Universidade Federal de Minas Gerais
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Comparing Models for $(\infty, 1)$-categories:

The Quillen Equivalence between Simplicial Sets and Simplicially Enriched Categories

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RESUMO

Neste trabalho estudamos o que são $(\infty, 1)$-categorias e como elas se relacionam com Teoria da Homotopia, duas possíveis formas de defini-las formalmente e como essas duas formas estão relacionadas por uma equivalência de Quillen. Além disso, tentamos apresentar os tópicos de uma forma intuitiva, para que nossas definições possam ser vistas como boas e que possamos sentir que elas capturam os conceitos que desejamos.

Palavras Chave: Homotopia, Quasi-categorias, Categorias Simpliciais, Categorias Modelo.

ABSTRACT

In this work we study what are $(\infty, 1)$-categories supposed to be and why they're related to Homotopy Theory, two possible ways of formally defining them and how these two ways are related by a Quillen equivalence. Moreover, we try to present the topics in an intuitive way, so that our definitions can be seen as good ones that capture the concepts we wanted them to.

Key Words: Homotopy, Quasi-categories, Simplicial categories, Model Categories.

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Part I

PRELIMINARIES

1 INTRODUCTION

1.1 FROM ISOMORPHISMS TO WEAK EQUIVALENCES

One of the main goals of Topology is to classify topological spaces up to homeomorphism and what are these homeomorphisms. That is, given two spaces X and Y, when is X homeomorphic to Y? Although easy to state, this problem is extremely hard, for homeomorphisms are hard to come by (are too strict). For instance, let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, then surely X is not homeomorphic to Y if $m \neq n$, this is true, however a proof of this is not that trivial. In fact, even this simple case involves sophisticated tools, such as homology and homotopy groups. Also, $\mathtt{Iso}(\mathbb{R}^n, \mathbb{R}^n)$, the set of homeomorphisms from $\mathbb{R}^n$ into itself, is not well understood to this day.

So it is natural to seek an easier (less strict) way of classifying spaces. That is, we want a weaker condition for a function $f : X \to Y$ to be considered an "equivalence of spaces". A good candidate for this are the homotopy equivalences. Recall that, from the intuitive point of view, two spaces are homeomorphic when one can be continuously deformed into the other. These deformations, however, are not arbitrary: they must preserve dimensions and are not allowed to puncture or rip the space and so on. Thus, the main idea to weaken the notion of homeomorphism is to take some other kind of deformation, one that has less restrictions. The question concerning $\mathbb{R}^n$ and $\mathbb{R}^m$ was precisely about dimension. Therefore, if we are looking for a deformation which allows us to deform R^n into R^m, it is natural to remove the need of dimension preservation.What we obtain from this is exactly what is called homotopy equivalence. More precisely, we say that X and Y are homotopy equivalent if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that, instead of having identities $g \circ f = \mathtt{id}_X$, and $f \circ g = \mathtt{id}_Y$, we have just homotopies $g \circ f \simeq \mathtt{id}_X$ and $f \circ g \simeq \mathtt{id}_Y$.

That gives a new, easier problem: given two spaces X and Y, when is X homotopic to Y? Note that if two spaces are homeomorphic, then they are homotopic, so this is in fact an "easier" problem. For instance, in the previous example, where $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, the answer is trivial, since they are both homotopic to a point.

Using tools from category theory, we can see this problem in another lens. Let **Top** denote the category of topological spaces, then an isomorphism in **Top** is simply a homeomorphism, to ask if X is homeomorphic to Y is the same as asking if there is an isomorphism $X \to Y$. In contrast, to ask if X is homotopic to Y is the same as asking if there is a homotopy equivalence to $X \to Y$.

One natural question is if we can realize the homotopy equivalences as actual isomorphisms in some other category in a "natural way". It turns out the answer is yes. Let $\mathbf{Top}(X, Y)$ be the set of continuous maps from X to Y. We know that the relation of "being homotopic as functions" is an equivalence relation on this set, so that $\mathbf{Top}(X,Y)/\sim$ is well defined. Also, we know that homotopic maps compose to homotopic maps, so that if $[f] \in \mathbf{Top}(X,Y)/\sim$ and $[g] \in \mathbf{Top}(Y,Z)/\sim$, it makes sense to define $[g] \circ [f] := [g \circ f]$. With this, we can define a new category, $H\mathbf{Top}$, whose objects are topological spaces and the morphisms from one space to another are homotopy classes of continuous maps from one space to the other. That is, $H\mathbf{Top}(X,Y) = \mathbf{Top}(X,Y)/\sim$. In this new category, an isomorphism is a pair of homotopy classes $[f] : X \leftrightarrows Y : [g]$ such that $[g] \circ [f] = [id_X]$ and $[f] \circ [g] = [id_Y]$. Unwinding this, we see that an isomorphism is the same as a homotopy equivalence!

This new category, $H\mathbf{Top}$, is special not only in that isomorphisms are precisely the homotopy equivalences, but in that it also satisfies a universal property. We have a "projection" functor $\gamma : \mathbf{Top} \to H\mathbf{Top}$ which is the identity in objects and that takes a function to its homotopy class. Now, let $F : \mathbf{Top} \to \mathscr{C}$ be any functor that sends homotopy equivalences into isomorphisms. Then there is a unique[1] factorization F through $H\mathbf{Top}$, more precisely, there is a unique functor $\tilde{F} : H\mathbf{Top} \to \mathscr{C}$ such that the following diagram

$$\begin{tikzcd} \mathbf{Top} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & & \mathscr{C} \\ & H\mathbf{Top} \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

commutes. For instance, Algebraic Topology takes full advantage of this, and studies functors that send homotopy equivalences into isomorphisms, such as the singular homology and cohomology functors.

Now, turning our eye to the general case, note that the above discussion makes sense for any category. Suppose that we have a category $\mathscr{C}$ and we want to know when two objects are isomorphic, but that this question is too hard. Then we might seek a weaker notion than that of isomorphisms. So, we want a collection W of morphisms of $\mathscr{C}$ such that all isomorphisms of $\mathscr{C}$ are in W. With this collection, we say that an object X is *weakly equivalent* to another object Y if there is a morphism $X \to Y$ in W. Typically, the collection W is called the collection of *weak equivalences* and its elements are called weak equivalences and the pair $(\mathscr{C}, W)$ is called a category with weak equivalences. In the case of topological spaces considered above, W was the collection of all homotopy equivalences.

Just as before, we might wonder if we can realize the weak equivalences as actual isomorphisms in some category which satisfies the same universal property. That is, if there is a category $h\mathscr{C}$ together with a functor $\gamma : \mathscr{C} \to \mathscr{C}[W^{-1}]$ such that every functor which inverts weak equivalences factors uniquely through $\mathscr{C}[W^{-1}]$. Again, as shown by Gabriel and Zisman in [12], the answer is yes! We call $\gamma : \mathscr{C} \to \mathscr{C}[W^{-1}]$ a *localization* for $(\mathscr{C}, W)$, which is also unique up to equivalence of categories.

1.2 FROM WEAK EQUIVALENCES TO MODEL CATEGORIES

As mentioned in the last section, given a category with weak equivalences $(\mathscr{C}, W)$ we can construct a localization $\gamma : \mathscr{C} \to \mathscr{C}[W^{-1}]$ for it. However, this process is far from perfect. The category $\mathscr{C}[W^{-1}]$ can be hard to compute and its construction uses another intermediate category which can be "very large". With that said, for $\mathbf{Top}$, the localization is quite simple, and we constructed it quite easily, why? First of all, recall that our notion of weak equivalence, namely homotopy equivalences, came from a simpler concept, that of homotopy of functions. Using that, we could simply "pass to the quotient" in the hom-sets.

[1]This factorization is actually unique up to natural isomorphism, but for now we set that aside.

This begs the question: given a category with weak equivalences $(\mathscr{C}, W)$, can we define a notion of homotopy between its morphisms such that the localization is simply the original category with hom-sets quotiented by the relation of being homotopic? Not always, but for several cases we can do that by means of what are called *model structures*.

A model structure in a category of weak equivalences $(\mathscr{C}, W)$ is simply a choice of two distinct classes of morphisms from $\mathscr{C}$, called fibrations and cofibrations, which satisfy a series of properties. In this text we'll use $(\mathfrak{C})\mathfrak{F}$ to denote the collection of (co)fibrations of a model structure. So, a *model category* is a tuple $(\mathscr{C}, W, \mathfrak{F}, \mathfrak{C})$ where $(\mathscr{C}, W)$ is a category of weak equivalences and $(\mathfrak{F}, \mathfrak{C})$ is a model structure for it.

Although it is not at all obvious how, these fibrations and cofibrations allow us to define homotopy between morphisms of $\mathscr{C}$. That is, for the class of fibrant-cofibrant objects[2], we can define equivalence relations on its hom-sets that behave well under composition. So, we can define a new category $h\mathscr{C}$ whose objects are the fibrant-cofibrant objects of $\mathscr{C}$ and whose morphisms are the homotopy classes of morphisms of $\mathscr{C}$. The category $h\mathscr{C}$ is called the *homotopy category* of $\mathscr{C}$. This is the same thing we did for **Top** and indeed, as we shall see, **Top** has a model structure which realizes its homotopy theory as just a particular case of the homotopy theory of model categories.

For a model category $(\mathscr{C}, W, \mathfrak{F}, \mathfrak{C})$ and $h\mathscr{C}$ its homotopy category, there is a functor $\gamma : \mathscr{C} \to h\mathscr{C}$ which is the identity on fibrant-cofibrant objects that exhibits $h\mathscr{C}$ as a localization for the category with equivalences $(\mathscr{C}, W)$. So, at least when our category with weak equivalences admits a model structure, there is a very concrete way of getting a localization.

Model structures are very successful but they don't encompass all categories with weak equivalences. For there to be a model structure and for it to be useful we need at least that our category have finite (co)limits and that its weak equivalences satisfy a closure condition. All of this will be seen in the next chapter.

1.3 FROM MODEL CATEGORIES TO $(\infty, 1)$-CATEGORIES

Recapitulating what we are doing, we started with a category $\mathscr{C}$ in which we wished to weaken the notion of isomorphisms, so we chose a class of morphisms W to be the "new isomorphisms" and, using model structures, defined homotopies between morphisms of $\mathscr{C}$. To do this, we needed extra structure, something *external* to the category. This shows that an ordinary category isn't equipped to handle the notion of homotopy by itself, so we add stuff to it until we can talk about homotopies of morphisms. Also, it wasn't always that we could do this, since the existence of model structures has obstructions.

A possible solution would be to have something similar to a category but such that homotopies are intrinsic to it, that is, that in its very definition we have a notion of homotopies between morphisms. Let's again look at **Top** for some insight.

A homotopy from $f : X \to Y$ to $g : X \to Y$ is a function $H : X \times I \to Y$ such that $H_0 = f$ and $H_1 = g$. We can think of H as a morphism from f to g, and write $H : f \Rightarrow g$. Following the idea that a homotopy is a morphism between functions, for any two spaces Y and X, we have and for any two functions $f, g \in \mathbf{Top}(X, Y)$ we have a set of morphisms (homotopies) $\mathbf{Top}(X, Y)(f, g)$. For any function f we have a constant homotopy $C_f : f \Rightarrow f$, which serves as an identity morphism from f to f. With this, we see that homotopies make the set $\mathbf{Top}(X, Y)$ some sort of category, but there is a problem with this idea. In a category we

[2]Don't worry about what this means now, take it as just the name of some specific class of objects, the precise definition can be found in 2.19.

must have composition of morphisms and this composition must be associative. If we have homotopies $H : f \to g$ and $H' : g \Rightarrow h$, we can define a homotopy $H' * H : f \Rightarrow h$ by

$$H' * H(x,t) := \begin{cases} H(x,2t) & \text{if } t \leq 1/2 \\ H'(x,2(t-1/2)) & \text{if } t \geq 1/2 \end{cases}$$

This serves as some kind of composition, but the problem arises when we want to compose three or more homotopies. Note that in this construction each homotopy runs through half of the interval $[0,1]$, that is, in the first half of I we use H and in the second H', if we where to compose this yet with another homotopy, H would be used in only a fourth of the interval. But, if we were to compose H with the composition of the other two, H would now be used in half the interval: this composition is not associative! More generally, the way we compose homotopies will always depend on how we parametrize the interval. We've concluded that in general

$$(H * K) * G \neq H * (K * G)$$

so that we can't hope for $\mathbf{Top}(X,Y)$ to be a category. However, what is true is that

$$(H * K) * G \simeq H * (K * G)$$

that is, the compositions are homotopic! Thinking of homotopies between homotopies as homotopies between morphisms in $\mathbf{Top}(X,Y)(f,g)$, we see that composition is defined "up to homotopy ”. So, we "went up" by a level in our problem. That is, if we have homotopies $H, K : f \Rightarrow g$, we may think of the set of morphisms $\mathbf{Top}(X,Y)(f,g)(H,K)$. Homotopies between homotopies are just homotopies (homotopies are just functions), so that their composition is again not associative but is associative up to homotopy, so we could keep "climbing" to morphisms of morphisms of morphisms of morphisms... ever throwing this associativity problem up to the next level. This is exactly what ∞-categories do.

Let's rename things: call a continuous function $f : X \to Y$ a 1-morphism, a homotopy $H : f \Rightarrow g$ a 2-morphism , a homotopy between homotopies a 3-morphism and so on. So, we conclude that for each two n-morphisms H and G we have a set $\mathbf{Top}_n(H,G)$ that is almost a category, with the exception that composition is only defined up to $(n+1)$-morphisms.

Besides that, note that every homotopy between functions is "invertible". For a homotopy $H : f \Rightarrow g$, we can define $\overline{H} : g \Rightarrow f$ by

$$\overline{H}(x,t) := H(x, 1-t).$$

With this, we can check that

$$\overline{H} * H \simeq C_f \quad H * \overline{H} \simeq C_g.$$

Since composition is only defined up to homotopy, this is the same as saying that their composition equals the identities. Thus, we conclude that every morphism between n-morphisms is an isomorphism whenever $n \geq 1$.

Observe now, that we can get the homotopy category of **Top**, $H\mathbf{Top}$, by declaring the objects of $H\mathbf{Top}$ to be same as **Top** and the set of morphism from X to Y to be the isomorphism classes of $\mathbf{Top}(X,Y)$ when viewed as this weird kind of category.

In the end, we ended up with a means to avoid the associativity issue and to represent the homotopies of **Top** as morphisms if morphisms in some thing that looks like a category. This "thing" that it looks like is precisely an $(\infty,1)$-category. Here the "∞" indicates that we have morphisms of all orders ($(n+1)$-morphisms between n-morphisms), and the index "1" means that for $n > 1$, any n-morphism is an isomorphism. In a $(\infty, 1)$-category we have the notion of homotopy equivalence and homotopies between 1-morphisms (the 2-morphisms), that we internalized these notions to some object. In any $(\infty, 1)$-category we can construct an ordinary category, its homotopy category, but just taking the homotopy classes of 1-morphisms to be the hom-sets, so that everything follows quite naturally. This is not a coincidence, since they envisioned it as the natural setting of homotopy theory. As we shall mention later, every category with weak equivalence can be made into an $(\infty, 1)$-category in such a manner that its homotopy category is the localization, further illustrating how useful they are.

1.4 QUASI-CATEGORIES AND SIMPLICIAL CATEGORIES

In the last section we said that the natural setting for homotopy theories are the $(\infty, 1)$-categories, and gave an heuristic definition: a category with $(n+1)$-morphisms between n-morphisms. However, to give a formal definition of what is an $(\infty, 1)$-category is not at all a trivial matter. Today, there are several possible definitions or "models"[3] [5], but here we'll talk about two of them: *quasi-categories* and *simplicial categories*.

The first one will talk about are the quasi-categories. A normal category $\mathscr{C}$ can be described as two sets: $\mathscr{C}_1 = \mathrm{Mor}(\mathscr{C})$ and $\mathscr{C}_0 = \mathrm{ob}(\mathscr{C})$. Also, this sets come equipped $s, t : \mathscr{C}_1 \rightrightarrows \mathscr{C}_0$ and $\mathrm{id} : \mathscr{C}_0 \to \mathscr{C}_1$, where s takes a morphism to its domain (its "source), t takes a morphism to its codomain (its target) and id take an object into its identity. As seen before, we want $n+1$-morphisms between n-morphisms. With that in mind, we have for each $n \in \mathbb{N}$ a set X_n of n-morphisms. A quasi-category is exactly this: a collection of sets indexed by the natural numbers together with a bunch of maps which either go up or down by one level. This collection of sets and maps has to satisfy a series of properties, so that they have the desired properties: we can compose morphisms, we have identities, we have the notion of homotopy, etc. . If we lose some of the mentioned properties, we get what is called a *simplicial set*, so that quasi-categories are special cases of these. Simplicial sets are easy to define and to work with, so that we have where to start. With quasi-categories, or with the other model we shall see, we don't get exactly "$n+1$-morphisms between n-morphisms", but something similar which has the desired properties, and this really isn't a problem, since the need for higher morphisms is only needed to deal with the fact that composition isn't associative.

The other model we shall see are the simplicial categories. This is a more abstract approach to the problem, but it's also quite natural after you get used to it. As said before, in an $(\infty, 1)$-category $\mathscr{C}$, for each pair x, y of objects we have a collection of 1-morphisms $\mathscr{C}(x, y)$. Since we have morphisms for each degree, we can also think of $\mathscr{C}(x, y)$ as a $(\infty, 1)$-category. Note now that since "1" in the $(\infty, 1)$ means that every morphisms of order higher than 1 are invertible, $\mathscr{C}(x, y)$ is such that every morphism is an isomorphism. Recall that a category in which all morphisms are isomorphisms is called a groupoid, so we call $\mathscr{C}(x, y)$ an ∞-groupoid. Thus, we conclude that for an $(\infty, 1)$-category, $\mathscr{C}(x, y)$ is an ∞-groupoid.

An ordinary category $\mathscr{C}$ such that for each pair of objects x, y, $\mathscr{C}(x, y)$ is an object of another category $\mathscr{D}$ is said to be *enriched over* $\mathscr{D}$. For instance, the category of vector spaces is enriched over itself since the space of linear transformation between two vector spaces is a vector space. With this in mind, suppose that we have a category of ∞-groupoids,

[3] Here the word "model" has nothing to do with model categories, but rather to something that is "modeling" a desired concept.

∞**Gpd**, then we may define an $(\infty, 1)$-category as a category enriched over ∞**Gpd**. As it turns out, we can model ∞-groupoids very well as simplicial sets which satisfy some additional properties, so that we may define an $(\infty, 1)$-category as a category enriched over the category of simplicial sets but such that each hom-set satisfies the required additional properties. Categories enriched over simplicial sets are what we call simplicial categories.

Following the ideas above we have two models for what is a $(\infty, 1)$-category, so it's natural to ask how they relate to each other. And the answer is quite nice. Just as with ordinary categories, model categories have a natural notion of equivalence between them, which is something that implies "homotopically equivalent". These equivalences are called *Quillen equivalences*. As mentioned, a quasi-category is a simplicial set and a category enriched over ∞**Gpd** is a simplicial category. As it turns out, there is one model structure for the category of simplicial sets whose weak equivalences are equivalences of quasi-categories and one model structure on the category of simplicial categories whose weak equivalences are the equivalences of ∞**Gpd**-enriched categories in such a manner that these categories are Quillen equivalent. Thus, we conclude that these two approaches, although different, are homotopically equivalent.

In what follows, we develop the formal theory of all that has been mentioned here and present further intuition on all of these things. Indeed, in the remainder of Part I we discuss the theory of model categories so that we may gain more intuition on Abstract Homotopy Theory and so that we can use its tools later. In Part II, we develop the theory of simplicial sets, use it to define the two models we'll be working with and build on the intuition of these models. Finally, on Part III, we construct the aforementioned model structures and present their Quillen Equivalence.

2

MODEL CATEGORIES

As we mentioned, model categories are great for studying homotopy theory in an abstract setting. They were introduced by Quillen in [26] to unify some homotopy theories, and to this day are extremely useful. To say how they work does not help much, and at first what we are doing may be a little off putting, but by the end of this chapter we hope that the reader understands where each little detail went for our constructions to work.

2.1 FACTORIZATION SYSTEMS

Definition 2.1 (Lifting Properties of Morphisms)**.** Let $\mathscr{C}$ be a category and $K \subset \mathrm{Mor}(\mathscr{C})$ be any collection of morphisms. A morphism

$$p : E \longrightarrow B$$

in $\mathscr{C}$ is said to have the right lifting property against a morphism $k : X \longrightarrow Y$ if for every commuting square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "k"'] & E \arrow[d, "p"] \\ Y \arrow[r] & B \end{tikzcd}$$

there exists a morphism $h : Y \to E$, called a lift, such that

$$\begin{tikzcd} X \arrow[r] \arrow[d, "k"'] & E \arrow[d, "p"] \\ Y \arrow[r] \arrow[ur, "h"] & B \end{tikzcd}$$

commutes. The morphism $p : E \longrightarrow B$ is said to have the right lifting property against K if it has it for every morphism $k \in K$. Dually, a morphism $E \to B$ is said to have the left lifting property against k if every commutative diagram of the form

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & X \arrow[d, "k"] \\ B \arrow[r] & Y \end{tikzcd}$$

admits a left lift $h : B \to Y$ such that

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & X \arrow[d, "k"] \\ B \arrow[r] \arrow[ur, "h"] & Y \end{tikzcd}$$

commutes. $E \to B$ has the left lifting property against K if it has against every morphism $k \in K$. We denote by $\mathrm{RLP}(K)$ the collection of morphisms that have the right lifting against K and $\mathrm{LLP}(K)$ the collection of morphisms that have the left lifting against K. □

With this definition we may define the following.

