UNIVERSIDADE FEDERAL DE MINAS GERAIS

Instituto de Ciências Exatas

Departamento de Matemática

VERTEX-SWITCHING RECONSTRUCTION AND RELATED PROBLEMS

Amanda Caroline Silva

Belo Horizonte, August 2019

Amanda Caroline Silva

VERTEX-SWITCHING RECONSTRUCTION AND RELATED PROBLEMS

Versão Final

Dissertação apresentada ao corpo docente de Pós-Graduação em Matemática do Instituto de Ciências Exatas da Universidade Federal de Minas Gerais, como parte dos requisitos para a obtenção do título de Mestre em Matemática.

Orientador: Bhalchandra Digambar Thatte

Belo Horizonte
2019

Silva, Amanda Caroline

S586v Vertex-switching reconstruction and related problems [manuscrito] / Amanda Caroline Silva – 2019.
83 f.: il. ; 29cm

Orientador: Bhalchandra Digambar Thatte.

Dissertação (mestrado) - Universidade Federal de Minas Gerais, Instituto de Ciências Exatas, Departamento de Matemática.
Referências: f. 80-81

1. Matemática – Teses. 2. Álgebra linear - Teses. 3. Teoria dos grafos – Teses. I. Thatte, Bhalchandra Digambar. II. Universidade Federal de Minas Gerais, Instituto de Ciências Exatas, Departamento de Matemática. III. Título.

CDU 51(043)

Ficha catalográfica elaborada pela bibliotecária Irénquer Vismeg Lucas Cruz - CRB 6ª Região / nº 819.

Universidade Federal de Minas Gerais
Departamento de Matemática
Programa de Pós-Graduação em Matemática

FOLHA DE APROVAÇÃO

Vertex switching reconstruction and related problems

AMANDA CAROLINE SILVA

Dissertação defendida e aprovada pela banca examinadora constituída pelos Senhores:

Prof. Bhalchandra Digambar Thatte
UFMG

Prof. John William MacQuarrie
UFMG

Prof. Maurício de Lemos Rodrigues Collares Neto
UFMG

Belo Horizonte, 28 de agosto de 2019.

Av. Antônio Carlos, 6627 – Campus Pampulha - Caixa Postal: 702
CEP-31270-901 - Belo Horizonte – Minas Gerais - Fone (31) 3409-5963
e-mail: pgmat@mat.ufmg.br - home page: http://www.mat.ufmg.br/pgmat

Primeiramente agradeço a Deus. Agradeço aos meus pais e irmãs por sempre me apoiarem e por todo amor e carinho dispensados. Ao meu orientador Bhalchandra Digambar Thatte por toda a dedicação. Aos amigos, professores e funcionários que eu tive o prazer de conhecer na UFMG. Aos meus amigos do CEFET-MG que se mantém presentes em todas as minhas conquistas. Ao suporte financeiro oferecido pelo PICME (CAPES).

RESUMO

Um grafo simples G é um par ordenado $(V(G), E(G))$, onde $V(G)$ é dito o conjunto de vértices do grafo G e $E(G)$ é o conjunto de subconjuntos de dois elementos de $V(G)$, chamado conjunto de arestas do grafo G. A análise do multiconjunto obtido ao se apagar um vértice de G, junto com as arestas incidentes a este vértice, nos leva a conjectura de reconstrução. Esta conjectura nos diz que a partir do multiconjunto supracitado podemos obter um grafo isomorfo ao grafo original, se o grafo original possui pelo menos três vértices. Esta conjectura foi proposta em 1941, por Kelly e por Ulam (veja [2]), e levou a diversos outros problemas de reconstrução, como a reconstrução por *vertex-switching*.

Um *vertex-switching*, ou apenas *switching*, de um grafo G é um grafo obtido a partir de G ao se deletar todas as arestas incidentes a um vértice específico e adicionar todas as arestas que são incidentes a este mesmo vértice no grafo complementar de G. Ao se realizar todos os possíveis *switchings* de um dado grafo obtemos um multiconjunto, dito *switching deck*. A partir deste multiconjunto obtemos o problema de reconstrução de grafos por vertex-switching.

Problema de reconstrução por *vertex-switching* 1 (Stanley [12]). *Sejam G e H grafos simples com o mesmo conjunto de vértices de cardinalidade diferente de* 4. *Se estes grafos possuem switching decks iguais então eles são isomorfos?*

O teorema a seguir é um importante resultado relacionado ao problema anterior.

Teorema 1 (Stanley [12]). *Seja G um grafo simples com conjunto de vértices de cardinalidade $n \not\equiv 0 \pmod 4$, então qualquer grafo com o mesmo switching deck que G é isomorfo a G.*

Em 1977, Bondy e Hemminger fizeram um *survey* com os principais resultados conhecidos a época (veja [2]). Porém, foi apenas em 1985 que Stanley propôs o problema de reconstrução por *vertex-switching*. Fortemente insirados pelo *survey* de Bondy e Hemminger apresentaremos os principais resultados obtidos para *vertex-switching*, porém este trabalho se diferencia deste *survey* por focar nos métodos de álgebra linear utilizados na resolução de problemas relacionados a reconstrução por *vertex-switching*, tendo como principal objetivo apresentar estas técnicas. Inclusive, apresentaremos a prova do Teorema 1.

O foco nas técnicas de álgebra linear também nos distancia dos artigos utilizados como base nesta dissertação. Visando estes métodos apresentaremos análogos para o Lema de Kelly (veja o Lema 1), tanto para a reconstrução por *vertex-switching* quanto para a reconstrução a partir de subgrafos obtidos ao se apagar uma aresta de um dado grafo.

Lema 1. *Sejam F e G grafos tais que o número de vértices de F é menor que o número de vértices de G. Então o número de subgrafos de G que são isomorfos a F é reconstrutível.*

Os resultados apresentados nesta dissertação podem ser dividido em duas partes. Na primeira, que consiste nos capítulos 2 e 3, apresentamos alguns problemas relacionados ao problema de reconstrução de grafos por *vertex-switching*. Nós começamos mostrando alguns resultados para 1-*vertex-switching*. Posteriormente generalizamos os resultados para o *switching* de conjuntos de cardinalidade arbitrária. Na segunda parte, que consiste no capítulo 4, apresentamos resultados relacionados a subgrafos obtidos pela remoção de uma aresta e generalizamos alguns deles para grafos obtidos ao se remover subconjuntos de arestas de cardinalidade arbitrária.

Palavras chave: Reconstrução de grafos. Álgebra linear. *Vertex-switching*. Recons-trução por arestas apagadas.

ABSTRACT

The aim of this work is to present linear algebraic methods for some problems related to the vertex-switching reconstruction problem of Stanley (1985). Kelly's Lemma for vertex-switching is the main result obtained in this area. The results in this dissertation can be divided into two parts. In the first, consisting of chapters 2 and 3, we present some problems related to the vertex-switching reconstruction problem. We begin presenting some results for one-vertex switching. Next we generalize the results for the switching of sets of arbitrary cardinality. In the second part, consisting of chapter 4, we present results related to edge-deleted subgraphs and generalize some of them for graphs obtained by deleting edge-subsets of arbitrary cardinality.

Keywords: Graph reconstruction. Linear algebra. Vertex-switching. Edge reconstruction.

List of Symbols

General Mathematical Notation

$\mathbb{C}$ Set of complex numbers

$\mathbb{N}$ Set of natural numbers

$\mathbb{Z}$ Set of integer numbers

$\mathbb{Z}[x,y]$ Set of polynomials in two variables with integer coefficients

$A+B$ Union of the sets A and B

$A-B$ Elements in the set A that are not in B

$A \triangle B$ Symmetric difference between the sets A and B

$f_i(z,n)$ Binary form of degree $\lfloor i/2 \rfloor$

$g_i(z,n)$ Polynomial of degree less than the degree of $f_i(z,n)$

$K_s^n(x)$ Krawtchouk polynomial

$P_s^n(y)$ Krawtchouk polynomial calculated in $(n-y)/2$

Structures and families of graphs

(X,Y) Bipartition of a graph with parts X and Y

$\binom{E(H)}{i}$ Collection of all i-subsets of $E(G)$

$\mathscr{R}$ Collection of graphs

$\mathscr{R}_n$ Set of representatives on n vertices

$\mathscr{R}_{n,m}$ Set of representatives on n vertices and m edges

$d_G(v)$ Degree of a vertex v in G

$E(G)$ Edge set of G

$E[T,U]$ Edges in $E(G)$ that are incident to a vertex in T and another vertex in U

$e_G(u,v)$ Number of edges between the vertices u and v

$G \cong H$ Graph G is isomorphic to the graph H

G^* Representative of a graph G

G^c Complement of a graph G

$i(S, G)$ Number of induced subgraphs of G isomorphic to S

J Johnson graph

K_n Complete graph on n vertices

$K_{m,n}$ Complete bipartite graph with parts of cardinality m and n

$N_G(v)$ Neighbours of a vertex v in G

$N_G(v+,u+)$ Set of vertices in G adjacent to both u and v

$N_G(v+,u-)$ Set of vertices in G adjacent to v, but not u

$N_G(v-,u+)$ Set of vertices in G adjacent to u, but not v

$N_G(v-,u-)$ Set of vertices in G adjacent to neither u nor v

P_n Path on n vertices

$s(S, G)$ Number of subgraphs of G isomorphic to S

$V(G)$ Vertex set of G

$X_{\mathscr{R}}$ Characteristic vector of a collection $\mathscr{R}$ of graphs

Graph General Operations

$G+e$ Addition of the edge e

$G+Y$ Addition of a set Y of edges

$G-e$ Deletion of the edge e

$G-v$ Deletion of the vertex v

$G-Y$ Deletion of a set Y of edges

Switching reconstruction

M Switching matrix

M_S Switching matrix of S

$t(S, G)$ Number of induced subgraphs isomorphic to S in the switching deck of G

$X_s(S \to G)$ Number of s-switchings isomorphic to H

G_v Switching of a vertex v

G_{uv} Switching of the vertices u and v

G_W Switching of G on a set W of vertices

$D(G)$ Switching deck of G

$D_s(G)$ s-switching deck of G

Edge reconstruction

ED_i	*i*-edge deck
MD_i	Modified *i*-deck
PD_i	Perturbed *i*-deck
d_i	Matrix related to the *i*-edge deck
Δ_i	Matrix related to the modified *i*-deck
D_i	Matrix related to the perturbed *i*-deck
L_k	Linear combination of operators constructed from Δ_1

Contents

Introduction

Let G be a graph. The multiset of all graphs obtained by deleting a vertex v of G together with all the edges incident with v is called the **collection of vertex-deleted subgraphs** of G. The **reconstruction conjecture** asserts that every finite simple graph, with at least three vertices, is determined, up to isomorphism, by its collection of unlabelled vertex-deleted subgraphs. This conjecture was first formulated in 1941, by Kelly and Ulam (see [2]).

In 1964, Harary proposed the edge reconstruction conjecture. We denote by $G-e$ the subgraph of G obtained by deleting the edge e. The **edge reconstruction conjecture** says that any finite simple graph with at least four edges is determined, up to isomorphism, by its collection of unlabelled edge-deleted subgraphs. This conjecture is weaker than the reconstruction conjecture.

In 1985, Stanley proposed the **vertex-switching reconstruction problem**. For a vertex $v \in V(G)$ the graph G_v, obtained from G by deleting all edges incidents to v and adding edges joining v to every vertex not adjacent to v in G, is called a vertex-switching. The vertex-switching reconstruction problem asks if every finite simple graph, with at least five vertices is determined, up to isomorphism, by its collection of unlabelled vertex-switching graphs.

In this dissertation, we will study problems related to the vertex-switching reconstruction problem and the edge reconstruction conjecture.

In Chapter 1, we present some basic definitions in graph theory and present some notation we will use in this work. Furthermore, we present the main terminology related to the vertex-switching reconstruction problem and the edge-reconstruction conjecture.

In Chapter 2, we present Kelly's Lemma for vertex-switching. Moreover, we show conditions under which disconnected graphs, regular graphs and triangle-free graphs are switching reconstructible; that is conditions under which those graphs are determined, up to isomorphism, by their collection of vertex-switching graphs.

In Chapter 3, we generalize some results obtained in chapter 2 for the switching of sets of arbitrary cardinality. By analysing the roots of Krawtchouk polynomial, we show a condition under which a graph is s-switching reconstructible, that is, conditions under which a graph is determined, up to isomorphism, by its collection of graphs obtained by realizing the switching of sets of vertices. Furthermore we show conditions under which the number of subgraphs in a given isomorphism class is s-switching reconstructible. In Chapter 4, by making use of similar techniques to those used in the previous chapters, we show a condition under which a graph is edge-reconstructible. We also relate different decks obtained from the edge-deleted subgraphs of a given graph. Furthermore, we generalize these results for graphs obtained by deletion of sets of edges with arbitrary cardinality.

Chapter 1

Basic definitions and notations

In this chapter we will present some basic definitions in Graph Theory and some terminology related to graph reconstruction. The definitions in Sections 1.1, 1.2 and 1.3 can be found in [1].

1.1 Graphs

A **simple graph** G is an ordered pair $(V(G), E(G))$, where $V(G)$ is called the **vertex set of** G, and $E(G)$ is a set of two-element subsets of $V(G)$, called the **edge set of** G. In this work, we consider only finite simple graphs that is, simple graphs with finite number of vertices and edges. The **order** of a graph G is the number of vertices and the **size** of G is the number of edges.

Let G be a graph and let u and $v \in V(G)$, if $e = \{u, v\} \in E(G)$ then u and v are called the **ends** of e, and e is said to **join** u and v. We denote by $e_G(u,v)$ the number of edges between u and v. Two vertices u and v are said to be **adjacent** if there exists an edge $e = \{u, v\} \in E(G)$. In this case we say that u and $v \in V(G)$ are **incident** with e, or e is **incident** to u and v. If there is no edge between u and v we say these vertices are **nonadjacent**. Two distinct vertices that are adjacent are called **neighbours**. The set of neighbours of a vertex $v \in V(G)$ is denoted by $N_G(v)$. The **degree** of v in G, denoted by $d_G(v)$, is the number of neighbours of v in G, that is $d_G(v) := |N_G(v)|$.

Let G be a graph on n vertices. The **adjacency matrix** of G is the $n \times n$ matrix whose rows and columns are indexed by the vertices on G and the uv entry is equal to $e_G(u,v)$.

For convenience, we also denote by uv the edge $\{u, v\}$.

1.2 Special types of graphs

A graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a **subgraph** of G. If $V(H) \subseteq V(G)$ and $E(H) = \{uv \in E(G) : u,\ v \in V(H)\}$ then H is called an **induced subgraph** of G.

A graph in which any two vertices are adjacent is called a **complete graph**. Up to isomorphism, there is a unique complete graph on n vertices, denoted by K_n, that has size $\binom{n}{2}$. A graph whose edge set is empty is called an **empty graph**.

The **complement of a graph** G is the graph with vertex set $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. We denote the complement of G by G^c.

A graph G is called **bipartite** if we can partition the vertex set of G in two blocks X and Y such that each edge of G has one end in X and the other in Y; such a partition (X, Y) is called a **bipartition** of G, and X and Y are called its **parts**. Let m, $n \in \mathbb{N}$. A bipartite graph with bipartition (X, Y) is a **complete bipartite graph** if every vertex in X is adjacent to every vertex in Y. Up to isomorphism, there is a unique complete bipartite graph with parts of cardinality m and n, denoted by $K_{m,n}$. A **star** is a complete bipartite graph with bipartition (X, Y) such that $|X| = 1$ or $|Y| = 1$.

A graph is k**-regular** if each vertex of the graph has degree k. A **regular graph** is a graph that is k-regular for some k.

A **path** is a graph whose vertex set can be arranged in a linear sequence such that any two vertices are adjacent if and only if they are consecutive in this sequence. Up to isomorphism, there is a unique path on n vertices, denoted by P_n.

A **cycle** is a graph, on three or more vertices, whose vertex set can be arranged in a cyclic sequence such that any two vertices are adjacent if and only if they are consecutive in this sequence. The **length** of a cycle is the number of its edges. Up to isomorphism, there is a unique cycle on n vertices. A cycle on three vertices is often called a **triangle**. A graph which contains no triangle is called **triangle-free**.

A graph G is **connected** if, for every partition of $V(G)$ into two nonempty subsets X and Y, there exists an edge with one end in X and one end in Y; otherwise the graph is called **disconnected**.

1.3 Isomorphism of graphs

Two graphs G and H are **isomorphic** if there exists a bijection $\psi : V(G) \to V(H)$ such that u and v are adjacent in G if and only if $\psi(u)$ and $\psi(v)$ are adjacent in H. In this case ψ is called an isomorphism; we denote the fact that G is isomorphic to H by $G \cong H$. An isomorphism of a graph to itself is called an **automorphism**.

The relation of isomorphism is an equivalence relation. We define $\mathscr{R}_n$ to be the set of n vertex graphs with exactly one representative from each isomorphism class. Let C be an isomorphism class. We denote by C^* the unique element in $\mathscr{R}_n$ that is isomorphic to any member of C. We define $\mathscr{R}_{n,m}$ to be the set of graphs on n vertices and m edges with exactly one representative from each isomorphism class. Given a graph G, by G^* we denote the unique element in $\mathscr{R}_{n,m}$ or in $\mathscr{R}_n$, according to context, isomorphic to G.

Example 1.3.1. We have $\mathscr{R}_3 = \{K_3, P_3, K_2 \cup K_1, K_1 \cup K_1 \cup K_1\}$ and $\mathscr{R}_{3,2} = \{P_3\}$.

We define an **unlabelled graph** as a representative of an equivalence class of isomorphism of graphs.

1.4 Graph reconstruction terminology

The definitions in this section can be found in [1, 2, 5, 4, 6, 12, 13].

