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ENHANCED NONQUADRATIC STABILIZATION OF DISCRETE-TIME TAKAGI-SUGENO FUZZY MODELS

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RESUMO

O tema principal abordado nesta dissertação diz respeito a estabilização não-quadrática de sistemas não-lineares a tempo discreto descritos por modelos fuzzy Takagi-Sugeno (TS). Uma das principais vantagens ao se usar a representação TS, além de sua capacidade de representar diferentes classes de sistemas não-lineares, é a possibilidade de se obter condições suficientes e convexas, descritas por desigualdades matriciais lineares (LMIs, do inglês Linear Matrix Inequalities). No entanto, um grau de conservadorismo embutido em tais condições está intimamente relacionado à escolha da função de Lyapunov candidata. Dentro do contexto de modelos TS a tempo discreto, condições multi-parametrizadas baseadas em funções de Lyapunov não-quadráticas com atraso têm se mostrado efetivas para redução do conservadorismo para o projeto de controle. Contudo, essa redução é normalmente alcançada ao custo do aumento excessivo da complexidade computacional. Portanto, os métodos propostos nesta dissertação são tais que o conservadorismo das condições de projeto de controladores fuzzy baseados em LMIs é reduzido sem um aumento substancial do custo computacional. As condições são obtidas para projeto de controladores sem atraso e com atraso e são estendidas para tratar o problema de atenuação de distúrbios. A efetividade dos métodos propostos é ilustrada por simulações numéricas.

Palavras-chave: Estabilização não-quadrática. Modelos fuzzy Takagi-Sugeno. Desigualdades matriciais lineares. Funções de Lyapunov não-quadráticas. Controle com atraso.

ABSTRACT

The main topic in this work is concerned to nonquadratic stabilization of discrete-time nonlinear systems described by Takagi-Sugeno (TS) fuzzy models. One of the advantages of employing the TS representation, besides the ability to represent different classes of nonlinear systems, is the possibility to derive sufficient and convex conditions described as Linear Matrix Inequalities (LMIs). However, the conservativeness of such conditions is closely related to the choice of a Lyapunov function candidate. Within the context of discrete-time TS models, multiple-parameterized conditions based on delayed nonquadratic Lyapunov functions have been shown to be effective in reducing control design conservatism. Nevertheless, this reduction is usually achieved at the cost of excessively increasing the computational complexity. Therefore, the methods proposed in this work are such that the conservativeness of LMI-based fuzzy control design conditions is reduced without substantially increasing the computational complexity. The conditions are obtained to design non-delayed and delayed controllers and extended to deal with the disturbance attenuation problem. The effectiveness of the proposed methods is illustrated by numerical simulations.

Keywords: Nonquadratic stabilization. Takagi-Sugeno fuzzy models. Linear matrix inequalities. Nonquadratic Lyapunov function. Delayed control.

LIST OF FIGURES

LIST OF TABLES

LIST OF ABBREVIATIONS

ANS	Asymptotically necessary and sufficient
LMI	Linear matrix inequality
LPV	Linear parameter-varying
MFS	Multidimensional fuzzy summation
PDC	Parallel distributed compensation
TS	Takagi-Sugeno

LIST OF SYMBOLS

$\mathcal{I}_p$	The set $\{1, 2, \ldots, p\}$ for a given natural number p
$\mathbb{I}_p$	The index set $\{\mathbf{i} = (i_i, i_2, \ldots i_p) : i_j \in \mathcal{I}_r, j \in \mathcal{I}_p\}$
$\mathbb{I}_p^+$	The set $\{\mathbf{i} \in \mathbb{I}_p : i_j \leq i_{j+1}, i_j \in \mathcal{I}_r, j \in \mathcal{I}_{p-1}\}$
$\mathscr{P}(\mathbf{i})$	The set of permutations of a multi-index $\mathbf{i} = (i_1, \ldots, i_p) \in \mathbb{I}_p$
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{R}^{m \times n}$	The set of real matrices of order m by n
$x^\top$ or $A^\top$	Transpose of a vector x or a matrix A
$\|x\|$	Euclidean norm of a vector x
$\begin{bmatrix} A & B \\ \star & C \end{bmatrix}$ or $\begin{bmatrix} A & \star \\ B^\top & C \end{bmatrix}$	Shorthand notation for $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$
$P \succ 0$ $(\succeq 0)$	P is a symmetric positive (semi-)definite matrix
I_n	Identity matrix of order n. If the order is clear in the context, it is omitted
0	Null matrix of arbitrary order

CONTENTS

1 INTRODUCTION

It is well known that a number of real-world systems can be modeled by a set of nonlinear differential equations usually derived from physical laws. These are referred to as *nonlinear systems*. Throughout several years, linear control techniques such as pole placement and PID control have been applied to stabilize nonlinear systems in industrial applications. Designers generally adopted linear techniques because of their easy design and long history of successful applications [1]. However, as this class of controllers are frequently designed for linear models derived from the linearization around an operating point of interest, their validity is restricted to a close vicinity around the operating point. Therefore, when state trajectories evolve far from the operating point, control performance can be seriously deteriorated [2]. It can occur when the operating range is large or the control objective is tracking time-varying references. In addition, depending on the kind of nonlinearity present in the model, the linearization procedure required for linear control design may not be applied. This is the case of systems with hard nonlinearities, *e.g.*, saturation, dead-zone, backlash and hysteresis [1].

Aiming to outperform the linear control limitations, nonlinear techniques have been proposed. However, many of them can be difficult to be designed for engineering applications involving complex nonlinear systems. In classical nonlinear techniques, such as feedback linearization, this difficulty is avoided by nonlinearity cancellation, which attempt to impose some predetermined dynamical behavior for the closed-loop system. However, as feedback linearization-based control depends on accurate models, the closed-loop performance is sensitive to the presence of structural uncertainties and external disturbances. Other techniques, like passivity-based control, aim at respecting the system's structure and fully exploit it in the control, but its design is based on the solution of partial differential equations, which can be difficult to solve [3].

Motivated by the proposal of fuzzy logic by [4], a new class of nonlinear controllers was initiated by [5], the so-called Mamdani-type fuzzy control. They are based on a fuzzy inference system constructed as a set of If-Then fuzzy rules whose both antecedent and consequent parts are defined in terms of fuzzy relations of linguistic variables. It allows easily introducing human knowledge on the control strategy, reducing the necessity, or even without requiring, a possibly nonlinear model for the system. Although there are works concerned with proving the stability of nonlinear systems in feedback with Mamdani-type controllers, algorithms of wide applicability are still an open problem [6].

To overcome the drawbacks on stability analysis of Mamdani-type controllers, Takagi-Sugeno (TS) fuzzy models were proposed by [7]. Differently from Mamdani inference systems, in which the consequent part is a relation of fuzzy sets, the consequent of TS models are defined by local functions. Specifically, nonlinear dynamical systems can be represented by TS fuzzy models defining the consequent parts as local linear state-space equations. Then, the overall nonlinear dynamics is inferred by a convex summation of these simpler linear subsystems or local models [8]. The influence of each local model for the current overall inferred nonlinear behavior is weighted by the membership degrees, which assume values within the real interval between 0 and 1. The convexity is ensured thanks to the additional property of the sum of all membership degrees be equal to 1.

By exploiting the convexity of TS models and Lyapunov theory [9], sufficient conditions for stability analysis and control design can be formulated in terms of Linear Matrix Inequalities (LMIs), which can be efficiently solved by existing semidefinite optimization software [10]. Therefore, TS fuzzy model-based control provides an interesting commitment between effectiveness and design complexity [11].

The first approaches on TS model-based control were derived using a common quadratic Lyapunov function and the Parallel Distributed Compensation (PDC) fuzzy control law [12]. In this approach, the stability has to be certified by a unique positive definite quadratic matrix, which composes the Lyapunov function [8]. It is clear that standard quadratic stability can be very conservative, specially in applications involving complex nonlinear systems, since in this case a high number of local models is required for TS modeling. However, the Lyapunov function candidate structure is not the only source of conservatism. The procedure to derive LMI-based conditions from membership-dependent stability/stabilization conditions can also introduce conservatism. Mainly for the continuous-time case, recent works have shown that introducing information from membership functions can lead to less conservative designs [13].

The search for conservativeness reduction of stability/stabilization conditions has motivated a lot of investigation in the TS/LMI framework. In early efforts, additional decision (or slack) variables were introduced to provide new degrees of freedom for the LMI optimization [14, 15, 8, 16]. Later, Asymptotically Necessary and Sufficient (ANS) conditions based on the Pólya's theorem were proposed [17, 18, 19]. The ANS approach is known to provide progressively less conservative LMI conditions. However, both aforementioned approaches can considerably increase the computational complexity possibly leading to numerical intractability.

When it comes to new classes of Lyapunov function candidates, one may cite piecewise [20, 21, 22, 23] and fuzzy ones, or nonquadratic functions, combined with a non-PDC control law. Although the latter class of Lyapunov functions has shown to be effective to reduce design conservatism for continuous-time TS models [24, 25, 26, 27, 28], notable improvements have been achieved in the discrete-time case [29, 30], specially with the multiple-parameterized approach [31, 32], which has been further generalized with the multi-instant approach [33, 34]. Nevertheless, once again, the computational complexity may also quickly increases when multiple-parameterized conditions are regarded.

Recent efforts have been directed to obtain less conservative conditions while avoiding excessive computational burden related to the solution of LMI conditions [35], as occurs when additional decision variables are used, or the degree of fuzzy summations is increased on ANS conditions, or in the multiple-parameterized approach.

Within this context, significant improvements have been obtained with delayed fuzzy controllers and nonquadratic Lyapunov functions. Differently from the conventional PDC and non-PDC fuzzy controllers, in which the control gains are blended by the convex sum of membership functions evaluated at the current sample time, information on past membership functions is introduced in delayed control, establishing new controller design possibilities that reach a wider class of TS systems [36, 37, 38, 33, 34]. The idea of including memory in the control/filtering scheme has also been successfully exploited in the works of [39] and [40].

In particular, [38] proposed general multiple-parameterized conditions based on delayed nonquadratic Lyapunov functions and fuzzy control law. The main difference of this approach with respect to the one in [31] is that it offers a unified framework to design non-delayed and delayed controllers. It is based on the application of the theory of multisets [41] for collecting all delays in the multidimensional fuzzy summation of the membership-dependent design conditions. This allows to easily construct new control laws so that several existing conditions in the literature for both non-delayed [12, 29, 30, 31] and delayed framework [36, 37] can be seen as particular cases of those proposed by [38].

1.1 Objectives and adopted methodology

This work tackles the problem of conservatism reduction of control design conditions for discrete-time TS fuzzy models. The main motivation is to derive less conservative results than those existing in the literature without excessively increasing the computational complexity.

To derive less conservative design conditions, the delayed control law and the two delayed nonquadratic Lyapunov function candidates considered in [38] are regarded in this work. One of these functions is mainly used to design non-delayed controllers whereas the second is employed for delayed control. Similar to the conditions of [38], we also use the theory of multisets to properly represent delays on multidimensional fuzzy summations. It allows to derive conditions which can easily handle both non-delayed and delayed controllers.

It is generally agreed that great improvements in the fuzzy control area were motivated by results on robust control. For instance, the nonquadratic framework proposed by [29] was motivated by the matrix transformation of [42] and the multiple-parameterized approach of [32] was based on the homogeneous polynomially parameter-dependent framework of [43]. In the same line, motivated by the recently appeared conditions of [44, 45] in the context of LPV systems, this work proposes new control design conditions for discrete-time TS fuzzy models.

The new conditions are derived from those of [38] by applying adequate matrix transformations based on the introduction of new decision variables similar to the works of [44, 45]. It is shown that the conditions of [38] are particular cases of those proposed here. As a consequence, our conditions also contain several other existing in the literature.

The proposed stabilization conditions are extended to cope with the disturbance attenuation control problem, which is based on the minimization of the l_2-gain upper-bound. It introduces a control design performance index instead of only finding a stabilizing controller.

1.2 Manuscript outline

This manuscript is organized as follows. Chapter 2 presents the literature review and provides the theoretical background to support our contributions. More specifically, the TS fuzzy model, classical fuzzy control conditions and the main concepts on multidimensional fuzzy summations are provided. The stabilization and l_2-gain performance conditions based on the delayed controllers of [38] are discussed in Chapter 3.

In Chapter 4, the main contributions of this work are presented. Two new stabilization conditions are proposed and extended for l_2-gain performance control. The effectiveness of our proposed conditions is illustrated with numerical simulations on stabilization of two physically-motivated systems recurrent in the fuzzy control literature: the inverted pendulum and truck-trailer systems. Finally, the conclusions of this work as well as future directions and related publications are presented in Chapter 5.

2 UNDERSTANDING MULTIPLE FUZZY SUMMATIONS

This chapter concerns Multidimensional Fuzzy Summations (MFS), a recurrent tool employed by recent works on relaxed stabilization conditions for discrete-time Takagi-Sugeno fuzzy models. The MFS usually arise when either multi-parameterized Lyapunov functions and state feedback fuzzy controllers or the Polya's theorem are considered to derive less conservative control design conditions. As MFS-based conditions depend on the membership functions, it is presented a procedure to rewrite them in terms of a finite set of LMIs, which allows to perform controller design by using convex optimization tools. Nevertheless, obtaining these LMIs constitutes one of the main challenges of this approach since the design conservativeness is closely related to the considered LMI relaxation. The procedure to obtain such LMIs is also discussed in this chapter.

2.1 Conventional control design conditions for TS models

This section describes the Takagi-Sugeno (TS) fuzzy model and introduces the discussion on fuzzy control design. The motivation related to the influence of increasing fuzzy summations on the design is introduced by comparing two well-known fuzzy controllers: the Parallel Distributed Compensation (PDC) and the non-PDC.

2.1.1 Discrete-time Takagi-Sugeno fuzzy models

Consider the discrete-time input-affine nonlinear system

$$x_{k+1} = f(x_k) + g(x_k)u_k, \tag{2.1}$$

where $f : \Omega \rightarrow \mathbb{R}^{n_x}$ and $g : \Omega \rightarrow \mathbb{R}^{n_u}$ are smooth functions in their arguments, $x \in \mathbb{R}^{n_x}$ is the state vector and $u \in \mathbb{R}^{n_u}$ is the input vector. The subspace $\Omega \subset \mathbb{R}^{n_x}$, $0 \in \Omega$, is considered in place of the entirely $\mathbb{R}^{n_x}$ to take into account difference equation solution and/or input constraints.

The most common methods[1] for representing a nonlinear system in the form of (2.1) by a TS fuzzy model are the sector nonlinearity and the linearization approaches [8, 46]. The former offers the possibility to exactly represent nonlinearities within a compact set $\Omega_x \subseteq \Omega$, with $0 \in \Omega_x$, while the latter provides approximate representations. However, for complex nonlinear systems, the number of fuzzy rules obtained from the sector nonlinearity approach can be excessive due to the exponential relation with the number of premise variables. As a consequence, conditions for stability/control design tend to be intractable for a large number of fuzzy rules. On the other hand, the number of fuzzy rules can be reduced with the linearization

[1] Details related to TS model construction methods can be found in [8] and [46].

approach, which results in simpler models. The tensor-product model transformation technique is another approach that can be employed to numerically obtain a TS fuzzy representation for a system. This technique can be applied as an alternative to obtaining TS models with a smaller number of fuzzy rules than the sector nonlinearity approach [47].

In spite of the approach employed to obtain the TS fuzzy model, it is defined by the following set of If–Then fuzzy rules:

$$\textbf{Model rule } i: \quad \begin{array}{l} \text{If } z_{k(1)} \text{ is } \mathcal{M}_1^i \text{ and } \ldots \text{ and } z_{k(n_z)} \text{ is } \mathcal{M}_{n_z}^i \\ \text{Then } x_{k+1} = A_i x_k + B_i u_k \end{array}, \quad i \in \mathcal{I}_r,$$

where $\mathcal{I}_r = \{1, \ldots, r\}$, r is the number of fuzzy model rules whose antecedent is defined by the premise variables $z_{k(j)} \in \mathbb{R}$, $j \in \mathcal{I}_{n_z}$, each one defined within a fuzzy set $\mathcal{M}_j^i$, $j \in \mathcal{I}_{n_z}$. The premise variables are gathered on the vector $z_k \in \mathbb{R}^{n_z}$. The fuzzy rule consequent is a linear state-space model with constant matrices $A_i \in \mathbb{R}^{n_x \times n_x}$ and $B_i \in \mathbb{R}^{n_x \times n_u}$. The consequent of the ith fuzzy rule can be referred as a subsystem or local model.

By employing the center-of-gravity method for defuzzification, the global TS model is inferred as follows:

$$x_{k+1} = \sum_{i=1}^{r} h_i(z_k)\Big(A_i x_k + B_i u_k\Big), \tag{2.2}$$

where

$$h_i(z_k) = \frac{w_i(z_k)}{\sum\limits_{i=1}^{r} w_i(z_k)}, \quad w_i(z_k) = \prod_{j=1}^{n_z} M_j^i(z_{k(j)}), \quad i \in \mathcal{I}_r, \tag{2.3}$$

being $M_j^i(z_j(k)) \in [0, 1]$ the membership degree of $z_{j(k)}$ with respect to $\mathcal{M}_j^i$. The normalized membership functions $h_i(z_k)$ satisfy the convex sum property:

$$\sum_{i=1}^{r} h_i(z_k) = 1 \quad \text{and} \quad h_i(z_k) \geq 0, \quad i \in \mathcal{I}_r. \tag{2.4}$$

Therefore, (2.2) can be viewed as a convex sum of local models weighted by the normalized membership degrees, which in practical aspects corresponds to the blending of simple local linear models that represent the state trajectories in different state-space partitions. Hereafter, the sum of membership degrees will be referred as *fuzzy summation*. Note that the computation of the membership degrees depend on the premise variables measurements at each sample time k. In addition, to apply state-feedback control, the state measurement is also required. For this reason, the following assumption is made.

Assumption 2.1. *The state and premise variable vectors x_k and z_k, respectively, are available for measurement at each sample time k.*

The process of obtaining an exact TS representation for a nonlinear system via the sector nonlinearity approach is illustrated in the next example. This system has been used in several works as a benchmark for comparing different methods in terms of conservatism reduction, for example in [29, 30, 31, 38, 35] and so on. This TS model will be also used on this manuscript for the same objective.

Example 2.1. *Consider the discrete-time nonlinear system*

$$\begin{aligned} x_{k+1(1)} &= x_{k(1)} - x_{k(1)}x_{k(2)} + (5 + x_{k(1)})u_k \\ x_{k+1(2)} &= -x_{k(1)} - 0.5x_{k(2)} + 2x_{k(1)}u_k, \end{aligned} \tag{2.5}$$

where $x_{k(1)} \in [-b, b]$, $b > 0 \in \mathbb{R}$. All nonlinearities of this system are related to the state variable $x_{k(1)}$, which allows one rewrite equations (2.5) as

$$\begin{bmatrix} x_{k+1(1)} \\ x_{k+1(2)} \end{bmatrix} = \begin{bmatrix} 1 & -x_{k(1)} \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} x_{k(1)} \\ x_{k(2)} \end{bmatrix} + \begin{bmatrix} 5 + x_{k(1)} \\ 2x_{k(1)} \end{bmatrix} u(k). \tag{2.6}$$

From the sector nonlinearity approach described in [8, Chapter 2], a TS fuzzy model can be obtained as follows. Define the antecedent variable $z_k = x_{k(1)}$. The parameter b is the maximum value that $x_k(1)$ is allowed to assume within the validity domain

$$\Omega_x = \{x \in \mathbb{R}^2 : |x_{k(1)}| \leq b\}.$$

Then, z_k can be described by the following convex sum:

$$z_k = bM_1^1(z_k) + (-b)M_1^2(z_k), \tag{2.7}$$

being $M_1^i \in [0, 1]$, $i \in \{1, 2\}$, the membership functions satisfying

$$M_1^1(z_k) + M_1^2(z_k) = 1. \tag{2.8}$$

Solving the linear system gathered by (2.7) and (2.8), the obtained membership functions are

$$M_1^1(z_k) = \frac{z_k + b}{2b} \quad \text{and} \quad M_1^2(z_k) = 1 - M_1^1(z_k). \tag{2.9}$$

Accordingly, the TS representation for the nonlinear system (2.5) is defined by the following fuzzy rules:

$$\textbf{Model rule } i: \quad \begin{array}{l} \text{If } z_{k(1)} \text{ is } \mathcal{M}_1^i \\ \text{Then } x_{k+1} = A_i x_k + B_i u_k \end{array}, \quad i \in \{1, 2\}, \tag{2.10}$$

where

$$A_1 = \begin{bmatrix} 1 & -b \\ -1 & -0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 5 + b \\ 2b \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & b \\ -1 & -0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 5 - b \\ -2b \end{bmatrix},$$

and $\mathcal{M}_1^i$, $i \in \{1, 2\}$, are fuzzy sets defined by the membership functions in (2.9).

Until now, we have seen that a nonlinear system in the form (2.1) can be represented as a TS fuzzy model by blending local linear models weighted by a fuzzy summation. However, the main question to be answered here is: *how the number of fuzzy summations grows up?* The first answer is given in the sequel, where a state-feedback fuzzy controller is considered for TS model stabilization.

2.1.2 Quadratic stabilization

The quadratic stabilization was the first proposed approach in the literature to design state-feedback fuzzy controllers for TS models. The aim is to design control gains so that the origin of the closed-loop TS model is asymptotically stable in the sense of Lyapunov (see Appendix A). The methodology to derive control design conditions is based on a common quadratic Lyapunov function and the PDC control law [48, 12]. Similar to the TS model, the PDC control law is defined by fuzzy rules whose consequent parts are local linear state-feedback controllers as follows:

$$\textbf{Control rule } i: \quad \begin{array}{l} \text{If } z_{k(1)} \text{ is } \mathcal{M}_1^i \text{ and } \ldots \text{ and } z_{k(n_z)} \text{ is } \mathcal{M}_{n_z}^i \\ \text{Then } u_k = -F_i x_k \end{array}, \quad i \in \mathcal{I}_r,$$

where $F_i \in \mathbb{R}^{n_u \times n_x}$, $i \in \mathcal{I}_r$, are constant control gains to be designed. Following the inference procedure previously described for the TS model, the global PDC control law is expressed as:

$$u_k = -\sum_{i=1}^{r} h_i(z_k) F_i x_k. \tag{2.11}$$

Note that the PDC control law shares the membership functions with the TS model. Then, information related to the system nonlinearities is introduced into the control scheme. Therefore, the PDC can be viewed as a nonlinear controller.