Definition 2.2 (Weak Factorization System)**.** Let $(\mathrm{LLP}, \mathrm{RLP})$ be a pair of collections of morphisms of $\mathscr{C}$ such that

(i) Every morphism $f : X \longrightarrow Y$ of $\mathscr{C}$ may be factored as composition of the form

$$f : X \xrightarrow{\in \mathrm{LLP}} X' \xrightarrow{\in \mathrm{RLP}} Y.$$

(ii) $\mathrm{LLP} = \mathrm{LLP}(\mathrm{RLP})$ and $\mathrm{RLP} = \mathrm{RLP}(\mathrm{LLP})$. That is to say that they are closed under having lifting properties against each other.

Then $(\mathrm{LLP}, \mathrm{RLP})$ is said to be a weak factorization system in $\mathscr{C}$. □

Example 2.3. In **Set**, the pair $(\mathrm{Mono}, \mathrm{Epi})$, of monomorphisms and epimorphisms, is a weak factorization system.

Example 2.4. In any category $\mathscr{C}$, $(\mathrm{Mor}(\mathscr{C}), \mathrm{Iso})$ and $(\mathrm{Iso}, \mathrm{Mor}(\mathscr{C}))$ are factorization system. These are the trivial factorization systems.

Proposition 2.5. *Let $\mathscr{C}$ be a category and $K \subset \mathrm{Mor}(\mathscr{C})$, and consider the collections $\mathrm{LLP}(K)$ and $\mathrm{RLP}(K)$, the following hold:*

(i) Both collections contain all isomorphisms of $\mathscr{C}$;

(ii) Both collections are closed under composition. $\mathrm{LLP}(K)$ is also closed under transfinite composition (see Definition 2.39);

(iii) Both collections are closed under forming retracts in the arrow category $\mathscr{C}^{\Delta[1]}$;

(iv) $\mathrm{LLP}(K)$ is closed under forming pushouts of morphisms and $\mathrm{RLP}(K)$ is closed under forming pullback of morphisms;

(v) $\mathrm{LLP}(K)$ is closed under forming coproducts in $\mathscr{C}^{\Delta[1]}$ and $\mathrm{RLP}(K)$ is closed under forming products in $\mathscr{C}^{\Delta[1]}$.

Proof: Throughout this proof k will be a morphism in K. In each of the cases we only prove one of the statements, since the other one will follow by duality. Indeed, having the right lifting property in $\mathscr{C}$ is the same as having the left lifting property in $\mathscr{C}^{\mathrm{op}}$.

(i) Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "k"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

be a commuting square, and i an isomorphism. Then we have left lifting in the diagram given by $B \xrightarrow{f \circ i^{-1}} X$.

(ii) Let $p_1, p_2 \in \mathrm{RLP}(K)$, and given a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, "k"'] & X \arrow[d, "p_1"] \\
 & X' \arrow[d, "p_2"] \\
B \arrow[r] & Y
\end{tikzcd}$$

we can compose the top horizontal morphism with p_1 to get a morphism (1) from A to X', using the lifting property of p_2 we get a morphism (2) from B to X', and finally use the lifting property of p_1

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, "k"'] \arrow[dr, dashed, "1"] & X \arrow[d, "p_1"] \\
 & X' \arrow[d, "p_2"] \\
B \arrow[r] \arrow[ur, dashed, "2"] \arrow[uur, dashed, "3"] & Y
\end{tikzcd}$$

showing that $p_2 \circ p_1$ has the right lifting property against morphisms of K. The fact $\mathrm{LLP}(K)$ is closed under transfinite composition follows from the fact that each of the successor morphisms arise from the composition and that the composition is we behaved under limit ordinals: just use the lift on each stage and use the universal product of the colimit to get a lift from the limit.

(iii) Let j be a retract of $i \in \mathrm{LLP}(K)$ in $\mathscr{C}^{\Delta[1]}$, thus we have a diagram of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"] \arrow[rr, bend left, equal] & C \arrow[r] \arrow[d, "i"] & A \arrow[d, "j"] \\
B \arrow[r] \arrow[rr, bend right, equal] & D \arrow[r] & B
\end{tikzcd}$$

where the horizontal arrows compose to identities. We need to show that $j \in \mathrm{LLP}(K)$. Plugging into this diagram a morphism of K

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"] & C \arrow[r] \arrow[d, "i"] & A \arrow[r] \arrow[d, "j"] & X \arrow[d, "k"] \\
B \arrow[r] & D \arrow[r] & B \arrow[r] & Y
\end{tikzcd}$$

we get a lift from D to X and then it suffices to compose the morphism from B to D to get a lift B to X

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"] & C \arrow[r] \arrow[d, "i"] & A \arrow[r] \arrow[d, "j"] & X \arrow[d, "k"] \\
B \arrow[r] \arrow[urrr, dashed] & D \arrow[r] \arrow[urr, dashed] & B \arrow[r] & Y
\end{tikzcd}$$

and since the composition of the initial horizontal arrows compose to identities, this is the desired lift.

(iv) Let the following be a pullback diagram of $p \in \mathrm{RLP}(K)$

$$\begin{tikzcd} Z \times_f X \arrow[r] \arrow[d, "f^*p"'] & X \arrow[d, "p"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

We need to show that f^*p has the right lifting property against morphisms of K. So plug a morphism k into the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "k"'] & Z \times_f X \arrow[r] \arrow[d, "f^*p"] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Z \arrow[r, "f"'] & Y \end{tikzcd}$$

and using the lift property of p we get a lift $\tilde{gf}$ from B to X, then, by the universal property of the pullback we get a lift $\tilde{g}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "k"'] & Z \times_f X \arrow[r] \arrow[d, "f^*p"] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[urr, dashed, "\tilde{gf}"] & Z \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "k"'] & Z \times_f X \arrow[r] \arrow[d, "f^*p"] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{g}"] \arrow[urr, dashed, "\tilde{gf}"] & Z \arrow[r, "f"'] & Y \end{tikzcd}$$

now need to show that $\tilde{g}$ commutes with the morphisms we want. Note that we have two cones with apex A, namely

- The first one with legs $A \to Z \times_f X \to X$, $A \xrightarrow{k} B \xrightarrow{g} Z$ and universal morphism $A \to Z \times_f Z$.
- The second one with legs $A \xrightarrow{k} B \xrightarrow{\tilde{g}} Z \times_f X \to X$, $A \xrightarrow{k} B \xrightarrow{g} Z$ and universal morphism $A \xrightarrow{k} B \xrightarrow{\tilde{g}} Z \times_f X$.

So be the commutativity we already had in our diagrams and the unicity of the universal morphisms we have that $A \xrightarrow{k} B \xrightarrow{\tilde{g}} Z \times_f X \to X = A \to Z \times_f X \to X$, which is what we needed.

(v) Limits in $\mathscr{C}^{\Delta[1]}$ are computed component wise. So let $\{A_i \xrightarrow{p_i} B_i \in LLP(K)\}_{i \in I}$, then the colimit is the universal morphism

$$\coprod_i A_i \xrightarrow{\coprod_i p_i} \coprod_i B_i$$

induced by the family of morphisms above. Now, if we have a diagram

$$\begin{tikzcd} \coprod_i A_i \arrow[r] \arrow[d, "\coprod_i p_i"'] & X \arrow[d, "k"] \\ \coprod_i B_i \arrow[r] & Y \end{tikzcd}$$

this is the same as saying that for each $i \in I$ we have a diagram

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "p_i"'] & X \arrow[d, "k"] \\ B_i \arrow[r] & Y \end{tikzcd} \Longrightarrow \begin{tikzcd} A_i \arrow[r] \arrow[d, "p_i"'] & X \arrow[d, "k"] \\ B_i \arrow[r] \arrow[ur, "h_i"] & Y \end{tikzcd}$$

which have a lift for each i, thus induced universal morphism

$$\begin{tikzcd} \coprod_i A_i \arrow[r] \arrow[d, "\coprod_i p_i"'] & X \arrow[d, "k"] \\ \coprod_i B_i \arrow[r] \arrow[ur, "\coprod_i h_i"] & Y \end{tikzcd}$$

is an appropriate left lifting. ■

Remark 2.6. The above result, although simple, is extremely useful and we'll come back to it over and over again. Whenever it is not clear why a morphism lifts against another, there is a good chance it follows from this.

Corollary 2.7. *Let $\mathscr{C}$ be a category with small colimits and $K \subset \mathrm{Mor}(\mathscr{C})$ a collection of its morphisms, then every morphism in $\mathsf{RLP}(K)$ has the right lifting properties against all K-relative cell complexes (see Definition 2.40) and their retracts.*

Proof: We have the collection $\mathsf{RLP}(K)$, now consider the collection $\mathsf{LLP}(\mathsf{RLP}(K))$, it's clear that $K \subset (\mathsf{LLP}(\mathsf{RLP}(K))$, so by the last proposition the result follows. ■

2.2 MODEL CATEGORIES

We are almost ready to define what a model category is and prove some results about them. As mentioned in the introduction, above all, model categories are categories with weak equivalences, so we must define them. Here we don't want any category with weak equivalences, but those which satisfy the *2-out-of-3* rule, which is required for a series of arguments to work.

Definition 2.8 (Categories with Weak Equivalence)**.** A category with weak equivalence is a category $\mathscr{C}$ together with a collection $\mathfrak{W} \subset \mathrm{Mor}(\mathscr{C})$ of its morphism such that

(i) $\mathfrak{W}$ contains all the isomorphisms of $\mathscr{C}$.

(ii) $\mathfrak{W}$ is closed by the *2-out-of-3* rule: for a commuting diagram

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur] \arrow[rr] & & Z \end{tikzcd}$$

if two of its morphisms are in W then so is the third.

The elements of the collection $\mathfrak{W}$ are called weak equivalences. □

Example 2.9. Any category can be thought of as a category with weak equivalence by taking $\mathfrak{W}$ to be the collection of isomorphisms of the category, since they satisfy the 2-out-of-3 rule trivially.

Example 2.10. Another useful way to get weak equivalences is by considering functors going out of our category. Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be any functor. Then the collection $\mathfrak{W}_F = \{f \mid Ff \text{ is an isomorphism }\}$ makes our category a category with weak equivalence, this follows trivially by the functorial properties.

Definition 2.11 (Model Categories)**.** A model category is a complete and cocomplete category $\mathscr{C}$ equipped with weak equivalences $\mathfrak{W} \subset \mathrm{Mor}(\mathscr{C})$ together with two additional collection of morphisms $\mathfrak{F}, \mathfrak{C} \subset \mathrm{Mor}(\mathscr{C})$ such that

1. $(\mathfrak{W} \cap \mathfrak{C}, \mathfrak{F})$ is a weak factorization system for $\mathscr{C}$.
2. $(\mathfrak{C}, \mathfrak{W} \cap \mathfrak{F})$ is a weak factorization system for $\mathscr{C}$.

The elements of $\mathfrak{C}$ are called cofibrations, the elements of $\mathfrak{F}$ are called fibrations, the elements of $\mathfrak{W} \cap \mathfrak{C}$ are called trivial cofibrations and the elements of $\mathfrak{W} \cap \mathfrak{F}$ are called trivial fibrations. □

Corollary 2.12. *The class of (trivial) cofibrations of a model category is closed under forming relative cell complexes.*

Proof: The class of (trivial) cofibrations are characterized by having the left lifting property against another class, thus this follows from Corollary 2.7. ■

Example 2.13. Every complete and cocomplete category admits the trivial model structure: $\mathfrak{F} = \mathfrak{C} = \mathrm{Mor}(\mathscr{C})$ and $\mathfrak{W} = \mathrm{Iso}$.

Remark 2.14. Just as we have duality for regular categories, we have duality for model categories. If $\mathscr{C}$ is a model category with the collections, $\mathfrak{F}_{\mathscr{C}}, \mathfrak{C}_{\mathscr{C}}, \mathfrak{W}_{\mathscr{C}}$, then we may give $\mathscr{C}^{op}$ a model structure by considering the collections

$$\mathfrak{C}_{\mathscr{C}^{op}} := \{f^{op} \ / \ f \in \mathfrak{F}_{\mathscr{C}}\},\ \mathfrak{F}_{\mathscr{C}^{op}} := \{f^{op} \ / \ f \in \mathfrak{C}_{\mathscr{C}}\} \text{ and } \mathfrak{W}_{\mathscr{C}^{op}} := \{w^{op} \ / \ w \in \mathfrak{W}_{\mathscr{C}}\}.$$

It is immediate that the above does indeed constitute a model category. So we see that fibrations are cofibrations in the opposite category and cofibrations are fibrations. This will be very useful, since in dual situations, which will be quite frequent, we need to prove only half of the statement.

Proposition 2.15. *Let $\mathscr{C}$ be a model category, then its collection of weak equivalences is closed under forming retracts.*

Proof: Given a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, equal] & X \arrow[r] \arrow[d, "w"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, equal] & Y \arrow[r] & B \end{tikzcd}$$

where the horizontal arrows compose to the identities, we have to show that $f \in \mathfrak{W}$. Suppose first that $f \in \mathfrak{F}$, and factor w in a cofibration and trivial fibration, then the cofibration is also trivial by the 2-out-of-3 rule. Write a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "id"'] & X \arrow[r] \arrow[d, "\in \mathfrak{W}\cap\mathfrak{C}"] & A \arrow[d, "id"] \\ A \arrow[r, "s"] \arrow[d, "f\in\mathfrak{F}"'] & X' \arrow[r, "t"] \arrow[d, "\in \mathfrak{W}\cap\mathfrak{F}"] & A \arrow[d, "f\in\mathfrak{F}"] \\ B \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

where s is uniquely determined and t is a lift of the top middle trivial cofibration against f with the second arrow given by the composition with the bottom middle trivial fibration. Since this diagram commutes, all horizontal arrows compose to identities, thus we exhibited f as retract of trivial fibration, thus $f \in \mathfrak{W}$ by Proposition 2.5. Now in the general case we can factor f as a trivial cofibration and a fibration, and then form a pushout in the top left square getting diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\in\mathfrak{W}\cap\mathfrak{C}"'] & X \arrow[r] \arrow[dd, "w"] & A \arrow[d] \\ A' \arrow[d, "\in\mathfrak{F}"'] & & A' \arrow[d] \\ B \arrow[r] & Y \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "\in\mathfrak{W}\cap\mathfrak{C}"'] & X \arrow[r] \arrow[d, "\in\mathfrak{W}\cap\mathfrak{C}"] & A \arrow[d, "\in\mathfrak{C}"] \\ A' \arrow[r] \arrow[d, "\in\mathfrak{F}"'] & X' \arrow[r] \arrow[d, "\in\mathfrak{W}"] & A' \arrow[d, "\in\mathfrak{F}"] \\ B \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

where the morphisms coming out of the pushout are the ones induced by its universal property. We know that the top vertical arrow is a trivial cofibration because they are closed under forming pushouts. We can take all horizontal morphisms to compose identities. The bottom middle vertical arrow is in $\mathfrak{W}$ by the 2-out-of-3 rule, so by the previous case, the left bottom vertical arrow is in $\mathfrak{W}$ and, again by the 2-out-of-3 rule, so is f. ■

Lemma 2.16 (The Retract Argument). *Let*

$$f : X \xrightarrow{i} Z \xrightarrow{p} Y$$

be a composition of morphisms. If f has the left lifting property against p, then f is a retract of i. Dually, if f has the right lifting property against i, then f is a retract of p.

Proof: We prove only the first statement, since the other is dual to it. Writing the factorization of f in the following form and lifting f through p

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \end{tikzcd} \Longrightarrow \begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, "h"] & Y \end{tikzcd}$$

we see that the second diagram is equivalent to

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "i"] & \\ Y \arrow[r, "h"'] & Z \arrow[r, "p"'] & Y \end{tikzcd}$$

where the vertical arrows are identities, so completing the diagram we get

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "f"] \\ Y \arrow[r, "h"'] & Z \arrow[r, "p"'] & Y \end{tikzcd}$$

exhibiting f as a retract of i. ■

2.3 HOMOTOPY

Having talked about model categories, we are now ready to define what homotopies are in an abstract setting.

Definition 2.17 (Cylinder and Path Objects)**.** Let $\mathscr{C}$ be a model category, and $X \in \mathscr{C}$ an object. Then a path object for X is an object $\mathrm{Path}(X)$ which factors the diagonal map

$$\Delta_X : X \xrightarrow[\in \mathfrak{W}]{i} \mathrm{Path}(X) \xrightarrow[\in \mathfrak{F}]{(p_0,p_1)} X \times X.$$

as a weak equivalence followed by a fibration. Dually, a cylinder object for X is an object $\mathrm{Cyl}(X)$ that factors the codiagonal map

$$\nabla_X : X \bigsqcup X \xrightarrow[\in \mathfrak{C}]{(i_0,i_1)} \mathrm{Cyl}(X) \xrightarrow[\in \mathfrak{W}]{p} X.$$

as a cofibration followed by a weak equivalence. Observe that by the definition of a model category these factorizations always exist, and are actually better then what we require, since there will be trivial (co)fibrations satisfying this. □

Remark 2.18. There is a model structure in **Top** such that a cylinder object for X is simply the product space $X \times I$[1] while the path object will be X^I with the compact open topology.

Definition 2.19 ((Co)Fibrant Objects)**.** Let $\mathscr{C}$ be a model category. An object $X \in \mathscr{C}$ is called fibrant if the map

$$X \longrightarrow *$$

is a fibration. Dually an object Y is cofibrant if

$$\varnothing \longrightarrow Y$$

is a cofibration. We write "fibrant-cofibrant object" for an object that is both fibrant and cofibrant. □

Lemma 2.20. *Let $\mathscr{C}$ be a model category. If X is a cofibrant object, for every cylinder object $\mathrm{Cyl}(X)$ of X the maps*

$$i_0, i_1 : X \longrightarrow \mathrm{Cyl}(X)$$

are trivial cofibrations. Dually, if X is fibrant, then for every path object $\mathrm{Path}(X)$ of X the maps

$$p_0, p_1 : \mathrm{Path}(X) \longrightarrow X$$

are trivial fibrations.

Proof: We prove the second statement, the other case is dual to it. Consider the commuting diagram

$$\begin{tikzcd}
 & & X \\
X \arrow[r, "i \in \mathfrak{W}"] \arrow[urr, bend left, "\mathrm{id}_X"] \arrow[drr, bend right, "\mathrm{id}_X"'] & \mathrm{Path}(X) \arrow[r, "(p_0,p_1)"'] \arrow[ur, dashed, "p_0"] \arrow[dr, dashed, "p_1"'] & X \times X \arrow[u, "\pi_0"'] \arrow[d, "\pi_1"] \\
 & & X
\end{tikzcd}$$

Then by the 2-out-of-3 rule p_0, p_1 are weak equivalences. Now to check that they are fibrations, note that the projections fit in the pullback diagram

$$\begin{tikzcd}
X \times X \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & X \arrow[d] \\
X \arrow[r] & *
\end{tikzcd}$$

[1] Well yes, but actually, no. There is a standard model structure in **Top**, and in it, in general the inclusion (i_0, i_1) won't be a cofibration. However, if X is a CW-complex, then $X \times I$ is indeed a cylinder object.

and by Proposition 2.5 they are fibrations since X is fibrant, so p_0, p_1 are compositions of fibrations, which are fibrations by the same Proposition. ■

Definition 2.21 (Left and Right Homotopy). Let $f, g : X \rightrightarrows Y$ be morphisms in a model category. A left homotopy between f and g is a morphism $\eta : \mathrm{Cyl}(X) \longrightarrow Y$ such that

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(X) \arrow[d, "\eta"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commutes. Dually, a right homotopy between f and g is a morphism $\xi : X \longrightarrow \mathrm{Path}(Y)$ such that

$$\begin{tikzcd} & X \arrow[dl] \arrow[d, "\xi"] \arrow[dr] & \\ Y & \mathrm{Path}(Y) \arrow[l, "p_0"] \arrow[r, "p_1"'] & Y \end{tikzcd}$$

commutes. We write $\eta : f \Rightarrow_L g$ to indicate that η is a left homotopy and $\xi : f \Rightarrow_R g$ to indicate a right homotopy. □

Definition 2.22 (Homotopy Equivalence). We say that two objects X, Y are left homotopic if there is a pair of morphisms $f : X \rightleftarrows Y : g$ such that there exist left homotopies $\eta_X : gf \Rightarrow_L \mathrm{id}_X$ and $\eta_Y : fg \Rightarrow_L \mathrm{id}_Y$. Dually, they are right homotopic if there exist right homotopies $\xi_X : gf \Rightarrow_R \mathrm{id}_X$ and $\xi_Y : fg \Rightarrow_R \mathrm{id}_Y$. □

Example 2.23. Let $\mathscr{C}$ be equipped with the trivial model structure, then f is (left or right) homotopic to g if, and only if, $f = g$. Thus, homotopy equivalences reduce to isomorphisms.