1.4.1 Switching reconstruction

A **switching** G_v is a graph obtained from G by deleting all edges incident to v and adding edges joining v to every vertex not adjacent to v in G.

The graph G_{uv} is obtained from G by switching first on u and then v. We have $G_{uv} = G_{vu}$.

Let $W \subseteq V(G)$. The **switching of** G **on** W is the graph G_W obtained from G by deleting all edges between W and $V(G)-W$ and adding all edges in G^c that are incident to a vertex in W and a vertex in $V(G)-W$. Set $s = |W|$, the graph G_W is also called a s**-switching of** G.

The multiset $D_s(G) := \{G_W^* : W \subseteq V(G), |W| = s\}$ is called the s**-switching deck of** G. The 1-switching deck of G is also called the **switching deck of** G, and is denoted by $D(G)$.

A graph G is s**-switching reconstructible** if any graph H such that $D_s(H) = D_s(G)$ is isomorphic to G. A parameter or property is s**-switching reconstructible** if its is completely determined by the s-switching deck. If $s = 1$ we can also say a graph, a parameter or a property is **switching reconstructible**.

Graphs G and H are **switching-equivalent** if there exists $W \subseteq V(G)$ such that $G_W \cong H$. The relation of switching equivalence is an equivalence relation. Hence it partitions $\mathscr{R}_n$ into equivalence classes called **switching classes**. Isomorphic graphs are trivially switching equivalents.

Vertex-switching reconstruction problem (Stanley [12]). *Let G and H be graphs with the same vertex set. Let $|V(G)| = |V(H)| \neq 4$. If $D(G) = D(H)$ then $G \cong H$?*

Example 1.4.1 ([12]). Let G be the empty graph on 4 vertices and let H be a cycle on 4 vertices, with $V = V(G) = V(H)$. For any $v \in V$, we have G_v and H_v are isomorphic to the star $K_{1,3}$.

Let S be a graph on k vertices. Let $s(S, G)$ denote the number of subgraphs of G that are isomorphic to S and $i(S, G)$ be the number of induced subgraphs of G that are isomorphic to S. Let $t(S, G)$ indicate the number of occurrences of S as an induced subgraph in the switching deck of G, that is $t(S, G) = \sum_{v \in V(G)} i(S, G_v)$. For a graph H on n vertices, we will denote by $X_s(G \to H)$ the total number of s-switchings of G that are isomorphic to H, that is $X_s(G \to H) = |\{G_S : S \subseteq V(G), |S| = s, G_S \cong H\}|$.

Example 1.4.2. All graphs has a graphical representation. In this representation each vertex is indicated by a point, and each edge is indicated by a line joining the points that represents its ends. Let G be a graph given by the graphical representation in Figure 1.1.

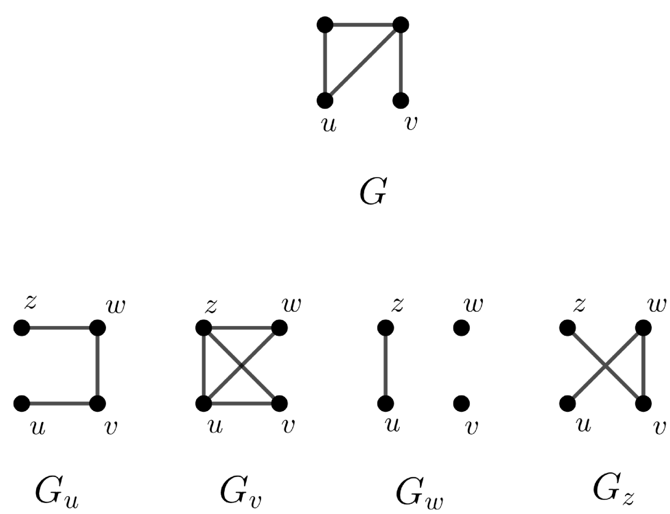

Figure 1.1: A representation of G and its switchings

Let S be isomorphic to P_2. We have $s(S, G) = 4$, but $i(S, G) = 1$. Furthermore $t(S, G) = 2 + 8 + 0 + 2 = 12$. Let H be isomorphic to P_3, then $X_1(G \to H) = 2$.

1.4.2 Edge reconstruction

Let $\mathscr{R}$ be a multiset of graphs with n vertices and m edges. The **characteristic vector** of $\mathscr{R}$, denoted by $X_{\mathscr{R}}$, is a vector with length equal to $|\mathscr{R}_{n,m}|$. Each entry of $X_{\mathscr{R}}$ is associated to one graph in $\mathscr{R}_{n,m}$ and is equal to the number of graphs in $\mathscr{R}$ that are isomorphic to the graph in $\mathscr{R}_{n,m}$ that corresponds to this entry. The characteristic vector of the singleton $\{G\}$ is denoted by X_G, this is a vector with only one entry equal to 1 and other entries equal to 0.

Let $A \subseteq E(G)$ such that $|A| = i$. We denote by $G - A$ the ***i*-edge deleted subgraph** of G obtained from G by deleting the edges in A. We have three different decks related to the i-edge deleted subgraphs:

- The ***i*-edge deck** of G, denoted by $\mathrm{ED}_i(G)$, is the multiset of all i-edge deleted subgraphs obtained from G;
- The **modified *i*-deck** of G, denoted by $\mathrm{MD}_i(G)$, is the multiset of all graphs obtained from G by deleting i edges and then adding i edges in all possible ways, where the added edges are not necessarily different from the deleted one;
- The **perturbed *i*-deck** of G, denoted by $\mathrm{PD}_i(G)$, is the multiset of all graphs obtained from G by deleting i edges and adding i different edges.

We define these decks for a collection of graphs $\mathscr{R}$ as the multiunion of the deck of each member of $\mathscr{R}$, e.g, $\mathrm{MD}_k(\mathscr{R}) = \bigcup_{H\in\mathscr{R}} \mathrm{MD}_k(H)$.

The operations of constructing the i-edge deck, the modified i- deck and the perturbed i-deck of a graph on n vertices and m edges can be represented, respectively, by the following matrices: d_i, Δ_i and D_i. The matrix d_i has rows indexed by graphs on $\mathscr{R}_{n,m-i}$, while the rows of Δ_i and D_i are indexed by graphs on $\mathscr{R}_{n,m}$. All these matrices have columns indexed by graphs on $\mathscr{R}_{n,m}$. Let $\{G_1, \ldots, G_{|\mathscr{R}_{n,m}|}\}$ be all the graphs in $\mathscr{R}_{n,m}$ and let $\{H_1, \ldots, H_{|\mathscr{R}_{n,m-i}|}\}$ be all the graphs in $\mathscr{R}_{n,m-i}$, the entries of these matrices are given by:

$$(d_i)_{kl} = |H \cong H_k : H \in \mathrm{ED}_i(G_l)|; \quad (1.1)$$
$$(\Delta_i)_{kl} = |H \cong G_k : H \in \mathrm{MD}_i(G_l)|; \quad (1.2)$$
$$(D_i)_{kl} = |H \cong G_k : H \in \mathrm{PD}_i(G_l)|. \quad (1.3)$$

Thus $d_i X_G$ is the vector that represents the i-edge deck of G and $\Delta_i X_G$ is the vector that represents the modified i-deck of G.

Remark 1.4.3. D_0 and Δ_0 are identity operators.

Let $\binom{E(G)}{i}$ be the collection of all i-subsets of $E(G)$. A graph H is called an ***i*-edge reconstruction** of G if there is a bijection

$$\psi : \binom{E(G)}{i} \longrightarrow \binom{E(H)}{i}$$

such that $G - E \cong H - \psi(E)$ for every $E \in \binom{E(G)}{i}$.

Let G be a graph. If every i-edge reconstruction of G is isomorphic to G, then G is ***i*-edge reconstructible**. We can make an analogous definition for a collection of graphs. A collection $\mathscr{R}$ of graphs is i-edge reconstructible if for every collection of graphs $\mathscr{P}$ such that $d_i X_{\mathscr{R}} = d_i X_{\mathscr{P}}$ we have $X_{\mathscr{R}} = X_{\mathscr{P}}$. In the same way a parameter is ***i*-edge reconstructible** if the parameter takes the same values on all i-edge reconstructions of G.

Edge reconstruction conjecture ([2])**.** *All graphs with at least four edges are* 1*-edge reconstructible.*

Example 1.4.4 ([2])**.** Let G and H be graphs on the same vertex set such that $G \cong P_3 \cup K_1$ and $H \cong K_2 \cup K_2$. Each graph in $\mathrm{ED}_1(G)$, as each graph in $\mathrm{ED}_1(H)$, is isomorphic to a graph on four vertices with a single edge. So G and H are not 1-edge reconstructible.

Example 1.4.5 ([2])**.** Let G and H be graphs on the same vertex set such that $G \cong K_{1,3}$ and $H \cong K_3 \cup K_1$. Each graph in $\mathrm{ED}_1(G)$, as each graph in $\mathrm{ED}_1(H)$, is isomorphic to $P_3 \cup K_1$. So G and H are not 1-edge reconstructible.

Chapter 2

Vertex-switching reconstruction

We consider the problem of reconstructing a graph, up to isomorphism, given its switching deck. With exception of the results indicated, all results in this chapter can be found in [5].

In this chapter, we will denote by G an arbitrary graph on n vertices.

2.1 Reconstructing the number of subgraphs

The goal of this section is to compute the number of subgraphs of G isomorphic to an arbitrary graph S given the switching deck of G, that is Kelly's Lemma for vertex-switching. But first, we will show that the number of triangles of G, when G has order not equal to four, is reconstructible.

In the next lemma we will relate $t(S, G)$ to all induced subgraphs of G on k vertices.

Lemma 2.1.1 (Elligham and Royle [5]). *For all graphs S on k vertices*

$$t(S, G) = (n-k)\, i(S, G) + \sum_{T \in \mathscr{R}_k} X_1(T \to S)\, i(T, G). \tag{2.1}$$

Proof. We will count all ways a graph isomorphic to S may appear in $D(G)$. Let T be an induced subgraph of G on k vertices. We have two cases.

Case. T is isomorphic to S

Let $v \in V(G)$ such that $v \notin V(T)$. So T is an induced subgraph of G_v and we have $n-k$ choices of v, obtaining the first term. If we may obtain S from a switching in S we have a contribution on the second term.

Case. T is not isomorphic to S.

The graph T contributes in the summation if and only if we may obtain S from a switching in T. Thus we have the other contributions of the last term. □

We know all graphs in $\mathscr{R}_k$. Given $T \in \mathscr{R}_k$ we may compute $X_1(T \to S)$, so we are able to present a condition to reconstruct $i(S,\, G)$. Let $\{S_1,\, S_2,\, \ldots,\, S_m\}$ be the switching class of S. Let $m_{ij} = X_1(S_j \to S_i)$, we define the **switching matrix** $M = M_S = (m_{ij})$. Taking Equation 2.1 for each S_j, in this switching class, we obtain a system of linear equations. Rewriting this system in the matrix form we obtain $t = ((n-k)I + M_S)i$, where $i = (i(S_1,\, G),\, i(S_2,\, G),\, \ldots,\, i(S_m,\, G))^t$ is unknown and $t = (t(S_1,\, G), t(S_2,\, G),\, \ldots, t(S_m,\, G))^t$ is known from the deck.

Example 2.1.2. The switching class of K_3 is $\{K_3, K_2 \cup K_1\}$. Since $X_1(K_3 \to K_3) = 0$, $X_1(K_2 \cup K_1 \to K_3) = 1$, $X_1(K_3 \to K_2 \cup K_1) = 3$, and $X_1(K_2 \cup K_1 \to K_2 \cup K_1) = 2$, thus we have the following system:

$$\begin{cases} t(K_3,G) & = (n-3)i(K_3,\ G) + X_1(K_2 \cup K_1 \to K_3)i(K_2 \cup K_1, G) \\ t(K_2 \cup K_1,G) & = (n-3)i(K_2 \cup K_1,\ G) + X_1(K_2 \cup K_1 \to K_2 \cup K_1)i(K_2 \cup K_1, G) \\ & \quad + X_1(K_3 \to K_2 \cup K_1)i(K_3, G) \end{cases}$$

$$\begin{cases} t(K_3,G) & = (n-3)i(K_3,\ G) + i(K_2 \cup K_1,\, G) \\ t(K_2 \cup K_1,G) & = (n-3)i(K_2 \cup K_1,\ G) + 2i(K_2 \cup K_1,\, G) + 3i(K_3,\, G) \end{cases}$$

By rewriting this equation in the matrix form, we obtain:

$$\begin{pmatrix} t(K_3, G) \\ t(K_2 \cup K_1, G) \end{pmatrix} = \begin{pmatrix} n-3 & 1 \\ 3 & n-3+2 \end{pmatrix} \begin{pmatrix} i(K_3, G) \\ i(K_2 \cup K_1, G), \end{pmatrix}$$

where $M_{K_3} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$.

Proposition 2.1.3 (Elligham and Royle [5]). *Let S be a graph on k vertices. The system $t = ((n-k)I + M_S)i$ has solution if and only if $k-n$ is not an eigenvalue of M_S.*

Proof. We have $(n-k)I + M_S$ is not invertible if and only if 0 is an eigenvalue of this matrix if and only if $n-k$ is an eigenvalue of M_S. □

We will illustrate the previous proposition by showing that the number of triangles is reconstructible.

Theorem 2.1.4 (Elligham and Royle [5]). *If $n \neq 4$, then the number of triangles of G is switching reconstructible.*

Proof. As we see in Example 2.1.2

$$\begin{pmatrix} t(K_3, G) \\ t(K_2 \cup K_1, G) \end{pmatrix} = \begin{pmatrix} n-3 & 1 \\ 3 & n-3+2 \end{pmatrix} \begin{pmatrix} i(K_3, G) \\ i(K_2 \cup K_1, G). \end{pmatrix}$$

For $n \notin \{0, 4\}$ the matrix $(n-3)I + M_{K_3}$ is invertible. Hence the number of triangles is reconstructible, that is we may obtain $i(K_3, G)$. □

We will investigate the eigenvalues of M, the switching matrix. For this we need some definitions and new notation. Let $k > 2$. Let $Y_1, Y_2, \ldots, Y_K$ be all graphs on the vertex set $\{1, 2, \ldots, k\}$, so $K = 2^{\binom{k}{2}}$. Let

$$y_{ij} = \begin{cases} 1, & \text{if } (Y_j)_v = Y_i \text{ for some } v \in \{1, 2, \ldots, k\} \\ 0, & \text{otherwise.} \end{cases}$$

The $K \times K$ matrix $Y = (y_{ij})$ has exactly k entries equal to 1 in each column.

Example 2.1.5. If $k = 2$, then there are two graphs on the vertex set $\{1, 2\}$. Let Y' be a 2×2 matrix whose rows and columns are indexed by these graphs. Each entry ij, of Y', is equal to 1 if we can obtain the graph associated to the ith row from a switching of the graph associated to the jth column; otherwise this entry is equal to 0. Since we can not obtain the original graph by realizing a switching in any vertex, this matrix is equal $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition 2.1.6. A **folded k-cube** is a graph whose vertices are the partition of a k-set into two subsets. Two partitions are **adjacent** if their common refinement contains a set of size one.

In order to prove Kelly's Lemma for vertex-switching we need know the eigenvalues of the folded k-cube. Next lemma present this eigenvalues. The proof of this result is beyond the scope of this work.

Lemma 2.1.7 ([3]). *Let $k > 2$. The eigenvalues of the adjacency matrix of the folded k-cube are*

$$\theta_j = k - 4j, 0 \leq j \leq \lfloor k/2 \rfloor$$

with multiplicity $f_j = \binom{k}{2j}$

For the proof of the previous lemma see [3].

Lemma 2.1.8 (Elligham and Royle [5]). *It is possible to reorder the graphs $Y_1, Y_2, \ldots, Y_K$ so that Y is a block diagonal matrix with $K/2^{k-1}$ blocks. Moreover each of the blocks has size 2^{k-1}, and is the adjacency matrix of a graph isomorphic to the folded k-cube.*

Proof. Given Y_i, there are 2^k subsets of vertices on which we can make a switching. Since switching a set W and switching $V(G) - W$ result in the same graph, we obtain at most 2^{k-1} distinct graphs. Let U and $V \subseteq \{1, 2, \ldots, k\}$ and let $U \triangle V$ be the symmetric difference between U and V, that is $U\triangle V=\{x\in U\cup V \mid x\notin U\cap V\}$. Since

$$\begin{aligned}(Y_i)_U &= (Y_i)_V\\ ((Y_i)_U)_V &= Y_i\\ (Y_i)_{U\triangle V} &= Y_i\end{aligned}$$

if and only if $U\triangle V=\emptyset$ or $V(G)$, that is $U=V$ or $U=V(G)-V$; we obtain exactly 2^{k-1} distinct graphs.

We will reorder the graphs $Y_1, Y_2, \ldots, Y_K$ as follows. We choose an arbitrary Y_{j_1} and consider a set A_1 of graphs obtained from Y_{j_1} by switching a subset of $\{1,2,\ldots,k\}$, so $|A_1|=2^{k-1}$. Then we choose some $Y_{j_2}\notin A_1$ and consider the set A_2 of graphs obtained from Y_{j_2} by switching a subset of $\{1,2,\ldots,k\}$. Next we choose some $Y_{j_3}\notin A_1\cup A_2$ and make the same thing we did with Y_{j_1} and Y_{j_2}. We repeat these steps always choosing Y_{j_k} that is not in any A_{j_i} constructed before. In this way we obtain $K/2^{k-1}$ sets $A_{j_i}'s$; with $|A_i|=2^{k-1}$, for every i.

Reordering the $Y_j's$, listing first all graphs that are in A_1, then the graphs that are in A_2, and so on, we obtain a block diagonal matrix.

