering γ_v of the edges around the vertex $v = S_i^\rho$ given by:

$$\rho(S_i), \rho^2(S_i), ..., \rho^{|S_i^\rho|-1}(S_i), \rho^{|S_i^\rho|}(S_i) = S_i.$$

Similarly, there is a cyclic ordering γ_v of the edges around the vertex $v = S_i^\sigma$ given by:

$$\sigma(S_i), \sigma^2(S_i), ..., \sigma^{|S_i^\sigma|-1}(S_i), \rho^{|S_i^\sigma|}(S_i) = S_i.$$

By [33, Theorem 5.10.37], B_N is the Brauer tree algebra of $\Gamma(B_N)$, meaning that:

1. There is a one-to-one correspondence between the edges of the tree and the elements of $\mathcal{S}_N$,
2. the top $P_{i_N}/rad(P_{i_N})$ of the indecomposable projective module P_{i_N} in B_N is isomorphic to the socle of P_{i_N},
3. the projective cover P_{i_N} of the simple module corresponding to the edge S_i is such that

$$rad(P_{i_N})/soc(P_{i_N}) \cong U_N^v(S_i) \oplus U_N^w(S_i)$$

   for two (possibly zero) uniserial modules $U_N^v(S_i)$ and $U_N^w(S_i)$ in B_N, where v, w are the vertices adjacent to the edge S_i,
4. if v is not the exceptional vertex and if v is adjacent to the edge S_i then $U_N^v(S_i)$ has $s(v) - 1$ composition factors, where $s(v)$ is the number of edges adjacent to v,

5. if v is the exceptional vertex with multiplicity m_N and if v is adjacent to S_i, then $U_N^v(S_i)$ has $m_N \cdot s(v) - 1$ composition factors.
6. if v is adjacent to S_i then the composition factors of $U_N^v(S_i)$ are described as

$$rad^j(U_N^v(S_i))/rad^{j+1}(U_N^v(S_i)) \cong \gamma_v^{j+1}(S_i),$$

for all j as long as j is smaller than the number of composition factors of $U_N^v(S_i)$.

Lemma 9.2.1. *Via the canonical identification $\mathcal{S} = \mathcal{S}_N$, the Brauer trees $\Gamma(B_N)$ are equal for each $N \in \mathcal{M}'$, except for the multiplicity m_N.*

Proof. Since (apart from the multiplicity) $\Gamma(B_N)$ is completely determined by $\rho_N = \rho, \sigma_N = \sigma, \mathcal{S}_N = \mathcal{S}$, for each $N \in \mathcal{M}'$, then the Brauer trees $\Gamma(B_N)$ are equal for each $N \in \mathcal{M}'$. By our conditions on $\mathcal{M}'$, either no $\Gamma(B_N)$ has an exceptional vertex, in which case there is nothing to check, or they all do. In this case, consider $N \leqslant M$ in $\mathcal{M}'$ and let v be the exceptional vertex of $\Gamma(B_M)$. By [33, Theorem 5.10.37], the modules $U_M^v(S_i)$ are the only modules having strictly more than the size of the corresponding orbit composition factors. But since $U_N^v(S_i)$ surjects onto $U_M^v(S_i)$, then $U_N^v(S_i)$ also has strictly more than the size of the corresponding orbit composition factors, and hence v is also the exceptional vertex of $\Gamma(B_N)$. □

Definition 9.2.2. *Define the Brauer tree of B to be $\Gamma(B) := \Gamma(B_N)$, for any $N \in \mathcal{M}'$, except for the multiplicity m of the exceptional vertex, which is $\frac{|D|-1}{|\mathcal{S}|}$ if D is finite, or ∞ if D is infinite.*

The next theorem is a pseudocompact version of [33, Theorem 5.10.37]:

Theorem 9.2.3. *Let B be a block of a profinite group G with cyclic defect group D. Then B is the Brauer tree algebra of the Brauer tree $\Gamma(B)$ in the following sense:*

1. *There is a one-to-one correspondence between the edges of $\Gamma(B)$ and the elements of $\mathcal{S}$.*
2. *the projective cover P_i of the simple module corresponding to the edge S_i is such that*

$$rad(P_i)/soc(P_i) \cong U^v(S_i) \oplus U^w(S_i)$$

for two (possibly zero) pro-uniserial modules $U^v(S_i)$ and $U^w(S_i)$, where v, w are the vertices adjacent to the edge S_i,

