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Geometric structures on 3-dimensional manifolds.

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Sergio Andrés Pinillos Prado*

Orientador: Ph.D. Mathematics, Nikolai Alexandrovitch Goussevskii

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ATA DA DEFESA DE DISSERTAÇÃO DE MESTRADO DO ALUNO SERGIO ANDRÉS PINILLOS PRADO, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DO INSTITUTO DE CIÊNCIAS EXATAS DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA NO DIA 22 DE JULHO DE 2021.

Aos vinte e dois dias do mês de julho de 2021, às 13h00, em reunião pública virtual na Plataforma Google Meet pelo link https://meet.google.com/hqm-rbbp-pbr (conforme mensagem eletrônica da Pró-Reitoria de Pós-Graduação de 26/03/2020, com orientações para a atividade de defesa de dissertação durante a vigência da Portaria nº 1819), reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de dissertação do aluno **Sergio Andrés Pinillos Prado**, intitulada: "*Geometric structures on 3-manifolds"*, requisito final para obtenção do Grau de mestre em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Nikolai Alexandrovitch Goussevskii, após dar conhecimento aos presentes do teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se reservadamente sem a presença do aluno e do público, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 22 de julho de 2021.

PROF. DR. NIKOLAI ALEXANDROVITCH GOUSSEVSKII
Orientador (UFMG)

PROF. DR. MAURÍCIO BARROS CORRÊA JÚNIOR
Examinador (UFMG)

PROF. DR. VICTOR GUERASSIMOV
Examinador (UFMG)

Universidade Federal de Minas Gerais
Departamento de Matemática
Programa de Pós-Graduação em Matemática

FOLHA DE APROVAÇÃO

Geometric structures on 3-manifolds

SERGIO ANDRÉS PINILLOS PRADO

Dissertação defendida e aprovada pela banca examinadora constituída pelos Senhores:

Prof. Nikolai Alexandrovitch Goussevskii
UFMG

Prof. Maurício Barros Corrêa Júnior
UFMG

Prof. Victor Guerassimov
UFMG

Belo Horizonte, 22 de julho de 2021.

Av. Antônio Carlos, 6627 – Campus Pampulha - Caixa Postal: 702
CEP-31270-901 - Belo Horizonte – Minas Gerais - Fone (31) 3409-5963
e-mail: pgmat@mat.ufmg.br - home page: http://www.mat.ufmg.br/pgmat

Abstract

The work is focused on the study of geometric structures on 3-dimensional manifolds. The main objective is the description of the eight three-dimensional geometries given by the Thurston's theorem:

There are eight three-dimensional model geometries (G, X), *as follows:*

(a) *If the point stabilizers are* 3*-dimensional,* X *is* $\mathbb{S}^3$, $\mathbb{R}^3$, $\mathbb{H}^3$.

(b) *If the point stabilizers are* 1*-dimensional,* X *fibers over one of the two dimensional model geometries, in a way that is invariant under* G. *There is a* G*-invariant Riemannian metric on* X *such that the connection orthogonal to the fibers has curvature* 0 *or* 1.

(b1) *If the curvature is zero,* X *is* $\mathbb{S}^2 \times \mathbb{R}$ *or* $\mathbb{H}^2 \times \mathbb{R}$.

(b2) *If the curvature is 1, we have* nilgeometry *(wich fibers over* $\mathbb{R}^2$*) or the geometry of* $\tilde{S}L(2, \mathbb{R})$

(c) *The only geometry with* 0*-dimensional stabilizers is* solvegeometry, *which fibers over the line.*

Moreover, we will also give examples of compact 3-dimensional manifolds modeled on each one of these geometries and we shall present some interesting examples of manifolds modeled in $\mathbb{H}^3$, the 3-hyperbolic space.
Keywords:3**-dimensional manifolds, model geometries,** 3**-hyperbolic space.**

Resumo

O trabalho é focado no estudo de estruturas geométricas sobre variedades de dimensão três. O objetivo principal é a descrição das oito geometrias dadas pelo teorema de Thurston: *Existem oito geometrias modelo de dimensão três* (G, X) *como se segue:*

(a) Se os estabilizadores ponto tiverem de dimensão três, X é $\mathbb{S}^3$, $\mathbb{R}^3$, $\mathbb{H}^3$.

(b) Se os estabilizadores ponto tiverem de dimensão um, X fibra sobre uma das geometrias de dimensão dois, de uma forma que é invariável pela ação de G. Além disso, há uma métrica Riemanniana invariante de G sobre X, de tal forma que a conexão ortogonal às fibras tem curvatura 0 *ou* 1.

(b1) Se a curvatura é zero, X é $\mathbb{S}^2 \times \mathbb{R}$ *ou* $\mathbb{H}^2 \times \mathbb{R}$.

(b2) Se a curvatura é 1*, têmos a* nilgeometria *(que fibra sobre* $\mathbb{R}^2$*) ou a geometria de* $\tilde{S}L(2, \mathbb{R})$

(c) A única geometria que tem estabilizadores ponto de dimensão zero é a geometria Sol*, que fibra sobre a linha.*

Além disso, também daremos exemplos de variedades compactas de dimensão três modeladas sobre cada uma daquelas geometrias e apresentaremos alguns exemplos interessantes de variedades modeladas em $\mathbb{H}^3$ o 3-espaço hiperbólico.

Palavras-chave: Variedades de dimensão três, geometrias modelo, 3-espaço hiperbólico.

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Contents

Introduction

The theory of 3-manifolds was revolutionised by Thurston. He showed that the geometry together with the topology have an important role in this theory. But, what does it mean the term *geometry* for Thurston? In our context we shall use the approach given by Klein, is to say simply that if X is a set and G is a group acting on X, then the geometry of the pair (G, X) is the study of those properties of X left invariant by G.

We define a model geometry (G, X) to be a connected and simply connected manifold X together with a maximal Lie group G of diffeomorphisms which acts transitively on X with compact stabilizers. By logical purposes we shall define a (G, X)-manifold using this fact, the manifold M is locally modeled on X, and the transition maps are given by elements of the group G. In other words, a (G, X)-manifold is a manifold modeled with the model geometry (G, X), sometimes, we call them just as geometries, without any distinction.

Thurston proved that there are eight three-dimensional model geometries (G, X). There are three obvious geometries which correspond directly to the two-dimensional ones, the constant curvature geometries $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, but it is easy to find closed 3-manifolds which are not modeled on any of these. For example $\mathbb{S}^2 \times \mathbb{S}^1$ is modeled on $\mathbb{S}^2 \times \mathbb{R}$ wich is not homeomorphic to $\mathbb{S}^3$ or $\mathbb{R}^3$. Its metric is the product of the standard metrics, but here the stabiliser of a point is not $O(3)$, as in the constant curvature case. In fact the isometry group of $\mathbb{S}^2 \times \mathbb{R}$ is the direct product of the isometry group of $\mathbb{S}^2$ and of $\mathbb{R}$, and so the stabiliser of a point is isomorphic to $O(2) \times \mathbb{Z}_2$.

In this context, in the first chapter, we will present the preliminaries to develop this work. In the following chapters, we will sketch Thurston's proof that there are only eight three-dimensional geometries. Then we will describe the eight three-dimensional geometries, and will give some examples of compact 3-manifolds modeled on each one of these geometries. To construct manifolds modeled with these geometries, we use two ways: the first one is studying the action of subgroups Γ of the isometry group of X, which acts freely and properly discontinuously. And the second one is through the Poincaré's polyhedron theorem (2.2.2). Finally, in the last chapter, we shall give some examples of 3-hyperbolic manifolds, that is one of the most difficult geometries. It is important to say that this dissertation is based on the *William Thurston's* work, summarised on his book *Three-Dimensional Geometry and Topology* [Thu97] and the article *The Geometries of* 3*-manifolds* by *Scott, P.* [Sco83].

Index of notations

$\mathbb{R}$, $\mathbb{C}$	The real and complex number fields, respectively.
$\mathbb{R}^n$	Vector space of n-tuples of real numbers $(x^1, \cdots, x^n)$.
$\mathbb{C}^n$	Vector space of n-tuples of complex numbers $(z^1, \cdots, z^n)$.
$GL(V)$	General linear group acting on a vector space V.
$\mathfrak{g}$	Given a Lie group, denotes its Lie algebra.
T_xM	Tangent space of M at x.
TM	Tangent bundle of M.
T^1M	Unit tangent bundle of M.
LM	Bundle of linear frames of M.
OM	Bundle of orthonormal linear frames of M.
L_a, R_a	Left and right translation by $a \in G$. (G Lie group).
ω	Connection form.
Ω	Curvature form.
$Isom(X)$	Isometry group of X.
$\widetilde{X}$	Universal cover of X.

Chapter 1

Preliminaries

The foundations of manifolds, Lie groups and Riemannian geometry that were used to develop this dissertation are found at [KN63],[War94], [Thu97] and [DC92]. Therefore, we shall only show the most important theorems that will be used through all the work. Another books and articles that could complement the study are in the bibliography .

1.1 Pseudogroup

Definition 1.1.1. A pseudogroup on a topological space X is a set $\mathcal{G}$ of homeomorphisms between open sets of X satisfying the following conditions:

(a) The domains of the elements $g \in \mathcal{G}$ cover X.

(b) The restriction of an element $g \in \mathcal{G}$ to any open set contained in its domain is also in $\mathcal{G}$.

(c) The composition $g_1 \circ g_2$ of two elements of $\mathcal{G}$, when defined, is in $\mathcal{G}$.

(d) The inverse of an element of $\mathcal{G}$ is in $\mathcal{G}$

(e) If $g : U \to V$ is an homeomorphism between open sets of X and U is covered by open sets U_α such that each restriction $g|_{U_\alpha}$ is in $\mathcal{G}$, then $g \in \mathcal{G}$.

Example 1.1.2. Let X be a non-empty topological space . The *trivial pseudogroup* is defined as $\mathcal{G} = \{i_X\}$, where i_X is the identity map of X.

Example 1.1.3. ($\mathcal{G}$-manifold) Let $\mathcal{G}$ be a pseudogroup on $\mathbb{R}^n$. An n-dimensional $\mathcal{G}$-manifold is a topological space M with a $\mathcal{G}$-atlas on it. A $\mathcal{G}$-atlas is a collection of $\mathcal{G}$-compatible coordinate charts whose domains cover M. A coordinate chart, or *local coordinate system*, is a pair (U_i, ϕ_i), where U_i is open in M and $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphism onto its image. Compatibility means that whenever two charts (U_i, ϕ_i) and (U_j, ϕ_j) intersect, the transition map

$$\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is in $\mathcal{G}$.

Example 1.1.4. (Differentiable manifolds). If $\mathcal{C}^r$, for $r \geq 1$, is the pseudogroup of $\mathcal{C}^r$ diffeomorphisms between open sets of $\mathbb{R}^n$ a $\mathcal{C}^r$-manifold is called a differentiable manifold (of class $\mathcal{C}^r$). $\mathcal{C}^\infty$ manifolds are also called *smooth manifolds*.

Convention. Manifolds are Hausdorff and have countable basis.

Example 1.1.5. Let $\mathcal{C}^\omega$ be the pseudogroup of real analytic diffeomorphisms between open subsets of $\mathbb{R}^n$. A $\mathcal{C}^\omega$-manifold is called a *real analytic manifold*.

Example 1.1.6. (foliations). Write $\mathbb{R}^n$ as the product $\mathbb{R}^{n-k} \times \mathbb{R}^k$ and let $\mathcal{G}$ be the pseudogroup generated by diffeomorphisms ϕ(between open subsets of $\mathbb{R}^n$), that have the form

$$\phi(x, y) = (\phi_1(x, y), \phi_2(x, y)),$$

for $x \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$. The pseudogroup $\mathcal{G}$ consists of all diffeomorphisms between open sets of $\mathbb{R}^n$ whose Jacobian at every point is a $n \times n$ matrix such that the lower left $(n-k) \times k$ block is 0. A $\mathcal{G}$-atlas maximal is called a *foliation* of codimension k (or dimension $n-k$).

One dimensional foliations exist on a great many manifolds: any nowhere vanishing vector field has an associated foliation, obtained by following the flow lines.

Example 1.1.7. When n is even, $\mathbb{R}^n$ can be identified with $\mathbb{C}^{n/2}$. Let **Hol** be the pseudogroup of biholomorphic maps between open subsets of $\mathbb{C}^{n/2}$. A **Hol**-manifold is called a complex manifold of dimension $n/2$, and also a Riemann surface when $n = 2$.

Definition 1.1.8. A *manifold with-boundary* is a space locally modeled on $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. A manifold (without boundary) that is compact is called a *closed manifold*.

Example 1.1.9. Let $\mathbb{B}^2 \subset \mathbb{R}^2$ denote the open unit disk. Its manifold boundary is empty but its topological boundary over $\mathbb{R}^2$ is $\mathbb{S}^1$. So, $\mathbb{B}^2$ is a manifold without boundary. At the other side if $M = \overline{\mathbb{B}^2}$, then M is a manifold with boundary where its topological and manifold boundary coincides.

Definition 1.1.10. According to the example 1.1.3, it'll be allowed $\mathcal{G}$ to be a pseudogroup on any connected manifold X. These manifolds are also called $\mathcal{G}$*-manifolds*.

Definition 1.1.11. Let X be a manifold and G be a Lie group acting on X via diffeomorphisms, then a manifold M is called a (G, X) manifold if satisfies the following properties:

(a) There is an open cover $\{U_\alpha\}$ of M and a family $\{\phi_\alpha : U_\alpha \to V_\alpha\}$ of diffeomorphisms onto open sets $V_\alpha \subset X$, and

(b) If $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a $g \in G$, such that $gx = \phi_\alpha \circ \phi_\beta^{-1}(x)$, for all $x \in U_\alpha \cap U_\beta$, in other words, each transition map is given by a restriction of an element of the group G.

Proposition 1.1.12. Let $\mathcal{G}_0$ be a set of homeomorphisms between open subsets of X. Then there is a unique minimal pseudogroup $\mathcal{G}$ that contains $\mathcal{G}_0$. It is said that $\mathcal{G}$ is generated by $\mathcal{G}_0$.

Proof. Let $\mathcal{M} = \{\mathcal{G} : \mathcal{G}\ \textit{is a pseudogroup on } X \textit{ that contains } \mathcal{G}_0\}$ and set $\langle \mathcal{G}_0 \rangle = \bigcap_{\mathcal{G}\in\mathcal{M}} \mathcal{G}$. We have to prove that $\langle \mathcal{G}_0 \rangle$ satisfies the conditions $(a), (b), (c)$ and (d) of the definition 1.1.

Since the elements of $\langle \mathcal{G}_0 \rangle$ belongs to $\mathcal{G}$ for all $\mathcal{G} \in \mathcal{M}$ then its domains cover X. Given $f \in \langle \mathcal{G}_0 \rangle$, $f \in \mathcal{G}$ for all $\mathcal{G} \in \mathcal{M}$, that is, the restriction $f|_U$ to any open set contained in its domains is also in $\mathcal{G}$, for all $\mathcal{G} \in \mathcal{M}$. Also, $f^{-1} \in \mathcal{G}$ for all $\mathcal{G} \in \mathcal{M}$, so $f \in \langle \mathcal{G}_0 \rangle$. If $U = \bigcup_{\alpha\in A} U_\alpha$ where U_α is an open set for all $\alpha \in A$ and $g : U \to V$ is a homeomorphism between open sets of X, $g|_{U_\alpha} \in \mathcal{G}$ for all $\mathcal{G} \in \mathcal{M}$, that is, $g \in \langle \mathcal{G}_0 \rangle$. Finally, given $f_1 : U \to V \in \langle \mathcal{G}_0 \rangle$ and $f_2 : U' \to V' \in \langle \mathcal{G}_0 \rangle$, such that $U \cap V'$ is non-empty, then $f_1 \circ f_2 \in \mathcal{G}$ for all $\mathcal{G} \in \mathcal{M}$. Hence $f_1 \circ f_2 \in \langle \mathcal{G}_0 \rangle$. Therefore, $\langle \mathcal{G}_0 \rangle$ is by definition the minimal pseudogroup that contains $\mathcal{G}_0$. □

Observation. A (G, X)-manifold is a $\mathcal{G}$-manifold, where $\mathcal{G}$ is the pseudogroup generated by restrictions of elements of G whenever G is a given group acting on a manifold X.

Example 1.1.13. (Euclidean manifolds). If G is the group of isometries of Euclidean space E^n, a (G, E^n)-manifold is called a *Euclidean* or *flat*, manifold.

Example 1.1.14. Consider E^n as an n-dimensional vector space and let $e^1, \cdots, e^n$ be any basis of E^n. Let G be the group generated by $e^1, \cdots, e^n$: $G = \{\sum m_i e^i |\ m_i\ \textit{are integers}\}$. The n-torus $T^n = E^n/G$ has a flat structure.

Example 1.1.15. If G is the orthogonal group $O(n+1)$ acting on the sphere $\mathbb{S}^n$, a $(G, \mathbb{S}^n)$-manifold is called spherical or elliptic.

Example 1.1.16. If G is the group of isometries of hyperbolic space $\mathbb{H}^n$, a $(G, \mathbb{H}^n)$-manifold is a hyperbolic manifold.

1.2 Discrete Groups

Definition 1.2.1. Let Γ be a group acting on a topological space X by homeomorphisms. Normally is considered that the action is *effective*; this means that if $gx = x$ for all $x \in X$ then $g = e$. Other properties that the action might have:

(i) The action is *free* if $gx = x$ for some $x \in X$ implies that $g = e$.

(ii) The actions is *discrete* if Γ is a discrete subset of the group of homeomorphisms of X, with the compact-open topology.(In the *compact-open topology* on a set $C(X, Y) = \{f : X \to Y; f\ \textit{continuous}\}$, a neighborhood basis of $f \in C(X, Y)$ is given by finite intersections of sets of the form $\{f' \in C(X, Y) : f'K \subset U\}$, for all $K \subset X$ compact and $U \subset Y$ open such that $fK \subset U$).

(iii) The action *has discrete orbits* if every $x \in X$ has a neighborhood U such that the set of $g \in \Gamma$ mapping x inside U is finite.

(iv) The action is *wandering* if every $x \in X$ has a neighborhood U such that the set of $g \in \Gamma$ for wich $gU \cap U \neq \emptyset$ is finite.

(v) Assume X is locally compact. The actions of Γ is *properly discontinuous* if for every compact subset K of X the set $\{g \in \Gamma; gK \cap K \neq \emptyset\}$ is finite.

Example 1.2.2. $GL(n,\mathbb{R})$ acts on $\mathbb{R}^n$ as usual $y^j = \sum_i A^i_j x_i$, for $i, j = 1, \cdots, n$. The action is effective, neither transitive nor free, because for any $A \in GL(n,\mathbb{R})$ $A(0) = 0$. If consider $\mathbb{R}^n - \{0\}$ the action is transitive.

Example 1.2.3. $SO(2)$ acts on $\mathbb{S}^1$ as usual $A \cdot x = Ax$ for $A \in SO(2)$. The action is effective, transitive and free.

Example 1.2.4. Let $X = \mathbb{R}^2 - \{(0,0)\}$ and let $\Gamma = \mathbb{Z}$ be the group of diffeomorphisms generated by $(x, y) \mapsto (2x, y/2)$. The action is free and for any $p \in X$ its orbit doesn't have its accumulation points in X. Consider the compact $\overline{B_1((0,0))} \cap X = K$, then the set $\{g \in \Gamma | gK \cap K \neq \emptyset\}$ is not finite. Therefore, the action isn't properly discontinuous and the space X/Γ isn't a Haussdorf space.

Proposition 1.2.5. Let (Hausdorff) Γ be a topological group acting on a topological space (locally compact and Hausdorff) X. Γ acts properly discontinuously if and only if the map $\phi : \Gamma \times X \to X \times X$, given by $(\gamma, x) \mapsto (\gamma x, x)$ is proper and Γ is discrete.

Proof. Suppose that Γ acts properly discontinuously and let $K \subset X$ be a compact set. So, $K \times K \subset X \times X$ is a compact set. Since Γ acts properly discontinuously then the set $\{\gamma \in \Gamma | \gamma(K) \cap K \neq \emptyset\}$ is finite.

$$\begin{aligned} \phi^{-1}(K \times K) &= \{(\gamma, x) \in \Gamma \times X | (\gamma x, x) \in K \times K\} \\ &= \{(\gamma, x) \in \Gamma \times X | \gamma x \in K \text{ and } x \in K\} \\ &= \{\gamma \in \Gamma | \gamma(K) \cap K \neq \emptyset\} \times K \end{aligned}$$

Let $\{U_\alpha \times V_\alpha\}_{\alpha \in A}$ be an open covering of $\{\gamma \in \Gamma | \gamma(K) \cap K \neq \emptyset\} \times K$. For each $\gamma_j \in \{\gamma \in \Gamma | \gamma(K) \cap K \neq \emptyset\}$, $j \in \{1, 2, \cdots, m\}$, there is a basis element, that is the union of finite intersections of elements of the form $\{f \in C(X, X) | f(K') \subset U\}$ for K' compact and U an open set in X, then for each U_α there is a basis element Γ_α.
By choosing the respective basis elements Γ_{α_j} for each γ_j, we have a finite subcovering $\{\Gamma_{\alpha_j}\}_{j=1}^m$ of $\{U_\alpha\}$ that covers $\{\gamma \in \Gamma | \gamma K \cap K \neq \emptyset\}$, and since K is compact we also have a finite subcovering $\{V_{\alpha_i}\}_{i=1}^n$ of $\{V_\alpha\}$ that covers K. Then $\{\Gamma_j \times V_{\alpha_i}\}$ is a finite subcovering of $\{U_\alpha \times V_\alpha\}$ that also covers $\{\gamma \in \Gamma | \gamma K \cap K \neq \emptyset\} \times K$,

$$\phi^{-1}(K \times K) \subset \bigcup_{i,j} \left(\Gamma_{\alpha_j} \times V_{\alpha_i}\right) \text{ for } i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m.$$

Then $\phi^{-1}(K \times K)$ is compact.
To prove that Γ is discrete, let $x \in X$ and $\Gamma x \subset X$ be its orbit. By the local compactness of X, there is a compact neighborhood $K \supseteq U \ni x$ with K compact and U open. If $U \cap gU \neq$ then $K \cap gK \neq \emptyset$, but since the action is properly discontinuous, then $U \cap gU \neq \emptyset$ for only finitely many $g \in \Gamma$. Since $U \cap \Gamma x$ is finite then the action has discrete orbits. Consider the

map $\Gamma/\Gamma_x \to \Gamma x$ given by $g\Gamma_x \mapsto gx$. This map is a homeomorphism for all $x \in X$. Therefore, Γ_x(the preimage of x) is an open finite neighborhood of e, it means that Γ_x is discrete and translating, the topological group Γ has an open cover by discrete sets, then Γ is discrete.

Conversely suppose that ϕ is a proper map and Γ a discrete group. Let $K \subset X$ be a compact set, then by hypothesis $\phi^{-1}(K \times K)$ is a compact set in $\Gamma \times X$. By projecting this compact set in Γ we get a compact set in a discrete group, therefore finite. So it means that the actions of Γ is properly discontinuous. □

Corollary 1.2.6. Let G be a Lie group and $\Gamma \subset G$ a discrete subgroup. Consider the action of Γ on G by left translation. Then

(a) The action of Γ is wandering, and

(b) The action of Γ is properly discontinuous.

Proof. (a) Since Γ is discrete, for the identity element of the group $e \in G$ there is a neighborhood U such that $\Gamma \cap U = \{e\}$. By using the continuity of the product map $\cdot : G \times G \to G$, for each $x, y \in U$ there exist neighborhoods V_x and V_y of x and y respectively such that $\cdot(V_x, V_y) \subset U$. Let $\hat{V} = V_x \cap V_y$ and let $V = \hat{V} \cap \hat{V}^{-1}$. Then, $V = V^{-1}$ and since $V \subset V_x \cap V_y$ we have that $\cdot(V \times V) = V^2 \subset U$. Let $\gamma \in \Gamma$ and $g \in G$ and suppose that $x \in \gamma V g \cap V g$, then

$$x = \gamma(yg) = y'g \text{ for } y, y' \in V.$$

Thus $\gamma y = y'$ and therefore $\gamma = y'y^{-1} \in VV^{-1} = V^2 \subset U$. Then $\gamma \in \Gamma \cap U$ and by the hypotheses $\gamma = e$. Therefore for $g \in G$ there exists a neighborhood Vg such that the set of $\gamma \in \Gamma$ for which $\gamma V g \cap V g \neq \emptyset$ is finite.

(b) Let $K \subset G \times G$ be compact and let $(\gamma_i, g_i)_{i \in \mathbb{N}}$ be a sequence in $\phi^{-1}(K) \subset \Gamma \times G$. Then since $(\gamma_i g_i, g_i)_{i \in \mathbb{N}} \in K$ is a sequence in a compact set, then there exist g and g' such that a subsequence converges

$$\gamma_i g_i \to g' \text{ and } g_i \to g.$$

Given U a neighborhood of g' and V a neighborhood of g there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\gamma_i g_i \in U \text{ for all } i \geq N_1,$$

and

$$g_i \in V \text{ for all } i \geq N_2.$$

If we consider the set $S = \{\gamma \in \Gamma | \gamma(V) \cap U \neq \emptyset\}$ then it is non-empty and must be finite. We shall show that for $i \geq \max\{N_1, N_2\}$, $\gamma_i \to e$, this proves that there is a convergent subsequence of (γ_i, g_i) that converges in $\phi^{-1}(K)$ and using the proposition 1.2.5 we shall have finished.

Let $\gamma_i \in S$ then $\gamma_i(v) = u$ for some $v \in V$ and $u \in U$. Hence $\gamma_i = uv^{-1} \in UV^{-1}$. Since $\gamma_i(UV^{-1}) \subseteq UV^{-1}$ then $\gamma_i \in UV^{-1}VU^{-1}$ and $\gamma_i(UV^{-1}VU^{-1}) \subseteq \{e\}$. Therefore $UV^{-1}VU^{-1}$ is a neighborhood that intersects Γ at the identity, and using the discreteness of Γ we have that $\gamma_i \to e$ or equivalently that $\gamma_i = e$ for $i \geq \max\{N_1, N_2\}$. Then the set S is finite and the sequence (γ_i, g_i) have a convergent subsequence, as we required.

□

Proposition 1.2.7. Let Γ be a group acting on a connected manifold X. The quotient X/Γ is a manifold with $X \to X/\Gamma$ a covering projection if and only if Γ acts freely and properly discontinuously.

Proof. Suppose that the action is free and properly discontinuous. Let x and y be points in X, such that neither $G(x) = \{gx | g \in \Gamma\}$ contain y nor $G(y)$ contains x. Let K_1 and K_2 be compact disjoint neighborhoods of x and y respectively and set $K = K_1 \cup K_2$. Then $K - \cup_{g \neq e} gK$ is still a union of a neighborhood of x with a neighborhood of y, and these neighborhoods project to disjoint neighborhoods in X/Γ. Given $x \in X$, take a neighborhood U of x that intersects only finitely many of its translates gU, it's possible because if the action is properly discontinuous, then the action is wandering. As the action is free and X is Hausdorff, can be chosen a smaller neighborhood of x, U' whose translates are all disjoints. Then each translate maps homeomorphically to its image in the quotient, so the image is evenly covered. Then the quotient map is a covering map and the space X/Γ is a manifold.

For the converse, let (x_1, x_2) be any pair of points in $X \times X$. If $x_2 \in G(x)$ then exists a $g \in \Gamma$ such that $x_2 = gx_1$, as $p : X \to X/\Gamma$ is a covering map, it will be taken a neighborhood U_1 of x_1 that projects homeomorphically to the quotient space, and let $U_2 = gU_1$. So, $gU_1 \cap U_2 \neq \emptyset$. If x_2 isn't on the orbit of x_1, by the Haussdorf property of X/Γ there exist disjoint neighborhoods of $p(x_1)$ and $p(x_2)$, so for at most one $g \in \Gamma$, $gU_1 \cap U_2 \neq \emptyset$, where U_1 and U_2 are neighborhoods of x_1 and x_2 respectively.