Lemma 2.24. *Let $f, g : X \rightrightarrows Y$ be morphisms in a model category. If there is a left homotopy $\eta : f \Rightarrow_L g$ between f and g and X is cofibrant, then there is also a right homotopy $\xi : f \Rightarrow_R g$ from f to g with respect to any chosen path object.*

Dually, if there is a right homotopy $\xi : f \Rightarrow_R g$ between f and g and Y is fibrant, then there is also a left homotopy $\eta : f \Rightarrow_L g$ from f to g with respect to any cylinder object.

Proof: The two cases are dual, so we prove the first one. By hypothesis we have a left homotopy

$$\eta : \mathrm{Cyl}(X) \longrightarrow Y$$

between $f, g : X \rightrightarrows Y$. Recall that we have the maps

$$\Delta_Y : Y \xrightarrow[\in \mathfrak{W}]{i} \mathrm{Path}(Y) \xrightarrow[\in \mathfrak{F}]{(p_0, p_1)} Y \times Y$$

and

$$\nabla_X : X \bigsqcup X \xrightarrow[\in \mathfrak{C}]{(i_0, i_1)} \mathrm{Cyl}(X) \xrightarrow[\in \mathfrak{W}]{p} X.$$

Lemma 2.20 implies that the diagram below has a lift h

$$\begin{tikzcd} X \arrow[r, "i \circ f"] \arrow[d, "i_0 \in \mathfrak{W} \cap \mathfrak{C}"'] & \mathrm{Path}(Y) \arrow[d, "(p_0, p_1) \in \mathfrak{F}"] \\ \mathrm{Cyl}(X) \arrow[r, "(f \circ p, \eta)"'] \arrow[ur, "h"] & Y \times Y. \end{tikzcd}$$

So take $\xi = h \circ i_1$ to be the desired right homotopy:

$$\begin{tikzcd}
 & X \arrow[r, "i \circ f"] \arrow[d, "i_0 \in \mathfrak{W} \cap \mathfrak{C}"'] & \mathrm{Path}(Y) \arrow[d, "(p_0, p_1) \in \mathfrak{F}"] \\
X \arrow[r, "i_1"'] \arrow[urr, dashed] & \mathrm{Cyl}(X) \arrow[r, "(f \circ p, \eta)"'] \arrow[ur, "h"] & Y \times Y.
\end{tikzcd}$$

This will do since the second component of the bottom horizontal arrow is the left homotopy from f to g. ■

Theorem 2.25. *Let $\mathscr{C}$ be a model category. If X is cofibrant and Y is fibrant, then the relations of left homotopy and right homotopy coincide in $\mathscr{C}(X, Y)$. Moreover, this is an equivalence relation.*

Proof: It follows from Lemma 2.24 that these relations are equal. Symmetry and reflexivity are trivial. To show transitivity, we first show that for a path object of a fibrant object,

$$Y \to \mathrm{Path}(Y) \xrightarrow{p_1, p_0} Y \times Y,$$

the fiber product

$$\begin{tikzcd}
\mathrm{Path}(Y) \times_Y \mathrm{Path}(Y) \arrow[r, "\rho_1"] \arrow[d, "\rho_0"'] & \mathrm{Path}(Y) \arrow[d, "p_1"] \\
\mathrm{Path}(Y) \arrow[r, "p_0"'] & Y
\end{tikzcd}$$

together with the maps $\tilde{p}_0 := \rho_1 \circ p_0$ and $\tilde{p}_1 := \rho_0 \circ p_1$ is again a path object for Y. Using Proposition 2.5 we have the following properties in the arrows of the equivalent pullback diagram

$$\begin{tikzcd}
\mathrm{Path}(Y) \times_Y \mathrm{Path}(Y) \arrow[r] \arrow[d] & \mathrm{Path}(Y) \times \mathrm{Path}(Y) \arrow[d, "\in \mathfrak{F}"] \\
Y \times Y \times Y \arrow[r, "(\mathrm{id}_Y, \Delta_Y, \mathrm{id}_Y)"] & Y \times Y \times Y \times Y.
\end{tikzcd}$$

We can complete this diagram and maintain its commutativity by using the universal property of the pullback:

$$\begin{tikzcd}
Y \arrow[r, "\Delta_Y"] \arrow[d] & Y \times Y \arrow[d] \\
\mathrm{Path}(Y) \times_Y \mathrm{Path}(Y) \arrow[r] \arrow[d, "\in \mathfrak{F}"'] & \mathrm{Path}(Y) \times \mathrm{Path}(Y) \arrow[d, "\in \mathfrak{F}"] \\
Y \times Y \times Y \arrow[r, "(\mathrm{id}_Y, \Delta_Y, \mathrm{id}_Y)"] \arrow[d, "(\pi_1, \pi_3)"'] & Y \times Y \times Y \times Y \arrow[d, "(\pi_1, \pi_4)"] \\
Y \times Y \arrow[r, "\mathrm{id}_{Y \times Y}"'] & Y \times Y.
\end{tikzcd}$$

With this we see that $(\tilde{p}_0, \tilde{p}_1)$ is a fibration, and using Lemma 2.20, Proposition 2.5 and the 2-out-of-3 rule we get that $Y \to \mathrm{Path}(Y) \times_Y \mathrm{Path}(Y)$ is a weak equivalence.

With this result, suppose that we have right homotopies $\eta : X \to \mathrm{Path}(Y)$ from f to g and

$\kappa : X \longrightarrow \text{Path}(Y)$ from g to h, then these homotopies fit in

$$\begin{tikzcd}
X \arrow[drr, bend left, "\eta"] \arrow[dr, dashed, "\exists!(\eta*\kappa)"] \arrow[ddr, bend right, "\kappa"'] & & \\
& \text{Path}(Y) \times_Y \text{Path}(Y) \arrow[r, "\rho_1"] \arrow[d, "\rho_0"'] & \text{Path}(Y) \arrow[d, "p_1"] \\
& \text{Path}(Y) \arrow[r, "p_0"'] & Y
\end{tikzcd}$$

Then, the induced universal morphism $(\eta * \kappa)$ is a right homotopy between f and h as we wished. ■

So, we've established that homotopy is an equivalence relation, at least for "good" objects, so we denote the equivalence class of a morphism $f : X \longrightarrow Y$ by $[f]$. This notation will be used several times in what is to come.

Proposition 2.26. *Let X, Y and Z be fibrant-cofibrant objects. Then for*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \underset{k}{\overset{h}{\rightrightarrows}} Z$$

with $[f] = [g]$ and $[h] = [k]$ we have that

$$[h \circ f] = [k \circ g]$$

Proof: First suppose that $f = g$, we just need to show that $kf \sim hf$. Since $\sim$ is the same for left and right homotopy, we may prove this by exhibiting a right homotopy between these morphisms. Let $\xi : Y \longrightarrow \text{Path}(Z)$ be a right homotopy between k and h, the diagram

$$\begin{tikzcd}
& & Z \\
X \arrow[r, "f"] \arrow[urr, bend left, "h \circ f"] \arrow[drr, bend right, "k \circ f"'] & Y \arrow[r, "\xi"] \arrow[ur, "h"] \arrow[dr, "k"'] & \text{Path}(Z) \arrow[u, "p_0"'] \arrow[d, "p_1"] \\
& & Z
\end{tikzcd}$$

tells us that $\xi \circ f$ is the desired right homotopy. If the case were that $h = k$ and that f and g were different, by the same argument but with a left homotopy (or by duality) we would have that $hf \sim hg$, so

$$hf \sim kf \sim kg$$

which concludes the proof. ■

With this last result we are ready to move on to the next step in our abstract construction, we will define the *homotopy category of a model category*!

2.4 LOCALIZATION AND THE HOMOTOPY CATEGORY

In this section we define what the *homotopy category* of a model category is and show that it corresponds to the *localization* of its underlying category with weak equivalences.

Definition 2.27 (Localization). Let $\mathscr{C}$ be a category with weak equivalences $\mathfrak{W}$. A localization at the weak equivalences in $\mathscr{C}$, is a category $\mathscr{C}[\mathfrak{W}^{-1}]$ together with a functor

$$\gamma : \mathscr{C} \longrightarrow \mathscr{C}[\mathfrak{W}^{-1}]$$

satisfying the following properties:

(i) The image of a weak equivalence by γ is an isomorphism.

(ii) For any category $\mathscr{D}$ and any functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ that F takes weak equivalences to isomorphisms, there exists a factorization of said functor up to a natural isomorphism α as

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\alpha"] & \mathscr{D} \\ \mathscr{C}[\mathfrak{W}^{-1}] \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

with the property that for any two factorizations $(\alpha_1, \tilde{F}_1)$ and $(\alpha_2, \tilde{F}_2)$ there exists a unique natural isomorphism $\zeta : \tilde{F}_1 \Rightarrow \tilde{F}_2$ making the following diagram

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "\alpha_1{,}\alpha_2 \Downarrow"] & & \mathscr{D} \\ & \mathscr{C}[\mathfrak{W}^{-1}] \arrow[ur, bend left, "\tilde{F}_1"] \arrow[ur, bend right, "\tilde{F}_2"'] \arrow[ur, phantom, "\zeta"] & \end{tikzcd}$$

commutes. That is to say that there exists a unique natural transformation $\zeta : \tilde{F}_1 \Rightarrow \tilde{F}_2$ that induces the natural transformation $\alpha_2 \circ \alpha_1^{-1}$.

More often than not we simply say the localization of a category instead of its localization at weak equivalences. □

Proposition 2.28. *If a category with weak equivalence $\mathscr{C}$ has two different localizations, then they are equivalent as categories.*

Proof: Let h_1 and h_2 be two different localizations, so we can factor γ_1 through h_2 and γ_2 through h_1

$$\begin{tikzcd} & h_2 \arrow[d, "h_1\gamma_2"] \\ \mathscr{C} \arrow[ur, "\gamma_2"] \arrow[r, "\gamma_1"'] \arrow[ur, phantom, shift right=2ex, "\alpha"] & h_1 \end{tikzcd} \qquad \begin{tikzcd} & h_1 \\ \mathscr{C} \arrow[ur, "\gamma_1"] \arrow[r, "\gamma_2"'] \arrow[ur, phantom, shift right=2ex, "\beta"] & h_2 \arrow[u, "h_2\gamma_1"'] \end{tikzcd}$$

thus, we have that $\gamma_2 \cong h_1\gamma_2 \circ \gamma_1$ and $\gamma_1 \cong h_2\gamma_1 \circ \gamma_2$, thus we get that

$$\gamma_2 \cong h_1\gamma_2 \circ h_2\gamma_1 \circ \gamma_2$$

thus we have a factorization for γ_2 through $h_1\gamma_2 \circ h_2\gamma_1$, but note that γ_2 can be factored through the identity, by the property of the localization functor, we must have that

$$h_1\gamma_2 \circ h_2\gamma_1 \cong \mathrm{id}_{h_2}.$$

The other side is analogous. ■

Example 2.29. Taking $\mathscr{C}$ to be equipped with the trivial model structure we see that the category $\mathscr{C}$ itself with the identity functor is a localization at its isomorphisms.

Definition 2.30 (The Homotopy Category)**.** Let $\mathscr{C}$ be a model category, then we define its homotopy category as the category $h\mathscr{C}$ whose objects are the cofibrant-fibrant objects of $\mathscr{C}$ and whose morphisms are homotopy equivalence classes of morphisms (either right or left, since we saw that they are equivalent to fibrant-cofibrant objects). The composition of morphisms is given by taking the class of the composition of representatives. □

Now what we want is to show that the above definition does indeed give us a localization for $\mathscr{C}$, so we prove some preliminary results.

The first thing to do is show that weak equivalences are isomorphisms in the homotopy category, as shown below.

Lemma 2.31 (Whitehead's Theorem)**.** *Let $\mathscr{C}$ be a model category, then weak equivalences between fibrant-cofibrant objects are homotopy equivalences.*

Proof: First, suppose that

$$f : X \longrightarrow Y$$

is a weak equivalence, so it factors as

$$X \xrightarrow{\in \mathfrak{C} \cap \mathfrak{W}} Z \xrightarrow{\in \mathfrak{F} \cap \mathfrak{W}} Y$$

by the 2-out-of-3 rule. Writing

$$\varnothing \xrightarrow{\in \mathfrak{C}} X \xrightarrow{\in \mathfrak{C} \cap \mathfrak{W}} Z \xrightarrow{\in \mathfrak{F} \cap \mathfrak{W}} Y \xrightarrow{\in \mathfrak{F}} *$$

we see that Z is also fibrant-cofibrant, so it suffices to show that trivial fibrations are homotopy equivalences. The case for trivial fibrations will follow by duality.

Assume that $f : X \longrightarrow Y$ is a trivial fibration, then by lifting

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, "\in \mathfrak{C}"'] & X \arrow[d, "f \in \mathfrak{F} \cap \mathfrak{W}"] \\ Y \arrow[r, equal] \arrow[ur, "f^{-1}"] & Y \end{tikzcd}$$

we get a right inverse for f. Now we need to show that $f^{-1}f \sim \mathrm{id}_X$. Taking a cylinder object

$$X \coprod X \xrightarrow[\in \mathfrak{C}]{(i_0, i_1)} \mathrm{Cyl}(X) \xrightarrow[\in \mathfrak{W}]{p} X$$

for X, we can form the following commutative square

$$\begin{tikzcd} X \coprod X \arrow[r, "(f^{-1} \circ f, \mathrm{id}_X)"] \arrow[d, "(i_0, i_1) \in \mathfrak{C}"'] & X \arrow[d, "f \in \mathfrak{W} \cap \mathfrak{F}"] \\ \mathrm{Cyl}(X) \arrow[r, "f \circ p"'] & Y \end{tikzcd}$$

whose lift from $\mathrm{Cyl}(X)$ to X is homotopy between $f^{-1}f$ and id_X. ■

Remark 2.32. The above result is actually Whitehead's Theorem for model categories. The classical result in topology can be seen as an application of this theorem.

Proposition 2.33. *Let $\mathscr{C}$ be a model category. For each object X of $\mathscr{C}$, choose* *(i) A factorization*

$$\varnothing \xrightarrow[\in \mathfrak{C}]{i_X} QX \xrightarrow[\in \mathfrak{W} \cap \mathfrak{F}]{p_X} X$$

and for X *cofibrant then* $p_X = \mathrm{id}_X$.

(ii) A factorization

$$X \xrightarrow[\in \mathfrak{W} \cap \mathfrak{C}]{j_X} SX \xrightarrow[\in \mathfrak{F}]{q_X} *$$

and for X *fibrant then* $j_X = \mathrm{id}_X$.

Let

$$\gamma_{SQ} : \mathscr{C} \longrightarrow h\mathscr{C}$$

be the rule which assigns to an object X *the object* SQX *and for a morphism* $f : X \longrightarrow Y$*, the class of the lift* SQf *obtained by*

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, "i_X \in \mathfrak{C}"'] & QY \arrow[d, "p_Y \in \mathfrak{W} \cap \mathfrak{F}"] \\ QX \arrow[r, "f \circ p_X"'] \arrow[ur, dashed, "Qf"] & Y \end{tikzcd} \qquad \begin{tikzcd} QX \arrow[r, "j_{QY} \circ Qf"] \arrow[d, "j_{QX}"'] & SQY \arrow[d, "q_{QY}"] \\ SQX \arrow[r] \arrow[ur, dashed, "SQf"] & * \end{tikzcd} .$$

Then γ_{SQ} *is a functor from* $\mathscr{C}$ *to its homotopy category* $h\mathscr{C}$.

Proof: First we need to show that SQX is fibrant and cofibrant. We have that

$$\varnothing \xrightarrow{\in \mathfrak{C}} QX \xrightarrow{\mathfrak{W} \cap \mathfrak{C}} SQX \xrightarrow{\in \mathfrak{F}} *$$

so SQX is fibrant and cofibrant. Now we need to show that different choices for lifts yield the same classes. Take different choices of the first lift, Qf_1 and Qf_2. For a cylinder object

$$QX \coprod QX \xrightarrow[\in \mathfrak{C}]{(i_0, i_1)} \mathrm{Cyl}(QX) \xrightarrow[\in \mathfrak{W}]{t} QX$$

the lift

$$\begin{tikzcd} QX \coprod QX \arrow[r, "(Qf_1, Qf_2)"] \arrow[d, "\in \mathfrak{C}"'] & QY \arrow[d, "p_Y \in \mathfrak{W} \cap \mathfrak{F}"] \\ \mathrm{Cyl}(QX) \arrow[r, "f \circ p_X \circ t"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

is a left homotopy between Qf_1 and Qf_2. This implies that $j_{QY} \circ Qf_1$ is left homotopic to $j_{QY} \circ Qf_2$, so there also a right homotopy $\xi : QX \longrightarrow \mathrm{Path}(SQY)$ between them. Plugging this information into a diagram

$$\begin{tikzcd} QX \arrow[r, "\xi"] \arrow[d, "\in \mathfrak{W} \cap \mathfrak{C}"'] & \mathrm{Path}(SQY) \arrow[d, "\in \mathfrak{F}"] \\ SQX \arrow[r, "(SQf_1, SQf_2)"'] \arrow[ur, dashed, "h"] & SQY \times SQY \end{tikzcd}$$

we get a lift h which is a right homotopy between SQf_1 and SQf_2. Lastly, we need to show that this rule is functorial. By the way we constructed the morphisms Qf and SQf, the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & QX \arrow[l, "p_X"'] \arrow[r, "j_{QX}"] \arrow[d, "Qf"] & SQX \arrow[d, "SQf"] \\ Y & QY \arrow[l, "p_Y"] \arrow[r, "j_{QY}"'] & SQY \end{tikzcd}$$

commutes. So for morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, the diagram

$$\begin{tikzcd}
X & QX \arrow[l, "p_X"'] \arrow[r, "j_{QX}"] & SQX \\
Y \arrow[from=1-1, "f"'] & QY \arrow[from=1-2, "Qf"] \arrow[l, "p_Y"] \arrow[r, "j_{QY}"'] & SQY \arrow[from=1-3, "SQf"] \\
Z \arrow[from=2-1, "g"'] & QZ \arrow[from=2-2, "Qg"] \arrow[l, "p_Z"] \arrow[r, "j_{QY}"'] & SQZ \arrow[from=2-3, "SQg"]
\end{tikzcd}$$

shows us that $SQg \circ SQf$ is a lift for $g \circ f$, which by the preceding discussion is homotopic to the chosen $SQ(g \circ f)$. ■

Now we are ready to show that $h\mathscr{C}$ is the localization of $\mathscr{C}$, but first we prove a useful Lemma that we will use later.

Proposition 2.34. *Let $X, Y \in \mathscr{C}$ be cofibrant fibrant respectively, then there is a natural bijection*

$$h\mathscr{C}(SX, QY) = \mathscr{C}(SX, QY)/\sim \xrightarrow{(j_X^*, p_{y*})} \mathscr{C}(X, Y)/\sim$$

where S, Q and j_X, p_Y are as in the proposition above and "$\sim$" is the realtion of being homotopic.

Proof: First note that the map defined can be decomposed as

$$\mathscr{C}(SX, QY)/\sim \xrightarrow{p_{y*}} \mathscr{C}(SX, Y)/\sim \xrightarrow{j_X^*} \mathscr{C}(X, Y)/\sim .$$

So it suffices to show that either one of the above maps is a bijection, since that the other is a bijection will follow by duality. We prove that the first one in the decomposition is a bijection. To show that it is surjective, note that the lifting property of

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, "\in\mathfrak{C}"'] & QY \arrow[d, "p_Y \in \mathfrak{W}\cap\mathfrak{F}"] \\
Y \arrow[r, "f"'] & Y
\end{tikzcd}$$

implies that every morphism $f : X \to Y$ can be obtained by post composition with p_y. To show that it is injective, suppose that $p_Y \circ f \sim p_Y \circ g$, so we have a left homotopy $\eta : p_Y \circ f \Rightarrow_L p_Y \circ g$, thus the lift in the diagram

$$\begin{tikzcd}
X \coprod X \arrow[r, "(f,g)"] \arrow[d, "\in\mathfrak{C}"'] & QY \arrow[d, "p_Y"] \\
\mathrm{Cyl}(X) \arrow[r, "\eta"'] & Y
\end{tikzcd}$$

shows that f and g were already homotopic, and the result follows. ■

Lemma 2.35. *Let $\mathscr{C}$ be a model category and $F : \mathscr{C} \to \mathscr{D}$ a functor that sends weak equivalences to isomorphisms. Then if $f, g : X \rightrightarrows Y$ are either left or right homotopic, then $F(g) = F(f)$.*

Proof: Let $\eta : f \Rightarrow_L g$ be a left homotopy. Then we have the diagram

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(X) \arrow[d, "\eta"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

so that $F(f) = F(\eta)F(i_0)$ and $F(g) = F(\eta)F(i_1)$. Now note that by the definition of a cylinder object of X

$$X \bigsqcup X \xrightarrow[\in \mathfrak{C}]{(i_0,i_1)} \mathrm{Cyl}(X) \xrightarrow[\in \mathfrak{W}]{p} X$$

we have that $F(p)F(i_0) = F(p)F(i_1)$. Since p is a weak equivalence, $F(p)$ is an isomorphism, and $F(i_0) = F(i_1)$, thus $F(f) = F(g)$. ∎

Theorem 2.36. *Let $\mathscr{C}$ be a model category. The category $h\mathscr{C}$ together with the functor*

$$\gamma_{SQ} : \mathscr{C} \longrightarrow h\mathscr{C}$$

is a localization of the underlying category with weak equivalence of $\mathscr{C}$ namely $(\mathscr{C}, \mathfrak{W})$.

