Julio Leo Fonseca Quispe

Horrocks-Mumford Holomorphic Distributions

Belo Horizonte - MG

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Thesis presented to the Department of Mathematics of the Federal University of Minas Gerais, as a requirement to obtain a Ph.D. degree in Mathematics.

UFMG

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Coorientador: Prof. Dr. José Omegar Calvo-Andrade

Belo Horizonte - MG

Fonseca Quispe, Julio Leo.

F676h Horrocks-Munford Holomorphic Distributions [manuscrito] / Julio Leo Fonseca Quispe. - 2020.
139 f. il.

Orientador: Maurício Barros Corrêa Júnior.
Coorientador: José Omegar Calvo-Andrade.
Tese (doutorado) - Universidade Federal de Minas Gerais, Instituto de Ciências Exatas, Departamento Matemática.
Referências: f.137-139
.
1. Matemática – Teses. 2. Fibrados vetoriais – Teses. 3. Teoría Global de Folheações e Distribuições Holomorfas – Teses. 4. Espaços de Moduli – Teses.
5. Classes de Chern (Geometria Algébrica) - Teses I. Corrêa Júnior ,Maurício Barros. II. Calvo- Andrade, José Omegar. III. Universidade Federal de Minas Gerais;Instituto de Ciências Exatas, Departamento de Matemática. IV.Título.

CDU 51(043)

Ficha catalográfica elaborada pela bibliotecária Belkiz Inez Rezende Costa
CRB 6ª Região nº 1510

Universidade Federal de Minas Gerais
Departamento de Matemática
Programa de Pós-Graduação em Matemática

FOLHA DE APROVAÇÃO

Horrocks-Mumford Holomorphic Distributions

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Belo Horizonte, 05 de março de 2020.

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ATA DA CENTÉSIMA QUADRAGÉSIMA QUINTA DEFESA DE TESE DO ALUNO JULIO LEO FONSECA QUISPE, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA, DO INSTITUTO DE CIÊNCIAS EXATAS, DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA NO DIA 05 DE MARÇO DE 2020.

Aos cinco dias do mês de março de 2020, às 10h00, na sala 3060, reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese do aluno **Julio Leo Fonseca Quispe**, intitulada: "*Horrocks-Mumford Holomorphic Distributions*", requisito final para obtenção do Grau de doutor em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Maurício Barros Corrêa Júnior, após dar conhecimento aos presentes o teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se sem a presença do aluno e do público, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 05 de março de 2020.

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Este trabajo está dedicado a mis queridos padres Félix y Doris.

AGRADECIMENTOS

Agradeço a Deus por ter me dado saúde, coragem, persistência e disposição para lutar todos os dias.

Aos meus pais Félix e Doris, por sempre me apoiar, acreditar e confiar em mim de maneira incondicional durante esse tempo na distância. Eles têm sido um pilar importante para mim nesse novo desafio, com seu amor incondicional e palavras constantes de apoio para o meu desenvolvimento pessoal e profissional. Aos meus irmãos Pepe e Lucy, e a família Wilmer, Alicia, Aracely e o novo integrante da família o pequeno Joaquín, obrigado pelo apoio e compreensão em cada dia de minha vida.

Aos meus orientadores Dr. Maurício Corrêa Jr. e Dr. José Omegar Calvo–Andrade, pelo seu permanente apoio, disponibilidade, profissionalismo, crítica exigente que foi fundamental durante o desenvolvimento deste trabalho, e sobre todo pela amizade.

Agradeço aos meus colegas do ICEx nesta caminhada do doutorado, ao pessoal do seminario dos Folheados, à Alan Ramos pela parceria e amizade de pesquisa, e aos meus amigos-irmãos de décadas Eduardo, Carlos, John, José, Kathi, Nelcy, Juan, Tauan, Seu Eduardo, Ana Catarina e sua família e toda a galera do Icex, obrigado todos vocês pelo constante apoio, força e carinho, e claro aos meus amigos do meu querido Mollendo: Santos e Jhon; obrigado todos eles, esses anos tornaram-se muito mais agradáveis. Agradeço aos amigos: Iris, Karina, Miguel, Ruth, Diosel y Silvia, pela companhia e amizade durante minha estadia no México e a toda a comunidade no CIMAT.

Não poderia deixar de agradecer aos professores Alberto Sarmiento, Walter Torres, Gilcione Costa, Arturo Fernández pelo incentivo a seguir nessa profissão, e aos professores da banca: Marcos Jardim, Israel Vainsencher, Márcio Soares e Miguel Arnulfo por suas valiosas críticas e considerações que contribuíram neste trabalho.

Às secretarias pela sempre disponibilidade e soporte durante estes anos.

À CAPES e ao CIMAT pelo apoio financeiro, durante meu doutorado e o período do doutorado sandwich, pois sem esse eu não teria feito uma pós-graduação.

It is my point of view that we now understand the geometry of this bundle quite well (of course this does not mean that other interesting and surprising facts might not be discovered in the future).

Klaus Hulek – 1995.

RESUMO

Nesta tese de doutorado nos dedicamos ao estudo de Distribuições Holomorfas de codimensão dois em $\mathbb{P}^4$ cujo feixe tangente e conormal é Horrocks-Mumford, isto é, um fibrado vetorial estável, em particular não decomponível de posto 2. Nosso primeiro objetivo é descrever a geometria do esquema singular dessas distribuições. Provamos que o esquema singular é uma curva suave aritmeticamente Buchsbaum, conexa e irredutível. Mostramos que tais distribuições não são integráveis. Finalmente, descrevemos o espaço de Moduli dessas distribuições, provando que tal espaço é uma variedade quasi-projectiva irredutível e calculamos sua dimensão.

Palavras-chave: Fibrado vetorial Horrocks-Mumford. Distribuições Holomorfas. Espaço de Moduli.

ABSTRACT

This thesis is devoted to the study of Codimension two Holomorphic Distributions on $\mathbb{P}^4$ whose tangent and conormal sheaves are Horrocks-Mumford, that is a stable vector bundle of rank 2, in particular non-decomposable. Our first goal is to describe the geometry of the singular scheme of these distributions. We prove that the singular scheme is a smooth, reduced, irreducible (hence connected) arithmetically Buchsbaum curve. We show that such distributions are non-integrable. Finally, we describe the Moduli space of these distributions, proving that such space is an irreducible quasi-projective variety and we calculate its dimension.

Keywords: Horrocks-Mumford Bundle. Holomorphic Distributions. Moduli space.

LIST OF TABLES

CONTENTS

INTRODUCTION

The main goal of this thesis is to study Codimension two Holomorphic Distributions on the projective space $\mathbb{P}^4$ whose normalization of the tangent and conormal sheaves are Horrocks-Mumford.

A (saturated) distribution $\mathscr{F}$ of codimension $k \geq 1$ on $\mathbb{P}^n$ is a short exact sequence of coherent sheaves:

$$\mathscr{F} : 0 \to T\mathscr{F} \to T\mathbb{P}^n \to N_{\mathscr{F}} \to 0 \ , \tag{1}$$

such that $N_{\mathscr{F}}$, called the normal sheaf of $\mathscr{F}$, is a nontrivial torsion free sheaf of rank k on $\mathbb{P}^n$. It follows that the sheaf $T\mathscr{F}$, called the tangent sheaf of $\mathscr{F}$, is a reflexive sheaf of rank $n-k$. The singular scheme of $\mathscr{F}$, defined by $\mathrm{Sing}(\mathscr{F}) := \mathrm{Sing}(N_{\mathscr{F}})$, is a closed subscheme on $\mathbb{P}^n$ of codimension at least 2. A foliation is an integrable distribution, that is a distribution whose tangent sheaf is closed under Lie brackets of vector fields, that is, $[T\mathscr{F}, T\mathscr{F}] \subset T\mathscr{F}$.

A dual perspective can be considered. A holomorphic distribution $\mathscr{F}$, of codimension $k \geq 1$ on $\mathbb{P}^n$, is given by an exact sequence of coherent sheaves:

$$\mathscr{F} : 0 \to N^*_{\mathscr{F}} \to \Omega^1_{\mathbb{P}^n} \to \mathcal{Q}_{\mathscr{F}} \to 0 \ , \tag{2}$$

where $\mathcal{Q}_{\mathscr{F}}$ is a nontrivial torsion free sheaf of rank $n-k$ on $\mathbb{P}^n$. It follows that the conormal sheaf $N^*_{\mathscr{F}}$ is a reflexive sheaf of rank k.

M. Corrêa, M. Jardim and R. Vidal have shown in [4] that, for distributions of arbitrary codimension on $\mathbb{P}^n$, the tangent and conormal sheaves split as a sum of line bundles, if and only if, the singular scheme is arithmetically Cohen-Macaualay. Subsequently, Marcos Jardim, José Omegar and Maurício Corrêa, in [5], studied the properties of the singular schemes and tangent sheaves of codimension 1 distributions on $\mathbb{P}^3$, and provided a classification to codimension 1 distributions of degree at most 2 with locally free tangent sheaves, and showed that codimension one distribution of arbitrary degree, with only isolated singularities have stable tangent sheaf. In addition, they described the Moduli space of distributions in terms of Grothendieck's Quot scheme for the tangent bundle and, in certain cases, they showed that Moduli space of codimension one distributions, in the projective space, is a non-singular irreducible quasi-projective variety. Recently, Maurício Corrêa, Marcos Jardim and Simone Marchesi in [7] classified foliations by curves up to degree 3 whose cornormal sheaf is locally free. In addition, they provided a slightly different dual construction more suitable to describe the Moduli Spaces of these foliations.

In 1973, G. Horrocks and D. Mumford showed the existence of a stable bundle E of rank 2 on $\mathbb{P}^4$, called the Horrocks-Mumford bundle [23]; this is the only known non-decomposable vector bundle of rank 2. Later, H. Sumihiro showed in [36] that $E(a)$ is generated by global sections, for every $a \geq 1$. Since $T\mathbb{P}^4(-1)$ and $\Omega^1_{\mathbb{P}^4}(2)$ are globally generated sheaves then $E(a) \otimes T\mathbb{P}^4(-1) = \mathcal{H}om(E(-a-4), T\mathbb{P}^4)$ and $E(a) \otimes \Omega^1_{\mathbb{P}^4}(2) = \mathcal{H}om(E(-a-7), \Omega^1_{\mathbb{P}^4})$ are also globally generated sheaves. In this work, we study holomorphic distribution on $\mathbb{P}^4$ whose tangent sheaf is $T\mathscr{F} = E(-a-4)$ and distributions whose conormal sheaf is $N^*_{\mathscr{F}} = E(-a-7)$ for every $a \geq 1$, so $T\mathscr{F}$ and $N^*_{\mathscr{F}}$ are non split stable sheaves. These distributions are induced by a Bertini-type Theorem 28 and, in consequence, the singular scheme $Z = \mathrm{Sing}(\mathscr{F})$ is a smooth closed subscheme of $\mathbb{P}^4$ with expected codimension 3. Distributions with split tangent and conormal sheaf were studied in [4] and it is natural to study distributions whose tangent or conormal sheaves are Horrocks-Mumford. More precisely, we prove the following result:

Proposition 1. *Let $\mathscr{F}_a$ is a codimension 2 holomorphic distributions (1). Let $Z_a = \mathrm{Sing}(\mathscr{F}_a)$ the singular scheme, for $a \geq 1$ then:*

1. $\deg(\mathscr{F}_a) = 2a + 5$.
2. $\deg(Z_a) = 4a^3 + 33a^2 + 77a + 46$.
3. $p_a(Z_a) = 9a^4 + 89a^3 + \frac{553}{2}a^2 + \frac{573}{2}a + 45$.

Proposition 2. *Let $\mathscr{F}_a$ is a codimension 2 holomorphic distributions (2). Let $Z_a = \mathrm{Sing}(\mathscr{F}_a)$ the singular scheme, for $a \geq 1$, then:*

1. $\deg(\mathscr{F}_a) = 2a + 6$.
2. $\deg(Z_a) = 4a^3 + 39a^2 + 113a + 92$.
3. $p_a(Z_a) = 9a^4 + 107a^3 + \frac{847}{2}a^2 + \frac{1261}{2}a + 260$.

Proposition 3. *The singular scheme $Z_a = \mathrm{Sing}(\mathscr{F}_a)$ is reduced and irreducible.*

Theorem 1 (A). *Let $\mathscr{F}_a$, for all $a \geq 1$, is a codimension 2 holomorphic distributions (1) on $\mathbb{P}^4$. Then the singular scheme $Z_a = \mathrm{Sing}(\mathscr{F})$ is a smooth curve but not arithmetically Buchsbaum nor arithmetically Cohen Macaulay, and the Rao module dimensions of the singular scheme, for $a \geq 1$, are:*

1. $\dim_{\mathbb{C}} R_{\mathscr{F}_1} \geq 184$.
2. $\dim_{\mathbb{C}} R_{\mathscr{F}_2} \geq 284$.
3. $\dim_{\mathbb{C}} R_{\mathscr{F}_3} \geq 369$.
4. $\dim_{\mathbb{C}} R_{\mathscr{F}_a} = 401$, $\forall a \geq 4$.

Theorem 2 (B)**.** *Let $\mathscr{F}_a$, for all $a \geq 1$, is a codimension 2 holomorphic distributions (2) on $\mathbb{P}^4$. Then the singular scheme $Z_a = \operatorname{Sing}(\mathscr{F})$ is a smooth curve but not arithmetically Buchsbaum nor arithmetically Cohen Macaulay, and the Rao module dimensions of the singular scheme, for $a \geq 1$, are:*

1. $\dim_{\mathbb{C}} R_{\mathscr{F}_1} \geq 184$.
2. $\dim_{\mathbb{C}} R_{\mathscr{F}_2} \geq 284$.
3. $\dim_{\mathbb{C}} R_{\mathscr{F}_3} \geq 369$.
4. $\dim_{\mathbb{C}} R_{\mathscr{F}_a} = 401$, $\forall a \geq 4$.

Theorem 3. *The First cohomology dimension of the singular scheme of Horrocks-Mumford distributions (1) and (2) is* ***27****.*

On the other hand, these distributions are non-integrable. In fact, if $\mathscr{F}$ is a foliation of codimension 2, taking a transversal section at a generic point $p \in Z = \operatorname{Sing}_3(\mathscr{F})$ then the Grothendieck residue $\operatorname{Res}(\mathscr{F}, c_1^3; p) = 0$. For the other hand, since $\mathscr{F}$ is a holomorphic foliation of codimension 2 then by Baum-Bott formula we have:

$$c_1^3(\det(N_{\mathscr{F}})) = \operatorname{Res}_{c_1^3}(\mathscr{F}\ |_{B_p}; p) \cdot [Z],$$

hence $0 < \deg(\det(N_{\mathscr{F}})) = \operatorname{Res}_{c_1^3}(\mathscr{F}\ |_{B_p}; p) \cdot \deg(Z)$, thus $\operatorname{Res}_{c_1^3}(\mathscr{F}\ |_{B_p}; p) \neq 0$. This is a contradiction, since the ampleness of $\det(N_{\mathscr{F}})$ implies that the cohomology class $c_1^3(\det(N_{\mathscr{F}}))$ is non zero. Therefore:

Theorem 4 (C)**.** *Let $\mathscr{F}_a$ be the Horrocks-Mumford distribution (1), then $\mathscr{F}_a$ is non-integrable, for $a \geq 1$.*

Theorem 5 (D)**.** *Let $\mathscr{F}_a$ be the Horrocks-Mumford distribution (2), then $\mathscr{F}_a$ is non-integrable, for $a \geq 1$.*

Finally, the authors, in [5] and [7], provided an explicit description of the Moduli Spaces of distributions by a forgetful morphism that associates each holomorphic distribution with its tangent and conormal sheaf. W. Decker, in [10], provided a description of the Moduli Space of the Horrocks-Mumford stable bundles so, by forgetful morphism, we show that the Moduli Space of the Horrocks-Mumford holomorphic distributions is an irreducible quasi-projective non-singular variety and calculate its dimension as a consequence of the Theorem on the dimension of the fibers.

Theorem 6 (E)**.** *The Moduli space $\mathscr{D}ist^{P,st}(2a+5, a^2+3a+6)$ of codimension two holomorphic distributions (1) is an irreducible, quasi-projective variety of dimension*

$$\frac{1}{3}a^4 + 7a^3 + \frac{277}{6}a^2 + \frac{199}{2}a + 43$$

for $a \geq 1$.

Theorem 7 (F). *The Moduli space* $\mathscr{D}ist^{P,st}(2a+6, a^2+9a+24)$ *of codimension two holomorphic distributions (2) is an irreducible, quasi-projective variety of dimension*

$$\frac{1}{3}a^4 + \frac{23}{3}a^3 + \frac{343}{6}a^2 + \frac{899}{6}a + 98$$

for $a \geq 1$.

Parte I

Preliminaries

1 A BRIEF TOUR THROUGH ALGEBRAIC GEOMETRY

In this chapter we present some definitions, properties and algebraic-geometric theorems that will be fundamental tools during the development of this thesis. The main references will be [16], [20], [21], [22], [25], [26], [31].

1.1 SHEAVES

1.1.1 Properties of coherent sheaves

As an important class of $\mathcal{O}_X-$modules, including locally free sheaves, we have the "coherent sheaves".

Definition 1. *An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be a coherent sheaf if each point in X admits a neighborhood U such that there exist an exact sequence of the form*

$$\mathcal{O}_U^q \xrightarrow{x} \mathcal{O}_U^p \to \mathcal{F}|_U \to 0, \tag{1.1}$$

with p and q positive integers.

Example 1. *If $\mathcal{E}$ and $\mathcal{F}$ are two coherent sheaves, then $\mathcal{E}\oplus\mathcal{F}$, $\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F})$ are coherent.*

Any $\mathcal{O}_Y-$sheaf $\mathcal{F}$ on a closed complex submanifold $Y \subset X$ can be considered as an $\mathcal{O}_X-$sheaf on X supported on Y. More precisely, one identifies $\mathcal{F}$ with its direct image $i_*\mathcal{F}$ under the inclusion $i : Y \to X$. The restriction of holomorphic functions yields a natural surjection $\mathcal{O}_X \to \mathcal{O}_Y$. This gives rise to the *structure sheaf sequence* of $Y \subset X$:

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

where $\mathscr{I}_Y$ is the *ideal sheaf* of all holomorphic functions vanishing on Y.

Theorem 8. *If $\mathcal{F}$ is a coherent holomorphic sheaf over a analytic variety X, then $\{x \in X \mid \mathcal{F}_x \neq 0\}$ is a analytic subvariety of X.*

Proof. Vide [31], pag 73. □

For general holomorphic sheaves $\mathcal{F}$ over a analytic variety X, the closure of the set $\{x \in X \mid \mathcal{F}_x \neq 0\}$ is called the *support* of the sheaf $\mathcal{F}$. For a coherent holomorphic sheaf $\mathcal{F}$ over a analytic variety X, the support of $\mathcal{F}$ is then always a analytic subvariety of X; moreover the support of $\mathcal{F}$ consists exactly of those points of X over which the sheaf has nonzero stalks.

Corollary 1. *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a homomorphism between two coherent holomorphic sheaves over a analytic variety X, and if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is the induced homomorphism on stalks over a point $x \in X$, then $\{x \in X \mid \varphi_x \text{ is not injective}\}$ and $\{x \in X \mid \varphi_x \text{ is not surjective}\}$ are analytic subvarieties of X.*

Definition 2. *For a coherent sheaf $\mathcal{F}$ we set*

$$\mathrm{Sing}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \textit{ is not } \mathcal{O}_{X,x} - \textit{free}\},$$

and call it the singular set of $\mathcal{F}$.

1.2 TORSION-FREE AND REFLEXIVE SHEAVES

In this section we mention some basic properties of torsion-free and reflexive sheaves. For more information, see [21] and [27].

Definition 3. *A coherent sheaf $\mathcal{F}$ over X is torsion free if every stalk $\mathcal{F}_x$ is a torsion free $\mathcal{O}_{X,x}-$module; i.e., $fa = 0$ for $f \in \mathcal{O}_{X,x}$, $a \in \mathcal{F}_x$ always implies $a = 0$ or $f = 0$.*

Example 2. *Every locally free sheaf is torsion free. Any coherent subsheaf of a torsion-free sheaf is again torsion-free.*

Theorem 9. *Let $\mathcal{F}$ be a torsion-free sheaf on X. Then*

$$\operatorname{codim} \mathrm{Sing}(\mathcal{F}) \geq 2.$$

Proof. Vide [31], pag. 75.

□

On the other hand, there is a natural map of $\mathcal{F}$ to its double dual $\mathcal{F}^{**}$.

Definition 4. *A coherent sheaf $\mathcal{F}$ on X is reflexive if the natural map $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism.*

Example 3. *Any locally free sheaf is reflexive, in particular vector bundles. On the other hand, any reflexive sheaf is torsion-free.*

Proposition 4. *A coherent sheaf $\mathcal{F}$ on a noetherian scheme X is reflexive if and only if it can be included in an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0,$$

where $\mathcal{E}$ is locally free and $\mathcal{G}$ is torsion-free.

Proof. Vide [21], pag. 124. □

Corollary 2. *The dual of any coherent sheaf is reflexive.*

Proof. Vide [21], pag. 124. □

1.3 CHERN CLASS

In this section we introduce these topological invariants axiomatically. For more details see [16].
Let E be a vector bundle on X. For each integer $i \geq 0$ we can assign a cohomology class $c_i(E) \in H^{2i}(X;\mathbb{Z})$ such that $c_0(E) = 1$. The *total Chern class* $c(E)$ is the sum

$$c(E) = 1 + c_1(E) + \cdots + c_r(E) \in H^*(X;\mathbb{Z}), \quad r = \operatorname{rank} E.$$

$c_r(E)$ is called the *top Chern class* of E. The class $c(E)$ is invertible in $H^*(X;\mathbb{Z})$.

Theorem 10 (Axioms for Chern class)**.** *The Chern classes satisfy the following properties:*

(a) (Vanishing) For all vector bundles E on X, all $i > \operatorname{rank} E$,

$$c_i(E) = 0.$$

(b) (Naturality) Let E be a vector bundle on X, $f : X' \to X$ a flat morphism. Then

$$c(f^*E) = f^*c(E) \in H^*(X', \mathbb{Z}).$$

(c) (Whitney sum) For any exact sequence

$$0 \to E \to F \to G \to 0,$$

of vector bundles on X, then

$$c(F) = c(E) \cdot c(G),$$

i.e,

$$c_k(F) = \sum_{i+j=k} c_i(E) \cdot c_j(G).$$

(d) *(Normalization) For the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$*

$$c(\mathcal{O}_{\mathbb{P}^n}(1)) = 1 + \boldsymbol{h} \in H^*(\mathbb{P}^n; \mathbb{Z}) \simeq \mathbb{Z}[\boldsymbol{h}]/\boldsymbol{h}^{n+1},$$

where $\boldsymbol{h}$ denotes the canonical generator of $H^2(\mathbb{P}^n; \mathbb{Z})$, the Poincaré dual of the homology class $[\mathbb{P}^{n-1}]$.

1.3.1 Splitting Principle

For a given vector bundle E over a manifold X we can find a space Y and a flat morphism $f : Y \to X$ such that :

1. f^*E decomposes over Y into a direct sum of line bundles (topologically!) $f^*E \simeq L_1 \oplus L_2 \oplus \cdots \oplus L_r$ and
2. $f^* : H^*(X; \mathbb{Z}) \to H^*(Y; \mathbb{Z})$ is injective.

So, any formula involving Chern classes need only be checked on sums of line bundles.

1.3.2 Applications of the Splitting Principle

If E splits into line bundles $L_1, ..., L_r$, then the L_i will be called *root bundles* of E and the $\alpha_i = c_1(L_i)$ the *Chern roots* of E. So,

$$c(E) = \prod_{i=1}^{r}(1 + \alpha_i).$$

Using the splitting principle, we can calculate the Chern class of tensor products, exterior products and dual of locally free sheaf.

- **Dual bundles:** If E has Chern roots $\alpha_1, ..., \alpha_r$, then E^* has Chern roots $-\alpha_1, ..., -\alpha_r$ and so

$$c_i(E^*) = (-1)^i c_i(E).$$

- **Tensor product:** Let $\alpha_1, ..., \alpha_r$ and $\beta_1, ..., \beta_s$ be the Chern roots of vector bundles E and F, respectively, then by the bilinearity of the tensor product it follows that the Chern roots of $E \otimes F$ are $\alpha_i + \beta_j$ $(1 \leq i \leq r, 1 \leq j \leq s)$. Hence

$$c(E\otimes F)=\prod_{1\le i\le r,1\le j\le s}(1+\alpha_i+\beta_j).$$

So, when F is a line bundle L with $c_1(L)=\beta$, we have:

$$c_k(E\otimes L)=\sum_{i=0}^{k}\binom{r-i}{k-i}c_i(E)\beta^{k-i}.$$

In particular, the first and top Chern classes of $E\otimes L$ is:

$$c_1(E\otimes L)=r\beta+c_1(E)\quad\text{and}\quad c_r(E\otimes L)=\sum_{i=0}^{r}c_{r-i}(E)\beta^i.$$

Example 4. *Let E is a rank 2 vector bundle. Then for all $k\in\mathbb{Z}$:*

$$c\big(E(k)\big)=1+\big(c_1(E)+2k\big)\cdot\boldsymbol{h}+\big(c_2(E)+k\cdot c_1(E)+k^2\big)\cdot\boldsymbol{h}^2. \tag{1.2}$$

Example 5. *Let E is a rank 3 vector bundle. Then for all $k\in\mathbb{Z}$:*

$$c\big(E(k)\big)=1+\big(c_1(E)+3k\big)\cdot\boldsymbol{h}+\big(c_2(E)+2k\cdot c_1(E)+3k^2\big)\cdot\boldsymbol{h}^2+\big(c_3(E)+k\cdot c_2(E)+k^2c_1(E)+k^3\big)\cdot\boldsymbol{h}^3.$$

- **Exterior Power:** The set of Chern roots of the exterior power $\bigwedge^p E$ coincides with the set of sums

$$\alpha_{i_1}+\alpha_{i_2}+...+\alpha_{i_p},\quad 1\le i_1<i_2<...<i_p\le r.$$

The total Chern class of $\bigwedge^p E$ is the given by the product

$$c(\bigwedge^p E)=\prod_{1\le i_1<i_2<...<i_p\le r}(1+\alpha_{i_1}+\alpha_{i_2}+...+\alpha_{i_p}).$$

Example 6. *Let us to calculate the Chern class of exterior power of a vector bundle E of rank 4 with Chern classes c_1, c_2, c_3, c_4.*

$$\begin{aligned}
c_1(\bigwedge^2 E(k)) &= 6k + 3c_1 \\
c_2(\bigwedge^2 E(k)) &= 15k^2 + 15kc_1 + 3c_1^2 + 2c_2 \\
c_3(\bigwedge^2 E(k)) &= 20k^3 + 30k^2c_1 + 12kc_1^2 + c_1^3 + 8kc_2 + 4c_1c_2 \\
c_4(\bigwedge^2 E(k)) &= 15k^4 + 30k^3c_1 + 18k^2c_1^2 + 3kc_1^3 + 12k^2c_2 + 12kc_1c_2 + 2c_1^2c_2 + c_2^2 + c_1c_3 - 4c_4 \\
c_5(\bigwedge^2 E(k)) &= 6k^5 + 15k^4c_1 + 12k^3c_1^2 + 8k^3c_2 + 3k^2c_1^3 + 12k^2c_1c_2 + 4kc_1^2c_2 + 2kc_2^2 + 2kc_1c_3 \\
&\quad - 8kc_4 + c_1c_2^2 + c_1^2c_3 - 4c_1c_4 \\
c_6(\bigwedge^2 E(k)) &= k^6 + 3k^5c_1 + 3k^4c_1^2 + 2k^4c_2 + k^3c_1^3 + 4k^3c_1c_2 + 2k^2c_1^2c_2 + k^2c_2^2 + k^2c_1c_3 \\
&\quad - 4k^2c_4 + kc_1c_2^2 + kc_1^2c_3 - 4kc_1c_4 + c_1c_2c_3 - c_1^2c_4 - c_3^2.
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
c_1(\bigwedge^3 E(k)) &= 4k + 3c_1 \\
c_2(\bigwedge^3 E(k)) &= 6k^2 + 9kc_1 + 3c_1^2 + c_2 \\
c_3(\bigwedge^3 E(k)) &= 4k^3 + 9k^2c_1 + 6kc_1^2 + c_1^3 + 2kc_2 + 2c_1c_2 - c_3 \\
c_4(\bigwedge^3 E(k)) &= k^4 + 3k^3c_1 + 3k^2c_1^2 + kc_1^3 + k^2c_2 + 2kc_1c_2 + c_1^2c_2 - kc_3 - c_1c_3 + c_4.
\end{aligned} \tag{1.4}$$

We notice that:

$$c_1(\bigwedge^4 E(k)) = k + c_1(E).$$

- **Symmetric Power:** In a similar way, it follows that the set of Chern roots of the symmetric power $S_p(E)$ coincides with the set of sums

$$\alpha_{i_1} + \alpha_{i_2} + ... + \alpha_{i_p}, \quad 1 \le i_1 \le i_2 \le ... \le i_p \le r.$$

The total Chern class of $S_p(E)$ is the given by the product

$$c(S_p(E)) = \prod_{1 \le i_1 \le i_2 \le ... \le i_p \le r} (1 + \alpha_{i_1} + \alpha_{i_2} + ... + \alpha_{i_p}).$$

Example 7. *Let us to compute the Chern classes of some symmetric powers of a vector bundle E of rank 2 with Chern classes c_1 and c_2.*

$$\begin{aligned} c_1(S_2(E)(k)) &= 3k + 3c_1 \\ c_2(S_2(E)(k)) &= 3k^2 + 6kc_1 + 2c_1^2 + 4c_2 \\ c_3(S_2(E)(k)) &= k^3 + 3k^2c_1 + 2kc_1^2 + 4kc_2 + 4c_1c_2. \end{aligned} \tag{1.5}$$

Proposition 5. *If E, F are vector bundles of ranks e and f respectively, then*

$$c_1(E \otimes F) = f \cdot c_1(E) + e \cdot c_1(F).$$

Proof. Vide [14], pag. 176. □

If $E = \bigoplus_{i=1}^{e} L_i$ and $F = \bigoplus_{i=1}^{f} M_j$ are direct sums of line bundles, we can write

$$c(E) = \prod_{i=1}^{e}(1 + \alpha_i) \quad \text{and} \quad c(F) = \prod_{i=1}^{f}(1 + \beta_j),$$

with $c_1(L_i) = \alpha_i$ and $c_2(M_j) = \beta_j$. Hence $c_1(E) = \alpha_1 + \cdots + \alpha_e$ and $c_1(F) = \beta_1 + \cdots + \beta_f$. We have:

$$E \otimes F = \bigoplus_{i,j=1,1}^{e,f} L_i \otimes M_j,$$

thus

$$c(E \otimes F) = \prod_{i,j=1,1}^{e,f} (1 + \alpha_i + \beta_j).$$

In particular, if $X = \mathbb{P}^4$, $e = 4$ and $f = 2$, then denoting by $c_i(E) = e_i$ and $c(F) = f_j$, we have:

$$\begin{aligned} c(E) &= \prod_{i=1}^{4}(1 + \alpha_i) \\ &= 1 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\mathbf{h} + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4)\mathbf{h}^2 \\ &+ (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4)\mathbf{h}^3 + (\alpha_1\alpha_2\alpha_3\alpha_4)\mathbf{h}^4 \\ &= 1 + e_1\mathbf{h} + e_2\mathbf{h}^2 + e_3\mathbf{h}^3 + e_4\mathbf{h}^4, \end{aligned}$$

and

$$\begin{aligned} c(F) &= \prod_{j=1}^{2}(1 + \beta_j) \\ &= 1 + (\beta_1 + \beta_2)\mathbf{h} + (\beta_1\beta_2)\mathbf{h}^2 \\ &= 1 + f_1\mathbf{h} + f_2\mathbf{h}^2. \end{aligned}$$

hence

$$\begin{aligned}
c(E \otimes F) = & \prod_{i,j=1,1}^{4,2} (1+\alpha_i+\beta_j) \\
& = (1+\alpha_1+\beta_1)(1+\alpha_2+\beta_1)(1+\alpha_3+\beta_1)(1+\alpha_4+\beta_1) \\
& \cdot (1+\alpha_1+\beta_2)(1+\alpha_2+\beta_2)(1+\alpha_3+\beta_2)(1+\alpha_4+\beta_2) \\
& = \big\{1 + (\alpha_1+\alpha_2+\alpha_3+\alpha_4+4\beta_1)\mathbf{h} \\
& + (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_4+\alpha_3\alpha_4+3\alpha_1\beta_1+3\alpha_2\beta_1+3\alpha_3\beta_1+3\alpha_4\beta_1+6\beta_1^2)\mathbf{h}^2 \\
& + (\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4+2\alpha_1\alpha_2\beta_1+2\alpha_1\alpha_3\beta_1+2\alpha_2\alpha_3\beta_1+2\alpha_1\alpha_4\beta_1 \\
& + 2\alpha_2\alpha_4\beta_1+2\alpha_3\alpha_4\beta_1+3\alpha_1\beta_1^2+3\alpha_2\beta_1^2+3\alpha_3\beta_1^2+3\alpha_4\beta_1^2+4\beta_1^3)\mathbf{h}^3 \\
& + \big(\alpha_1\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_2\alpha_3\beta_1+\alpha_1\alpha_2\alpha_4\beta_1+\alpha_1\alpha_3\alpha_4\beta_1+\alpha_2\alpha_3\alpha_4\beta_1+\alpha_1\alpha_2\beta_1^2 \\
& + \alpha_1\alpha_3\beta_1^2+\alpha_2\alpha_3\beta_1^2+\alpha_1\alpha_4\beta_1^2+\alpha_2\alpha_4\beta_1^2+\alpha_3\alpha_4\beta_1^2+\alpha_1\beta_1^3+\alpha_2\beta_1^3+\alpha_3\beta_1^3+\alpha_4\beta_1^3+\beta_1^4\big)\mathbf{h}^4\big\} \\
& \cdot \big\{1 + (\alpha_1+\alpha_2+\alpha_3+\alpha_4+4\beta_2)\mathbf{h} \\
& + (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_4+\alpha_3\alpha_4+3\alpha_1\beta_2+3\alpha_2\beta_2+3\alpha_3\beta_2+3\alpha_4\beta_2+6\beta_2^2)\mathbf{h}^2 \\
& + (\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4+2\alpha_1\alpha_2\beta_2+2\alpha_1\alpha_3\beta_2+2\alpha_2\alpha_3\beta_2+2\alpha_1\alpha_4\beta_2 \\
& + 2\alpha_2\alpha_4\beta_2+2\alpha_3\alpha_4\beta_2+3\alpha_1\beta_2^2+3\alpha_2\beta_2^2+3\alpha_3\beta_2^2+3\alpha_4\beta_2^2+4\beta_2^3)\mathbf{h}^3 \\
& + \big(\alpha_1\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_2\alpha_3\beta_2+\alpha_1\alpha_2\alpha_4\beta_2+\alpha_1\alpha_3\alpha_4\beta_2+\alpha_2\alpha_3\alpha_4\beta_2+\alpha_1\alpha_2\beta_2^2 \\
& + \alpha_1\alpha_3\beta_2^2+\alpha_2\alpha_3\beta_2^2+\alpha_1\alpha_4\beta_2^2+\alpha_2\alpha_4\beta_2^2+\alpha_3\alpha_4\beta_2^2+\alpha_1\beta_2^3+\alpha_2\beta_2^3+\alpha_3\beta_2^3+\alpha_4\beta_2^3+\beta_2^4\big)\mathbf{h}^4\big\} \\
& = 1 + (2\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+4\beta_1+4\beta_2)\mathbf{h} + \cdots \\
& = 1 + (2e_1+4f_1)\mathbf{h} + (e_1^2+7e_1f_1+6f_1^2+2e_2+4f_2)\mathbf{h}^2 \\
& + (3e_1^2f_1+9e_1f1^2+4f_1^3+2e_1e_2+6e_2f_1+6e_1f_2+12f_1f_2+2e_3)\mathbf{h}^3 \\
& + (3e_1^2f_1^2+5e_1f_1^3+f_1^4+5e_1e_2f_1+7e_2f_1^2+3e_1^2f_2+15e_1f_1f_2+12f_1^2f_2 \\
& + e_2^2+2e_1e_3+5e_3f_1+2e_2f_2+6f_2^2+2e_4)\mathbf{h}^4.
\end{aligned}$$

Thus

$$\begin{aligned}
c_1(E\otimes F) &= 2e_1+4f_1 \\
c_2(E\otimes F) &= e_1^2+7e_1f_1+6f_1^2+2e_2+4f_2 \\
c_3(E\otimes F) &= 3e_1^2f_1+9e_1f1^2+4f_1^3+2e_1e_2+6e_2f_1+6e_1f_2+12f_1f_2+2e_3 \\
c_4(E\otimes F) &= 3e_1^2f_1^2+5e_1f_1^3+f_1^4+5e_1e_2f_1+7e_2f_1^2+3e_1^2f_2+15e_1f_1f_2+12f_1^2f_2 \\
&\quad + e_2^2+2e_1e_3+5e_3f_1+2e_2f_2+6f_2^2+2e_4.
\end{aligned} \tag{1.6}$$

Next, let us to present an exact sequence that relates to the tangent bundle of $\mathbb{P}^n$.