3. *if v is not the exceptional vertex and if v is adjacent to the edge S_i then $U^v(S_i)$ has $s(v) - 1$ composition factors, where $s(v)$ is the number of edges adjacent to v,*

4. *if v is the exceptional vertex with multiplicity m, and if v is adjacent to S_i, then $U^v(S_i)$ has $m \cdot s(v) - 1$ composition factors if m is finite, or infinitely many if $m = \infty$.*

5. *if v is adjacent to S_i then the composition factors of $U^v(S_i)$ are described as*

$$rad^j(U^v(S_i))/rad^{j+1}(U^v(S_i)) \cong \gamma_v^{j+1}(S_i),$$

for all j as long as j is smaller than the number of composition factors of $U^v(S_i)$.

6. *The socle of P_i is zero if, and only if, S_i is adjacent to a vertex of infinite multiplicity. Otherwise, $soc(P_i) \cong S_i$.*

Proof. Recall that we work with the notation fixed in Remark 9.1.1. Property 1 follows by the construction of $\Gamma(B)$. Property 2 follows from the construction of $\Gamma(B)$ and Proposition 9.1.11.

Let v be a not exceptional vertex of $\Gamma(B)$. Since by [33, Theorem 5.10.37], B_N is a Brauer tree algebra with Brauer tree $\Gamma(B_N)$, and the Brauer trees $\Gamma(B_N)$ are equal for every $N \in \mathcal{M}'$ by construction, then, for each $N \in \mathcal{M}'$, the number of composition factors of $U_N^v(S_i)$ is $s(v) - 1$. Since for each $N \in \mathcal{M}'$, $Fact(U^v(S_i)) = Fact(U_N^v(S_i))$, and since each element of $Fact(U_N^v(S_i))$ appear only once as a composition factor of $U_N^v(S_i)$, then, $U^v(S_i)$ has $s(v) - 1$ composition factors. Hence Property 3 follows.

If v is the exceptional vertex of $\Gamma(B)$, and the multiplicity m is finite, then there is $N' \in \mathcal{M}'$ such that $I_{N'}U^v(S_i) \subseteq rad^n(U^v(S_i))$, where $n = m \cdot s(v)$. Since for each

$N \leqslant N'$, the module $U_N^v(S_i)$ has $m \cdot s(v) - 1$ composition factors, then $U^v(S_i)$ has $m \cdot s(v) - 1$ composition factors.

If m is infinite, for each $n \in \mathbb{N}$, there is $N \in \mathcal{M}'$ such that $m_N \cdot s(v) - 1 > n$. Since $U^v(S_i)$ surjects onto $U_N^v(S_i)$, it follows that $U^v(S_i)$ has at least n composition factors for every n, and hence it has infinitely many composition factors. Hence Property 4 follows.

Assume that $U^v(S_i)$ has a finite number n of composition factors. Since $rad^n(U^v(S_i))$ is open in $U^v(S_i)$ by Propostition 9.1.11, there is $N' \in \mathcal{M}'$ such that $I_{N'}U^v(S_i) \subseteq rad^n(U^v(S_i))$. Since for each $N \leqslant N'$, $rad^j(U_N^v(S_i))/rad^{j+1}(U_N^v(S_i)) \cong \gamma_v^{j+1}(S_i)$, then

$$rad^j(U^v(S_i))/rad^{j+1}(U^v(S_i)) \cong \varprojlim_{N \leqslant N'} rad^j(U_N^v(S_i))/rad^{j+1}(U_N^v(S_i)) \cong \gamma_v^{j+1}(S_i),$$

for all j as long as j is smaller than n.

If v is an exceptional vertex with infinite multiplicity, then, by Item 4., $U^v(S_i)$ has infinitely many composition factors. So for each $n \in \mathbb{N}$ there is $N \in \mathcal{M}'$ with $m_N \cdot s(v) - 1 > n$. Then $rad^j(U_N^v(S_i))/rad^{j+1}(U_N^v(S_i)) \cong \gamma_v^{j+1}(S_i)$, for all j as long as j is smaller than $m_N \cdot s(v) - 1$. So

$$rad^j(U^v(S_i))/rad^{j+1}(U^v(S_i)) \cong \varprojlim_{N \leqslant N'} rad^j(U_N^v(S_i))/rad^{j+1}(U_N^v(S_i)) \cong \gamma_v^{j+1}(S_i),$$

for all $j \in \mathbb{N}$. Hence Property 5 follows.