Proof: To see that γ_{SQ} sends weak equivalences to isomorphisms, observe that the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & QX \arrow[l, "p_X"'] \arrow[r, "j_{QX}"] \arrow[d, "Qf"] & SQX \arrow[d, "SQf"] \\ Y & QY \arrow[l, "p_Y"] \arrow[r, "j_{QY}"'] & SQY \end{tikzcd}$$

assuming that f is a weak equivalence, together with the 2-out-of-3 rule implies that $SQf : SQX \longrightarrow SQY$ is a weak equivalence. Since SQX and SQY are fibrant and cofibrant, Whitehead's Theorem (Lemma 2.31) implies that SQf is a homotopy equivalence, which is an isomorphism in the homotopy category $h\mathscr{C}$.

Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor that sends weak equivalences to isomorphisms. We need to factor it as $(\alpha, \tilde{F})$ as in Definition 2.27. By construction, γ_{SQ} is the identity on fibrant-cofibrant objects, so if such factorization exists, for $f : X \to Y$ with X and Y fibrant and cofibrant it must satisfy that

$$\begin{tikzcd} \tilde{F}(X) \arrow[r, "\text{iso}"] \arrow[d, "\tilde{F}([f])"'] & F(X) \arrow[d, "F(f)"] \\ \tilde{F}(Y) \arrow[r, "\text{iso}"'] & F(Y) \end{tikzcd}$$

in other words, $\tilde{F}([f]) \cong F(f)$, so we set $\tilde{F}(X) = F(X)$ and $\tilde{F}([f]) = F(f)$, by Lemma 2.35, this rule is well defined and it is functorial. Also, for every morphism $f : X \longrightarrow Y$ between any objects, we have, by choice, a morphism $SQf : SQX \longrightarrow SQY$, and these exhaust all morphisms in the homotopy category, in particular, $\tilde{F}([SQf]) = F(SQf)$. Since the objects of $h\mathscr{C}$ are fibrant-cofibrant objects, this already fixes the values of $\tilde{F}$ for all of $h\mathscr{C}$. Now we need to show that there is in fact a natural isomorphism $\alpha : F \Rightarrow \tilde{F} \circ \gamma_{SQ}$ filling the vertical arrows of the above diagram. Applying F to the first diagram , we get

$$\begin{tikzcd} F(X) \arrow[d, "F(f)"'] & F(QX) \arrow[l, "F(p_X)"'] \arrow[r, "F(j_{QX})"] \arrow[d, "F(Qf)"] & F(SQX) \arrow[d, "F(SQf)"] \\ F(Y) & F(QY) \arrow[l, "F(p_Y)"] \arrow[r, "F(j_{QY})"'] & F(SQY). \end{tikzcd}$$

By hypothesis, F sends weak equivalences to isomorphisms, so all horizontal arrows above are isomorphisms, so we can define α by defining its components as

$$\alpha_X := F(j_{QX}) \circ F(p_X)^{-1}.$$

So we get the diagram

$$\begin{tikzcd}
F(X) \arrow[r] \arrow[d, "F(f)"'] \arrow[rrr, bend left, "\alpha_X"] & F(QX) \arrow[r] & F(SQX) \arrow[r, equal] \arrow[d, "F(SQf)"] & \tilde{F}(L_{SQ}(X)) \arrow[d, "\tilde{F}(L_{SQ}(f))"] \\
F(Y) \arrow[r] \arrow[rrr, bend right, "\alpha_Y"'] & F(QY) \arrow[r] & F(SQY) \arrow[r, equal] & \tilde{F}(L_{SQ}(X))
\end{tikzcd}$$

which shows that α is the desired natural transformation. Note that the uniqueness of $\tilde{F}$ up to unique natural isomorphism is satisfied trivially since γ_{SQ} fixes objects. ■

Remark 2.37. Now we know that $h\mathscr{C}$ is a localization of $\mathscr{C}$ at its weak equivalences $\mathfrak{W}$, so from now on we will omit the subscript SQ, and just talk about the localization functor $\gamma : \mathscr{C} \longrightarrow h\mathscr{C}$.

Let $\mathscr{C}$ be a model category. We have the following inclusion of categories with weak equivalences

$$\begin{tikzcd}
& \mathscr{C}_{fc} \arrow[dl, hook] \arrow[dr, hook] & \\
\mathscr{C}_f \arrow[dr, hook] & & \mathscr{C}_c \arrow[dl, hook] \\
& \mathscr{C} &
\end{tikzcd}$$

where $\mathscr{C}_{fc}$ is the full subcategory of fibrant-cofibrant objects, $\mathscr{C}_f$ is the full subcategory of fibrant objects and $\mathscr{C}_c$ is the full subcategory of cofibrant objects. Their weak equivalence structures are inherited from the equivalences $\mathfrak{W}$. □

Proposition 2.38. *The homotopy category of $\mathscr{C}$ is also a localization for the categories with weak equivalence $\mathscr{C}_f$, $\mathscr{C}_{fc}$ and $\mathscr{C}_c$.*

Proof: This follows from the proof of Theorem 2.36, the construction is the same, but in this case, one of the "replacements" (Q or S) is the identity. ■

2.5 COFIBRANT GENERATION

In this section we briefly discuss the notion of a *Cofibrantly Generated Model Category*. Sometimes, we wish there would be a much smaller class of cofibrations such that all other cofibrations were somehow encoded into such a collection. This is what we will study in this section.

First of all, we need to understand which construction we will use the get more cofibrations out of a given class, this why we need the following:

Definition 2.39 (Transfinite Composition). Let α be an ordinal, then we can view it as a partial order category. Let $\mathscr{C}$ be a category and $K \subset \mathrm{Mor}(\mathscr{C})$ be any class of morphisms of $\mathscr{C}$. Let

$$F : \alpha \longrightarrow \mathscr{C}$$

be a diagram such that

(i) For any successor ordinal $\beta \to \beta+1$ in α, we have that the morphism $F_\beta \to F_{\beta+1}$ is in $K \subset \mathrm{Mor}(\mathscr{C})$.

(ii) For any limit ordinal $\gamma < \alpha$, the funtor F restricted to the subcategory γ

$$F : \gamma \longrightarrow \mathscr{C}$$

has as its colimit F_γ.

Under these conditions we call the induced morphism

$$F_0 \longrightarrow F_\alpha := \operatorname{colim} F$$

the transfinite composition of an α-indexed sequence of morphisms of K. □

Most of the time we will only be interested in transfinite compositions indexed by $\mathbb{N}$, so the second item won't come into play very often. Now we work toward the definition of a CW-complex.

Definition 2.40 (Cell Complexes)**.** Let $\mathscr{C}$ be a category with colimits, and let $K \subset \mathrm{Mor}(\mathscr{C})$ be a collection of morphisms of $\mathscr{C}$. We say that a map $f : X \longrightarrow Y$ in $\mathscr{C}$ is a K-relative cell complex if it can be exhibited as a transfinite composition of pushouts of coproducts of morphisms from K. Explicitly, we have a transfinite sequence of morphisms $X = X_0 \to X_1 \to X_2 \to \ldots \to X_\alpha = Y$ such that its transfinite composition is f and each succession morphism is of the form

$$\begin{tikzcd} \coprod_i A_i \arrow[r] \arrow[d, "\sqcup_i"'] & X_n \arrow[d] \\ \coprod_i B_i \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X_{n+1} \end{tikzcd}$$

where $A_i \to B_i$ are morphisms in K. An object X is called a K-cell complex if the map $\emptyset \to X$ is a K-relative cell complex. The morphisms in K are called the "cells" of the complex and we can think of the pushouts as steps "adding cells" to the object. □

This all may seem a little abstract, but what we mean by "$f : X \to Y$ is a relative cell complex" is that the object Y can be obtained from X by gluing cells at X in a possibly infinite (the need for transfinite composition) sequence of steps! The condition that X be a cell complex is just saying that we can build X using *only* cells, with no need of a "starting" object!

Example 2.41 (CW-Complexes)**.** Let $I_{\mathbf{Top}} = \{S^{n-1} \hookrightarrow D^n\}$[2] be the set of inclusions of the spheres into disks. A space X is called a CW-complex if it is a $I_{\mathbf{Top}}$-cell complex, its transfinite composition is at most countable (*i.e.* the indexing ordinal is at most $\mathbb{N}$), and the space X_{k+1} is obtained by only attaching cells of dimension $k+1$ to X_k (the dimension of the cell is the dimension of the disk).

Definition 2.42. A model category $\mathscr{C}$ is said to be cofibrantly generated if there exist two collections of morphisms

$$I, J \subset \mathrm{Mor}(\mathscr{C})$$

satisfying the following properties:

(i) All morphisms I and J have small domains relative to themselves (see Definition A.52).

[2]Here we consider $S^{-1} \hookrightarrow D^0$ as the map $\emptyset \to *$.

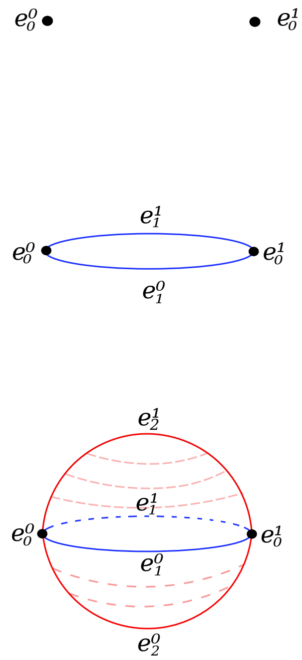


Figure 2.1: Constructing a sphere as a CW-complex.

(ii) The cofibrations $\mathscr{C}$ are exactly the retracts of I-relative cell complexes and the trivial cofibrations are the retracts of J-relative cell complexes.

The collection I is said to be the collection of generating cofibrations and the collection J is said to be the collection of generating trivial cofibrations. □

Theorem 2.43. *For a cofibrantly generated model category $\mathscr{C}$ with generating cofibrations $\mathrm{I}, \mathrm{J} \subset \mathrm{Mor}(\mathscr{C})$, we have the following:*

(i) $\mathfrak{C} = \mathrm{LLP}(\mathrm{RLP}(\mathrm{I}))$;

(ii) $\mathfrak{W} \cap \mathfrak{C} = \mathrm{LLP}(\mathrm{RLP}(\mathrm{J}))$;

(iii) $\mathfrak{F} = \mathrm{RLP}(\mathrm{J})$;

(iv) $\mathfrak{W} \cap \mathfrak{F} = \mathrm{RLP}(\mathrm{I})$.

Proof: Since $\mathrm{I} \subset \mathfrak{C} = \mathrm{LLP}(\mathrm{RLP}(\mathrm{I}))$, by Proposition 2.5, we have that $\mathfrak{C} \subset \mathrm{LLP}(\mathrm{RLP}(\mathrm{I}))$, since $\mathfrak{C}$ is by definition the retract of I-relative cell complexes. Now suppose that $i \in \mathrm{LLP}(\mathrm{RLP}(\mathrm{I}))$, then by the small object argument, i factors as

$$i : X \xrightarrow{\in \mathrm{cell}(\mathrm{I})} X' \xrightarrow{\in \mathrm{RLP}(\mathrm{I})} Y$$

so by the retract argument (2.16), i is a retract of an I-relative cell complex, this proves the first item. The second item follows directly from the closure properties of a model category. The remaining two items are analogous, just replace the letter I with J and all the same arguments are valid. ■

Corollary 2.44. *In a cofibrant generated model category with generating cofibrations* I *the cofibrant objects are the retracts of cell complexes of* I.

To conclude this chapter we give a recipe to create cofibrantly generated model categories.

Theorem 2.45. *Let $\mathscr{C}$ be a category with weak equivalences $\mathfrak{W}$. Let $\mathrm{I}, \mathrm{J} \subset \mathrm{Mor}(\mathscr{C})$ and suppose that* *(i) The domains of morphisms in* I *are small relative to* I *and the morphisms in* J *are small relative to* J*.*

(ii) Every J*-relative cell complex is both an* I*-relative cell complex and a weak equivalence.*

(iii) Every morphism in RLP(I) *is in* RLP(J) *and is a weak equivalence.*

(iv) A morphism that is in RLP(J) *morphism and is a weak equivalence is in* RLP(I)*.*

Then $\mathscr{C}$ has the structure of a cofibrantly generated model category with I *being the generating cofibrations and* J *the trivial generating cofibrations.*

Proof: Explicitly, the model structure is given by

$$\mathfrak{C} = \mathtt{cell}(I) \text{ and } \mathfrak{F} = \mathsf{RLP}(J).$$

Since the domains of morphisms in I and J are small, by the Quillen Small Object Argument (see Theorem A.53), we know that all morphisms in $\mathscr{C}$ factor as (J)I-relative cell complexes followed by a morphisms $\mathsf{RLP}(J)$ and $\mathsf{RLP}(I)$. We must show that the collections $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$ and $(\mathfrak{C}, \mathfrak{F} \cap \mathfrak{W})$ are factorization systems. We first show the closure under lifting properties.

Let's start with $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$. We first show that $\mathfrak{C} \cap \mathfrak{W} = \mathtt{cell}(J)$. By item (ii), $\mathtt{cell}(J) \subset \mathfrak{C} \cap \mathfrak{W}$, suppose then that $f \in \mathfrak{C} \cap \mathfrak{W}$. We may factor it as

$$f : X \xrightarrow{i \in \mathtt{cell}(J)} X' \xrightarrow{p \in \mathfrak{F}} Y.$$

By Item (ii) and the 2-out-of-3 rule every morphism above is a weak equivalence, so that p is in $\mathsf{RLP}(J)$ and is a weak equivalence, so by item (iv), f has the lifting property against p, and by the retract argument f is a retract of i and so it is a J-relative cell complex. Now we show the closure properties of $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$. Since $\mathfrak{C} \cap \mathfrak{W} = \mathtt{cell}(J)$, by Corollary 2.7, $\mathfrak{F} = \mathsf{RLP}(\mathfrak{C} \cap W)$. It is trivial that $\mathtt{cell}(J) \subset \mathsf{LLP}(\mathfrak{F})$. If $f \in \mathsf{LLP}(\mathfrak{F})$, then we may factor it as we did above, and the retract argument gives us that f is a J-relative cell complex.

It remains to show that $(\mathfrak{C}, \mathfrak{F} \cap \mathfrak{W})$ is a factorization system. By item (iv), $\mathfrak{F} \cap \mathfrak{W} \subset \mathsf{RLP}(\mathfrak{C})$ and by Item (iii) $\mathsf{RLP}(\mathfrak{C}) \subset \mathfrak{F} \cap \mathfrak{W}$ so that $\mathfrak{F} \cap \mathfrak{W} = \mathsf{RLP}(\mathfrak{C})$. By the previous equality we know that $\mathfrak{C} \subset \mathsf{LLP}(\mathfrak{F} \cap \mathfrak{W})$, we may factor a morphism $f \in \mathsf{LLP}(\mathfrak{F} \cap \mathfrak{W})$ as

$$f : X \xrightarrow{i \in \mathtt{cell}(I)} X' \xrightarrow{p \in \mathsf{RLP}(I)} Y.$$

By item (ii), $p \in \mathfrak{F} \cap \mathfrak{W}$, and so by the retract argument f is retract of i, thus it is an I-relative cell complex.

To see that this collections indeed factor all morphisms, note that we showed that $\mathtt{cell}(J) = \mathfrak{C} \cap \mathfrak{W}$ and that $\mathtt{fib} \cap \mathfrak{W} = \mathsf{RLP}(I)$, so the initial factorizations given by I and J are already the desired ones. ■

2.6 EXAMPLES OF MODEL CATEGORIES

So, we've discussed what are model categories, now we need some examples to show why they really work. There are a huge number of such examples, but here we give just two. With this we hope to illustrate how model categories capture essential ideas about homotopy. Since the goal here is just to give examples, we won't give proofs for all the claims in this chapter.

TOPOLOGICAL SPACES

Firstly, we talk about **Top**, the precursor of all of this. Above all, a model category is a category with weak equivalences, so we must choose which morphisms are going to play this role. We have two obvious choices: homotopy equivalences and weak homotopy equivalences.

Recall that a homotopy equivalence is a continuous function $f : X \to Y$ together with a continuous function $g : Y \to X$ such that

$$g \circ f \sim \mathrm{id}_X \text{ and } f \circ g \sim \mathrm{id}_Y$$

where the relation "$\sim$" is the relation "is homotopic to".

A weak homotopy equivalence is a map $f : X \to Y$ that induces a bijection on the set of connected components of X and Y and is such that

$$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

is an isomorphism of groups for every positive integer n and for every $x \in X$.

Every homotopy equivalence is a weak homotopy equivalence, but the converse does not hold in general, for instance, there are weakly contractible spaces which are not contractible.

The most common choice of weak equivalences is that of weak homotopy equivalences, because we want to differentiate spaces up to their homotopy groups. This allows us to work with only CW-complexes (for reasons we'll give shortly), which are much better behaved then generic topological spaces. Thus, we shall give a model structure with these weak equivalences. However, there are model structures with homotopy equivalences as weak equivalences, an example of this can be found at [35].

Definition 2.46. Call a function $f : X \to Y$ a weak equivalence if it is a weak homotopy equivalence. Denote the collection of such maps by $\mathfrak{W}_{\mathbf{Top}}$. □

Now, we have to define the fibrations and cofibrations. We do this by defining two sets, one of generating cofibrations and one of generating trivial cofibrations.

Definition 2.47. Define the collections $I_{\mathbf{Top}}, J_{\mathbf{Top}} \subset \mathrm{Mor}(\mathbf{Top})$ as

$$I_{\mathbf{Top}} := \{S^{n-1} \hookrightarrow D^n\} \text{ and } J_{\mathbf{Top}} := \{D^n \hookrightarrow D^n \times I\}$$

where $I = [0, 1]$ and D^n is the disk. Clearly the elements of $J_{\mathbf{Top}}$ are (weak) homotopy equivalences. □

Since we're claiming these are generating cofibrations, we know what the the fibrations must be:

Definition 2.48. A function $f : X \to Y$ is called a Serre fibration if $f \in \mathrm{RLP}(J_{\mathbf{Top}})$. Explicitly, f is a Serre fibration if any diagram of the form

$$\begin{tikzcd} D^n \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ D^n \times I \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

admits a lift h. Denote the collection of such fibrations by $\mathfrak{F}_{\mathbf{Top}}$. □

Here we make a pause of the development of model structures to explain a bit more of why lifting properties have to do with homotopy. By definition, a Serre fibration has the right lifting property against $J_{\mathbf{Top}}$, and so, it has the right lifting property against all relative cell complexes which are also cofibrations. In particular it has the right lifting property against the inclusion of CW-complexes which are also weak homotopy equivalences. The map

$$S^n \hookrightarrow S^n \times I$$

is a weak homotopy equivalence, thus a diagram of the form

$$\begin{tikzcd} S^n \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ S^n \times I \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

always has a lift h. This says that for every homotopy between elements of $\mathbf{Top}(S^n, Y)$, if one of these elements factors through f, then so does the homotopy: this is a homotopy lifting property. With arguments like this we can get more and more intuition, but we'll say no more about them here.

With these collections $\mathfrak{W}_{\mathbf{Top}}$, $\mathfrak{F}_{\mathbf{Top}}$, $I_{\mathbf{Top}}$ and $J_{\mathbf{Top}}$ were are ready to define the Serre-Quillen Model Structure:

Theorem 2.49. *Let $\mathfrak{C}_{\mathbf{Top}} := \mathrm{cell}(I_{\mathbf{Top}})$. Then, $(\mathbf{Top}, \mathfrak{F}_{\mathbf{Top}}, \mathfrak{C}_{\mathbf{Top}}, \mathfrak{W}_{\mathbf{Top}})$ is a cofibrantly generated model category with generating cofibrations $I_{\mathbf{Top}}$ and $J_{\mathbf{Top}}$.*

Corollary 2.50. *The cofibrant objects of the Serre-Quillen Model Structure are the retracts of cell complexes. Also, trivially, all objects of this model structure are fibrant.*

This Theorem is a classical result due to Quillen and can be found in almost any reference on model categories. For instance see [32].

Now we shall look at the homotopy category of **Top**. First of all, recall what the Cellular Approximation theorem tells us:

Theorem 2.51. *Every topological space is weakly homotopic to a CW-complex.*

Thus, if we consider the full subcategory $h\mathbf{Top}_{CW} \hookrightarrow h\mathbf{Top}$, where $h\mathbf{Top}$ is the homotopy category of **Top** with the Serre-Quillen model structure, the inclusion

$$h\mathbf{Top}_{CW} \hookrightarrow h\mathbf{Top}$$

is essentially surjective, and so it is an equivalence of categories.

Now, for any CW-complex X, $X \times I$ is a CW-complex and $X \coprod X \hookrightarrow X \times I$ is a cofibration. Moreover, $X \times I \to X$ is a trivial fibration, so that $X \times I$ is a cylinder object for X. Thus, a left homotopy from $f : X \to Y$ to $g : X \to Y$ is a map $h : X \times I \to Y$ such that

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "f"'] & X \times I \arrow[d, "h"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

commutes. With this, we conclude that a left homotopy between maps from CW-complexes are simply homotopies in the classical sense. Thus, the homotopy category of **Top** is the full subcategory of CW-complexes with homotopy classes of functions as the morphisms, in particular, weak homotopy equivalences between CW-complexes are homotopy equivalences. This explains why we need only to worry about CW-complexes.

Before we conclude this section, we show how this model structure isn't actually just a complicated way to say that homotopies are just left and right homotopies, that is, we'll give an example of a space X such that $X \times I$ is a cylinder object.