Let Y_{i_j} be a graph on the vertex set $\{1,2,\ldots,k\}$. Let A_j be the set related to Y_{i_j}. For each graph $Y_{l_j}\in A_j$ we associate a partition $\{W_l,W_l^c\}$, where $W_l\subseteq\{1,2,\ldots,k\}$ is the set required to switch from Y_{l_j} to Y_{i_j}. A graph Y_{m_j} can be obtained from a 1-vertex switching of a graph Y_{r_j} if and only if the common refinement of the partitions $\{W_m,W_m^c\}$ and $\{W_r,W_r^c\}$ contains a set of size one. This is the definition of the adjacency matrix of the folded k-cube (see 2.1.6). □

Corollary 2.1.9 (Elligham and Royle [5]). *The eigenvalues of Y are*

$$\theta_j = k-4j,\ \textit{for}\ 0\le j\le\lfloor k/2\rfloor.$$

For each j the multiplicity of θ_j is $K\binom{k}{2j}/2^{k-1}$.

Proof. The result follows from Lemma 2.1.7 and Lemma 2.1.8. □

Now we will relate the eigenvalues of Y to the eigenvalues of M. Consider the symmetric group S_k acting on the set $\{Y_1, Y_2, \ldots, Y_K\}$, by permutation of the vertices of the Y_i's. The orbits of this group partitions this set into isomorphism classes. Let $C_1, C_2, \ldots, C_p$ be these isomorphism classes. Let

$$p_{ij} = \begin{cases} 1, & \text{if } Y_j \in C_i \\ 0, & \text{otherwise.} \end{cases}$$

$P = (p_{ij})$ is a $p \times K$ matrix with rows indexed by $C_1, C_2, \ldots, C_p$ and columns indexed by $Y_1, Y_2, \ldots, Y_K$. Let $X = (x_{ij})$, where $x_{ij} = X_1(C_j^* \to C_i^*)$.

Corollary 2.1.10 (Elligham and Royle [5])**.** *X is a block diagonal matrix, with each block corresponding to a switching matrix M_S of a given switching class S.*

Proof. Reordering the isomorphism classes $C_1, C_2, \ldots, C_p$, as we did with $Y_1, Y_2, \ldots, Y_K$ in the proof of Lemma 2.1.8, we obtain the result. □

Now we will associate the matrix Y to the matrix X.

Lemma 2.1.11 (Elligham and Royle [5])**.** *$PY = XP$. Furthermore any eigenvalue of Y is also an eigenvalue of X.*

Proof. We will consider the ij entry of each matrix. First consider $(PY)_{ij}$:

$$(PY)_{ij} = \sum_{l=1}^{K} p_{il} y_{lj}.$$

We have $p_{il} = 1$ if and only if $Y_l \in C_i$ and $y_{lj} = 1$ if and only if $(Y_j)_v = Y_l$ for some $v \in \{1, 2, \ldots, k\}$. So $(PY)_{ij}$ counts the number of switchings of Y_j that are in the isomorphism class C_i.

$$(XP)_{ij} = \sum_{l=1}^{K} x_{il} p_{lj}.$$

We have $p_{lj} = 1$ if and only if $Y_j \in C_l$ and $x_{il} = X_1(C_l^* \to C_i^*)$. So $(XP)_{ij}$ counts the number of switchings of Y_j that are isomorphic to C_i^*. Then $XP = PY$.

Let λ be an eigenvalue of Y associated to the eigenvector v. We have

$$\begin{aligned} XPv &= PYv \\ &= P\lambda v \\ &= \lambda Pv. \end{aligned}$$

So, we found that Pv is an eigenvector of X corresponding to λ. □

Next Theorem was proposed by Stanley, but we will present the proof of Elligham and Royle (see [5]).

Theorem 2.1.12 (Stanley [12])**.** *If $n \not\equiv 0 \pmod 4$, then G is vertex-switching reconstructible.*

Proof. Let H be a graph nonisomorphic to G, such that $D(H) = D(G)$. So $G \in C_i$ and $H \in C_j$, with $i \neq j$. Hence two columns of X are identical and 0 is an eigenvalue of X. Since the eigenvalues of X are of the form $n - 4j$ we conclude that $n \equiv 0 \pmod 4$. □

Theorem 2.1.13 (Elligham and Royle [5])**.** *Let S be a graph on k vertices. If $k < n/2$, then the number of induced subgraphs of G isomorphic to S is vertex-switching reconstructible.*

Proof. Let M_S be the switching matrix of the switching class of S. We have $n > 2k$, then

$$k - n < k - 2k \leq k - 4\lfloor k/2 \rfloor. \tag{2.2}$$

Let λ be an eigenvalue of M_S. Since M_S is a block of Y, from Lemma 2.1.11 we obtain λ is also an eigenvalue of the matrix X, and from Corollary 2.1.9 we have $\lambda = k -$

$4j$, for some $0 \le j \le \lfloor k/2 \rfloor$. So, we have

$$\begin{aligned} &0 \ge -4j \ge -4\lfloor k/2 \rfloor, \text{ and therefore} \\ &k \ge \lambda \ge k - 4\lfloor k/2 \rfloor. \end{aligned} \tag{2.3}$$

Looking at Expressions 2.2 and 2.3 we obtain $k-n$ is not an eigenvalue of M_S. So, the number of induced subgraphs of G isomorphic to S is reconstructible, from Proposition 2.1.3. □

Next corollary is a version of Kelly's Lemma for vertex-switching.

Corollary 2.1.14 (Elligham and Royle [5]). *Let S be a graph on k vertices. If $k < n/2$, then the number of subgraphs of G isomorphic to S is vertex-switching reconstructible.*

Proof. Let S be a graph on k vertices. From Theorem 2.1.13 the number of induced subgraphs isomorphic to a given graph T is reconstructible, that is $i(T,G)$ is reconstructible. Since $s(S,G) = \sum_{T \in \mathscr{R}_k} s(S,T) i(T,G)$ we have the result. □

Corollary 2.1.15 (Stanley). *If $n > 4$, then the number of edges of G is vertex-switching reconstructible.*

2.2 Reconstruction of special classes of graphs

In this section we will show that disconnected graphs, regular graphs and triangle-free graphs, on n vertices, are switching reconstructible when $n \neq 4$.

Lemma 2.2.1 (Krasikov and Roditty). *Let H be a graph of order n, nonisomorphic to G, such that $D(G) = D(H)$. If $n \neq 4$, then for every $u \in V(G)$ there exists $v \in V(G)$, with $u \neq v$, such that $G_{uv} \cong H$. Moreover $d_G(v) + d_G(u) = n - 2 + 2e_G(v,u)$.*

Proof. Let H be a graph of order n, nonisomorphic to G, such that $D(G) = D(H)$. Without loss of generality, we may assume $V(G) = V(H) = V$. Let $u \in V$. That implies for each graph in $D(G)$ there is a graph in $D(H)$ that is isomorphic to that first graph, that is there exists $v \in V$, such that $G_u \cong H_v$, and so $G_{uv} \cong H_{vv} \cong H$. As $H \not\cong G$, we have $u \neq v$.

Now we will prove the second part of the lemma. From Corollary 2.1.15, we have

$$|E(G)| = |E(H)| = |E(G_{uv})|. \tag{2.4}$$

Furthermore,

$$|E(G_u)| = |E(G)| - d_G(u) + (n-1) - d_G(u), \tag{2.5}$$

where $d_G(u)$ and $(n-1)-d_G(u)$ are, respectively, the number of edges deleted and added in the switching of G on u. From Equation 2.5:

$$\begin{aligned} |E(G_{uv})| &= |E(G_u)| - d_{G_u}(v) + [(n-1) - d_{G_u}(v)] && (2.6) \\ &= |E(G)| + 2(n-1) - 2d_G(u) - 2d_{G_u}(v). && (2.7) \end{aligned}$$

We have

$$\begin{aligned} d_{G_u}(v) &= \begin{cases} d_G(v) - 1, & \text{if } e_G(u,v) = 1 \\ d_G(v) + 1, & \text{if } e_G(u,v) = 0 \end{cases} \\ &= d_G(v) + 1 - 2e_G(u,v). \end{aligned}$$

From Equation 2.7, we have

$$|E(G_{uv})| = |E(G)| + 2(n-1) - 2d_G(u) - 2d_G(v) - 2 + 4e_G(u,v)$$

$$d_G(u) + d_G(v) = n - 2 + 2e_G(u,v), \text{ from Equation 2.4.}$$ □

Definition 2.2.2. Let u and $v \in V(G)$, we have a partition of $V(G) - \{u,v\}$ into the following sets:

- $N_G(v+,u+)$ is the set of vertices adjacent to both u and v;
- $N_G(v+,u-)$ is the set of vertices adjacent to v, but not u;
- $N_G(v-,u+)$ is the set of vertices adjacent to u, but not v;

- $N_G(v-,u-)$ is the set of vertices adjacent to neither u nor v.

Claim 2.2.3 (Elligham and Royle [5]). *For u and v as in Lemma 2.2.1, we have*

$$|N_G(v+,u+)| = |N_G(v-,u-)|.$$

Proof. From Lemma 2.2.1, the number of edges joining u and v to other vertices of G equals $n-2$, that is $0 = d_G(u) - d_G(v) - [(n-2) + 2e_G(u,v)]$. From Corollary 2.1.15, we know $|E(G)| = |E(G_{uv})|$, we obtain

$$\begin{aligned} 2|N_G(v+,u+)| + |N_G(v+,u-)| + |N_G(v-,u+)| &= n-2, \text{ from } G; \\ 2|N_G(v-,u-)| + |N_G(v+,u-)| + |N_G(v-,u+)| &= n-2, \text{ from } G_{uv}. \end{aligned}$$

That implies $|N_G(v+,u+)| = |N_G(v-,u-)|$. □

Claim 2.2.4 (Elligham and Royle [5]). *For u and v as in Lemma 2.2.1, we have $N_G(v+,u+) \neq \emptyset$.*

Proof. Suppose $N_G(v+,u+) = \emptyset$, from Claim 2.2.3, we have $N_G(v-,u-) = \emptyset$. Hence, $\psi : V \to V$ defined by

$$\psi(u) = v$$
$$\psi(v) = u$$
$$\psi(w) = w, \text{ for every } w \in V \setminus \{u,v\};$$

is an isomorphism between G and G_{uv}. That implies $G_{uv} \cong G$, but $G_{uv} \cong H \ncong G$, a contradiction. □

Theorem 2.2.5 (Krasikov). *Let G be a disconnected graph of order $n \neq 4$. Then G is vertex-switching reconstructible.*

Proof. Let H be a graph of order n, nonisomorphic to G, such that $D(G) = D(H)$. Without loss of generality, we may assume $V(G) = V(H) = V$. Let C be the component of G

with fewest vertices and let $v \in V(C)$. From Lemma 2.2.1, there exists $u \in V$ such that $G_{vu} \cong H$. By Claim 2.2.4, we have $u \in V(C)$. The $n-2$ edges that are joining u and v to the other vertices of G must be incident to at least $\frac{n-2}{2}$ other vertices; we have exactly this number when all vertices adjacent to u are also adjacent to v. Thus the component C has at least $\frac{n-2}{2}+2=\frac{n+2}{2}$ vertices. That is a contradiction, since C is the component with fewest vertices. Thus we have the result. □

Theorem 2.2.6 (Elligham and Royle [5]). *Let G and H be graphs such that $D(G) = D(H)$. Let $|V(G)| = n \neq 4$. If G is regular then $G \cong H$.*

Proof. Let G be an r-regular graph of order $n \neq 4$ and let $v \in V(G)$. We have $d_{G_v}(v) = (n-1)-r$. The vertices that were adjacent to v in G have degree $r-1$ in G_v and the vertices that were not adjacent to v in G have degree $r+1$ in G_v. We have three different cases:

Case. $n-1-r \neq r-1$ and $n-1-r \neq r+1$.

In this case we can easily identify v. Because $G_{vv} = G$, the graph G is reconstructible.

Case. $n-1-r = r-1$.

We have $r = n/2$. In this case $d_{G_v}(v) = r-1$ and v is adjacent to all vertices of degree $r+1$. Let u be a vertex of degree $r-1$ that is adjacent to all vertices of degree $r+1$ in G_v. Hence $\psi : V(G) \to V(G)$ defined by

$$\psi(u) = v$$
$$\psi(v) = u$$
$$\psi(w) = w, \text{ for every } w \in V(G) - \{u, v\};$$

is an automorphism of G_v. Then $G_{vv} \cong G_{vu}$. Thus G is reconstructible.

Case. $n-1-r = r+1$.

We have $r = \frac{n-2}{2}$. In G_v, the vertex v has degree $r+1$ and is adjacent to all vertices of degree $r+1$. Let u be a vertex in G_v of degree $r+1$ that is adjacent to all other vertices of degree $r+1$ in G_v. Hence $\psi : V(G) \to V(G)$ defined by

$$\psi(u) = v$$
$$\psi(v) = u$$
$$\psi(w) = w, \text{ for every } w \in V(G) - \{u,v\};$$

is an automorphism of G_v. Then $G_{vv} \cong G_{vu}$. Thus G is reconstructible. $\square$

Theorem 2.2.7 (Elligham and Royle [5])**.** *Let G be a triangle-free graph of order $n \neq 4$. Then G is vertex-switching reconstructible.*

Proof. Suppose that G is not reconstructible and let H be a graph of order n, nonisomorphic to G, such that $D(G) = D(H)$. Without loss of generality, we may assume $V(G) = V(H) = V$. From Theorem 2.1.12, $n \equiv 0 \pmod 4$, and then $n \geq 8$. As the number of triangles is reconstructible (see Theorem 2.1.4), H is also triangle-free.

Let v be an arbitrary vertex in V and let $u \in V$ such that $G_{vu} \cong H$, the existence of u affirmed in Lemma 2.2.1. Since $N_G(v+,u+) \neq \emptyset$ (by Claim 2.2.4), we have $e_G(u,v) = 0$. Let

$$A = N_G(v+,u+), \qquad B = N_G(v+,u-),$$
$$C = N_G(v-,u+), \qquad D = N_G(v-,u-).$$

For T and U subsets of V; we will denote the set of edges in $E(G)$ that are incident to a vertex in T and to another vertex in U by $E[T,U]$. We have $E[A,A] = E[A,B] = E[B,B] = E[A,C] = E[C,C] = \emptyset$, because G is triangle-free. Furthermore, we have $E[B,D] = E[C,D] = E[D,D] = \emptyset$, since $H \cong G_{uv}$ is also triangle-free. Therefore, the only edges that are neither incident to u nor to v are in some of the sets: $E[A,D]$ or $E[B,C]$. In order to obtain a contradiction, that is obtain G is reconstructible, we will divide into two cases:

Case. G has a cycle of odd length.

We claim that $G - v$ is bipartite; for example $(\{u\} \cup B \cup D,\ A \cup C)$ is a bipartition.

Hence, if K is a cycle of odd length in G, then $v \in V(K)$. Since v is arbitrary, we have $V(G) = V(K)$ and that implies $|V(K)| \equiv 0 \pmod 4$, a contradiction, since $|V(K)|$ is odd.

Case. G has no cycle of odd length.

Let $a \in A$; the existence of a is guaranteed by Claim 2.2.4. If there is an edge $\{b,c\}$ such that $b \in B$ and $c \in C$, then we have a cycle of odd length $avbcua$. Thus we conclude that $E[B,C] = \emptyset$. Hence each edge of G is in the set $E[\{v\},A \cup B]$, the set $E[\{u\},A \cup C]$, or the set $E[A,D]$. Consequently each vertex of B has v as its only neighbour and each vertex of C has u as its only neighbour. We have two subcases:

Subcase. $B = C = \emptyset$, for every v.

In the proof of Claim 2.2.3 we see that

$$\begin{aligned} n-2 &= 2|A| + |B| + |C| = 2|A| \\ |A| &= \frac{n-2}{2}. \end{aligned}$$

As $B = \emptyset$ and $N_G(v) = A$, we have $d_G(v) = \frac{n-2}{2}$. Since v is arbitrary, G is a regular graph and as stated by Theorem 2.2.6 is reconstructible, a contradiction!

Subcase. B or $C \neq \emptyset$, for some v.

Without loss of generality, we may assume that $B \neq \emptyset$. Let $v' \in B$ and let $u' \in V$ such that $G_{v'u'} \cong H$. According to Claim 2.2.4, we have $A' = N_G(v'+,u'+) \neq \emptyset$, furthermore since v is the only neighbour of v', then $A' = \{v'\}$ and $B' = N_G(v'+,u'-) = \emptyset$. In consonance with Claim 2.2.3, we have $D' = N_G(v'-,u'-) = \{x\}$, for some $x \in V$. Hence all edges of G are the edge $\{v,v'\}$, the edge $\{v,u'\}$, edges joining u' to $C' =$

$N_G(v'-, u'+)$ and maybe $\{v, x\}$. Hence $\psi : V \to V$, defined by

$$\begin{aligned}
\psi(v) &= x, \\
\psi(x) &= v, \\
\psi(v') &= u', \\
\psi(u') &= v', \\
\psi(c) &= c, \text{ for every } c \in C';
\end{aligned}$$

is an isomorphism from G to $G_{v'u'} \cong H$, a contradiction!

So, G is reconstructible. □

Chapter 3

s-Vertex-switching reconstruction

The main goal of this chapter is to present conditions under which a graph is s-switching reconstructible. For this we will make use of the Krawtchouk polynomial and will analyse its roots in a given interval. We will also present conditions under which the number of induced subgraphs of a given graph is s-switching reconstructible, a generalization of Theorem 2.1.13. With exception of the results indicated, all results in this chapter can be found in [8].

In this chapter, we will denote by G an arbitrary graph on n vertices.