Now let K be any compact of X. Since $K \times K$ is compact, there is a finite covering of $K \times K$ by product neighborhoods of the form $U_1 \times U_2$, where U_1 has at most one image under Γ intersecting U_2. Therefore the set $\{g \in \Gamma | gK \cap K \neq \emptyset\}$ is finite, and Γ acts freele and properly discontinuously. □

Lemma 1.2.8. *Let G act transitively on an analytic manifold X. Then X admits a G-invariant Riemannian metric if and only if, for some $x \in X$, the image of the stabilizer S_x of x in $GL(T_xX)$ has compact closure.*

Proof. Since the Riemannian metric is preserved by G, then the map $f : S_x \to GL(T_xX)$ defined by $g \mapsto (L_g)_*$ maps S_x to a subgroup of $O(T_xX)$, which is compact. Conversely fix x and suppose the image of S_x has compact closure H_x. Let Q be any positive definite form on

T_xX. Any compact topological group can be given a finite measure, called *Haar measure* that is invariant under left or right translations of the group. Define

$$(u, v) = \int_{H_x} Q\left((L_g)_*u, (L_g)_*v\right)dg, \text{ where } u, v \in T_xX$$

where dg is the *Haar measure* on H_x. Then (u, v) is an inner product on T_xX invariant under the action of S_x . Since G acts transitively, this inner product can be propagated to T_yX for any $y \in X$, and we thus get a G-invariant Riemannian metric on X. □

1.3 Bundles and connections

Definition 1.3.1. (fiber bundle). If G is a topological group acting on a topological space X, a (G, X)-bundle (or more formally a *fiber bundle* with structure group G and fiber X) consists of the following data: a *total space* E, a *base space* B, a continuous map $p : E \to B$, called the *bundle projection*, and a *local trivialization*, explained below. It can also be said that the space E fibers over B with fiber X.

A local trivialization is a covering of B by a collection of open sets U_i, and for each U_i a homeomorphism $\phi_i : p^{-1}(U_i) \to U_i \times X$ which gives p when it is composed with the projection $U_i \times X \to U_i$. The ϕ_i are required to be such that for each intersecting U_i and U_j, for each intersecting U_i and U_j, the composition

$$\psi_{ij} = \phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times X \to (U_i \cap U_j) \times X$$

has the form

$$\psi_{ij}(u, x) = (u, \gamma_{ij}(u)x),$$

where $\gamma_{ij} : U_i \cap U_j \to G$ is continuous.

Example 1.3.2. (product bundle). The *product* (G, X)-bundle over a space B is $B \times X$.

Example 1.3.3. (mapping torus). If M is a smooth manifold and $\phi : M \to M$ is a diffeomorphism, the mapping torus M_ϕ is obtained from the cylinder $M \times [0, 1]$ by identifying the two ends via the map ϕ. Clearly, M_ϕ is an M-bundle over the circle.

Example 1.3.4. The cylinder is an $(\{e\}, \mathbb{R})$-bundle over $\mathbb{S}^1$. Also, if $\phi : [0, 1] \to [0, 1]$ is the identity map on $[0, 1]$ then $[0, 1]_\phi$ is the cylinder, obtained using the mapping torus.

Example 1.3.5. The Möbius band is a $(\mathbb{Z}_2, \mathbb{R})$-bundle over $\mathbb{S}^1$. Consider the diffeomorphism $\phi : [0, 1] \to [0, 1]$ given by $\phi(x) = 1 - x$. Then $[0, 1]_\phi$ is the Möbius band, obtained using the mapping torus.

Example 1.3.6. If X is a vector space and $G = GL(X)$ is its group of linear automorphisms, then a $(G, X) - bundle$ is called a *vector bundle.*

Definition 1.3.7. A *principal bundle* is one in wich the fiber is the structure group itself, and the action is by left translations.

Example 1.3.8. Let M be a n-manifold. A linear frame u at a point $x \in M$ is a basis $X_1, \cdots, X_n$ of the tangent space T_xM. Let LM be the set of all linear frames u at all points of M and let p be the mapping of LM onto M wich maps a linear frame u at x into x. The general linear group $GL(n, \mathbb{R})$ acts freely on LM by $Y_i = \displaystyle\sum_{j=1}^{n} a_i^j X_j$, where $a = (a_i^j) \in GL(n, \mathbb{R})$. Every frame u at $x \in U$ can be expressed uniquely in the form $u = (X_1, \cdots, X_n)$ with $X_i = \displaystyle\sum_{j=1}^{n} X_i^j \left(\partial/\partial x^j\right)$ for $(x^1, \cdots, x^n)$ a local coordinate system in a coordinate neighborhood U in M, where (X_i^j) is a non-singular matrix. This shows that $p^{-1}(U)$ is in $1:1$ correspondence with $U \times GL(n; \mathbb{R})$. So, LM is a principal G-bundle over M.

We'll construct a fiber bundle called a *fiber bundle associated with P and standard fibre F* as follows. Let P be a principal G-bundle and F a manifold on which G acts on the right: $F \times G \to F$, via the map $(\zeta, g) \mapsto \zeta g$. On the product manifold $P \times F$, let G be acting as follows: an element $g \in G$ maps $(u, \zeta) \in P \times F$ into $(g^{-1}u, \zeta g)$. The quotient space $P \times F$ by this group action is denoted by $E = P \times_G F$. The mapping $P \times F \to B$ wich maps (u, ζ) into $p(u)$ induces a mapping $p_E : E \to B$, called the projection of E over B. In fact, if $[(u_1, \zeta_1)] = [(u_2, \zeta_2)]$, then exists a $g \in G$ such that $(g^{-1}u_1, \zeta_1 g) = (u_2, \zeta_2)$, so $[u_1] = [u_2]$ over P and therefore $p_E([(u_1, \zeta_1)]) = p(u_1) = p(u_2) = p_E([(u_2, \zeta_2)]$. Then, p_E is well defined.

For each $x \in B$, the set $p_E^{-1}(x)$ is called the fibre of E over x. Every point $x \in B$ has a neighborhood U such that U is homeomorphic to $U \times G$. Identifying $p^{-1}(U)$ with $U \times G$, we see that the action of G on $p^{-1}(U) \times F$ on the right is given by

$$(b, g, \zeta) \mapsto (b, h^1 g, \zeta h) \text{ for } (b, g, \zeta) \in U \times G \times F \text{ and } h \in G.$$

It follows that the homeomorphism $p^{-1}(U) \simeq U \times G$ induces an homeomorphism $p_E^{-1}(U) \simeq U \times F$.

Definition 1.3.9. The fiber bundle that was constructed or more precisely the (G, F)-manifold E is called the fibre bundle over the base B, with fibre F and group G, which is associated with the principal fiber bundle P.

Example 1.3.10. (*Tangent bundle*). Let $GL(n; \mathbb{R})$ be the structure group acting on $\mathbb{R}^n$, if $e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^n$ then $e_i \mapsto \sum_j a_j^i e_i \in \mathbb{R}^n$ for $a = (a_j^i) \in GL(n; \mathbb{R}^n)$. Then for $u \in LM$, $ua : \mathbb{R}^n \to T_xM$ is the composition of the following maps, $a(e_i) = \sum_j a_j^i e_i$ and $u(e_i) = X_i \in T_xM$,

$$\mathbb{R}^n \xrightarrow{a} \mathbb{R}^n \xrightarrow{u} T_xM \; .$$

So, the tangent bundle $TM = \{[(u, X)] | u \in LM, X \in \mathbb{R}^n\}$ with the projection map $p_{TM} : TM \to M$, given by $p_{TM}([(u, X)]) = p(u) = x$, is the bundle associated with LM with standar fibre $\mathbb{R}^n$ and structure group $\mathbb{R}^n$. The fiber of TM over $x \in M$ may be considered as T_xM.

Example 1.3.11. The *tangent sphere bundle* of a differentiable n-manifold M is obtained by collapsing each ray in TM to a point, so the fiber becomes the $(n-1)$-sphere $\mathbb{S}^{n-1}$. If M has a Riemannian metric, the tangent sphere bundle can be thought as the subset of TM consisting of tangent vectors of unit length. In this case it will be called the *unit tangent bundle* to M, and denote it T^1M. In other words, T^1M is a fiber bundle over M,where the fiber is the unit sphere in TM and M is a Riemannian manifold.

Definition 1.3.12. (**Connections**) Let $p : E \to B$ be a smooth (G, X)-bundle, that is, E, B and X are all smooth manifolds, of dimensions $m+n$, n and m respectively, p is a smooth map, G is a Lie group acting smoothly on X and the transition maps γ_{ij} defining E are smooth. A *connection* for E is an n-plane field τ, transverse to the fibers and satisfying a (G, X)-compatibility: for any fiber E_x, there must exist some smooth local coordinate chart E for E such that τ is tangent to the horizontal directions. In other words, given $p \in E$ and V_p the subspace of T_pE consisting of vectors tangent to a fiber over p, a connection τ in E is an assignment of a subspace H_p de T_pE such that:

(a) $T_pE = H_p \oplus V_p$

(b) $H_{gp} = (L_g)_*(H_p)$ where $g \in G$.

(c) The distribution $p \mapsto H_p$ is differentiable.

The condition of compatibility could be expressed in the following way: Given any smooth n-plane field τ transverse to the fibers and given a path α in the base between points x and y, there is a map between some subset of the fiber E_x to some subset of E_y. This map called the *parallel translation* along α, is obtained by lifting α to a path $\hat{\alpha}$ always tangent to τ.

Example 1.3.13. Let M be a submanifold of some Euclidean space $\mathbb{R}^n$, with its inherited Riemannian metric. Given two nearly parallel m-planes $P, P' \subset \mathbb{R}^n$, the orthogonal projection from one to the other is nearly an isometry: it distorts the metric by a factor no greater than the cosine of the angle between the planes. It can be taken a one-parameter family of m-planes that mediate between P and P', and look at the composition of orthogonal projections from $P = P_0$ to P_1 to P_2 and so on to $P_N = P'$, where $P_1, \cdots, P_N - 1$ are elements of the family taken in order, the distortion decreases as the subdivision gets finer, and in the limit we get an actual isometry. Applying this to the family of tangent planes to M along a path α, we get a flow of euclidean isometries, with trajectories orthogonal to the planes. This flow defines parallel translation for a certain Euclidean connection on TM. The *Levi-Civita* connection is obtained by normalizing the Euclidean connection by translations to keep the origin fixed, converting it into an orthogonal connection.

Definition 1.3.14. Let τ be a connection in E. For each $X \in T_pE$, $\omega(X)$ is defined to be the unique $A \in \mathfrak{g}$ such that $(A^*)_p$ is equal to the vertical component of X. The form ω is called the connection form of the given connection τ.

Remark.

(a) Given $a_t \in G$ the integral curve of the vector field A that begins at $a_0 = e$. a_1 is denoted by $\exp A$ and therefore $a_t = \exp(tA)$. The mapping $\exp : \mathfrak{g} \to G$ given by $A \mapsto \exp(A)$ is called the *exponential mapping*. ([KN63, p.39])

(b) If G acts on B on the right, the action of the 1-parameter subgroup $a_t = \exp(tA)$ on B induces a vector field on B wich will be denoted by A^* and called the fundamental vector field corresponding to A. So, $A \mapsto (A^*)_p$ is a linear mapping of $\mathfrak{g}$ onto V_p for each $p \in E$. ([KN63, p.42])

Example 1.3.15. Let τ be the plane field in $\mathbb{R}^3$ that associates to (x, y, z) the plane spanned by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$. So, τ is a connection for the $(\mathbb{R}, \mathbb{R})$-bundle $\pi_z : \mathbb{R}^3 \to \mathbb{R}^2$, where π_z is projection along the z-axis. We can write τ in a dual way, as the kernel of the 1-form $\omega = -xdy + dz$.

Example 1.3.16. Let τ be the plane field in $\mathbb{R}^3$ considered in the example 1.3.15, where the 1-connection form ω, is given by the expression $\omega = -xdy + dz$. Then the curvature form of ω is given by $\Omega = d\omega = -dx \wedge dy = -\pi_z^* dA$, where dA is the area form in the plane and π_z^* denotes the pullback under π_z.

Definition 1.3.17. The *curvature* Ω is a $\mathfrak{g}$-valued two-form: given two vectors X and Y at a point $p \in M$, map the unit square into M so that the vectors $(1, 0)$ and $(0, 1)$ at the origin are taken to X and Y. For each s and t, let $P(s, t)$ be the Parallel translation of the Levi-Civita connection around the boundary of the rectangle $[0, s] \times [0, t]$. Then the second derivative of this map, with respect to s and t, is the curvature $\Omega(X, Y)$.

In the case of the Levi-Civita connection for a Riemannian surface, there is only a one-dimensional vector space of tangent two-vectors and a one-dimensional vector space of rotations over the fiber(that is, the Lie algebra of $O(2)$ is one-dimensional). Gaussian curvature of a surface is the curvature of the Levi-Civita connection expressed in a canonical basis: the area form and the unit speed rotation.

More generally, for an n-dimensional Riemannian manifold, the *sectional curvature* at a point p with respect to a tangent plane P is obtained by restricting the domain of the curvature form to two-vectors in P, and projecting the image to the Lie subalgebra of infinitesimal rotations of P.

1.4 Contact structures in three dimensions

The *contact structures* in three dimensions are related to a kind of plane field, that is as non-integrable as possible.

Definition 1.4.1. Le τ be a plane field in $\mathbb{R}^3$. A *contact diffeomorphism* between subsets of $\mathbb{R}^3$ is one that preserves τ. The *contact pseudogroup* **Con** is the pseudogroup of contact diffeomorphisms between open subsets of $\mathbb{R}^3$, and a *contact structure* on a three-dimensional manifold is a **Con**-structure.

Example 1.4.2. Let τ be the plane field in $\mathbb{R}^3$ sppaned by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$. To understand τ, it helps to think about *Legendrian curves*, which are curves in $\mathbb{R}^3$ whose tangent vectors are always contained in τ. Let π_x be the projection to the yz-plane along parallel lines to the x-axis. These lines are Legendrian curves. Let γ be any Legendrian curve and its projection $\pi_x(\gamma)$. At any time when the derivative $\frac{d\pi_x(\gamma(t))}{dt}$ is non-zero, it is the slope of this tangent vector in the

yz-plane. Now let ϕ be any diffeomorphism of the yz-plane. Then there is a unique contact diffeomorphism $\hat{\phi}$, defined on most $\mathbb{R}^3$, that preserves the foliation by lines parallel to the x-axis, and projects ϕ under π_x. Given $p = (x, y, z) \in \mathbb{R}^3$, the projection of τ to the yz-plane is a line of slope x. The derivative of ϕ maps the vector $(1, x)$ at (y, z) to some other vector; we set the x-coordinate of $\hat{\phi}(p)$ to the slope of this vector. If the slope is infinite, $\hat{\phi}$ is undefined at p. By construction, $\hat{\phi}$ preserves τ. If ϕ maps vertical lines to vertical lines, the slope is never infinite, and we get a contact automorphism of $\mathbb{R}^3$. For example the map

$$(x, y, z) \mapsto (x + x_0, y + y_0, z + z_0 + x_0 y)$$

preserves τ and preserves the foliation of $\mathbb{R}^3$ by the curves parallel to the x-axis and also the foliation by vertical lines.

Example 1.4.3. (area-preserving automorphisms lift). Let ϕ be a diffeomorphism of the xy-plane that preserves area (or multiplies it by a constant factor). Then there is contact automorphism $\widetilde{\phi}$ of $\mathbb{R}^3$ that preserves the foliation by vertical lines, and projects to ϕ under π_z. Moreover, any two such maps differ by a vertical translation.

For given an arbitrary point $p \in \mathbb{R}^3$, we can connect p to a fixed point $q \in \mathbb{R}^3$ by a smooth Legendrian curve γ. If $\widetilde{\phi}$ is to map Legendrian curves to Legendrian curves, the only possible candidate for $\widetilde{\phi}(p)$ is the end point of the Legendrian lift of $\phi \circ \pi_z \gamma$ that starts at $\widetilde{\phi}(q)$. This endpoint does not depend on the choice of γ: the lifts of two curves with the same endpoints have the same endpoints if and only if the signed area enclosed by the curves is zero, and this area property is preserved by ϕ.

This shows the existence and uniqueness of $\widetilde{\phi}$, and also that every contact automorphism of $\mathbb{R}^3$ that preserves the foliation by vertical lines is of this type.

Chapter 2

The Eight Model Geometries

2.1 Model Geometry

In this section first we shall sketch the Thurston's proof about the existence of eight 3-dimensional geometries and second we shall describe each one of these eight geometries. In this context, there seems to be three distinct, but related approaches to geometry which could be taken. Often these are combined in various ways. The first approach is the classical one exemplified by Euclid in which discusses only points, lines, incidence relations, angles and length. The second approach is the differential geometry. Here the geometry of the Euclidean plane $\mathbb{R}^2$ is recovered from the standard Riemannian metric on $\mathbb{R}^2$ given by the expression $g = dx \otimes dx + dy \otimes dy$. A third approach , formulated by Klein, where X is a set and G is a group acting on X, the geometry of the pair (X, G) is the study of those properties of X left invariant by G.

For logical purposes, we shall pick only one definition.

Definition 2.1.1. (model geometry) A model geometry (G, X) is a manifold X together with a Lie group G of diffeomorphisms of X, such that:

(a) X is connected and simply connected;

(b) G acts transitively on X, with compact point stabilizers;

(c) G is not contained in any larger group of diffeomorphisms of X with compact stabilizers of points; and

(d) there exists at least one compact manifold modeled on (G, X)

Example 2.1.2. Consider $X = \mathbb{R}^2$ and $G = Isom(\mathbb{R}^2)$, and $G_1 = \mathbb{R}^2$, where G_1 acts on itself by translations. The geometry (G_1, X) must be ignored by the condition (c) at the definition, because clearly $G_1 \subset G$. So, the model geometry is given by (G, X). In this case, the covering projection $p : \mathbb{R}^2 \to T^2$ gives to T^2 the geometry of (G, X).

Theorem 2.1.3. *The only simply connected, complete Riemannian n-manifolds with constant sectional curvature are $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$.*

Proof. See [DC92, p.163]. □

Theorem 2.1.4. *There are precisely three two-dimensional model geometries: spherical, Euclidean and hyperbolic.*

Proof. Let (G, X) be a 2-dimensional model geometry . Then at each point there exists a tangent plane where the sectional curvature, in this case, coincides with the Gaussian curvature. Since G acts transitively on X, and the Riemannian metric on X is G-invariant, the curvature at a point is taken by the transitive action to any other point, and since the curvature is invariant by the metric, then the curvature must be equal at all the points in X. Thus, by theorem 2.1.3, X only could be $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$. □

We obtain the classification of the 3-dimensional model geometries by considering the size of G_x. But all the work will be focused on the connected component of the identity that we shall denote it by G' by the following facts that we will prove:

1. The action of G' is still transitive and the stabilizers G'_x, for $x \in X$ are connected.
2. $G'_x/(G'_x)_0$ form a covering space of X, where $(G'_x)_0$ is the component of the identity of G'_x
3. Since X is simply connected the covering is trivial.
4. G'_x is a connected closed subgroup of $SO(3)$, and therefore is also a Lie group. So there are only three possiblities: $SO(3)$, $SO(2)$ and the trivial group.

Since the stabilizer G_x is a Lie group of the same dimension, we obtain the classification of the 3-dimensional geometries by considering $G_x = SO(3)$, $G_x = SO(2)$ and G_x as the trivial group.

Lemma 2.1.5. *Let (G, X) be a model geometry and G' be the identity component of G. Then the action of G' is still transitive*

Proof. Since G acts transitively, G/G_x is diffeomorphic to X. Moreover, G' is an open normal subgroup of G and hence $G'G_x$ is an open subgroup of G. Let $f : G \to G/G_x$ be the projection map, and consider the image of $G'G_x$ by f, it is an open set, because it is the image of its cosets in G. Then we have got a collection of disjoint open sets in G/G_x, however since X is connected and diffeomorphic to G/G_x then there is only one open set, so $G'G_x = G$.

Let x, y be points in X. As G acts transitively on X, there is a $g \in G$ such that $gx = y$. By the fact that $G'G_x = G$, then $y = gx = h'fx = h'x$, for some $h' \in G'$ and $f \in G_x$. Therefore, G' acts transitively on X. □

Lemma 2.1.6. *Let (G, X) be a model geometry and G' be the connected component of the identity. Then G'_x is connected.*

Proof. The lemma 2.1.5 implies that the quotient G'/G'_x is diffeomorphic to X. Let $(G'_x)_0$ be the connected component of the identity in G'_x. If we consider the quotient of G'_x by its connected component of the identity, then, the map $p : G'_x/(G'_x)_0 \to X$ will be surjective. Since X is simply connected this covering must be trivial, and for each $x \in X$ the fiber $G'_x/(G'_x)_0$ is exactly one point. So, $G'_x = (G'_x)_0$. □

The lemmas 2.1.5, 2.1.6 and 1.2.8 prove that G'_x is a connected closed subgroup of $SO(3)$. Moreover, by using the fact that a closed subgroup of a Lie group is also a Lie group, and therefore a manifold, therefore there are only three possibilities: $SO(3)$, $SO(2)$ and the trivial group. The stabilizer G_x is a Lie group of the same dimension.

Theorem 2.1.7. (Thurston) *There are eight* 3*-dimensional model geometries* (G, X)*, as follows:*

(a) If the point stabilizers are 3*-dimensional,* X *is* $\mathbb{S}^3$, $\mathbb{R}^3$, $\mathbb{H}^3$.

(b) If the point stabilizers are 1*-dimensional,* X *fibers over one of the two dimensional model geometries, in a way that is invariant under* G*. There is a* G*-invariant Riemannian metric on* X *such that the connection orthogonal to the fibers has curvature* 0 *or* 1*.*

(b1) If the curvature is zero, X *is* $\mathbb{S}^2 \times \mathbb{R}$ *or* $\mathbb{H}^2 \times \mathbb{R}$*.*

(b2) If the curvature is 1, we have nilgeometry *(wich fibers over* $\mathbb{R}^2$*) or the geometry of* $\tilde{S}L(2, \mathbb{R})$

(c) The only geometry with 0*-dimensional stabilizers is* solvegeometry*, which fibers over the line.*

Proof. Let g be a G-invariant Riemannian metric on X.

(a) If the connected component of the identity G' of G acts with stabilizer $SO(3)$, any tangent plane at any point can be taken by G to any tangent two-plane at any other point. If we think of each tangent plane as a great circle of $\mathbb{S}^2$, and since the action of $SO(3)$ on $\mathbb{S}^2$ is by rotations, then any great circle can be taken to any other great circle. Moreover, since the metric, and hence the curvature is G-invariant, then X has constant sectional curvature. As in the 2-dimensional case, it follows that the geometry is spherical, Euclidean or hyperbolic. The full group of isometries G contains G' with index 2, and can be obatined by adding any orientation-reversing isometry.

(b) If G' acts with stabilizer $SO(2)$, then the tangent space T_xX contains a one-dimensional subspace which is fixed under the action of G'_x for any $x \in X$. Let $p \in X$ and V_p a tangent vector in the one-subspace. Since G acts transitively, $g_* : TX \to TX$ takes V_p overall $g \in G$ and $g_*(V_p)$ is fixed if $g \in G'$. It gives us a G'-invariant vector field V on X. The integral curves of V form a G'-invariant one-dimensional foliation $\mathfrak{F}$. Let ϕ_t be the flow of V.

Lemma 2.1.8. *If* $g' \in G'$ *then the flow of* V *commutes with* G'*.*

Proof. Consider on V the expression $\psi_t(p) = g \circ \phi_t \circ g^{-1}(p)$, so, $\psi_t(gp) = g \circ \phi_t \circ (g^{-1}g)(p) = g \circ \phi_t(p)$. Besides $\psi_0(gp) = gp$ and $\psi'_0(p) = (g \circ \phi_t)'|_{t=0}(p) = g_*(V_p) = V_{gp}$. Then ψ_t is a flow of V at the time t, but since the flow is unique then $\psi_t = \phi_t = g \circ \phi_t \circ g^{-1}$. □

Also, if an element of G' fixes some point on a leaf F of $\mathfrak{F}$, it fixes any other point on F: Given x, y two points in F, there exists a t such that $\phi_t(x) = y$. Then, for any $g \in G'_x$, $gy = g\phi_t(x) = \phi_t(gx) = y$, so $G'_x = G'_y$. It proves that all the elements in the same leaf, have the same stabilizer. This also implies that if an element of G' takes a point $x \in F$ to another point $y \in F$, it commutes with any element of the stabilizer G'_x.

Now, let F be a leaf and $x \in F$ a point fixed, and let g_t be an element of G taking $\phi_t(x)$ back to x. Then $g_t \circ \phi_t$ fixes x and $(g_t \circ \phi_t)_*$ is a linear automorphism of T_xM. The differential is the identity along the axis of the action of G'_x. Moreover, it commutes with rotations around this axis, that is, with elements of G'_x. Then it must be itself a rotation around this axis, possibly composed with an expansion or contraction, but since by assumption there is compact manifold modeled on (G, X) then the vector field inherited from X must preserve the volume. So, an expansion or contraction is ruled out. Now, it will be proved that $g_t \circ \phi_t$ is a rotation around this axis. Suppose that ω and V are the volume form and the vector field on M, respectively, inherited from X, both invariant under G'. Then for $p \in M$ there exists a $g \in G'$ such that $gp = q$ where $q \in M$. So,

$$(\phi_t^*(\omega))_q = (\phi_t^*(\omega_{gp})) = \omega_{\phi_t \circ g(p)},$$

Moreover, ϕ_t commutes with the elements of G' then $\phi_t \circ g = g \circ \phi_t$ and

$$\omega_{\phi_t \circ g(p)} = \omega_{g \circ \phi_t(p)} = g^*(\omega_{\phi_t(p)}) = ((g \circ \phi_t)^*(\omega))_p.$$

Therefore, $(\phi_t^*(\omega))_q = (\phi_t^*(\omega))_p$ for any $p, q \in M$, because the flow of the vector field and the volume form are both invariant under G'. Since the *divergence* of the vector field V on the manifold M with volume form ω, div V, is the Lie derivative $L_V\omega$, that measures how much V expands or contracts volume, then $(L_V\omega)_p = (L_V\omega)_q$ for any $p, q \in M$, that is, div V is constant over M. Moreover, since the manifold modeled on (G, X) is compact, the vector field must preserve the total volume, and must preserve the volume at every point. Therefore, the div $V = 0$ and it implies that $(g_t \circ \phi_t)_*$ is an isometry on T_xM and thus $(\phi_t)_*$ is also an isometry. Since x was arbitrary, the flow of the vector field V is by isometries.

Let x be a point in a leaf F_1 and y be a point in a leaf F_2. Since X is Hausdorff there exist U_x and U_y neighborhoods of x and y respectively. Then set $W_x = G_xF_1$, where $G_x = \{g \in G | gx \in U_x\}$, and $W_y = G_yF_2$ in a similar way. Therefore, since the action of G' commutes with the flow of the vector field V, W_x and W_y are two disjoint neighborhoods for F_1 and F_2 respectively. So, $X/\mathfrak{F}$ is a Hausdorff space.

By considering a point on a leaf with its respective neighborhood and the fact that any leaf is invariant under a subgroup G'_x isomorphic to $SO(2)$, it is then possible to conclude that the leaf doesn't accumulate on itself, but it is an embedded image of either $\mathbb{S}^1$ or $\mathbb{R}$. In fact, suppose by contradiction that the leaf F isn't an embedded image of either $\mathbb{S}^1$ or $\mathbb{R}$ and consider that F is an integral curve of V for all t, then it must approach a point p without passing for it or accumulate somewhere on itself. In the first case, the point p belongs to some leaf $F_0 \neq F$ but then F and F_0 wouldn't have disjoint neighborhoods, contradicting the fact $X/\mathfrak{F}$ is a Hausdorff space. In the other case, suppose that q is an accumulation point in F and let $(g_n)_{n\in\mathbb{N}}$ be a sequence in G' such that $g_n \to e$. If p_1 is point in the leaf for some t_1 such that at the first time that the integral curve pass for p is in t_0, where $t_0 < t_1$, then the group action moves the curve "ahead" and $g_n p \to p_1$ but this contradicts the fact that $g_n p \to ep = p$. So, the leaf F is an embedded image of $\mathbb{S}^1$ or $\mathbb{R}$.

