UNIVERSIDADE FEDERAL DE MINAS GERAIS
INSTITUTO DE CIÊNCIAS EXATAS

Tese de Doutorado

Upper block triangular matrix algebras graded by finite cyclic groups: the factorability of their graded T-ideals and the minimal varieties

Marcos Antônio da Silva Pinto

Belo Horizonte - MG
2020

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Tese apresentada ao corpo docente de Pós-Graduação em Matemática do Instituto de Ciências Exatas da Universidade Federal de Minas Gerais como parte dos requisitos para a obtenção do título de Doutor em Matemática.

Orientadora: Viviane Ribeiro Tomaz da Silva

Universidade Federal de Minas Gerais
29 de Outubro de 2020

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Pinto, Marcos Antônio da Silva.

P659u Upper block triangular matrix algebras graded by finite cyclic groups: the factorability of their graded T-ideals and the minimal varieties/ Marcos Antônio da Silva Pinto – 2020.
xxx, 88 f. il.

Orientadora: Viviane Ribeiro Tomaz da Silva.
Tese (doutorado) - Universidade Federal de Minas Gerais, Instituto de Ciências Exatas, Departamento de Matemática.
Referências: f.86-88.

1. Matemática – Teses. 2. Algebra abstrata – Teses. 3. Grupos finitos – Teses. 4. Fatoração (Matematica) – Teses. 5. Variedades (Matematica) I. Silva, Viviane Ribeiro Tomaz da. II. Universidade Federal de Minas Gerais, Instituto de Ciências Exatas, Departamento de Matemática. IV.Título.

CDU 51 (043)

Ficha catalográfica elaborada pela bibliotecária Belkiz Inez Rezende Costa
CRB 6ª Região nº 1510

Universidade Federal de Minas Gerais
Departamento de Matemática
Programa de Pós-Graduação em Matemática

FOLHA DE APROVAÇÃO

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Digitalizado com CamScanner

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Agradecimentos

A Deus, pela presença constante em minha vida e por todas as bênçãos. Agradeço ao Pai Celestial por me conceder a oportunidade de realizar este sonho. A Nossa Senhora Aparecida, por me cobrir com o teu manto sagrado e por sempre interceder por mim.

À minha orientadora, Viviane Ribeiro Tomaz da Silva, por acreditar em mim e por todo o apoio. Agradeço pelo conhecimento transmitido, pela disponibilidade, pela atenção, bem como pelas suas importantes contribuições e o seu comprometimento para a realização deste trabalho. Seus ensinamentos foram e serão fundamentais para a minha caminhada.

Ao professor Onofrio Mario Di Vincenzo pelas importantes contribuições dadas a este trabalho.

Aos professores membros da banca, Ana Cristina Vieira, Claudemir Fidelis Bezerra Júnior, Irina Sviridova e Plamen Emilov Kochloukov, por todas as sugestões e observações feitas neste texto.

Aos meus pais, Carlos e Dulce, por estarem comigo a todo momento e por me apoiarem durante toda essa trajetória. Dedico essa conquista à vocês, meus exemplos de vida.

À minha namorada, Lourainy, pelo amor e por todo o seu carinho. Agradeço por me ouvir e por me incentivar. Sua presença foi essencial para a concretização deste sonho e seguiremos juntos até o fim.

À toda minha família e à todos os meus amigos. A caminhada torna-se mais leve com vocês ao meu lado.

À CAPES e à FAPEMIG, pelo apoio financeiro.

Resumo

Seja F um corpo algebricamente fechado de característica zero e seja G um grupo cíclico finito. Neste trabalho, todas as F-álgebras são assumidas como associativas. Dadas F-álgebras G-simples de dimensão finita $A_1, \dots, A_m$, tomadas como subálgebras graduadas de álgebras de matrizes com algumas graduações elementares, considere a álgebra de matrizes bloco triangular superior $A := (UT(A_1, \dots, A_m), \widetilde{\alpha})$ munida com uma G-graduação elementar induzida por uma aplicação $\widetilde{\alpha}$ (definida "colando" as graduações das A_i's). Nesta tese, abordamos dois tópicos principais: a *propriedade de fatorabilidade* relacionada ao T_G-ideal $\mathrm{Id}_G(A)$ das identidades polinomiais G-graduadas satisfeitas por A e as *variedades minimais* de PI-álgebras associativas G-graduadas sobre F, de posto finito, com respeito a um dado G-expoente.

Mais precisamente, provamos que qualquer F-álgebra G-simples de dimensão finita, anteriormente descrita por Bahturin, Sehgal e Zaicev (para qualquer grupo arbitrário), pode ser vista, para grupos cíclicos, como uma subálgebra graduada de uma álgebra de matriz munida com uma graduação elementar. Além disso, se G é um p-grupo cíclico, com p sendo um primo arbitrário, estabelecemos que $\mathrm{Id}_G(A)$ é fatorável se, e somente se, existe no máximo um índice $i \in \{1, \dots, m\}$ tal que A_i não é G-regular se, e somente se, existe uma única classe de isomorfismo de G-graduações para A. Isto é uma generalização dos resultados apresentados por Avelar, Di Vincenzo e da Silva, quando G tem ordem 2, que já contrastavam com o caso ordinário, investigado por Giambruno e Zaicev. Vale ressaltar que usamos técnicas diferentes daquelas empregadas em tais casos. Ainda, generalizando o conceito de G-regularidade, introduzimos a definição de *α-regularidade* e estabelecemos interessantes relações entre tal conceito e os chamados *subgrupos invariantes*. Finalmente, quando G não é necessariamente um p-grupo, apresentamos condições necessárias e suficientes a fim de obter que $\mathrm{Id}_G((UT(A_1, A_2), \widetilde{\alpha}))$ é fatorável, requerindo que A_1 e A_2 sejam α_1-regular e α_2-regular, respectivamente.

Em relação às variedades minimais, provamos que elas são geradas por adequadas álgebras de matrizes bloco triangulares superiores G-graduadas $(UT(A_1, \dots, A_m), \widetilde{\alpha})$. Por outro lado, assumindo algumas condições sobre essas álgebras, provamos que as variedades geradas por algumas delas são minimais. Estes problemas foram explorados, no caso ordinário, por Giambruno e Zaicev, e, quando G é de ordem prima, por Di Vincenzo, da Silva e Spinelli.

Palavras-chave: álgebras graduadas, grupos cíclicos finitos, fatorabilidade, variedades minimais.

Abstract

Let F be an algebraically closed field of characteristic zero and G be a finite cyclic group. In this work, all the F-algebras are assumed to be associative. Given finite dimensional G-simple F-algebras $A_1, \ldots, A_m$, taken as graded subalgebras of matrix algebras with some elementary gradings, consider the upper block triangular matrix algebra $A := (UT(A_1, \ldots, A_m), \widetilde{\alpha})$ endowed with an elementary G-grading induced by a map $\widetilde{\alpha}$ (defined by gluing the gradings of the A_i's). In this thesis, we approach two main topics: the *factoring property* related to the T_G-ideal $\mathrm{Id}_G(A)$ of the G-graded polynomial identities satisfied by A and the *minimal varieties* of associative G-graded PI-algebras over F, of finite basic rank, with respect to a given G-exponent.

More precisely, we prove that any finite dimensional G-simple F-algebra, previously described by Bahturin, Sehgal and Zaicev (for any arbitrary group), can be seen, for cyclic groups, as a graded subalgebra of a matrix algebra endowed with an elementary grading. Moreover, if G is a cyclic p-group, with p being an arbitrary prime, we establish that $\mathrm{Id}_G(A)$ is factorable if, and only if, there exists at most one index $i \in \{1, \ldots, m\}$ such that A_i is not G-regular if, and only if, there exists a unique isomorphism class of G-gradings for A. This is a generalization of the results presented by Avelar, Di Vincenzo and da Silva, when G has order 2, which already contrasted with the ordinary case, investigated by Giambruno and Zaicev. It is worth highlighting that we use different techniques from those employed in such cases. Still, by generalizing the concept of G-regularity, we introduce the definition of *α-regularity* and we establish nice connections between such concept and the so-called *invariance subgroups*. Finally, when G is not necessarily a p-group, we present necessary and sufficient conditions in order to obtain that $\mathrm{Id}_G((UT(A_1, A_2), \widetilde{\alpha}))$ is factorable, by requiring that A_1 and A_2 are α_1-regular and α_2-regular, respectively.

Regarding the minimal varieties, we prove that they are generated by suitable G-graded upper block triangular matrix algebras $(UT(A_1, \ldots, A_m), \widetilde{\alpha})$. On the other hand, by assuming some conditions over these algebras, we show that the varieties generated by some of them are minimal. These problems was explored, in ordinary case, by Giambruno and Zaicev, and, when G is of prime order, by Di Vincenzo, da Silva and Spinelli.

Keywords: graded algebras, finite cyclic groups, factorability, minimal varieties.

Resumo estendido

Nas últimas décadas, o estudo das álgebras satisfazendo identidades polinomiais, nomeadamente PI-álgebras, tem se desenvolvido em grande escala. Existe um número crescente de pesquisas envolvendo tais álgebras, o que explicita a importância dessa teoria no âmbito matemático. Nesse sentido, os resultados apresentados nesta tese contribuem significativamente com os trabalhos na área de álgebra e, particularmente, com aqueles relativos às PI-álgebras. É importante ressaltar que esses resultados foram desenvolvidos em um trabalho conjunto com a minha orientadora de doutorado, Professora Viviane Ribeiro Tomaz da Silva, e com o Professor Onofrio Mario Di Vincenzo (Università degli Studi della Basilicata - Itália).

Seja F um corpo algebricamente fechado de característica zero e considere G um grupo cíclico finito. Ao longo deste trabalho, todas as F-álgebras são assumidas como associativas. Dedicamos a primeira parte desta tese ao estudo da *propriedade de fatorabilidade* associada aos T_G-ideais de identidades polinomiais G-graduadas satisfeitas por álgebras de matrizes bloco triangulares superiores G-graduadas $UT_G(A_1, \ldots, A_m)$, onde $A_1, \ldots, A_m$ são álgebras G-simples de dimensão finita sobre F. Nossos resultados obtidos neste parte já foram publicados e podem ser encontrados em [22].

Em segundo lugar, o presente trabalho é devotado a explorar as variedades de PI-álgebras associativas G-graduadas, de posto finito. Mais precisamente, propomos descrever aquelas variedades que são minimais, de um dado G-expoente, por meio de álgebras geradoras adequadas relacionadas às álgebras de matrizes bloco triangulares superiores. Por outro lado, impondo algumas condições extras sobre $UT_G(A_1, \ldots, A_m)$, provamos que tais álgebras de matrizes bloco triangulares superiores G-graduadas geram variedades minimais. Os resultados obtidos nesta parte se encontram no artigo [31] submetido para publicação.

Neste resumo, damos as principais definições relacionadas à PI-teoria, bem como as notações que serão utilizadas ao longo deste texto. Contextualizamos os tópicos abordados, dando mais detalhes sobre nossos principais objetivos e suas relevâncias, e discutimos sobre as ferramentas de estudo empregadas. Finalizamos este resumo listando os assuntos abordados em cada capítulo desta tese.

Seja A uma álgebra associativa sobre um corpo F de característica zero e seja G um grupo abeliano finito. Dizemos que A é uma *álgebra G-graduada* se $A = \oplus_{g\in G} A_g$ (soma direta como espaço vetorial), onde, para cada $g \in G$, A_g é um subespaço vetorial de A, e $A_g A_h \subseteq A_{gh}$, para todo $g, h \in G$. Cada subespaço A_g é chamado uma *componente graduada* de grau g de A. Além disso, um elemento $a \in A_g$ é dito ser *homogêneo* de grau g e o seu grau é denotado por $|a|_A$. Quando a álgebra graduada A é unitária e todos os seus elementos homogêneos não-nulos são invertíveis, dizemos que A é uma *álgebra de divisão graduada*. Uma subálgebra (subespaço vetorial, ideal, respectivamente) V de uma álgebra G-graduada A que admite a decomposição $V = \bigoplus_{g\in G}(V \cap A_g)$ é chamada uma *subálgebra graduada (subespaço vetorial graduado, ideal graduado,* respectivamente) de A. É notória a relevância das álgebras graduadas nas pesquisas dos últimos 20 anos (veja, por exemplo, [1, 5, 9, 10, 29, 32]). Ainda, dadas duas álgebras graduadas $A = \oplus_{g\in G} A_g$ e $B = \oplus_{g\in G} B_g$, se existe um isomorfismo de álgebras $\varphi : A \to B$ tal que $\varphi(A_g) = B_g$, para todo $g \in G$, então dizemos que A é *G-isomorfa* à B, em outras palavras, A e B são isomorfas como álgebras G-graduadas.

Uma importante e bem conhecida álgebra com a qual lidamos nesta tese é a álgebra $M_k(F)$ de matrizes $k \times k$ sobre F, simplesmente denotada por M_k. Munimos essa álgebra com uma graduação adequada, a saber, uma *graduação elementar* da seguinte forma: fixada uma k-upla $\widetilde{g} = (g_1, \ldots, g_k) \in G^k$, tal graduação consiste em definir, para cada $h \in G$, $(M_k)_h := \mathrm{span}_F\{e_{ij} \mid g_i^{-1} g_j = h\}$, onde, para cada $i, j \in \{1, \ldots, k\}$, e_{ij} denota a (i, j)-matriz unitária de M_k. Note que, para cada $i, j \in \{1, \ldots, k\}$, a matriz unitária e_{ij} é homogênea com grau $g_i^{-1} g_j$. Por outro lado, em [13], foi afirmado que se as matrizes unitárias e_{ij} são homogêneas, para todo $i, j \in \{1, \ldots, k\}$, então a G-graduação sobre M_k é elementar. Vale observar que, no caso em que F é um corpo algebricamente fechado, as graduações elementares são essenciais na classificação de todas as G-graduações de M_k (veja [9]). Ainda, qualquer graduação elementar sobre a álgebra de matrizes M_k é induzida por uma aplicação $\alpha : \{1, \ldots, k\} \to G$, se definimos $|e_{ij}|_{M_k} = \alpha(i)^{-1}\alpha(j)$, para todo $i, j \in \{1, \ldots, k\}$. Aqui, a notação (M_k, α) indica que a álgebra M_k está munida da graduação elementar induzida pela aplicação α. Finalmente, dada a álgebra de matrizes (M_k, α), definimos a *aplicação peso* $w_\alpha : G \to \mathbb{N}$ como $w_\alpha(h) := |\{i \mid 1 \le i \le k,\ \alpha(i) = h\}|$, e o *subgrupo invariante*, relacionado à (M_k, α), como

$$\mathcal{H}_\alpha := \{h \in G \mid w_\alpha(hg) = w_\alpha(g), \quad \text{para todo } g \in G\}.$$

Tal subgrupo foi introduzido por Di Vincenzo e Spinelli, em [24], e é uma ferramenta crucial ao longo do nosso trabalho.

Ressaltamos que, quando F é algebricamente fechado, as álgebras de matrizes M_k são as únicas *álgebras simples* de dimensão finita, a menos de isomorfismo. Em relação ao contexto G-graduado, dizemos que uma álgebra G-graduada A é *G-simples* se $A^2 \neq 0$ e A não possui ideais graduados não-triviais. Mesmo neste caso, as álgebras de matrizes desempenham um papel fundamental na classificação das F-álgebras G-simples de dimensão finita, onde F é um corpo algebricamente fechado. Mais precisamente, em [10], Bahturin, Sehgal e Zaicev trabalhando em um contexto geral, obtiveram para grupos abelianos finitos que qualquer F-álgebra G-simples de dimensão finita é G-isomorfa à uma álgebra G-graduada dada por um produto tensorial entre M_k e uma álgebra de divisão graduada.

Além disso, observamos que a classificação anterior pode ser reescrita quando estamos lidando com alguns grupos particulares. Por exemplo, se F é um corpo algebricamente fechado e $G = C_2$, um grupo cíclico de ordem 2, em [35], é estabelecido que as F-álgebras G-simples de dimensão finita (bem conhecidas como as *superálgebras simples*) são, a menos de G-isomorfismo, iguais à:

(*i*) $M_{k,l} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, onde $k \geq l \geq 0$, $k \neq 0$, $A \in M_k$, $D \in M_l$, $B \in M_{k\times l}$ e $C \in M_{l\times k}$, munida da graduação $(M_{k,l})_0 := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ e $(M_{k,l})_1 := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$;

(*ii*) $M_n(F \oplus cF)$, onde $c^2 = 1$, com a graduação $(M_n(F \oplus cF))_0 := M_n$ e $(M_n(F \oplus cF))_1 := cM_n$.

Vale dizer que, em ambos os casos acima, conforme explicitaremos na Seção 1.1, podemos ver tais superálgebras simples como subálgebras graduadas de álgebras de matrizes munidas de uma graduação elementar. Ainda, assumindo que o corpo F é algebricamente fechado, também temos uma descrição das F-álgebras G-simples de dimensão finita, quando G é um grupo de ordem prima p (veja [21]).

Nesta tese, generalizamos tais resultados para o caso em que $G = C_n$ é um grupo cíclico finito de ordem n, exibindo uma caracterização das F-álgebras G-simples de dimensão finita vistas como subálgebras graduadas de álgebras de matrizes munidas de graduações elementares. Além disso, aplicando resultados de Aljadeff e Haile, apresentados em [3], estabelecemos interessantes condições a fim de obter um G-isomorfismo entre essas álgebras G-simples.

Neste momento, lidando em um contexto mais geral, dadas subálgebras graduadas $A_1, \ldots, A_m$ de álgebras de matrizes $(M_{d_1}, \alpha_1), \ldots, (M_{d_m}, \alpha_m)$, respectivamente, considere a álgebra de matrizes bloco triangular superior $UT(A_1, \ldots, A_m)$. De maneira natural, munimos tal álgebra $UT(A_1, \ldots, A_m)$ com a G-graduação elementar $\widetilde{\alpha}$ obtida "colando" as graduações elementares $\alpha_1, \ldots, \alpha_m$ dadas, e escreveremos a álgebra G-graduada assim obtida como $(UT(A_1, \ldots, A_m), \widetilde{\alpha})$ ou simplesmente $UT_G(A_1, \ldots, A_m)$.

As álgebras de matrizes bloco triangulares superiores aparecem em vários trabalhos, sendo um objeto significativo de estudo para muitos pesquisadores. Por exemplo, Valenti e Zaicev provaram que, a menos de isomorfismo graduado, todas as G-graduações da álgebra

$UT(F, \ldots, F)$ são, na verdade, G-graduações elementares (quando G é um grupo qualquer, não necessariamente finito e abeliano, e F é um corpo qualquer) (veja [34]). Recentemente, em [11], Borges e Diniz descreveram as G-graduações de álgebras de matrizes bloco triangulares superiores adequadas, no caso em que G é um grupo abeliano (não necessariamente finito) e F é um corpo algebricamente fechado de característica zero. Esta descrição também envolve as graduações elementares. Além disso, em [36], Yasumura estudou as G-graduações sobre as álgebras de matrizes bloco triangulares superiores, quando G é um grupo qualquer (não necessariamente finito e abeliano) e F é um corpo de característica zero, ou característica grande o suficiente, não necessariamente algebricamente fechado.

Seja F um corpo algebricamente fechado de característica zero. Assumindo que o grupo G é cíclico finito e considerando nossa descrição de cada F-álgebra G-simples A_i de dimensão finita como uma subálgebra graduada de uma álgebra de matrizes munida de graduação elementar, nesta tese, focamos nossos estudos nas álgebras $UT_G(A_1, \ldots, A_m)$. Em particular, propomos investigar propriedades relacionadas ao conjunto de todas as identidades polinomiais G-graduadas satisfeitas por $UT_G(A_1, \ldots, A_m)$. A fim de apresentar esses conceitos e clarificar nossos objetivos, precisamos estabelecer algumas definições e notações.

Primeiramente, lembramos que, de maneira natural, podemos definir $F\langle X; G\rangle$ como a *álgebra G-graduada associativa livre unitária livremente gerada por* $X_G := \cup_{g\in G} X_g$, onde $X_g := \{x_1^g, x_2^g, \ldots\}$ são conjuntos enumeráveis disjuntos de variáveis não comutativas, com $g \in G$. Dada uma álgebra graduada $A = \oplus_{g\in G} A_g$, um elemento $f = f(x_1^{g_{i_1}}, \ldots, x_n^{g_{i_n}})$ de $F\langle X; G\rangle$ é uma *identidade polinomial G-graduada* de A se $f(a_1, \ldots, a_n) = 0$, para todo $a_1 \in A_{g_{i_1}}, \ldots, a_n \in A_{g_{i_n}}$. O conjunto de todas as identidades polinomiais G-graduadas de A será denotado por $\mathrm{Id}_G(A)$. É bem conhecido que $\mathrm{Id}_G(A)$ é um *T_G-ideal* (ou um *T-ideal graduado*) de $F\langle X; G\rangle$, isto é, $\mathrm{Id}_G(A)$ é um ideal graduado, estável sob todos endomorfismos G-graduados de $F\langle X; G\rangle$. Lembramos que o chamado *caso ordinário* corresponde à $G = \{1_G\}$. Finalmente, se uma álgebra G-graduada A satisfaz uma identidade polinomial ordinária não-trivial (isto é, se existe um polinômio não nulo $f(x_1, \ldots, x_n) \in F\langle X\rangle$ tal que $f(a_1, \ldots, a_n) = 0$, para todo $a_i \in A$), então A é chamada uma *PI-álgebra G-graduada.*

Fixado um T_G-ideal I de $F\langle X; G\rangle$, é interessante e útil coletar todas as álgebras G-graduadas A satisfazendo $I \subseteq \mathrm{Id}_G(A)$. Para este fim, definimos a *variedade de álgebras G-graduadas* $\mathcal{V}^G$, determinada por I, como $\mathcal{V}^G := \mathcal{V}^G(I) = \{A \mid I \subseteq \mathrm{Id}_G(A)\}$ e denotamos seu T_G-ideal I como $\mathrm{Id}_G(\mathcal{V}^G)$. Se A é uma álgebra G-graduada tal que $\mathrm{Id}_G(\mathcal{V}^G) = \mathrm{Id}_G(A)$, então dizemos que a variedade $\mathcal{V}^G$ é *gerada por A* e escrevemos $\mathcal{V}^G = \mathrm{var}_G(A)$. As variedades exploradas ao longo dos capítulos desta tese serão aquelas geradas por uma PI-álgebra G-graduada finitamente gerada. Tais variedades serão chamadas de *posto finito*. Lembramos que, como foi mostrado em [5], sobre corpos algebricamente fechados de característica zero qualquer variedade de álgebras G-

graduadas de posto finito é gerada por uma PI-álgebra G-graduada de dimensão finita, quando G é um grupo finito. Tal fato também foi provado, independentemente, em [33] para grupos abelianos finitos.

Dentre os elementos da álgebra livre $F\langle X;G\rangle$, os chamados *polinômios multilineares* merecem um destaque especial em virtude de suas aplicabilidades na solução de vários problemas da PI-teoria. É bem conhecido que, sobre corpos de característica zero, o T_G-ideal $\mathrm{Id}_G(A)$ de uma álgebra graduada A é completamente determinado pelos polinômios multilineares que ele contém. Alguns exemplos de polinômios multilineares são os *polinômios de Capelli* e os *polinômios standard*, os quais serão utilizados ao longo deste trabalho. Dada uma álgebra graduada A e um inteiro $n \geq 1$, se consideramos P_n^G como o F-espaço vetorial gerado pelos polinômios multilineares de grau n de $F\langle X;G\rangle$, então o inteiro não-negativo $c_n^G(A) := \dim_F \frac{P_n^G}{P_n^G \cap \mathrm{Id}_G(A)}$ mede o crescimento das identidades polinomiais G-graduadas de A. Tal inteiro é chamado *n-ésima codimensão G-graduada* de A.

No caso em que A é uma PI-álgebra G-graduada, $\{c_n^G(A)\}_{n\geq 1}$ é limitada exponencialmente ([28]) e, nesta situação, definimos $\exp_G(A) := \lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$ como o *G-expoente* de A. Em 2011, Aljadeff, Giambruno e La Mattina provaram que o G-expoente existe e é um inteiro não-negativo, quando A é uma álgebra G-graduada de dimensão finita sobre um corpo algebricamente fechado de característica zero (veja [2]). Além disso, neste caso, eles apresentaram um método de como calcular o G-expoente de A. Mais precisamente, considere a generalização da decomposição de Wedderburn-Malcev de A, dada por $A = A_1 \oplus \cdots \oplus A_m + J(A)$, onde $A_1, \ldots, A_m$ são F-álgebras G-simples e $J(A)$, o radical de Jacobson de A, é um ideal graduado. Então, o G-expoente de A é o número $q := \max \dim_F(A_{r_1} \oplus \cdots \oplus A_{r_l})$, onde $A_{r_1}, \ldots, A_{r_l}$ são subálgebras G-simples distintas do conjunto $\{A_1, \ldots, A_m\}$ que satisfazem $A_{r_1}J(A)A_{r_2}J(A)\cdots A_{r_{l-1}}J(A)A_{r_l} \neq 0$.

No âmbito das variedades $\mathcal{V}^G$ geradas por uma PI-álgebra G-graduada A, definimos sua n-ésima *codimensão G-graduada* e seu *G-expoente* como sendo, respectivamente, a n-ésima codimensão G-graduada e o G-expoente de A. Em outras palavras, $c_n^G(\mathcal{V}^G) := c_n^G(A)$, para todo $n \geq 1$, e $\exp_G(\mathcal{V}^G) := \exp_G(A)$. Em particular, neste trabalho, estamos interessados em estudar as variedades $\mathcal{V}^G$ de PI-álgebras G-graduadas de posto finito tais que $\exp_G(\mathcal{V}^G) = d$ e para toda subvariedade própria $\mathcal{U}^G$ de $\mathcal{V}^G$ é válido que $\exp_G(\mathcal{U}^G) < d$. Essas variedades são chamadas *minimais de G-expoente d*.

Em relação ao caso ordinário, em [27], Giambruno e Zaicev mostraram que uma variedade $\mathcal{V}$ de posto finito, de um dado expoente, é minimal se, e somente se, $\mathcal{V}$ é gerada por uma álgebra de matrizes bloco triangular superior $UT(d_1, \ldots, d_m)$, de tamanho $d_1, \ldots, d_m$. Ainda, neste mesmo artigo, os autores provaram que o T-ideal de $UT(d_1, \ldots, d_m)$ satisfaz a *propriedade de* *fatorabilidade*, ou seja, $\mathrm{Id}(UT(d_1,\ldots,d_m))$ se decompõe em

$$\mathrm{Id}(UT(d_1,\ldots,d_m)) = \mathrm{Id}(M_{d_1})\cdots\mathrm{Id}(M_{d_m}).$$

Vale enfatizar que a fim de obter a decomposição acima, os autores aplicaram os significantes resultados desenvolvidos por Lewin em [30]. Tais resultados são considerados os passos cruciais na investigação do T-ideal de identidades polinomiais de álgebras de matrizes bloco triangulares superiores.

A propriedade de fatorabilidade é também um problema relevante quando consideramos álgebras com algumas estruturas adicionais. Por exemplo, para álgebras com involução, Di Vincenzo e La Scala obtiveram interessantes resultados sobre a propriedade de fatorabilidade relacionada aos T_*-ideais de algumas álgebras de matrizes bloco triangulares superiores $UT_*(A_1,\ldots,A_m)$, onde $A_1,\ldots,A_m$ são álgebras $*$-simples de dimensão finita (veja [20]).

Para um grupo cíclico finito G e dada uma m-upla $(A_1,\ldots,A_m)$ de álgebras G-simples de dimensão finita, consideramos a álgebra de matrizes bloco triangular superior G-graduada $UT_G(A_1,\ldots,A_m)$, munida de uma graduação elementar. Neste trabalho, estamos interessados em explorar a propriedade de fatorabilidade relacionada ao T_G-ideal $\mathrm{Id}_G(UT_G(A_1,\ldots,A_m))$. Mais precisamente, pretendemos estabelecer condições necessárias e suficientes a fim de obter que o T_G-ideal $\mathrm{Id}_G(UT_G(A_1,\ldots,A_m))$ se fatore em

$$\mathrm{Id}_G(UT_G(A_1,\ldots,A_m)) = \mathrm{Id}_G(A_1)\cdots\mathrm{Id}_G(A_m).$$

Destacamos que o conceito de *G-regularidade*, introduzido por Di Vincenzo e La Scala em [19], é uma importante ferramenta conectada à fatorabilidade do T_G-ideal de $UT_G(A_1,\ldots,A_m)$. Este conceito está relacionado a subálgebras graduadas B de álgebras de matrizes (munidas de graduações elementares) e leva em conta aplicações adequadas definidas sobre álgebras genéricas G-graduadas associadas à B, bem como todos os elementos do grupo G. No mesmo artigo, no caso em que G é um grupo abeliano finito e $A_1 \subseteq (M_{d_1},\alpha_1)$, $A_2 \subseteq (M_{d_2},\alpha_2)$ são subálgebras graduadas, os autores provaram que se uma das álgebras A_1 e A_2 é G-regular, então $\mathrm{Id}_G(UT_G(A_1,A_2)) = \mathrm{Id}_G(A_1)\mathrm{Id}_G(A_2)$. Além disso, se o grupo G tem ordem prima, eles estabeleceram que o T_G-ideal $\mathrm{Id}_G(UT_G(M_{d_1},M_{d_2}))$ é fatorável se, e somente se, uma das álgebras M_{d_1} ou M_{d_2} é G-regular. Enfatizamos que os resultados de Lewin, dados em [30], foram essenciais na obtenção destas afirmações. Ademais, vale dizer que a G-regularidade tem sido explorada em muitos trabalhos recentes (veja, por exemplo, [7, 12, 15, 16, 23]).

No caso em que $G = C_2$, um grupo cíclico de ordem 2, e $A_1,\ldots,A_m$ são álgebras G-simples de dimensão finita, a fatorabilidade dos T_G-ideais $\mathrm{Id}_G(UT_G(A_1,\ldots,A_m))$ foi desenvolvida, em [7], por Avelar, Di Vincenzo e da Silva. Foi provado que o T_G-ideal $\mathrm{Id}_G(UT_G(A_1,\ldots,A_m))$

é fatorável se, e somente se, existe no máximo um índice $i \in \{1, \ldots, m\}$ tal que A_i é uma superálgebra simples não-G-regular. Além disso, eles mostraram que tais afirmações são equivalentes à existência de uma única classe de isomorfismo de G-graduações para $UT_G(A_1, \ldots, A_m)$.

Nesta tese, generalizamos as equivalências acima, obtendo as afirmações similares para o caso em que G é um p-grupo cíclico, onde p é um primo arbitrário. Mais precisamente, provamos o seguinte resultado:

Teorema A. *Seja p um número primo e seja G um p-grupo cíclico. Dadas álgebras G-simples de dimensão finita $A_1, \ldots, A_m$, considere $A = UT_G(A_1, \ldots, A_m)$. As seguintes afirmações são equivalentes:*

(i) O T_G-ideal de A é fatorável;

(ii) Existe no máximo um índice $\ell \in \{1, \ldots, m\}$ tal que A_ℓ é uma álgebra G-simples não-G-regular;

(iii) Existe uma única classe de isomorfismo de G-graduações para A.

Destacamos que, para obter o teorema acima, aplicamos técnicas diferentes daquelas empregadas no caso C_2. Um papel crucial é desempenhado pelos subgrupos invariantes $\mathcal{H}_{\widetilde{\alpha}}^{(l)}$ relacionados às álgebras G-simples A_l que aparecem nos blocos diagonais de $(UT(A_1, \ldots, A_m), \widetilde{\alpha})$. Na sequência, diremos algumas palavras sobre a G-regularidade e sua conexão com os subgrupos invariantes.

Primeiramente, em [19], Di Vincenzo e La Scala caracterizaram as álgebras de matrizes (M_k, α) que são G-regulares através de propriedades relacionadas às aplicações α. Mais precisamente, é válido que (M_k, α) é G-regular se, e somente se, existe $c \in \mathbb{N}^*$ tal que $w_\alpha(h) = c$, para todo $h \in G$. Além disso, eles obtiveram uma caracterização das superálgebras simples C_2-regulares, mostrando que $M_{k,l}$ é C_2-regular se, e somente se, $k = l$, enquanto $M_n(F \oplus cF)$ é C_2-regular, para todo $n \geq 1$.

Para qualquer grupo cíclico finito G, uma vez que estamos considerando cada álgebra G-simples de dimensão finita como uma subálgebra graduada de uma álgebra de matrizes munida de uma graduação elementar, propomos caracterizar as álgebras G-simples G-regulares de dimensão finita. Acontece que, neste caso, estabelecemos uma interessante conexão entre tais álgebras G-regulares e os subgrupos invariantes. Mais precisamente, provamos que uma álgebra G-simples de dimensão finita, sobre um corpo algebricamente fechado, é G-regular se, e somente se, o subgrupo invariante relacionado à essa álgebra G-simples coincide com o grupo G.

Como consequência desta caracterização, obtemos importantes resultados quando lidamos com as álgebras de matrizes bloco triangulares superiores G-graduadas $(UT(A_1, \ldots, A_m), \widetilde{\alpha})$.

Em particular, se G é um p-grupo cíclico, com p sendo um número primo, provamos que a G-regularidade de A_a ou A_b é equivalente à $\mathcal{H}_{\widetilde{\alpha}}^{(a)}\mathcal{H}_{\widetilde{\alpha}}^{(b)} = G$. Mais ainda, estabelecemos interessantes e úteis relações entre os subgrupos invariantes $\mathcal{H}_{\widetilde{\alpha}}^{(l)}$, a existência de uma única classe de isomorfismos de G-graduações para $UT_G(A_1, \ldots, A_m)$ e os *T_G-ideais indecomponíveis* associados às identidades polinomiais G-graduadas de $UT_G(A_1, \ldots, A_m)$. Consequentemente, tais fatos se revelaram como pontos cruciais para concluir nossos resultados principais sobre a propriedade de fatorabilidade de $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, no caso em que G é um p-grupo cíclico.

Contudo, se o grupo cíclico finito G não é um p-grupo, então as equivalências relacionadas à propriedade de fatorabilidade dos T_G-ideais $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, descritas anteriormente, não são mais necessariamente válidas. Mais precisamente, construímos uma adequada álgebra de matrizes bloco triangular superior G-graduada $A = (UT(A_1, A_2), \widetilde{\alpha})$ tal que $\mathrm{Id}_G(A)$ é fatorável, mas com ambas A_1 e A_2 não sendo álgebras G-simples G-regulares. Acontece que embora essas álgebras não sejam G-regulares, elas pertencem à uma nova classe de subálgebras graduadas de (M_k, α), a saber, as *subálgebras graduadas α-regulares*. Tal conceito generaliza a definição de subálgebras graduadas G-regulares, já que também consideramos aplicações adequadas definidas sobre álgebras genéricas G-graduadas, mas associadas aos elementos pertencendo à imagem de α (ao invés de estarem necessariamente associadas à todos os elementos de G). Neste contexto, assumindo que G é um grupo cíclico finito, estabelecemos que qualquer álgebra G-simples de dimensão finita (a qual é uma subálgebra graduada de (M_k, α)) é α-regular se, e somente se, a imagem de α coincide com uma classe lateral do subgrupo invariante relacionado à essa álgebra G-simples em G. Além disso, estabelecemos condições necessárias e suficientes a fim de obter que o T_G-ideal $\mathrm{Id}_G(UT_G(A_1, A_2))$ é fatorável, no caso em que G é um grupo cíclico finito e as álgebras G-simples A_1 e A_2 são α_1-regular e α_2-regular, respectivamente.

Voltando à nossa discussão sobre as variedades minimais e as álgebras de matrizes bloco triangulares superiores G-graduadas, vamos pontuar algumas observações e resultados. Como já mencionamos anteriormente, no caso ordinário, qualquer variedade minimal de PI-álgebras associativas sobre F, de posto finito, com um dado expoente, é gerada por uma álgebra de matrizes bloco triangular superior $UT(d_1, \ldots, d_m)$, e a recíproca é verdadeira (veja [27]). Recentemente, em [17], para G sendo um grupo de ordem prima, Di Vincenzo, da Silva e Spinelli provaram que uma variedade de PI-álgebras G-graduadas de posto finito é minimal de G-expoente d se, e somente se, ela é gerada por uma álgebra G-graduada $UT_G(A_1, \ldots, A_m)$ satisfazendo $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$, onde $A_1, \ldots, A_m$ são álgebras G-simples de dimensão finita. Para álgebras munidas de outras estruturas adicionais veja, por exemplo, [18] e [20].

No caso em que G é um grupo cíclico finito, seja $\mathcal{V}^G$ uma variedade de PI-álgebras G-graduadas associativas sobre F, de posto finito, de um dado G-expoente d. Nesta tese, mostramos que se $\mathcal{V}^G$ é minimal, então ela é gerada por uma álgebra de matrizes bloco triangular superior G-graduada $UT_G(A_1, \ldots, A_m)$ adequada satisfazendo $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$, onde $A_1, \ldots, A_m$ são álgebras G-simples de dimensão finita. Por outro lado, dada uma m-upla $(A_1, \ldots, A_m)$ de álgebras G-simples de dimensão finita e considerando $A = UT_G(A_1, \ldots, A_m)$, resta provar a recíproca do resultado acima. Neste texto, estabelecemos o seguinte resultado:

Teorema B. *Seja G um grupo cíclico finito. Dadas álgebras G-simples de dimensão finita $A_1, \ldots, A_m$, considere $A = UT_G(A_1, \ldots, A_m)$. Assuma que pelo menos uma das seguintes propriedades é válida:*

(*i*) *$m = 1$ ou 2;*

(*ii*) *existe $\ell \in \{1, \ldots, m\}$ tal que o subgrupo invariante relacionado à álgebra G-simples A_ℓ é $\{1_G\}$;*

(*iii*) *os subgrupos invariantes relacionados às álgebras G-simples $A_1, \ldots, A_m$ são todos (exceto para no máximo um) iguais à G.*

Então $\mathrm{var}_G(A)$ *é minimal com* $\exp_G(A) = \dim_F(A_1 \oplus \cdots \oplus A_m)$.

Ainda, assumindo pelo menos uma das condições acima, concluímos também que quaisquer duas álgebras de matrizes bloco triangulares superiores G-graduadas, munidas de graduações elementares, são G-isomorfas se, e somente se, elas satisfazem as mesmas identidades polinomiais G-graduadas. Neste sentido, contribuímos com o problema do isomorfismo no contexto da PI-teoria. Mais pesquisas relacionadas à este problema podem ser encontradas em [3, 8, 14, 17, 18, 24, 29].

Observamos que obter tais resultados anteriormente citados significa dar um passo importante no estudo das variedades minimais de PI-álgebras G-graduadas, de posto finito, com G sendo um grupo abeliano finito arbitrário. Além disso, vale mencionar que para alcançar essas afirmações, uma ferramenta crucial usada são os chamados *polinômios de Kemer* associados às álgebras $UT_G(A_1, \ldots, A_m)$. Esses polinômios desempenham um papel importante na PI-teoria (veja, por exemplo, [4, 5, 17]).

Esta tese está estruturada por meio de cinco capítulos. No Capítulo 1, assumimos que G é um grupo abeliano finito e lembramos alguns dos principais tópicos associados à teoria das álgebras satisfazendo identidades polinomiais. Começamos definindo álgebras G-graduadas e exibindo alguns exemplos. Em especial, construímos cuidadosamente a álgebra de matrizes bloco triangular superior G-graduada $UT_G(A_1, \ldots, A_m)$, onde $A_1, \ldots, A_m$ são subálgebras graduadas de álgebras de matrizes munidas de graduações elementares. Além disso, apresentamos a definição dos T_G-ideais de identidades polinomiais G-graduadas, as codimensões G-graduadas, o G-expoente, as variedades minimais e as álgebras G-graduadas minimais.

No Capítulo 2, também assumimos que o grupo G é abeliano finito e lembramos a definição de G-regularidade e fatorabilidade dos T_G-ideais $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, onde $A_1, \ldots, A_m$ são subálgebras graduadas de álgebras de matrizes munidas de graduações elementares. Além disso, investigamos a propriedade de fatorabilidade quando lidamos com álgebras de matrizes bloco triangulares superiores G-graduadas tendo dois blocos, no caso em que A_1 e A_2 são subálgebras graduadas de álgebras de matrizes munidas de graduações elementares. Feito isso, introduzimos as subálgebras graduadas α-regulares de uma álgebra de matrizes (M_k, α) e o conceito de subgrupos invariantes. Finalizamos este capítulo relacionando as álgebras de matrizes (M_k, α) que são α-regulares com os seus subgrupos invariantes.

No Capítulo 3, assumimos que G é um grupo cíclico finito. A primeira seção deste capítulo é dedicada à caracterização das F-álgebras G-simples de dimensão finita como subálgebras graduadas de álgebras de matrizes munidas de apropriadas graduações elementares. Na sequência, estabelecemos interessantes condições necessárias e suficientes a fim de existir um isomorfismo graduado entre duas tais álgebras G-simples, bem como importantes resultados técnicos relacionados à elas. Finalmente, abordamos a noção de G-regularidade e α-regularidade quando associadas às álgebras G-simples de dimensão finita, e também conectamos tais conceitos com os subgrupos invariantes.

O Capítulo 4 tem como objetivo apresentar um dos principais resultados desta tese. Mais precisamente, aquele que estabelece condições necessárias e suficientes para a fatorabilidade do T_G-ideal $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, no caso em que G é um p-grupo cíclico, com p sendo um número primo, e $A_1, \ldots, A_m$ são álgebras G-simples de dimensão finita. Apresentamos algumas condições suficientes para a existência de uma única classe de isomorfismo de G-graduações para $UT_G(A_1, \ldots, A_m)$, bem como para $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$ ser indecomponível. Tais condições estão intimamente ligadas com os subgrupos invariantes relacionados aos blocos G-simples $A_1, \ldots, A_m$. Finalizamos este capítulo discutindo a propriedade de fatorabilidade dos T_G-ideais $\mathrm{Id}_G(UT_G(A_1, A_2))$, no caso em que G não é necessariamente um p-grupo cíclico, e as álgebras G-simples A_1 e A_2 são α_1-regular e α_2-regular, respectivamente.

No Capítulo 5, o grupo G é cíclico finito e exploramos as variedades minimais de PI-álgebras G-graduadas associativas sobre F, de posto finito, com um dado G-expoente. Na primeira seção, estabelecemos que tais variedades minimais são geradas por álgebras de matrizes bloco triangulares superiores G-graduadas adequadas. Nas seções seguintes, introduzimos os polinômios de Kemer para as álgebras $UT_G(A_1, \ldots, A_m)$. Além disso, usando tais polinômios, estabelecemos importantes propriedades estruturais entre duas álgebras de matrizes bloco triangulares superiores G-graduadas. Finalmente, concluímos que $\mathrm{var}_G(UT_G(A_1, \ldots, A_m))$ é minimal, quando a álgebra $UT_G(A_1, \ldots, A_m)$ satisfaz pelo menos uma das importantes condições dadas por (i), (ii) ou (iii).

Nas Considerações Finais, apresentamos uma revisão geral de alguns dos principais resultados abordados ao longo desta tese. Em particular, destacamos a caracterização das álgebras G-simples de dimensão finita, a propriedade de fatorabilidade do T_G-ideal $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, no caso em que G é um p-grupo cíclico, e as afirmações obtidas quando trabalhamos com as variedades minimais de PI-álgebras G-graduadas associativas, de posto finito. Além disso, dedicamos esta parte final para discutir sobre alguns resultados cuja demonstração foi feita, nesta tese, diferentemente daquela apresentada em [22]; mencionando ainda outros resultados obtidos em [22].

Introduction

In the last decades, the study of the algebras satisfying polynomial identities, namely PI-algebras, has been developed on a large scale. There is a growing number of researches involving such algebras, which explicite the importance of this theory in the mathematical ambit. In this sense, the results present in this thesis contribute, in a positive way, with the works in the area of algebra and, particularly, with those concerning to PI-algebras. It is important highlighting that these results were developed in a joint work with my doctoral advisor, Professor Viviane Ribeiro Tomaz da Silva, and with Professor Onofrio Mario Di Vincenzo (Università degli Studi della Basilicata - Italy).