3.1 Krawtchouk polynomials

Definition 3.1.1. A **binary form** of degree k in the variables x and y is a polynomial of the form $\sum_{i=0}^{k} a_i x^{k-i} y^i$.

For $s \in \mathbb{N}$ we define the **Krawtchouk polynomial** by

$$K_s(x;n,q) := \sum_{i=0}^{s} (-1)^i \binom{x}{i} \binom{n-x}{s-i} (q-1)^{s-i},$$

where q is some prime power and n is some positive integer. The Krawtchouk polynomial is a polynomial with integral domain. For the special case $q = 2$ we will denote the Krawtchouk polynomial by $K_s^n(x)$, we have

$$K_s^n(x) = \sum_{i=0}^{s} (-1)^i \binom{x}{i} \binom{n-x}{s-i}.$$

We set $P_s^n(y) = K_s^n((n-y)/2)$.

The proof of the following result can be found in [12].

Theorem 3.1.2 (Stanley [12])**.** *If $K_s^n(x)$ has no even integer roots in the interval $[0,n]$, then G is s-vertex switching reconstructible.*

We will not present the proof of the above result since it require some techniques that are beyond the scope of this work. The proof of the this result can be found in [12].

Now we will present some properties of the Krawtchouk polynomial. In order to prove these properties we need some propositions that follow from the definition of the Krawtchouk polynomial.

Proposition 3.1.3 ([15])**.**

$$\sum_{k=0}^{\infty} K_k(x;n,q) z^k = [1+(q-1)z]^{n-x}(1-z)^x, \tag{3.1}$$

where $|z| < 1$.

Proof. The Taylor series of $[1+(q-1)z]^{n-x}$ and $(1-z)^x$ are respectively

$$[1+(q-1)z]^{n-x} = \sum_{i=0}^{\infty} (q-1)^i \binom{n-x}{i} z^i, \tag{3.2}$$

$$(1-z)^x = \sum_{j=0}^{\infty} (-1)^j \binom{x}{j} z^j. \tag{3.3}$$

We have

$$[1+(q-1)z]^{n-x}(1-z)^x = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^j\binom{x}{j}\binom{n-x}{k-j}(q-1)^{k-j}\right)z^k$$
$$= \sum_{k=0}^{\infty} K_k(x;n,q)z^k. \qquad \square$$

Proposition 3.1.4 ([15]). *The Krawtchouk polynomials satisfy the recurrence relation*

$$(s+1)K_{s+1}(x;n,q) = [s+(q-1)(n-s)-qx]K_s(x;n,q)-(q-1)(n-s+1)K_{s-1}(x;n,q).$$

For $q=2$ we have

$$(s+1)K^n_{s+1}(x) = (n-2x)K^n_s(x)-(n-s+1)K^n_{s-1}(x).$$

Proof. In order to prove this recurrence relation we need make some manipulation of equation 3.1 and use Taylor series. We present the resulting calculations now.

By differentiating both sides of Equation 3.1 with respect to z we obtain

$$\sum_{k=1}^{\infty} kz^{k-1}K_k(x;n,q) = (n-x)(q-1)[1+(q-1)z]^{n-x-1}(1-z)^x - x[1+(q-1)z]^{n-x}(1-z)^{x-1}.$$

By taking $l=k-1$, we have

$$\sum_{l=0}^{\infty}(l+1)z^l K_{l+1}(x;n,q) = \left[\frac{(n-x)(q-1)}{1+(q-1)z} - \frac{x}{(1-z)}\right][1+(q-1)z]^{n-x}(1-z)^x.$$

Multiplying both sides by $[1+(q-1)z](1-z)$ and taking some common factors

$$\sum_{l=0}^{\infty}(l+1)[1+(q-2)z-(q-1)z^2]z^l K_{l+1}(x;n,q) =$$
$$[(q-1)(n-x-nz)-x][1+(q-1)z]^{n-x}(1-z)^x.$$

Looking at the Taylor series of $[1+(q-1)z]^{n-x}$ and $(1-z)^x$ (see Equations 3.2

and 3.3), comparing the coefficients of z^s we obtain

$$\begin{aligned}(s+1)K_{s+1}(x) &= -(q-2)sK_s(x)+(q-1)(s-1)z^2K_{s-1}(x)\\ &\quad +[(n-x)(q-1)-x]\left[\sum_{j=0}^{s}(-1)^j\binom{n-x}{s-j}\binom{x}{j}(q-1)^{s-j}\right]\\ &\quad -n(q-1)\left[\sum_{j=0}^{s-1}(-1)^j\binom{n-x}{s-j-1}\binom{x}{j}(q-1)^{s-j-1}\right].\end{aligned}$$

From the Definition of the Krawtchouk polynomial 3.1.1, we have

$$\begin{aligned}(s+1)K_{s+1}(x) &= [(n-x)(q-1)-x-(q-2)s]K_s(x)+(q-1)(s-1-n)K_{s-1}(x)\\ &= [s+(q-1)(n-s)-qx]K_s(x)-(q-1)(n-s+1)K_{s-1}(x).\end{aligned}$$

By taking $q=2$, in the previous expression, we obtain

$$(s+1)K^n_{s+1}(x)=[s+(n-s)-2x]K^n_s(x)-(n-s+1)K^n_{s-1}(x).$$ □

Corollary 3.1.5 ([8]). *Let $x=\frac{n-y}{2}$. The $P^n_s(y)$ satisfy the recurrence relation*

$$\begin{aligned}(s+1)P^n_{s+1}(y) &= yP^n_s(y)-(n-s+1)P^n_{s-1}(y),\\ P^n_0(y)&=1,\ P^n_1(y)=y.\end{aligned} \tag{3.4}$$

Proof. Taking $x=\frac{n-y}{2}$ in Proposition 3.1.4, and making use of the definition of P^n_s, we obtain

$$\begin{aligned}(s+1)K^n_{s+1}((n-y)/2) &= yK^n_s((n-y)/2)-(n-s+1)K^n_{s-1}((n-y)/2)\\ (s+1)P^n_{s+1}(y) &= yP^n_s(y)-(n-s+1)P^n_{s-1}(y),\\ P^n_0(y) &= \binom{\frac{n-y}{2}}{0}\binom{\frac{n+y}{2}}{0}\\ &=1;\end{aligned}$$

$$
\begin{aligned}
P_1^n(y) &= \binom{\frac{n-y}{2}}{0}\binom{\frac{n+y}{2}}{1} - \binom{\frac{n-y}{2}}{1}\binom{\frac{n+y}{2}}{0} \\
&= y.
\end{aligned}
$$

□

Taking $z = y^2$ in Equation 3.4, and by induction on s, we may rewrite $P_s^n(y)$ as the sum of a binary form and a polynomial of degree less than that of the binary form in question.

Lemma 3.1.6 ([8]). *Let $x = \frac{n-y}{2}$ and $z = y^2$. Then we have*

$$
\begin{aligned}
P_{2s}^n(y) &= f_{2s}(z,n) + g_{2s}(z,n) \\
P_{2s+1}^n(y) &= z^{1/2}(f_{2s+1}(z,n) + g_{2s+1}(z,n)),
\end{aligned}
$$

where $f_i(z,n)$ is a binary form of degree $\lfloor i/2 \rfloor$ and $g_i(z,n)$ is a polynomial of degree less than the degree of $f_i(z,n)$.

Proof. We prove the result by induction on s. The majority part of the calculus presented in this proof are taking by rewriting the equations as sums of f_k's and g_k's.

For $s = 1$

$$
\begin{aligned}
2P_2^n(y) &= yP_1^n(y) - nP_0^n(y), \text{ from Equation 3.4.} \\
P_2^n(y) &= \frac{y^2 - n}{2} \\
P_2^n(y) &= \frac{z-n}{2} \\
P_2^n(y) &= f_2(z,n), \text{where } f_2(z,n) = \tfrac{z-n}{2} \text{ is a binary form of degree 1.}
\end{aligned}
$$

$$
\begin{aligned}
3P_3^n(y) &= yP_2^n(y) - (n-2+1)P_1^n(y), \text{ from Equation 3.4.} \\
3P_3^n(y) &= \frac{y^3 - ny}{2} - ny + y \\
P_3^n(y) &= \frac{y^3 - 3ny + 2y}{6}
\end{aligned}
$$

$$P_3^n(y) = \sqrt{z}\,\frac{z-3n+2}{6}$$
$$P_3^n(y) = \sqrt{z}\left(f_3(z,n)+g_3(z,n)\right),$$

where $f_3(z,n) = \frac{z-3n}{6}$ is a binary form of degree 1 and $g_3(z,n) = \frac{1}{3}$ is a polynomial of degree 0.

Suppose the hypothesis is valid for every $s < k$. Let $s = k$. From Corollary 3.1.5, we have

$$(2k)P_{2k}^n(y) = yP_{2k-1}^n(y) - (n-2k)P_{2k-2}^n(y).$$

By the induction hypothesis, we obtain

$$(2k)P_{2k}^n(y) = z(f_{2k-1}(z,n)+g_{2k-1}(z,n)) - (n-2k)(f_{2(k-1)}(z,n)+g_{2(k-1)}(z,n)),$$

where $f_{2k-1}(z,n) = \sum_{i=0}^{k-1} a_i z^{k-i-1} n^i$ and $g_{2k-1}(z,n)$ is a polynomial of degree less than $k-1$. Furthermore, $f_{2(k-1)}(z,n) = \sum_{i=0}^{k-1} b_i z^{k-i-1} n^i$, and $g_{2k-1}(z,n)$ is a polynomial of degree less than $k-1$. Hence, we have

$$(2k)P_{2k}^n(y) = \sum_{i=0}^{k-1} a_i z^{k-i} n^i + zg_{2k-1}(z,n) - \sum_{i=0}^{k-1} b_i z^{k-i-1} n^{i+1} + 2k\sum_{i=0}^{k-1} b_i z^{k-i-1} n^i$$
$$- ng_{2k-1}(z,n) + 2kg_{2k-1}(z,n).$$

Let $g_{2k}(z,n) = zg_{2k-1}(z,n) - ng_{2k-1}(z,n) + 2kg_{2k-1}(z,n)$. By rewriting $\sum\limits_{i=0}^{k-1} b_i z^{k-i-1} n^{i+1}$ as $\sum\limits_{i=0}^{k} d_i z^{k-i} n^i$, where $d_i = b_{i+1}$, for $i > 0$ and $d_0 = 0$, we obtain

$$(2k)P_{2k}^n(y) = \sum_{i=0}^{k-1} a_i z^{k-i} n^i - \sum_{i=0}^{k} d_i z^{k-i} n^i + \sum_{i=0}^{k-1} 2kb_i z^{k-i-1} n^i + g_{2k}(z,n)$$
$$P_{2k}^n(y) = f_{2k}(z,n) + g_{2k}(z,n),$$

where $f_{2k}(z,n)$ is a binary form of degree k and $g_{2k}(z,n)$ is a polynomial of degree less than k.

Analogously, we have

$$P^n_{2k+1}(y) = z^{1/2}(f_{2k+1}(z,n) + g_{2k+1}(z,n)). \qquad \square$$

For $s \in \mathbb{N}$, we set $Q_{2s} = (2s)!f_{2s}(z,n)$ and $Q_{2s+1} = (2s+1)!z^{1/2}f_{2s+1}(z,n)$.

Corollary 3.1.7 ([8]). *$Q_s(z,n)$ satisfy the recurrence relation.*

$$\begin{aligned} Q_{s+1}(z,n) &= z^{1/2}Q_s(z,n) - nsQ_{s-1}(z,n), \\ Q_0(z,n) &= 1,\ Q_1(z,n) = z^{1/2}. \end{aligned} \tag{3.5}$$

Proof. We split into two cases according to parity of s and looking at the recurrence relation of P^n_k, the equations in Lemma 3.1.6 and the definition of Q_i.

Case. s is even.

Let $s = 2k$ for some $k \in \mathbb{N}$. From Equation 3.4, we obtain

$$(2k+1)P^n_{2k+1}(y) = yP^n_{2k}(y) - (n-2k+1)P^n_{2k-1}(y).$$

By making use of the equations in Lemma 3.1.6, we obtain

$$\begin{aligned} (2k+1)z^{1/2}(f_{2k+1}(z,n) + g_{2k+1}(z,n)) &= z^{1/2}(f_{2k}(z,n) + g_{2k}(z,n)) \\ &\quad - (n-2k+1)z^{1/2}(f_{2k-1}(z,n) + g_{2k-1}(z,n)). \end{aligned}$$

By looking at the terms of $k+1/2$, we obtain

$$(2k+1)z^{1/2}f_{2k+1}(z,n) = z^{1/2}f_{2k}(z,n) - nz^{1/2}f_{2k-1}(z,n).$$

From the definition of Q_i, we have

$$\frac{Q_{2k+1}}{(2k)!} = \frac{z^{1/2}Q_{2k}}{(2k)!} - \frac{nQ_{2k-1}}{(2k-1)!}.$$

Hence, we have

$$Q_{2k+1} = z^{1/2}Q_{2k} - n(2k)Q_{2k-1}.$$

Case. s is odd.

Let $s = 2k+1$ for some $k \in \mathbb{N}$. From the Equation of recurrence 3.4 we obtain

$$(2k)P^n_{2k}(y) = yP^n_{2k-1}(y) - (n-2k+2)P^n_{2k-2}(y).$$

By making use of the equations in Lemma 3.1.6, we have

$$(2k)(f_{2k}(z,n) + g_{2k}(z,n)) = z(f_{2k-1}(z,n) + g_{2k-1}(z,n)) - \\ (n-2k+2)(f_{2k-2}(z,n) + g_{2k-2}(z,n)).$$

By looking at the terms of degree k, we obtain

$$(2k)f_{2k}(z,n) = zf_{2k-1}(z,n) - nf_{2k-2}(z,n).$$

From the definition of Q_i, we obtain

$$\frac{Q_{2k}}{(2k-1)!} = \frac{z^{1/2}Q_{2k-1}}{(2k-1)!} - \frac{nQ_{2k-2}}{(2k-2)!}.$$

Hence, we have

$$Q_{2k} = z^{1/2}Q_{2k-1} - n(2k-1)Q_{2k-2}. \qquad \square$$

Proposition 3.1.8 ([8]).

$$Q_{2s}(z,n) = \sum_{i=0}^{s}(-1)^i\binom{2s}{2i}(2i-1)!!z^{s-i}n^i, \text{ and}$$
$$Q_{2s+1}(z,n) = z^{1/2}\sum_{i=0}^{s}(-1)^i\binom{2s+1}{2i}(2i-1)!!z^{s-i}n^i,$$

where $(2m-1)!! = \prod_{j=1}^{m}(2j-1)$, $(-1)!! = 1$.

Proof. We will prove the result by induction on s.

For $s = 0$

$$\begin{aligned} Q_0(z,n) &= 1 \\ &= (-1)^0\binom{2s}{0}(-1)!!z^0n^0 \\ &= Q_{2\cdot 0}(z,n), \\ Q_1(z,n) &= z^{1/2} \\ &= z^{1/2}(-1)^0\binom{2s+1}{0}(-1)!!z^0n^0 \\ &= Q_{2\cdot 0+1}(z,n). \end{aligned}$$

Suppose the result is valid for $s < k$, the result is obtained from some algebraic manipulation. For $s = k$, from the recurrence relation of Q_s (see Equation 3.5), we have

$$Q_{2s}(z,n) = z^{1/2}Q_{2s-1}(z,n) - n(2s-1)Q_{2(s-1)}(z,n).$$

By the induction hypothesis, we have

$$Q_{2s}(z,n) = z\sum_{i=0}^{s-1}(-1)^i\binom{2s-1}{2i}(2i-1)!!z^{s-1-i}n^i$$

$$
\begin{aligned}
&- n(2s-1)\sum_{i=0}^{s-1}(-1)^i\binom{2s-2}{2i}(2i-1)!!z^{s-1-i}n^i \\
&= \sum_{i=0}^{s-1}(-1)^i\binom{2s-1}{2i}(2i-1)!!z^{s-i}n^i \\
&- \sum_{i=0}^{s-1}(-1)^i\binom{2s-2}{2i}(2s-1)(2i-1)!!z^{s-1-i}n^{i+1}.
\end{aligned}
$$

Taking $j = i+1$ in the second sum, we obtain

$$
\begin{aligned}
Q_{2s}(z,n) &= \sum_{i=0}^{s-1}(-1)^i\binom{2s-1}{2i}(2i-1)!!z^{s-i}n^i \\
&+ \sum_{j=1}^{s}(-1)^j\binom{2s-2}{2j-2}(2s-1)(2j-3)!!z^{s-j}n^j.
\end{aligned}
$$

Since $\binom{n}{k} = 0$, for $j < 0$, we have

$$
\begin{aligned}
Q_{2s}(z,n) &= \sum_{i=0}^{s}(-1)^i z^{s-i}n^i\left[\binom{2s-1}{2i}(2i-1)!! + \binom{2s-2}{2i-2}(2s-1)(2i-3)!!\right] \\
&= \sum_{i=0}^{s}(-1)^i z^{s-i}n^i\left[\frac{(2s-1)!(2i-1)!!}{(2s-2i-1)!(2i)!} + \frac{(2s-1)!(2i-3)!!}{(2s-2i)!(2i-2)!}\right] \\
&= \sum_{i=0}^{s}(-1)^i z^{s-i}n^i\frac{(2s-1)!}{(2s-2i-1)!}\left[\frac{1}{2^i\cdot i!} + \frac{1}{2^{i-1}(i-1)!(2s-2i)}\right] \\
&= \sum_{i=0}^{s}(-1)^i z^{s-i}n^i\frac{(2s-1)!}{(2s-2i-1)!}\left(\frac{2s}{2^i\cdot i!(2s-2i)}\right) \\
&= \sum_{i=0}^{s}(-1)^i z^{s-i}n^i\frac{(2s)!(2i-1)!!}{(2s-2i)!(2i)!} \\
&= \sum_{i=0}^{s}(-1)^i\binom{2s}{2i}(2i-1)!!z^{s-i}n^i.
\end{aligned}
$$