Therefore the quotient space $X/\mathfrak{F}$ is a two dimensional manifold Y. Since V acts by isometries, Y inherits a Riemannian metric from X (just ignore the component of the metric of X in direction of the leaves) and a transitive action of G' by isometries. Also Y is connected because X is. In the same way if $[x] \in Y$ and we consider a loop based at $[x]$, then it has multiple preimages in X. If we choose any of these paths in X, it has its ending points at the same leaf of x and since X is simply connected and F is path-connected, then the path is homotopic to a path between the two ending points, lying entirely inside F. Thus the identity map and the loop are homotopic and Y is simply connected. By the proof of the Theorem 2.1.4, Y must be one of the two-dimensional model geometries: $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$. In addition, X is a principal fiber bundle over Y, with fiber and structure group equal to $\mathbb{S}^1$ or $\mathbb{R}$.

The plane field τ orthogonal to $\mathfrak{F}$ is a connection for this bundle. Since the group of isometries of X acts transitively, τ has constant curvature.

(b_1) If the curvature is zero, the plane field is integrable (see [Thu97, p. 177]), and τ defines a foliation transverse to the fibers. Since Y is simply connected, the bundle is trivial (see [Thu97, p. 163]). There are three possibilites, depending on Y:

- If $Y = \mathbb{S}^2$, we obtain the model geometry $\mathbb{S}^2 \times \mathbb{R}$.

∘ If $Y = \mathbb{R}^2$, then $X = \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$. Thus G' (and hence G) is contained in a bigger group of isometries, we don't get a new model geometry.

- If $Y = \mathbb{H}^2$, we obtain the model geometry $\mathbb{H}^2 \times \mathbb{R}$.

(b_2) If the curvature of τ is non-zero, the plane field is non-integrable. After rescaling our metric in the direction of the fibers and choosing appropiate orientations for the base and the fiber, we can assume the curvature is 1. If Y has non-zero curvature, X can be taken as the tangent circle bundle of Y (or rather, its universal cover). The group is made of derivatives of isometries of Y, together with rotations of unit tangent vector keeping the base point fixed.

- If $Y = \mathbb{R}^2$, we obtain *nilgeometry.* This can be defined as the group of contact automorphisms that are lifts of isometries of the xy-plane.

∘ If $Y = \mathbb{S}^2$, the tangent circle bundle is $SO(3)$, whose universal cover is $\mathbb{S}^3$. For G, we get the group of isometries of $\mathbb{S}^3$ that preserve the Hopf fibration. This is not a maximal group acting with compact stabilizers, so it is not a model geometry.

- If $Y = \mathbb{H}^2$, the unit tangent bundle is $PSL(2,\mathbb{R})$, the group of orientation-preserving isometries of $\mathbb{H}^2$. Passing to the universal cover, we get $X = \widetilde{SL}(2,\mathbb{R})$.

(c) If G' acts with trivial stabilizers, it can be identified with its single orbit $X = G'/G'_x$, so X is itself a Lie group. The work is reduced to investigate connected and simply connected 3-dimensional Lie groups, asking which ones admit a discrete cocompact subgroup and are not subsumed by one of the preceding seven geometries. A Lie group where every left-invariant vector field preserves volume is called *unimodular*. From now on G will be assumed unimodular, so that tr ad $V = 0$ for all V in the Lie algebra $\mathfrak{g}$ of G. Being skew-symmetric, the product on $\mathfrak{g}$ can be thought of as a map $\Lambda^2\mathfrak{g} \to \mathfrak{g}$. For G three dimensional, if we fix a positive quadratic form and an orientation for $\mathfrak{g}$, we get an identification between $\Lambda^2\mathfrak{g}$ and $\mathfrak{g}$, sending $V \wedge W$ to the cross product $V \times W$. Then the Lie bracket can be seen as a linear map $L : \mathfrak{g} \to \mathfrak{g}$, wich turns out to be symmetric with respect to the quadratic form: $L(V) \cdot W = V \cdot L(W)$ for all $V, W \in \mathfrak{g}$. In fact, let $\{e_1, e_2, e_3\}$ be a fixed, orthonormal, positively oriented basis for $\mathfrak{g}$, and let $L = (l_{ij})$ be the matrix of L with respect to that basis. The unimodularity condition tr ad $V = 0$, applied to $V = e_1$, is given by $\sum_{i=1,2,3} [e_1, e_i] \cdot e_i = \text{tr ad } e_1 = 0$, or

$$L(e_3) \cdot e_2 - L(e_2) \cdot e_3 = l_{23} - l_{32} = 0.$$

Repeating this for the other basis vectors,

$$\begin{aligned} \text{tr ad } e_2 &= l_{31} - l_{13} = 0, \\ \text{tr ad } e_3 &= l_{12} - l_{21} = 0. \end{aligned}$$

Thus G is unimodular if and only if the matrix of L is symmetric, in other words, if and only if the map L is self-adjoint.

Every symmetric linear transformation has an orthonormal basis of eigenvectors. Changing $\{e_1, e_2, e_3\}$ to such basis, the matrix of L becomes diagonal with entries $c_i = l_{ii}$. In other words, $[e_i, e_{i+1}] = c_{i+2}e_{i+2}$, where the subscripts are taken modulo 3. Suppose the quadratic form is altered, but the bracket product is fixed. If the new quadratic form is chosen in such way that the basis $\{a_1e_1, a_2e_2, a_3e_3\}$ were orthonormal, then

$$[a_ie_i, a_{i+1}e_{i+1}] \cdot a_{i+2}e_{i+2} = c_{i+2}(a_ia_{i+1}/a_{i+2})a_{i+2}e_{i+2} \cdot a_{i+2}e_{i+2} \tag{2.1.1}$$

so, the new structure constants c_i will be given by the expression $c_i(a_{i+1}a_{i+2}/a_i)$.

Thus if the basis vectors are multiplied by *arbitrary positive numbers* then its underlying Lie algebra wouldn't be changed. Up to isomorphism, then there are six possibilites for $\mathfrak{g}$, and therefore for the Lie group G. Only one gives rise to a new geometry, (see [Joh76, p. 307]):

- ○ $c_1, c_2, c_3 > 0$ gives $G = SU(2)$: group of unitary matrices of determinant 1; homeomorphic to the unit 3-sphere.

- $c_1, c_2 > 0$ and $c_3 < 0$ gives $G = \widetilde{SL}(2,\mathbb{R})$ or $G = O(1,2)$.
- $c_1, c_2 > 0$ and $c_3 = 0$ gives $G = E(2)$ the group of rigid motions of the 2-Euclidean space.
- $c_1 > 0$, $c_2 < 0$, and $c_3 = 0$ gives $G = E(1,1)$ the group of rigid motions of *Minkowski* 2-*space*. This group is a semidirect group of subgroups isomorphic to $\mathbb{R}^2$ and to $\mathbb{R}$, where each $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ by the matrix $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. Thus G is an extension $0 \to \mathbb{R}^2 \to G \to \mathbb{R} \to 1$, consisting of maps of the form

$$(x, y, t) \mapsto (e^{t_0}x + x_0, e^{-t_0}y + y_0, t + t_0),$$

for arbitrary real x_0, y_0 and t_0.

- $c_1 > 0$ and $c_2 = c_3 = 0$ gives the Heisenberg group.
- $c_1 = c_2 = c_3 = 0$ gives $G = \mathbb{R}^3$.

□

2.2 The eight Geometries

This section will present the eight geometries found by Thurston, its Riemannian metrics, isometries and besides we shall give examples that support the item (d) given in the definition 2.1.1. One of the most important theorems that we shall present is the *Poincaré's fundamental polyhedron theorem*, it will allow us to construct some examples in each geometry.

Definition 2.2.1. Let X be one of the spaces $\mathbb{E}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, $Isom(X)$ its isometries group and let G be a discrete subgroup of $Isom(X)$. A polyhedron D is a fundamental polyhedron for G if the following hold:

(i) For every non trivial $g \in G$, $gD \cap D = \emptyset$.

(ii) For every $x \in X$, there is a $g \in G$ such that $g(x) \in \overline{D}$.

(iii) The sides of D are paired by elements of G, this is, for every side $s \in D$ there is a side s' and there is a $g \in G$ such that $g(s) = s'$. These satisfy the conditions: $g_{s'} = g_s^{-1}$. The element g_s is called a side pairing transformation.

(iv) For any compact set $K \subset X$, we have that $gD \cap K \neq \emptyset$ for finitely many $g \in G$.

Theorem 2.2.2. Poincaré's fundamental polyhedron theorem. *Suppose that X is one of the spaces $\mathbb{E}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$ with $n > 1$, $Isom(X)$ is the group of isometries of X and let Γ be the group, generated by the identification of the sides of D. Assume that we are given a polyhedron D, where the sides of D are pairwise identified by elements of $Isom(X)$. That is we assume that for each side s of D, there is a side s', not necessarily distinct from s, and there is an element $g_s \in Isom(X)$, satisfying the following conditions:*

(i) $g_s(s) = s'$.

(ii) $g_{s'} = g_s^{-1}$. *If there is a side* s *with* $s' = s$ *then* $g_s^2 = 1$, *this relation is called a* reflection relation.
The isometries g_s *are* side pairing transformations.

(iii) $g_s(D) \cap D = \emptyset$.
The side pairing transformations induce an equivalence relation on $\overline{D}$ *where each point of* D *is equivalent only to itself. Let* $\overline{D}/\Gamma$ *be the space of equivalence classes, with the usual topology, so that the projection* $p : D \to \overline{D}/\Gamma$ *is continuous and open. If* D *is to be a fundamental polyhedron for* Γ *then*

(iv) *For every* $z \in \overline{D}/\Gamma$, $p^{-1}(z)$ *is a finite set.*

For each edge $e = e_1$, *let* $\{e_1, \cdots, e_k\}$ *be the ordered set of edges in the cycle containing* e, *and let* $g_1, \cdots, g_k$ *be the corresponding side pairing transformations. Then the* cycle transformation $h = h(e) = g_k \circ \cdots \circ g_1$ *keeps* e *invariant.* h *depends on a choice of a side abutting* e; *if we choose the other side to start with, then we obtain* h^{-1} *as the cycle transformation.*

(v) *For each edge* e, *there is a positive integer* t *so that* $h^t = 1$. *The relations in* Γ, *of the form* $h^t = 1$, *are called the* cycle relations.

Let $\alpha(e)$ *be the angle, measured from inside* D, *at the edge* e.

(vi) $\sum_{m=1}^{k} \alpha(e_m) = 2\pi/t$.

(vii) $\overline{D}/\Gamma$ *is complete.*

Then Γ, *the group generated by the side pairing transformations is discrete,* D *is a fundamental polyhedron for* Γ, *and the reflections and cycle relations form a complete set of relations for* Γ.

Proof. See [Mas88, p.75] □

Special Case. For the condition (vii) we are concerned with the completeness of the quotient space obtained by gluing the sides together. Note that we have not excluded the possibility that sides of D extend out to infinity.Suppose two sides are tangent at a point x_1 on the sphere at infinity. That is, the two sides do not intersect in X but both get arbitrarily close to x_1. Note that this can only occur if $X = \mathbb{H}^n$. Call one of the sides s_1, let g_1 be the corresponding side pairing transformation with $g_1(s_1) = s_1'$, and let $x_2 = g_1(x_1)$. If x_2 is a point of tangency of s_1' and some other side s_2, then let g_2 be the side pairing transformation with $g_2(s_2) = s_2'$, let $x_3 = g_2(x_2)$, and so on. If we eventually have $x_k = x_1$, for some finite k (i.e. $h = g_k \circ \cdots \circ g_1$ leaves x_1 invariant), then we call x_1 an infinite tangency point and h the *infinite cycle transformation* at x_1. We thus require:

Every infinite cycle transformation at every infinite tangency point is parabolic.

2.2.1 The geometry of $\mathbb{R}^3$.

$\mathbb{R}^3$ itself is a Riemannian manifold endowed with the Riemannian metric, $ds^2 = dx \otimes dx + dy \otimes dy + dz \otimes dz$. With respect to the group of isometries of $\mathbb{R}^3$ we have that any isometry α of $\mathbb{R}^3$ can be expressed as $\alpha(x) = Ax + b$ where $A \in O(3)$ and b is a vector in $\mathbb{R}^3$.

Proposition 2.2.3. The map $\alpha \mapsto A$ defines a surjective homomorphism $Isom(\mathbb{R}^3) \to O(3)$. Then we have an exact sequence

$$0 \to \mathbb{R}^3 \longrightarrow Isom(\mathbb{R}^3) \longrightarrow O(3) \longrightarrow 1.$$

Proof. Clearly for any $A \in O(3)$, the map $Ax + b$ is an isometry of $\mathbb{R}^3$ where b is a vector in $\mathbb{R}^3$. Moreover if α_1 and α_2 are two isometries of $\mathbb{R}^3$ then $\alpha_1(x) = A_1x + b_1$ and $\alpha_2(x) = A_2x + b_2$ for $A_1, A_2 \in O(3)$ and $b_1, b_2 \in \mathbb{R}^3$. Then $(\alpha_1 \circ \alpha_2)(x) = A_1A_2(x + b_2) + b_1 = A_1A_2x + b_3$, where $b_3 = A_1A_2b_2 + b_2$ is a vector in $\mathbb{R}^3$. Then $\alpha_1 \circ \alpha_2 \mapsto A_1A_2$, it means that the map is a surjective homomorphism as we required. Moreover its kernel is given by $\{\alpha \in Isom(\mathbb{R}^3) | \alpha \mapsto I_{3\times3} \in O(3)\}$,hence the kernel of this mapping is equal to the group of translations of $\mathbb{R}^3$. Then, this gives the exact sequence

$$0 \to \mathbb{R}^3 \longrightarrow Isom(\mathbb{R}^3) \longrightarrow O(3) \longrightarrow 1.$$

□

The geometric description of isometries of $\mathbb{R}^3$ differs from that in dimension two, here we have screw motions, these are orientation preserving isometries, consisting of the composite of translation with a rotation about a line left invariant by the translation. Thus both translations and rotations are special cases of this type. We can now present an example of a compact 3-manifold modeled on $\mathbb{R}^3$.

Example 2.2.4. Let P be a cube in $\mathbb{R}^3$. Define Γ as the side-pairing for P by pairing the opposite sides of P via translations. A cube has four edges around each face and each edge is along two faces, so it has $4 \times 6/2 = 12$ edges. The edges are glued together in 3 groups of four, as follows:

Let e_{ij} be the edge that lies between the faces i and j. So we get these edge identifications

- $\{e_{AC'}, e_{A'C'}, e_{CA'}, e_{AC}\}$, $h = g_C^{-1} g_A g_C g_A^{-1}$,
- $\{e_{AB}, e_{A'B}, e_{A'B'}, e_{AB'}\}$, $h = g_B g_A g_B^{-1} g_A^{-1}$, and
- $\{e_{B'C}, e_{B'C'}, e_{BC'}, e_{BC}\}$, $h = g_B^{-1} g_C g_B g_C^{-1}$.

Since each angle measured inside P is $\pi/2$ then the angles between the faces around each edge of the glued-up manifold is 2π. Then by the Poincare's theorem 2.2.2 we can obtain a presentation for Γ, moreover, we can conclude that Γ is a discrete group and that P is a fundamental polyhedron for Γ. In other words, the quotient of $\mathbb{R}^3$ by Γ is obtained by P and its side-pairing defined by Γ. This manifold is known as the *three-torus* (Figure 2.1) and the presentation for Γ is given by the cycle relations, in fact $\Gamma = \{g_A, g_B, g_C | \ [g_i, g_j] = 1, i \neq j \ and \ i, j \in \{A, B, C\}\}$.

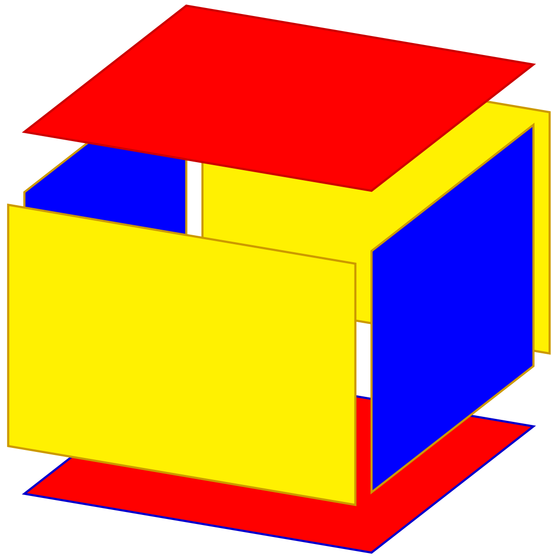

Figure 2.1: **The *three-torus*.** Each face of the cube is identified with its opposite face via translations. The image shows the identification of each face. The euclidean manifold $\mathbb{R}^3/\Gamma$ is the three-torus.

2.2.2 The geometry of $\mathbb{H}^3$.

We can think of $\mathbb{H}^3$ as the half 3-space $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 |\ z > 0\}$ with its Riemannian metric defined by $ds^2 = \dfrac{dx \otimes dx + dy \otimes dy + dz \otimes dz}{z^2}$. Then its vertical straight lines are geodesics in this space, that is lines $x = y =$constant, are geodesics of $\mathbb{H}^3$. Let p and q be two points in $\mathbb{R}^3_+$ with the same x and y coordinate, and let z_0 and z_1 be its z-coordinates respectively. Then the length of the vertical straight line segment γ from p to q is

$$\int_\gamma ds = \int_{z_0}^{z_1} \frac{1}{z} dz = log \left| \frac{z_1}{z_0} \right|.$$

If l is a different path from p to q, we can parametrize l by t so that l has length

$$\int_{t_0}^{t_1} \frac{1}{z} \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right) + \left(\frac{dz}{dt} \right)^2 \right)^{1/2} dt.$$

As dx/dt or dy/dt is non zero at some point then

$$\text{length of } l > \int_{t_0}^{t_1} \frac{1}{z} \left(\left(\frac{dz}{dt} \right)^2 \right)^{1/2} dt \geq \text{length of } \gamma,$$

Therefore all straight vertical lines in $\mathbb{R}^3_+$ are geodesics of $\mathbb{H}^3$.

Definition 2.2.5. If $S \subset \mathbb{R}^n$ is an $(n-1)$-sphere in Euclidean space, the *inversion* i_S in S is the unique map from the complement of the center of S into itsel that fixes every point of S, exchanges the interior and exterior of S and takes spheres orthogonal to S to themselves.
The image $i_S(P)$ of a point P in the sphere S with center O and radius r is the point on the ray $\overrightarrow{OP}$ such that $OP \cdot OP' = r^2$.

Theorem 2.2.6. *The inversion in a sphere $S \subset \mathbb{R}^3$ with centre c on the xy-plane defines an isometry of $\mathbb{H}^3$.*

Proof. Let $i_S(w) = c + \dfrac{r^2(w-c)}{||w-c||^2}$ be the inversion in a sphere S with centre c on the xy-plane and radius $r > 0$. By using coordinates for $c = (x_0, y_0, 0)$ and $w = (x, y, z) \in \mathbb{R}^3_+$, then

$$\begin{aligned} di_S^2 &= \frac{dx \otimes dx + dy \otimes dy + dz \otimes dz}{(r^2 z/||w-c||^2)^2}(r||w-c||)^4 \\ &= \frac{dx \otimes dx + dy \otimes dy + dz \otimes dz}{z^2} \end{aligned}$$

Hence i_S is an isometry of $\mathbb{H}^3$. □

Now, we can say that the geodesics of $\mathbb{H}^3$ are the vertical straight lines and the arcs of circles wich meet the xy-plane orthogonally.
The inversion in a sphere in $\mathbb{R}^n$ doesn't map its center anywhere, but if we consider the one-point compactification $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$ which is homeomorphic to the sphere $\mathbb{S}^n$, then the inversion i_S can be extended to map the center of S to ∞ and vice versa, so it becomes a homeomorphism of $\widehat{\mathbb{R}^n}$.

Definition 2.2.7. Using the stereographic projection, it is natural to think of lines and planes as circles and spheres passing through ∞. In this way we define the inversion in a plane $P(a,t) = \{x \in \mathbb{R}^n | (x,a) = t\} \cup \{\infty\}$, where (x,a) denotes the usual inner product in $\mathbb{R}^n$ defined as $\sum x_i a_i$ for $x = (x_1, \cdots, x_n)$ and $a = (a_1, \cdots, a_n)$, as follows:

$$i_{P(a,t)}(Q) = Q - 2[(Q,a) - t]a/|a|^2$$

Obs. The inversion in spheres which are orthogonal to the bounding plane are the hyperbolic reflections.

Theorem 2.2.8. *Let $\mathbb{H}^n$ be the upper half space and let $O(n)$ be the group of isometries of the tangent space to $\mathbb{H}^n$. Then*

(i) Given two points in upper half-space, $\mathbb{H}^n$, there exists a composition of hyperbolic reflections that will map one to the other.

(ii) Any element of $O(n)$ can be realized by a composition of hyperbolic reflections.

(iii) The whole group of isometries of $\mathbb{H}^n$ is generated by reflections.

Proof. (i) Let p, q be two points in $\mathbb{H}^n$. Suppose that $p = (x, t_1)$ and $q = (y, t_2)$ for some $x, y \in \mathbb{R}^{n-1}$ and $t_1, t_2 > 0$. If we consider the inversion in the plane $(\hat{x}, a) = 0$ followed by the inversion in the plane $(\hat{x}, a) = \frac{1}{2}|a|^2$, we shall obtain the translation $T_a(p) = p + a$. It means that if $a = (-x, 0)$ then $T_a(p) = (0, t_1)$. Then we shall use the inversion in the unit sphere followed by the inversion in the sphere of radius $\sqrt{\lambda}$ centred in the origin, this composition will give us an euclidean similarity as follows:

$$s_\lambda(x) = i_{S_{\sqrt{\lambda}}}(i_S(x)) = i_{S_{\sqrt{\lambda}}}(x/|x|^2) = \left(\frac{\sqrt{\lambda}}{|x|/|x|^2}\right)^2 x/|x|^2 = \lambda x.$$

By choosing $\lambda = \frac{t_2}{t_1}$, we shall get that $s_{\frac{t_2}{t_1}}(0, t_1) = (0, t_2)$. And applying again a translation with $b = (y, 0)$ then $T_b(0, t_2) = q$. Since translations and euclidean similarities are composition of hyperbolic reflections, then given two any points p, q in $\mathbb{H}^n$, we have proved that always is possible to map one to the other using composition of hyperbolic reflections. In our case, $q = T_{(y,0)} \circ s_{\frac{t_2}{t_1}} \circ T_{(-x,0)}(p)$.

(ii) We shall show by induction that $O(n)$ is generated by reflections, and that any reflection in $O(n)$ can be realized by a reflection on a sphere orthogonal to the given space $\mathbb{R}^{n-1}$ in the upper half-space model. For $n = 1$, $O(1) = \pm 1$, where -1 is a reflection about 0. Then $O(1)$ is generated by reflections. Suppose that our hypotheses is true for $n-1$ and let $f \in O(n)$. If $f = id$ then for any reflection r, we have that $f = r^2$. Suppose that $f \neq id$, if $f(v) = v$ for some $v \in \mathbb{R}^n \setminus \{0\}$, then let H be the hyperplane orthogonal to $\mathbb{R}v$. So, $\mathbb{R}^n = \mathbb{R}v \oplus H$. Besides if $y \in f(H)$ then there is $w \in H$ such that $f(w) = y$ and $(v, w) = 0$. Since $f \in O(n)$, f preserves the inner product and

$$(f(v), f(w)) = (v, w) \text{ and moreover } (v, y) = (f(v), f(w)) = (v, w) = 0.$$

Then $y \in H$, so $f(H) \subseteq H$. Since f is not the identity $f \neq id_H$, and besides $dim(H) = n-1$. By induction on H, there are $r_1, \cdots, r_k$, reflections about some hyperplanes H_i, $i = 1, \cdots, k$, such that

$$f|_H = r_k \circ \cdots \circ r_1$$

If $H = H_i \oplus H_i^\perp$ and $K_i = H_i \oplus \mathbb{R}v$, since H and $\mathbb{R}v$ are orthogonal K_i is an hyperplane and

$$\mathbb{R}^n = K_i \oplus H_i^\perp = H_i \oplus \mathbb{R}v \oplus H_i^\perp.$$

For every $u = h + \lambda v \in H \oplus \mathbb{R}v$, since

$$r_i(h) = p_{H_i}(h) - p_{H_i^\perp}(h),$$

we can extend on $\mathbb{R}^n$ as follows,

$$r_i(u) = p_{H_i}(h) + \lambda v - p_{H_i^\perp}(h), \; for \; h \in H, \; v \in \mathbb{R}v,$$

and

$$r_i(h + \lambda v) = p_{K_i}(h + \lambda v) - p_{H_i^\perp}(h + \lambda v),$$

which defines a reflection on K_i. Now since f is the identity on $\mathbb{R}v$, then

$$f = r_k \circ \cdots \circ r_1.$$

On the other hand if $f(x) \neq x$ for all $x \in \mathbb{R}^n \setminus \{0\}$ then taking $v = f(x) - x$, let H be the orthogonal hyperplane to $f(x) - x$. Since f preserves distances then

$$|f(x)| = |x|,$$

then $s(x) = f(x)$, where s is the reflection on H. Hence,

$$(s \circ f)(x) = s(f(x)) = s(s(x)) = x,$$

but since $s^2 = id$ then $s \circ f$ cannot be the identity since this would imply that $f = s$, where s is the identity on H, contradicting the fact that $f(x) \neq x$ for all x. Thus, taking $T(x) = (s \circ f)(x)$ we are again working with the first part, where $T(H) \subseteq H$, and therefore there are $r_1, \cdots, r_k$ reflections such that $s \circ f = r_k \circ \cdots \circ r_1$, so

$$f = s \circ r_k \circ \cdots \circ r_1.$$

(iii) Let $f \in Isom(\mathbb{H}^n)$ and p be a point in $\mathbb{H}^n$. If $f(p) \neq p$, there is an hyperbolic reflection g_1 such that $g_1 \circ f(p) = p$. Since $g_1 \circ f$ is an isometry, then $g_1 \circ f$ will preserve the metric and therefore for any $v, w \in T_p\mathbb{H}^n$,

$$(v, w) = ((g_1 \circ f)_*)_p(v), ((g_1 \circ f)_*)_p(w)).$$

So, $g_1 \circ f$ is an orthogonal application and therefore, there is a $g \in O(n)$ such that $((g \circ g_1 \circ f)_*)_p = 1$. The isometry $g \circ g_1 \circ f$ preserves every ray emanating from p and preserves every sphere orthogonal to $\mathbb{R}^{n-1}$ that pass for p with its respective direction, then $g \circ g_1 \circ f = id$. Hence $f = g_1^{-1} \circ g^{-1} = r_k \circ \cdots \circ r_1 \circ g^{-1}$, where r_i, $i = 1, \cdots, k$, are hyperbolic reflections, because $g_1 \in O(n)$ and we had proved that every element in $O(n)$ is generated by hyperbolic reflections. Since f was arbitrary, any isometry of $\mathbb{H}^n$ is generated by hyperbolic reflections.

□

We can use the upper half-space model to study the isometry group of the hyperbolic space. Consider a reflection of $\mathbb{H}^3$ given by inversion in a 2-sphere S orthogonal to the space $\mathbb{R}^2$. The restriction of this inversion with respect to the sphere at infinity $S^2_\infty = \mathbb{R}^2 \cup \infty$ is just the inversion in the sphere $S \cap S^2_\infty$, and every inversion of S^2_∞ can be so expressed.

Definition 2.2.9. A transformation of S^2_∞ that can be expressed as a composition of inversions is known as *Möbius transformation*, and the group of all such transformations is the Möbius group, denoted by Möb_2.