After feeding back (2.11) into (2.2), the following closed-loop dynamics is obtained:

$$x_{k+1} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z_k) h_j(z_k) \Big(A_i - B_i F_j\Big) x_k. \tag{2.12}$$

Closed-loop stability is based on adequately designing PDC gains such that the origin of (2.12) is asymptotically stable. A sufficient stabilization condition for assuring that is stated in the following theorem. The proof is presented here to illustrate the methodology employed to derive Lyapunov-based design conditions.

Theorem 2.1 (Quadratic stabilization [8])**.** *The origin of the closed-loop system* (2.12) *is asymptotically stable if there exist matrices $X = X^\top \succ 0$ and M_j, $j \in \mathcal{I}_r$, such that*

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z_k) h_j(z_k) \begin{bmatrix} -X & \star \\ A_i X - B_i M_j & -X \end{bmatrix} \prec 0. \tag{2.13}$$

If the above inequality is satisfied, the PDC gains are obtained by $F_j = M_j X^{-1}$, $j \in \mathcal{I}_r$.

Proof. Consider the following quadratic Lyapunov function candidate:

$$V(x_k) = x_k^\top P x_k, \quad P = P^\top \succ 0. \tag{2.14}$$

Taking its difference along trajectories of the closed-loop system (2.12), one has

$$V(x_{k+1}) - V(x_k) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r}\sum_{m=1}^{r} h_i(z_k)h_j(z_k)h_l(z_k)h_m(z_k)x_k^\top \left[\Big(A_i - B_iF_j\Big)^\top P\Big(A_l - B_lF_m\Big)\right] x_k - x_k^\top P x_k.$$

The asymptotic stability in the sense of Lyapunov is assured if

$$x_k^\top \left[\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r}\sum_{m=1}^{r} h_i(z_k)h_j(z_k)h_l(z_k)h_m(z_k)\Big(A_i - B_iF_j\Big)^\top P\Big(A_l - B_lF_m\Big) - P\right] x_k < 0$$

holds. By applying a Schur complement argument, the last inequality is equivalently fulfilled if

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z_k)h_j(z_k)\begin{bmatrix} -P & \star \\ P\Big(A_i - B_iF_j\Big) & -P \end{bmatrix} \prec 0.$$

By defining $X = P^{-1}$, $M_j = F_jX$, and applying a congruence transformation in the above condition with diag(X, X), it results in (2.13). This completes the proof. □

At this point, a suitable design condition has not been found since the negativity of (2.13) depends on the adequate choice of the control gains so that the *double fuzzy summation* be negative. The next lemma depicts LMI-based sufficient conditions to ensure negativeness of a given double fuzzy summation.

Lemma 2.1 (Relaxation of [12]). *The negativity of the double fuzzy summation*

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z_k)h_j(z_k)\Gamma_{(i,j)} \prec 0 \tag{2.15}$$

is fulfilled if the following LMIs hold.

$$\begin{aligned} \Gamma_{(i,i)} &\prec 0, \\ \Gamma_{(i,j)} + \Gamma_{(j,i)} &\prec 0, \quad i < j, \end{aligned} \tag{2.16}$$

for all $i, j \in \mathcal{I}_r$.

Proof. Inequality (2.15) can be written as follows:

$$\sum_{i=1}^{r} h_i^2(z_k)\Gamma_{(i,i)} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} h_i(z_k)h_j(z_k)\left(\Gamma_{(i,j)} + \Gamma_{(j,i)}\right) \prec 0.$$

Therefore, the conditions in (2.16) are sufficient to ensure the negativity of (2.15). □

The procedure to obtain LMI-based conditions to ensure negativity of (2.13) using Lemma 2.1 is illustrated in the next example.

Example 2.2. *From* (2.13), *define*

$$\Gamma_{(i,j)} = \begin{bmatrix} -X & \star \\ A_iX - B_iM_j & -X \end{bmatrix}, \quad i, j \in \mathcal{I}_r,$$

with $M_j = F_jX$. By Lemma 2.1, the negativity of (2.13) *is ensured if* (2.16) *hold. In case of feasibility, the controller gains are obtained by $F_j = M_jX^{-1}$, $j \in \mathcal{I}_r$, and the Lyapunov function matrix by $P = X^{-1}$.*

To illustrate the condition design application, consider the two-ruled TS model (2.10)*. The goal is to obtain the maximum variation for parameter b such that there exists a feasible solution, that is, there exist a state-feedback control guaranteeing the asymptotic stability for the closed-loop TS fuzzy model. The set of LMIs to be solved in this case is*

$$\Gamma_{(1,1)} \prec 0, \qquad \Gamma_{(1,2)} + \Gamma_{(2,1)} \prec 0, \qquad \Gamma_{(2,2)} \prec 0. \tag{2.17}$$

The maximum parameter b obtained solving the LMIs above is $b = 1.36$.

The first answer for the question of how fuzzy summations are increased was given. In this case, a double fuzzy summation was obtained after feeding back the PDC fuzzy controller (2.11) on the TS model (2.2). It is important to note that quadratic stabilization conditions require a common positive definite matrix to be found to ensure negativity of all LMIs in (2.16), which leads to notable conservativeness mainly in the case of designing fuzzy controllers for TS models derived from complex nonlinear systems. An usual approach to reduce such conservatism is by introducing new degrees of freedom on LMIs via slack variables [14, 15, 16]. However, this increases the computational burden. Another way to improve stability/stabilization conditions is based on nonquadratic Lyapunov function candidates, which is discussed in the sequel.

2.1.3 Nonquadratic stabilization

The main source of conservativeness on quadratic stabilization is due to the use of common quadratic Lyapunov functions in the form of (2.14) to derive design conditions. This is mainly because a unique symmetric positive definite matrix P should be assigned in the optimization procedure so that the stability of all closed-loop local models be ensured. Aiming to reduce such conservatism, new classes of Lyapunov functions have been proposed establishing new possibilities for fuzzy control design. Here, our attention will be directed to the following Lyapunov function candidate:

$$V(x_k) = x_k^\top \left(\sum_{i=1}^{r} h_i(z_k) P_i \right) x_k, \tag{2.18}$$

where $P_i = P_i^\top \succ 0$, $i \in \mathcal{I}_r$. This is the so-called fuzzy Lyapunov function, which sometimes is called parameterized or, in a more general nomenclature, *nonquadratic Lyapunov function*. It has been applied in both continuous-time [49, 50, 51, 52, 53, 54] and discrete-time [55, 56, 57] cases. In contrast to (2.14), (2.18) is constructed based on the fuzzy summation of symmetric positive definite matrices P_i, which introduces more degrees of freedom for the solution of LMI-based conditions. Based on the Lyapunov function (2.18), another useful Lyapunov function candidate is the following:

$$V(x_k) = x_k^\top \left(\sum_{i=1}^{r} h_i(z_k) P_i \right)^{-1} x_k, \tag{2.19}$$

also with $P_i = P_i^\top \succ 0$. It was applied, for instance by [58, 29, 25, 59].

The nonquadratic framework is mainly based on the work of [29], where a new fuzzy controller called non-PDC and nonquadratic Lyapunov function candidates were proposed. The non-PDC control law is defined as follows:

$$u_k = -\left(\sum_{i=1}^{r} h_i(z_k) F_i \right) \left(\sum_{i=1}^{r} h_i(z_k) H_i \right)^{-1} x_k, \tag{2.20}$$

where $H_i \in \mathbb{R}^{n_x \times n_x}$, $i \in \mathcal{I}_r$. The main difference between this control law and the PDC are the new degrees of freedom introduced by matrices H_i. For a shorthand notation, the following definitions are considered along this section.

Definition 2.1. *Let X_i, $i \in \mathcal{I}_r$, be constant matrices of arbitrary dimension. Their fuzzy summation at sample time k is denoted as $X_z = \sum_{i=1}^{r} h_i(z_k) X_i$. The fuzzy summation at time $k+1$ is denoted as $X_{z+} = \sum_{i=1}^{r} h_i(z_{k+1}) X_i$.*

After substituting (2.20) into (2.2), the closed-loop system is:

$$x_{k+1} = \left(A_z - B_z F_z H_z^{-1} \right) x_k. \tag{2.21}$$

To derive non-PDC design conditions so that the origin of (2.21) be asymptotically stable, the following nonquadratic Lyapunov function candidate was also considered in the work of [29]:

$$V(x_k) = x_k^\top H_z^{-\top} P_z H_z^{-1} x_k, \tag{2.22}$$

where $P_i = P_i^\top \succ 0 \in \mathbb{R}^{n_x \times n_x}$, $i \in \mathcal{I}_r$, and matrices H_i are the same of the non-PDC controller. The control design condition in this case is stated in the following theorem. Its proof is shown to illustrate the methodology to prove sufficiency of control design conditions.

Theorem 2.2 (Nonquadratic stabilization [29])**.** *If there exist matrices $P_i = P_i^\top \succ 0$, F_i and H_i, $i \in \mathcal{I}_r$, such that*

$$\begin{bmatrix} -P_z & \star \\ A_z H_z - B_z F_z & P_{z+} - H_{z+} - H_{z+}^\top \end{bmatrix} \prec 0 \tag{2.23}$$

holds, the origin of (2.21) *is asymptotically stable.*

Proof. The idea is to prove that if (2.23) is fulfilled, then there exist control gains F_i and H_i, $i \in \mathcal{I}_r$, so that the origin of (2.21) is asymptotically stable. Assuming that (2.23) holds, then

$$H_{z+} + H_{z+}^{\top} \succ P_{z+} \succ 0,$$

which implies that H_{z+} and H_z are invertible matrices. By applying a congruence transformation in the inequality (2.23) multiplying with diag$(H_z^{-\top}, H_{z+}^{-\top})$ on the left and its transpose on the right, one has

$$\begin{bmatrix} -H_z^{-T} P_z H_z^{-1} & \star \\ H_{z+}^{-T}\left(A_z - B_z F_z H_z^{-1}\right) & H_{z+}^{-T} P_{z+} H_{z+}^{-1} - H_{z+}^{-1} - H_{z+}^{-T} \end{bmatrix} \prec 0.$$

Multiplying the last inequality with $[\mathrm{I} \quad A_z - B_z F_z H_z^{-1}]^{\top}$ on the left and its transpose on the right, leads to

$$\left(A_z - B_z F_z H_z^{-1}\right)^{\top} H_{z+}^{-\top} P_{z+} H_{z+}^{-1} \left(A_z - B_z F_z H_z^{-1}\right) - H_z^{-\top} P_z H_z^{-1} \prec 0.$$

By pre and post-multiplying by $x_k^{\top}$ and its transpose, respectively, it implies that

$$V(x_{k+1}) - V(x_k) < 0.$$

Then, the designed control gains ensure the origin of the closed-loop system (2.21) is asymptotically stable. This completes the proof. □

Notice that condition (2.23) involves *three* fuzzy summations. In comparison to (2.13), the number of fuzzy summations was increased and new degrees of freedom introduced by variables P_i and H_i, $i \in \mathcal{I}_r$. However, in the same way, the condition is given in terms of the membership functions, thus requiring a procedure to derive LMI-based conditions. This procedure is shown in the following lemma, which is an extension of the Wang's relaxation in Lemma 2.1.

Lemma 2.2 (see [29]). *The negativity of the triple fuzzy summation*

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(z_k) h_j(z_k) h_l(z_{k+1}) \Gamma_{(i,j,l)} \prec 0 \tag{2.24}$$

is fulfilled if the following LMIs hold.

$$\begin{aligned} \Gamma_{(i,i,l)} &\prec 0, \quad i, l \in \mathcal{I}_r \\ \Gamma_{(i,j,l)} + \Gamma_{(j,i,l)} &\prec 0, \quad i, j > i, l \in \mathcal{I}_r. \end{aligned} \tag{2.25}$$

The conservativeness reduction provided by the nonquadratic framework when compared to the quadratic one is studied in the next example.

Example 2.3. *In this example, we proceed similar to Example 2.2. From* (2.23)*, define*

$$\Gamma_{(i,j,l)} = \begin{bmatrix} -P_j & \star \\ A_i H_j - B_i F_j & -H_l - H_l^\top + P_l \end{bmatrix}, \quad i,j,l \in \mathcal{I}_r,$$

From Lemma 2.2, the negativity of (2.23) *is ensured if* (2.25) *hold. In this case, the set of LMIs to be solved is*

$$\begin{aligned} &\Gamma_{(1,1,1)} \prec 0, \qquad \Gamma_{(1,2,1)} + \Gamma_{(2,1,1)} \prec 0, \qquad \Gamma_{(2,2,1)} \prec 0, \\ &\Gamma_{(1,1,2)} \prec 0, \qquad \Gamma_{(1,2,2)} + \Gamma_{(2,1,2)} \prec 0, \qquad \Gamma_{(2,2,2)} \prec 0. \end{aligned} \tag{2.26}$$

Considering the maximum variation for the parameter b such that there exists a feasible solution for Theorem 2.2, the maximum b obtained solving the above set of LMIs is $b = 1.539$. In comparison to the value of $b = 1.36$ obtained in Example 2.2, clearly the condition based on the nonquadratic framework is less conservative.

This section has presented the definition of TS fuzzy models and a brief review on conventional design conditions of both PDC and non-PDC fuzzy controllers for discrete-time TS fuzzy models. The main feature of TS models is they can represent nonlinear dynamics within a given validity region by a fuzzy summation of linear local models. When the PDC fuzzy controller was introduced, the number of fuzzy summations was increased to 2 and a 3-dimensional fuzzy summation was obtained with the non-PDC and a nonquadratic Lyapunov function. It can be noticed that the number of fuzzy summations is related to the introduction of new degrees of freedom for the LMI-based conditions, which allows reducing conservatism. This fact was illustrated with the presented numerical example.

Broadly speaking, one can expect from the previous analysis that conservativeness can be further reduced if the fuzzy summations dimension are increased even more. This subject is discussed in the next section, where the main approaches to derive multidimensional fuzzy summation based conditions are introduced.

2.2 Reducing conservativeness with multiple fuzzy summation

This section introduces the two main approaches employed to increase fuzzy summation dimension. The first is based on Polya's theorem, which exploits the convexity properties of fuzzy summations while the second is based on defining multiple-parameterized Lyapunov functions and control laws, which naturally conducts to multiple fuzzy summations based conditions. The advantages and disadvantages of each approach are discussed in this section. Before proceeding with this discussion, some useful notations and definitions on multidimensional fuzzy summations and multi-indexes are provided in the sequel.

2.2.1 Exploiting multiple fuzzy summations and multi-indexes

As a generalization for condition (2.15), the negativity of a p-dimensional fuzzy summation, *i.e*, a Multidimensional Fuzzy Summation (MFS), can be expressed as follows

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\dots\sum_{i_p=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)\dots h_{i_p}(z_k)\Gamma_{(i_1,i_2,\dots,i_p)} \prec 0. \tag{2.27}$$

Note that the particular case with $p = 2$ reduces condition (2.27) to (2.15). Aiming to improve notation on MFS, we adopt the multi-index notation previously presented in [17, 18].

Definition 2.2 (Index set and multi-indexes)**.** *The index set contain all p-dimensional indexes, or multi-indexes, and is defined as:*

$$\mathbb{I}_p = \{\mathbf{i} = (i_i, i_2, \dots i_p) : i_j \in \mathcal{I}_r, j \in \mathcal{I}_p\}. \tag{2.28}$$

A p-dimensional index $\mathbf{i} = (i_i, i_2 \dots, i_p)$ is generically called multi-index.

Then, a MFS can be written in terms of multi-indexes as shown in Definition 2.3.

Definition 2.3. *The MFS of matrices $\Gamma_{(i_1,i_2,\dots i_p)}$ can be defined in terms of multi-indexes as:*

$$\begin{aligned}\sum_{\mathbf{i}\in\mathbb{I}_p} h_{\mathbf{i}}(z_k)\Gamma_{\mathbf{i}} &= \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\dots\sum_{i_p=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)\dots h_{i_p}(z_k)\Gamma_{(i_1,i_2,\dots,i_p)}\\ &= \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\dots\sum_{i_p=1}^{r}\prod_{j=1}^{p} h_{i_j}(z_k)\Gamma_{(i_1,i_2,\dots,i_p)}.\end{aligned} \tag{2.29}$$

As has been discussed along this chapter, the main task on fuzzy controller design is obtaining LMI-based conditions to ensure negativity of fuzzy summations. For the sake of motivation, consider the following negativity condition of a 2-dimensional fuzzy summation expanded for $r = 2$:

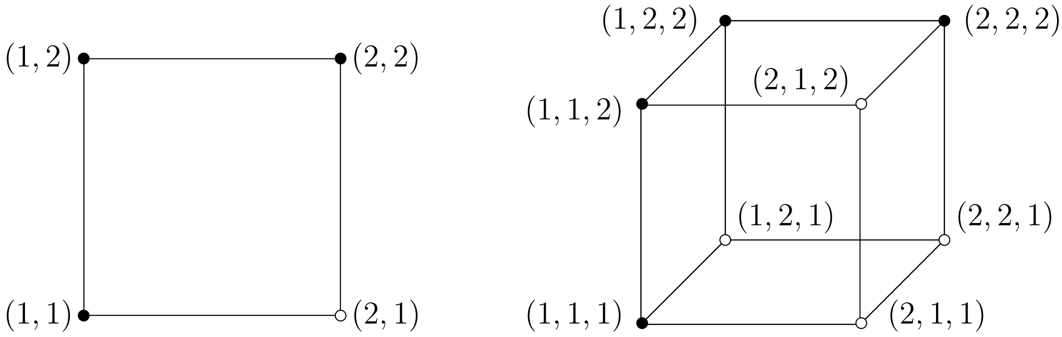
$$h_1^2(z_k)\Gamma_{(1,1)} + h_1(z_k)h_2(z_k)\left(\Gamma_{(1,2)} + \Gamma_{(2,1)}\right) + h_2^2(z_k)\Gamma_{(2,2)} \prec 0. \tag{2.30}$$

The indexes in the above summation can be viewed as vertices coordinates of a square, as illustrated in Figure 2.1(a). The vertices in **black** are called *upper-triangle* indexes. Notice the LMIs in (2.17), used to ensure negativity of (2.30), are composed by the upper-triangle indexes and its permutations.

Now, consider the following expansion for $r = 2$ of the negativity of a 3-dimensional fuzzy summation:

$$\begin{aligned}&\sum_{i_1=1}^{2}\sum_{i_1=1}^{2}\sum_{i_3=1}^{2} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_k)\Gamma_{(i_1,i_2,i_3)} =\\ &h_1^3(z_k)\Gamma_{(1,1,1)} + h_1^2(z_k)h_2(z_k)\Big(\Gamma_{(1,1,2)} + \Gamma_{(1,2,1)} + \Gamma_{(2,1,1)}\Big)\\ &\quad + h_1(z_k)h_2^2(z_k)\Big(\Gamma_{(1,2,2)} + \Gamma_{(2,1,2)} + \Gamma_{(2,2,1)}\Big) + h_2^3(z_k)\Gamma_{(2,2,2)} \prec 0.\end{aligned} \tag{2.31}$$

A similar geometric interpretation can be given looking for the indexes as vertices of a cube, as shown in Figure 2.1(b). Again, LMIs to ensure negativity of (2.31) can be obtained using only the upper-triangle indexes and its permutations. Of course, this geometric interpretation is lost for higher dimensional summations.



(a) Indexes for the 2-dimensional fuzzy summation (2.15).

(b) Indexes for the 3-dimensional fuzzy summation (2.31).

Figure 2.1 – Geometric interpretation for the indexes on fuzzy summations. The vertices in **black** represent the upper-triangle indexes.

Motivated by the aforementioned discussion, the set of upper-triangle indexes is mathematically formulated in Definition 2.4. This notion will be useful to derive LMI-based conditions to ensure negativity of MFS-dependent design conditions.

Definition 2.4 (Upper-triangle index set)**.** *The set of p-dimensional upper-triangle indexes is defined as:*

$$\mathbb{I}_p^+ = \{\mathbf{i} \in \mathbb{I}_p : i_j \leq i_{j+1}, i_j \in \mathcal{I}_r, j \in \mathcal{I}_{p-1}\}. \tag{2.32}$$

It is worth to mention that $\mathbb{I}_p^+ \subset \mathbb{I}_p$.

Both fuzzy summations (2.15) and (2.31) depend only on membership degrees at the current time sample k, which is different to (2.24) that depends on both the current and the future time sample $k+1$. The above definitions of index sets and upper-triangle indexes are valid only for the case of MFS whose membership degrees are in the same sample time. Then, different index sets should be assigned to represent membership degrees evaluated at other sample times.

Moreover, as LMI-based conditions to ensure negativity of fuzzy summations can be obtained by the upper-triangle indexes and its permutations, it is useful to consider the following set of index permutations.

Definition 2.5 (Set of index permutations)**.** *Given a multi-index $\mathbf{i} \in \mathbb{I}_p$, its set of permutations is denoted by $\mathscr{P}(\mathbf{i}) \subset \mathbb{I}_p$. This definition can be directly extended for a n-tuple of multi-indexes $(\mathbf{i}_{p_1}, \ldots, \mathbf{i}_{p_n}) \in \mathbb{I}_{p_1} \times \ldots \times \mathbb{I}_{p_n}$, which is denoted by $\mathscr{P}(\mathbf{i}_{p_1}, \ldots, \mathbf{i}_{p_n})$.*

Example 2.4. *Consider the multi-index* $\mathbf{i} = (1, 1, 2)$*. The related set of permutations is* $\mathscr{P}(\mathbf{i}) = \{(1, 1, 2), (1, 2, 1), (2, 1, 1)\}$*. Moreover, consider the* 2*-tuple composed by the multi-indexes* $\mathbf{i} = (1, 2)$ *and* $j = 1$*. The permutation set is* $(\mathbf{i}, j)$ *is* $\mathscr{P}(\mathbf{i}, j) = \{(1, 2, 1), (2, 1, 1)\}$*.*

Based on the introduced definitions, LMI-based condition to ensure negativity of a MFS whose membership degrees are dependent on both actual and future time sample $k + 1$ is stated in the next lemma.