Proposition 6 (Euler Sequence)**.** *On $\mathbb{P}^n$ there exists a natural short exact sequence of holomorphic vector bundles*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T\mathbb{P}^n \to 0,$$

called the Euler sequence.

Let $\Omega^p_{\mathbb{P}^n}$ be the sheaf of germs of holomorphic p-forms on $\mathbb{P}^n$, then $\Omega^1_{\mathbb{P}^n} = (T\mathbb{P}^n)^*$.
After dualizing the Euler exact sequence and by taking the p-th exterior power, we obtain the following exact sequence, see [31, pag. 3]:

$$0 \to \Omega^p_{\mathbb{P}^n}(p) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus\binom{n+1}{p}} \to \Omega^{p-1}_{\mathbb{P}^n}(p) \to 0. \tag{1.7}$$

In particular, the canonical bundle of $\mathbb{P}^n$ is

$$\omega_{\mathbb{P}^n} = \det \Omega^1_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

The classes $c_i(TX)$, TX the tangent bundle of X, will be called *Chern classes* of X, and to simplify notation we write $c_i(X)$ instead of $c_i(TX)$. So, the *total Chern class* of X, $c(X)$, is the class $c(TX)$.

Example 8 (Chern classes of $\mathbb{P}^n$)**.** *By the Euler exact sequence, the additivity formula implies that*

$$\begin{aligned} c(T\mathbb{P}^n) &= (1+\boldsymbol{h})^{n+1} \\ &= 1 + \binom{n+1}{1}\boldsymbol{h} + \binom{n+1}{2}\boldsymbol{h}^2 + \cdots + \binom{n+1}{n}\boldsymbol{h}^n, \end{aligned} \tag{1.8}$$

where $\boldsymbol{h} = [H]$ is the class of a hyperplane H. Hence:

$$c_i(T\mathbb{P}^n) = \binom{n+1}{i}.$$

In particular, $c_1(T\mathbb{P}^n) = (n+1)\cdot\boldsymbol{h}$. Furthermore, since $\Omega^1_{\mathbb{P}^n} = (T\mathbb{P}^n)^*$ then

$$c_i(\Omega^1_{\mathbb{P}^n}) = (-1)^i\binom{n+1}{i}.$$

In particular, $c_1(\Omega^1_{\mathbb{P}^n}) = -(n+1)$.

For any vector bundle E on $\mathbb{P}^n$, the Chern classes $c_i(E) \in H^{2i}(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$ will be regarded as integers.
The Chern character. Let E be a vector bundles of rank r on a variety X of dimension n and let $c(E) = \prod_{i=1}^r (1+\alpha_i\mathbf{h})$. Then we define the *exponential Chern character*

$$\mathrm{ch}(E) = \sum_{i=1}^r e^{\alpha_i}.$$

Using these definitions and denoting by $c_i = c_i(E)$, we can show that:

$$\mathrm{ch}(E) = \mathrm{rank}(E)+c_1+\frac{1}{2}(c_1^2-2c_2)+\frac{1}{6}(c_1^3-3c_1c_2+3c_3)+\frac{1}{24}(c_1^4-4c_1^2c_2+4c_1c_3+2c_2^2-4c_4)+\cdots.$$

The Todd class:
We define the *Todd class* of E to be

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1-e^{-\alpha_i}}.$$

Then,

$$\mathrm{td}(E) = 1+\frac{1}{2}c_1+\frac{1}{12}(c_1^2+c_2)+\frac{1}{24}c_1c_2-\frac{1}{720}(c_1^4-4c_1^2c_2-3c_2^2-c_1c_3+c_4)+\cdots.$$

Observation: Let X be a smooth projective variety and $\mathcal{F}$ a coherent sheaf of X, then we can resolve $\mathcal{F}$ by locally free sheaves; that is, we can find an exact sequence

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

in that all the sheaves $\mathcal{E}_i$ are locally free. We can use this to extend the definitions of Chern classes and Character class to all coherent sheaves. We define the Chern class of $\mathcal{F}$ by:

$$c(\mathcal{F}) = \prod_{i=0}^{n} c(\mathcal{E}_i)^{(-1)^i}.$$

1.4 STABLE REFLEXIVE SHEAVES

Vector bundles are divided into two quite distinct types, stable and unstable.

Definition 5. *Let E a torsion-free sheaf on $\mathbb{P}^n$ and let $L = \mathcal{O}_{\mathbb{P}^n}(1)$ be very ample line bundle. The slope of E with respect to L, denoted $\mu(E)$, is given by*

$$\mu(E) := \frac{c_1(E)\cdot L^{n-1}}{\mathrm{rank}\, E}.$$

Definition 6 (Mumford-Takemoto)**.** *A reflexive coherent sheaf E on $\mathbb{P}^n$ is stable (resp. semistable) if for every coherent subsheaf F of E, with $0 < \mathrm{rank}\, F < \mathrm{rank}\, E$,*

$$\mu(F) < \mu(E),$$

(resp. $\leq$).

Definition 7. *The normalization E_η of a torsion-free sheaf E of rank 2 on $\mathbb{P}^n$ is defined by:*

$$E_\eta = \begin{cases} E\big(-\frac{c_1(E)}{2}\big), & \textit{if} \quad c_1(E) \quad \textit{even} \\ E\big(-\frac{(c_1(E)+1)}{2}\big), & \textit{if} \quad c_1(E) \quad \textit{odd.} \end{cases}$$

Let us to give a stability criterion for reflexive sheaves of rank 2 over $\mathbb{P}^n$.

Lemma 1. *Let E be a rank 2 reflexive sheaf on $\mathbb{P}^n$. Then E is stable if and only if E_η has no sections:*

$$H^0(\mathbb{P}^n, E_\eta) = 0.$$

Proof. Vide [31], pag. 84. □

Now, let us give the relationship between stability and simplicity.

Definition 8. *A holomorphic vector bundle E is called simple if $h^0(\mathbb{P}^n, E^* \otimes E) = 1$.*

Since $E^* \otimes E = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(E, E) = \mathcal{E}nd(E)$, a bundle is simple if and only if its only endomorphisms are homotheties. So:

Theorem 11. *Stable bundles are simple.*

Proof. Vide [31], pag. 87. □

For holomorphic vector bundles of rank 2 over $\mathbb{P}^n$ we also have the converse of the theorem:

Theorem 12. *Every simple rank two vector bundle over $\mathbb{P}^n$ is stable.*

Proof. Vide [31], pag. 87. □

Definition 9. *A holomorphic vector bundle E on $\mathbb{P}^n$ is decomposable if it is isomorphic to a direct sum $F \oplus G$, where F and G are two proper sub-bundles $F, G \subset E$. Otherwise, E is called indecomposable.*

If E is decomposable then it has non-trivial endomorphisms, given by different homotheties on both factors. Thus simple bundles are always indecomposable.
We say that a holomorphic vector bundle on $\mathbb{P}^n$ of rank r *splits* when it can be represented as a direct sum of r holomorphic line bundles

$$E = \mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(a_r).$$

Therefore, a rank 2 stable vector bundle on $\mathbb{P}^n$ is non split.

1.5 SHEAF COHOMOLOGY

Cohomology was invented, partly, to solve various problems, for example:

- If X is a variety, $\dim H^0(X, \mathcal{O}_X)$ is the number of connected components of X.
- If X is a curve on $\mathbb{P}^n$, $\dim H^0(\mathbb{P}^n, \mathscr{I}_X(q))$ is the number of (independent) degree q hypersurfaces containing X (we make an analysis of this shortly).

In this work we do not develop the general theory on cohomology and Cech cohomology, for more details see [20, Chapter III]. We recall some important properties that will be useful for the purposes of it.
For a sheaf $\mathcal{F}$ we associate its i-th cohomology group, denoted by $H^i(X, \mathcal{F})$, defined for $i \geq 0$ and such that $H^0(X, \mathcal{F})$ is the space of global sections of $\mathcal{F}$.
Their most important property is that, given an exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

there is a long exact sequence:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(X, \mathcal{E}) & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{G}) & \xrightarrow{\delta^0} \\
\longrightarrow & H^1(X, \mathcal{E}) & \longrightarrow & H^1(X, \mathcal{F}) & \longrightarrow & H^1(X, \mathcal{G}) & \xrightarrow{\delta^1} \\
\longrightarrow & H^2(X, \mathcal{E}) & \longrightarrow & H^2(X, \mathcal{F}) & \longrightarrow & H^2(X, \mathcal{G}) & \dashrightarrow \\
 & & & \vdots & & & \\
\dashrightarrow & H^n(X, \mathcal{E}) & \longrightarrow & H^n(X, \mathcal{F}) & \longrightarrow & H^n(X, \mathcal{G}). &
\end{array}$$

In addition, it follows from the definition of Čech cohomology [20] that it commutes with direct sums, i.e.,

$$H^p(X, \bigoplus_{i \in I} \mathcal{F}_i) = \bigoplus_{i \in I} H^p(X, \mathcal{F}_i).$$

The usefulness of cohomology, and particularly the long exact sequence, depends largely on our ability to prove that certain cohomology groups vanish, obtaining group isomorphisms, that will allow us to study geometrical properties of schemes. Next, let us to enunciate some vanishing results.

Theorem 13 (Vanishing Theorem of Grothendieck)**.** *Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.*

Proof. Vide [20], pag. 208. □

On the other hand, duality is an indispensable tool both computationally and conceptually. Next, let us to enunciate the following result about duality for vector bundles. See [31, pag. 4].

Proposition 7 (Serre's duality)**.** *If X is an n-dimensional variety with canonical line bundle ω_X, then we have for any holomorphic vector bundle E over X:*

$$H^q(X,E)^* \cong H^{n-q}(X, E^* \otimes \omega_X).$$

If $X = \mathbb{P}^n$, the cohomology ring $H^*(\mathbb{P}^n, \mathbb{Z})$ is isomorphic to the ring $\mathbb{Z}[\mathbf{h}]/\mathbf{h}^{n+1}$, where $\mathbf{h}$ is the Poincaré dual of a hyperplane, and $h^i(\mathbb{P}^n, E) := \dim_{\mathbb{C}} H^i(\mathbb{P}^n, E)$ for a vector bundle E on $\mathbb{P}^n$.

Serre's duality implies that:

$$h^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(k)) = h^{n-q}(\mathbb{P}^n, \Omega^{n-p}_{\mathbb{P}^n}(-k)).$$

The values of $h^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(k))$ are given by the *Bott's formula*. For more details see [31, pag. 4]:

$$h^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(k)) = \begin{cases} \binom{k+n-p}{k}\binom{k-1}{p} & \text{for } q=0, \quad 0 \le p \le n, k > p \\ 1 & \text{for } k=0, \quad 0 \le p = q \le n \\ \binom{-k+p}{-k}\binom{-k-1}{n-p} & \text{for } q=n, \quad 0 \le p \le n, k < p-n \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

On the other hand, as for submanifolds, one has for any closed subvariety $Y \subset \mathbb{P}^n$ a short exact sequence

$$0 \to \mathscr{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0.$$

Twist by $\mathcal{O}_{\mathbb{P}^n}(q)$, where $q \in \mathbb{Z}$, we get the exact sequence:

$$0 \to \mathscr{I}_Y(q) \to \mathcal{O}_{\mathbb{P}^n}(q) \to \mathcal{O}_Y(q) \to 0.$$

Hence, applying cohomology we have:

$$0 \to H^0(\mathbb{P}^n, \mathscr{I}_Y(q)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q)) \to H^0(X, \mathcal{O}_Y(q)),$$

is exact, as $\mathcal{O}_Y(q)$ has support Y. Now since

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q)) = \begin{cases} (0) & \text{if } \ q < 0 \\ \mathbb{C}^{\binom{n+q}{q}} & \text{i.e., all forms of degree } q, \text{ if } q \ge 0, \end{cases}$$

then we deduce that:

$$H^0(\mathbb{P}^n, \mathscr{I}_Y(q)) = \{\text{all polynomials of degree } q \text{ vanishing on Y}\} \cup \{0\},$$

that is, all hypersurfaces $V \subseteq \mathbb{P}^n$ such that $Y \subseteq V$ (and 0).
Therefore, to give $\xi \in H^0(\mathbb{P}^n, \mathscr{I}_Y(q))$ is to give a hypersurface of $\mathbb{P}^n$ containing Y. So,

$$H^0(\mathbb{P}^n, \mathscr{I}_Y(q)) = (0) \text{ if and only if no hypersurface of degree } q \text{ contains } Y.$$

1.5.1 Hirzebruch-Riemann–Roch

Definition 10. *Let $X \subseteq \mathbb{P}^n$ be a projective variety and $\mathcal{F}$ be a coherent sheaf on X. Then*

$$\chi(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i h^i(X, \mathcal{F}), \tag{1.10}$$

is the Euler characteristic of $\mathcal{F}$.

In addition, given an exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

then

$$\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{G}).$$

Example 9. *Let $Y \subset \mathbb{P}^n$ be a irreducible projective curve. If $d = h^0(\mathcal{O}_Y)$ then*

$$\chi(\mathcal{O}_Y(t)) = dt + \chi(\mathcal{O}_Y)$$

i.e, is a polynomial of degree 1 in t.

Theorem 14 (Hirzebruch-Riemann–Roch)**.** *If X is a smooth projective variety of dimension n and $\mathcal{F}$ a coherent sheaf on X, then:*

$$\chi(\mathcal{F}) = \int_X (\text{ch}(\mathcal{F}) \cdot \text{td}(X))_n. \tag{1.11}$$

Proof. Vide [20], pag. 432. □

Consequently,

Theorem 15. *Let $\mathcal{F}$ be a coherent sheaf of rank r on $\mathbb{P}^4$ with Chern class c_1, c_2, c_3, c_4. Then:*

$$\chi(\mathcal{F}) = r + \binom{c_1+4}{4} + \frac{1}{12}c_2^2 - \frac{35}{12}c_2 - \frac{1}{6}c_1^2c_2 - \frac{5}{4}c_1c_2 + \frac{1}{6}c_1c_3 + \frac{5}{4}c_3 - \frac{1}{6}c_4 - 1. \tag{1.12}$$

We define the following geometric invariant.

Definition 11. *Let X be a scheme of codimension k on $\mathbb{P}^n$. We call positive integer $p_a(X) = (-1)^k(\chi(\mathcal{O}_X) - 1)$ the arithmetic genus of X. In particular, if X is a curve, we have:*

$$p_a(X) = h^1(X, \mathcal{O}_X).$$

Definition 12. *Let $\mathcal{F}$ be a coherent sheaf and*

$$P_{\mathcal{F}}(t) := \chi(\mathcal{F}(t)), \tag{1.13}$$

for all $t \in \mathbb{Z}$. We say that $P_{\mathcal{F}}(t)$ is Hilbert's polynomial of $\mathcal{F}$.

Proposition 8. *Let $X \subseteq \mathbb{P}^n$ be a projective variety and $\mathcal{F}$ be a coherent sheaf on X. Then $P_{\mathcal{F}}$ is a polynomial function in t of degree $\dim \mathcal{F}$.*

For more details, see [26, pag. 10].
In particular, $P_{\mathcal{F}}$ can be uniquely written in the form

$$P_{\mathcal{F}}(t) = \sum_{i=0}^{\dim \mathcal{F}} \alpha_i(\mathcal{F}) \cdot \frac{t^i}{i!}$$

with rational coefficients $\alpha_i(\mathcal{F})$, for $i = 0, ..., \dim \mathcal{F}$. Furthermore, if $\mathcal{F} \neq 0$ the leading coefficient $\alpha_{\dim \mathcal{F}}(\mathcal{F})$, called the *multiplicity*, is always positive. Note that $\alpha_{\dim X}(\mathcal{O}_X)$ is the degree of X with respect to $\mathcal{O}(1)$.

Definition 13. *If $\mathcal{F}$ is a coherent sheaf of dimension $d = \dim X$, then*

$$\operatorname{rank}(\mathcal{F}) = \frac{\alpha_d(\mathcal{F})}{\alpha_d(\mathcal{O}_X)}, \tag{1.14}$$

is called rank of $\mathcal{F}$.

Example 10. *Let Y be a curve. Note that $\operatorname{rank}(\mathscr{I}_Y) = 1$. From the exact sequence*

$$0 \to \mathscr{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0,$$

hence $P_{\mathscr{I}_Y}(t) = P_{\mathcal{O}_{\mathbb{P}^n}}(t) - P_{\mathcal{O}_Y}(t)$. So:

$$\begin{aligned} P_{\mathscr{I}_Y}(t) &= \binom{t+n}{n} - (\deg(Y)t + \chi(\mathcal{O}_Y)) \\ &= \frac{t^n + \cdots + \left(n! \cdot \sum_{k=1}^n \frac{1}{k}\right)t + n!}{n!} - \big(\deg(Y)t + \chi(\mathcal{O}_Y)\big) \\ &= 1\frac{t^n}{n!} + \cdots + \left(\sum_{k=1}^n \frac{1}{k} - \deg(Y)\right)t + 1 - \chi(\mathcal{O}_Y). \end{aligned}$$

Then, by Proposition 8, we have $\dim \mathscr{I}_Y = n$. So, by definition 13:

$$\operatorname{rank}(\mathscr{I}_Y) = \frac{\alpha_n(\mathscr{I}_Y)}{\alpha_n(\mathcal{O}_{\mathbb{P}^n})} = \frac{1}{1} = 1. \tag{1.15}$$

Theorem 16 (Grothendieck-Riemann-Roch)**.** *Let $f : X \to X'$ be a smooth projective morphism of non-singular projective varieties and T_f the relative tangent bundle of f. Then for any $\alpha \in K(X)$ the following relation holds in $A(X') \otimes \mathbb{Q}$:*

$$\mathrm{ch}(f_!\alpha) = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}(T_f)),$$

where $K(X)$ is the Grothendieck group.

Proof. Vide [20], pag. 436. □

The *degree* $\deg(\alpha)$ of a k-cycle $\alpha \in A^k(X)$ is defined as the integer

$$\deg(\alpha) = \int_X c_1(\mathcal{O}_X(1))^k \cap \alpha.$$

As a consequence of the Grothendieck-Riemann-Roch Theorem, we have:

Proposition 9. *Let X be a smooth quasi-projective algebraic variety and Y a closed subvariety of X of codimension k . Let $A^k(Y)$ denotes the group of cycles of codimension k on Y modulo rational equivalence. Then:*

1. $c_j(\mathcal{O}_Y) = 0, \quad$ *for* $\quad 0 < j < k.$
2. $c_k(\mathcal{O}_Y) = (-1)^{k-1}(k-1)! \cdot [Y],$

where $[Y] \in A^k(Y)$.

Proof. Vide [28], pag. 157. □

Under the same hypotheses and using the exact sequence:

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

we can relate the k-th Chern class of ideal sheaf $\mathscr{I}_Y$ and its fundamental class.

Corollary 3. *Let X be a projective variety of dimension n, Y be a non-singular variety of codimension k and $i : Y \to X$, a closed imbedding. Then for any $[Y] \in A^k(Y)$ holds:*

$$c_k(\mathscr{I}_Y) = (-1)^k(k-1)! \cdot [Y].$$

1.6 SHEAVES GENERATED BY GLOBAL SECTIONS

Definition 14. *A coherent analytic sheaf $\mathcal{F}$ is said to be generated by global sections if the canonical homomorphism of sheaves*

$$\varphi : H^0(\mathbb{P}^n, \mathcal{F}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}, \quad \varphi_x(s \otimes h) = hs_x,$$

is surjective.

Example 11. $\mathcal{O}_{\mathbb{P}^4}(1)$, $T\mathbb{P}^4(-1)$, $\Omega^1_{\mathbb{P}^4}(2)$.

In effect:

- Consider the Euler exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5} \to T\mathbb{P}^4 \to 0,$$

  twisting by $\mathcal{O}_{\mathbb{P}^4}(-1)$ we have:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1) \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} \to T\mathbb{P}^4(-1) \to 0,$$

  with long exact sequence of cohomology:

$$0 \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-1)) \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4})^{\oplus 5} \to H^0(\mathbb{P}^4, T\mathbb{P}^4(-1)) \to H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-1)) \to \cdots.$$

  Since $H^i(\mathcal{O}_{\mathbb{P}^4}(-1)) \simeq 0$ for all i then $H^0(\mathbb{P}^4, T\mathbb{P}^4(-1)) \simeq H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4})^{\oplus 5}$ hence:

$$\varphi : H^0(\mathbb{P}^4, T\mathbb{P}^4(-1)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n} \to T\mathbb{P}^4(-1),$$

  is a surjective map.

- Since the dual and tensor product of two sheaves generated by global sections is generated by global sections, then:

$$(T\mathbb{P}^4(-1))^* \otimes \mathcal{O}_{\mathbb{P}^4}(1) = \Omega^1_{\mathbb{P}^4}(1) \otimes \mathcal{O}_{\mathbb{P}^4}(1) = \Omega^1_{\mathbb{P}^4}(2),$$

  is generated by global sections too.

1.7 DEGENERACY LOCI OF MAPS

In this section we recall some basic results on an important type of determinantal variety, the degeneracy loci of morphisms. For more details see [32].

1.7.1 Degeneracy loci of maps of vector bundles

Let E, F be vector bundle on the variety X such that $\operatorname{rank} E = e$ and $\operatorname{rank} F = f$. Let $\varphi : E \to F$ be a morphism.

Definition 15.

$$D_k(\varphi) := \{x \in X \ ; \ \operatorname{rank}(\varphi_x) \leq k\}$$

is called the k-th degeneracy locus of φ.

$D_k(\varphi)$ has a natural subscheme structure of X defined by the ideal generated by the $(k+1)\times(k+1)$ minors of a matrix representation of φ.

Definition 16. *If* $\operatorname{codim}_X D_k(\varphi) = (e-k)(f-k)$ *we say that* $D_k(\varphi)$ *has the expected codimension.*

If $k = \min\{e, f\}-1$ then $D_k(\varphi) := \operatorname{Sing}(\varphi)$ is maximal degeneracy loci and $\operatorname{codim}_X \operatorname{Sing}(\varphi) = f - e + 1$ is the expected codimension.

Theorem 17 (Bertini type)**.** *Let* E*,* F *be vector bundles on a variety* X *such that* $\operatorname{rank} E = m$*,* $\operatorname{rank} F = n$*. Let* $E^* \otimes F$ *be generated by the global sections. If* $\varphi : E \to F$ *is a generic morphism, then one of the following holds:*

- $D_k(\varphi)$ *is empty.*
- $D_k(\varphi)$ *has the expected codimension* $(m-k)(n-k)$ *and* $\operatorname{Sing}(D_k(\varphi)) \subset D_{k-1}(\varphi)$*.*

In particular if $\dim X < (m-k+1)(n-k+1)$ *then* $D_k(\varphi)$ *is empty or smooth when* φ *is generic.*

Proof. Vide [32], pag. 16. □

Theorem 18. *Let* X *be an irreducible complex projective variety of dimension* n *and let* $\varphi : E \to F$ *be a homomorphism of vector bundles on* X *of ranks* e *and* f*. Assume that the vector bundle* $E^* \otimes F = \operatorname{Hom}(E, F)$ *is ample. Then:*

- $D_k(\varphi)$ *is non-empty if* $n \geq (e-k)(f-k)$*.*
- $D_k(\varphi)$ *is connected when* $n > (e-k)(f-k)$*.*

Proof. Vide [17], pag. 273. □

1.7.2 The Eagon-Northcott resolution

Let $\mathcal{E}$ and $\mathcal{G}$ be locally free sheaves on X of rank e and g respectively and $\varphi : \mathcal{E} \to \mathcal{G}$ a generically surjective morphism. By taking maximum exterior power on both sides, we obtain a map: $\bigwedge^g \varphi : \bigwedge^g \mathcal{E} \to \det \mathcal{G}$. It corresponds to a global section

$$\omega_\varphi \in H^0(X, \bigwedge^g (\mathcal{E}^*) \otimes \det \mathcal{G}).$$

Definition 17. *The degeneracy scheme* $\operatorname{Sing}(\varphi)$ *of the map* $\varphi : \mathcal{E} \to \mathcal{G}$ *is the zero scheme of the associated global section* $\omega_\varphi \in H^0(X, \bigwedge^g(\mathcal{E}^*) \otimes \det \mathcal{G})$*.*

Note also that ω_φ can be considered as a map $\bigwedge^g \mathcal{E} \otimes \det(\mathcal{G})^* \to \mathcal{O}_X$; the image of such map is the ideal sheaf of $\mathrm{Sing}(\varphi)$.
Suppose that $Y = \mathrm{Sing}(\varphi) \subset X$ has pure expected dimension, i.e., Y has pure codimension equal to $e-g+1$. Then the ideal sheaf of Y admits a special resolution, called the *Eagon-Northcott resolution*:

$$\begin{aligned} 0 \to \bigwedge^{e} \mathcal{E} \otimes S_{e-g}(\mathcal{G}^*) \otimes \det(\mathcal{G}^*) \to \bigwedge^{e-1} \mathcal{E} \otimes S_{e-g-1}(\mathcal{G}^*) \otimes \det(\mathcal{G}^*) \to \cdots \\ \cdots \to \bigwedge^{g+1} \mathcal{E} \otimes \mathcal{G}^* \otimes \det(\mathcal{G}^*) \to \bigwedge^{g} \mathcal{E} \otimes \det(\mathcal{G}^*) \to \mathscr{I}_Y \to 0. \end{aligned} \tag{1.16}$$

For more details, see [13].

1.8 ACM AND AB SCHEMES

In this section we define the arithmetically Cohen-Macaulay and arithmetically Buchsbaum schemes. For more details on Buchsbaum and Cohen-Macaulay rings, see [2] and [37].

Definition 18. *A closed subscheme $X \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay (aCM for short) if its homogeneous coordinate ring $S(X) = k[x_0, ..., x_n]/I(X)$ (where $I(X)$ is the satured ideal of X) is a Cohen-Macaulay ring. This is equivalent to saying $H^i_*(\mathcal{O}_X) = 0$ for $1 \le i \le \dim X - 1$ and $H^1_*(\mathscr{I}_X) = 0$.*

For any coherent sheaf $\mathcal{F}$ we are using the notation $H^i_*(\mathcal{F})$ as the sum $\bigoplus_{k\in\mathbb{Z}} H^i(\mathcal{F}(k))$ and as usual $\mathscr{I}_X$ will denote as the sheaf of ideals associated to the variety X.

Definition 19. *A closed subscheme X on $\mathbb{P}^n$ is arithmetically Buchsbaum (aB for short) if its homogeneous coordinate ring is a Buchsbaum ring. Every aCM scheme is aB, but the converse is not true.*

In this work we use the following cohomological characterization of arithmetically Buchsbaum schemes. For more details, see: [37].

Proposition 10 (Stückrad, Vogel)**.** *If $X \subset \mathbb{P}^n$ is closed subscheme such that:*

1. *The multiplication map $H^p(\mathscr{I}_X(i)) \xrightarrow{x} H^p(\mathscr{I}_X(i+1))$ is zero for every section $x \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, $i \in \mathbb{Z}$ and $1 \le p \le \dim X$;*
2. *$h^p(\mathscr{I}_X(i)), h^q(\mathscr{I}_X(j)) \neq 0$ for $1 \le p < q \le \dim X$, implies $(p+i)-(q+j) \neq 1$; then X is arithmetically Buchsbaum.*

2 THE HORROCKS-MUMFORD BUNDLE

2.1 HOLOMORPHIC VECTOR BUNDLES ON $\mathbb{P}^n$

It is know that given cohomology class $(a, b) \in H^2(\mathbb{P}^4, \mathbb{Z}) \oplus H^4(\mathbb{P}^4, \mathbb{Z})$, there exits a rank 2 *topological* complex E with Chern class $c(E) = 1 + a\mathbf{h} + b\mathbf{h}^2$, if and only if the Schwarzenberger's condition

$$S_4^2 : b \cdot (b + 1 - 3a - 2a^2) \equiv 0 \mod (12)$$

is satisfied.
The problem lies in the existence of a holomorphic structure in a given topological vector bundle E.
In order to present the Horrocks-Mumford bundle, we recall some known results about the existence of vector bundles on $\mathbb{P}^n$. For more details, see: [24, pag. 139].

2.1.1 The projective line $\mathbb{P}^1$

On the projective line all holomorphic vector bundles split as a sum of line bundles.

Theorem 19 (Grothendieck's)**.** *Every holomorphic r-bundle E on $\mathbb{P}^1$ is of the form:*

$$E = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r),$$

with uniquely determined numbers $a_1, ..., a_r \in \mathbb{Z}$ with $a_1 \geq a_2 \geq \cdots \geq a_r$.

2.1.2 The projective plane $\mathbb{P}^2$

Topological $\mathbb{C}^2$-bundles over $\mathbb{P}^2$ were classified by Wu:

Theorem 20. *There is a bijection between the isomorphism classes of topological $\mathbb{C}^2$-bundles on $\mathbb{P}^2$ and $\mathbb{Z}^2$ given by associating to each vector bundle E its Chern classes $(c_1(E), c_2(E))$.*

Do these vector bundles admit a holomorphic structure?. A positive answer was given by Schwarzenberger.

Theorem 21. *Every topological $\mathbb{C}^2$-bundle on $\mathbb{P}^2$, and hence every complex topological vector bundle on $\mathbb{P}^2$ admits a holomorphic structure.*

As a consequence of this result, there are many bundles of rank 2 on $\mathbb{P}^2$.

2.1.3 Projective space $\mathbb{P}^3$

Here the situation is similar to $\mathbb{P}^2$. The topological $\mathbb{C}^2$-bundles on $\mathbb{P}^3$ were classified by Atiyah and Rees who also proved the following.

Theorem 22. *Every topological $\mathbb{C}^2$-bundle on $\mathbb{P}^3$ admits an holomorphic structure.*

This was generalized by Vogelaar to higher rank.

Theorem 23. *Every topological $\mathbb{C}^3$-bundle on $\mathbb{P}^3$, and hence every complex topological bundle on $\mathbb{P}^3$, carries at least one holomorphic structure.*

2.1.4 Higher dimensional projective spaces $\mathbb{P}^n$, $n \geq 5$

If the characteristic of the base field is different from 2, then no indecomposable rank 2 bundles on $\mathbb{P}^n$, $n \geq 5$ are known. (Horrocks constructed one indecomposable rank 2 bundle on $\mathbb{P}^5$ in characteristic 2).
There are two conjectures concerning the non-existence of rank 2 bundles on $\mathbb{P}^n$, $n \geq 4$:

Conjecture 1 (Grauert-Schneider)**.** *Every unstable rank 2 bundle on $\mathbb{P}^n$, $n \geq 4$, is the sum of two line bundles.*

Conjecture 2 (Hartshorne)**.** *Every rank 2 vector bundle on $\mathbb{P}^n$, $n \geq 6$, split as a sum of line bundles.*

The latter conjecture was originally formulated in terms of complete intersections.

2.1.5 Projective space $\mathbb{P}^4$

In 1972, G. Horrocks and D. Mumford proved in [23] the existence of a indecomposable vector bundle of rank 2 on $\mathbb{P}^4$, namely the **Horrocks-Mumford bundle** E, which comes from an abelian surface of degree 10. This is essentially the only known indecomposable 2-bundle on $\mathbb{P}^4$. The Chern classes of this bundle are:

$$c_1(E) = 5, \qquad c_2(E) = 10. \tag{2.1}$$

Hence, $E^* = E(-5)$ and $\det E = \mathcal{O}_{\mathbb{P}^4}(5)$.

2.2 HORROCKS-MUMFORD BUNDLE

There are several methods to build the Horrocks-Mumford bundle. In this thesis we go to adopt a construction using monads. For more details about monads, see: [31], [34].

Definition 20. *A monad is a complex of holomorphic vector bundles*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

which is exact except possibly at B.