By theorem 2.2.8(iii), all hyperbolic isometries can be generated by reflections. It follows that the isometry group of $\mathbb{H}^3$ is isomorphic to Möb_2. Another model of hyperbolic space that we could use is known as the the *Poincaré ball model*, is what we get by taking the unit ball D^3 in $\mathbb{R}^3$ and declaring to be hyperbolic geodesics all those arcs of circles orthogonal to the boundary of D^3. Here the hyperbolic reflections are the inversions in 2-spheres orthogonal to ∂D^3.

Proposition 2.2.10. There is an homeomorphism from the *Poincaré ball model* to the upper half space.

Proof. Consider the maps: $(x, y, z) \mapsto \frac{1}{2}(x, y, z)$, $(x, y, z) \mapsto (x, y, z + 1/2)$ and $i_{\mathbb{S}^2}$, the inversion in the unit 2-sphere. Composing these maps we get a homeomorphism from D^3 to $\{(x, y, z) \in \mathbb{R}^3 | z > 1\}$.The translation $(x, y, z) \mapsto (x, y, z-1)$ composed with the last homeomorphism gives us the required homeomorphism. □

If this composed map is called q then the isometry group of the *Poincaré ball model* is given by $q^{-1}\text{Möb}_2 q$, where Möb_2 is the isometry group of the upper half space. The Riemannian metric in this model is given by the following relations: $\widetilde{x} = 2x/r$, $\widetilde{y} = 2y/r$, $\widetilde{z} = (2(z+1) - r)/r$, where $r = ||(x, y, z + 1)||^2$. So,

$$\begin{aligned} ds^2 &= \frac{1}{\widetilde{z}^2} d\widetilde{x} \otimes d\widetilde{x} + d\widetilde{y} \otimes d\widetilde{y} + d\widetilde{z} \otimes d\widetilde{z} \\ &= \frac{(4/r^2)dx \otimes dx + dy \otimes dy + dz \otimes dz}{(2(z+1) - r)^2/r^2} \\ &= \frac{4}{(1 - (x^2 + y^2 + z^2))^2} dx \otimes dx + dy \otimes dy + dz \otimes dz. \end{aligned}$$

By our construction an isometry of $\mathbb{H}^3$ is in the group Möb_2 of Möbius transformations. A Möbius transformation that preserves orientation in $\mathbb{C} \cup \{\infty\}$ is a map of the form $z \mapsto \dfrac{az+b}{cz+d}$, where $ad - bc \neq 0$ and we identify $\mathbb{C}$ with the xy-plane. This group of orientation preserving isometries is naturally isomorphic to $PSL(2, \mathbb{C})$ and extends its natural action on $\mathbb{R}^3_+$ by the action $w \mapsto \dfrac{aw+b}{cw+d}$, where w is a quaternion of the form $x + yi + zj$, $z > 0$. In this way such isometries fixes one or two points of the sphere at infinity.

Definition 2.2.11. If a non-trivial orientation-preserving isometry α of $\mathbb{H}^3$ has an axis that is fixed pointwise, it is called *elliptic*, or sometimes it is also known as a *rotation* about its axis.

If α is an orientation preserving isometry of $\mathbb{H}^3$ fixing two points x and y at infinity, it is called *hyperbolic*. Then α is a screw motion whose invariant axis is the geodesic joining x and y.

If α fixes a single point at infinity, it is called *parabolic*. We can conjugate α in $PSL(2, \mathbb{C})$ so that this fixed point is ∞. Now α is of the form $w \mapsto w + b$, and the group of all *parabolic* isometries fixing ∞ is clearly isomorphic to $\mathbb{R}^2$.

Lemma 2.2.12. *If α is an isometry of $\mathbb{H}^3$ let $\text{fix}(\alpha)$ denote the set of points on the sphere at infinity which are fixed by α.*

(i) If α and β are two non-trivial orientation preserving isometries of $\mathbb{H}^3$, then α and β commute if and only if $\text{fix}(\alpha) = \text{fix}(\beta)$.

(ii) If α is a non-trivial orientation preserving isometry of $\mathbb{H}^3$, then the group $C(\alpha)$ of all orientation preserving isometries wich commute with α is abelian and isomorphic to $\mathbb{R}^2$ or $\mathbb{S}^1 \times \mathbb{R}$.

Proof. The points fixed by $\beta\alpha\beta^{-1}$ are the points $\beta(\text{fix}(\alpha))$, because $\beta\alpha\beta^{-1}(\beta(w)) = \beta(\alpha(w)) = \beta(w)$ for any $w \in \text{fix}(\alpha)$. Moreover if $\alpha\beta = \beta\alpha$ then $\beta(\text{fix}(\alpha)) = \text{fix}(\beta\alpha\beta^{-1}) = \text{fix}(\alpha)$.

If α is elliptic and β commutes with α, then β must also fix the same axis rotation of α, then β is also a rotation about this axis. Hence $\text{fix}(\alpha) = \text{fix}(\beta)$ and $C(\alpha)$ is isomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

If α is parabollic, we can suppose that its fixed point is ∞. If β is another isometry that commutes with α then $\alpha\beta(\infty) = \beta\alpha(\infty) = \beta(\infty)$, so $\beta(\infty) \in \text{fix}(\alpha)$, it means that $\beta(\infty) = \infty$, and therefore β is also of the form $w \mapsto w + c$. Clearly $C(\alpha)$ is abelian an isomorphic to $\mathbb{R}^2$ and $\text{fix}(\alpha) = \text{fix}(\beta)$.

If α is hyperbolic, let x, y be its fixed points on $\mathbb{S}^2$ and σ be the geodesic which joins x and y and that is left invariant by α. If β is another isometry that commutes with α, then β fixes x and y, or β interchanges x and y. If β interchanges x with y then we can think of β as a rotation through π over the great circle that joins x and y, but in this case β has a unique axis fixed pointwise, but this is not possible by the previous paragraph, so β must be a screw motion, and fixes x and y. Then $\text{fix}(\alpha) = \text{fix}(\beta)$ and further $C(\alpha)$ is abelian and isomorphic to $S^1 \times \mathbb{R}$.

□

Before continuing with the other geometries, we shall present one example of a compact 3-manifold modeled on $\mathbb{H}^3$ in the *Poincaré ball model*.

Example 2.2.13. (**The Seifert-Weber dodecahedral space)** Let D be a dodecahedron and define the gluing of its opposite faces using a translation and a rotation of $3\pi/5$ in the clockwise direction from front to back (Figure 2.2). The $12 \cdot 5/2 = 30$ edges are glued together in six groups of five, as follows:

Let e_{ij} be the edge that lies between the faces i and j. So, we get these edge identifications:

- $\{e_{AD}, e_{A'B'}, e_{BC}, e_{C'E'}, e_{ED'}\}$ and $h = g_D^{-1} g_E^{-1} g_C g_B^{-1} g_A$,
- $\{e_{AB}, e_{A'C'}, e_{CF}, e_{F'D}, e_{D'B'}\}$ and $h = g_B^{-1} g_D g_F g_C^{-1} g_A$,
- $\{e_{AE'}, e_{A'D'}, e_{DB}, e_{B'F}, e_{F'E}\}$ and $h = g_E g_F g_B g_D^{-1} g_A$,
- $\{e_{AF}, g_{A'E}, e_{E'D}, e_{D'C}, e_{C'F'}\}$ and $h = g_F^{-1} g_C g_D g_E g_A$,
- $\{e_{AC}, e_{A'F'}, e_{FE'}, e_{EB}, e_{B'C'}\}$ and $h = g_C^{-1} g_B g_E^{-1} g_F^{-1} g_A$,
- $\{e_{BF'}, e_{B'E'}, e_{EC}, e_{C'D}, e_{D'F}\}$ and $h = g_F g_D g_C g_E^{-1} g_B$.

The group Γ generated by the side-pairing is given by the cycle relations described before, and its presentation is given by

$$\Gamma = \{g_A, g_B, g_C, g_D, g_E, g_F \mid g_D^{-1} g_E^{-1} g_C g_B^{-1} g_A,\ g_B^{-1} g_D g_F g_C^{-1} g_A,\ g_E g_F g_B g_D^{-1} g_A,\ g_F^{-1} g_C g_D g_E g_A,\ g_C^{-1} g_B g_E^{-1} g_F^{-1} g_A,\ g_F g_D g_C g_E^{-1} g_B\,\}.$$

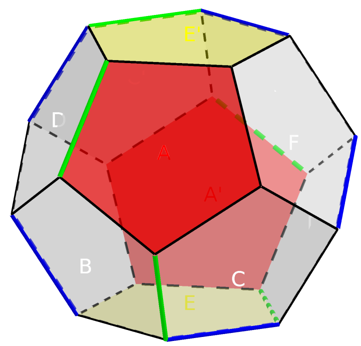

Figure 2.2: **The Seifert-Weber dodecahedral space.** If opposite faces of a dodecahedron are glued by three-tenths of a clockwise revolution, the edges are glued in quintuples, for example the green and blue edges correspond to the cycle of the edge e_{AD} and $e_{BF'}$ respectively. The gluing can be realized geometrically if we use a hyperbolic dodecahedron in the Poincaré ball model. The figure shows the identification of the faces A with A' (red faces), E with E' (yellow faces) and two cycles of edges. The resulting space is the Seifert-Weber dodecahedral space.

By identifying $g_A = a_4$, $g_B = a_2^{-1}$, $g_C = a_5$, $g_D = a_1^{-1}$, $g_E = a_3^{-1}$, $g_F = a_6$, we obtain that Γ has the following presentation

$$a_1a_2a_5a_3a_4 = 1,\ a_2a_1^{-1}a_6a_5^{-1}a_4 = 1,\ a_3^{-1}a_6a_2^{-1}a_1a_4 = 1,$$
$$a_6^{-1}a_5a_1^{-1}a_3^{-1}a_4 = 1,\ a_4^{-1}a_6a_3^{-1}a_2a_5 = 1,\ a_6a_1^{-1}a_5a_5a_3a_2^{-1} = 1.$$

This presentation for Γ coincides up to isomorphism with the presentation of the fundamental group of the hyperbolic dodecahedron [Vin69].

To prove this dodecahedron is modeled in $\mathbb{H}^3$, by the theorem 2.2.2, we require that the angles between the faces around each edge of the glued-up manifold add up to 2π, so they would each equal to $2\pi/5$. The angles of a Euclidean dodecahedron are much larger than the $2\pi/5$ needed to do the gluing geometrically.

By inscribing the dodecahedron in the *Poincaré ball model* of hyperbolic space, planes are represented as sectors of spheres orthogonal to the boundary of the ball, and the angle between two hyperbolic planes is the same as the angle between the two spheres.

The ideal dodecahedron in $\mathbb{H}^3$ whose vertices are on S^2_∞ has $\pi/3$ dihedral angles and for a very small dodecahedron its dihedral angles are approximately $116.565°$. This deformation is continuous and therefore intermediate between a very small hyperbolic dodecahedron and a very large dodecahedron with angles tending toward $\pi/3$ there is a dodecahedron whose dihedral angles are exactly $2\pi/5$, as required. Then, this dodecahedron can be glued to make a geometric form called the *Seifert-Weber dodecahedral* space. Moreover Γ is discrete and D is a fundamental polyhedron for Γ. In other words our 3-manifold defined by the quotient $M = \mathbb{H}^3/\Gamma$ is modeled on $\mathbb{H}^3$ and can obtained by D and its side-pairing.

2.2.3 The geometry of $\mathbb{S}^3$.

We shall think of $\mathbb{S}^3$ as the unit sphere in $\mathbb{R}^4$, or identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ as the set of ordered pairs (z_1, z_2) such that $|z_1|^2 + |z_2|^2 = 1$. With the induced metric from the euclidean metric on $\mathbb{R}^4$, the isometry group of $\mathbb{S}^3$ is $O(4)$. A path γ is a geodesic if and only if there is a 2-dimensional plane Π in $\mathbb{R}^4$ passing through the origin such that $\gamma \subset \Pi \subset \mathbb{R}^4$. Assuming the basic existence and local uniqueness results about geodesics on any complete Riemannian manifold, if a and b are two points in $\mathbb{S}^3$ close enough then, there exists a unique geodesic arc γ from a to b. Let Π a 2-plane in $\mathbb{R}^4$ through a, b and the origin, and let Σ be any 3-plane containing Π. Making a reflexion on Σ is clear that all the points in the plane are fixed by this isometry and in particular γ is invariant, hence γ lies in Σ . Then, since Π was arbitrary, it follows that γ lies in Π as we required.

Identifying the three-sphere as a topological group, we could think of $\mathbb{S}^3$ as the *unit quaternions.* The basis of the quaternions $\mathcal{Q}$ is traditionally denoted $\{1, i, j, k\}$, with the following product, where 1 is the identity:

$$i^2 = j^2 = k^2 = -1$$
$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik.$$

The *conjugate* of a quaternion $q = a + bi + cj + dk$, is $\overline{q} = a - bi - cj - dk$. The product $q\overline{q}$ is a positive real number and its square root is the *norm* of q, denoted by $|q|$. A quaternion of norm 1 is a *unit quaternion.* Any non-zero quaternion q has and inverse $q^{-1} = |q|^{-2}\overline{q}$. And if p and q are quaternions then $\overline{pq} = \overline{q}\overline{p}$ and $|pq| = |p||q|$. With this, we have proved the first part in the following theorem.

Theorem 2.2.14. *The three-sphere has the structure of a non-conmutative group, with center* $\{\pm 1\}$. *Left or right multiplication gives a self-action on* $\mathbb{S}^3$ *by orientation preserving isometries. Conjugation gives a self-action by isometries, which, in addition, takes any two-sphere with center* 1 *onto itsel. The quotient* $\mathbb{S}^3/\{\pm 1\}$ *is isomorphic to* $SO(3)$.

Proof. Let r be a unit quaternion and let p and q two points in $\mathbb{S}^3$. Since $L_r p = rp$ and $R_r p = pr$ are well defined then $|L_r(p) - L_r(q)| = |r||p - q| = |p - q|$. Hence, this map is an isometry and in the same way, the right multiplication also is an isometry. They are orientation-preserving by continuity, because $\mathbb{S}^3$ is connected.

The conjugation $f_r(q) = rqr^{-1}$ is also an isometry, that fixes 1, $f_r(1) = r1r^{-1} = 1$. Then, the conjugation will leave invariant the set of points at constant distance from 1, which are two-spheres.

We define $\rho : \mathbb{S}^3 \to SO(3)$ by letting $\rho(q)$ be the isometry of $\mathbb{S}^3$ sending x to qxq^{-1}. The kernel of this mapping is the centre of $\mathbb{S}^3$, $Z(\mathbb{S}^3) = \{x \in \mathbb{S}^3;\ qxq^{-1} = x\}$. As $\mathbb{S}^3$ is 3-dimensional and the kernel of ρ is finite, the image of ρ must be a 3-dimensional subgroup of $SO(3)$. As $SO(3)$ is a connected 3-dimensional group, ρ must have image $SO(3)$. Then $\mathbb{S}^3/Z(\mathbb{S}^3) \cong SO(3)$.

Let $q = a + bi + cj + dk$ be a quaternion, then $qi = ai - b - ck + dj$ and $iq = ai - b + ck - dj$. Then, $qi = iq$ if and only if $d = c = 0$. In analogous way if we take the other elements of the basis j, k repeating this product by the left and the right, then $a = 0$ or $b = 0$ or $c = 0$ or $d = 0$. It means that the only unit quaternions that commute with any other unit quaternion are 1 and -1. Therefore $Z(\mathbb{S}^3) = \{\pm 1\}$ as required. □

Now, the description of the sphere via complex numbers $(z_1, z_2) \in \mathbb{C}^2$ can be identified as the quaternion $z_1 + z_2 j$ motivating the following construction. Each complex line (one-dimensional subspace) in $\mathbb{C}^2$ intersects $\mathbb{S}^3$ in a great circle, called a *Hopf circle*. Since exactly one Hopf circle passes through each point of $\mathbb{S}^3$, the family of Hopf circles fills up $\mathbb{S}^3$, and the circles are in one-to-one correspondence with the complex lines of $\mathbb{C}^2$. Formally we get a fiber bundle $p : \mathbb{S}^3 \to \mathbb{S}^2$ with fiber $\mathbb{S}^1$.

Theorem 2.2.15. *The maps $g_t : \mathbb{S}^3 \to \mathbb{S}^3$ given by multiplication by e^{it}, for $t \in \mathbb{R}$, are isometries, that leave the fibers of the Hopf map invariant $p : \mathbb{S}^3 \to \mathbb{S}^2$, where $p(z_1, z_2) = z_1/z_2$.*

Proof. Let $p, q \in \mathbb{S}^3$, then $|g_t(p) - g_t(q)| = |e^{it}||p - q| = |p - q|$, hence these maps are isometries for any $t \in \mathbb{R}$. Moreover, if $(z_1, z_2) \in \mathbb{S}^3$, $g_t(z_1, z_2) = (e^{it}z_1, e^{it}z_2)$ and $p^{-1}(\{\lambda\})$ is the fiber over λ, then the circle $z_1/z_2 = \lambda$ is left invariant by g_t because $g_t(z_1, z_2) = (e^{it}z_1, e^{it}z_2)$ and thus the fiber $\lambda = z_1/z_2 = e^{it}z_1/e^{it}z_2$, as required. □

This theorem proves that $\mathbb{S}^3$ has isometries that don't have an axis: the motion near any point is like the motion any other point. The one-parameter family $\{g_t\}$ is called the Hopf flow. Now we shall present an example of a 3-manifold modeled with the geometry of the 3-sphere.

Example 2.2.16. **(Poincaré dodecahedral space)** Let D be a dodecahedron and define the gluing of its opposite faces using a translation and a rotation of $\pi/5$ in the clockwise direction from front to back (Figure 2.3).
A dodecahedron has five edges around each face and each edge is along two faces, so it has $12 \cdot 5/2 = 30$ edges. The edges are glued together in ten groups of three, as follows:

Let e_{ij} be the edge that lies between the faces i and j. So, we get these edge identifications:

- $\{e_{AC'}, e_{A'B'}, e_{BC}\}$ and $h = g_C g_B g_A$,
- $\{e_{AE}, e_{A'D'}, e_{DE'}\}$ and $h = g_E^{-1} g_D^{-1} g_A$,
- $\{e_{AD}, e_{A'F}, e_{F'D'}\}$ and $h = g_D^{-1} g_F g_A$,
- $\{e_{AF'}, e_{A'C}, e_{C'F}\}$ and $h = g_F g_C g_A$,
- $\{e_{AB'}, e_{A'E'}, e_{EB}\}$ and $h = g_B g_E^{-1} g_A$,
- $\{e_{B'F'}, e_{BD'}, e_{DF}\}$ and $h = g_F g_D^{-1} g_B^{-1}$,
- $\{e_{B'E'}, e_{BF}, e_{F'E}\}$ and $h = g_E g_F g_B^{-1}$,
- $\{e_{B'D}, e_{BC'}, e_{CD'}\}$ and $h = g_D^{-1} g_C^{-1} g_B^{-1}$,
- $\{e_{DC'}, e_{D'E}, e_{EC}\}$ and $h = g_C g_E g_D$,
- $\{e_{C'E}, e_{CF'}, e_{FE'}\}$ and $h = g_E^{-1} g_F^{-1} g_C^{-1}$.

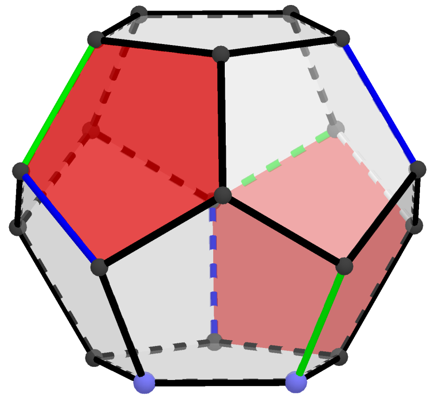

Figure 2.3: **The Poincaré dodecahedral space.** Each pentagonal face is identified with its opposite face by one-tenth of a clockwise revolution, the resulting space is the Poincaré dodecahedral space. The image shows the way as the edges are glued in triples in this pattern. The green and blue edges are examples of this fact.

The $12 \cdot 5/3 = 20$ vertices of the dodecahedron are glued in five groups of four, and the space obtained by this gluing is a manifold since it is locally homeomorphic to Euclidean space. The side pairing is given by the relations in front of each set of edges that were identified. To work more comfortable, we shall write the cycle relations $h = g_C g_B g_A$ as CBA, then the group defined by the side pairing, is given by the following relations:

$$CBA = 1,$$
$$E^{-1}D^{-1}A = 1,$$
$$D^{-1}FA = 1,$$
$$FCA = 1,$$
$$BE^{-1}A = 1,$$
$$FD^{-1}B^{-1} = 1.$$

From the last two relations we get $F = BD$ and $E = AB$, using these to eliminate F and E we get

$$CBA = 1,$$
$$B^{-1}A^{-1}D^{-1}A = 1,$$
$$D^{-1}BDA = 1,$$
$$BDCA = 1.$$

Using the first relation $A = B^{-1}C^{-1}$, and eliminating A we get

$$B^{-1}CBD^{-1}B^{-1}C^{-1} = 1,$$
$$D^{-1}BDB^{-1}C^{-1} = 1,$$
$$BDCB^{-1}C^{-1} = 1.$$

From the first of these relations $D = B^{-1}C^{-1}B^{-1}CB$, and eliminating D from the last two relations we get

$$B^{-1}C^{-1}BCBC^{-1}B^{-1} = 1 \text{ equivalent to } B^2 = C^{-1}BCBC^{-1}$$

and

$$B(B^{-1}C^{-1}B^{-1}CB)CB^{-1}C^{-1} = 1 \text{ equivalent to } C^2 = B^{-1}CBCB^{-1}.$$

Hence, the group defined by the side pairing is

$$\Gamma = \langle B, C \mid CB^2C = BCB,\ BC^2B = CBC\rangle.$$

By introducing a new generator $\tau = BC^{-1}$ and eliminating C using the relation $C = \tau^{-1}B$, we get

$$(\tau^{-1}B)^2 = B^{-1}\tau^{-1}B^2\tau^{-1} \text{ and } B^2 = B^{-1}\tau B\tau^{-1}B\tau.$$

Then

$$\tau B\tau^{-1}B\tau^{-1}B\tau = B^{-1}\tau B\tau^{-1}B\tau, \tag{2.2.1}$$

and from the relation (2.2.1) we get

$$B\tau = \tau^2 B^{-1} \tag{2.2.2}$$

and

$$\tau^2 = B\tau B \tag{2.2.3}$$

By using the relation (2.2.2)

$$(B\tau)^2 = (\tau^2 B^{-1})(B\tau) = \tau^3. \tag{2.2.4}$$

And, by using

$$B^2 = B^{-1}\tau B\tau^{-1}B\tau, \tag{2.2.5}$$

and (2.2.2), we get

$$\begin{aligned} B^4 &= B\tau B\tau^{-1}B\tau \\ &= \tau^2 B^{-1}B\tau^{-1}\tau^2 B^{-1} \\ &= \tau^3 B^{-1}. \end{aligned} \tag{2.2.6}$$

Then, $B^5 = (B\tau)^2 = \tau^3$. In this way, we have found another presentation for Γ, given by $\Gamma = \langle \tau, B \mid B^5 = (B\tau)^2 = \tau^3\rangle$. The relations that define this group with this presentation is known as the *binary icosahedral group*.
Observation. The groups that we have found are isomorphic.

To prove that this manifold has the geometry of $\mathbb{S}^3$ we require that the angles between the faces around each edge of the glued-up manifold add up to 2π, so they would each equal to $2\pi/3$. By inscribing the dodecahedron on the $S(e_4, r)$ sphere ($e_4 = (0,0,0,1)$) in $\mathbb{S}^3$ we know that when r tends to zero then the dihedral angles of the dodecahedron are very close to the Euclidean angles wich is approximately 116.565, but when r increases continuously until $\pi/2$, then the dodecahedron is geometrically a two sphere: the angles are π. In this way, we have expressed the dihedral angle as a function depending on the radius of the sphere where it is inscribed, taking values in $(116.565, 180]$, so there must exist a radius where the dodecahedron has dihedral angles equal to $120° = 2\pi/3$. By taking this dodecahedron with the gluing defined before, we have obtained a 3-manifold $M = D/\Gamma$, where Γ is the group defined by the side pairing, and M has the geometry of the 3-sphere. By using the Poincaré's theorem we have proved that Γ is a discrete group of $Isom(\mathbb{S}^3)$ and D is a fundamental polyhedron for Γ. In other words we also could say that the quotient $\mathbb{S}^3/\Gamma$ can be obtained from D by the side pairing.

2.2.4 The geometry of $\mathbb{S}^2 \times \mathbb{R}$.

The space $\mathbb{S}^2 \times \mathbb{R}$ is endowed with the product of the standard metrics in $\mathbb{S}^2$ and $\mathbb{R}$ respectively. There are seven 3-manifolds withouth boundary, including $S^2 \times \mathbb{R}$ itself, with geometric structure modeled on $\mathbb{S}^2 \times \mathbb{R}$.The isometry group $Isom(\mathbb{S}^2 \times \mathbb{R})$ can be identified with the product $Isom(\mathbb{S}^2) \times Isom(\mathbb{R})$.

To give an example of a compact manifold M modeled in $(Isom(\mathbb{S}^2 \times \mathbb{R}), \mathbb{S}^2 \times \mathbb{R})$, we shall consider that $\mathbb{S}^2 \times \mathbb{R}$ is a covering space of this manifold M, and that there is a subgroup Γ of the group of isometries $Isom(\mathbb{S}^2 \times \mathbb{R})$ acting freely and properly discontinuously such that $M = (\mathbb{S}^2 \times \mathbb{R})/\Gamma$. Then $p : \mathbb{S}^2 \times \mathbb{R} \to M$ will be the respective covering map and M the 3-manifold, as we required.

Example 2.2.17. Let Γ be the group generated by $(id_{\mathbb{S}^2}, T_a)$ where $T_a(x) = x + a$, is a translation. This group acts freely, because given $(x, y) \in \mathbb{S}^2 \times \mathbb{R}$ and $(g_1, g_2) \in \Gamma$ if $(g_1, g_2)(x, y) = (x, y)$ then $g_1(x) = x$ and $g_2(y) = y$, the first case where $g_1(x) = x$ is trivial because $g_1 = id_{\mathbb{S}^2}$ in the other case $g_2 = T_{ma}$ for some $m \in \mathbb{Z}$ then $T_{ma}(y) = y$ if and only if $m = 0$. Then $(g_1, g_2) = (id_{\mathbb{S}^2}, id_{\mathbb{R}})$, i.e. Γ acts freely. To prove that Γ acts properly discontinuously, let $K_1 \times K_2 \subset \mathbb{S}^2 \times \mathbb{R}$ be a compact subset, then for (g_1, g_2) we have the following, $(g_1, g_2)K_1 \times K_2 = K_1 \times g_2 K_2$, if $(g_1, g_2)K_1 \times K_2 \cap K_1 \times K_2 \neq \emptyset$ then by the definition of the group Γ there is a $n_0 \in \mathbb{Z}$ such that $g_2^{n_0} K_2 \cap K_2 \neq \emptyset$, where $g_2^n = g_2 \circ g_2 \circ \cdots \circ g_2$, n- times. Therefore $\{(g_1, g_2), (g_1, g_2^2), \cdots, (g_1, g_2^n)\}$ is a finite set as required. On the other hand, if $g_2^n K_2 \cap K_2 = \emptyset$ for every $n \in \mathbb{Z}$, then, the set $\{g \in \Gamma | gK_1 \times K_2 \cap K_1 \times K_2 \neq \emptyset\} = \emptyset$, it means that it is finite, as we required. Then Γ acts freely and properly discontinuously on $\mathbb{S}^2 \times \mathbb{R}$.