Similarly, we have

$$
Q_{2s+1}(z,n) = z^{1/2}Q_{2s}(z,n) - n\cdot 2sQ_{2s-1}(z,n).
$$

By the induction hypothesis

$$\begin{aligned}Q_{2s+1}(z,n) &= z^{1/2}\sum_{i=0}^{s}(-1)^i\binom{2s}{2i}(2i-1)!!z^{s-i}n^i \\ &-n\cdot 2s\cdot z^{1/2}\sum_{i=0}^{s-1}(-1)^i\binom{2s-1}{2i}(2i-1)!!z^{s-1-i}n^i \\ &= z^{1/2}\sum_{i=0}^{s}(-1)^i\binom{2s}{2i}(2i-1)!!z^{s-i}n^i \\ &-z^{1/2}\sum_{i=0}^{s-1}(-1)^i\binom{2s-1}{2i}2s(2i-1)!!z^{s-1-i}n^{i+1}.\end{aligned}$$

Taking $i+1=j$ in the second sum, we have

$$\begin{aligned}Q_{2s+1}(z,n) &= z^{1/2}\sum_{i=0}^{s}(-1)^i\binom{2s}{2i}(2i-1)!!z^{s-i}n^i \\ &+z^{1/2}\sum_{j=1}^{s}(-1)^j\binom{2s-1}{2j-2}2s(2j-3)!!z^{s-j}n^j.\end{aligned}$$

Since $\binom{n}{k}=0$, for $k<0$, we have

$$\begin{aligned}Q_{2s+1}(z,n) &= z^{1/2}\sum_{i=0}^{s}(-1)^i z^{s-i}n^i\left[\binom{2s}{2i}(2i-1)!!+\binom{2s-1}{2i-2}2s(2i-3)!!\right] \\ &= z^{1/2}\sum_{i=0}^{s}(-1)^i z^{s-i}n^i\left(\frac{(2s)!(2i-1)!!}{(2s-2i)!(2i)!}+\frac{(2s)!(2i-3)!!}{(s-2i+1)!(2i-2)!}\right) \\ &= z^{1/2}\sum_{i=0}^{s}(-1)^i z^{s-i}n^i\frac{(2s)!}{(2s-2i)!}\left(\frac{1}{2^i\cdot i!}+\frac{1}{(s-2i+1)2^{i-1}(i-1)!}\right) \\ &= z^{1/2}\sum_{i=0}^{s}(-1)^i z^{s-i}n^i\frac{(2s)!}{(2s-2i)!}\frac{s+1}{(s-2i+1)2^i\cdot i!} \\ &= z^{1/2}\sum_{i=0}^{s}(-1)^i z^{s-i}n^i\frac{(2s+1)!(2i-1)!!}{(2s-2i+1)!(2i)!} \\ &= z^{1/2}\sum_{i=0}^{s}(-1)^i\binom{2s+1}{2i}(2i-1)!!z^{s-i}n^i.\end{aligned}$$

□

To prove that Q_{2s} is irreducible, we will make use of Eisenstein criterion. We will not present the proof of this criterion, since it uses techniques that are beyond the scope of this work. This proof can be found in [7], but first we will present a definition.

Definition 3.1.9. The polynomial $f(x) = a^n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n, \ldots, a_0 \in \mathbb{Z}$ is called **primitive** if the greatest common divisor of the terms $a_n, \ldots, a_0$ is 1.

Proposition 3.1.10 (Eisenstein Criterion)**.** *Let $f(x) = a^n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with integer coefficients. Suppose there exists some prime p such that $p \nmid a_n$, but p divides the other coefficients $a_{n-1}, \ldots, a_0$ and $p^2 \nmid a_0$. Then $f(x)$ is irreducible over the rationals. If $f(x)$ is primitive, it is irreducible in $\mathbb{Z}[x]$.*

Lemma 3.1.11 ([8])**.** *Q_s is a polynomial with integer coefficients.*

Proof. Looking at the definition of the Krawtchouk polynomial (see Definition 3.1.1) and rewriting the $K_s^n(x)$ we obtain

$$K_s^n(x) = \sum_{i=0}^{s} \frac{1}{s!}\binom{s}{i} x \cdot \ldots \cdot (x-i+1) \cdot (n-x) \cdot \ldots \cdot [n-x-(s-i)+1].$$

From the definition of $P_s^n(y)$ (see Definition 3.1.1), we have $P_s^n(y) = (1/s!)(y^s + \alpha_{s-1}(n)y^{s-1} + \cdots + \alpha_0(n))$, with $\alpha_i(n) \in \mathbb{Z}$. From the definition of Q_s, we conclude that the coefficients of Q_s are integer numbers. □

Proposition 3.1.12 ([8])**.** *$Q_s(z,n)$ is irreducible, for $s \geq 6$.*

Proof. We will prove that Q_k is irreducible when k is even. The case k odd is similar. Let $k = 2s$, for some $s \geq 3$. We have

$$Q_{2s}(z,n) = \sum_{i=0}^{s} a_i z^{s-i} n^i, \text{ where } a_i = (-1)^i \binom{2s}{2i}(2i-1)!!,\ 0 \leq i \leq s.$$

Let p be the largest prime less than $2s$, which must be odd since $s \geq 3$. We have $2p > 2s$ and therefore p is the only multiple of p that is a term of $(2s-1)!!$. So $p^2 \nmid (2s-1)!!$, thus $p^2 \nmid a_s$. We have $a_0 = 1$, that implies $p \nmid a_0$. Now we will look at the coefficients a_i, for $0 < i < s$, we have two cases.

Case. $2i-1 \geq p$. Since p is odd and that implies $p \mid (2i-1)!!$. So $p \mid a_i$.
Case. $2i-1 < p$. In this case $p \mid a_i$, since $p \mid \binom{2s}{2i}$.

Since $a_0 = 1$, clearly $Q_{2s}(z,n)$ is primitive. □

In the next theorem we present a property of a diophantine equation with integer coefficients. The proof of this result is beyond the scope of this work, but can be found in [11].

Theorem 3.1.13 (Siegel [11]). *Let*

$$y^2 = a_0x^n + a_1x^{n-1} + \ldots + a_n \tag{3.6}$$

be an equation with integer coefficients. If the right-hand side of this equation has at least three different roots over $\mathbb{C}$, then 3.6 has only finitely many integer solutions.

The proof of the following result is also beyond the scope of this work and can be found in [10].

Theorem 3.1.14 ([10]). *Let $\mathbb{Z}[x,y]$ be the set of polynomials in two variables with integer coefficients. If $f \in \mathbb{Z}[x,y]$ is an irreducible binary form with degree at least three and $g \in \mathbb{Z}[x,y]$ has degree less than the degree of f then $f(x,y) = g(x,y)$ has only finitely many integer solutions.*

3.2 s-vertex-switching reconstruction

Theorem 3.2.1 (Krasikov and Roditty [8]). *G is s-vertex-switching reconstructible if*

i. $s = 1$ *and* $n \not\equiv 0 \pmod 4$*;*

ii. $s = 2$ *and* $n \neq t^2$*, where* $t \equiv 0, 1 \pmod 4$*;*

iii. $s = 3$ *and* $n \not\equiv 0 \pmod 4$, *and* $n \neq \frac{t^2+2}{3}$, *where* $t \equiv 1, 2, 5, 10 \pmod{12}$.

Furthermore, for each $s \geq 4$, *there exists* $N_s \in \mathbb{Z}$ *such that G is s-vertex-switching reconstructible if*

iv. $n > N_s$, *for s even;*

v. $n > N_s$ *and* $n \not\equiv 0 \pmod 4$, *for s odd.*

Proof. Let G be not reconstructible. From Theorem 3.1.2, there exists an even integer $x \in [0, n]$ such that $K_s^n(x) = 0$ and observe that $y \equiv n \pmod 4$. Let $x = \frac{n-y}{2}$. We have six cases.

Case. $s = 1$.

We proved this case in Chapter 2 (see Theorem 2.2.6), but we will present a new proof of it, since this will help us to understand the next cases.

By calculating $K_1^n((n-y)/2)$, we obtain

$$\begin{aligned} 0 &= K_1^n((n-y)/2) \\ &= P_1^n(y) \\ &= y \text{ (see Corollary 3.1.5).} \end{aligned}$$

Since $y \equiv n \pmod 4$, we have $n \equiv 0 \pmod 4$.

Case. $s = 2$. By calculating $K_2^n((n-y)/2)$, we obtain

$$\begin{aligned} 0 &= K_2^n((n-y)/2) \\ &= P_2^n(y) \\ &= \frac{y^2 - n}{2} \text{ (see Lemma 3.1.6).} \end{aligned}$$

Therefore, we have $n = y^2 \equiv y \pmod 4$. Then $y \equiv 0, 1 \pmod 4$.

Case. $s = 3$

By calculating $K_3^n((n-y)/2)$, we obtain

$$\begin{aligned} 0 &= K_3^n((n-y)/2) \\ &= P_3^n(y) \\ &= \frac{y}{6}(y^2-3n+2) \text{ (see Lemma 3.1.6).} \end{aligned}$$

Therefore $y=0$ or $y^2-3n+2=0$. In the first case $n \equiv 0 \pmod 4$, or $y^2-3n+2=0$. In the second case we have $\frac{y^2+2}{3} \in \mathbb{Z}$. And so, we have

$$\begin{aligned} y^2 &\equiv 1 \pmod 3 \\ y &\equiv 1,2 \pmod 3. \end{aligned}$$

Moreover, we have

$$\begin{aligned} y^2 &\equiv 3y-2 \pmod 4 \\ y &\equiv 1,2 \pmod 4. \end{aligned}$$

Thus $y \equiv 1,2,5,10 \pmod{12}$.

Case. $s=4$

By calculating $K_4^n((n-y)/2)$, we obtain

$$\begin{aligned} 0 &= K_4^n((n-y)/2) \\ &= P_4^n(y) \\ &= \frac{yP_3^n(y)-(n-2)P_2^n(y)}{4}, \text{ from Corollary 3.1.5.} \\ 0 &= \frac{\frac{y^2}{6}(y^2-3n+2)-(n-2)\frac{y^2-n}{2}}{4} \\ &= \frac{y^4+(8-6n)y^2+(3n^2-6n)}{24}. \end{aligned}$$

We have

$$n = (y^2+1) \pm \sqrt{\frac{6y^4-6y^2+9}{9}}.$$

Since n is integer, we have

$$6y^4-6y^2+9 = k^2, \text{ for some } k \in \mathbb{Z}. \tag{3.7}$$

From Theorem 3.1.13, Equation 3.7 has only finitely many solutions. Thus $K_4^n(x)$ has even roots only for finitely many n's And so, from Theorem 3.1.2, only a finitely number of graphs are not s-switching reconstructible.

Case. $s = 5$.

We have

$$\begin{aligned}
0 &= K_5^n((n-y)/2) \\
&= P_5^n(y) \\
&= \frac{yP_4^n(y)-(n-3)P_3^n(y)}{5}, \text{ from Corollary 3.1.5.} \\
0 &= \frac{y\left[\frac{y^4+(8-6n)y^2+(3n^2-6n)}{24}\right]-(n-3)\left[\frac{y}{6}(y^2-3n+2)\right]}{5} \\
&= \frac{y\left(15n^2-10ny^2-50n+y^4+20y^2+24\right)}{5!}.
\end{aligned}$$

Therefore $y = 0$ and that implies $n \equiv 0 \pmod 4$ or $y^4+(20-10n)y^2+15n^2-50n+24 = 0$. In the second case we have

$$n = \frac{y^2+5}{3} \mp \frac{\sqrt{10y^4-50y^2+265}}{15},$$

where $\sqrt{10y^4-50y^2+265} \in \mathbb{Z}$. From Theorem 3.1.13, has only finitely many y's that satisfy the previous condition.

Case. $s \geq 6$

Since G is not reconstructible, therefore by Theorem 3.1.2 $K_s^n(x)$ has an even root.

From Lemma 3.1.6, we have

$$\begin{cases} 0 = f_s(z,n) + g_s(z,n), & \text{if } s \text{ is even,} \\ y = 0 \text{ or } 0 = f_s(z,n) + g_s(z,n), & \text{if } s \text{ is odd.} \end{cases}$$

From Proposition 3.1.12, we have $Q_k(z,n)$ is irreducible for every $k \geq 6$. From the definition of Q_k, we obtain $f_{2s}(z,n) = \frac{Q_{2s}(z,n)}{(2s)!}$ and $f_{2s+1}(z,n) = \frac{z^{-1/2}Q_{2s+1}(z,n)}{(2s+1)!}$. Then $f_s(z,n)$ is irreducible, for every s. We can apply Theorem 3.1.14, and so the equation $0 = f_s(z,n) + g_s(z,n)$ has only finitely many integer solutions. From the case $y = 0$ when s is odd we obtain $n \equiv 0 \pmod 4$, since $n \equiv y \pmod 4$. □

3.3 Reconstructing the number of subgraphs

In this section, we will present a condition guaranteeing that the number of induced subgraphs of G, isomorphic to a given graph S is s-switching reconstructible. For this we need to define matrices related to classes of switching, similarly to the matrix M in Chapter 2.

Definition 3.3.1. Let $r \in \mathbb{N}$ and let $\{H^1, H^2, \ldots\} = \mathscr{R}_r$. Let $A_k^r(ij)$ be a matrix whose rows and columns are indexed by the elements of $\mathscr{R}_r$. Each ij entry is equal

$$\begin{aligned} a_{ij} &= |\{W \subseteq V(H^j) : H^j_W \cong H^i, |W| = k\}| \\ &= X_k(H^j \to H^i). \end{aligned}$$

Remark 3.3.2. We have $A_0^r = A_r^r = I_R$, where $R = |\mathscr{R}_r|$, since switching a set W and switching $V(G) - W$ results in the same graph. Moreover, we have $A_1^r = X$, where X is the matrix defined in Chapter 2.

Proposition 3.3.3 ([8]). *Two nonisomorphic graphs have the same s-switching deck if and only if the corresponding columns of A_s^n are equal.*

Proof. The ijth entry of A_s^n is the number of graphs isomorphic to H^i in the s-switching deck of H^j. □

Definition 3.3.4. We define the matrix B_s^r to be

$$B_s^r = \sum_{k=0}^{r} \binom{n-r}{s-k} A_k^r.$$

Let b_{ij} be ijth entry of B_s^r.

Lemma 3.3.5 (Levy-Desplanques Theorem)**.** *Let $A = (a_{ij})$ be an $n \times n$ matrix with non-negative entries such that $a_{ii} > \sum_{j \neq i} a_{ji}$, that is a diagonally dominant matrix. Then* $\det A \neq 0$.

Proof. By induction on n. For $n = 1$ the result is trivial. Suppose the result valid for $n \leq k$. Let $n = k+1$. When we perform operations $R_i - \frac{a_{i1}}{a_{11}} R_1 \to R_i$ we obtain the new entries $a'_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}$, for all $i > 1$ and for all j. We have

$$\begin{aligned} a'_{jj} - \sum_{\substack{i \neq j \\ i \neq 1}} a'_{ij} &= a_{jj} - \frac{a_{j1}}{a_{11}} a_{1j} - \sum_{\substack{i \neq j \\ i \neq 1}} \left(a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \right) \\ &= a_{jj} - \sum_{\substack{i \neq j \\ i \neq 1}} a_{ij} + \frac{\sum_{i \neq 1} a_{i1} - 2a_{j1}}{a_{11}} a_{1j} \\ &> a_{jj} - \sum_{i \neq j} a_{ij}, \text{ since } a_{11} > \sum_{i \neq 1} a_{i1}. \\ a'_{jj} - \sum_{\substack{i \neq j \\ i \neq 1}} a'_{ij} &> 0. \end{aligned}$$

We have $\det A = a_{11} \cdot \det A_{1,1}$, where $A_{1,1}$ is the matrix obtained from A by deleting the first row and the first column. We have $A_{1,1}$ is an $k \times k$ matrix. From the induction hypothesis, we have $\det(A) \neq 0$. □

Theorem 3.3.6 ([8])**.** *Let H be a graph of order r. If*

$$\binom{n-r}{s} + \binom{n-r}{s-r} > \frac{1}{2} \binom{n}{s}$$

then $i(H,G)$ is s-vertex-switching reconstructible.

Proof. Let s and r be integers such that $\binom{n-r}{s}+\binom{n-r}{s-r} > \frac{1}{2}\binom{n}{s}$. First, we will prove B_s^r is invertible. The sum over each column of A_k^r is $\binom{r}{k}$. Thus the sum of any column of B_s^r is given by

$$\sum_j b_{ji} = \sum_{k=0}^{r} \binom{n-r}{s-k}\binom{r}{k} = \binom{n}{s}. \tag{3.8}$$

Furthermore, for each $i,\ j \leq R$ we have

$$[B_s^r]_{ij} \geq \binom{n-r}{s}[A_0^r]_{ij} + \binom{n-r}{s-r}[A_r^r]_{ij} = \left[\binom{n-r}{s} + \binom{n-r}{s-r}\right][I_R]_{ij},$$

Hence, we have

$$b_{ii} \geq \binom{n-r}{s} + \binom{n-r}{s-r} > \frac{1}{2}\binom{n}{s}, \text{ for all } i.$$
$$b_{ii} > \sum_{j\neq i} b_{ji}, \text{ from Equation 3.8.}$$

From Lemma 3.3.5, we have B_s^r is invertible.