Let F be an algebraically closed field of characteristic zero and consider G a finite cyclic group. Throughout this work, all the F-algebras are assumed to be associative. We dedicate the first part of this thesis to studying the *factoring property* associated to the T_G-ideals of G-graded polynomial identities satisfied by the G-graded upper block triangular matrix algebras $UT_G(A_1, \ldots, A_m)$, where $A_1, \ldots, A_m$ are finite dimensional G-simple algebras over F. Our results obtained in this part have already been published and can be found in [22].

Secondly, the present work is devoted to exploring the varieties of associative G-graded PI-algebras over F, of finite basic rank. More precisely, we propose to describe those varieties which are minimal, of a given G-exponent, by means of suitable generating algebras related to upper block triangular matrix algebras. On the other hand, by imposing some extra conditions on $UT_G(A_1, \ldots, A_m)$, we prove that such G-graded upper block triangular matrix algebras generate minimal varieties. The results obtained in this part are in the paper [31] submitted for publication.

In this introduction, we give the main definitions related to the PI-theory, as well as the notations which will be used along this text. We contextualize the topics addressed, giving more details about our main aims and their relevance, and we discourse regarding the study tools employed. We finish this introduction listing the subjects covered in each chapter of this thesis.

Let A be an associative algebra over a field F of characteristic zero and G be a finite abelian group. We say that A is a *G-graded algebra* if $A = \oplus_{g \in G} A_g$ (direct sum as vector space), where, for each $g \in G$, A_g is a vector subspace of A, and $A_gA_h \subseteq A_{gh}$, for all $g, h \in G$. Each subspace A_g is called a *graded component* of degree g of A. Moreover, an element $a \in A_g$ is said to be *homogeneous* of degree g and its degree is denoted by $|a|_A$. When the graded algebra A is unitary and all its non-zero homogeneous elements are invertible, we say that A is a *graded skew field.* A subalgebra (vector subspace, ideal, respectively) V of a G-graded algebra A which admits the decomposition $V = \bigoplus_{g\in G}(V \cap A_g)$ is called a *graded subalgebra (graded vector subspace, graded ideal,* respectively) of A. It is notorious the relevance of the graded algebras in researches over the last 20 years (see, for instance, [1, 5, 9, 10, 29, 32]). Given two graded algebras $A = \oplus_{g\in G}A_g$ and $B = \oplus_{g\in G}B_g$, if there exists an algebra isomorphism $\varphi : A \to B$ such that $\varphi(A_g) = B_g$, for all $g \in G$, then we say that A is *graded-isomorphic* to B, in other words, A and B are isomorphic like G-graded algebras.

An important and well known algebra which we deal in this thesis is the $k \times k$ matrix algebra $M_k(F)$ over F, shortly denoted by M_k. We endow it with a suitable grading, namely, an *elementary grading* in the following way: fixed a k-tuple $\widetilde{g} = (g_1, \dots, g_k) \in G^k$, such grading consists in defining, for each $h \in G$, $(M_k)_h := \mathrm{span}_F\{e_{ij} \mid g_i^{-1}g_j = h\}$, where, for each $i, j \in \{1, \dots, k\}$, e_{ij} denotes the (i, j)-matrix unit of M_k. Notice that, for each $i, j \in \{1, \dots, k\}$, the matrix unit e_{ij} is homogeneous with degree $g_i^{-1}g_j$. On the other hand, in [13], it was proved that if the matrix units e_{ij} are homogeneous, for all $i, j \in \{1, \dots, k\}$, then the G-grading on M_k is elementary. It is worth mentioning that in case F is an algebraically closed field, the elementary gradings are essential in the classification of all G-gradings of M_k (see [9]). Still, any elementary grading on the matrix algebra M_k is induced by a map $\alpha : \{1, \dots, k\} \to G$, if we define $|e_{ij}|_{M_k} = \alpha(i)^{-1}\alpha(j)$, for all $i, j \in \{1, \dots, k\}$. Here, the notation (M_k, α) indicates that the algebra M_k is equipped with the elementary grading induced by the map α. Finally, given the matrix algebra (M_k, α), we set the *weight map* $w_\alpha : G \to \mathbb{N}$ as $w_\alpha(h) := |\{i \mid 1 \leq i \leq k,\ \alpha(i) = h\}|$, and the *invariance subgroup*, related to (M_k, α), as

$$\mathcal{H}_\alpha := \{h \in G \mid w_\alpha(hg) = w_\alpha(g), \quad \text{for all } g \in G\}.$$

Such subgroup was introduced by Di Vincenzo and Spinelli, in [24], and it is a crucial tool throughout our work.

We highlight that, when F is algebraically closed, the matrix algebras M_k are the unique finite dimensional *simple algebras*, up to isomorphism. Regarding to G-graded context, we say that a G-graded algebra A is *G-simple* if $A^2 \neq 0$ and A has no non-trivial graded ideals. Even in this case, the matrix algebras also play a fundamental role in the classification of the finite dimensional G-simple F-algebras, where F is an algebraically closed field. More precisely, in [10], Bahturin, Sehgal and Zaicev by working in a context general, obtained for finite abelian groups that any finite dimensional G-simple F-algebra is graded-isomorphic to a G-graded algebra given by a tensor product of M_k and a graded skew field.

Furthermore, we remark that the previously classification can be rewritten when we are dealing with some particular groups. For instance, if F is an algebraically closed field and $G = C_2$, a cyclic group of order 2, in [35], it is proved that the finite dimensional G-simple F-algebras (also known as the *simple superalgebras*) are, up to graded isomorphism, equal to:

(i) $M_{k,l} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $k \geq l \geq 0$, $k \neq 0$, $A \in M_k$, $D \in M_l$, $B \in M_{k\times l}$ and $C \in M_{l\times k}$, endowed with the grading $(M_{k,l})_0 := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and $(M_{k,l})_1 := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$;

(ii) $M_n(F \oplus cF)$, where $c^2 = 1$, with the grading $(M_n(F \oplus cF))_0 := M_n$ and $(M_n(F \oplus cF))_1 := cM_n$.

It is worth saying that, in both above cases, as we will explicit in Section 1.1, we can see such simple superalgebras as graded subalgebras of matrix algebras endowed with an elementary grading. Still, by assuming that the field F is algebraically closed, we also have a description of the finite dimensional G-simple F-algebras, when G is a group of prime order p (see [21]).

In this thesis, we generalize such results for the case $G = C_n$, a finite cyclic group of order n, by exhibiting a characterization of the finite dimensional G-simple F-algebras seen as graded subalgebras of matrix algebras endowed with elementary gradings. Furthermore, by applying results of Aljadeff and Haile, presented in [3], we establish nice conditions in order to obtain a graded isomorphism between these G-simple algebras.

At this moment, dealing in a more general context, given graded subalgebras $A_1, \ldots, A_m$ of matrix algebras $(M_{d_1}, \alpha_1), \ldots, (M_{d_m}, \alpha_m)$, respectively, consider the upper block triangular matrix algebra $UT(A_1, \ldots, A_m)$. Naturally, we endow such algebra $UT(A_1, \ldots, A_m)$ with the elementary G-grading $\widetilde{\alpha}$ obtained by gluing the given elementary gradings $\alpha_1, \ldots, \alpha_m$, and we will write the G-graded algebra obtained in this way as $(UT(A_1, \ldots, A_m), \widetilde{\alpha})$ or simply by $UT_G(A_1, \ldots, A_m)$.

The upper block triangular matrix algebras appear in several works, being a significant object of study for many researchers. For instance, Valenti and Zaicev proved that, up to graded isomorphism, all the G-gradings of the algebra $UT(F, \ldots, F)$ are, actually, elementary G-gradings (when G is an any group, not necessarily finite and abelian, and F is an any field) (see [34]). Recently, in [11], Borges and Diniz described the G-gradings of suitable upper block triangular matrix algebras, in case G is an abelian group (not necessarily finite) and F is an algebraically closed field of characteristic zero. This description also involves the elementary gradings. Moreover, in [36], Yasumura studied the G-gradings on the algebra of upper block triangular matrices, when G is an any group (not necessarily finite and abelian) and F is a field of characteristic either zero or large enough, not necessarily algebraically closed.

Let F be an algebraically closed field of characteristic zero. By assuming that the group G is finite cyclic and considering our description of each finite dimensional G-simple F-algebra A_i as a graded subalgebra of a matrix algebra endowed with elementary grading, in this thesis, we focus our studies on the algebras $UT_G(A_1, \ldots, A_m)$. In particular, we propose to investigate properties related to the set of all G-graded polynomial identities satisfied by $UT_G(A_1, \ldots, A_m)$. In order to present these concepts and to clarify our aims, we need to establish some definitions and notations.

Firstly, we recall that, in a natural way, we can define $F\langle X; G\rangle$ as the *unitary free associative G-graded algebra freely generated by* $X_G := \cup_{g\in G} X_g$, where $X_g := \{x_1^g, x_2^g, \ldots\}$ are disjoint countable sets of non-commutative variables, with $g \in G$. Given a graded algebra $A = \oplus_{g\in G} A_g$, an element $f = f(x_1^{g_{i_1}}, \ldots, x_n^{g_{i_n}})$ of $F\langle X; G\rangle$ is a *G-graded polynomial identity* of A if $f(a_1, \ldots, a_n) = 0$, for all $a_1 \in A_{g_{i_1}}, \ldots, a_n \in A_{g_{i_n}}$. The set of all the G-graded polynomial identities of A will be denoted by $\mathrm{Id}_G(A)$. It is well known that $\mathrm{Id}_G(A)$ is a *T_G-ideal* (or a *graded T-ideal*) of $F\langle X; G\rangle$, that is, $\mathrm{Id}_G(A)$ is a graded ideal, stable under all G-graded endomorphism of $F\langle X; G\rangle$. We recall that the so-called *ordinary case* corresponds to $G = \{1_G\}$. Finally, if a G-graded algebra A satisfies a non-trivial ordinary polynomial identity (that is, if there exists a non-zero polynomial $f(x_1, \ldots, x_n) \in F\langle X\rangle$ such that $f(a_1, \ldots, a_n) = 0$, for all $a_i \in A$), then A is called a *G-graded PI-algebra*.

Fixed a T_G-ideal I of $F\langle X; G\rangle$, it is interesting and useful to collect all the G-graded algebras A satisfying $I \subseteq \mathrm{Id}_G(A)$. To this end, we set the *variety of G-graded algebras* $\mathcal{V}^G$, determined by I, as $\mathcal{V}^G := \mathcal{V}^G(I) = \{A \mid I \subseteq \mathrm{Id}_G(A)\}$ and we denote its T_G-ideal I as $\mathrm{Id}_G(\mathcal{V}^G)$. If A is a G-graded algebra such that $\mathrm{Id}_G(\mathcal{V}^G) = \mathrm{Id}_G(A)$, thus we say that the variety $\mathcal{V}^G$ is *generated by A* and we write $\mathcal{V}^G = \mathrm{var}_G(A)$. The varieties explored along the chapters of this thesis will be those generated by a finitely generated G-graded PI-algebra. Such varieties will be called of *finite basic rank*. We recall that, as shown in [5], over algebraically closed fields of characteristic zero any variety of G-graded algebras of finite basic rank is generated by a finite dimensional G-graded PI-algebra, when G is a finite group. Such fact also was proved, independently, in [33] for finite abelian groups.

Among the elements of the free algebra $F\langle X; G\rangle$, the so-called *multilinear polynomials* deserve a great prominence due to their applicability in the solution of several problems of the PI-theory. It is well known that, over fields of characteristic zero, the T_G-ideal $\mathrm{Id}_G(A)$ of a graded algebra A is completely determined by the multilinear polynomials it contains. Some examples of multilinear polynomials are the *Capelli polynomials* and the *standard polynomials*, which will be used throughout this work. Given a graded algebra A and an integer $n \geq 1$, if we consider P_n^G as the F-vector space spanned by the multilinear polynomials of degree n of $F\langle X;G\rangle$, then the non-negative integer $c_n^G(A) := \dim_F \frac{P_n^G}{P_n^G \cap \mathrm{Id}_G(A)}$ measures the growth of the G-graded polynomial identities of A. Such integer is called nth *G-graded codimension* of A.

In case A is a G-graded PI-algebra, $\{c_n^G(A)\}_{n\geq 1}$ is exponentially bounded ([28]) and, in this situation, we define $\exp_G(A) := \lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$ as the *G-exponent* of A. In 2011, Aljadeff, Giambruno and La Mattina proved that this G-exponent exists and is a non-negative integer, when A is a finite dimensional G-graded algebra over an algebraically closed field of characteristic zero (see [2]). In addition, in this case, they presented a method of how to calculate the G-exponent of A. More precisely, consider the generalization of the decomposition of Wedderburn-Malcev of A, given by $A = A_1 \oplus \cdots \oplus A_m + J(A)$, where $A_1, \ldots, A_m$ are G-simple F-algebras (need not be ideals in A) and $J(A)$, the Jacobson radical of A, is a graded ideal given by a direct sum of vector spaces. Thus, the G-exponent of A is the number $q := \max \dim_F(A_{r_1} \oplus \cdots \oplus A_{r_l})$, where $A_{r_1}, \ldots, A_{r_l}$ are distinct G-simple subalgebras of the set $\{A_1, \ldots, A_m\}$ which satisfy $A_{r_1}J(A)A_{r_2}J(A)\cdots A_{r_{l-1}}J(A)A_{r_l} \neq 0$.

Within the scope of the varieties $\mathcal{V}^G$ generated by a G-graded PI-algebra A, we define its nth *G-graded codimension* and its *G-exponent* as being, respectively, the nth G-graded codimension and the G-exponent of A. In other words, $c_n^G(\mathcal{V}^G) := c_n^G(A)$, for all $n \geq 1$, and $\exp_G(\mathcal{V}^G) := \exp_G(A)$. In particular, in this work, we are interested in studying the varieties $\mathcal{V}^G$ of G-graded PI-algebras of finite basic rank such that $\exp_G(\mathcal{V}^G) = d$ and for every proper subvariety $\mathcal{U}^G$ of $\mathcal{V}^G$ it is valid that $\exp_G(\mathcal{U}^G) < d$. These varieties are called *minimal of G-exponent d*.

Concerning the ordinary case, in [27], Giambruno and Zaicev showed that a variety $\mathcal{V}$ of finite basic rank, of a given exponent, is minimal if, and only if, $\mathcal{V}$ is generated by an upper block triangular matrix algebra $UT(d_1, \ldots, d_m)$, of size $d_1, \ldots, d_m$. In this same paper, they proved that the T-ideal of $UT(d_1, \ldots, d_m)$ satisfies the *factoring property*, that is, $\mathrm{Id}(UT(d_1, \ldots, d_m))$ decomposes into

$$\mathrm{Id}(UT(d_1, \ldots, d_m)) = \mathrm{Id}(M_{d_1}) \cdots \mathrm{Id}(M_{d_m}).$$

It is worth emphasizing that in order to obtain the above decomposition, the authors applied the important results established by Lewin in [30]. Such results are considered the crucial steps in the investigation of the T-ideal of polynomial identities of upper block triangular matrix algebras.

The factoring property is also a relevant problem when we consider algebras with some additional structures. For instance, for algebras with involution, Di Vincenzo and La Scala obtained interesting results about the factoring property related to the T_*-ideals of some upper block triangular matrix algebras $UT_*(A_1, \ldots, A_m)$, where $A_1, \ldots, A_m$ are finite dimensional $*$-simple algebras (see [20]).

For a finite cyclic group G and an m-tuple $(A_1,\dots,A_m)$ of finite dimensional G-simple algebras, consider the G-graded upper block triangular matrix algebra $UT_G(A_1,\dots,A_m)$, endowed with an elementary grading. In this work, we are interested in exploring the factoring problem related to the T_G-ideal $\mathrm{Id}_G(UT_G(A_1,\dots,A_m))$. More precisely, we intend to establish necessary and sufficient conditions in order to obtain that the T_G-ideal $\mathrm{Id}_G(UT_G(A_1,\dots,A_m))$ factorizes into

$$\mathrm{Id}_G(UT_G(A_1,\dots,A_m)) = \mathrm{Id}_G(A_1)\cdots\mathrm{Id}_G(A_m).$$

We highlight that the concept of *G-regularity*, introduced by Di Vincenzo and La Scala in [19], is an important tool connected to the factorability of the T_G-ideal of $UT_G(A_1,\dots,A_m)$. This concept is related to graded subalgebras B of matrix algebras (endowed with elementary gradings) and takes into account suitable maps defined on G-graded generic algebras associated to B, as well as all the elements of the group G. In the same paper, in case G is a finite abelian group and $A_1\subseteq(M_{d_1},\alpha_1)$, $A_2\subseteq(M_{d_2},\alpha_2)$ are graded subalgebras, the authors proved that if one of A_1 and A_2 is G-regular, then $\mathrm{Id}_G(UT_G(A_1,A_2))=\mathrm{Id}_G(A_1)\mathrm{Id}_G(A_2)$. Furthermore, if G has prime order, they stated that the T_G-ideal $\mathrm{Id}_G(UT_G(M_{d_1},M_{d_2}))$ is factorable if, and only if, one of the algebras M_{d_1} or M_{d_2} is G-regular. We emphasize that the results of Lewin, given in [30], were essential in obtaining these statements. Moreover, it is worth saying that the G-regularity has been explored in many recent works (see, for instance, [7, 12, 15, 16, 23]).

In case $G=C_2$, a cyclic group of order 2, and $A_1,\dots,A_m$ are finite dimensional G-simple algebras, the factorability of the T_G-ideals $\mathrm{Id}_G(UT_G(A_1,\dots,A_m))$ was developed, in [7], by Avelar, Di Vincenzo and da Silva. They proved that the T_G-ideal $\mathrm{Id}_G(UT_G(A_1,\dots,A_m))$ is factorable if, and only if, there exists at most one index $i\in\{1,\dots,m\}$ such that A_i is a non-G-regular simple superalgebra. Moreover, they obtained that such statements are equivalent to the existence of a unique isomorphism class of G-gradings for $UT_G(A_1,\dots,A_m)$.

In this thesis, we generalize the above equivalences obtaining the similar ones for the case G is a cyclic p-group, where p is an arbitrary prime. More precisely, we prove the following result:

Theorem A. *Let p be a prime number and let G be a cyclic p-group. Given finite dimensional G-simple algebras $A_1,\dots,A_m$, consider $A=UT_G(A_1,\dots,A_m)$. The following statements are equivalent:*

(i) The T_G-ideal of A is factorable;

(ii) There exists at most one index $\ell\in\{1,\dots,m\}$ such that A_ℓ is a non-G-regular G-simple algebra;

(iii) There exists a unique isomorphism class of G-gradings for A.

We highlight that, in order the above theorem, we apply different techniques from those employed in case C_2. A crucial role is played by the invariance subgroups $\mathcal{H}_{\widetilde{\alpha}}^{(l)}$ related to the finite dimensional G-simple algebras A_l appearing in the diagonal blocks of $(UT(A_1, \ldots, A_m), \widetilde{\alpha})$. In the sequel, let us say some words about the G-regularity and its connection with the invariance subgroups.

Firstly, in [19], Di Vincenzo and La Scala characterized the matrix algebras (M_k, α) which are G-regular through properties related to the maps α. More precisely, (M_k, α) is G-regular if, and only if, there exists $c \in \mathbb{N}^*$ such that $w_\alpha(h) = c$, for all $h \in G$. Also, they obtained a characterization of the C_2-regular simple superalgebras, showing that $M_{k,l}$ is C_2-regular if, and only if, $k = l$, whereas $M_n(F \oplus cF)$ is C_2-regular, for all $n \geq 1$.

For any finite cyclic group G, since we are seeing each finite dimensional G-simple algebra as a graded subalgebra of a matrix algebra endowed with an elementary grading, we characterize the finite dimensional G-regular G-simple algebras. It turns out that, in this case, we establish a connection between such G-regular algebras and the invariance subgroups. More precisely, we prove that a finite dimensional G-simple algebra, over an algebraically closed field, is G-regular if, and only if, the invariance subgroup related to this G-simple algebra coincides with the group G.

As a consequence of this characterization, we obtain important results when we deal with the G-graded upper block triangular matrix algebras $(UT(A_1, \ldots, A_m), \widetilde{\alpha})$. In particular, if G is a cyclic p-group, with p being a prime number, we prove that the G-regularity of A_a or A_b is equivalent to $\mathcal{H}_{\widetilde{\alpha}}^{(a)}\mathcal{H}_{\widetilde{\alpha}}^{(b)} = G$. Additionally, we establish interesting and useful relations between the invariance subgroups $\mathcal{H}_{\widetilde{\alpha}}^{(l)}$, the existence of a unique isomorphism class of G-gradings for $UT_G(A_1, \ldots, A_m)$ and the *indecomposable T_G-ideals* associated to the G-graded polynomial identities of $UT_G(A_1, \ldots, A_m)$. Consequently, such facts reveal as crucial points to concluding our main results about the factoring property of $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, in case G is a cyclic p-group.

However, if the finite cyclic group G is not a p-group, thus the equivalences related to the factoring property of the T_G-ideals $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, described above, are no longer necessarily valid. More precisely, we build a suitable G-graded upper block triangular matrix algebra $A = (UT(A_1, A_2), \widetilde{\alpha})$ such that $\mathrm{Id}_G(A)$ is factorable, but with both A_1 and A_2 not being G-regular G-simple algebras. It turns out that although these algebras are not G-regular, they belong to a new class of graded subalgebras of (M_k, α), namely, the *α-regular graded subalgebras*. Such concept generalizes the definition of G-regular graded subalgebras, once we also consider suitable maps defined on G-graded generic algebras but associated to the elements belonging to the image of α (instead of being necessarily associated to all the elements of G). In this context, by assuming that G is a finite cyclic group, we obtain that any finite dimensional

G-simple algebra (which is a graded subalgebra of (M_k, α)) is α-regular if, and only if, the image of α coincides with a coset of invariance subgroup related to this G-simple algebra in G. Moreover, we establish necessary and sufficient conditions in order to obtain that the T_G-ideal $\mathrm{Id}_G(UT_G(A_1, A_2))$ is factorable, in case G is a finite cyclic group and the G-simple algebras A_1 and A_2 are α_1-regular and α_2-regular, respectively.

Coming back to our discussion about the minimal varieties and the G-graded upper block triangular matrix algebras, let us point out some remarks and results. As we have already mentioned above, in the ordinary case, any minimal variety of associative PI-algebras over F, of finite basic rank, with a given exponent, is generated by an upper block triangular matrix algebra $UT(d_1, \ldots, d_m)$, and the reciprocal is true (see [27]). Recently, in [17], for G being a group of prime order, Di Vincenzo, da Silva and Spinelli proved that a variety of G-graded PI-algebras of finite basic rank is minimal of G-exponent d if, and only if, it is generated by a G-graded algebra $UT_G(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$, where $A_1, \ldots, A_m$ are finite dimensional G-simple algebras. For algebras endowed with other additional structures see, for instance, [18] and [20].

In case G is a finite cyclic group, let $\mathcal{V}^G$ be a variety of associative G-graded PI-algebras over F, of finite basic rank, of a given G-exponent d. In this thesis, we show that if $\mathcal{V}^G$ is minimal, thus it is generated by a suitable G-graded upper block triangular matrix algebra $UT_G(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$, where $A_1, \ldots, A_m$ are finite dimensional G-simple algebras. On the other hand, given an m-tuple $(A_1, \ldots, A_m)$ of finite dimensional G-simple algebras and by considering $A = UT_G(A_1, \ldots, A_m)$, remains to prove the reciprocal of the above result. In this text, we establish the following result:

Theorem B. *Let G be a finite cyclic group. Given finite dimensional G-simple F-algebras $A_1, \ldots, A_m$, consider $A := (UT(A_1, \ldots, A_m), \widetilde{\alpha})$. Assume that at least one of the following properties hold:*

(i) $m = 1$ *or* 2*;*

(ii) there exists $\ell \in \{1, \ldots, m\}$ such that the invariance subgroup related to the G-simple algebra A_ℓ is $\{1_G\}$;

(iii) the invariance subgroups related to the G-simple algebras $A_1, \ldots, A_m$ are all (except for at most one) equal to G.

Then $\mathrm{var}_G(A)$ *is minimal with* $\exp_G(A) = \dim_F(A_1 \oplus \cdots \oplus A_m)$.

Still, under at least one of the above conditions we also conclude that any two G-graded upper block triangular matrix algebras, endowed with elementary gradings, are graded-isomorphic if, and only if, they satisfy the same G-graded polynomial identities. In this sense, we contribute to the isomorphism problem in the context of the PI-theory. More research related to this problem can be found in [3, 8, 14, 17, 18, 24, 29].

We remark that getting such results previously cited means taking an important step in the study of the minimal varieties of G-graded PI-algebras, of finite basic rank, with G being an arbitrary finite abelian group. Moreover, it is worth mentioning that in order to obtain these statements, a crucial tool used are the so-called *Kemer polynomials* associated to the algebras $UT_G(A_1, \ldots, A_m)$. These polynomials play an important role in PI-theory (see, for instance, [4, 5, 17]).

This thesis is structured by means of five chapters. In Chapter 1, we assume that G is a finite abelian group and we recall some of the main topics associated to the theory of the algebras satisfying polynomial identities. We start by defining G-graded algebras and by exhibiting some examples. In particular, we construct carefully the G-graded upper block triangular matrix algebra $UT_G(A_1, \ldots, A_m)$, where $A_1, \ldots, A_m$ are graded subalgebras of matrix algebras endowed with elementary gradings. We present the definition of the T_G-ideals of G-graded polynomial identities, the G-graded codimensions, the G-exponent, the minimal varieties and the minimal G-graded algebras.

In Chapter 2, we also assume that the group G is finite abelian and we recall the definition of G-regularity and factorability of the T_G-ideals $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, where $A_1, \ldots, A_m$ are graded subalgebras of matrix algebras endowed with elementary gradings. Moreover, we investigate the factoring property when we deal with G-graded upper block triangular matrix algebras having two blocks, in case A_1 and A_2 are graded subalgebras of matrix algebras endowed with elementary gradings. That done, we introduce the α-regular graded subalgebras of a matrix algebra (M_k, α) and the concept of invariance subgroups. We finish the chapter by relating the matrix algebras (M_k, α) which are α-regular with their invariance subgroups.

In Chapter 3, we assume that G is a finite cyclic group. The first section of this chapter is dedicated to the characterization of the finite dimensional G-simple F-algebras as graded subalgebras of matrix algebras endowed with appropriate elementary gradings. In the sequel, we establish necessary and sufficient conditions in order to have a graded isomorphism between two such G-simple algebras, as well as important technical results related to them. Finally, we approach the notion of G-regularity and α-regularity when associated to the finite dimensional G-simple algebras, and we also connect such concepts with the invariance subgroups.

Chapter 4 aims to present one of the main results of this thesis. More precisely, it presents one which establishes necessary and sufficient conditions for the factorability of the T_G-ideal $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, in case G is a cyclic p-group, with p being a prime number, and $A_1, \ldots, A_m$ are finite dimensional G-simple algebras. We present some sufficient conditions for the existence of a unique isomorphism class of G-gradings for $UT_G(A_1, \ldots, A_m)$, as well as for $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$ to be indecomposable. Such conditions are closely connected with the invariance subgroups related to the G-simple blocks $A_1, \ldots, A_m$. We finish this chapter by discussing the factoring property of the T_G-ideals $\mathrm{Id}_G(UT_G(A_1, A_2))$, in case G is not necessarily a cyclic p-group, and the G-simple algebras A_1 and A_2 are α_1-regular and α_2-regular, respectively.

In Chapter 5, the group G is finite cyclic and we explore the minimal varieties of associative G-graded PI-algebras over F, of finite basic rank, with a given G-exponent. In the first section, we prove that such minimal varieties are generated by suitable G-graded upper block triangular matrix algebras. In the following sections, we introduce the Kemer polynomials for the algebras $UT_G(A_1, \ldots, A_m)$. Furthermore, by using such polynomials, we establish important structural properties between any two G-graded upper block triangular matrix algebras. Finally, we conclude that $\mathrm{var}_G(UT_G(A_1, \ldots, A_m))$ is minimal, when the algebra $UT_G(A_1, \ldots, A_m)$ satisfies at least one of the important conditions given by $(i), (ii)$ or (iii).

In Final Considerations, we present a general review of some of the main results addressed throughout this thesis. In particular, we talk about the characterization of the finite dimensional G-simple algebras, the factoring property of the T_G-ideal $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, in case G is a cyclic p-group, and about the statements obtained when we work with the minimal varieties of associative G-graded PI-algebras, of finite basic rank. Moreover, we dedicate this final part to discussing about some results whose proofs were done, in this thesis, differently from those presented in [22]; also mentioning other results obtained in [22].

Chapter 1

Preliminaries and the algebras $UT_G(A_1, \dots, A_m)$

Let G be a finite abelian group and let F be a field of characteristic zero. In this chapter, we shall give a general review of several concepts related to PI-theory. In particular, we will present the definition of G-graded F-algebras and we will give some important examples, with special emphasis on the G-graded upper block triangular matrix algebra $UT_G(A_1, \dots, A_m)$, where $A_1, \dots, A_m$ are graded subalgebras of matrix algebras endowed with elementary gradings. Moreover, we will recall the definition of the T_G-ideal of G-graded polynomial identities, the sequence of G-graded codimensions and the G-exponent of a G-graded algebra. Furthermore, we will introduce the minimal varieties and the minimal G-graded algebras, and we will establish relevant connections between these concepts. We highlight that, throughout this thesis, all algebras which we consider are associative and over F.

1.1 G-graded algebras

Firstly, for any positive integers u and v such that $u \leq v$, let us define

$$[u, v] := \{u, u + 1, \dots, v - 1, v\}.$$

Given a finite set $\mathbf{X}$, we denote by $\mathrm{Sym}(\mathbf{X})$ the *symmetric group on* $\mathbf{X}$, whose elements are all bijective functions from $\mathbf{X}$ to $\mathbf{X}$. If $\mathbf{X} = [1, u]$, for some positive integer u, then we write $\mathrm{Sym}(\mathbf{X}) = \mathrm{Sym}(u)$.

An algebra A is said to be *G-graded* if, for each $g \in G$, there exists a vector subspace A_g of

A such that A decomposes into a direct sum of vector subspaces

$$A = \bigoplus_{g \in G} A_g$$

satisfying

$$A_g A_h \subseteq A_{gh}, \quad \text{for all } g, h \in G.$$

For each $g \in G$, we refer to the subspace A_g as *graded component* of degree g of A. In particular, if $G = C_2$, a cyclic group of order 2, thus the G-graded algebras are known as *superalgebras*.

Given a G-graded algebra $A = \oplus_{g \in G} A_g$ and an element a of A, then a can be written uniquely as

$$a = a_{g_1} + a_{g_2} + \cdots + a_{g_n},$$

where $a_{g_i} \in A_{g_i}$, for every $i \in [1, n]$ and $g_i \neq g_j$ for all $i, j \in [1, n]$, with $i \neq j$. If $a = a_g$ for some $g \in G$, then we say that a is *homogeneous* of degree g. In this case, we denote the degree of a homogeneous element $a = a_g \in A_g$ by $|a|_A$ and thus $|a|_A = g$. Denoting by 1_G the identity element of G, the G-grading of A is called *trivial* if A_g is equal to zero, for all $g \neq 1_G$. Notice that every algebra A admits at least the trivial G-grading. Moreover, if A is unitary and all non-zero homogeneous elements of A are invertible, then A is called a *graded skew field*.

We define the *support* of a G-graded algebra A as

$$\mathrm{Supp}(A) := \{g \in G \mid A_g \neq 0\}.$$

We remark that, in general, $\mathrm{Supp}(A)$ is not a subgroup of G.

If a vector subspace V of A is of the form

$$V = \bigoplus_{g \in G} (V \cap A_g),$$

then we say that V has a G-grading induced from A and we shall refer to the subspace V as *G-graded* (or, shortly, as *graded*). Similarly, we define *G-graded subalgebras* and *G-graded two-sided ideals* of A.

Let $A = \oplus_{g \in G} A_g$ and $B = \oplus_{g \in G} B_g$ be two G-graded algebras and $\varphi : A \to B$ a homomorphism of algebras. We say that φ is a *homomorphism of G-graded algebras* (or a *G-graded homomorphism*) if $\varphi(A_g) \subseteq B_g$, for all $g \in G$. In particular, φ is said to be a *G-graded embedding* if φ is a G-graded injective homomorphism. Moreover, if φ is an isomorphism of algebras and $\varphi(A_g) = B_g$, for all $g \in G$, then φ is called an *isomorphism of G-graded algebras* (or a *G-graded isomorphism*) and, in this case, we say that A is *graded-isomorphic* to B and we write

$A \cong_G B$. Furthermore, the G-graded homomorphisms $\varphi : A \to A$ are called *G-graded endomorphisms*, and the G-graded isomorphisms $\varphi : A \to A$ are called *G-graded automorphisms of A*.

At this point, we present an important example of G-graded algebra, with a suitable grading, which will be essential throughout this work. Let $M_k(F)$ be the $k \times k$ matrix algebra over F. When convenient, such matrix algebra will be simply denoted by M_k, as well as the vector space $M_{u\times v}(F)$ of all matrices, over F, with u rows and v columns, will be denoted by $M_{u\times v}$. Moreover, for each $i \in [1, u]$ and $j \in [1, v]$, we denote by e_{ij} the (i, j)-matrix unit of $M_{u\times v}$. We notice that $M_k = M_{k\times k}$.

Fixed any k-tuple $\widetilde{g} = (g_1, \ldots, g_k) \in G^k$, we define a G-grading on $A := M_k$ by setting

$$A_h := \operatorname{span}_F\{e_{ij} \mid g_i^{-1}g_j = h\}, \quad \text{for each } h \in G.$$

We refer to this G-grading as an *elementary G-grading* (or, shortly, an *elementary grading*). Note that, by the definition, for each $i, j \in [1, k]$, the matrix unit e_{ij} is homogeneous with degree $g_i^{-1}g_j$. Conversely, if all matrix units e_{ij} are homogeneous, then the G-grading on A is elementary (see [13]). We remark that, any elementary grading on A is induced by a map $\alpha : [1, k] \to G$, by setting the degree of e_{ij} equal to

$$\alpha(i)^{-1}\alpha(j), \quad \text{for all } i, j \in [1, k].$$

In this case, we shall denote the matrix algebra A endowed with the elementary grading induced by the map α (or by the k-tuple $\widetilde{g}$) as (A, α) (or as $(A, \widetilde{g})$) and we denote by $|a|_\alpha$ the degree of the homogeneous element a in A.

Moreover, we denote by $\mathcal{I}_\alpha$ the image of α, that is,

$$\mathcal{I}_\alpha := \alpha([1, k]),$$

and we define the *weight map* $w_\alpha : G \to \mathbb{N}$ as

$$w_\alpha(h) := |\{i \mid 1 \le i \le k,\ \alpha(i) = h\}|.$$

We remark that $w_\alpha(h) = 0$, when $h \notin \mathcal{I}_\alpha$. Hence $\mathcal{I}_\alpha = \{h \in G \mid w_\alpha(h) \neq 0\}$. Moreover, if there exists $c \in \mathbb{N}^*$ such that

$$w_\alpha(h) = c, \quad \text{for all } h \in \mathcal{I}_\alpha,$$

thus we say that *all fibers of the map α are equipotent.* Finally, a G-grading of a graded subalgebra B of M_k is called *elementary* if it is the restriction of an elementary G-grading of

M_k.

A useful and important class of G-graded algebras studied by several authors are the so-called *G-simple algebras*. Given a G-graded algebra $A = \oplus_{g\in G} A_g$, we say that A is *G-simple* if $A^2 \neq 0$ and A has no non-trivial graded ideals. Throughout this work, we will deal with these algebras in several results.

We remember that, if F is algebraically closed and $G = C_2 = \{\bar{0}, \bar{1}\}$, a cyclic group of order 2, thus the finite dimensional G-simple algebras, known as the *simple superalgebras*, are graded-isomorphic to one of the following superalgebras (see [35]):

(*i*) $M_{k,l} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $k \geq l \geq 0$, $k \neq 0$, $A \in M_k$, $D \in M_l$, $B \in M_{k\times l}$ and $C \in M_{l\times k}$, endowed with the grading $(M_{k,l})_{\bar{0}} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and $(M_{k,l})_{\bar{1}} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$;

(*ii*) $M_n(F \oplus cF)$, where $c^2 = 1$, with the grading $(M_n(F\oplus cF))_{\bar{0}} := M_n$ and $(M_n(F\oplus cF))_{\bar{1}} := cM_n$.

Notice that the superalgebras involved in this classification can be seen endowed with elementary gradings. In fact, in case (*i*), its elementary C_2-grading can be induced by the map $\alpha : [1, k+l] \to G$ such that $\alpha(i) = \bar{0}$, if $i \in [1, k]$, and $\alpha(i) = \bar{1}$, if $i \in [k+1, k+l]$. On the other hand, in case (*ii*), it is enough to note that we can see such algebra as a graded subalgebra of $M_{n,n}$ through the application on the elements of $M_n(F \oplus cF)$ to $M_{n,n}$:

$$C + cD \mapsto \begin{pmatrix} C & D \\ D & C \end{pmatrix},$$

where $C, D \in M_n$.

In case $G = C_p = \{\bar{0}, \bar{1}, \ldots, \overline{p-1}\}$ is a group of prime order p and the field F is algebraically closed, Di Vincenzo, da Silva and Spinelli ([17]) obtained a characterization of the finite dimensional G-simple F-algebras by applying the results established by Bahturin, Sehgal and Zaicev, in [9], and assertions stated by Di Vincenzo and Nardozza, in [21]. More precisely, they defined the following graded subalgebra of M_p with the elementary grading induced by the map $\alpha : [1, p] \to G$ such that $\alpha(i) = \overline{i-1}$:

$$D_p := \left\{ \begin{pmatrix} d_0 & d_1 & \cdots & d_{p-2} & d_{p-1} \\ d_{p-1} & d_0 & \ddots & & d_{p-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ d_2 & & \ddots & \ddots & d_1 \\ d_1 & d_2 & \cdots & d_{p-1} & d_0 \end{pmatrix} \mid d_0, d_1, \ldots, d_{p-1} \in F \right\},$$

and proved that any finite dimensional G-simple algebra is graded-isomorphic to one of the following G-graded algebras:

(i) M_k with an elementary grading;

(ii) the graded subalgebra $M_k(D_p)$ of M_{kp} endowed with an elementary grading,

for some positive integer k.

In this thesis, when the group G is a finite cyclic group, we will present a characterization of the finite dimensional G-simple F-algebras as graded subalgebras of matrix algebras endowed with some elementary gradings (see Section 3.1).

In the sequel, we will construct, for any m-tuple $(A_1, \ldots, A_m)$ of graded subalgebras of $(M_{d_1}, \widetilde{\alpha}_1), \ldots, (M_{d_m}, \widetilde{\alpha}_m)$, the G-graded upper block triangular matrix algebra $UT_G(A_1, \ldots, A_m)$.

Firstly, given the matrix algebras $M_{d_1}, \ldots, M_{d_m}$, let $\mathbf{U} := UT(d_1, \ldots, d_m)$ be the corresponding upper block triangular matrix algebra, of size $d_1, \ldots, d_m$, that is,

$$UT(d_1, \ldots, d_m) = \begin{pmatrix} M_{d_1} & M_{d_1 \times d_2} & \cdots & M_{d_1 \times d_m} \\ 0 & M_{d_2} & \cdots & M_{d_2 \times d_m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{d_m} \end{pmatrix}.$$

Let us write any of its elements as blocks (a_{ij}), where $i, j \in [1, m]$ and moreover

$$a_{ij} \in M_{d_i \times d_j} \text{ if } 1 \leq i \leq j \leq m \text{ and } a_{ij} = 0_{M_{d_i \times d_j}} \text{ otherwise.}$$

For each $l \in [1, m]$, let us define

$$\eta_0 := 0, \ \eta_l := \sum_{\iota=1}^{l} d_\iota \quad \text{and} \quad \mathbf{Bl}_l := [\eta_{l-1} + 1, \eta_l].$$

Still, fixed $1 \leq u \leq v \leq m$, for each $i \in [1, d_u], \ j \in [1, d_v]$, we denote the matrix unit of M_{η_m}, corresponding to the position (i, j) of the block

$$\mathbf{U}_{u,v} := \{(a_{st}) \in \mathbf{U} \mid a_{st} = 0_{M_{d_s \times d_t}}, \text{ for all } (s, t) \neq (u, v)\},$$

by

$$\mathbf{E}_{ij}^{(u,v)} := \mathbf{E}_{\eta_{u-1}+i, \eta_{v-1}+j},$$

where $\mathbf{E}_{\eta_{u-1}+i, \eta_{v-1}+j}$ is the $(\eta_{u-1} + i, \eta_{v-1} + j)$-matrix unit of M_{η_m}. By a direct computation we obtain

$$\mathbf{E}_{ij}^{(u,v)} \mathbf{E}_{i'j'}^{(u',v')} = \delta_{vu'} \delta_{ji'} \mathbf{E}_{ij'}^{(u,v')}, \tag{1.1}$$

where $\delta_{vu'}$ and $\delta_{ji'}$ is the *Kronecker delta*.

Now, given an m-tuple $(A_1, \ldots, A_m)$ of graded subalgebras of $(M_{d_1}, \widetilde{\alpha}_1), \ldots, (M_{d_m}, \widetilde{\alpha}_m)$, we define

$$UT(A_1, \ldots, A_m) := \{(a_{ij}) \in \mathbf{U} \mid a_{ll} \in A_l,\ l \in [1, m] \text{ and } a_{ij} \in M_{d_i \times d_j},\ 1 \le i < j \le m\}.$$

Let $\mathbf{A} := UT(A_1, \ldots, A_m)$. For every $1 \le u \le v \le m$, we set the block

$$\mathbf{A}_{u,v} := \mathbf{A} \cap \mathbf{U}_{u,v}.$$

Assume that each A_l is a graded subalgebra of M_{d_l} with respect to the elementary grading defined by $\widetilde{\alpha}_l$. We define the map $\widetilde{\alpha} : [1, \eta_m] \to G$ as

$$\widetilde{\alpha}(i) = \widetilde{\alpha}_l(i - \eta_{l-1}),$$

where $l \in [1, m]$ is the unique integer such that $i \in \mathbf{Bl}_l$. Let us consider in the matrix algebra M_{η_m} the elementary grading defined by the map $\widetilde{\alpha}$. Clearly $\mathbf{A}_{l,l}$ and $UT(A_1, \ldots, A_m)$ *are G-graded subalgebras of* $(M_{\eta_m}, \widetilde{\alpha})$, for all $l \in [1, m]$ and, moreover, $\mathbf{A}_{l,l}$ is graded-isomorphic to the given G-graded subalgebra A_l of $(M_{d_l}, \widetilde{\alpha}_l)$.

We say that an elementary G-grading $\widetilde{\beta}$ on M_{η_m} is $\widetilde{\alpha}$-*admissible* if, and only if, $\mathbf{A}_{l,l}$ is a graded subalgebra of $(M_{\eta_m}, \widetilde{\beta})$ for all $l \in [1, m]$ and, moreover, $\mathbf{A}_{l,l}$ (with the grading induced by $\widetilde{\beta}$) is graded-isomorphic to the given G-graded subalgebra A_l of $(M_{d_l}, \widetilde{\alpha}_l)$. In this thesis, we are also interested in describing conditions of the existence of a G-graded isomorphism between $(UT(A_1, \ldots, A_m), \widetilde{\alpha})$ and $(UT(A_1, \ldots, A_m), \widetilde{\beta})$, for any $\widetilde{\alpha}$-*admissible* grading $\widetilde{\beta}$, in case $A_1, \ldots, A_m$ are finite dimensional G-simple algebras.

Although the grading $\widetilde{\alpha}$ depends strongly on the sequence $(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_m)$, when convenient we will indicate the G-graded algebra $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$ simply by $UT_G(A_1, \ldots, A_m)$. Given $1 \le u \le v \le m$, we denote

$$A^{[u,v]} := (UT(A_u, \ldots, A_v), \widetilde{\alpha}_{[u,v]}),$$

where the map $\widetilde{\alpha}_{[u,v]} : [1, \eta_v - \eta_{u-1}] \to G$ is defined as $\widetilde{\alpha}_{[u,v]}(i) = \widetilde{\alpha}(\eta_{u-1} + i)$.