By the definition of Γ, $\mathbb{S}^2 \times \mathbb{R}/\Gamma \cong \mathbb{S}^2 \times \mathbb{S}^1$, and $p : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2 \times \mathbb{R}/\Gamma$ is the respective covering map. Using the proposition 1.2.7 we have proved that $\mathbb{S}^2 \times \mathbb{S}^1$ is a 3-manifold which is compact, since $\mathbb{S}^2$ and $\mathbb{S}^1$ are both compact. By considering $\pi : (\mathbb{S}^2 \times \mathbb{R})/\Gamma \to \mathbb{S}^1$ by $\pi([(x, y)]) = [y]$ and $\phi : (\mathbb{S}^2 \times \mathbb{R})/\Gamma \to \mathbb{S}^1 \times \mathbb{S}^2$ by $\phi([x, y]) = ([y], [x]) = ([y], x)$, we have that the following diagram commutes

$$\begin{array}{ccc} \mathbb{S}^2 \times \mathbb{R}/\Gamma & \xrightarrow{\phi} & \mathbb{S}^1 \times \mathbb{S}^2 \\ {\scriptstyle\pi}\downarrow & \swarrow {\scriptstyle p_1} & \\ \mathbb{S}^1 \ , & & \end{array}$$

which makes $\mathbb{S}^2 \times \mathbb{S}^1$ a trivial bundle over $\mathbb{S}^1$.

Example 2.2.18. Let Γ be the group generated by (α, T_a) where α is the antipodal map and T_a a translation. In this case $[(x, y)] = \{(x, y + a), (-x, y + 2a), (x, y + 3a), \cdots\}$, then again we have that $\mathbb{S}^2 \times \mathbb{R}/\Gamma \cong \mathbb{S}^2 \times \mathbb{S}^1$ but in this case it isn't a trivial bundle over $\mathbb{S}^1$, so, we shall denote it by $\mathbb{S}^2 \widehat{\times} \mathbb{S}^1$.

Example 2.2.19. Let Γ be the group generated by $(\alpha, id_{\mathbb{R}})$ where α is the antipodal map. If $(g_1, g_2)(x, y) = (x, y)$ for $(x, y) \in \mathbb{S}^2 \times \mathbb{R}$ and $(g_1, g_2) \in \Gamma$ then $g_1(x) = x$ and $g_2(y) = y$. By definition $g_2 = id_{\mathbb{R}}$ then for g_1 we have only two cases, the first one is $\alpha(x) = -x$ or $\alpha^2(x) = x$. Then $g_1 = \alpha^2 = id_{\mathbb{S}^2}$. Hence, Γ acts freely. Now, let $K_1 \times K_2 \subset \mathbb{S}^2 \times \mathbb{R}$ be a compact subset. Besides the set $\{(g_1, g_2) \in \Gamma | (g_1, g_2)K_1 \times K_2 \cap K_1 \times K_2 \neq \emptyset\}$ is always finite, since $g_1 = id_{\mathbb{S}^2}$ or $g_1 = -id_{\mathbb{S}^2}$. Therefore Γ acts freely and properly discontinuously on $\mathbb{S}^2 \times \mathbb{R}$. By using the proposition 1.2.7 $p : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2 \times \mathbb{R}/\Gamma$ is a covering map and $\mathbb{S}^2 \times \mathbb{R}/\Gamma \cong \mathbb{P}^2_{\mathbb{R}} \times \mathbb{R}$ is a 3-manifold. If we consider $\phi : \mathbb{S}^2 \times \mathbb{R}/\Gamma \to \mathbb{P}^2_{\mathbb{R}} \times \mathbb{R}$ given by $\phi([(x, y)]) = ([x], [y]) = ([x], y)$ and $\pi : \mathbb{S}^2 \times \mathbb{R}/\Gamma \to \mathbb{P}^2_{\mathbb{R}}$ given by $\pi([(x, y)]) = [x]$, then the following diagram commutes,

$$\begin{array}{ccc} \mathbb{S}^2 \times \mathbb{R}/\Gamma & \xrightarrow{\phi} & \mathbb{P}^2_{\mathbb{R}} \times \mathbb{R} \\ {\scriptstyle \pi} \downarrow & \swarrow {\scriptstyle p_1} & \\ \mathbb{P}^2_{\mathbb{R}}\ , & & \end{array}$$

which makes $\mathbb{P}^2_{\mathbb{R}} \times \mathbb{R}$ a trivial $\mathbb{R}$-bundle over $\mathbb{P}^2_{\mathbb{R}}$.

Example 2.2.20. Let Γ be the group generated by (α, β) where α is the antipodal map and β is a reflection. Then $[(x, y)] = \{(x, y), (-x, \beta(y))\}$ it means that we get analogously $\mathbb{S}^2 \times \mathbb{R}/\Gamma \cong \mathbb{P}^2_{\mathbb{R}} \widehat{\times} \mathbb{R}$ a non-trivial line bundle over the projective plane $\mathbb{P}^2_{\mathbb{R}}$.

Example 2.2.21. Let Γ be the group generated by $(\alpha, id_{\mathbb{R}})$ and $(id_{\mathbb{S}^2}, T_a)$ where α is the antipodal map and T_a a translation. Let (x, y) be a point in $\mathbb{S}^2 \times \mathbb{R}$,then

$$(\alpha, id_{\mathbb{R}}) \circ (id_{\mathbb{S}^2}, T_a)(x, y) = (\alpha, id_{\mathbb{R}})(x, y + a) = (-x, y + a)$$

and

$$(id_{\mathbb{S}^2}, T_a) \circ (\alpha, id_{\mathbb{R}})(x, y) = (id_{\mathbb{S}^2}, T_a)(-x, y) = (-x, y + a).$$

Since the generators of the group Γ commute then the identifications in both components happen independently, and therefore $\mathbb{S}^2 \times \mathbb{R}/\Gamma \cong \mathbb{P}^2_{\mathbb{R}} \times \mathbb{S}^1$. Besides, if $(g_1, g_2) \in \Gamma$ and $(g_1, g_2)(x, y) = (x, y)$ then $(g_1, g_2) = (id_{\mathbb{S}^2}, id_{\mathbb{R}})$, i.e., Γ acts freely. Let $K_1 \times K_2 \subset \mathbb{S}^2 \times \mathbb{R}$ be a compact subset, if $(g_1, g_2)K_1 \times K_2 \cap K_1 \times K_2 \neq \emptyset$ then by the definition of Γ if (g_1, g_2) is such that $g_1 = \alpha$ then $g_2 = id_{\mathbb{R}}$ or $g_2 = T_a$ in any of these cases $K_1 = \mathbb{S}^2$ because in other case is possible that $g_1 K_1 \cap K_1 = \emptyset$, but even if it happens, then the set $\{g \in \Gamma | K \subset \mathbb{S}^2 \times \mathbb{R}\ \textit{compact},\ gK \cap K \neq \emptyset\}$ is finite. If $K_1 = \mathbb{S}^2$ then for $g_2 = id_{\mathbb{R}}$ is trivial that $(g_1, g_2)K_1 \times K_2 \cap K_1 \times K_2 \neq \emptyset$. For $g_2 = T_a$ if the intersection is non-empty then there is a $n_0 \in \mathbb{Z}$ such that $(g_1, g_2)^{n_0} \in K_1 \times K_2$, where $(g_1, g_2)^n = (g_1, g_2) \circ \cdots \circ (g_1, g_2)$, n-times. On the other hand, if $g_1 = id_{\mathbb{S}^2}$ then $g_2 = T_a$ and analogously there is a $n_0 \in \mathbb{Z}$ such that $(g_1, g_2)^{n_0} \in K_1 \times K_2$, but even if it doesn't happen for any $n \in \mathbb{Z}$ then the set $\{g \in \Gamma | K \subset \mathbb{S}^2 \times \mathbb{R}\ \textit{compact},\ gK \cap K \neq \emptyset\}$ is finite. Hence, Γ acts freely and properly discontinuously on $\mathbb{S}^2 \times \mathbb{R}$. By using the proposition 1.2.7 $p : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2 \times \mathbb{R}/\Gamma$ is a covering map and $\mathbb{S}^2 \times \mathbb{R}/\Gamma \cong \mathbb{P}^2_{\mathbb{R}} \times \mathbb{S}^1$ is a manifold with the geometry of $\mathbb{S}^2 \times \mathbb{R}$.

2.2.5 The geometry of $\mathbb{H}^2 \times \mathbb{R}$.

The space $\mathbb{H}^2 \times \mathbb{R}$ is a geometry really different to the last one, there are infinitely many 3-manifolds modeled with this geometry. For example, the product of any compact hyperbolic surface with $\mathbb{R}$ or $\mathbb{S}^1$ is modeled with this geometry. However, there are still many similarities. The isometry group of $\mathbb{H}^2 \times \mathbb{R}$ is naturally isomorphic to $Isom(\mathbb{H}^2) \times Isom(\mathbb{R})$, and this space is endowed with the product metric of the respective Riemannian metrics in $\mathbb{H}^2$ and $\mathbb{R}$ respectively.

Lemma 2.2.22. *If M is a 3-manifold modeled with the geometry of $\mathbb{H}^2 \times \mathbb{R}$, then M admits a foliation by lines or circles.*

Proof. Let $(\alpha, \beta) \in Isom(\mathbb{H}^2 \times \mathbb{R})$ and $\{x\} \times \mathbb{R}$ be the foliation by lines of $\mathbb{H}^2 \times \mathbb{R}$. Then $(\alpha, \beta)\{x\} \times \mathbb{R} = \{\alpha(x)\} \times \beta(\mathbb{R})$, and thus the foliation is left invariant by this group. Since M is modeled by this geometry there is a discrete subgroup Γ of $Isom(\mathbb{H}^2 \times \mathbb{R})$ such that $M = \mathbb{H}^2 \times \mathbb{R}/\Gamma$ and $p : \mathbb{H}^2 \times \mathbb{R} \to M$ is a covering map, then this foliation by lines $\{x\}$ descends to a foliation by lines or circles. $\square$

Analogously to the geometry of $\mathbb{S}^2 \times \mathbb{R}$, we shall present one example of a compact 3-manifold modeled with this geometry.

Example 2.2.23. Let $X \subset \mathbb{H}^2$ be a regular octagon, and let f_1, f_2, f_3, f_4 be the isometries of $\mathbb{H}^2$ such that $f_1(e_{AB}) = e_{CD}$, $f_2(e_{ED}) = e_{BC}$, $f_3(e_{EF}) = e_{HG}$, $f_4(e_{AH}) = e_{FG}$. (Figure 2.4) The vertices are glued in one group of eight as follows:

$$(E, e_{ED}) \xrightarrow{f_2} (B, e_{BC})$$

$$(B, e_{AB}) \xrightarrow{f_1} (C, e_{DC})$$

$$(C, e_{BC}) \xrightarrow{f_2^{-1}} (D, e_{ED})$$

$$(D, e_{DC}) \xrightarrow{f_1^{-1}} (A, e_{AB})$$

$$(A, e_{AH}) \xrightarrow{f_4} (F, e_{FG})$$

$$(F, e_{EF}) \xrightarrow{f_3} (G, e_{HG})$$

$$(G, e_{FG}) \xrightarrow{f_4^{-1}} (H, e_{AH})$$

$$(H, e_{HG}) \xrightarrow{f_3^{-1}} (E, e_{EF})$$

$$(E, e_{ED}).$$

The group generated by the edge-pairing is given by the relation $f_3^{-1}f_4^{-1}f_3f_4f_1^{-1}f_2^{-1}f_1f_2 = [f_3^{-1}, f_4^{-1}][f_1^{-1}, f_2^{-1}] = 1$, then $G = \{f_1, f_2, f_3, f_4 | [f_3^{-1}, f_4^{-1}][f_1^{-1}, f_2^{-1}] = 1\}$.

Since the octagon is regular, each angle is $\frac{\pi}{4}$, and the eight vertices are glued in only one vertice in $\mathbb{H}^2/G$, with angle 2π. Using the theorem 2.2.2 for $n = 2$, we have that the group generated by the isometries $f_1,\ f_2,\ f_3,\ f_4$ of $\mathbb{H}^2$ is a discrete subgroup of $Isom(\mathbb{H}^2)$, and X is a fundamental region for G. Then $\mathbb{H}^2/G$ can be obtained from X and its edge-pairing.

Let Γ be the group generated by (f_i, T_a), $i = 1, 2, 3, 4$, where $f_i \in G$, the group obtained by the edge-pairing of X. Given $(\alpha, \beta) \in \Gamma$ if $(\alpha, \beta)(x, y) = (x, y)$ for some $(x, y) \in \mathbb{H}^2 \times \mathbb{R}$, then $(\alpha, \beta) = (id_{\mathbb{H}^2}, id_{\mathbb{R}})$ since G is free and $T_a(y) = y + a = y$ implies $a = 0$. Since Γ is generated by G and the translation T_a we have that each one of these groups is discrete on the isometry groups of $\mathbb{H}^2$ and $\mathbb{R}$ respectively, then Γ is a discrete subgroup of $Isom(\mathbb{H}^2 \times \mathbb{R})$. By using the corollary 1.2.6 we have that Γ acts properly discontinuously on $X \times \mathbb{R}$. By the proposition 1.2.7 the map $p : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}/\Gamma$ is a covering map and $\mathbb{H}^2 \times \mathbb{R}/\Gamma \cong M_2 \times \mathbb{S}^1$ is manifold, that is compact, where M_2 is the compact orientable surface of genus 2. Moreover, $M_2 \times \mathbb{S}^1$ has the geometry of $\mathbb{H}^2 \times \mathbb{R}$ as we required.

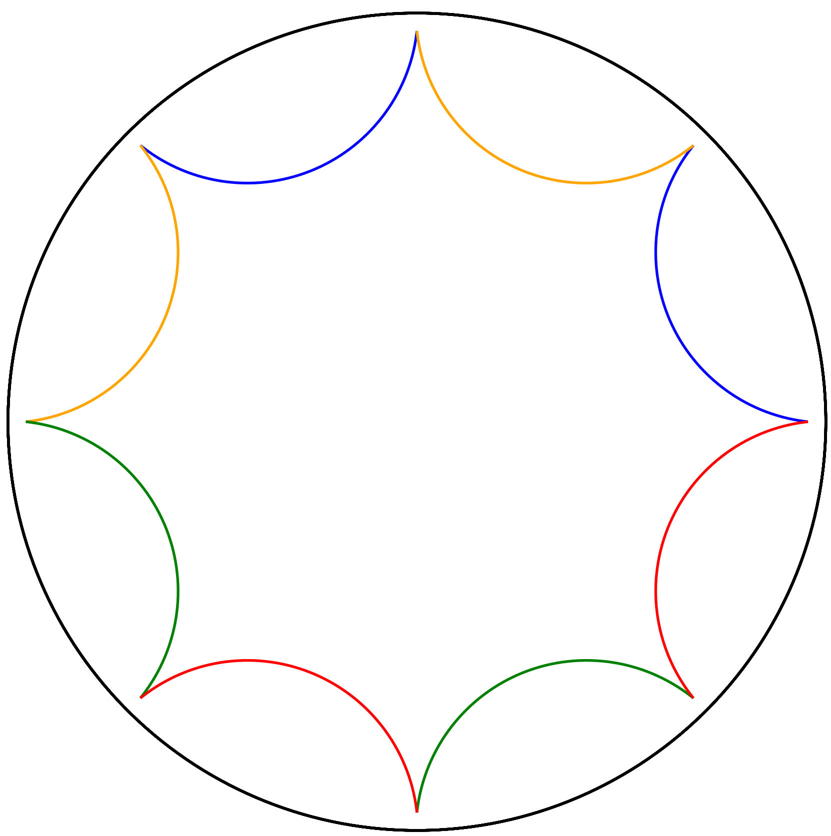

Figure 2.4: Hyperbolic regular octagon X with its edge-pairing.

Example 2.2.24. Let Γ be the group generated by $(f_i, id_{\mathbb{R}})$, where $f_i \in G$, the group obtained by the edge-pairing of X in the example 2.2.23. Then Γ acts freely and properly discontinuously on $\mathbb{H}^2 \times \mathbb{R}$, the map $p : \mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2 \times \mathbb{R}/\Gamma$ is a covering map and $\mathbb{H}^2 \times \mathbb{R}/\Gamma \cong M_2 \times \mathbb{R}$ is a manifold with the geometry of $\mathbb{H}^2 \times \mathbb{R}$. If $\phi : \mathbb{H}^2 \times \mathbb{R}/\Gamma \to M2 \times \mathbb{R}$ is given by $\phi([(x, y)]) = ([x], [y]) = ([x], y)$ and $\pi : \mathbb{H}^2 \times \mathbb{R}/\Gamma \to M_2$ is given by $\pi([(x, y)]) = [x]$ then the following diagram commutes

$$\begin{array}{ccc} \mathbb{H}^2 \times \mathbb{R}/\Gamma & \xrightarrow{\phi} & M_2 \times \mathbb{R} , \\ {\scriptstyle \pi}\downarrow & \swarrow{\scriptstyle p_1} & \\ M_2 & & \end{array}$$

and therefore $M_2 \times \mathbb{R}$ is a trivial $\mathbb{R}$-bundle over M_2.

2.2.6 The *nilgeometry*

The *nilgeometry* can be defined in terms of our model contact structure τ of example 1.4.2 as the group of contact automorphisms that are lifts of isometries of the xy-plane (example 1.4.3). Let G be the group of isometries of *nilgeometry* and $\phi \in Isom(\mathbb{R}^2)$. Using the example 1.4.3 we can construct the automorphism of contact that project to ϕ. Let γ be a Legendrian curve for τ whose projection to the xy-plane $\mathbb{R}^2$ is a line segment with endpoints (x_0, y_0) and (x_1, y_1), so to find γ explicitly, we have to remember the fact that $\tau = \langle \partial/\partial x, \partial/\partial y + x\partial/\partial z\rangle$ and that $\gamma' \in \tau$. So,

$$\gamma(t) = ((x_1 - x_0)t + x_0, (y_1 - y_0)t + y_0, z(t)) \tag{2.2.7}$$

and

$$\gamma'(t) = (x_1 - x_0, y_1 - y_0, z'(t)) = (x_1 - x_0)\frac{\partial}{\partial x} + (y_1 - y_0)\left(\frac{\partial}{\partial y} + z'(t)/(y_1 - y_0)\frac{\partial}{\partial z}\right). \tag{2.2.8}$$

By using the equation (2.2.8) and the fact that $\gamma' \in \tau$ we have that

$$\begin{aligned} z(t) &= \int (y_1 - y_0)((x_1 - x_0)t + x_0)dt \\ &= (y_1 - y_0)\left((x_1 - x_0)\frac{t^2}{2} + x_0 t\right) + C. \end{aligned} \tag{2.2.9}$$

Since $\gamma(0) = (x_0, y_0, z_0)$ then $z(0) = C = z_0$, and

$$\gamma(t) = \left((x_1 - x_0)t + x_0, (y_1 - y_0)t + y_0, (y_1 - y_0)\left((x_1 - x_0)\tfrac{t^2}{2} + x_0 t\right) + z_0\right).$$

Then the point $q = (x_1, y_1, z_1) \in \mathbb{R}^3$ joined to p by γ has a difference in elevation given by

$$z_1 - z_0 = z(1) - z(0) = \tfrac{1}{2}(y_1 - y_0)(x_0 + x_1).$$

For given $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ we found a point $q = (x_1, y_1, \frac{1}{2}(y_1 - y_0)(x_1 + x_0) + z_0) \in \mathbb{R}^3$ fixed and joined by a Legendrian curve γ, so if $\phi(x, y) = (\phi_1(x, y), \phi_2(x, y))$ is the isometry given, we have that the lifting of $\phi \circ \pi_z \circ \gamma$ is the contact automorphism we required. So,

$$\phi(\pi_z(\gamma)) = (\phi_1(x(t), y(t)), \phi_2(x(t), y(t))),$$

and

$$\widetilde{\phi \circ \pi_z \circ \gamma}(t) = ((\phi_1(x_0,y_0) - \phi_1(x_1,y_1))t + \phi_1(x_1,y_1), (\phi_2(x_0,y_0) - \phi_2(x_1,y_1))t + \phi_2(x_1,y_1),$$
$$(\phi_2(x_0,y_0) - \phi_2(x_1,y_1)) \left(\phi_1(x_1,y_1)t + (\phi_1(x_0,y_0) - \phi_1(x_1,y_1))\frac{t^2}{2} \right) + z_1 \Big).$$

Then

$$\widetilde{\phi}(q) = \widetilde{\phi \circ \pi_z \circ \gamma}(0) = (\phi_1(x_1,y_1), \phi_2(x_1,y_1), z_1) \tag{2.2.10}$$

and

$$\widetilde{\phi}(p) = \widetilde{\phi \circ \pi_z \circ \gamma}(1) = \left(\phi_1(x_0,y_0), \phi_2(x_0,y_0), \frac{1}{2}(\phi_2(x_0,y_0) - \phi_2(x_1,y_1))(\phi_1(x_0,y_0) + \phi_1(x_1,y_1)) + z_1 \right), \tag{2.2.11}$$

so

$$\begin{gathered} \widetilde{\phi} : \mathbb{R}^3 \to \mathbb{R}^3 \\ (x_0,y_0,z_0) \mapsto \left(\phi_1(x_0,y_0), \phi_2(x_0,y_0), \tfrac{1}{2}(\phi_2(x_0,y_0) - \phi_2(x_1,y_1))(\phi_1(x_0,y_0) + \phi_1(x_1,y_1)) + z_1 \right). \end{gathered}$$

As a special case if $\phi(x,y) = (x+x', y+y')$ is a translation in $\mathbb{R}^2$, then

$$\widehat{\phi}(x,y,z) = (x+x', y+y', \tfrac{1}{2}(y-y_1)(x+x_1) + x'y - x'y_1 + z_1).$$

Therefore

$$\widetilde{\phi}(x,y,z) = (x+x', y+y', z+x'y+z'),$$

where x', y', z' are arbitrary real numbers, is an isometry of the *nilgeometry*.

Proposition 2.2.25. Let H be the group of isometries of *nilgeometry* that project over a translation of $\mathbb{R}^2$. Then $H \cong \mathbf{H}$, where $\mathbf{H}$ is the *Heisenberg group* of real upper triangular 3×3 matrices with ones on the diagonal.

Proof. Consider the action of H on $\mathbb{R}^3$, $H \times \mathbb{R}^3 \to \mathbb{H}$ given by $(\widetilde{\phi}, (x,y,z)) \mapsto (x+x', y+y', z+xy'+z')$ for x', y', z' arbitrary real numbers but fixed. If for some $(x_0,y_0,z_0) \in \mathbb{R}^3$, we have $(\widetilde{\phi}, (x_0,y_0,z_0)) \mapsto (x_0,y_0,z_0)$ then

$$\left.\begin{aligned} x_0 + x' &= x_0 \\ y_0 + y' &= y_0 \\ z_0 + x_0y' + z' &= z_0 \end{aligned}\right\} \begin{aligned} x' &= 0 \\ y' &= 0 \\ z' &= 0. \end{aligned}$$

Then $\widetilde{\phi} = id_{\mathbb{R}^3}$. Besides given $(x_0,y_0,z_0) \in \mathbb{R}^3$ and $(x_1,y_1,z_1) \in \mathbb{R}^3$ exist $x' = x_1 - x_0$, $y' = y_1 - y_0$ and $z' = z_1 - z_0 - (x_1 - x_0)y_0$ such that

$$\begin{aligned} \widetilde{\phi}(x_0,y_0,z_0) &= (x_0+x', y_0+y', z_0+z'+x'y_0) \\ &= (x_1,y_1,z_1), \end{aligned}$$

then, ϕ acts transitively on $\mathbb{R}^3$. So, if we consider $\mathbb{R}^3$ as a group with multiplication given by $(x',y',z')(x,y,z) = (x+x', y+y', z+x'y+z')$ then the map $f : \mathbb{R}^3 \to H$ given by $f((x',y',z')) =$

$\widetilde{\phi}_{(x',y'z')}$, where $\widetilde{\phi}_{(x',y',z')}(x,y,z) = (x+x', y+y', z+z'+x'y)$ is well defined, $Ker(f) = \{id_{\mathbb{R}^3}\}$ and $f(\mathbb{R}^3) = H$. Moreover,

$$\begin{aligned} f((x_0,y_0,z_0)(x_1,y_1,z_1))(x,y,z) &= \widetilde{\phi}_{(x_1+x_0,y_1+y_0,z_1+x_0y_1+z_0)}(x,y,z) \\ &= (x+x_1+x_0, y+y_1+y_0, z+z_1+x_0y_1+z_0+(x_1+x_0)y) \\ &= \widetilde{\phi}_{(x_0,y_0,z_0)}(x+x_1, y+y_1, z+z_1+x_1y) \\ &= \widetilde{\phi}_{(x_0,y_0,z_0)} \circ \widetilde{\phi}_{(x_1,y_1,z_1)}(x,y,z), \end{aligned}$$

and since (x,y,z) is arbitrary f is an homomorphism of groups and $\mathbb{R}^3 \cong H$. Now, if we identify a point $(x,y,z) \in \mathbb{R}^3$ with a matrix $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ in the *Heisenberg group*, so,

$$(x',y',z')(x,y,z) \longleftrightarrow \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

and is also true that $H \cong \mathbf{H}$. □

Observation. By identifying $\mathbf{H}$ with $\mathbb{R}^3$ then the action of H on $\mathbb{R}^3$ describes a left invariant Riemannian metric . In fact, if we choose $ds^2 = dx \otimes dx + dy \otimes dy + dz \otimes dz$ at the origin, then the corresponding invariant metric on $\mathbb{R}^3$ is given by $ds^2 = dx \otimes dx + dy \otimes dy + (dz - xdy) \otimes (dz - xdy)$. Moreover the *Heisenber group* is a Lie group.

Example 2.2.26. If $h_a, h_b \in H$ project to translations by vectors $a = (a_0, a_1, a_2)$ and $b = (b_0, b_1, b_2)$, the commutator

$$\begin{aligned} [h_a, h_b] &= h_a h_b h_a^{-1} h_b^{-1} \\ &= h_{aba^{-1}b^{-1}} \\ &= h_{aba^{-1}(-b_0,-b_1,b_0b_1-b2)} \\ &= h_{ab(-a_0-b_0,-a_1-b_1,b_0b_1-b_2+a_0a_1-a_2+a_0b_1)} \\ &= h_{a(-a_0,-a_1,a_0a_1-a_2+a_0b_1-a_1b_0} \\ &= h_{(0,0,a_0b_1-a_1b_0)} \end{aligned}$$

is a vertical translation $(0, 0, a_0b_1 - a_1b_0)$, by a distance equal to the signed area of the parallelogram with sides $(a_0, a_1, 0)$ and $(b_0, b_1, 0)$. Moreover the vertical translations form the center of H, because for any $h_{(x',y',z')} \in H$ if $h_{(0,0,z)}$ is a vertical translation then

$$h_{(x'y'z')} h_{(0,0,z)} = h_{(x',y',z'+z)} = h_{(0,0,z)} h_{(x',y',z')},$$

and if $h_{(x',y'z')} h_{(a,b,c)} = h_{(a,b,c)} h_{(x',y',z')}$ then for all x', y', $x'b = ay'$. So, $a = b = 0$, and $h_{(a,b,c)} = h_{(0,0,c)}$ is a vertical translation.

Example 2.2.27. (**Discrete subgroup of** G**).** Let h_a, h_b be defined as in the example 2.2.26 and consider $a = (1, 0, 0)$ and $b = (0, 1, 0)$. Then, the group generated by h_a and h_b is known as the integer *Heisenberg group*.

So, $h_a^{-1} = h_{-a}$, $h_b^{-1} = h_{-b}$ and if $n \in \mathbb{Z}$ then $h_a^n = h_{na}$ and $h_b^n = h_{nb}$. Moreover $[h_a, h_b] = h_{(0,0,1)}$ is a vertical translation and therefore for any $m_1, m_2, m_3 \in \mathbb{Z}$ we have that $h_{(m_1,m_2,m_3)} = h_a^{m_1} h_b^{m_2} [h_a, h_b]^{m_3}$. These relations describe this subgroup completely.