In order to prove that $i(H,G)$ is s-vertex-switching reconstructible we define the vector

$$i(G) := (i_1, i_2, \ldots), \text{where } i_j = i(H^j, G) \text{ and } H^j \text{ is a graph in } \mathscr{R}_r. \tag{3.9}$$

We also define

$$i(D_s(G)) := \sum_{F\in D_s(G)} i(F).$$

Let F be an induced subgraph of G of order r and let $Z \subset V(F)$, such that $|Z| = k$. Let G_W be an s-switching such that $W \cap V(F) = Z$. We have $\binom{n-r}{s-k}$ choices for W such that the induced subgraph of G_W on vertex set $V(F)$ equals F_W.

From Equation 3.9, we see that the lth component of the vector $\binom{n-r}{s-k}A_k^r\, i(G)$ is

$$\binom{n-r}{s-k}\left(\sum_{j>1}\left|\{Z \subseteq V(H^j) : H_z^j \cong H^l, |Z| = k\}\right| i(H^j, G)\right).$$

That is the number of induced subgraphs isomorphic to H^l in $D_s(G)$ that are obtained from the k-switching of subgraphs of G with r vertices. Then, from the Definition 3.3.4, we have the equation $B_s^r\, i(G) = i(D_s(G))$. Since B_s^r is invertible $i(G)$ is reconstructible. □

Theorem 3.3.7 (Krasikov and Roditty [8])**.** *Let $s \geq 2$. Let H be a graph on $r = 2$ or $r = 3$ vertices. Then $i(H,G)$ is s-vertex-switching reconstructible if $s \neq \binom{\sqrt{n}}{2}$ and $s \neq \binom{\sqrt{n}+1}{2}$, where $\sqrt{n} \in \{2, 3, \dots\}$.*

Proof. Suppose $n \geq \max\,(r,s)$. From the proof of Theorem 3.3.6, if B_s^r is invertible, then $i(H, G)$ is determined by the s-switching deck. First we consider the case $r = 2$. We will look under which conditions B_s^2 is not invertible. We have $\mathscr{R}_2 = \{K_1 \cup K_1, K_2\}$. Furthermore, we have $X_1(K_1 \cup K_1 \to K_2) = X_1(K_2 \to K_1 \cup K_1) = 2$ and $X_1(K_1 \cup K_1 \to K_1 \cup K_1) = X_1(K_2 \to K_2) = 0$. Hence

$$A_1^2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Since $A_0^2 = A_2^2 = I_2$, we have

$$\begin{aligned}
B_s^2 &= \sum_{k=0}^{2}\binom{n-2}{s-k}A_k^2 \\
&= \binom{n-2}{s}I_2 + \binom{n-2}{s-1}A_1^2 + \binom{n-2}{s-2}I_2 \\
&= \left[\binom{n-2}{s} + \binom{n-2}{s-2}\right]I_2 + \binom{n-2}{s-1}A_1^2 \\
&= \begin{pmatrix} a & 2b \\ 2b & a \end{pmatrix}, \text{ taking } a = \tbinom{n-2}{s} + \tbinom{n-2}{s-2} \text{ and } b = \tbinom{n-2}{s-1}.
\end{aligned}$$

Suppose B_s^2 is not invertible. Then

$$\begin{aligned} 0 &= \det B_s^2 \\ &= a^2 - 4b^2 \\ &= (a-2b)(a+2b). \end{aligned}$$

Therefore $a+2b=0$ or $a-2b=0$. If $a+2b=0$, then $a=0$ and $b=0$, which implies that $n<s$. If $a-2b=0$, we have

$$\begin{aligned} 0 &= \binom{n-2}{s} + \binom{n-2}{s-2} - 2\binom{n-2}{s-1} \\ &= \frac{(n-2)\dots(n-s+1)[(n-s-1)(n-s)+s(s-1)-2s(n-s)]}{s!}. \end{aligned}$$

That implies

$$\begin{aligned} 0 &= (n-3s-1)(n-s)+s(s-1) \\ &= n^2 - n(4s+1) + 4s^2 \\ &= (n-2s)^2 - n. \end{aligned}$$

Thus $n \geq 2$, and we may write $n = t^2$ and obtain

$$\begin{aligned} (t^2-2s)^2 &= t^2 \\ t^2 &= 2s \pm t \\ 2s &= t^2 \pm t. \end{aligned}$$

That implies $n = t^2$ for some $t = 2, 3, 4, \dots$, and so, we have there exists $t \geq 2$ such that $n = t^2$ and $s = \binom{t}{2}$ or $s = \binom{t+1}{2}$.

Now consider the case $r = 3$. We will look under which conditions B_s^3 is not invertible. We have $\mathscr{R}_3 = \{K_1 \cup K_1 \cup K_1, K_2 \cup K_1, P_3, K_3\}$. The nonzero entries of A_1^3 are given by

$$X_1(K_1 \cup K_1 \cup K_1 \to P_3) = 3, \qquad X_1(P_3 \to P_3) = 2,$$
$$X_1(P_3 \to K_1 \cup K_1 \cup K_1) = 1, \qquad X_1(K_2 \cup K_1 \to K_3) = 1,$$
$$X_1(K_2 \cup K_1 \to K_2 \cup K_1) = 2, \qquad X_1(K_3 \to K_2 \cup K_1) = 3.$$

Moreover, we have $A_0^3 = A_3^3 = I$ and $A_1^3 = A_2^3$, since realize switching in a set or in its complement result in the same graph. We have

$$A_1^3 = A_2^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & c \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} B_s^3 &= \sum_{k=0}^{3} \binom{n-3}{s-k} A_k^3 \\ &= \binom{n-3}{s} I_3 + \binom{n-3}{s-1} A_1^3 + \binom{n-3}{s-2} A_2^3 + \binom{n-3}{s-3} I_3. \end{aligned}$$

Taking $c = \binom{n-3}{s} + \binom{n-3}{s-3}$ and $d = \binom{n-3}{s-1} + \binom{n-3}{s-2}$, we obtain

$$B_s^3 = \begin{pmatrix} c & 0 & d & 0 \\ 0 & c+2d & 0 & 3d \\ 3d & 0 & c+2d & 0 \\ 0 & d & 0 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned} \det B_s^3 &= c\left[(c+2d)^2 c - 3d^2(c+2d)\right] + d\left[-3d \cdot c(c+2d) + 9d^3\right] \\ &= c^2(c+2d)^2 - 6d^2 c(c+2d) + 9d^4 \end{aligned}$$

$$= \left[c(c+2d) - 3d^2\right]^2$$
$$= (c+3d)^2(c-d)^2.$$

We have

$$c = \binom{n-3}{s} + \binom{n-3}{s-3}$$
$$= \binom{n-3}{s} + \binom{n-3}{s-1} + \binom{n-3}{s-2} + \binom{n-3}{s-3} - \binom{n-3}{s-1} - \binom{n-3}{s-2}, \text{ from Pascal's rule.}$$
$$c = \binom{n-2}{s} + \binom{n-2}{s-2} - \binom{n-2}{s-1}$$
$$= a - b.$$

Since $\binom{n-3}{s-1} + \binom{n-3}{s-2} = \binom{n-2}{s-1}$, we have $d = b$. So, we have

$$\det B_s^3 = (c+3d)^2(c-d)^2$$
$$= (a+2b)^2(a-2b)^2, \text{taking } c = a-b \text{ and } d = b.$$

Suppose $\det B_s^3 = 0$, we have the same conditions in the case $r = 2$, hence there exists $t \geq 2$ such that $n = t^2$ and $s = \binom{t}{2}$ or $s = \binom{t+1}{2}$. We may rewrite s as $s = \binom{\sqrt{n}}{2}$ or $s = \binom{\sqrt{n}+1}{2}$, where $\sqrt{n} \in \{2, 3, \ldots\}$. □

Lemma 3.3.8. *[8] The matrices A_k^r satisfy the recurrence relation*

$$(k+1)A_{k+1}^r = A_k^r A_1^r - (r-k+1)A_{k-1}^r.$$

Proof. We will look at the ijth entry of each term in this expression. The ijth entry of $A_k^r A_1^r$ counts the number of ways to obtain H^i from H^j by first performing a switch of a unique vertex s and then realize a switching of a set W of k vertices. We have two cases.

Case. $s \notin W$.

In this case the result is a $(k+1)$-switching. Let W' be a set of cardinality $k+1$ such that $H_{W'}^i \cong H^j$. We have $k+1$ choices for s. So we have the left hand side.

Case. $s \in W$.

In this case the result is a $(k-1)$-switching. Let W' be a set of cardinality $k-1$ such that $H^i_{W'} \cong H^j$. We have $r-(k-1) = r-k+1$ choices for $s \notin W'$, to obtain $W = W' \cup \{s\}$. So we have the last term of the right hand side. □

Lemma 3.3.9 (Krasikov and Roditty [8]). *Let H be a graph of order r. Then $D_s(G)$ determines the number of induced subgraphs of G isomorphic to H if and only if no eigenvalue of A^r_1 is a root y of*

$$R^r_s(y) = \sum_{k=0}^{s} \binom{n-r}{s-k} P^r_k(y). \tag{3.10}$$

Proof. The recurrence relation of $P^n_s(y)$ (see Corollary 3.1.5) is equivalent to the recurrence relation of A^r_k (see Lemma 3.3.8). Hence, we can see B^r_s as the polynomial $R^r_s(A^r_1)$. Then B^r_s is invertible if and only if no eigenvalue of A^r_1 is a root of $R^r_s(y)$. □

Making use of Lemma 3.3.9 we will prove a theorem presented in [4]. This theorem presents another condition under which the number of induced subgraphs of G isomorphic to a given graph is reconstructible.

Theorem 3.3.10 (Ellingham [4]). *Let H be a graph of order r. The number of induced subgraphs of G isomorphic to H is s-vertex switching reconstructible if K^n_s has no even roots in the interval $[0,r]$.*

Proof. By Remark 3.3.2, we have $A^r_1 = X$, where X is the matrix defined in Chapter 2. Corollary 2.1.10 says that X is a block diagonal matrix whose blocks correspond to switching matrices of switching classes. From Corollary 2.1.9 and Lemma 2.1.11, the eigenvalues of A^r_1 are of the form

$$\theta_j = r - 4j, \text{ for } 0 \le j \le \lfloor r/2 \rfloor.$$

By substituting the eigenvalues of A^r_1 in R^r_s in Equation 3.10, for all j, such that

$0 \geq 2j \geq r$, we obtain

$$R_s^r(r-4j) = \sum_{k=0}^{s} \binom{n-r}{s-k} P_k^r(r-4j).$$

Since $P_k^r(y) = K_k^r((r-y)/2)$, we have

$$R_s^r(r-4j) = \sum_{k=0}^{s} \binom{n-r}{s-k} K_k^r(2j).$$

By substituting the expression of $K_k^r(2j)$, from Definition 3.1.1, we obtain

$$R_s^r(r-4j) = \sum_{k=0}^{s} \binom{n-r}{s-k} \sum_{t=0}^{k} (-1)^t \binom{2j}{t} \binom{r-2j}{k-t}.$$

Changing the order of the sums, we obtain

$$R_s^r(r-4j) = \sum_{t=0}^{s} (-1)^t \binom{2j}{t} \sum_{k=t}^{s} \binom{n-r}{s-k} \binom{r-2j}{k-t}.$$

Taking $i = k-t$, we have

$$R_s^r(r-4j) = \sum_{t=0}^{s} (-1)^t \binom{2j}{t} \sum_{i=0}^{s-t} \binom{n-r}{s-t-i} \binom{r-2j}{i}.$$

From the identity $\sum_{i=0}^{n} \binom{x}{i}\binom{y}{n-i} = \binom{x+y}{n}$, we have

$$\begin{aligned} R_s^r(r-4j) &= \sum_{t=0}^{k} (-1)^t \binom{2j}{t} \binom{n-2j}{s-t} \\ &= K_s^n(2j). \end{aligned}$$

That means $r-4j$ is not a root of $R_s^r(x)$ if $K_s^n(y)$ has no even root in the interval $[0,n]$. From Lemma 3.3.9 this occur if and only if the number of induced subgraphs of G isomorphic to H is s-switching reconstructible. □

Chapter 4

Edge reconstruction

In this chapter we consider the problem of reconstructing a graph, up to isomorphism, given its i-edge deck. Moreover, we will show that we can construct the i-edge deck of a given graph from the modified i-deck of the same graph. The results in Section 4.1, Section 4.2, Section 4.3 and Section 4.4 can be found in [13]. With exception of the results indicated all results in Sections 4.5 and 4.6 can be found in [14].

In this chapter, we will denote by G an arbitrary graph with n vertices and m edges. We will also assume $N = \binom{n}{2}$, and so $|E(G^c)| = N - m$. Now we present some definitions.

Remark 4.0.1. MD_i is constructed from ED_i just taking each graph in ED_i and adding i edges in all possible ways.

We want to construct ED_i from MD_i.

4.1 Properties of the perturbed i-deck

The next lemma relates Δ_k to D_i's, where $0 \le i \le k$ and consequently relates MD_k to PD_i's, where $0 \le i \le k$. We prove the next result by analysing the ways a graph H in the perturbed i-deck can appear in the modified k-deck.

Lemma 4.1.1 ([13]). *Let $k \geq 0$. We have*

$$\Delta_k = \sum_{i=0}^{k} \binom{m-i}{k-i} D_i.$$

Proof. Let $k \geq 0$ and $0 \leq i \leq k$ be natural numbers. Let H be a graph on $\mathrm{PD}_i(G)$. We may write $H = G - A + B$, for some $A \subseteq E(G)$ and $B \subseteq E(G^c)$ with cardinality i, where $G - A + B$ is the graph obtained from G by deleting the edges in A and adding the edges in B. We may rewrite $H = G - (A+C) + (B+C)$, where $|C| = k - i$. The graph H is in $\mathrm{MD}_k(G)$ for all possible $C \subseteq E(G)$ such that $A \cap C = \emptyset$. There are $\binom{m-i}{k-i}$ sets that satisfy these conditions. □

In the next lemma we count how many times a graph in $\mathrm{PD}_1(\mathrm{PD}_i(G))$ appears in $\mathrm{PD}_k(G)$, for an arbitrary k.

Lemma 4.1.2 ([13]). *Let $i \geq 1$. We have*

$$D_1 D_i = (m-i+1)(N-m-i+1)D_{i-1} + i(N-2i)D_i + (i+1)^2 D_{i+1}.$$

Proof. Let $X \subseteq E(G)$ and $Y \subseteq E(G^c)$ with cardinality $1 \leq i \leq \min(m, N-m)$. Then $G - X + Y \in \mathrm{PD}_i(G)$. Let $e \in E(G) - X + Y$ and $f \notin E(G) - X + Y$. Then the graph $H = G - X + Y - e + f$ satisfies $H \in \mathrm{PD}_1(\mathrm{PD}_i(G))$. We will assume $E(G) = \{e_1, e_2, \ldots, e_m\}$ and $E(G^c) = \{f_1, f_2, \ldots, f_{N-m}\}$. Without loss of generality, suppose $X = \{e_1, e_2, \ldots, e_i\}$ and $Y = \{f_1, f_2, \ldots, f_i\}$. We have four cases.

Case. $e \in Y$ and $f \in X$

$H = G - (X - f) + (Y - e)$ and $|X - f| = |Y - e| = i - 1$ that implies $H \in \mathrm{PD}_{i-1}(G)$.

Without loss of generality, suppose $e = f_i$ and $f = e_i$. We have $H = G - \{e_1, e_2, \ldots, e_{i-1}\} + \{f_1, f_2, \ldots, f_{i-1}\}$. If $\alpha \in G - \{e_1, e_2, \ldots, e_{i-1}\}$ and $\beta \in G^c - \{f_1, f_2, \ldots, f_{i-1}\}$ then $H = G - \{e_1, e_2, \ldots, e_{i-1}, \alpha\} + \{f_1, f_2, \ldots, f_{i-1}, \beta\}$. So there are $m - i - 1$ choices for α and $N - m - i + 1$ choices for β, then we get the first term.

Case. $e \in E(G) - X$ and $f \in X$

$H = G - (X + e - f) + Y$ and $|X + e - f| = i \implies H \in \mathrm{PD}_i(G)$.

Suppose $e = e_{i+1},\ f = e_1$, so $H = G - \{e_2, e_3, \ldots, e_{i+1}\} + \{f_1, f_2, \ldots, f_i\}$ without loss of generality. If $j \in \{2, 3, \ldots, i+1\}$ and $k \in G^c - \{1, i+2, i+3, \ldots, m\}$ then $H = G - (X - e_1 + e_{i+1} + e_j - e_k) + Y - e_j + e_k$. There are i choices for j and $m - i$ choices for k. In total there are $i(m-i)$ forms.

Case. $e \in Y$ and $f \in E(G^c) - Y$

$$H = G - X + (Y - e + f) \text{ and } |Y - e + f| = i \implies H \in \mathrm{PD}_i(G)$$

Without loss of generality, suppose $e = f_1,\ f = f_{i+1}$, then $H = G - \{e_1, e_2, \ldots, e_i\} + \{f_2, f_3, \ldots, f_{i+1}\}$. If $j \in \{2, 3, \ldots, i+1\}$ and $k \in \{1, i+2, i+3, \ldots, N-m\}$ then $H = G - X + (Y - f_1 + f_{i+1} - f_j + f_k) + f_j - f_k$. So there are i choices for j and $N - m - i$ choices for k. In total there are $i(N-m-i)$ choices.

Case. $e \in E(G) - X$ and $f \in E(G^c) - Y$

$$H = G - (X + e) + (Y + f) \text{ and } |X + e| = |Y + f| = i + 1 \implies H \in \mathrm{PD}_{i+1}(G)$$

Without loss of generality, suppose $e = e_{i+1}$ and $f = f_{i+1}$. We have $H = G - \{e_1, e_2, \ldots, e_{i+1}\} + \{f_1, f_2, \ldots, f_{i+1}\}$. If j and k are in the set $\{1, 2, \ldots, i+1\}$ then $H = G - (X + e_{i+1} - e_j) + (Y + f_{i+1} - f_k) - e_j + f_k$. So there are $i+1$ choices for either e_j and f_k, thus we obtain the last term.