The sheaf $F = \operatorname{Ker}\beta / \operatorname{Im}\alpha$ is a holomorphic vector bundle, called the cohomology of the monad, with $\operatorname{rank} F = \operatorname{rank} B - \operatorname{rank} A - \operatorname{rank} C$ and Chern class

$$c(F) = c(B) \cdot c(A)^{-1} \cdot c(C)^{-1}.$$

The Horrocks-Mumford bundle is given by the monad:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} \xrightarrow{\alpha} \Omega^2_{\mathbb{P}^4}(2)^{\oplus 2} \xrightarrow{\beta} \mathcal{O}^{\oplus 5}_{\mathbb{P}^4} \to 0, \tag{2.2}$$

such that the cohomology of monad 2.2 is:

$$E_\eta = \operatorname{Ker}\beta / \operatorname{Im}\alpha,$$

where $F := E_\eta = E(-3)$ is the normalized of Horrocks-Mumford bundle, such that

$$c(F) = 1 - \mathbf{h} + 4\mathbf{h}^2.$$

With associated display:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} & \longrightarrow & \mathcal{K} & \longrightarrow & E_\eta & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} & \longrightarrow & \Omega^2_{\mathbb{P}^4}(2)^{\oplus 2} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}^{\oplus 5}_{\mathbb{P}^4} & = & \mathcal{O}^{\oplus 5}_{\mathbb{P}^4} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

(2.3)

where $\mathcal{K} := \operatorname{Ker}\beta$ and $\mathcal{Q} := \operatorname{Coker}\alpha$.

2.2.1 Properties of Horrocks-Mumford bundle

- The Horrocks-Mumford bundle E is a stable sheaf. In fact, since $E_\eta = E(-3)$ then it follows of the cohomology table (1) that $h^0(\mathbb{P}^4, E(-3)) = 0$, i.e., E_η has no sections, therefore by Lemma 1 we have E is a stable vector bundle. Similarly $E(k)$ are stable, for all $k \in \mathbb{Z}$.

- This is essentially the only known indecomposable rank 2 vector bundle on $\mathbb{P}^4$, [11]. Certainly this is not entirely true, we can twist E by a linear bundle or may pull E back under a finite morphism $\pi : \mathbb{P}^4 \to \mathbb{P}^4$.

- In addition, since E is stable then E is simple, so holds that $h^0(\mathbb{P}^n, E^* \otimes E) = 1$.

- In particular, the Horrocks-Mumford bundle E is non-split.

The following result will be useful in this work to describe our study problem.

Proposition 11 (H. Sumihiro - 1998). *$E(1)$ is generated by global sections and it is 1-ample in the sense of A. Sommese. Hence $E(a)$ is very ample if and only if $a \geq 2$.*

Proof. Vide [36], pag. 427. □

In particular $E(a)$, for $a \geq 1$, is a vector bundle generated by global sections.

2.2.2 Horrocks-Mumford bundle cohomologies

The dimensions of the cohomology groups of the Horrocks-Mumford bundle are described in Table (1).

Table 1 – Table of $\dim H^i(E(k))$

k	H^0	H^1	H^2	H^3	H^4
$k \geq 1$	$\frac{((k+5)^2-1)((k+5)^2-24)}{12}$	0	0	0	0
0	4	2	0	0	0
-1	0	10	0	0	0
-2	0	10	0	0	0
-3	0	5	0	0	0
-4	0	0	0	0	0
-5	0	0	2	0	0
-6	0	0	0	0	0
-7	0	0	0	5	0
-8	0	0	0	10	0
-9	0	0	0	10	0
-10	0	0	0	2	4
$k \leq -11$	0	0	0	0	$\frac{((k+5)^2-1)((k+5)^2-24)}{12}$

Hence, by Hirzebruch–Riemann–Roch theorem:

$$\chi\big(E(k)\big) = \frac{1}{12}\big((k+5)^2 - 1\big)\big((k+5)^2 - 24\big), \quad k \in \mathbb{Z}. \tag{2.4}$$

For more details, see [23, pag. 74].

2.2.3 The Moduli space of the Horrocks-Mumford bundle

Let $\mathscr{M}_{\mathbb{P}^4}(-1,4)$ be the Moduli space that describes the Horrocks-Mumford bundle. $H^1(\mathbb{P}^4, \mathcal{H}om(F,F))$ is isomorphic to the Zariski tangent space of $\mathscr{M}_{\mathbb{P}^4}(-1,4)$ in the point F defined by $[F]$ and from $H^2(\mathbb{P}^4, \mathcal{H}om(F,F)) = 0$ follows $\mathscr{M}_{\mathbb{P}^4}(-1,4)$ is smooth in $[F]$. The following results are provided in [10, pag. 104] and [11, pag. 218]:

Theorem 24 (W. Decker, 1984)**.** *Let F be the Horrocks-Mumford bundle. Then:*

- $h^1(\mathbb{P}^4, \mathcal{H}om(F,F)) = 24$*,* $h^1(\mathbb{P}^4, \mathcal{H}om(F,F)) = 5$*,* $h^1(\mathbb{P}^4, \mathcal{H}om(F,F)(k)) = 0$ *for* $k \leq -2$ *and*

$$h^2(\mathbb{P}^4, \mathcal{H}om(F,F)) = 2.$$

- $\mathscr{M}_{\mathbb{P}^4}(-1,4)$ *is smooth in* $[F]$ *with dimension* 24*.*

3 HOLOMORPHIC DISTRIBUTIONS AND FOLIATIONS

3.1 HOLOMORPHIC DISTRIBUTIONS AND FOLIATIONS

In this section we will recall some definitions about holomorphic distributions and foliations.

Definition 21. *Let X be a complex manifold of complex dimension $n = k+s$. A saturated codimension k singular holomorphic distribution on X is given by a short exact sequence of analytic coherent sheaves*

$$\mathscr{F} : 0 \to T_{\mathscr{F}} \xrightarrow{\varphi} TX \xrightarrow{\pi} N_{\mathscr{F}} \to 0, \tag{3.1}$$

where $T_{\mathscr{F}}$ is a coherent sheaf of rank $r := \dim(X) - k$, and $N_{\mathscr{F}}$ is a non-trivial torsion free sheaf of rank k on X.

The sheaves $T_{\mathscr{F}}$ and $N_{\mathscr{F}}$ are called the *tangent* and the *normal sheaves* of $\mathscr{F}$, respectively. Note that by Proposition 4 $T_{\mathscr{F}}$ must be reflexive.
By taking maximum exterior power of the map $\varphi : T\mathscr{F} \to TX$, we have:

$$\bigwedge^{n-k} \varphi : \bigwedge^{n-k} T\mathscr{F} \to \bigwedge^{n-k} TX,$$

i.e.,

$$\bigwedge^{n-k} \varphi : \det(T\mathscr{F}) \to \bigwedge^{n-k} TX.$$

This induces a global section $\omega \in H^0(X, \bigwedge^{n-k} TX \otimes \det(T\mathscr{F})^*)$.
Since

$$\bigwedge^{n-k} TX \simeq \Omega_X^k \otimes \det(TX),$$

then

$$\mathbb{P}H^0(X, \bigwedge^{n-k} TX \otimes \det(T\mathscr{F})^*) \simeq \mathbb{P}H^0(X, \Omega_X^k \otimes \det(TX) \otimes \det(T\mathscr{F})^*) = \mathbb{P}H^0(X, \Omega_X^k \otimes L),$$

where $L = \det(TX) \otimes \det(T\mathscr{F})^* = \det(N_{\mathscr{F}})$. Therefore, the space of the holomorphic distributions of codimension k in X may be identified by a class of sections

$$[\omega] \in \mathbb{P}H^0(X, \Omega_X^k(c_1(N_{\mathscr{F}}))).$$

Let us now define the *singular scheme* of $\mathscr{F}$. In order to do this, by taking the maximal exterior power of the dual morphism $\varphi^\vee : \Omega_X^1 \to T_{\mathscr{F}}^*$ we obtain the morphism: $\psi = \bigwedge^{n-k} \varphi^* : \Omega_X^{n-k} \to \det(T_{\mathscr{F}})^*$. The image of such morphism is the ideal sheaf $\mathscr{I}_Z$ with support on a closed subscheme $Z \subset X$ twisted by $\det(T_{\mathscr{F}})^*$, i.e.:

$$\psi : \Omega_X^{n-k} \to \mathrm{Im}(\psi) = \mathscr{I}_Z \otimes \det(T_{\mathscr{F}})^* \subset \mathcal{O}_X \otimes (\det(T_{\mathscr{F}})^*).$$

Definition 22. $\mathrm{Sing}(\mathscr{F}) := Z \subset X$ *the singular scheme of* $\mathscr{F}$*.*

A distribution $\mathscr{F}$ is said to be *locally free* if $T_{\mathscr{F}}$ is a locally free sheaf. Note that if the tangent sheaf $T_{\mathscr{F}}$ is locally free, then Z coincides, as a set, with the singular set of the normal sheaf. In fact [31]:

$$\mathrm{Sing}(N_{\mathscr{F}}) := \bigcup_{p=1}^{\dim(X)-1} \mathrm{Supp}(\mathcal{E}xt^p(N_{\mathscr{F}}, \mathcal{O}_X)),$$

where by $\mathrm{Supp}(-)$ we mean the set-theoretical support of a sheaf. If $T_{\mathscr{F}}$ is locally free, then $\mathcal{E}xt^p(N_{\mathscr{F}}, \mathcal{O}_X) = 0$ for $p \geq 2$, therefore:

$$\begin{aligned}
\mathrm{Sing}(N_{\mathscr{F}}) &= \mathrm{Supp}\left(\mathcal{E}xt^1(N_{\mathscr{F}}, \mathcal{O}_X)\right) \\
&= \{x \in X \mid \varphi(x) \text{ is not injective}\} \\
&= \{x \in X | (N_{\mathscr{F}})_x \text{ is not free } \mathcal{O}_x \text{ - module}\} \\
&= \mathrm{Supp}(\mathcal{O}_Z) \\
&= Z.
\end{aligned}$$

Since $N_{\mathscr{F}}$ is a coherent sheaf on a projective complex manifold, then the singular scheme $\mathrm{Sing}(\mathscr{F})$ of the distribution $\mathscr{F}$ is a closed analytic subvariety of X. Moreover, since $N_{\mathscr{F}}$ is a torsion free sheaf then by Theorem 9 we have $\mathrm{codim}(\mathrm{Sing}(\mathscr{F})) \geq 2$.

Next, let define the notion of integrability. A *foliation* is an *integrable* distribution, i.e, a distribution:

$$\mathscr{F} : 0 \to T_{\mathscr{F}} \xrightarrow{\varphi} TX \to N_{\mathscr{F}} \to 0,$$

whose tangent sheaf is closed under the Lie bracket of vector fields, i.e.,

$$[\varphi(T_{\mathscr{F}}), \varphi(T_{\mathscr{F}})] \subset \varphi(T_{\mathscr{F}}).$$

A codimension k holomorphic foliation on X may be represented by a class of sections $[\omega] \in \mathbb{P}H^0(X, \Omega_X^k \otimes \det(TX) \otimes \det(T_{\mathscr{F}})^*)$, such that the *singular set*

$$\mathrm{Sing}(\omega) = \{p \in X \mid \omega(p) = 0\} \quad \text{has} \quad \mathrm{codim}(\mathrm{Sing}(\omega)) \geq 2.$$

The integrability condition can also be defined as follows. Let $U \subset X$ a open set and $\omega \in \Omega^k_X(U)$. For any $p \in U \setminus \mathrm{Sing}(\omega)$ there exist a neighborhood V of p, $V \subset U$, and 1-forms $\omega_1, ..., \omega_k \in \Omega^1_X(V)$ such that:

$$\omega\mid_V = \omega_1 \wedge \cdots \wedge \omega_k.$$

We say that ω satisfies the integrability condition if and only if, $d\omega_j \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0$, $\forall j = 1, ..., k$.
Alternatively, a foliation $\mathscr{F}$, of codimension k, can be induced by a exact sequence

$$0 \to N^*_{\mathscr{F}} \to \Omega^1_X \to \mathcal{Q}_{\mathscr{F}} \to 0,$$

where $\mathcal{Q}_{\mathscr{F}}$ is a torsion free sheaf of rank $n-k$ and the sheaf $N^*_{\mathscr{F}}$ is called the *conormal sheaf* of $\mathscr{F}$. Moreover, the singular set of $\mathscr{F}$ is $\mathrm{Sing}(\mathcal{Q}_{\mathscr{F}})$. We say that this foliation is induced by the cotangent sheaf.
In particular to the case $X = \mathbb{P}^n$, a singular holomorphic foliation $\mathscr{F}$ on $\mathbb{P}^n$, of codimension $k \geq 1$, is given by a locally decomposable and integrable twisted k-form

$$\omega \in H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n} \otimes L),$$

where $L = \det(N_{\mathscr{F}})$.
Next, we define the main invariant of a distribution, the *degree* of $\mathscr{F}$. The degree of $\mathscr{F}$, denoted by $\deg(\mathscr{F})$, is by definition the degree of the zero locus of $i^*\omega$, where $i : \mathbb{P}^k \hookrightarrow \mathbb{P}^n$ is a linear embedding of a generic k-plane. Since $\Omega^k_{\mathbb{P}^k} = \mathcal{O}_{\mathbb{P}^n}(-k-1)$ so it follows at once that $L = \mathcal{O}_{\mathbb{P}^n}(\deg(\mathscr{F}) + k + 1)$. In particular L is ample. Thus:

Definition 23. *The degree of the foliation $\mathscr{F}$, denoted by* $\deg(\mathscr{F})$, *is defined by*

$$\deg(\mathscr{F}) = c_1(N_{\mathscr{F}}) - \mathrm{codim}(\mathscr{F}) - 1.$$

On the other hand, the vector space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\deg(\mathscr{F}) + k + 1))$ can be canonically identified by the vector space of k-forms on $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $d+1$, whose contraction $i_{\mathcal{R}}\omega = 0$ with the radial vector field $\mathcal{R} = \sum_{i=0}^n x_i \cdot \frac{\partial}{\partial x_i}$.
Since $\mathscr{F}$ is a holomorphic foliation of codimension k of degree $d = \deg(\mathscr{F})$ then $\mathscr{F}$ is induced by a k-form $\omega \in H^0(\Omega^k_{\mathbb{P}^n}(d+k+1))$. Now, it follows from the Euler sequence:

$$0 \to \Omega^k_{\mathbb{P}^n}(k) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus\binom{n+1}{k}} \to \Omega^{k-1}_{\mathbb{P}^n}(k) \to 0,$$

twisting by $\mathcal{O}_{\mathbb{P}^n}(d+1)$:

$$0 \to \Omega^k_{\mathbb{P}^n}(d+k+1) \to \mathcal{O}_{\mathbb{P}^n}(d+1)^{\oplus\binom{n+1}{k}} \to \Omega^{k-1}_{\mathbb{P}^n}(d+k+1) \to 0,$$

hence by long exact sequence of cohomology and using Bott's formula, we have:

$$0 \to H^0(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(d+k+1)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d+1))^{\oplus\binom{n+1}{k}} \to H^0(\mathbb{P}^n, \Omega^{k-1}_{\mathbb{P}^n}(d+k+1)) \to 0,$$

i.e, a section ω of $\Omega^k_{\mathbb{P}^n}(d+k+1)$ can be thought of as a polynomial k-form on $\mathbb{C}^{n+1}$

$$\omega(z) = \sum_{i_1,\dots,i_k} a_{i_1\cdots i_k}(z) dz_1 \wedge \cdots \wedge dz_k,$$

with homogeneous coefficients of degree $\deg(a_{i_1\cdots i_k}(z)) = d+1$. Since $Z_a = \{a_{i_1\cdots i_k}(z) = 0\}$ then Z_a is contained in a hypersurface of degree $d+1$.

3.2 RELATIONS BETWEEN THE SINGULAR SCHEME AND THE TANGENT SHEAF

L. Giraldo and A. Pan-Collantes characterized when the tangent sheaf, of a codimension one holomorphic foliation in $\mathbb{P}^3$, is locally free or split (as a sum of linear bundles) in terms of the geometry of their Singular scheme.

Theorem 25 (L. Giraldo, A. Pan-Collantes - 2009)**.** *The tangent sheaf $T\mathscr{F}$ is locally free if and only if $Z = \mathrm{Sing}(\mathscr{F})$ is a curve.*

Proof. Vide [19], pag. 848. □

Theorem 26 (L. Giraldo, A. Pan-Collantes - 2009)**.** *Suppose $\deg(\mathscr{F}) > 1$. Then $T\mathscr{F}$ splits if and only if Z is an arithmetically Cohen-Macaulay curve.*

Proof. Vide [19], pag. 849. □

Shortly after, M. Corrêa, M. Jardim and R. Vidal generalized these facts for distributions in $\mathbb{P}^n$ whose tangent sheaf splits as a sum of linear bundles.

Theorem 27 (M. Corrêa, M. Jardim, R. Vidal - 2015)**.** *Let $\mathscr{F}$ be a distribution on $\mathbb{P}^n$ of codimension k, such that the tangent sheaf $T\mathscr{F}$ is locally free, and whose singular locus has the expected dimension $n-k-1$. If $T\mathscr{F}$ splits as a sum of line bundles, then $\mathrm{Sing}(\mathscr{F})$ is arithmetically Cohen-Macaulay. Conversaly, if $k = 1$ and $\mathrm{Sing}(\mathscr{F})$ is arithmetically Cohen-Macaulay, then $T\mathscr{F}$ splits as a sum of line bundles.*

Proof. Vide [8], pag. 2. □

Remark: The goal of this thesis is to consider codimension two holomorphic distributions on $\mathbb{P}^4$ whose tangent and conormal sheaf are Horrocks-Mumford, that is, the tangent and conormal sheaf of the distribution is a stable vector bundle of rank 2, non-split. We are going to study the geometry of its singular scheme and its Moduli space.

3.3 A BERTINI TYPE THEOREM FOR REFLEXIVE SHEAVES

In [6], the authors gave a generalization of a Bertini-type Theorem for reflexive sheaves in order to construct new distributions.

Theorem 28 (O. Calvo Andrade, M. Corrêa, M. Jardim - 2018)**.** *Let* $\mathcal{G}$ *be a globally generated reflexive sheaf on a projective variety* X *such that* $\operatorname{rank}\mathcal{G} \leq \dim X - 1 \geq 2$. *If* $\mathcal{T}X \otimes L$ *is globally generated, for some line bundle* L*, then* $\mathcal{G}^* \otimes L^*$ *is the tangent sheaf of a distribution on* X *of codimension* $n - \operatorname{rank}\mathcal{G}$.

Proof. Vide [6], pag. 61. □

From now on we only consider codimension two distributions on $X = \mathbb{P}^4$.

Parte II

Horrocks-Mumford Holomorphic Distributions

4 GEOMETRIC PROPERTIES OF THE SINGULAR SCHEME

In this chapter, we study the geometry of the singular scheme of Horrocks-Mumford holomorphic distributions. We establish relationships between the numerical invariants of the tangent and conormal sheaves and the numerical invariants of the singular scheme. We determine the genus and degree of the singular scheme and we show that the singular scheme of these distributions of high degree is an arithmetically Buschbaum, smooth, reduced and irreducible curve; and hence without isolated singularities. The techniques used are based on the study of the long exact cohomology sequence of (4.18) and (4.30) together with the knowledge of cohomologies of the Horrocks-Mumford bundle.

4.1 VANISHING LEMMAS

The following Vanishing Lemmas are consequences of having considered the Euler's exact sequence twisted by a convenient sheaf and its long exact sequence of Cohomology. Using the Cohomologies table (1) of the Horrocks-Mumford bundle and in some cases using *Macaulay2* (see script in the Appendix 8), we obtain:

Lemma 2. *Let E be the Horrocks-Mumford bundle. Then for $j = 1, 2, 3$ and $k \in \mathbb{Z}$ we have:*

1. $h^0(\Omega^j_{\mathbb{P}^4} \otimes E(k)) = 0$ *for* $k \leq j$.
2. $h^1(\Omega^j_{\mathbb{P}^4} \otimes E(k)) = 0$ *for* $k \leq j - 4$ *or* $k \geq j + 1$.
3. $h^2(\Omega^j_{\mathbb{P}^4} \otimes E(k)) = 0$ *for* $k \leq j - 6$ *or* $k \geq j - 3$.
4. $h^3(\Omega^j_{\mathbb{P}^4} \otimes E(k)) = 0$ *for* $k \leq j - 10$ *or* $k \geq j - 5$.
5. $h^4(\Omega^j_{\mathbb{P}^4} \otimes E(k)) = 0$ *for* $k \geq j - 9$.

Proof. Let us consider the Euler exact sequence:

$$0 \to \Omega^1_{\mathbb{P}^4}(1) \to \mathcal{O}^{\oplus 5}_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(1) \to 0.$$

Then, twisting by $E(k-1)$ we have:

$$0 \to \Omega^1_{\mathbb{P}^4} \otimes E(k) \to E(k-1)^{\oplus 5} \to E(k) \to 0. \tag{4.1}$$

Now, taking long exact sequence of cohomology, we have:

$$0 \to H^0(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^0(\mathbb{P}^4, E(k-1))^{\oplus 5} \to H^0(\mathbb{P}^4, E(k)) \to \cdots$$

and by cohomology table 1 of the Horrocks-Mumford bundle, we have: $h^0(\mathbb{P}^4, E(k-1)) = 0$ for $k \leq 0$ then there is an isomorphism such that $h^0(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0$ for $k \leq 0$, and by *Macaulay2*, see Appendix 8, this sheaf is null for $k = 1$, so:

$$h^0(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq 1.$$

Let us consider the long sequence of cohomology of sequence (4.1), we have:

$$\cdots \to H^0(E(k-1))^{\oplus 5} \to H^0(\mathbb{P}^4, E(k)) \to H^1(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^1(\mathbb{P}^4, E(k-1))^{\oplus 5} \to \cdots.$$

Since $h^0(E(k-1)) = 0 = h^1(E(k-1))$ for $k \leq -3$, then $h^1(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = h^0(E(k))$ for $k \leq -3$. But $h^0(E(k)) = 0$ for $k \leq -1$, then

$$h^1(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -3,$$

and by *Macaulay2* this sheaf is also null for $k \geq 2$, thus:

$$h^1(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -3 \quad \text{or} \quad k \geq 2.$$

Let us consider the long sequence of cohomology of sequence (4.1), we have:

$$\cdots \to H^1(\mathbb{P}^4, E(k)) \to H^2(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^2(\mathbb{P}^4, E(k-1))^{\oplus 5} \to H^2(\mathbb{P}^4, E(k)) \to \cdots.$$

It follows from the cohomology table 1 that: $h^1(\mathbb{P}^4, E(k)) = 0 = h^2(\mathbb{P}^4, E(k))$ since $k \leq -6$ or $k = -4$ or $k \geq 1$, hence there is a isomorphism

$$H^2(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \simeq H^2(\mathbb{P}^4, E(k-1))^{\oplus 5},$$

for $k \leq -6$ or $k = -4$ or $k \geq 1$, i.e., $h^2(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) = 5 \cdot h^2(\mathbb{P}^4, E(k-1))$ for $k \leq -6$ or $k = -4$ or $k \geq 1$. And since $h^2(E(k-1)) = 0$ for $k \leq -5$ or $k \geq -3$, we have:

$$h^2(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -6 \quad \text{or} \quad k \geq 1. \tag{4.2}$$

On the other hand, considering the sequence:

$$\cdots \to H^1(\mathbb{P}^4, E(k-1))^{\oplus 5} \to H^1(\mathbb{P}^4, E(k)) \to H^2(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^2(\mathbb{P}^4, E(k-1))^{\oplus 5} \to \cdots$$

Since $h^1(E(k-1)) = 0 = h^2(E(k-1))$ for $k \leq -5$ or $k = -3$ or $k \geq 2$, then

$$h^2(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = h^1(E(k)) \ \text{ for } \ k \leq -5 \ \text{ or } \ k = -3 \ \text{ or } \ k \geq 2.$$

But $h^1(E(k)) = 0$ for $k \leq -4$ or $h \geq 1$, thus:

$$h^2(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -5 \quad \text{or} \quad k \geq 2. \tag{4.3}$$

From (4.2) and (4.3), we have:

$$h^2(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -5 \quad \text{or} \quad k \geq 1.$$

And using *Macaulay2* this sheaf is also null for $k = -2, -1, 0$, thus:

$$h^2(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -5 \quad \text{or} \quad k \geq -2.$$

Let us consider the long exact sequence of cohomology of sequence (4.1), we have:

$$\cdots \to H^2(\mathbb{P}^4, E(k)) \to H^3(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^3(\mathbb{P}^4, E(k-1))^{\oplus 5} \to H^3(\mathbb{P}^4, E(k)) \to \cdots.$$

It follows from the cohomology table 1 that: $h^2(\mathbb{P}^4, E(k)) = 0 = h^3(\mathbb{P}^4, E(k))$ since $k \leq -11$ or $k = -6$ or $k \geq -4$, then there is a isomorphism

$$H^3(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \simeq H^3(\mathbb{P}^4, E(k-1))^{\oplus 5},$$

for $k \leq -11$ or $k = -6$ or $k \geq -4$, i.e., $h^3(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) = 5 \cdot h^3(\mathbb{P}^4, E(k-1))$ for $k \leq -11$ or $k = -6$ or $k \geq -4$. And since $h^3(E(k-1)) = 0$ for $k \leq -10$ or $k \geq -5$, we have:

$$h^3(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -11 \quad \text{or} \quad k \geq -4. \tag{4.4}$$

On the other hand, considering the sequence:

$$\cdots \to H^2(\mathbb{P}^4, E(k-1))^{\oplus 5} \to H^2(\mathbb{P}^4, E(k)) \to H^3(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^3(\mathbb{P}^4, E(k-1))^{\oplus 5} \to \cdots.$$

Since $h^2(E(k-1)) = 0 = h^3(E(k-1))$ for $k \leq -10$ or $k = -5$ or $k \geq -3$, then $h^3(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = h^2(E(k))$ for $k \leq -10$ or $k = -5$ or $k \geq -3$. But $h^2(E(k)) = 0$ for $k \leq -6$ or $h \geq -4$, thus:

$$h^3(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -10 \quad \text{or} \quad k \geq -3. \tag{4.5}$$

From (4.4) and (4.5) we have:

$$h^3(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -10 \quad \text{or} \quad k \geq -4.$$

And using *Macaulay2* this sheaf is also null for $k = -9$, thus:

$$h^3(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -9 \quad \text{or} \quad k \geq -4.$$

Let us consider the long exact sequence of cohomology of sequence (4.1), we have:

$$\cdots \to H^3(\mathbb{P}^4, E(k)) \to H^4(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^4(\mathbb{P}^4, E(k-1))^{\oplus 5} \to H^4(\mathbb{P}^4, E(k)) \to \cdots .$$

From the cohomology table 1 we have: $h^3(\mathbb{P}^4, E(k)) = 0 = h^4(\mathbb{P}^4, E(k))$ when $k \geq -6$, then there is a isomorphism $H^4(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \simeq H^4(\mathbb{P}^4, E(k-1))^{\oplus 5}$ for $k \geq -6$, i.e., $h^4(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) = 5 \cdot h^4(\mathbb{P}^4, E(k-1))$ for $k \geq -6$. Since $h^4(E(k-1)) = 0$ for $k \geq -8$, we have:

$$h^4(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \geq -6.$$

And using *Macaulay2* this sheaf is also null for $k = -7, -8$, thus:

$$h^4(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \geq -8. \tag{4.6}$$

Let us consider the Euler exact sequence:

$$0 \to \Omega^2_{\mathbb{P}^4}(2) \to \mathcal{O}^{\oplus 10}_{\mathbb{P}^4} \to \Omega^1_{\mathbb{P}^4}(2) \to 0,$$

and twisting by $E(k-2)$, we have:

$$0 \to \Omega^2_{\mathbb{P}^4} \otimes E(k) \to E(k-2)^{\oplus 10} \to \Omega^1_{\mathbb{P}^4} \otimes E(k) \to 0. \tag{4.7}$$

Now, let us consider the long exact sequence of cohomology of sequence (4.7), we have:

$$0 \to H^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4} \otimes E(k)) \to H^0(\mathbb{P}^4, E(k-2))^{\oplus 10} \to H^0(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4} \otimes E(k)) \to \cdots ,$$

since $h^0(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0$ for $k \leq 1$ then there is an isomorphism such that

$$h^0(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 10 \cdot h^0(E(k-2)),$$

for $k \leq 1$. But by cohomology table 1, we have $h^0(E(k-2)) = 0$ for $k \leq 1$. Thus

$$h^0(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq 1.$$

And using *Macaulay2* this sheaf is also null for $k = 2$, thus:

$$h^0(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq 2.$$

Taking the long exact sequence of cohomology of sequence (4.7), we have:

$$\cdots \to H^0(E(k-2))^{\oplus 10} \to H^0(\Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^1(\Omega^2_{\mathbb{P}^4} \otimes E(k)) \to H^1(E(k-2))^{\oplus 10} \to \cdots .$$

Since $h^0(k-2) = 0 = h^1(k-2)$ for $k \leq -2$, then $h^1(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = h^0(\Omega^1_{\mathbb{P}^4} \otimes E(k))$ for $k \leq -2$. But $h^0(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0$ for $k \leq 1$, thus:

$$h^1(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -2.$$

And using *Macaulay2* this sheaf is also null for $k \geq 3$, thus:

$$h^1(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -2 \quad \text{or} \quad k \geq 3.$$

Considering the long exact sequence of cohomology of sequence (4.7), we have:

$$\cdots \to H^1(E(k-2))^{\oplus 10} \to H^1(\Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^2(\Omega^2_{\mathbb{P}^4} \otimes E(k)) \to H^2(E(k-2))^{\oplus 10} \to \cdots .$$

Since $h^1(k-2) = 0 = h^2(k-2)$ for $k \leq -4$ or $k = -2$ or $k \geq 3$, then

$$h^2(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = h^1(\Omega^1_{\mathbb{P}^4} \otimes E(k)),$$

for $k \leq -4$ or $k = -2$ or $k \geq 3$. But $h^1(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0$ for $k \leq -3$ or $k \geq 2$, then:

$$h^2(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -4 \quad \text{or} \quad k \geq 3.$$

And using *Macaulay2* this sheaf is also null for $k = -1, 0, 1, 2$, thus:

$$h^2(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -4 \quad \text{or} \quad k \geq -1.$$

Considering the long exact sequence of cohomology of sequence (4.7), we have:

$$\cdots \to H^2(E(k-2))^{\oplus 10} \to H^2(\Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^3(\Omega^2_{\mathbb{P}^4} \otimes E(k)) \to H^3(E(k-2))^{\oplus 10} \to \cdots .$$

Since $h^2(E(k-2)) = 0 = h^3(E(k-2))$ for $k \leq -9$ or $k = -4$ or $k \geq -2$, then $h^3(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(k))$ for $k \leq -9$ or $k = -4$ or $k \geq -2$. But $h^2(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0$ for $k \leq -5$ or $k \geq -2$. Then:

$$h^3(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -9 \quad \text{or} \quad k \geq -2.$$

And using *Macaulay2* this sheaf is also null for $k = -8, -3$, thus:

$$h^3(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -8 \quad \text{or} \quad k \geq -3.$$

Considering the long exact sequence of cohomology of sequence (4.7), we have:

$$\cdots \to H^3(E(k-2))^{\oplus 10} \to H^3(\Omega^1_{\mathbb{P}^4} \otimes E(k)) \to H^4(\Omega^2_{\mathbb{P}^4} \otimes E(k)) \to H^4(E(k-2))^{\oplus 10} \to \cdots .$$

Since $h^3(E(k-2)) = 0 = h^4(E(k-2))$ for $k \geq -4$, then

$$h^4(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = h^3(\Omega^1_{\mathbb{P}^4} \otimes E(k)),$$

for $k \geq -4$. But $h^3(\Omega^1_{\mathbb{P}^4} \otimes E(k)) = 0$ for $k \leq -9$ or $k \geq -4$. Thus:

$$h^4(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \geq -4.$$

And using *Macaulay2* this sheaf is also null for $k = -7, -6, -5$, thus:

$$h^4(\Omega^2_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \geq -7.$$

Let us consider the Euler exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1) \to \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} \to \Omega^3_{\mathbb{P}^4}(4) \to 0,$$

and twisting by $E(k-4)$, we have:

$$0 \to E(k-5) \to E(k-4)^{\oplus 5} \to \Omega^3_{\mathbb{P}^4} \otimes E(k) \to 0. \tag{4.8}$$

Now, let us consider the long exact sequence of cohomology of sequence (4.8), obtaining:

$$0 \to H^0(E(k-5)) \to H^0(E(k-4))^{\oplus 5} \to H^0(\Omega^3_{\mathbb{P}^4} \otimes E(k)) \to H^1(E(k-5)) \to \cdots .$$

Since $h^0(E(k-5)) = 0 = h^1(E(k-5))$ for $k \leq 1$, then $h^0(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 5 \cdot h^0(E(k-4))$ for $k \leq 1$. But $h^0(E(k-4)) = 0$ for $k \leq 3$. Then:

$$h^0(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq 1.$$

By *Macaulay2* this sheaf is also null for $k = 2, 3$, thus:

$$h^0(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq 3.$$

Taking the long exact sequence of cohomology of sequence (4.8), we have:

$$\cdots \to H^1(E(k-5)) \to H^1(E(k-4))^{\oplus 5} \to H^1(\Omega^3_{\mathbb{P}^4} \otimes E(k)) \to H^2(E(k-5)) \to \cdots .$$

Since $h^1(E(k-5)) = 0 = h^2(E(k-5))$ for $k \leq -1$ or $k = 1$ or $k \geq 6$, then there is an isomorphism such that $h^1(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 5 \cdot h^1(E(k-4))$ for $k \leq -1$ or $k = 1$ or $k \geq 6$. But $h^1(E(k-4) = 0$ for $k \leq 0$ or $k \geq 5$, thus:

$$h^1(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -1 \quad \text{or} \quad k \geq 6. \tag{4.9}$$

On the other hand, considering the sequence:

$$\cdots \to H^1(E(k-4))^{\oplus 5} \to H^1(\Omega^3_{\mathbb{P}^4} \otimes E(k)) \to H^2(E(k-5)) \to H^2(E(k-4))^{\oplus 5} \to \cdots .$$

Since $h^1(E(k-4)) = 0 = h^2(E(k-4))$ for $k \leq -2$ or $k = 0$ or $k \geq 5$, then

$$h^1(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = h^2(E(k-5)) \ \text{for} \ k \leq -2 \ \text{or} \ k = 0 \ \text{or} \ k \geq 5.$$

But $h^2(E(k-5)) = 0$ for $k \leq -1$ or $k \geq 1$. Thus

$$h^1(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -2 \quad \text{or} \quad k \geq 5. \tag{4.10}$$

Now, from (4.9) and (4.10), we have:

$$h^1(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -1 \quad \text{or} \quad k \geq 5.$$

And using *Macaulay2* this sheaf is also null for $k = 4$, thus:

$$h^1(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -1 \quad \text{or} \quad k \geq 4.$$

Taking the long exact sequence of cohomology of sequence (4.8), obtaining:

$$\cdots \to H^2(E(k-5)) \to H^2(E(k-4))^{\oplus 5} \to H^2(\Omega^3_{\mathbb{P}^4} \otimes E(k)) \to H^3(E(k-5)) \to \cdots .$$

Since $h^2(E(k-5)) = 0 = h^3(E(k-5))$ for $k \leq -6$ or $k = -1$ or $k \geq 1$, then there is an isomorphism such that $h^2(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 5 \cdot h^2(E(k-4))$ for $k \leq -6$ or $k = -1$ or $k \geq 1$. But $h^2(E(k-4) = 0$ for $k \leq -2$ or $k \geq 0$, thus:

$$h^2(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -6 \quad \text{or} \quad k \geq 1. \tag{4.11}$$

On the other hand, considering the sequence:

$$\cdots \to H^2(E(k-4))^{\oplus 5} \to H^2(\Omega^3_{\mathbb{P}^4} \otimes E(k)) \to H^3(E(k-5)) \to H^3(E(k-4))^{\oplus 5} \to \cdots .$$

Since $h^2(E(k-4)) = 0 = h^3(E(k-4))$ for $k \leq -7$ or $k = -2$ or $k \geq 0$, then

$$h^2(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = h^3(E(k-5)) \quad \text{for} \quad k \leq -7 \quad \text{or} \quad k = -2 \quad \text{or} \quad k \geq 0.$$

But $h^3(E(k-5)) = 0$ for $k \leq -6$ or $k \geq -1$. Thus

$$h^2(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -7 \quad \text{or} \quad k \geq 0. \tag{4.12}$$

Now, from (4.11) and (4.12), we have:

$$h^2(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -6 \quad \text{or} \quad k \geq 0.$$

And using *Macaulay2* this sheaf is also null for $k = -5, -4, -3$, thus:

$$h^2(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -3 \quad \text{or} \quad k \geq 0.$$

Considering the long exact sequence of cohomology of sequence (4.8), obtaining:

$$\cdots \to H^3(E(k-4))^{\oplus 5} \to H^3(\Omega^3_{\mathbb{P}^4} \otimes E(k)) \to H^4(E(k-5)) \to H^4(E(k-4))^{\oplus 5} \to \cdots .$$

Since $h^3(E(k-4)) = 0 = h^4(E(k-4))$ for $k \geq -2$, then $h^3(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = h^4(E(k-5))$ for $k \geq -2$. But $h^4(E(k-5)) = 0$ for $k \geq -4$. Thus

$$h^3(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \geq -2.$$

And using *Macaulay2* this sheaf is also null for $k \leq -7$, thus:

$$h^3(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \geq -2 \quad \text{or} \quad k \leq -7.$$

Taking the long exact sequence of cohomology of sequence (4.8), obtaining:

$$\cdots \to H^4(E(k-5)) \to H^4(E(k-4))^{\oplus 5} \to H^4(\Omega^3_{\mathbb{P}^4} \otimes E(k)) \to 0.$$

Since $h^4(E(k-5)) = 0$ for $k \geq -4$, then $h^4(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 5 \cdot h^4(E(k-4))$ for $k \geq -4$. But $h^4(E(k-5)) = 0$ for $k \geq -5$. Then:

$$h^4(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \geq -4.$$

And by *Macaulay2* this sheaf is also null for $k = -6, -5$, thus:

$$h^4(\Omega^3_{\mathbb{P}^4} \otimes E(k)) = 0 \quad \text{for} \quad k \geq -6.$$

□

Lemma 3. *Let E be the Horrocks-Mumford bundle and $\mathcal{K}$ be the sheaf in the display 2.3. Then for $k \in \mathbb{Z}$ we have:*

1. $h^0(\mathcal{K} \otimes E(k)) = 0$ *for* $k \leq -1$.
2. $h^1(\mathcal{K} \otimes E(k)) = 0$ *for* $k \leq -4$.
3. $h^2(\mathcal{K} \otimes E(k)) = 0$ *for* $k \leq -6$ *or* $k \geq 1$.
4. $h^3(\mathcal{K} \otimes E(k)) = 0$ *for* $k \leq -10$ *or* $k \geq -4$.
5. $h^4(\mathcal{K} \otimes E(k)) = 0$ *for* $k \geq -6$.