Lemma 2.3 (see [31]). *Consider the* n*-dimensional fuzzy summation*

$$\sum_{\mathbf{i}\in\mathbb{I}_p} \sum_{\mathbf{j}\in\mathbb{I}_{n-p}} h_{\mathbf{i}}(z_k) h_{\mathbf{j}}(z_{k+1}) \Gamma_{(\mathbf{i},\mathbf{j})} = \sum_{\mathbf{k}\in\mathbb{I}_p^+} \sum_{\mathbf{l}\in\mathbb{I}_{n-p}^+} \Xi_{(\mathbf{i},\mathbf{j})}, \tag{2.33}$$

where $\Xi_{(\mathbf{i},\mathbf{j})} = \sum_{\mathbf{i}\in\mathscr{P}(\mathbf{k})} \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{l})} \Gamma_{(\mathbf{i},\mathbf{j})}$*. Its negativity is ensured if*

$$\Xi_{(\mathbf{i},\mathbf{j})} \prec 0,$$

for all $\mathbf{i} \in \mathbb{I}_p^+$*,* $\mathbf{j} \in \mathbb{I}_{n-p}^+$*.*

Note that Lemmas 2.1 and 2.2 are particular cases of the last one. They are recovered by setting, respectively, $(n, p) = (3, 2)$ and $(n, p) = (2, 0)$.

2.2.2 Dimension expansion via Polya's theorem

In the last decade, several research efforts were made to reduce conservatism of LMI-based conditions to design PDC controllers within the quadratic framework [8, 14, 15, 16]. In summary, less conservative conditions to check negativity of fuzzy summations for assuring asymptotic stability of the closed-loop fuzzy system were obtained by introducing extra slack variables to the optimization problem. However, these conditions were only sufficient, which means that their feasibility still remained subject to the system to be stabilized. This limitation motivated the investigation of sufficient and necessary conditions.

The problem of finding sufficient and necessary conditions for that purpose was addressed later in the works of [60, 17] by applying Polya's Theorem, which provide progressively less conservative sufficient conditions obtained by increasing a complexity parameter namely m related to the fuzzy summation dimension. This strategy is based on the evident fact that [17]:

$$\sum_{i=1}^{r} h_i(z_k) = \left(\sum_{i=1}^{r} h_i(z_k)\right)^m = \sum_{\mathbf{i}\in\mathbb{I}_m} h_{\mathbf{i}}(z_k) = 1, \quad \forall m \in \mathbb{Z}_{\geq 0}, \tag{2.34}$$

where $\mathbb{Z}_{\geq 0}$ is the set of non-negative integers. The above equality is a direct consequence of the convexity properties in (2.4). To exploit the idea, consider the fuzzy summation in (2.15) equivalently rewritten as follows:

$$\left(\sum_{i=1}^{r} h_i(z_k)\right)^{m-2} \left(\sum_{i_1=1}^{r} \sum_{i_2=1}^{r} h_{i_1}(z_k) h_{i_2}(z_k) \Gamma_{(i_1,i_2)}\right) = \sum_{\mathbf{i}\in\mathbb{I}_m} h_{\mathbf{i}}(z_k) \Gamma_{(i_1,i_2)} \prec 0, \tag{2.35}$$

where $m \geq 2 \in \mathbb{Z}_{\geq 0}$. Then, by increasing m, a 2-dimensional fuzzy summation can be expanded to any desired dimension. Following similar arguments as those in Lemma 2.3, we have the following sufficient condition to ensure negativity of (2.35):

$$\Xi_{\mathbf{i}} = \sum_{\mathbf{j} \in \mathscr{P}(\mathbf{i})} h_{\mathbf{j}}(z_k)\Gamma_{(j_1,j_2)} \prec 0, \quad \forall \mathbf{i} \in \mathbb{I}_m^+. \tag{2.36}$$

The condition (2.36) is less conservative as n increases. Then, for a large enough dimensionality expansion ($m \to \infty$), condition (2.36) tend to become equivalent to (2.35) [17]. Accordingly, the following theorem can be stated.

Theorem 2.3 (see [17]). *Given matrices $\Gamma_{(i_1,i_2)}$ satisfying* (2.15)*, there exists a finite m so that* (2.36) *holds, i.e.,* (2.36) *becomes necessary and sufficient for some finite m.*

The application of Theorem 2.3 is illustrated in the following example.

Example 2.5. *To illustrate the application of Theorem 2.3, consider the dimensionality expansion for $m = 3$:*

$$h_1^3(z_k)\Xi_{(1,1,1)} + h_1^2(z_k)h_2(z_k)\Xi_{(1,1,2)} + h_1(z_k)h_2^2(z_k)\Xi_{(1,2,2)} + h_2^3(z_k)\Xi_{(2,2,2)} \prec 0,$$

where

$$\Xi_{(1,1,1)} = \Gamma_{(1,1)}, \qquad \Xi_{(1,1,2)} = \Gamma_{(1,1)} + \Gamma_{(1,2)} + \Gamma_{(2,1)},$$
$$\Xi_{(2,2,2)} = \Gamma_{(2,2)}, \qquad \Xi_{(1,2,2)} = \Gamma_{(2,2)} + \Gamma_{(1,2)} + \Gamma_{(2,1)}.$$

In this case, the negativity is ensured solving the following set of LMIs:

$$\Xi_{(1,1,1)} \prec 0, \quad \Xi_{(1,1,2)} \prec 0, \quad \Xi_{(1,2,2)} \prec 0, \quad \Xi_{(2,2,2)} \prec 0. \tag{2.37}$$

From conditions $\Xi_{(1,1,1)} \prec 0$ and $\Xi_{(2,2,2)} \prec 0$, it follows that $\Gamma_{(1,1)} \prec 0$ and $\Gamma_{(2,2)} \prec 0$. Therefore, introducing these terms in $\Xi_{(1,1,2)}$ and $\Xi_{(1,2,2)}$ allows to relax the inequalities $\Xi_{(1,1,2)} \prec 0$ and $\Xi_{(1,2,2)} \prec 0$. For this reason, the above LMIs are less conservative than those in (2.17).

In Example 2.2, the maximum b for feasibility of condition (2.17) *was $b = 1.36$. It corresponds to the particular case of $m = 2$. To evaluate the conservatism reduction achieved with the Polya's theorem application, we compute the maximum parameter b for different values of m and the related number of solved LMIs. The results are summarized in Table 2.1.*

Table 2.1 – Comparison among maximum b for LMI feasibility obtained with different values of m in condition (2.36).

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$
b	1.360	1.618	1.624	1.671	1.691	1.698	1.713	1.716
LMIs	3	4	5	6	7	8	9	10

From the results in Table 2.1, it is clear that conservativeness can be progressively reduced when parameter m is increased. However, this gain is achieved at the cost of increasing the number of LMIs to be solved.

2.2.3 Multiple-parameterized approach

Although ANS conditions proposed by [60, 17] provided reduction of conservativeness associated with the quadratic framework, these approaches are also based on a common quadratic Lyapunov function as (2.14). This also conducts to a certain conservatism, which can be reduced using nonquadratic, or *parameter-dependent*, Lyapunov functions.

This motivated a generalization for the nonquadratic framework discussed in Section 2.1.3, the so-called Homogeneous Polynomially Nonquadratic approach [32, 31]. It is based on generalizations for both non-PDC controller (2.20) and nonquadratic Lyapunov functions. In spite of similarities between the works of [32] and [31], the main difference between them is that the former adopted homogeneous polynomials notation and the latter multi-index notation. In addition, both works considered dimensionality expansion via Polya's theorem.

Following the multi-index notation, the multiple-parameterized non-PDC control law is given by:

$$u_k = -\left(\sum_{\mathbf{i}\in\mathbb{I}_l} h_{\mathbf{i}}(z_k)F_{\mathbf{i}}\right)\left(\sum_{\mathbf{i}\in\mathbb{I}_p} h_{\mathbf{i}}(z_k)H_{\mathbf{i}}\right)^{-1} x_k. \tag{2.38}$$

The classical non-PDC control law can be recovered simply choosing $l = p = 1$.

The Lyapunov function candidate considered in [32] generalizes (2.19) as:

$$V(x_k) = x_k^\top \left(\sum_{\mathbf{i}\in\mathbb{I}_q} h_{\mathbf{i}}(z_k)P_{\mathbf{i}}\right)^{-1} x_k \tag{2.39}$$

and the one of [31] generalizes (2.22) as:

$$V(x_k) = x_k^\top \left(\sum_{\mathbf{i}\in\mathbb{I}_p} h_{\mathbf{i}}(z_k)H_{\mathbf{i}}\right)^{-\top}\left(\sum_{\mathbf{i}\in\mathbb{I}_q} h_{\mathbf{i}}(z_k)P_{\mathbf{i}}\right)\left(\sum_{\mathbf{i}\in\mathbb{I}_p} h_{\mathbf{i}}(z_k)H_{\mathbf{i}}\right)^{-1} x_k. \tag{2.40}$$

The method of [32] is not discussed in this section. Here, we focus on the condition of [31] since it is a direct generalization of Theorem 2.2. The condition of [31] without slack variables and extra dimensionality expansion is stated in the next theorem. Note that LMI-based conditions to ensure (2.41) can be directly derived from Lemma 2.3.

Theorem 2.4 (adapted from [31])**.** *Given dimensions* (l, p, q)*, let* $\alpha_0 := \max(l+1, p+1, q)$ *and* $\alpha_1 := \max(p, 1)$*. If there exist matrices* $P_{\mathbf{i}} = P_{\mathbf{i}}^\top \succ 0$, $\mathbf{i} \in \mathbb{I}_q$, $F_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{I}_l$*, and* $H_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{I}_p$*,* *such that*

$$\sum_{\mathbf{i}\in\mathbb{I}_{\alpha_0}}\sum_{\mathbf{j}\in\mathbb{I}_{\alpha_1}} h_{\mathbf{i}}(z_k)h_{\mathbf{j}}(z_{k+1})$$
$$\begin{bmatrix} -P_{(i_1,\dots,i_q)} & \star \\ A_{i_1}H_{(i_2,\dots,i_{p+1})} - B_{i_1}F_{(i_2,\dots,i_{l+1})} & P_{(j_1,\dots,j_q)} - H_{(j_1,\dots,j_p)} - H^{\top}_{(j_1,\dots,j_p)} \end{bmatrix} \prec 0 \tag{2.41}$$

holds, then the origin of (2.21) *is asymptotically stable.*

Proof. The proof is omitted since it follows similar steps as Theorem 2.2. □

The application of the last condition is shown in the next example.

Example 2.6. *In this example, we investigate the relation of parameters (l, p, q) with the conservatism reduction provided by Theorem 2.4. For that, consider the TS model* (2.5)*. We want to find the maximum b for feasibility. For simplicity, it is assumed $p = l$. The results for different choices of p and q are depicted in Table 2.2.*

Table 2.2 – Comparison among maximum b for LMI feasibility obtained with different values of (q, p), $p = l$, in Theorem 2.4.

	$p=l=1$	$p=l=2$	$p=l=3$	$p=l=4$	$p=l=5$
$q=1$	1.539	1.677	1.725	1.752	1.767
$q=2$	1.547	1.693	1.733	1.754	1.770
$q=3$	1.665	1.693	1.735	1.754	1.771
$q=4$	1.684	1.719	1.736	1.754	1.771
$q=5$	1.708	1.738	1.749	1.762	1.771

The maximum value of b, 1.771, is obtained with $p = l = 5$ and $q = 3$. Even increasing q from 3 to 5, the value of b is maintained, indicating that there is a kind of limit for which conservativeness can be reduced. However, this value is still greater than the largest in Table 2.1 obtained with dimensionality expansion via Polya's Theorem. It illustrates that conservativeness could be further reduced with the multiple-parameterized nonquadratic Lyapunov function and control law.

Remark 2.1. *ANS conditions based on multiple-parameterized nonquadratic Lyapunov functions and non-PDC control law were derived by [18] for discrete-time TS fuzzy models, and by [19] for continuous-time TS models. In comparison to the ANS condition presented in Section 2.2.2, in place of only multiplying the negativity condition by $\left(\sum_{i=1}^{r} h_i(z_k)\right)^{m_0}$, a term of the form*

$$\left(\sum_{i=1}^{r} h_i(z_k)\right)^{m_0}\left(\sum_{i=1}^{r} h_i(z_{k+1})\right)^{m_1}$$

was introduced. In the discrete-time case, it conducts to $\alpha_0 = \max(l+1, p+1, q) + m_0$ and $\alpha_1 = \max(p, q) + m_1$ in Theorem 2.4. In this case, both fuzzy summation dimensions in the current and future sample time $k+1$ can be increased.

This section has presented the main MFS-based approaches to provide less conservative control design conditions for discrete-time TS fuzzy models. As discussed, by expanding MFS dimension, the number of LMIs is increased and, consequently, the related computational burden to solve them. This conducts to a new question: *how to obtain less conservative design conditions without excessively increase the computational complexity?* One answer for this question will be given in Chapter 3, where a new approach based on the use of fuzzy controllers and Lyapunov functions with *delayed* membership functions is introduced.

2.3 Conclusion

This chapter has discussed the dimensionality expansion of fuzzy summation-based conditions. It was shown that nonlinear systems can be described by TS fuzzy models by means of a fuzzy summation of local linear models. When a PDC control was used, the resulting closed-loop system was represented by a 2-fuzzy summation, one due to the model and other to the PDC control law. After this, it was shown that the number of fuzzy summations was increased to 3 when the non-PDC controller was used. As illustrated by a numerical example, the non-PDC conditions provided less conservative outcomes than the PDC condition, which indicate the possibility to reduce design conservativeness as the fuzzy dimension is increased. This fact was put in evidence with application of Polya's theorem and multiple-parameterized approach, where numerical simulations illustrated that conservativeness could be further reduced by increasing the fuzzy summation dimension. These multidimensional fuzzy conditions constitute the foundation of the recent fuzzy control design conditions.

3 DELAYED CONTROL OF DISCRETE-TIME TS MODELS

This chapter presents recent control design conditions for stabilization of discrete-time TS fuzzy models. They are based on the multiple-parameterized approach. However, differently from conditions in Chapter 2, here, an MFS can depend on delayed membership functions. The theory of multisets is used to represent the multiple delays present in a general MFS. Based on that, design conditions for both non-delayed and delayed controllers can be easily designed. As far as multisets are used, some definitions and notations on this topic are provided in the beginning of this chapter. In addition, the extension of the stabilization conditions to deal with the disturbance attenuation problem is also addressed here.

3.1 Multiple fuzzy summations: a multiset point of view

This section introduces useful definitions, notations and operations related to multisets. A multiset (mset) is an unordered collection of elements that may appear repeated times [41]. The concept of msets can be understood as a generalization for standard sets, in which elements are allowed to appear only one time. A general definition for msets is given as follows.

Definition 3.1 (Multisets, see [41])**.** *Let $D = \{d_1, d_2, \dots, d_p\}$ be a set. An mset G_D over D is a cardinal-valued function $G_D : D \mapsto \mathbb{N}$ such that $d_j \in D$ implies a cardinal $\mathbf{1}_{G_D}(d_j) > 0$. The value $\mathbf{1}_{G_D}(d_j)$ denotes the number of times that d_j occurs in G_D. Here, msets will be represented by the set of pairs as follows:*

$$G_D = \{\langle \mathbf{1}_{G_D}(d_1), d_1\rangle, \dots, \langle \mathbf{1}_{G_D}(d_p), d_p\rangle\}.$$

Each pair $\langle \mathbf{1}_{G_D}(d_j), d_j\rangle$ is defined by the multiplicity $\mathbf{1}_{G_D}(d_j)$ and the corresponding d_j.

Although msets have been mainly applied in the areas of mathematics and computer science [41], recently, this representation has been shown to be useful in engineering applications. More specifically, within the context of fuzzy control, msets have been used to collect arbitrary delays present in the membership functions of MFS. This notion was initially proposed by [38] and later adopted in the works of [26, 27] and [61]. The representation of general MFS in terms of msets is given as follows.

Definition 3.2. *Consider an n_P-dimensional MFS evaluated at the sample time k:*

$$\mathbb{P}_{G_0^P} = \sum_{i_1=1}^{r} h_{i_1}(z_{k+d_1}) \dots \sum_{i_{n_p}=1}^{r} h_{i_{n_p}}(z_{k+d_{n_p}}) P_{(i_1,\dots,i_{n_P})}.$$

The delays present in this MFS of matrices $P_{(i_1,\dots,i_{n_P})}$ are collected in the mset

$$G_0^P = \{d_1, \dots, d_{n_P}\},$$

where the subscript "0" denotes that the MFS has been evaluated at the current sample time k, the superscript, "P" in this case, corresponds to the matrices of the MFS and $d_j \in \mathbb{Z}$, $j \in \mathcal{I}_{n_P}$, are the delays in the premise variable of each sum.

If the MFS is evaluated at the sample time $k+T$, $T \in \mathbb{Z}$, the mset of delays is denoted $G_T^P = \{d_1+T, \ldots, d_{n_P}+T\}$ and the corresponding MFS as $\mathbb{P}_{G_T^P}$.

Differently from the MFS in (2.29), here, each membership function can be computed in terms of a *delayed* premise variable evaluated at $k+d_i$, *i.e.*, z_{k+d_i}. Therefore, the above definition is more general than Definition 2.3, since it allows representing membership functions with arbitrary delays in a more compact and elegant way. To further exploit this idea, the following definitions on msets adopted from [41] and [38] are presented.

Definition 3.3 (Cardinality of msets)**.** *The cardinality of the mset G_d, denoted $|G_d|$, is the total number of possibly repeated elements in G_D. It is computed as*

$$|G_D| = \sum_{j=1}^{p} \mathbf{1}_{G_D}(d_j).$$

Definition 3.4 (Operations on msets)**.** *The main operations on msets are defined as follows:*

a) The union of two msets G_A and G_B is the mset

$$G_A \cup G_B = \{d \in G_A \cup G_B : \mathbf{1}_{G_A \cup G_B}(d) = \max\{\mathbf{1}_{G_A}(d), \mathbf{1}_{G_B}(d)\}\}.$$

b) The intersection of two msets G_A and G_B is the mset

$$G_A \cap G_B = \{d \in G_A \cap G_B : \mathbf{1}_{G_A \cap G_B}(d) = \min\{\mathbf{1}_{G_A}(d), \mathbf{1}_{G_B}(d)\}\}.$$

c) The sum of two msets G_A and G_B is the mset

$$G_A \oplus G_B = \{d \in G_A \oplus G_B : \mathbf{1}_{G_A \oplus G_B}(d) = \mathbf{1}_{G_A}(d) + \mathbf{1}_{G_B}(d)\}.$$

In addition, aiming to link the notations on msets introduced in this section and those of multi-indexes presented in Chapter 2, the following definitions are considered.

Definition 3.5 (Index set)**.** *The index set related to the MFS $\mathbb{P}_{G_0^P}$ is defined as:*

$$\mathbb{I}_{G_0^P} = \{(i_1, \ldots, i_{n_P}) : i_j \in \mathcal{I}_r,\ j \in \mathcal{I}_{n_p}\},$$

where $|G_0^P| = n_p$. It contains all indexes that appear in the MFS.

Definition 3.6 (Projection of a multi-index to msets)**.** *The projection of a multi-index $\mathbf{i} \in \mathbb{I}_{G_A}$ to the mset G_B, denoted $\mathrm{pr}_{G_B}^{\mathbf{i}}$, is the part of $\mathbf{i}$ that corresponds to the delays in $G_A \cap G_B$.*

The following example illustrates the definitions and operations related to msets.

Example 3.1. *Consider the MFS with mset of delays $G_0^P = \{\langle 2, 0\rangle, -1, -2\}$:*

$$\mathbb{P}_{G_0^P} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r}\sum_{i_4=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_{k-1})h_{i_4}(z_{k-2})P_{(i_1,i_2,i_3,i_4)}.$$

At the sample time $k+T$, the multiset of delays is $G_T^P = \{\langle 2, T\rangle, T-1, T-2\}$. The multiplicity of the elements of G_0^P are $\mathbf{1}_{G_0^P}(0) = 2$, $\mathbf{1}_{G_0^P}(-1) = 1$ and $\mathbf{1}_{G_0^P}(-2) = 1$, and the cardinality of G_0^P is $|G_0^P| = 4$.

To illustrate the operations on msets, consider also $G_0^H = \{\langle 2, 0\rangle, \langle 2, -1\rangle\}$. The union of these msets is $G_0^P \cup G_0^H = \{\langle 2, 0\rangle, \langle 2, -1\rangle, -2\}$, the intersection is $G_0^P \cap G_0^H = \{\langle 2, 0\rangle, -1\}$ and the sum $G_0^P \oplus G_0^H = \{\langle 4, 0\rangle, \langle 3, -1\rangle, -2\}$.

The projection of the multi-index $\mathbf{i} = (1, 2, 3, 4)$, $\mathbf{i} \in \mathbb{I}_{G_0^P}$, to the multiset of delays $G_c = \{-1, -2\}$ is $\text{pr}^{\mathbf{i}}_{G_C} = (3, 4)$. Note that the projection of a multi-index may be not unique, for example, the projection of $\mathbf{i} = (1, 2, 3, 4) \in \mathbb{I}_{G_0^P}$ to $G_D = \{0, -1\}$ is either $\text{pr}^{\mathbf{i}}_{G_D} = (1, 3)$ or $\text{pr}^{\mathbf{i}}_{G_C} = (2, 3)$.

Remark 3.1. *The TS fuzzy model* (2.2) *can be rewritten using the notation on msets as follows:*

$$x_{k+1} = \mathbb{A}_{G_0^A} x_k + \mathbb{B}_{G_0^B} u_k, \tag{3.1}$$

where $G_0^A = G_0^B = \{0\}$.

3.2 Reducing conservativeness with delayed control

In this section, two sets of fuzzy control design conditions proposed by [38] are presented. Firstly, the multiple-parameterized framework discussed in Section 2.2.3 is generalized by allowing arbitrary delays. In the sequel, conditions for delayed control design are presented. Finally, a numerical example is concerned to illustrate the conservatism reduction acquired with the use of delayed conditions.

Here, the following control law is considered [38]:

$$u_k = -\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H} x_k, \tag{3.2}$$

where $\mathbb{F}_{G_0^F}$ and $\mathbb{H}_{G_0^H}$ are MFS of matrices $F_{\mathbf{i}} \in \mathbb{R}^{n_u \times n_x}$, $\mathbf{i} \in \mathbb{I}_{G_0^F}$, and $H_{\mathbf{i}} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{i} \in \mathbb{I}_{G_0^H}$, respectively. This control law generalizes the conventional fuzzy controllers presented in Chapter 2. For instance, the PDC control law in (2.11) can be recovered choosing $G_0^F = \{0\}$ and $G_0^H = \{\emptyset\}$, and the non-PDC in (2.20) by $G_0^F = \{0\}$ and $G_0^H = \{0\}$.

After substituting the control law (3.2) into system (3.1), the following closed-loop system is obtained:

$$x_{k+1} = \left(\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H}\right) x_k. \tag{3.3}$$

Similar to Chapter 2, the control goal here is to design gains $F_{\mathbf{i}}$ and $H_{\mathbf{i}}$ such that the origin of the closed-loop system (3.3) be asymptotically stable in the sense of Lyapunov.