1.2 G-graded polynomial identities and T_G-ideals

In this section, we present some of the main concepts of the PI-theory which will be used along this work. In particular, we recall the definition of T_G-ideal of G-graded polynomial identities satisfied by G-graded algebras.

Consider disjoint countable sets $X_g := \{x_1^g, x_2^g, \ldots\}$ of non-commutative variables, with $g \in G$. Define $X_G := \cup_{g\in G} X_g$ and let $F\langle X; G\rangle$ be the unitary free associative algebra freely generated by X_G. The algebra $F\langle X; G\rangle$ has a natural G-grading, where the variables from X_g have degree g and the unit of $F\langle X; G\rangle$ has degree 1_G in this G-grading. Given a monomial $m = x_1^{g_{i_1}} x_2^{g_{i_2}} \cdots x_n^{g_{i_n}}$ in $F\langle X; G\rangle$, we define the *homogeneous degree* of m as

$$|m|_{F\langle X;G\rangle} := |x_1^{g_{i_1}} x_2^{g_{i_2}} \cdots x_n^{g_{i_n}}|_{F\langle X;G\rangle} = g_{i_1} g_{i_2} \cdots g_{i_n}.$$

We refer to this algebra as the *free G-graded algebra* over F.

Let $f = f(x_1^{g_{i_1}}, \ldots, x_n^{g_{i_n}})$ be an element in $F\langle X; G\rangle$. If the variable $x_j^{g_{i_j}}$ appears once in each monomial of f, then we say that f is a *linear polynomial in $x_j^{g_{i_j}}$*. If f is linear in all its variables $x_1^{g_{i_1}}, \ldots, x_n^{g_{i_n}}$, we call f a *multilinear polynomial of degree n*.

We say that $f = f(x_1^{g_{i_1}}, \ldots, x_n^{g_{i_n}}) \in F\langle X; G\rangle$ is a *G-graded polynomial identity* of a G-graded algebra $A = \oplus_{g\in G} A_g$ if

$$f(a_1, \ldots, a_n) = 0, \quad \text{for all } a_1 \in A_{g_{i_1}}, \ldots, a_n \in A_{g_{i_n}}.$$

A G-graded ideal I of $F\langle X; G\rangle$ is called a *T_G-ideal* (or a *graded T-ideal*) if I is stable under all G-graded endomorphism of $F\langle X; G\rangle$. Moreover, we define $\mathrm{Id}_G(A)$ as the set of all the G-graded polynomial identities satisfied by A, or, in shortly,

$$\mathrm{Id}_G(A) = \{f \in F\langle X; G\rangle \mid f \text{ is a } G\text{-graded polynomial identity for } A\}.$$

It follows that $\mathrm{Id}_G(A)$ is a T_G-ideal of $F\langle X; G\rangle$ and, once F is a field of characteristic zero, similarly to the ordinary case, $\mathrm{Id}_G(A)$ is completely determined by the multilinear polynomials it contains. Also, we say that A is a *G-graded F-algebra with a polynomial identity* (or simply a *G-graded PI-algebra*) if A satisfies a non-trivial ordinary polynomial identity, that is, if there exists a non-zero polynomial $f(x_1, \ldots, x_n) \in F\langle X\rangle$ such that $f(a_1, \ldots, a_n) = 0$, for all $a_i \in A$.

In the sequel, we present some definitions related to T_G-ideals, which can be found in [6] and [7].

Definition 1.2.1. Let I be a T_G-ideal of the free graded algebra $F\langle X; G\rangle$.

(*i*) We say that I is a *verbally prime* T_G-ideal if for any T_G-ideals I_1 and I_2 of $F\langle X; G\rangle$ such that $I_1 I_2 \subseteq I$, we have $I_1 \subseteq I$ or $I_2 \subseteq I$.

(*ii*) If there exist T_G-ideals $I_1 \neq I$ and $I_2 \neq I$ such that $I = I_1 I_2$, then I is called a *decomposable* T_G-ideal. Otherwise, we say that I is *indecomposable*.

The next step will be to prove that the T_G-ideal $\mathrm{Id}_G(A)$ is indecomposable whenever A is a G-simple algebra. To this end, we introduce the definition of verbally prime algebras and, in the sequel, we characterize such algebras by means of some properties related to the suitable G-graded ideals.

Let $A = \oplus_{g\in G} A_g$ be a G-graded algebra. We say that A is *verbally prime* if the T_G-ideal $\mathrm{Id}_G(A)$ is verbally prime. Given a T_G-ideal I of $F\langle X; G\rangle$, we define

$$I(A)_G := \{f(a_1, \dots, a_n) \mid f = f(x_1^{g_{i_1}}, \dots, x_n^{g_{i_n}}) \in I \text{ and } a_1 \in A_{g_{i_1}}, \dots, a_n \in A_{g_{i_n}}\}.$$

It is clear that $I(A)_G$ is a G-graded ideal of A. Furthermore, we notice that $I(A)_G = 0$ if, and only if, $I \subseteq \mathrm{Id}_G(A)$. Given T_G-ideals I_1 and I_2 of $F\langle X; G\rangle$, it holds

$$I_1 I_2(A)_G = I_1(A)_G I_2(A)_G. \tag{1.2}$$

As an immediate consequence of the previous definitions and remarks, we obtain the following:

Lemma 1.2.2. *Let $A = \oplus_{g\in G} A_g$ be a G-graded algebra. Then A is verbally prime if, and only if, for any T_G-ideals I_1 and I_2 of $F\langle X; G\rangle$ such that $I_1(A)_G I_2(A)_G = 0$, we have $I_1(A)_G = 0$ or $I_2(A)_G = 0$, or both.*

Finally, as an application of the above lemma, we can prove the next statement.

Lemma 1.2.3. *Let $A = \oplus_{g\in G} A_g$ be a G-simple algebra. Then A is verbally prime. Consequently,* $\mathrm{Id}_G(A)$ *is indecomposable.*

Proof. Consider I_1 and I_2 T_G-ideals of $F\langle X; G\rangle$ such that $I_1(A)_G I_2(A)_G = 0$. Assume that $I_1(A)_G \neq 0$ and $I_2(A)_G \neq 0$. Since A is G-simple, it follows that $I_1(A)_G = I_2(A)_G = A$ and then

$$0 \neq A^2 = I_1(A)_G I_2(A)_G = 0,$$

which is an absurd. Therefore, $I_1(A)_G = 0$ or $I_2(A)_G = 0$, and, by invoking Lemma 1.2.2, we have that A is verbally prime, as desired.

The fact that A is verbally prime is enough to obtain that $\mathrm{Id}_G(A)$ is indecomposable. □

We will see in Example 5.3.3 an indecomposable T_G-ideal which can not be generated by a finite dimensional G-simple algebra.

In order to finish this section, we present two results. The first is associated to G-graded embeddings between finite dimensional G-simple F-algebras and it was stated by O. David, in [14]. The second establishes an important property involving products of T_G-ideals related to algebras $A = (UT(A_1, \dots, A_m), \widetilde{\alpha})$ seen in Section 1.1.

Theorem 1.2.4 (Theorem 1 of [14]). *Let G be an abelian group and F be an algebraically closed field of characteristic zero. Consider two finite dimensional G-simple F-algebras A and B. There exists a G-graded embedding $\varphi : A \to B$ if, and only if,* $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$.

Lemma 1.2.5. *Let $A_1, \ldots, A_m$ be graded subalgebras of $(M_{d_1}, \widetilde{\alpha}_1), \ldots, (M_{d_m}, \widetilde{\alpha}_m)$, respectively. Given $A = UT_G(A_1, \ldots, A_m)$ and $u \geq 1$ an integer, then for any integers $c_1, c_2, \ldots, c_u$ such that* $1 \leq c_1 < c_2 < \cdots < c_u < m$,

$$\mathrm{Id}_G(A^{[1,c_1]})\mathrm{Id}_G(A^{[c_1+1,c_2]}) \cdots \mathrm{Id}_G(A^{[c_u+1,m]}) \subseteq \mathrm{Id}_G(A).$$

Proof. Notice that if $m = 1$, thus the statement is trivial. Assume $m \geq 2$ and take graded polynomials

$$f_1 \in \mathrm{Id}_G(A^{[1,c_1]}),\ f_2 \in \mathrm{Id}_G(A^{[c_1+1,c_2]}),\ \ldots\ ,\ f_u \in \mathrm{Id}_G(A^{[c_u+1,m]}).$$

Given $i \in [1, u]$, we remark that any graded evaluation $\rho_i : F\langle X; G\rangle \to A$, of the polynomial f_i in A, satisfies

$$\rho_i(f_i) = \sum_{1 \leq p \leq q \leq m} a_{pq}^{(i)},$$

where $a_{pq}^{(i)} \in A_{p,q}$ are such that $a_{pq}^{(i)} = 0_A$, for all $c_{i-1} + 1 \leq p \leq q \leq c_i$, with $c_0 := 0$. Therefore, since $A_{i,j}A_{i',j'} = \delta_{j,i'}A_{i,j'}$, we conclude that $f_1 f_2 \cdots f_u \in \mathrm{Id}_G(A)$. □

1.3 Kemer polynomials

Let I be a T_G-ideal of identities of a finite dimensional G-graded algebra. In this section, we will define the so-called *Kemer polynomials* for I based on [5]. To this end, assume that $G = \{g_1, \ldots, g_n\}$. In addition, since F is a field of characteristic zero, we have that I is generated by multilinear graded polynomials f which are *strongly homogeneous*, that is, every monomial in f has the same homogeneous degree in the G-grading.

Definition 1.3.1. Let $f \in F\langle X; G\rangle$ be a multilinear G-graded polynomial which is strongly homogeneous. Given $g \in G$, let $S_g = \{x_1^g, \ldots, x_m^g\}$ be a subset of X_g and consider $Y_G := X_G \setminus S_g$ the set of the remaining variables. We say that f is *alternating in the set* S_g (or that the variables of S_g *alternate in* f) if there exists a (multilinear, strongly homogeneous) G-graded polynomial $h(S_g; Y_G) = h(x_1^g, \ldots, x_m^g; Y_G)$ such that

$$f(x_1^g, \ldots, x_m^g; Y_G) = \sum_{\sigma \in \mathrm{Sym}(m)} (-1)^\sigma h(x_{\sigma(1)}^g, \ldots, x_{\sigma(m)}^g; Y_G).$$

Moreover, if $S_{g_{i_1}}, \ldots, S_{g_{i_p}}$ are p disjoint sets of variables of X_G, where $S_{g_{i_j}} \subset X_{g_{i_j}}$, for all $j \in [1, p]$, we say that f is *alternating* in $S_{g_{i_1}}, \ldots, S_{g_{i_p}}$ if f is alternating in each set $S_{g_{i_j}}$.

Let us consider polynomials which alternate in ν disjoint sets of the form S_g, for all $g \in G$. If the sets S_g have the same cardinality, say d_g, for every $g \in G$, then we say that f is *ν-fold $(d_{g_1}, \ldots, d_{g_n})$-alternating.* Moreover, we need to consider polynomials which, in addition to the alternating in such above sets, they alternate in t disjoint sets $K_g \subset X_g$, and also disjoint to the previous sets, such that $|K_g| = d_g + 1$ (where the elements g's that correspond to the K_g's need not be different).

Definition 1.3.2. Let $X_{l,g} = \{x_1^g, \ldots, x_l^g\}$ be a set of l variables of degree g and let $Y = \{y_1, \ldots, y_l\}$ be a set of l ungraded variables. The *g-Capelli polynomial* $c_{l,g}$ (of degree $2l$) is the polynomial obtained by alternating the set x_i^g's in the monomial $x_1^g y_1 x_2^g y_2 \cdots x_l^g y_l$, that is,

$$c_{l,g} := \sum_{\sigma \in \mathrm{Sym}(l)} (-1)^\sigma x^g_{\sigma(1)} y_1 x^g_{\sigma(2)} y_2 \cdots x^g_{\sigma(l)} y_l.$$

The g-Capelli polynomial $c_{l,g}$ is in the T_G-ideal I if all the G-graded polynomials obtained from $c_{l,g}$ through substitutions of the form $y_i \mapsto y_i^h$, for some $h \in G$, are in I.

Remark 1.3.3. Since I is a T_G-ideal of identities of a finite dimensional G-graded algebra, then by Lemma 3.4 of [5], for every $g \in G$, there exists an integer l_g such that the T_G-ideal I contains $c_{l_g,g}$.

Corollary 1.3.4 (Corollary 3.5 of [5])**.** *Let I be a T_G-ideal of identities of a finite dimensional G-graded algebra. If f is a multilinear G-graded polynomial, strongly homogeneous and alternating on a set S_g of cardinality l_g, then $f \in I$. Consequently there exists an integer m_g which bounds (from above) the cardinality of the g-alternating sets in any G-graded polynomial which is not in I.*

In order to introduce the Kemer polynomials for I, by considering $\mathbb{N}^n = \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{n \text{ times}}$, let us define a partial order $\preceq$ on $\mathbb{N}^n \times \mathbb{N}$. Firstly, given $\delta = (\delta_1, \ldots, \delta_n)$ and $\rho = (\rho_1, \ldots, \rho_n)$ elements of $\mathbb{N}^n$, we write $(\delta_1, \ldots, \delta_n) \preccurlyeq (\rho_1, \ldots, \rho_n)$ if, and only if, $\delta_i \leq \rho_i$, for all $i \in [1, n]$. Now, given (δ, s) and (ρ, s') elements of $\mathbb{N}^n \times \mathbb{N}$, we write $(\delta, s) \preceq (\rho, s')$ if, and only if, either

(i) $\delta \prec \rho$, that is, $\delta \preccurlyeq \rho$ and, for some j, $\delta_j < \rho_j$, or

(ii) $\delta = \rho$ and $s \leq s'$.

In the sequel, we will define the *Kemer points* of I, which will be denoted by Kemer(I). Such Kemer points will be given by a finite set of points in $\mathbb{N}^n \times \mathbb{N}$. We start by defining the set $\mathrm{Ind}(I)_0$ as:

$$\mathrm{Ind}(I)_0 := \{\delta \in \mathbb{N}^n \mid \text{for each } \nu \in \mathbb{N},\ \exists\ f \notin I \text{ such that } f \text{ is } \nu\text{-fold } \delta\text{-alternating}\}.$$

In virtue of Corollary 1.3.4, we have that the set $\mathrm{Ind}(I)_0$ is bounded (finite). Furthermore, if $\delta \in \mathrm{Ind}(I)_0$, then $\delta' \in \mathrm{Ind}(I)_0$, for any $\delta' \preccurlyeq \delta$ (see Lemma 3.7 of [5]). Now, given $\nu \in \mathbb{N}$, we set

$$\Delta_\nu := \{\delta \in \mathbb{N}^n \mid \exists\ f \notin I \text{ such that } f \text{ is } \nu\text{-fold } \delta\text{-alternating}\}.$$

Notice that

$$\cap_{\nu\in\mathbb{N}}\Delta_\nu = \mathrm{Ind}(I)_0.$$

On the other hand, if $\nu \leq \nu'$, thus $\Delta_{\nu'} \subseteq \Delta_\nu$, and once each Δ_ν is finite (see Corollary 1.3.4), the chain

$$\Delta_1 \supseteq \Delta_2 \supseteq \cdots$$

stabilizes, that is, there exists $\gamma \in \mathbb{N}$ such that

$$\Delta_\nu = \Delta_\gamma, \quad \text{for all } \nu \geq \gamma, \tag{1.3}$$

and, hence, $\mathrm{Ind}(I)_0 = \Delta_\gamma$.

Let Δ_ν^0 be the *extremal points* of Δ_ν, that is, the points $\delta \in \Delta_\nu$ such that for any $\rho \in \Delta_\nu$ satisfying $\delta \preccurlyeq \rho$, it is valid that $\rho = \delta$. Note that $\Delta_\nu^0 = \Delta_\gamma^0$, for all $\nu \geq \gamma$.

We also set

$$\Omega_\nu := \{f \in F\langle X; G\rangle \mid\ f \notin I \text{ and } f \text{ is } \nu\text{-fold } \delta\text{-alternating, for some } \delta \in \Delta_\nu\}.$$

Clearly $\Omega_\nu = \cup_{\delta\in\Omega_\nu}\Omega_{\delta,\nu}$, where

$$\Omega_{\delta,\nu} := \{f \in F\langle X; G\rangle \mid\ f \notin I \text{ and } f \text{ is } \nu\text{-fold } \delta\text{-alternating}\}.$$

At this stage, fixed $\nu \in \mathbb{N}$, $\delta = (\delta_{g_1}, \ldots, \delta_{g_n}) \in \Delta_\nu$ and $f \in \Omega_{\delta,\nu}$, let $s_I(\delta, \nu, f)$ be the number of alternating g-homogeneous sets (any $g \in G$) of disjoint variables, of cardinality $\delta_g + 1$. We claim that if γ satisfies (1.3), then for any fixed pair (δ, ν) with $\delta \in \Delta_\gamma^0$ and $\nu \geq \gamma$, we have $\{s_I(\delta, \nu, f)\}_{f\in\Omega_{\delta,\nu}}$ is bounded. Actually, in this case, if $\{s_I(\delta, \nu, f)\}_{f\in\Omega_{\delta,\nu}}$ is not bounded, thus there exists a sequence of polynomials $f_1, f_2, \ldots$ in $\Omega_{\delta,\nu}$ such that $s_i = s_I(\delta, \nu, f_i)$ and $\lim_{i\to\infty} s_i = \infty$. Since the group G is finite we obtain, by the pigeonhole principle, that there exist $g \in G$ and a subsequence $f_{i_1}, f_{i_2}, \ldots$ such that $\lim_{k\to\infty} s_{i_k,g} = \infty$, where $s_{i_k,g}$ is the number of alternating g-homogeneous sets of cardinality $\delta_g + 1$ in f_{i_k}. However, this implies that the point δ' defined as $\delta'_g = \delta_g + 1$ and $\delta'_h = \delta_h$, for $h \neq g$, belongs to Δ_ν (actually, it is enough to take k such that $s_{i_k,g} \geq \nu$ and thus f_{i_k} is ν-fold δ'-alternating). Once $\nu \geq \gamma$, we have $\delta \in \Delta^0_\gamma = \Delta^0_\nu$, and thus since $\delta \preccurlyeq \delta'$ and $\delta' \neq \delta$, we obtain a contradiction.

Let $s_I(\delta, \nu) = \max\{s_I(\delta, \nu, f)\}_{f \in \Omega_{\delta,\nu}}$. Since the sequence $s_I(\delta, \nu)$ is monotonically decreasing as a function of ν, there exists an integer $\mu = \mu(I, \nu) \geq \gamma$ for which the sequence stabilizes, that is, $s_I(\delta, \nu)$ is constant for all $\nu \geq \mu$. In this sense, we set

$$s_I(\delta) := \lim_{\nu \to \infty} s_I(\delta, \nu) = s_I(\delta, \mu).$$

Once the set Δ^0_γ is finite and $\delta \in \Delta^0_\gamma$, take μ to be the maximum of all μ's considered above.

Given a T_G-ideal I of identities of a finite dimensional G-graded algebra, we define the *Kemer set* of I as the set of points:

$$\mathrm{Kemer}(I) := \{(\delta, s_I(\delta)) \mid \delta \in \Delta^0_\gamma\}.$$

The elements of Kemer(I) are called *Kemer points* of I.

Finally, we present the definition of Kemer polynomials for a T_G-ideal I.

Definition 1.3.5. Let I be a T_G-ideal of identities of a finite dimensional G-graded algebra.

(*i*) Let $(\delta, s_I(\delta))$ be a Kemer point of the T_G-ideal I. A graded polynomial f is said to be a *Kemer polynomial for the point* $(\delta, s_I(\delta))$ if f is not in I and it has at least μ-folds of alternating g-sets of cardinality δ_g (*small sets*) for all $g \in G$ and $s_I(\delta)$ homogeneous sets of disjoint variables Y_g (some g in G) of cardinality $\delta_g + 1$ (*big sets*).

(*ii*) A polynomial f is *Kemer* for the T_G-ideal I if it is Kemer for a Kemer point of I.

Note that a polynomial f cannot be Kemer simultaneously for different Kemer points of I. In fact, assume that $(\delta, s_I(\delta))$ and $(\delta', s_I(\delta'))$ are both points for a Kemer polynomial f of I, with $\delta \neq \delta'$. Consider δ'' defined as $\delta''_g = \max\{\delta_g, \delta'_g\}$, for all $g \in G$. Consequently, we have $\delta'' \in \Delta_\mu$, with $\delta, \delta' \preccurlyeq \delta''$ and $\delta'' \neq \delta$ or $\delta'' \neq \delta'$, and this contradicts the fact that δ, δ' are extremal points of $\Delta_\mu = \Delta_\gamma$.

Let A be a finite dimensional G-graded F-algebra. We say that (δ, l) is a *Kemer point of* A if (δ, l) is a Kemer point of $\mathrm{Id}_G(A)$. Let us finish this section by investigating the Kemer points of the algebra A. First, we recall that, by the generalization of the Wedderburn-Malcev Theorem (see [28]), it is valid that

$$A = A_{ss} + J(A),$$

where $A_{ss} = A_1 \oplus \cdots \oplus A_m$ (direct sum as algebras) is a maximal semisimple graded subalgebra of A, with $A_1, \ldots, A_m$ G-simple algebras. Moreover $J := J(A)$, the Jacobson radical of A, is a graded ideal.

By denoting the nilpotency index of J by n_A, we define the $(n+1)$-tuple

$$G - \mathrm{Par}(A) := (\dim_F(A_{ss})_{g_1}, \ldots, \dim_F(A_{ss})_{g_n}, n_A - 1) \in \mathbb{N}^n \times \mathbb{N}.$$

In the sequel, we present a relation between the Kemer points of A and $G - \mathrm{Par}(A)$.

Proposition 1.3.6 (Proposition 4.4 of [5]). *If $(\delta, l) = (\delta_{g_1}, \ldots, \delta_{g_n}, l)$ is a Kemer point of A, then $(\delta, l) \preceq G - \mathrm{Par}(A)$.*

As a corollary we obtain that:

Corollary 1.3.7. *If $G - \mathrm{Par}(A)$ is a Kemer point of A, then it is the unique Kemer point of A.*

Proof. Denote $I := \mathrm{Id}_G(A)$. Let $(\delta, s_I(\delta))$ be a Kemer point of A and assume that $G - \mathrm{Par}(A) = (\delta', s_I(\delta'))$ is a Kemer point of A. By invoking the above proposition, it follows that $(\delta, s_I(\delta)) \preceq (\delta', s_I(\delta'))$, that is, either

(*i*) $\delta \prec \delta'$, or

(*ii*) $\delta = \delta'$ and $s_I(\delta) \leq s_I(\delta')$.

Since $\delta, \delta' \in \Delta_\gamma^0$, it follows that condition (*i*) is not satisfied. Thus $\delta = \delta'$ and hence $s_I(\delta) = s_I(\delta')$, which implies

$$(\delta, s_I(\delta)) = (\delta', s_I(\delta')) = G - \mathrm{Par}(A).$$

□

In Chapter 5, we will construct the Kemer polynomials for the G-graded upper block triangular matrix algebra $UT_G(A_1, \ldots, A_m)$, in case G is a finite cyclic group and $A_1, \ldots, A_m$ are finite dimensional G-simple algebras.

1.4 G-graded codimension, G-exponent and varieties

We start this section by presenting the concept of G-graded codimension of a G-graded algebra. To this end, for all $n \geq 1$, we consider P_n^G as the F-vector space generated by the multilinear polynomials of degree n of $F\langle X;G\rangle$ in the variables x_i^g, for $g \in G$ and $i \in [1,n]$. Given a G-graded algebra A, we define

$$c_n^G(A) := \dim_F \frac{P_n^G}{P_n^G \cap \mathrm{Id}_G(A)}$$

and we refer to this non-negative integer as the nth *G-graded codimension* of A.

Let A be a G-graded PI-algebra. It is well known that its sequence of G-graded codimensions $\{c_n^G(A)\}_{n\geq 1}$ is exponentially bounded (see Lemma 10.1.3 of [28]). We define the *G-graded exponent*, or simply *G-exponent*, of the G-graded PI-algebra A as

$$\exp_G(A) := \lim_{n\to\infty} \sqrt[n]{c_n^G(A)}.$$

If A is finite dimensional and the field F is algebraically closed, then such G-exponent exists and is a non-negative integer (see [2]). In the sequel, we exhibit a way to calculate the G-exponent, which was presented by Aljadeff, Giambruno and La Mattina in [2].

Given a finite dimensional G-graded F-algebra A, by the previous section, we have $A = A_{ss} + J = A_1 \oplus \cdots \oplus A_m + J$, where $A_1, \ldots, A_m$ are G-simple algebras and J, the Jacobson radical of A, is a graded ideal. Consider all products

$$A_{r_1} J A_{r_2} J \cdots A_{r_{l-1}} J A_{r_l} \neq 0, \tag{1.4}$$

where $A_{r_1}, \ldots, A_{r_l}$ are distinct G-simple subalgebras of the set $\{A_1, \ldots, A_m\}$. We define

$$q := \max \dim_F (A_{r_1} \oplus \cdots \oplus A_{r_l})$$

as being the maximum dimension among the dimension of all the subalgebras $A_{r_1} \oplus \cdots \oplus A_{r_l}$ such that $A_{r_1}, \ldots, A_{r_l}$ satisfy condition (1.4). Therefore, it holds

$$q = \exp_G(A). \tag{1.5}$$

At this stage, given a T_G-ideal I of $F\langle X;G\rangle$, we define the *variety of G-graded algebras* $\mathcal{V}^G$ (determined by I) as the class of all G-graded algebras A such that $I \subseteq \mathrm{Id}_G(A)$. In short,

$$\mathcal{V}^G := \mathcal{V}^G(I) = \{A \mid I \subseteq \mathrm{Id}_G(A)\}.$$

We denote the T_G-ideal I of $F\langle X;G\rangle$ associated to $\mathcal{V}^G$ as $\mathrm{Id}_G(\mathcal{V}^G)$. If $\mathrm{Id}_G(\mathcal{V}^G) = \mathrm{Id}_G(A)$, for a G-graded algebra A, then we say that the variety $\mathcal{V}^G$ is *generated by* A and write

$$\mathcal{V}^G = \mathrm{var}_G(A).$$

Moreover, in this case, if A is a finitely generated G-graded PI-algebra, then $\mathcal{V}^G$ is called a variety of *finite basic rank*. If F is an algebraically closed field of characteristic zero, we can assume that any variety $\mathcal{V}^G$ of finite basic rank is generated by a finite dimensional G-graded PI-algebra (see [5] or [33]).

In case $\mathcal{V}^G = \mathrm{var}_G(A)$, the variety generated by a G-graded PI-algebra A, we set the nth *G-graded codimension* and the *G-exponent* of the variety $\mathcal{V}^G$, respectively, as

$$c_n^G(\mathcal{V}^G) := c_n^G(A), \text{ for every } n \geq 1, \text{ and } \exp_G(\mathcal{V}^G) := \exp_G(A).$$

1.5 Minimal varieties and minimal G-graded algebras

In this section, firstly, we recall the concept of minimal varieties of G-graded PI-algebras of a given G-exponent. In the sequel, we will give the definition of minimal G-graded F-algebras, which is a natural generalization of the well known *minimal superalgebras*, introduced by Giambruno and Zaicev in [26].

Definition 1.5.1. Let $\mathcal{V}^G$ be a variety of G-graded PI-algebras. We say that $\mathcal{V}^G$ is *minimal* of G-exponent d if $\exp_G(\mathcal{V}^G) = d$ and for every proper subvariety $\mathcal{U}^G$ of $\mathcal{V}^G$ one has that $\exp_G(\mathcal{U}^G) < d$.

Let $\mathcal{V}$ be a variety of associative PI-algebras over F. In [27], Giambruno and Zaicev described the minimal varieties $\mathcal{V}$ of finite basic rank, of a given exponent, by means of suitable generating algebras. More precisely, they showed that such variety $\mathcal{V}$ is minimal if, and only if, $\mathcal{V}$ is generated by an upper block triangular matrix algebra $UT(d_1, \ldots, d_m)$. Denote by C_n the finite cyclic group of order n. If $n = p$ is an arbitrary prime number, in 2019, Di Vincenzo, da Silva and Spinelli proved that a variety $\mathcal{V}^{C_p}$ of C_p-graded PI-algebras of finite basic rank, with respect to a given C_p-exponent, is minimal if, and only if, $\mathcal{V}^{C_p}$ is generated by a C_p-graded algebra $UT_{C_p}(A_1, \ldots, A_m)$, where $A_1, \ldots, A_m$ are finite dimensional C_p-simple algebras (see [17]).

Let F be an algebraically closed field. In this work, more precisely in Chapter 5, we will take a new step towards the classification of such minimal varieties in case n is any positive integer. In particular, we will prove that they are generated by a suitable C_n-graded upper block triangular matrix algebra $UT_{C_n}(A_1, \ldots, A_m)$, with $A_1, \ldots, A_m$ being finite dimensional C_n-simple algebras. Moreover, by assuming that $UT_{C_n}(A_1, \ldots, A_m)$ satisfies at least one of the conditions $(i), (ii)$ or (iii) of Theorem 5.3.7, we also will show that $\mathrm{var}_{C_n}(UT_{C_n}(A_1, \dots, A_m))$ is minimal.

Definition 1.5.2. A G-graded algebra A is said *minimal* if it is finite dimensional and either A is a G-simple algebra or $A = A_{ss} + J(A)$ where

(i) $A_{ss} = A_1 \oplus \cdots \oplus A_m$, with $A_1, \dots, A_m$ G-simple algebras and $m \geq 2$;

(ii) there exist homogeneous elements $w_{12}, \dots, w_{m-1,m} \in J(A)$ and minimal homogeneous idempotents $e_1 \in A_1, \dots, e_m \in A_m$ such that

$$e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1}, \text{ for all } i \in [1, m-1]$$

and

$$w_{12} w_{23} \cdots w_{m-1,m} \neq 0_A;$$

(iii) $w_{12}, \dots, w_{m-1,m}$ generate $J(A)$ as a two-sided ideal of A.

Clearly any minimal G-graded algebra A admits a vector space decomposition given by

$$A = \bigoplus_{1 \leq i \leq j \leq m} A_{ij},$$

where

$$A_{ij} := \begin{cases} A_i & \text{if } i = j, \\ A_i w_{i,i+1} A_{i+1} \cdots A_{j-1} w_{j-1,j} A_j & \text{if } i < j. \end{cases}$$

Moreover $J(A) = \oplus_{i<j} A_{ij}$ and $A_{ij} A_{i'j'} = \delta_{ji'} A_{ij'}$. Still, for all $1 \leq u \leq v \leq m$, we define

$$A^{[u,v]} := \bigoplus_{u \leq i \leq j \leq v} A_{ij},$$

and, for each $1 < \ell < m$, we set

$$A^{(\check{\ell})} := \bigoplus_{\substack{1 \leq i \leq j \leq m \\ i \neq \ell \neq j}} A'_{ij},$$

where

$$A'_{ij} := \begin{cases} A_i w_{i,i+1} A_{i+1} \cdots A_{\ell-1} w_{\ell-1,\ell} w_{\ell,\ell+1} A_{\ell+1} \cdots A_{j-1} w_{j-1,j} A_j & \text{if } i < \ell < j, \\ A_{ij} & \text{otherwise.} \end{cases}$$

Example 1.5.3. Let $G = C_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$, a cyclic group of order 4. Moreover, consider

$A_1 = (M_2, \widetilde{\alpha}_1)$, $A_2 = (M_2, \widetilde{\alpha}_2)$ and $A_3 = (M_3, \widetilde{\alpha}_3)$, where

$$(\widetilde{\alpha}_1(1), \widetilde{\alpha}_1(2)) = (\bar{0}, \bar{1}),\ (\widetilde{\alpha}_2(1), \widetilde{\alpha}_2(2)) = (\bar{1}, \bar{2}) \quad \text{and} \quad (\widetilde{\alpha}_3(1), \widetilde{\alpha}_3(2), \widetilde{\alpha}_3(3)) = (\bar{1}, \bar{2}, \bar{3}).$$

Finally, let $A = (UT(A_1, A_2, A_3), \widetilde{\alpha})$.

For each $l \in [1,3]$, take the minimal homogenenous idempotents as

$$e_l := \mathbf{E}_{11}^{(l,l)}$$

and, for each $l \in [1,2]$, take the homogeneous radical elements as

$$w_{l,l+1} := \mathbf{E}_{11}^{(l,l+1)}.$$

Clearly A is a G-graded minimal algebra.

Moreover, in this case, the decomposition of A in the form $A = \bigoplus_{1\leq i\leq j\leq m} A_{ij}$ can be given as

$$A = \left(\begin{array}{cc|cc|ccc} F & F & F & F & F & F & F \\ F & F & F & F & F & F & F \\ \hline 0 & 0 & F & F & F & F & F \\ 0 & 0 & F & F & F & F & F \\ \hline 0 & 0 & 0 & 0 & F & F & F \\ 0 & 0 & 0 & 0 & F & F & F \\ 0 & 0 & 0 & 0 & F & F & F \end{array}\right),$$

where, for each $1 \leq i \leq j \leq 3$, A_{ij} corresponds to the block of the position (i,j). For instance, $A_{23} = \mathrm{span}_F\{\mathbf{E}_{35}, \mathbf{E}_{36}, \mathbf{E}_{37}, \mathbf{E}_{45}, \mathbf{E}_{46}, \mathbf{E}_{47}\}$. Furthermore, we have, for instance,

$$A^{[1,2]} = \left(\begin{array}{cc|cc|ccc} F & F & F & F & 0 & 0 & 0 \\ F & F & F & F & 0 & 0 & 0 \\ \hline 0 & 0 & F & F & 0 & 0 & 0 \\ 0 & 0 & F & F & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad A^{(\check{2})} = \left(\begin{array}{cc|cc|ccc} F & F & 0 & 0 & F & F & F \\ F & F & 0 & 0 & F & F & F \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & F & F & F \\ 0 & 0 & 0 & 0 & F & F & F \\ 0 & 0 & 0 & 0 & F & F & F \end{array}\right).$$

Remark 1.5.4. Let $A = A_{ss} + J = A_1 \oplus \cdots \oplus A_m + J$ be a minimal G-graded algebra. Since, from Definition 1.5.2, $A_1 J A_2 J \cdots A_{m-1} J A_m \neq 0$, we conclude, by invoking (1.5), that

$$\exp_G(A) = \dim_F(A_1 \oplus \cdots \oplus A_m) = \dim_F A_{ss}.$$

Finally, we present some important technical results related to minimal varieties and minimal G-graded algebras. The first is a natural extension of Lemma 8.1.4 given in [28].

Lemma 1.5.5. *Let A be a finite dimensional G-graded F-algebra. Then there exists a minimal G-graded algebra $B \subseteq A$ such that* $\exp_G(B) = \exp_G(A)$.

Proof. Firstly, we remember that, by the generalization of the Wedderburn-Malcev Theorem, we have $A = A_{ss}+J$, where $A_{ss} = A_1\oplus\cdots\oplus A_m$ (direct sum as algebras) is a maximal semisimple graded subalgebra of A, with $A_1, \ldots, A_m$ G-simple algebras. Moreover J, the Jacobson radical of A, is a graded ideal.

Consider $n \leq m$ such that

$$A_{r_1} J A_{r_2} J \cdots A_{r_{n-1}} J A_{r_n} \neq 0 \tag{1.6}$$

and $\dim_F(A_{r_1} \oplus\cdots\oplus A_{r_n})$ is maximum, where $A_{r_1}, \ldots, A_{r_n}$ are distinct G-simple subalgebras of the set $\{A_1, \ldots, A_m\}$. Thus, there exist $x_1, \ldots, x_{n-1} \in J$ and $a_1 \in A_{r_1}, \ldots, a_n \in A_{r_n}$ satisfying

$$a_1x_1a_2 \cdots a_{n-1}x_{n-1}a_n \neq 0.$$

For each i, we can write $x_i = \sum_{g\in G} x_i^g$, with $x_i^g \in J_g$, and $a_i = \sum_{g\in G} a_i^g$, with $a_i^g \in (A_{r_i})_g$. Hence, there exist $\varepsilon_1, \eta_1, \ldots, \eta_{n-1}, \varepsilon_n \in G$ such that

$$a_1^{\varepsilon_1} x_1^{\eta_1} a_2^{\varepsilon_2} \cdots a_{n-1}^{\varepsilon_{n-1}} x_{n-1}^{\eta_{n-1}} a_n^{\varepsilon_n} \neq 0.$$

This means that we can assume the elements $x_1, \ldots, x_{n-1}, a_1, \ldots, a_n$ as being homogeneous.

Let $1_1, \ldots, 1_n$ be the units of the algebras $A_{r_1}, \ldots, A_{r_n}$, respectively. Then,

$$1_1(a_1x_1a_2)1_2(x_2a_3)1_3 \cdots 1_{n-1}(x_{n-1}a_n)1_n \neq 0.$$

Now, we remark that, for each $j \in [1, n]$, there exist minimal graded idempotents $e_{j1}, \ldots, e_{jk_j} \in (A_{r_j})_{1_G}$ such that $1_j = e_{j1} + \cdots + e_{jk_j}$. Thus

$$(e_{11} + \cdots + e_{1k_1})(a_1x_1a_2)(e_{21} + \cdots + e_{2k_2})(x_2a_3) \cdots (x_{n-1}a_n)(e_{n1} + \cdots + e_{nk_n}) \neq 0,$$

which implies that there exist minimal graded idempotents $e_1 \in A_{r_1}, \ldots, e_n \in A_{r_n}$ such that

$$e_1(a_1x_1a_2)e_2(x_2a_3)e_3 \cdots e_{n-1}(x_{n-1}a_n)e_n \neq 0.$$

At this stage, we define the following homogeneous elements:

$$w_{12} := e_1(a_1x_1a_2)e_2,\ w_{23} := e_2(x_2a_3)e_3,\ w_{34} := e_3(x_3a_4)x_4, \ldots, w_{n-1,n} := e_{n-1}(x_{n-1}a_n)e_n.$$

Since J is a two-sided ideal of A, one has that $w_{i,i+1} \in J$, for all $i \in [1, n-1]$. Moreover, we have

$$e_1 w_{12} = e_1 e_1 (a_1 x_1 a_2) e_2 = e_1 (a_1 x_1 a_2) e_2 = e_1 (a_1 x_1 a_2) e_2 e_2 = w_{12} e_2,$$

$$e_i w_{i,i+1} = e_i e_i (x_i a_{i+1}) e_{i+1} = e_i (x_i a_{i+1}) e_{i+1} = e_i (x_i a_{i+1}) e_{i+1} e_{i+1} = w_{i+1} e_{i+1}, \quad \text{for all } i \in [2, n-1],$$

$$w_{12} \cdots w_{n-1,n} = e_1 (a_1 x_1 a_2) e_2 (x_2 a_3) e_3 \cdots e_{n-1} (x_{n-1} a_n) e_n \neq 0.$$

Let $B := A_{r_1} \oplus \cdots \oplus A_{r_n} + J(B)$ be the algebra generated by $A_{r_1}, \ldots, A_{r_n}, w_{12}, \ldots, w_{n-1,n}$. Notice that $B \subseteq A$ and $J(B)$ is generated by the elements $w_{12}, \ldots, w_{n-1,n}$. Therefore, according to Definition 1.5.2, (1.6) and (1.5), we conclude that B is a minimal G-graded algebra such that $\exp_G(B) = \exp_G(A)$. □

Theorem 1.5.6. *Let $\mathcal{V}^G$ be a variety of G-graded PI-algebras of finite basic rank. If $\mathcal{V}^G$ is minimal of G-exponent d, then there exists a minimal G-graded algebra A such that*

$$\mathcal{V}^G = \operatorname{var}_G(A).$$

Proof. The fact that the variety $\mathcal{V}^G$ is of finite basic rank allows us to conclude that, from Theorem 1.1 of [5], there exists a finite dimensional G-graded algebra B over F such that

$$\mathcal{V}^G = \operatorname{var}_G(B) \quad \text{and} \quad \exp_G(B) = d.$$

By invoking Lemma 1.5.5, it follows that there exists a minimal G-graded algebra $A \subseteq B$ such that $\exp_G(A) = \exp_G(B)$. Thus $\operatorname{Id}_G(B) \subseteq \operatorname{Id}_G(A)$ and, hence, $A \in \mathcal{V}^G = \operatorname{var}_G(B)$. Once $\mathcal{V}^G$ is minimal and $\exp_G(A) = \exp_G(B)$, we conclude the proof of the theorem. □

As an application of the above theorem, we will see in Section 5.1 that if G is a finite cyclic group, then any minimal variety of G-graded PI-algebras of finite basic rank, of G-exponent d, is generated by a suitable G-graded algebra $UT_G(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$, where $A_1, \ldots, A_m$ are finite dimensional G-simple algebras.

Chapter 2

Factorability and α-regularity

In this chapter, F will denote a field of characteristic zero and G will be a finite abelian group. Here, we will start the study of one of the main topics of this work. More precisely, we will define the factoring property of the ideals of graded polynomial identities satisfied by the graded upper block triangular matrix algebras $UT_G(A_1, \ldots, A_m)$, in case $A_1, \ldots, A_m$ are graded subalgebras (not necessarily G-simple) of matrix algebras endowed with elementary gradings, and we will present some results. Moreover, we will introduce the definition of α-regularity, which is a generalization of the concept of G-regularity, and we will obtain some relevant connections between these points with the invariance subgroups. It is worth saying that the new results presented in this chapter have been recently published in [22], in a joint work with Professor Viviane Ribeiro Tomaz da Silva and Professor Onofrio Mario Di Vincenzo. Furthermore, some of these results present an alternative proof of that shown in [22].

2.1 G-regularity and factorability

We start this section recalling the concept of G-regularity, which was introduced by Di Vincenzo and La Scala in [19]. To this end, let A be a finite dimensional G-graded algebra. We define a *G-graded generic algebra associated to A*, which will be denoted by $\mathrm{Gen}_G(A)$, as being a G-graded algebra isomorphic to $F\langle X; G\rangle/\mathrm{Id}_G(A)$. This is the analogous construction of the generic matrix algebra (see Section 7.2 of [25]).

Consider $\{v_1, \ldots, v_n\}$ a linear homogeneous basis of A, that is, a basis of A formed by homogeneous elements. We define

$$P(A) := F[x_i^{(l)} \mid i \in [1, n] \text{ and } l \geq 1]$$

as the polynomial ring in the countable set of commuting variables $x_i^{(l)}$ and we refer to this ring as the *polynomial ring associated to the finite dimensional G-graded algebra A*. Moreover, consider the tensor product $A \otimes P(A)$ with the natural grading induced from that of A, that is,

$$A \otimes P(A) = \bigoplus_{g \in G} (A_g \otimes P(A)).$$

Since $P(A)$ is commutative and F is an infinite field, it is valid that $\mathrm{Id}_G(A \otimes P(A)) = \mathrm{Id}_G(A)$. At this stage, we consider in $A \otimes P(A)$ the graded subalgebra $\bar{A}$ generated by the homogeneous elements

$$a_{l,g} := \sum_{\substack{i \in [1,n] \\ |\upsilon_i|_A = g}} x_i^{(l)} \upsilon_i, \qquad \text{for all } l \geq 1 \text{ and } g \in G.$$

Remark that we omit the symbol for the tensor product in the above elements. It is well known that $\bar{A}$ is a G-graded generic algebra associated to A, that is, $\bar{A}$ is isomorphic to $F\langle X; G\rangle / \mathrm{Id}_G(A)$.

Our next step is to define G-regular graded subalgebras of matrix algebras and then, in Chapter 3, we will classify the finite dimensional G-simple algebras that are G-regular, in case G is a finite cyclic group.