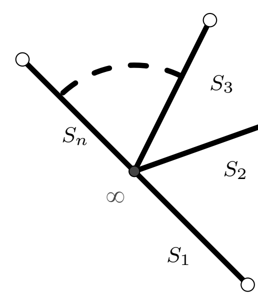
If neither of the vertices adjacent to S_i has infinite multiplicity, then P_i is finite dimensional, since it has a finite number of composition factors. So P_i is isomorphic to P_{i_N} for some $N \in \mathcal{M}'$. Then $P_i/rad(P_i) = P_{i_N}/rad(P_{i_N}) \cong soc(P_{i_N}) = soc(P_i)$.

If some vertex adjacent to S_i has infinite multiplicity, then for $N \leqslant M \in \mathcal{M}'$ the map $\varphi_{MN} : soc(P_{i_N}) \to soc(P_{i_M})$ is the zero map whenever $|DN/N| > |DM/M|$. So $soc(P_i) = \varprojlim_N soc(P_{i_N}) = 0$. Hence Property 6. follows. □

Theorem 9.2.4. *Let B be a block of a profinite group G with cyclic defect group D. Then D is a finite group if and only if $dim(B) < \infty$.*

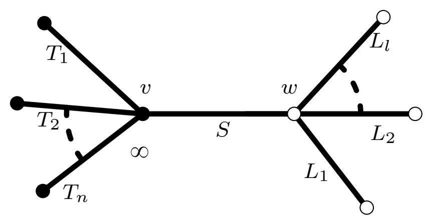
Proof. By Theorem 9.2.3, B is a Brauer tree algebra with Brauer tree $\Gamma(B)$ with $|\mathcal{S}|$ edges and exceptional vertex with multiplicity $m = \frac{|D|-1}{|\mathcal{S}|}$ if D is finite or ∞ is D is infinite. But, $dim(B) < \infty$ if, and only if, every indecomposable projective module has finite dimension, since, by Lemma 9.1.2, $\mathcal{P}$ is a finite set and each $P_i \in \mathcal{P}$ has multiplicity $dim_k(S_i)$ in B by Lemma 9.1.6. This is equivalent to B being a Brauer tree algebra with Brauer tree $\Gamma(B)$ whose exceptional vertex has multiplicity $m \in \mathbb{N}$. So, $dim(B) < \infty$ if, and only if, D is a finite group. □

The next result shows that a block with infinite cyclic defect group has a Brauer tree of star type: That is, a Brauer tree with exceptional vertex of infinite multiplicity with all edges emanating from this vertex, as in the following diagram:



Theorem 9.2.5. *Let B be a block of a profinite group G with infinite cyclic defect group D. Then $\Gamma(B)$ is of star type.*

Proof. By Theorem 9.2.3, B is a Brauer tree algebra with Brauer tree $\Gamma(B)$ with exceptional vertex of multiplicity $m = \infty$. Assume that $\Gamma(B)$ is not star type. Then $\Gamma(B)$ has a subgraph of the form



where n could be 0 but l is at least 1.

For each $M \in \mathcal{M}'$, denote by $X_M^v,\ X_M^w$ the uniserial submodules of the projective cover P_M of S corresponding to the vertices v, w respectively, so that $X_M^v + X_M^w = rad(P_M)$ and $X_M^v \cap X_M^w = soc(P_M)$. Fix $M \in \mathcal{M}'$. Since $m = \infty$, there exists $N \in \mathcal{M}'$ contained in M such that the projection $\varphi_{MN} : X_N^v \to X_M^v$ is not an isomorphism.

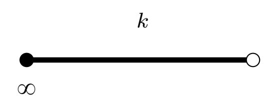
Since $ker(\varphi_{MN}) \cap X_N^v \neq 0$, it follows that φ_{MN} must map $soc(X_N^v) = soc(P_N)$ to 0. But on the other hand $\varphi_{MN} : X_N^w \to X_M^w$ is an isomorphism. This can be seen as follow: The modules X_N^w, X_M^w are isomorphic uniserial modules, which are not simple by hypothesis, since they have at least S and L_1 as composition factors. Hence $soc(X_N^w) = soc(P_N) \subseteq rad(X_N^w)$ and similarly with M. The surjective map $X_N^w/soc(P_N) \to X_M^w/soc(P_M)$ induced by φ_{MN} thus induces a surjective map $X_N^w/rad(X_N^w) \to X_M^w/rad(X_M^w)$. So, by [19, Proposition 4.5.1], $\varphi_{MN} : X_N^w \to X_M^w$ is surjective, and hence an isomorphism.