Remark 3.2. *The msets G_0^F and G_0^H must not contain positive delays, since incorporating future premise variables would conduct a non-causal closed-loop dynamics.*

3.2.1 Generalized control design conditions

Here, the two design conditions proposed by [38] are presented. Each one is derived with a different nonquadratic Lyapunov function candidate. Although both non-delayed and delayed controllers can be designed with these conditions, the former is mainly employed for non-delayed control design while the second stands for delayed control design. It will be shown that designing these classes of controllers with each of these conditions allows reducing the number of required LMIs for design.

Non-delayed conditions

The design conditions of non-delayed controllers is based on the following Lyapunov function candidate:

$$V_1(x_k) = x_k^\top \mathbb{H}_{G_0^H}^{-\top} \mathbb{P}_{G_0^P} \mathbb{H}_{G_0^H}^{-1} x_k, \tag{3.4}$$

where $P_{\mathbf{i}} = P_{\mathbf{i}}^\top \succ 0 \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{i} \in \mathbb{I}_{G_0^P}$, and $\mathbb{H}_{G_0^H}$ being the same MFS in (3.3). The control design obtained with (3.4) is stated in the following theorem. It generalizes Theorem 2.4 in the sense that arbitrary delays can be regarded.

Theorem 3.1 (see [38]). *Given $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_1^H$, the origin of the closed-loop system* (3.3) *is asymptotically stable if there exist matrices $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}^{\mathbf{i}}_{G_j^P}$, and $H_{\mathbf{i}_j^H}$, $\mathbf{i}_j^H = \mathrm{pr}^{\mathbf{i}}_{G_0^H}$, $j = 0, 1$, and $F_{\mathbf{i}_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}^{\mathbf{i}}_{G_0^F}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, such that*

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{H}_{G_1^H}^\top - \mathbb{H}_{G_1^H} + \mathbb{P}_{G_1^P} \end{bmatrix} \prec 0. \tag{3.5}$$

Proof. Consider the Lyapunov function candidate (3.4). Taking its difference along trajectories of the closed-loop system (3.3), results in:

$$V_1(x_{k+1}) - V_1(x_k) = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}^\top \begin{bmatrix} -\mathbb{H}_{G_0^H}^{-\top}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & -\mathbb{H}_{G_1^H}^{-\top}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}.$$

In addition, the closed-loop system (3.3) can be rewritten as:

$$\begin{bmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -\mathrm{I} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = 0. \tag{3.6}$$

Using the Finsler's Lemma, condition $V_1(x_{k+1}) - V_1(x_k) < 0$ is equivalent to

$$\begin{bmatrix} -\mathbb{H}_{G_0^H}^{-\top}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & -\mathbb{H}_{G_1^H}^{-\top}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbb{H}_{G_1^H}^{-\top} \end{bmatrix} \begin{bmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -\mathrm{I} \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbb{A}_{G_0^A}^\top - \mathbb{H}_{G_0^H}^{-\top}\mathbb{F}_{G_0^F}^\top\mathbb{B}_{G_0^B}^\top \\ -\mathrm{I} \end{bmatrix} \begin{bmatrix} 0 & \mathbb{H}_{G_1^H}^{-1} \end{bmatrix} \prec 0,$$

which leads to

$$\begin{bmatrix} -\mathbb{H}_{G_0^H}^{-\top}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & \star \\ \mathbb{H}_{G_1^H}^{-\top}\mathbb{A}_{G_0^A} - \mathbb{H}_{G_1^H}^{-\top}\mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -\mathbb{H}_{G_1^H}^{-\top}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} - \mathbb{H}_{G_1^H}^{-\top} - \mathbb{H}_{G_1^H}^{-1} \end{bmatrix} \prec 0. \tag{3.7}$$

By applying congruence transformation multiplying (3.7) with

$$\begin{bmatrix} \mathbb{H}_{G_0^H}^\top & 0 \\ 0 & \mathbb{H}_{G_1^H}^\top \end{bmatrix}$$

on the left and its transpose on the right, it results in (3.5), which completes the proof. □

The classical design conditions described in Chapter 2 can be easily reconstructed from Theorem 3.1. For example, the PDC design based on a quadratic Lyapunov function (Theorem 2.1) is obtained choosing $G_0^F = \{0\}$, $G_0^H = \{\emptyset\}$ and $G_0^P = \{\emptyset\}$. The non-PDC design in Theorem 2.2 corresponds to $G_0^F = G_0^H = \{0\}$ and $G_0^P = \{0\}$. Moreover, [36, Thm. 1] is obtained with $G_0^P = \{-1\}$ and $G_0^H = G_0^F = \{0, -1\}$. The last condition concerns delayed control.

Delayed conditions

To derive the second control design condition, the following nonquadratic Lyapunov function candidate is considered:

$$V_2(x_k) = x_k^\top \mathbb{P}_{G_0^P}^{-1} x_k, \tag{3.8}$$

where $P_\mathbf{i} = P_\mathbf{i}^\top \succ 0 \in \mathbb{R}^{n_x \times n_x}$, for $\mathbf{i} \in \mathbb{I}_{G_0^P}$. The conditions in this case are stated in Theorem 3.2.

Theorem 3.2 (see [38]). *Given $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A)$, the origin of the closed-loop system* (3.3) *is asymptotically stable if there exist matrices $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}_{G_j^P}^{\mathbf{i}}$, $j = 0, 1$, $F_{\mathbf{i}_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}_{G_0^F}^{\mathbf{i}}$, and $H_{\mathbf{i}_0^H}$, $\mathbf{i}_0^H = \mathrm{pr}_{G_0^H}^{\mathbf{i}}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, such that*

$$\begin{bmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^\top + \mathbb{P}_{G_0^P} & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{P}_{G_1^P} \end{bmatrix} \prec 0. \tag{3.9}$$

Proof. Consider the Lyapunov function candidate (3.8). Taking its difference along trajectories of the closed-loop system (3.3), it results:

$$V_2(x_{k+1}) - V_2(x_k) = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}^\top \begin{bmatrix} -\mathbb{P}^{-1}_{G_0^P} & 0 \\ 0 & \mathbb{P}^{-1}_{G_1^P} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}.$$

From Finsler's Lemma, it is possible to write

$$\begin{bmatrix} -\mathbb{P}^{-1}_{G_0^P} & 0 \\ 0 & \mathbb{P}^{-1}_{G_1^P} \end{bmatrix} + \mathcal{M} \begin{bmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H} & -\mathrm{I} \end{bmatrix} + \begin{bmatrix} \mathbb{A}^\top_{G_0^A} - \mathbb{H}^{-T}_{G_0^H}\mathbb{F}^\top_{G_0^F}\mathbb{B}^\top_{G_0^B} \\ -\mathrm{I} \end{bmatrix} \mathcal{M}^\top \prec 0,$$

where $\mathcal{M} = \begin{bmatrix} 0 \\ \mathbb{P}^{-1}_{G_1^P} \end{bmatrix}$. By congruence transformation with $\begin{bmatrix} \mathbb{H}_{G_0^H} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{bmatrix}$, it leads to

$$\begin{bmatrix} -\mathbb{H}^\top_{G_0^H}\mathbb{P}^{-1}_{G_0^P}\mathbb{H}_{G_0^H} & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{P}_{G_1^P} \end{bmatrix} \prec 0, \tag{3.10}$$

Using $-\mathbb{H}^\top_{G_0^H}\mathbb{P}^{-1}_{G_0^P}\mathbb{H}_{G_0^H} \preceq -\mathbb{H}_{G_0^H} - \mathbb{H}^\top_{G_0^H} + \mathbb{P}_{G_0^P}$, results in (3.9) and completes proof. □

The condition in (3.9) also generalizes existing results in the literature concerning non-delayed control. For example, [29, Thm. 4] is obtained with $G_0^P = \{0\}$, $G_0^H = G_0^F = \{0\}$ and $\mathbb{H}_{G_0^H} = \mathbb{P}_{G_0^P}$ and [30, Thm. 3] with $G_0^P = \{0, 0\}$ and $G_0^H = G_0^F = \{0\}$. For delayed control design, [37, Thm. 1] can be obtained with $G_0^P = -1$ and $G_0^F = G_0^H = \{0, -1\}$.

Remark 3.3. *The number of decision variables in both Theorems 3.1 and 3.4 can be computed as:*

$$N_{d_1} = r^{|G_0^P|}\frac{n_x + 1}{2}n_x + r^{|G_0^H|}n_x^2 + r^{|G_0^F|}n_x n_u. \tag{3.11}$$

The conditions (3.5) and (3.9) clearly depend on the adequate choice of the msets of delays. In the sequel, the procedure proposed by [38] to choose the msets of delays such that, for a fixed number of sums, the number of LMIs and, consequently, the computational complexity can be reduced is discussed.

3.2.2 Choosing msets of delays

Consider $G_0^P = \{\emptyset\}$ and $G_0^F = G_0^H = \{\langle 2, -1 \rangle\}$. In this case, condition (3.9) is

$$\begin{bmatrix} -\mathbb{H}_{\{-1,-1\}} - \mathbb{H}^\top_{\{-1,-1\}} + P & \star \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{-1,-1\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{-1,-1\}} & -P \end{bmatrix} \prec 0,$$

which is equivalent to a linear control, since it has to be solved for every index [38]. To conclude, G_V should include $\{\langle 2, 0 \rangle\}$ in order to allow that LMI relaxations such as Lemma 2.1 can be applied.

Assuming classical TS fuzzy models, the msets $G_0^A = G_0^B = \{0\}$ are fixed. Then, it is apparent that the terms $\mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H}$ and $\mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}$, which appear in both conditions (3.5) and (3.9), play a similar role. Therefore, the convenient choice $G_0^F = G_0^H$ can be made without loss of generality.

Now, consider the msets $G_0^F = G_0^H = \{0, -1\}$, $G_0^P = \{\langle 2, -1\rangle\}$, which implies the following summations:

$$\mathbb{F}_{G_0^F} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r} h_{i_1}(z_k)h_{i_2}(z_{k-1})F_{(i_1,i_2)}, \quad \mathbb{H}_{G_0^H} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r} h_{i_1}(z_k)h_{i_2}(z_{k-1})H_{(i_1,i_2)},$$
$$\mathbb{P}_{G_0^P} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r} h_{i_1}(z_{k-1})h_{i_2}(z_{k-1})P_{(i_1,i_2)}.$$

In this case, condition (3.5) corresponds to

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r}\sum_{i_4=1}^{r}\sum_{i_5=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_{k-1})h_{i_4}(z_{k-1})h_{i_5}(z_{k+1})$$
$$\begin{bmatrix} -P_{(i_3,i_4)} & \star \\ A_{i_1}H_{(i_2,i_3)} - B_{i_1}F_{(i_2,i_3)} & -H_{(i_1,i_5)} - H_{(i_1,i_5)}^\top + P_{(i_1,i_2)} \end{bmatrix} \prec 0,$$

and condition (3.9) to

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r}\sum_{i_4=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_{k-1})h_{i_4}(z_{k-1})$$
$$\begin{bmatrix} -H_{(i_2,i_3)} - H_{(i_2,i_3)}^\top + P_{(i_3,i_4)} & \star \\ A_{i_1}H_{(i_2,i_3)} - B_{i_1}F_{(i_2,i_3)} & -P_{(i_1,i_2)} \end{bmatrix} \prec 0.$$

With this choice of msets, condition (3.5) is a 5-dimensional fuzzy summation, while (3.9) is a 4-dimensional fuzzy summation. It implies that a smaller number of LMIs is required to ensure negativity of (3.9).

Following the arguments of [38] to choose msets of delays, assume that $|G_0^P| = 1$ and G_0^F and G_0^H containing $\{0\}$ in order to allow the use of relaxations. For condition (3.5), we have

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \star \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{H}_{\{0\}} & -\mathbb{H}_{\{1\}} - \mathbb{H}_{\{1\}}^\top + \mathbb{P}_{G_1^P} \end{bmatrix} \prec 0,$$

which has at least 3 fuzzy summations, independently of $|G_0^P|$. For arbitrary dimensions of msets of delays G_0^P and $G_0^F = G_0^H$, it follows that it is possible to choose $G_0^P = \{\langle \mathbf{1}_{G_0^P}(0), 0\rangle\}$ and $G_0^F = G_0^H = \{\langle \mathbf{1}_{G_0^H}(0), 0\rangle\}$. For instance, consider $|G_0^P| = |G_0^H| = |G_0^F| = n_P$. This leads to the $(2n_P + 1)$-dimensional mset $G_V = \{0, \langle n_P, 0\rangle, \langle n_P, 1\rangle\}$. Note however that including negative delays in this condition implies in increasing the related number of fuzzy summations, which means that the number of LMIs is also increased.

For condition (3.9) with $|G_0^P| = 1$ and G_0^F and G_0^H containing $\{0\}$, we have

$$\begin{bmatrix} -\mathbb{H}_{\{0\}} - \mathbb{H}_{\{0\}}^\top + \mathbb{P}_{G_0^P} & \star \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{H}_{\{0\}} & -\mathbb{P}_{G_1^P} \end{bmatrix} \prec 0,$$

which has at least 2 summations. To obtain 3 sums, one can choose either $G_0^P = \{0\}$:

$$\begin{bmatrix} -\mathbb{H}_{\{0\}} - \mathbb{H}_{\{0\}}^\top + \mathbb{P}_{\{0\}} & \star \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{H}_{\{0\}} & -\mathbb{P}_{\{1\}} \end{bmatrix} \prec 0,$$

or $G_0^P = \{-1\}$:

$$\begin{bmatrix} -\mathbb{H}_{\{0\}} - \mathbb{H}_{\{0\}}^\top + \mathbb{P}_{\{-1\}} & \star \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{H}_{\{0\}} & -\mathbb{P}_{\{0\}} \end{bmatrix} \prec 0.$$

Notice also that for $G_0^P = \{-1\}$, another degree of freedom can be introduced in G_0^H and G_0^F without increasing the sum dimension, so that $G_0^H = G_0^F = \{0, -1\}$. Therefore, the 3-dimensional fuzzy summation is preserved but conservatism can be reduced. This is the main advantage of delayed control when compared to the non-delayed approach. Extending the analysis for arbitrary dimensions of G_0^P and G_0^H, we have $G_0^P = \{\langle \mathbf{1}_{G_0^P}(-1), -1\rangle\}$ and $G_0^F = G_0^H = \{\langle \mathbf{1}_{G_0^H}(0), 0\rangle, \langle \mathbf{1}_{G_0^H}(-1), -1\rangle\}$. For instance, if $|G_0^P| = |G_0^H| = |G_0^F| = n_P$, the $(2n_P + 1)$-dimensional mset $G_V = \{0, \langle \mathbf{1}_{G_0^P}(0), 0\rangle, \langle \mathbf{1}_{G_0^P}(-1), -1\rangle\}$ is obtained.

The use of delayed control is more recent and allows handling new control laws that are not possible to be designed with the previous existing conditions [38]. The advantages of using delayed conditions for conservatism reduction are illustrated in the next example.

Example 3.2. *To illustrate the conservatism reduction achieved with delayed control, we consider again the TS model* (2.5)*. The analysis in this example is limited to 3-dimensional fuzzy summations in* (3.9)*. The msets of delays are $G_0^P = \{-1\}$ and $G_0^F = G_0^H = \{0, -1\}$. In this case, condition* (3.9) *can be written as*

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_{k-1})\Upsilon_{(i_1,i_2,i_3)} \prec 0,$$

where

$$\Upsilon_{(i_1,i_2,i_3)} = \begin{bmatrix} -H_{(i_2,i_3)} - H_{(i_2,i_3)}^\top + P_{i_3} & \star \\ A_{i_1}H_{(i_2,i_3)} - B_{i_1}F_{(i_2,i_3)} & -P_{i_2} \end{bmatrix} \prec 0.$$

By expanding the above delayed summation for $r = 2$, one has

$$\begin{aligned} &h_1^2(z_k)h_1(z_{k-1})\Upsilon_{(1,1,1)} + h_1^2(z_k)h_2(z_{k-1})\Upsilon_{(1,1,2)}+ \\ &h_1(z_k)h_2(z_k)h_1(z_{k-1})\left(\Upsilon_{(1,2,1)} + \Upsilon_{(2,1,1)}\right) + h_2^2(z_k)h_1(z_{k-1})\Upsilon_{(2,2,1)}+ \\ &h_1(z_k)h_2(z_k)h_1(z_{k-1})\left(\Upsilon_{(1,2,2)} + \Upsilon_{(2,1,2)}\right) + h_2^2(z_k)h_2(z_{k-1})\Upsilon_{(2,2,2)} \prec 0. \end{aligned}$$

LMI-based conditions to ensure negativity of this expansion can be obtained directly applying the LMI relaxation in Lemma 2.2. Solving such LMIs, the maximum b in system (2.5) *for which LMIs are feasible is $b = 1.553$. In comparison to Example 2.3, the number of solved LMIs is the same, but a greater value for b is obtained thanks to the extra degrees introduced in $\mathbb{H}_{G_0^H}$ and $\mathbb{F}_{G_0^F}$.*

As shown in the last example, less conservative results can be obtained with the use of delayed controllers when compared to the conventional non-delayed conditions. However, LMIs were obtained only for the 3-dimensional case. In the sequel, we propose a procedure to derive LMI-based conditions to ensure negativity of MFS with arbitrary delays, *i.e.*, either non-delayed or delayed membership functions.

3.2.3 Deriving LMI-based conditions from general MFS

The control design conditions discussed in this section are given as multiple fuzzy summations possibly with delayed membership functions. It implies that their feasibility depends on the relaxation employed to derive a finite set of LMI conditions to design the control gains. In this subsection, a novel methodology to derive LMI-based design conditions is proposed. This approach can be viewed as a generalization of Lemma 2.3 in the sense that it can be applied to both non-delayed and delayed MFS-based conditions with arbitrary delays. This result is stated in the following lemma.

Lemma 3.1. *The negativity condition of the multidimensional fuzzy summation:*

$$\sum_{\mathbf{i}_1 \in \mathbb{I}_{G_{d_1}}} \cdots \sum_{\mathbf{i}_q \in \mathbb{I}_{G_{d_q}}} h_{\mathbf{i}_1}(z_{k+d_1}) \dots h_{\mathbf{i}_q}(z_{k+d_q}) \Gamma_{(\mathbf{i}_1 \dots \mathbf{i}_q)} \prec 0, \tag{3.12}$$

with $G_V = G_{d_1} \oplus \ldots \oplus G_{d_q}$, $G_{d_k} = \{\langle \mathbf{1}_{G_V}(d_k), d_k \rangle\}$, for all $k \in \mathcal{I}_q$, is fulfilled if

$$\sum_{\mathbf{i}_1 \in \mathscr{P}(\mathbf{j}_1)} \cdots \sum_{\mathbf{i}_q \in \mathscr{P}(\mathbf{j}_q)} \Gamma_{(\mathbf{i}_1 \dots \mathbf{i}_q)} \prec 0, \quad \mathbf{j}_k \in \mathbb{I}^+_{G_{d_k}}, \quad k \in \mathcal{I}_q.$$

Proof. Consider the negativity condition for a multiple fuzzy summation with an arbitrary multiset of delays as (3.12):

$$\sum_{\mathbf{i}_1 \in \mathbb{I}_{G_{d_1}}} \cdots \sum_{\mathbf{i}_q \in \mathbb{I}_{G_{d_q}}} h_{\mathbf{i}_1}(z_{k+d_1}) \dots h_{\mathbf{i}_q}(z_{k+d_q}) \Gamma_{\mathbf{i}_1 \dots \mathbf{i}_q} \prec 0.$$

The above condition is equivalent to

$$\sum_{\mathbf{j}_1 \in \mathbb{I}^+_{G_{d_1}}} \cdots \sum_{\mathbf{j}_q \in \mathbb{I}^+_{G_{d_q}}} h_{\mathbf{j}_1}(z_{k+d_1}) \dots h_{\mathbf{j}_q}(z_{k+d_q}) \left(\sum_{\mathbf{i}_1 \in \mathscr{P}(\mathbf{j}_1)} \cdots \sum_{\mathbf{i}_q \in \mathscr{P}(\mathbf{j}_q)} \Gamma_{\mathbf{i}_1 \dots \mathbf{i}_q} \right) \prec 0$$

$$\Leftrightarrow \quad \sum_{\mathbf{i}_1 \in \mathscr{P}(\mathbf{j}_1)} \cdots \sum_{\mathbf{i}_q \in \mathscr{P}(\mathbf{j}_q)} \Gamma_{\mathbf{i}_1 \dots \mathbf{i}_q} \prec 0, \quad \mathbf{j}_k \in \mathbb{I}^+_{G_{d_k}}, \; k \in \mathcal{I}_q.$$

This concludes the proof. □

Remark 3.4. *Consider the negativity condition of an MFS with mset of delays* $G_V = G_{d_1} \oplus \ldots \oplus G_{d_q}$ *such as* (3.12)*. The number of LMIs obtained with Lemma 3.1 can be computed as:*

$$\prod_{j=1}^{q} |\mathbb{I}^+_{G_{d_j}}|, \qquad \textit{where } |\mathbb{I}^+_{G_{d_j}}| = \frac{(r + |G_{d_k}| - 1)!}{|G_{d_k}|!(r-1)!}.$$

The procedure to obtain LMI-based conditions from arbitrary MFS with Lemma 3.1 is illustrated in the following example.

Example 3.3. *Consider a 4-dimensional fuzzy summation with mset of delays*

$$G_V = \{\langle 2, 0\rangle, \langle 2, 1\rangle\}.$$

Its is clear that $G_V = G_0 \oplus G_1$, *with* $G_0 = \{\langle 2, 0\rangle\}$, $G_1 = \{\langle 2, 1\rangle\}$*. From Lemma 3.1, its negativity is ensured if the following set of 9 LMIs hold:*

$$\begin{aligned}
&\Gamma_{(1,1,1,1)} \prec 0, \quad \Gamma_{(2,2,2,2)} \prec 0, \quad \Gamma_{(1,1,1,2)} + \Gamma_{(1,1,2,1)} \prec 0, \quad \Gamma_{(1,1,2,2)} \prec 0,\\
&\Gamma_{(1,2,1,1)} + \Gamma_{(2,1,1,1)} \prec 0, \quad \Gamma_{(1,2,1,2)} + \Gamma_{(2,1,1,2)} + \Gamma_{(1,2,2,1)} + \Gamma_{(2,1,2,1)} \prec 0,\\
&\Gamma_{(1,2,2,2)} + \Gamma_{(2,1,2,2)} \prec 0, \quad \Gamma_{(2,2,1,1)} \prec 0, \quad \Gamma_{(2,2,1,2)} + \Gamma_{(2,2,2,1)} \prec 0.
\end{aligned}$$

Remark 3.5. *The number of LMI rows in both Theorems 3.1 and 3.2 can be computed as:*

$$N_l = 2n_x \prod_{j=1}^{q} |\mathbb{I}^+_{G_{d_j}}|,$$

with G_{d_j} *obtained from the decomposition of* G_V *as in Lemma 3.1.*

Remark 3.6. *The computational complexity of interior-point-based methods, which is the case of LMI solvers, can be estimated by* $\log_{10}(N_d^3 N_l)$*, where* N_d *is the number of decision variables and* N_l *is the number of LMI rows [31, 62]. This computation will be used to perform a quantitative comparison of computational complexity among different design conditions.*

Now, we are in position to apply both conditions (3.5) and (3.9) to any dimension and delays in G_V, since it is possible to derive LMIs for ensuring their negativity using Lemma 3.1. The results related to the application of these conditions are presented in the next example.