We shall prove that this subgroup is discrete. Let $h_{(a_n,b_n,c_n)}$ be a sequence in $\langle h_a, h_b\rangle$ such that $h_{(a_n,b_n,c_n)} \to h_{(a,b,c)}$, in other words $h_{(a_n,b_n,c_n)} h^{-1}_{(a,b,c)} \to e$. For given $\epsilon = 1/2$, there is a $N \in \mathbb{N}$ such that $|h_{(a_n-a, b_n-b, c_n+(ab-c)-a_n b)} - e| < \epsilon$ for $n \geq N$. Then $a_n - a \to 0$, $b_n - b \to 0$ and $c_n + (ab - c) - a_n b \to 0$ for $n \geq N$. Since $a_n, b_n, c_n, a, b, c \in \mathbb{Z}$ and the distance between two different integers is greater or equal than one, then it is only possible that $a_n = a$, $b_n = b$ and $c_n = c$ for $n \geq N$. Then, the sequence is almost constant and we can conclude that $\langle h_a, h_b\rangle$ is a discrete subgroup of the group of isometries of *nilgeometry*. Let us denote the integer *Heisenberg group* by $\mathbf{H}_{\mathbb{Z}}$.

Example 2.2.28. (Compact 3-manifold modeled on the *nilgeometry*).

Let $M = \mathbb{R}^3/\mathbf{H}_{\mathbb{Z}}$, since $\mathbf{H}_{\mathbb{Z}}$ is a discrete subgroup of the group of isometries of *nilgeometry*, by using the corollary 1.2.6, the integer *Heisenberg group* acts properly discontinuously, moreover it acts freely and therefore $\mathbb{R}^3/\mathbf{H}_{\mathbb{Z}}$ is a manifold with covering map $p : \mathbb{R}^3 \to \mathbb{R}^3/\mathbf{H}_{\mathbb{Z}}$. If $(x, y, z) \sim (x', y', z')$ then there is $h_{(m_1,m_2,m_3)} \in \mathbf{H}_{\mathbb{Z}}$ such that $h_{(m_1,m_2,m_3)}(x, y, z) = (x', y', z')$, so, $(x', y', z') = (x+m_1, y+m_2, z+m_3+m_1 y)$. Suppose without loss of generality that $m_1, m_2, m_3 > 0$ and consider the parallelepiped $[0, m_1] \times [0, m_2] \times [0, m_3] \subset \mathbb{R}^3$. By choosing a constant value for the last coordinate we have the equation $z = k - (m_3 + xy)$, where $k \in [m_3, 2m_3]$, it means that for the planes $x = 0$ and $x = m_1$ we have an identification given by $z = k - m_3$ in $x = 0$ and $z = k - (m_3 + m_1 y)$ in $x = m_1$. Then the quotient of $\mathbb{R}^3$ by $\mathbf{H}_{\mathbb{Z}}$ is also the quotient of the parallepiped by the cyclic group generated by the $(x, y, z) \mapsto (x + m_1, y + m_2, z + m_3 + m_1 y)$. So, the map $f : [0, m_1] \times [0, m_2] \times [0, m_3] \to M$ given by $(x, y, z) \mapsto [(x, y, z)]$ describes the quotient manifold M and by definition it is compact. (Figure 2.5)

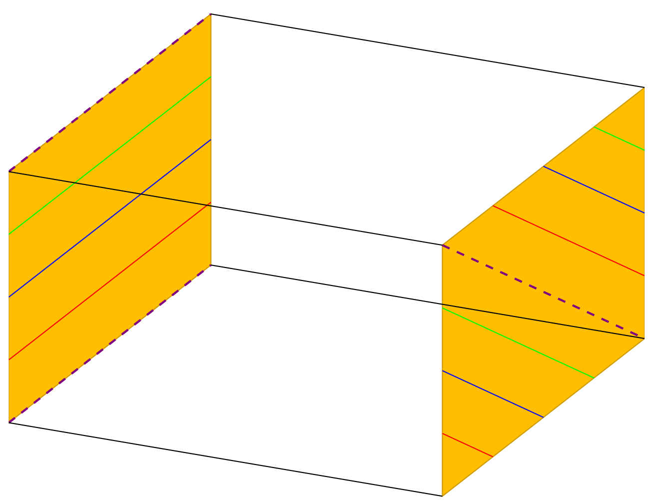

Figure 2.5: **The quotient of $\mathbb{R}^3$ by $\mathbf{H}_{\mathbb{Z}}$.** The quotient of $\mathbb{R}^3$ by the integer *Heisenberg group* is also the quotien of the parallelepiped of $[0, m_1] \times T^2$ by the cyclic group generated by the map $(x, y, z) \mapsto (x + m_1, y + m_2, z + m_3 + m_1 y)$. The figure shows $[0, m_1] \times T^2$, the two squares represent the toruses at $x = 0$ and $x = m_1$. We get our manifold by taking the region between the two toruses and identifying the toruses.

The example 2.2.28 gives us a manifold that is circle bundle over the torus. Moreover the

Heisenberg group is nilpotent, in fact, it is the only three-dimensional, non abelian connected and simply connected Lie group. This explains the term "*nilgeometry*".

2.2.7 The geometry of $\widetilde{SL}(2,\mathbb{R})$

The universal covering of the 3-dimensional Lie group $SL(2,\mathbb{R}) = \{A \in GL(2,\mathbb{R}) | det(A) = 1\}$ is denoted by $\widetilde{SL}(2,\mathbb{R})$. To describe a Riemannian metric on $\widetilde{SL}(2,\mathbb{R})$ we shall consider the following facts.

Proposition 2.2.29. For given any $A \in PSL(2,\mathbb{R})$, the action on $\mathbb{H}^2$ given by $A(z) = \dfrac{az+b}{cz+d}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, induces an action on $T\mathbb{H}^2$, via the differential map (A_*) at z.

Proof. Let $z \in \mathbb{H}^2$, $v \in T_z\mathbb{H}^2$ and let γ be a path in $\mathbb{H}^2$ such that $\gamma(t_0) = z$ and $\gamma'(t_0) = v$. Then

$$\begin{aligned}
(A_*)_z(v) &= \left.\frac{d}{dt}\right|_{t=t_0} (A \circ \gamma)\, t \\
&= \left.\frac{d}{dt}\right|_{t=t_0} \frac{a\gamma(t)+b}{c\gamma(t)+d} \\
&= \left.\frac{a\gamma'(t)(c\gamma(t)+d) - (a\gamma(t)+b)c\gamma'(t)}{(c\gamma(t)+d)^2}\right|_{t=t_0} \\
&= \frac{(ad-bc)\gamma'(t_0)}{(c\gamma(t_0)+d)^2} \\
&= \frac{1}{(cz+d)^2} v \\
&= A'(z)v.
\end{aligned}$$

Then

$$PSL(2,\mathbb{R}) \times T(\mathbb{H}^2) \to T(\mathbb{H}^2)$$

is an action given by $(A,(z,v)) \mapsto (A(z), A'(z)v)$ for $(z,v) \in T(\mathbb{H}^2)$ where $v \in T_z\mathbb{H}^2$. □

Corollary 2.2.30. The action of $PSL(2,\mathbb{R})$ on $\mathbb{H}^2$ is transitive.

Proof. For any $z_1, z_2 \in \mathbb{H}^2$ there is a matrix $A \in PSL(2,\mathbb{R})$ such that $Az_1 = i$. Of course if $z_1 = x_1 + iy_1$ then $A(z_1) = \frac{az_1+b}{cz_1+d} = i$ and if $c = 0$, then $a \neq 0$, $d \neq 0$, $ad = 1$ and

$$\frac{a(x_1)+b}{d} + \frac{iay_1}{d} = i,$$

so, $b = -ax_1$ and $y_1 = d^2$. Therefore the matrix $A = \begin{pmatrix} 1/\sqrt(y_1) & -x_1/\sqrt{y_1} \\ 0 & \sqrt{y_1} \end{pmatrix} \in PSL(2,\mathbb{R})$ and $A(z_1) = i$. Analogously, for z_2 there exists another matrix $B \in PSL(2,\mathbb{R})$ such that $B(z_2) = i$. Then, $B^{-1}A(z_1) = z_2$ as we required. □

Theorem 2.2.31. *The action of $PSL(2,\mathbb{R})$ on $T^1\mathbb{H}^2$ is free and transitive. Then $PSL(2,\mathbb{R}) \cong T^1\mathbb{H}^2$.*

Proof. Let $z_0 = i$ and $v_0 = (0,1)_i$ be a tangent vector at i. Let $(z,v) \in T^1\mathbb{H}^2$ and $A \in PSL(2,\mathbb{R})$. For (z,v) there exists a geodesic $\sigma \in \mathbb{H}^2$ such that $\sigma(t_0) = z$ and $\sigma'(t_0) = v$. Let L denote the positive imaginary axis. Since L is a geodesic in $\mathbb{H}^2$ and $PSL(2,\mathbb{R})$ is the group of preserving-orientations isometries, there exists an $A \in PSL(2,\mathbb{R})$ that maps L in σ and maps z_0 to z, in other words $A(L) = \sigma$ and $A(z_0) = z$. Then, $(A_*)_{z_0}(v_0) = v$ as we required, this proves the existence and uniqueness of A. Therefore, we can conclude that the action is free. The map $A \mapsto (A_*)_i(i) = (A(i), A'(i)i)$ gives us the identification we required, so $PSL(2,\mathbb{R}) \cong T^1\mathbb{H}^2$. □

$T^1\mathbb{H}^2$ is submanifold of $T\mathbb{H}^2$, and by definition it is a bundle over $\mathbb{H}^2$ with fibre $\mathbb{S}^1$. Clearly any isometry f of $\mathbb{H}^2$ gives us an isometry f_* of $T\mathbb{H}^2$ that is also an isometry of $T^1\mathbb{H}^2$. Moreover by the theorem 2.2.31 we have an identification of $T^1\mathbb{H}^2$ and $PSL(2,\mathbb{R})$, and therefore a metric on $PSL(2,\mathbb{R})$. This metric will give us a metric on $\widetilde{SL}(2,\mathbb{R})$, because $PSL(2,\mathbb{R})$ is doubly covered by $SL(2,\mathbb{R})$ and $\widetilde{SL}(2,\mathbb{R})$ is the universal covering of $SL(2,\mathbb{R})$. So, using the pullback we can define a metric on $\widetilde{SL}(2,\mathbb{R})$, as we required. But, there is still something unsolved and is the fact that by the construction $\widetilde{SL}(2,\mathbb{R})$ has the structure of a bundle over $\mathbb{H}^2$ with fiber $\mathbb{R}$. We could think of that $\widetilde{SL}(2,\mathbb{R})$ has a relation with $\mathbb{H}^2 \times \mathbb{R}$, even when topologically this bundle must be trivial, we shall show that $\widetilde{SL}(2,\mathbb{R})$ is not isometric to $\mathbb{H}^2 \times \mathbb{R}$.

Let $\pi : T(\mathbb{H}^2) \to \mathbb{H}^2$ be the projection given by its smooth structure. For each $v \in T(\mathbb{H}^2)$ we define the vertical space V_v as the tangent vectors to the fiber through v and let σ be a geodesic in $\mathbb{H}^2$ through x, such that its lift $\widetilde{\sigma}$ to $T(\mathbb{H}^2)$ passes through v. Then, define the horizontal space H_v as the tangent plane of the union of all such paths, one for each geodesic of $\mathbb{H}^2$ through x. Clearly $\widetilde{\sigma}$ is is the parallel transportation along σ and defines an isomorphism between the fiber through x, $T_x(\mathbb{H}^2)$ and the fiber through any point of σ, $T_{\sigma(t)}\mathbb{H}^2$. Clearly this defines a connection τ for $T(\mathbb{H}^2)$.

Definition 2.2.32. Let M be a 2-dimensional Riemannian manifold and $x \in M$. If l is a loop based at x, the parallel translation along l induces a rotation of the fiber T_xM called the *holonomy* of l and the angle of rotation is called the *holonomy angle*.

Example 2.2.33. If σ is a loop, then the parallel translation induces a rotation on T_xM. We called this rotation the *holonomy* of σ and the angle of rotation is the *holonomy angle*. If $M = \mathbb{R}^2$ is clear that the holonomy of any loop is trivial.

Example 2.2.34. Let $\triangle$ be a geodesic triangle in $\mathbb{H}^2$, and denote by σ the loop defined the edges of the geodesic triangle. The holonomy angle is $\pi - (\alpha + \beta + \gamma)$. And the holonomy in this case is not trivial.

Proposition 2.2.35. Let τ be the connection defined for $T(\mathbb{H}^2)$. Then, the horizontal plane field is not integrable.

Proof. Let $S \subset T(\mathbb{H}^2)$ be the surface formed by the union of all such paths through v, used to define the horizontal space H_v. The projection map $\pi : S \to \mathbb{H}^2$ is a covering map. Hence, as $\mathbb{H}^2$ is simply connected, S meets each fiber of $T(\mathbb{H}^2)$ exactly once. Moreover if γ is a loop based at

x with non-trivial holonomy then its parallel translation will find the fibre over x in two points. But it is a contradiction. Therefore the horizontal plane field on $T(\mathbb{H}^2)$ is not integrable. $\square$

We shall define an inner product on V_v taking the inner product of $T_x\mathbb{H}^2$, because by definition V_v is a tangent plane to T_xM. Hence, the induced Riemannian metric on T_xM will make T_xM isometric to $\mathbb{R}^2$. The inner product on H_v is chosen to be essentially the same as that on T_xM. Precisely, given the connection τ, the projection $\pi : T(\mathbb{H}^2) \to \mathbb{H}^2$ gives us a linear isomorphism between H_v and T_xM. Since $T_vT\mathbb{H}^2 = H_v \oplus V_v$, then an inner product on this space induce a Riemannian metric on $T(\mathbb{H}^2)$, that we shall choose in such way that H_v and V_v are orthogonal. In other words if $w_i, w_j \in T_vT\mathbb{H}^2$

$$\begin{aligned}
\langle w_i, w_j \rangle_{T_vT\mathbb{H}^2} &= \langle h_i + k_i, h_j + k_j \rangle_{T_vT\mathbb{H}^2} \\
&= \langle h_i, h_j \rangle_{T_vT\mathbb{H}^2} + \langle h_i, k_j \rangle_{T_vT\mathbb{H}^2} + \langle k_i, h_j \rangle_{T_vT\mathbb{H}^2} + \langle k_i, k_j \rangle_{T_vT\mathbb{H}^2} \\
&= \langle h_i, h_j \rangle_{H_v} + \langle k_i, k_j \rangle_{V_v}.
\end{aligned}$$

Since $\widetilde{SL}(2,\mathbb{R})$ is a $\mathbb{R}$-bundle over $\mathbb{H}^2$, we call again this fibers vertical. The horizontal plane field on $T^1\mathbb{H}^2$ gives a plane field on $\widetilde{SL}(2,\mathbb{R})$ which we again call horizontal. As the projection map $p : \widetilde{SL}(2,\mathbb{R}) \to T^1\mathbb{H}^2$ is a local isometry, this plane field is non-integrable. This shows directly that $\mathbb{SL}(2,\mathbb{R})$ is not isometric to $\mathbb{H}^2 \times \mathbb{R}$ by any homeomorphism preserving fibers.

Now we have the required metric on $PSL(2,\mathbb{R})$. Moreover, $PSL(2,\mathbb{R})$ acts on $T^1\mathbb{H}^2$ by isometries, when $T^1\mathbb{H}^2$ has the metric we have described. The induced metric on $PSL(2,\mathbb{R})$ is invariant under left multiplication, because $PSL(2,\mathbb{R})$ acts on $\mathbb{H}^2$ on the left. So, the metric on $\widetilde{SL}(2,\mathbb{R})$ is also invariant under left multiplication.

Definition 2.2.36. The group $Isom(\widetilde{SL}(2,\mathbb{R}))$ is given by the group $\widetilde{SL}(2,\mathbb{R})$ acting by left multiplication together all the isometries in $\mathbb{H}^2$ whose lifts fixes the fibers.

Example 2.2.37. Let X be the regular octagon in $\mathbb{H}^2$ and let Γ be the side-pairing defined by the isometries defined in the example 2.2.23, here, we proved $\mathbb{H}^2/\Gamma$ is a compact orientable hyperbolic surface, called the genus 2 surface, M_2. Moreover, since $T^1\mathbb{H}^2 \cong PSL(2,\mathbb{R})$, then

$$PSL(2,\mathbb{R})/\widehat{\Gamma} \cong T^1\mathbb{H}^2/\widehat{\Gamma} \cong T^1\mathbb{H}^2/\Gamma,$$

where $\widehat{\Gamma}$ is the group of isometries in $T\mathbb{H}^2$ induced by the isometries in Γ. Therefore T^1M_2 is a compact 3-manifold modeled on $\widetilde{SL}(2,\mathbb{R})$.

$$\begin{array}{ccc}
\widetilde{SL}(2,\mathbb{R}) & & \\
\downarrow & & \\
SL(2,\mathbb{R}) & & \\
\downarrow & & \\
PSL(2,\mathbb{R}) \cong T^1\mathbb{H}^2 & \longrightarrow & T^1M_2
\end{array}$$

Example 2.2.38. Analogously to the example 2.2.37, for given a compact hyperbolic surface M, its unit tangent bundle T^1M is a compact 3-manifold modeled on $\widetilde{SL}(2,\mathbb{R})$.

2.2.8 The geometry of *Sol*.

The Lie group *Sol*, is defined as a split extension of $\mathbb{R}$ by $\mathbb{R}^2$. Thus we have the exact sequence

$$0 \longrightarrow \mathbb{R}^2 \longrightarrow Sol \longrightarrow \mathbb{R} \longrightarrow 0,$$

where $\mathbb{R}$ acts over $\mathbb{R}^2$ via the map $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $(t,(x,y)) \mapsto (e^t x, e^{-t} y)$. For a given non-zero t, the map $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(e^t x, e^{-t} y)$ is a linear isomorphism, where its transformation is given by the matrix

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

whose determinant is one and his eigenvectors are e^t and e^{-t}.

Definition 2.2.39. If we identify *Sol* with $\mathbb{R}^3$, we can define the multiplication " $\cdot$ " of *Sol* by

$$(x,y,z)\cdot(x',y',z') = (x + e^z x', y + e^{-z} y', z + z').$$

Proposition 2.2.40. $\mathbb{R}^2$ is a normal subgroup of $\mathbb{R}^3$ and $(0,0,0)$ is the identity of this group.

Proof. Clearly

$$\begin{aligned}(0,0,0)\cdot(x,y,z) &= (0 + e^0 x, 0 + e^0 y, 0 + z) \\ &= (x,y,z) \\ &= (x + e^z 0, y + e^z 0, z + 0) \\ &= (x,y,z)\cdot(0,0,0).\end{aligned}$$

Moreover for any $(x,y,z) \in \mathbb{R}^3$, $(x,y,z)^{-1} = (-xe^{-z}, -ye^{z}, -z)$. Then,

$(x,y,z)\cdot(x',y',0)\cdot(x,y,z)^{-1} = (x + e^z x', y + e^{-z} y', z)(-xe^{-z}, -ye^z, -z) = (e^z x', e^{-z} y', 0) \in \mathbb{R}^2$.

Then $\mathbb{R}^2 \triangleleft \mathbb{R}^3$ and $(0,0,0)$ is the identity of $(\mathbb{R}^3, \cdot)$. □

Definition 2.2.41. Let $\mathbb{R}^3$ be the group defined with the multiplication $(x,y,z)\cdot(x',y',z') = (x + e^z x', y + e^{-z} y', z + z')$ and $ds^2 = dx \otimes dx + dy \otimes dy + dz \otimes dz$ be the metric at the origin. Then we can define the left-invariant metric on $\mathbb{R}^3$ by

$$ds^2 = e^{-2z} dx \otimes dx + e^{2z} dy \otimes dy + dz \otimes dz.$$

By considering the map $\varphi_t(x,y) = (e^t x, e^{-t} y)$ we can obtain some information about the identity component of *Sol* and its full group of isometries of *Sol*. Of course, the eigenspaces generated by this map are given by the following relations

$$\begin{pmatrix} 0 & 0 \\ 0 & e^{-t} - e^t \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} e^t - e^{-t} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then $E(e^t) = \{(x,0)|x \in \mathbb{R}\}$ and $E(e^{-t}) = (E(e^t))^{\perp} = \{(0,y)|y \in \mathbb{R}\}$.

So there are independent reflection in the two eigenspaces in the $\mathbb{R}^2$ direction. And one can also reflect in the $\mathbb{R}$ direction while at the same time interchanging the two eigenspaces. So the stabiliser at the origin consists of maps of $\mathbb{R}^3$ given by $(x,y,z) \mapsto (\pm\, x, \pm\, y, z)$ and $(x,y,z) \mapsto (\pm\, y, \pm\, x, -z)$, so the index of Sol' in Sol is 8.

Proposition 2.2.42. These eight maps are isometries of Sol.

Proof. We shall verify that one of these maps is an isometry, analogously the other maps also are. By taking the map $(x,y,z) \mapsto (y,x,-z)$ we have that our new coordinates are $x^1 = y$, $x^2 = x$, $x^3 = -z$. Then

$$\begin{aligned} ds^2 &= e^{-2(-z)}dy \otimes dy + e^{2(-z)}dx \otimes dx + dz \otimes dz \\ &= e^{-2z}dx \otimes dx + e^{2z}dy \otimes dy + dz \otimes dz. \end{aligned}$$

Then, the map is an isometry as we required. □

Example 2.2.43. Let M_ϕ be the mapping torus defined at the example 1.3.3, and let ϕ be the diffeomorphism from the torus to itsel given by $(x,y) \mapsto (2x+y, x+y)$. This linear automorphism has matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Its eigenvalues are $\lambda = \frac{3\pm\sqrt{5}}{2}$ and its eigenspaces are given by the following relations

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $E\left(\frac{3-\sqrt{5}}{2}\right) = \left(E\left(\frac{3+\sqrt{5}}{2}\right)\right)^{\perp}$. Then $E\left(\frac{3+\sqrt{5}}{2}\right) = \left\{\left(\left(\frac{1-\sqrt{5}}{2}\right)t, t\right)|t \in \mathbb{R}\right\}$ and $E\left(\frac{3-\sqrt{5}}{2}\right) = \left\{\left(t, \left(\frac{\sqrt{5}-1}{2}\right)t\right)|t \in \mathbb{R}\right\}$.

By arranging the universal cover of the torus T^2 so that the two of the eigenspaces of ϕ line up with x- and y-axes, since the eigenvalues of ϕ are reciprocal to each other, there is some t_0 such that the transformation

$$\psi : (x,y,t) \mapsto (e^{t_0}x, e^{-t_0}y, t+t_0)$$

of $\mathbb{R}^3$ induces the given automorphism ϕ between $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{t_0\}$. Let Γ be the group of automorphisms of Sol generated by ψ together with unit translations along the x- and y-axes.

For a given $m \in \mathbb{Z}$, $\psi^m(x,y,t) = (e^{mt_0}x, e^{-mt_0}, t+mt_0)$. If $f_m = \psi^m$ and $f_m \to id$, where id is the identity automorphism of Sol, then for the open neighborhood of id, $f(\{(x,y,t)\}, (0,1/2)^3) = \{f \in C(\mathbb{R}^2, \mathbb{R}^2)|f(x,y,t) \subset (0,1/2)^3\}$, in the compact-open topology, where $(x,y,t) \in (0,1/2)^3 = (0,1/2) \times (0,1/2) \times (0,1/2) \subset \mathbb{R}^3$ is fixed, we have that there must be a $N \in \mathbb{N}$, such that $f_m \in f(\{(x,y,t)\}, (0,1/2)^3)$ whenever $m \geq N$. So, $(e^{mt_0}x, e^{-mt_0}y, t+mt_0) \in (0,1/2)^3$ whenever $m \geq N$.

Then,

$$0 < t + mt_0 \leq \tfrac{1}{2},\ e^{mt_0}x < \tfrac{1}{2} \text{ and } e^{-mt_0}y < \tfrac{1}{2} \text{ for } m \geq N.$$

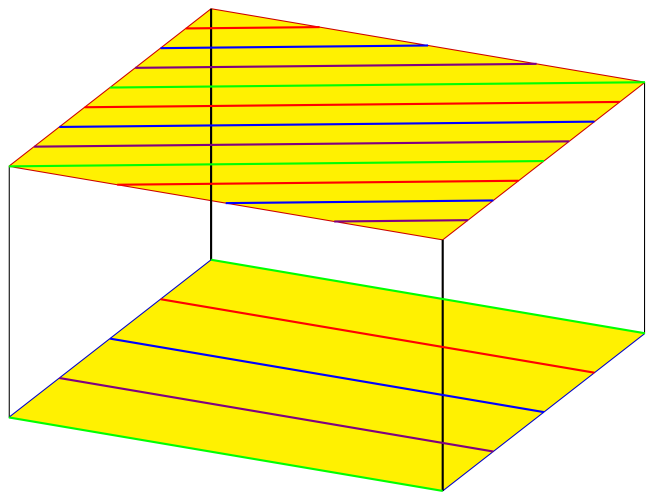

Figure 2.6: **The torus mapping T^2_ϕ for** $\phi(x, y) = (2x + y, x + y)$. The torus mapping T^2_ϕ is also the quotient of $\mathbb{R}^3$ by the discrete group Γ. The figure shows the parallelepiped $T^2 \times [0, t_0]$, where the two shaded squares represent the toruses at $z = 0$ and $z = t_0$. We get $\mathbb{R}^3/\Gamma$ by taking the region between the two toruses and identifying the torus so the shadings match.

There must be some natural number N_0 with $N_0 \geq N$ such that $e^{-N_0 t_0} > \frac{1}{y}$ or $e^{N_0 t_0} > \frac{1}{x}$. Then $e^{-mt_0} y > 1 > \frac{1}{2}$ (or $e^{mt_0} > \frac{1}{2}$) for $m \geq N_0$. On the other hand, also there must be a natural N_1 such that $t + N_1 t_0 > 1$ or $t + N_1 t_0 < 0$ for $N_1 \geq m$. Hence, $f_m = id$ for $m \geq \min\{N_0, N_1\}$, and f_m is an almost constant sequence.

Moreover, let $g_m = T^m_{(1,0,0)}$ (or $g_m = T^m_{(0,1,0)}$) be the translation m-times in the x-(or y-)axis, and suppose that $g_m \to id$. Let $f(\{(x, y, t)\}, (0, 1/2)^3)$ be an open neighborhood of id. So, $(mx, y, t) \in (0, 1/2)^3$ (or $(x, my, t) \in (0, 1/2)^3$) whenever $m \geq N$ for some natural $N \in \mathbb{N}$. Then $0 < mx < 1/2$ (or $0 < my < 1/2$). There must be a natural N_0 with $N_0 \geq N$ such that $N_0 x > 1$ (or $N_0 y > 1$). Then $N_0 x > 1$ (or $N_0 y > 1$), so, $g_m \to id$ if and only if $g_m = id$ for $m \geq N_0$. Then Γ is a group endowed with the discrete topology.

Therefore M_ϕ is a compact Sol-manifold: it is the quotient of $\mathbb{R}^3$ by the discrete group Γ (Figure 2.6).

Chapter 3

3-dimensional hyperbolic manifolds

3.1 Gieseking Manifold

This section is dedicated to present in some detail an example found by Gieseking (1912). It is an example of a cusped, non-compact 3-dimensional hyperbolic manifold. Consider the ideal hyperbolic tetrahedron T_0 on the upper half model for the hyperbolic 3-space $\mathbb{H}^3$ having vertices 0, 1, ∞ and $-\omega$ (Figure 3.1) and the isometries of $\mathbb{H}^3$ defined by

$$(-\omega, 0, \infty) \xrightarrow{U} (-\omega, 1, 0) \text{ , and } (1, 0, \infty) \xrightarrow{V} (-\omega, 1, \infty) \text{ .}$$

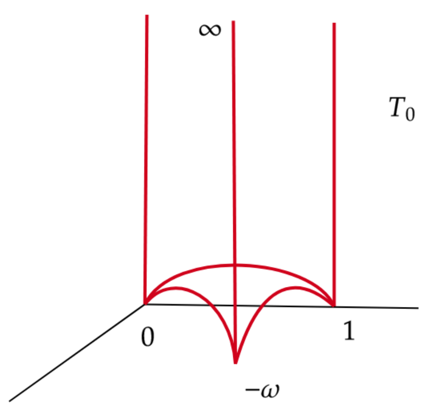

Figure 3.1: Ideal hyperbolic tetrahedron T_0.