Let A be a graded subalgebra of (M_k, α). Given $g \in G$, define the following linear map:

$$\begin{array}{rccc} \pi_g : & M_k \otimes P(A) & \rightarrow & M_k \otimes P(A) \\ & \displaystyle\sum_{i,j} a_{ij} e_{ij} & \mapsto & \displaystyle\sum_{i,j;\ \alpha(i) = g} a_{ij} e_{ij}, \end{array} \tag{2.1}$$

where, for each $i, j \in [1, k]$, e_{ij} denote the (i, j)-matrix unit of M_k. We remark that π_g is the zero map in case $g \notin \mathcal{I}_\alpha = \alpha([1, k])$. It is valid that

$$\bar{A} = \mathrm{Gen}_G(A) \subseteq A \otimes P(A) \subseteq M_k \otimes P(A),$$

and then we define the map

$$\widehat{\pi}_g : \bar{A} \rightarrow M_k \otimes P(A)$$

as being the restriction of π_g to $\bar{A}$.

Similarly, we define the map:

$$\begin{array}{rccc} \pi_g^* : & M_k \otimes P(A) & \rightarrow & M_k \otimes P(A) \\ & \displaystyle\sum_{i,j} a_{ij} e_{ij} & \mapsto & \displaystyle\sum_{i,j;\ \alpha(j) = g} a_{ij} e_{ij}, \end{array} \tag{2.2}$$

where $i, j \in [1, k]$, and we consider

$$\widehat{\pi}_g^* : \bar{A} \to M_k \otimes P(A)$$

its restriction to $\bar{A}$.

Definition 2.1.1. Let A be a graded subalgebra of M_k endowed with an elementary grading α. We say that A is a G-*regular* subalgebra of (M_k, α) if the maps $\widehat{\pi}_g$ are injective, for all $g \in G$.

Equivalently one could define G-regular subalgebras of (M_k, α) requiring that the maps $\widehat{\pi}_g^*$ are injective, for all $g \in G$ (see Proposition 4.2 of [19]).

The next result establishes when matrix algebras are G-regular.

Theorem 2.1.2 (Theorem 5.4 of [19])**.** *Let G be a finite abelian group. Let (M_k, α) be a G-graded matrix algebra. Then (M_k, α) is G-regular if, and only if, the map α is surjective and all its fibers are equipotent.*

As consequence of some results of [19], we have also a characterization of the simple superalgebras that are C_2-regular. More precisely, the authors stated that $M_{k,l}$ is C_2-regular if, and only if, $k = l$, whereas $M_n(F \oplus cF)$ is C_2-regular for all $n \geq 1$.

In the sequel, we present two definitions related to the factoring property associated to the T_G-ideals of the G-graded upper block triangular matrix algebras.

Definition 2.1.3. Let $A = UT_G(A_1, \ldots, A_m)$. We say that the T_G-ideal $\mathrm{Id}_G(A)$ is *weakly factorable* if there exist $1 \leq c_1 < c_2 < \cdots < c_u < m$ such that

$$\mathrm{Id}_G(A) = \mathrm{Id}_G(A^{[1,c_1]})\mathrm{Id}_G(A^{[c_1+1,c_2]}) \cdots \mathrm{Id}_G(A^{[c_u+1,m]}).$$

In particular, if $\mathrm{Id}_G(A)$ satisfies

$$\mathrm{Id}_G(A) = \mathrm{Id}_G(A_1)\mathrm{Id}_G(A_2) \cdots \mathrm{Id}_G(A_m),$$

then we say that $\mathrm{Id}_G(A)$ is *factorable*.

There exist some studies involving the factoring property (see, for instance, [7, 12, 15, 19, 27]). Let us start discussing such problem for G-graded upper block triangular matrix algebras having exactly two blocks.

Let R be the G-graded upper block triangular matrix algebra

$$R := \begin{pmatrix} A & U \\ 0 & B \end{pmatrix},$$

where $A \subseteq (M_m, \alpha)$, $B \subseteq (M_n, \beta)$ are graded subalgebras and $U = M_{m\times n}$. Denote $P := P(R)$ as the polynomial ring associated to the finite dimensional algebra R. As in Section 4 of [19], we consider a linear homogeneous basis of R given by the disjoint union of some homogeneous basis of A, B and the canonical basis $\{\mathbf{E}_{ij} \mid i \in [1, m], j \in [m+1, m+n]\}$ of U. In this way, the algebra $R \otimes P$ contains

$$\bar{A} = \mathrm{Gen}_G(A), \quad \bar{B} = \mathrm{Gen}_G(B) \quad \text{and} \quad \bar{R} = \mathrm{Gen}_G(R).$$

Let us define the following algebra:

$$R^* := \begin{pmatrix} \bar{A} & \bar{U} \\ 0 & \bar{B} \end{pmatrix},$$

where $\bar{U}$ is the graded $(\bar{A}\text{-}\bar{B})$-bimodule contained in $R \otimes P$ generated by the homogeneous elements

$$u_{l,g} := \sum_{\substack{i\in[1,m], j\in[1,n] \\ |\mathbf{E}_{i,m+j}|_R = g}} x^{(l)}_{i,m+j} \mathbf{E}_{i,m+j}, \qquad \text{for all } l \geq 1 \text{ and } g \in G.$$

Notice that R^* is a graded subalgebra of $R \otimes P$. Moreover, from Proposition 4.1 of [19], one has that

$$\mathrm{Id}_G(R) = \mathrm{Id}_G(\bar{R}) \subseteq \mathrm{Id}_G(R^*).$$

Still, as a consequence of Lewin's Theorem (see [30]), the authors proved in [19] the important statement:

Lemma 2.1.4 (Corollary 3.2 of [19])**.** *If the set $\{u_{l,g}\}$ is a countable free set of homogeneous elements such that $|u_{l,g}|_{R^*} = |x_l|_{F\langle X;G\rangle}$ for all $l \geq 1$, then $\mathrm{Id}_G(R^*) = \mathrm{Id}_G(A)\mathrm{Id}_G(B)$.*

The next result states that the G-regularity of only one of the G-graded algebras A or B is a sufficient condition for the factorability of the T_G-ideal $\mathrm{Id}_G(R)$.

Theorem 2.1.5 (Theorem 4.5 of [19])**.** *Let G be a finite abelian group. Let R be the G-graded upper block triangular matrix algebra*

$$R := \begin{pmatrix} A & U \\ 0 & B \end{pmatrix},$$

where $A \subseteq (M_m, \alpha)$, $B \subseteq (M_n, \beta)$ are graded subalgebras and $U = M_{m\times n}$. If one of A and B is G-regular then the T_G-ideal $\mathrm{Id}_G(R)$ factorizes as

$$\mathrm{Id}_G(R) = \mathrm{Id}_G(A)\mathrm{Id}_G(B).$$

In case G is a group of prime order, with A and B matrix algebras, Di Vincenzo and La Scala also obtained the following:

Theorem 2.1.6 (Theorem 5.8 of [19]). *Let R be the G-graded upper block triangular matrix algebra*

$$R := \begin{pmatrix} A & U \\ 0 & B \end{pmatrix},$$

where $A = (M_m, \alpha)$, $B = (M_n, \beta)$ and $U = M_{m\times n}$. If the finite group G has prime order, then the T_G-ideal $\mathrm{Id}_G(R)$ factorizes as $\mathrm{Id}_G(R) = \mathrm{Id}_G(A)\mathrm{Id}_G(B)$ if, and only if, one of the algebras A or B is G-regular.

2.2 Establishing weaker conditions for the factorability

At the end of the previous section, we exhibited some results on the factoring property which are related to the concept of G-regularity. In this section, we have as main goal to present new results, concerning also the factoring property, which require weaker conditions than the G-regularity, but regarding both G-graded algebras A and B. To this end, we will use the same notations introduced in Section 2.1.

Theorem 2.2.1 (Theorem 3.2 of [22]). *Let G be a finite abelian group. Let R be the G-graded upper block triangular matrix algebra*

$$R := \begin{pmatrix} A & U \\ 0 & B \end{pmatrix},$$

where $A \subseteq (M_m, \alpha)$, $B \subseteq (M_n, \beta)$ are graded subalgebras and $U = M_{m\times n}$. Suppose that, for all $g \in G$, there exist $i \in [1, m]$ and $j \in [1, n]$ such that

(i) $\alpha(i)^{-1}\beta(j) = g$;

(ii) The map $\widehat{\pi}^*_{\alpha(i)}$ defined on $\mathrm{Gen}_G(A)$ is injective;

(iii) The map $\widehat{\pi}_{\beta(j)}$ defined on $\mathrm{Gen}_G(B)$ is injective.

Then the T_G-ideal $\mathrm{Id}_G(R)$ factorizes as

$$\mathrm{Id}_G(R) = \mathrm{Id}_G(A)\mathrm{Id}_G(B).$$

Proof. In order to prove the result, it is enough to show that the elements $u_{l,g}$ form, for all $l \geq 1$ and $g \in G$, a countable free set in the graded ($\bar{A}$-$\bar{B}$)-bimodule $\bar{U}$. If this is the case, by invoking Lemmas 2.1.4 and 1.2.5, we obtain that

$$\mathrm{Id}_G(R) \subseteq \mathrm{Id}_G(R^*) = \mathrm{Id}_G(\bar{A})\mathrm{Id}_G(\bar{B}) = \mathrm{Id}_G(A)\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(R)$$

and, hence, $\mathrm{Id}_G(R) = \mathrm{Id}_G(A)\mathrm{Id}_G(B)$.

First we remark that, by item (i), $u_{l,g} \neq 0$, for all $l \geq 1$ and $g \in G$. Suppose that $\sum_{l,g,p}(a_{lgp})u_{l,g}(b_{lgp}) = 0$, with $a_{lgp} \in \bar{A}$ and $b_{lgp} \in \bar{B}$, for all l, g, p. Notice that, for all l and g, the non-zero entries of $u_{l,g}$ are distinct variables, and thus we need to show that each $u_{l,g} =: u$ is torsion-free. Therefore, assume that $\sum_p (a_p)u(b_p) = 0$, with $(a_p) \neq 0$ and (b_p) being linearly independent, for all p. It holds

$$\sum_p \sum_{r,s} (a_p)_{qr} u_{rs} (b_p)_{sv} = 0,$$

for any pair of indices (q, v). Since the non-zero entries u_{rs} of u are variables that are different from those in $(a_p)_{qr}$ and $(b_p)_{sv}$ and, by definition of u, one has that the position u_{rs} is non-zero if, and only if, $\alpha(r)^{-1}\beta(s) = |u|_{R^*}$, we can suppose that

$$\sum_p (a_p)_{qr}(b_p)_{sv} = 0, \tag{2.3}$$

for any quadruple (q, r, s, v) such that $\alpha(r)^{-1}\beta(s) = |u|_{R^*}$.

Let us fix a pair (i, j) such that $i \in [1, m]$, $j \in [1, n]$ and the conditions $(i), (ii), (iii)$ are satisfied for $g = |u|_{R^*}$. Then

$$\alpha(i)^{-1}\beta(j) = |u|_{R^*}.$$

On the other hand, once $(a_1) \neq 0$, item (ii) guarantees $\widehat{\pi}^*_{\alpha(i)}(a_1) \neq 0$ and this implies that there exist indices $\bar{q}, \bar{r} \in [1, m]$ such that

$$\alpha(\bar{r}) = \alpha(i) \quad \text{and} \quad (a_1)_{\bar{q}\bar{r}} \neq 0.$$

In particular, from (2.3), we obtain

$$\sum_p (a_p)_{\bar{q}\bar{r}}(b_p)_{sv} = 0,$$

for all indices $v \in [1,n]$ and all $s \in [1,n]$ such that $\beta(s) = \beta(j)$. Consequently, it follows that

$$\sum_{\substack{s,v \\ \beta(s)=\beta(j)}} \sum_p (a_p)_{\bar{q}\bar{r}} (b_p)_{sv} \mathbf{E}_{sv} = 0,$$

and thus

$$\sum_p (a_p)_{\bar{q}\bar{r}} \widehat{\pi}_{\beta(j)}(b_p) = 0.$$

Finally, since (b_p) are linearly independents, we conclude, by item (iii), that $\widehat{\pi}_{\beta(j)}(b_p)$ are also linearly independents. But, the fact that $(a_1)_{\bar{q}\bar{r}} \neq 0$ give us a contradiction, as desired. □

As a consequence of the above theorem we obtain the following:

Corollary 2.2.2 (Corollary 3.3 of [22])**.** *Let G be a finite abelian group. Let R be the G-graded upper block triangular matrix algebra*

$$R := \begin{pmatrix} A & U \\ 0 & B \end{pmatrix},$$

where $A \subseteq (M_m, \alpha)$, $B \subseteq (M_n, \beta)$ are graded subalgebras and $U = M_{m \times n}$. Suppose that

(i) $G = \{\alpha(i)^{-1}\beta(j) \mid i \in [1,m]$ and $j \in [1,n]\}$;

(ii) The maps $\widehat{\pi}^*_{\alpha(i)}$ defined on $\mathrm{Gen}_G(A)$ are injective, for all $i \in [1,m]$;

(iii) The maps $\widehat{\pi}_{\beta(j)}$ defined on $\mathrm{Gen}_G(B)$ are injective, for all $j \in [1,n]$.

Then the T_G-ideal $\mathrm{Id}_G(R)$ factorizes as

$$\mathrm{Id}_G(R) = \mathrm{Id}_G(A)\mathrm{Id}_G(B).$$

We notice that the conditions $(i), (ii)$ and (iii) of the above corollary are weaker than the G-regularity condition. Actually, we select the rows (or the columns) whose indices correspond only to the values assumed by the maps that define the elementary gradings. Moreover, we require that the maps $\widehat{\pi}_\bullet$ (or $\widehat{\pi}^*_\bullet$) corresponding to these selections are injective. This motivates us to introduce a new definition which will be presented in the next section.

2.3 α-regularity and invariance subgroups

The concept of α-regularity appears as a natural extension of the definition of G-regular subalgebras. We start by establishing such definition for graded subalgebras of (M_k, α). We recall that $\mathcal{I}_\alpha = \alpha([1,k])$, that is, $\mathcal{I}_\alpha$ is the image of the map $\alpha : [1,k] \to G$. Moreover, we remark that the map $\widehat{\pi}_g$ is not injective if, and only if, there exists a polynomial $f \notin \mathrm{Id}_G(A)$ such that $\pi_g(\rho(f)) = 0$, for every G-graded evaluation $\rho : F\langle X; G\rangle \to A$. We can assume that the polynomial f is homogeneous in the free algebra $F\langle X; G\rangle$ and let $|f|_{F\langle X;G\rangle} = h$ be its degree. Then $\widehat{\pi}^*_{hg}(\rho(f)) = 0$ for every G-graded evaluation ρ and, hence, $\widehat{\pi}^*_{hg}$ is a not injective either. Clearly g and hg are both elements of the set $\mathcal{I}_\alpha$ or both do not belong to $\mathcal{I}_\alpha$. In this way, we present the following definition:

Definition 2.3.1. Let A be a graded subalgebra of (M_k, α) endowed with an elementary grading. We say that A is *α-regular* if the maps $\widehat{\pi}_g$ are injective, for all $g \in \mathcal{I}_\alpha$, or equivalently if the maps $\widehat{\pi}^*_g$ are injective, for all $g \in \mathcal{I}_\alpha$.

In the sequel, given $A := (M_k, \alpha)$, we will prove some results which establish connections between the maps $\widehat{\pi}_\bullet$ and $\widehat{\pi}^*_\bullet$, defined on $\mathrm{Gen}_G(A)$, and the image of the map α. We will also see important relations between these concepts and the so-called *invariance subgroups*. Such subgroups were introduced by Di Vincenzo and Spinelli in [24].

By considering (M_k, α) and the weight map $w_\alpha : G \to \mathbb{N}$ introduced in Section 1.1, we set

$$\mathcal{H}_\alpha := \{h \in G \mid w_\alpha(hg) = w_\alpha(g), \quad \text{for all } g \in G\}.$$

The subgroup $\mathcal{H}_\alpha$ is the *invariance subgroup* related to the algebra (M_k, α).

Proposition 2.3.2. *Let G be a finite abelian group and consider $A = (M_k, \alpha)$. The following statements are equivalent:*

(i) The maps $\widehat{\pi}_h$ defined on $\mathrm{Gen}_G(A)$ are injective, for all $h \in \mathcal{I}_\alpha$;

(ii) The maps $\widehat{\pi}^*_h$ defined on $\mathrm{Gen}_G(A)$ are injective, for all $h \in \mathcal{I}_\alpha$;

(iii) There exist a subgroup H of G and an element $g \in G$ such that

$$\mathcal{I}_\alpha = gH$$

and all fibers of the map α are equipotent.

Proof. First, let us prove that (i) implies (iii). Suppose that the maps $\widehat{\pi}_h$ defined on $\mathrm{Gen}_G(A)$ are injective, for all $h \in \mathcal{I}_\alpha$. Then there exist a subset $S = \{g_1, \ldots, g_s\}$ of G and an element $g \in G$ such that

$$\mathcal{I}_\alpha = gS \quad \text{and} \quad 1_G \in S.$$

Thus, in order to conclude that S is a subgroup of G, it is enough to show that $g_i^{-1}g_j \in S$, for all $g_i, g_j \in S$.

Fix arbitrary elements $g_i, g_j \in S$. Clearly, there exist indices $u, v \in [1, k]$ such that

$$|\mathbf{E}_{uv}|_A = g_i^{-1}g_j.$$

Consequently, there exists a non-zero homogeneous element $a' \in \mathrm{Gen}_G(A)$ such that

$$a' = \sum_{\substack{l,t \\ |\mathbf{E}_{lt}|_A = g_i^{-1}g_j}} f_{lt}\mathbf{E}_{lt}, \quad \text{with } f_{lt} \in P(A).$$

Since $g \in \mathcal{I}_\alpha$, then $\widehat{\pi}_g$ is injective, which yields

$$\widehat{\pi}_g(a') = \sum_{\substack{\alpha(l)=g;\ t \\ |\mathbf{E}_{lt}|_A = g_i^{-1}g_j}} f_{lt}\mathbf{E}_{lt} \neq 0$$

and this implies that there exist $l, t \in [1, k]$, such that $\alpha(l) = g$, satisfying

$$g_i^{-1}g_j = |\mathbf{E}_{lt}|_A = \alpha(l)^{-1}\alpha(t).$$

Once $\alpha(t) = gg_{t'}$, for some $t' \in [1, s]$, we conclude that $g_i^{-1}g_j = g_{t'} \in S$, and then S is a subgroup of G.

Now, let us assume that the fibers of the map α are not equipotent. Then, by denoting, for each $i \in [1, s]$, $q_i := w_\alpha(gg_i)$, let us suppose, without loss of generality, that $q_1 > q_\ell$, for some $\ell \in [2, s]$. Consider the graded standard polynomial

$$S_{2q_\ell} := S_{2q_\ell}(y_1, \ldots, y_{2q_\ell}) = \sum_{\sigma \in \mathrm{Sym}(2q_\ell)} (-1)^\sigma y_{\sigma(1)} \cdots y_{\sigma(2q_\ell)},$$

where $y_1, \ldots, y_{2q_\ell}$ are homogeneous variables of degree 1_G. It follows that if $\rho : F\langle X; G\rangle \to A$ is an arbitrary graded evaluation, then $\rho(S_{2q_\ell})$ is a homogeneous element in A of degree 1_G.

We remark that the following direct sum (as algebras) holds:

$$A_{1_G} = A_{1_G}^{(gg_1)} \oplus \cdots \oplus A_{1_G}^{(gg_s)},$$

where, for each $i \in [1, s]$,

$$A_{1_G}^{(gg_i)} := \mathrm{span}_F\{\mathbf{E}_{uv} \mid \alpha(u) = \alpha(v) = gg_i\}.$$

Then, we can apply Amitsur-Levitzki theorem and conclude that $\rho(S_{2q_\ell})$ has zero component in $A_{1_G}^{(gg_\ell)}$ as direct summand of A_{1_G}, for any graded evaluation $\rho : F\langle X;G\rangle \to A$.

On the other hand, since $q_1 > q_\ell$, again by Amitsur-Levitzki theorem, there exists a graded evaluation $\rho' : F\langle X;G\rangle \to A$ such that $\rho'(S_{2q_\ell})$ is also a homogeneous element in A of degree 1_G which has non-zero component in $A_{1_G}^{(gg_1)}$. Therefore, the graded standard polynomial S_{2q_ℓ} defines a non-zero element a' in $\mathrm{Gen}_G(A)$ such that $\widehat{\pi}_{gg_\ell}(a') = 0$, which implies $\widehat{\pi}_{gg_\ell}$ is not injective.

In order to prove that (iii) implies (i), assume that there exist a subgroup $H = \{h_1, \dots, h_s\}$ of G and an element $g \in G$ such that $\mathcal{I}_\alpha = gH$ and all fibers of the map α are equipotent. Then, by denoting, for each $i \in [1,s]$, $q_i := w_\alpha(gh_i)$, it follows that

$$q_1 = \cdots = q_s.$$

Fix $\ell \in [1,s]$ and an element a' in $\mathrm{Gen}_G(A)$ satisfying $\widehat{\pi}_{gh_\ell}(a') = 0$. We claim that $a' = 0$.

In fact, let $\varphi : F\langle X;G\rangle \to \mathrm{Gen}_G(A)$ be the canonical G-graded epimorphism such that $\ker(\varphi) = \mathrm{Id}_G(A)$. Take $f \in F\langle X;G\rangle$ such that $\varphi(f) = a'$ and fix $\rho : F\langle X;G\rangle \to A$ an arbitrary graded evaluation. Thus, we obtain that

$$\rho(f) = \sum_{i,j} d_{ij}\mathbf{E}_{ij}, \text{ with } d_{pj} = 0, \text{ for all } p \in [1,k] \text{ satisfying } \alpha(p) = gh_\ell.$$

Fix an arbitrary $\ell' \in [1,s]$ and consider

$$\bar{g} := h_{\ell'}^{-1}h_\ell. \tag{2.4}$$

Since H is a subgroup of G it follows that $\bar{g} \in H$. Thus, there exists $\theta \in \mathrm{Sym}(s)$ such that

$$h_l\bar{g} = h_{\theta(l)}, \quad \text{for all } l \in [1,s],$$

and, in particular,

$$\theta(\ell') = \ell.$$

Moreover, the equalities $q_1 = q_2 = \cdots = q_s$ guarantee the existence of σ in $\mathrm{Sym}(k)$ satisfying

$$\alpha(\sigma(\iota)) = \bar{g}\alpha(\iota), \quad \text{for all } \iota \in [1,k]. \tag{2.5}$$

Finally, define the map

$$\begin{array}{rccc} \Gamma : & A & \to & A \\ & \mathbf{E}_{uv} & \mapsto & \mathbf{E}_{\sigma(u)\sigma(v)}. \end{array}$$

It is clear that Γ is a graded isomorphism. Furthermore, we remark that $\Gamma\rho : F\langle X; G\rangle \to A$ is still a graded evaluation and

$$\Gamma(\rho(f)) = \sum_{i,j} d_{ij}\mathbf{E}_{\sigma(i)\sigma(j)}.$$

Since $\widehat{\pi}_{gh_\ell}(a') = 0$, by combining (2.4) and (2.5), we obtain that

$$d_{pj} = 0, \quad \text{for all } p \in [1,k] \text{ satisfying } \alpha(p) = gh_{\ell'}.$$

Therefore, once $gh_{\ell'}$ is arbitrary, we conclude that $d_{ij} = 0$, for every $i, j \in [1,k]$. Then, $f \in \mathrm{Id}_G(A)$ and this implies $a' = 0$, as desired.

The proof that the statements (ii) and (iii) are equivalent is analogous. □

We remark that, as a consequence of Proposition 2.3.2, if (M_k, α) is α-regular and we multiply the elements $\alpha(1), \ldots, \alpha(k)$ by a suitable element of G, then we obtain an H-grading on (M_k, α) such that (M_k, α) is H-regular according with Definition 2.1.1. In particular, in case $H = G$, the notion of α-regularity coincides with G-regularity. The next step is to establish a connection between α-regularity and the invariance subgroup $\mathcal{H}_\alpha$. First we state the following lemma which depends only on the map $\alpha : [1,k] \to G$.

Lemma 2.3.3 (Lemma 3.6 of [22])**.** *Let G be a finite abelian group and consider a map α : $[1,k] \to G$. Then the following statements are equivalent:*

(i) There exist a subgroup H of G and an element $g \in G$ such that $\mathcal{I}_\alpha = gH$ and all fibers of the map α are equipotent;

(ii) There exists an element $g \in G$ such that

$$\mathcal{I}_\alpha = g\mathcal{H}_\alpha.$$

Proof. First, suppose that there exist a subgroup $H = \{h_1, \ldots, h_s\}$ of G and an element $g \in G$ such that $\mathcal{I}_\alpha = gH$ and all fibers of the map α are equipotent, that is,

$$w_\alpha(gh_i) = w_\alpha(gh_j), \quad \text{for all } i, j \in [1,s].$$

Take an arbitrary element $h_l \in H$. Let us prove that h_l satisfies

$$w_\alpha(h_l\bar{g}) = w_\alpha(\bar{g}), \quad \text{for all } \bar{g} \in G,$$

and consequently $h_l \in \mathcal{H}_\alpha$. If $\bar{g} \in \mathcal{I}_\alpha$, then $\bar{g} = gh_i$, for some $i \in [1,s]$, and since H is a subgroup of G it follows that $h_l h_i \in H$. Thus, since G is abelian,

$$w_\alpha(h_l \bar{g}) = w_\alpha(h_l g h_i) = w_\alpha(g h_l h_i) = w_\alpha(g h_i) = w_\alpha(\bar{g}).$$

On the other hand, if $\bar{g} \notin \mathcal{I}_\alpha$, then $w_\alpha(\bar{g}) = 0$. In this case, it is easy to verify that $w_\alpha(h_l \bar{g}) = 0$.

Now, take $\tilde{h} \in \mathcal{H}_\alpha$. Then, one has that

$$w_\alpha(g\tilde{h}) = w_\alpha(\tilde{h} g 1_G) = w_\alpha(g 1_G) \neq 0 \tag{2.6}$$

and this allows us to conclude that $\tilde{h} \in H$. Therefore, we obtain that $H = \mathcal{H}_\alpha$.

Reciprocally, assume that $\mathcal{I}_\alpha = g\mathcal{H}_\alpha$. It is valid that $\mathcal{H}_\alpha$ is a subgroup of G. Moreover, for any $\widetilde{h} \in \mathcal{H}_\alpha$, (2.6) holds. Therefore all fibers of the map α are equipotent. $\square$

As a consequence of Proposition 2.3.2 and Lemma 2.3.3, we obtain the following nice characterization of the graded matrix algebras (M_k, α) which are α-regular.

Theorem 2.3.4 (Theorem 3.7 of [22])**.** *Let G be a finite abelian group. Then (M_k, α) is α-regular if, and only if, there exists an element $g \in G$ such*

$$\mathcal{I}_\alpha = g\mathcal{H}_\alpha.$$

Chapter 3

C_n-simple algebras

In Chapter 1, we presented, when G is a group of order 2 and even in case G is a group of any prime order, the description of the finite dimensional G-simple algebras as graded subalgebras of matrix algebras endowed with elementary gradings obtained in [35] and [17], respectively. In this sense, if $G = C_n$ is any finite cyclic group of order n, the first aim of this chapter consists in describing the finite dimensional G-simple algebras as graded subalgebras of matrix algebras endowed with some elementary gradings. In the sequel, we will present some necessary and sufficient conditions in order to obtain a graded isomorphism between such G-simple algebras and we will study its regularity. The new results establish here count with the collaboration of Professor Viviane Ribeiro Tomaz da Silva and Professor Onofrio Mario Di Vincenzo, and can be found in [22]. It is worth highlighting that the proofs of some of these results are different from those presented in [22].

3.1 The characterization of the C_n-simple algebras

Let G be an arbitrary group. Consider $R = F[G]$ the group algebra over F and let $B = \{\mathbf{r}_g \mid g \in G\}$ be a basis for R, with the product of its elements being $\mathbf{r}_g\mathbf{r}_h = \mathbf{r}_{gh}$, for all $g, h \in G$. We endow R with the *canonical G-grading* $R = \oplus_{g\in G} R_g$, where, for each $g \in G$, $R_g = \text{span}_F\{\mathbf{r}_g\}$. Notice that all homogeneous non-zero elements of R are invertible and, hence, R is a graded skew field. Assume now that the product of the basis elements of R is defined as

$$\mathbf{r}_g\mathbf{r}_h = \sigma(g, h)\mathbf{r}_{gh},$$

where $\sigma(g, h) \in F^*$, for all $g, h \in G$. For such product to be associative the map $\sigma : G\times G \to F^*$ has to satisfy

$$\sigma(g, h)\sigma(gh, l) = \sigma(h, l)\sigma(g, hl), \quad \text{for all } g, h, l \in G.$$

In this case, the map σ is said a *2-cocycle* on G with values in F^* and the associative algebra

$$F^\sigma[G] := \mathrm{span}_F\{\mathbf{r}_g \mid g \in G\}$$

is called the *twisted group algebra* defined by σ. We remark that if $\sigma = 1$, thus $F^\sigma[G]$ is the ordinary group algebra R.

Such algebras are related with the description of the finite dimensional G-simple F-algebras, presented by Bahturin, Sehgal and Zaicev, in [10]. In that paper, the authors proved the following result:

Theorem 3.1.1 (Theorems 2 and 3 of [10])**.** *Let G be an arbitrary group and F an algebraically closed field such that either* $\mathrm{char}F = 0$ *or* $\mathrm{char}F = p > 0$ *is coprime with the order of each finite subgroup of G. Consider A a finite dimensional F-algebra. Then A is a G-simple algebra if, and only if, A is graded-isomorphic to $M_k \otimes D \cong M_k(D)$, where $D = \oplus_{h\in H} D_h$ is a graded skew field with* $\mathrm{Supp}(D) = H$ *being a subgroup of G, and M_k has an elementary G-grading defined by a k-tuple $(g_1, \dots, g_k) \in G^k$ such that*

$$|e_{ij} \otimes d_h|_{M_k(D)} = g_i^{-1} h g_j,$$

for each matrix unit $e_{ij} \in M_k$ and each homogeneous element $d_h \in D_h$. Moreover, D is isomorphic to a twisted group algebra $F^\sigma[H]$ with canonical H-grading, where $\sigma : H \times H \to F^*$ is a 2-cocycle on H.

From now on, unless otherwise is stated, F is an algebraically closed field of characteristic zero and ϵ is a primitive nth root of the unity in F^*. Moreover, we consider $G := C_n = \langle \epsilon \rangle$, the finite cyclic group generated by ϵ.

The aim of this section is presenting, by applying results of [10], a characterization of the finite dimensional G-simple F-algebras as graded subalgebras of matrix algebras endowed with some elementary gradings.

First, given a finite dimensional G-simple F-algebra A, from Theorem 3.1.1, one has that A is graded-isomorphic to $M_k \otimes D \cong M_k(D)$, where $D = \oplus_{h\in H} D_h$ is a graded skew field with $\mathrm{Supp}(D) = H$ being a subgroup of G. Then $|H| = r$ and $H = \langle \epsilon^s \rangle = \{1_G, \epsilon^s, (\epsilon^s)^2, \dots, (\epsilon^s)^{r-1}\}$, for some positive integers r, s such that $|G| = n = r \cdot s$.

According to Lemma 3 of [10], $\dim_F D_h = 1$, for all $h \in H$. Therefore, we have that

$$D_{\epsilon^s} = Fa, \quad \text{for some } a \in D_{\epsilon^s}.$$

It is easy to verify that $D_{(\epsilon^s)^t} = Fa^t$, for all $t \geq 1$. Then, we obtain

$$Fa^r = D_{(\epsilon^s)^r} = D_{1_G} = F1_D$$

and this implies that there exists $\gamma \in F^*$ such that $a^r = \gamma$. Since F is algebraically closed, also there exists $\gamma' \in F^*$ such that $(\gamma')^r = \gamma$. By setting $b := (\gamma')^{-1}a$ we get $b^r = 1_D$ and we conclude that

$$D = D_{1_G} \oplus D_{\epsilon^s} \oplus D_{(\epsilon^s)^2} \oplus \cdots \oplus D_{(\epsilon^s)^{r-1}} = F \oplus Fb \oplus Fb^2 \oplus \cdots \oplus Fb^{r-1},$$

with b^t being homogeneous of degree $(\epsilon^s)^t$, for all $t \in [0, r-1]$.

Consider the matrix algebra M_r with elementary grading induced by the r-tuple

$$\widetilde{\epsilon}_r := (1_G, \epsilon^s, (\epsilon^s)^2, \ldots, (\epsilon^s)^{r-1}) \in G^r$$

and, for each $i, j \in [1, r]$, denote by E_{ij} the (i, j)-matrix unit of M_r (it is worth remarking that we are using E_{ij} for the matrix units of M_r in order to distinguish them of the matrix units e_{uv} of M_k, introduced in Section 1.1).

Consider the permutation

$$\varsigma := (1\ 2 \cdots r)$$

and set

$$E := \sum_{l=0}^{r-1} E_{\varsigma^l(1),\varsigma^l(2)} = E_{12} + E_{23} + E_{34} + \cdots + E_{r-1,r} + E_{r1}.$$

It holds that $E^t = \sum_{l=0}^{r-1} E_{\varsigma^l(1),\varsigma^l(t+1)}$, for all $t \in [0, r-1]$ and $E^r = E^0$. Furthermore, the set

$$\{E^t \mid t \in [0, r-1]\}$$

is linearly independent and $|E^t|_{M_r} = (\epsilon^s)^t$, for all $t \in [0, r-1]$.

Let us denote by D_r the graded subalgebra of $(M_r, \widetilde{\epsilon}_r)$ generated by the elements $\{E^t \mid t \in [0, r-1]\}$, that is, D_r is defined as

$$D_r := \left\{ \begin{pmatrix} d_0 & d_1 & \cdots & d_{r-2} & d_{r-1} \\ d_{r-1} & d_0 & \ddots & & d_{r-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ d_2 & & \ddots & \ddots & d_1 \\ d_1 & d_2 & \cdots & d_{r-1} & d_0 \end{pmatrix} \mid d_0, d_1, \ldots, d_{r-1} \in F \right\},$$

with its natural grading induced by the r-tuple $\widetilde{\epsilon}_r = (1_G, \epsilon^s, (\epsilon^s)^2, \ldots, (\epsilon^s)^{r-1})$. Clearly D_r is a finite dimensional graded skew algebra and $\mathrm{Supp}(D_r, \widetilde{\epsilon}_r) = \langle \epsilon^s \rangle$.

In the next lemma we stated that D is graded-isomorphic to D_r and, consequently, we will obtain that D can be seen as a graded subalgebra of the matrix algebra $(M_r, \widetilde{\epsilon}_r)$.

Lemma 3.1.2 (Lemma 4.1 of [22]). *Let $G = \langle \epsilon \rangle$ be a cyclic group such that $|G| = n = r \cdot s$, for some positive integers r and s. Moreover, let $D = \oplus_{h \in H} D_h$ be a graded skew field, with* $\mathrm{Supp}(D) = H = \langle \epsilon^s \rangle$*, and consider the matrix algebra $(M_r, \widetilde{\epsilon}_r)$. Then D is graded-isomorphic to $D_r \subseteq (M_r, \widetilde{\epsilon}_r)$.*

Proof. From the above discussions one has that there exists $b \in D$ such that

$$D = F \oplus Fb \oplus Fb^2 \oplus \cdots \oplus Fb^{r-1}$$

and, for each $t \in [0, r-1]$, b^t is homogeneous of degree $(\epsilon^s)^t$. Define the map

$$\begin{array}{rccc} \Gamma: & F \oplus Fb \oplus Fb^2 \oplus \cdots \oplus Fb^{r-1} & \to & D_r \\ & d_0 + d_1 b + d_2 b^2 + \cdots + d_{r-1} b^{r-1} & \mapsto & \begin{pmatrix} d_0 & d_1 & \cdots & d_{r-1} \\ d_{r-1} & d_0 & \cdots & d_{r-2} \\ \vdots & \vdots & \ddots & \vdots \\ d_1 & d_2 & \cdots & d_0 \end{pmatrix} \end{array}.$$

Clearly Γ is an isomorphism of algebras. Since, for each $t \in [0, r-1]$, b^t and E^t are homogeneous of degree $(\epsilon^s)^t$, and $\Gamma(b^t) = E^t$, we obtain that Γ is a graded isomorphism and this concludes the proof of the lemma. □

Now, given the matrix algebra M_k, with the elementary grading defined by a k-tuple $\widetilde{g} = (g_1, \ldots, g_k) \in G^k$, consider the tensor product $M_k \otimes D_r$. The set

$$\mathcal{B} := \{e_{ij} \otimes E^t \mid i, j \in [1, k], t \in [0, r-1]\}$$

is a basis of $M_k \otimes D_r$, which will be called the *canonical basis* of $M_k \otimes D_r$, where, for each $i, j \in [1, k]$, e_{ij} denote the (i, j)-matrix unit of the matrix algebra M_k. At light of Theorem 3.1.1, we endow $M_k \otimes D_r$ with the grading such that

$$|e_{ij} \otimes E^t|_{M_k \otimes D_r} = g_i^{-1} g_j (\epsilon^s)^t, \quad \text{for all } i, j \in [1, k], t \in [0, r-1].$$

In particular, $\mathcal{B}$ is a homogeneous basis of $M_k \otimes D_r$.

On the other hand, consider the finite dimensional algebra $M_k(D_r) \subseteq M_{kr}$ with the elementary grading induced by the (kr)-tuple

$$\widetilde{g} \odot \widetilde{\epsilon}_r := (g_1, \epsilon^s g_1, \dots, (\epsilon^s)^{r-1} g_1, \dots, g_k, \epsilon^s g_k, \dots, (\epsilon^s)^{r-1} g_k),$$

where r, s are such that $|G| = n = r \cdot s$.

Clearly, since G is an abelian group, $M_k \otimes D_r$ is graded-isomorphic to $(M_k(D_r), \widetilde{g} \odot \widetilde{\epsilon}_r)$ and we can identify any element

$$\sum_{\substack{i,j \in [1,k] \\ t \in [0,r-1]}} d_t^{(ij)} (e_{ij} \otimes E^t) \in M_k \otimes D_r$$

with

$$\left(\begin{array}{ccccc|c|ccccc}
d_0^{(11)} & d_1^{(11)} & \cdots & d_{r-2}^{(11)} & d_{r-1}^{(11)} & & d_0^{(1k)} & d_1^{(1k)} & \cdots & d_{r-2}^{(1k)} & d_{r-1}^{(1k)} \\
d_{r-1}^{(11)} & d_0^{(11)} & \ddots & & d_{r-2}^{(11)} & & d_{r-1}^{(1k)} & d_0^{(1k)} & \ddots & & d_{r-2}^{(1k)} \\
\vdots & \ddots & \ddots & \ddots & \vdots & \cdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
d_2^{(11)} & & \ddots & \ddots & d_1^{(11)} & & d_2^{(1k)} & & \ddots & \ddots & d_1^{(1k)} \\
d_1^{(11)} & d_2^{(11)} & \cdots & d_{r-1}^{(11)} & d_0^{(11)} & & d_1^{(1k)} & d_2^{(1k)} & \cdots & d_{r-1}^{(1k)} & d_0^{(1k)} \\
\hline
 & & \vdots & & & \ddots & & & \vdots & & \\
\hline
d_0^{(k1)} & d_1^{(k1)} & \cdots & d_{r-2}^{(k1)} & d_{r-1}^{(k1)} & & d_0^{(kk)} & d_1^{(kk)} & \cdots & d_{r-2}^{(kk)} & d_{r-1}^{(kk)} \\
d_{r-1}^{(k1)} & d_0^{(k1)} & \ddots & & d_{r-2}^{(k1)} & & d_{r-1}^{(kk)} & d_0^{(kk)} & \ddots & & d_{r-2}^{(kk)} \\
\vdots & \ddots & \ddots & \ddots & \vdots & \cdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
d_2^{(k1)} & & \ddots & \ddots & d_1^{(k1)} & & d_2^{(kk)} & & \ddots & \ddots & d_1^{(kk)} \\
d_1^{(k1)} & d_2^{(k1)} & \cdots & d_{r-1}^{(k1)} & d_0^{(k1)} & & d_1^{(kk)} & d_2^{(kk)} & \cdots & d_{r-1}^{(kk)} & d_0^{(kk)}
\end{array}\right) \in M_k(D_r),$$

where $d_t^{(ij)} \in F$, for all $i, j \in [1, k]$ and $t \in [0, r-1]$. We notice that $(M_k(D_r), \widetilde{g} \odot \widetilde{\epsilon}_r)$ is a graded subalgebra of $(M_{kr}, \widetilde{g} \odot \widetilde{\epsilon}_r)$.

Let us divide M_{kr} into $r \times r$ blocks, labeled with pairs (u, v) such that $u, v \in [1, k]$, that is,

$$M_{kr} = \{(b_{uv})_{u,v \in [1,k]} \mid b_{uv} \in M_r, \text{ for all } u, v \in [1, k]\}$$

and, for each $i, j \in [1, k]$, let us define the block

$$B_{ij} := \{(b_{uv})_{u,v \in [1,k]} \mid b_{uv} = 0, \text{ for all } (u, v) \neq (i, j)\}.$$

For each $i, j \in [1, k]$, $d, p \in [1, r]$, we denote the matrix unit of M_{kr}, corresponding to the position (d, p) of the block B_{ij}, by

$$E_{dp}^{(i,j)_r} := \mathrm{E}_{(i-1)r+d,(j-1)r+p},$$

where E_{lq} is the (l,q)-matrix unit of M_{kr} and the index r emphasizes that each block is a $r \times r$ matrix.

We remark that, by taking the permutation

$$\tau := (0\ 1 \cdots r-1),$$

if $i,j \in [1,k]$ and $t \in [0,r-1]$ then, in the block B_{ij} of $M_k(D_r) \subseteq M_{kr}$, the elements appearing at positions $(l+1, \tau^l(t)+1)$, with $l \in [0,r-1]$, are the same.

Therefore, with the previously seen identification, we can write each element $e_{ij} \otimes E^t$ of $\mathcal{B}$ as a sum of r distinct matrices in M_{kr}:

$$e_{ij} \otimes E^t = \sum_{l=0}^{r-1} E^{(i,j)_r}_{l+1,\tau^l(t)+1} = \sum_{l=0}^{r-1} \mathrm{E}_{(i-1)r+l+1,(j-1)r+\tau^l(t)+1}. \tag{3.1}$$

We observe that the left (and right) indices of the matrix units $\mathrm{E}_{i_pj_p}$ appearing in the above sum are pairwise distinct. Furthermore, for all $i_p, j_p \in [1,kr]$, there exists an unique canonical basis element $e_{ij} \otimes E^t$ of $\mathcal{B}$ such that $\mathrm{E}_{i_pj_p}$ appears in the sum of $e_{ij} \otimes E^t$. Then, when it is convenient, we will denote

$$e_{ij} \otimes E^t = \overline{\mathrm{E}}_{(i-1)r+1,(j-1)r+t+1} = \cdots = \overline{\mathrm{E}}_{(i-1)r+r,(j-1)r+\tau^{r-1}(t)+1}. \tag{3.2}$$

Moreover if $r=1$ then $M_k(D_r)$ is M_k and $e_{ij} \otimes E^0 = \mathrm{E}_{ij}$.

We are in position to state the main result of this section, which classifies all finite dimensional G-simple algebras as graded subalgebras of matrix algebras, in case G is a finite cyclic group.

Theorem 3.1.3 (Theorem 4.2 of [22])**.** *Let F be an algebraically closed field of characteristic zero and $G = \langle \epsilon \rangle$ a cyclic group, with ϵ being a primitive* nth *root of the unity in F^*. Then any finite dimensional G-simple algebra is graded-isomorphic to a graded subalgebra*

$$(M_k(D_r), \widetilde{g} \odot \widetilde{\epsilon}_r) \subseteq (M_{kr}, \widetilde{g} \odot \widetilde{\epsilon}_r),$$

whose grading is induced by the (kr)-tuple

$$\widetilde{g} \odot \widetilde{\epsilon}_r := (g_1, \epsilon^s g_1, \ldots, (\epsilon^s)^{r-1} g_1, \ldots, g_k, \epsilon^s g_k, \ldots, (\epsilon^s)^{r-1} g_k),$$

where the tuples $\widetilde{g} = (g_1, \ldots, g_k) \in G^k$ and $\widetilde{\epsilon}_r = (1_G, \epsilon^s, (\epsilon^s)^2, (\epsilon^s)^3, \ldots, (\epsilon^s)^{r-1})$ induce the elementary gradings in M_k and M_r, respectively, and r, s are such that $n = r \cdot s$.