It follow that $soc(X_N^w) = soc(P_N)$ is not sent to 0 under φ_{MN}, which yields a contradiction. □

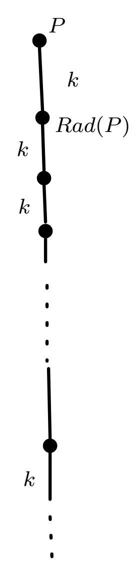
9.2.1 Examples:

(1) Let G be an infinite cyclic pro-p group, $k[[G]]$ is the inverse limit of finite group algebras $k[G/N]$ of the finite cyclic p-groups G/N. Each $k[G/N]$ is a Brauer tree algebra with Brauer tree Γ_N with two vertices, one exceptional, and one edge, where the exceptional vertex has multiplicity $m_N = |DN/N| - 1 = |G/N| - 1$ (cf. [1, V, §17]).

$k[[G]]$ is indecomposable as an algebra, since each $k[G/N]$ is indecomposable. So, $k[[G]]$ has only one block, B, and this block has cyclic defect group G. Then B is a Brauer tree algebra with Brauer tree Γ having two vertices, one exceptional, and one edge, where the exceptional vertex has infinite multiplicity. So the Brauer tree of $k[[G]]$ is the tree below,

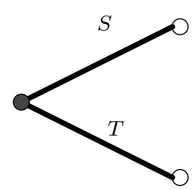


The unique indecomposable projective module P, corresponding to the unique edge of the Brauer tree can be represented by the following diagram:



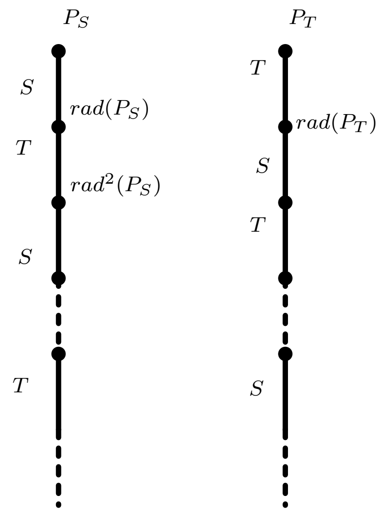
(2) Let k be a field of characteristic 5. For each $n \geqslant 1$, let $G_n =< a, b \mid a^{5^n} = b^2 = 1, bab = a^{-1} >$ be the dihedral group of order $2 \cdot 5^n$ and let C_{5^n} be the cyclic

subgroup of order 5^n generated by a. Observe that C_{5^n} is normal in G_n and $C_{G_n}(C_{5^n}) = C_{5^n}$, so, by [3, Proposition 6.2.2], $k[G_n]$ has only one block, with defect group C_{5^n}. By 2.3.1, $k[G_n]$ is a Brauer tree algebra for a Brauer tree with $|\mathcal{S}_n|$ edges and multiplicity $\frac{|C_{5^n}|-1}{|\mathcal{S}_n|}$. Its Brauer tree $\Gamma(k[G_n])$ has 2 edges, because, by [20, Theorem 11.1.3], $|\mathcal{S}_n| = |E_n| = |N_{G_n}(C_{5^n})/C_{G_n}(C_{5^n})| = 2$, where E_n is the inertial quotient of the block $k[G_n]$ (cf. [20, Definition 6.7.7]). The exceptional vertex has multiplicity $m_n = \frac{|C_{5^n}|-1}{2}$. By [1, V, §17, p.123] the Brauer tree is star type since C_{5^n} is a cyclic normal 5-Sylow subgroup in G_n.



The simple modules S, T corresponding to the edges are 1 dimensional. One of them is the trivial module $S = k$ and the other $T =< t >$ is a simple module with the action of $k[G_n]$ given by $a \cdot t = t$ and $b \cdot t = -t$.

Now, let $G = \varprojlim_n G_n$ and $C = \varprojlim_n C_{5^n} \cong \mathbb{Z}_5$. Consider $k[[G]] = \varprojlim_n k[G_n]$, an indecomposable algebra with cyclic defect group C. Then $k[[G]]$ is a Brauer tree algebra of Brauer tree star type, with 2 edges S, T and exceptional vertex with infinite multiplicity. The indecomposable projective modules corresponding to S an T can be represented by the following diagram:


P_S
S
$rad(P_S)$
T
$rad^2(P_S)$
S
T
P_T
T
$rad(P_T)$
S
T
S

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