Proof. Taking the exact sequence of the display 2.3

$$0 \to \mathcal{K} \to \Omega^2_{\mathbb{P}^4}(2)^{\oplus 2} \to \mathcal{O}^{\oplus 5}_{\mathbb{P}^4} \to 0,$$

and twisting by $E(k)$, we have:

$$0 \to \mathcal{K} \otimes E(k) \to \left(\Omega^2_{\mathbb{P}^4} \otimes E(k+2)\right)^{\oplus 2} \to E(k)^{\oplus 5} \to 0. \tag{4.13}$$

Now, let us consider the long exact sequence of cohomology of sequence (4.13):

$$0 \to H^0(\mathcal{K} \otimes E(k)) \to H^0(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))^{\oplus 2} \to H^0(E(k))^{\oplus 5} \to \cdots.$$

Since $h^0(E(k+2) \otimes \Omega^2_{\mathbb{P}^4}) = 0$ for $k \leq 0$ then

$$h^0(\mathcal{K} \otimes E(k)) = 0 \quad \text{for} \quad k \leq 0.$$

From the long exact sequence of cohomology of sequence (4.13):

$$\cdots \to H^0(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))^{\oplus 2} \to H^0(E(k))^{\oplus 5} \to H^1(\mathcal{K} \otimes E(k)) \to H^1(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))^{\oplus 2} \to \cdots.$$

Since $h^0(\Omega^2_{\mathbb{P}^4} \otimes E(k+2)) = 0 = h^1(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))$ for $k \leq -4$ then there is an isomorphism such that $h^1(\mathcal{K} \otimes E(k)) = 5 \cdot h^0(E(k))$ for $k \leq -4$. But $h^0(E(k)) = 0$ for $k \leq -1$. Thus

$$h^1(\mathcal{K} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -4.$$

From the long exact sequence of cohomology of sequence (4.13):

$$\cdots \to H^1(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))^{\oplus 2} \to H^1(E(k))^{\oplus 5} \to H^2(\mathcal{K} \otimes E(k)) \to H^2(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))^{\oplus 2} \to \cdots.$$

Since $h^1(\Omega^2_{\mathbb{P}^4} \otimes E(k+2)) = 0 = h^2(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))$ for $k \leq -6$ or $k \geq 1$ then there is an isomorphism such that $h^2(\mathcal{K} \otimes E(k)) = 5 \cdot h^1(E(k))$ for $k \leq -6$ or $k \geq 1$. But $h^1(E(k)) = 0$ for $k \leq -4$ or $k \geq 1$. Thus

$$h^2(\mathcal{K} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -6 \quad \text{or} \quad k \geq 1.$$

Regarding the long exact sequence of cohomology of sequence (4.13):

$$\cdots \to H^2(E(k))^{\oplus 5} \to H^3(\mathcal{K} \otimes E(k)) \to H^3(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))^{\oplus 2} \to H^3(E(k))^{\oplus 5} \to \cdots.$$

Since $h^2(E(k)) = 0 = h^3(E(k))$ for $k \leq -11$ or $k = -6$ or $k \geq -4$ then there is an isomorphism such that $h^3(\mathcal{K} \otimes E(k)) = 2 \cdot h^3(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))$ for $k \leq -11$ or $k = -6$ or $k \geq -4$. But $h^3(\Omega^2_{\mathbb{P}^4} \otimes E(k+2)) = 0$ for $k \leq -10$ or $k \geq -5$. Thus

$$h^3(\mathcal{K} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -10 \quad \text{or} \quad k \geq -4. \tag{4.14}$$

We note also that if we consider the long exact sequence:

$$\cdots \to H^2(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))^{\oplus 2} \to H^2(E(k))^{\oplus 5} \to H^3(\mathcal{K} \otimes E(k)) \to H^3(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))^{\oplus 2} \to \cdots.$$

Since $h^2(\Omega^2_{\mathbb{P}^4} \otimes E(k+2)) = 0 = h^3(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))$ for $k \leq -10$ or $k \geq -3$ then there is an isomorphism such that $h^3(\mathcal{K} \otimes E(k)) = 5 \cdot h^2(E(k))$ for $k \leq -10$ or $k \geq -3$. But $h^2(E(k)) = 0$ for $k \leq -6$ or $k \geq -4$. Thus

$$h^3(\mathcal{K} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -10 \quad \textit{or} \quad k \geq -3. \tag{4.15}$$

From (4.14) and (4.15), we have:

$$h^3(\mathcal{K} \otimes E(k)) = 0 \quad \text{for} \quad k \leq -10 \quad \text{or} \quad k \geq -4.$$

Concerning the long exact sequence of cohomology of sequence (4.13):

$$\cdots \to H^3(E(k))^{\oplus 5} \to H^4(\mathcal{K} \otimes E(k)) \to H^4(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))^{\oplus 2} \to H^4(E(k))^{\oplus 5} \to \cdots .$$

Since $h^3(E(k)) = 0 = h^4(E(k))$ for $k \geq -6$ then there is an isomorphism such that $h^4(\mathcal{K} \otimes E(k)) = 2 \cdot h^4(\Omega^2_{\mathbb{P}^4} \otimes E(k+2))$ for $k \geq -6$. But $h^4(\Omega^2_{\mathbb{P}^4} \otimes E(k+2)) = 0$ for $k \geq -9$. Thus

$$h^4(\mathcal{K} \otimes E(k)) = 0 \quad \text{for} \quad k \geq -6.$$

□

Lemma 4. *Let E be the Horrocks-Mumford bundle and $k \in \mathbb{Z}$, then:*

1. $h^0(E \otimes E(k)) = 0$ *for* $k \leq -6$.
2. $h^1(E \otimes E(k)) = 0$ *for* $k \leq -7$ *or* $k \geq 1$.
3. $h^2(E \otimes E(k)) = 0$ *for* $k \leq -11$ *or* $k \geq -4$.
4. $h^3(E \otimes E(k)) = 0$ *for* $k \leq -16$ *or* $k \geq -8$.
5. $h^4(E \otimes E(k)) = 0$ *for* $k \geq -9$.

Proof. Let the short exact sequence of the display 2.3:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} \to \mathcal{K} \to E(-3) \to 0.$$

Twisting by $E(k+3)$, we have:

$$0 \to E(k+2)^{\oplus 5} \to \mathcal{K} \otimes E(k+3) \to E \otimes E(k) \to 0. \tag{4.16}$$

Now, let us consider the long exact sequence of cohomology of sequence (4.16):

$$0 \to H^0(E(k+2))^{\oplus 5} \to H^0(\mathcal{K} \otimes E(k+3)) \to H^0(E \otimes E(k)) \to H^1(E(k+2))^{\oplus 5} \to \cdots .$$

Since $h^0(E(k+2)) = 0 = h^1(E(k+2))$ for $k \leq -6$, then $h^0(E \otimes E(k)) = h^0(\mathcal{K} \otimes E(k+3))$ for $k \leq -6$. But $h^0(\mathcal{K} \otimes E(k+3)) = 0$ for $k \leq -4$. Thus:

$$h^0(E \otimes E(k)) = 0 \quad \text{for} \quad k \leq -6.$$

From the long exact sequence of cohomology of sequence (4.16):

$$\cdots \to H^1(E(k+2))^{\oplus 5} \to H^1(\mathcal{K} \otimes E(k+3)) \to H^1(E \otimes E(k)) \to H^2(E(k+2))^{\oplus 5} \to \cdots.$$

Since $h^1(E(k+2)) = 0 = h^2(E(k+2))$ for $k \leq -8$ or $k = -6$ or $k \geq -1$, then $h^1(E \otimes E(k)) = h^1(\mathcal{K} \otimes E(k+3))$ for $k \leq -8$ or $k = -6$ or $k \geq -1$. But $h^1(\mathcal{K} \otimes E(k+3)) = 0$ for $k \leq -7$, then:

$$h^1(E \otimes E(k)) = 0 \quad \text{for} \quad k \leq -8.$$

And using *Macaulay2* this sheaf is also null for $k = -7$ or $k \geq 1$, thus:

$$h^1(E \otimes E(k)) = 0 \quad \text{for} \quad k \leq -7 \quad \text{or} \quad k \geq 1.$$

Now, let us consider the long exact sequence of cohomology of sequence (4.16):

$$\cdots \to H^2(E(k+2))^{\oplus 5} \to H^2(\mathcal{K} \otimes E(k+3)) \to H^2(E \otimes E(k)) \to H^3(E(k+2))^{\oplus 5} \to \cdots.$$

Since $h^2(E(k+2)) = 0 = h^3(E(k+2))$ for $k \leq -13$ or $k = -8$ or $k \geq -6$, then

$$h^2(E \otimes E(k)) = h^2(\mathcal{K} \otimes E(k+3)),$$

for $k \leq -13$ or $k = -8$ or $k \geq -6$. But $h^2(\mathcal{K} \otimes E(k+3)) = 0$ for $k \leq -9$ or $k \geq -2$, then:

$$h^2(E \otimes E(k)) = 0 \quad \text{for} \quad k \leq -13 \quad \text{or} \quad k \geq -2.$$

And using *Macaulay2* this sheaf is also null for $k = -4, -3$, thus:

$$h^2(E \otimes E(k)) = 0 \quad \text{for} \quad k \leq -11 \quad \text{or} \quad k \geq -4.$$

By Serre's duality we have $h^3(E \otimes E(k)) = h^1(E \otimes E(-k-15)) = 0$ for $k \leq -16$ or $k \geq -8$. Thus:

$$h^3(E \otimes E(k)) = 0 \quad \text{for} \quad k \leq -16 \quad \text{or} \quad k \geq -8.$$

Finally, using Serre's duality, we have to: $h^4(E \otimes E(k)) = h^0(E \otimes E(-k-15)) = 0$ for $k \geq -9$. Thus:

$$h^4(E \otimes E(k)) = 0 \quad \text{for} \quad k \geq -9.$$

□

Table 2 – Table of $\dim H^i(E \otimes E(k))$

k	H^0	H^1	H^2	H^3	H^4
$k \geq 1$	$\frac{k^4+30k^3+290k^2+975k+624}{6}$	0	0	0	0
0	136	32	0	0	0
-1	70	85	0	0	0
-2	35	100	0	0	0
-3	15	85	0	0	0
-4	5	55	0	0	0
-5	1	24	2	0	0
-6	0	5	10	0	0
-7	0	0	20	0	0
-8	0	0	20	0	0
-9	0	0	10	5	0
-10	0	0	2	24	1
-11	0	0	0	55	5
-12	0	0	0	85	15
-13	0	0	0	100	35
-14	0	0	0	85	70
-15	0	0	0	32	136
$k \leq -16$	0	0	0	0	$\frac{k^4+30k^3+290k^2+975k+624}{6}$

In addition, using Serre's duality and Theorem 24, we build the table (2).
Hence, by Hirzebruch–Riemann–Roch Theorem, we have:

$$\chi\big(E \otimes E(k)\big) = \frac{1}{6}(k^4 + 30k^3 + 290k^2 + 975k + 624), \quad k \in \mathbb{Z}. \tag{4.17}$$

4.2 HORROCKS-MUMFORD HOLOMORPHIC DISTRIBUTIONS AS SUBSHEAVES OF TANGENT BUNDLE

In this section, we study codimension 2 holomorphic distributions induced by a Bertini-type Theorem 28. We determine the genus, degree and the Rao module of the singular scheme and we show that the singular scheme of these distributions of high degree is a smooth, reduced and irreducible curve; and hence without isolated singularities. The techniques used are based on the study of the long exact cohomology sequence of (4.18) together with the knowledge of cohomologies of the Horrocks-Mumford bundle.
By Proposition [36], $E(a)$, for $a \geq 1$, is a vector bundle generated by global sections.
Since $T\mathbb{P}^4(-1)$ is globally generated then by Bertini type Theorem 28 such that $\mathcal{G} = E(a)$, $a \geq 1$, and $L = \mathcal{O}_{\mathbb{P}^4}(-1)$, we have

$$T_{\mathscr{F}a} = E(-a-4), \quad (a \geq 1)$$

is the tangent sheaf of a codimension 2 holomorphic distribution on $\mathbb{P}^4$:

$$\mathscr{F}_a : 0 \to E(-a-4) \xrightarrow{\varphi} T\mathbb{P}^4 \to N_{\mathscr{F}_a} \to 0, \quad (a \geq 1). \tag{4.18}$$

of degree $d_a := \deg(\mathscr{F}_a) = 2a+5$.
In addition, since $E(a) \otimes T\mathbb{P}^4(-1) = \mathcal{H}om(E(-a-4)), T\mathbb{P}^4)$ is globally generated then by Theorem 17 the generic morphism $\varphi : E(-a-4) \to T\mathbb{P}^4$ satisfies $\mathrm{Sing}(\mathscr{F}_a) := \mathrm{Sing}(\varphi)$ is a closed analytical subvariety on $\mathbb{P}^4$ of expected codimension $\mathrm{codim}(\mathrm{Sing}(\mathscr{F}_a)) = 3$.
We begin by calculating the Chern classes of the ideal sheaf and calculate the degree of these distributions.

Proposition 12. *Let $\mathscr{F}_a$ be the holomorphic distributions family (4.18) above. Then, for $a \geq 1$ we have:*

1. *The Chern classes of the normal sheaf are:*
   - $c_1(N_{\mathscr{F}_a}) = 2a+8$.
   - $c_2(N_{\mathscr{F}_a}) = 3a^2+19a+28$.
   - $c_3(N_{\mathscr{F}_a}) = 4a^3+33a^2+77a+46$.
   - $c_4(N_{\mathscr{F}_a}) = 5a^4+50a^3+150a^2+125a-25$.
2. *The Chern classes of ideals sheaf of the singular scheme $Z_a = \mathrm{Sing}(\mathscr{F}_a)$ are:*
   - $c_1(\mathscr{I}_{Z_a}) = 0$.
   - $c_2(\mathscr{I}_{Z_a}) = 0$.
   - $c_3(\mathscr{I}_{Z_a}) = -8a^3-66a^2-154a-92$.
   - $c_4(\mathscr{I}_{Z_a}) = -54a^4-594a^3-2154a^2-2874a-954$.
3. *The degree of the distribution $\mathscr{F}_a$ is $\deg(\mathscr{F}_a) = 2a+5$.*

Proof. Let $q_i = c_i(N_{\mathscr{F}_a})$, it follows from $c(T\mathbb{P}^4) = c(E(-a-4)) \cdot c(N_{\mathscr{F}_a})$ that:

$$1+5\mathbf{h}+10\mathbf{h}^2+10\mathbf{h}^3+5\mathbf{h}^4 = (1+(-2a-3)\mathbf{h}+(a^2+3a+6)\mathbf{h}^2)\cdot(1+q_1\mathbf{h}+q_2\mathbf{h}^2+q_3\mathbf{h}^3+q_4\mathbf{h}^4).$$

Then:

$$\begin{aligned} c(N_{\mathscr{F}_a}) &= 1+(2a+8)\mathbf{h}+(3a^2+19a+28)\mathbf{h}^2+(4a^3+33a^2+77a+46)\mathbf{h}^3 \\ &\quad +(5a^4+50a^3+150a^2+125a-25)\mathbf{h}^4. \end{aligned}$$

Let $\varphi : E(-a-4) \to T\mathbb{P}^4$ be the map that induces the distribution (4.18). Then, the Eagon-Northcott resolution associated with the dual map $\varphi^\vee : \Omega^1_{\mathbb{P}^4} \to E(a-1)$ is:

$$0 \to S_2(E(-4-a))(-2a-8) \to \Omega^3_{\mathbb{P}^4} \otimes E(-7-3a) \to \Omega^2_{\mathbb{P}^4}(-3-2a) \to \mathscr{I}_{Z_a} \to 0. \tag{4.19}$$

Using this resolution of ideal sheaf $\mathscr{I}_{Z_a}$ by locally free sheaves, let us calculate its Chern class:

$$c(\mathscr{I}_{Z_a}) = c(S_2(E(-4-a))(-2a-8)) \cdot c(\Omega^3_{\mathbb{P}^4} \otimes E(-7-3a))^{-1} \cdot c(\Omega^2_{\mathbb{P}^4}(-3-2a)).$$

Now, by the Chern class formulas of the tensor product, exterior product and symmetric product in Examples 4, 6, and 7 , respect., and by equations (1.6), we have:

$$\begin{aligned} c(S_2(E(-4-a))(-2a-8)) = 1 &+ (-12a-33)\mathbf{h} + (48a^2+264a+378)\mathbf{h}^2 \\ &+ (-64a^3-528a^2-1521a-1496)\mathbf{h}^3, \end{aligned}$$

$$\begin{aligned} c(\Omega^2_{\mathbb{P}^4}(-3-2a)) = 1 &+ (-12a-33)\mathbf{h} + (60a^2+330a+455)\mathbf{h}^2 \\ &+ (-160a^3-1320a^2-3640a-3355)\mathbf{h}^3 \\ &+ (240a^4+2640a^3+10920a^2+20130a+13950)\mathbf{h}^4, \end{aligned}$$

$$\begin{aligned} c(\Omega^3_{\mathbb{P}^4} \otimes E(-7-3a)) = 1 &+ (-24a-66)\mathbf{h} + (252a^2+1386a+1922)\mathbf{h}^2 \\ &+ (-160a^3-1320a^2-3640a-3355)\mathbf{h}^3 \\ &+ (240a^4+2640a^3+10920a^2+20130a+13950)\mathbf{h}^4, \end{aligned}$$

then:

$$c(\mathscr{I}_{Z_a}) = 1+(-8a^3-66a^2-154a-92)\mathbf{h}^3+(-54a^4-594a^3-2154a^2-2874a-954)\mathbf{h}^4, \quad (4.20)$$

for $a \geq 1$.

Finally, since $c_1(N_{\mathscr{F}_a}) = 2a+8$ and $\mathscr{F}$ is a codimension 2 holomorphic distribution, then by Definition 23 we have:

$$\deg(\mathscr{F}_a) = (2a+8) - 2 - 1 = 2a+5.$$

□

Now, let us determine the numerical invariants of the singular scheme through the sheaf of ideals.

Proposition 13 (Numerical invariants of the singular locus)**.** *Let* $Z_a = \mathrm{Sing}(\mathscr{F}_a)$ *the singular scheme, then for* $a \geq 1$ *we have:*

1. $\deg(Z_a) = 4a^3 + 33a^2 + 77a + 46$.
2. $p_a(Z_a) = 9a^4 + 89a^3 + \frac{553}{2}a^2 + \frac{573}{2}a + 45$.

Proof. By (4.20) we have $c_3(\mathscr{I}_{Z_a}) = -8a^3-66a^2-154a-92$, since Z_a is a pure codimension 3 projective curve then by Corollary 3 we have:

$$-8a^3 - 66a^2 - 154a - 92 = -2 \cdot \deg(Z_a),$$

hence $\deg(Z_a) = 4a^3 + 33a^2 + 77a + 46$.
On the other hand, since $Z_a \subset \mathbb{P}^4$ is a projective curve of pure dimension 1 and $\operatorname{rank}(\mathscr{I}_{Z_a}) = 1$, then by Proposition 9 and Theorem 14 we have:

$$\begin{aligned}
\chi(\mathscr{I}_{Z_a}) &= \int_{\mathbb{P}^4} \left(\operatorname{ch}(\mathscr{I}_{Z_a}) \cdot \operatorname{td}(\mathbb{P}^4)\right)_4 \\
&= 1 + \frac{5}{4}c_3(\mathscr{I}_{Z_a}) - \frac{1}{6}c_4(\mathscr{I}_{Z_a}) \\
&= 1 + \frac{5}{4}(-8a^3 - 66a^2 - 154a - 92) - \frac{1}{6}(-54a^4 - 594a^3 - 2154a^2 - 2874a - 954) \\
&= 9a^4 + 89a^3 + \frac{553}{2}a^2 + \frac{573}{2}a + 45.
\end{aligned}$$

Now, consider the exact sequence:

$$0 \to \mathscr{I}_{Z_a} \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{Z_a} \to 0. \tag{4.21}$$

Hence:

$$\chi(\mathcal{O}_{Z_a}) = \chi(\mathcal{O}_{\mathbb{P}^4}) - \chi(\mathscr{I}_{Z_a}).$$

Then,

$$\begin{aligned}
p_a(Z_a) &= 1 - \chi(\mathcal{O}_{Z_a}) \\
&= 1 - \chi(\mathcal{O}_{\mathbb{P}^4}) + \chi(\mathscr{I}_{Z_a}) \\
&= \chi(\mathscr{I}_{Z_a}) \\
&= 9a^4 + 89a^3 + \frac{553}{2}a^2 + \frac{573}{2}a + 45.
\end{aligned}$$

□

Proposition 14. *The singular scheme* $Z_a = \operatorname{Sing}(\mathscr{F}_a)$ *is reduced and irreducible.*

Proof. We claim that Z_a is connected. In fact, since Z_a has pure expected codimension 3 then the ideal sheaf admits the Eagon-Northcott resolution (1.16). Consider this complex associated to the morphism $\varphi^\vee : \Omega^1_{\mathbb{P}^4} \to E(a-1)$:

$$0 \to S_2(E(-a-4))(-2a-8) \to \Omega^3_{\mathbb{P}^4} \otimes E(-3a-7) \to \Omega^2_{\mathbb{P}^4}(-2a-3) \xrightarrow{\alpha} \mathscr{I}_{Z_a} \to 0. \tag{4.22}$$

breaking into short exact sequences and passing to cohomology

$$0 \to S_2(E(-a-4))(-2a-8) \to \Omega^3_{\mathbb{P}^4} \otimes E(-3a-7) \to K \to 0 \tag{4.23}$$

and

$$0 \to K \to \Omega^2_{\mathbb{P}^4}(-2a-3) \to \mathscr{I}_{Z_a} \to 0 \tag{4.24}$$

where $K = \operatorname{Ker}\alpha$.
From exact sequence (4.23) passing to cohomology

$$\cdots \to H^i(S_2(E(-a-4))(-2a-8)) \to H^i(\Omega^3_{\mathbb{P}^4} \otimes E(-3a-7)) \to H^i(K) \to \cdots.$$

By Lemma 2 and since $a \geq 1$ we have

$$H^2(K) \simeq H^3(S_2(E(-a-4))(-2a-8)).$$

On the other hand, if V is a vector space over $\mathbb{C}$ then $V \otimes_{\mathbb{C}} V = S_2(V) \oplus \bigwedge^2(V)$, i.e, every 2-tensor may be written uniquely as a sum of a symmetric and an alternating tensor, therefore the second tensor power of a vector bundle decomposes as the direct sum of the symmetric and alternating squares as vector bundles. So, twisting by $\mathcal{O}_{\mathbb{P}^4}(-2a-8)$ we have:

$$E \otimes E(-4a-16) \simeq S_2(E(-a-4))(-2a-8) \oplus \mathcal{O}_{\mathbb{P}^4}(-4a-11),$$

thus

$$h^i(E \otimes E(-4a-16)) = h^i(S_2(E(-a-4))(-2a-8)) + h^i(\mathcal{O}_{\mathbb{P}^4}(-4a-11)), \quad i = 0, ..., 4.$$

Hence, since $a \geq 1$ then using Lemma 4 and by Bott's formula [31, pag. 4] we have

$$h^3(S_2(E(-a-4))(-2a-8)) = h^2(K) = 0.$$

From exact sequence (4.24) passing to cohomology

$$\cdots \to H^i(\Omega^2_{\mathbb{P}^4}(-2a-3)) \to H^i(\mathscr{I}_{Z_a}) \to H^{i+1}(K) \to H^{i+1}(\Omega^2_{\mathbb{P}^4}(-2a-3)) \to \cdots$$

Since $a \geq 1$ using Bott's formula,

$$H^2(K) \simeq H^1(\mathscr{I}_{Z_a}),$$

hence $h^1(\mathscr{I}_{Z_a}) = 0$. From exact sequence

$$0 \to \mathscr{I}_{Z_a} \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{Z_a} \to 0$$

passing to cohomology, we have:

$$0 \to H^0(\mathscr{I}_{Z_a}) \to H^0(\mathcal{O}_{\mathbb{P}^4}) \to H^0(\mathcal{O}_{Z_a}) \to H^1(\mathscr{I}_{Z_a}) \to \cdots$$

since $h^i(\mathscr{I}_{Z_a}) = 0$ for $i = 0, 1$ hence $h^0(\mathcal{O}_{\mathbb{P}^4}) = h^0(\mathcal{O}_{Z_a}) = 1$. Therefore Z_a is connected. Furthermore by Theorem 17 Z_a is a smooth scheme then Z_a is regular and hence normal scheme. In particular, Z_a is a reduced scheme.
Finally, since Z_a is smooth and connected then is a irreducible scheme.

□

4.2.1 Rao Module dimension of Singular scheme

Let $\mathscr{F}$ be a holomorphic distribution on $\mathbb{P}^4$. Consider the following graded module

$$R_{\mathscr{F}} := H^1_*(\mathscr{I}_Z) = \bigoplus_{l\in\mathbb{Z}} H^1(\mathscr{I}_Z(l));$$

called the *Rao module*.
Since $\mathscr{F}$ is locally free then $R_{\mathscr{F}}$ is finite dimensional and $M_{\mathscr{F}}$ is always finite dimensional. For more details, see [7].
Next, we will determine the Rao Module dimensions for the Horrocks-Mumford distributions.
Let $\mathscr{F}_a$ is a Horrocks-Mumford distribution

$$\mathscr{F}_a : 0 \to E(-a-4) \xrightarrow{\varphi} T\mathbb{P}^4 \to N_{\mathscr{F}_a} \to 0.$$

Consider the Eagon-Northcott complex associated to the morphism
$\varphi^\vee : \Omega^1_{\mathbb{P}^4} \to E(a-1)$

$$0 \to S_2(E(-a-4))(-2a-8) \to \Omega^3_{\mathbb{P}^4} \otimes E(-3a-7) \to \Omega^2_{\mathbb{P}^4}(-2a-3) \to \mathscr{I}_{Z_a} \to 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^4}(q)$

$$0 \to S_2(E(-a-4))(q-2a-8) \to \Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7) \to \Omega^2_{\mathbb{P}^4}(q-2a-3) \to \mathscr{I}_{Z_a}(q) \to 0. \tag{4.25}$$

In order to calculate $h^1(\mathscr{I}_{Z_a}(q))$ in 4.25, for all $q \in \mathbb{Z}$ and $a \geq 1$, we have the following Lemmas:

Lemma 5.

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-16)) \neq 0$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \neq 2a+3\} \cap \{q \leq 3a \quad or \quad q \geq 3a+7\} \cap \{4a < q < 4a+8\}.$$

Proof. In fact, by breaking complex 4.25 in the exact sequences:

$$0 \to S_2(E(-a-4))(q-2a-8) \to \Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7) \to K(q) \to 0, \quad (4.26)$$

$$0 \to K(q) \to \Omega^2_{\mathbb{P}^4}(q-2a-3) \to \mathscr{I}_{Z_a}(q) \to 0. \quad (4.27)$$

with exact long sequences of cohomology, respectively:

$$0 \to H^0(S_2(E(-a-4))(q-2a-8)) \to H^0(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to H^0(K(q)) \to$$
$$\to H^1(S_2(E(-a-4))(q-2a-8)) \to H^1(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to H^1(K(q)) \to$$
$$\to H^2(S_2(E(-a-4))(q-2a-8)) \to H^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to H^2(K(q)) \to$$
$$\to H^3(S_2(E(-a-4))(q-2a-8)) \to H^3(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to H^3(K(q)) \to$$
$$\to H^4(S_2(E(-a-4))(q-2a-8)) \to H^4(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to H^4(K(q)) \to 0$$

and

$$0 \to H^0(K(q)) \to H^0(\Omega^2_{\mathbb{P}^4}(q-2a-3)) \to H^0(\mathscr{I}_{Z_a}(q)) \to$$
$$\to H^1(K(q)) \to H^1(\Omega^2_{\mathbb{P}^4}(q-2a-3)) \to H^1(\mathscr{I}_{Z_a}(q)) \to$$
$$\to H^2(K(q)) \to H^2(\Omega^2_{\mathbb{P}^4}(q-2a-3)) \to H^2(\mathscr{I}_{Z_a}(q)) \to$$
$$\to H^3(K(q)) \to H^3(\Omega^2_{\mathbb{P}^4}(q-2a-3)) \to H^3(\mathscr{I}_{Z_a}(q)) \to$$
$$\to H^4(K(q)) \to H^4(\Omega^2_{\mathbb{P}^4}(q-2a-3)) \to H^4(\mathscr{I}_{Z_a}(q)) \to 0$$

Let us study the long exact sequence of cohomology of sequence (4.39):

$$\cdots \to H^1(\Omega^2_{\mathbb{P}^4}(q-2a-3)) \to H^1(\mathscr{I}_{Z_a}(q)) \to H^2(K(q)) \to H^2(\Omega^2_{\mathbb{P}^4}(q-2a-3)) \to \cdots,$$

and by Bott's formula: $h^1(\Omega^2_{\mathbb{P}^4}(q-2a-3)) = 0 = h^2(\Omega^2_{\mathbb{P}^4}(q-2a-3))$ for $q \neq 2a+3$, then

$$h^1(\mathscr{I}_{Z_a}(q)) = h^2(K(q)) \quad \text{for} \quad q \neq 2a+3 \quad \text{and} \quad a \geq 1. \quad (4.28)$$

Now, from

$$\cdots \to H^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to H^2(K(q)) \to$$
$$\to H^3(S_2(E(-a-4))(q-2a-8)) \to H^3(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to \cdots$$

by Lemma 2 we have:

$$h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) = 0 = h^3(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7))$$

for all $q \in \mathbb{Z}$ such that $q \in \mathbb{Z}$ such that

$$\{q \leq 3a+4 \ \text{ or } \ q \geq 3a+7\} \cap \{q \leq 3a \ \text{ or } \ q \geq 3a+5\}$$

i.e., for all $q \in \mathbb{Z}$ such that $\{q \leq 3a \ \text{ or } \ q \geq 3a+7\}$ hence

$$h^2(K(q)) = h^3(S_2(E(-a-4))(q-2a-8))$$

for all $q \in \mathbb{Z}$ such that $\{q \leq 3a \ \text{ or } \ q \geq 3a+7\}$.
Thus

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(S_2(E(-a-4))(q-2a-8))$$

for all $q \in \mathbb{Z}$ such that $\{q \neq 2a+3\} \cap \{q \leq 3a \ \text{ or } \ q \geq 3a+7\}$.
By decomposition of tensor product:

$$E(-a-4) \otimes E(-a-4) \simeq S_2(E(-a-4)) \oplus \bigwedge^2 E(-a-4),$$

i.e,

$$E \otimes E(-2a-8) \simeq S_2(E(-a-4)) \oplus \mathcal{O}_{\mathbb{P}^4}(-2a-3),$$

twisting by $\mathcal{O}_{\mathbb{P}^4}(q-2a-8)$ we have:

$$E \otimes E(q-4a-16) \simeq S_2(E(-a-4))(q-2a-8) \oplus \mathcal{O}_{\mathbb{P}^4}(q-4a-11),$$

so

$$h^i(E \otimes E(q-4a-16)) = h^i(S_2(E(-a-4))(q-2a-8)) + h^i(\mathcal{O}_{\mathbb{P}^4}(q-4a-11)), \quad i = 0, ..., 4.$$

Now, by Bott's formula (1.9) we have that $h^3(\mathcal{O}_{\mathbb{P}^4}(q-4a-11)) = 0$ thus:

$$h^3(E \otimes E(q-4a-16)) = h^3(S_2(E(-a-4))(q-2a-8)),$$

So

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-16))$$

for all $q \in \mathbb{Z}$ such that $\{q \neq 2a+3\} \cap \{q \leq 3a \ \text{ or } \ q \geq 3a+7\}$.
By table 2, $h^3(E \otimes E(q-4a-16) \neq 0$ for all $q \in \mathbb{Z}$ such that $4a < q < 4a+8$, thus

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-16)) \neq 0$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \neq 2a+3\} \cap \{q \leq 3a \ \text{ or } \ q \geq 3a+7\} \cap \{4a < q < 4a+8\}.$$

□

Lemma 6.