Example 3.4. *In this example, we apply conditions* (3.5) *and* (3.9) *with different choices of msets for both control law and Lyapunov function. The aim here is to evaluate the influence of increasing the fuzzy summation dimension in the case of both non-delayed and delayed control.*

To perform such comparison, we consider again the problem of finding the maximum value of b *in system* (2.5) *such that the optimization problem to solve LMIs is feasible. The results are depicted in Table 3.1.*

Table 3.1 – Comparison among maximum b for LMI problem feasibility and computational complexity obtained with different choices of G_0^P and $G_0^F = G_0^H$ in Theorems 3.1 and 3.2. The largest obtained value is in **bold**.

G_0^P	$G_0^F = G_0^H$	$\lvert G_V \rvert$	b	N_l	N_{d_1}	$\log_{10}(N_{d_1}^3 N_l)$
Theorem 3.1 - Non-delayed control						
$\{0\}$	$\{0\}$	3	1.539	24	18	5.1460
$\{\langle 2,0\rangle\}$	$\{0\}$	4	1.547	36	24	5.6969
$\{\langle 2,0\rangle\}$	$\{\langle 2,0\rangle\}$	5	1.693	48	36	6.3501
$\{\langle 3,0\rangle\}$	$\{\langle 2,0\rangle\}$	6	1.693	64	48	6.8499
Therem 3.2 - Delayed control						
$\{-1\}$	$\{0,-1\}$	3	1.553	24	30	5.8116
$\{-1\}$	$\{\langle 2,0\rangle,-1\}$	4	1.693	36	60	6.8908
$\{-1\}$	$\{\langle 3,0\rangle,-1\}$	5	1.735	40	102	7.6279
$\{-1\}$	$\{\langle 4,0\rangle,-1\}$	6	**1.757**	48	198	8.5712

Note that some of the reported results in Table 3.1 were previously presented in this manuscript, such as the 3-dimensional condition ($\lvert G_V\rvert = 3$) of Theorem 3.1 which corresponds to the condition of [29] studied in Example 2.2. In addition, the conditions for higher dimensions obtained with Theorem 3.1 can be directly related to the multiple-parameterized approach in Theorem 2.4. For example, the case of $q = 2$ and $p = l = 1$ in (2.41) is equivalent to condition (3.5) with $G_0^P = \{\langle 2,0\rangle\}$ and $G_0^F = G_0^H = \{0\}$. In a similar way, this equivalence can be stated for other cases. To sum up, if none delays are introduced in Theorem 3.1, it is equivalent to Theorem 2.4.

When it comes to delayed control design with (3.2), to obtain a 4-dimensional fuzzy summation ($\lvert G_V\rvert = 4$) we can choose either $G_0^P = \{-1\}$ and $G_0^F = G_0^H = \{\langle 2,0\rangle,-1\}$ or $G_0^P = \{\langle 2,-1\rangle\}$ and $G_0^F = G_0^H = \{0,\langle 2,-1\rangle\}$. For simplicity, the Lyapunov fuzzy dimension is restricted to 1. Then, to increase the fuzzy summation dimensionality, e.g., to 5, we choose $G_0^F = G_0^H = \{\langle 3,0\rangle,-1\}$, and so on.

To perform the comparison in terms of computational complexity, consider the 5-dimensional non-delayed condition, whose maximum b for feasibility is 1.693 is obtained with computational complexity of 6.3501. Even increasing the fuzzy dimension to 6, the same value for b is obtained with computational complexity of 6.8499. On the other hand, this value is also obtained with only 4 fuzzy summations in the delayed condition, but with a greater computational complexity of 6.8908.

Comparing the results obtained with Theorems 3.1 and 3.2, it is clear that, for any fuzzy sum dimension, delayed control leads to less conservative outcomes. Furthermore, in Table 2.2, a value for b greater than $b = 1.757$ is only achieved when $\lvert G_V\rvert \geq 9$, which corresponds to choose $q = 5$ and $p = l = 4$ or $1 \leq q \leq 5$ and $p = l = 5$ in Theorem 2.4. From the aforementioned discussion, it is possible to conclude that delayed control can provide *less conservative results but requiring a greater computational complexity than conventional nonquadratic framework based on non-delayed control.*

3.3 Disturbance attenuation: l_2-gain performance control

This section concerns the problem of l_2-gain performance control for reducing the effect of energy-bounded disturbances in the output channel of an input-affine discrete-time nonlinear system. Consider the following TS fuzzy model representation for a discrete-time input-affine nonlinear system subject to disturbances:

$$\begin{aligned} x_{k+1} &= \mathbb{A}_{G_0^A} x_k + \mathbb{B}_{G_0^B} u_k + \mathbb{E}_{G_0^E} w_k \\ y_k &= \mathbb{C}_{G_0^C} x_k + \mathbb{D}_{G_0^D} u_k + \mathbb{K}_{G_0^K} w_k, \end{aligned} \tag{3.13}$$

where $y_k \in \mathbb{R}^{n_y}$ is the output vector, $w_k \in \mathbb{R}^{n_w}$ is the vector of energy-bounded disturbances, *i.e.*, $\|w\|_{l_2} < \infty$. In addition, the matrices $\mathbb{A}_{G_0^A}$ and $\mathbb{B}_{G_0^B}$ are the same as in (3.1) and $\mathbb{E}_{G_0^E}$, $\mathbb{C}_{G_0^C}$, $\mathbb{D}_{G_0^D}$, $\mathbb{K}_{G_0^K}$ are, respectively, the ordinary fuzzy summations of matrices $E_i \in \mathbb{R}^{n_x \times n_w}$, $C_i \in \mathbb{R}^{n_y \times n_x}$, $D_i \in \mathbb{R}^{n_y \times n_u}$, $K_i \in \mathbb{R}^{n_y \times n_w}$, $i \in \mathcal{I}_r$, with $G_0^E = G_0^C = G_0^D = G_0^K = \{0\}$.

After substituting the control law (3.2) into system (3.13), the following closed-loop dynamics is obtained:

$$\begin{aligned} x_{k+1} &= \Big(\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1}\Big) x_k + \mathbb{E}_{G_0^E} w_k \\ y_k &= \Big(\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D} \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1}\Big) x_k + \mathbb{K}_{G_0^K} w_k. \end{aligned} \tag{3.14}$$

Similar to the $\mathscr{H}_\infty$ control in the context of linear systems, the l_2-gain performance control is considered to reduce the effect of disturbances in the output channel of discrete-time nonlinear systems [63] by minimizing the l_2-gain. Here, to design control gains of controller (3.2) such that the origin of the closed-loop system (3.14) is asymptotically stable and the induced l_2-gain is minimized, we consider arguments of dissipative analysis (see Appendix B).

The aim here is to derive sufficient conditions to ensure the closed-loop system (3.14) be dissipative with respect to the supply rate

$$\mathcal{S}(w, y) = -y_k^\top y_k + \gamma^2 w_k^\top w_k. \tag{3.15}$$

The idea is to consider the Lyapunov functions (3.4) and (3.8) as storage functions for the dissipativity analysis. With this supply rate, the conditions derived here also provide an upper-bound for the l_2-gain of system (3.14). These can thus be seen as extensions of Theorems 3.1 and 3.2 that take into account the l_2-gain performance index. Such extensions are stated in the sequel.

Non-delayed conditions

The condition for the case of Lyapunov function (3.8) considered as storage function is stated in the following theorem.

Theorem 3.3 (adapted from [38]). *Let* $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_1^H \cup G_0^K \cup G_0^E$ *be given. If there exist a scalar* $\gamma > 0$ *and matrices* $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}^{\mathbf{i}}_{G_j^P}$, $H_{\mathbf{i}_j^H}$, $\mathbf{i}_j^H = \mathrm{pr}^{\mathbf{i}}_{G_0^H}$, $j = 0, 1$, *and* $F_{i_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}^{\mathbf{i}}_{G_0^F}$, $\mathbf{i}_0^H = \mathrm{pr}^{\mathbf{i}}_{G_0^H}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, *such that*

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \star & \star & \star \\ 0 & -\gamma^2 \mathrm{I} & \star & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \mathbb{E}_{G_0^E} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^\top + \mathbb{P}_{G_1^P} & \star \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F} & \mathbb{K}_{G_0^K} & 0 & -\mathrm{I} \end{bmatrix} \prec 0 \qquad (3.16)$$

holds. Then, the closed-loop system (3.14) *is asymptotically stable and the* l_2*-gain has upper bound* γ.

Proof. Assume that condition (3.16) is fulfilled, so $\mathbb{H}_{G_1^H} + \mathbb{H}_{G_1^H}^\top \succ \mathbb{P}_{G_1^P} \succ 0$, which ensure that matrices $\mathbb{H}_{G_j^H}$, $j \in \{0, 1\}$, are invertible. Then, it is possible to apply the congruence transformation in the above condition with diag$(\mathbb{H}_{G_0^H}^{-\top}, \mathrm{I}, \mathbb{H}_{G_1^H}^{-\top}, \mathrm{I})$, which results in:

$$\begin{bmatrix} -\mathbb{H}_{G_0^H}^{-\top}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & \star & \star & \star \\ 0 & -\gamma^2\mathrm{I} & \star & \star \\ \mathbb{H}_{G_1^H}^{-\top}\left(\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\right) & \mathbb{H}_{G_1^H}^{-\top}\mathbb{E}_{G_0^E} & -\mathbb{H}_{G_1^H}^{-1} - \mathbb{H}_{G_1^H}^{-\top} + \mathbb{H}_{G_1^H}^{-\top}\mathbb{P}_{G_1^P}^{-1}\mathbb{H}_{G_1^H}^{-1} & \star \\ \mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & \mathbb{K}_{G_0^K} & 0 & -\mathrm{I} \end{bmatrix} \prec 0. \qquad (3.17)$$

Multiplying the last inequality with

$$\begin{bmatrix} \mathrm{I} & 0 & \mathbb{A}_{cl}^\top & \mathbb{C}_{cl}^\top \\ 0 & \mathrm{I} & \mathbb{E}_{G_0^E}^\top & \mathbb{K}_{G_0^K}^\top \end{bmatrix},$$

on the left and its transpose on the right, where $\mathbb{A}_{cl} = \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}$ and $\mathbb{C}_{cl} = \mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}$, leads to

$$\begin{bmatrix} -\mathbb{H}_{G_0^H}^{-\top}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} + \mathbb{A}_{cl}^\top\mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}\mathbb{A}_{cl}^\top + \mathbb{C}_{cl}^\top\mathbb{C}_{cl} & \star \\ \mathbb{K}_{G_0^K}^\top\mathbb{C}_{cl} + \mathbb{E}_{G_0^E}^\top\mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}\mathbb{A}_{cl} & \begin{pmatrix} -\gamma^2\mathrm{I} + \mathbb{K}_{G_0^K}^\top\mathbb{K}_{G_0^K} + \\ \mathbb{E}_{G_0^E}^\top\mathbb{H}_{G_1^H}^{-\top}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}\mathbb{E}_{G_0^E} \end{pmatrix} \end{bmatrix} \prec 0$$

By pre an post-multiplication, respectively, with $\left[x_k^\top, w_k^\top\right]$ and its transpose, leads to:

$$x_{k+1}^\top\mathbb{H}_{G_1^H}^{-\top}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}x_{k+1} - x_k^\top\mathbb{H}_{G_0^H}^{-\top}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1}x_k + y_k^\top y_k - \gamma^2 w_k^\top w_k < 0$$
$$V_1(x_{k+1}) - V_1(x_k) < -y_k^\top y_k + \gamma^2 w_k^\top w_k.$$

Now, we consider two cases. The first is for $w_k \equiv 0$, for all k. It implies that $\Delta V_1 < 0$ and the origin of the closed-loop system (3.14) is asymptotically stable. For the second case, when $w_k \neq 0$, take the sum over $k \in \{0, \dots, \tau - 1\}$, $\tau \in \mathbb{N}$. It results in

$$V_1(x_\tau) < V_1(x_0) + \sum_{i=0}^{\tau-1}\left(-y_i^\top y_i + \gamma^2 w_i^\top w_i\right).$$

This proves the closed-loop system (3.14) is strictly dissipative with respect to the supply rate (3.15). Using the fact $V(x_k) > 0$ and taking $\tau \to \infty$, it follows that

$$\sum_{i=1}^{\infty} y_i^\top y_i < \sum_{i=1}^{\infty} \gamma^2 w_i^\top w_i + V_1(x_0) \quad \Rightarrow \quad \|y\|_{l_2} < \gamma \|w\|_{l_2} + \sqrt{V_1(x_0)}.$$

Therefore, the upper-bound for the l_2-gain is γ, which completes the proof. □

Note that ensuring strictly dissipativity of system (3.14) implies the origin is asymptotically stable. The proof of this fact can be found in [9, Lemma 6.7, p. 243].

Theorem 3.3 has provided a condition to ensure the asymptotic stability of (3.14) and obtain an upper-bound for the l_2-gain. However, it is of interest to minimize γ such that the influence of disturbances into the output channel be minimized. Such minimization can be performed by the optimization problem stated in the following lemma.

Lemma 3.2. *Let G_V be given as in Theorem 3.3. If there exist a scalar $\mu = \gamma^2$ and matrices $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top$, $\mathbf{i}_j^P = \mathrm{pr}_{G_j^P}^{\mathbf{i}}$, $H_{\mathbf{i}_j^H}$, $\mathbf{i}_j^H = \mathrm{pr}_{G_0^H}^{\mathbf{i}}$, $j = 0, 1$, and $F_{i_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}_{G_0^F}^{\mathbf{i}}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, such that the optimization problem*

$$\min_{P_{\mathbf{i}_0^P}, P_{\mathbf{i}_1^P}, H_{\mathbf{i}_0^H}, H_{\mathbf{i}_1^H}, F_{\mathbf{i}_0^F}} \mu, \qquad \textit{s.t. } (3.16),$$

is feasible. Then, the origin of the closed-loop system (3.14) *is asymptotically stable and $\gamma = \sqrt{\mu}$ is the minimal upper bound for the l_2-gain.*

Proof. The proof is a direct consequence of Theorem 3.3. □

Delayed conditions

The condition for which the Lyapunov function (3.8) is considered as storage function is stated in the following theorem.

Theorem 3.4. *Given $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_0^K \cup G_0^E$, If there exist a scalar $\gamma > 0$ and matrices $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}_{G_j^P}^{\mathbf{i}}$, $j = 0, 1$, $F_{i_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}_{G_0^F}^{\mathbf{i}}$, and $H_{i_0^H}$, $\mathbf{i}_0^H = \mathrm{pr}_{G_0^H}^{\mathbf{i}}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, such that*

$$\begin{bmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^\top + \mathbb{P}_{G_0^P} & \star & \star & \star \\ 0 & -\gamma^2 \mathrm{I} & \star & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \mathbb{E}_{G_0^E} & -\mathbb{P}_{G_1^P} & \star \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F} & \mathbb{K}_{G_0^K} & 0 & -\mathrm{I} \end{bmatrix} < 0 \tag{3.18}$$

holds, then the origin of the closed-loop system (3.14) *is asymptotically stable and the l_2-gain has upper bound γ.*

Proof. Choosing the storage function as (3.4) and appropriate congruence transformations, the proof follows similar steps as the one of Theorem 3.3. □

The optimization problem to minimize the upper-bound of the l_2-gain using conditions of Theorem 3.4 is stated in the following lemma.

Lemma 3.3. *Let G_V be given as in Theorem 3.4. If there exist a scalar $\mu = \gamma^2$ and matrices $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}^{\mathbf{i}}_{G_j^P}$, $j = 0, 1$, $H_{\mathbf{i}_0^H}$, $\mathbf{i}_0^H = \mathrm{pr}^{\mathbf{i}}_{G_0^H}$, and $F_{i_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}^{\mathbf{i}}_{G_0^F}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, such that the optimization problem*

$$\min_{P_{\mathbf{i}_0^P}, P_{\mathbf{i}_1^P}, H_{\mathbf{i}_0^H}, F_{\mathbf{i}_j^F}} \mu, \quad \textit{s.t. } (3.16) \tag{3.19}$$

is feasible. Then, the closed-loop system (3.14) *is dissipative with respect to the supply rate* (3.15) *and $\gamma = \sqrt{\mu}$ is the minimal upper bound for the l_2-gain.*

Proof. The proof is consequence of Theorem 3.4. □

As long as Lemmas 3.2 and 3.3 involve ensuring negativity of MFS, Lemma 3.1 is used to derive LMI-based conditions. The application of these conditions is illustrated in the next example.

Example 3.5 (see [38])**.** *Consider the following TS fuzzy model:*

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{r} h_i(z_k)\left(A_i x_k + B_i u_k + E w_k\right) \\ y_k &= x_k, \end{aligned} \tag{3.20}$$

where

$$A_1 = \begin{bmatrix} 1 & -1.4 \\ -1 & -0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1.4 \\ -1 & -0.5 \end{bmatrix}, \quad E = \begin{bmatrix} -0.1357 & 0.1 \\ -0.1 & -0.039 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 6.4 \\ 2.8 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3.6 \\ -2.8 \end{bmatrix}.$$

The aim here is to compare the two l_2-gain control performance methods. The comparison is made in terms of the minimal upper-bound for the l_2-gain obtained with Lemmas 3.2 and 3.3, which correspond, respectively, to the non-delayed and delayed control design. For simplicity, it is considered only 3-dimensional fuzzy summations conditions. Then, the msets for Lemma 3.2 are $G_0^P = G_0^F = G_0^H = \{0\}$ and for Lemma 3.3 are $G_0^P = \{0\}$, $G_0^F = G_0^H = \{0, -1\}$. The minimal upper-bounds for the l_2-gain obtained after solving the optimization problem in theses lemmas are depicted in Table 3.2.

The smallest γ is obtained with Lemma 3.3. As expected, delayed control has provided less conservative results.

Table 3.2 – Comparison of minimal upper-bounds for the l_2-gain obtained with Lemmas 3.2 and 3.3.

	γ
Lemma 3.2	1.426
Lemma 3.3	1.327

3.4 Conclusion

This chapter has presented a general framework for the design of nonquadratic conditions for TS fuzzy models. Two methods have been presented, each one related to a Lyapunov function candidate. The main feature of them is the ability offered by the msets notation to easily incorporate delayed control conditions, a recent approach proposed to reduce conservatism in the conventional nonquadratic framework. It has been shown that the presented controllers proposed by [38] include those previous reported in the literature. In addition, based on the dissipativity analysis of nonlinear systems, it was possible to incorporate the l_2-gain performance index to the design conditions.

4 ENHANCED CONTROL DESIGN CONDITIONS

In this chapter, the main contributions of this manuscript are presented. Here, two new conditions are proposed to design both non-delayed and delayed fuzzy controllers. They are formulated for general multidimensional fuzzy summations, similar to the conditions of [38] described in Chapter 3. The main objective here is to obtain less conservative results with reduced computational complexity. In addition, the disturbance attenuation problem is also regarded here. In this case, two more results are given. It is demonstrated that the conditions of [38] for both stabilization and disturbance attenuation are particular cases of those proposed here.

4.1 Improving existing design conditions

This section presents two new control design conditions. Motivated by the work of [44] in the context of LPV systems, the proposed conditions are obtained from appropriate matrix transformations based on the inclusion of new decision variables, which introduces new degrees of freedom for the optimization problem to design the control gains. Similar to the conditions of [38] described in Chapter 3, those proposed here are such that both non-delayed and delayed controllers can easily be handled. It is shown that less conservative results can be obtained without increasing the computational complexity in comparison to recent results in the literature.

4.1.1 New control design conditions

Following the structure of Chapter 3, we consider two cases: the first is based on the Lyapunov function candidate in (3.4) whereas the second is based on the Lyapunov function in (3.8). In both cases, non-delayed and delayed controllers can be designed. However, to obtain a reduced number of fuzzy summations, the first case is mainly for non-delayed control design and the second case for delayed control.

Non-delayed conditions

The condition based on the Lyapunov function candidate (3.4) is stated in the following theorem.

Theorem 4.1. *Let* $G_V = G_0^P \cup G_1^P \cup (G_0^Y \oplus G_0^B) \cup (G_0^Z \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_1^H$ *be given. If there exist matrices* $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}_{G_j^P}^{\mathbf{i}}$ *and* $H_{\mathbf{i}_j^H}$, $\mathbf{i}_j^H = \mathrm{pr}_{G_0^H}^{\mathbf{i}}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, $F_{\mathbf{i}_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}_{G_0^F}^{\mathbf{i}}$, $Y_{\mathbf{i}_0^Y}$, $\mathbf{i}_0^Y = \mathrm{pr}_{G_0^Y}^{\mathbf{i}}$, *and* $Z_{\mathbf{i}_0^Z}$, $\mathbf{i}_0^Z = \mathrm{pr}_{G_0^Z}^{\mathbf{i}}$, *such that* (4.1) *holds. Then, the origin of* *the closed-loop system* (3.3) *is asymptotically stable.*

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \star & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} & \begin{pmatrix} -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^\top + \mathbb{P}_{G_1^P} \\ -\mathbb{B}_{G_0^B}\mathbb{Y}_{G_0^Y} - \mathbb{Y}_{G_0^Y}^\top\mathbb{B}_{G_0^B}^\top \end{pmatrix} & \star \\ \mathbb{F}_{G_0^F} & -\mathbb{Y}_{G_0^Y} + \mathbb{Z}_{G_0^Z}^\top\mathbb{B}_{G_0^B}^\top & \mathbb{Z}_{G_0^Z} + \mathbb{Z}_{G_0^Z}^\top \end{bmatrix} \prec 0 \tag{4.1}$$

Proof. Assume that condition (4.1) is fulfilled. As a result, the following inequality holds.

$$-\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^\top + \mathbb{P}_{G_1^P} - \mathbb{B}_{G_0^B}\mathbb{Y}_{G_0^Y} - \mathbb{Y}_{G_0^Y}^\top\mathbb{B}_{G_0^B}^\top \prec 0.$$

It is equivalent to $\left(\mathbb{H}_{G_1^H} + \mathbb{B}_{G_0^B}\mathbb{Y}_{G_0^Y}\right) + \left(\mathbb{H}_{G_1^H} + \mathbb{B}_{G_0^B}\mathbb{Y}_{G_0^Y}\right)^\top \succ \mathbb{P}_{G_1^P} \succ 0$. It ensures that $\left(\mathbb{H}_{G_1^H} + \mathbb{B}_{G_0^B}\mathbb{Y}_{G_0^Y}\right)$ is invertible. From the Matrix Inversion Lemma, one can conclude that the matrices $\mathbb{H}_{G_1^H}$ and, consequently, $\mathbb{H}_{G_0^H}$ are also invertible.