Let A be a weakly contractible space which is not contractible[3]. Then the map $A \longrightarrow *$ is a trivial Serre fibration. Since A is not contractible, a constant map is not homotopic to the identity map, that is, the maps $\mathrm{id}, \mathrm{const} : A \longrightarrow A$ are not homotopic. Now consider the commutative diagram

$$\begin{tikzcd} A \coprod A \arrow[r, "(\mathrm{id},\mathrm{const})"] \arrow[d, "(i_0,i_1)"'] & A \arrow[d, "\in \mathfrak{W}_{\mathbf{Top}} \cap \mathfrak{F}_{\mathbf{Top}}"] \\ A \times I \arrow[r] & * \end{tikzcd}$$

Since any lift in this diagram would be a homotopy between id and const, there is no lift, which implies that (i_0, i_1) is not a cofibration.

CATEGORIES

There is an obvious notion of what a weak equivalence between categories should be: equivalence of categories. So, let $\mathfrak{W}_{\mathbf{Cat}}$ be the collection of equivalences of categories. Now we must look at the fibrations and cofibrations.

Let $\mathscr{J}$ denote the walking isomorphism[4]. Call $F : \mathscr{C} \to \mathscr{D}$ an isofibrations if it has the right lifting property against $* \to \mathscr{J}$. This is equivalent to the diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\ \mathscr{J} \arrow[r] \arrow[ur, dashed, "H"] & \mathscr{D} \end{tikzcd}$$

having a lift H. This is simply saying that F lifts isomorphisms: a map $\mathscr{J} \to \mathscr{D}$ as in the diagram is simply a choice of isomorphism $f : F(x) \to y$, then a lift is simply a choice of isomorphism $g : x \to z$ such that $F(g) = f$. Denote by $\mathfrak{F}_{\mathbf{Cat}}$ the class of isofibrations. Note that trivial isofibrations are equivalences of categories which are surjective on objects

Call a functor $F : \mathscr{C} \to \mathscr{D}$ an isocofibration if it is injective on objects. With this we are ready to define a model structure on **Cat**.

Theorem 2.52. $(\mathbf{Cat}, \mathfrak{F}_{\mathbf{Cat}}, \mathfrak{C}_{\mathbf{Cat}}, \mathfrak{W}_{\mathbf{Cat}})$ *is a model category.*

Remark 2.53. This same model structure works for the full subcategory **Grpd** of groupoids.

This model structure is called the canonical model structure on the category of categories. The "canonical" comes from the fact that this is the unique model structure that has as its weak equivalences the equivalences of categories. A detailed proof of this fact can be found here [30].

[3] For instance the double comb space.

[4] The groupoid with two objects and exactly one morphism in each hom-set. This is also known as the "free standing isomorphism".

A cool feature of this model structure is that all objects are fibrant-cofibrant, as one would hope, since we want to distinguish *any* category up to equivalence. Thus in the homotopy category of **Cat** two objects are isomorphic if and only if they are equivalent as categories.

Now, let's look at the homotopies of this model structure. For any category $\mathscr{C}$, consider the inclusion $\mathscr{C} \coprod \mathscr{C} \hookrightarrow \mathscr{C} \times \mathscr{J}$. Clearly this map is an isocofibration. Now, note that the projection $p : \mathscr{C} \times \mathscr{J} \to \mathscr{C}$ is an equivalence of categories since each hom-set of $\mathscr{J}$ consists of a single element. This means that $\mathscr{C} \times \mathscr{J}$ is a cylinder object for $\mathscr{C}$. Thus, a homotopy between the functors $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ is a functor $H : \mathscr{C} \times \mathscr{J} \to \mathscr{D}$ such that the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathscr{C} \times \mathscr{J} \arrow[d, "H"] & \mathscr{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\
& \mathscr{D} &
\end{tikzcd}$$

commutes. Now observe that this map $H : \mathscr{C} \times \mathscr{J} \to \mathscr{D}$ is the same as a natural isomorphism $\eta : F \Rightarrow G$. To see this, note that $\mathscr{J} = \{\bullet \xrightarrow{j} \bullet\}$, so by the commutativity of the diagram above we get that for a morphism $f : x \to y$ in $\mathscr{C}$, we get

$$H(f, j) = F(x) \to G(x)$$

where the above morphism is an isomorphism. Functoriality assures us that these morphisms assemble into a natural transformation. The reciprocal is analogous.

With this we conclude that h**Cat** is the category whose objects are categories and

$$h\mathbf{Cat}(\mathscr{C}, \mathscr{D}) = \mathbf{Cat}(\mathscr{C}, \mathscr{D}) / \sim$$

where $F \sim G$ if and only if they are isomorphic. In particular, we see that isomorphisms in h**Cat** are the equivalences of categories.

3

QUILLEN ADJUNCTIONS AND EQUIVALENCES

Now that we know what model categories are we would like to have a way to "compare" them. Remember that model categories arise as a means to study homotopy, in the sense that we have a nice way of getting a homotopy category and that we are only worried about the differences between any two objects up to homotopy equivalences.

As with any other kind of objects, we compare categories through morphisms, so, by functors. Model categories are categories with extra structure, so it is natural to wish that "morphisms" between them preserve such extra structure in some sense, that is exactly what Quillen adjunctions do. As we'll see below, a Quillen adjunction between two model categories will induce an adjunction between their homotopy categories, giving us an interaction between the homotopy theories defined by each category. When this induced adjunction between the homotopy categories is in fact an equivalence of categories, we may say that the two model categories have the same homotopy theory, so a Quillen equivalence is simply a precise formulation of what it means for a homotopy theory to be the same as another.

3.1 DERIVED FUNCTORS

Definition 3.1 (Homotopical Functors)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be categories with weak equivalence, then a functor

$$F : \mathscr{C} \longrightarrow \mathscr{D}$$

is homotopical if it sends weak equivalences to weak equivalences. □

Definition 3.2 (Derived Functors)**.** Given a homotopical functor

$$F : \mathscr{C} \longrightarrow \mathscr{D}$$

between categories $\mathscr{C}$ and $\mathscr{D}$ whose localizations exist. Then the derived functor of F is the functor

$$hF : \mathscr{C}[\mathfrak{W}_{\mathscr{C}}^{-1}] \longrightarrow \mathscr{D}[\mathfrak{W}_{\mathscr{D}}^{-1}]$$

given by a factorization of F in

$$\begin{tikzcd}
\mathscr{C} \arrow[r] \arrow[d, "\gamma_{\mathscr{C}}"'] \arrow[dr, phantom, "\swarrow\cong"] & \mathscr{D} \arrow[d, "\gamma_{\mathscr{D}}"] \\
\mathscr{C}[\mathfrak{W}_{\mathscr{C}}^{-1}] \arrow[r] & \mathscr{D}[\mathfrak{W}_{\mathscr{D}}^{-1}]
\end{tikzcd}$$

which exists by the definition of the localization functor. □

Sometimes a functor of interest (in a model category) is not homotopical, but its restriction to one of the categories $\mathscr{C}_f$ or $\mathscr{C}_c$ is, so it is useful to generalize the definition above to such cases. These are called left and right derived functors.

Definition 3.3 (Left and Right Derived Functors)**.** Let $\mathscr{C}$ be a model category and $\mathscr{D}$ a category with weak equivalences. Let

$$F : \mathscr{C} \longrightarrow \mathscr{D}$$

be a functor from $\mathscr{C}$ to $\mathscr{D}$. If F restricted to the category of fibrant objects $\mathscr{C}_f$ is homotopical, then the functor represented in the bottom horizontal arrows

$$\begin{tikzcd}
\mathscr{C}_f \arrow[r, hook] \arrow[d, "\gamma_{\mathscr{C}_f}"'] & \mathscr{C} \arrow[r, "F"] \arrow[d, phantom, "\overset{\cong}{\Leftarrow}"] & \mathscr{D} \arrow[d, "\gamma_{\mathscr{D}}"] \\
\mathscr{C}_f[\mathfrak{W}^{-1}] \arrow[r, "\cong"] \arrow[rr, bend right, "\mathbb{R}F"'] & h\mathscr{C} \arrow[r] & h\mathscr{D}
\end{tikzcd}$$

is called the right derived functor of F as is denoted by $\mathbb{R}F$. Dually, if F restricted to the category of cofibrant objects $\mathscr{C}_c$ is homotopical, then the functor represented in the bottom horizontal arrows

$$\begin{tikzcd}
\mathscr{C}_c \arrow[r, hook] \arrow[d, "\gamma_{\mathscr{C}_c}"'] & \mathscr{C} \arrow[r, "F"] \arrow[d, phantom, "\overset{\cong}{\Leftarrow}"] & \mathscr{D} \arrow[d, "\gamma_{\mathscr{D}}"] \\
\mathscr{C}_c[\mathfrak{W}^{-1}] \arrow[r, "\cong"] \arrow[rr, bend right, "\mathbb{L}F"'] & h\mathscr{C} \arrow[r] & h\mathscr{D}
\end{tikzcd}$$

is called the left derived functor of F as is denoted by $\mathbb{L}F$. □

For a morphism $f : X \to Y$ in a model category $\mathscr{C}$ we may construct an object $\mathrm{Cyl}(f)$, the mapping cylinder of f, via the following pushout:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_1"'] & Y \arrow[d, "(i_1)_* f"] \\
\mathrm{Cyl}(X) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Cyl}(f).
\end{tikzcd}$$

In **Top**, considering the cylinder object as $X \times I$, this would amount to gluing $X \times I$ to Y along the image of f in Y. See Figure 3.1.

Dual to the mapping cylinder, we have the mapping cocylinder, obtained by the pullback with a path object $\mathrm{Path}(Y)$:

$$\begin{tikzcd}
\mathrm{Path}(f) \arrow[r] \arrow[d, "(p_1)^* f"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\
\mathrm{Path}(Y) \arrow[r, "p_1"'] & Y.
\end{tikzcd}$$

This objects will be used later, but for now they will only be used in the statement of the Factorization Lemma below.

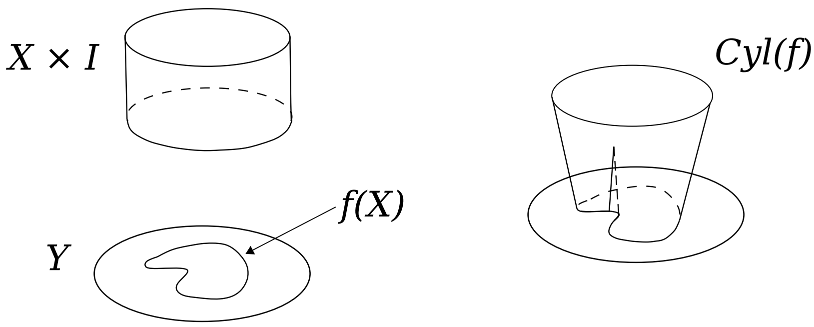


Figure 3.1: The Mapping Cylinder of a continuous function f.

Lemma 3.4 (Factorization Lemma). *Let $\mathscr{C}_c$ be the category of cofibrant objects and let $f : X \longrightarrow Y$ be any morphism in $\mathscr{C}_c$. Then, we have*

(i) The map $X \xrightarrow{i_0} \mathrm{Cyl}(X) \xrightarrow{(i_1)_* f} \mathrm{Cyl}(f)$ *is a cofibration.*

(ii) f *factors through the above cofibration as* $f : X \xrightarrow{\in \mathfrak{C}} \mathrm{Cyl}(X) \xrightarrow{w \in \mathfrak{W}} Y$ *where w is a weak equivalence right inverse to some trivial cofibration.*

Dually, if $\mathscr{C}_f$ is a category of fibrant objects and $f : X \longrightarrow Y$ is any morphism in $\mathscr{C}_f$, we have

(i) The map $\mathrm{Path}(f) \xrightarrow{(p_1)^* f} \mathrm{Path}(Y) \xrightarrow{p_0} Y$ *is a fibration.*

(ii) f *factors through the above fibration as* $f : X \xrightarrow{w \in \mathfrak{W}} \mathrm{Path}(f) \xrightarrow{\in \mathfrak{F}} Y$ *where w is a weak equivalence left inverse to some trivial fibration.*

Proof: We prove the second statement since they are dual. Note that we may decompose the diagram

$$\begin{tikzcd}
\mathrm{Path}(f) \arrow[r, "\in \mathfrak{W} \cap \mathfrak{F}"] \arrow[d, "(p_1)^* f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\mathrm{Path}(Y) \arrow[r, "p_1"'] \arrow[d, "p_0 \in \mathfrak{W} \cap \mathfrak{F}"'] & Y \\
Y &
\end{tikzcd}$$

into

$$\begin{tikzcd}
\mathrm{Path}(f) \arrow[r] \arrow[d, "(p_1)^* f"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times Y \arrow[r, "\pi_1"'] \arrow[d, "(f, \mathrm{id}_Y)"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
\mathrm{Path}(Y) \arrow[r, "(p_1, p_0)"'] \arrow[d, "p_0"'] & Y \times Y \arrow[r] \arrow[dl, "\pi_2"] & Y \\
Y & &
\end{tikzcd}$$

Since X is fibrant the projection π_2 is a fibration (being the pullback of a fibration), the same is true for $\mathrm{Path}(f) \longrightarrow X \times Y$ for each square is a pullback. Thus we have exhibited the left vertical morphism (the one that we want to show is a fibration) as

$$\mathrm{Path}(f) \xrightarrow{\in \mathfrak{F}} X \times Y \xrightarrow{(f, \mathrm{id}_Y)} Y \times Y \xrightarrow{\pi_2} Y$$

but observing that $\pi_2 \circ (f, \mathrm{id}_Y) = \pi_2$ (which is a fibration), shows that our map is the composition of fibrations and so it is a fibration. Now to see the second statement, note that by the universal property of the pullback we have maps factoring the identity of X fitting the diagram

$$\begin{tikzcd}
X \arrow[r, "w"] \arrow[d, "f"'] \arrow[rr, bend left, equal] & \mathrm{Path}(f) \arrow[r, "\in \mathfrak{W} \cap \mathfrak{F}"'] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r] \arrow[rr, bend right, equal] \arrow[dr, bend right, equal] & \mathrm{Path}(Y) \arrow[r, "p_1"] \arrow[d, "p_0"] & Y \\
& Y &
\end{tikzcd}$$

In this diagram we see that w is our desired morphism and that it is in fact a weak equivalence by the 2-out-of-3 rule. ■

Theorem 3.5 (Ken Brown's Lemma)**.** *Let $\mathscr{C}$ be a model category, and C_f and C_c be as defined above. Let $\mathscr{D}$ be a category with weak equivalences, then the following statements hold.*

(i) A functor

$$F : \mathscr{C}_f \longrightarrow \mathscr{D}$$

that sends trivial fibrations into weak equivalences is homotopical.

(ii) A functor

$$F : \mathscr{C}_c \longrightarrow \mathscr{D}$$

that sends trivial cofibrations into weak equivalences is homotopical.

Proof: We prove the first statement since they are dual. Let $f : X \to Y$ be a weak equivalence in $\mathscr{C}_f$, we will show that $F(f)$ is a weak equivalence in D. Using the same procedure as the Factorization Lemma, we have a diagram

$$\begin{tikzcd}
\mathrm{Path}(f) \arrow[r, "\in \mathfrak{W} \cap \mathfrak{F}"] \arrow[d, "(p_1)^* f \in \mathfrak{W}"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f \in \mathfrak{W}"] \\
\mathrm{Path}(Y) \arrow[r, "p_1 \in \mathfrak{W} \cap \mathfrak{F}"'] \arrow[d, "p_0 \in \mathfrak{W} \cap \mathfrak{F}"'] & Y \\
Y &
\end{tikzcd}$$

where p_1 is a trivial cofibration because Y is fibrant, and $(p_1)^* f$ is a weak equivalence by the 2-out-of-3 rule. By this diagram we see that $(p_1)^* f \circ p_0 = \tilde{f}$ is a weak equivalence, but by the Factorization Lemma, this composite is also a fibration, thus $\tilde{f} \in \mathfrak{W} \cap \mathfrak{F}$. We know that f may be written as

$$f : X \xrightarrow{w} \mathrm{Path}(f) \xrightarrow{\tilde{f}} Y$$

with w a weak equivalence left inverse to some trivial fibration $v : \mathrm{Path}(f) \to X$. So, we have that

$$\mathrm{id}_{\mathrm{Path}(f)} : \mathrm{Path}(f) \xrightarrow{v \in \mathfrak{W} \cap \mathfrak{F}} X \xrightarrow{w \in \mathfrak{W}} \mathrm{Path}(f).$$

Now applying F to the above expression we get that

$$\mathrm{id}_{F(\mathrm{Path}(f))} : F(\mathrm{Path}(f)) \xrightarrow{F(v)} F(x) \xrightarrow{F(w)} F(\mathrm{Path}(f)).$$

By hypothesis, F sends trivial fibrations to weak equivalences so that $F(v) \in \mathfrak{W}_{\mathscr{D}}$, so, since all isomorphisms are weak equivalences, by the 2-out-of-3 rule, $F(w) \in \mathfrak{W}_{\mathscr{D}}$. Thus, since $\tilde{f}$ is also a trivial fibration we have that $F(\tilde{f} \circ w) \in \mathfrak{W}_{\mathscr{D}}$. ■

Corollary 3.6. *Let $\mathscr{C}, \mathscr{D}$ be model categories, and*

$$F : \mathscr{C} \longrightarrow \mathscr{D}$$

be a functor. If F *preserves cofibrant objects and trivial cofibrations between them, then its left derived functor exists. Dually, if* F *preserves fibrant objects and trivial fibrations between them, then its right derived functor exists.*

Remark 3.7. In the above corollary the condition that the functor preserves (co)fibrant objects is not required, but with it we can think of the restriction of F as functor taking values in the full subcategories $\mathscr{D}_f$ and $\mathscr{D}_c$:

$$F : \mathscr{C}_f \longrightarrow \mathscr{D}_f \text{ or } F : \mathscr{C}_c \longrightarrow \mathscr{D}_c.$$

3.2 QUILLEN ADJUNCTIONS

Now we study the notion of Quillen Adjunctions, and prove that a Quillen adjunction between model categories induces an adjunction between their homotopy categories. To define a Quillen adjunction, we need the following first:

Lemma 3.8. *Let* $L : \mathscr{D} \rightleftarrows \mathscr{C} : R$ *be an adjunction between model categories with* L *left adjoint of* R*, then the following are equivalent:*

(i) L *preserves cofibrations and* R *preserves fibrations.*

(ii) L *preserves trivial cofibrations and* R *preserves trivial fibrations.*

(iii) L *preserves cofibrations and trivial cofibrations.*

(iv) R *preserves fibrations and trivial fibrations.*

Proof: First we claim that a left adjoint L preserves trivial cofibrations precisely if and only if its right adjoint preserves fibrations. Indeed, let $A \xrightarrow{f} B$ be a trivial cofibration in $\mathscr{D}$ and $X \xrightarrow{g} Y$ be a fibration in $\mathscr{C}$, then, we have diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & R(X) \arrow[d, "Rg"] \\ B \arrow[r] & R(Y) \end{tikzcd} \qquad \begin{tikzcd} L(A) \arrow[r] \arrow[d, "Lf"'] & X \arrow[d, "g"] \\ L(B) \arrow[r] & Y \end{tikzcd}$$

showing that if L preserves trivial cofibrations, then $R(X) \xrightarrow{Rg} R(Y)$ has the right lifting property against all trivial cofibrations in $\mathscr{D}$ (you can "take" the lift to the other side), so it is a fibration, the converse is analogous. Of course this claim dualizes to give us the following: a left adjoint L preserves cofibrations precisely if and only if its right adjoint preserves trivial fibrations.

Now note that applying these two results for the items of this Lemma we get the desired result. ■

Definition 3.9 (Quillen Adjunctions). Let $L : \mathscr{D} \rightleftarrows \mathscr{C} : R$ be an adjunction between model categories with L left adjoint to R. This adjunction is said to be a Quillen adjunction if any, therefore all, of the items in Lemma 3.8 is satisfied. The functors L, R are called the left/right Quillen functors of the adjunction. □

Remark 3.10. Note that by Ken Brown's Lemma, a left Quillen functor admits a left derived functor while a right Quillen functor admits a right derived functor.

Lemma 3.11. *Let*

$$\begin{tikzcd} \mathscr{D} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "R"] \end{tikzcd}$$

be a Quillen adjunction, then for $X \in \mathscr{D}$ *cofibrant, for a cylinder object* $\mathrm{Cyl}(X)$ *of* X, $L(\mathrm{Cyl}(X))$ *is a cylinder object for* $L(X)$. *Dually, for* $Y \in \mathscr{C}$ *fibrant, for a path object* $\mathrm{Path}(Y)$ *of* Y, $R(\mathrm{Path}(X))$ *is a path object for* $R(X)$.