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-16)) = 0,$$

for all $q \in \mathbb{Z}$ *and* $a \geq 1$ *such that*

$$\{q \neq 2a+3\} \cap \{q \leq 3a \quad or \quad q \geq 3a+7\} \cap \{q \leq 4a \quad or \quad q \geq 4a+8\}.$$

Proof. In fact, by table 2 we have $h^3(E \otimes E(q-4a-16)) = 0$ for $q \in \mathbb{Z}$ such that $\{q \leq 4a \ \text{ or } \ q \geq 4a+8\}$ then

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-16)) = 0,$$

for all $q \in \mathbb{Z}$ such that $\{q \neq 2a+3\} \cap \{q \leq 3a \ \text{ or } \ q \geq 3a+7\} \cap \{q \leq 4a \ \text{ or } \ q \geq 4a+8\}$. □

Lemma 7.

$$h^1(\mathscr{I}_{Z_a}(q)) = h^1(\Omega^2_{\mathbb{P}^4}(q-2a-3)) = 0,$$

for all $q \in \mathbb{Z}$ *such that*

$$\{q \leq 4a \quad or \quad q \geq 4a+8\} \cap \{q \leq 3a+4 \quad or \quad q \geq 3a+11\}.$$

Proof. In fact, from

$$\cdots \to H^1(K(q)) \to H^1(\Omega^2_{\mathbb{P}^4}(q-2a-3)) \to H^1(\mathscr{I}_{Z_a}(q)) \to H^2(K(q)) \to \cdots$$

we have:

$$h^1(\mathscr{I}_{Z_a}(q)) = h^1(\Omega^2_{\mathbb{P}^4}(q-2a-3))$$

if and only if $h^1(K(q)) = 0 = h^2(K(q))$, and from

$$\cdots \to H^1(S_2(E(-a-4))(q-2a-8)) \to H^1(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to H^1(K(q)) \to$$
$$\to H^2(S_2(E(-a-4))(q-2a-8)) \to H^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to H^2(K(q)) \to$$
$$\to H^3(S_2(E(-a-4))(q-2a-8)) \to H^3(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to H^3(K(q)) \to \cdots$$

we have

$$h^1(K(q)) = 0 = h^2(K(q))$$

if and only if $h^i(S_2(E(-a-4))(q-2a-8)) = 0$ for all $i = 1, 2, 3$ and
$h^i(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) = 0$ for all $i = 1, 2$.
By table 2 we have that $h^i(S_2(E(-a-4))(q-2a-8)) = h^i(E \otimes E(q-4a-16)) = 0$ for all $i = 1, 2, 3$ and $q \in \mathbb{Z}$ such that $\{q \leq 4a \ \text{ or } \ q \geq 4a+8\}$.

By Lemma 2 we have that $h^i(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) = 0$ for all $i = 1, 2$ and $q \in \mathbb{Z}$ such that $\{q \leq 3a+6 \ \text{ or } \ q \geq 3a+11\} \cap \{q \leq 3a+4 \ \text{ or } \ q \geq 3a+7\}$, i.e., $q \in \mathbb{Z}$ such that $\{q \leq 3a+4 \ \text{ or } \ q \geq 3a+11\}$.
So

$$h^1(\mathscr{I}_{Z_a}(q)) = h^1(\Omega^2_{\mathbb{P}^4}(q-2a-3))$$

for all $q \in \mathbb{Z}$ such that $\{q \leq 4a \ \text{ or } \ q \geq 4a+8\} \cap \{q \leq 3a+4 \ \text{ or } \ q \geq 3a+11\}$.
By Bott's formula, $h^1(\Omega^2_{\mathbb{P}^4}(q-2a-3)) = 0$, therefore

$$h^1(\mathscr{I}_{Z_a}(q)) = 0$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \leq 4a \ \text{ or } \ q \geq 4a+8\} \cap \{q \leq 3a+4 \ \text{ or } \ q \geq 3a+11\}.$$

□

Lemma 8.

$$h^1(\mathscr{I}_{Z_a}(q)) = 0$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\big\{\{q \neq 2a+3\} \cap \{q \leq 3a \ \textit{ or } \ q \geq 3a+7\} \cap \{q \leq 4a \ \textit{ or } \ q \geq 4a+8\}\big\}$$
$$\cup \big\{\{q \leq 3a+4 \ \textit{ or } \ q \geq 3a+11\} \cap \{q \leq 4a \ \textit{ or } \ q \geq 4a+8\}\big\}.$$

Proof. Putting together Lemmas 6 and 7, we have:

$$h^1(\mathscr{I}_{Z_a}(q)) = 0$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\big\{\{q \neq 2a+3\} \cap \{q \leq 3a \ \text{ or } \ q \geq 3a+7\} \cap \{q \leq 4a \ \text{ or } \ q \geq 4a+8\}\big\}$$
$$\cup \big\{\{q \leq 3a+4 \ \text{ or } \ q \geq 3a+11\} \cap \{q \leq 4a \ \text{ or } \ q \geq 4a+8\}\big\}.$$

□

Lemma 9.

$$h^1(\mathscr{I}_{Z_a}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) + h^3(E \otimes E(q-4a-16))$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \neq 2a+3\} \cap \{q \leq 3a \ \textit{ or } \ q \geq 3a+5\} \cap \{q \leq 4a+5 \ \textit{ or } \ q \geq 4a+12\}.$$

Proof. By equation 4.28 we have $h^1(\mathscr{I}_{Z_a}(q)) = h^2(K(q))$ for $q \neq 2a+3$.
Now, from

$$\cdots \to H^2(S_2(E(-a-4))(q-2a-8)) \to H^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to H^2(K(q)) \to$$
$$\to H^3(S_2(E(-a-4))(q-2a-8)) \to H^3(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to \cdots$$

We note that:

$$h^2(K(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) + h^3(S_2(E(-a-4)(q-2a-8)))$$

if and only if $h^2(S_2(E(-a-4))(q-2a-8)) = 0$ and $h^3(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) = 0$.
On the one hand, by table 2 we have

$$h^2(S_2(E(-a-4)(q-2a-8))) = h^2(E \otimes E(q-4a-16)) = 0$$

if and only if $q \leq 4a+5$ or $q \geq 4a+12$.
On the other hand, by Lemma 3 we have

$$h^3(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) = 0$$

if and only if $q \leq 3a$ or $q \geq 3a+5$.
From here, putting these last equations together, we have:

$$h^1(\mathscr{I}_{Z_a}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) + h^3(S_2(E(-a-4)(q-2a-8)))$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \neq 2a+3\} \cap \{q \leq 3a \quad \text{or} \quad q \geq 3a+5\} \cap \{q \leq 4a+5 \quad \text{or} \quad q \geq 4a+12\}.$$

□

Lemma 10.

$$h^1(\mathscr{I}_{Z_a}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)),$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \neq 2a+3\} \cap \{q \leq 4a \quad \textit{or} \quad q \geq 4a+12\}.$$

Proof. By equation 4.28 we have $h^1(\mathscr{I}_{Z_a}(q)) = h^2(K(q))$ for $q \neq 2a+3$.
Now, from

$$\cdots \to H^2(S_2(E(-a-4))(q-2a-8)) \to H^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7)) \to$$
$$\to H^2(K(q)) \to H^3(S_2(E(-a-4))(q-2a-8)) \to \cdots$$

We note that:

$$h^2(K(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7))$$

if and only if $h^i(S_2(E(-a-4)(q-2a-8))) = h^i(E \otimes E(q-4a-16)) = 0$ for $i = 2, 3$. By table 2 we have

$$h^2(E \otimes E(q-4a-16)) = 0 = h^3(E \otimes E(q-4a-16))$$

if and only if $q \leq 4a$ or $q \geq 4a+12$.
Hence, $h^1(\mathscr{I}_{Z_a}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7))$ for all $q \in \mathbb{Z}$ and $a \geq 1$ such that $\{q \neq 2a+3\} \cap \{q \leq 4a \ \text{ or } \ q \geq 4a+12\}$.

□

Theorem 29. *Let $\mathscr{F}_a$ is a codimension 2 holomorphic distributions (4.18) on $\mathbb{P}^4$. Then:*

1. $\dim_{\mathbb{C}} R_{\mathscr{F}_1} = h^1(\mathscr{I}_{Z_1}(5)) + h^1(\mathscr{I}_{Z_1}(6)) + h^1(\mathscr{I}_{Z_1}(7)) + 184.$
2. $\dim_{\mathbb{C}} R_{\mathscr{F}_2} = h^1(\mathscr{I}_{Z_2}(9)) + h^1(\mathscr{I}_{Z_2}(10)) + 284.$
3. $\dim_{\mathbb{C}} R_{\mathscr{F}_3} = h^1(\mathscr{I}_{Z_3}(13)) + 369.$
4. $\dim_{\mathbb{C}} R_{\mathscr{F}_a} = 401, \ \forall a \geq 4.$

Proof. For $a = 1$.
By Lemma 8 we have $h^1(\mathscr{I}_{Z_1}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 4$ or $q \geq 12$.
By Lemma 9 we have $h^1(\mathscr{I}_{Z_1}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-10)) + h^3(E \otimes E(q-20))$ for $q = 8, 9$.
Hence

$$h^1(\mathscr{I}_{Z_1}(8)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-2)) + h^3(E \otimes E(-12)).$$

Now, by Lemma 2 we have $h^i(\Omega^3_{\mathbb{P}} \otimes E(-2)) = 0$ for all $i = 0, 1, 3, 4$, so

$$\chi(\Omega^3_{\mathbb{P}} \otimes E(-2)) = h^2(\Omega^3_{\mathbb{P}} \otimes E(-2))$$

since

$$c(\Omega^3_{\mathbb{P}} \otimes E(-2)) = 1 - 26\mathbf{h} + 312\mathbf{h}^2 - 2238\mathbf{h}^3 + 10455\mathbf{h}^4$$

then by Riemann-Roch Theorem we have $\chi(\Omega^3_{\mathbb{P}} \otimes E(-2)) = 5$, so

$$h^2(\Omega^3_{\mathbb{P}} \otimes E(-2)) = 5.$$

By table 2 we have $h^3(E \otimes E(-12)) = 85$. Thus:

$$\begin{aligned} h^1(\mathscr{I}_{Z_4}(17)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-2)) + h^3(E \otimes E(-12)) \\ &= 5 + 85 \\ &= 90. \end{aligned}$$

Similarly, by Lemma 2 we have $h^i(\Omega^3_{\mathbb{P}} \otimes E(-1)) = 0$ for all $i = 0, 1, 3, 4$, so

$$\chi(\Omega^3_{\mathbb{P}} \otimes E(-1)) = h^2(\Omega^3_{\mathbb{P}} \otimes E(-1))$$

since

$$c(\Omega^3_{\mathbb{P}} \otimes E(-1)) = 1 - 18\mathbf{h} + 158\mathbf{h}^2 - 856\mathbf{h}^3 + 3105\mathbf{h}^4$$

then by Riemann-Roch Theorem we have $\chi(\Omega^3_{\mathbb{P}} \otimes E(-1)) = 10$, so

$$h^2(\Omega^3_{\mathbb{P}} \otimes E(-1)) = 10.$$

By table 2 we have $h^3(E \otimes E(-11)) = 55$. Thus:

$$\begin{aligned} h^1(\mathscr{I}_{Z_1}(9)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-1)) + h^3(E \otimes E(-11)) \\ &= 10 + 55 \\ &= 65. \end{aligned}$$

By Lemma 5 we have $h^1(\mathscr{I}_{Z_1}(q)) = h^3(E \otimes E(q-20)) \neq 0$ for all $q \in \mathbb{Z}$ such that $10 \leq q < 12$. So, by table 2 we have:

$$\begin{aligned} h^1(\mathscr{I}_{Z_1}(10)) &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_1}(11)) &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned} \dim_{\mathbb{C}} R_{\mathscr{F}_1} &= \dim \bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{I}_{Z_1}(l)) \\ &= \sum_{l=5}^{11} h^1(\mathscr{I}_{Z_1}(l)) \\ &= h^1(\mathscr{I}_{Z_1}(5)) + h^1(\mathscr{I}_{Z_1}(6)) + h^1(\mathscr{I}_{Z_1}(7)) + 90 + 65 + 24 + 5 \\ &= h^1(\mathscr{I}_{Z_1}(5)) + h^1(\mathscr{I}_{Z_1}(6)) + h^1(\mathscr{I}_{Z_1}(7)) + 184. \end{aligned}$$

For $a = 2$.
By Lemma 8 we have $h^1(\mathscr{I}_{Z_2}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 8$ or $q \geq 16$.
By Lemma 9 we have $h^1(\mathscr{I}_{Z_2}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-13)) + h^3(E \otimes E(q-24))$ for $q = 11, 12$.
Hence

$$\begin{aligned} h^1(\mathscr{I}_{Z_2}(11)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-2)) + h^3(E \otimes E(-13)) \\ &= 5 + 100 \\ &= 105. \end{aligned}$$

similarly,

$$\begin{aligned} h^1(\mathscr{I}_{Z_2}(12)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-1)) + h^3(E \otimes E(-12)) \\ &= 10 + 85 \\ &= 95. \end{aligned}$$

By Lemma 5 we have $h^1(\mathscr{I}_{Z_2}(q)) = h^3(E \otimes E(q-24)) \neq 0$ for all $q \in \mathbb{Z}$ such that $13 \leq q < 16$. So, by table 2 we have:

$$\begin{aligned} h^1(\mathscr{I}_{Z_2}(13)) &= h^3(E \otimes E(-11)) \\ &= 55. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_2}(14)) &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_2}(15)) &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned} \dim_{\mathbb{C}} R_{\mathscr{F}_2} &= \dim \bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{I}_{Z_2}(l)) \\ &= \sum_{l=9}^{15} h^1(\mathscr{I}_{Z_2}(l)) \\ &= h^1(\mathscr{I}_{Z_2}(9)) + h^1(\mathscr{I}_{Z_2}(10)) + 105 + 95 + 55 + 24 + 5 \\ &= h^1(\mathscr{I}_{Z_2}(9)) + h^1(\mathscr{I}_{Z_2}(10)) + 284. \end{aligned}$$

For $a = 3$.
By Lemma 8 we have $h^1(\mathscr{I}_{Z_3}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 12$ or $q \geq 20$.
By Lemma 9 we have $h^1(\mathscr{I}_{Z_3}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-16)) + h^3(S_2(E(-7)(q-14)))$ for $q = 14, 15$. Hence

$$\begin{aligned} h^1(\mathscr{I}_{Z_3}(14)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-2)) + h^3(E \otimes E(-14)) \\ &= 5 + 85 \\ &= 90. \end{aligned}$$

similarly,

$$\begin{aligned} h^1(\mathscr{I}_{Z_3}(15)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-1)) + h^3(E \otimes E(-13)) \\ &= 10 + 100 \\ &= 110. \end{aligned}$$

By Lemma 5 we have $h^1(\mathscr{I}_{Z_3}(q)) = h^3(E \otimes E(q-28)) \neq 0$ for all $q \in \mathbb{Z}$ such that $16 \leq q < 20$. So, by table 2 we have:

$$\begin{aligned} h^1(\mathscr{I}_{Z_3}(16)) &= h^3(E \otimes E(-12)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_3}(17)) &= h^3(E \otimes E(-11)) \\ &= 55. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_3}(18)) &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_3}(19)) &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned} \dim_{\mathbb{C}} R_{\mathscr{F}_3} &= \dim \bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{I}_{Z_3}(l)) \\ &= \sum_{l=13}^{19} h^1(\mathscr{I}_{Z_3}(l)) \\ &= h^1(\mathscr{I}_{Z_3}(13)) + 90 + 110 + 85 + 55 + 24 + 5 \\ &= h^1(\mathscr{I}_{Z_3}(13)) + 369. \end{aligned}$$

For $a = 4$.
By Lemma 8 we have $h^1(\mathscr{I}_{Z_4}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 16$ or $q \geq 24$.

By Lemma 9 we have $h^1(\mathscr{I}_{Z_4}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-19)) + h^3(S_2(E(-8)(q-16)))$ for $q = 17, 18$. Hence

$$\begin{aligned} h^1(\mathscr{I}_{Z_4}(17)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-2)) + h^3(E \otimes E(-15)) \\ &= 5 + 32 \\ &= 37. \end{aligned}$$

similarly,

$$\begin{aligned} h^1(\mathscr{I}_{Z_4}(18)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-1)) + h^3(E \otimes E(-14)) \\ &= 10 + 85 \\ &= 95. \end{aligned}$$

By Lemma 5 we have $h^1(\mathscr{I}_{Z_4}(q)) = h^3(E \otimes E(q-32)) \neq 0$ for all $q \in \mathbb{Z}$ such that $19 \leq q < 24$. So, by table 2 we have:

$$\begin{aligned} h^1(\mathscr{I}_{Z_4}(19)) &= h^3(E \otimes E(-13)) \\ &= 100. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_4}(20)) &= h^3(E \otimes E(-12)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_4}(21)) &= h^3(E \otimes E(-11)) \\ &= 55. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_4}(22)) &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_4}(23)) &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned} \dim_{\mathbb{C}} R_{\mathscr{F}_4} &= \dim \bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{I}_{Z_4}(l)) \\ &= \sum_{l=17}^{23} h^1(\mathscr{I}_{Z_4}(l)) \\ &= 37 + 95 + 100 + 85 + 55 + 24 + 5 \\ &= 401. \end{aligned}$$

For $a = 5$.
By Lemma 8 we have $h^1(\mathscr{I}_{Z_5}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 19$ or $q \geq 28$.
By Lemma 9 we have $h^1(\mathscr{I}_{Z_5}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-22)) + h^3(E \otimes E(q-36)))$ for $q = 20, 21$.
Hence

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(20)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-2)) + h^3(E \otimes E(-16))) \\ &= 5 + 0 \\ &= 5. \end{aligned}$$

and

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(21)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-1)) + h^3(E \otimes E(-15))) \\ &= 10 + 32 \\ &= 42. \end{aligned}$$

By Lemma 5 we have $h^1(\mathscr{I}_{Z_5}(q)) = h^3(E \otimes E(q-36)) \neq 0$ for all $q \in \mathbb{Z}$ such that $22 \leq q < 28$. So, by table 2 we have:

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(22)) &= h^3(E \otimes E(-14)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(23)) &= h^3(E \otimes E(-13)) \\ &= 100. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(24)) &= h^3(E \otimes E(-12)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(25)) &= h^3(E \otimes E(-11)) \\ &= 55. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(26)) &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(27)) &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned}\dim_{\mathbb{C}} R_{\mathscr{F}_5} &= \dim \bigoplus_{l\in\mathbb{Z}} H^1(\mathscr{I}_{Z_5}(l)) \\ &= \sum_{l=20}^{27} h^1(\mathscr{I}_{Z_5}(l)) \\ &= 5+42+85+100+85+55+24+5 \\ &= 401.\end{aligned}$$

For $a = 6$.
By Lemma 8 we have $h^1(\mathscr{I}_{Z_6}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 22$ or $q \geq 32$.
By Lemma 10 we have $h^1(\mathscr{I}_{Z_6}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-25))$ for $q = 23, 24$. Hence

$$\begin{aligned}h^1(\mathscr{I}_{Z_6}(23)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-2)) \\ &= 5.\end{aligned}$$

and

$$\begin{aligned}h^1(\mathscr{I}_{Z_6}(24)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-1)) \\ &= 10.\end{aligned}$$

By Lemma 5 we have $h^1(\mathscr{I}_{Z_6}(q)) = h^3(E \otimes E(q-40)) \neq 0$ for all $q \in \mathbb{Z}$ such that $25 \leq q < 32$. So, by table 2 we have:

$$\begin{aligned}h^1(\mathscr{I}_{Z_6}(25)) &= h^3(E \otimes E(-15)) \\ &= 32.\end{aligned}$$

$$\begin{aligned}h^1(\mathscr{I}_{Z_6}(26)) &= h^3(E \otimes E(-14)) \\ &= 85.\end{aligned}$$

$$\begin{aligned}h^1(\mathscr{I}_{Z_6}(27)) &= h^3(E \otimes E(-13)) \\ &= 100.\end{aligned}$$

$$\begin{aligned}h^1(\mathscr{I}_{Z_6}(28)) &= h^3(E \otimes E(-12)) \\ &= 85.\end{aligned}$$

$$\begin{aligned}h^1(\mathscr{I}_{Z_6}(29)) &= h^3(E \otimes E(-11)) \\ &= 55.\end{aligned}$$

$$h^1(\mathscr{I}_{Z_6}(30)) = h^3(E \otimes E(-10)) \\ = 24.$$

$$h^1(\mathscr{I}_{Z_6}(31)) = h^3(E \otimes E(-9)) \\ = 5.$$

Therefore

$$\begin{aligned} \dim_{\mathbb{C}} R_{\mathscr{F}_6} &= \dim \bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{I}_{Z_6}(l)) \\ &= \sum_{l=23}^{31} h^1(\mathscr{I}_{Z_6}(l)) \\ &= 5 + 10 + 32 + 85 + 100 + 85 + 55 + 24 + 5 \\ &= 401. \end{aligned}$$

For $a = 7$.
By Lemma 8 we have $h^1(\mathscr{I}_{Z_7}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 25$ or $q = 28$ or $q \geq 36$.
By Lemma 10 we have $h^1(\mathscr{I}_{Z_7}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-28))$ for $q = 26, 27$. Hence

$$h^1(\mathscr{I}_{Z_7}(26)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-2)) \\ = 5.$$

and

$$h^1(\mathscr{I}_{Z_7}(27)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-1)) \\ = 10.$$

By Lemma 5 we have $h^1(\mathscr{I}_{Z_7}(q)) = h^3(E \otimes E(q-44)) \neq 0$ for all $q \in \mathbb{Z}$ such that $28 < q < 36$. So, by table 2 we have:

$$h^1(\mathscr{I}_{Z_7}(29)) = h^3(E \otimes E(-15)) \\ = 32.$$

$$h^1(\mathscr{I}_{Z_7}(30)) = h^3(E \otimes E(-14)) \\ = 85.$$

$$h^1(\mathscr{I}_{Z_7}(31)) = h^3(E \otimes E(-13)) \\ = 100.$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(32)) &= h^3(E \otimes E(-12)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(33)) &= h^3(E \otimes E(-11)) \\ &= 55. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(34)) &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(35)) &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned} \dim_{\mathbb{C}} R_{\mathscr{F}_7} &= \dim \bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{I}_{Z_7}(l)) \\ &= \sum_{l=26}^{35} h^1(\mathscr{I}_{Z_7}(l)) \\ &= 5 + 10 + 0 + 32 + 85 + 100 + 85 + 55 + 24 + 5 \\ &= 401. \end{aligned}$$

For $a \geq 7$.
By Lemma 8 we have $h^1(\mathscr{I}_{Z_a}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 3a+4$ or $3a+7 \leq q \leq 4a$ or $q \geq 4a+8$.
By Lemma 10 we have $h^1(\mathscr{I}_{Z_a}(q)) = h^2(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-7))$ for all $q \in \mathbb{Z}$ such that $\{q \neq 2a+3\} \cap \{q \leq 4a \;\text{ or }\; q \geq 4a+12\}$, in particular for $q = 3a+5, 3a+6$.
Hence

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(3a+5)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(3a+5-3a-7)) \\ &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-2)) \\ &= 5. \end{aligned}$$

and

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(3a+6)) &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(3a+6-7+3a)) \\ &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-1)) \\ &= 10. \end{aligned}$$

By Lemma 5 we have $h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-16)) \neq 0$ for all $q \in \mathbb{Z}$ such that $4a < q < 4a+8$. So, by table 2 we have:

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+1)) &= h^3(E \otimes E(4a+1-4a-16)) \\ &= h^3(E \otimes E(-15)) \\ &= 32. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+2)) &= h^3(E \otimes E(4a+2-4a-16)) \\ &= h^3(E \otimes E(-14)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+3)) &= h^3(E \otimes E(4a+3-4a-16)) \\ &= h^3(E \otimes E(-13)) \\ &= 100. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+4)) &= h^3(E \otimes E(4a+4-4a-16)) \\ &= h^3(E \otimes E(-12)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+5)) &= h^3(E \otimes E(4a+5-4a-16)) \\ &= h^3(E \otimes E(-11)) \\ &= 55. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+6)) &= h^3(E \otimes E(4a+6-4a-16)) \\ &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+7)) &= h^3(E \otimes E(4a+7-4a-16)) \\ &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned}\dim_{\mathbb{C}} R_{\mathscr{F}_a} &= \dim \bigoplus_{l\in\mathbb{Z}} H^1(\mathscr{I}_{Z_a}(l)) \\ &= \sum_{l=3a+5}^{4a+7} h^1(\mathscr{I}_{Z_a}(l)) \\ &= \sum_{l=3a+5}^{3a+6} h^1(\mathscr{I}_{Z_a}(l)) + 0 + \cdots + 0 + \sum_{l=4a+1}^{4a+7} h^1(\mathscr{I}_{Z_a}(l)) \\ &= 5 + 10 + 0 + \cdots + 0 + 32 + 85 + 100 + 85 + 55 + 24 + 5 \\ &= 401.\end{aligned}$$

□

4.2.2 First cohomology dimension:

Let $\mathscr{F}$ be a holomorphic distribution on $\mathbb{P}^4$. Consider the following graded module

$$M_{\mathscr{F}} := H^1_*(T\mathscr{F}) = \bigoplus_{l\in\mathbb{Z}} H^1(T\mathscr{F}(l));$$

called the *first cohomology module* of $\mathscr{F}$ as a sub-sheaf of the tangent sheaf.
Since $\mathscr{F}$ is locally free then $M_{\mathscr{F}}$ is always finite dimensional. For more details, see [7].
We are going to calculate the first cohomology dimension for Horrocks-Mumford distributions as subsheaves of tangent.
Since $T\mathscr{F}_a = E(-a-4)$ then $H^1(T\mathscr{F}_a(l)) = H^1(E(-a-4+l))$. By table 1 we have $h^1(T\mathscr{F}_a(l)) = h^1(E(-a-4+l)) \neq 0$ if, and only if, $a+1 \leq l \leq a+4$. Hence:

$$\begin{aligned}\dim_{\mathbb{C}} M_{\mathscr{F}_a} &= \dim \bigoplus_{l\in\mathbb{Z}} H^1(E(-a-4+l)) \\ &= \sum_{l=a+1}^{a+4} h^1(E(-a-4+l)) \\ &= h^1(E(-3)) + h^1(E(-2)) + h^1(E(-1)) + h^1(E) \\ &= 5 + 10 + 10 + 2 \\ &= 27.\end{aligned}$$

Next, we describe the cohomology of the normal sheaf of these distributions.

Proposition 15. *Let $\mathscr{F}_a$ is a codimension 2 holomorphic distributions family (4.18) on $\mathbb{P}^4$. Then:*

1. $h^0(N_{\mathscr{F}_a}(q)) = 0$ *for* $q \leq -2$.

2. *$h^1(N_{\mathscr{F}_a}(q)) = 0$ for $q \leq a-2$ or $q \geq a$.*

3. *$h^2(N_{\mathscr{F}_a}(q)) = 0$*

   - *If $a = 1$ and $q \leq -6$ or $q \geq -1$.*
   - *If $a = 2$ and $q \leq -5$ or $q \geq 0$.*
   - *If $a \geq 3$ and $q < -5$ or $-5 < q \leq a-7$ or $q \geq a-2$.*

4. *$h^3(N_{\mathscr{F}_a}(q)) = 0$ for $q \geq a-5$.*

5. *$h^4(N_{\mathscr{F}_a}(q)) = 0$ for $q \geq a-5$.*

Proof. From the $\mathscr{F}_a$ distribution (4.18):

$$0 \to E(-4-a) \to T\mathbb{P}^4 \to N_{\mathscr{F}_a} \to 0,$$

twisting by $\mathcal{O}_{\mathbb{P}^4}(q)$, we have:

$$0 \to E(q-4-a) \to T\mathbb{P}^4(q) \to N_{\mathscr{F}_a}(q) \to 0.$$

Let us consider the long exact sequence of cohomology:

$$\cdots \to H^0(T\mathbb{P}^4(q)) \to H^0(N_{\mathscr{F}_a}(q)) \to H^1(E(q-4-a)) \to H^1(T\mathbb{P}^4(q)) \to \cdots.$$

By Serre's duality and Bott's formula we have $h^0(T\mathbb{P}^4(q)) = 0 = h^1(T\mathbb{P}^4(q))$ if and only if $h^4(\Omega^1_{\mathbb{P}^4}(-q-5)) = 0 = h^3(\Omega^1_{\mathbb{P}^4}(-q-5))$ for $q \leq -2$, thus $h^0(N_{\mathscr{F}_a}(q)) = h^1(E(q-4-a))$ for $q \leq -2$. But $h^1(E(q-4-a)) = 0$ for $q \leq a$ or $q \geq a+5$. Therefore $h^0(N_{\mathscr{F}_a}(q)) = 0$ for $q \leq -2$.
Let us consider the long exact sequence of cohomology:

$$\cdots \to H^1(T\mathbb{P}^4(q)) \to H^1(N_{\mathscr{F}_a}(q)) \to H^2(E(q-4-a)) \to H^2(T\mathbb{P}^4(q)) \to \cdots.$$

By Serre's duality and Bott's formula we have $h^1(T\mathbb{P}^4(q)) = 0 = h^2(T\mathbb{P}^4(q))$ if and only if $h^3(\Omega^1_{\mathbb{P}^4}(-q-5)) = 0 = h^2(\Omega^1_{\mathbb{P}^4}(-q-5))$ for $q \in \mathbb{Z}$, thus $h^1(N_{\mathscr{F}_a}(q)) = h^2(E(q-4-a))$ for $q \in \mathbb{Z}$. But $h^2(E(q-4-a)) = 0$ for $q \leq a-2$ or $q \geq a$. Therefore $h^1(N_{\mathscr{F}_a}(q)) = 0$ for $q \leq a-2$ or $q \geq a$.
Let us consider the long exact sequence of cohomology:

$$\cdots \to H^2(T\mathbb{P}^4(q)) \to H^2(N_{\mathscr{F}_a}(q)) \to H^3(E(q-4-a)) \to H^3(T\mathbb{P}^4(q)) \to \cdots.$$

By Serre's duality and Bott's formula we have $h^2(T\mathbb{P}^4(q)) = 0 = h^3(T\mathbb{P}^4(q))$ if and only if $h^2(\Omega^1_{\mathbb{P}^4}(-q-5)) = 0 = h^1(\Omega^1_{\mathbb{P}^4}(-q-5))$ for $q \neq -5$, thus $h^2(N_{\mathscr{F}_a}(q)) = h^3(E(q-4-a))$ for $q \neq -5$. But $h^3(E(q-4-a)) = 0$ for $q \leq a-7$ or $q \geq a-2$. Therefore for $a = 1$ we have $h^2(N_{\mathscr{F}_a}(q)) = 0$ for $q \leq -6$ or $q \geq -1$. For $a = 2$ we have $h^2(N_{\mathscr{F}_a}(q)) = 0$ for

$q < -5$ or $q \geq 0$. For $a \geq 3$ we have $h^2(N_{\mathscr{F}_a}(q)) = 0$ for $q < -5$ or $-5 < q \leq a - 7$ or $q \geq a - 2$.
Let us consider the long sequence of cohomology:

$$\cdots \to H^3(T\mathbb{P}^4(q)) \to H^3(N_{\mathscr{F}_a}(q)) \to H^4(E(q-4-a)) \to H^4(T\mathbb{P}^4(q)) \to \cdots.$$

By Serre's duality and Bott's formula we have $h^3(T\mathbb{P}^4(q)) = 0 = h^4(T\mathbb{P}^4(q))$ if and only if $h^1(\Omega^1_{\mathbb{P}^4}(-q-5)) = 0 = h^0(\Omega^1_{\mathbb{P}^4}(-q-5))$ for $-6 \leq q < -5$ or $-5 < q$, thus

$$h^3(N_{\mathscr{F}_a}(q)) = h^4(E(q-4-a)) \text{ for } -6 \leq q < -5 \text{ or } -5 < q.$$

But $h^4(E(q-5)) = 0$ for $q \geq a-5$. Therefore $h^3(N_{\mathscr{F}_a}(q)) = 0$ for $q \geq a-5$.
Let us consider the long exact sequence of cohomology:

$$\cdots \to H^4(E(q-4-a)) \to H^4(T\mathbb{P}^4(q)) \to H^4(N_{\mathscr{F}_a}(q)) \to 0.$$

Since $h^4(E(q-5)) = 0$ for $q \geq a-5$ then $h^4(N_{\mathscr{F}_a}(q)) = h^4(T\mathbb{P}^4(q))$ for $q \geq a-5$. Now, by Serre's duality and Bott's formula, we have $h^4(T\mathbb{P}^4(q)) = h^0(\Omega^1_{\mathbb{P}^4}(-q-5)) = 0$ for $q \geq -6$. Therefore $h^4(N_{\mathscr{F}_a}(q)) = 0$ for $q \geq a-5$.

□

In particular, for $a = 1$ we have:

Corollary 4. *Let $\mathscr{F}$ is a distribution*

$$\mathscr{F} : 0 \to E(-5) \to T\mathbb{P}^4 \to N_{\mathscr{F}} \to 0, \tag{4.29}$$

and $Z = \mathrm{Sing}(\mathscr{F})$. Then:

1. *The Chern classes of the normal bundle are:*

$$c_1(N_{\mathscr{F}}) = 10 \;,\quad c_2(N_{\mathscr{F}}) = 50 \;,\quad c_3(N_{\mathscr{F}}) = 160 \;,\quad c_4(N_{\mathscr{F}}) = 305.$$

2. *The degree of distribution $\mathscr{F}$ is $\deg(\mathscr{F}) = 7$.*

3. *The Chern classes of ideals sheaf of the singular scheme are:*

$$c_1(\mathscr{I}_Z) = 0 \;,\quad c_2(\mathscr{I}_Z) = 0 \;,\quad c_3(\mathscr{I}_Z) = -320 \;,\quad c_4(\mathscr{I}_Z) = -6630.$$

4. $\deg(Z) = 160$.