By applying a congruence transformation in (4.1) with diag$(\mathbb{H}_{G_0^H}^{-\top}, \mathrm{I}, \mathrm{I})$, it results

$$\begin{bmatrix} -\mathbb{H}_{G_0^H}^{-\top}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & \star & \star \\ \mathbb{A}_{G_0^A} & \begin{pmatrix} -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^\top + \mathbb{P}_{G_1^P} \\ -\mathbb{B}_{G_0^B}\mathbb{Y}_{G_0^Y} - \mathbb{Y}_{G_0^Y}^\top\mathbb{B}_{G_0^B}^\top \end{pmatrix} & \star \\ \mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -\mathbb{Y}_{G_0^Y} + \mathbb{Z}_{G_0^Z}^\top\mathbb{B}_{G_0^B}^\top & \mathbb{Z}_{G_0^Z} + \mathbb{Z}_{G_0^Z}^\top \end{bmatrix} \prec 0.$$

Multiplying the last inequality with

$$\begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & -\mathbb{B}_{G_0^B} \end{bmatrix}$$

on the left and its transpose on the right, turns into

$$\begin{bmatrix} -\mathbb{H}_{G_0^H}^{-\top}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & \star \\ \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^\top + \mathbb{P}_{G_1^P} \end{bmatrix} \prec 0.$$

By applying a congruence transformation in the above inequality with diag$(\mathrm{I}, \mathbb{H}_{G_1^H}^{-\top})$, one has

$$\begin{bmatrix} -\mathbb{H}_{G_0^H}^{-\top}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & \star \\ \mathbb{H}_{G_1^H}^{-\top}\mathbb{A}_{G_0^A} - \mathbb{H}_{G_1^H}^{-\top}\mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -\mathbb{H}_{G_1^H}^{-1} - \mathbb{H}_{G_1^H}^{-\top} + \mathbb{H}_{G_1^H}^{-\top}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} \end{bmatrix} \prec 0.$$

Finally, multiplying the last inequality with $[\mathrm{I} \quad (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})^\top]$ on the left and its transpose on the right leads to

$$\left(\mathbb{A}_{G_0^A}^\top - \mathbb{H}_{G_0^H}^{-\top}\mathbb{F}_{G_0^F}^\top\mathbb{B}_{G_0^B}^\top\right)\mathbb{H}_{G_1^H}^{-\top}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}\left(\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\right) - \mathbb{H}_{G_0^H}^{-\top}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} \prec 0.$$

By pre and post-multiplying the above inequality with $x_k^\top$ and by its transpose, respectively, it follows that

$$V_1(x_{k+1}) - V_1(x_k) < 0.$$

This proves that the origin of the closed-loop system (3.3) is asymptotically stable, which completes the proof. □

Remark 4.1. *It is possible to prove the equivalence between conditions* (3.5) *and* (4.1)*. To show that, consider a sufficiently large scalar* $\rho > 0$ *such that* (3.5) *can be rewritten as*

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \star & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{H}_{G_1^H}^\top - \mathbb{H}_{G_1^H} + \mathbb{P}_{G_1^P} & \star \\ \mathbb{F}_{G_0^F} & 0 & -\rho \mathrm{I} \end{bmatrix} \prec 0. \tag{4.2}$$

Using similar arguments as [44], define the new variables

$$\mathbb{Z}_{G_0^Z} = -\frac{\rho}{2}\mathrm{I} \quad \textit{and} \quad \mathbb{Y}_{G_0^Y} = -\mathbb{Z}_{G_0^Z}\mathbb{B}_{G_0^B}^\top$$

such that (4.2) *turns into*

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \star & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{H}_{G_1^H}^\top - \mathbb{H}_{G_1^H} + \mathbb{P}_{G_1^P} & \star \\ \mathbb{F}_{G_0^F} & -\mathbb{Y}_{G_0^Y} - \mathbb{Z}_{G_0^Z}\mathbb{B}_{G_0^B}^\top & \mathbb{Z}_{G_0^Z} + \mathbb{Z}_{G_0^Z}^\top \end{bmatrix} \prec 0.$$

Multiplying the last inequality with

$$\begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & \mathbb{B}_{G_0^B} \\ 0 & 0 & \mathrm{I} \end{bmatrix},$$

on the left and its transpose on the right, it results in (4.1)*.*

To prove the converse, multiply (4.3) *with*

$$\begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & -\mathbb{B}_{G_0^B} \end{bmatrix},$$

on the left and its trasnpose on the right, which directly results in (3.5)*. From this result, condition* (3.5) *can be viewed as a particular case of* (4.1)*.*

Delayed conditions

The proposed condition for the case of Lyapunov function candidate (3.8) is depicted in the following theorem.

Theorem 4.2. *Let* $G_V = G_0^P \cup G_1^P \cup (G_0^Y \oplus G_0^B) \cup (G_0^Z \oplus G_0^B) \cup (G_0^H \oplus G_0^A)$ *be given. If there exist matrices* $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}^{\mathbf{i}}_{G_j^P}$, $j = 0, 1$, $F_{\mathbf{i}_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}^{\mathbf{i}}_{G_0^F}$, $H_{\mathbf{i}_0^H}$, $\mathbf{i}_0^H = \mathrm{pr}^{\mathbf{i}}_{G_0^H}$, $Y_{\mathbf{i}_0^Y}$, $\mathbf{i}_0^Y = \mathrm{pr}^{\mathbf{i}}_{G_0^Y}$ *and* $Z_{\mathbf{i}_0^Z}$, $\mathbf{i}_0^Z = \mathrm{pr}^{\mathbf{i}}_{G_0^Z}$, $\mathbf{i} \in \mathbb{I}_{G_V}$*, such that* (4.3) *holds. Then, the origin of the closed-loop system* (3.3) *is asymptotically stable.*

$$\begin{bmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^\top + \mathbb{P}_{G_0^P} & \star & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} & -\mathbb{P}_{G_1^P} - \mathbb{B}_{G_0^B}\mathbb{Y}_{G_0^Y} - \mathbb{Y}_{G_0^Y}^\top\mathbb{B}_{G_0^B}^\top & \star \\ \mathbb{F}_{G_0^F} & -\mathbb{Y}_{G_0^Y} + \mathbb{Z}_{G_0^Z}^\top\mathbb{B}_{G_0^B}^\top & \mathbb{Z}_{G_0^Z} + \mathbb{Z}_{G_0^Z}^\top \end{bmatrix} \prec 0. \tag{4.3}$$

Proof. Assume that condition (4.3) holds. It follows that $\mathbb{H}_{G_1^H} + \mathbb{H}_{G_1^H}^\top \succ \mathbb{P}_{G_1^P} \succ 0$. From property $-\mathbb{H}_{G_0^H}^\top \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} \preceq -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^\top + \mathbb{P}_{G_0^P}$, it results:

$$\begin{bmatrix} -\mathbb{H}_{G_0^H}^\top \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} & \star & \star \\ \mathbb{A}_{G_0^A} \mathbb{H}_{G_0^H} & -\mathbb{P}_{G_1^P} - \mathbb{B}_{G_0^B} \mathbb{Y}_{G_0^Y} - \mathbb{Y}_{G_0^Y}^\top \mathbb{B}_{G_0^B}^\top & \star \\ \mathbb{F}_{G_0^F} & -\mathbb{Y}_{G_0^Y} + \mathbb{Z}_{G_0^Z}^\top \mathbb{B}_{G_0^B}^\top & \mathbb{Z}_{G_0^Z} + \mathbb{Z}_{G_0^Z}^\top \end{bmatrix} \prec 0.$$

By applying a congruence transformation in the above inequality with $\mathrm{diag}(\mathbb{H}_{G_0^H}^{-\top}, \mathrm{I}, \mathrm{I})$, one has

$$\begin{bmatrix} -\mathbb{P}_{G_0^P}^{-1} & \star & \star \\ \mathbb{A}_{G_0^A} & -\mathbb{P}_{G_1^P} - \mathbb{B}_{G_0^B} \mathbb{Y}_{G_0^Y} - \mathbb{Y}_{G_0^Y}^\top \mathbb{B}_{G_0^B}^\top & \star \\ \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1} & -\mathbb{Y}_{G_0^Y} + \mathbb{Z}_{G_0^Z}^\top \mathbb{B}_{G_0^B}^\top & \mathbb{Z}_{G_0^Z} + \mathbb{Z}_{G_0^Z}^\top \end{bmatrix} \prec 0,$$

that after multiplied by

$$\begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & -\mathbb{B}_{G_0^B} \end{bmatrix}$$

on the left and its transpose on the right leads to

$$\begin{bmatrix} -\mathbb{P}_{G_0^P}^{-1} & \mathbb{A}_{G_0^A}^\top - \mathbb{H}_{G_0^H}^{-\top} \mathbb{F}_{G_0^F}^\top \mathbb{B}_{G_0^B}^\top \\ \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1} & -\mathbb{P}_{G_1^P} \end{bmatrix} \prec 0.$$

Applying a Schur complement argument in the above condition, one has

$$\left(\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1}\right)^\top \mathbb{P}_{G_1^P}^{-1} \left(\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1}\right) - \mathbb{P}_{G_0^P}^{-1} \prec 0.$$

Pre and post-multiplying with $x_k^\top$ and by its transpose, respectively, it results that

$$V_2(x_{k+1}) - V_2(x_k) < 0.$$

This proves that the origin of the closed-loop system (3.3) is asymptotically stable. □

Remark 4.2. *It is possible to prove that* (3.9) *is equivalent to* (4.3) *following similar steps as Remark 4.1.*

Remark 4.3. *The number of decision variables and LMI rows used to estimate the computational complexity of conditions in Theorems 4.1 and 4.2 are computed as follows:*

$$N_{d_2} = N_{d_1} + r^{|G_0^Y|} n_x n_u + r^{|G_0^Z|} n_u^2,$$
$$N_l = (2n_x + n_u) \prod_{j=1}^{q} |\mathbb{I}_{G_{d_j}}^+|,$$

where N_{d_1} *is defined in* (3.11) *and corresponds to the number of decision variables involved in Theorems 3.3 and 3.4. The term* G_{d_j} *is obtained from the decomposition of* G_V *by applying Lemma 3.1.*

4.1.2 Choosing msets of delays

Here, the procedure to choose msets of delays discussed in Chapter 3 for Theorems 3.1 and 3.2 is extended for Theorems 4.1 and 4.2 so that the number of LMIs in each case is not excessively increased.

Assuming classical TS models, $G_0^A = G_0^B = \{0\}$, the terms $\mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H}$, $\mathbb{B}_{G_0^B}\mathbb{Y}_{G_0^Y}$ and $\mathbb{B}_{G_0^B}\mathbb{Z}_{G_0^Z}$ play similar roles in both conditions (4.1) and (4.3). Therefore, without loss of generality, the convenient choice $G_0^H = G_0^Y = G_0^Z$ can be made. It implies that $G_0^H = G_0^Y = G_0^F = G_0^Z$. Thus, in comparison to conditions of [38] discussed in Chapter 3, new degrees of freedom are introduced keeping the number of fuzzy summations the same. For motivation, consider the non-delayed condition (3.5) with the msets of delays $G_0^P = \{\langle 2, 0\rangle\}$ and $G_0^F = G_0^H = \{\langle 2, 0\rangle\}$, which corresponds to:

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r}\sum_{i_4=1}^{r}\sum_{i_5=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_k)h_{i_4}(z_{k+1})h_{i_5}(z_{k+1})$$
$$\begin{bmatrix} -P_{(i_2,i_3)} & \star \\ A_{i_1}H_{(i_2,i_3)} - B_{i_1}F_{(i_2,i_3)} & -H_{(i_4,i_5)} - H_{(i_4,i_5)}^{\top} + P_{(i_4,i_5)} \end{bmatrix} \prec 0.$$

If the same msets of delays are considered in condition (4.1) with $G_0^H = G_0^Y = G_0^F = G_0^Z$, it follows that:

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r}\sum_{i_4=1}^{r}\sum_{i_5=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_k)h_{i_4}(z_{k+1})h_{i_5}(z_{k+1})$$
$$\begin{bmatrix} -P_{(i_2,i_3)} & \star & \star \\ A_{i_1}H_{(i_2,i_3)} & \begin{pmatrix} -H_{(i_4,i_5)} - H_{(i_4,i_5)}^{\top} + P_{(i_4,i_5)} \\ -B_{i_1}Y_{(i_2,i_3)} - Y_{(i_2,i_3)}^{\top}B_{i_1}^{\top} \end{pmatrix} & \star \\ F_{(i_2,i_3)} & -Y_{(i_2,i_3)} + Z_{(i_2,i_3)}^{\top}Y_{i_1}^{\top} & Z_{(i_2,i_3)} + Z_{(i_2,i_3)}^{\top} \end{bmatrix} \prec 0.$$

Both conditions have the same 5-dimensional fuzzy summation. Thus, the procedure to derive LMIs from them is the same. As a generalization, the mset of delays in Theorem 4.1 for arbitrary dimensions can be chosen as follows: $G_0^P = \{\langle \mathbf{1}_{G_0^P}(0), 0\rangle\}$ and $G_0^F = G_0^H = G_0^Y = G_0^Z = \{\langle \mathbf{1}_{G_0^F}(0), 0\rangle\}$.

Proceeding in a similar way for the delayed condition (3.9) with msets of delays $\mathbb{P}_{G_0^P} = \{\langle 2, -1\rangle\}$ and $\mathbb{F}_{G_0^F} = \mathbb{H}_{G_0^H} = \{\langle 2, 0\rangle, \langle 2, -1\rangle\}$, it results:

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r}\sum_{i_4=1}^{r}\sum_{i_5=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_k)h_{i_4}(z_{k-1})h_{i_5}(z_{k-1})$$
$$\begin{bmatrix} -H_{(i_2,i_3,i_4,i_5)} - H_{(i_2,i_3,i_4,i_5)}^{\top} + P_{(i_4,i_5)} & \star \\ A_{i_1}H_{(i_2,i_3,i_4,i_5)} - B_{i_1}F_{(i_2,i_3,i_4,i_5)} & -P_{(i_2,i_3)} \end{bmatrix} \prec 0.$$

Choosing $G_0^H = G_0^Y = G_0^F = G_0^Z$ for the msets of delays in (4.3):

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r}\sum_{i_4=1}^{r}\sum_{i_5=1}^{r} h_{i_1}(z_k)h_{i_2}(z_k)h_{i_3}(z_k)h_{i_4}(z_{k-1})h_{i_5}(z_{k-1})$$

$$\begin{bmatrix} \begin{pmatrix} -H_{(i_2,i_3,i_4,i_5)} \\ -H_{(i_2,i_3,i_4,i_5)}^{\top} + P_{(i_4,i_5)} \end{pmatrix} & \star & \star \\ A_{i_1} H_{(i_2,i_3,i_4,i_5)} & \begin{pmatrix} -P_{(i_2,i_3)} - B_{i_1} Y_{(i_2,i_3,i_4,i_5)} \\ -Y_{(i_2,i_3,i_4,i_5)}^{\top} B_{i_1}^{\top} \end{pmatrix} & \star \\ F_{(i_2,i_3,i_4,i_5)} & -Y_{(i_2,i_3,i_4,i_5)} + Z_{(i_2,i_3,i_4,i_5)}^{\top} B_{i_1}^{\top} & Z_{(i_2,i_3,i_4,i_5)} + Z_{(i_2,i_3,i_4,i_5)}^{\top} \end{bmatrix} \prec 0.$$

Then, the msets of delays in Theorem 4.2 is generalized for arbitrary dimensions choosing $G_0^P = \{\langle \mathbf{1}_{G_0^P}(-1), -1\rangle\}$ and $G_0^F = G_0^H = G_0^Y = G_0^Z = \{\{\langle \mathbf{1}_{G_0^F}(0), 0\rangle\}, \{\langle \mathbf{1}_{G_0^F}(-1), -1\rangle\}\}$.

With these msets of delays, it is clear that the number of fuzzy summations in conditions (4.1) and (4.3) is not increased when compared, respectively, to conditions (3.5) and (3.9). Moreover, Lemma 3.1 can also be applied to derive LMI-based conditions for Theorems 4.1 and 4.2.

Remark 4.4. *In this section, from appropriate matrix transformations, it was shown the equivalence between the proposed conditions and those of [38] when these are given in terms of multidimensional fuzzy summations. However, this form is not appropriate for control design purpose. Thus, Lemma 2.3 is applied to derive a finite set of LMIs. When conditions are rewritten in terms of LMIs, the equivalence does not hold anymore and the proposed ones in this work can lead to less conservative results.*

The reduction of conservativeness provided by the proposed conditions is illustrated in the next example.

Example 4.1. *This example illustrates the application of Theorems 4.1 and 4.2 with different msets of delays. The aim is to evaluate the conservatism reduction provided by the proposed approach when compared to those existing in the literature and also compare them in terms of computational complexity. For that, the system* (2.5) *is considered again. The maximal* b *for feasibility and the numerical complexity for the proposed conditions with different msets of delays are shown in Table 4.1.*

As expected, design conservativeness is mainly reduced with delayed control. For instance, with only 3 fuzzy summations, $|G_V| = 3$*, the maximum* b *obtained with the non-delayed approach is 1.758, while 1.786 is obtained with the delayed condition. However, both values for* b *are greater than those achieved with the conditions of [38] in Table 3.1, which are 1.539 for the non-delayed case and 1.553 for the delayed. It shows that the proposed conditions can indeed lead to less conservative results.*

Following the comparison to Table 3.1, the maximum value obtained with the 6-dimensional non-delayed condition, $b = 1.693$*, is smaller than the one obtained with our*

Table 4.1 – Comparison among maximum b for feasibility obtained with different choices of G_0^P and $G_0^F = G_0^H = G_0^Y = G_0^Z$ in Theorems 4.1 and 4.2. The greatest value is in **bold**.

G_0^P	$G_0^F = G_0^H = G_0^Y = G_0^Z$	$\|G_V\|$	b	N_l	N_{d_2}	$\log_{10}(N_{d_2}^3 N_l)$
			Theorem 4.1 - Non-delayed condition			
$\{0\}$	$\{0\}$	3	1.758	30	24	5.6177
$\{\langle 2,0\rangle\}$	$\{0\}$	4	1.759	45	30	6.0858
$\{\langle 2,0\rangle\}$	$\{\langle 2,0\rangle\}$	5	1.766	60	48	6.8219
$\{\langle 3,0\rangle\}$	$\{\langle 2,0\rangle\}$	6	1.768	80	60	7.2375
			Therem 4.2 - Delayed condition			
$\{-1\}$	$\{0,-1\}$	3	1.786	30	42	6.3469
$\{-1\}$	$\{\langle 2,0\rangle,-1\}$	4	1.794	45	84	7.4261
$\{-1\}$	$\{\langle 3,0\rangle,-1\}$	5	1.803	50	150	8.2272
$\{-1\}$	$\{\langle 4,0\rangle,-1\}$	6	**2.041**	60	294	9.1832

proposal for only 3 fuzzy summations, which is $b = 1.758$. *The same occurs in the non-delayed case, where the maximum* b *obtained with condition* (3.9), $b = 1.757$, *is smaller than the 3-dimensional case of condition* (4.3), $b = 1.786$. *The proposed conditions are also compared with other in the recent literature, as shown in Table 4.2. The value of* $b = 2.041$ *obtained with Theorem 4.2 is greater than those reported in the recent literature and it is obtained requiring less computational complexity.*

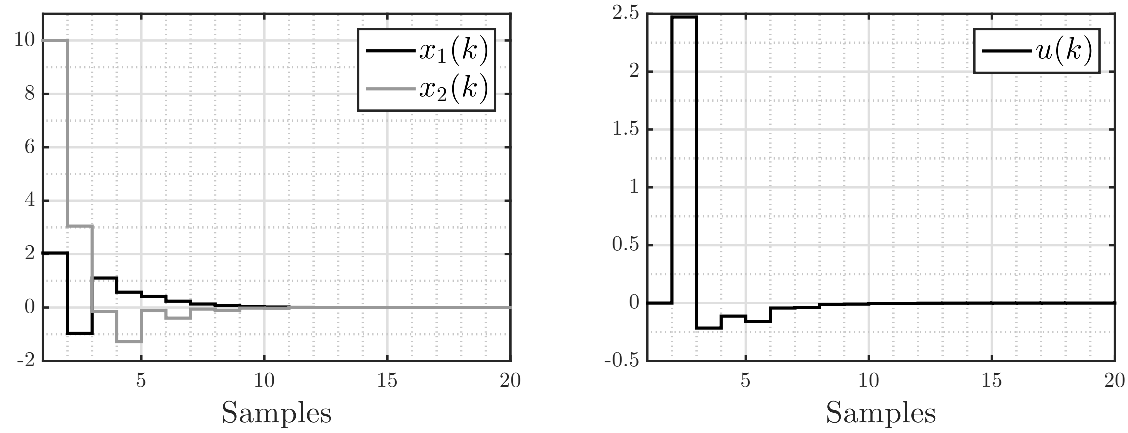
Table 4.2 – Comparison among maximum b for feasibility and computational complexity for different approaches in the literature.

	b	$\log_{10}(N_d^3 N_l)$
[64]	1.774	12.318
[65]	1.817	17.3221
[35]	1.819	13.4150
[31]	1.821	11.3152
[66]	1.823	13.691
[34]	1.828	14.2160
[33]	1.875	9.3324
Theorem 4.2	2.041	9.1832

To conclude, Figure 4.1 depicts the state trajectories of the closed-loop system (3.3) *and the control signal for the controller* (3.2) *designed with Theorem 4.2 for* $b = 2.041$.