Proof. Let A be a finite dimensional G-simple algebra. From Theorem 3.1.1, it follows that there exists a graded skew field $D = \oplus_{h\in H} D_h$, with $\mathrm{Supp}(D) = H$ being a subgroup of G and $r = |H|$, such that A is isomorphic to $M_k \otimes D$, and M_k has an elementary grading induced by a k-tuple $\widetilde{g} = (g_1, \dots, g_k) \in G^k$.

Now, by invoking Lemma 3.1.2, we can suppose that $D = D_r$ is a graded subalgebra of $(M_r, \widetilde{\epsilon}_r)$. Therefore, by our previous discussions, we can concluded that A is graded-isomorphic to the graded subalgebra $(M_k(D_r), \widetilde{g} \odot \widetilde{\epsilon}_r)$ of $(M_{kr}, \widetilde{g} \odot \widetilde{\epsilon}_r)$. □

Given a positive integer $l \geq 1$, we define the *Capelli polynomial of rank* l (or the lth *Capelli polynomial*) as

$$Cap_l(x_1, \dots, x_l; x_{l+1}, \dots, x_{2l+1}) := \sum_{\sigma \in \mathrm{Sym}(l)} (-1)^{\sigma} x_{l+1} x_{\sigma(1)} x_{l+2} \cdots x_{2l} x_{\sigma(l)} x_{2l+1}.$$

We finish this section by presenting an important property of such polynomial associated to finite dimensional G-simple algebras.

Lemma 3.1.4. *Let $G = \langle \epsilon \rangle$ be a cyclic group and consider $A = (M_k(D_r), \widetilde{g} \odot \widetilde{\epsilon}_r)$. The Capelli polynomial $Cap_l(x_1, \dots, x_l; x_{l+1}, \dots, x_{2l+1})$ is an ordinary polynomial identity for A if, and only if, $l > k^2$.*

Proof. For each positive integer $l \geq 1$, let us write f_l as being the Capelli polynomial of rank l and we denote the evaluation of each variable x_i, at elements of the canonical basis of A, by $\bar{x}_i$.

First, suppose that $l = k^2$. In this case, assume that $\bar{x}_1, \dots, \bar{x}_{k^2}$ is equal to $e_{11} \otimes E^0, \dots, e_{1k} \otimes E^0, \dots, e_{k1} \otimes E^0, \dots, e_{kk} \otimes E^0$, respectively, $\bar{x}_{k^2+1} = e_{11} \otimes E^0$, $\bar{x}_{2k^2+1} = e_{k1} \otimes E^0$, and for all remaining $\bar{x}_i$'s we consider the evaluation such that the monomial of f_{k^2} associated to $\sigma = 1$ is the unique monomial whose evaluation is non-zero. Thus

$$Cap_{k^2}(\bar{x}_1, \dots, \bar{x}_{k^2}; \bar{x}_{k^2+1}, \dots, \bar{x}_{2k^2+1}) = e_{11} \otimes E^0$$

and this yields us that $f_{k^2} \notin \mathrm{Id}(A)$. Similarly, we obtain that $f_l \notin \mathrm{Id}(A)$ in case $l < k^2$.

On the other hand, assume $l > k^2$. Since the algebra D_r is commutative, for each $i \in [1, 2l+1]$, by considering an evaluation by canonical basis elements $\bar{x}_i = e_{p_i q_i} \otimes E^{t_i}$, it follows that

$$\bar{f}_l = Cap_l(\bar{x}_1, \dots, \bar{x}_l; \bar{x}_{l+1}, \dots, \bar{x}_{2l+1}) = Cap_l(e_{p_1 q_1}, \dots, e_{p_l q_l}; e_{p_{l+1} q_{l+1}}, \dots, e_{p_{2l+1} q_{2l+1}}) \otimes E^t,$$

for some $t \in [0, r-1]$. The fact that the Capelli polynomial is alternating in the variables $x_1, \dots, x_l$ and multilinear guarantees us that $Cap_l(e_{p_1 q_1}, \dots, e_{p_l q_l}; e_{p_{l+1} q_{l+1}}, \dots, e_{p_{2l+1} q_{2l+1}}) = 0$ and, hence, $\bar{f}_l \in \mathrm{Id}(A)$, as desired. □

3.2 C_n-simple algebras and the isomorphism problem

In this section, we will establish conditions in order to obtain a graded isomorphism between finite dimensional C_n-simple F-algebras. Moreover, we will explore the isomorphism problem regarding such algebras.

Let $A = (M_k(D_r), \widetilde{g} \odot \widetilde{\epsilon}_r)$. If $\alpha : [1, k] \to G$ is the map corresponding to the elementary grading $\widetilde{g} = (g_1, \dots, g_k)$ defined on M_k, we denote by $\alpha \odot \widetilde{\epsilon}_r : [1, kr] \to G$ the map corresponding to the grading $\widetilde{g} \odot \widetilde{\epsilon}_r$ defined on A, that is,

$$((\alpha \odot \widetilde{\epsilon}_r)(1), \dots, (\alpha \odot \widetilde{\epsilon}_r)(kr)) = (g_1, \epsilon^s g_1, \dots, (\epsilon^s)^{r-1} g_1, \dots, g_k, \epsilon^s g_k, \dots, (\epsilon^s)^{r-1} g_k).$$

In this case, we write

$$A = (M_k(D_r), \widetilde{g} \odot \widetilde{\epsilon}_r) = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r).$$

Analogously to matrix algebras, we set

$$\mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r} := (\alpha \odot \widetilde{\epsilon}_r)([1, kr]),$$

and we also define the *weight map* $w_{\alpha \odot \widetilde{\epsilon}_r} : G \to \mathbb{N}$ as

$$w_{\alpha \odot \widetilde{\epsilon}_r}(g) := |\{i \mid 1 \le i \le kr, \ (\alpha \odot \widetilde{\epsilon}_r)(i) = g\}|.$$

Notice that $\mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r} = \{g \in G \mid w_{\alpha \odot \widetilde{\epsilon}_r}(g) \neq 0\}$. Moreover, we set

$$\mathcal{H}_{\alpha \odot \widetilde{\epsilon}_r} := \{h \in G \mid w_{\alpha \odot \widetilde{\epsilon}_r}(hg) = w_{\alpha \odot \widetilde{\epsilon}_r}(g), \quad \text{for all } g \in G\}.$$

The subgroup $\mathcal{H}_{\alpha \odot \widetilde{\epsilon}_r}$ is said the *invariance subgroup* related to the G-simple algebra A. We remark that

$$H_r := \langle \epsilon^s \rangle \subseteq \mathcal{H}_{\alpha \odot \widetilde{\epsilon}_r}.$$

In [3] Aljadeff and Haile established suitable properties which determine G-simple algebras up to graded isomorphism (for any group G). In the sequel, we present such properties in case G is finite cyclic.

Let $G = \langle \epsilon \rangle$ be a cyclic group and consider the finite dimensional G-simple algebras

$$A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r) \quad \text{and} \quad B = (M_h(D_t), \beta \odot \widetilde{\epsilon}_t).$$

First we remark that the presentation P_A, introduced by Aljadeff and Haile in Definition 1.2 of [3], of $A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$ is determined by r and $(\alpha(1), \dots, \alpha(k)) = (g_1, \dots, g_k)$, because

$H_r = \langle \epsilon^s \rangle$ is the unique subgroup of G of order r and by Lemma 3.1.2 there exists, up to graded isomorphism, a unique graded skew field D_r having H_r as a support. Hence we can write

$$P_A = (r; \alpha) = (r; (g_1, \ldots, g_k)).$$

In the same way, we can write the presentation P_B of $B = (M_h(D_t), \beta \odot \widetilde{\epsilon}_t)$ as $P_B = (t; \beta) = (t; (g'_1, \ldots, g'_h))$. Moreover, in our case, the *basic moves* of type (1), (2) or (3), introduced in Lemma 1.3 of [3], correspond to the actions described in the following items:

(i) Permuting the elements in the k-tuple $(g_1, \ldots, g_k)$, that is, consider the presentation

$$(r; \alpha \cdot \nu) := (r; (g_{\nu(1)}, \ldots, g_{\nu(k)})),$$

where ν is an arbitrary element of the symmetric group $\mathrm{Sym}(k)$;

(ii) Given $i \in [1, k]$, replacing the entry g_i by any element hg_i of $H_r g_i$, that is, consider the presentation $(r; (g_1, \ldots, g_{i-1}, hg_i, g_{i+1}, \ldots, g_k))$;

(iii) Given $g \in G$, multiplying the elements in the k-tuple $(g_1, \ldots, g_k)$ by g, that is, consider the presentation

$$(r; l_g \cdot \alpha) := (r; (gg_1, \ldots, gg_k)),$$

where l_g is the left multiplication by g on G.

As in [3], we say that the presentations P_A of the G-simple algebra A and P_B of the G-simple algebra B are *equivalent* if one is obtained from the other by a finite sequence of basic moves (items (i),(ii) or (iii) above). It follows from Lemma 1.3 and Proposition 3.1 of [3] that the algebras A and B are graded-isomorphic if, and only if, they have equivalent presentations.

Now let us consider the map $\overline{\alpha} : [1, k] \to G/H_r$ defined by

$$\overline{\alpha}(i) := H_r \alpha(i).$$

Let $\overline{\beta}$ be the map induced by β in the corresponding way. We remark that any basic move of type (ii) on the presentation P_A has no effect on the map $\overline{\alpha}$. Therefore the presentations P_A and P_B are equivalent if, and only if, $k = h$, $r = t$ and there exist $g \in G$, $\nu \in \mathrm{Sym}(k)$ such that $\overline{\beta} = l_{H_r g} \cdot \overline{\alpha} \cdot \nu$. This last condition is satisfied if, and only if, one has $w_{\overline{\beta}} = w_{l_{H_r g} \cdot \overline{\alpha}}$, that is:

$$w_{\overline{\beta}}(H_r g x) = w_{\overline{\alpha}}(H_r x), \text{ for all } x \in G.$$

Since $w_{\overline{\alpha}}(H_r x) = w_{\alpha\odot\widetilde{\epsilon}_r}(x)$ and $w_{\overline{\beta}}(H_r x) = w_{\beta\odot\widetilde{\epsilon}_r}(x)$, for all $x \in G$, we conclude that

$$w_{\beta\odot\widetilde{\epsilon}_r}(gx) = w_{\alpha\odot\widetilde{\epsilon}_r}(x), \quad \text{for all } x \in G.$$

Finally, the above equality guarantees us that

$$\mathcal{I}_{\beta\odot\widetilde{\epsilon}_r} = g\mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r} \quad \text{and} \quad \mathcal{H}_{\beta\odot\widetilde{\epsilon}_r} = \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r}.$$

We summarize all this information in the following statement:

Proposition 3.2.1 (Proposition 4.3 of [22]). *Let $G = \langle\epsilon\rangle$ be a cyclic group and consider the finite dimensional G-simple algebras*

$$A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r) \quad \textit{and} \quad B = (M_h(D_t), \beta \odot \widetilde{\epsilon}_t).$$

Then B is graded-isomorphic to A if, and only if, $k = h$, $r = t$ and there exists $g \in G$ such that

$$w_{\beta\odot\widetilde{\epsilon}_r}(gx) = w_{\alpha\odot\widetilde{\epsilon}_r}(x), \quad \textit{for all } x \in G.$$

In this case, one has that

$$\mathcal{I}_{\beta\odot\widetilde{\epsilon}_r} = g\mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r} \quad \textit{and} \quad \mathcal{H}_{\beta\odot\widetilde{\epsilon}_r} = \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r}.$$

Furthermore, as consequence of the previous results, we obtain the following:

Corollary 3.2.2 (Corollary 3.3 of [31]). *Let $G = \langle\epsilon\rangle$ be a cyclic group. Consider two finite dimensional G-simple algebras*

$$A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r) \quad \textit{and} \quad B = (M_h(D_t), \beta \odot \widetilde{\epsilon}_t)$$

such that $\dim_F B = \dim_F A$.

The following statements are equivalent:

(i) $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$*;*

(ii) B is graded-isomorphic to A;

(iii) there exists $g \in G$ such that $w_{\beta\odot\widetilde{\epsilon}_t}(gx) = w_{\alpha\odot\widetilde{\epsilon}_r}(x)$, for all $x \in G$.

Proof. First, if item (i) is valid, since $\dim_F B = \dim_F A$ we obtain, from Theorem 1.2.4, that B is graded-isomorphic to A. On the other hand, if item (ii) holds, thus it is clear that $\mathrm{Id}_G(B) = \mathrm{Id}_G(A)$, and hence we conclude the equivalence between (i) and (ii).

Now, by invoking Proposition 3.2.1, it follows that item (ii) implies (iii). Finally, if item (iii) is true, then $ht = kr$. Once $h^2t = \dim_F B = \dim_F A = k^2 r$, one has that $h = k$ and $t = r$. Thus, it is enough to apply Proposition 3.2.1 in order to conclude the proof. $\square$

Let $A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$ with presentation $P_A = (r; (g_1, \ldots, g_k))$. We consider $\overline{\mathbf{T}}_A \subseteq [1, k]$ such that $\mathcal{I}_{\overline{\alpha}} = \overline{\alpha}(\overline{\mathbf{T}}_A)$, with $\overline{\alpha}(i) \neq \overline{\alpha}(j)$ for all $i, j \in \overline{\mathbf{T}}_A$, $i \neq j$.

Moreover, we consider $\mathbf{T}_A \subseteq [1, kr]$ such that $\mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r} = (\alpha \odot \widetilde{\epsilon}_r)(\mathbf{T}_A)$, with $(\alpha \odot \widetilde{\epsilon}_r)(i) \neq (\alpha \odot \widetilde{\epsilon}_r)(j)$ for all $i, j \in \mathbf{T}_A$, $i \neq j$. Note that we could take, for instance, $\mathbf{T}_A = \{(i-1)r + j \mid i \in \overline{\mathbf{T}}_A,\ j \in [1, r]\}$. Let us write $\mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r} = \{\mathbf{h}_i \mid i \in \mathbf{T}_A\}$.

Given $g \in G$, by setting

$$A_{1_G}^{(g)} := \operatorname{span}_F\{e_{pq} \otimes E^l \mid H_r g_p = H_r g_q = H_r g \text{ and } g_p^{-1} (\epsilon^s)^l g_q = 1_G\},$$

the following direct sum (as algebras) holds:

$$A_{1_G} = \bigoplus_{i \in \overline{\mathbf{T}}_A} A_{1_G}^{(g_i)}.$$

We finish this section by presenting a technical lemma and an important remark, which will be useful in the next chapters.

Lemma 3.2.3 (Lemma 5.1 of [22])**.** *Let $G = \langle \epsilon \rangle$ be a cyclic group and consider*

$$A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r),$$

with presentation $P_A = (r; (g_1, \ldots, g_k))$. Fix $a \in [1, k]$ such that

$$w_{\alpha\odot\widetilde{\epsilon}_r}(g_a) = \max\{w_{\alpha\odot\widetilde{\epsilon}_r}(h) \mid h \in \mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r}\}.$$

Then there exists a homogeneous multilinear polynomial $\Psi_A \in F\langle X; G\rangle$ of degree 1_G such that

- (i) *$\Psi_A \notin \mathrm{Id}_G(A)$ and, for all $\ell \in [1, k]$, such that $H_r g_\ell = H_r g_a$, there exists a suitable non-zero graded evaluation $\rho : F\langle X; G\rangle \to A$, at elements of the canonical basis of A, with*

$$\rho(\Psi_A) = e_{\ell\ell} \otimes E^0.$$

- (ii) *If ρ is a graded evaluation of Ψ_A, at elements of the canonical basis of A, then*

$$\rho(\Psi_A) \in \bigoplus_{i \in \overline{\mathbf{T}}_A;\ g_i \in \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r} g_a} A_{1_G}^{(g_i)}.$$

Proof. Define, for every $i \in \mathbf{T}_A$, $t_i := w_{\alpha\odot\widetilde{\epsilon}_r}(g_a)w_{\alpha\odot\widetilde{\epsilon}_r}(\mathbf{h}_i)$, and consider the following polynomial

$$\psi_i := \sum_{\sigma\in \mathrm{Sym}(t_i)} (-1)^\sigma u^{(i)}_{\sigma(1)} v^{(i)}_1 u^{(i)}_{\sigma(2)} v^{(i)}_2 \cdots u^{(i)}_{\sigma(t_i)} v^{(i)}_{t_i},$$

where the sets $\{u^{(i)}_1, \dots, u^{(i)}_{t_i}\}$ and $\{v^{(i)}_1, \dots, v^{(i)}_{t_i}\}$, with $i \in \mathbf{T}_A$, are pairwise disjoint sets of homogeneous variables of degree

$$|u^{(i)}_l|_{F\langle X;G\rangle} := g_a^{-1}\mathbf{h}_i \quad \text{and} \quad |v^{(i)}_l|_{F\langle X;G\rangle} := \mathbf{h}_i^{-1} g_a,$$

for all $l \in [1, t_i]$. Then define the polynomial

$$\Psi_A := \Pi_{i\in\mathbf{T}_A}\psi_i.$$

Notice that each ψ_i is a homogeneous multilinear graded polynomial of degree 1_G and thus the same holds for Ψ_A.

Take an integer $\ell \in [1, k]$ such that $H_r g_\ell = H_r g_a$. We claim that, for all $i \in \mathbf{T}_A$, there exists a graded evaluation ρ_i of ψ_i, at elements of the canonical basis of A, such that

$$\rho_i(\psi_i) = e_{\ell\ell} \otimes E^0.$$

Indeed, we remark that there are $w_{\alpha\odot\widetilde{\epsilon}_r}(g_a)$ elements of the coset $H_r g_a$ appearing in $\widetilde{g}$, whereas $w_{\alpha\odot\widetilde{\epsilon}_r}(\mathbf{h}_i)$ elements of the coset $H_r\mathbf{h}_i$ appearing in $\widetilde{g}$. Thus, just write all the $t_i = w_{\alpha\odot\widetilde{\epsilon}_r}(g_a)w_{\alpha\odot\widetilde{\epsilon}_r}(\mathbf{h}_i)$ elements e_{pq} of the basis of M_k, such that $H_r g_p = H_r g_a$ and $H_r g_q = H_r\mathbf{h}_i$, in some sequence $e_{p_1q_1}, \dots, e_{p_{t_i}q_{t_i}}$, with $p_1 = \ell$. Then, by writing, for each $l \in [1, t_i]$, $g_{p_l} = (\epsilon^s)^{a_l} g_a$ and $g_{q_l} = (\epsilon^s)^{b_l}\mathbf{h}_i$, consider the following evaluations in the variables $u^{(i)}_l$ and $v^{(i)}_l$:

$$\begin{array}{lcll} u^{(i)}_l & \mapsto & e_{p_lq_l} \otimes E^{a_l-b_l}, & \text{for all } l \in [1, t_i], \\ v^{(i)}_l & \mapsto & e_{q_lp_{l+1}} \otimes E^{b_l-a_{l+1}}, & \text{for all } l \in [1, t_i - 1], \\ v^{(i)}_{t_i} & \mapsto & e_{q_{t_i}\ell} \otimes E^{b_{t_i}-\ell}, & \end{array}$$

and we obtain $\rho_i(\psi_i) = e_{\ell\ell} \otimes E^0$.

Therefore, by considering, for each $i \in \mathbf{T}_A$, the above evaluates ρ_i in ψ_i we get a graded evaluation ρ of Ψ_A, at elements of the canonical basis of A, resulting in $e_{\ell\ell} \otimes E^0$, and thus we concluded the proof of item (i).

In order to prove item (ii), we remember that

$$A_{1_G} = \bigoplus_{i\in\overline{\mathbf{T}}_A} A^{(g_i)}_{1_G},$$

where, for each $i \in \overline{\mathbf{T}}_A$, $A_{1_G}^{(g_i)} := \mathrm{span}_F\{e_{pq} \otimes E^l \mid H_r g_p = H_r g_q = H_r g_i \text{ and } g_p^{-1}(\epsilon^s)^l g_q = 1_G\}$. Once Ψ_A is a homogeneous multilinear polynomial of degree 1_G, we can suppose that if ρ is a non-zero graded evaluation of Ψ_A, then ρ must be in a unique component of the sum in A_{1_G}. Assume that $\rho(\Psi_A) \in A_{1_G}^{(g_b)}$, for some g_b such that $H_r g_b \neq H_r g_a$. Consequently, each ψ_i has also a non-zero graded evaluation in $A_{1_G}^{(g_b)}$, and then each product $u_{\sigma(l)}^{(i)} v_l^{(i)}$ appearing in this ψ_i has non-zero graded evaluations resulting in linear combinations of elements $e_{pq} \otimes E^{c-d}$, such that $H_r g_p = H_r g_q = H_r g_b$, with $g_p = (\epsilon^s)^c g_p$ and $g_q = (\epsilon^s)^d g_b$.

Thus, for all $i \in \mathbf{T}_A$, we must evaluate the $t_i = w_{\alpha\odot\widetilde{\epsilon}_r}(g_a) w_{\alpha\odot\widetilde{\epsilon}_r}(\mathbf{h}_i)$ alternating variables $u_l^{(i)}$ of the polynomial ψ_i in

$$A_{g_a^{-1}\mathbf{h}_i}^{(g_b)} := \mathrm{span}_F\{e_{pq} \otimes E^{c'-d'} \mid g_p = (\epsilon^s)^{c'} g_b \ \text{ and } \ g_q = (\epsilon^s)^{d'} g_b g_a^{-1}\mathbf{h}_i\}.$$

We observe that $\dim_F(A_{g_a^{-1}\mathbf{h}_i}^{(g_b)}) = w_{\overline{\alpha}}(H_r g_b) w_{\overline{\alpha}}(H_r g_b g_a^{-1}\mathbf{h}_i) = w_{\alpha\odot\widetilde{\epsilon}_r}(g_b) w_{\alpha\odot\widetilde{\epsilon}_r}(g_b g_a^{-1}\mathbf{h}_i)$ and by using the fact that the variables $u_l^{(i)}$ are alternating and $w_{\alpha\odot\widetilde{\epsilon}_r}(g_a)$ is maximum, one has that

$$w_{\alpha\odot\widetilde{\epsilon}_r}(g_b) w_{\alpha\odot\widetilde{\epsilon}_r}(g_b g_a^{-1}\mathbf{h}_i) = \dim_F(A_{g_a^{-1}\mathbf{h}_i}^{(g_b)}) \geq t_i = w_{\alpha\odot\widetilde{\epsilon}_r}(g_a) w_{\alpha\odot\widetilde{\epsilon}_r}(\mathbf{h}_i) \geq w_{\alpha\odot\widetilde{\epsilon}_r}(g_b) w_{\alpha\odot\widetilde{\epsilon}_r}(\mathbf{h}_i),$$

and hence $w_{\alpha\odot\widetilde{\epsilon}_r}(g_b g_a^{-1}\mathbf{h}_i) \geq w_{\alpha\odot\widetilde{\epsilon}_r}(\mathbf{h}_i)$, for all $i \in \mathbf{T}_A$. Then

$$kr \geq \sum_{i\in\mathbf{T}_A} w_{\alpha\odot\widetilde{\epsilon}_r}(g_b g_a^{-1}\mathbf{h}_i) \geq \sum_{i\in\mathbf{T}_A} w_{\alpha\odot\widetilde{\epsilon}_r}(\mathbf{h}_i) = kr,$$

and this implies that $w_{\alpha\odot\widetilde{\epsilon}_r}(g_b g_a^{-1}\mathbf{h}_i) = w_{\alpha\odot\widetilde{\epsilon}_r}(\mathbf{h}_i)$, for every $i \in \mathbf{T}_A$. Such equality allows us to conclude that $g_b g_a^{-1} \in \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r}$ and therefore $\rho(\Psi_A) \in \bigoplus_{i\in\overline{\mathbf{T}}_A;\ g_i\in\mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r} g_a} A_{1_G}^{(g_i)}$, as desired. □

Remark 3.2.4. By using the same notations which were introduced in the above lemma, let $A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$ with the following presentation $P_A = (r; (g_1, \ldots, g_k))$. Consider $B = (M_k(D_r), \beta \odot \widetilde{\epsilon}_r)$ and suppose that there exists $\eta \in G$ such that

$$\beta \odot \widetilde{\epsilon}_r = l_\eta \cdot (\alpha \odot \widetilde{\epsilon}_r).$$

This implies that $P_B = (r; (\eta g_1, \ldots, \eta g_k))$ is a presentation of B, still $w_\beta(\eta g_i) = w_\alpha(g_i)$, for all $i \in [1,k]$, and $\mathcal{H}_{\beta\odot\widetilde{\epsilon}_r} = \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r}$. Moreover, if $a \in [1,k]$ is such that $w_{\alpha\odot\widetilde{\epsilon}_r}(g_a) = \max\{w_{\alpha\odot\widetilde{\epsilon}_r}(h) \mid h \in \mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r}\}$, thus $w_{\beta\odot\widetilde{\epsilon}_r}(\eta g_a) = \max\{w_{\beta\odot\widetilde{\epsilon}_r}(h) \mid h \in \mathcal{I}_{\beta\odot\widetilde{\epsilon}_r}\}$ and the corresponding polynomials Ψ_A and Ψ_B coincide. Therefore, if ρ is any graded evaluation of Ψ_A in B, one has that

$$\rho(\Psi_A) \in \bigoplus_{i\in\overline{\mathbf{T}}_A;\, g_i\in\mathcal{H}_{\beta\odot\widetilde{\epsilon}_r} g_a} (B_{1_G})^{(\eta g_i)}.$$

3.3 C_n-simple algebras and $(\alpha \odot \widetilde{\epsilon}_r)$-regularity

In Section 3.1, we described the finite dimensional C_n-simple F-algebras as graded subalgebras of matrix algebras endowed with some elementary gradings. In this section, we will deal with the $(\alpha \odot \widetilde{\epsilon}_r)$-regularity of these algebras.

Firstly, given $A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$, we remember that the maps $\widehat{\pi}_\bullet, \widehat{\pi}^*_\bullet : \bar{A} \to M_{kr} \otimes P(A)$ are, respectively, the restrictions of $\pi_\bullet$ and $\pi^*_\bullet$, given by (2.1) and (2.2), to $\bar{A} = \mathrm{Gen}_G(A)$, where $P(A)$ is the polynomial ring associated to A (see Section 2.1). In the sequel, we generalize Proposition 2.3.2 for G-simple algebras, in case G is a finite cyclic group.

Proposition 3.3.1. *Let $G = \langle \epsilon \rangle$ be a cyclic group and consider*

$$A = (M_k(D_r), \widetilde{g} \odot \widetilde{\epsilon}_r) = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r),$$

with presentation $P_A = (r; (g_1, \ldots, g_k))$.

The following statements are equivalent:

(i) The maps $\widehat{\pi}_h$ defined on $\mathrm{Gen}_G(A)$ are injective, for all $h \in \mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r}$;

(ii) The maps $\widehat{\pi}^*_h$ defined on $\mathrm{Gen}_G(A)$ are injective, for all $h \in \mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r}$;

(iii) There exist a subgroup H of G and an element $g \in G$ such that

$$\mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r} = gH,$$

and all fibers of the map $\alpha \odot \widetilde{\epsilon}_r$ are equipotent, that is, there exists $c \in \mathbb{N}^*$ such that $w_{\alpha \odot \widetilde{\epsilon}_r}(h) = c$, for all $h \in \mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r}$.

Proof. The proof is analogous to that of Proposition 2.3.2. Here we will only deduce the implication of item (iii) to (i) since it contains important details to be highlighted.

Suppose that $\mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r} = gH$, for some subgroup H of G and some element $g \in G$. Moreover, assume that all fibers of the map $\alpha \odot \widetilde{\epsilon}_r$ are equipotent, that is, there exists $c \in \mathbb{N}^*$ such that $w_{\alpha \odot \widetilde{\epsilon}_r}(h) = c$, for all $h \in \mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r}$. Then, it follows that $w_{\overline{\alpha}}(H_r h) = c$, for all $h \in \mathcal{I}_\alpha$.

We claim that, for each $l \in \overline{\mathbf{T}}_A$, $\widehat{\pi}_{g_l}$ is injective if, and only if, $\widehat{\pi}_{(\epsilon^s)^t g_l}$ is injective, for every $t \in [0, r-1]$.

Indeed, let $\varphi : F\langle X; G \rangle \to \bar{A} = \mathrm{Gen}_G(A)$ be the canonical G-graded epimorphism such that $\ker(\varphi) = \mathrm{Id}_G(A)$, and fix $\rho : F\langle X; G \rangle \to A$ an arbitrary graded evaluation.

Given $l \in \overline{\mathbf{T}}_A$, assume that $\widehat{\pi}_{g_l}$ is injective. Suppose, if possible, that there exists $t' \in [1, r-1]$ such that $\widehat{\pi}_{(\epsilon^s)^{t'} g_l}$ is not injective. Thus there exists a non-zero element a' in $\bar{A}$ satisfying

$\widehat{\pi}_{(\epsilon^s)^{t'}g_l}(a') = 0$. Take $f \in F\langle X; G\rangle$ such that $\varphi(f) = a'$. Hence, one has that

$$\rho(f) = \sum_{i,j,t} d_t^{(ij)}(e_{ij} \otimes E^t), \quad d_t^{(ij)} \in F,$$

with $d_t^{(pj)} = 0$, for all $p \in [1, k]$ such that $H_r g_p = H_r g_l$, and for all $j \in [1, k]$ and $t \in [0, r-1]$. In particular, this implies that $\widehat{\pi}_{g_l}(a') = 0$, a contradiction. Therefore, $\widehat{\pi}_{(\epsilon^s)^t g_l}$ is injective, for every $t \in [0, r-1]$. Since the reciprocal is trivial, we conclude the claim.

Therefore, in order to conclude that (i) is valid, it is enough to show that fixed $\ell \in \overline{\mathbf{T}}_A$ and an element a' in $\bar{A}$ satisfying $\widehat{\pi}_{g_\ell}(a') = 0$, one has that $a' = 0$.

To this end, define, for each $\gamma \in \overline{\mathbf{T}}_A$,

$$\mathcal{T}_\gamma := \{i \in [1, k] \mid H_r g_i = H_r g_\gamma\},$$

and, for each $\delta \in [1, k]$, set

$$Bl_\delta := [(\delta - 1)r + 1, \delta r].$$

As above, take $f \in F\langle X; G\rangle$ such that $\varphi(f) = a'$. We obtain that if $\rho : F\langle X; G\rangle \to A$ is an arbitrary graded evaluation, then

$$\rho(f) = \sum_{i,j,t} d_t^{(ij)} \overline{\mathrm{E}}_{(i-1)r+1,(j-1)r+t+1},$$

with

$$d_t^{(pj)} = 0, \quad \forall p \in [1, k] \text{ satisfying } H_r g_p = H_r g_\ell, \forall j \in [1, k], \forall t \in [0, r-1].$$

Fix an arbitrary $\ell' \in \overline{\mathbf{T}}_A$ and consider

$$\bar{g} := g_{\ell'}^{-1} g_\ell. \tag{3.3}$$

Since $\mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r} = gH$ and H is a subgroup of G, it follows that $\bar{g} \in H$ and there exists $\overline{\theta} \in \mathrm{Sym}(\overline{\mathbf{T}}_A)$ such that

$$H_r g_{\bar{\theta}(l)} = H_r \bar{g} g_l, \quad \text{for all } l \in \overline{\mathbf{T}}_A,$$

and, in particular,

$$\overline{\theta}(\ell') = \ell.$$

Moreover, the fact that all the fibers of the map $\alpha \odot \widetilde{\epsilon}_r$ are equipotent guarantees the existence of $\theta \in \mathrm{Sym}(k)$ satisfying

$$H_r g_{\theta(l)} = H_r \bar{g} g_l, \quad \text{for all } l \in [1, k],$$

such that the restriction of θ to $\overline{\mathbf{T}}_A$ coincides with $\overline{\theta}$ and $\theta(\mathcal{T}_\gamma) = \mathcal{T}_{\overline{\theta}(\gamma)}$, for all $\gamma \in \overline{\mathbf{T}}_A$.

Now, from the above discussions, we have that there exists σ in $\mathrm{Sym}(kr)$ satisfying

$$\sigma(Bl_\delta) = Bl_{\theta(\delta)}, \quad \text{for all } \delta \in [1,k], \tag{3.4}$$

and

$$(\alpha \odot \widetilde{\epsilon}_r)(\sigma(\iota)) = \bar{g}(\alpha \odot \widetilde{\epsilon}_r)(\iota), \quad \text{for all } \iota \in [1,kr]. \tag{3.5}$$

Finally, define the map

$$\begin{array}{rccc} \Gamma: & (M_{kr}, \alpha \odot \widetilde{\epsilon}_r) & \to & (M_{kr}, \alpha \odot \widetilde{\epsilon}_r) \\ & \mathrm{E}_{uv} & \mapsto & \mathrm{E}_{\sigma(u)\sigma(v)}. \end{array}$$

Clearly Γ is a graded isomorphism. Furthermore, given $i,j \in [1,k]$ and $t \in [0,r-1]$, one has that

$$\Gamma(\overline{\mathrm{E}}_{(i-1)r+1,(j-1)r+t+1}) \underset{(3.1)}{=} \Gamma\left(\sum_{l=0}^{r-1} \mathrm{E}_{(i-1)r+l+1,(j-1)r+\tau^l(t)+1}\right) = \sum_{l=0}^{r-1} \mathrm{E}_{\sigma((i-1)r+l+1),\sigma((j-1)r+\tau^l(t)+1)}.$$

Since there exist unique δ_1 and δ_2 such that, for all $l,t \in [0,r-1]$,

$$(i-1)r+l+1 \in Bl_{\delta_1} \quad \text{and} \quad (j-1)r+\tau^l(t)+1 \in Bl_{\delta_2},$$

it follows from (3.4) that

$$\sigma((i-1)r+l+1) \in Bl_{\theta(\delta_1)}, \quad \sigma((j-1)r+\tau^l(t)+1) \in Bl_{\theta(\delta_2)} \tag{3.6}$$

and thus

$$\Gamma(\overline{\mathrm{E}}_{(i-1)r+1,(j-1)r+t+1}) = \overline{\mathrm{E}}_{\sigma((i-1)r+1),\sigma((j-1)r+t+1)}.$$

This implies that the map Γ induces a graded isomorphism on A.

We notice that $\Gamma\rho : F\langle X;G\rangle \to A$ is still a graded evaluation and

$$\Gamma(\rho(f)) = \Gamma\left(\sum_{i,j,t} d_t^{(ij)}\overline{\mathrm{E}}_{(i-1)r+1,(j-1)r+t+1}\right) = \sum_{i,j,t} d_t^{(ij)}\overline{\mathrm{E}}_{\sigma((i-1)r+1),\sigma((j-1)r+t+1)}.$$

Since $\widehat{\pi}_{g_\ell}(a') = 0$, by combining (3.3), (3.5) and (3.6), we obtain that

$$d_t^{(pj)} = 0, \quad \forall p \in [1,k] \text{ satisfying } H_r g_p = H_r g_{\ell'}, \forall j \in [1,k], \forall t \in [0,r-1].$$

Thus, once $g_{\ell'}$ is arbitrary, we conclude that $d_t^{(ij)} = 0$, for every i, j, t. Consequently, $f \in \mathrm{Id}_G(A)$ and this implies $a' = 0$, as desired. $\square$

The next result classifies the G-simple $(\alpha \odot \widetilde{\epsilon}_r)$-regular algebras.

Theorem 3.3.2 (Theorem 4.8 of [22])**.** *Let $G = \langle \epsilon \rangle$ be a cyclic group and consider*

$$A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r).$$

Then A is $(\alpha \odot \widetilde{\epsilon}_r)$-regular if, and only if, there exists $g \in G$ such that

$$\mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r} = g\mathcal{H}_{\alpha \odot \widetilde{\epsilon}_r}.$$

In this case, $[\mathcal{H}_{\alpha \odot \widetilde{\epsilon}_r} : H_r] = |\overline{\mathbf{T}}_A|$ and all fibers of the map $\alpha \odot \widetilde{\epsilon}_r$ are equipotent.

Proof. Proposition 3.3.1 and Lemma 2.3.3 guarantee that A is $(\alpha \odot \widetilde{\epsilon}_r)$-regular if, and only if, $\mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r} = g\mathcal{H}_{\alpha \odot \widetilde{\epsilon}_r}$. In this case, it follows that $|\mathcal{H}_{\alpha \odot \widetilde{\epsilon}_r}| = |\mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r}| = r|\overline{\mathbf{T}}_A|$ and consequently $[\mathcal{H}_{\alpha \odot \widetilde{\epsilon}_r} : H_r] = |\overline{\mathbf{T}}_A|$. Finally, Proposition 3.3.1 guarantees the existence of $c \in \mathbb{N}^*$ such that $c = w_{\alpha \odot \widetilde{\epsilon}_r}(h)$, for all $h \in \mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r}$. $\square$

As a direct consequence of the previous theorem, we have the following result.

Corollary 3.3.3 (Corollary 4.9 of [22])**.** *Let $G = \langle \epsilon \rangle$ be a cyclic group and consider $A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$. Then A is a G-regular subalgebra of the matrix algebra $(M_{kr}, \alpha \odot \widetilde{\epsilon}_r)$ if, and only if,*

$$\mathcal{H}_{\alpha \odot \widetilde{\epsilon}_r} = G.$$

Given $A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$ and $B = (M_k(D_r), \beta \odot \widetilde{\epsilon}_r)$, we finish this section by stating that if B is graded-isomorphic to A, then B is $(\beta \odot \widetilde{\epsilon}_r)$-regular if, and only if, A is $(\alpha \odot \widetilde{\epsilon}_r)$-regular. Moreover, in this case, we establish interesting relations between the images of the maps $\alpha \odot \widetilde{\epsilon}_r$ and $\beta \odot \widetilde{\epsilon}_r$.

Proposition 3.3.4 (Proposition 4.10 of [22])**.** *Let $G = \langle \epsilon \rangle$ be a cyclic group and consider*

$$A = (M_k(D_r), \alpha \odot \widetilde{\epsilon}_r) \quad \text{and} \quad B = (M_k(D_r), \beta \odot \widetilde{\epsilon}_r).$$

Suppose that B is graded-isomorphic to A. Then B is $(\beta \odot \widetilde{\epsilon}_r)$-regular if, and only if, A is $(\alpha \odot \widetilde{\epsilon}_r)$-regular.

In this case, if $g_\alpha, g_\beta \in G$ are such that $\mathcal{I}_{\alpha \odot \widetilde{\epsilon}_r} = g_\alpha \mathcal{H}_{\alpha \odot \widetilde{\epsilon}_r}$ and $\mathcal{I}_{\beta \odot \widetilde{\epsilon}_r} = g_\beta \mathcal{H}_{\beta \odot \widetilde{\epsilon}_r}$, then $g \in G$ is such that

$$w_{\beta \odot \widetilde{\epsilon}_r}(gx) = w_{\alpha \odot \widetilde{\epsilon}_r}(x), \quad \text{for all } x \in G,$$

if, and only if,

$$g \in g_\beta \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r} g_\alpha^{-1} = g_\beta \mathcal{H}_{\beta\odot\widetilde{\epsilon}_r} g_\alpha^{-1}.$$

Proof. Since B is graded-isomorphic to A, it follows from Proposition 3.2.1 that there exists $g \in G$ such that

$$\mathcal{I}_{\beta\odot\widetilde{\epsilon}_r} = g\mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r} \quad \text{and} \quad \mathcal{H}_{\beta\odot\widetilde{\epsilon}_r} = \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r}.$$

By combining the above equalities with Theorem 3.3.2, we conclude that B is $(\beta \odot \widetilde{\epsilon}_r)$-regular if, and only if, A is $(\alpha \odot \widetilde{\epsilon}_r)$-regular.

Now, assume that $g_\alpha, g_\beta \in G$ are such that

$$\mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r} = g_\alpha \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r} \quad \text{and} \quad \mathcal{I}_{\beta\odot\widetilde{\epsilon}_r} = g_\beta \mathcal{H}_{\beta\odot\widetilde{\epsilon}_r}.$$

If $g \in G$ is such that $w_{\beta\odot\widetilde{\epsilon}_r}(gx) = w_{\alpha\odot\widetilde{\epsilon}_r}(x)$, for all $x \in G$, then, in particular,

$$w_{\beta\odot\widetilde{\epsilon}_r}(gg_\alpha) = w_{\alpha\odot\widetilde{\epsilon}_r}(g_\alpha) \neq 0$$

and this implies that $gg_\alpha = g_\beta h$, for some $h \in \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r}$. Hence, $g \in g_\beta \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r} g_\alpha^{-1}$.

Conversely, assume that $g \in g_\beta \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r} g_\alpha^{-1}$, that is, $g = g_\beta h g_\alpha^{-1}$, for some $h \in \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r}$. In this case, it is valid that

$$w_{\beta\odot\widetilde{\epsilon}_r}(gx) = w_{\alpha\odot\widetilde{\epsilon}_r}(x), \quad \text{for all } x \in G.$$

In fact, if $x \in \mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r}$, then $w_{\alpha\odot\widetilde{\epsilon}_r}(x) \neq 0$ and $x = g_\alpha \widetilde{h}$, for some $\widetilde{h} \in \mathcal{H}_{\alpha\odot\widetilde{\epsilon}_r}$. Thus

$$w_{\beta\odot\widetilde{\epsilon}_r}(gx) = w_{\beta\odot\widetilde{\epsilon}_r}(g_\beta h g_\alpha^{-1} g_\alpha \widetilde{h}) = w_{\beta\odot\widetilde{\epsilon}_r}(g_\beta h\widetilde{h}) \neq 0.$$

By using the fact that there exists $c \in \mathbb{N}^*$ such that $w_{\alpha\odot\widetilde{\epsilon}_r}(y) = w_{\beta\odot\widetilde{\epsilon}_r}(z) = c$, for all $y \in \mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r}$ and $z \in \mathcal{I}_{\beta\odot\widetilde{\epsilon}_r}$, we conclude that $w_{\beta\odot\widetilde{\epsilon}_r}(gx) = w_{\alpha\odot\widetilde{\epsilon}_r}(x)$. On the other hand, if $x \notin \mathcal{I}_{\alpha\odot\widetilde{\epsilon}_r}$, it is easy to verify that $w_{\beta\odot\widetilde{\epsilon}_r}(gx) = w_{\alpha\odot\widetilde{\epsilon}_r}(x) = 0$. $\square$

Chapter 4

The factorability of the T_{C_n}-ideals $\mathrm{Id}_{C_n}(UT_{C_n}(A_1, \ldots, A_m))$

Let F be an algebraically closed field of characteristic zero. Consider ϵ a primitive nth root of the unity in F^* and $G = \langle \epsilon \rangle = C_n$, the finite cyclic group generated by ϵ. Moreover, consider $A_1, \ldots, A_m$ finite dimensional G-simple algebras. If p is a prime number and G is a p-group, that is, the order of G is a power of p, in this chapter, we will present necessary and sufficient conditions to the factorability of the T_G-ideals $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$ of the G-graded upper block triangular matrix algebras $UT_G(A_1, \ldots, A_m)$ endowed with elementary G-gradings.

We will see that such factorability is associated to the concept of G-regularity of the G-simple algebras $A_1, \ldots, A_m$ and the number of non-isomorphic G-gradings on $UT_G(A_1, \ldots, A_m)$. Such statements are similar to those obtained by Avelar, Di Vincenzo and da Silva, in case $n = 2$ (see [7]). Nevertheless, it is worth saying that in our works we use different techniques from those applied in [7]. In particular, the invariance subgroups associated to the G-simple blocks $A_1, \ldots, A_m$ are important and crucial tools in obtaining several results.