Moreover,

$$(1, 0, \infty) \xrightarrow{\text{Rot}_{1\infty}} (1, -\omega, \infty) \xrightarrow{\text{Ref}_{0\infty}} (-\omega, 1, \infty)$$
$$\text{and}$$
$$(-\omega, 0, \infty) \xrightarrow{\text{Rot}_{-\omega 0}} (-\omega, 0, 1) \xrightarrow{\text{Ref}_{-\omega\infty}} (-\omega, 1, 0) \text{ .}$$

Then, $U = \text{Ref}_{-\omega\infty} \circ \text{Rot}_{-\omega 0}$ and $V = \text{Ref}_{0\infty} \circ \text{Rot}_{1\infty}$, where Rot_{ab} is a rotation respect the edge e_{ab} and Ref_{ab} is a reflection respect the edge e_{ab}.

Now, if we consider the same tetrahedron T_0, but in the *Poincaré ball model* for the 3-hyperbolic space $\mathbb{H}^3$ we can map U over the tetrahedron to obtain an adjacent tetrahedron T_0' such that:

The face $(-\omega, 0, \infty)$ maps to the face $(-\omega, 1, 0)$ and let $U(1) = p$ (Figure 3.2). With this new tetrahedron, we glue the faces in pairs respecting not only face labels but also directions, for example in the face $(-\omega, 0, \infty)$ we have the three edges $(-\omega, 0)$ (from $-\omega$ to 0), $(-\omega, \infty)$ (from $-\omega$ to ∞) and $(0, \infty)$ (from 0 to ∞). Let $S' = (\omega, 0, p)$, $S = (1, 0, \infty)$, $T' = (p, 0, 1)$, $T = (-\omega, 1, \infty)$, $E = (-\omega, 0, \infty)$ and $E' = (\omega, p, 1)$ (Figure 3.3).

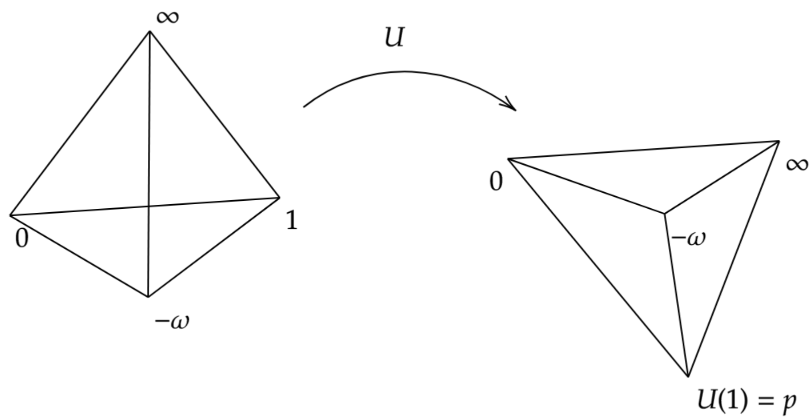

Figure 3.2: Mapping of the tetrahedron T_0 over the tetrahedron T_0'.

Define the identification of the faces as follows:

$$S' : (-\omega, p, 0) \xrightarrow{U^{-1}} (-\omega, 1, \infty) \xrightarrow{V^{-1}} (1, 0, \infty) : S \, ,$$
$$E' : (-\omega, p, 1) \xrightarrow{U^{-1}} (-\omega, 1, 0) \xrightarrow{U^{-1}} (-\omega, 0, \infty) : E \, ,$$
$$T' : (p, 1, 0) \xrightarrow{U^{-1}} (1, 0, \infty) \xrightarrow{V} (-\omega, 1, \infty) : T \, .$$

And in this way, we got the edge identifications as follows:
$$(-\omega, \infty) \xrightarrow{V^{-1}} (1, \infty) \xrightarrow{V^{-1}} (0, \infty) \xrightarrow{U} (1, 0) \xrightarrow{V} (-\omega, 1) \xrightarrow{U^{-1}} (-\omega, 0) \xrightarrow{U^{-1}} (-\omega, \infty) \, ,$$
where the cycle transformation is $h = U^{-2}VUV^{-2}$. Analogously we can find the infinite cycle transformation by using the figure and the identification of the faces, just as follows:

$$e_{-\omega\infty} \xrightarrow{U^2} e_{-\omega 1} \xrightarrow{UV^{-1}} e_{p1} \xrightarrow{U^{-2}} e_{0\infty} \xrightarrow{UV} e_{p0} \xrightarrow{VU^{-1}} e_{-\omega\infty} \, ,$$

$h = V^2U^{-1}V^{-1}U^2$. By the special case of the Poincaré's theorem 2.2.2 the group Γ is given by the last infinite cycle transformation $U^{-2}VUV^{-2} = 1$. $\Gamma = \langle U, V : VU = U^2V^2 \rangle$ is a discrete subgroup of $Isom(\mathbb{H}^3)$, all the vertex are all identified, and since the six edges of T_0 are all identified and its dihedral angles must be $\pi/3$ then each edge in the manifold add up to 2π then T_0 is a fundamental polyhedron for Γ. The identification of the faces of T_0 define a cusped, non-compact hyperbolic manifold called the *Gieseking* manifold.

Now, we shall show that the Gieseking manifold is double covered by the complement of the figure-eight knot $\mathbb{S}^3 \backslash K_8$. In this way, we will find the first the fundamental group of $\mathbb{S}^3 \backslash K_8$ using

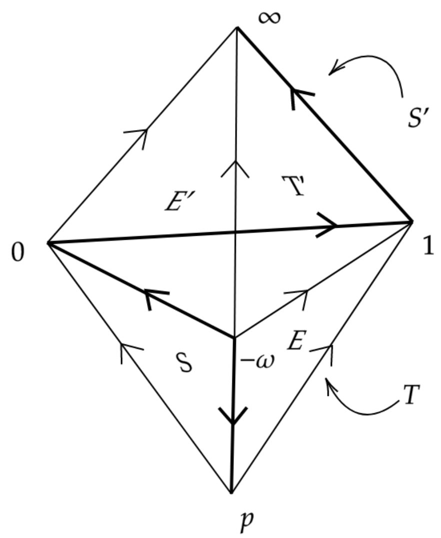

Figure 3.3: Gluing pattern of the tetrahedra T_0 and T_0'.

the Wirtinger algorithm [Rol03, p.56]. We have a presentation with generators x_1, x_2, x_3, x_4 and relations

1. $x_3^{-1} x_4 x_3 = x_1$
2. $x_1^{-1} x_2 x_1 = x_3$
3. $x_2 x_3 x_2^{-1} = x_4$
4. $x_4 x_1 x_4^{-1} = x_2$

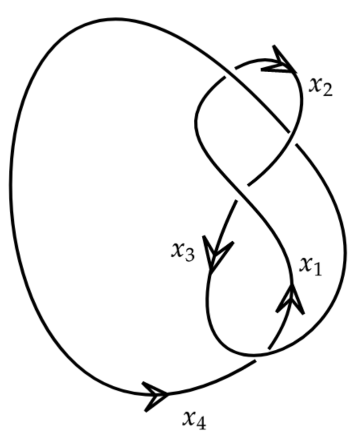

Figure 3.4: Figure-eight knot with its generators x_1, x_2, x_3, x_4 for the Wirtinger algorithm.

We may simplify using 2., 3. and 4. as follows:

$$\begin{aligned} x_2 &= x_4 x_1 x_4^{-1} \\ &= x_2 x_3 x_2^{-1} x_1 x_2 x_3^{-1} x_2^{-1} \\ &= x_2 x_1^{-1} x_2 x_1 x_2^{-1} x_1 x_2 x_1^{-1} x_2^{-1} x_1 x_2^{-1}. \end{aligned}$$

So, $1 = x_1^{-1} x_2 x_1 x_2^{-1} x_1 x_2 x_1^{-1} x_2^{-1} x_1 x_2^{-1}$. By taking $x = x_2^{-1}$ and $y = x_1$ we obtain an equivalent presentation

$$\pi_1(\mathbb{S}^3 \setminus K_8) = \langle x, y | xy^{-1}x^{-1}yxyx^{-1}y^{-1}xy = 1 \rangle.$$

According to [Mag74, p.155] a subgroup D of index 2 in the group Γ is generated by the elements

$$U^2 = x,\ V^2 = y$$

with the single defining relation

$$xy^{-1}x^{-1}yxyx^{-1}y^{-1}xy = 1.$$

It means that $\pi_1(\mathbb{S}^3 \setminus K_8)$ is a subgroup of index 2 in Γ. Since Γ is a presentation of the Gieseking manifold's fundamental group, it proves that the Gieseking manifold is double covered by a manifold homeomorphic to the complement of figure-eight knot as required. Finally we will present a fundamental polyhedron for this subgroup D.

Figure 3.5 shows one three-dimensional gluing pattern. Start with two tetrahedra T and T' with labeled faces and directed edges divided into two types. Then glue faces in pairs, respecting not only face labels but also edge types and directions. Let the two tetrahedra be regular ideal tetrahedra in hyperbolic space and let e_{ij} be the edge that lies between the face i and j. We get the following cycle relations:

- $e_{AB} \xrightarrow{g_{A'}} e_{A'C'} \xrightarrow{g_C} e_{CD} \xrightarrow{g_{D'}} e_{D'A'} \xrightarrow{g_A} e_{AC} \xrightarrow{g_{C'}} e_{C'B'} \xrightarrow{g_B} e_{AB}$, $h = g_B g_C^{-1} g_A g_D^{-1} g_C g_A^{-1}$.
- $e_{AC} \xrightarrow{g_{A'}} e_{A'D'} \xrightarrow{g_D} e_{CD} \xrightarrow{g_{C'}} e_{C'A'} \xrightarrow{g_A} e_{AB} \xrightarrow{g_{B'}} e_{C'B'} \xrightarrow{g_C} e_{AC}$, $h = g_C g_B^{-1} g_A g_C^{-1} g_D g_A^{-1}$.
- $e_{AD} \xrightarrow{g_{A'}} e_{A'B'} \xrightarrow{g_B} e_{BD} \xrightarrow{g_{D'}} e_{D'C'} \xrightarrow{g_C} e_{CB} \xrightarrow{g_{B'}} e_{D'B'} \xrightarrow{g_D} e_{AD}$, $h = g_D g_B^{-1} g_C g_D^{-1} g_B g_A^{-1}$.

Then, the group defined by the side-pairing relations is given by

$$BC^{-1}AD^{-1}CA^{-1} = 1,\ CB^{-1}AC^{-1}DA^{-1} = 1, \text{ and } DB^{-1}CD^{-1}BA^{-1} = 1.$$

From the first one relation, we get that $B^{-1} = C^{-1}AD^{-1}CA^{-1}$. Moreover, this relation is equal to the second one, so, we will use the last one, as follows:

$$\begin{aligned} 1 &= DB^{-1}CD^{-1}BA^{-1} && (3.1.1) \\ &= DC^{-1}AD^{-1}CA^{-1}CD^{-1}AC^{-1}DA^{-1}CA^{-1}. && (3.1.2) \end{aligned}$$

By taking $x = DA^{-1}$ and $y = CD^{-1}$ we obtain an equivalent presentation

$$DC^{-1}AD^{-1}CA^{-1}CD^{-1}AC^{-1}DA^{-1}CA^{-1} = y^{-1}x^{-1}yxyx^{-1}y^{-1}xyx = 1.$$

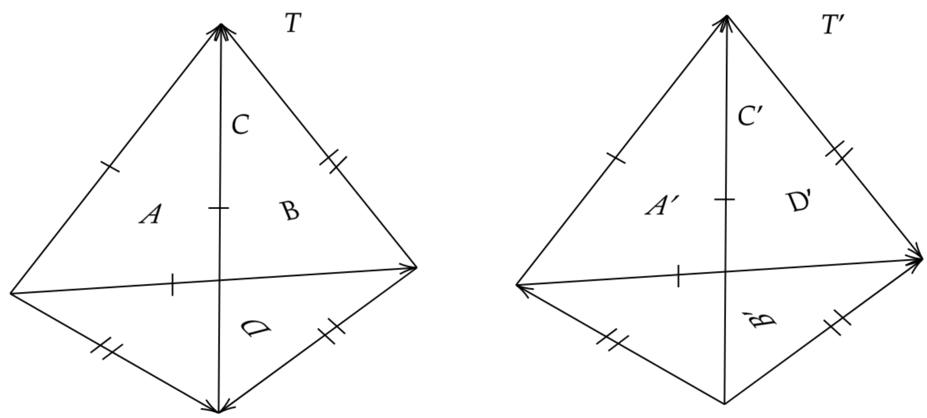

Figure 3.5: Gluing pattern for the tetrahedra T and T'. Each face has a label, and faces with the same label are identified in a way that is unambiguously determined by the requirement that edge types(one or two tiny segments crossing it) and directions match.

Thus, the group defined by the side-pairing is

$$\hat{\Gamma} = \langle x, y | xy^{-1}x^{-1}yxyx^{-1}y^{-1}xy = 1 \rangle.$$

Note that the gluing pattern gives us the identification of the sides in two sets with six edges each one. Since the hyperbolic tetrahedra are regular, each dihedral angle is $\pi/3$, so with the gluing, the dihedral angle of the edges identificated add up to 2π. Then, by the Poincaré's polyhedron theorem 2.2.2, we can conclude that this gluing is a fundamental polyhedron for $\hat{\Gamma}$ whose presentation is exactly the group D as required. Note that, combinatorially, a regular ideal tetrahedron is a simplex with its vertices deleted.

3.1.1 Hyperbolic structure on the complement of the figure-eight knot

This part is based on the study of knots, specifically on the figure-eight knot to give it a hyperbolic structure. Along this part of the text, we will consider $\mathbb{H}^3$ as the half 3-space. In addition, we refer the reader to the following articles [Fox62, Ril72, Ril75].

Definition 3.1.1. k is a *knot* if there exists a homeomorphism of the unit circle $\mathbb{S}^1$ into 3-dimensional sphere $\mathbb{S}^3$ (or in the 3-dimensional space $\mathbb{R}^3$) whose image is k. It is often usefull to work in $\mathbb{R}^3$ and view $\mathbb{S}^3$ as the one point compactification of $\mathbb{R}^3$. Two knots k_1 and k_2 are equivalent if there exists a homeomorphism of $\mathbb{S}^3$ onto itself which maps k_1 onto k_2. Equivalent knots are said to be of the same *type*, and each equivalence class of knots is a *knot type*.

Example 3.1.2. The knots equivalent to the unknotted circle $x^2 + y^2 = 1$, $z = 0$, are called *trivial* and constitute the trivial type. The informal statement that the figure-eight knot and the unknot are different is rigorously expressed by saying that they belong to distinct knot types.

Definition 3.1.3. A *polygonal knot* is one which is the union of a finite number of closed straight-line segments called edges, whose endpoints are the *vertices* of the knot. A knot is *tame* if it is equivalent to a polygonal knot; otherwise it is *wild*.

Example 3.1.4. The figure-eight knot is tame.

Definition 3.1.5. An *isotopic deformation* of a topological space X is a family of homeomorphisms h_t, $0 \le t \le 1$, of X onto itself such that h_0 is the identity, and the function H defined by $H(t,p) = h_t(p)$ is simultaneously continuous in t and p. Knots k_1 and k_2 are said to belong to the same *isotopy type* if there exists an isotopic deformation $\{h_t\}$ of $\mathbb{R}^3$ such that $h_1 k_1 = k_2$.

The letter k denotes a tame knot in $\mathbb{S}^3$, $\overline{K}$ denotes the isotopy type of k, and K denotes the type of k. We write πK for the group of K, i.e. $\pi K = \pi_1(\mathbb{S}^3 \setminus k; *)$.

Definition 3.1.6. An element of πK is called *peripheral* if, for every neighborhood of k, it is representable as a loop of the form $\gamma\alpha\gamma^{-1}$ where γ is a path from the base point to a point of $W - k$ and α is a loop in $W - k$. An element determined by the boundary of a small disk pierced once by k is a *meridian*. An element determined by a curve that runs parallel to k and is homologous to 0 in the complement of k is called a *longitude*. Any maximal peripheral subgroup of πK is generated by a meridian and a longitude, and any two maximal peripheral subgroups are conjugate.

Definition 3.1.7. We define the marked group of $\overline{K}$, denoted $\pi\overline{K}$, to be πK with the conjugacy class of maximal peripheral subgroups specified.

To mark πK consider an over presentation

$$\pi K = \langle x_1, \cdots, x_n | r_2, \cdots, r_n \rangle.$$

The first named over generator x_1 is a meridian of πK to which corresponds a unique longitude $\gamma \in \pi K$ that commutes with x_1 and is determined by a loop running "parallel" to k in the sense of the orientation of k. The subgroup $\langle x_1, \gamma \rangle$ is then a maximal peripheral subgroup of πK which determines the marked group $\pi\overline{K}$. We write

$$\pi\overline{K} = \langle x_1, \cdots, x_n | r_2, \cdots, r_n | \gamma \rangle$$

for this marked group, and in each specific case γ would be written out as a word in $x_1, \cdots, x_n$.

If k_8 denotes the figure-eight knot, we want to find a presentation for the image group $\theta(\pi K_8)$ where θ is a parabolic representation $\theta : \pi K_8 \to PSL(2, \mathbb{C})$. These projective representations can be divided into two types *parabolic* or *non-parabolic*, according as the image of any over generator of πK is parabolic or not. We have the following presentation for the figure-eight knot group

$$\pi K_8 = \left\langle x_1^{-1} x_2 x_1 x_2^{-1} x_1 x_2 x_1^{-1} x_2^{-1} x_1 x_2^{-1} = 1 \right\rangle.$$

Let $\omega = \frac{-1+\sqrt{-3}}{2}$, and define a map θ from πK_8 into $SL(2, \mathbb{C})$ by

$$\theta(x_1) = A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \theta(x_2) = B = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}.$$

The complement in $\mathbb{S}^3$ of an open tubular neighborhood of the figure-eight knot k_8 is a compact 3-manifold X_1 whose boundary ∂X_1 is a torus. The marked knot group πK_1 is $\pi_1(X_1)$ with a particular peripherical subgroup distinguished, and has a presentation

$$\pi \overline{K_8} = \left\langle x_1, x_2 | w x_1 w^{-1} = x_2 | \gamma \right\rangle, \tag{3.1.3}$$

in which

$$w = x_1^{-1} x_2 x_1 x_2^{-1},\ \tilde{w} = x_1 x_2^{-1} x_1^{-1} x_2,\ \gamma = \tilde{w}^{-1} w.$$

Define

$$W = \theta(w) = \begin{pmatrix} 0 & \omega \\ -\omega^2 & 1-\omega \end{pmatrix},\ \tilde{W} = \theta(\tilde{w}) = \begin{pmatrix} 0 & -\omega \\ \omega^2 & 1-\omega \end{pmatrix},$$

Then on writing $A\{t\} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ we find

$$\theta(\gamma^{-1}) = W^{-1}\tilde{W} = A\{2+4\omega\}. \tag{3.1.4}$$

So, $A\{2+4\omega\}$ is a translation in a direction perpendicular to that of A. We write $\mathfrak{G} = \langle A, B \rangle = \theta(\pi K_8)$.

Note that by definition θ is a homomorphism since the relation in πK_8 is satisfied by the matrices A and B. In fact,

$$A^{-1}BAB^{-1}AB = \begin{pmatrix} -\omega^2 & \omega \\ \omega^3 - \omega & -\omega^2 - \omega + 1 \end{pmatrix}. \tag{3.1.5}$$

On the other hand

$$BA^{-1}BA = \begin{pmatrix} 1+\omega & \omega \\ -\omega^2 - 2\omega & -\omega^2 - \omega + 1 \end{pmatrix}. \tag{3.1.6}$$

Since $\omega^3 - \omega = \omega(\omega^2 - 1) = \omega(-1-\omega-1) = \omega(-\omega-2)$. Then $A^{-1}BAB^{-1}AB = BA^{-1}BA$ as required. To determine the faithfulness and discreteness Riley constructs a fundamental polyhedron for the action of $\theta(\pi K_8)$ on $\mathbb{H}^3$.

Theorem 3.1.8. *A presentation for* $\mathfrak{G}$ *is*
Generators A, B
Assistan generator $W := A^{-1}BAB^{-1}$,
Relation $WAW^{-1} = B$.
Consequently, the parabolic representation $\theta : \pi K_8 \to \mathfrak{G}$ *is an isomorphism.*

Proof. See [Ril75]. □

Definition 3.1.9. Given $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C})$ with $c \neq 0$ we define the isometric circle $I_0(T) \subset \mathbb{C} \times \{0\}$ as the set of $(z,0) \in \mathbb{C} \times \{0\}$ such that $|z + c^{-1}d| = |c^{-1}|$. We call the hyperbolic plane $I(T)$ whose Euclidean boundary is $I_0(T)$ the *isometric sphere* of T. When $c = 0$, T has no isometric sphere.

Example 3.1.10. Let $B = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}$ be a matrix in $PSL(2,\mathbb{C})$ acting on $\mathbb{C}P^1$. Its isometric circle is given by

$$I_0(B) = \{(z,0) \in \mathbb{C} \times \{0\}; |z - \overline{\omega}| = 1\}.$$

Definition 3.1.11. Let $\mathfrak{G}$ be a discrete subgroup of $PSL(2,\mathbb{C})$, $\mathfrak{G}$ acts discontinuously on $\mathbb{H}^3$ and $\mathfrak{G}_\infty$ acts discontinuously on $\mathbb{C}\times\{0\}$. Hence $\mathfrak{G}$ has an open fundamental domain $D(\mathfrak{G}) \subset \mathbb{H}^3$. A *Ford domain* is the portion of $\mathbb{H}^3$ outside all isometric spheres of $\mathfrak{G}$ which (Euclidean)-projects onto a fundamental domain of $\mathfrak{G}_\infty$.

To determine the faithfulness and discreteness Riley constructed a fundamental polyhedron for the action of $\theta(\pi K_8)$ on $\mathbb{H}^3$.

Consider the isometric spheres of $W^{\pm 1}$, $B^{\pm 1}$ whose radius is 1. Their centers are

$$\begin{aligned} I(B) &: (\omega^2, 0), & I(W) &: (1 + 2\omega, 0), \\ I(B^{-1}) &: (1 + \omega, 0), & I(W^{-1}) &: (0, 0). \end{aligned}$$

Every $\alpha \in \mathbb{Z}[\omega]$ is congruent to one of these centres $\mod (1, 2 + 4\omega)$, so α is the centre of an isometric sphere of $\mathfrak{G}$ of radius 1. The collection $\mathcal{L}$ of all these spheres is a regular triangular lattice of spheres which is stable under $\langle A, A\{2 + 4\omega\}\rangle$. Each sphere of $\mathcal{L}$ meets six other spheres of $\mathcal{L}$ along the edges of a regular hyperbolic hexagon, and the angles of intersection are all $2\pi/3$. Furthermore, the interior of the closure of the union of the (Euclidean)-projections on $\mathbb{C} \times \{0\}$ of the hexagons on $I(B^{\pm 1})$, $I(W^{\pm 1})$ is a fundamental region D_∞ of $\langle A, A\{2 + 4\omega\}\rangle$. Let D be the portion of $\mathbb{H}^3$ lying above all spheres in $\mathcal{L}$ which (Euclidean)-projects onto D_∞. The euclidean-vertical sides of D meet the spherical sides in the angle $\pi/3$. We refer to the figure 3.6 which depicts the projection on $\mathbb{C} \times \{0\}$ of the objects that we defined.

D was the fundamental domain for $\mathfrak{G}$ that Riley constructed. Note that the quotient manifold has the same fundamental group and peripheral structure as the figure-eight knot complement, in the Waldhausen's language there exists an isomorphism from the fundamental groups which respect the peripheral structure. And again if we invoke the Waldhausen's work [Wal68], we can affirm that there exist a homeomorphism between these manifolds.

Corollary 3.1.12. The identified polyhedron D^* is homeomorphic to the knot complement $\mathbb{S}^3 \setminus k_8$.

Proof. See [Ril75]. □

With this we have constructed a quotient manifold which admits a hyperbolic structure and is homeomorphic to the complement of the figure eight-knot as required.

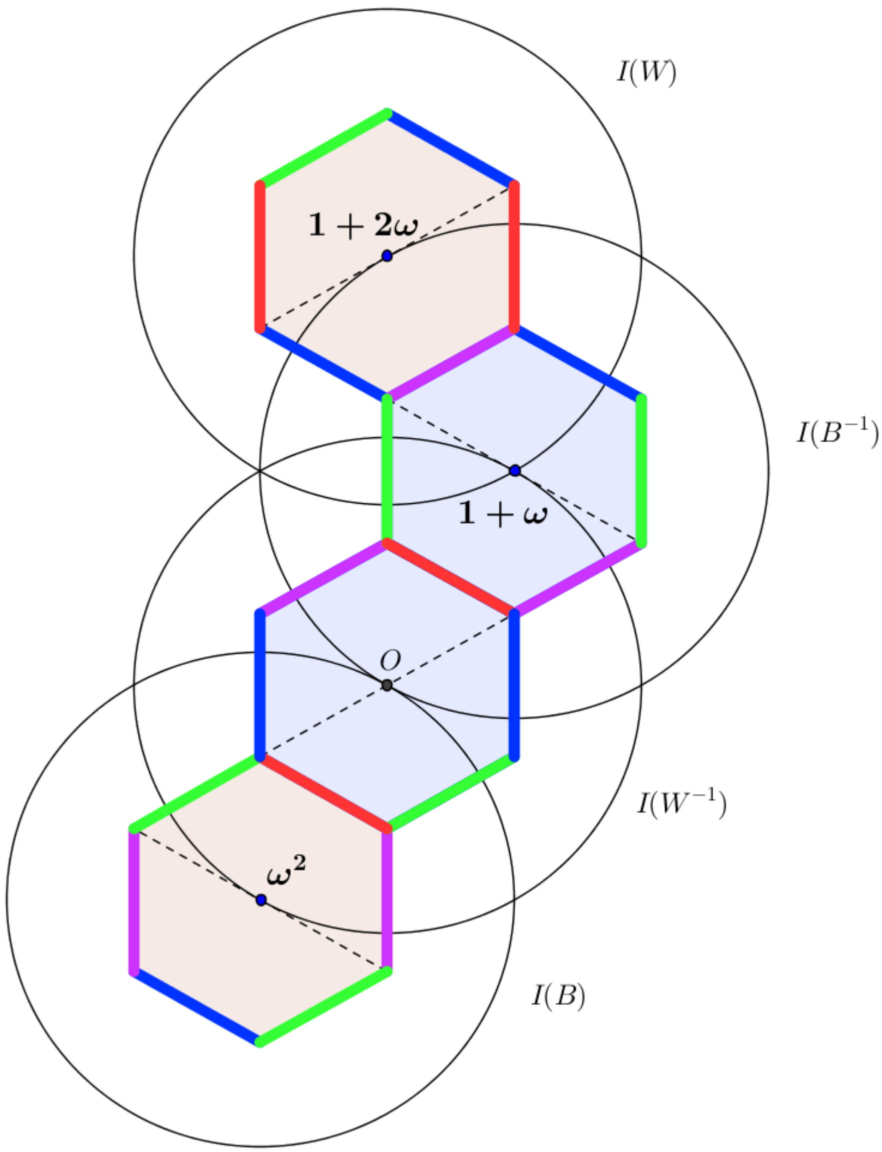

Figure 3.6: Fundamental Domain D of $\mathfrak{G}$ (euclidean)-projected on $\mathbb{C} \times \{0\}$. The cycle transformation corresponding to the purple edges is given by $B^{-1}W^{-1}A^{-1}BA = 1$.

3.2 The hyperbolic dodecahedra spaces

This section will provide us examples of hyperbolic dodecahedra obtained through gluings. First, we will apply the Mostow's Rigidity theorem to obtain the Seifert Weber dodecahedral as the orbit space of the kernel of a mapping onto the alternating group A_5. Finally, we shall show how different gluing patterns in the same hyperbolic polyhedron could give us non-homeomorphic hyperbolic manifolds. This section is based overall on the article [Bes71].