Proof: The two statements are dual, so we prove the first. Since L is left adjoint, it preserves colimits, and also cofibrations, for it is a left Quillen functor, so we get

$$L(X \sqcup X \xrightarrow{\in \mathfrak{C}} \mathrm{Cyl}(X)) = (L(X) \sqcup L(X) \xrightarrow{\in \mathfrak{C}} L(\mathrm{Cyl}(X)))$$

and from Lemma 2.20 and the again by the fact the L is a left Quillen functor, we have that

$$i_0, i_1 : L(X) \longrightarrow L(X) \sqcup L(X)$$

are trivial cofibrations, so by the 2-out-of-3 rule the morphism $L(\mathrm{Cyl}(X)) \longrightarrow L(X)$ is a weak equivalence. ■

Theorem 3.12. *Let*

$$\begin{tikzcd} \mathscr{D} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "R"] \end{tikzcd}$$

be a Quillen adjunction, then the left derived functor of L *is left adjoint to the right derived functor of* R, *that is*

$$\begin{tikzcd} h\mathscr{D} \arrow[r, bend left, "\mathbb{L}L"] \arrow[r, phantom, "\perp"] & h\mathscr{C} \arrow[l, bend left, "\mathbb{R}R"] \end{tikzcd}$$

Proof: First note that by Remark 3.10 these functors are indeed well defined. By Proposition 2.34 it is sufficient to show that for X cofibrant and Y fibrant, we have a natural bijection

$$\mathscr{C}(LX, Y)/\sim \;\cong\; \mathscr{D}(X, RY)/\sim$$

since those translate to natural bijections in the homotopy category. This bijection already exists before taking the equivalence classes, since we already have an adjunction $L \dashv R$, so we need to show that it carries on to the equivalence classes. To see this, note that by our previous Lemma, if $\mathrm{Cyl}(X)$ is a cylinder for X, $L(\mathrm{Cyl}(X))$ is a cylinder for $L(X)$, so we have that left homotopies

$$\eta^\sharp : L\mathrm{Cyl}(X) \longrightarrow Y$$

are in bijection with homotopies

$$\eta^\flat : \mathrm{Cyl}(X) \longrightarrow RY$$

by the definition of adjoint functors, and the result follows. ■

Lemma 3.13. *Let* $\begin{tikzcd} \mathscr{D} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "R"] \end{tikzcd}$ *be a Quillen adjunction. Then the following statements are equivalent.*

(i) The right derived functor

$$\mathbb{R}R : h\mathscr{C} \longrightarrow h\mathscr{D}$$

of R *is an equivalence of categories.*

(ii) The left derived functor

$$\mathbb{L}L : h\mathscr{D} \longrightarrow h\mathscr{C}$$

of L *is an equivalence of categories.*

(iii) For any cofibrant object $D \in \mathscr{D}$*, the "derived unit"*

$$D \xrightarrow{\alpha_D} RL(D) \xrightarrow{R(j_{L(D)})} R(S(L(D)))$$

where α *is the unit of the adjunction and* S *is a fibrant replacement of* L(D) *is a weak equivalence and for every fibrant object* $C \in \mathscr{C}$ *the derived counit*

$$L(Q(R(C))) \xrightarrow{L(p_{R(C)})} L(R(C)) \xrightarrow{\beta_C} C$$

with Q *being any cofibrant replacement is a weak equivalence.*

(iv) A morphism

$$w^\sharp : D \longrightarrow R(C)$$

is a weak equivalence if and only if its transposed morphism

$$w^\flat : L(D) \longrightarrow C$$

is a weak equivalence.

Proof: That (i) $\iff$ (ii) follows immediately from Theorem 3.12, since adjunctions preserve equivalences of categories, that is, any adjunct of a equivalence of categories is also an equivalence of categories. Now to see that (ii) $\iff$ (iii), remember that an adjunction pair is an equivalence of categories precisely if the unit and counit are natural isomorphisms, so we need to show that derived unit and counits represent the the counit and unit of $\mathbb{L}L \dashv \mathbb{R}R$, since them being weak equivalences by hypothesis will imply that they are isomorphisms in the homotopy category. To see this, observe that the diagram

$$\begin{tikzcd}
\mathscr{D}_c \arrow[r, "L"] \arrow[d, "\gamma_S"'] \arrow[dr, phantom, "\swArrow\cong"] & \mathscr{C} \arrow[d, "\gamma_{SQ}"] \\
h\mathscr{D} \arrow[r, "\mathbb{L}L"'] & h\mathscr{C}
\end{tikzcd}$$

with γ_S, γ_{SQ} the fibrant/cofibrant replacement functors, gives us that

$$\mathbb{L}LD \cong SLSD \cong SLD$$

where the second isomorphism holds because L is a Quillen functor. The unit of $\mathbb{L}L \dashv \mathbb{R}R$ on $SD \in h\mathscr{C}$ is the image of the identity under the bijection

$$\mathrm{Hom}_{h\mathscr{C}}(\mathbb{L}LSD, \mathbb{L}LSD) \cong \mathrm{Hom}_{h\mathscr{D}}(D, \mathbb{R}R\mathbb{L}LSD).$$

So, by the proof of Theorem 3.12 and the bijection in Proposition 2.34, we have that this bijection is the same as the one in $L \dashv R$

$$\mathrm{Hom}_{h\mathscr{C}}(SLD, SLD) \xrightarrow{(j^*_{LD}, \mathrm{id}_*)} \mathrm{Hom}_{\mathscr{C}}(Ld, SLd)/\sim$$

under the equivalence relation, so we have that the derived adjunction unit of

$$LD \xrightarrow{j_{LD}} SLD \xrightarrow{id} SLd$$

is

$$D \xrightarrow{\alpha} RLD \xrightarrow{Rj_{LD}} RSLD.$$

Now it remains to show (iii) $\iff$ (iv). First we show that (iv) $\implies$ (iii). Let

$$D \xrightarrow{j_D} SLD$$

be a weak equivalence, then its transpose is

$$D \xrightarrow{\alpha} RLD \xrightarrow{Rj_D} RSLD$$

with α being the adjunction unit. By the assumption in (iv), this a weak equivalence, which is the requirement of (iii), the case for the counit is dual. Finally, to see that (iii) $\implies$ (iv), let $f : LD \longrightarrow D$ be weak equivalence with D cofibrant and C fibrant. The transpose of f fits in the top horizontal arrows of the diagram below

$$\begin{tikzcd}
D \arrow[r, "\alpha"] \arrow[d, equal] & RLD \arrow[r, "Rf"] \arrow[d, "Rj_{LD}"] & RC \arrow[d, "Rj_C"] \\
D \arrow[r, "\in \mathfrak{W}"'] & RSLD \arrow[r, "RSf"'] & RSC
\end{tikzcd}$$

where Sf is any lift as we've done before. The bottom left horizontal morphism is the derived unit which is a weak equivalence by hypothesis, and since f is a weak homotopy equivalence, so is Sf, since R preserves trivial fibrations, we know it that it is homotopical (Ken Brown's Lemma) in the category of fibrant objects, so RSf and Rj_C are also weak equivalences, so by the 2-out-of-3 rule so is the top composite horizontal morphism, which is the transpose of f. ■

Definition 3.14 (Quillen Equivalences). If $\begin{tikzcd} \mathscr{D} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, bend left, "R"] \end{tikzcd}$ is a Quillen adjunction, then this adjunction is said to be a Quillen Equivalence, denoted

$$\begin{tikzcd} \mathscr{D} \arrow[r, bend left, "L"] \arrow[r, phantom, "\wr\|"] & \mathscr{C} \arrow[l, bend left, "R"] \end{tikzcd}$$

if any of the conditions of the above Lemma is satisfied. □

Remark 3.15. The important thing to take away from this is that our notion of equivalence of model categories will be that of a Quillen equivalence, since it translates to an equivalence on the homotopy category of the model categories.

Part II

THE MODELS

4

SIMPLICIAL SETS

Simplicial sets are objects which serve as a model for spaces built by simplices. So, we shall have points, 1-dimensional edges connecting these points, 2-dimensional faces connecting the edges and so on. This already has the looks of "$(n+1)$-morphisms between n-morphisms" and this is indeed the case. However, to make all of this work we need to be technical and specify how the different dimensional simplices interact. This is done by means of face and coface maps which satisfy some properties, namely the *simplicial identities*. In this chapter we present the basics of the theory of simplicial sets, so that we can use them to define higher categories.

4.1 SIMPLICIAL OBJECTS

Definition 4.1. Let $[n]$ be the linearly ordered set $\{0 < 1 <, \dots, < n\}$ The simplicial category, denoted Δ, is the category whose objects are $[n]$, with $n \geq 0$, and whose morphisms are non-decreasing functions. □

Definition 4.2 (Cofaces and Codegeneracy)**.** The map $\delta^i : [n-1] \to [n]$ given by

$$\delta^i(j) := \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

is called the i-th coface map.

The map $\sigma^i : [n+1] \to [n]$ defined by

$$\sigma^i(j) := \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

is called the i-th codegeneracy map. □

Theorem 4.3 (Cosimplicial Identities)**.** *The following identities hold:*

(i) $\delta^j \delta^i = \delta^i \delta^{j-1}$ *if* $j > i$*;*

(ii) $\sigma^j \sigma^i = \sigma^i \sigma^{j+1}$ *if* $j \geq i$*;*

(iii) $\sigma^j \delta^i = \begin{cases} \delta^i \sigma^{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1 \\ \delta^i \sigma^{j+1} & \text{if } i > j+1 \end{cases}$

Proof: We show only the second case since they are analogous.By computing we get that

$$\sigma^j\sigma^i(k) = \begin{cases} k & \text{if } k \le i \\ k-1 & \text{if } i < k \le j+1 \\ k-2 & \text{if } k > j+1 \end{cases}$$

and also that

$$\sigma^i\sigma^{j+1}(k) = \begin{cases} k & \text{if } k \le i \\ k-1 & \text{if } i \le k \le j+1 \\ k-2 & \text{if } k > j+1 \end{cases}$$

and so they are equal. ■

Definition 4.4. Let $\mathscr{C}$ be any category, then a simplicial object in $\mathscr{C}$ is a contravariant functor $F : \Delta^{op} \to \mathscr{C}$. Given a simplicial object in $\mathscr{C}$ we call the map $d_i := F(\delta^i)$ the i-th face map and $s_i = F(\sigma^i)$ the i-th degeneracy map. □

Corollary 4.5 (Simplicial Identities). *Let $F : \Delta^{op} \to \mathscr{C}$ be a simplicial object, then the following identities hold*

(i) $d_i d_j = d_{j-1} d_i$ *if* $j > i$;

(ii) $s_i s_j = s_{j+1} \sigma_i$ *if* $j \ge i$;

(iii) $d_i s_j = \begin{cases} s_{j-1} d_i & \textit{if } i < j \\ \mathrm{id} & \textit{if } i = j, j+1 \\ s_{j+1} d_i & \textit{if } i > j+1. \end{cases}$

Proof: This is immediate from functoriality and the cosimplicial identities. ■

Faces and degeneracies are important because they generate the simplicial object, in the sense that every morphism in $F(f)$ is the composition of faces and degeneracies, or equivalently, every map in Δ is the composition of cofaces and codegeneracies.

Proposition 4.6. *Let $f : [m] \to [n]$ be a morphism in Δ, then f can be written as the composition $\sigma^{i_1} \cdots \sigma^{i_k} \delta^{j_1} \cdots \delta^{j_l}$.*

Proof: It suffices to show that every injective map is a compositions of coface maps and that every surjective map is a compositions of degeneracy maps, since every function can be factored as an injective one followed by a surjective one. We prove the first statement, their proofs are similar. If $f : [m] \to [n]$ is injective, then it is strictly increasing, let $i_j \in [n]$ denote the image of j by f. We need a composition of cofaces that maps 0 to i_0, one candidate for this is the map

$$\delta^{i_0-1} \cdots \delta^0.$$

It takes 0 to 1, then 1 to 2 and so on, until $i_0 - 1$ is taken to i_0. By the end of this we have that 1 is in $i_0 + 1$, but it needs to go to i_1, so we may apply δ^{i_0+1} to move it up by one and keep 0 at i_0, so we have that in the composition

$$\delta^{i_1-1} \cdots \delta^{i_0+1} \delta^{i_0-1} \cdots \delta^0$$

sends 0 to i_0 and 1 to i_1. Continuing this process we get a decomposition of f by coface maps. ■

Definition 4.7. A simplicial set is a simplicial object in the category of sets, that is, a functor $X : \Delta^{op} \to \textbf{Set}$. □

Remark 4.8. Since the objects of Δ^{op} are the ordered sets $[n]$, we see that in particular a simplicial set consists of sets $X_n := X([n])$. By the above discussion, a simplicial set is equivalently a choice of set X_n for each non-negative integer together with maps

$$d_i : X_n \longrightarrow X_{n-1} \quad \text{and} \quad s_i : X_n \longrightarrow X_{n+1}$$

satisfying the Simplicial Identities. This is done by setting $X(\delta^i) = d_i$ and $X(\sigma^j) = s_j$, and by the previous proposition this defines the action of X in all morphisms $f : [n] \to [m]$.

4.2 SIMPLICIAL SETS

As mentioned above, a simplicial set is a bunch of sets together with a bunch of maps. The reason why they became interesting is because they are a good “combinatorial model" for topological spaces, so it is not surprising that they are useful for doing homotopy theory. Here we give basic definitions and hint at how they represent topological spaces in some way.

Definition 4.9. We denote by **sSet** the functor category $\mathrm{Func}(\Delta^{op}, \mathbf{Set})$. □

Since **Set** is a complete and cocomplete category so is **sSet** (it is a presheaf category), and its (co)limits are computed level-wise, this is very important, since later we shall give this category the structure of a model category.

Definition 4.10 (Simplices)**.** Let X be a simplicial set, then an n-simplex is an element of X_n, and X_n is called the set of n-simplices. The elements of X_0 are also called vertices for reasons that will become clear later. □

Among the simplicial sets we have important ones, namely the standard simplices, which are simply the representable simplicial sets.

Definition 4.11 (Standard Simplices)**.** For $n \geq 0$, we call the simplicial set

$$\Delta^n := \Delta(-, [n])$$

the standard n-simplex.

Example 4.12. For $n = 0$, $\Delta^0_k := \Delta([k], [0])$, thus for every k, Δ^0_k has exactly one element, furthermore for any other simplicial set X, there exists exactly one natural transformation $f : X \longrightarrow \Delta^0$, and so Δ^0 is the final object in **sSet**.

By the Yoneda Lemma, we have a natural bijection

$$\mathbf{sSet}(\Delta^n, X) \cong X_n$$

where $f \in \mathbf{sSet}(\Delta^n, X)$ is determined by its value $f(id_n) \in X_n$, thus a natural transformation $f : \Delta^n \to X$ corresponds naturally to some n-simplex, moreover we have a nice property shown below (namely the naturality of the Yoneda Lemma).

Proposition 4.13. *Let X be a simplicial set, $f : [m] \to [n]$ a morphism in Δ and $\sigma \in X_n$, then the natural transformation corresponding to $Xf(\sigma) \in X_m$ is the morphism*

$$\Delta^m \xrightarrow{f} \Delta^n \xrightarrow{\sigma} X$$

where σ is the morphism corresponding to $\sigma \in X_n$.

Proof: It suffices to show that $(\sigma \circ f)(\mathrm{id}_m) = Xf(\sigma)$. Tracing the identity id_n along the naturality square

$$\begin{tikzcd} \Delta^n_n \arrow[r, "\sigma"] \arrow[d, "f^*"'] & X_n \arrow[d, "Xf"] \\ \Delta^n_m \arrow[r, "\sigma"'] & X_n \end{tikzcd}$$

note that by going down and then right we get $\sigma(f)$ which is equal to $(\sigma \circ f)(\mathrm{id}_m)$, now by going right and then down we get $Xf(\sigma(\mathrm{id}_n)) = Xf(\sigma)$. ■

Definition 4.14. A simplicial subset of a simplicial X is a simplicial set Y such that for every n we have that $Y_n \subset X_n$ and $Yf = Xf|_{Y_n}$. Equivalently, it is for each n a subset $Y_n \subset X_n$ such that $d_i(Y_n) \subset Y_{n-1}$ and $s_i(Y_n) \subset Y_{n+1}$. □

We have two main examples of important simplicial subsets: horns and boundaries. We start by talking about boundaries.

Definition 4.15 (Boundaries). Let Δ^n be the standard n-simplex, then the boundary of Δ^n is the simplicial subset defined by

$$(\partial\Delta^n)_k := \{f : [k] \to [n] \mid [n] \not\subset f([k])\}.$$

It is immediate that this in fact defines a simplicial subset. For a simplicial set X, an n-boundary in X is a morphism $f : \Delta^n \to X$. □

Remark 4.16. The boundary of Δ^n is equivalently described as the simplicial subset generated by the face maps $\{d_0, \dots, d_n : \Delta^n \to \Delta^{n-1}\}$.

Theorem 4.17. *The n-boundaries of X correspond bijectively with ordered sets of $(n-1)$-simplices $(\sigma_0, \dots, \sigma_n)$ satisfying the following property:*

$$d_j(\sigma_k) = d_{k-1}(\sigma_j) \ \ \textit{whenever} \ \ k > j.$$

Proof: For $f : \partial\Delta^n \to X$ define the map

$$f \mapsto (f \circ \delta^0, \dots, f \circ \delta^n)$$

where $f \circ \delta^i$ is the $n-1$-simplex represented by

$$\Delta^{n-1} \xrightarrow{\delta^i} \partial\Delta^n \xrightarrow{f} X.$$

We show that this map is injective and surjective. To see injectivity, suppose that $f \circ \delta^i = g \circ \delta^i$ for all i. For $h \in \partial\Delta^n_k$, h factors through $[n-1]$ since it is not surjective, that is, there exists $\tilde{h} : [k] \to [n-1]$ and j such that

$$h = \delta^j \circ \tilde{h}.$$

So

$$f(h) = f(\delta^j \circ \tilde{h}) = (f \circ \delta^j)(\tilde{h}) = (g \circ \delta^j)(\tilde{h}) = g(h)$$

and thus our map is injective. To see that is satisfies the mentioned property, note that

$$d_j(\sigma_k) = d_j(f \circ \delta^k) = f \circ (\delta^k \circ \delta^j) = f \circ (\delta^j \circ \delta^{k-1}) = d_{k-1}(f \circ \delta^j) = d_{k-1}(\sigma_j).$$

where we are using the cosimplicial identities.

Now, let $(\sigma_0, \ldots, \sigma_n)$ be $n-1$-simplices satisfying

$$d_j(\sigma_k) = d_{k-1}(\sigma_j) \text{ whenever } k > j.$$

We must define a transformation

$$f : \partial\Delta^n \longrightarrow X$$

such that $\sigma_i = f \circ \delta^i$. Copying what we did above, for

$$g \in \partial\Delta^n_k$$

we set

$$f(g) = (Xh)(\sigma_i)$$

where h is such that

$$g = \delta^i \circ \tilde{h}.$$

If this map is well defined, naturality will follow. To see that it is, if it were the case that

$$g = \delta^j \circ \hat{h}'$$

If $j = i$, then $\hat{h} = \tilde{h}$ (coface maps are injective), so without loss of generality assume $j > i$, then, g factors as

$$g = \delta^j \circ \delta^i \circ \tilde{h}$$

which by cosimplicial identities is $g = \delta^i \circ \delta^{j-1} \circ \tilde{h}$. Thus, $h' = \delta^i \circ \tilde{h}$ and $h = \delta^{j-1} \circ \tilde{h}$. Computing,w e get that

$$X(h')(\sigma_j) = X(\delta^i \circ \tilde{h}) = X(\tilde{h}) d_i \sigma_j = X(\tilde{h}) d_{j-1} \sigma_i = X(\delta^{j-1} \circ \tilde{h}) = X(h)(\sigma_i)$$

where in the third equality we used the property we assumed that $(\sigma_0, \ldots, \sigma_n)$ has. ■

Let X be a simplicial set, for $\sigma \in X_n$, define the sequence of $(n-1)$-simplices

$$(d_0\sigma, \ldots, d_n\sigma).$$

By the simplicial identities, we have that

$$d_j(d_i\sigma) = d_{i-1}(d_j\sigma)$$

whenever $i > j$, so that it defines a boundary in X, so we may think of it as the boundary of σ. We might wonder if every boundary arises this way, much like we wonder about if a topological space has trivial homology, and as with topological spaces, the answer is no. Below, where we explore concrete examples of simplicial sets, we will see boundaries that don't cant be constructed like this.

If it is the case that a boundary $f : \partial\Delta^n \to X$ arises as the faces of an n-simplex σ, we say that σ is a filler for f. This name is clear, but see that this condition, by the Yoneda Lemma, is the same as saying that the diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "f"] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, dashed, "\sigma"'] &
\end{tikzcd}$$

admits a lift. This diagram hints at a model structure similar to Quillen's model structure for topological spaces, which is indeed the case as we shall see later.

An analogous construction is that of horns, that are essentially boundaries with one less simplex.

Definition 4.18 (Horns)**.** Let Δ^n be the standard n-simplex, then the i-horn of Δ^n is the simplicial subset defined by

$$(\Lambda_i^n)(k) := \{f : [k] \to [n] \mid [n] \not\subset f([m]) \cup \{i\}\}.$$

Again it is clear that this indeed constitutes a simplicial subset. A horn of a simplicial set X is a morphism $f : \Lambda_i^n \to X$. □

Remark 4.19. The i-horn of Δ^n is equivalently defined as the simplicial subset generated by all the face maps $d_j : \Delta^n \longrightarrow \Delta^{n-1}$ with the exception of d_i.

Clearly we have the inclusion, $\Lambda_i^n \hookrightarrow \partial\Delta^n \hookrightarrow \Delta^n$. As for boundaries, we have a nice characterization of when a horn admits a filling to a map out the whole standard simplex.