When we take the sum of the choices of the second and the third cases we get the second term of the equation. □

Example 4.1.3. We have $D_1 D_0(G) = D_1(G)$.

In the next results we will consider L_k as a linear combination of operators constructed from Δ_1. The next result relates D_k to Δ_1.

Lemma 4.1.4 ([13])**.** *Let* $k \geq 0$. *We have*

$$D_k = L_k + (-1)^k \binom{m}{k} D_0.$$

Proof. By induction on n. For $k = 0$, the result is trivial. Suppose that the result is valid for $k \leq r$. In the next counts we do some algebraic manipulation in order to obtain the result. From Lemma 4.1.2, for $k = r+1$

$$D_{r+1} = \frac{1}{(r+1)^2}[D_1 D_r - (m-r+1)(N-m-r+1)D_{r-1} + r(N-2r)D_r].$$

By the induction hypothesis, we have

$$\begin{aligned}(r+1)^2 D_{r+1} = &D_1 \left(L_r + (-1)^r \binom{m}{r} D_0\right) \\ &- (m-r+1)(N-m-r+1)\left[L_{r-1} + (-1)^{r-1}\binom{m}{r-1} D_0\right] \\ &+ r(N-2r)\left[L_r + (-1)^r \binom{m}{r} D_0\right].\end{aligned}$$

From Lemma 4.1.1, $D_1 = \Delta_1 - mD_0$. As $D_0^2 = D_0$, we have

$$\begin{aligned}(r+1)^2 D_{r+1} = &D_1 L_r + (-1)^r \binom{m}{r}\Delta_1 - (m-r+1)(N-m-r+1)L_{r-1} \\ &+ r(N-2r)L_r - m(-1)^r \binom{m}{r} D_0 \\ &- (m-r+1)(N-m-r+1)(-1)^{r-1}\binom{m}{r-1} D_0 \\ &- r(N-2r)(-1)^r \binom{m}{r} D_0.\end{aligned}$$

Denoting by L_{r+1} the sum of the terms that not depend on D_0, we obtain

$$\begin{aligned}(r+1)^2 D_{r+1} = &L_{r+1} + (-1)^{r+1} r(N-2r)L_r - m(-1)^r \binom{m}{r} D_0 \\ &- (-1)^{r+1}(m-r+1)(N-m-r+1)(-1)^{r-1}\binom{m}{r-1} D_0 \\ &- (-1)^{r+1} r(N-2r)(-1)^r \binom{m}{r} D_0 \\ = &L_{r+1} + (-1)^{r+1}\left[(r(N-2r)+m)\binom{m}{r}\right] D_0\end{aligned}$$

$$-(-1)^{r+1}\left[(m-r+1)(N-m-r+1)\binom{m}{r-1}\right]D_0.$$

By making use of the identity $k\binom{n}{k}=(n-k+1)\binom{n}{k-1}$, we have

$$\begin{aligned}(r+1)^2D_{r+1} &= L_{r+1}\\ &\quad+(-1)^{r+1}\left[(r(N-2r)+m)\binom{m}{r}+r(m+r-N-1)\binom{m}{r}\right]D_0\\ &= L_{r+1}+(-1)^{r+1}(-r^2+m+rm-r)\binom{m}{r}D_0\\ &= L_{r+1}+(-1)^{r+1}(m-r)(r+1)\binom{m}{r}D_0.\end{aligned}$$

Using the identity $\binom{m}{r}=\frac{r+1}{m-r}\binom{m}{r+1}$, we obtain

$$(r+1)^2D_{r+1}=L_{r+1}+(-1)^{r+1}(r+1)^2\binom{m}{r+1}D_0.$$

Then $D_{r+1}=L'_{r+1}+(-1)^{r+1}\binom{m}{r+1}D_0$, where L'_{r+1} is a linear combination of operators constructed from Δ_1. □

4.2 A property of the modified i-deck

In the next theorem we present a relation between the modified *i*-deck of a graph and the modified 1-deck of this same graph.

Theorem 4.2.1 (Thatte [13])**.** *If $i\geq 1$, then Δ_i can be written in terms of Δ_1.*

Proof. From Lemma 4.1.1, we have

$$\begin{aligned}\Delta_i &= \sum_{k=0}^{i}\binom{m-k}{i-k}D_k\\ &= \sum_{k=0}^{i}\binom{m-k}{i-k}\left(L_k+(-1)^k\binom{m}{k}D_0\right),\text{ from Lemma 4.1.4,}\end{aligned}$$

$$= \sum_{k=0}^{i} L_k^{'} + \sum_{k=0}^{i} (-1)^k \binom{m-k}{i-k} \binom{m}{k} D_0, \text{ taking } L_k^{'} = \tbinom{m-k}{i-k} L_k$$
$$= \sum_{k=0}^{i} L_k^{'} + \sum_{k=0}^{i} (-1)^k \binom{m}{i} \binom{i}{k} D_0, \text{ since } \tbinom{n}{r}\tbinom{n-r}{m-r} = \tbinom{n}{m}\tbinom{m}{r},$$
$$= \sum_{k=0}^{i} L_k^{'} + \binom{m}{i} \sum_{k=0}^{i} (-1)^k \binom{i}{k} D_0$$
$$= \sum_{k=0}^{i} L_k^{'}, \text{since } \textstyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \text{ if } n > 0.$$

Then we have the result. □

4.3 Proving that the 1-edge deck can be constructed from the modified 1-deck

In the next results we will consider $\mathscr{Q}$ and $\mathscr{R}$ as arbitrary collections of graphs with n vertices and m edges.

Lemma 4.3.1 ([13]). *Let $m > N/2$. If $d_1 X_{\mathscr{R}} = d_1 X_{\mathscr{Q}}$ or $\Delta_1 X_{\mathscr{R}} = \Delta_1 X_{\mathscr{Q}}$, then $X_{\mathscr{R}} = X_{\mathscr{Q}}$, where $X_{\mathscr{R}}$ is the characteristic vector of $\mathscr{R}$.*

Proof. As we may obtain MD_i from ED_i, see Remark 4.0.1, we will prove the implication for $\Delta_1 X_{\mathscr{R}} = \Delta_1 X_{\mathscr{Q}}$. We will invert the equality presented in Lemma 4.1.1. Using Theorem 4.2.1 we obtain $\Delta_i X_{\mathscr{R}} = \Delta_i X_{\mathscr{Q}}$, for every $i \geq 1$. We have

$$D_k = \sum_{i=0}^{k} (-1)^{k+i} \binom{m-i}{k-i} \Delta_i. \tag{4.1}$$

By induction on k. For $i = 0$, the result is trivial. Suppose the result for $k < i$. From

Lemma 4.1.1, we have

$$\begin{aligned}\Delta_i &= \sum_{k=0}^{i} \binom{m-k}{i-k} D_k \\ &= \sum_{k=0}^{i-1} \binom{m-k}{i-k} \sum_{j=0}^{k} (-1)^{k+j} \binom{m-j}{k-j} \Delta_j + D_i.\end{aligned}$$

We will calculate the terms of Δ_l, for $l < i$ in the previous expression. For this we will take $j = l$ in the second sum and a satisfactory inferior limit in the first sum, we obtain

$$\sum_{k=l}^{i-1} \binom{m-k}{i-k} (-1)^{k+l} \binom{m-l}{k-l} = \sum_{k=l}^{i-1} (-1)^{k+l} \binom{m-k}{i-k} \binom{m-l}{k-l}.$$

Applying the identity $\binom{n}{r} = \binom{n}{n-r}$ we obtain

$$\sum_{k=l}^{i-1} \binom{m-k}{i-k} (-1)^{k+l} \binom{m-l}{k-l} = \sum_{k=l}^{i-1} (-1)^{k+l} \binom{m-l}{m-k} \binom{m-k}{i-k}.$$

By making use of the identity $\binom{n}{m}\binom{m}{r} = \binom{n}{r} = \binom{n-r}{m-r}$, we have

$$\sum_{k=l}^{i-1} \binom{m-k}{i-k} (-1)^{k+l} \binom{m-l}{k-l} = \sum_{k=l}^{i-1} (-1)^{k+l} \binom{m-l}{m-i} \binom{i-l}{i-k}.$$

Applying the identity $\binom{n}{r} = \binom{n}{n-r}$, we have

$$\sum_{k=l}^{i-1} \binom{m-k}{i-k} (-1)^{k+l} \binom{m-l}{k-l} = \binom{m-l}{m-i} \sum_{k=l}^{i-1} (-1)^{k+l} \binom{i-l}{k-l}.$$

Taking $j = k - l$, we obtain

$$\sum_{k=l}^{i-1} \binom{m-k}{i-k} (-1)^{k+l} \binom{m-l}{k-l} = \binom{m-l}{m-i} \sum_{j=0}^{i-l-1} (-1)^{j+2l} \binom{i-l}{j}$$

$$= \binom{m-l}{m-i}(-1)^{2l}\sum_{j=0}^{i-l-1}(-1)^j\binom{i-l}{j}.$$

As $\sum_{i=0}^{n}(-1)^j\binom{n}{j} = 0$, if $n > 0$, we have

$$\sum_{k=l}^{i-1}\binom{m-k}{i-k}(-1)^{k+l}\binom{m-l}{k-l} = \binom{m-l}{m-i}(-1)^{2l}(-1)^{i-l+1}\binom{i-l}{i-l}.$$

Since $\binom{n}{n} = 1$ we obtain

$$\sum_{k=l}^{i-1}\binom{m-k}{i-k}(-1)^{k+l}\binom{m-l}{k-l} = -(-1)^{i+l}\binom{m-l}{m-i}.$$

So, we have

$$\begin{aligned} D_i &= \sum_{l=0}^{i-1}(-1)^{i+l}\binom{m-l}{i-l}\Delta_l + \Delta_i \\ &= \sum_{l=0}^{i-1}(-1)^{i+l}\binom{m-l}{i-l}\Delta_l + (-1)^{2i}\binom{m-i}{i-i}\Delta_i \\ &= \sum_{l=0}^{i}(-1)^{i+l}\binom{m-l}{i-l}\Delta_l. \end{aligned}$$

This prove Equation 4.1.

As $m > N/2$, we have D_m is the null matrix. Since $N - m < N - N/2 = N/2 < m$, if we remove m edges from a graph of m edges we will not have m distinct edges to add. Then

$$0 = \sum_{i=0}^{m}(-1)^{m+i}\Delta_i,$$

and therefore, isolating the Δ_0 term, we obtain

$$\Delta_0 = \sum_{i=1}^{m}(-1)^{i+1}\Delta_i.$$

From Remark 1.4.3, we have

$$\begin{aligned}
X_{\mathscr{R}} &= \Delta_0 X_{\mathscr{R}} \\
&= \left(\sum_{i=0}^{m} (-1)^{i+1} \Delta_i \right) X_{\mathscr{R}} \\
&= \sum_{i=0}^{m} (-1)^{i+1} \Delta_i X_{\mathscr{R}} \\
&= \sum_{i=0}^{m} (-1)^{i+1} \Delta_i X_{\mathscr{Q}} \text{ by Theorem 4.2.1 and the hypothesis.} \\
X_{\mathscr{R}} &= \Delta_0 X_{\mathscr{Q}} \\
&= X_{\mathscr{Q}}.
\end{aligned}$$

□

Now we will prove, by making use of Lemma 4.3.1, that if two collections of graphs have the same modified i-deck, then they have the same i-edge deck.

Theorem 4.3.2 (Thatte [13]). *If $\Delta_1 X_{\mathscr{R}} = \Delta_1 X_{\mathscr{Q}}$, then $d_1 X_{\mathscr{R}} = d_1 X_{\mathscr{Q}}$.*

Proof. We have two cases.

Case. $m > N/2$.

From Lemma 4.3.1, we have $X_{\mathscr{R}} = X_{\mathscr{Q}}$ and then $d_1 X_{\mathscr{R}} = d_1 X_{\mathscr{Q}}$.

Case. $m \leq N/2$. Let $\mathscr{R}' = \{F^c : F \in \mathrm{ED}_1(\mathscr{R})\}$ and $\mathscr{Q}' = \{F^c : F \in \mathrm{ED}_1(\mathscr{Q})\}$. We will prove the next equivalence:

$$\Delta_1 X_{\mathscr{R}} = \Delta_1 X_{\mathscr{Q}} \iff d_1 X_{\mathscr{R}'} = d_1 X_{\mathscr{Q}'}.$$

Let $F \in \mathscr{R}$ and let $e \in E(F)$ and $f \notin E(F) - e$. We have $F - e + f \in \mathrm{MD}_1(\mathscr{R})$, then $(F - e + f)^c = (F - e)^c - f \in \mathrm{PD}_1(\mathscr{R})$. Since $F - e \in \mathrm{ED}_1(\mathscr{R})$ that implies $(F - e)^c \in \mathscr{R}'$. Analogously, for any $F \in \mathscr{Q}$ we have $(F - e)^c \in \mathscr{Q}'$. Thus we get the equivalence.

Furthermore $m \leq N/2$ implies $N - m + 1 > N/2$. So, we have $d_1 X_{\mathscr{R}'} = d_1 X_{\mathscr{Q}'}$ that implies $X_{\mathscr{R}'} = X_{\mathscr{Q}'}$. From the definition of $\mathscr{R}'$ and $\mathscr{Q}'$, we have $d_1 X_{\mathscr{R}} = d_1 X_{\mathscr{Q}}$. □

From the definition of Δ_1 and d_1 we obtain the next Corollary.

Corollary 4.3.3 ([13])**.** *We can construct the* 1*-edge-deck of a graph from the modified* 1*-deck of the same graph.*

4.4 1-edge reconstructibility

In 1972, Lovász presented the next result in [9]. This result present a condition that a graph is 1-edge reconstructible.

Theorem 4.4.1 (Lovász [9])**.** *Let $E(G) = \{e_1, e_2, \ldots, e_m\}$ and $E(H) = \{f_1, f_2, \ldots, f_m\}$. If $G - e_i \cong H - f_i$, for each $1 \leq i \leq m$, and $m > N/2$ then $G \cong H$.*

The goal of this section is present the proof of Theorem 4.4.7 that is a version of the Lovász's result for any collection of graphs. For this we need to define a special graph.

Definition 4.4.2 ([3])**.** A **Johnson graph** is a graph whose vertex set is the family of subsets of cardinality m of a set of cardinality N. Two vertices U and V of this graph are adjacent if and only if $|U \cap V| = m - 1$. Let J be the adjacency matrix of the Johnson graph with parameters N and m.

Now we will present a property of J that was proved in [3]. This proof make use of tools techniques that was not covered in this work, so we prefer omit it.

Proposition 4.4.3. *[3] The eigenvalues of J are*

$$\theta_j = (m - j)(N - m - j) - j, \text{ where } 0 \leq j \leq min(m, N - m).$$

For each j the multiplicity of θ_j is $f_j = \binom{N}{j} - \binom{N}{j-1}$.

Definition 4.4.4. We define the matrix B to be the square matrix whose rows and columns are indexed by labelled graphs with m edges and vertex set $\{v_1, v_2, \ldots, v_n\}$. Let

$G_1, G_2, \ldots, G_M$ be those graphs. Let d_{ij} be the ijth entry of B. We set

$$d_{ij} = |\{\{e,f\} : e \in E(G_j),\, f \notin E(G_j) - e \text{ and } G_j - e + f = G_i\}|.$$

Let A be the square matrix with rows and columns indexed by the graphs in $\mathscr{R}_{n,m}$, the set of representatives on n vertices and m edges. The entries of A are defined similarly as the entries of matrix B, that is each ij entry of A is the number of ways to obtain a graph isomorphic to H_i^* from H_j^* by removing an edge and adding an edge not necessarily different from the first. We have $A = \Delta_1$.

Let P' be the matrix with rows indexed by the graphs in $\mathscr{R}_{n,m}$ and columns indexed by the labelled graphs with m edges and vertex set $\{v_1, v_2, \ldots, v_n\}$ with entries q_{ij}, where

$$q_{ij} = \begin{cases} 1, & \text{if } G_j \cong H_i^* \\ 0, & \text{otherwise,} \end{cases}$$

where $H_i^* \in \mathscr{R}_{n,m}$ is the graph associated to the ith row of P'.

Lemma 4.4.5.

$$d_{ij} = \begin{cases} m, & \textit{if } i = j \\ 0 \textit{ or } 1, & \textit{otherwise.} \end{cases}$$

Proof. If we remove an arbitrary edge of G_j and add that same edge, we obtain G_j and that is the only way to obtain G_j from G_j. Since the edge is arbitrary, we conclude that all entries in the main diagonal are equal to m. As all G_j's are labelled, the other entries of G are 0 or 1. □

Lemma 4.4.6. *$AP = PB$ and every eigenvalue of A is also an eigenvalue of B.*

Proof. The proof of this lemma is analogous to the proof of Lemma 2.1.11. □

Theorem 4.4.7. *Let $\mathscr{Q}$ and $\mathscr{R}$ be collections of graphs with n vertices and m edges. If $d_1X_{\mathscr{Q}} = d_1X_{\mathscr{R}}$ and $m > N/2$, then $X_{\mathscr{Q}} = X_{\mathscr{R}}$.*

Proof. From the definition of B and J (see 4.4.2 and 4.4.4), we have $B = mI + J$. From Proposition 4.4.3, we obtain $-m$ is an eigenvalue of J if and only if $m \le N/2$. From

Proposition 4.4.3, we have

$$\begin{aligned}\theta_j = -m &\iff j = m \\ &\iff m \leq N - m \\ &\iff 2m \leq N \\ &\iff m \leq N/2.\end{aligned}$$

As $m > N/2$, all the eigenvalues of B are nonzero. From 4.4.6, all eigenvalues of $A = \Delta_1$ are nonzero. Thus, if $\Delta_1(X_{\mathscr{R}} - X_{\mathscr{Q}}) = 0$, then $X_{\mathscr{R}} - X_{\mathscr{Q}} = 0$. But $\Delta_1(X_{\mathscr{R}} - X_{\mathscr{Q}}) = 0$ implies $d_1(X_{\mathscr{R}} - X_{\mathscr{Q}}) = 0$, from 4.3.2, and then we have the result. □

4.5 Relating the modified decks of a graph

The next lemma relates the modified $(i+1)$-deck to the modified i-deck.