4.2 Improving l_2-gain performance control

In this section, the proposed approach is extended to deal with the l_2-gain performance control problem. The aim is to obtain new conditions to design the fuzzy controller (3.2) so



(a) States **(b)** Control signal

Figure 4.1 – Closed-loop trajectories of system (2.5) with $b = 2.041$ in feedback with controller (3.2) designed with Theorem 4.2.

that the upper-bound for the l_2-gain of system (3.14) be minimized. From the results in the last section, it is expected that the conditions proposed here result in a more efficient disturbance attenuation than conditions in Theorems 3.3 and 3.4. Similar to the last section, two more conditions are proposed here, one related to the Lyapunov function candidate (3.4), which leads to non-delayed conditions, and other to the Lyapunov candidate (3.8) that corresponds to delayed conditions.

Non-delayed condition

The condition based on the Lyapunov function candidate (3.4) is stated in the next theorem.

Theorem 4.3. *Let* $G_V = G_0^P \cup G_1^P \cup (G_0^B \oplus G_0^Y) \cup (G_0^H \oplus G_0^A) \cup (G_0^Z \oplus G_0^B) \cup G_1^H \cup G_0^K \cup G_0^E$ *be given. If there exist a scalar* $\gamma > 0$ *and matrices* $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}^{\mathbf{i}}_{G_j^P}$, $H_{\mathbf{i}_j^H}$, $\mathbf{i}_j^H = \mathrm{pr}^{\mathbf{i}}_{G_j^H}$, $j = 0, 1$, $F_{i_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}^{\mathbf{i}}_{G_0^F}$, $Y_{i_0^Y}$, $\mathbf{i}_0^Y = \mathrm{pr}^{\mathbf{i}}_{G_0^Y}$, $W_{i_0^W}$, $\mathbf{i}_0^W = \mathrm{pr}^{\mathbf{i}}_{G_0^W}$, *and* $Z_{i_0^Z}$, $\mathbf{i}_0^Z = \mathrm{pr}^{\mathbf{i}}_{G_0^Z}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, *such that* (4.4) *holds. Then, the closed-loop system* (3.14) *is dissipative with respect to the supply rate* (3.15) *and the* l_2*-gain has upper bound* γ*.*

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \star & \star & \star & \star \\ 0 & -\gamma^2 \mathrm{I} & \star & \star & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} & \mathbb{E}_{G_0^E} & \psi_{33} & \star & \star \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} & \mathbb{K}_{G_0^K} & \psi_{43} & \psi_{44} & \star \\ \mathbb{F}_{G_0^F} & 0 & \psi_{53} & \psi_{54} & \mathbb{Z}_{G_0^Z} + \mathbb{Z}_{G_0^Z}^\top \end{bmatrix} \prec 0, \tag{4.4}$$

where $\psi_{43} = -\mathbb{D}_{G_0^D}\mathbb{Y}_{G_0^Y} - \mathbb{W}_{G_0^W}^\top \mathbb{B}_{G_0^B}^\top$, $\psi_{33} = -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^\top + \mathbb{P}_{G_1^P} - \mathbb{B}_{G_0^B}\mathbb{Y}_{G_0^Y} - \mathbb{Y}_{G_0^Y}^\top \mathbb{B}_{G_0^B}^\top$, $\psi_{44} = -\mathrm{I} - \mathbb{D}_{G_0^D}\mathbb{W}_{G_0^W} - \mathbb{W}_{G_0^W}^\top \mathbb{D}_{G_0^D}^\top$, $\psi_{53} = -\mathbb{Y}_{G_0^Y} + \mathbb{Z}_{G_0^Z}^\top \mathbb{B}_{G_0^B}^\top$, $\psi_{54} = -\mathbb{W}_{G_0^W} + \mathbb{Z}_{G_0^Z}^\top \mathbb{D}_{G_0^D}^\top$.

Proof. Assume that condition (4.4) holds. Similar to the proof of Theorem 4.1, it is possible to show that $\mathbb{H}_{G_1^H}$ and $\mathbb{H}_{G_0^H}$ are invertible.

Multiplying (4.4) with

$$\begin{bmatrix} \mathbb{H}_{G_0^H}^{-\top} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{I} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{I} & 0 & -\mathbb{B}_{G_0^B} \\ 0 & 0 & 0 & \mathrm{I} & -\mathbb{D}_{G_0^D} \end{bmatrix}$$

on the left and its transpose on the right, one has

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \star & \star & \star \\ 0 & -\gamma^2\mathrm{I} & \star & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \mathbb{E}_{G_0^E} & -\mathbb{H}_{G_1^H}^T\mathbb{P}_{G_1^P}^{-1}\mathbb{H}_{G_1^H} & \star \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F} & \mathbb{K}_{G_0^K} & 0 & -\mathrm{I} \end{bmatrix} \prec 0.$$

Following the same steps as Theorem 3.3, it is possible to conclude that condition (4.4) ensures the origin of the closed-loop system (3.14) is asymptotically stable and the l_2-gain has upper bound γ. □

The optimization problem related to the l_2-gain upper-bound minimization using condition (4.4) is depicted in Lemma 4.1.

Lemma 4.1. *Let G_V be given as in Theorem 4.3. If there exist a scalar $\mu = \gamma^2$ and matrices $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}^{\mathbf{i}}_{G_j^P}$, $H_{\mathbf{i}_j^H}$, $\mathbf{i}_j^H = \mathrm{pr}^{\mathbf{i}}_{G_j^H}$, $j = 0, 1$, $F_{i_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}^{\mathbf{i}}_{G_0^F}$, $Y_{i_0^Y}$, $\mathbf{i}_0^Y = \mathrm{pr}^{\mathbf{i}}_{G_0^Y}$, $W_{i_0^W}$, $\mathbf{i}_0^W = \mathrm{pr}^{\mathbf{i}}_{G_0^W}$, and $Z_{i_0^Z}$, $\mathbf{i}_0^Z = \mathrm{pr}^{\mathbf{i}}_{G_0^Z}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, such that the optimization problem*

$$\min_{P_{\mathbf{i}_0^P}, P_{\mathbf{i}_1^P}, H_{\mathbf{i}_0^H}, H_{\mathbf{i}_1^H}, F_{\mathbf{i}_j^F}, Y_{\mathbf{i}_1^Y}, W_{\mathbf{i}_1^W}, Z_{\mathbf{i}_1^Z}} \quad \mu, \qquad \textit{s.t. } (4.4)$$

is feasible. Then, the closed-loop system (3.14) *is dissipative with respect to the supply rate* (3.15) *and $\gamma = \sqrt{\mu}$ is the minimal upper-bound for the l_2-gain.*

Proof. The proof is consequence of Theorem 4.3. □

Remark 4.5. *Similar to Remark 4.1, it is possible to show that condition* (3.16) *is equivalent to* (4.4)*. Following the arguments of [45], a sufficiently large scalar $\rho > 0$ can be selected such that condition* (3.16) *can be rewritten as*

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \star & \star & \star & \star \\ 0 & -\gamma^2\mathrm{I} & \star & \star & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \mathbb{E}_{G_0^E} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^\top + \mathbb{P}_{G_1^P} & \star & \star \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F} & \mathbb{K}_{G_0^K} & 0 & -\mathrm{I} & \star \\ \mathbb{F}_{G_0^F} & 0 & 0 & 0 & -\rho\mathrm{I} \end{bmatrix} \prec 0. \tag{4.5}$$

Defining the new variables

$$\mathbb{Z}_{G_0^Z} = -\frac{\rho}{2}\mathrm{I}, \quad \mathbb{Y}_{G_0^Y} = -\mathbb{Z}_{G_0^Z}\mathbb{B}_{G_0^B}^\top, \quad \mathbb{W}_{G_0^W} = -\mathbb{Z}_{G_0^Z}\mathbb{D}_{G_0^D}^\top,$$

the inequality (4.5) *is equivalent to*

$$\begin{bmatrix} -\mathbb{P}_{G_0^P} & \star & \star & \star & \star \\ 0 & -\gamma^2 \mathrm{I} & \star & \star & \star \\ \psi_{31} & \mathbb{E}_{G_0^E} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^\top + \mathbb{P}_{G_1^P} & \star & \star \\ \psi_{41} & \mathbb{K}_{G_0^K} & 0 & -\mathrm{I} & \star \\ \mathbb{F}_{G_0^F} & 0 & -\mathbb{Y}_{G_0^Y} - \mathbb{Z}_{G_0^Z}\mathbb{B}_{G_0^B}^\top & -\mathbb{W}_{G_0^W} - \mathbb{Z}_{G_0^Z}\mathbb{D}_{G_0^D}^\top & \mathbb{Z}_{G_0^Z} + \mathbb{Z}_{G_0^Z}^\top \end{bmatrix} \prec 0,$$

where $\psi_{31} = \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}$ *and* $\psi_{41} = \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}$. *By multiplying the above inequality with*

$$\begin{bmatrix} \mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{I} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{I} & 0 & \mathbb{B}_{G_0^B} \\ 0 & 0 & 0 & \mathrm{I} & \mathbb{D}_{G_0^D} \\ 0 & 0 & 0 & 0 & \mathrm{I} \end{bmatrix}$$

on the left and its transpose on the right, results in (4.4).

The converse is shown by multiplying (4.4) *with*

$$\begin{bmatrix} \mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{I} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{I} & 0 & -\mathbb{B}_{G_0^B} \\ 0 & 0 & 0 & \mathrm{I} & -\mathbb{D}_{G_0^D} \end{bmatrix},$$

on the left and its transpose on the right, which results in (3.16).

Delayed condition

The condition based on the Lyapunov function candidate (3.8) as storage function is depicted in the following theorem. This condition corresponds to the use of delayed control.

Theorem 4.4. *Let* $G_V = G_0^P \cup G_1^P \cup (G_0^B \oplus G_0^Y) \cup (G_0^H \oplus G_0^A) \cup (G_0^Z \oplus G_0^B) \cup G_0^K \cup G_0^E$ *be given. If there exist a scalar* $\gamma > 0$ *and matrices* $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top \succ 0$, $\mathbf{i}_j^P = \mathrm{pr}^{\mathbf{i}}_{G_j^P}$, $j = 0, 1$, $H_{\mathbf{i}_0^H}$, $\mathbf{i}_0^H = \mathrm{pr}^{\mathbf{i}}_{G_0^H}$, $F_{i_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}^{\mathbf{i}}_{G_0^F}$, $Y_{i_0^Y}$, $\mathbf{i}_0^Y = \mathrm{pr}^{\mathbf{i}}_{G_0^Y}$, $W_{i_0^W}$, $\mathbf{i}_0^W = \mathrm{pr}^{\mathbf{i}}_{G_0^W}$, *and* $Z_{i_0^Z}$, $\mathbf{i}_0^Z = \mathrm{pr}^{\mathbf{i}}_{G_0^Z}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, *such that* (4.6) *holds. Then, the closed-loop system* (3.14) *is dissipative with respect to the supply rate* (3.15) *and the* l_2*-gain has upper bound* γ.

$$\begin{bmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^\top + \mathbb{P}_{G_0^P} & \star & \star & \star & \star \\ 0 & -\gamma^2 \mathrm{I} & \star & \star & \star \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} & \mathbb{E}_{G_0^E} & \psi_{33} & \star & \star \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} & \mathbb{K}_{G_0^K} & \psi_{43} & \psi_{44} & \star \\ \mathbb{F}_{G_0^F} & 0 & \psi_{53} & \psi_{54} & \mathbb{Z}_{G_0^Z} + \mathbb{Z}_{G_0^Z}^\top \end{bmatrix} \prec 0, \tag{4.6}$$

where $\psi_{43} = -\mathbb{D}_{G_0^D}\mathbb{Y}_{G_0^Y} - \mathbb{W}_{G_0^W}^\top \mathbb{B}_{G_0^B}^\top$, $\psi_{33} = -\mathbb{P}_{G_1^P} - \mathbb{B}_{G_0^B}\mathbb{Y}_{G_0^Y} - \mathbb{Y}_{G_0^Y}^\top \mathbb{B}_{G_0^B}^\top$,
$\psi_{44} = -\mathrm{I} - \mathbb{D}_{G_0^D}\mathbb{W}_{G_0^W} - \mathbb{W}_{G_0^W}^\top \mathbb{D}_{G_0^D}^\top$, $\psi_{53} = -\mathbb{Y}_{G_0^Y} + \mathbb{Z}_{G_0^Z}^\top \mathbb{B}_{G_0^B}^\top$, $\psi_{54} = -\mathbb{W}_{G_0^W} + \mathbb{Z}_{G_0^Z}^\top \mathbb{D}_{G_0^D}^\top$.

Proof. The proof follows similar steps as the one of Theorem 4.3 choosing the storage function as (3.4) and appropriate congruence transformations. □

The optimization problem related to the l_2-gain upper-bound minimization using condition (4.6) is depicted in Lemma 4.2.

Lemma 4.2. *Let G_V be given as in Theorem 4.4. If there exist a scalar $\mu = \gamma^2$ and matrices $P_{\mathbf{i}_j^P} = P_{\mathbf{i}_j^P}^\top$, $\mathbf{i}_j^P = \mathrm{pr}^{\mathbf{i}}_{G_j^P}$, $j = 0, 1$, $H_{\mathbf{i}_0^H}$, $\mathbf{i}_0^H = \mathrm{pr}^{\mathbf{i}}_{G_0^H}$, $F_{i_0^F}$, $\mathbf{i}_0^F = \mathrm{pr}^{\mathbf{i}}_{G_0^F}$, $Y_{i_0^Y}$, $\mathbf{i}_0^Y = \mathrm{pr}^{\mathbf{i}}_{G_0^Y}$, $W_{i_0^W}$, $\mathbf{i}_0^W = \mathrm{pr}^{\mathbf{i}}_{G_0^W}$, and $Z_{i_0^Z}$, $\mathbf{i}_0^Z = \mathrm{pr}^{\mathbf{i}}_{G_0^Z}$, $\mathbf{i} \in \mathbb{I}_{G_V}$, such that the optimization problem*

$$\min_{P_{\mathbf{i}_0^P}, P_{\mathbf{i}_1^P}, H_{\mathbf{i}_0^H}, F_{\mathbf{i}_j^F}, Y_{\mathbf{i}_1^Y}, W_{\mathbf{i}_1^W}, Z_{\mathbf{i}_1^Z}} \mu, \quad \text{s.t. (4.6)} \tag{4.7}$$

is feasible. Then, the closed-loop system (3.14) *is dissipative with respect to the supply rate* (3.15) *and $\gamma = \sqrt{\mu}$ is the minimal upper bound for the l_2-gain.*

Proof. The proof is consequence of Theorem 4.4. □

Remark 4.6. *Using similar arguments as Remark 4.5, it is possible to show that condition* (3.18) *is equivalent to* (4.6).

Remark 4.7. *The same conclusion presented in Remark 4.4 follows for Theorems 4.3 and 4.4.*

The following example is considered to illustrate the effectiveness of proposed conditions in providing smaller l_2-gain upper-bounds than those existing.

Example 4.2 (see [38])**.** *In this example we compare the two proposed l_2-gain control performance methods. To compare the results obtained here with those presented in Table 3.2, the TS fuzzy model* (3.20) *is considered. In addition, the same delays are considered, namely, $G_0^P = G_0^F = G_0^H = \{0\}$ for Lemma 4.1 and $G_0^P = \{0\}$, $G_0^F = G_0^H = \{0, -1\}$ for Lemma 4.2. The minimal upper-bound for the l_2-gain obtained with theses lemmas are depicted in Table 4.3.*

Table 4.3 – Comparison of l_2-gain upper-bounds obtained with Lemmas 4.1 and 4.2.

	γ
Lemma 4.1	0.511
Lemma 4.2	0.5061

In comparison to conditions of [38], whose results are in Table 3.2, the proposed approach led to smaller l_2-gain upper-bounds. It illustrates the effectiveness in providing less conservatism outcomes.

4.3 Numerical simulations

This section presents the application of the proposed control design conditions in physically-motivated models. The aim is to illustrate the proposal effectiveness to deal with practical control problems. For that, two systems are considered: the inverted pendulum and the truck-trailer. In the inverted pendulum application, the msets of delays are chosen such that Theorem 4.1 represents a new non-PDC design condition. The results are compared with the classical non-PDC design of [29]. In the truck-trailer application, the fuzzy controller is designed considering the l_2-gain performance design in order to attenuate energy-bounded disturbances acting on the system. In this case, the condition of Lemma 4.2 is considered for delayed control design. The results are compared with previous methods applied to the same problem.

4.3.1 Inverted pendulum system

The aim of this section is to apply Theorem 4.1 with the msets of delays $G_0^P = \{0\}$, $G_0^G = G_0^F = G_0^Y = G_0^Z = \{0\}$. It corresponds to a novel non-PDC design approach. Thus, LMI-based conditions can be directly derived using Lemma 3.1. The results are compared with the classical non-PDC design condition given in Theorem 2.2, or, equivalently, in Theorem 3.1 with the msets of delays $G_0^P = \{0\}$, $G_0^G = G_0^F = \{0\}$.

To illustrate the effectiveness of proposed approach, the state-feedback control of an inverted pendulum controlled by a DC motor via a gear train is considered. The continuous-time nonlinear dynamic equations which describe this system are [64]:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= \frac{g}{l}\sin x_1(t) + \frac{NK_m}{ml^2}x_3(t)\\ \dot{x}_3(t) &= -\frac{K_bN}{L_a}x_2(t) - \frac{R_a}{L_a}x_3(t) + \frac{1}{L_a}u(t),\end{aligned} \tag{4.8}$$

where x_1 is the angular position, x_2 is the angular velocity, x_3 is the DC motor armature current and u is the control input. In addition, K_m is the motor constant, K_b is the back electromotive force constant and N is the gear ratio. The considered parameters are the same as [64]: $g = 9.8$ m/s^2, $l = 1$ m, $m = 1$ kg, $N = 10$, $K_m = 0.1$ Nm/A, $K_b = 0.1$ Vs/rad, $R_a = 1\ \Omega$ and $L_a = 0.5$ mH.

By employing Euler's discretization method in the differential equations (4.8), the following discrete-time nonlinear equations are obtained:

$$\begin{aligned}x_{k+1(1)} &= x_{k(1)} + Tx_{k(2)}\\ x_{k+1(2)} &= x_{k(2)} + T(9.8\sin x_{k(1)} + x_{k(3)})\\ x_{k+1(3)} &= x_{k(3)} + 2T(-x_{k(2)} - x_{k(3)} + u_k),\end{aligned} \tag{4.9}$$

with $T = 0.1$ s assumed as sampling time. It is possible to derive a TS model to represent system (4.9) by applying the sector nonlinearity approach [8]. Considering the premise variable $z_k = \sin x_{k(1)}$ within the set $\Omega_x = \{x \in \mathbb{R}^3 : |x_{k(1)}| \leq \pi, \forall k \in \mathbb{N}\}$, the obtained TS model is based on the following membership functions [64]:

$$M_1^1(z_k) = \begin{cases} \dfrac{\sin x_{k(1)}}{x_{k(1)}}, & x_{k(1)} \neq 0, \\ 1, & x_{k(1)} = 0 \end{cases} \quad \text{and} \quad M_1^2(z_k) = 1 - M_1^1(z_k),$$

and local models:

$$A_1 = \begin{bmatrix} 1 & 0.1 & 0 \\ 0.98 & 1 & 0.1 \\ 0 & -0.2 & 0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0.1 & 0 \\ 0 & 1 & 0.1 \\ 0 & -0.2 & 0.8 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.2 \end{bmatrix}.$$

By solving[1] the LMI-based conditions derived from Theorems 4.1 and 2.2, the designed control gains are listed as follows.

Control gains obtained with Theorem 2.2:

$$F_1 = \begin{bmatrix} 0.0282 \\ 0.1319 \\ 8.1496 \end{bmatrix}^\top, F_2 = \begin{bmatrix} 0.0626 \\ 0.1624 \\ 8.0471 \end{bmatrix}^\top, H_1 = \begin{bmatrix} 0.0639 & -0.1479 & 0.044 \\ -0.1559 & 0.4819 & -0.1638 \\ -0.1330 & -0.1594 & 1.9663 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.0639 & -0.1479 & 0.044 \\ -0.1559 & 0.4819 & -0.1638 \\ -0.1330 & -0.1594 & 1.9663 \end{bmatrix}.$$

Control gains obtained with Theorem 4.1:

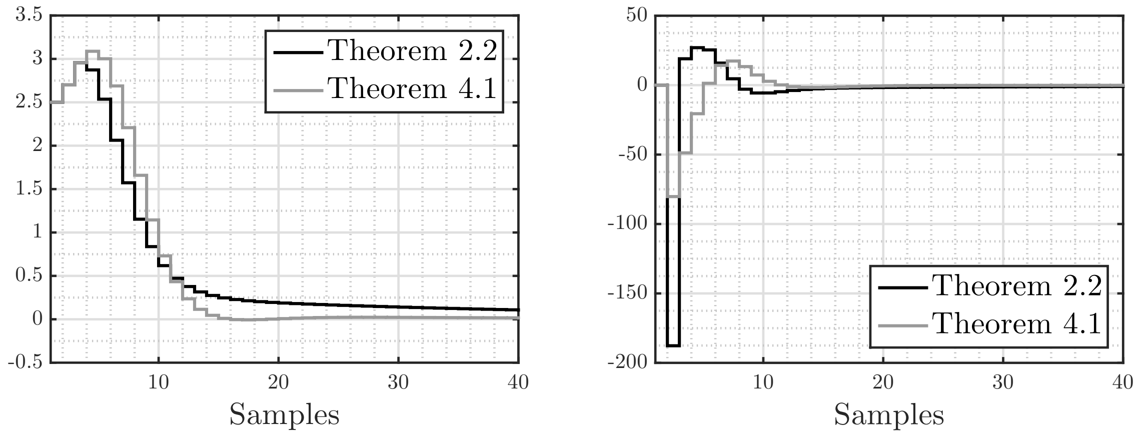
$$F_1 = \begin{bmatrix} 0.1230 \\ -0.0294 \\ -0.2640 \end{bmatrix}^\top, F_2 = \begin{bmatrix} -0.1679 \\ 0.6099 \\ 0.3120 \end{bmatrix}^\top, H_1 = \begin{bmatrix} 0.0381 & -0.0945 & -0.0222 \\ -0.0939 & 0.3422 & -0.3279 \\ -0.0661 & -0.2391 & 1.7638 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.0362 & -0.0835 & -0.038 \\ -0.0726 & 0.2778 & -0.2981 \\ -0.0517 & -0.2815 & 1.7816 \end{bmatrix}.$$

Assuming the initial state $x_0 = [2.5, 2, -0.24]^\top$ and the above designed controllers, the time responses of both state x_1 and control input u are shown in Figure 4.2.