Theorem 4.20. *The i-horns in X correspond bijectively with the ordered sets of $(n-1)$-simplices $(\sigma_0, \dots, \sigma_{i-1}, \bullet, \sigma_{i+1}, \dots \sigma_n)$ satisfying the following property:*

$$d_j(\sigma_k) = d_{k-1}(\sigma_j) \ \textit{whenever} \ k > j, \ \ k \neq i \neq j.$$

The proof of the result above is almost the same as that for boundaries. As before, we know that n-simplex $\sigma \in X_n$ defines a horn, but the converse doesn't always hold.

Example 4.21. There are horns in Δ^n that don't have a filling for every $n \geq 1$.This happens because Δ^n is the *nerve of a category*, a concept we discuss below.

4.3 EXAMPLE: THE NERVE OF A CATEGORY

In this section we define what is the nerve of a category, and show that it extends to a functor **Cat** $\to$ **sSet**.

$[n]$ can be regarded as a category (as can any poset). From this point of view, a functor $F : [n] \to [m]$ is simply a non decreasing function. Thus, we may think of Δ as a full subcategory of **Cat**. So, for any category $\mathscr{C}$, we can think of the functor category $\mathbf{Cat}([n], \mathscr{C})$. For a non-decreasing map, $f : [n] \to [m]$, we have a map

$$f^* : \mathbf{Cat}([m], \mathscr{C}) \longrightarrow \mathbf{Cat}([n], \mathscr{C}).$$

Since $\mathbf{Cat}(-, \mathscr{C})$ is functorial, we have that the maps $d_i = \mathbf{Cat}(\delta^i, \mathscr{C})$ and $s_i = \mathbf{Cat}(\sigma^i, \mathscr{C})$ satisfy the simplicial identities. So setting

$$(\mathcal{N}\mathscr{C})_n := \mathbf{Cat}([n], \mathscr{C})$$

we get a simplicial set $\mathcal{N}\mathscr{C}$. It is easy to see that for any functor $F : \mathscr{C} \to \mathscr{D}$ we get a natural transformation $\mathcal{N}F : \mathcal{N}\mathscr{C} \to \mathcal{N}\mathscr{D}$, thus we have a functor

$$\mathcal{N} : \mathbf{Cat} \longrightarrow \mathbf{sSet}.$$

from the category of categories to the category of simplicial sets.

Definition 4.22 (Nerve of a Category)**.** The functor

$$\mathcal{N} : \mathbf{Cat} \longrightarrow \mathbf{sSet}.$$

is called the nerve functor. The simplicial set $\mathcal{N}\mathscr{C}$ is called the nerve of $\mathscr{C}$. □

Now we must unravel what the nerve of a category represents. First, note that the category $[n]$ is a chain of morphisms
$$0 \to 1 \to 2 \to \ldots \to n.$$
Thus a functor $F : [n] \to \mathscr{C}$ is simply a chain of composable morphisms
$$F(0) \xrightarrow{f_1} F(1) \xrightarrow{f_2} F(2) \to \ldots \xrightarrow{f_n} F(n).$$
Looking at $[n]$ as a category, we see that $\delta^i : [n-1] \to [n]$ has as its image
$$0 \to 1 \ldots \to i-1 \to i+1 \ldots \to n.$$
so that $d_i(F) = F \circ \delta_i : [n-1] \to \mathscr{C}$ is the chain
$$F(0) \xrightarrow{f_1} F(1) \to \ldots \to F(i-1) \xrightarrow{f_i \circ f_{i-1}} F(i+1) \to \ldots \xrightarrow{f_n} F(n)$$
where if $i = 0$ or $i = 1$ we simply remove the first or last morphism from the chain.

The discussion above shows us that a n-simplex of $\mathcal{N}\mathscr{C}$ is simply a chain of n composable morphisms in $\mathscr{C}$, and the face maps simply compose two of the morphisms of the chain to get a smaller chain. Similar reasoning shows us that the degeneracy maps simply insert identities into the chain, that is, if $F : [n] \to \mathscr{C}$ is given by
$$F(0) \xrightarrow{f_1} F(1) \xrightarrow{f_2} F(2) \to \ldots \xrightarrow{f_n} F(n)$$
then $s_i(F)$ is given by
$$F(0) \xrightarrow{f_1} F(1) \to \ldots \to F(i) \xrightarrow{id_{F(i)}} F(i) \to \ldots \xrightarrow{f_n} F(n).$$

Note that the 0-simplices of $\mathcal{N}\mathscr{C}$ is are the objects of $\mathscr{C}$ and a 1-simplex of $\mathcal{N}\mathscr{C}$ is simply a morphism $f : x \to y$, the face maps determine its domain and codomain:
$$d_1 f = x \text{ and } d_0 f = y.$$
It's clear that the nerve of category carries all of the information of the category, but does it hold anything else? The answer is no, as we see below:

Proposition 4.23. *The nerve functor*
$$\mathcal{N} : \mathbf{Cat} \longrightarrow \mathbf{sSet}$$
is fully faithful.

Proof: For a functor $F : \mathscr{C} \to \mathscr{D}$, its nerve $\mathcal{N}F : \mathcal{N}\mathscr{C} \to \mathcal{N}\mathscr{C}$ is characterized by
$$(x_1 \xrightarrow{f_1} x_2 \to \ldots \to x_{n-1} \xrightarrow{f_{n-1}} x_n) \mapsto (F(x_1) \xrightarrow{Ff_1} F(x_2) \to \ldots \to x_{n-1} \xrightarrow{Ff_{n-1}} F(x_n))$$
thus it's clear that if $\mathcal{N}G = \mathcal{N}F$ then $F = G$, since their actions are equal, in particular, in 0-simplices and 1-simplices.

Now we must show that every natural transformation
$$f : \mathcal{N}\mathscr{C} \longrightarrow \mathcal{N}\mathscr{D}$$
arises from a functor $F : \mathscr{C} \to \mathscr{D}$. We will define a functor that induces f. For $x \in \mathscr{C}$, $x \in \mathcal{N}\mathscr{C}_0$, thus we may define $F(x) = f_0(x)$, similarly, a morphism $x \xrightarrow{g} y = \sigma \in \mathcal{N}\mathscr{C}_1$, so

we set $F(g) = f_1(\sigma)$. Now it remains to show that this indeed defines a functor. First, by naturality of f, we get that

$$d_i(f_1(\sigma)) = f_0(d_i(\sigma))$$

so that $F(\sigma) : F(x) \to F(y)$, since $d_0(\sigma) = y$ and $d_1(\sigma) = x$. For $x \in X$, $id_x : x \to x \in \mathcal{N}\mathscr{C}_1$ is equally expressed by $s_0 x \in \mathcal{N}\mathscr{C}_1$, again the naturality implies that

$$f_1(s_0 x) = s_0 f_0(x) = id_{f(x)}.$$

Now, for $\sigma = x \xrightarrow{g} y \xrightarrow{h} z$, we need to show that $f_1(h) \circ f_1(g) = f_1(h \circ g)$. Note that $f_1(h) \circ f_1(g) = d_1(\gamma)$ where $\gamma = f_0(x) \xrightarrow{f_1(g)} f_0(y) \xrightarrow{f_1(h)} f_0(z)$ and that $\gamma = f_2(x \xrightarrow{g} y \xrightarrow{h} z)$, so we get that

$$f_1(h) \circ f(g) = d_1(f_2(\sigma)) = f_1(d_1\sigma) = f_1(h \circ g).$$

Since a morphism of nerves is determined by its action on 0 and 1-simplices we see that F induces f. ■

Remark 4.24. Since the nerve functor is fully-faithful it is customary to write $\mathscr{C}$ to mean $\mathcal{N}\mathscr{C}$, but we will not do this here.

Now let's see what a boundary $\partial\Delta^2 \to \mathcal{N}\mathscr{C}$ is. As we saw, this may be expressed as three morphisms (1-simplices) (f_0, f_1, f_2) satisfying

$$\begin{cases} d_0 f_1 = d_0 f_0 \\ d_0 f_2 = d_1 f_0 \\ d_1 f_2 = d_1 f_1. \end{cases}$$

So this is to say that a boundary is simply three morphisms forming a (not necessarily commuting) diagram

$$\begin{tikzcd} & y \arrow[dr, "f_0"] & \\ x \arrow[ur, "f_2"] \arrow[rr, "f_1"'] & & z \end{tikzcd}$$

If it were the case that this boundary arose from a 2-simplex

$$\sigma = x \xrightarrow{f} y \xrightarrow{g} z$$

we'd have that $f_0 = d_0(\sigma) = g$, $f_2 = f$ and that $f_1 = g \circ f$. So, in general a boundary won't have a filler unless every triangle in $\mathscr{C}$ is commutative.

With horns the situation is a little more interesting. For $n = 2$, we have three types of horns: Λ^2_0, Λ^2_1 and Λ^2_2. Using our characterization of horns, these correspond to diagrams of the form

$$\begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur] \arrow[rr] & & z \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur] \arrow[rr, dashed] & & z \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur, dashed] \arrow[rr] & & z \end{tikzcd}$$

respectively, where the dotted arrow indicates a missing arrow. A filler of these diagrams to a simplex would be an arrow fitting in the place of a dotted arrow making the diagram commute. Note that in the first and last diagram there is no reason for a filler to exist, but for the middle one (corresponding to Λ^2_1) we may replace the dotted arrow with with the composition of the legs of the triangle, thus getting an extension to a 2-simplex, furthermore, this extension is clearly unique since composition of morphisms is unique.

Definition 4.25. A horn Λ^n_k is said to be an inner horn if $0 < k < n$. □

Above we concluded that every horn of type Λ^2_1 in the nerve of a category admits a unique filler. This is in fact a specific case of a much more general result: if we have a horn of type Λ^n_k in the nerve of a category with $0 < k < n$, that is, an inner horn, then it admits a unique filler to an n-simplex. We dedicate the next part of this section to proofing this result.

Lemma 4.26. *Every inner horn $\Lambda^3_K \to \mathcal{N}\mathcal{C}$ admits a unique filler.*

Proof: The proof is done by "brute" force, and is analogous to the case $n = 2$. ■

Theorem 4.27. *Every inner horn $\Lambda^n_K \to \mathcal{N}\mathcal{C}$ admits a unique filler.*

Proof: Let $\sigma_0 : \Lambda^n_k \to \mathcal{N}\mathcal{C}$ be an inner horn. We know that the result is valid for $n \leq 3$. So assume that $n > 3$. In this case, every 1-simplex and 0-simplex of Δ^n belong to Λ^n_k. Define $x_j := \sigma_0(j) \in \mathcal{C}$. Every $i \leq j$ define a 1-simplex (i, j) of Δ^n with $d_0(i, j) = j$ and $d_1(i, j) = 1$, thus $f_{j,i} = \sigma_0((i, j)) : x_i \to x_j$. Define the functor $F : [n] \to \mathcal{C}$ by

$$j \mapsto x_j$$

$$(i \leq j) \mapsto f_{j,i}.$$

If this rule is indeed a functor then F is an n-simplex which extends σ_0.

Note that if $i = j$ then the 1-simplex corresponding to it is $s_0 i$, thus by naturality of σ_0, $\sigma_0(s_0 i) = s_0(x_i) = id_{x_i}$. Now we must show that for every $j \leq i \leq l$ we have that

$$f_{l,i} \circ f_{i,j} = f_{l,j}.$$

The triple (j, i, l) determines an unique 2-simplex τ, the simplex that satisfies $d_0(j, i, l) = (i, l)$, $d_1(j, i, l) = (j, l)$ and $d_2(j, i, l) = (j, i)$. Note that $\tau \in \Lambda^n_k$, since $n \geq 3$, thus the naturality of σ_0 gives us

$$f_{l,i} \circ f_{i,j} = d_1(\sigma_0(\tau)) = \sigma_0(d_1(\tau)) = \sigma_0((j, l)) = f_{l,j}$$

and we showed that F is a functor. ■

Corollary 4.28. *Every inner horn of Δ^n admits a unique filler.*

Proof: Δ^n is equivalently the nerve of $[n]$ viewed as a poset category. ■

We now know that $\mathcal{N}\mathcal{C}$ admits a filler for every inner horn, so given a simplicial set X for it to be a nerve this property must hold. One might ask what property, besides that one, X must have in order for it to be the nerve of a category, and the answer is none! We won't show this here but the reciprocal of Theorem 4.27 is true, so that a simplicial set is the nerve of a category if and only every inner horn admits a filler [23].

Well, it is rather easy to see that $\mathcal{N}$ preserves products, and so it is natural to wonder if preserves other limits, or even better, if it has a left adjoint. The last thing we show in this section is that the Nerve Functor does indeed admit a left adjoint.

Theorem 4.29. *The Nerve Functor*

$$\mathcal{N} : \mathbf{Cat} \longrightarrow \mathbf{sSet}$$

admits a left adjoint

$$\tau_1 : \mathbf{sSet} \longrightarrow \mathbf{Cat}.$$

Proof: Let X be a simplicial set, we must define a category $\tau_1 X$. Taking a hint from the nerve, we set the objects of $\tau_1 X$ to be the 0-simplices of X, that is

$$\mathrm{ob}(\tau_1 X) = X_0.$$

In the nerve of a category the morphisms are exactly the 1-simplices, so we could say that a morphism from $x \to y$ is a 1-simplex $f \in X_1$ such that $d_1 f = x$ and $d_0 f = y$ and that identities are the simplices $s_0 x$. That is not enough, since a category must have composition, so we could say that the morphisms are the free compositions generated by the 1-simplices. Again, that doesn't work, so we impose the following condition: for every $\sigma \in X_2$, we require that

$$d_1(\sigma) = d_0\sigma \circ d_2\sigma.$$

Note that it makes sense to talk about composition of simplices because we are seeing them as morphisms in a category.

The action of τ_1 in morphisms is obvious and well defined since natural transformations commute with face and degeneracy maps. Now we must exhibit a natural bijection

$$\mathbf{sSet}(X, \mathcal{N}\mathscr{C}) \cong \mathbf{Cat}(\tau_1 X, \mathscr{C}).$$

An obvious map would the one takes $f : X \to \mathcal{N}\mathscr{C}$, and takes it to the functor $F : \tau_1 X \to \mathscr{C}$ given by

$$F(x) = f_0(x)$$
$$F([h]) = f_1(h)$$

which can easily be checked to be well defined. Suppose that $f, g : X \to \mathcal{N}\mathscr{C}$ induce the same functor $\tau_1 f = \tau_1 g$, then they must be equal in the zero simplices of X. Also, by definition, if $f_1 h \neq f_1 k$ for h, k 1-simplices, then $F(h) \neq F(k)$, so that also f equals g in the 1-simplices of X, and this is enough to show that $f = g$. In fact, if they differ in an n-simplex, then we would have different chains of n morphisms in $\mathscr{C}$, and by applying the face maps we would be able to differentiate the 1-simplices.

Now we must show that our rule is also surjective. Let $F : \tau_1 X \to \mathscr{C}$ be any functor, so we must have a natural transformation $f : X \to \mathcal{N}\mathscr{C}$ such that

$$f_0(x) = F(x)$$
$$f_1([g]) = F(g).$$

So we define $f\sigma$, where σ is an n-simplex, by

$$f(\sigma) = (d_0^{n-2}\sigma) \circ (d_0^{n-2} d_n \sigma) \circ \cdots \circ (d_2 d_3 \ldots d_n \sigma) \in \mathcal{N}\mathscr{C}_n.$$

This rule is well defined and induces F. ■

Above we defined some sort of relation of when two morphisms are equal under τ_1 that was dependent on the 2-simplices of X. Because of that, we may think of the 2-simplices as something similar to homotopies between morphisms. This will become clear when we discuss quasi-categories and their homotopy categories.

Definition 4.30. For a simplicial set X, we call the category $\tau_1 X$ the fundamental category of X. □

Remark 4.31. For $\mathcal{N}\mathscr{C}$ the nerve of a category, it is true that its fundamental category is $\mathscr{C}$, *i.e.*, $\tau_1 \mathcal{N}\mathscr{C} \cong \mathscr{C}$.

Later on we will construct a model structure on **sSet**, the Joyal Model Structure, and the functor τ_1 will play an important role in defining the weak equivalences of said model structure.

4.4 KAN COMPLEXES AND TOPOLOGICAL SPACES

In this section we shall study Kan complexes, which play the role of ∞-groupoids in the setting of quasi-categories, and later we shall see that Kan complexes are the same thing as topological spaces, at least homotopically.

Definition 4.32 (Kan Complexes)**.** A Kan complex is a simplicial set K such that every horn $\Lambda^n_k \to K$ has filler. $\square$

The above definition is equivalent to saying that every diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & K \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\sigma"] & * \end{tikzcd}$$

admits a lift h. This is already foreshadowing Kan complexes as the fibrant objects of some model structure.

In the remainder of this section we will define the singular complex of a topological space and prove that it is a Kan complex. Further intuition on why this is so incredible will come later.

For $n \in \mathbb{N}$, let $|\Delta^n|$ be the topological standard n-simplex, that is, the subspace

$$|\Delta^n| := \{(x_0, \dots, x_n) \in \mathbb{R}^n \mid x_i \in [0,1], \sum x_i = 1\}.$$

For a map $f : [n] \to [m]$, define the continuous map

$$|f| : |\Delta^n| \to |\Delta^m|$$
$$(x_0, \dots, x_n) \mapsto (\sum_{f(x_i)=0} x_i, \dots, \sum_{f(x_i)=m} x_i)$$

which is clearly well defined.

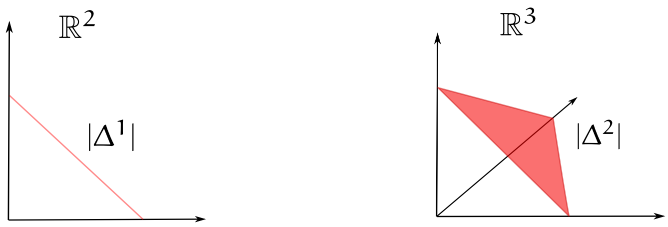


Figure 4.1: Illustration of $|\Delta^1|$ and $|\Delta^2|$.

With this we may define the following:

Definition 4.33. Let X be a topological space, define the functor $\mathbf{Sing}(-) : \mathbf{Top} \to \mathbf{sSet}$ by

$$\mathbf{Sing}(X)_n = \mathbf{Top}(|\Delta^n|, X)$$

and for $f \in \mathbf{Sing}(X)_n$ and $\alpha : [m] \to [n]$

$$\mathbf{Sing}(X)(\alpha)(f) = f \circ |\sigma| : |\Delta^m| \to X.$$

Post composition with $f : X \to Y$ defines the action of $\mathbf{Sing}(-)$ on morphisms. The simplicial set $\mathbf{Sing}(X)$ is called the singular complex of X. $\square$

Now, note that $\mathbf{Sing}(X)_0$ is simply the underlying set of X and $\mathbf{Sing}(X)_1$ is the set of paths in X. For an arbitrary simplicial set X, we say that $x, y \in X_0$ are in the same path component if they are in the same path component in $\tau_1 X$. This induces a functor

$$\pi_0 : \mathbf{sSet} \longrightarrow \mathbf{Set}$$

that takes a simplicial set into its set of path components. Since $\mathbf{Sing}(X)_1$ are the paths in X, we see that we may factor

$$\pi_0 : \mathbf{Top} \xrightarrow{\mathbf{Sing}(-)} \mathbf{sSet} \xrightarrow{\pi_0} \mathbf{Set}.$$

Thus, $\mathbf{Sing}(X)$ is connected if, and only if, X is a path connected space: $\mathbf{Sing}(X)$ carries the 0-dimensional homotopy information of X! As shall see later, it actually contains all of the homotopy information of X, but for now this is a nice detail.

In the definition of $\mathbf{Sing}(-)$ we defined $|\Delta^n|$ for each n and for each map $f : [n] \to [m]$ a continuous map $|f| : |\Delta^n| \to |\Delta|^m$. This clearly defines a functor

$$|-| : \Delta \longrightarrow \mathbf{Top}.$$

So, by abstract nonsense, since $\mathbf{Top}$ is cocomplete, we have that $|-|$ has a left Kan extension along the Yoneda embedding $y : \Delta \to \mathbf{sSet}$, that is,

$$\begin{tikzcd} \Delta \arrow[rr, "|-|"] \arrow[dr, hook, "y"'] & \Downarrow \mathrm{id}_{|-|} & \mathbf{Top} \\ & \mathbf{sSet} \arrow[ur, dashed, "\mathrm{Lan}_y |-|"'] & \end{tikzcd}$$

Definition 4.34. Define $|-| : \mathbf{sSet} \to \mathbf{Top}$ to be the the Kan extension $\mathrm{Lan}_y |-|$. This functor is called the geometric realization functor [1]. □

Arbitrary left Kan extensions along Yoneda embeddings

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, hook, "y"'] & \Downarrow \mathrm{id}_F & \mathscr{D} \\ & \mathbf{Set}^{\mathscr{C}^{op}} \arrow[ur, "\mathrm{Lan}_y F"'] & \end{tikzcd}$$

have right adjoints $R : \mathscr{D} \to \mathrm{Func}(\mathscr{C}^{op}, \mathbf{Set})$ given by

$$R(x) = \mathscr{D}(F(-), x)$$

and whose action on a morphism $f : x \to y$ is simply post-composition with it.