Still, if $m = 2$ and by requiring some assumptions on the G-simple algebras A_1 and A_2, we also will establish conditions for the factorability of $\mathrm{Id}_G(UT_G(A_1, A_2))$, even when G is not necessarily a p-group. In this case, we will see that the factorability of the T_G-ideal of the algebra $UT_G(A_1, A_2)$ is not necessarily related with the concept of G-regularity.

The results cited above were obtained with the participation of Professor Viviane Ribeiro Tomaz da Silva and Professor Onofrio Mario Di Vincenzo, and are available in [22]. Furthermore, in order to achieve these results, we will employ, in this chapter, some different tools from those used in our paper ([22]). In particular, the indecomposable T_G-ideals allowed us to exhibit some alternative proofs for our results.

4.1 The algebra $UT_{C_n}(A_1, \ldots, A_m)$ and the invariance subgroups

In this section, we will focus on the G-graded upper block triangular matrix algebras $UT_G(A_1, \ldots, A_m)$, where $A_1, \ldots, A_m$ are finite dimensional G-simple algebras. In order to obtain relations between $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$ and the invariance subgroups of the G-simple components $A_1, \ldots, A_m$, we will establish some technical results associated to such algebras $UT_G(A_1, \ldots, A_m)$.

First, fix an m-tuple $(A_1, \ldots, A_m)$ of finite dimensional G-simple F-algebras. In light of Theorem 3.1.3, we may assume that

$$A_l = (M_{k_l}(D_{r_l}), \widetilde{g}_l \odot \widetilde{\epsilon}_{r_l}) = (M_{k_l}(D_{r_l}), \alpha_l \odot \widetilde{\epsilon}_{r_l}) = (M_{k_l}(D_{r_l}), \widetilde{\alpha}_l),$$

where $\widetilde{\alpha}_l := \alpha_l \odot \widetilde{\epsilon}_{r_l}$ and $\widetilde{g}_l := (g_{l1}, g_{l2}, \ldots, g_{lk_l})$ is such that $P_{A_l} = (r_l; \widetilde{g}_l)$ is a presentation of A_l. We remember that the tuples $\widetilde{g}_l$ and $\widetilde{\epsilon}_{r_l} = (1_G, \epsilon^{s_l}, \ldots, (\epsilon^{s_l})^{r_l-1})$ induce, respectively, the elementary gradings in M_{k_l} and D_{r_l}.

Consider the G-graded upper block triangular matrix algebra $A := (UT(A_1, \ldots, A_m), \widetilde{\alpha})$ (see Section 1.1). In this case, for each $l \in [1, m]$, it follows that

$$\eta_l = \sum_{\iota=1}^{l} k_\iota r_\iota \quad \text{and} \quad \mathbf{Bl}_l := [\eta_{l-1} + 1, \eta_l].$$

Moreover, for each $l \in [1, m]$, given $g \in G$ we set

$$w_{\widetilde{\alpha}}^{(l)}(g) := |\{i \mid i \in \mathbf{Bl}_l \text{ and } \widetilde{\alpha}(i) = g\}|$$

and we denote by $\mathcal{H}_{\widetilde{\alpha}}^{(l)}$ the *invariance subgroup* of the G-simple algebra $A_{l,l}$, that is,

$$\mathcal{H}_{\widetilde{\alpha}}^{(l)} := \{h \in G \mid w_{\widetilde{\alpha}}^{(l)}(hg) = w_{\widetilde{\alpha}}^{(l)}(g), \text{ for all } g \in G\}.$$

Remark 4.1.1. The set formed by the elements

- $\mathbf{E}_{ij}^{(u,v)}$, for all $1 \leq u < v \leq m$, with $i \in [1, k_u r_u]$, $j \in [1, k_v r_v]$;

- $(e_{ij} \otimes E^t)^{(u,u)}$, for all $u \in [1, m]$, with $i, j \in [1, k_u]$ and $t \in [0, r_u - 1]$

is a homogeneous basis of the vector space A, called its *canonical basis*. Such basis will be denoted by $\mathbf{B}$. Notice that $\mathbf{B}$ is a multiplicative basis of A (since, given $b_1, b_2 \in \mathbf{B}$, if $b_1 b_2 \neq 0_A$, then $b_1 b_2 \in \mathbf{B}$).

The next result resembles Lemma 3.3 of [17] and presents important properties related to elements of the basis $\mathbf{B}$.

Lemma 4.1.2 (Lemma 3.4 of [31])**.** *Let $b_1, \ldots, b_l \in \mathbf{B}$ and assume that $b := b_1 \cdots b_l \neq 0_A$.*

(i) If $b \in A_\ell$, then $b_i \in A_\ell$ for every $i \in [1, l]$, and $b = (e_{ij} \otimes E^t)^{(\ell,\ell)}$, for some $i, j \in [1, k_\ell]$ and $t \in [0, r_\ell - 1]$. Moreover, if $b^\pi := b_{\pi(1)} \cdots b_{\pi(l)} \neq 0_A$ for some $\pi \in \mathrm{Sym}(l)$, then $b^\pi = (e_{ij} \otimes E^t)^{(\ell,\ell)}$ when $i \neq j$, whereas $b^\pi = (e_{l'l'} \otimes E^t)^{(\ell,\ell)}$ for some $l' \in [1, k_\ell]$ otherwise.

(ii) If $b \in J(A)$, then there exist $1 \leq u < v \leq m$, $i \in [1, k_u r_u]$ and $j \in [1, k_v r_v]$ such that $b = \mathbf{E}_{ij}^{(u,v)}$. Moreover if $b^\pi := b_{\pi(1)} \cdots b_{\pi(l)} \neq 0_A$ for some $\pi \in \mathrm{Sym}(l)$, then $b^\pi = \mathbf{E}_{ij}^{(u,v)}$.

Proof. First, remember that $\mathbf{B}$ is a multiplicative basis of A. The initial statement given in item (i) follows directly by applying (1.1) and the fact that $A = A_{ss} + J(A)$, where $A_{ss} = A_1 \oplus \cdots \oplus A_m$ is a direct sum as algebras.

Now, for each $\iota \in [1, l]$, by writing $b_\iota = (e_{i_\iota j_\iota} \otimes E^{t_\iota})^{(\ell,\ell)}$, we have

$$\begin{aligned} (e_{ij} \otimes E^t)^{(\ell,\ell)} &= b = b_1 b_2 \cdots b_l = (e_{i_1 j_1} \otimes E^{t_1})^{(\ell,\ell)} \cdot (e_{i_2 j_2} \otimes E^{t_2})^{(\ell,\ell)} \cdots (e_{i_l j_l} \otimes E^{t_l})^{(\ell,\ell)} \\ &= (e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_l j_l} \otimes E^{t_1 + t_2 + \cdots + t_l})^{(\ell,\ell)}. \end{aligned}$$

Thus, since $(e_{i'j'} \otimes E^{t'})^{(u',u')} \cdot (e_{i''j''} \otimes E^{t''})^{(u'',u'')} = \delta_{u'u''} \delta_{j'i''} (e_{i'j''} \otimes E^{t'+t''})^{(u',u'')}$, it follows that

$$t_1 + t_2 + \cdots + t_l \equiv t \pmod{r_\ell}, \; i_1 = i, \; j_l = j \text{ and } j_\iota = i_{\iota+1}, \quad \text{for all } \iota \in [1, l-1].$$

Notice that the rows $i_2, \ldots, i_l$ and the columns $j_1, \ldots, j_{l-1}$ such that $j_\iota = i_{\iota+1}$ are appearing in pairs. This means that if $i \neq j$, then

$$|\{\iota \in [1, l] \mid i_\iota = i\}| = 1 + |\{\iota \in [1, l] \mid j_\iota = i\}| \text{ and } |\{\iota \in [1, l] \mid j_\iota = j\}| = 1 + |\{\iota \in [1, l] \mid i_\iota = j\}|.$$

Therefore, if $i \neq j$ and $b^\pi \neq 0_A$, we conclude that $b^\pi = (e_{ij} \otimes E^t)^{(\ell,\ell)}$. Similarly, in case $i = j$, we have $b^\pi = (e_{l'l'} \otimes E^t)^{(\ell,\ell)}$, for some $l' \in [1, k_\ell]$.

Finally, we can argue analogously to what was done above and to conclude the proof of (ii).

□

In the sequel, we present a technical lemma which is crucial for our aims.

Lemma 4.1.3 (Lemma 5.4 of [22])**.** *The Capelli polynomial $Cap_l(x_1, \ldots, x_l; x_{l+1}, \ldots, x_{2l+1})$ is an ordinary polynomial identity for the upper block triangular matrix algebra $UT(A_1, \ldots, A_m)$ if, and only if, $l \geq m + \sum_{i=1}^m k_i^2$. In particular, if $m \geq 2$, define $t := m - 1 + \sum_{i=1}^m k_i^2$, for any $u \in [1, \eta_1]$ and $v \in [\eta_{m-1} + 1, \eta_m]$ there exists an evaluation of $Cap_t(x_1, \ldots, x_t; x_{t+1}, \ldots, x_{2t+1})$ in $UT(A_1, \ldots, A_m)$, at canonical basis elements, equal to $\mathbf{E}_{uv}^{(1,m)}$.*

Proof. First, given $l \geq 1$, we set $f_l := Cap_l(x_1, \ldots, x_l; x_{l+1}, \ldots, x_{2l+1})$ and $A := UT(A_1, \ldots, A_m)$. In order to conclude the proof of the lemma, we will show that $f_l \notin \mathrm{Id}(A)$ if, and only if, $l \leq t_{1m} := m - 1 + \sum_{1=1}^{m} k_i^2$. To this end, let us apply an induction on m.

The case $m = 1$ is guaranteed by Lemma 3.1.4.

Assume that $m \geq 2$ and suppose that $f_{l'} \notin \mathrm{Id}(UT(A_{i_1}, \ldots, A_{i_p}))$, with $1 \leq i_1 < i_2 < \cdots < i_p \leq m$ and $1 \leq p \leq m-1$, if, and only if, $l' \leq p - 1 + \sum_{s=1}^{p} k_{i_s}^2$. We start by assuming that $f_l \notin \mathrm{Id}(A)$. Since f_l is multilinear, it follows that there exists a non-zero evaluation of f_l, at canonical basis elements of A, which we will denote by $\bar{f}_l$.

Notice that, if $\bar{x}_1, \ldots, \bar{x}_{2l+1} \notin J(A)$, then there exists $\ell \in [1, m]$ such that $\bar{x}_1, \ldots, \bar{x}_{2l+1} \in A_\ell$ and, hence, we finish by applying the case $m = 1$. Therefore, assume that there exists at least one positive integer $\ell \in [1, 2l+1]$ such that $\bar{x}_\ell \in A_{i,j} \subseteq J(A)$, with $i < j$.

If $\ell \in [1, l]$, then we can suppose that

$$\bar{x}_{l+1}\bar{x}_1\bar{x}_{l+2}\cdots\bar{x}_{l+\ell}\bar{x}_\ell\bar{x}_{l+\ell+1}\cdots\bar{x}_l\bar{x}_{2l+1} \neq 0$$

and this implies

$$\bar{x}_{l+1}\bar{x}_1\bar{x}_{l+2}\cdots\bar{x}_{l+\ell} \in A_{i',i}, \quad \text{with } 1 \leq i' \leq i,$$

and

$$\bar{x}_{l+\ell+1}\cdots\bar{x}_l\bar{x}_{2l+1} \in A_{j,j'}, \quad \text{with } j \leq j' \leq m.$$

Consequently, we have $\bar{x}_1, \ldots, \bar{x}_{\ell-1}, \bar{x}_{l+1}, \ldots, \bar{x}_{l+\ell} \in A^{[i',i]}$ and $\bar{x}_{\ell+1}, \ldots, \bar{x}_l, \bar{x}_{l+\ell+1}, \ldots, \bar{x}_{2l+1} \in A^{[j,j']}$.

We remark that, given $\sigma \in \mathrm{Sym}(l)$, if either $\sigma(\ell) \neq \ell$, or there exists $q \in [1, \ell-1]$ such that $\sigma(q) \in [\ell+1, m]$, or there exists $q \in [\ell+1, m]$ such that $\sigma(q) \in [1, \ell-1]$, thus we obtain

$$\bar{x}_{l+1}\bar{x}_{\sigma(1)}\bar{x}_{l+2}\cdots\bar{x}_{2l}\bar{x}_{\sigma(l)}\bar{x}_{2l+1} = 0,$$

since $i' \leq i < j \leq j'$ and $A_{r,s}A_{r',s'} = \delta_{sr'}A_{r,s'}$, for all $r, s, r', s' \in [1, m]$. Therefore, we can write

$$\bar{f}_l = f_{\ell-1}(\bar{x}_1, \ldots, \bar{x}_{\ell-1}, \bar{x}_{l+1}, \ldots, \bar{x}_{l+\ell})\bar{x}_\ell f_{l-\ell}(\bar{x}_{\ell+1}, \ldots, \bar{x}_l, \bar{x}_{l+\ell+1}, \ldots, \bar{x}_{2l+1}),$$

where $0 \neq \bar{f}_{\ell-1} \in A^{[i',i]}$ and $0 \neq \bar{f}_{l-\ell} \in A^{[j,j']}$. Once $A^{[i',i]} \cong UT(A_{i'}, \ldots, A_i)$ and $A^{[j,j']} \cong UT(A_j, \ldots, A_{j'})$, with $1 \leq i - i' + 1 \leq m - 1$ and $1 \leq j' - j + 1 \leq m - 1$, by applying the induction hypothesis one has that

$$\ell - 1 \leq i - i' + \sum_{s=i'}^{i} k_s^2 \quad \text{and} \quad l - \ell \leq j' - j + \sum_{s=j}^{j'} k_s^2,$$

and hence

$$\ell - 1 \leq i - i' + \sum_{s=1}^{i} k_s^2 \quad \text{and} \quad l - \ell \leq j' - j + \sum_{s=j}^{m} k_s^2.$$

This allows us to obtain

$$l - 1 \leq j' - j + i - i' + \sum_{s=1}^{i} k_s^2 + \sum_{s=j}^{m} k_s^2,$$

and, since $i - j \leq -1$, we conclude that

$$l \leq j' - i' + \sum_{s=1}^{m} k_s^2 \leq m - 1 + \sum_{s=1}^{m} k_s^2,$$

as desired.

On the other hand, if $\ell \in [l+1, 2l+1]$, then $\ell = l + \ell'$, with $\ell' \in [1, l+1]$. We remember that there exist diagonal elements $e_i \in A_{i,i}$ and $e_j \in A_{j,j}$, in the canonical basis of A, such that $\bar{x}_\ell = e_i \bar{x}_\ell e_j$, with $i < j$, and thus, similarly to the previous case, we can write

$$\bar{f}_l = f_{\ell'-1}(\bar{x}_1, \ldots, \bar{x}_{\ell'-1}, \bar{x}_{l+1}, \ldots, \bar{x}_{l+\ell'-1}, e_i)\bar{x}_\ell f_{l-\ell'+1}(\bar{x}_{\ell'}, \ldots, \bar{x}_l, e_j, \bar{x}_{l+\ell'+1}, \ldots, \bar{x}_{2l+1})$$

and we are done.

Conversely, assume that $l \leq t_{1m}$. In order to conclude the proof, we show that $f_l \notin \mathrm{Id}(A)$. We define $t_{10} := -1$ and, for each $\ell \in [1, m]$, we consider the following evaluation given by k_ℓ^2 distinct elements of the canonical basis of A_ℓ:

$$\begin{aligned} \bar{x}_{t_{1,\ell-1}+2} \cdots \bar{x}_{t_{1,\ell}} &= (e_{11} \otimes E^0)^{(\ell,\ell)} \cdot (e_{12} \otimes E^0)^{(\ell,\ell)} \cdot (e_{22} \otimes E^0)^{(\ell,\ell)} \cdot (e_{21} \otimes E^0)^{(\ell,\ell)} \cdot \\ &\quad (e_{13} \otimes E^0)^{(\ell,\ell)} \cdot (e_{33} \otimes E^0)^{(\ell,\ell)} \cdot (e_{32} \otimes E^0)^{(\ell,\ell)} \cdot (e_{23} \otimes E^0)^{(\ell,\ell)} \cdot \\ &\quad (e_{31} \otimes E^0)^{(\ell,\ell)} \cdot (e_{14} \otimes E^0)^{(\ell,\ell)} \cdot (e_{44} \otimes E^0)^{(\ell,\ell)} \cdots (e_{k_\ell 1} \otimes E^0)^{(\ell,\ell)} \\ &= (e_{11} \otimes E^0)^{(\ell,\ell)}. \end{aligned}$$

Still, for all $\ell \in [1, m-1]$, consider $\bar{x}_{t_{1,\ell}+1} = \mathbf{E}_{11}^{(\ell,\ell+1)}$. Thus, given $u \in [1, \eta_1]$ and $v \in [\eta_{m-1} + 1, \eta_m]$, there exist suitable diagonal elements $\bar{x}_{l+1}, \ldots, \bar{x}_{2l+1}$, of the canonical basis of A, such that

$$\bar{x}_{l+1}\bar{x}_1\bar{x}_{l+2} \cdots \bar{x}_{2l}\bar{x}_l\bar{x}_{2l+1} = \mathbf{E}_{uv}^{(1,m)},$$

and, for every $\sigma \in \mathrm{Sym}(l)$, with $\sigma \neq 1$, we have

$$\bar{x}_{l+1}\bar{x}_{\sigma(1)}\bar{x}_{l+2} \cdots \bar{x}_{2l}\bar{x}_{\sigma(l)}\bar{x}_{2l+1} = 0.$$

Then $f_l(\bar{x}_1, \cdots, \bar{x}_{2l+1}) = \mathbf{E}_{uv}^{(1,m)}$ and this means that $f_l \notin \mathrm{Id}(A)$. $\square$

From now on, let us fix an m-tuple $(A_1, \ldots, A_m)$ of finite dimensional G-simple F-algebras and consider $A := (UT(A_1, \ldots, A_m), \widetilde{\alpha})$ and $B := (UT(A_1, \ldots, A_m), \widetilde{\beta})$ such that $\widetilde{\beta}$ is $\widetilde{\alpha}$-admissible. Moreover, for each $l \in [1, m]$, let us assume that $(A_l, \widetilde{\alpha}_l)$ and $(A_l, \widetilde{\beta}_l)$ have the following presentations:

$$P_{(A_l, \widetilde{\alpha}_l)} = (r_l; (g_{l1}, \ldots, g_{lk_l})) \quad \text{and} \quad P_{(A_l, \widetilde{\beta}_l)} = (r_l; (\widetilde{g}_{l1}, \ldots, \widetilde{g}_{lk_l})).$$

The next result relates the invariance subgroups of the G-simple algebras $A_{l,l}$ and the ideals of G-graded polynomial identities of the algebras A and B. Such result is a generalization of our Lemma 5.5, stated in [22].

Proposition 4.1.4. *Let $G = \langle \epsilon \rangle$ be a cyclic group. Suppose that $m \geq 2$,*

$$\widetilde{\beta}_1 = l_h \cdot \widetilde{\alpha}_1 \quad \textit{and} \quad \widetilde{\beta}_m = l_\eta \cdot \widetilde{\alpha}_m,$$

for some $h, \eta \in G$ such that $h^{-1}\eta \notin \mathcal{H}_{\widetilde{\alpha}}^{(1)} \mathcal{H}_{\widetilde{\alpha}}^{(m)}$. Then $\mathrm{Id}_G(B) \nsubseteq \mathrm{Id}_G(A)$ and $\mathrm{Id}_G(A) \nsubseteq \mathrm{Id}_G(B)$.

Proof. In order to obtain that $\mathrm{Id}_G(B) \nsubseteq \mathrm{Id}_G(A)$, let us construct a suitable graded polynomial f such that $f \in \mathrm{Id}_G(B)$ and $f \notin \mathrm{Id}_G(A)$. The proof of $\mathrm{Id}_G(A) \nsubseteq \mathrm{Id}_G(B)$ follows in an analogous way.

First, let us suppose, without loss of generality, that

$$w_{\alpha_1 \odot \widetilde{\epsilon}_{r_1}}(g_{11}) = \max\{w_{\alpha_1 \odot \widetilde{\epsilon}_{r_1}}(h) \mid h \in \mathcal{I}_{\alpha_1 \odot \widetilde{\epsilon}_{r_1}}\}$$

and

$$w_{\alpha_m \odot \widetilde{\epsilon}_{r_m}}(g_{m1}) = \max\{w_{\alpha_m \odot \widetilde{\epsilon}_{r_m}}(h) \mid h \in \mathcal{I}_{\alpha_m \odot \widetilde{\epsilon}_{r_m}}\}.$$

Denote $t_{1m} := m - 1 + \sum_{i=1}^m k_i^2$. By invoking Lemma 4.1.3, there exists an evaluation of the polynomial $Cap_{t_{1m}}(x_1, \ldots, x_{t_{1m}}; x_{t_{1m}+1}, \ldots, x_{2t_{1m}+1})$ in the algebra $UT(A_1, \ldots, A_m)$, at its canonical basis elements, resulting in $\mathbf{E}_{1,\eta_{m-1}+1}$. Now, consider the multilinear graded polynomial $Cap_{t_{1m}}(u_1, \ldots, u_{t_{1m}}; u_{t_{1m}+1}, \ldots, u_{2t_{1m}+1})$ built in a such way that each homogeneous variable u_i has the degree, induced by $\widetilde{\alpha}$, of the canonical basis element used in the above evaluation. Then $Cap_{t_{1m}}(u_1, \ldots, u_{t_{1m}}; u_{t_{1m}+1}, \ldots, u_{2t_{1m}+1})$ has a graded evaluation in A equal to $\mathbf{E}_{11}^{(1,m)} = \mathbf{E}_{1,\eta_{m-1}+1}$. Still, once

$$|\mathbf{E}_{11}^{(1,m)}|_A = |\mathbf{E}_{1,\eta_{m-1}+1}|_A = \widetilde{\alpha}(1)^{-1}\widetilde{\alpha}(\eta_{m-1}+1) = g_{11}^{-1} g_{m1}$$

it follows that $Cap_{t_{1m}}(u_1, \ldots, u_{t_{1m}}; u_{t_{1m}+1}, \ldots, u_{2t_{1m}+1})$ has homogeneous degree being $g_{11}^{-1} g_{m1}$

as an element of $F\langle X;G\rangle$.

Thus, by item (i) of Lemma 3.2.3, there exist homogeneous multilinear polynomials Ψ_{A_1} and Ψ_{A_m}, in pairwise disjoint sets of homogeneous variables (and also distinct from those of the set $\{u_1,\ldots,u_{2t_{1m}+1}\}$), with evaluations $\rho_1: F\langle X;G\rangle\to A$ and $\rho_m: F\langle X;G\rangle\to A$, such that

$$\rho_1(\Psi_{A_1})=(e_{11}\otimes E^0)^{(1,1)}\quad\text{and}\quad\rho_m(\Psi_{A_m})=(e_{11}\otimes E^0)^{(m,m)}.$$

Therefore, by setting

$$f:=\Psi_{A_1}Cap_{t_{1m}}(u_1,\ldots,u_{t_{1m}};u_{t_{1m}+1},\ldots,u_{2t_{1m}+1})\Psi_{A_m},$$

we have that $f\notin \mathrm{Id}_G(A)$.

Our next step is to show that $f\in\mathrm{Id}_G(B)$. We start by remarking that any non-zero graded evaluation of $Cap_{t_{1m}}(u_1,\ldots,u_{t_{1m}};u_{t_{1m}+1},\ldots,u_{2t_{1m}+1})$ in B must give elements of $J(B)^{m-1}$ which are linear combinations of matrix units $\mathbf{E}_{pq}$ of homogeneous degree equal to $g_{11}^{-1}g_{m1}$, that is, matrices $\mathbf{E}_{pq}\in J(B)^{m-1}$ such that

$$\widetilde{\beta}(p)^{-1}\widetilde{\beta}(q)=g_{11}^{-1}g_{m1}.$$

Thus, in order to have that f is not a graded identity of B, the homogeneous multilinear polynomials Ψ_{A_1} and Ψ_{A_m} must be evaluated, respectively, in A_1 and A_m.

If ρ_1 and ρ_m are graded evaluations, respectively, of Ψ_{A_1} and Ψ_{A_m} in, respectively, A_1 and A_m (with the grading induced by $\widetilde{\beta}$), since $\widetilde{\beta}_1=l_h\cdot\widetilde{\alpha}_1$ and $\widetilde{\beta}_m=l_\eta\cdot\widetilde{\alpha}_m$, from Remark 3.2.4, such evaluations satisfy

$$\rho_1(\Psi_{A_1})\in\bigoplus_{i\in\overline{\mathbf{T}}_{A_1};\,g_{1i}\in\mathcal{H}_{\widetilde{\beta}}^{(1)}g_{11}}(A_1)_{1_G}^{(hg_{1i})}\quad\text{and}\quad\rho_m(\Psi_{A_m})\in\bigoplus_{j\in\overline{\mathbf{T}}_{A_m};\,g_{mj}\in\mathcal{H}_{\widetilde{\beta}}^{(m)}g_{m1}}(A_m)_{1_G}^{(\eta g_{mj})}.$$

In particular, the evaluation of Ψ_{A_1} results in linear combinations of basis canonical elements $(e_{uv}\otimes E^{a-b})^{(1,1)}\in((A_1)_{1_G}^{(hg_{1i})},\widetilde{\beta}_1=\beta_1\odot\widetilde{\epsilon}_{r_1})$ such that

$$\beta_1(u)=h(\epsilon^{s_1})^a g_{1i}\ \text{ and }\ \beta_1(v)=h(\epsilon^{s_1})^b g_{1i},\ \text{for some } a,b\in[0,r_1-1],$$

and once $g_{1i}\in\mathcal{H}_{\widetilde{\beta}}^{(1)}g_{11}$, we have

$$\beta_1(u)=h(\epsilon^{s_1})^a h_{1i}g_{11}\ \text{ and }\ \beta_1(v)=h(\epsilon^{s_1})^b h_{1i}g_{11},\ \text{for some }\ h_{1i}\in\mathcal{H}_{\widetilde{\beta}}^{(1)};$$

whereas, one has that the evaluation of Ψ_{A_m} results in linear combinations of basis canonical elements $(e_{uv} \otimes E^{c-d})^{(m,m)} \in ((A_m)_{1_G}^{(\eta g_{mj})}, \widetilde{\beta}_m = \beta_m \odot \widetilde{\epsilon}_{r_m})$ such that

$$\beta_m(u) = \eta(\epsilon^{s_m})^c h_{mj} g_{m1} \text{ and } \beta_m(v) = \eta(\epsilon^{s_m})^d h_{mj} g_{m1}, \quad \text{for some} \quad h_{mj} \in \mathcal{H}_{\widetilde{\beta}}^{(m)},$$

with $c, d \in [0, r_m - 1]$.

Thus, from above discussions, we have that there exist $l_1 \in [0, r_1 - 1]$ and $l_m \in [0, r_m - 1]$ such that

$$g_{11}^{-1} g_{m1} = \widetilde{\beta}(p)^{-1} \widetilde{\beta}(q) = (h(\epsilon^{s_1})^b h_{1i} g_{11} (\epsilon^{s_1})^{l_1})^{-1} \eta(\epsilon^{s_m})^c h_{mj} g_{m1} (\epsilon^{s_m})^{l_m}$$

which implies that $h^{-1}\eta = (\epsilon^{s_1})^{b+l_1} h_{1i} (\epsilon^{s_m})^{-(c+l_m)} h_{mj}^{-1}$. Since $\langle \epsilon^{s_1} \rangle \subseteq \mathcal{H}_{\widetilde{\beta}}^{(1)}$ and $\langle \epsilon^{s_m} \rangle \subseteq \mathcal{H}_{\widetilde{\beta}}^{(m)}$, we conclude that $h^{-1}\eta \in \mathcal{H}_{\widetilde{\beta}}^{(1)} \mathcal{H}_{\widetilde{\beta}}^{(m)} = \mathcal{H}_{\widetilde{\alpha}}^{(1)} \mathcal{H}_{\widetilde{\alpha}}^{(m)}$, a contradiction with our hypotheses. Hence, this forces $f \in \mathrm{Id}_G(B)$, as required, and then $\mathrm{Id}_G(B) \nsubseteq \mathrm{Id}_G(A)$. □

We can generalize the previous proposition in the following manner:

Proposition 4.1.5. *Let $G = \langle \epsilon \rangle$ be a cyclic group. Assume that, for $1 \leq a < b \leq m$,*

$$\widetilde{\beta}_a = l_h \cdot \widetilde{\alpha}_a \quad \text{and} \quad \widetilde{\beta}_b = l_\eta \cdot \widetilde{\alpha}_b,$$

for some $h, \eta \in G$ such that $h^{-1}\eta \notin \mathcal{H}_{\widetilde{\alpha}}^{(a)} \mathcal{H}_{\widetilde{\alpha}}^{(b)}$. Then $\mathrm{Id}_G(B) \nsubseteq \mathrm{Id}_G(A)$ and $\mathrm{Id}_G(A) \nsubseteq \mathrm{Id}_G(B)$.

Proof. Firstly, Proposition 4.1.4 guarantees that there exists a multilinear graded polynomial f such that

$$f \notin \mathrm{Id}_G(A^{[a,b]}) \quad \text{and} \quad f \in \mathrm{Id}_G(B^{[a,b]}).$$

Define $t_{1a} := a - 1 + \sum_{j=1}^{a} k_j^2$ and $t_{bm} := m - b + \sum_{j=b}^{m} k_j^2$. It follows from Lemma 4.1.3 that we can build graded multilinear polynomials $f_{t_{1a}} := Cap_{t_{1a}}(u_1, \ldots, u_{t_{1a}}; u_{t_{1a}+1}, \ldots, u_{2t_{1a}+1})$ and $f_{t_{bm}} := Cap_{t_{bm}}(v_1, \ldots, v_{t_{bm}}; v_{t_{bm}+1}, \ldots, v_{2t_{bm}+1})$, in pairwise disjoint sets of homogeneous variables (also distinct from those involved in f), such that $f_{t_{1a}} \notin \mathrm{Id}_G(A^{[1,a]})$ and $f_{t_{bm}} \notin \mathrm{Id}_G(A^{[b,m]})$.

Therefore, by considering new distinct variables x_g, $\widetilde{x}_g$, for each $g \in G$, and setting

$$\widetilde{f} := f_{t_{1a}} \left(\sum_{g \in G} x^g \right) f \left(\sum_{g \in G} \widetilde{x}^g \right) f_{t_{bm}},$$

it is easy to see that $\widetilde{f} \notin \mathrm{Id}_G(A)$.

Finally, we claim that $\widetilde{f} \in \mathrm{Id}_G(B)$. Indeed, from Lemma 4.1.3, one has

$$f_{t_{1a}} \in \mathrm{Id}_G(B^{[1,l]}), \quad \text{for all } l < a,$$

and

$$f_{t_{bm}} \in \mathrm{Id}_G(B^{[l',m]}), \quad \text{for all } l' > b.$$

Therefore, in order to obtain a non-zero evaluation, we must evaluate the polynomial $f_{t_{1a}}$ in $B^{[1,l]}$, for some $l \geq a$, whereas $f_{t_{bm}}$ in $B^{[l',m]}$, for some $l' \leq b$, and thus $(\sum x^g) f (\sum \widetilde{x}^g)$ in $B^{[l,l']}$. Then, once

$$B^{[l,l']} \subseteq B^{[a,b]} \quad \text{and} \quad f \in \mathrm{Id}_G(B^{[a,b]}),$$

we obtain $\widetilde{f} \in \mathrm{Id}_G(B)$ and, hence, $\mathrm{Id}_G(B) \nsubseteq \mathrm{Id}_G(A)$. Analogously, we conclude that $\mathrm{Id}_G(A) \nsubseteq \mathrm{Id}_G(B)$. □

Finally, when $m = 2$, we can also relate, in a special case, the invariance subgroups of the G-simple algebras A_1 and A_2 with the factoring property.

Proposition 4.1.6 (Theorem 5.9 of [22]). *Let $G = \langle \epsilon \rangle$ be a cyclic group. If $m = 2$ and $(A_i, \widetilde{\alpha}_i)$ is $\widetilde{\alpha}_i$-regular, for all $i \in [1, 2]$, then the T_G-ideal $\mathrm{Id}_G(A)$ is factorable if, and only if, $\mathcal{H}^{(1)}_{\widetilde{\alpha}} \mathcal{H}^{(2)}_{\widetilde{\alpha}} = G$.*

Proof. If $\mathcal{H}^{(1)}_{\widetilde{\alpha}} \mathcal{H}^{(2)}_{\widetilde{\alpha}} \neq G$, by the previous proposition there exists an $\widetilde{\alpha}$-admissible G-grading $\widetilde{\beta}$ on $UT(A_1, A_2)$ such that for the corresponding G-graded algebra B we have $\mathrm{Id}_G(A) \nsubseteq \mathrm{Id}_G(B)$. Hence $\mathrm{Id}_G(A) \neq \mathrm{Id}_G(A_1)\mathrm{Id}_G(A_2)$, since by Lemma 1.2.5, $\mathrm{Id}_G(A_1)\mathrm{Id}_G(A_2) \subseteq \mathrm{Id}_G(B)$.

Conversely, if $\mathcal{H}^{(1)}_{\widetilde{\alpha}} \mathcal{H}^{(2)}_{\widetilde{\alpha}} = G$ the result follows by Corollary 2.2.2 and Theorem 3.3.2. □

4.2 The factorability and the indecomposable T_{C_n}-ideals

Let $A_1, \ldots, A_m$ be finite dimensional G-simple F-algebras and consider the G-graded upper block triangular matrix algebra $UT_G(A_1, \ldots, A_m)$. In Section 1.2, we presented the definition of decomposable and indecomposable T_G-ideals. These concepts are important tools in obtaining results related to the factorability of the T_G-ideal $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$. We recall here that the notion of weakly factorable appears in Definition 2.1.3. The first result associated to decomposable T_G-ideals is the following:

Proposition 4.2.1. *Let $G = \langle \epsilon \rangle$ be a cyclic group and $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$. The T_G-ideal $\mathrm{Id}_G(A)$ is decomposable if, and only if, $m \geq 2$ and $\mathrm{Id}_G(A)$ is weakly factorable.*

Proof. If $m = 1$, from Lemma 1.2.3, it follows that $\mathrm{Id}_G(A)$ is indecomposable.

Let us study the case $m \geq 2$. If $\mathrm{Id}_G(A)$ is weakly factorable, then there exist integers $1 \leq c_1 < c_2 < \cdots < c_u < m$ such that

$$\mathrm{Id}_G(A) = \mathrm{Id}_G(A^{[1,c_1]})\mathrm{Id}_G(A^{[c_1+1,c_2]}) \cdots \mathrm{Id}_G(A^{[c_u+1,m]}).$$

By invoking Lemma 1.2.5, we can conclude that

$$\mathrm{Id}_G(A^{[1,c_1]})\mathrm{Id}_G(A^{[c_1+1,c_2]})\cdots\mathrm{Id}_G(A^{[c_u+1,m]}) \subseteq \mathrm{Id}_G(A^{[1,c_1]})\mathrm{Id}_G(A^{[c_1+1,m]}) \subseteq \mathrm{Id}_G(A),$$

and, hence, we obtain

$$\mathrm{Id}_G(A) = \mathrm{Id}_G(A^{[1,c_1]})\mathrm{Id}_G(A^{[c_1+1,m]}).$$

By combining the facts that $A^{[1,c_1]} \cong UT_G(A_1,\ldots,A_{c_1})$ and $A^{[c_1+1,m]} \cong UT_G(A_{c_1+1},\ldots,A_m)$ with Lemma 4.1.3, one has that

$$\mathrm{Id}_G(A^{[1,c_1]}) \neq \mathrm{Id}_G(A) \quad \text{and} \quad \mathrm{Id}_G(A^{[c_1+1,m]}) \neq \mathrm{Id}_G(A).$$

Consequently, $\mathrm{Id}_G(A)$ is a decomposable T_G-ideal.

Conversely, assume that $\mathrm{Id}_G(A)$ is decomposable. Thus $m \geq 2$ and there exist T_G-ideals $I_1 \neq \mathrm{Id}_G(A)$ and $I_2 \neq \mathrm{Id}_G(A)$ such that

$$\mathrm{Id}_G(A) = I_1I_2.$$

We claim that, for any $v \in [1, m-1]$,

$$\text{either } I_1 \subseteq \mathrm{Id}_G(A^{[1,v]}) \text{ or } I_2 \subseteq \mathrm{Id}_G(A^{[v,m]}).$$

In fact, suppose, if possible, that there exist

$$f_1 \in I_1 \setminus \mathrm{Id}_G(A^{[1,v]}) \quad \text{and} \quad f_2 \in I_2 \setminus \mathrm{Id}_G(A^{[v,m]}),$$

for some $v \in [1, m-1]$. This means that there exist graded evaluations $\rho_1 : F\langle X;G\rangle \to A^{[1,v]}$ and $\rho_2 : F\langle X;G\rangle \to A^{[v,m]}$ such that

$$\rho_1(f_1) = a \neq 0 \quad \text{and} \quad \rho_2(f_2) = b \neq 0.$$

In this case, we remark that there exist $\omega \in A$ such that $a\omega b \neq 0$. Then, the polynomial $f_1(\sum_{g\in G} x^g)f_2$ is not a graded polynomial identity for A and also it satisfies $f_1(\sum_{g\in G} x^g)f_2 \in I_1I_2 = \mathrm{Id}_G(A)$, a contradiction.

Moreover, by using the above claim, it is easy to verify that $I_1 \subseteq \mathrm{Id}_G(A^{[1,1]})$. Consider $\ell := \max\{v \mid I_1 \subseteq \mathrm{Id}_G(A^{[1,v]})\}$. Thus $I_1 \not\subseteq \mathrm{Id}_G(A^{[1,\ell+1]})$ and we notice that $\ell \neq m$. By applying again the above claim, it follows that $I_2 \subseteq \mathrm{Id}_G(A^{[\ell+1,m]})$. Therefore, we have

$$\mathrm{Id}_G(A) = I_1I_2 \subseteq \mathrm{Id}_G(A^{[1,\ell]})\mathrm{Id}_G(A^{[\ell+1,m]}) \subseteq \mathrm{Id}_G(A)$$

and, hence, $\mathrm{Id}_G(A) = \mathrm{Id}_G(A^{[1,\ell]})\mathrm{Id}_G(A^{[\ell+1,m]})$. Then $\mathrm{Id}_G(A)$ is weakly factorable, as desired. $\square$

The next result presents a sufficient condition for the T_G-ideal of $UT_G(A_1, \ldots, A_m)$ be indecomposable. Such condition is related to the invariance subgroups of the G-simple components A_1 and A_m.

Proposition 4.2.2. *Let $G = \langle \epsilon \rangle$ be a cyclic group. If $m \geq 2$ and $\mathcal{H}^{(1)}_{\widetilde{\alpha}}\mathcal{H}^{(m)}_{\widetilde{\alpha}} \neq G$, then the T_G-ideal* $\mathrm{Id}_G((UT(A_1, \ldots, A_m), \widetilde{\alpha}))$ *is indecomposable.*

Proof. First, let us denote $A := (UT(A_1, \ldots, A_m), \widetilde{\alpha})$. Suppose, by contradiction, that $\mathrm{Id}_G(A)$ is decomposable. From Proposition 4.2.1, there exist $1 \leq c_1 < c_2 < \cdots < c_u < m$ such that

$$\mathrm{Id}_G(A) = \mathrm{Id}_G(A^{[1,c_1]})\mathrm{Id}_G(A^{[c_1+1,c_2]}) \cdots \mathrm{Id}_G(A^{[c_u+1,m]}).$$

Consider the G-graded upper block triangular matrix algebra $B = (UT(A_1, \ldots, A_m), \widetilde{\beta})$ where

$$\widetilde{\beta}(j) = \begin{cases} \widetilde{\alpha}(j) & \text{if } 1 \leq j \leq \eta_{c_u} \\ l_\epsilon \cdot \widetilde{\alpha}(j) & \text{if } \eta_{c_u} + 1 \leq j \leq \eta_m. \end{cases}$$

Once $\widetilde{\beta}(j) = \widetilde{\alpha}(j)$, for all $j \in [1, \eta_{c_u}]$, one has that $(UT(A_1, \ldots, A_{c_u}), \widetilde{\beta}) = (UT(A_1, \ldots, A_{c_u}), \widetilde{\alpha})$ and hence $\mathrm{Id}_G(B^{[c_{i-1}+1,c_i]}) = \mathrm{Id}_G(A^{[c_{i-1}+1,c_i]})$, for all $i \in [1, u]$, by setting $c_0 := 0$. Still, the fact that $(UT(A_{c_u+1}, \ldots, A_m), \widetilde{\beta}) = (UT(A_{c_u+1}, \ldots, A_m), \widetilde{\alpha})$ yields us

$$\mathrm{Id}_G(B^{[c_u+1,m]}) = \mathrm{Id}_G(A^{[c_u+1,m]}).$$

Therefore, by applying Lemma 1.2.5, we have

$$\begin{aligned} \mathrm{Id}_G(A) &= \mathrm{Id}_G(A^{[1,c_1]})\mathrm{Id}_G(A^{[c_1+1,c_2]}) \cdots \mathrm{Id}_G(A^{[c_u+1,m]}) \\ &= \mathrm{Id}_G(B^{[1,c_1]})\mathrm{Id}_G(B^{[c_1+1,c_2]}) \cdots \mathrm{Id}_G(B^{[c_u+1,m]}) \subseteq \mathrm{Id}_G(B), \end{aligned}$$

a contradiction with Proposition 4.1.4 (here $h = 1_G$ and $\eta = \epsilon$). Then, we conclude that $\mathrm{Id}_G(A)$ is indecomposable and the proof of the theorem is complete. $\square$

We remark that if (B, β) is a finite dimensional G-simple algebra, then B is G-regular if, and only if, $\mathcal{H}_\beta = G$ (see Corollary 3.3.3). Thus given $m \geq 2$, and considering an m-tuple $(A_1, \ldots, A_m)$ of finite dimensional G-simple algebras, if $\mathcal{H}^{(1)}_{\widetilde{\alpha}}\mathcal{H}^{(m)}_{\widetilde{\alpha}} \neq G$, where $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$, then A_1 and A_m are both non-G-regular G-simple algebras. However, in general the converse may not be valid. In the sequel, if p is a prime number, we will obtain that, for p-groups, the condition $\mathcal{H}^{(1)}_{\widetilde{\alpha}}\mathcal{H}^{(m)}_{\widetilde{\alpha}} \neq G$ is equivalent to requiring that A_1 and A_m are both non-G-regular G-simple algebras.

Theorem 4.2.3. *Let $G = \langle \epsilon \rangle$ be a cyclic p-group, where p is a prime number. Assume that $m \geq 2$ and $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$. The following statements are equivalent:*

(i) $\mathcal{H}_{\widetilde{\alpha}}^{(1)} \mathcal{H}_{\widetilde{\alpha}}^{(m)} \neq G$*;*

(ii) *A_1 and A_m are both non-G-regular G-simple algebras;*

(iii) *The T_G-ideal of A is indecomposable.*

Proof. First, the implication of item (i) to (ii) is trivial, since if A_1 or A_m is G-regular then, by Corollary 3.3.3, $\mathcal{H}_{\widetilde{\alpha}}^{(1)} = G$ or $\mathcal{H}_{\widetilde{\alpha}}^{(m)} = G$, which is contrary to the fact that $\mathcal{H}_{\widetilde{\alpha}}^{(1)} \mathcal{H}_{\widetilde{\alpha}}^{(m)} \neq G$.

In order to prove the converse, assume that statement (ii) holds, with $\mathcal{H}_{\widetilde{\alpha}}^{(i)} = \langle \epsilon^{c_i} \rangle$, for $i \in [1, m]$, and suppose, by contradiction, that $\mathcal{H}_{\widetilde{\alpha}}^{(1)} \mathcal{H}_{\widetilde{\alpha}}^{(m)} = G$. Thus $\epsilon \in \mathcal{H}_{\widetilde{\alpha}}^{(1)} \mathcal{H}_{\widetilde{\alpha}}^{(m)}$, and we can write

$$\epsilon = (\epsilon^{c_1})^{l_1} (\epsilon^{c_m})^{l_m},$$

for some integers l_1, l_m. Since A_1 and A_m are both non-G-regular G-simple algebras, it follows that $\mathcal{H}_{\widetilde{\alpha}}^{(1)} \neq G$ and $\mathcal{H}_{\widetilde{\alpha}}^{(m)} \neq G$, and then p divides c_1 and c_m. From the above equality, one has that p divides $(c_1 l_1 + c_m m_m - 1)$, and consequently p divides 1, an absurd. Therefore, we conclude that statements (i) and (ii) are equivalents.