5. $p_a(Z) = 706$.

Corollary 5. *Let $\mathscr{F}$ is a codimension 2 holomorphic distribution (4.29) on $\mathbb{P}^4$. Then the singular scheme $Z = \mathrm{Sing}(\mathscr{F})$ is reduced and irreducible.*

Corollary 6. *Let $\mathscr{F}_a$ is a codimension 2 holomorphic distributions (4.29) on $\mathbb{P}^4$. Then:*

1. *The singular scheme Z is not a arithmetically Cohen-Macaulay nor arithmetically Buchsbaum curve.*
2. $\dim_{\mathbb{C}} R_{\mathscr{F}_1} \geq 184$.

Corollary 7. *Let $\mathscr{F}$ is a codimension 2 holomorphic distribution (4.29) on $\mathbb{P}^4$. Then:*

1. $h^0(N_{\mathscr{F}}(q)) = 0$ *for* $q \leq -2$.
2. $h^1(N_{\mathscr{F}}(q)) = 0$ *for* $q \leq -1$ *or* $q \geq 1$.
3. $h^2(N_{\mathscr{F}}(q)) = 0$ *for* $q \leq -6$ *or* $q \geq -1$.
4. $h^3(N_{\mathscr{F}}(q)) = 0$ *for* $q \geq -4$.
5. $h^4(N_{\mathscr{F}}(q)) = 0$ *for* $q \geq -4$.

4.3 HORROCKS-MUMFORD HOLOMORPHIC DISTRIBUTIONS AS SUBSHEAVES OF COTANGENT BUNDLE

In this section, we study codimension 2 holomorphic distributions induced by a Bertini-type Theorem. The conormal sheaf of these distributions are Horrocks-Mumford bundles, that is, stable and non-split bundles as the sum of line bundles.

From Proposition 11 we know that $E(a)$ is a globally generated rank 2 bundle on $\mathbb{P}^4$, for all $a \geq 1$. Since $\Omega^1_{\mathbb{P}^4}(2)$ is a sheaf generated by global sections then by Bertini type Theorem 28 such that $\mathcal{G} = E(a)$ and $L = \mathcal{O}_{\mathbb{P}^4}(2)$, we have that

$$N^*_{\mathscr{F}_a} = E(-a-7), \quad (a \geq 1)$$

is the conormal sheaf of a codimension 2 holomorphic distribution on $\mathbb{P}^4$:

$$\mathscr{F}_a : 0 \to E(-a-7) \xrightarrow{\varphi} \Omega^1_{\mathbb{P}^4} \to \mathcal{Q}_{\mathscr{F}_a} \to 0, \quad (a \geq 1). \tag{4.30}$$

In this work we will denote these distributions by *Horrocks-Mumford distributions induced by the cotangent bundle.*

In addition, since $E(a) \otimes \Omega^1_{\mathbb{P}^4}(2) = \mathcal{H}om(E(-a-7)), \Omega^1_{\mathbb{P}^4})$ is a sheaf generated by global sections then by Theorem 17 the generic morphism $\varphi : E(-a-7) \to \Omega^1_{\mathbb{P}^4}$ satisfies $\mathrm{Sing}(\mathscr{F}_a) := \mathrm{Sing}(\varphi)$ is a smooth scheme of expected codimension $\mathrm{codim}(\mathrm{Sing}(\mathscr{F}_a)) = 3$. Next, we calculate the Chern classes of the ideal sheaf and calculate the degree of these distributions.

Proposition 16. *Let $\mathscr{F}_a$ be the distribution (4.30) above. We have:*

1. *The Chern classes of the conormal sheaf are:*
   - $c_1(N^*_{\mathscr{F}_a}) = -2a - 9$.
   - $c_2(N^*_{\mathscr{F}_a}) = a^2 + 9a + 24$.
2. *The Chern classes of ideals sheaf of the singular scheme are:*
   - $c_1(\mathscr{I}_{Z_a}) = 0$.
   - $c_2(\mathscr{I}_{Z_a}) = 0$.
   - $c_3(\mathscr{I}_{Z_a}) = -8a^3 - 78a^2 - 226a - 184$.
   - $c_4(\mathscr{I}_{Z_a}) = -54a^4 - 702a^3 - 3126a^2 - 5478a - 2934$.
3. *The degree of the holomorphic distributions family $\mathscr{F}_a$ is* $\deg(\mathscr{F}_a) = 2a + 6$.

Proof. Since $N^*_{\mathscr{F}_a} = E(-a-7)$, $a \geq 1$, then

$$c(N^*_{\mathscr{F}_a}) = 1 + (-2a-9)\mathbf{h} + (a^2 + 9a + 24)\mathbf{h}^2$$

Let $\varphi : E(-a-7) \to \Omega^1_{\mathbb{P}^4}$ be the map that induces the distribution (4.30). Then, the Eagon-Northcott resolution associated with the dual map $\varphi^\vee : T\mathbb{P}^4 \to E(a+2)$ is:

$$0 \to S_2(E(-a-7))(-2a-4) \to \bigwedge^3 T\mathbb{P}^4 \otimes E(-3a-16) \to \bigwedge^2 T\mathbb{P}^4(-2a-9) \to \mathscr{I}_{Z_a} \to 0.$$

And since: $\bigwedge^3 T\mathbb{P}^4 = \Omega^1_{\mathbb{P}^4}(5)$ and $\bigwedge^2 T\mathbb{P}^4 = \Omega^2_{\mathbb{P}^4}(5)$, then replacing, we have:

$$0 \to S_2(E(-a-7))(-2a-4) \to \Omega^1_{\mathbb{P}^4} \otimes E(-3a-11) \to \Omega^2_{\mathbb{P}^4}(-2a-4) \to \mathscr{I}_{Z_a} \to 0. \quad (4.31)$$

Using this resolution of ideal sheaf $\mathscr{I}_{Z_a}$ by locally free sheaves, let us calculate its Chern class:

$$c(\mathscr{I}_{Z_a}) = c(S_2(E(-7-a))(-2a-4)) \cdot c(\Omega^1_{\mathbb{P}^4} \otimes E(-3a-11))^{-1} \cdot c(\Omega^2_{\mathbb{P}^4}(-2a-4)).$$

Hence, by the Chern class formulas of the tensor product, exterior product and symmetric product in Examples 4, 6 and 7, respectively, we have:

$$\begin{aligned} c(S_2(E(-a-7))(-2a-4)) = 1 &+ (-12a-39)\mathbf{h} + (48a^2 + 312a + 522)\mathbf{h}^2 \\ &+ (-64a^3 - 624a^2 - 2088a - 2392)\mathbf{h}^3, \end{aligned}$$

$$\begin{aligned} c(\Omega^1_{\mathbb{P}^4} \otimes E(-3a-11)) = 1 &+ (-24a-78)\mathbf{h} + (252a^2 + 1638a + 2678)\mathbf{h}^2 \\ &+ (-1512a^3 - 14742a^2 - 48204a - 52856)\mathbf{h}^3 \\ &+ (5670a^4 + 73710a^3 + 361530a^2 + 792840a + 655905)\mathbf{h}^4, \end{aligned}$$

$$
\begin{aligned}
c(\Omega^2_{\mathbb{P}^4}(-2a-4)) = 1 &+ (-12a-39)\mathbf{h} + (60a^2+390a+635)\mathbf{h}^2 \\
&+ (-160a^3-1560a^2-5080a-5525)\mathbf{h}^3 \\
&+ (240a^4+3120a^3+15240a^2+33150a+27090)\mathbf{h}^4,
\end{aligned}
$$

then:

$$
c(\mathscr{I}_{Z_a}) = 1+(-8a^3-78a^2-226a-184)\mathbf{h}^3+(-54a^4-702a^3-3126a^2-5478a-2934)\mathbf{h}^4, \tag{4.32}
$$

for $a \geq 1$.
Finally, since $c_1(N^*_{\mathscr{F}}) = -2a-9$ and $\mathscr{F}_a$ is a codimension 2 holomorphic distributions family, then by Definition 23:

$$
\deg(\mathscr{F}_a) = -(-2a-9)-2-1 = 2a+6.
$$

□

Let us determine the numerical invariants of the singular scheme.

Proposition 17 (Numerical invariants of the singular locus)**.** *Let $Z_a = \mathrm{Sing}(\mathscr{F}_a)$ the singular scheme, for $a \geq 1$, then:*

1. $\deg(Z_a) = 4a^3+39a^2+113a+92.$
2. $p_a(Z_a) = 9a^4+107a^3+\frac{847}{2}a^2+\frac{1261}{2}a+260.$

Proof. By (4.32) we have $c_3(\mathscr{I}_{Z_a}) = -8a^3-78a^2-226a-184$, since Z_a is a pure codimension 3 projective curve then by Corollary 3 we have:

$$
-8a^3-78a^2-226a-184 = -2\cdot\deg(Z_a),
$$

hence $\deg(Z_a) = 4a^3+39a^2+113a+92$.
Since $Z_a \subset \mathbb{P}^4$ is a pure dimension 1 projective and $\mathrm{rank}(\mathscr{I}_{Z_a}) = 1$, then by Proposition 9 and by Hirzebruch-Riemann-Roch Theorem 14 we have:

$$
\begin{aligned}
\chi(\mathscr{I}_{Z_a}) &= \int_{\mathbb{P}^4} \left(\mathrm{ch}(\mathscr{I}_{Z_a})\cdot \mathrm{td}(\mathbb{P}^4)\right)_4 \\
&= 1+\frac{5}{4}c_3(\mathscr{I}_{Z_a}) - \frac{1}{6}c_4(\mathscr{I}_{Z_a}) \\
&= 1+\frac{5}{4}(-8a^3-78a^2-226a-184) - \frac{1}{6}(-54a^4-702a^3-3126a^2-5478a-2934) \\
&= 9a^4+107a^3+\frac{847}{2}a^2+\frac{1261}{2}a+260.
\end{aligned}
$$

Now, consider the exact sequence:

$$0 \to \mathscr{I}_{Z_a} \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{Z_a} \to 0. \tag{4.33}$$

Hence:

$$\chi(\mathcal{O}_{Z_a}) = \chi(\mathcal{O}_{\mathbb{P}^4}) - \chi(\mathscr{I}_{Z_a}).$$

Then,

$$\begin{aligned} p_a(Z_a) &= 1 - \chi(\mathcal{O}_{Z_a}) \\ &= 1 - \chi(\mathcal{O}_{\mathbb{P}^4}) + \chi(\mathscr{I}_{Z_a}) \\ &= \chi(\mathscr{I}_{Z_a}) \\ &= 9a^4 + 107a^3 + \frac{847}{2}a^2 + \frac{1261}{2}a + 260. \end{aligned}$$

□

Proposition 18. *The singular scheme* $Z_a = \mathrm{Sing}(\mathscr{F}_a)$ *is reduced and irreducible.*

Proof. We claim that Z_a is connected. Let $\varphi : E(-a-7) \to \Omega^1_{\mathbb{P}^4}$ be the map that induces the distribution (4.30). Then, the Eagon-Northcott resolution associated with the dual map $\varphi^\vee : T\mathbb{P}^4 \to E(a+2)$ is:

$$0 \to S_2(E(-a-7))(-2a-4) \to \bigwedge^3 T\mathbb{P}^4 \otimes E(-3a-16) \to \bigwedge^2 T\mathbb{P}^4(-2a-9) \to \mathscr{I}_{Z_a} \to 0.$$

And since: $\bigwedge^3 T\mathbb{P}^4 = \Omega^1_{\mathbb{P}^4}(5)$ and $\bigwedge^2 T\mathbb{P}^4 = \Omega^2_{\mathbb{P}^4}(5)$, then replacing, we have:

$$0 \to S_2(E(-a-7))(-2a-4) \to \Omega^1_{\mathbb{P}^4} \otimes E(-3a-11) \to \Omega^2_{\mathbb{P}^4}(-2a-4) \xrightarrow{\alpha} \mathscr{I}_{Z_a} \to 0. \tag{4.34}$$

Breaking it down into the short exact sequences

$$0 \to S_2(E(-a-7))(-2a-4) \to \Omega^1_{\mathbb{P}^4} \otimes E(-3a-11) \to K \to 0, \tag{4.35}$$

and

$$0 \to K \to \Omega^2_{\mathbb{P}^4}(-2a-4) \to \mathscr{I}_{Z_a} \to 0. \tag{4.36}$$

where $K = \mathrm{Ker}\,\alpha$.

From exact sequence 4.35 passing to cohomology

$$\cdots \to H^i(S_2(E(-a-7))(-2a-4)) \to H^i(\Omega^1_{\mathbb{P}^4} \otimes E(-3a-11)) \to H^i(K) \to \cdots.$$

By Lemma 2 and since $a \geq 1$ we have

$$H^2(K) \simeq H^3(S_2(E(-a-7))(-2a-4)).$$

By decomposition of tensor product:

$$E(-a-7) \otimes E(-a-7) \simeq S_2(E(-a-7)) \oplus \bigwedge^2 E(-a-7),$$

i.e,

$$E \otimes E(-2a-14) \simeq S_2(E(-a-7)) \oplus \mathcal{O}_{\mathbb{P}^4}(-2a-9),$$

twisting by $\mathcal{O}_{\mathbb{P}^4}(-2a-4)$ we have:

$$E \otimes E(-4a-18) \simeq S_2(E(-a-7))(-2a-4) \oplus \mathcal{O}_{\mathbb{P}^4}(-4a-13),$$

so

$$h^i(E \otimes E(-4a-18)) = h^i(S_2(E(-a-7))(-2a-4)) + h^i(\mathcal{O}_{\mathbb{P}^4}(-4a-13)), \quad i = 0, ..., 4.$$

Hence, since $a \geq 1$ then using Lemma 4 and Bott's formula we have

$$h^3(S_2(E(-a-7))(-2a-4)) = h^2(K) = 0.$$

From exact sequence 4.36 passing to cohomology

$$\cdots \to H^i(\Omega^2_{\mathbb{P}^4}(-2a-4)) \to H^i(\mathscr{I}_{Z_a}) \to H^{i+1}(K) \to H^{i+1}(\Omega^2_{\mathbb{P}^4}(-2a-4)) \to \cdots.$$

by Bott's formula

$$H^2(K) \simeq H^1(\mathscr{I}_{Z_a})$$

So $h^1(\mathscr{I}_{Z_a}) = 0$.

From exact sequence

$$0 \to \mathscr{I}_{Z_a} \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{Z_a} \to 0$$

passing to cohomology, we have:

$$0 \to H^0(\mathscr{I}_{Z_a}) \to H^0(\mathcal{O}_{\mathbb{P}^4}) \to H^0(\mathcal{O}_{Z_a}) \to H^1(\mathscr{I}_{Z_a}) \to \cdots$$

since $h^i(\mathscr{I}_{Z_a}) = 0$ for $i = 0, 1$ hence $h^0(\mathcal{O}_{\mathbb{P}^4}) = h^0(\mathcal{O}_{Z_a}) = 1$. Therefore Z_a is connected.
Since Z_a is a smooth scheme then is a reduced scheme.
To conclude, since Z_a is smooth and connected then is a irreducible scheme.

□

4.3.1 Rao Module dimension of Singular scheme:

Let $\mathscr{F}_a$ is a Horrocks-Mumford distribution

$$\mathscr{F}_a : 0 \to E(-a-7) \xrightarrow{\varphi} \Omega^1_{\mathbb{P}^4} \to \mathcal{Q}_{\mathscr{F}_a} \to 0.$$

Let us consider the Eagon-Northcott complex associated to the morphism $\varphi^\vee : T\mathbb{P}^4 \to E(a+2)$

$$0 \to S_2(E(-a-7))(-2a-4) \to \Omega^1_{\mathbb{P}^4} \otimes E(-3a-11) \to \Omega^2_{\mathbb{P}^4}(-2a-4) \to \mathscr{I}_{Z_a} \to 0.$$

Twist by $\mathcal{O}_{\mathbb{P}^4}(q)$ and break it down into the short exact sequences

$$0 \to S_2(E(-a-7))(q-2a-4) \to \Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11) \to \Omega^2_{\mathbb{P}^4}(q-2a-4) \to \mathscr{I}_{Z_a}(q) \to 0. \tag{4.37}$$

In order to calculate $h^1(\mathscr{I}_{Z_a}(q))$ in 4.37, for all $q \in \mathbb{Z}$ and $a \geq 1$, we have the following Lemmas:

Lemma 11.

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-18)) \neq 0$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \neq 2a+4\} \cap \{q \leq 3a+2 \quad or \quad q \geq 3a+9\} \cap \{4a+2 < q < 4a+10\}.$$

Proof. In fact, by breaking complex 4.37 in the exact sequences:

$$0 \to S_2(E(-a-7))(q-2a-4) \to \Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11) \to K(q) \to 0, \tag{4.38}$$

$$0 \to K(q) \to \Omega^2_{\mathbb{P}^4}(q-2a-4) \to \mathscr{I}_{Z_a}(q) \to 0. \tag{4.39}$$

with exact long sequences of cohomology, respectively:

$$\begin{aligned}
0 &\to H^0(S_2(E(-a-7))(q-2a-4)) \to H^0(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to H^0(K(q)) \to \\
&\to H^1(S_2(E(-a-7))(q-2a-4)) \to H^1(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to H^1(K(q)) \to \\
&\to H^2(S_2(E(-a-7))(q-2a-4)) \to H^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to H^2(K(q)) \to \\
&\to H^3(S_2(E(-a-7))(q-2a-4)) \to H^3(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to H^3(K(q)) \to \\
&\to H^4(S_2(E(-a-7))(q-2a-4)) \to H^4(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to H^4(K(q)) \to 0
\end{aligned}$$

and

$$0 \to H^0(K(q)) \to H^0(\Omega^2_{\mathbb{P}^4}(q-2a-4)) \to H^0(\mathscr{I}_{Z_a}(q)) \to$$
$$\to H^1(K(q)) \to H^1(\Omega^2_{\mathbb{P}^4}(q-2a-4)) \to H^1(\mathscr{I}_{Z_a}(q)) \to$$
$$\to H^2(K(q)) \to H^2(\Omega^2_{\mathbb{P}^4}(q-2a-4)) \to H^2(\mathscr{I}_{Z_a}(q)) \to$$
$$\to H^3(K(q)) \to H^3(\Omega^2_{\mathbb{P}^4}(q-2a-4)) \to H^3(\mathscr{I}_{Z_a}(q)) \to$$
$$\to H^4(K(q)) \to H^4(\Omega^2_{\mathbb{P}^4}(q-2a-4)) \to H^4(\mathscr{I}_{Z_a}(q)) \to 0$$

Let us study the long exact sequence of cohomology of sequence (4.39):

$$\cdots \to H^1(\Omega^2_{\mathbb{P}^4}(q-2a-4)) \to H^1(\mathscr{I}_{Z_a}(q)) \to H^2(K(q)) \to H^2(\Omega^2_{\mathbb{P}^4}(q-2a-4)) \to \cdots,$$

and by Bott's formula: $h^1(\Omega^2_{\mathbb{P}^4}(q-2a-4)) = 0 = h^2(\Omega^2_{\mathbb{P}^4}(q-2a-4))$ for $q \neq 2a+4$, then

$$h^1(\mathscr{I}_{Z_a}(q)) = h^2(K(q)) \quad \text{for} \quad q \neq 2a+4 \quad \text{and} \quad a \geq 1. \tag{4.40}$$

Now, from

$$\cdots \to H^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to H^2(K(q)) \to$$
$$\to H^3(S_2(E(-a-7))(q-2a-4)) \to H^3(\Omega^3_{\mathbb{P}^4} \otimes E(q-3a-11)) \to \cdots$$

by Lemma 2 we have:

$$h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) = 0 = h^3(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11))$$

for all $q \in \mathbb{Z}$ such that

$$\{q \leq 3a+6 \quad \text{or} \quad q \geq 3a+9\} \cap \{q \leq 3a+2 \quad \text{or} \quad q \geq 3a+7\}$$

i.e., for all $q \in \mathbb{Z}$ such that $\{q \leq 3a+2 \quad \text{or} \quad q \geq 3a+9\}$ hence

$$h^2(K(q)) = h^3(S_2(E(-a-7))(q-2a-4))$$

for all $q \in \mathbb{Z}$ such that $\{q \leq 3a+2 \quad \text{or} \quad q \geq 3a+9\}$.
Thus

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(S_2(E(-a-7))(q-2a-4))$$

for all $q \in \mathbb{Z}$ such that $\{q \neq 2a+4\} \cap \{q \leq 3a+2 \quad \text{or} \quad q \geq 3a+9\}$.
Now, by decomposing the tensor product, we have:

$$E(-a-7) \otimes E(-a-7) \simeq S_2(E(-a-7)) \oplus \bigwedge^2 E(-a-7),$$

i.e,

$$E \otimes E(-2a-14) \simeq S_2(E(-a-7)) \oplus \mathcal{O}_{\mathbb{P}^4}(-2a-9),$$

twisting by $\mathcal{O}_{\mathbb{P}^4}(q-2a-4)$ we have:

$$E \otimes E(q-4a-18) \simeq S_2(E(-a-7))(q-2a-4) \oplus \mathcal{O}_{\mathbb{P}^4}(q-4a-13),$$

so

$$h^i(E\otimes E(q-4a-18)) = h^i(S_2(E(-a-7))(q-2a-4)) + h^i(\mathcal{O}_{\mathbb{P}^4}(q-4a-13)), \quad i=0,...,4.$$

Now, by Bott's formula (1.9) we have that $h^3(\mathcal{O}_{\mathbb{P}^4}(q-4a-13)) = 0$ thus:

$$h^3(E \otimes E(q-4a-18)) = h^3(S_2(E(-a-7))(q-2a-4)),$$

So

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-18))$$

for all $q \in \mathbb{Z}$ such that $\{q \neq 2a+4\} \cap \{q \leq 3a+2 \ \text{ or } \ q \geq 3a+9\}$.
By table 2, $h^3(E \otimes E(q-4a-18) \neq 0$ for all $q \in \mathbb{Z}$ such that $4a+2 < q < 4a+10$, thus

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-18)) \neq 0$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \neq 2a+4\} \cap \{q \leq 3a+2 \ \text{ or } \ q \geq 3a+9\} \cap \{4a+2 < q < 4a+10\}.$$

□

Lemma 12.

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-18)) = 0,$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \neq 2a+4\} \cap \{q \leq 3a+2 \ \textit{ or } \ q \geq 3a+9\} \cap \{q \leq 4a+2 \ \textit{ or } \ q \geq 4a+10\}.$$

Proof. In fact, by table 2 we have $h^3(E \otimes E(q-4a-18)) = 0$ for $q \in \mathbb{Z}$ such that $\{q \leq 4a+2 \ \text{ or } \ q \geq 4a+10\}$ then

$$h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-18)) = 0,$$

for all $q \in \mathbb{Z}$ such that $\{q \neq 2a+4\} \cap \{q \leq 3a+2 \ \text{ or } \ q \geq 3a+9\} \cap \{q \leq 4a+2 \ \text{ or } \ q \geq 4a+10\}$.

□

Lemma 13.

$$h^1(\mathscr{I}_{Z_a}(q)) = h^1(\Omega^2_{\mathbb{P}^4}(q-2a-4)) = 0,$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \leq 4a+2 \ \textit{ or } \ q \geq 4a+10\} \cap \{q \leq 3a+6 \ \textit{ or } \ q \geq 3a+13\}.$$

Proof. In fact, from

$$\cdots \to H^1(K(q)) \to H^1(\Omega^2_{\mathbb{P}^4}(q-2a-4)) \to H^1(\mathscr{I}_{Z_a}(q)) \to H^2(K(q)) \to \cdots$$

we have:

$$h^1(\mathscr{I}_{Z_a}(q)) = h^1(\Omega^2_{\mathbb{P}^4}(q-2a-4))$$

if and only if $h^1(K(q)) = 0 = h^2(K(q))$.
From

$$\cdots \to H^1(S_2(E(-a-7))(q-2a-4)) \to H^1(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to H^1(K(q)) \to$$
$$\to H^2(S_2(E(-a-7))(q-2a-4)) \to H^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to H^2(K(q)) \to$$
$$\to H^3(S_2(E(-a-7))(q-2a-4)) \to H^3(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to H^3(K(q)) \to \cdots$$

we have

$$h^1(K(q)) = 0 = h^2(K(q))$$

if and only if $h^i(S_2(E(-a-7))(q-2a-4)) = 0$ for all $i = 1,2,3$ and
$h^i(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) = 0$ for all $i = 1,2$.
By table 2 we have that $h^i(S_2(E(-a-7))(q-2a-4)) = h^i(E \otimes E(q-4a-18)) = 0$ for all $i = 1,2,3$ and $q \in \mathbb{Z}$ such that $\{q \leq 4a+2 \;\text{ or }\; q \geq 4a+10\}$.
By Lemma 2 we have that $h^i(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) = 0$ for all $i = 1,2$ and $q \in \mathbb{Z}$ such that $\{q \leq 3a+8 \;\text{ or }\; q \geq 3a+13\} \cap \{q \leq 3a+6 \;\text{ or }\; q \geq 3a+9\}$, i.e., $q \in \mathbb{Z}$ such that $\{q \leq 3a+6 \;\text{ or }\; q \geq 3a+13\}$.
So

$$h^1(\mathscr{I}_{Z_a}(q)) = h^1(\Omega^2_{\mathbb{P}^4}(q-2a-4))$$

for all $q \in \mathbb{Z}$ such that $\{q \leq 4a+2 \;\text{ or }\; q \geq 4a+10\} \cap \{q \leq 3a+6 \;\text{ or }\; q \geq 3a+13\}$.
By Bott's formula, $h^1(\Omega^2_{\mathbb{P}^4}(q-2a-4)) = 0$, therefore

$$h^1(\mathscr{I}_{Z_a}(q)) = 0$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \leq 4a+2 \;\text{ or }\; q \geq 4a+10\} \cap \{q \leq 3a+6 \;\text{ or }\; q \geq 3a+13\}.$$

□

Lemma 14.

$$h^1(\mathscr{I}_{Z_a}(q)) = 0$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\Big\{\{q \neq 2a+4\} \cap \{q \leq 3a+2 \;\textit{ or }\; q \geq 3a+9\} \cap \{q \leq 4a+2 \;\textit{ or }\; q \geq 4a+10\}\Big\}$$
$$\cup \Big\{\{q \leq 4a+2 \;\textit{ or }\; q \geq 4a+10\} \cap \{q \leq 3a+6 \;\textit{ or }\; q \geq 3a+13\}\Big\}.$$

Proof. Putting together Lemmas 12 and 13, we have:

$$h^1(\mathscr{I}_{Z_a}(q)) = 0$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\big\{\{q \neq 2a+4\} \cap \{q \leq 3a+2 \text{ or } q \geq 3a+9\} \cap \{q \leq 4a+2 \text{ or } q \geq 4a+10\}\big\}$$
$$\cup \big\{q \leq 4a+2 \text{ or } q \geq 4a+10\} \cap \{q \leq 3a+6 \text{ or } q \geq 3a+13\}\big\}.$$

□

Lemma 15.

$$h^1(\mathscr{I}_{Z_a}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) + h^3(E \otimes E(q-4a-18))$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \neq 2a+4\} \cap \{q \leq 3a+2 \text{ or } q \geq 3a+7\} \cap \{q \leq 4a+7 \text{ or } q \geq 4a+14\}.$$

Proof. By equation 4.40 we have $h^1(\mathscr{I}_{Z_a}(q)) = h^2(K(q))$ for $q \neq 2a+4$.
Now, from

$$\cdots \to H^2(S_2(E(-a-7))(q-2a-4)) \to H^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to H^2(K(q)) \to$$
$$\to H^3(S_2(E(-a-7))(q-2a-4)) \to H^3(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to \cdots$$

We note that:

$$h^2(K(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) + h^3(S_2(E(-a-7)(q-2a-4)))$$

if and only if $h^2(S_2(E(-a-7))(q-2a-4)) = 0$ and $h^3(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) = 0$.
On the one hand, by table 2 we have

$$h^2(S_2(E(-a-7)(q-2a-4))) = h^2(E \otimes E(q-4a-18)) = 0$$

if and only if $q \leq 4a+7$ or $q \geq 4a+14$.
On the other hand, by Lemma 2 we have

$$h^3(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) = 0$$

if and only if $q \leq 3a+2$ or $q \geq 3a+7$.
From here, putting these last equations together, we have:

$$h^1(\mathscr{I}_{Z_a}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) + h^3(S_2(E(-a-7)(q-2a-4)))$$

for all $q \in \mathbb{Z}$ and $a \geq 1$ such that

$$\{q \neq 2a+4\} \cap \{q \leq 3a+2 \text{ or } q \geq 3a+7\} \cap \{q \leq 4a+7 \text{ or } q \geq 4a+14\}.$$

□

Lemma 16.

$$h^1(\mathscr{I}_{Z_a}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)),$$

for all $q \in \mathbb{Z}$ *and* $a \geq 1$ *such that*

$$\{q \neq 2a+4\} \cap \{q \leq 4a+2 \quad or \quad q \geq 4a+14\}.$$

Proof. By equation 4.40 we have $h^1(\mathscr{I}_{Z_a}(q)) = h^2(K(q))$ for $q \neq 2a+4$.
Now, from

$$\cdots \to H^2(S_2(E(-a-7))(q-2a-4)) \to H^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11)) \to$$
$$\to H^2(K(q)) \to H^3(S_2(E(-a-7))(q-2a-4)) \to \cdots$$

We note that:

$$h^2(K(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11))$$

if and only if $h^i(S_2(E(-a-7)(q-2a-4))) = h^i(E \otimes E(q-4a-18)) = 0$ for $i = 2, 3$.
By table 2 we have

$$h^2(E \otimes E(q-4a-18)) = 0 = h^3(E \otimes E(q-4a-18))$$

if and only if $q \leq 4a+2$ or $q \geq 4a+14$.
Hence, $h^1(\mathscr{I}_{Z_a}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11))$ for all $q \in \mathbb{Z}$ and $a \geq 1$ such that $\{q \neq 2a+4\} \cap \{q \leq 4a+2 \quad \text{or} \quad q \geq 4a+14\}$.

□

We state our second main result that describes the geometry of the singular scheme whose conormal sheaf is Horrocks-Mumford.

Theorem 30. *Let* $\mathscr{F}_a$ *is a codimension* 2 *holomorphic distributions (4.30) on* $\mathbb{P}^4$*. Then:*

1. $\dim_{\mathbb{C}} R_{\mathscr{F}_1} = h^1(\mathscr{I}_{Z_1}(7)) + h^1(\mathscr{I}_{Z_1}(8)) + h^1(\mathscr{I}_{Z_1}(9)) + 184.$
2. $\dim_{\mathbb{C}} R_{\mathscr{F}_2} = h^1(\mathscr{I}_{Z_2}(11)) + h^1(\mathscr{I}_{Z_2}(12)) + 284.$
3. $\dim_{\mathbb{C}} R_{\mathscr{F}_3} = h^1(\mathscr{I}_{Z_3}(15)) + 369.$
4. $\dim_{\mathbb{C}} R_{\mathscr{F}_a} = 401,\ \forall a \geq 4.$

Proof. For $a = 1$.
By Lemma 14 we have $h^1(\mathscr{I}_{Z_1}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 6$ or $q \geq 14$.
By Lemma 15 we have $h^1(\mathscr{I}_{Z_1}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-14)) + h^3(E \otimes E(q-22))$ for $q = 10, 11$.
Hence

$$h^1(\mathscr{I}_{Z_1}(10)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) + h^3(E \otimes E(-12)).$$

Now, by Lemma 2 we have $h^i(\Omega^1_{\mathbb{P}} \otimes E(-4)) = 0$ for all $i = 0, 1, 3, 4$, so

$$\chi(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-4))$$

since

$$c(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) = 1 - 22\mathbf{h} + 228\mathbf{h}^2 - 1434\mathbf{h}^3 + 5955\mathbf{h}^4$$

then by Riemann-Roch Theorem we have $\chi(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) = 10$, so

$$h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) = 10.$$

By table 2 we have $h^3(E \otimes E(-12)) = 85$. Thus:

$$\begin{aligned} h^1(\mathscr{I}_{Z_1}(10)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) + h^3(E \otimes E(-12)) \\ &= 10 + 85 \\ &= 95. \end{aligned}$$

Similarly, by Lemma 2 we have $h^i(\Omega^1_{\mathbb{P}} \otimes E(-3)) = 0$ for all $i = 0, 1, 3, 4$, so

$$\chi(\Omega^1_{\mathbb{P}^4} \otimes E(-3)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-3))$$

since

$$c(\Omega^1_{\mathbb{P}^4} \otimes E(-3)) = 1 - 14\mathbf{h} + 102\mathbf{h}^2 - 472\mathbf{h}^3 + 1505\mathbf{h}^4$$

then by Riemann-Roch Theorem we have $\chi(\Omega^1_{\mathbb{P}} \otimes E(-3)) = 5$, so

$$h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-3)) = 5.$$

By table 2 we have $h^3(E \otimes E(-11)) = 55$. Thus:

$$\begin{aligned} h^1(\mathscr{I}_{Z_1}(11)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-3)) + h^3(E \otimes E(-11)) \\ &= 5 + 55 \\ &= 60. \end{aligned}$$

By Lemma 11 we have $h^1(\mathscr{I}_{Z_1}(q)) = h^3(E \otimes E(q-22)) \neq 0$ for all $q \in \mathbb{Z}$ such that $12 \leq q < 14$. So, by table 2 we have:

$$\begin{aligned} h^1(\mathscr{I}_{Z_1}(12)) &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_1}(13)) &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned}
\dim_{\mathbb{C}} R_{\mathscr{F}_1} &= \dim \bigoplus_{l\in\mathbb{Z}} H^1(\mathscr{I}_{Z_1}(l)) \\
&= \sum_{l=7}^{13} h^1(\mathscr{I}_{Z_1}(l)) \\
&= h^1(\mathscr{I}_{Z_1}(7)) + h^1(\mathscr{I}_{Z_1}(8)) + h^1(\mathscr{I}_{Z_1}(9)) + 95 + 60 + 24 + 5 \\
&= h^1(\mathscr{I}_{Z_1}(7)) + h^1(\mathscr{I}_{Z_1}(8)) + h^1(\mathscr{I}_{Z_1}(9)) + 184.
\end{aligned}$$

For $a = 2$.
By Lemma 14 we have $h^1(\mathscr{I}_{Z_2}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 10$ or $q \geq 18$.
By Lemma 15 we have $h^1(\mathscr{I}_{Z_2}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-17)) + h^3(E \otimes E(q-26))$ for $q = 13, 14$.
Hence

$$\begin{aligned}
h^1(\mathscr{I}_{Z_2}(13)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) + h^3(E \otimes E(-13)) \\
&= 10 + 100 \\
&= 110.
\end{aligned}$$

similarly,

$$\begin{aligned}
h^1(\mathscr{I}_{Z_2}(14)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-3)) + h^3(E \otimes E(-12)) \\
&= 5 + 85 \\
&= 90.
\end{aligned}$$

By Lemma 11 we have $h^1(\mathscr{I}_{Z_2}(q)) = h^3(E \otimes E(q-26)) \neq 0$ for all $q \in \mathbb{Z}$ such that $15 \leq q < 18$. So, by table 2 we have:

$$\begin{aligned}
h^1(\mathscr{I}_{Z_2}(15)) &= h^3(E \otimes E(-11)) \\
&= 55.
\end{aligned}$$

$$\begin{aligned}
h^1(\mathscr{I}_{Z_2}(16)) &= h^3(E \otimes E(-10)) \\
&= 24.
\end{aligned}$$

$$\begin{aligned}
h^1(\mathscr{I}_{Z_2}(17)) &= h^3(E \otimes E(-9)) \\
&= 5.
\end{aligned}$$

Therefore

$$\begin{aligned}\dim_{\mathbb{C}} R_{\mathscr{F}_2} &= \dim \bigoplus_{l\in\mathbb{Z}} H^1(\mathscr{I}_{Z_2}(l)) \\ &= \sum_{l=11}^{17} h^1(\mathscr{I}_{Z_2}(l)) \\ &= h^1(\mathscr{I}_{Z_2}(11)) + h^1(\mathscr{I}_{Z_2}(12)) + 110 + 90 + 55 + 24 + 5 \\ &= h^1(\mathscr{I}_{Z_2}(9)) + h^1(\mathscr{I}_{Z_2}(10)) + 284.\end{aligned}$$

For $a = 3$.
By Lemma 14 we have $h^1(\mathscr{I}_{Z_3}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 14$ or $q \geq 22$.
By Lemma 15 we have $h^1(\mathscr{I}_{Z_3}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-20)) + h^3(E \otimes E(q-30))$ for $q = 16, 17$.
Hence

$$\begin{aligned}h^1(\mathscr{I}_{Z_3}(16)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) + h^3(E \otimes E(-14)) \\ &= 10 + 85 \\ &= 95.\end{aligned}$$

similarly,

$$\begin{aligned}h^1(\mathscr{I}_{Z_3}(17)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-3)) + h^3(E \otimes E(-13)) \\ &= 5 + 100 \\ &= 110.\end{aligned}$$

By Lemma 11 we have $h^1(\mathscr{I}_{Z_3}(q)) = h^3(E \otimes E(q-30)) \neq 0$ for all $q \in \mathbb{Z}$ such that $18 \leq q < 22$. So, by table 2 we have:

$$\begin{aligned}h^1(\mathscr{I}_{Z_3}(18)) &= h^3(E \otimes E(-12)) \\ &= 85.\end{aligned}$$

$$\begin{aligned}h^1(\mathscr{I}_{Z_3}(19)) &= h^3(E \otimes E(-11)) \\ &= 55.\end{aligned}$$

$$\begin{aligned}h^1(\mathscr{I}_{Z_3}(20)) &= h^3(E \otimes E(-10)) \\ &= 24.\end{aligned}$$

$$\begin{aligned}h^1(\mathscr{I}_{Z_3}(21)) &= h^3(E \otimes E(-9)) \\ &= 5.\end{aligned}$$

Therefore

$$\begin{aligned}\dim_{\mathbb{C}} R_{\mathscr{F}_3} &= \dim \bigoplus_{l\in\mathbb{Z}} H^1(\mathscr{I}_{Z_3}(l))\\ &= \sum_{l=15}^{21} h^1(\mathscr{I}_{Z_3}(l))\\ &= h^1(\mathscr{I}_{Z_3}(15)) + 95 + 105 + 85 + 55 + 24 + 5\\ &= h^1(\mathscr{I}_{Z_3}(13)) + 369.\end{aligned}$$

For $a = 4$.