Lannér enumerated the hyperbolic tetrahedra possesing the property of having all dihedral angles equal to an integer submultiple of π. Letting π/λ_i and π/μ_i, $i = 1, 2, 3$, be the angles at opposite edges of the tetrahedron, where π/λ_i, $i = 1, 2, 3$, are the angles at the edges of a face, he showed there are precisely nine such non-congruent tetrahedra described by its dihedral angles $[\lambda_1, \lambda_2, \lambda_3 : \mu_1, \mu_2, \mu_3]$.

$$T1[2,2,3:3,5,2],\ T2[2,2,3:2,5,3],\ T3[2,2,4:2,3,5],$$
$$T4[2,2,5:2,3,5],\ T5[2,3,3:2,3,4],\ T6[2,3,4:2,3,4],$$
$$T7[2,3,3:2,4,5],\ T8[2,3,4:2,3,5],\ T9[2,3,5:2,3,5].$$

The canonical presentation of the associated hyperbolic tetrahedral group is

$$a^{\lambda_1} = b^{\lambda_2} = c^{\lambda_3} = (bc)^{\mu_1} = (ca)^{\mu_2} = (ab)^{\mu_3} = 1.$$

Consider the group Γ corresponding to $T4$:

$$a^2 = b^2 = c^5 = (bc)^2 = (ca)^3 = (ab)^5 = 1.$$

Let N be a proper normal subgroup of Γ and r, s, and t be the respective images of a, b, and c under the canonical homomorphism $\phi : \Gamma \to \Gamma/N$. Then r, s, t generate Γ/N and the relations $r^2 = s^2 = t^5 = (st)^2 = (tr)^3 = (rs)^5 = 1$ hold. If $a \in N$ then $r = 1$ and by considering the relations we have the following:

Since $t^5 = (tr)^3 = 1$ implies that $t^2 = 1$ then $t = t^2 * t = 1$, so $t = 1$. Moreover $1 = s^2 = s^5$ implies that $s^3 = 1$, so $s = s * s^2 = 1$.

Then $N = \Gamma$, but it is a contradiction since we assume Γ as a proper normal subgroup. It implies that $a \notin N$. By a similar argument b, c, bc, ca, ab are not in N, and since these are the unique elliptic elements, then N is torsion-free.

The smallest non-trivial group onto which Γ can be mapped homomorphically is the alternating group A_5, a homomorphism $\psi : \Gamma \to A_5$ being given by

$$\psi(a) = (15)(34),\ \psi(b) = (14)(23),\ \psi(c) = (12345).$$

The kernel of ψ is determined by the Reidemeister-Schreier method to be the group Γ_0 on six generators and defining relations

$$\begin{aligned} abcde &= 1, & cxad^{-1}e^{-1} &= 1,\\ axdb^{-1}c^{-1} &= 1, & dxbe^{-1}a^{-1} &= 1,\\ bxec^{-1}d^{-1} &= 1, & exca^{-1}b^{-1} &= 1. \end{aligned}$$

Obs. *The Reidemeister-Schreier method [Bes71, MKS04] also is accesible via a computer, in our case we use GAP (computer algebra system), which can be found at* `https://www.gap-system.org/`. *We considered* $S = \{c^k, c^l ac^k, c^l ac^2 ac^k, c^l ac^2 ac^3 a\}_{l,k=0,1,\cdots,4.}$ *as the Schreier system of representations of the right cosets with respect to* Γ_0. *Excluding unnecessary generators and relations* Γ_0 *is generated by transformations*

$$a_k = c^{k-1} ac^2 ac^4 b (c^{k+1} a)^{-1},\ k = 1, 2, \cdots, 5, \quad a_6 = c^2 ac^2 ac^3 ab.$$

This group Γ_0 is in fact, the fundamental group of the Seifert Weber dodecahedral space (example 2.2.13).

Theorem 3.2.1. *(* **Mostow's Rigidity Theorem** *) If two hyperbolic manifolds of finite volume, with dimension $n \geq 3$, have isomorphic fundamental groups, then they must necessarily be isometric to each other.*

Proof. See [Mos73, Joh82]. □

Since $\mathbb{H}^3/\Gamma_0$ and the Seifert Weber dodecahedral space have isomorphic fundamental groups, by the Mostow's Rigidity theorem 3.2.1, we can conclude that the orbit space of the kernel of ψ is isometric to the Seifert Weber dodecahedral space as required.

To finish this section, consider the following example, where our main objective is to construct hyperbolic 3-manifolds from the dodecahedron with dihedral angles $2\pi/5$.

Example 3.2.2. Let X be a hyperbolic dodecahedral (Figure 3.7) with dihedral angles $2\pi/5$ and the following gluing patterns.

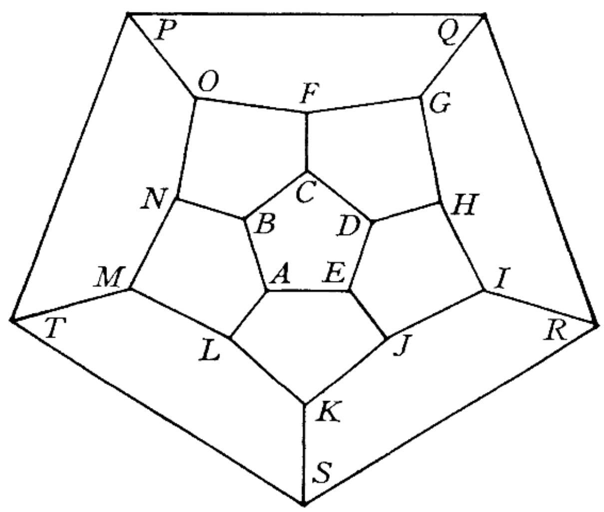

Figure 3.7: Dodecahedron X.

$$a : ABCDE \to TPQRS \qquad b : AEJKL \to QGHIR$$
$$c : BALMN \to JKSRI \qquad d : CBNOF \to MTSKL$$
$$e : DCFGH \to NOPTM \qquad f : EDHIJ \to POFGQ.$$

These identifications give us that the 30 edges are glued in 6 groups of 5, where e_{ij} is the edge corresponding to the segment ij at the respective face of the hyperbolic dodecahedron.

- $\{e_{AB}, e_{TP}, e_{GF}, e_{IH}, e_{KJ}\}$, $h = cbfea^{-1}$.
- $\{e_{BC}, e_{PQ}, e_{EJ}, e_{GH}, e_{TM}\}$, $h = de^{-1}b^{-1}fa^{-1}$.
- $\{e_{CD}, e_{QR}, e_{AL}, e_{KS}, e_{ON}\}$, $h = edc^{-1}ba^{-1}$.
- $\{e_{DE}, e_{RS}, e_{ML}, e_{CF}, e_{OP}\}$, $h = fe^{-1}dca^{-1}$.
- $\{e_{EA}, e_{ST}, e_{NB}, e_{IJ}, e_{GQ}\}$, $h = bf^{-1}c^{-1}da^{-1}$.

- $\{e_{HD}, e_{MN}, e_{RI}, e_{KL}, e_{OF}\}$, $h = fdbc^{-1}e^{-1}$.

At the glued-up manifold M, the edges add up to 2π and by the Poincare's polyhedron theorem, we have got a hyperbolic 3-manifold, where $M \cong \mathbb{H}^3/\Gamma$ and the hyperbolic dodecahedral is fundamental for Γ, the discrete group given by the cycle relations, i.e.,

$$\Gamma = \{a, b, c, d, e, f | cbfea^{-1}, de^{-1}b^{-1}fa^{-1}, edc^{-1}ba^{-1}, fe^{-1}dca^{-1}, bf^{-1}c^{-1}da^{-1}, fdbc^{-1}e^{-1}\}.$$

Moreover, since our objective is to show that even with the same manifold, different gluing patterns could give us different hyperbolic 3-manifolds, we will study the first homology group of the manifold. By considering the first homology group as the abelianization of the first fundamental group, we will use the following map $\Gamma \to \Gamma/[\Gamma,\Gamma]$. Let $A_1, A_2, \cdots A_6$ be the homological classes given by the map, for the elements $a, b, \cdots, f$. Then from the cycle relations we obtained relations for the elements $A_1, \cdots, A_6$ that generate the first homology group:

$$\begin{aligned} A_2 - A_3 + A_4 - A_5 + A_6 &= 0, \\ -A_1 + A_3 + A_4 - A_5 + A_6 &= 0, \\ -A_1 - A_2 + A_4 - A_5 + A_6 &= 0, \\ -A_1 + A_2 + A_3 + A_5 + A_6 &= 0, \\ -A_1 + A_2 - A_3 + A_4 - A_6 &= 0, \\ -A_1 + A_2 - A_3 + A_4 + A_5 &= 0. \end{aligned}$$

This satifies the following matrix relation

$$\begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 & -1 & 1 \\ -1 & -1 & 0 & 1 & -1 & 1 \\ -1 & 1 & 1 & 0 & 1 & 1 \\ -1 & 1 & -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.2.1}$$

By elementary transformations the system (3.2.1), reduces to the following:

(I) $A_2 - A_3 + A_4 + 2A_6 = 0$,

(II) $-A_1 + A_3 + A_4 - A_5 + A_6 = 0$,

(III) $-A_2 - A_3 = 0$

(IV) $2A_1 + A_3 + A_4 + 2A_6 = 0$,

(V) $A_5 + A_6 = 0$.

(VI) $-A_1 + A_2 - A_3 + A_4 + A_6 = 0$

From the equations (I), (III) we get

$$\text{(VII)}\ \ 2A_2 + A_4 + 2A_6 = 0.$$

From the equations (II), (III) and (V) we get

$$\text{(VIII)}\ \ -A_1 + -A_2 + A_4 + 2A_6 = 0.$$

From the equations (IV) and (III) we get

$$\text{(IX)}\ \ -2A_1 - A_2 + A_4 + 2A_6 = 0.$$

From the equations (III) and (VI) we get

$$\text{(X)}\ \ -A_1 + 2A_2 + A_4 + A_6 = 0.$$

By adding the inverse of the equation (IX) to the equation (VIII) we get

$$\text{(XI)}\ \ A_1 = 0.$$

Then, from the equations (IX), (VII) and $A_1 = 0$ we get

$$\text{(XII)}\ \ 3A_2 = 0.$$

From the equations (VIII), (X) and $3A_2 = 0$ we get

$$\text{(XIII)}\ \ A_6 = 0.$$

From $A_6 = 0$ and the equation (IX) we get that

$$\text{(XIV)}\ \ A_2 = A_4.$$

To summarize, the matrix relation (3.2.1) reduces to

$$3A_2 = 3A_4 = 0,\ A_1 = A_6 = 0,\ A_3 = -A_2 \text{ and } A_5 = -A_6 = 0.$$

Then $H_1(M) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

The table 3.1 shows that there are another six non-homeomorphic 3-manifolds, which are obtained from the dodecahedron with dihedral angles $2\pi/5$ by identifying pair of its faces. We refer again to the Figure 3.7.

Obs. The presentation and the homology groups in the table 3.1, were got by an analogous process that we explained with more detail at the example 3.2.2.

In conclusion, at the table 3.1 we have shown that although the identification has been made using the same dodecahedron, we could obtain non-homeomorphic hyperbolic 3-manifolds only changing the gluing pattern on its faces, since these manifolds are determined by their fundamental groups.

Identified faces	Fundamental group (Presentation)	Homology group
$a: ABCDE \to RSTPQ$; $b: AEJKL \to QGHIR$ $c: BALMN \to RIJKS$; $d: CBNOF \to SKLMT$ $e: DCFGH \to TMNOP$; $f: EDHIJ \to POFGQ$	$ac^{-1}dbc^{-1} = 1 = ad^{-1}ecd^{-1}$, $ae^{-1}fde^{-1} = 1 = af^{-1}bef^{-1}$, $ab^{-1}cfb^{-1} = 1 = dcbfe$.	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$
$a: ABCDE \to PQRST$; $b: AEJKL \to IRQGH$ $c: BALMN \to KSRIJ$; $d: CBNOF \to MTSKL$ $e: DCFGH \to OPTMN$; $f: EDHIJ \to GQPOF$	$af^{-1}edc^{-1} = 1 = ab^{-1}fed$, $ac^{-1}bfe^{-1} = 1 = ad^{-1}cbf^{-1}$, $ae^{-1}dcb^{-1} = 1 = becfd$.	$\mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$
$a: ABCDE \to PQRST$; $b: AEJKL \to FGQPO$ $c: BALMN \to DHGFC$; $d: CBNOF \to JIHDE$ $e: GHIRQ \to STMLK$; $f: IJKSR \to PONMT$	$ab^{-1}fdc^{-1} = 1 = aebf^{-1}d^{-1}$, $afe^{-1}d^{-1}c = 1 = ae^{-1}c^{-1}bd$, $af^{-1}ecb^{-1} = 1 = dbefc$.	$\mathbb{Z}_5 \oplus \mathbb{Z}_{15}$
$a: ABCDE \to TPQRS$; $b: AEJKL \to HIRQG$ $c: BALMN \to KSRIJ$; $d: CBNOF \to LMTSK$ $e: DCFGH \to TMNOP$; $f: EDHIJ \to POFGQ$	$ae^{-1}fdc^{-1} = 1 = dcb^{-1}fa^{-1}$, $ed^{-1}cba^{-1} = 1 = ac^{-1}be^{-1}f$, $d^{-1}ecfb^{-1} = 1 = bfeda^{-1}$.	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$
$a: ABCDE \to TPQRS$; $b: AEJKL \to IRQGH$ $c: BALMN \to JKSRI$; $d: CBNOF \to SKIMT$ $e: DCFGH \to TMNOP$; $f: EDHIJ \to FGQPO$	$cbf^{-1}ea^{-1} = 1 = dc^{-1}bfa^{-1}$, $ab^{-1}fde^{-1} = 1 = fedca^{-1}$, $ad^{-1}ecb^{-1} = 1 = dbef^{-1}c^{-1}$.	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$
$a: ABCDE \to KJEAL$; $b: DCFGH \to JKSRI$ $c: BALMN \to KLMTS$; $d: EDHIJ \to QPOFG$ $e: CBNOF \to HIRQG$; $f: MNOPT \to PTSRQ$	$ab^{-1}a^2c^{-1} = 1 = eded^{-1}a^{-1}$, $a^2cab^{-1} = 1 = ac^2fd^{-1}$, $ac^{-1}ab^{-1}a = 1 = fc^{-1}f^2e^{-1}$.	$\mathbb{Z}_{35}$

Table 3.1: Hyperbolic 3-Manifolds obtained from the dodecahedron (dihedral angles: $2\pi/5$) by the identification in pairs of its faces.

Chapter 4

A brief historical note

Before finishing the dissertation, I want to present that was known as the Thurston's *geometrisation conjecture* and some important facts that have made possible the development of the theory of 3-manifolds. Let me begin by talking about the Poincaré conjecture.

Poincaré was interested in the topological properties that could characterize a 3-sphere. In 1900, Poincaré had claimed that the *Homology* was enough to tell when a 3-manifold was a 3-sphere. But, in 1904, he presented a counterexample in a paper for his first claim. Currently, the counterexample is called the *Poincaré homology sphere* and is exactly the Poincaré dodecahedral space, the elliptic 3-manifold that we constructed in the example 2.2.16. This manifold has exactly the same homology of the 3-sphere. To establish the difference between this manifold and the 3-sphere, Poincaré introduced a new topological invariant, *the fundamental group*, in this way, he showed that the Poincaré homology sphere has a fundamental group of order 120 while the 3-sphere has trivial fundamental group. Finally, at the same paper, Poincaré wondered whether a 3-manifold with the homology of a 3-sphere and also trivial fundamental group had to be a 3-sphere and in spite of he never declared that this additional property could characterize the 3-sphere, the statement that it does is known as the Poincaré conjecture.

Poincaré conjecture. Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

After nearly a century the conjecture was proved by *Grigori Perelman* in 3 papers in 2002 and 2003 on arXiv. The proof was built by using the *Ricci flow with surgery*, an idea introduced at the program of *Richard S. Hamilton* [Ham82] that hadn't been proved in three dimensions. In these papers, Perelman completed the proof [Per02, Per03b, Per03a].

The prime decomposition

The geometrisation conjecture asserts that any 3-manifold can be cut in geometric pieces. We will first begin by describing how to decompose the manifold to understand the conjecture. An application of this conjecture is precisely the proof of the Poincaré conjecture.

When we write $M_1 \# M_2$ to denote the connected sum of M_1 and M_2, note that $M \# \mathbb{S}^3$ is always homeomorphic to M. We say that a 3-manifold M is prime if any expression of M as $M_1 \# M_2$ has M_1 or M_2 homeomorphic to $\mathbb{S}^3$. A theorem of Kneser [Kne29] asserts that any compact 3-manifold can be expressed as a finite connected sum of primes, and Milnor [Joh62] showed that the factors involved are unique if M is orientable. The non-orientable case is explained by the homeomorphism $N\#\mathbb{S}^1 \times \mathbb{S}^2 \approx N\#\mathbb{S}^1 \tilde{\times} \mathbb{S}^2$ when N is non-orientable.

Theorem 4.0.1. *Every compact oriented 3-manifold M with (possibly empty) boundary decomposes into prime manifolds:*

$$M = M_1\# \cdots \# M_k.$$

This list of prime factors is unique up to permutations and adding/removing copies of S^3.

A detailed proof of the theorem can be found at [Hat07].

On the other hand, we say that a 3-manifold is irreducible if every embedded sphere bounds an embedded ball. In general these definitions are nearly equivalent for the following facts:

- Every irreducible 3-manifold M is prime.
- Every prime orientable 3-manifolds are irreducible, except one: $\mathbb{S}^1 \times \mathbb{S}^2$.

This prime decomposition is obtained by cutting M along a separating 2-sphere and then adding a 3-ball to each of the manifolds obtained. Beyond the prime decomposition there is a further decomposition of irreducible compact orientable 3-manifolds, splitting along tori rather than spheres.

Firstly, we need to define some properties of embedded surfaces on 3-manifolds. Let M be a 3-manifold and let $\mathcal{S} \hookrightarrow M$ be an embedded connected surface. An embedded disk $D \hookrightarrow M$ with $D \cap \mathcal{S} = \partial D$ is called a *compressing disk* for $\mathcal{S}$. Now, an *incompressible surface* on a 3-manifold is a connected surface $\mathcal{S} \subset M$ other than the 2-sphere or the 2-disk if for each compressing disk $D \subset M$ for $\mathcal{S}$ there is a disk $D' \subset \mathcal{S}$ such that $\partial D = \partial D'$. The incompressible surfaces are interesting because, if they are removed from an irreducible manifold, it remains irreducible. Also, we can say that $\mathcal{S}$ is an incompressible surface if the map $\iota : \pi_1(\mathcal{S}) \hookrightarrow \pi_1(M)$ is injective. This less intuitive way to define it comes from the fact that if $D \subset M$ is a compressing disk, then ∂D is nullhomotopic in M, hence also in $\mathcal{S}$ if the map ι is injective.

The incompressible surfaces we are interested in are incompressible tori. A compact 3-manifold M is said to be *atoroidal* if we cannot embed any incompressible tori in it. The following theorem, known as the *JSJ splitting theorem*, was discovered in the 1970s by W. Jaco and P. Shale from one side, and K. Johannson independently [JS76].

Theorem 4.0.2. (JSJ splitting theorem) *Let M be an irreducible, compact and orientable* 3-*manifold. There exists a finite collection $\mathcal{T} = \{T_1, \cdots, T_k\}$ of disjoint incompressible tori such that each component $M \setminus \mathcal{T}$ is either atoroidal or a Seifert manifold, and a minimal such collection $\mathcal{T}$ is unique up to isomorphism.*

Note that the JSJ decomposition is obtained by cutting along tori, it means that each component of $M \setminus \mathcal{T}$ is a manifold M_i with boundary $\partial M_i = T_i$, for some $T_i \in \mathcal{T}$.

To summarize, given the prime decomposition of a compact, orientable 3-manifold $M = P_1\#\cdots\#P_n$, as every prime component P_i is either $\mathbb{S}^1 \times \mathbb{S}^2$ or an irreducible manifold N_i, we can write $M = N_1\#\cdots\#N_r\#\mathbb{S}^1 \times \mathbb{S}^2\#\cdots\#\mathbb{S}^1 \times \mathbb{S}^2$. Moreover the irreducible components N_i can be split along tori, by a JSJ splitting, so that each subcomponent is either atoroidal or a Seifert manifold (or both). The interior of such subcomponents is a manifold. This final decomposition is known as *canonical decomposition* and through this decomposition we have obtained the canonical pieces into wich a 3-manifold decomposes, that can be Seifert manifolds or atoroidal manifolds.

The geometrisation conjecture

In 1982, Thurston gave a first version of the Geometrisation conjecture:

Thurston's Geometrisation Conjecture. *Every compact, orientable 3-manifold decomposes canonically into pieces whose interior is either Seifert fibred or hyperbolic.*

The canonical decomposition he refers is the last decomposition that we described above. Nevertheless, this statement does not mention geometric structure. Nowadays, the geometrisation conjecture is enunciated as follows:

Geometrisation Conjuecture. *The interior of any compact, orientable 3-manifold M can be split along a finite collection of disjoint embedded spheres and incompressible tori into a canonical collection of 3-submanifolds $M_1, \cdots, M_n$ such that, for each i, the manifold obtained from M_i by capping off all sphere components by balls admits a geometric structure.*

The geometrisation conjecture was motivated because some partial results were known before. In 1981, Thurston proved that any Haken manifold is hyperbolic if, and only if, it is atoroidal, where a *Haken* manifold is a compact, orientable, irreducible 3-manifold that contains an irreducible surface. Clearly a Haken manifold satisfies the geometrisation conjecture.

The JSJ splitting theorem 4.0.2 asserts that the interior of the pieces obtained in the canonical decomposition are either Seifert or atoroidal. Moreover, any Seifert manifold admits a geometric structure [Sco83], so in order to prove the geometrisation conjecture one only needs to prove that atoroidal manifolds have either an elliptic or hyperbolic geometry, according to whether the fundamental group is finite or not. Then, the geometrisation conjecture splits in two simpler conjectures as follows.

Conjecture. *M is a closed, orientable 3-dimensional manifold modeled on $\mathbb{S}^3$ if and only if $\pi_1(M)$ is finite.*

Conjecture. *M is a closed, orientable hyperbolic 3-dimensional manifold if and only if it is atoroidal and has infinite fundamental group.*

In 2003, Grigori Perelman presented a proof of the geometrisation conjecture, based on the Richard Hamilton notes about the Ricci flow. He showed that any compact orientable manifold M decomposes as

$$M = M_1 \# \cdots \# M_r \# E_1 \# \cdots \# E_k \# \mathbb{S}^1 \times \mathbb{S}^2 \# \cdots \# \ \mathbb{S}^1 \times \mathbb{S}^2$$

where E_i are manifolds modeled on $\mathbb{S}^3$ and each M_i admits a torus decomposition as $H_i \sqcup G_i$, with H_i an hyperbolic manifold and G_i a graph manifold. (A *Graph manifold* is a 3-dimensional compact, orientable manifold that is a union of Seifert manifolds along toral boundaries).

Note that the geometrisation theorem, together with the Thurston's classification theorem, gives a complete understanding of the geometric structures on compact, orientable 3-dimensional manifolds. We can say that such a manifold can be decomposed into canonical pieces whose interior can be modelled in one and only one of the eight Thurston's geometries. Moreover the canonical decomposition described is sufficient but no necessary, since *Sol* geometry does not appear. Any manifold modeled on Sol is neither Seifert nor hyperbolic, although it has a geometric structure.

Moreover, the Poincaré conjecture results as a consequence of the geometrisation theorem:

Proof of the Poincaré Conjecture. Let M be a compact, simply connected manifold. As $\pi_1(M)$ is trivial, it has no incompressible tori since the fundamental group of the torus is $\mathbb{Z} \oplus \mathbb{Z}$. Hence, the canonical decomposition of M must be trivial, and thus by the geometrisation theorem M admits a geometric structure. Again, since M is simply connected, it is its own universal cover, so the M must be the model geometry of M. Then M is one of the eight Thurston geometries, but the only compact one is $\mathbb{S}^3$ as required. $\square$

Bibliography

[ADP99] R.C. Alperin, W. Dicks, and J. Porti. The boundary of the gieseking tree in hyperbolic three-space. *Topology and its applications*, (93):219–259, 1999.

[Bes71] L. A. Best. On torsion-free discrete subgroups of psl(2, c) with compact orbit space. *Canadian Journal of Mathematics*, 23(3):451–460, 1971.

[DC76] Manfredo P. Do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice-Hall, Inc., 1976.

[DC92] Manfredo P. Do Carmo. *Riemannian Geometry*. Series: Mathematics (Boston, Mass.), 1992.

[Fox62] R. H. Fox. A quick trip through knot theory. *Topology of 3-manifolds and related topics*, 1962.

[GP74] Victor Guillemin and Allan Pollack. *DIFFERENTIAL TOPOLOGY*. Prentice-HaIl, Inc., 1974.

[Ham82] Richard S. Hamilton. Three-manifolds with positive ricci curvature. *Journal of Differential Geometry*, 17(2):255–306, 1982.

[Hat07] Allen Hatcher. Notes on basic 3-manifold topology, 2007. `http://pi.math.cornell.edu/~hatcher/3M/3Mdownloads.html`.

[Joh62] Milnor John. A unique decomposition theorem for 3-manifolds. *American Journal of Mathematics*, 84(1):1–7, 1962.

[Joh76] Milnor John. Curvatures of left invariant metrics on lie groups. *Advances in Mathematics*, 21:293–329, 1976.

[Joh82] Milnor John. Hyperbolic geometry: The first 150 years. *Bulletin (New Series) of the American Mathematical Society*, 6(1):9–24, 1982.

[JS76] William Jaco and Peter B. Shalen. Seifert fibered spaces in irreducible, sufficiently-large 3-manifolds. *Bulletin of the American Mathematical Society*, 82(5):765–767, 1976.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*, volume I. Interscience Publishers, 1963.

[KN69] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*, volume II. Interscience Publishers, 1969.

[Kne29] Hellmuth Kneser. Geschlossene flächen in dreidimensionalen mannigfaltigkeiten. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 38:248–259, 1929.

[Mag74] Wilhem Magnus. *Noneuclidean Tesselations and their groups*. Academic Press, 1974.

[Mas88] Bernard Maskit. *Kleinian Groups*. Springer-Verlag, second edition, 1988.

[MKS04] W. Magnus, A. Karrass, and D. Solitar. *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations*. Dover books on mathematics. Dover Publications, 2004.

[Mos73] G.D. Mostow. *Strong Rigidity of Locally Symmetric Spaces*. Number 78 in Annals of Mathematics Studies. Princeton University Press., 1973.

[Mun02] James R. Munkres. *Topology*. Prentice Hall, Inc., second edition, 2002.

[Per02] Grigori Perelman. The entropy formula for the ricci flow and its geometric applications. *arXiv:math/0211159*, 2002. `https://arxiv.org/abs/math/0211159`.

[Per03a] Grigori Perelman. Finite extinction time for the solutions to the ricci flow on certain three-manifolds. *arXiv:math/0307245*, 2003. `https://arxiv.org/abs/math/0307245`.

[Per03b] Grigori Perelman. The ricci flow with surgery on three-manifolds. *arXiv:math/0303109*, 2003. `https://arxiv.org/abs/math/0303109`.

[Ril72] Robert Riley. Parabolic representations of knot groups, i. *Proceedings of the London Mathematical Society*, s3-24(2):217–242, 03 1972.

[Ril75] Robert Riley. A quadratic parabolic group. *Mathematical Proceedings of the Cambridge Philosophical Society*, 77(2):281–288, 1975.

[Rol03] Dale Rolfsen. *Knots and Links*. American Mathematical Society, 2003.

[Sco83] P. Scott. The geometries of 3-manifolds. *Bulletin of the London Mathematical Society*, 15:401–487, 1983.

[Thu97] William Thurston. *Three-Dimensional Geometry and Topology*, volume 1. Princeton University Press, 1997.

[Vin69] È. B. Vinberg. Some example of crystallographic groups in LobaČeskiĭ spaces. *Mathematics of the USSR-Sbornik*, 7(4):617–622, 1969.

[Wal68] Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Annals of Mathematics*, 87(1):56–88, 1968.

[War94] W. Frank Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Springer Undergraduate Mathematics Series, 1994.