We remark that Proposition 4.2.2 guarantees that (i) implies (iii). Thus, in order to finish the proof of the theorem, let us show that (iii) implies (ii). For such, it is enough to notice that if A_1 is G-regular, then, by applying Theorem 2.1.5, we have

$$\mathrm{Id}_G(A) = \mathrm{Id}_G(A_1) \mathrm{Id}_G(UT_G(A_2, \ldots, A_m)).$$

Similarly, we concluded if A_m is G-regular. □

In the sequel, we give a generalization of Theorems 4.6 and 4.9 of [7]. We highlight that its proof is a direct consequence of Proposition 4.1.5 and Theorems 2.1.5 and 4.2.3.

Theorem 4.2.4. *Let $G = \langle \epsilon \rangle$ be a cyclic p-group, where p is a prime number, and consider an m-tuple $(A_1, \ldots, A_m)$ of finite dimensional G-simple algebras. Let $A = UT_G(A_1, \ldots, A_m)$. Then, either $\mathrm{Id}_G(A)$ is an indecomposable T_G-ideal (related to minimal graded algebras) or $\mathrm{Id}_G(A)$ can be written as a product of indecomposable T_G-ideals.*

More precisely, if there exists at most one index $\ell \in [1, m]$ such that A_ℓ is a non-G-regular G-simple algebra, then

$$\mathrm{Id}_G(A) = \mathrm{Id}_G(A_1) \mathrm{Id}_G(A_2) \cdots \mathrm{Id}_G(A_m).$$

Otherwise, we can set u and v as the first and the last index (with $1 \leq u < v \leq m$), respectively, such that A_u and A_v are non-G-regular G-simple algebras. In this way, the decomposition of $\mathrm{Id}_G(A)$ as a product of indecomposable T_G-ideals is given by

$$\mathrm{Id}_G(A) = \mathrm{Id}_G(A_1)\cdots\mathrm{Id}_G(A_{u-1})\mathrm{Id}_G(UT_G(A_u,\ldots,A_v))\mathrm{Id}_G(A_{v+1})\cdots\mathrm{Id}_G(A_m).$$

As a consequence, we obtain:

Corollary 4.2.5. *Let $G = \langle\epsilon\rangle$ be a cyclic p-group, where p is a prime number, and consider an m-tuple $(A_1,\ldots,A_m)$ of finite dimensional G-simple algebras. Let $A = UT_G(A_1,\ldots,A_m)$. The T_G-ideal $\mathrm{Id}_G(A)$ is factorable if, and only if, there exists at most one index $\ell \in [1,m]$ such that A_ℓ is a non-G-regular G-simple algebra.*

4.3 The factorability and the isomorphism

Let $A_1,\ldots,A_m$ be finite dimensional G-simple F-algebras. We start this section by giving a result which establishes a relation between the invariance subgroups associated to $A_1,\ldots,A_m$ and the number of non-isomorphic G-gradings on $UT_G(A_1,\ldots,A_m)$. Moreover, we present necessary and sufficient conditions to the factorability of the T_G-ideals $\mathrm{Id}_G(UT_G(A_1,\ldots,A_m))$, in case G is a cyclic p-group, with p an arbitrary prime. Such statement is one of the main results of this thesis. Finally, we explore the factorability of $\mathrm{Id}_G(UT_G(A_1,A_2))$, where G is not necessarily a cyclic p-group.

Here, we also consider $A := (UT(A_1,\ldots,A_m),\widetilde{\alpha})$ and $B := (UT(A_1,\ldots,A_m),\widetilde{\beta})$ such that $\widetilde{\beta}$ is $\widetilde{\alpha}$-admissible. For each $l \in [1,m]$, we assume that $(A_l,\widetilde{\alpha}_l)$ and $(A_l,\widetilde{\beta}_l)$ have the following presentations: $P_{(A_l,\widetilde{\alpha}_l)} = (r_l;(g_{l1},\ldots,g_{lk_l}))$ and $P_{(A_l,\widetilde{\beta}_l)} = (r_l;(\widetilde{g}_{l1},\ldots,\widetilde{g}_{lk_l}))$.

Proposition 4.3.1. *Let $G = \langle\epsilon\rangle$ be a cyclic group. If $\mathcal{H}^{(a)}_{\widetilde{\alpha}}\mathcal{H}^{(b)}_{\widetilde{\alpha}} \neq G$ for some $1 \leq a < b \leq m$, then there exists at least an $\widetilde{\alpha}$-admissible G-grading $\widetilde{\beta}$ such that $A = (UT(A_1,\ldots,A_m),\widetilde{\alpha})$ and $B = (UT(A_1,\ldots,A_m),\widetilde{\beta})$ are non-isomorphic as G-graded algebras.*

Proof. Since $\mathcal{H}^{(a)}_{\widetilde{\alpha}}\mathcal{H}^{(b)}_{\widetilde{\alpha}} \neq G$, for any $h,\eta \in G$ such that $h^{-1}\eta \notin \mathcal{H}^{(a)}_{\widetilde{\alpha}}\mathcal{H}^{(b)}_{\widetilde{\alpha}}$, let $\widetilde{\beta}$ be the G-grading defined on $UT(A_1,\ldots,A_m)$ satisfying

$$\widetilde{\beta}_a = l_h \cdot \widetilde{\alpha}_a \quad \text{and} \quad \widetilde{\beta}_b = l_\eta \cdot \widetilde{\alpha}_b.$$

By invoking Proposition 4.1.5, it follows that $\mathrm{Id}_G(B) \nsubseteq \mathrm{Id}_G(A)$ and $\mathrm{Id}_G(A) \nsubseteq \mathrm{Id}_G(B)$. Consequently, A and B are non-isomorphic as G-graded algebras. □

The next lemma gives us an important condition in order to obtain a graded isomorphism between the algebras $A = (UT(A_1,\ldots,A_m),\widetilde{\alpha})$ and $B = (UT(A_1,\ldots,A_m),\widetilde{\beta})$.

Lemma 4.3.2 (Lemma 3.6 of [31]). *Let $G = \langle \epsilon \rangle$ be a cyclic group. If there exists $g \in G$ such that*

$$w_{\widetilde{\beta}}^{(l)}(gx) = w_{\widetilde{\alpha}}^{(l)}(x), \quad \textit{for all } l \in [1, m] \textit{ and } x \in G,$$

then B is graded-isomorphic to A.

Proof. For each $l \in [1, m]$, the hypothesis guarantees us that there exists a permutation $\theta_l \in \mathrm{Sym}(k_l)$ such that

$$H_{r_l}\widetilde{g}_{l\theta_l(i)} = H_{r_l} g g_{li}, \quad \text{for all } i \in [1, k_l].$$

Given $\delta \in [1, k_l]$, let us define $Bl_{l\delta} := [(\delta - 1)r_l + 1, \delta r_l]$. Then, for each $l \in [1, m]$, there exists $\sigma_l \in \mathrm{Sym}(k_l r_l)$ such that

$$\sigma_l(Bl_{li}) = Bl_{l\theta_l(i)}, \quad \text{for all } i \in [1, k_l],$$

and

$$\widetilde{\beta}_l(\sigma_l(\iota)) = g\widetilde{\alpha}_l(\iota), \quad \text{for all } \iota \in [1, k_l r_l].$$

Define the map

$$\begin{array}{rccc} \Gamma: & (M_{\eta_m}, \widetilde{\alpha}) & \to & (M_{\eta_m}, \widetilde{\beta}) \\ & \mathbf{E}_{ij}^{(u,v)} & \mapsto & \mathbf{E}_{\sigma_u(i)\sigma_v(j)}^{(u,v)}. \end{array}$$

It is easy to verify that Γ is a graded isomorphism which induces a graded isomorphism between the algebras A and B, as desired. $\square$

Now, by dealing with the concept of G-regularity, we have the following result about the uniqueness of G-gradings on A up to isomorphisms of G-graded algebras.

Proposition 4.3.3 (Proposition 5.7 of [22]). *Let $G = \langle \epsilon \rangle$ be a cyclic group. If there exists at most one index $\ell \in [1, m]$ such that A_ℓ is a non-G-regular G-simple algebra, then for all $\widetilde{\alpha}$-admissible G-grading $\widetilde{\beta}$, the corresponding algebra $(UT(A_1, \ldots, A_m), \widetilde{\beta})$ is graded-isomorphic to A.*

Proof. If $\widetilde{\beta}$ is $\widetilde{\alpha}$-admissible, we will show that $B = (UT(A_1, \ldots, A_m), \widetilde{\beta})$ is graded-isomorphic to $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$.

First, suppose that A_l is a G-regular G-simple algebra, for all $l \in [1, m]$. Then, for all $l \in [1, m]$ and $x \in G$, the following equality

$$w_{\widetilde{\beta}}^{(l)}(gx) = w_{\widetilde{\alpha}}^{(l)}(x)$$

is valid for any choice of $g \in G$.

Consequently, fixed a such element $g \in G$, the assertion comes from Lemma 4.3.2.

It remains to study the case in which there exists an unique $\ell \in [1, m]$ such that A_ℓ is a non-G-regular G-simple algebra. In this case, since $(A_\ell, \widetilde{\beta}_\ell)$ is graded-isomorphic to $(A_\ell, \widetilde{\alpha}_\ell)$, by Proposition 3.2.1 there exists $g_\ell \in G$ such that

$$w^{(\ell)}_{\widetilde{\beta}}(g_\ell x) = w^{(\ell)}_{\widetilde{\alpha}}(x), \quad \text{for all } x \in G.$$

Therefore, once A_l is G-regular, for all $l \in [1, m]$, with $l \neq \ell$, we obtain that

$$w^{(l)}_{\widetilde{\beta}}(g_\ell x) = w^{(l)}_{\widetilde{\alpha}}(x), \quad \text{for all } l \in [1, m] \text{ and } x \in G.$$

Then, by considering $g := g_\ell$, by invoking Lemma 4.3.2, we conclude that B is graded-isomorphic to A. □

At this stage, as a consequence of the results presented in this work, we state the generalization of Theorem 4.9 of [7] for the case where G is a finite cyclic p-group, with p being a prime number.

Theorem 4.3.4 (Theorem 5.8 of [22]). *Let p be a prime number and let $G = \langle \epsilon \rangle$ be a cyclic p-group. Given $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$, the following statements are equivalent:*

(i) The T_G-ideal of A is factorable;

(ii) There exists at most one index $\ell \in [1, m]$ such that A_ℓ is a non-G-regular G-simple algebra;

(iii) For all $\widetilde{\alpha}$-admissible G-grading $\widetilde{\beta}$, the algebra $(UT(A_1, \ldots, A_m), \widetilde{\beta})$ is graded-isomorphic to A.

Proof. From Corollary 4.2.5 one has the equivalence of (i) and (ii). Moreover, Proposition 4.3.3 guarantees that item (ii) implies (iii).

In order to prove that (iii) implies (ii), notice that if there exist indices $1 \leq a < b \leq m$ such that the G-simple algebras A_a and A_b are both non-G-regular, then by Corollary 3.3.3 we have $\mathcal{H}^{(a)}_{\widetilde{\alpha}} \neq G \neq \mathcal{H}^{(b)}_{\widetilde{\alpha}}$. Since G is a cyclic p-group, as in the proof of Theorem 4.2.3, it follows that $\mathcal{H}^{(a)}_{\widetilde{\alpha}} \mathcal{H}^{(b)}_{\widetilde{\alpha}} \neq G$. Then, Proposition 4.3.1 guarantees that there exist at least an $\widetilde{\alpha}$-admissible G-grading $\widetilde{\beta}$, such that the corresponding algebra $(UT(A_1, \ldots, A_m), \widetilde{\beta})$ is not graded-isomorphic to A. □

In the sequel, we examine the case when $m = 2$, and $(A_i, \widetilde{\alpha}_i)$ is $\widetilde{\alpha}_i$-regular for all $i \in [1, 2]$. In this situation, we can characterize the factoring property for $\mathrm{Id}_G(A)$ removing the hypothesis that G is a cyclic p-group.

Theorem 4.3.5 (Theorem 5.9 of [22]). *Let $G = \langle \epsilon \rangle$ be a cyclic group and $A = (UT(A_1, A_2), \widetilde{\alpha})$. If $(A_i, \widetilde{\alpha}_i)$ is $\widetilde{\alpha}_i$-regular, for all $i \in [1, 2]$, then the following statements are equivalent:*

(*i*) *The* T_G*-ideal* $\mathrm{Id}_G(A)$ *is factorable;*

(*ii*) $\mathcal{H}^{(1)}_{\widetilde{\alpha}}\mathcal{H}^{(2)}_{\widetilde{\alpha}} = G$;

(*iii*) *For all* $\widetilde{\alpha}$*-admissible* G*-grading* $\widetilde{\beta}$*, the algebra* $(UT(A_1, A_2), \widetilde{\beta})$ *is graded-isomorphic to* A.

Proof. First, by invoking Proposition 4.1.6, it follows that items (i) and (ii) are equivalent. Now, assume that statement (ii) holds and let $B = (UT(A_1, A_2), \widetilde{\beta})$ such that $\widetilde{\beta}$ is $\widetilde{\alpha}$-admissible. Let us prove that B is graded-isomorphic to A and, hence, we obtain item (iii).

Since $(A_i, \widetilde{\alpha}_i)$ is $\widetilde{\alpha}_i$-regular, for all $i \in [1, 2]$, one has, from Proposition 3.3.4, that $(A_i, \widetilde{\beta}_i)$ is $\widetilde{\beta}_i$-regular, for all $i \in [1, 2]$. Thus, by applying Proposition 3.2.1 and Theorem 3.3.2, it follows that there exist elements $g_{\widetilde{\alpha}_1}, g_{\widetilde{\beta}_1}, g_{\widetilde{\alpha}_2}, g_{\widetilde{\beta}_2} \in G$ such that, for each $i \in [1, 2]$, $\mathcal{I}_{\widetilde{\alpha}_i} = g_{\widetilde{\alpha}_i}\mathcal{H}^{(i)}_{\widetilde{\alpha}}$ and $\mathcal{I}_{\widetilde{\beta}_i} = g_{\widetilde{\beta}_i}\mathcal{H}^{(i)}_{\widetilde{\beta}}$, still $\mathcal{H}^{(i)}_{\widetilde{\alpha}} = \mathcal{H}^{(i)}_{\widetilde{\beta}}$. Suppose, without loss of generality, that $g_{\widetilde{\alpha}_1} = g_{\widetilde{\beta}_1} = 1_G$.

From Proposition 3.3.4, for any elements $\bar{g}_1 \in \mathcal{H}^{(1)}_{\widetilde{\alpha}}$ and $\bar{g}_2 \in g_{\widetilde{\beta}_2}\mathcal{H}^{(2)}_{\widetilde{\alpha}}g^{-1}_{\widetilde{\alpha}_2}$, we have, for each $i \in [1, 2]$, $w^{(i)}_{\widetilde{\beta}}(\bar{g}_i g) = w^{(i)}_{\widetilde{\alpha}}(g)$, for all $g \in G$. Since $\mathcal{H}^{(1)}_{\widetilde{\alpha}}\mathcal{H}^{(2)}_{\widetilde{\alpha}} = G$, there exist $h_1 \in \mathcal{H}^{(1)}_{\widetilde{\alpha}}, h_2 \in \mathcal{H}^{(2)}_{\widetilde{\alpha}}$ such that $g_{\widetilde{\beta}_2}g^{-1}_{\widetilde{\alpha}_2} = h_1h_2$ and thus

$$h_1 = g_{\widetilde{\beta}_2}h_2^{-1}g^{-1}_{\widetilde{\alpha}_2} \in \mathcal{H}^{(1)}_{\widetilde{\alpha}} \cap g_{\widetilde{\beta}_2}\mathcal{H}^{(2)}_{\widetilde{\alpha}}g^{-1}_{\widetilde{\alpha}_2}.$$

Therefore, it follows that $w^{(l)}_{\widetilde{\beta}}(h_1 g) = w^{(l)}_{\widetilde{\alpha}}(g)$, for all $l \in [1, 2]$ and $g \in G$. Finally, such equality yields that B is graded-isomorphic to A (see Lemma 4.3.2).

Conversely, if item (iii) is valid, then, by Proposition 4.3.1, one has that $\mathcal{H}^{(1)}_{\widetilde{\alpha}}\mathcal{H}^{(2)}_{\widetilde{\alpha}} = G$. □

We finish this section by remarking that if G is not a p-group, then Theorem 4.3.4 is not valid. More precisely, items (i) and (iii) of Theorem 4.3.4 may not be equivalent to item (ii). Indeed, assume, for instance, that $G = C_6$, a cyclic group of order 6. Let $A_1 = (D_2, \widetilde{\alpha}_1)$ and $A_2 = (D_3, \widetilde{\alpha}_2)$, where

$$(\widetilde{\alpha}_1(1), \widetilde{\alpha}_1(2)) = (1_G, \epsilon^3) \quad \text{and} \quad (\widetilde{\alpha}_2(1), \widetilde{\alpha}_2(2), \widetilde{\alpha}_2(3)) = (1_G, \epsilon^2, \epsilon^4).$$

Moreover, consider $A = (UT(A_1, A_2), \widetilde{\alpha})$.

It is easy to verify that

$$\mathcal{I}_{\widetilde{\alpha}_1} = \mathcal{H}^{(1)}_{\widetilde{\alpha}} = \langle \epsilon^3 \rangle \quad \text{and} \quad \mathcal{I}_{\widetilde{\alpha}_2} = \mathcal{H}^{(2)}_{\widetilde{\alpha}} = \langle \epsilon^2 \rangle.$$

This means that the G-simple algebras $(A_i, \widetilde{\alpha}_i)$ are $\widetilde{\alpha}_i$-regular, but not G-regular, for all $i \in [1, 2]$ (see Theorem 3.3.2 and Corollary 3.3.3). Finally, once $\mathcal{H}^{(1)}_{\widetilde{\alpha}}\mathcal{H}^{(2)}_{\widetilde{\alpha}} = G$, by invoking Theorem 4.3.5, one has that the T_G-ideal $\mathrm{Id}_G(A)$ is factorable and for all $\widetilde{\alpha}$-admissible G-grading $\widetilde{\beta}$, the algebra $(UT(A_1, A_2), \widetilde{\beta})$ is graded-isomorphic to A.

Chapter 5

Minimal varieties and the algebras $UT_{C_n}(A_1, \ldots, A_m)$

Throughout this chapter, F will denote an algebraically closed field of characteristic zero. Moreover, we consider ϵ a primitive nth root of the unity in F^* and $G = \langle \epsilon \rangle = C_n$, the cyclic group generated by ϵ. We dedicate the last chapter of this thesis to studying the minimal varieties of G-graded PI-algebras of finite basic rank, with respect to a given G-exponent. We will show that they are generated by suitable G-graded upper block triangular matrix algebras. Moreover, given finite dimensional G-simple F-algebras $A_1, \ldots, A_m$, let $A := UT_G(A_1, \ldots, A_m)$. By imposing some conditions on A, we will prove that, in this case, $\mathrm{var}_G(A)$ is minimal. The new results established in this chapter count with the collaboration of Professor Viviane Ribeiro Tomaz da Silva and are in the paper [31] submitted for publication.

5.1 Minimal C_n-graded algebras and minimal varieties

In this section, we will prove that any minimal variety of G-graded PI-algebras of finite basic rank, of a given G-exponent, is generated by a suitable G-graded upper block triangular matrix algebra. To this end, we fix an m-tuple $(A_1, \ldots, A_m)$ of finite dimensional G-simple F-algebras and we consider the G-graded upper block triangular matrix algebra $A := (UT(A_1, \ldots, A_m), \widetilde{\alpha})$ (as in Section 4.1). Let us start proving that A is a minimal G-graded algebra.

Proposition 5.1.1 (Proposition 4.3 of [31]). *Let $G = \langle \epsilon \rangle$ be a cyclic group. The G-graded upper block triangular matrix algebra $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$ is a minimal G-graded algebra, whose lth G-simple component of its maximal semisimple graded subalgebra is isomorphic to $(M_{k_l}(D_{r_l}), \widetilde{\alpha}_l)$.*

Proof. If $m = 1$, then A is a G-simple algebra and we are done.

Suppose $m \geq 2$. In this case, for each $l \in [1, m]$, it is enough to take the minimal homogenenous idempotents as

$$e_l := (e_{11} \otimes E^0)^{(l,l)} = \overline{\mathbf{E}}_{11}^{(l,l)} = \mathbf{E}_{11}^{(l,l)} + \cdots + \mathbf{E}_{r_l r_l}^{(l,l)}$$

and, for each $l \in [1, m-1]$, take the homogeneous radical elements as

$$w_{l,l+1} := \mathbf{E}_{11}^{(l,l+1)}.$$

□

Let $A = A_{ss} + J(A)$ be a minimal G-graded algebra, where its maximal semisimple subalgebra $A_{ss} = A_1 \oplus \cdots \oplus A_m$, with $A_1, \ldots, A_m$ being G-simple algebras, and $J(A)$, the Jacobson radical of A, is a graded ideal. For each $i \in [1, m]$, there exist positive integers k_i and r_i such that A_i is graded-isomorphic to a graded subalgebra of $M_{k_i r_i}$, endowed with an elementary grading given by a suitable map $\widetilde{\alpha}_i : [1, k_i r_i] \to G$ (see Theorem 3.1.3).

Consider the G-graded upper block triangular matrix algebra $(UT(A_1, \ldots, A_m), \widetilde{\alpha})$. By using the same notations for the homogeneous radical elements, which appear in Definition 1.5.2, define the map

$$\begin{array}{rccc} \widetilde{\alpha}_A : & [1, \eta_m] & \to & G \\ & i & \mapsto & |w_{12} w_{23} \cdots w_{l-1,l}|_A \widetilde{\alpha}_l(1)^{-1} \widetilde{\alpha}(i), \end{array}$$

where $l \in [1, m]$ is the unique integer such that $i \in \mathbf{Bl}_l$ and $|w_{01}|_A := 1_G$.

In the sequel, we shall assume that $UT(A_1, \ldots, A_m)$ is endowed with the grading induced by the map $\widetilde{\alpha}_A$. In this case, let us denote such graded algebra by $(UT(A_1, \ldots, A_m), \widetilde{\alpha}_A)$, where the index A emphasizes that the grading on $UT(A_1, \ldots, A_m)$ depends of that of A.

Definition 5.1.2. The G-graded algebra $(UT(A_1, \ldots, A_m), \widetilde{\alpha}_A)$ is said to be *the upper block triangular matrix algebra related to the minimal G-graded algebra $A = A_1 \oplus \cdots \oplus A_m + J(A)$.*

Now, we can state the following result which relates the varieties generated by a minimal G-graded algebra $A = A_1 \oplus \cdots \oplus A_m + J(A)$ and by $(UT(A_1, \ldots, A_m), \widetilde{\alpha}_A)$.

Proposition 5.1.3 (Proposition 4.8 of [31])**.** *Let $G = \langle \epsilon \rangle$ be a cyclic group and $A = A_{ss} + J(A)$ be a minimal G-graded algebra such that $A_{ss} = A_1 \oplus \cdots \oplus A_m$. Then $(UT(A_1, \ldots, A_m), \widetilde{\alpha}_A)$ belongs to* $\mathrm{var}_G(A)$*. In particular, if* $\mathrm{var}_G(A)$ *is minimal, then*

$$\mathrm{var}_G(A) = \mathrm{var}_G((UT(A_1, \ldots, A_m), \widetilde{\alpha}_A)).$$

Proof. First, we write $\mathcal{A} := (UT(A_1, \ldots, A_m), \widetilde{\alpha}_A)$. In order to conclude the result it is enough to show that $\mathrm{Id}_G(A) \subseteq \mathrm{Id}_G(\mathcal{A})$. To this end, let us apply the process of induction on m.

If $m = 1$, then $\mathcal{A} = UT(A_1)$ is graded-isomorphic to $A_1 = A$ and we are done.

Assume that $m \geq 2$ and suppose that, for every $d \in [0, m-1]$, one has $\mathrm{Id}_G(A^{[1,d]}) \subseteq \mathrm{Id}_G(\mathcal{A}^{[1,d]})$ (remember that the algebras $A^{[1,d]}$ and $\mathcal{A}^{[1,d]}$ were defined in Section 1.5). In this case, let us to prove that the inclusion $F\langle X;G\rangle\backslash\mathrm{Id}_G(\mathcal{A}) \subseteq F\langle X;G\rangle\backslash\mathrm{Id}_G(A)$ is valid. Take a polynomial $f = f(x_1^{g_1}, \ldots, x_p^{g_p}) \in F\langle X;G\rangle\backslash\mathrm{Id}_G(\mathcal{A})$. Since $\mathrm{char}F = 0$ we can assume that f is multilinear. Moreover, there exist elements $b_1, \ldots, b_p$, in the canonical basis of $\mathcal{A}$, with $|b_i|_{\mathcal{A}} = g_i$, for all $i \in [1,p]$, such that $f(b_1, \ldots, b_p) \neq 0_{\mathcal{A}}$.

Considere $\ell := |\{i \mid b_i \in J(\mathcal{A}),\ i \in [1,p]\}|$. The fact that $J(\mathcal{A})$ is a nilpotent ideal of index $m-1$, yields us $\ell \leq m-1$.

First, let us study when $\ell < m-1$. In this case, there exists $i \in [1, m-1]$ such that, for every $j \in [i+1, m]$, follows that $b_l \notin \mathcal{A}_{i,j}$, for all $l \in [1,p]$.

Notice that, if there exists $q \in [1,p]$ such that $b_q \in \mathcal{A}_{u,i}$, for some $u \in [1,i]$, thus the elements $b_1, \ldots, b_p$ are in

$$\bigoplus_{1 \leq u \leq v \leq i} \mathcal{A}_{u,v} \cong UT(A_1 \ldots, A_i)$$

with the induced G-graded. Otherwise, the elements $b_1, \ldots, b_p$ are in

$$\bigoplus_{\substack{1 \leq u \leq v \leq m \\ u \neq i \neq v}} \mathcal{A}_{u,v} \cong UT(A_1 \ldots, A_{i-1}, A_{i+1}, \ldots, A_m)$$

with the induced G-graded. Hence, either

$$f \notin \mathrm{Id}_G(UT(A_1 \ldots, A_i)) \quad \text{or} \quad f \notin \mathrm{Id}_G(UT(A_1 \ldots, A_{i-1}, A_{i+1}, \ldots, A_m)).$$

In both cases, once the G-graded algebras $UT(A_1 \ldots, A_i)$ and $UT(A_1 \ldots, A_{i-1}, A_{i+1}, \ldots, A_m)$ are related, respectively, to the graded subalgebras $A^{[1,i]}$ and $A^{(\hat{i})}$ of A (see the notation introduced in Section 1.5), we conclude, by the induction hypotheses, that $f \in F\langle X;G\rangle\backslash\mathrm{Id}_G(A)$, as desired.

Now, assume that $\ell = m-1$. Then there exist $t_1, \ldots, t_{m-1} \in [1,p]$ such that

$$b_{t_1} = \mathbf{E}_{i_1 j_2}^{(1,2)}, \ldots, b_{t_{m-1}} = \mathbf{E}_{i_{m-1} j_m}^{(m-1,m)},$$

where $i_l \in [1, k_l r_l]$ and $j_{l+1} \in [1, k_{l+1} r_{l+1}]$, for all $l \in [1, m-1]$, and all the elements of the set $\{b_1, \ldots, b_p\}\backslash\{b_{t_1}, \ldots, b_{t_{m-1}}\}$ are in the diagonal blocks of $\mathcal{A}$. Since f is multilinear, by invoking Lemma 4.1.2, one has that

$$f(b_1, \ldots, b_p) = \gamma \mathbf{E}_{ij}^{(1,m)}$$

for some $i \in [1, k_1 r_1]$, $j \in [1, k_m r_m]$ and $\gamma \in F^*$. Assume, without loss of generality, that

$b := b_1 \cdots b_p = \mathbf{E}_{ij}^{(1,m)}$. Still, by setting $j_1 := i$ and $i_m := j$, let us consider $b_0 := \overline{\mathbf{E}}_{1j_1}^{(1,1)}$ and $b_{p+1} := \overline{\mathbf{E}}_{i_m1}^{(m,m)}$. Thus, by denoting $t_0 := 0$ and $t_m := p+1$, it follows that

$$b_{t_{l-1}+1} \cdots b_{t_l-1} = \overline{\mathbf{E}}_{j_l i_l}^{(l,l)}, \quad \text{for all } l \in [1,m],$$

and

$$b_0 f(b_1,\ldots,b_p) b_{p+1} = \overline{\mathbf{E}}_{1i}^{(1,1)} (\gamma \mathbf{E}_{ij}^{(1,m)}) \overline{\mathbf{E}}_{j1}^{(m,m)} = \gamma \mathbf{E}_{11}^{(1,m)}.$$

At this point, for each $l \in [1, m-1]$, consider $v_l \in A_l$ and $z_{l+1} \in A_{l+1}$ the elements corresponding to $\overline{\mathbf{E}}_{i_l1}^{(l,l)}$ and $\overline{\mathbf{E}}_{1j_{l+1}}^{(l+1,l+1)}$ in the graded isomorphisms $A_l \cong \mathcal{A}_{l,l}$ and $A_{l+1} \cong \mathcal{A}_{l+1,l+1}$, respectively (see Proposition 5.1.1). Define

$$a_{t_l} := v_l w_{l,l+1} z_{l+1},$$

where $w_{l,l+1}$ is the lth homogeneous radical element of A. Then, it follows that, for each $l \in [1, m-1]$,

$$|a_{t_l}|_A = |v_l|_A |w_{l,l+1}|_A |z_{l+1}|_A = |\overline{\mathbf{E}}_{i_l1}^{(l,l)}|_{\mathcal{A}} |\mathbf{E}_{11}^{(l,l+1)}|_{\mathcal{A}} |\overline{\mathbf{E}}_{1j_{l+1}}^{(l+1,l+1)}|_{\mathcal{A}} = |\mathbf{E}_{i_l j_{l+1}}^{(l,l+1)}|_{\mathcal{A}} = |b_{t_l}|_{\mathcal{A}}.$$

Thus, one has that $b_i \in \mathcal{A}_{l,l}$, for all $i \in [t_{l-1}+1, t_l-1]$. Similarly to what was done above, let us consider $a_i \in A_{l,l}$ to be the element corresponding to $b_i \in \mathcal{A}_{l,l}$, $z_1 := a_0$ corresponding to b_0 in $\mathcal{A}_{1,1}$ and $v_m := a_{p+1}$ corresponding to b_{p+1} in $\mathcal{A}_{m,m}$.

Now, we remark that, for all $\pi \in \mathrm{Sym}(p)$, it is valid

$$a_0 a_{\pi(1)} \cdots a_{\pi(p)} a_{p+1} \neq 0_A \text{ if, and only if, } b_{\pi(1)} \cdots b_{\pi(p)} \neq 0_{\mathcal{A}}.$$

Indeed, let us suppose first that $a_0 a_{\pi(1)} \cdots a_{\pi(p)} a_{p+1} \neq 0_A$. In this case, $\pi(t_l) = t_l$, for all $l \in [1, m-1]$; and, for each $l' \in [1,m]$, if $t_{l'-1} < l < t_{l'}$, then $t_{l'-1} < \pi(l) < t_{l'}$. Therefore

$$\begin{aligned}
0_A \neq a_0 a_{\pi(1)} \cdots a_{\pi(p)} a_{p+1} &= z_1 a_{\pi(1)} \cdots a_{\pi(p)} v_m \\
&= z_1 a_{\pi(1)} \cdots a_{\pi(t_1-1)} a_{t_1} a_{\pi(t_1+1)} \cdots a_{\pi(t_2-1)} a_{t_2} \cdots a_{t_{m-1}} a_{\pi(t_{m-1}+1)} \cdots a_{\pi(p)} v_m \\
&= z_1 a_{\pi(1)} \cdots a_{\pi(t_1-1)} v_1 w_{12} z_2 a_{\pi(t_1+1)} \cdots a_{\pi(t_2-1)} v_2 w_{23} z_3 \cdots v_{m-1} w_{m-1,m} z_m a_{\pi(t_{m-1}+1)} \cdots a_{\pi(p)} v_m.
\end{aligned}$$

Such fact implies the following equivalent statements:

(i) $z_l a_{\pi(t_{l-1}+1)} \cdots a_{\pi(t_l-1)} v_l \neq 0_A, \quad$ for all $l \in [1,m]$;

(ii) $\overline{\mathbf{E}}_{1j_l}^{(l,l)} b_{\pi(t_{l-1}+1)} \cdots b_{\pi(t_l-1)} \overline{\mathbf{E}}_{i_l1}^{(l,l)} \neq 0_{\mathcal{A}}, \quad$ for all $l \in [1,m]$;

(iii) $b_{\pi(t_{l-1}+1)} \cdots b_{\pi(t_l-1)} = \overline{\mathbf{E}}^{(l,l)}_{j_l i_l}$, for all $l \in [1, m]$;

(iv) $\overline{\mathbf{E}}^{(1,1)}_{1j_1} b_{\pi(1)} \cdots b_{\pi(t_1-1)} \mathbf{E}^{(1,2)}_{i_1 j_2} \cdots \mathbf{E}^{(m-1,m)}_{i_{m-1} j_m} b_{\pi(t_{m-1}+1)} \cdots b_{\pi(p)} \overline{\mathbf{E}}^{(m,m)}_{i_m 1} = \mathbf{E}^{(1,m)}_{11}$;

(v) $b_{\pi(1)} \cdots b_{\pi(p)} \neq 0_{\mathcal{A}}$.

Reciprocally, if $b_{\pi(1)} \cdots b_{\pi(p)} \neq 0_{\mathcal{A}}$, thus by using the above statements $(i) - (v)$, it follows that $z_l a_{\pi(t_{l-1}+1)} \cdots a_{\pi(t_l-1)} v_l \neq 0_A$, for all $l \in [1, m]$. Once, from Proposition 5.1.1, for each $l \in [1, m]$, the minimal homogeneous idempotent $e_l \in A_l$ corresponds to $\overline{\mathbf{E}}^{(l,l)}_{11}$, we have that the product $z_l a_{\pi(t_{l-1}+1)} \cdots a_{\pi(t_l-1)} v_l$ coincides with e_l, for all $l \in [1, m]$. Hence

$$a_0 a_{\pi(1)} \cdots a_{\pi(p)} a_{p+1} = z_1 a_{\pi(1)} \cdots a_{\pi(p)} v_m = e_1 w_{12} e_2 \cdots e_{m-1} w_{m-1,m} e_m = w_{12} \cdots w_{m-1,m} \neq 0_A,$$

as desired.

Therefore, by applying the previous claim, we can conclude that $a_0 f(a_1, \ldots, a_p) a_{p+1} \neq 0_A$, and this implies in $f \in F\langle X; G\rangle \setminus \mathrm{Id}_G(A)$. Then, in case $\ell = m - 1$, we conclude also that $\mathcal{A} \in \mathrm{var}_G(A)$.

Finally, the fact that $\exp_G(A) = \exp_G(\mathcal{A})$ guarantees us $\mathrm{var}_G(A) = \mathrm{var}_G(\mathcal{A})$, in case $\mathrm{var}_G(A)$ is minimal, and the proof is completed. □

We finish this section by presenting the following important result:

Theorem 5.1.4 (Theorem 4.9 of [31]). *Let $G = \langle \epsilon \rangle$ be a cyclic group and $\mathcal{V}^G$ be a variety of G-graded PI-algebras of finite basic rank. If $\mathcal{V}^G$ is minimal of G-exponent d, then it is generated by a G-graded upper block triangular matrix algebra $UT_G(A_1, \ldots, A_m)$ such that $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.*

Proof. It is enough to apply Theorem 1.5.6 and Proposition 5.1.3. □

5.2 Kemer polynomials for the algebras $UT_{C_n}(A_1, \ldots, A_m)$

The so-called *Kemer polynomials*, seen in Section 1.3, are important tools in the solution of many problems of PI-theory. Fixed an m-tuple $(A_1, \ldots, A_m)$ of finite dimensional G-simple F-algebras, let us consider $A := UT_G(A_1, \ldots, A_m)$ (as in Section 4.1). In this section, our main aim is constructing such Kemer polynomials for the G-graded algebra A.

First, in order to simplify the notation, for each $l \in [1, m]$ and $g \in G$, let us define

$$d^A_l := \dim_F A_l, \quad d^A_{l,g} := \dim_F (A_l)_g \quad \text{and} \quad d^A_{ss,g} := \dim_F (A_{ss})_g = \sum_{l \in [1,m]} d^A_{l,g}.$$

At this moment, we presented some preliminary constructions involving the product of the canonical basis elements of A.

Assume that $m = 1$. In this case, A is a G-simple algebra and, by invoking Theorem 3.1.3, A is graded-isomorphic to $M_k(D_r) \subseteq M_{kr}$.

We consider, firstly, the case $r = 1$, that is, $A \cong_G M_k$. We remark that it is possible to write the canonical element e_{11} from a product of the k^2 distinct canonical basis elements of A of the form e_{ij}. Fixed a such product, we shall refer to it as the *standard total product* (of basis elements) of A.

Now, assume that $r \geq 2$. For each $t \in [0, r-1]$, we obtain a product of all the k^2 distinct basis elements of A of the form $e_{ij} \otimes E^t$ resulting in $e_{11} \otimes (E^t)^{k^2}$, where the elements e_{ij} compose the standard total product of M_k. Realizing this same process for all $t \in [0, r-1]$, we obtain the following product from all the rk^2 distinct canonical basis elements of A:

$$\begin{aligned}\Pi_{t\in[0,r-1]}(e_{11} \otimes (E^t)^{k^2}) &= e_{11} \otimes (E^{\sum_{t\in[0,r-1]} t})^{k^2} \\ &= \begin{cases} e_{11} \otimes E^{r/2} & \text{if } r \text{ is even and } k \text{ is odd,} \\ e_{11} \otimes E^0 & \text{otherwise.} \end{cases}\end{aligned}$$

We also refer to this product as the *standard total product* (of basis elements) of A. Moreover, we can write $e_{11} \otimes E^{r/2} = \overline{\mathbf{E}}_{1,\frac{r}{2}+1}$ and $e_{11} \otimes E^0 = \overline{\mathbf{E}}_{11}$.

Now, let us define a suitable monomial of $F\langle X; G\rangle$, where all its variables are distinct, constructed in a such way that each element appearing in the standard total product of A is replaced by a variable of X_G of the same degree. We denote such monomial by m_A. Observe that m_A has rk^2 variables. If we evaluate in m_A the same canonical basis elements of A which were used for its construction, then we say that such an evaluation is *standard total* and we denote it by $\overline{m}_A$.

For any $\iota \geq m \geq 1$ and $l \in [1, m]$, consider ι copies of m_{A_l} in pairwise disjoint sets of graded variables. For each $i \in [1, \iota]$, denote by $m_{A_l}^{(i)}$ the ith copy of m_{A_l}. Moreover, we denote by $S(l, i)$ the set of the variables of $m_{A_l}^{(i)}$ and by $S(l, i, g)$ the set of the variables of degree g in $S(l, i)$. Observe that $S(l, i) = \cup_{g\in G} S(l, i, g)$ and

$$|S(l, i)| = d_l^A \quad \text{and} \quad |S(l, i, g)| = d_{l,g}^A.$$

For all $i \in [1, \iota]$ and $g \in G$, define $T(i, g) := \cup_{l\in[1,m]} S(l, i, g)$ and, thus, it follows that

$$|T(i, g)| = \sum_{l\in[1,m]} d_{l,g}^A = d_{ss,g}^A.$$

We observe that

$$\overline{m}_{A_l}^{(1)}\cdots\overline{m}_{A_l}^{(\iota)} = \begin{cases} \overline{\mathbf{E}}_{1,\frac{r_l}{2}+1}^{(l,l)} & \text{if } r_l \text{ is even and } k_l \text{ and } \iota \text{ are both odd,}\\ \overline{\mathbf{E}}_{11}^{(l,l)} & \text{otherwise.}\end{cases}$$

In first case we shall say that (l,ι,r_l) *is an exception.*

Now, for each $j\in[1,m-1]$, take the homogeneous radical element $\mathbf{E}_{\frac{r_j}{2}+1,1}^{(j,j+1)}$ of A if (j,ι,r_j) is an exception, and $\mathbf{E}_{11}^{(j,j+1)}$, otherwise. Let us denote the homogeneous degree of such radical element by g_j. Now, consider a variable z_j with degree g_j such that the set $(\cup_{i\in[1,\iota],l\in[1,m]}S(l,i))\cup(\cup_{j\in[1,m-1]}\{z_j\})$ is formed by elements which are all distinct. Define $Z_j := T(j,g_j)\cup\{z_j\}$ and, clearly, $|Z_j| = d_{ss,g_j}^{A}+1$. Still, setting

$$\pi_{A,\iota} := m_{A_1}^{(1)}\cdots m_{A_1}^{(\iota)}z_1 m_{A_2}^{(1)}\cdots m_{A_2}^{(\iota)}z_2\cdots z_{m-1}m_{A_m}^{(1)}\cdots m_{A_m}^{(\iota)},$$

it is easy to observe that there exists a graded evaluation of $\pi_{A,\iota}$ by canonical basis elements of A, giving $\mathbf{E}_{1,\frac{r_m}{2}+1}^{(1,m)}$ if (m,ι,r_m) is an exception, and $\mathbf{E}_{11}^{(1,m)}$, otherwise.

Consider the monomial $\widetilde{\pi}_{A,\iota}$ obtained from $\pi_{A,\iota}$ by putting $\iota(\dim_F A_{ss})+m$ pairwise different variables of degree 1_G, which do not appear in $\pi_{A,\iota}$, bordering each variable of $\pi_{A,\iota}$.

For each $l\in[1,m]$ and $i\in[1,\iota]$, define $Y(l,i)$ to be the set of all the variables which were placed on the left of the variables of the set $S(l,i)$, and consider $\widetilde{y}_l$ the variable placed on the right of the monomial $m_{A_l}^{(\iota)}$. Finally, for each $l\in[1,m]$, define $Y_l := \cup_{i\in[1,\iota]}Y(l,i)\cup\{\widetilde{y}_l\}$, and for each $j\in[1,m-1]$ we alternate in the monomial $\widetilde{\pi}_{A,\iota}$ the variables of the set Z_j and, for each $i\in[m,\iota]$ and $g\in G$, those of $T(i,g)$, respectively.

When we finish this process, let us denote by $f_{A,\iota}$ the graded polynomial obtained and we will call it the *Kemer polynomial* for A. Actually, we will show that $f_{A,\iota}$ is not a graded identity for $A = UT_G(A_1,\ldots,A_m)$ and thus $G-\mathrm{Par}(A)$, defined in Section 1.3, is a Kemer point of A and, hence, is the unique Kemer point of A by Corollary 1.3.7.

Lemma 5.2.1 (Lemma 5.1 of [31]). *Let $G=\langle\epsilon\rangle$ be a cyclic group and $A = UT_G(A_1,\ldots,A_m)$. For every $\iota\geq m$ the graded polynomial $f_{A,\iota}$ is not a G-graded polynomial identity for the G-graded algebra A.*

Proof. First, for all $l\in[1,m]$ e $i\in[1,\iota]$, let us consider the standard total evaluation $\overline{S(l,i)}$ of the monomial $m_{A_l}^{(i)}$ in A.