By Lemma 14 we have $h^1(\mathscr{I}_{Z_4}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 18$ or $q \geq 26$.

By Lemma 15 we have $h^1(\mathscr{I}_{Z_4}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-23)) + h^3(E \otimes E(q-34))$ for $q = 19, 20$. Hence

$$\begin{aligned}h^1(\mathscr{I}_{Z_4}(19)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) + h^3(E \otimes E(-15))\\ &= 10 + 32\\ &= 42.\end{aligned}$$

similarly,

$$\begin{aligned}h^1(\mathscr{I}_{Z_4}(20)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-3)) + h^3(E \otimes E(-14))\\ &= 5 + 85\\ &= 90.\end{aligned}$$

By Lemma 11 we have $h^1(\mathscr{I}_{Z_4}(q)) = h^3(E \otimes E(q-34)) \neq 0$ for all $q \in \mathbb{Z}$ such that $21 \leq q < 26$. So, by table 2 we have:

$$\begin{aligned}h^1(\mathscr{I}_{Z_4}(21)) &= h^3(E \otimes E(-13))\\ &= 100.\end{aligned}$$

$$\begin{aligned}h^1(\mathscr{I}_{Z_4}(22)) &= h^3(E \otimes E(-12))\\ &= 85.\end{aligned}$$

$$\begin{aligned}h^1(\mathscr{I}_{Z_4}(23)) &= h^3(E \otimes E(-11))\\ &= 55.\end{aligned}$$

$$\begin{aligned}h^1(\mathscr{I}_{Z_4}(24)) &= h^3(E \otimes E(-10))\\ &= 24.\end{aligned}$$

$$h^1(\mathscr{I}_{Z_4}(25)) = h^3(E \otimes E(-9))$$
$$= 5.$$

Therefore

$$\dim_{\mathbb{C}} R_{\mathscr{F}_4} = \dim \bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{I}_{Z_4}(l))$$
$$= \sum_{l=19}^{25} h^1(\mathscr{I}_{Z_4}(l))$$
$$= 42 + 90 + 100 + 85 + 55 + 24 + 5$$
$$= 401.$$

For $a = 5$.
By Lemma 14 we have $h^1(\mathscr{I}_{Z_5}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 21$ or $q \geq 30$.
By Lemma 15 we have $h^1(\mathscr{I}_{Z_5}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-26)) + h^3(E \otimes E(q-38)))$ for $q = 22, 23$. Hence

$$h^1(\mathscr{I}_{Z_5}(22)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) + h^3(E \otimes E(-16)))$$
$$= 10 + 0$$
$$= 10.$$

and

$$h^1(\mathscr{I}_{Z_5}(23)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-3)) + h^3(E \otimes E(-15)))$$
$$= 5 + 32$$
$$= 37.$$

By Lemma 5 we have $h^1(\mathscr{I}_{Z_4}(q)) = h^3(E \otimes E(q-38)) \neq 0$ for all $q \in \mathbb{Z}$ such that $24 \leq q < 30$. So, by table 2 we have:

$$h^1(\mathscr{I}_{Z_5}(24)) = h^3(E \otimes E(-14))$$
$$= 85.$$

$$h^1(\mathscr{I}_{Z_5}(25)) = h^3(E \otimes E(-13))$$
$$= 100.$$

$$h^1(\mathscr{I}_{Z_5}(26)) = h^3(E \otimes E(-12))$$
$$= 85.$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(27)) &= h^3(E \otimes E(-11)) \\ &= 55. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(28)) &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_5}(29)) &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned} \dim_{\mathbb{C}} R_{\mathscr{F}_5} &= \dim \bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{I}_{Z_5}(l)) \\ &= \sum_{l=22}^{29} h^1(\mathscr{I}_{Z_5}(l)) \\ &= 10 + 37 + 85 + 100 + 85 + 55 + 24 + 5 \\ &= 401. \end{aligned}$$

For $a = 6$.
By Lemma 14 we have $h^1(\mathscr{I}_{Z_6}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 24$ or $q \geq 34$.
By Lemma 16 we have $h^1(\mathscr{I}_{Z_6}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-29))$ for $q = 25, 26$. Hence

$$\begin{aligned} h^1(\mathscr{I}_{Z_6}(25)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-4)) \\ &= 10. \end{aligned}$$

and

$$\begin{aligned} h^1(\mathscr{I}_{Z_6}(26)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-3)) \\ &= 5. \end{aligned}$$

By Lemma 11 we have $h^1(\mathscr{I}_{Z_4}(q)) = h^3(E \otimes E(q-42)) \neq 0$ for all $q \in \mathbb{Z}$ such that $27 \leq q < 34$. So, by table 2 we have:

$$\begin{aligned} h^1(\mathscr{I}_{Z_6}(27)) &= h^3(E \otimes E(-15)) \\ &= 32. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_6}(28)) &= h^3(E \otimes E(-14)) \\ &= 85. \end{aligned}$$

$$h^1(\mathscr{I}_{Z_6}(29)) = h^3(E \otimes E(-13))$$
$$= 100.$$

$$h^1(\mathscr{I}_{Z_6}(30)) = h^3(E \otimes E(-12))$$
$$= 85.$$

$$h^1(\mathscr{I}_{Z_6}(31)) = h^3(E \otimes E(-11))$$
$$= 55.$$

$$h^1(\mathscr{I}_{Z_6}(32)) = h^3(E \otimes E(-10))$$
$$= 24.$$

$$h^1(\mathscr{I}_{Z_6}(33)) = h^3(E \otimes E(-9))$$
$$= 5.$$

Therefore

$$\begin{aligned}\dim_{\mathbb{C}} R_{\mathscr{F}_6} &= \dim \bigoplus_{l\in\mathbb{Z}} H^1(\mathscr{I}_{Z_6}(l)) \\ &= \sum_{l=25}^{33} h^1(\mathscr{I}_{Z_6}(l)) \\ &= 10+5+32+85+100+85+55+24+5 \\ &= 401.\end{aligned}$$

For $a = 7$.
By Lemma 14 we have $h^1(\mathscr{I}_{Z_7}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 27$ or $q = 30$ or $q \geq 38$.
By Lemma 16 we have $h^1(\mathscr{I}_{Z_7}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-32))$ for $q = 28, 29$. Hence

$$h^1(\mathscr{I}_{Z_7}(28)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-4))$$
$$= 10.$$

and

$$h^1(\mathscr{I}_{Z_7}(29)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(-3))$$
$$= 5.$$

By Lemma 11 we have $h^1(\mathscr{I}_{Z_7}(q)) = h^3(E \otimes E(q-46)) \neq 0$ for all $q \in \mathbb{Z}$ such that $30 < q < 38$. So, by table 2 we have:

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(31)) &= h^3(E \otimes E(-15)) \\ &= 32. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(32)) &= h^3(E \otimes E(-14)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(33)) &= h^3(E \otimes E(-13)) \\ &= 100. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(34)) &= h^3(E \otimes E(-12)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(35)) &= h^3(E \otimes E(-11)) \\ &= 55. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(36)) &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_7}(37)) &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned} \dim_{\mathbb{C}} R_{\mathscr{F}_7} &= \dim \bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{I}_{Z_7}(l)) \\ &= \sum_{l=28}^{37} h^1(\mathscr{I}_{Z_7}(l)) \\ &= 10 + 5 + 0 + 32 + 85 + 100 + 85 + 55 + 24 + 5 \\ &= 401. \end{aligned}$$

For $a \geq 7$.
By Lemma 14 we have $h^1(\mathscr{I}_{Z_7}(q)) = 0$ for all $q \in \mathbb{Z}$ such that $q \leq 3a+6$ or $3a+9 \leq q \leq 4a+2$ or $q \geq 4a+10$.

By Lemma 16 we have $h^1(\mathscr{I}_{Z_7}(q)) = h^2(\Omega^1_{\mathbb{P}^4} \otimes E(q-3a-11))$ for all $q \in \mathbb{Z}$ such that $\{q \neq 2a+4\} \cap \{q \leq 4a+2 \ \text{ or } \ q \geq 4a+14\}$, in particular for $q = 3a+7, 3a+8$. Hence

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(3a+7)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(3a+7-3a-11)) \\ &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-4)) \\ &= 10. \end{aligned}$$

and

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(3a+8)) &= h^2(\Omega^1_{\mathbb{P}^4} \otimes E(3a+8-3a-11)) \\ &= h^2(\Omega^3_{\mathbb{P}^4} \otimes E(-3)) \\ &= 5. \end{aligned}$$

By Lemma 11 we have $h^1(\mathscr{I}_{Z_a}(q)) = h^3(E \otimes E(q-4a-18)) \neq 0$ for all $q \in \mathbb{Z}$ such that $4a+2 < q < 4a+10$. So, by table 2 we have:

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+3)) &= h^3(E \otimes E(4a+3-4a-18)) \\ &= h^3(E \otimes E(-15)) \\ &= 32. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+4)) &= h^3(E \otimes E(4a+4-4a-18)) \\ &= h^3(E \otimes E(-14)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+5)) &= h^3(E \otimes E(4a+5-4a-18)) \\ &= h^3(E \otimes E(-13)) \\ &= 100. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+6)) &= h^3(E \otimes E(4a+6-4a-18)) \\ &= h^3(E \otimes E(-12)) \\ &= 85. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+7)) &= h^3(E \otimes E(4a+7-4a-18)) \\ &= h^3(E \otimes E(-11)) \\ &= 55. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+8)) &= h^3(E \otimes E(4a+8-4a-18)) \\ &= h^3(E \otimes E(-10)) \\ &= 24. \end{aligned}$$

$$\begin{aligned} h^1(\mathscr{I}_{Z_a}(4a+9)) &= h^3(E \otimes E(4a+9-4a-18)) \\ &= h^3(E \otimes E(-9)) \\ &= 5. \end{aligned}$$

Therefore

$$\begin{aligned} \dim_{\mathbb{C}} R_{\mathscr{F}_a} &= \dim \bigoplus_{l \in \mathbb{Z}} H^1(\mathscr{I}_{Z_a}(l)) \\ &= \sum_{l=3a+7}^{4a+9} h^1(\mathscr{I}_{Z_a}(l)) \\ &= \sum_{l=3a+7}^{3a+8} h^1(\mathscr{I}_{Z_a}(l)) + 0 + \cdots + 0 + \sum_{l=4a+3}^{4a+9} h^1(\mathscr{I}_{Z_a}(l)) \\ &= 10 + 5 + 0 + \cdots + 0 + 32 + 85 + 100 + 85 + 55 + 24 + 5 \\ &= 401. \end{aligned}$$

□

4.3.2 First cohomology dimension:

Let $\mathscr{F}$ be a holomorphic distribution on $\mathbb{P}^4$. Consider the following graded module

$$M^o_{\mathscr{F}} := H^1_*(N^*_{\mathscr{F}}) = \bigoplus_{l \in \mathbb{Z}} H^1(N^*_{\mathscr{F}}(l));$$

called the *first cohomology module* of $\mathscr{F}$ as a sub-sheaf of the cotangent.
Since $N^*_{\mathscr{F}_a} = E(-a-7)$ then $H^1(N^*_{\mathscr{F}_a}(l)) = H^1(E(-a-7+l))$. By table 1 we have $h^1(N^*_{\mathscr{F}_a}(l)) = h^1(E(-a-7+l)) \neq 0$ if, and only if, $a+4 \leq l \leq a+7$. Hence:

$$\begin{aligned} \dim_{\mathbb{C}} M^o_{\mathscr{F}_a} &= \dim \bigoplus_{l \in \mathbb{Z}} H^1(E(-a-7+l)) \\ &= \sum_{l=a+4}^{a+7} h^1(E(-a-7+l)) \\ &= h^1(E(-3)) + h^1(E(-2)) + h^1(E(-1)) + h^1(E) \\ &= 5 + 10 + 10 + 2 \\ &= 27. \end{aligned}$$

Next, let us describe the cohomologies of the normal sheaf.

Proposition 19. *Let $\mathscr{F}_a$ be codimension 2 holomorphic distributions family (4.30) on $\mathbb{P}^4$. Then:*

1. $h^0(N^*_{\mathscr{F}_a}(q)) = 0$ *for* $q \leq a+6$.
2. $h^1(N^*_{\mathscr{F}_a}(q)) = 0$ *for* $q \leq a+3$ *or* $q \geq a+8$.
3. $h^2(N^*_{\mathscr{F}_a}(q)) = 0$ *for* $q \leq a+1$ *or* $q \geq a+3$.
4. $h^3(N^*_{\mathscr{F}_a}(q)) = 0$ *for* $q \leq a-4$ *or* $q \geq a+1$.
5. $h^4(N^*_{\mathscr{F}_a}(q)) = 0$ *for* $q \geq a-2$.

Proof. From the $\mathscr{F}_a$ distribution (4.30):

$$0 \to E(-a-7) \to \Omega^1_{\mathbb{P}^4} \to \mathcal{Q}_{\mathscr{F}_a} \to 0,$$

twisting by $\mathcal{O}_{\mathbb{P}^4}(q)$, we have:

$$0 \to E(q-a-7) \to \Omega^1_{\mathbb{P}^4}(q) \to \mathcal{Q}_{\mathscr{F}_a}(q) \to 0.$$

Hence $N^*_{\mathscr{F}_a}(q) = E(-a-7+q)$. Therefore $H^i(N^*_{\mathscr{F}_a}(q)) = H^i(E(-a-7+q))$, for all $i = 0, ..., 4$.
Since $h^0(N^*_{\mathscr{F}_a}(q)) = h^0(E(-a-7+q))$ then by table 1 we have $h^0(N^*_{\mathscr{F}_a}(q)) = 0$ if and only if $q \leq a+6$.
Similarly with the other cases.

□

In particular, for $a = 1$ we have:

Corollary 8. *Let $\mathscr{F}$ is a distribution*

$$\mathscr{F} : 0 \to E(-8) \xrightarrow{\varphi} \Omega^1_{\mathbb{P}^4} \to \mathcal{Q}_{\mathscr{F}} \to 0, \tag{4.41}$$

and $Z = \mathrm{Sing}(\mathscr{F})$. *Then*

1. *The Chern classes of the conormal sheaf are:*

$$c_1(N^*_{\mathscr{F}}) = -11 \ , \quad c_2(N^*_{\mathscr{F}}) = 35.$$

2. *The Chern classes of ideals sheaf of the singular scheme are:*

$$c_1(\mathscr{I}_Z) = 0 \ , \quad c_2(\mathscr{I}_Z) = 0 \ , \quad c_3(\mathscr{I}_Z) = -496 \ , \quad c_4(\mathscr{I}_Z) = -12294.$$

3. *The degree of distribution $\mathscr{F}$ is* $\deg(\mathscr{F}) = 3$.

4. $\deg(Z) = 248$.

5. $p_a(Z) = 1430$.

Corollary 9. *Let $\mathscr{F}$ is a codimension 2 holomorphic distribution (4.41) on $\mathbb{P}^4$. The singular scheme $Z = \mathrm{Sing}(\mathscr{F})$ is reduced and irreducible.*

Theorem 31. *Let $\mathscr{F}$ is a codimension 2 distribution (4.41) on $\mathbb{P}^4$. Then:*

1. $\dim_{\mathbb{C}} R_{\mathscr{F}_1} \geq 184$.

2. *The singular scheme $Z = \mathrm{Sing}(\mathscr{F})$ is not a arithmetically Buchsbaum nor arithmetically Cohen Macaulay curve.*

Corollary 10. *Let $\mathscr{F}$ is a codimension 2 distribution holomorphic (4.41) on $\mathbb{P}^4$. Then:*

1. $h^0(N^*_{\mathscr{F}_a}(q)) = 0$ *for* $q \leq 7$.

2. $h^1(N^*_{\mathscr{F}_a}(q)) = 0$ *for* $q \leq 4 \quad$ *or* $\quad q \geq 9$.

3. $h^2(N^*_{\mathscr{F}_a}(q)) = 0$ *for* $q \leq 2 \quad$ *or* $\quad q \geq 4$.

4. $h^3(N^*_{\mathscr{F}_a}(q)) = 0$ *for* $q \leq -3 \quad$ *or* $\quad q \geq 2$.

5. $h^4(N^*_{\mathscr{F}_a}(q)) = 0$ *for* $q \geq -1$.

5 ON THE NON-INTEGRABILITY OF DISTRIBUTIONS

This section is devoted to the study of the non-integrability of the Horrocks-Mumford distributions. The argument is based on the study of the Baum-Bott residue, in terms of the Grothendieck residue, and of the ampleness of the normal sheaf.
Let us denote by $\mathrm{Sing}_{k+1}(\mathscr{F})$ the subset of $\mathrm{Sing}(\mathscr{F})$ composed by analytic subsets of codimension $k+1$. It is called *the singular set of $\mathscr{F}$ with expected codimension.* In [9, Theorem 1.2], the authors determine Baum-Bott residues for $\mathscr{F}$ with respect to homogeneous symmetric polynomials of degree $k+1$ in terms of the Grothendieck residue of an one-dimensional foliation on a $(k+1)$-dimensional disc transversal to a $(k+1)$-codimensional component of the singular set of $\mathscr{F}$. More precisely:

Theorem 32 (M. Corrêa; F. Lourenço - 2019)**.** *Let $\mathscr{F}$ be a singular holomorphic foliation of codimension k on a compact complex manifold X such that* $\mathrm{codim}(\mathrm{Sing}(\mathscr{F})) \geq k+1$ *and Z be an irreducible component of* $\mathrm{Sing}_{k+1}(\mathscr{F})$*, the singular set of $\mathscr{F}$ with expected codimension. Then*

$$\mathrm{Res}(\mathscr{F}, \varphi; Z) = \mathrm{Res}_{\varphi}(\mathscr{F}\mid_{B_p}; p) \cdot [Z],$$

where $\mathrm{Res}_{\varphi}(\mathscr{F}\mid_{B_p}; p)$ *represents the Grothendieck residue at p of the one dimensional foliation $\mathscr{F}\mid_{B_p}$ on a $(k+1)-$dimensional transversal ball B_p.*

In order to show the non-integrability of the Horrocks-Mumford distributions, we have the following result:

Lemma 17. *Let $\mathscr{F}$ be a holomorphic foliation of codimension $k \geq 2$ on a complex manifold X, such that* $\mathrm{codim}(\mathrm{Sing}(\mathscr{F})) \geq k+1$*. If the conormal sheaf $N^*_{\mathscr{F}}$ is locally free and* $\det(N_{\mathscr{F}})$ *is ample, then* $\mathrm{Sing}_{k+1}(\mathscr{F})$ *can not be irreducible.*

Proof. Suppose by contradiction that $\mathrm{Sing}_{k+1}(\mathscr{F}) := Z$ is irreducible and take $p \in Z$ be a generic point, i.e., p is a point where Z is smooth. Since the conormal sheaf $N^*_{\mathscr{F}}$ is locally free then $\mathscr{F}$ is given by a locally decomposable holomorphic twisted and integrable k-form $\omega \in H^0(X, \Omega^k_X \otimes \det(N_{\mathscr{F}}))$. Take a neighborhood U of $p \in Z$ then there exist holomorphic 1-forms $\omega_1, ..., \omega_k \in H^0(U, \Omega^1_U)$ such that

$$\omega|_U = \omega_1 \wedge \cdots \wedge \omega_k$$

and

$$d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0, \quad \forall i = 1, ..., k.$$

Since $\text{codim}(\text{Sing}(\mathscr{F})) \geq 3$ then by Malgrange's Theorem, [29] and [30], there are $f_1, ..., f_k \in \mathcal{O}_n$ and $h \in \mathcal{O}_n^*$ such that

$$\omega = h \cdot df_1 \wedge \cdots \wedge df_k,$$

Hence $d\omega = dh \wedge df_1 \wedge \cdots \wedge df_k = \frac{dh}{h} \wedge \big(h.df_1 \wedge \cdots \wedge df_k\big) = \theta \wedge \omega$, where $\theta = \frac{dh}{h}$ is the trace of the Bott connection.

Consider B_p a ball centered at p, of dimension $k+1$ sufficiently small and transversal to Z in p. Then we can integrate the De Rham's class over an oriented $(k+1)$-sphere $L_p \subset B_p^*$ positively linked with $S(B_p)$. It follows from [3] and [9] that:

$$\text{Res}(\mathscr{F}, c_1^{k+1}, Z) = \text{Res}_{c_1^{k+1}}(\mathscr{F} \mid_{B_p}; p) \cdot [Z] = \frac{1}{(2\pi i)^{k+1}} \int_{L_p} \theta \wedge (d\theta)^k \cdot [Z],$$

is the Baum-Bott residue for $\mathscr{F}$ along Z with respect to c_1^{k+1}.

Since $h \in \mathcal{O}_n^*$ then $\theta = \frac{dh}{h}$ is a holomorphic 1-form, hence $\int_{L_p} \theta \wedge (d\theta)^k = 0$, so $\text{Res}_{c_1^{k+1}}(\mathscr{F} \mid_{B_p}; p) = 0$.

On the other hand, since $\mathscr{F}$ is a holomorphic foliation of codimension k then by Baum-Bott formula, [3], we have:

$$c_1^{k+1}(\det(N_{\mathscr{F}})) = \text{Res}_{c_1^{k+1}}(\mathscr{F} \mid_{B_p}; p) \cdot [Z],$$

where $[Z] \in H^{2k+2}(X, \mathbb{C})$ is the fundamental class of the irreducible component Z of $\text{Sing}_{k+1}(\mathscr{F})$, and the sum is done over all irreducible components of $\text{Sing}_{k+1}(\mathscr{F})$. Now, taking product both sides by $\mathbf{h}^{n-k-1}$, taking degree and since $\det(N_{\mathscr{F}})$ is ample sheaf, we have:

$$\int_X c_1^{k+1}(\det(N_{\mathscr{F}})) \cdot \mathbf{h}^{n-k-1} = \text{Res}_{c_1^{k+1}}(\mathscr{F} \mid_{B_p}; p) \cdot \int_X [Z] \cdot \mathbf{h}^{n-k-1},$$

thus:

$$0 < \deg(\det(N_{\mathscr{F}})) = \text{Res}_{c_1^{k+1}}(\mathscr{F} \mid_{B_p}; p) \cdot \deg(Z).$$

Hence $\text{Res}_{c_1^{k+1}}(\mathscr{F} \mid_{B_p}; p) \neq 0$. But this is a contradiction, since the ampleness of $\det(N_{\mathscr{F}})$ implies that the cohomology class $c_1^{k+1}(\det(N_{\mathscr{F}}))$ is non zero. □

With this in mind, we have the following result.

Theorem 33. *Let $\mathscr{F}_a$ be the Horrocks-Mumford distribution (4.18), then $\mathscr{F}_a$ is non-integrable, for $a \geq 1$.*

Proof. Suppose by contradiction that $\mathscr{F}_a$ is a codimension two holomorphic foliation and let $\text{Sing}_3(\mathscr{F}_a)$ is a irreducible component of codimension 3. By Proposition 14 and Theorem 17 we have that $\text{Sing}(\mathscr{F}_a)$ is a reduced, irreducible scheme of pure codimension 3, hence $\text{Sing}(\mathscr{F}_a) = \text{Sing}_3(\mathscr{F}_a)$. Since

$$c_1(\det(N_{\mathscr{F}_a})) = (2a+8) > 0,$$

for all $a \geq 1$, then $\det(N_{\mathscr{F}})$ is ample sheaf, hence by Lemma 17 the Singular scheme $\mathrm{Sing}(\mathscr{F}_a)$ is not irreducible, but this contradicts Proposition 14.

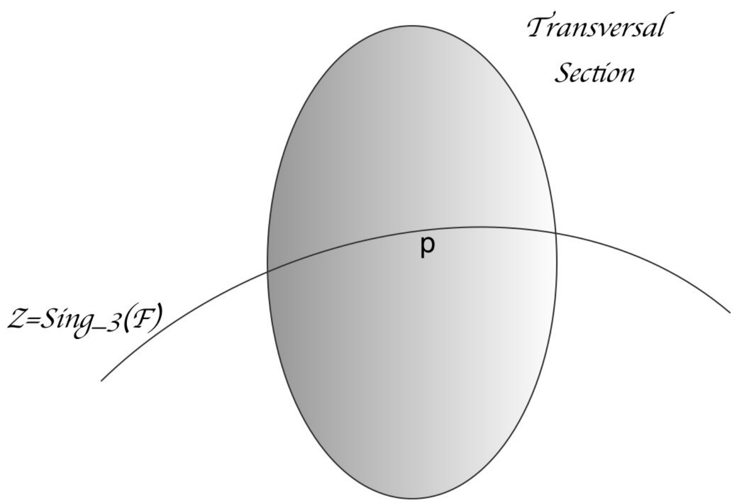


Figure 1 – Transversal section.

□

Similarly we have:

Theorem 34 (D)**.** *Let $\mathscr{F}_a$ be the Horrocks-Mumford distribution (4.30), then $\mathscr{F}_a$ is non-integrable, for $a \geq 1$.*

Proof. Suppose $\mathscr{F}_a$ is a codimension two holomorphic foliation. Since $c_1(\det(N_{\mathscr{F}_a})) = (2a+9) > 0$, for all $a \geq 1$, then by Lemma 17 we have that Z_a cannot be irreducible, for all $a \geq 1$, but it is a contradiction since $Z_a = \mathrm{Sing}_3(\mathscr{F}_a)$ is smooth, irreducible, pure codimension 3.

□

6 MODULI SPACES OF HORROCKS-MUMFORD DISTRIBUTIONS

Moduli spaces arose to give a solution to the classification problems, specifically in algebraic geometry. In this chapter we study the problem of classifying the Horrocks-Mumford distributions, fixing numerical invariants as data. This is the idea of constructing the moduli space of holomorphic distribution, meaning an algebraic variety that parametrizes the holomorphic distributions.

For more details on the general theory of Moduli spaces, see: [6], [7], [10], [22], [26] and [31].

6.1 MODULI SPACES OF DISTRIBUTIONS

In [6], the authors described the Moduli Space of holomorphic distributions of codimension one on $\mathbb{P}^3$, in terms of Grothendieck's Quot-scheme for the tangent bundle, and determined under what conditions this variety is smooth, irreducible and they calculated its dimension as a consequence of the Theorem on the dimension of the fibers.

Lemma 18 (O. Calvo Andrade, M. Corrêa, M. Jardim - 2018)**.** *Let $\mathscr{M}^{P,r,st}$ denote the open subset of $\mathscr{M}^P$ consisting of stable reflexive sheaves. Assume that the forgetful morphism $\varpi : \mathscr{D}ist^{P,st} \to \mathscr{M}^{P,r,st}$ is surjective, and that $\mathscr{M}^{P,r,st}$ is irreducible. If $\dim \mathrm{Hom}(F, TX)$ is constant for all $[F] \in \mathscr{M}^{P,r,st}$, then $\mathscr{D}ist^{P,st}$ is irreducible and*

$$\dim \mathscr{D}ist^{P,st} = \dim \mathscr{M}^{P,r,st} + \dim \mathrm{Hom}(F, TX) - 1.$$

If, in addition, $\mathrm{Ext}^2(T\mathscr{F}, T\mathscr{F}) = 0$ for every $[\mathscr{F}] \in \mathscr{D}ist^{P,st}$, then $\mathscr{D}ist^{P,st}$ is nonsingular and

$$\dim \mathscr{D}ist^{P,st} = \dim \mathrm{Ext}^1(T\mathscr{F}, T\mathscr{F}) + \dim \mathrm{Hom}(T\mathscr{F}, T\mathscr{F}) - 1.$$

Proof. Vide [6], pag. 12. □

In [7] the authors studied foliations by curves on $\mathbb{P}^3$ with locally free conormal sheaf and describe their moduli spaces. We adapt this general theory to describe the Moduli spaces of the Horrocks-Mumford distributions as subsheaves of the tangent and cotangent bundle.

6.2 MODULI SPACES OF THE HORROCKS-MUMFORD DISTRIBUTIONS AS SUBSHEAVES OF THE TANGENT BUNDLE

Having identified the tangent sheaf $T\mathscr{F}_a = E(-a-4)$ of the Horrocks-Mumford holomorphic distributions

$$\mathscr{F}_a : 0 \to E(-a-4) \to T\mathbb{P}^4 \to N_{\mathscr{F}} \to 0 \ , \quad a \geq 1,$$

we study the variety that parameterizes them, the associated Moduli space.
Let $P = P_{\mathscr{F}_a}(t) = \chi(T\mathscr{F}_a(t))$ be the Hilbert polynomial of the stable bundle $T\mathscr{F}_a = E(-a-4)$, and let $d_a = \deg(\mathscr{F}_a)$ and $c = c_2(T\mathscr{F}_a) = c_2(E(-a-4))$, for $a \geq 1$. Then by Riemann-Roch Theorem for rank 2 vector bundle $E(-a-4+t)$ we write:

$$\begin{aligned} P &:= P_{\mathscr{F}_a}(t) \\ &= 2 + \frac{25}{12}(2-d_a+2t) + \frac{35}{24}\big((2-d_a+2t)^2 - 2(c-2at+t^2-3t)\big) \\ &\quad + \frac{5}{12}\big((2-d_a+2t)^3 - 3(2-d_a+2t)(c-2at+t^2-3t)\big) \\ &\quad + \frac{1}{24}\big((2-d_a+2t)^4 - 4(2-d_a+2t)^2(c-2at+t^2-3t) + 2(c-2at+t^2-3t)^2\big). \end{aligned}$$

We are going to denoted by

$$\mathscr{D}ist^P(d_a, c) := \mathscr{D}ist^P(2a+5, a^2+3a+6)$$

the Moduli spaces of Horrocks-Munford Holomorphic Distribution as subsheaves of tangent bundle, with Hilbert polynomial P and degree $d_a = \deg(\mathscr{F}_a)$, and by

$$\mathscr{M}_{\mathbb{P}^4}(2-d_a, c) := \mathscr{M}_{\mathbb{P}^4}(-2a-3, a^2+3a+6)$$

denote the Moduli space of the stable tangent sheaves of the Horrocks-Mumford distributions (4.18), with Chern classes $c_1 = 2-d_a$ and $c_2 = c$.
Let us consider the forgetful morphism

$$\begin{aligned} \varpi_a : \mathscr{D}ist^P(d_a, c) &\longrightarrow \mathscr{M}_{\mathbb{P}^4}(2-d_a, c) \\ [\mathscr{F}_a] &\longmapsto \quad [E(-a-4)] \end{aligned}$$

Twisting by $\mathcal{O}_{\mathbb{P}^4}(a+1)$ we obtain the isomorphism $\mathscr{M}_{\mathbb{P}^4}(-3-2a, a^2+3a+6) \simeq \mathscr{M}_{\mathbb{P}^4}(-1,4)$. By Theorem 24 $\mathscr{M}_{\mathbb{P}^4}(-1,4)$ is a Zariski open then $\mathscr{M}_{\mathbb{P}^4}(2-d_a, c)$ is an irreducible, nonsingular variety of dimension 24. In addition, by Bertini type Theorem 28 each $E(-a-4)$

is the tangent sheaf of a distribution $\mathscr{F}_a$, therefore ϖ is surjective, for all $a \geq 1$. Thus the image $\operatorname{Im}\varpi = \mathscr{M}_{\mathbb{P}^4}(-1,4)$ is irreducible.