It can be observed in Figure 4.2 that the non-PDC designed with the proposed condition could effectively stabilize the inverted-pendulum system.

[1] The LMIs are solved in Matlab using Yalmip [10] and SeDuMi solver.



(a) State $x_{k(1)}$ **(b)** Control signal u_k

Figure 4.2 – Closed-loop trajectories of inverted pendulum system controlled by the non-PDC designed with Theorems 2.2 (black) and 4.1 (gray) applied to the inverted pendulum system. (a) state x_1; (b) control signal.

4.3.2 Truck-trailer system

This section presents the application of the proposed approach considering the l_2-gain control performance on the truck-trailer system. This problem has been addressed, for instance, in the works of [67, 68, 69] and [62]. However, delayed control was not applied in this context, which motivates this application. Instead of the model formulated by [70], the following simplified dynamics used by [69, 62] is considered:

$$\begin{aligned} x_{k+1(1)} &= (1 - vT/L)x_{k(1)} + (vT/l)u_k \\ x_{k+1(2)} &= (vT/L)x_{k(1)} + x_{2(k)} + 0.2w_k \\ x_{k+1(3)} &= x_{k(3)} + vT\sin\left(x_{k(2)} + (vT/2L)x_{k(1)}\right) + 0.1w_k \\ y_k &= 7x_{k(1)} - 2x_{x_2(k)} + 0.03x_{k(3)}, \end{aligned} \tag{4.10}$$

where x_1 is the difference of angles between the truck and the trailer, x_2 the angle of the trailer compared to the horizontal axis, x_3 the position related to the vertical axis of the back of the trailer, u is the steering angle applied to the front wheels of the truck, w is the angle disturbance and T is the sampling time. The system is illustrated in Figure 4.3.

Following the same modeling procedure as in [69, 68] and [62], where the sector nonlinearity approach was applied considering the premise variable $z_k = x_{k(2)} + (vT/2L)x_{k(1)}$ within the interval $|z| \leq 179.427$, the nonlinear truck-trailer model can be represented as a TS fuzzy model with the following membership functions:

$$M_1^1(z_k) = 1 - M_1^2(z_k), \quad \text{and} \quad M_1^2(z_k) = \begin{cases} \dfrac{z_k - \sin z_k}{z_k(1-g)}, & z_k \neq 0, \\ 1, & z_k = 0 \end{cases},$$

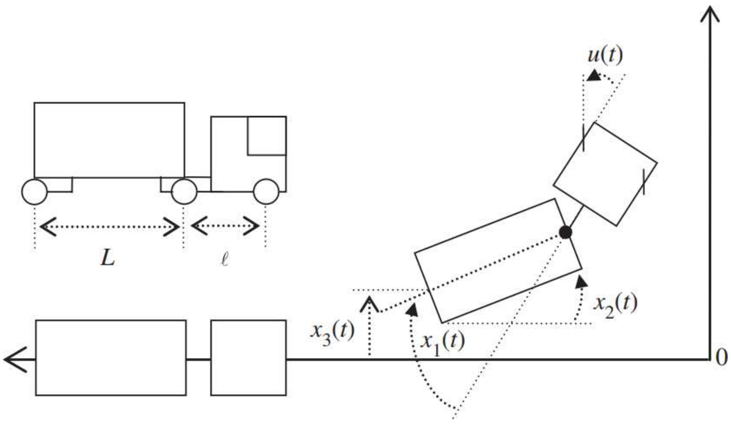


Figure 4.3 – Illustration of truck-trailer system. Extracted from [69].

and local models:

$$A_1 = \begin{bmatrix} 1 - vT/L & 0 & 0 \\ vT/L & 1 & 0 \\ v^2T^2/(2L) & vT & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 - vT/L & 0 & 0 \\ vT/L & 1 & 0 \\ gv^2T^2/(2L) & gvT & 1 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} vT/l \\ 0 \\ 0 \end{bmatrix},$$

$$E_1 = E_2 = \begin{bmatrix} 0 \\ 0.2 \\ 0.1 \end{bmatrix}, \quad C_1 = C_2 = \begin{bmatrix} 7 & -2 & 0.03 \end{bmatrix},$$

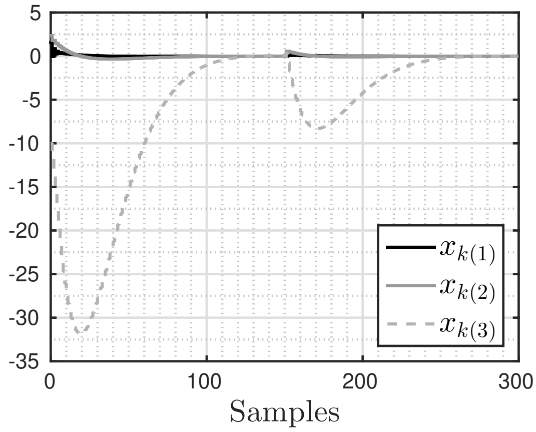
where $g = 10^{-2}/\pi$. The model parameters are selected as $l = 2.8$ m, $L = 5.5$ m, $v = -1.0$ m/s and $T = 2.0$ s. Considering the proposed delayed condition in (4.6) with msets of delays $G_0^P = \{-1\}$, $G_0^F = G_0^H = G_0^Y = G_0^W = G_0^Z = \{0, -1\}$, this choice of msets leads to a 3-dimensional fuzzy summation based condition. In order to minimize the l_2-gain upper-bound, Lemma 4.2 is applied with LMIs derived applying Lemma 2.3. Solving the optimization problem, the obtained upper-bound is $\gamma = 0.397$.

To illustrate the effectiveness, a numerical simulation is performed considering the initial condition $x_0 = [\pi/2,\ 3\pi/4,\ -10]^\top$ and the following energy-bounded disturbance:

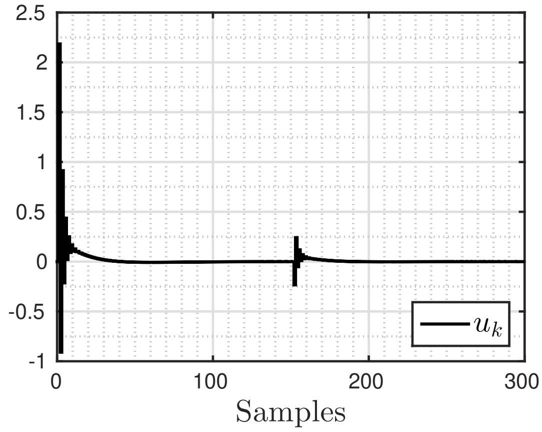
$$w_k = \begin{cases} 2.8, & 150 \leq k \leq 151, \\ 0, & k < 150 \text{ or } k > 151, \end{cases}$$

which corresponds to a perturbation with duration of 2 s. The state response of the closed-loop system and the control signal are shown in Figure 4.4. It is possible to note that the state trajectories converge to zero in the interval $1 \leq k \leq 150$. At the sample time $k = 150$, when the states are perturbed by w_k, the control acts to attenuate its effect and the states converge again to the origin until the end of the simulation.

This section has presented numerical simulations on stabilization of two physically motivated systems. In the inverted pendulum application, the results showed the new non-PDC design was able to stabilize the system requiring less control effort, which means less required energy. When it comes to the truck-trailer application, the proposed delayed conditions provided



(a) States x_k



(b) Control signal u_k

Figure 4.4 – Closed-loop trajectories of truck-trailer system in feedback with controller (2.20) designed with Lemma 4.2 applied to the truck-trailer system. (a) states; (b) control signal.

control gains such that the influence of energy-bounded disturbances are attenuated. Therefore, the numerical simulations presented here indicate the proposal effectiveness to cope with practical control problems.

4.4 Conclusion

The main contributions of this work have been presented in this chapter. They are related to new conditions able to reduce fuzzy control design conservativeness without increasing the number of fuzzy summations in comparison to the conditions of [38]. As a consequence, less conservative results could be obtained with a reduced computational complexity. In addition, using appropriate matrix transformations, it was shown that the conditions proposed here can be viewed as generalizations for those of [38], which in its turn generalize other previous existing conditions. Therefore, the proposal constitutes a new framework for model-based fuzzy control. The effectiveness was illustrated with numerical simulations on the application to practical systems.

5 ENDING COMMENTS

This work has proposed new sufficient conditions to design fuzzy controllers for stabilization of discrete-time nonlinear systems described by Takagi-Sugeno (TS) fuzzy models. The proposed design conditions are based on a general multidimensional fuzzy control law and nonquadratic Lyapunov functions, which allows easily designing both non-delayed and delayed fuzzy controllers in a multiple-parameterized setting. Then, the proposed conditions are obtained regarding appropriate congruence transformations based on the introduction of new decision variables. These are extended to cope with the disturbance attenuation problem based on the minimization of the l_2-gain performance, which improves the practical usage.

However, the proposed conditions are initially written in terms of multidimensional fuzzy summation-dependent LMI constraints that cannot be solved directly by numerical solvers. Aiming to perform the design in an appropriate way, it was proposed a methodology to derive a finite set of LMI-based conditions to design the fuzzy controllers.

In comparison to recent related approaches for fuzzy control design, there are two important aspects to be pointed out with respect to the conditions proposed here: (i) it was theoretically demonstrated that they contain other conditions in the literature as particular cases; (ii) it was shown via numerical simulations that less conservative designs can be obtained with a reduced computational complexity.

5.1 Future directions

Delayed control laws for stabilization of time-delayed nonlinear discrete-time systems.

In practical problems such as networked control, the influence of communication limitation in the closed-loop dynamics is generally modeled as a time-delayed state. The idea is to evaluate the possibility to obtain less conservative designs using the delayed control laws with multisets of delays selected in terms of the time-delay induced by communications constraints, for example.

Deriving new fuzzy static output feedback control design conditions.

Here, matrix transformations similar to those employed to derive the proposed state-feedback design conditions could be applied to obtain less conservative designs for fuzzy static output feedback controllers.

5.2 Publications

The publications related to the contributions of this work are listed below.

a) P. H. S. Coutinho and R. M. Palhares. Estabilização de modelos fuzzy Takagi-Sugeno a tempo discreto: reduzindo o conservadorismo no controle não-PDC. In *XXII Congresso Brasileiro de Automática*. SBA, João Pessoa, PB, 2018.

b) P. H. S. Coutinho, J. Lauber, M. Bernal, and R. M. Palhares. Efficient LMI conditions for enhanced stabilization of discrete-time Takagi-Sugeno models via delayed nonquadratic Lyapunov functions. *IEEE Transactions on Fuzzy Systems*, Early Access:1–10, 2019. DOI: 10.1109/TFUZZ.2019.2892341.

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APPENDICES

APPENDIX A – LYAPUNOV STABILITY THEORY FOR DISCRETE-TIME NONLINEAR SYSTEMS

This appendix regards the stability analysis of discrete-time nonlinear systems. The study is motivated by the Lyapunov stability analysis of continuous-time nonlinear systems and the established concepts and definitions are presented focusing on the discrete-time domain.

Consider the continuous-time autonomous nonlinear system

$$\dot{x}(t) = f(t, x(t)), \tag{A.1}$$

where $x \in \mathbb{R}^n$ is the vector state and $f : [t_0, \infty) \times D \to \mathbb{R}^n$ is a locally Lipschitz function over a domain $D \subset \mathbb{R}^n$ with the origin is contained in D. The points at which the vector field $f(t, x(t))$ vanishes for all $t \geq 0$, i.e,

$$f(t, \bar{x}) = 0, \quad \forall t \geq t_0, \tag{A.2}$$

are called equilibrium points of (A.1). The main goal in this section is decide about the stability of a given equilibrium point $\bar{x}$. Without loss of generality, the origin ($x = 0$) may be considered as equilibrium point for stability analysis; this is not restrictive because it is possible to translate the state space origin to the equilibrium point of interest [71]. Furthermore, as *the stability is a property of the equilibrium point*, it is not affected by geometrical transformations.

The stability of equilibrium points of nonlinear systems is frequently studied based on the concepts developed by Lyapunov in 1892 [72, 9]. The stability in the sense of Lyapunov is defined as follows.

Definition A.1 ([9]). *The equilibrium point $x = 0$ of (A.1) is stable if, for each $\varepsilon > 0$ and any t_0 there is $\delta(\varepsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0. \tag{A.3}$$

By the above definition, trajectories starting in a given initial condition, say $x(t_0) = x_0 \in B_\delta$, with $B_\delta = \{x \in \mathbb{R}^n \mid \|x(t)\| < \delta\}$, will never leave $B_\varepsilon = \{x \in \mathbb{R}^n \mid \|x(t)\| < \varepsilon\}$ for all $t \geq t_0$, as illustrated in Figure A.1. In other words, trajectories starting sufficiently close to the equilibrium point (origin, in this case) remains close to it. In view of Definition A.1, the equilibrium point is *unstable* if it is not stable. If the nonlinear system is autonomous, i.e, $\dot{x} = f(x)$, then the stability condition does not depend on time t, implying that for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that (A.3) is satisfied independently of t_0, this is called *uniform stability*. In addition, if δ could be chosen such that

$$\|x(t_0)\| < \delta \Rightarrow \lim_{t \to \infty} \|x(t)\| = 0, \tag{A.4}$$

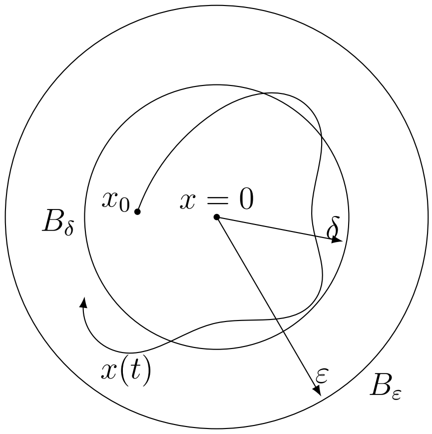

Figure A.1 – Illustration of the stability concept in the sense of Lyapunov.

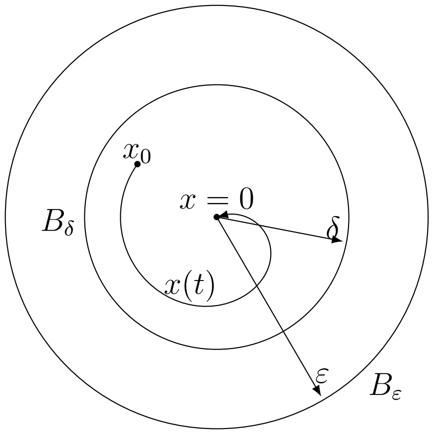

Figure A.2 – Illustration of asymptotic stability in the sense of Lyapunov.

then the equilibrium point is said *asymptotically stable*; it is illustrated in Figure A.2. In a similar way, if the system is autonomous, the equilibrium is *uniformly asymptotically stable*.

Notice the above definitions concludes about the local stability of equilibrium points since the assumptions are valid in a domain $D \subset \mathbb{R}^n$. When these conditions hold for all $x(t_0) \in \mathbb{R}^n$, the stability is global. Nonlinear systems generally present multiple equilibrium points (stable or unstable), conducting to local stability analysis. In contrast, linear systems of the form $\dot{x} = Ax$ have a unique isolated equilibrium point at the origin; thus conclusions of its stability are global.

Until now, discussions have been concerned about stability analysis of continuous-time nonlinear systems. From now on, the focus will be on the stability analysis of discrete-time nonlinear systems, in special, autonomous discrete-time nonlinear systems of the form

$$x_{k+1} = f(x_k), \tag{A.5}$$

where k is the sample time, $f : D \to \mathbb{R}^n$ is continuous in $D \subset \mathbb{R}^n$ and $0 \in D$. With these assumptions about f, all the stability definitions previously presented in the continuous-time domain are valid for equilibrium points of discrete-time nonlinear systems. It is summarized in the following definition.

Definition A.2. *The equilibrium point $x = 0$ of (A.5) is*
(i) stable if, for each $\varepsilon > 0$ there is $\delta(\varepsilon)$ such that

$$\|x_0\| < \delta \Rightarrow \|x_k\| < \varepsilon, \quad \forall k \geq 0;$$

(ii) unstable if it is not stable;
(iii) asymptotically stable if it is stable and δ can be chosen such that

$$\|x_0\| < \delta \Rightarrow \lim_{k \to \infty} \|x_k\| = 0.$$

Similar to the continuous-time case, it was considered (without loss of generality) the origin as equilibrium point whose stability one wants to study. However, in both cases, conclusions about stability requires the solution of the system equations, which can be quite difficult for general nonlinear systems.

Another approach to study the stability of equilibrium points is the second method of Lyapunov, also called Lyapunov's direct method. This method enables one to determine the stability of an equilibrium point without explicitly obtaining trajectories of the system by employing what can be viewed as a generalization of energy dissipation concept [71]. The main idea is to consider a (locally) energy-like function of the state, $V(x)$, called Lyapunov function candidate and evaluate if it monotonically decreases to zero as the time passes. This implies the "total energy" tends to zero and trajectories tends to equilibrium. In the continuous-time domain, the decreasing of such function is verified if $\dot{V}(x) < 0$ and in the discrete-time domain if $V(x_{k+1}) - V(x_k) < 0$. If such conditions hold the equilibrium is asymptotically stable. Lyapunov's direct method for discrete-time nonlinear systems can be stated as follows:

Theorem A.1 (adapted from [73, 9, Chapter 4])**.** *Let $x = 0$ be an equilibrium point for (A.5) and $D \subset \mathbb{R}^n$ be a domain containing the origin $x = 0$ and $V : D \to \mathbb{R}$ be a continuous function such that*

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \in D - \{0\} \tag{A.6}$$

$$V(f(x)) - V(x) \leq 0, \quad \forall x \in D, \tag{A.7}$$

then the equilibrium is stable. Moreover if

$$V(f(x)) - V(x) < 0, \quad \forall x \in D - \{0\}, \tag{A.8}$$

then the equilibrium is asymptotically stable.

Definition A.3. *If a continuous function $V : D \to \mathbb{R}$ satisfies (A.6), it is called a Lyapunov function candidate; if it also satisfies (A.7), then it is called Lyapunov function.*

The stability analysis using the Lyapunov's second method requires appropriately choosing a Lyapunov function candidate. This is the main task of this approach since there is no general methodology for that. Frequently, quadratic functions are considered as Lyapunov function candidates for studying the stability of equilibrium points of nonlinear systems because of their simplicity of construction. In particular, there are powerful results on linear systems theory in which sufficient and necessary stability conditions can be obtained using quadratic Lyapunov functions for both continuous and discrete-time domains.

APPENDIX B – DISSIPATIVITY ANALYSIS OF DISCRETE-TIME NONLINEAR SYSTEMS

This appendix presents the main concepts on dissipativity analysis of nonlinear discrete-time systems. The discussion presented here are mainly adopted from [74, 75].

Consider the following class of input-affine nonlinear discrete-time systems:

$$\begin{aligned} x_{k+1} &= f(x_k) + g(x_k)u_k \\ y_k &= h(x_k) + j(x_k)u_k, \end{aligned} \tag{B.1}$$

where $x_k \in \Omega \subseteq \mathbb{R}^{n_x}$, $u_k \in U \subseteq \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$ and the functions $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ and $j(\cdot)$ are smooth mappings. It is assumed that $f(0) = 0$ and $h(0) = 0$.

Definition B.1 (see [74])**.** *The system* (B.1) *is dissipative with respect to the supply rate* $\mathcal{S}(u, y)$ *if the dissipation inequality*

$$0 \leq \sum_{i=k_0}^{k-1} \mathcal{S}(u_i, y_i) \tag{B.2}$$

is satisfied for all $k - 1 \geq k_0$ *and all* $u \in U$ *with* $x_{k_0} = 0$ *along trajectories of* (B.1)*. The system is lossless with respect to the supply rate* $\mathcal{S}(u, y)$ *if it is dissipative with respect to the supply rate* $\mathcal{S}(u, y)$ *and the dissipation inequality* (B.2) *is satisfied as an equality for all* $k - 1 \geq k_0$ *and all* $u \in U$ *with* $x_{k_0} = x_k = 0$ *along trajectories of the system.*

The available storage $V_a(x_0)$ of the discrete-time nonlinear dynamical system (B.2) if defined by

$$\begin{aligned} V_a(x_0) &= - \inf_{u(\cdot), K \geq 0} \sum_{k=0}^{K-1} \mathcal{S}(u_k, y_k) \\ &= \sup_{u(\cdot), K \geq 0} \left[- \sum_{k=0}^{K-1} \mathcal{S}(u_k, y_k) \right], \end{aligned} \tag{B.3}$$

where x_k, $k \geq k_0$, is the solution to (B.1) with admissible input $u \in U$. Note that $V_a(x) \geq 0$ for all $x \in D$.

Definition B.2 ([74])**.** *Consider the system* (B.1)*. A continuous nonnegative function* $V_s : D \to \mathbb{R}$ *satisfying* $V_s(0) = 0$ *and*

$$V_s(x_k) \leq V_s(x_{k_0}) + \sum_{i=k_0}^{k-1} \mathcal{S}(u_i, y_i), \quad k - 1 \geq k_0, \tag{B.4}$$

for all k_0, $k \in \mathbb{Z}_+$*, where* x_k, $k \in \mathbb{Z}_+$*, is the solution of* (B.1) *with* $u \in U$*, is called a storage function for* (B.1)*.*

The function V_s can be viewed as a generalized energy function for a dissipative system. Therefore, inequality (B.4) represents the internal energy stored by the system so that the variation $V_s(x_{k+1}) - V_s(x_k)$ should be less or equal to the supplied energy. It conducts to the following definition of dissipative systems [76].

Theorem B.1 (see [74])**.** *Consider the system* (B.1)*. The system is dissipative with respect to the supply rate $\mathcal{S}(u, y)$ if and only if the available system storage $V_a(x_0)$ given by* (B.2) *is finite for all $x_0 \in D$ and $V_a(0) = 0$. Moreover, if $V_a(0) = 0$ and $V_a(x_0)$ is finite for all $x_0 \in D$, then $V_a(x)$, $x \in D$ is a storage function for the system. All storage functions $V_s(x)$ satisfy*

$$0 \leq V_a(x) \leq V_s(x), \quad x \in D. \tag{B.5}$$

The system (B.1) is passive if it is dissipative with respect to the supply rate $\mathcal{S}(u, y) = u^\top y$. It is strictly passive if $V_s(x_{k+1}) - V_s(x_k) < u^\top y$ for all admissible inputs $u \in U$ unless $x_k \equiv 0$, $\forall\, k \in \mathbb{Z}_+$. In addition, the system is said lossless if $V_s(x_{k+1}) - V_s(x_k) = u^\top y$ for all admissible inputs $u \in U$ and all k [75].