We remark that, for each variable $v_a^{(i)}\in S(l,i)$, it is valid $\overline{v}_a^{(i)} = (e_{pq}\otimes E^t)^{(l,l)}$, for some $p,q\in[1,k_l]$ and $t\in[0,r_l-1]$. Thus, evaluate the variable $y_a^{(i)}\in Y(l,i)$, appearing on the left of $v_a^{(i)}$, by $(e_{pp} \otimes E^0)^{(l,l)}$. Since the evaluation $\overline{y}_1^{(i)}\overline{v}_1^{(i)} \cdots \overline{y}_{d_l^A}^{(i)}\overline{v}_{d_l^A}^{(i)}$ is equal to

$$\begin{cases} (e_{11} \otimes E^{\frac{r_l}{2}})^{(l,l)} &= \overline{\mathbf{E}}_{1,\frac{r_l}{2}+1}^{(l,l)} & \text{if } r_l \text{ is even and } k_l \text{ is odd,} \\ (e_{11} \otimes E^0)^{(l,l)} &= \overline{\mathbf{E}}_{11}^{(l,l)} & \text{otherwise,} \end{cases}$$

we evaluate the variable $\widetilde{y}_l$ by $\overline{\mathbf{E}}_{\frac{r_l}{2}+1,\frac{r_l}{2}+1}^{(l,l)}$, if r_l is even and k_l is odd, and by $\overline{\mathbf{E}}_{11}^{(l,l)}$, otherwise. Notice that $\overline{\mathbf{E}}_{\frac{r_l}{2}+1,\frac{r_l}{2}+1}^{(l,l)} = \overline{\mathbf{E}}_{11}^{(l,l)} = (e_{11} \otimes E^0)^{(l,l)}$. Finally, for all $j \in [1, m-1]$, consider $\overline{z}_j = \mathbf{E}_{11}^{(j,j+1)}$, if (j, ι, r_j) is not an exception, and $\overline{z}_j = \mathbf{E}_{\frac{r_j}{2}+1,1}^{(j,j+1)}$, otherwise. Therefore, we have an evaluation of $\widetilde{\pi}_{A,\iota}$ in A being:

$$\begin{cases} \mathbf{E}_{11}^{(1,m)} & \text{if } (m, \iota, r_m) \text{ is not an exception,} \\ \mathbf{E}_{1,\frac{r_m}{2}+1}^{(1,m)} & \text{otherwise.} \end{cases}$$

Denote such evaluation by $\overline{S}_A$.

Given $i \in [m, \iota]$ and $g \in G$, consider a permutation σ of the variables of $\widetilde{\pi}_{A,\iota}$ which possibly moves only the variables of $T(i, g)$. It is valid that, if the evaluation of the monomial $\sigma(\widetilde{\pi}_{A,\iota})$ in A by $\overline{S}_A$ is non-zero, then σ is the identity permutation. In fact, we notice first that $aa' = 0_A$ for all $a \in A_l$ and $a' \in A_{l'}$, with $l \neq l'$, and $z_1, \ldots, z_{m-1}$ are not moved by σ. Hence, σ permutes only the variables of the set $S(l, i, g)$, for each $l \in [1, m]$. Still, σ does not move the variables of Y_l. In other words, in each monomial of $f_{A,\iota}$ the variables of the set Y_l appear in the same order. This implies that, once we have fixed, by the above choice, the elements in $\overline{Y}_l$, then, by using the fact that the evaluation of the monomial $\sigma(\widetilde{\pi}_{A,\iota})$ in A by $\overline{S}_A$ is non-zero, it follows that the evaluation $\overline{v}_a^{(i)}$ is uniquely determined by such choice of elements in $\overline{Y}_l$, as well the homogeneous degree of $v_a^{(i)}$. Consequently, this discussion guarantees us that σ is the identity permutation.

Moreover, given $j \in [1, m-1]$, we can argue analogously and obtain also that, if ν is a non-trivial permutation of the variables of Z_j in $\widetilde{\pi}_{A,\iota}$, then the evaluation $\nu(\widetilde{\pi}_{A,\iota})$ by $\overline{S}_A$ is zero. Consequently, $\widetilde{\pi}_{A,\iota}$ is the unique monomial of $f_{A,\iota}$ which is non-zero under the evaluation by $\overline{S}_A$, and this implies that $f_{A,\iota} \notin \mathrm{Id}_G(A)$, as desired. □

5.3 Minimal varieties of C_n-graded PI-algebras

Let $\mathcal{V}^G$ be a variety of G-graded PI-algebras of finite basic rank. We stated in Theorem 5.1.4 that if $\mathcal{V}^G$ is minimal of G-exponent d, then $\mathcal{V}^G$ is generated by a suitable G-graded algebra $UT_G(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$. On the other hand, in this section, we present some important classes of G-graded upper block triangular matrix algebras

$UT_G(A_1,\dots,A_m)$ which generate minimal varieties. To this end, fix two tuples $(A_1,\dots,A_m)$ and $(B_1,\dots,B_{m'})$ of finite dimensional G-simple F-algebras and consider

$$A = UT_G(A_1,\dots,A_m) \quad \text{and} \quad B = UT_G(B_1,\dots,B_{m'}).$$

In our first result, we will establish some conditions related to the structures of A and B, in case $\exp_G(B) = d^B_{ss} \leq d^A_{ss} = \exp_G(A)$.

Lemma 5.3.1 (Lemma 6.1 of [31])**.** *Let $G = \langle \epsilon \rangle$ be a cyclic group and consider two G-graded upper block triangular matrix algebras $A = UT_G(A_1,\dots,A_m)$ and $B = UT_G(B_1,\dots,B_{m'})$. Assume that $d^B_{ss} \leq d^A_{ss}$ and consider $\iota := m + m' - 1$. If $f_{A,\iota}$ is not a G-graded identity for B, then the following properties hold:*

(i) $d^B_{ss,g} = d^A_{ss,g}$, *for all* $g \in G$;

(ii) $m' = m$;

(iii) $d^B_{l,g} = d^A_{l,g}$, *for all* $l \in [1,m]$ *and* $g \in G$.

Proof. By hypothesis, the multilinear graded polynomial $f_{A,\iota} \notin \mathrm{Id}_G(B)$. Thus we can assume, without loss of generality, that there exists a non-zero graded evaluation $\overline{S}_B$, by canonical basis elements of B, in the monomial $\widetilde{\pi}_{A,\iota}$ of $f_{A,\iota}$.

It is easy to check that, since $J(B)$ is nilpotent of index m', there exists $\ell \in [m,\iota]$ such that all the variables of the sets $\cup_{g\in G} T(\ell,g) = \cup_{l\in[1,m]} S(l,\ell)$ and $\cup_{l\in[1,m]} Y(l,\ell)$ are evaluated only by semisimple elements in $\overline{S}_B$. Thus, once $f_{A,\iota}$ alternates the variables in the set $T(\ell,g)$, for all $g \in G$, one has that $d^A_{ss,g} = |T(\ell,g)| \leq d^B_{ss,g}$, for all $g \in G$. Then

$$d^A_{ss} = \sum_{g\in G} d^A_{ss,g} \leq \sum_{g\in G} d^B_{ss,g} = d^B_{ss} \leq d^A_{ss},$$

which implies that $d^B_{ss,g} = d^A_{ss,g}$, for all $g \in G$.

We remark that $\cup_{g\in G}\overline{T(\ell,g)} = \cup_{l\in[1.m]}\overline{S(l,\ell)}$ is an evaluation of the product $m^{(\ell)}_{A_1}\cdots m^{(\ell)}_{A_m}$ which involves all, and only, the canonical basis elements of B_{ss} and each one of this elements exactly once. Thus, for each $l \in [1,m]$, the monomial $m^{(\ell)}_{A_l}$ must be evaluated in a unique block of $B_{ss} = B_1 \oplus \cdots \oplus B_{m'}$. Consequently, we obtain that $m' \leq m$.

Furthermore, by remembering that, for each $j \in [1, m-1]$, the polynomial $f_{A,\iota}$ alternates in the set Z_j, whose cardinality is $d^A_{ss,g_j} + 1$, by applying item (i) it follows that $|Z_j| = d^A_{ss,g_j} + 1 = d^B_{ss,g_j} + 1$. This implies that we must have at least $m-1$ canonical basis elements of $J(B)$ in $\overline{S}_B$. Since $J(B)$ is nilpotent of index m', we have $m - 1 < m'$ and thus $m \leq m'$. By combining such inequality with $m' \leq m$ we conclude that $m' = m$.

At this stage, we notice that, for all $l \in [1, m]$, the monomial $m_{A_l}^{(\ell)}$ must be necessarily evaluated in B_l. Thus, the fact that $\overline{S(l,\ell)}$ is a total evaluation of $m_{A_l}^{(\ell)}$, by canonical basis elements of B_l, allows us to conclude that, for all $g \in G$, the number of variables in $m_{A_l}^{(\ell)}$ of degree g coincides with the number of canonical basis elements in B_l of degree g, that is, $d_{l,g}^B = d_{l,g}^A$, for all $l \in [1, m]$ and $g \in G$. Hence the proof of the lemma is completed. □

As a consequence, we have the following:

Proposition 5.3.2 (Proposition 6.2 of [31])**.** *Let $G = \langle \epsilon \rangle$. Consider the G-graded upper block triangular matrix algebras $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$ and $B = (UT(B_1, \ldots, B_{m'}), \widetilde{\beta})$ such that $\exp_G(B) = \exp_G(A)$.*

If $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$, then $m' = m$ and $d_l^B = d_l^A$, for all $l \in [1, m]$. Moreover, $\mathrm{Id}_G(B_l) \subseteq \mathrm{Id}_G(A_l)$, for all $l \in [1, m]$, and, consequently, $(B_l, \widetilde{\beta}_l)$ is graded-isomorphic to $(A_l, \widetilde{\alpha}_l)$, for all $l \in [1, m]$.

Proof. Since $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$, it follows, by applying Lemma 5.2.1, that $f_{A,\iota} \notin \mathrm{Id}_G(B)$, for all $\iota \geq m$. Thus, taking $\iota := m + m' - 1$, once $d_{ss}^B = \exp_G(B) = \exp_G(A) = d_{ss}^A$, by Lemma 5.3.1, one has that

$$m' = m \quad \text{and} \quad d_{l,g}^B = d_{l,g}^A, \quad \text{for all } l \in [1, m] \text{ and } g \in G,$$

which implies

$$d_l^B = d_l^A, \quad \text{for all } l \in [1, m].$$

Now, if $m = 1$, then the inclusion $\mathrm{Id}_G(B_1) \subseteq \mathrm{Id}_G(A_1)$ it is clear.

Assume that $m \geq 2$. Consider the graded subalgebras $A^{[1,m-1]}$ and $B^{[1,m-1]}$ of the G-graded algebras A and B, respectively. We claim that $\mathrm{Id}_G(B^{[1,m-1]}) \subseteq \mathrm{Id}_G(A^{[1,m-1]})$. Indeed, let us suppose that there exists a polynomial

$$f_1 \in \mathrm{Id}_G(B^{[1,m-1]}) \setminus \mathrm{Id}_G(A^{[1,m-1]}).$$

Consider the Kemer polynomial $f_{A^{[m-1,m]},2}$, whose variables can be assumed to be pairwise disjoint from those involved in f_1. By invoking Lemma 5.2.1, it follows that

$$f_{A^{[m-1,m]},2} \notin \mathrm{Id}_G(A^{[m-1,m]}).$$

Moreover, it is valid that

$$d_{ss}^{A^{[m-1,m]}} = d_{m-1}^A + d_m^A = d_{m-1}^B + d_m^B > d_m^B = d_{ss}^{B_m},$$

which allows us to conclude, in virtue of Lemma 5.3.1, that

$$f_{A^{[m-1,m]},2} \in \mathrm{Id}_G(B_m).$$

Now, by considering new graded variables x^g, for each $g \in G$, and setting

$$\widetilde{f} := f_1 \left(\sum_{g \in G} x^g \right) f_{A^{[m-1,m]},2},$$

we obtain that $\widetilde{f} \notin \mathrm{Id}_G(A)$. On the other hand, we have that

$$\widetilde{f} \in \mathrm{Id}_G(B^{[1,m-1]})\mathrm{Id}_G(B_m) \subseteq \mathrm{Id}_G(B).$$

By combining the above inclusion with the fact that $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$, we get a contradiction.

Similarly, we conclude that $\mathrm{Id}_G(B^{[2,m]}) \subseteq \mathrm{Id}_G(A^{[2,m]})$. In this way, by applying the above same arguments, we obtain that

$$\mathrm{Id}_G(B^{[l,l']}) \subseteq \mathrm{Id}_G(A^{[l,l']}), \quad \text{for all } 1 \leq l \leq l' \leq m, \tag{5.1}$$

and this implies that $\mathrm{Id}_G(B_l) \subseteq \mathrm{Id}_G(A_l)$, for all $l \in [1, m]$, as desired.

The final part follows in virtue of Theorem 3.2.2, once $d_l^B = d_l^A$, for all $l \in [1, m]$. □

Example 5.3.3. Considere $G = C_4 = \langle \epsilon \rangle$, a cyclic group of order 4, and let $A_1 = (D_2, \widetilde{\alpha}_1)$ and $A_2 = (D_2, \widetilde{\alpha}_2)$, where

$$(\widetilde{\alpha}_1(1), \widetilde{\alpha}_1(2)) = (\widetilde{\alpha}_2(1), \widetilde{\alpha}_2(2)) = (1_G, \epsilon^2).$$

Moreover, consider $A = (UT(A_1, A_2), \widetilde{\alpha})$.

It is easy to verify that

$$\mathcal{H}_{\widetilde{\alpha}}^{(1)} = \mathcal{H}_{\widetilde{\alpha}}^{(2)} = \langle \epsilon^2 \rangle$$

and this implies $\mathcal{H}_{\widetilde{\alpha}}^{(1)} \mathcal{H}_{\widetilde{\alpha}}^{(2)} \neq G$. Then, by Theorem 4.2.3, one has that $\mathrm{Id}_G(A)$ is indecomposable. We claim that $\mathrm{var}_G(A)$ can not be generated by a finite dimensional G-simple algebra.

Indeed, let us suppose that there exists a finite dimensional G-simple algebra A' such that $\mathrm{var}_G(A) = \mathrm{var}_G(A')$. Hence, we have $\mathrm{Id}_G(A) = \mathrm{Id}_G(A')$ and $\exp_G(A) = \exp_G(A')$. Therefore, since $A = (UT(A_1, A_2), \widetilde{\alpha})$ and A' is a G-simple algebra, we obtain a contradiction from Lemma 5.3.1.

At light of Proposition 5.3.2, given two tuples $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_m)$ of finite dimensional G-simple F-algebras, in our next results, we will always assume that

$$A := (UT(A_1, \ldots, A_m), \widetilde{\alpha}) \quad \text{and} \quad B := (UT(B_1, \ldots, B_m), \widetilde{\beta})$$

are such that

$$\exp_G(B) = \exp_G(A) \quad \text{and} \quad \mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A). \tag{5.2}$$

Hence, by invoking also Proposition 3.2.1, for each $l \in [1, m]$, $(B_l, \widetilde{\beta}_l) = (M_{k_l}(D_{r_l}), \widetilde{\beta}_l)$ is graded-isomorphic to $(A_l, \widetilde{\alpha}_l) = (M_{k_l}(D_{r_l}), \widetilde{\alpha}_l)$ and $\mathcal{H}^{(l)}_{\widetilde{\beta}} = \mathcal{H}^{(l)}_{\widetilde{\alpha}}$. Still, let us assume that $(A_l, \widetilde{\alpha}_l)$ and $(B_l, \widetilde{\beta}_l)$ have the following presentations:

$$P_{(A_l, \widetilde{\alpha}_l)} = (r_l; (g_{l1}, \ldots, g_{lk_l})) \quad \text{and} \quad P_{(B_l, \widetilde{\beta}_l)} = (r_l; (\widetilde{g}_{l1}, \ldots, \widetilde{g}_{lk_l})).$$

We remark that, in particular, B is graded-isomorphic to A in case $m = 1$. In the next result, we will show that if $m = 2$, then the above graded algebras B and A are also graded-isomorphic. To this end, the main strategy is guaranteeing that there exists $g \in G$ such that $w^{(l)}_{\widetilde{\beta}}(gx) = w^{(l)}_{\widetilde{\alpha}}(x)$, for all $l \in [1, m]$ and $x \in G$ (see Lemma 4.3.2).

Proposition 5.3.4 (Proposition 6.3 of [31])**.** *Let $G = \langle \epsilon \rangle$ be a cyclic group. Consider the G-graded upper block triangular matrix algebras*

$$A = (UT(A_1, A_2), \widetilde{\alpha}) \quad \textit{and} \quad B = (UT(B_1, B_2), \widetilde{\beta})$$

satisfying $\exp_G(B) = \exp_G(A)$ *and* $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$. *Then B is graded-isomorphic to A.*

Proof. First, let us suppose, without loss of generality, that

$w_{\alpha_1 \odot \widetilde{\epsilon}_{r_1}}(g_{11}) = \max\{w_{\alpha_1 \odot \widetilde{\epsilon}_{r_1}}(h) \mid h \in \mathcal{I}_{\alpha_1 \odot \widetilde{\epsilon}_{r_1}}\}$ and $w_{\alpha_2 \odot \widetilde{\epsilon}_{r_2}}(g_{21}) = \max\{w_{\alpha_2 \odot \widetilde{\epsilon}_{r_2}}(h) \mid h \in \mathcal{I}_{\alpha_2 \odot \widetilde{\epsilon}_{r_2}}\}$.

Set $t_{12} := 1 + k_1^2 + k_2^2$. In virtue of Lemma 4.1.3, there exists an evaluation of the polynomial $Cap_{t_{12}}(x_1, \ldots, x_{t_{12}}; x_{t_{12}+1}, \ldots, x_{2t_{12}+1})$ in the algebra $UT(A_1, A_2)$, at its canonical basis elements, resulting in $\mathbf{E}_{1,\eta_1+1}$. Let us consider the multilinear graded polynomial $Cap_{t_{12}}(u_1, \ldots, u_{t_{12}}; u_{t_{12}+1}, \ldots, u_{2t_{12}+1})$ built in a such way that each homogeneous variable u_i has the degree, induced by $\widetilde{\alpha}$, of the canonical basis elements used in the above evaluation. Then $Cap_{t_{12}}(u_1, \ldots, u_{t_{12}}; u_{t_{12}+1}, \ldots, u_{2t_{12}+1})$ has a graded evaluation in the algebra A equal to $\mathbf{E}^{(1,2)}_{11} = \mathbf{E}_{1,\eta_1+1}$. Since

$$|\mathbf{E}^{(1,2)}_{11}|_A = |\mathbf{E}_{1,\eta_1+1}|_A = \widetilde{\alpha}(1)^{-1}\widetilde{\alpha}(\eta_1 + 1) = g_{11}^{-1} g_{21},$$

one has that $Cap_{t_{12}}(u_1, \ldots, u_{t_{12}}; u_{t_{12}+1}, \ldots, u_{2t_{12}+1})$ has homogeneous degree equal to $g_{11}^{-1} g_{21}$ as an element of $F\langle X;G\rangle$.

Thus, by item (i) of Lemma 3.2.3, there exist homogeneous multilinear polynomials Ψ_{A_1} and Ψ_{A_2}, in pairwise disjoint sets of homogeneous variables (and also distinct from those of the set $\{u_1,\ldots,u_{2t_{12}+1}\}$), with evaluations $\rho_1 : F\langle X;G\rangle \to A$ and $\rho_2 : F\langle X;G\rangle \to A$, such that

$$\rho_1(\Psi_{A_1}) = (e_{11}\otimes E^0)^{(1,1)} = \overline{\mathbf{E}}_{11}^{(1,1)}$$

and

$$\rho_2(\Psi_{A_2}) = (e_{11}\otimes E^0)^{(2,2)} = \overline{\mathbf{E}}_{11}^{(2,2)}.$$

In this way, by setting

$$f := \Psi_{A_1} Cap_{t_{12}}(u_1,\ldots,u_{t_{12}};u_{t_{12}+1},\ldots,u_{2t_{12}+1})\Psi_{A_2},$$

we get that f has homogeneous degree equal to $g_{11}^{-1}g_{21}$ as an element of $F\langle X;G\rangle$ and $f\notin \mathrm{Id}_G(A)$.

At this stage, notice that the hypothesis $\mathrm{Id}_G(B)\subseteq \mathrm{Id}_G(A)$ yields that $f\notin \mathrm{Id}_G(B)$. Any non-zero graded evaluation of the polynomial $Cap_{t_{12}}(u_1,\ldots,u_{t_{12}};u_{t_{12}+1},\ldots,u_{2t_{12}+1})$ in B must give elements of $J(B)$. Hence, the homogeneous multilinear polynomials Ψ_{A_1} e Ψ_{A_2} must be evaluated, respectively, in B_1 and B_2.

Now, from Proposition 5.3.2 and Corollary 3.2.2, it follows that, for each $l\in[1,2]$, there exists an element $\bar{g}_l\in G$ such that

$$w_{\widetilde{\beta}}^{(l)}(\bar{g}_l x) = w_{\widetilde{\alpha}}^{(l)}(x), \quad \text{for all } x\in G. \tag{5.3}$$

In this situation, we consider the new graded algebra $B' = (UT(B_1',B_2'),\widetilde{\beta}')$ such that $B_l' = B_l$ and $\widetilde{\beta}_l' := l_{\bar{g}_l}\cdot\widetilde{\alpha}_l$, for all $l\in[1,2]$. We remark that $w_{\widetilde{\beta}'}^{(l)}(x) = w_{\widetilde{\beta}}^{(l)}(x)$, for all $l\in[1,2]$ and $x\in G$, and by Lemma 4.3.2 it follows that B' is graded-isomorphic to B. Thus, in the sequel, we may assume that $B=B'$, that is,

$$\widetilde{\beta}_l = l_{\bar{g}_l}\cdot\widetilde{\alpha}_l, \quad \text{for all } l\in[1,2].$$

Then, if ρ_1 and ρ_2 are graded evaluations, respectively, of Ψ_{A_1} and Ψ_{A_2} in, respectively, B_1 and B_2 (with the grading induced by $\widetilde{\beta}$), from Remark 3.2.4, such evaluations satisfy

$$\rho_1(\Psi_{A_1}) \in \bigoplus_{i\in\overline{\mathbf{T}}_{A_1};\, g_{1i}\in\mathcal{H}_{\widetilde{\beta}}^{(1)}g_{11}} (B_1)_{1_G}^{(\bar{g}_1 g_{1i})} \quad \text{and} \quad \rho_2(\Psi_{A_2}) \in \bigoplus_{j\in\overline{\mathbf{T}}_{A_2};\, g_{2j}\in\mathcal{H}_{\widetilde{\beta}}^{(2)}g_{21}} (B_2)_{1_G}^{(\bar{g}_2 g_{2j})}.$$

In particular, the evaluation of Ψ_{A_1} results in linear combinations of basis canonical elements

$(e_{u_1v_1} \otimes E^{a_1-b_1})^{(1,1)} \in ((B_1)_{1_G}^{(\bar{g}_1 g_{1i})}, \widetilde{\beta}_1 = \beta_1 \odot \widetilde{\epsilon}_{r_1})$ such that

$$\beta_1(u_1) = \bar{g}_1(\epsilon^{s_1})^{a_1} g_{1i} \text{ and } \beta_1(v_1) = \bar{g}_1(\epsilon^{s_1})^{b_1} g_{1i}, \text{ for some } a_1, b_1 \in [0, r_1 - 1],$$

and once $g_{1i} \in \mathcal{H}_{\widetilde{\beta}}^{(1)} g_{11}$, we have

$$\beta_1(u_1) = \bar{g}_1(\epsilon^{s_1})^{a_1} h_{1i} g_{11} \text{ and } \beta_1(v_1) = \bar{g}_1(\epsilon^{s_1})^{b_1} h_{1i} g_{11}, \text{ for some } h_{1i} \in \mathcal{H}_{\widetilde{\beta}}^{(1)};$$

whereas, one has that, the evaluation of Ψ_{A_2} results in linear combinations of basis canonical elements $(e_{u_2v_2} \otimes E^{a_2-b_2})^{(2,2)} \in ((B_2)_{1_G}^{(\bar{g}_2 g_{2j})}, \widetilde{\beta}_2 = \beta_2 \odot \widetilde{\epsilon}_{r_2})$ such that

$$\beta_2(u_2) = \bar{g}_2(\epsilon^{s_2})^{a_2} h_{2j} g_{21} \text{ and } \beta_2(v_2) = \bar{g}_2(\epsilon^{s_2})^{b_2} h_{2j} g_{21}, \quad \text{for some } h_{2j} \in \mathcal{H}_{\widetilde{\beta}}^{(2)},$$

with $c, d \in [0, r_2 - 1]$.

Thus, from the above discussions, once $f \notin \mathrm{Id}_G(B)$ and its homogeneous degree, as an element of $F\langle X; G\rangle$, is $g_{11}^{-1} g_{21}$, it follows that there exist $l_1 \in [0, r_1 - 1]$ and $l_2 \in [0, r_2 - 1]$ such that

$$g_{11}^{-1} g_{21} = \widetilde{\beta}((u_1-1)r_1+l_1+1)^{-1} \widetilde{\beta}((v_2-1)r_2+l_2+1) = (\bar{g}_1(\epsilon^{s_1})^{a_1} h_{1i} g_{11} (\epsilon^{s_1})^{l_1})^{-1} \bar{g}_2(\epsilon^{s_2})^{b_2} h_{2j} g_{21} (\epsilon^{s_2})^{l_2}.$$

Hence

$$\bar{g}_1(\epsilon^{s_1})^{a_1+l_1} h_{1i} = \bar{g}_2(\epsilon^{s_2})^{b_2+l_2} h_{2j}.$$

Define $g := \bar{g}_1(\epsilon^{s_1})^{a_1+l_1} h_{1i} = \bar{g}_2(\epsilon^{s_2})^{b_2+l_2} h_{2j}$. By using that $\langle \epsilon^{s_1}\rangle \subseteq \mathcal{H}_{\widetilde{\beta}}^{(1)}$, $\langle \epsilon^{s_2}\rangle \subseteq \mathcal{H}_{\widetilde{\beta}}^{(2)}$ and (5.3), it is easy to verify that

$$w_{\widetilde{\beta}}^{(l)}(gx) = w_{\widetilde{\alpha}}^{(l)}(x), \quad \text{for all } l \in [1, 2] \text{ and } x \in G,$$

and, consequently, B is graded-isomorphic to A (see Lemma 4.3.2). $\square$

At this stage, we present a new important condition in order to obtain a graded isomorphism between A and B.

Proposition 5.3.5 (Proposition 6.4 of [31]). *Let $G = \langle \epsilon \rangle$ be a cyclic group and let $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$ and $B = (UT(B_1, \ldots, B_m), \widetilde{\beta})$ satisfying $\exp_G(B) = \exp_G(A)$ and $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$.*

If there exists $\ell \in [1, m]$ such that

$$\mathcal{H}_{\widetilde{\beta}}^{(\ell)} = \mathcal{H}_{\widetilde{\alpha}}^{(\ell)} = \{1_G\},$$

then B is graded-isomorphic to A.

Proof. Once $\exp_G(B) = \exp_G(A)$ and $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$, by Proposition 5.3.2 and Corollary 3.2.2, for each $l \in [1, m]$, there exists $\overline{g}_l \in G$ such that

$$w^{(l)}_{\widetilde{\beta}}(\overline{g}_l x) = w^{(l)}_{\widetilde{\alpha}}(x), \quad \text{for all } x \in G.$$

We claim that

$$(\bar{g}_l)^{-1}\bar{g}_{l'} \in \mathcal{H}^{(l)}_{\widetilde{\alpha}}\mathcal{H}^{(l')}_{\widetilde{\alpha}}, \quad \text{for all } 1 \leq l < l' \leq m.$$

In fact, suppose that there exist $1 \leq l < l' \leq m$ such that $(\bar{g}_l)^{-1}\bar{g}_{l'} \notin \mathcal{H}^{(l)}_{\widetilde{\alpha}}\mathcal{H}^{(l')}_{\widetilde{\alpha}}$. Moreover, let us assume, without loss of generality, that $\widetilde{\beta}_l = l_{\bar{g}_l} \cdot \widetilde{\alpha}_l$ and $\widetilde{\beta}_{l'} = l_{\bar{g}_{l'}} \cdot \widetilde{\alpha}_{l'}$. Thus, from Proposition 4.1.4, one has $\mathrm{Id}_G(B^{[l,l']}) \nsubseteq \mathrm{Id}_G(A^{[l,l']})$ and $\mathrm{Id}_G(A^{[l,l']}) \nsubseteq \mathrm{Id}_G(B^{[l,l']})$, a contradiction with what was established in (5.1).

Now, we remark that if $\ell > 1$, then

$$(\bar{g}_l)^{-1}\bar{g}_\ell \in \mathcal{H}^{(l)}_{\widetilde{\alpha}}\mathcal{H}^{(\ell)}_{\widetilde{\alpha}} = \mathcal{H}^{(l)}_{\widetilde{\alpha}}, \quad \text{for all } l \in [1, \ell - 1],$$

whereas if $\ell < m$, thus

$$(\bar{g}_\ell)^{-1}\bar{g}_{l'} \in \mathcal{H}^{(\ell)}_{\widetilde{\alpha}}\mathcal{H}^{(l')}_{\widetilde{\alpha}} = \mathcal{H}^{(l')}_{\widetilde{\alpha}}, \quad \text{for all } l' \in [\ell + 1, m].$$

Therefore, for each $l \neq \ell$, there exists $h_l \in \mathcal{H}^{(l)}_{\widetilde{\alpha}} = \mathcal{H}^{(l)}_{\widetilde{\beta}}$ such that $\bar{g}_l = \bar{g}_\ell h_l$. Hence,

$$w^{(l)}_{\widetilde{\beta}}(\bar{g}_\ell x) = w^{(l)}_{\widetilde{\alpha}}(x), \quad \text{for all } l \in [1, m] \text{ and } x \in G,$$

and, from Lemma 4.3.2, B is graded-isomorphic to A. □

We remark that Propositions 5.3.4 and 5.3.5 generalize Theorem 3.3 of [24], where the authors deal with the G-graded upper block triangular matrix algebras $UT_G(A_1, \ldots, A_m)$, with $A_i = M_{k_i}$, for all $i \in [1, m]$.

Now, we will prove that if there exists at most one index $\ell \in [1, m]$ such that B_ℓ and A_ℓ are non-G-regular G-simple algebras, then B is graded-isomorphic to A.

Proposition 5.3.6 (Proposition 6.5 of [31])**.** *Let $G = \langle \epsilon \rangle$. Consider the G-graded upper block triangular matrix algebras $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$ and $B = (UT(B_1, \ldots, B_m), \widetilde{\beta})$ satisfying $\exp_G(B) = \exp_G(A)$ and $\mathrm{Id}_G(B) \subseteq \mathrm{Id}_G(A)$.*

If $\mathcal{H}^{(l)}_{\widetilde{\beta}} = \mathcal{H}^{(l)}_{\widetilde{\alpha}} = G$, for all (except for at most one) $l \in [1, m]$, then B is graded-isomorphic to A.

Proof. The result follows by applying Corollary 3.3.3 and Proposition 4.3.3. $\square$

Note that as a consequence of Propositions 5.3.5 and 5.3.6, in order to have that $B \cong_G A$, in addition to (5.2), it is enough to require that the invariance subgroups $\mathcal{H}^{(l)}_{\widetilde{\beta}}$ and $\mathcal{H}^{(l)}_{\widetilde{\alpha}}$ are $\{1_G\}$ or G (not all necessarily the same), for all $l \in [1, m]$. In particular, if $G = C_p$, with p being a prime number, thus we have that B is graded-isomorphic to A. Such case was developed by Di Vincenzo, da Silva and Spinelli, in [17].

Finally, we are in position to announce the main result of this section. It represents our contribution to the study of the minimal varieties of associative G-graded PI-algebras, of finite basic rank, with respect to a given G-exponent, when G is a finite cyclic group. More precisely, we exhibit some important conditions, related to the structure of $A = (UT(A_1, \ldots, A_m), \widetilde{\alpha})$ and the invariance subgroups $\mathcal{H}^{(l)}_{\widetilde{\alpha}}$, which are sufficient to concluding that $\mathrm{var}_G(A)$ is minimal.

We remark that, in view of the diversity of the possibilities for the invariance subgroups when we work with arbitrary finite cyclic groups (which are not of prime order), determining if $\mathrm{var}_G(A)$ is minimal or not is an engaging problem that still remains open. Nevertheless, the next theorem completely solves such problem for instance in the following cases:

- A has two blocks;

- all (except for at most one) the G-simple components of A are G-regular;

- $G = C_p$, with p being a prime number (in this case, see also [17]).

Theorem 5.3.7 (Theorem 6.6 of [31])**.** *Let F be an algebraically closed field of characteristic zero and $G = \langle \epsilon \rangle$ be a cyclic group, with ϵ being a primitive* nth *root of the unity in F^*. Given finite dimensional G-simple F-algebras $A_1, \ldots, A_m$, let $A := (UT(A_1, \ldots, A_m), \widetilde{\alpha})$. Assume that at least one of the following properties hold:*

(i) $m = 1$ *or* 2*;*

(ii) there exists $\ell \in [1, m]$ such that $\mathcal{H}^{(\ell)}_{\widetilde{\alpha}} = \{1_G\}$;

(iii) $\mathcal{H}^{(l)}_{\widetilde{\alpha}} = G$, for all (except for at most one) $l \in [1, m]$.

Then $\mathrm{var}_G(A)$ *is minimal with* $\exp_G(A) = \dim_F(A_1 \oplus \cdots \oplus A_m)$.

Proof. In order to conclude that $\mathrm{var}_G(A)$ is minimal, take a subvariety $\mathcal{U}^G \subseteq \mathrm{var}_G(A)$ such that $\exp_G(\mathcal{U}^G) = \exp_G(\mathrm{var}_G(A))$.

First, the fact that $\mathrm{var}_G(A)$ satisfies some Capelli identities (see Lemma 4.1.3) allows us to state, from Section 7.1 of [5], that $\mathcal{U}^G$ has finite basic rank. As consequence, by Theorem 1.1 of [5], one has that $\mathcal{U}^G$ is generated by a finite dimensional G-graded algebra $\bar{A}$.

Now, we notice that, in virtue of Lemma 1.5.5, there exists a minimal G-graded algebra $\tilde{A}$ such that $\mathrm{Id}_G(\bar{A}) \subseteq \mathrm{Id}_G(\tilde{A})$ and $\exp_G(\tilde{A}) = \exp_G(\bar{A})$. In particular, by invoking Proposition 5.1.3, it follows that there exists a G-graded algebra $B := (UT(B_1, \ldots, B_{m'}), \widetilde{\beta})$ such that $\mathrm{Id}_G(\tilde{A}) \subseteq \mathrm{Id}_G(B)$ and $\exp_G(B) = \exp_G(\tilde{A})$. Consequently,

$$\mathrm{Id}_G(A) \subseteq \mathrm{Id}_G(B) \quad \text{and} \quad \exp_G(A) = \exp_G(B).$$

Therefore, in this situation, Propositions 5.3.2 and 3.2.1 give us that $m' = m$ and $\mathcal{H}^{(l)}_{\widetilde{\beta}} = \mathcal{H}^{(l)}_{\widetilde{\alpha}}$, for all $l \in [1, m]$. Then, if one of statements $(i) - (iii)$ it is valid, Propositions 5.3.4 to 5.3.6 guarantee us that B is graded-isomorphic to A. Hence, $\mathrm{Id}_G(A) = \mathrm{Id}_G(B)$ and thus we obtain that $\mathrm{var}_G(A)$ is minimal. $\square$

We finish this chapter by highlighting that the results obtained in this section contribute to the isomorphism problem when associated with the theory of the G-graded PI-algebras. More precisely, given finite dimensional G-simple F-algebras $A_1, \ldots, A_m$, we have that any G-graded upper block triangular matrix algebras $UT_G(A_1, \ldots, A_m)$ satisfying conditions $(i) - (iii)$ of Theorem 5.3.7 are graded-isomorphic if, and only if, the T_G-ideal of their G-graded polynomial identities is the same.

Final Considerations

Throughout this thesis, we have addressed several important topics of PI-theory. In particular, in case F is an algebraically closed field of characteristic zero and $G = C_n = \langle \epsilon \rangle$ is a finite cyclic group of order n, we explored the G-graded upper block triangular matrix algebras $UT_G(A_1, \ldots, A_m)$ and the T_G-ideal $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$ of its G-graded polynomial identities, when $A_1, \ldots, A_m$ are finite dimensional G-simple F-algebras. Regarding this study, the first and crucial step realized was the description of the finite dimensional G-simple F-algebras as graded subalgebras of matrix algebras endowed with some elementary gradings.

Moreover, if the cyclic group G is a p-group, with p being an arbitrary prime number, we investigated the factoring problem related to the T_G-ideal $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$, by establishing necessary and sufficient conditions in order to have that $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m)) = \mathrm{Id}_G(A_1) \cdots \mathrm{Id}_G(A_m)$. More precisely, we proved that $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$ is factorable if, and only if, there exists at most one index $\ell \in [1, m]$ such that A_ℓ is a non-G-regular G-simple algebra if, and only if, there exists a unique isomorphism class of G-gradings for $UT_G(A_1, \ldots, A_m)$.

As previously seen throughout the text, the invariance subgroups related to the finite dimensional G-simple algebras $A_1, \ldots, A_m$ played an essential role in obtaining the above equivalences. It is worth saying that these statements were published in [22] together with some new results and alternative proofs from those presented in this thesis. In the sequel, in order to explicite some of these differences, let us recall some definitions and notations.

Firstly, given a finite dimensional G-simple F-algebra $(M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$ endowed with an elementary grading, we recall that $\alpha : [1, k] \to G$ is the map which induces the elementary grading on the matrix algebra M_k. Moreover, by remembering that $H_r = \langle \epsilon^s \rangle$, with $r \cdot s = n$, we consider the map $\overline{\alpha} : [1, k] \to G/H_r$ as

$$\overline{\alpha}(i) = H_r \alpha(i), \quad \text{for all } i \in [1, k].$$

It turns out that the G/H_r-graded matrix algebra $(M_k, \overline{\alpha})$ has important and useful connections with the G-graded algebra $(M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$. We start by comparing the graded multilinear polynomial identities of $(M_k, \overline{\alpha})$ and $(M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$. To this end, given any graded multilinear polynomial

$$f(x_1, x_2, \dots, x_m) = \sum_{\sigma \in \mathrm{Sym}(m)} c_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)} \quad \text{of } F\langle X; G\rangle, \quad \text{with } c_\sigma \in F,$$

let us define in the free G/H_r-graded algebra $F\langle \dot{X}; G/H_r\rangle$ the following graded polynomial

$$f_{H_r}(\dot{x}_1, \dot{x}_2, \dots, \dot{x}_m) = \sum_{\sigma \in \mathrm{Sym}(m)} c_\sigma \dot{x}_{\sigma(1)} \dot{x}_{\sigma(2)} \cdots \dot{x}_{\sigma(m)},$$

where $|\dot{x}_i|_{F\langle \dot{X}; G/H_r\rangle} = H_r |x_i|_{F\langle X; G\rangle}$, for all $i \in [1, m]$.

In the next statement, we enunciate the nice relation, obtained in [22], between the graded ideals $\mathrm{Id}_G(M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$ and $\mathrm{Id}_{G/H_r}(M_k, \overline{\alpha})$.

Proposition 1 (Proposition 4.6 of [22])**.** *Let $G = \langle \epsilon \rangle$ be a cyclic group and let $\overline{f}$ and f be graded multilinear polynomials in the free algebras $F\langle \dot{X}; G/H_r\rangle$ and $F\langle X; G\rangle$, respectively, such that $f_{H_r} = \overline{f}$. Then*

$$f \in \mathrm{Id}_G(M_k(D_r), \alpha \odot \widetilde{\epsilon}_r) \Longleftrightarrow \overline{f} \in \mathrm{Id}_{G/H_r}(M_k, \overline{\alpha}).$$

Therefore, at light of the above result, investigating the G/H_r-graded multilinear polynomial identities of the matrix algebra $(M_k, \overline{\alpha})$ allows us to obtain information about the elements of the T_G-ideal of G-graded polynomial identities of $(M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$. In this sense, with the appropriate adaptations, we prove Lemma 3.2.3 in [22] by working with the matrix algebra $(M_k, \overline{\alpha})$ and by invoking results given by Di Vincenzo and Spinelli, in [24], where the authors deal with matrix algebras endowed with elementary gradings.

In addition, we remark that, while in this thesis we prove some of the results directly for the finite dimensional G-simple F-algebras $M_k(D_r)$, in the paper [22] we chose to prove suitable results only for the matrix algebras (M_k, α) and, once done, we work with the algebras $M_k(D_r)$ (by dealing with $(M_k, \overline{\alpha})$).

It is worth highlighting another interesting bridge between the G-simple algebras $(M_k, \overline{\alpha})$ and $(M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$, which explores the regularity of these algebras and was crucial for our aims.

Theorem 1 (Proposition 4.7 of [22])**.** *Let $G = \langle \epsilon \rangle$ be a cyclic group. The G-graded algebra $(M_k(D_r), \alpha \odot \widetilde{\epsilon}_r)$ is $(\alpha \odot \widetilde{\epsilon}_r)$-regular if, and only if, $(M_k, \overline{\alpha})$ is $\overline{\alpha}$-regular.*

Now, we would like to point out some remarks and results about the minimal varieties of associative G-graded PI-algebras over F, of finite basic rank, of a given G-exponent. We recall that such subject was approached in Chapter 5. There we showed that these varieties are generated by suitable G-graded upper block triangular matrix algebras. On the other hand, given finite dimensional G-simple F-algebras $A_1, \ldots, A_m$, we considered the G-graded upper block triangular matrix algebra $UT_G(A_1, \ldots, A_m)$. By imposing some extras conditions on $UT_G(A_1, \ldots, A_m)$, we proved that, in this case, $\mathrm{var}_G(UT_G(A_1, \ldots, A_m))$ is minimal. More precisely, if $UT_G(A_1, \ldots, A_m)$ satisfies at least one the following conditions:

(i) $m = 1$ or 2;

(ii) there exists $\ell \in \{1, \ldots, m\}$ such that the invariance subgroup related to the G-simple algebra A_ℓ is $\{1_G\}$;

(iii) the invariance subgroups related to the G-simple algebras $A_1, \ldots, A_m$ are all (except for at most one) equal to G,

then it is valid that $UT_G(A_1, \ldots, A_m)$ generates a minimal variety.

It is worth remembering that, in order to achieve the above results, we proved first that any two G-graded upper block triangular matrix algebras endowed with elementary gradings, satisfying one of the above conditions, are graded-isomorphic if, and only if, they satisfy the same G-graded polynomial identities. Moreover, we emphasize that when we deal with a finite cyclic group G (which is not of order p, with p prime), obtaining that the $\mathrm{var}_G(UT_G(A_1, \ldots, A_m))$ is or not minimal, for any G-graded upper block triangular matrix algebra $UT_G(A_1, \ldots, A_m)$, it is an engaging problem, which is still open. In this case, our results indicate that the behavior of the invariance subgroups related to the finite dimensional G-simple algebras $A_1, \ldots, A_m$ is a crucial and important point in solving a such problem.

Since the factorability and the minimal varieties were the main topics addressed in this thesis, we would like to end by asking us what connections can be obtained between these concepts from our results. In this sense, we highlight the case G is a cyclic p-group, with p being a prime number. We remark that if the T_G-ideal $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$ decomposes into $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m)) = \mathrm{Id}_G(A_1) \cdots \mathrm{Id}_G(A_m)$, then, from Theorem 4.3.4, there exists a unique isomorphism class of G-gradings for $UT_G(A_1, \ldots, A_m)$. Consequently, in this situation, we conclude that the factorability of $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$ is a sufficient condition in order to have that $\mathrm{var}_G(UT_G(A_1, \ldots, A_m))$ is minimal.

On the other hand, the reciprocal is not true. Indeed, for instance when G is a group of prime order, any $\mathrm{var}_G(UT_G(A_1, \ldots, A_m))$ is minimal (see [17] or items (ii) and (iii) above). However, whenever there exist $1 \leq a < b \leq m$ such that the G-simple algebras A_a and A_b are both non-G-regular, by invoking Theorem 4.3.4, it follows that $\mathrm{Id}_G(UT_G(A_1, \ldots, A_m))$ is not factorable.

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