Theorem 35 (E)**.** *The Moduli space $\mathscr{D}ist^P(2a+5, a^2+3a+6)$ of codimension two holomorphic distributions (4.18) is an irreducible quasi-projective variety of dimension*

$$\frac{1}{3}a^4 + 7a^3 + \frac{277}{6}a^2 + \frac{199}{2}a + 43$$

for $a \geq 1$.

Proof. The fibers of ϖ_a over a point $[E(-a-4)] \in \mathscr{M}_{\mathbb{P}^4}(2-d_a, c)$ is the set $\mathscr{D}ist(E(-a-4))$ of all distributions whose tangent sheaf is Horrocks-Mumford:

$$\mathscr{D}ist(E(-a-4)) := \{\varphi \in \mathbb{P}\operatorname{Hom}(E(-a-4), T\mathbb{P}^4);\ \ker\varphi = 0 \text{ and } \operatorname{Coker}\varphi \text{ is torsion free}\}.$$

That is an open subset of $\mathbb{P}\operatorname{Hom}(E(-a-4), T\mathbb{P}^4)$, see [6, Section 2.3]. Hence

$$\dim \mathscr{D}ist(E(-a-4)) = \dim \operatorname{Hom}(E(-a-4), T\mathbb{P}^4) - 1.$$

We claim that $\dim \operatorname{Hom}(E(-a-4), T\mathbb{P}^4) = h^0(E(a-1) \otimes T\mathbb{P}^4)$ is constant.
Indeed, for this purpose, let us consider the Euler exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5} \to T\mathbb{P}^4 \to 0,$$

twisting by $E(a-1)$ we have:

$$0 \to E(a-1) \to E(a)^{\oplus 5} \to E(a-1) \otimes T\mathbb{P}^4 \to 0,$$

with long exact sequence of cohomology :

$$0 \to H^0(E(a-1)) \to H^0(E(a))^{\oplus 5} \to H^0(E(a-1) \otimes T\mathbb{P}^4) \to \cdots$$

$$\cdots \to H^4(E(a-1)) \to H^4(E(a))^{\oplus 5} \to H^4(E(a-1) \otimes T\mathbb{P}^4) \to 0.$$

Since $h^i(E(a)) = 0$ for $i = 1, 2, 3, 4$ and $h^i(E(a-1)) = 0$ for $i = 2, 3, 4$, then

$$h^i(E(a-1) \otimes T\mathbb{P}^4) = 0,$$

for $i = 1, 2, 3, 4$, therefore:

$$\begin{aligned} 5\chi(E(a)) &= \chi(E(a-1)) + \chi(E(a-1) \otimes T\mathbb{P}^4) \\ &= \chi(E(a-1)) + h^0(E(a-1) \otimes T\mathbb{P}^4). \end{aligned}$$

So by (2.4) we have:

$$\begin{aligned} h^0(E(a-1)\otimes T\mathbb{P}^4) &= 5\cdot\chi(E(a)) - \chi(E(a-1)) \\ &= 5\cdot\frac{1}{12}\big((a+5)^2-1\big)\big((a+5)^2-24\big) - \frac{1}{12}\big((a+4)^2-1\big)\big((a+4)^2-24\big) \\ &= \frac{1}{3}a^4 + 7a^3 + \frac{277}{6}a^2 + \frac{199}{2}a + 20, \end{aligned}$$

for $a \geq 1$.
So, all the fiber in $\mathscr{D}ist(E(-a-4))$ has the same dimension. Since $\operatorname{Im}\varpi_a = \mathscr{M}_{\mathbb{P}^4}(2-d_a, c)$ is irreducible then by Theorem on the dimension of fibers (see [35, pag. 77]), we have that $\mathscr{D}ist^P(d_a, c)$ is a irreducible variety.
Finally, by Theorem on the dimension of fibers (see [35, pag. 76]), Lemma 18 and Theorem 24, we have:

$$\begin{aligned} \dim \mathscr{D}ist^{P,st}(d_a, c) &= \dim \mathscr{M}_{\mathbb{P}^4}(2-d_a, c) + \dim \mathscr{D}ist(E(-a-4)) \\ &= \dim \mathscr{M}_{\mathbb{P}^4}(2-d_a, c) + \dim \operatorname{Hom}(E(-a-4), T\mathbb{P}^4) - 1 \\ &= 24 + h^0(E(a-1)\otimes T\mathbb{P}^4) - 1 \\ &= \frac{1}{3}a^4 + 7a^3 + \frac{277}{6}a^2 + \frac{199}{2}a + 43, \end{aligned}$$

for $a \geq 1$.

□

In particular, for $a = 1$ we have:

Corollary 11. *The Moduli space $\mathscr{D}ist^{P,st}(7, 10)$ of codimension two holomorphic distribution (4.29) is an irreducible, non-singular quasi-projective variety of dimension **196**.*

6.3 MODULI SPACES OF THE HORROCKS-MUMFORD DISTRIBUTIONS AS SUBSHEAVES OF COTANGENT BUNDLE

Similarly to the previous case, and having identified the stable conormal sheaf $N^*_{\mathscr{F}_a} = E(-a-7)$ of the Horrocks-Mumford distributions (4.30), in this section we analyze their Moduli spaces.
Consider the Horrocks-Mumford distribution induced by the cotangent bundle:

$$\mathscr{F}_a : 0 \to E(-a-7) \to \Omega^1_{\mathbb{P}^4} \to \mathcal{Q}_{\mathscr{F}} \to 0 \quad , \quad a \geq 1.$$

Let $P = P_{\mathscr{F}_a}(t) = \chi(N^*_{\mathscr{F}_a}(t))$ the Hilbert polynomial of $N^*_{\mathscr{F}_a} = E(-a-7)$, and let $d_a = \deg(\mathscr{F}_a)$ and let $c = c_2(N^*_{\mathscr{F}_a}) = c_2(E(-a-7))$, for $a \geq 1$. By Theorem 15 for rank 2 vector bundle $E(-a-7+t)$ we write:

$$\begin{aligned}P &:= P_{\mathscr{F}_a}(t)\\ &= 2+\frac{25}{12}(-d_a-3+2t)+\frac{35}{24}\big((-d_a-3+2t)^2-2(c-2at+t^2-9t)\big)\\ &\quad+\frac{5}{12}\big((-d_a-3+2t)^3-3(-d_a-3+2t)(c-2at+t^2-9t)\big)\\ &\quad+\frac{1}{24}\big((-d_a-3+2t)^4-4(-d_a-3+2t)^2(c-2at+t^2-9t)+2(c-2at+t^2-9t)^2\big).\end{aligned}$$

Let $\mathscr{D}ist^P(d_a,c)$ its Moduli spaces with Hilbert polynomial P and degree $d_a=\deg(\mathscr{F}_a)$, i.e,

$$\mathscr{D}ist^P(d_a,c) := \mathscr{D}ist^P(2a+6, a^2+9a+24)$$

and let $\mathscr{M}_{\mathbb{P}^4}(-d_a-3,c)$ denote the Moduli space of stable conormal sheaves of Horrocks-Mumford distributions 4.30, with Chern classes $c_1=-d_a-3$ and $c_2=c$, i.e,

$$\mathscr{M}_{\mathbb{P}^4}(-d_a-3,c)=\mathscr{M}_{\mathbb{P}^4}(-2a-9,a^2+9a+24).$$

For all $a\geq 1$, let us consider the forgetful morphism

$$\begin{aligned}\varpi_a^o : \mathscr{D}ist^P(d_a,c) &\longrightarrow \mathscr{M}_{\mathbb{P}^4}(-d_a-3,c)\\ [\mathscr{F}_a] &\longmapsto \quad [E(-a-7)].\end{aligned}$$

Twisting by $\mathcal{O}_{\mathbb{P}^4}(a+4)$ we have $\mathscr{M}_{\mathbb{P}^4}(-2a-9,a^2+9a+24)\simeq\mathscr{M}_{\mathbb{P}^4}(-1,4)$. By Theorem 24 $\mathscr{M}_{\mathbb{P}^4}(-1,4)$ is a Zariski open then $\mathscr{M}_{\mathbb{P}^4}(-d_a-3,c)$ is an irreducible, nonsingular variety of dimension 24. In addition, by Bertini type Theorem 28 each $E(-a-7)$ is the conormal sheaf of a distribution $\mathscr{F}_a$, therefore ϖ is surjective, for all $a\geq 1$.

Theorem 36 (F)**.** *The Moduli space $\mathscr{D}ist^P(2a+6,a^2+9a+24)$ of codimension two holomorphic distributions (4.30) is an irreducible quasi-projective variety of dimension*

$$\frac{1}{3}a^4+\frac{23}{3}a^3+\frac{343}{6}a^2+\frac{899}{6}a+98,$$

for $a\geq 1$.

Proof. The fibres over a point $[E(-a-7)]\in\mathscr{M}_{\mathbb{P}^4}(-d_a-3,c)$ is the set $\mathscr{D}ist(E(-a-7))$ of all distributions whose conormal sheaf is Horrocks-Mumford:

$$\mathscr{D}ist(E(-a-7)) := \{\varphi\in\mathbb{P}\operatorname{Hom}(E(-a-7),\Omega^1_{\mathbb{P}^4});\ \ker\varphi=0 \text{ and } \operatorname{Coker}\varphi \text{ is torsion free}\},$$

that is an open subset of $\mathbb{P}\operatorname{Hom}(E(-a-7),\Omega^1_{\mathbb{P}^4})$, hence

$$\dim\mathscr{D}ist(E(-a-7))=\dim\operatorname{Hom}(E(-a-7),\Omega^1_{\mathbb{P}^4})-1.$$

We claim that $\dim \mathrm{Hom}(E(-a-7), \Omega^1_{\mathbb{P}^4})$ is constant.
Indeed, for this purpose, let us consider the Euler exact sequence:

$$0 \to \Omega^1_{\mathbb{P}^4} \to \mathscr{O}_{\mathbb{P}^4}(-1)^{\oplus 5} \to \mathscr{O}_{\mathbb{P}^4} \to 0,$$

twisting by $E(a+2)$ we have:

$$0 \to E(a+2) \otimes \Omega^1_{\mathbb{P}^4} \to E(a+1)^{\oplus 5} \to E(a+2) \to 0,$$

with long exact sequence of cohomology :

$$0 \to H^0(E(a+2) \otimes \Omega^1_{\mathbb{P}^4}) \to H^0(E(a+1))^{\oplus 5} \to H^0(E(a+2)) \to \cdots$$

$$\cdots \to H^4(E(a+2) \otimes \Omega^1_{\mathbb{P}^4}) \to H^4(E(a+1))^{\oplus 5} \to H^4(E(a+2)) \to 0.$$

Since $h^i(E(a+1)) = 0$ for $i = 1,2,3,4$ and $h^i(E(a+2)) = 0$ for $i = 1,2,3,4$, then $h^i(E(a+2) \otimes \Omega^1_{\mathbb{P}^4}) = 0$ for $i = 2,3,4$ and $h^1(E(a+2) \otimes \Omega^1_{\mathbb{P}^4}) = 0$ for $a \leq -5$ and $a \geq 0$, therefore for all $a \geq 1$ we have:

$$\begin{aligned} 5\chi(E(a+1)) &= \chi(E(a+2) \otimes \Omega^1_{\mathbb{P}^4}) + \chi(E(a+2)) \\ &= h^0(E(a+2) \otimes \Omega^1_{\mathbb{P}^4}) + h^0(E(a+2)). \end{aligned}$$

So, by (2.4):

$$\begin{aligned} h^0(E(a+2) \otimes \Omega^1_{\mathbb{P}^4}) &= 5\chi(E(a+1)) - \chi(E(a+2)) \\ &= 5 \cdot \frac{1}{12}\big((a+6)^2 - 1\big)\big((a+6)^2 - 24\big) - \frac{1}{12}\big((a+7)^2 - 1\big)\big((a+7)^2 - 24\big) \\ &= \frac{1}{3}a^4 + \frac{23}{3}a^3 + \frac{343}{6}a^2 + \frac{899}{6}a + 75, \end{aligned}$$

for $a \geq 1$.
So, all the fiber $\mathscr{D}ist(E(-a-7))$ for $[E(-a-7)] \in \mathscr{M}_{\mathbb{P}^4}(-d_a-3, c)$ has the same dimension.
Since $\mathscr{M}_{\mathbb{P}^4}(-d_a-3, c)$ is irreducible then by Theorem of dimension of the fibers we have that $\mathscr{D}ist^{P,st}(d_a, c)$ is a irreducible variety.
Finally, by Lemma 18 and Theorem 24 we have:

$$\begin{aligned} \dim \mathscr{D}ist^{P,st}(d_a, c) &= \dim \mathscr{M}_{\mathbb{P}^4}(-d_a - 3, c) + \mathscr{D}ist(E(-a-7)) \\ &= \dim \mathscr{M}_{\mathbb{P}^4}(-d_a - 3, c) + \dim \mathrm{Hom}(E(-a-7), \Omega^1_{\mathbb{P}^4}) - 1 \\ &= 24 + h^0(E(a+2) \otimes \Omega^1_{\mathbb{P}^4}) - 1 \\ &= \frac{1}{3}a^4 + \frac{23}{3}a^3 + \frac{343}{6}a^2 + \frac{899}{6}a + 98, \end{aligned}$$

for $a \geq 1$.

□

In particular, for $a = 1$ we have:

Corollary 12. *The Moduli space $\mathscr{D}ist^{P,st}(8,34)$ of codimension two holomorphic distribution (4.41) is an irreducible, non-singular quasi-projective variety of dimension **313**.*

Parte III

Final considerations

7 THE $a = 0$ CASE

7.1 ON THE INJECTIVITY

By Lemma 2, we note that $\dim \operatorname{Hom}(E(-a-4), T\mathbb{P}^4) = h^0(E(a-1) \otimes T\mathbb{P}^4) = 0$ for $a \leq -1$. On the other hand, twisting Euler's sequence by $E(-1)$ and considering the long exact sequence in cohomology, then by Serre's duality and Lemma 2 we have $\dim \operatorname{Hom}(E(-4), T\mathbb{P}^4) = h^0(E(-1) \otimes T\mathbb{P}^4) = 20$. So, it is natural to ask: are there distributions when $a = 0$?, i.e., let:

$$\mathscr{F} : 0 \to E(-4) \xrightarrow{\varphi} T\mathbb{P}^4 \to N_{\mathscr{F}} \to 0.$$

Question: Is $\mathscr{F}$ a distribution?

Lemma 19. *Let E be the Horrocks-Mumford bundle. Then there are morphisms such that $\varphi : E(-4) \to T\mathbb{P}^4$ is injective.*

Proof. Let $\varphi \in \operatorname{Hom}(E(-4), T\mathbb{P}^4)$ be a non trivial morphism $\varphi : E(-4) \to T\mathbb{P}^4$. If φ is not injective then $\operatorname{Ker} \varphi \neq 0$, i.e, we have:

$$\operatorname{Ker} \varphi \hookrightarrow E(-4) \to T\mathbb{P}^4,$$

hence $\operatorname{rank}(\operatorname{Ker} \varphi) = 1$, so $\operatorname{Ker} \varphi$ is a reflexive sheaf, hence a line bundle $\operatorname{Ker} \varphi = \mathcal{O}_{\mathbb{P}^4}(k)$, obtaining the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^4}(k) & \longrightarrow & E(-4) & \xrightarrow{\quad\varphi\quad} & T\mathbb{P}^4 \\ & & & \searrow \quad \nearrow^{\tau} & \\ & & & \mathscr{I}_Z(-k-3) & \end{array}$$

Since τ induces a non-trivial section in $H^0(\mathbb{P}^4, T\mathbb{P}^4(k+3))$, then by Bott's formula 1.9 we have $h^0(T\mathbb{P}^4(k+3)) = h^4(\Omega^1_{\mathbb{P}^4}(-8-k)) \neq 0$ if, and only if, $k \geq -4$. On the other hand, since E is stable then $h^0(E_\eta) = h^0(E(-3)) = 0$, i.e, $E(-3)$ has no global section, so there is no injective map $\mathcal{O}_{\mathbb{P}^4} \hookrightarrow E(-3)$ hence $\mathcal{O}_{\mathbb{P}^4}(k) \hookrightarrow E(-4)$ for $k \leq -2$. Therefore $k \in \{-4, -3, -2\}$.

- If $k = -4$ and by $h^0(E) = 4 \neq 0$ then the map $\varphi : E(-4) \to \mathbb{P}^4$ has $\operatorname{Ker} \varphi \neq 0$.
- If $k = -3$ or -2 then $h^0(E(k)) = 0$ hence $\operatorname{Ker} \varphi = 0$ thus φ is injective.

Now, set

$$\mathcal{I}_1 = \{\varphi \in \mathrm{Hom}(E(-4), T\mathbb{P}^4), \text{such that} \operatorname{Ker}\varphi = 0\},$$

then $\dim \mathcal{I}_1 = h^0(E(-1) \otimes T\mathbb{P}^4) - h^0(E) = 20 - 4 = 16$. So, there are injective morphisms $\varphi : E(-4) \to T\mathbb{P}^4$.

$\square$

Similarly, by Lemma 2 we have:

$$\dim \mathrm{Hom}(E(-a-7), \Omega^1_{\mathbb{P}^4}) = h^0(E(a+2) \otimes \Omega^1_{\mathbb{P}^4}) = 0,$$

for $a \leq -1$. However for $a = 0$, twisting Euler's sequence by $E(2)$ and considering the long exact sequence in cohomology, then by Lemma 2 we have $\dim \mathrm{Hom}(E(-7), \Omega^1_{\mathbb{P}^4}) = h^0(E(2) \otimes \Omega^1_{\mathbb{P}^4}) = 75$. So, it is natural to ask: are there distributions when $a = 0$? , i.e., let:

$$\mathscr{F} : 0 \to E(-7) \xrightarrow{\varphi} \Omega^1_{\mathbb{P}^4} \to N_{\mathscr{F}} \to 0.$$

Question: Is $\mathscr{F}$ a distribution?

Lemma 20. *Let E be the Horrocks-Mumford bundle. Then there are morphisms such that $\varphi : E(-7) \to \Omega^1_{\mathbb{P}^4}$ is injective.*

Proof. Let $\varphi \in \mathrm{Hom}(E(-7), \Omega^1_{\mathbb{P}^4})$ be a non trivial morphism $\varphi : E(-7) \to \Omega^1_{\mathbb{P}^4}$. If φ is not injective, then $\operatorname{Ker}\varphi \neq 0$, i.e, we have:

$$\operatorname{Ker}\varphi \hookrightarrow E(-7) \to \Omega^1_{\mathbb{P}^4},$$

then $\operatorname{rank}(\operatorname{Ker}\varphi) = 1$, so $\operatorname{Ker}\varphi$ is a reflexive sheaf, hence a line bundle $\operatorname{Ker}\varphi = \mathcal{O}_{\mathbb{P}^4}(k)$, obtaining the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^4}(k) & \longrightarrow & E(-7) & \xrightarrow{\qquad\varphi\qquad} & \Omega^1_{\mathbb{P}^4} \\ & & & \searrow \qquad \nearrow^{\tau} & \\ & & & \mathscr{I}_Z(-k-9) & \end{array}$$

Since τ induces a non-trivial section in $H^0(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(k+9))$, then by Bott's formula 1.9 we have $h^0(\Omega^1_{\mathbb{P}^4}(k+9)) \neq 0$ if, and only if, $k \geq -7$. On the other hand, since E is stable then $h^0(E_\eta) = h^0(E(-3)) = 0$, i.e, $E(-3)$ has no global section, so there is no injective map $\mathcal{O}_{\mathbb{P}^4} \hookrightarrow E(-3)$ that induces $\mathcal{O}_{\mathbb{P}^4}(k) \hookrightarrow E(-7)$ for $k \leq -5$. Therefore $k \in \{-7, -6, -5\}$.

- If $k = -7$ and by $h^0(E) = 4 \neq 0$ then the map $\varphi : E(-7) \to \Omega^1_{\mathbb{P}^4}$ has $\operatorname{Ker}\varphi \neq 0$.

- If $k = -6$ or -5 then $h^0(E(k)) = 0$ hence $\operatorname{Ker}\varphi = 0$ thus φ is injective.

Now, set

$$\mathcal{I}_2 = \{\varphi \in \mathrm{Hom}(E(-7), \Omega^1_{\mathbb{P}^4}), \text{such that} \operatorname{Ker}\varphi = 0\},$$

then $\dim \mathcal{I}_2 = h^0(E(-1) \otimes \Omega^1_{\mathbb{P}^4}) - h^0(E) = 75 - 4 = 71$. So, there are injective morphisms $\varphi : E(-7) \to \Omega^1_{\mathbb{P}^4}$.

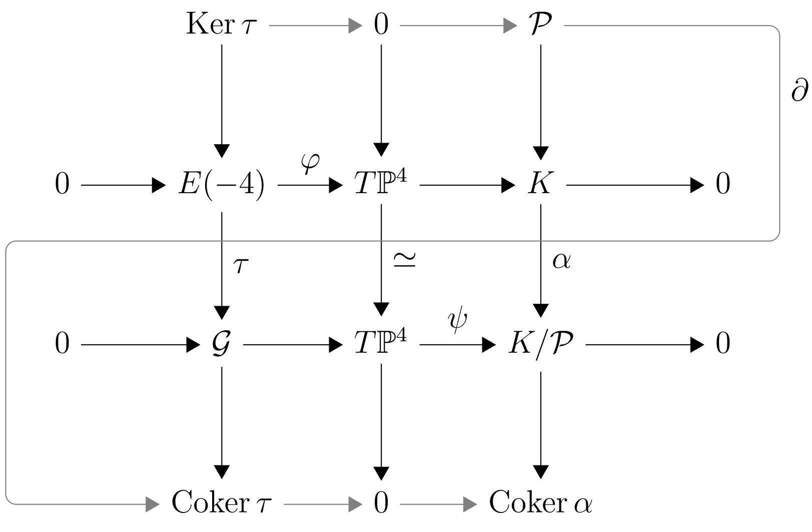
□

7.2 ON THE SATURATED DISTRIBUTION

Let $\varphi \in \mathcal{I}_1$, i.e, $\varphi : E(-4) \to T\mathbb{P}^4$ is a injective morphism, suppose that $K := \operatorname{Coker}\varphi$ is not torsion free. Let $\mathcal{P}$ be the maximal torsion subsheaf of K, then $K/\mathcal{P}$ is torsion free. We affirm that $\mathcal{P}$ has codimension one. Let ψ be the composed epimorphism $T\mathbb{P}^4 \to K \to K/\mathcal{P}$ and let $\mathcal{G} = \operatorname{Ker}\psi$, then we have

$$0 \to \mathcal{G} \to T\mathbb{P}^4 \xrightarrow{\psi} K/\mathcal{P} \to 0, \tag{7.1}$$

hence, since $T\mathbb{P}^4$ is locally free and $K/\mathcal{P}$ is torsion free sheaf then by Proposition 4 we have that $\mathcal{G}$ is a reflexive sheaf. So, by the Snake Lemma



we have:

$$0 \to \operatorname{Ker}\tau \to 0 \to \mathcal{P} \xrightarrow{\partial} \operatorname{Coker}\tau \to 0$$

hence $\operatorname{Ker}\tau = 0$ and $\mathcal{P} \simeq \operatorname{Coker}\tau$, so we obtain a exact sequence:

$$0 \to E(-4) \xrightarrow{\tau} \mathcal{G} \to \mathcal{P} \to 0.$$

If $\operatorname{codim}\mathcal{P} \geq 2$ then $\mathcal{P}^* = 0$ so $\mathcal{G}^* \simeq (E(-4))^*$, hence

$$E(-4) \simeq (E(-4))^{**} \simeq \mathcal{G}^{**} \simeq \mathcal{G},$$

because $\mathcal{G}$ is reflexive sheaf. Then since $\mathcal{P} = 0$ hence $K/\mathcal{P} = K$, i.e, K will be a torsion free sheaf, but this contradicts our hypothesis, therefore $\operatorname{codim} \mathcal{P} = 1$. Since $\mathcal{P}$ is a torsion sheaf then $\deg(\mathcal{P}) > 0$, so $c_1(\mathcal{P}) > 0$. Thus from

$$0 \to E(-4) \to \mathcal{G} \to \mathcal{P} \to 0,$$

then $c_1(\mathcal{G}) = c_1(E(-4)) + c_1(\mathcal{P})$, hence $c_1(\mathcal{G}) > -3$.
Additionally,

$$\mathscr{G} : 0 \to \mathcal{G} \to T\mathbb{P}^4 \to K/\mathcal{P} \to 0,$$

is a codimension 2 saturated distribution of $\mathscr{F}$, with tangent sheaf $\mathcal{G}$, then

$$\deg(\mathscr{G}) = d < 5 = \deg(\mathscr{F}).$$

Now, from:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(-4) & \xrightarrow{\varphi} & T\mathbb{P}^4 & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\tau} & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & T\mathbb{P}^4 & \xrightarrow{\psi} & K/\mathcal{P} & \longrightarrow & 0 \end{array}$$

since $\mathcal{P} \simeq \operatorname{Coker} \tau$ is a torsion sheaf, we obtain the injective map

$$0 \to \operatorname{Hom}(G, T\mathbb{P}^3) \xrightarrow{\circ\tau} \operatorname{Hom}(E(-k), T\mathbb{P}^3) \to \cdots$$

given by composition with τ. Hence $h^0(\mathcal{G}^* \otimes T\mathbb{P}^4) \leq h^0(E(-1) \otimes T\mathbb{P}^4) = 20$, i.e.,

$$h^0(\mathcal{G}^* \otimes T\mathbb{P}^4) \leq 20.$$

- If $h^0(\mathcal{G}^* \otimes T\mathbb{P}^4) = 20$ then $\operatorname{Hom}(\mathcal{G}, T\mathbb{P}^4) \simeq \operatorname{Hom}(E(-4), T\mathbb{P}^4)$, i.e, every injective morphism $\varphi : E(-4) \to T\mathbb{P}^4$ factors through τ, hence $\operatorname{Coker} \varphi$ cannot be torsion free because it contains $\operatorname{Coker} \varphi$ as a subsheaf.

- If $h^0(\mathcal{G}^* \otimes T\mathbb{P}^4) < 20$ then $\operatorname{Hom}(\mathcal{G}, T\mathbb{P}^4) \neq \operatorname{Hom}(E(-4), T\mathbb{P}^4)$, i.e, there is morphism $\varphi \in \operatorname{Hom}(E(-4), T\mathbb{P}^4)$ which does not factor through $\mathcal{G}$. By Lemma 19 we can choose φ to be injective. Now, if $\operatorname{Coker} \varphi$ is not torsion free, we would get in contradiction with the existence of the sequence (7.1).

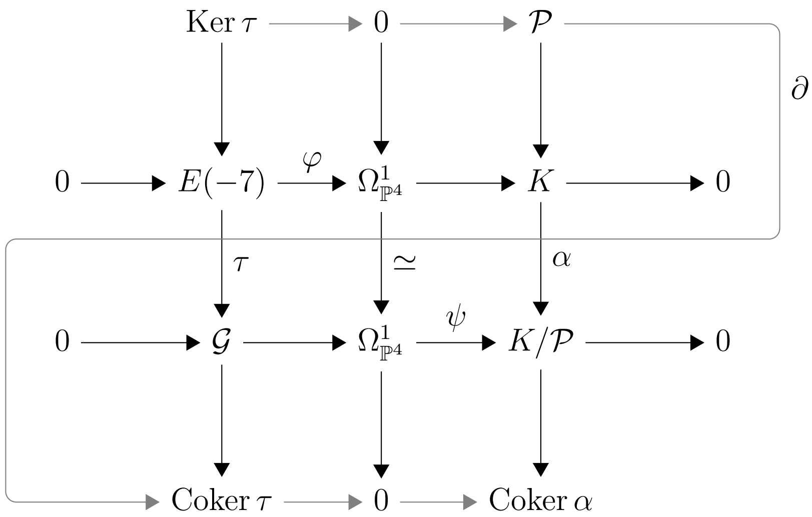
Observation: So, to finish this analysis, we need to determine the tangent sheaf $\mathcal{G}$, that is, we need to classify codimension 2 holomorphic distributions of low degree whose tangent sheaf is locally free.

Using the same argument as in the previous case, if $\varphi \in \mathcal{I}_2$, i.e, $\varphi : E(-7) \to \Omega^1_{\mathbb{P}^4}$ is a injective morphism, suppose that $K := \operatorname{Coker} \varphi$ is not torsion free. Let $\mathcal{P}$ be the maximal torsion subsheaf of K, then $K/\mathcal{P}$ is torsion free.

We affirm that $\mathcal{P}$ has codimension one. Let ψ be the composed epimorphism $\Omega^1_{\mathbb{P}^4} \to K \to K/\mathcal{P}$ and let $\mathcal{G} = \operatorname{Ker}\psi$, then we have

$$0 \to \mathcal{G} \to \Omega^1_{\mathbb{P}^4} \xrightarrow{\psi} K/\mathcal{P} \to 0, \tag{7.2}$$

hence, since $\Omega^1_{\mathbb{P}^4}$ is locally free and $\mathcal{P}$ is torsion free sheaf then by Proposition 4 we have that $\mathcal{G}$ is a reflexive sheaf. So, by the Snake Lemma



we have:

$$0 \to \operatorname{Ker}\tau \to 0 \to \mathcal{P} \xrightarrow{\partial} \operatorname{Coker}\tau \to 0$$

hence $\operatorname{Ker}\tau = 0$ and $\mathcal{P} \simeq \operatorname{Coker}\tau$, so we obtain a exact sequence:

$$0 \to E(-7) \xrightarrow{\tau} \mathcal{G} \to \mathcal{P} \to 0.$$

If $\operatorname{codim}\mathcal{P} \geq 2$ then $\mathcal{P}^* = 0$ so $\mathcal{G}^* \simeq (E(-7))^*$, hence

$$E(-7) \simeq (E(-7))^{**} \simeq \mathcal{G}^{**} \simeq \mathcal{G},$$

because $\mathcal{G}$ is reflexive sheaf. Then since $\mathcal{P} = 0$ hence $K/\mathcal{P} = K$, i.e, K will be a torsion free sheaf, but this contradicts our hypothesis, therefore $\operatorname{codim}\mathcal{P} = 1$. Since $\mathcal{P}$ is a torsion sheaf then $\deg(\mathcal{P}) > 0$, so $c_1(\mathcal{P}) > 0$. Thus from

$$0 \to E(-7) \to \mathcal{G} \to \mathcal{P} \to 0,$$

then $c_1(\mathcal{G}) = c_1(E(-7)) + c_1(\mathcal{P})$, hence $c_1(\mathcal{G}) > -9$.
Additionally,

$$\mathscr{G} : 0 \to \mathcal{G} \to \Omega^1_{\mathbb{P}^4} \to K/\mathcal{P} \to 0,$$

is a dimension 2 saturated distribution of $\mathscr{F}$, with the conormal sheaf $\mathcal{G}$, then

$$\deg(\mathscr{G}) = d < 6 = \deg(\mathscr{F}).$$

Now, from:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(-7) & \xrightarrow{\varphi} & \Omega^1_{\mathbb{P}^4} & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau} & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \Omega^1_{\mathbb{P}^4} & \xrightarrow{\psi} & K/\mathcal{P} & \longrightarrow & 0 \end{array}$$

since $\operatorname{Coker}\tau$ is a torsion sheaf, we obtain the injective map

$$0 \to \operatorname{Hom}(\mathcal{G}, \Omega^1_{\mathbb{P}^4}) \xrightarrow{\circ\tau} \operatorname{Hom}(E(-7), \Omega^1_{\mathbb{P}^4}) \to \cdots$$

given by composition with τ. Hence $h^0(\mathcal{G}^* \otimes \Omega^1_{\mathbb{P}^4}) \leq h^0(E(2) \otimes \Omega^1_{\mathbb{P}^4}) = 75$, so

$$h^0(\mathcal{G}^* \otimes \Omega^1_{\mathbb{P}^4}) \leq 75.$$

- If $h^0(\mathcal{G}^* \otimes \Omega^1_{\mathbb{P}^4}) = 75$ then $\operatorname{Hom}(\mathcal{G}, \Omega^1_{\mathbb{P}^4}) \simeq \operatorname{Hom}(E(-7), \Omega^1_{\mathbb{P}^4})$, i.e, every injective morphism $\varphi : E(-7) \to \Omega^1_{\mathbb{P}^4}$ factors through τ, hence $\operatorname{Coker}\varphi$ cannot be torsion free because it contains $\operatorname{Coker}\varphi$ as a subsheaf.

- If $h^0(\mathcal{G}^* \otimes \Omega^1_{\mathbb{P}^4}) < 75$ then $\operatorname{Hom}(\mathcal{G}, \Omega^1_{\mathbb{P}^4}) \neq \operatorname{Hom}(E(-7), \Omega^1_{\mathbb{P}^4})$, i.e, there is morphism $\varphi \in \operatorname{Hom}(E(-7), \Omega^1_{\mathbb{P}^4})$ which does not factor through $\mathcal{G}$. By Lemma 20 we can choose φ to be injective. Now, if $\operatorname{Coker}\varphi$ is not torsion free, we would get in contradiction with the existence of the sequence (7.2).

Observation: Then, we need to determine the conormal sheaf $\mathcal{G}$, that is, we need to classify dimension 2 holomorphic distributions of low degree whose conormal sheaf is locally free.

Parte IV

Appendix

8 APPENDIX

Here, we present scripts for the software *Macaulay2*, which can found in https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2-1.12/share/doc/Macaulay2/Schubert2/html/___The_sp__Horrocks-__Mumford_spbundle.html, to perform some tensor product cohomology calculations with Horrocks-Mumford bundle. You may open a session in http://habanero.math.cornell.edu:3690/ and perform the computations just by cut&paste.

8.0.1 Some calculations of Cohomology of tensor products

Example 12. *By Macaulay2 we have* $h^0(E(1) \otimes \Omega^1_{\mathbb{P}^4}) = 0$.

```
R = QQ[x_0..x_4];

a = {1,0,0,0,0}
b = {0,1,0,0,1}
c = {0,0,1,1,0}

M1 = matrix table(5,5, (i,j)-> x_((i+j)%5)*a_((i-j)%5))
M2 = matrix table(5,5, (i,j)-> x_((i+j)%5)*b_((i-j)%5))
M3 = matrix table(5,5, (i,j)-> x_((i+j)%5)*c_((i-j)%5))

M = M1 | M2 | M3;
betti (C=res coker M)

N = transpose submatrix(C.dd_3,{10..28},{2..36});
betti (D=res coker N)
Pfour = Proj(R)
F = sheaf(coker D.dd_3);


E=F(2);

X=Proj R
T=tangentSheaf X ;
C=cotangentSheaf X;
C2=exteriorPower(2,C);
```

```
C3=exteriorPower(3,C);
HH^0(E(1)**C)
```

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