Lemma 4.5.1 ([14]). *Let* $i \geq 0$.

$$\Delta_{i+1} = \frac{1}{(i+1)^2}[i(2m - N - i - 1)\Delta_0 + \Delta_1]\Delta_i.$$

Proof. In order to prove this lemma we will look at a result that relates D_i's and a result that relate Δ_i to D_i and then make some calculus to simplify the expression obtained. From Lemma 4.1.2, we have

$$(i+1)^2 D_{i+1} = D_1 D_i - (m - i + 1)(N - m - i + 1)D_{i-1} - i(N - 2i)D_i. \tag{4.2}$$

From Lemma 4.1.1, we have

$$D_i = \Delta_i - \sum_{j=0}^{i-1} \binom{m-j}{i-j} D_j, \tag{4.3}$$

$$D_{i+1} = \Delta_{i+1} - \sum_{j=0}^{i} \binom{m-j}{i-j+1} D_j. \tag{4.4}$$

Substituting Equation 4.3 and Equation 4.4 in Equation 4.2, we obtain

$$
\begin{aligned}
(i+1)^2\left(\Delta_{i+1}-\sum_{j=0}^{i}\binom{m-j}{i-j+1}D_j\right) &= D_1\left(\Delta_i-\sum_{j=0}^{i-1}\binom{m-j}{i-j}D_j\right).\\
&-(m-i+1)(N-m-i+1)D_{i-1}\\
&-i(N-2i)\left(\Delta_i-\sum_{j=0}^{i-1}\binom{m-j}{i-j}D_j\right)
\end{aligned}
$$

$$
\begin{aligned}
(i+1)^2\Delta_{i+1} = (i+1)^2\sum_{j=0}^{i}\binom{m-j}{i-j+1}D_j + D_1\Delta_i - \sum_{j=0}^{i-1}\binom{m-j}{i-j}D_1D_j\\
-(m-i+1)(N-m-i+1)D_{i-1} - i(N-2i)\left(\Delta_i-\sum_{j=0}^{i-1}\binom{m-j}{i-j}D_j\right).
\end{aligned}
$$

From Lemma 4.1.1, we have $\Delta_1 = mD_0 + D_1$ and $\Delta_0 = D_0$, hence $D_1 = \Delta_1 - m\Delta_0$. Substituting this expression and D_1D_j, giving by Lemma 4.1.2, in the previous equation, and knowing $D_1D_0 = D_1$, we obtain

$$
\begin{aligned}
(i+1)^2\Delta_{i+1} = (\Delta_1 - m\Delta_0)\Delta_i + (i+1)^2\sum_{j=0}^{i}\binom{m-j}{i-j+1}D_j\\
-\sum_{j=0}^{i-1}\binom{m-j}{i-j}(m-j+1)(N-m-j+1)D_{j-1}\\
-\sum_{j=0}^{i-1}\binom{m-j}{i-j}j(N-2j)D_j - \sum_{j=0}^{i-1}\binom{m-j}{i-j}(j+1)^2D_{j+1} - \binom{m}{i}D_1\\
-(m-i+1)(N-m-i+1)D_{i-1} - i(N-2i)\left(\Delta_i-\sum_{j=0}^{i-1}\binom{m-j}{i-j}D_j\right).
\end{aligned}
$$

Taking $k = j-1$ in the second sum and $r = j+1$ in the fourth term, we obtain

$$
(i+1)^2\Delta_{i+1} = (\Delta_1 - m\Delta_0)\Delta_i + (i+1)^2\sum_{j=0}^{i}\binom{m-j}{i-j+1}D_j
$$

$$
\begin{aligned}
&-\sum_{k=1}^{i-2}\binom{m-k-1}{i-k-1}(m-k)(N-m-k)D_k-\sum_{j=1}^{i-1}\binom{m-j}{i-j}j(N-2j)D_j-\\
&-\sum_{r=1}^{i}\binom{m-r+1}{i-r+1}r^2D_r-\binom{m}{i}D_1-(m-i+1)(N-m-i+1)D_{i-1}\\
&-i(N-2i)\left(\Delta_i-\sum_{j=0}^{i-1}\binom{m-j}{i-j}D_j\right)\\
&=(\Delta_1-m\Delta_0)\Delta_i+(i+1)^2\sum_{j=0}^{i}\binom{m-j}{i-j+1}D_j\\
&-\sum_{j=0}^{i-1}\binom{m-j-1}{i-j-1}(m-j)(N-m-j)D_j-\sum_{j=0}^{i-1}\binom{m-j}{i-j}j(N-2j)D_j\\
&-\sum_{j=0}^{i}\binom{m-j+1}{i-j+1}j^2D_j+i(N-2i)\sum_{j=0}^{i-1}\binom{m-j}{i-j}D_j-i(N-2i)\Delta_i.
\end{aligned}
$$

Using the identities $(m-j)\binom{m-j-1}{i-j-1}=(i-j)\binom{m-j}{i-j}$, $\binom{m-j+1}{i-j+1}=\binom{m-j}{i-j}\frac{m-j+1}{i-j+1}$ and $\binom{m-j}{i-j+1}=\binom{m-j}{i-j}\frac{m-i}{i-j+1}$, we have

$$
\begin{aligned}
(i+1)^2\Delta_{i+1}&=[\Delta_1-m\Delta_0-i(N-2i)]\Delta_i\\
&+\sum_{j=0}^{i-1}\binom{m-j}{i-j}\frac{(i+1)^2(m-i)-j^2(m-j+1)}{i-j+1}D_j\\
&+\sum_{j=0}^{i-1}\binom{m-j}{i-j}[i(N-2i)-(i-j)(N-m-j)-j(N-2j)]D_j\\
&+[(i+1)^2(m-i)-(m-i+1)i^2]D_i.
\end{aligned}
$$

Making use of Equation 4.3, we obtain

$$
\begin{aligned}
(i+1)^2\Delta_{i+1}&=[\Delta_1-m\Delta_0-i(N-2i)]\Delta_i\\
&+\sum_{j=0}^{i-1}\binom{m-j}{i-j}\frac{(i+1)^2(m-i)-j^2(m-j+1)}{i-j+1}D_j
\end{aligned}
$$

$$+\sum_{j=0}^{i-1}\binom{m-j}{i-j}[m(i-j)+ij+j^2-2i^2]D_j$$
$$+[(2i+1)(m-i)-i^2]\left[\Delta_i-\sum_{j=0}^{i-1}\binom{m-j}{i-j}D_j\right].$$

Since Δ_0 is the identity operator, we have

$$\begin{aligned}
(i+1)^2\Delta_{i+1} &= \{\Delta_1+[-m-i(N-2i)+(2i+1)(m-i)-i^2]\Delta_0\}\Delta_i \\
&+\sum_{j=0}^{i-1}\binom{m-j}{i-j}\frac{(i+1)^2(m-i)-j^2(m-i)}{i-j+1}D_j \\
&+\sum_{j=0}^{i-1}\binom{m-j}{i-j}[m(i-j)+ij-i^2-(2i+1)(m-i)]D_j \\
&= [\Delta_1+(2mi-iN-i-i^2)\Delta_0]\Delta_i \\
&+\sum_{j=0}^{i-1}\binom{m-j}{i-j}\frac{(i+1)^2(m-i)-j^2(m-i)}{i-j+1}D_j \\
&+\sum_{j=0}^{i-1}\binom{m-j}{i-j}[m(i-j)+ij-i^2-(2i+1)(m-i)]D_j \\
&= [i(2m-N-i-1)\Delta_0+\Delta_1]\Delta_i \\
&+\sum_{j=0}^{i-1}\binom{m-j}{i-j}\frac{(i^2+2i+1-j^2)(m-i)}{i-j+1}D_j \\
&+\sum_{j=0}^{i-1}\binom{m-j}{i-j}[(m-i)(i-j)-(2i+1)(m-i)]D_j \\
&= [i(2m-N-i-1)\Delta_0+\Delta_1]\Delta_i \\
&+\sum_{j=0}^{i-1}\binom{m-j}{i-j}\frac{[(i^2+2i+1-j^2)-(i+j+1)(i-j+1)](m-i)}{i-j+1}D_j.
\end{aligned}$$

As the terms in the numerator of the fraction are equal to zero, we have

$$(i+1)^2\Delta_{i+1}=[i(2m-N-i-1)\Delta_0+\Delta_1]\Delta_i. \qquad \square$$

The next result proves that if G and H have the same modified k-deck, then they have the same modified i-deck, for every $i \geq k$.

Corollary 4.5.2 ([14]). *Let $k \geq 0$. If $\Delta_k(X_G - X_H) = 0$ then $\Delta_i(X_G - X_H) = 0$, for every $i \geq k$.*

Proof. By induction on i. For $i = k$, the result is trivial. Suppose the result is valid for l, with $l \geq k$. From Lemma 4.5.1, we have

$$\Delta_{l+1}(X_G - X_H) = \frac{1}{(l+1)^2}\{l(2m - N - i - 1)\Delta_0 + \Delta_1\}\Delta_l(X_G - X_H).$$

The right hand side is equal to 0, by the induction hypothesis. So we have the result. □

By taking $\mathscr{R} = \{G\}$ and $\mathscr{Q} = \{H\}$ we obtain Lovász's result 4.4.1 as an immediate consequence of Theorem 4.4.7.

4.6 Proving that the i-edge deck can be constructed from the modified i-deck

In this section we will consider $\mathscr{Q}$ and $\mathscr{R}$ as collections of graphs with n vertices and m edges. To be able to prove the main goal of this chapter, we need present the k-edge version of Lovász result. The proof of this result uses techniques that we was not presented in this work, because of this we will not present that. This proof can be found in [6].

Lemma 4.6.1 (Lovász's Theorem: k-edge version [6]). *Let $k \geq 1$ be such that $2m - k + 1 > N$. If $d_k X_{\mathscr{R}} = d_k X_{\mathscr{Q}}$, then $X_{\mathscr{R}} = X_{\mathscr{Q}}$.*

Lemma 4.6.2 (Kelly's Lemma: Edge version). *Let F be a graph such that $|E(F)| < |E(G)|$. Then $s(F, G)$ is edge reconstructible.*

Proof. Let $i \geq 1$. Let H be an i-edge reconstruction of G. Then there exists a bijection $\psi : \binom{E(G)}{i} \to \binom{E(H)}{i}$ such that $G - E \cong H - \psi(E)$, for every $E \in \binom{E(G)}{i}$.

Let F be a graph such that $|E(F)| < |E(G)|$. From the definition of $s(F, G)$, we obtain

$$s(F, G) = \sum_{E \in \binom{E(G)}{i}} s(F, G - E).$$

Since $G - E \cong H - \psi(E)$, for every $E \in \binom{E(G)}{i}$, we have

$$s(F, G) = \sum_{E \in \binom{E(G)}{i}} s(F, H - \psi(E)).$$

From the definition of $s(F, H)$ and the fact that $\mathrm{ED}_i(G) = \mathrm{ED}_i(H)$, we obtain

$$s(F, G) = s(F, H). \qquad \square$$

Remark 4.6.3. Remember that $\mathrm{ED}_r(\mathscr{R})$ represents the multiunion of r-edge decks of graphs of $\mathscr{R}$.

The next theorem shows that if the modified k-deck of $\mathscr{R}$ is equal the modified k-deck of $\mathscr{Q}$, then the k-edge deck of $\mathscr{R}$ is equal the k-edge deck of $\mathscr{Q}$.

Theorem 4.6.4 (Thatte [14])**.** *Let $k \geq 1$. If $\Delta_k X_{\mathscr{R}} = \Delta_k X_{\mathscr{Q}}$, then $d_k X_{\mathscr{R}} = d_k X_{\mathscr{Q}}$.*

Proof. By induction on k. In Theorem 4.3.2, we proved the result for $k = 1$. Suppose the result is valid for $k \leq l - 1$ and suppose $\Delta_k X_{\mathscr{R}} = \Delta_k X_{\mathscr{Q}}$.

Let $\mathscr{R}' = \{F^c;\ F \in \mathrm{ED}_r(\mathscr{R})\}$ and $\mathscr{Q}' = \{F^c;\ F \in \mathrm{ED}_r(\mathscr{Q})\}$. Similarly as in the proof of Theorem 4.3.2, we obtain $\Delta_r X_{\mathscr{R}} = \Delta_r X_{\mathscr{Q}}$ if and only if $d_r X_{\mathscr{R}'} = d_r X_{\mathscr{Q}'}$. The graphs in $\mathscr{R}'$ and $\mathscr{Q}'$ have $N - m + r$ edges. From 4.6.1, if $2(N - m + r) - r + 1 > N$ then $X_{\mathscr{R}'} = X_{\mathscr{Q}'}$ and that implies $d_r X_{\mathscr{R}} = d_r X_{\mathscr{Q}}$.

So, we will suppose $2(N - m + r) - r + 1 \leq N$, then $2m - r - 1 \geq N$. From the proof of Theorem 4.4.7, we have $B = mI + J$. From Proposition 4.4.3, the eigenvalues of

B are $m+(m-j)(N-m-j)-j$, with $0 \leq j \leq \min(m,N-m)$.

From Lemma 4.5.1, we obtain

$$\Delta_r = \frac{1}{r^2}[(r-1)(2m-N-r)\Delta_0 + \Delta_1]\Delta_{r-1}. \tag{4.5}$$

From 4.4.6, all eigenvalues of $A = \Delta_1$ are also eigenvalues of B, so the eigenvalues of $(r-1)(2m-N-r)\Delta_0 + \Delta_1$ are $(m-j)(N-m-j+1)+(r-1)(2m-N-r)$. Now we will prove, by contradiction, that they are nonzero eigenvalues. Suppose that $m+(m-j)(N-m-j)-j+(r-1)(2m-N-r)=0$, which implies that $(r-1)(2m-N-r)=0$. So $r-1=0$ or $(2m-N-r)\leq 0$. Since we have already proved the case $r=1$, we assume $(2m-N-r)\leq 0$.

$$2m-N-r \leq 0$$
$$2m-r-1 < N,$$

which is a contradiction. So $(r-1)(2m-N-r)\Delta_0+\Delta_1$ is invertible.

From Equation 4.5, knowing $(r-1)(2m-N-r)\Delta_0+\Delta_1$ is invertible, we have $\Delta_{r-1}(X_{\mathscr{R}} - X_{\mathscr{Q}}) = 0$. From the induction hypothesis, we have $d_{r-1} - 0$. From the edge version of Kelly's Lemma 4.6.2, we have $d_r(X_{\mathscr{R}} - X_{\mathscr{Q}}) = 0$. Hence the result follows. □

Bibliography

[1] U.S.R. Murty Adrian Bondy. *Graph theory*. Graduate texts in mathematics 244. Springer, 3rd corrected printing. edition, 2008.

[2] John Adrian Bondy and Robert L Hemminger. Graph reconstruction—a survey. *Journal of Graph Theory*, 1(3):227–268, 1977.

[3] AE Brouwer, AM Cohen, and A Neumaier. *Distance-Regular Graphs*. Springer, 1989.

[4] Mark N Ellingham. Vertex-switching reconstruction and folded cubes. *journal of combinatorial theory, Series B*, 66(2):361–364, 1996.

[5] Mark N Ellingham and Gordon F Royle. Vertex-switching reconstruction of subgraph numbers and triangle-free graphs. *Journal of Combinatorial Theory, Series B*, 54(2):167–177, 1992.

[6] Christopher David Godsil, Ilia Krasikov, and Yehuda Roditty. Reconstructing graphs from their k-edge deleted subgraphs. *Journal of Combinatorial Theory, Series B*, 43(3):360–363, 1987.

[7] Israel N Herstein. *Topics in algebra*. John Wiley & Sons, 1975.

[8] Ilia Krasikov and Yehuda Roditty. Switching reconstruction and diophantine equations. *Journal of Combinatorial Theory, Series B*, 54(2):189–195, 1992.

[9] L Lovász. A note on the line reconstruction problem. *Journal of Combinatorial Theory, Series B*, 13(3):309–310, dec 1972.

[10] Andrzej Schinzel. *An improvement of Runge's theorem on diophantine equations*. ex Aedibus Academicis, 1969.

[11] Carl Ludwig Siegel, K. Chandrasekharan, and H. Maaß. The integer solutions of the equation $y^2 = ax^n + bx^{n-1} + \cdots + k$. In *Gesammelte Abhandlungen*, pages 207–208. Springer Berlin Heidelberg, 1966.

[12] Richard P Stanley. Reconstruction from vertex-switching. *Journal of Combinatorial Theory, Series B*, 38(2):132–138, 1985.

[13] Bhalchandra D Thatte. A reconstruction problem related to balance equations. *Discrete Mathematics*, 176(1-3):279–284, 1997.

[14] Bhalchandra D Thatte. A reconstruction problem related to balance equations ii: The general case. *Discrete mathematics*, 194(1-3):281–284, 1999.

[15] J. H. van Lint. *Introduction to coding theory*. Springer Berlin Heidelberg, 1992.

Index