In our case, we see that $R = \mathbf{Sing}(-)$, so that we have an adjunction

$$|-| : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top} : \mathbf{Sing}(-)$$

Remark 4.35. Right adjoints arising this way are often called "nerve" functors while the left adjoints are the "realization" functors. We could've used this same construction in the definition of the $\tau_1 \dashv \mathcal{N}$ adjunction, but the explicit construction helps in the understanding of a simplicial set as some sort of category.

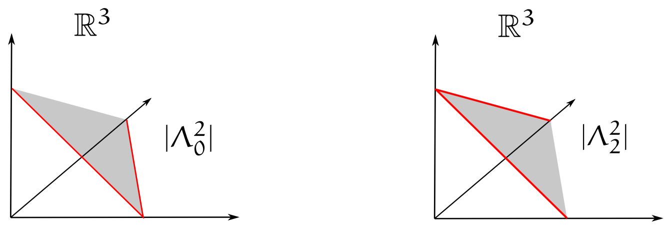


Figure 4.2: Representations of $|\Lambda^2_0|$ and $|\Lambda^2_1|$ (in red).

It can be shown that the map $|\Lambda^n_k| \to |\Delta^n|$ is isomorphic to the inclusion

$$\{(x_0, \dots, x_n) \in |\Delta^n| \mid \exists\, j \neq k \text{ with } x_j = 0\} \hookrightarrow |\Delta^n|.$$

With this, we are ready to show that $\mathbf{Sing}(X)$ is a Kan complex.

Theorem 4.36. *Let X be a topological space, then $\mathbf{Sing}(X)$ is a Kan complex.*

Proof: By the adjunction $|-| \dashv \mathbf{Sing}(-)$, we have an equivalence of the lifting properties

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f^\flat"] \arrow[d, hook] & \mathbf{Sing}(X) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} |\Lambda^n_k| \arrow[r, "f^\sharp"] \arrow[d, hook] & X \\ |\Delta^n| \arrow[ur, dashed] & \end{tikzcd}$$

By the previous observation, we can identify $|\Lambda^n_k|$ with the boundary of $|\Delta^n|$ minus a side, so that the the map $|\Delta^n_k| \to |\Delta^n|$ admits a retraction $r : |\Delta^n| \to |\Lambda^n_k|$. Then the map $h = f^\sharp \circ r$ is a lift for the second diagram, so that $h^\flat$ is a lift in the first one. ■

Note that, as one might guess, boundaries of the singular complexes don't always have a filler. The boundaries of the standard n-simplex is the boundary of the standard n-simplex, thus the diagram

$$\begin{tikzcd} |\partial\Delta^n| \arrow[r] \arrow[d, hook] & X \\ \Delta^n & \end{tikzcd}$$

always has a lift if and only if X is weakly-contractible.

[1]There is an explicit definition of the realization which is much more "geometric", for details see [11], Definition 4.1.

5

QUASI-CATEGORIES

Given the nerve of a category $\mathcal{N}\mathscr{C}$ we may recover the category by saying that the objects of $\mathscr{C}$ are the 0-simplices of $\mathcal{N}\mathscr{C}$ and that the morphisms of $\mathscr{C}$ are the 1-simplices. The identity of $x \in \mathcal{N}\mathscr{C}_0$ is given by $s_0 x \in \mathcal{N}\mathscr{C}_1$. Two morphisms $f, g \in \mathcal{N}\mathscr{C}_1$ (1-simplices) are composable if $d_1 g = d_0 f$, thus $(f, \bullet, g)$ defines a horn $\Lambda^2_1 \to \mathcal{N}\mathscr{C}$ which admits a unique filler $\sigma \in \mathcal{N}\mathscr{C}_2$, and so we may express the composition $g \circ f$ as $d_1\sigma$. With this in mind we define the following:

Definition 5.1 (Quasi-Categories)**.** A quasi-category[1] is a simplicial set X such that every inner horn has a filler. □

Remark 5.2. By definition, every Kan complex is a quasi-category.

The reason why this is a good definition will become clear as we continue.

5.1 OBJECTS AND MORPHISMS

Before we give some examples, let's define what are the objects and morphisms – notions that any aspiring category should have – of a quasi-category.

Definition 5.3. Let X be a quasi-category, then its objects are its 0-simplices, X_0, and its morphisms are the 1-simplices, X_1. □

Remark 5.4. A quasi-category X is in particular a simplicial set, so we may compute its fundamental category $\tau_1 X$. The objects of $\tau_1 X$ are the same as X, but now its morphisms are those of X modulo an equivalence relation. Below, we will see that this equivalence relation is the same as homotopy equivalence in a quasi category.

As in ordinary categories, morphisms in X come equipped with domains and codomains. The domain of a morphism $f \in X_1$ is the 0-simplex $d_1 f$ and its codomain is $d_0 f$, just like when we talked about the nerve of category. We will write $f : x \to y$ to indicate that f is a morphism from x to y. For $x \in X_0$, its identity morphism is given by $s_0 x : x \to x$.

Now let's look at some examples, first, unsurprisingly, categories are quasi-categories:

Example 5.5. For any category $\mathscr{C}$, its nerve $\mathcal{N}\mathscr{C}$ is a quasi-category by Theorem 4.27. The objects and morphisms of $\mathcal{N}\mathscr{C}$ are the same as of $\mathscr{C}$. For an object $x \in \mathcal{N}\mathscr{C}_0$, $s_0 x = \mathrm{id}_x$, this serves as an intuition of why identities are defined this way.

[1] It is not unusual to call quasi-categories simply ∞-categories since quasi-categories are good candidates to represent $(\infty, 1)$-categories, but since throughout this text we will talk about other models, we will refrain to use this terminology.

Example 5.6. For a topological space X, $\mathbf{Sing}(X)$ is a Kan complex, therefore a quasi-category. Its objects are the points of X and its morphisms are paths in X. For $x \in \mathbf{Sing}(X)_0$, $s_0 x$ is the constant path in x.

We know what objects and morphisms are in a quasi-category, but we are still missing a key feature: the composition. For nerves of ordinary categories, to define the composition is a rather trivial matter: the composition is the composition.

Now let's look at $\mathbf{Sing}(X)$. Suppose that we want to see $\mathbf{Sing}(X)$ as an ordinary category. A morphism would be a path, and we want two morphisms to be composable if one ends where the other starts. Well, we know how to concatenate paths, so it's quite natural to imagine the composition as simply being the concatenation of the two paths, the problem is that concatenation depends on the parametrization of the paths.

One way to solve this is by only defining composition up to homotopy, but if we want an ordinary category, this would not make sense without choosing specific morphisms to be the compositions, which would never work. This is where the structure of a quasi-category comes in handy, because they allow us to work with such things without requiring to make arbitrary choices.

5.2 COMPOSITION OF MORPHISMS

As we hinted above, composition will only be defined up to homotopy, thus before defining what are compositions, we will define what are homotopies.

Definition 5.7. Let X be a quasi-category, and $f, g : x \rightrightarrows y$ be morphisms of X, then a homotopy from f to g is a 2-simplex $\sigma \in X_2$ such that $d_0(\sigma) = \mathrm{id}_x$, $d_1\sigma = g$ and $d_2\sigma = f$, *i.e.*,

$$\begin{tikzcd} & y \arrow[dr, equal, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

f is said to be homotopic to g if there is a homotopy from f to g. □

Example 5.8. Let $\mathcal{N}\mathscr{C}$ be the nerve of category, then two morphisms are homotopic if and only if they are equal.

Example 5.9. A homotopy between two morphisms in $\mathbf{Sing}(X)$ is simply a homotopy between paths.

As one should hope, homotopy is an equivalence relation nn the set of morphisms between any two objects:

Theorem 5.10. *Let X be a quasi-category and let $x, y \in X_0$ be any two objects, then being homotopic is an equivalence relation on the set of morphisms from x to y.*

Before we prove the above result we state a simple Lemma:

Lemma 5.11. *Let X be a quasi-category, and let $f, g, h : x \to y$, then if f is homotopic to both g and h, then g is homotopic to h.*

Proof: Let σ_2 and σ_3 be the homotopies from f to g and h respectively, and let σ_0 be the 2-simplex $\Delta^2 \to \Delta^0 \xrightarrow{y} X$, then $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ is a horn $\Lambda^3_1 \to X$, this can be depicted in the following diagram

$$\begin{tikzcd}
 & y \arrow[dr, equal] \arrow[ddr, equal] & \\
 & & y \arrow[d, equal] \\
x \arrow[uur, "f"] \arrow[urr, "g"] \arrow[rr, "h"'] & & y
\end{tikzcd}$$

Now, since X is a quasi-category, this horn has a filler $\tau \in X_3$, and for this 3-simplex, $d_1\tau$ is a homotopy from g to h. ■

Proof of Theorem 5.10: Let "$\sim$" denote the relation of being homotopic. First, note that for $f \in X_1$, $s_1 f$ is a homotopy from f to itself, so $f \sim f$.

Now we must show symmetry and transitivity. Suppose that $f \sim g$, we know that $f \sim f$, so by the previous lemma $g \sim f$. Now, if $f \sim g$ and $g \sim h$, then $g \sim f$ and $g \sim h$, so, again by the previous lemma, $f \sim h$. ■

Finally we are ready to define the composition of morphisms, but first let's look at the case of nerves of categories. If $\mathcal{N}\mathscr{C}$ is the nerve of a category $\mathscr{C}$, then its morphisms are the morphisms of $\mathscr{C}$. In $\mathscr{C}$, two morphisms f and g are composable if $\operatorname{dom} g = \operatorname{cod} f$, so that we get a horn

$$(g, \bullet, f) : \Lambda^2_1 \longrightarrow \mathcal{N}\mathscr{C}.$$

The composition of f and g is therefore the map $d_1\sigma$ where σ is the unique filler of the horn $(g, \bullet, f)$ to a 2-simplex. This makes the following definition extremely natural:

Definition 5.12. Let $f, g \in X_1$ be morphisms of a quasi-category satisfying $d_0 f = d_1 g$, then a composition of f and g is a morphism $h : d_1 f \to d_0 g$ such that there exists a 2-simplex σ satisfying $d_0\sigma = g$, $d_1\sigma = h$ and $d_2\sigma = f$. In terms of diagrams:

$$\begin{tikzcd}
 & d_0 f \arrow[dr, "g"] \arrow[d, phantom, "\sigma"] & \\
d_1 f \arrow[ur, "f"] \arrow[rr, "h"'] & {} & d_0 g.
\end{tikzcd}$$

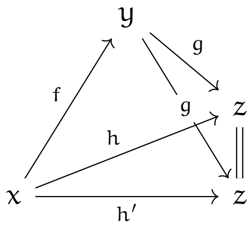
The simplex σ is said to *witness* h as the composition of f and g. □

Remark 5.13. The hypothesis that X is a quasi-category is fundamental, since the horn $(g, \bullet, f)$ always has a filler σ such that $(d_1\sigma, \sigma)$ is a composition: being a quasi-category ensures that compositions exist!

Since a compositions are horn filler, they always exist, but they are not unique in general. For example, in **Sing**(X), any two always to concatenate paths give a composition, in this case, they are not equal, but surely they are homotopic. The same happens for quasi-categories in general:

Proposition 5.14. *Let h, h' be two compositions for $x \xrightarrow{f} y \xrightarrow{g} z$, then h and h' are homotopic. Conversely, if h is a composition and h' is homotopic to h, then h' is also a composition.*

Proof: Let σ and σ' witness h and h' and the composition of f and g. Then we may form the inner horn $(s_1 g, \bullet, \sigma', \sigma)$. This horn is depicted in the diagram below where the base of the tetrahedron is the missing face.



This horn is a map

$$(s_1 g, \bullet, \sigma', \sigma) : \Lambda^3_1 \longrightarrow X$$

and since X is a quasi-category we have a filler $\tau \in X_3$ such that $\eta = d_1\tau$ is the missing face of the tetrahedron, *i.e.*, $d_0\eta = id_z$, $d_1\eta = h'$ and $d_2\eta = h$, so that η is a homotopy between h and h'.

To see the converse, note that in our diagram we would again be missing the face with edges (f, g, h') while the bottom face would be given by the homotopy from h to h', so by the same argument we'd have a inner horn whose filler τ has as $d_2\tau$ a 2-simplex witnessing h' as a composition. ■

5.3 THE HOMOTOPY CATEGORY OF A QUASI-CATEGORY

In this section we'll show that there is a nice way to forget all higher structure of a quasi-category by contracting homotopy equivalent morphisms. This construction is as follows:

Definition 5.15 (Homotopy Category)**.** Let X be a quasi-category, then denote by hX the category whose objects are the 0-simplices of X and whose morphisms are homotopy classes of morphisms of X. If $[f]$ and $[g]$ are homotopy classes of composable morphisms, then we define their composition as $[f] \circ [g] = [h]$ where h is a composition for f and g. For $x \in hX$, its identity is given by $[s_0 x]$. □

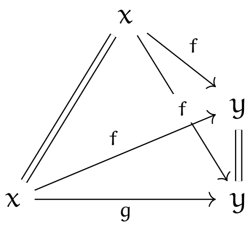
For the definition above to make sense we must show that the composition is well defined, that it is associative and that $[s_0 x]$ is the identity. This is what we show below.

Proposition 5.16. *Let f, f', g and g' be composable morphisms in a quasi-category X and let h and h' be compositions of f and g and f' and g'. If f is homotopic to f' and g is homotopic to g', then h' is homotopic to h.*

In the proof of this result we'll use the following lemma:

Lemma 5.17. *Let f, g be morphisms of a quasi-category. Then f and g are homotopic if and only if there exists a 2-simplex σ such that $d_0\sigma = f$, $d_1\sigma = g$ and $d_2\sigma = id_x$.*

Proof: Consider the following diagram:



This gives a horn $(s_1 f, \bullet, \sigma, s_0 f)$, since X is a quasi-category we have a filler τ such that $d_1\tau$ is a homotopy from f to g. The converse is similar. ■

Proof of Proposition 5.16: We'll show that both h and h' are homotopic to h'', where h'' is a composition of f and g'. As before, we have a diagram

$$\begin{tikzcd}
 & y \arrow[dr, "g"] \arrow[ddr, "g'"'] & \\
 & & z \arrow[d, equal] \\
x \arrow[uur, "f"] \arrow[urr, "h"] \arrow[rr, "h''"'] & & z
\end{tikzcd}$$

so that a filler will give a homotopy from h to h''. To see that h' is homotopic to h'', we note that the following diagram

$$\begin{tikzcd}
 & x \arrow[dr, "f'"] \arrow[ddr, "h'"'] \arrow[ddl, equal] & \\
 & & y \arrow[d, "g'"] \\
x \arrow[urr, "f"] \arrow[rr, "h''"'] & & z
\end{tikzcd}$$

is a horn whose filler gives a homotopy from h' to h'' by the previous lemma. ■

Proposition 5.18. *It holds that* $[s_0x] \circ [f] = [f]$ *and that* $[g] \circ [s_0x] = [g]$ *and that* $[f] \circ ([g] \circ [h]) = ([f] \circ [g]) \circ [h]$.

Proof: The first fact is trivial since f is homotopic to itself so that this homotopy witnesses f as the composition of f and s_0x.

Let (k, σ_0) be a composition for h and g, (l, σ_3) a composition for f and g and (j, σ_2) be a composition for f and k. This means that we need to show that j is also a composition for h and l. This gives the diagram

$$\begin{tikzcd}
 & y \arrow[dr, "g"] \arrow[ddr, "k"'] & \\
 & & z \arrow[d, "h"] \\
x \arrow[uur, "f"] \arrow[urr, "l"] \arrow[rr, "j"'] & & w
\end{tikzcd}$$

so we see that $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ defines an inner horn $\Lambda^3_1 \to X$, so that a filler τ for it is such that $d_1\tau$ witnesses j as a composition for h and l. ■

With this we finally conclude that hX is indeed a category. Not only that, if $F : X \to Y$ is a morphism between quasi-categories, then we have an induced functor in the homotopy categories. In fact, if σ is a homotopy from f to g, then, by naturality, $F\sigma$ is a homotopy from Ff to Fg, the argument for compositions is the same. By recalling the definition of the fundamental category of a simplicial set (Definition 4.30) we conclude the next result.

Corollary 5.19. *We have a functor* $h : q\mathbf{Cat} \to \mathbf{Cat}$ *that takes a quasi-category into its homotopy category. Furthermore, this functor coincides with the fundamental category functor, i.e.,* $\tau_1|_{q\mathbf{Cat}} = h$.

Now let's look at what is the homotopy category of our main examples, $\mathcal{N}\mathcal{C}$ and $\mathbf{Sing}(X)$. The case of $\mathcal{N}\mathcal{C}$ is not interesting, since $h\mathcal{N}\mathcal{C} = \mathcal{C}$. The case of $\mathbf{Sing}(X)$ is much more interesting. By recalling that a morphism in **Sing**(X) is simply a path connecting two points and that composition is the concatenation of paths, we get that its homotopy category is the fundamental groupoid of X, that is, $h\mathbf{Sing}(X) = \Pi_1(X)$. What this is telling us is that the fundamental groupoid functor factors as

$$\Pi_1 : \mathbf{Top} \xrightarrow{\mathbf{Sing}(-)} q\mathbf{Cat} \xrightarrow{h} \mathbf{Cat}.$$

This again illustrates how **Sing**(X) contains much more information than the fundamental gruopoid, since h can be thought as a forgetful functor, as it destroys all information of the higher simplices of a quasi-category.

5.4 FUNCTORS BETWEEN QUASI-CATEGORIES

By Proposition 4.23, we know that for given categories $\mathscr{C}, \mathscr{D} \in \mathbf{Cat}$, we have a natural bijection

$$\{F : \mathscr{C} \to \mathscr{D}\} \leftrightsquigarrow \{f : \mathcal{N}\mathscr{C} \to \mathcal{N}\mathscr{D}\}$$

Thus, the following is a straight-forward generalization:

Definition 5.20. A functor between two quasi-categories is a natural transformation between them. Thus the set $\mathbf{sSet}(X, Y)$ is the set of functors from X to Y. □

This definition is very simple and we already have plenty of examples. For instance every continuous function $f : X \to Y$ between topological spaces gives a functor $\mathbf{Sing}(f) : \mathbf{Sing}(X) \to \mathbf{Sing}(Y)$. Also, recall that for a quasi-category X, $hX = \tau_1 X$ so that we have a bijection

$$\{f : X \to \mathcal{N}\mathscr{C}\} \leftrightsquigarrow \{F : hX \to \mathscr{C}\}$$

for every category $\mathscr{C}$.

One problem with this definition is that we'd like $\mathbf{sSet}(X, Y)$ to be a quasi-category, just as $\mathbf{Cat}(\mathscr{C}, \mathscr{D})$ is a category. We solve this by defining for each $n \in \mathbb{N}$

$$\mathrm{Func}(X, Y)_n := \mathbf{sSet}(X \times \Delta^n, Y).$$

This defines a functor $\mathrm{Func}(X, Y) : \Delta^{op} \to \mathbf{sSet}$, *i.e.*, a simplicial set. Sometimes we will denote $\mathrm{Func}(X, Y)$ by Y^X, to indicate it as an exponential object. Note that for $n = 0$, $\mathrm{Func}(X, Y)_0 = \mathbf{sSet}(X, Y)$, so that $\mathrm{Func}(X, Y)$ carries all information about the functors from X to Y. Since **sSet** is a presheaf category, there is a natural bijection

$$\mathbf{sSet}(X \times Y, Z) \cong \mathbf{sSet}(X, \mathrm{Func}(Y, Z))$$

given explicitly by $f \mapsto \tilde{f}$ where, for $(x : \Delta^n \to X) \in X_n$

$$\tilde{f}(x) = Y \times \Delta^n \xrightarrow{(\mathrm{id}_Y, x)} Y \times X \xrightarrow{f} Z.$$

This next result is due to Joyal, and we skip its proof until we've seen more about lifting properties of simplicial sets (Corollary 7.19).

Theorem 5.21. *Let* A *be a simplicial set and let* X *be a quasi-category, then* $\mathrm{Func}(A, X)$ *is a quasi-category.*

What this is saying is that for X, Y quasi-categories we have a quasi-category $\mathrm{Func}(X, Y)$ whose objects (0-simplices) are functors between them. Note that by removing "quasi-" from the previous phrase we get the typical statement of **Cat** being enriched over itself!

Definition 5.22. Let X be a quasi-category and let $F, G : A \rightrightarrows X$ be functors, then a natural transformation from F to G is a morphism from F to G in the quasi-category $\mathrm{Func}(X, Y)$. In other words, a morphism $h : X \times \Delta^1 \to Y$ such that $h|_{X\times\{0\}} = F$ and $h|_{X\times\{1\}} = G$. A natural transformation is called a natural isomorphism if the 1-simplex $\Delta^1 \to \mathrm{Func}(X, Y)$ corresponds to an isomorphism in the sense of Definition 5.26. □

Example 5.23. Now we give some examples. Let $\mathcal{N}\mathscr{C}$ and $\mathcal{N}\mathscr{D}$ be ordinary categories, then a functor

$$F : \mathcal{N}\mathscr{C} \times \Delta^n \longrightarrow \mathcal{N}\mathscr{D}$$

is the same as

$$F : \mathcal{N}(\mathscr{C} \times [n]) \longrightarrow \mathcal{N}\mathscr{D}$$

since $\Delta^n = \mathcal{N}[n]$ and the Nerve functor is right adjoint. Again, this is the same as

$$F : \mathscr{C} \times [n] \longrightarrow \mathscr{D}$$

since **Cat** is cartesian closed, this is the same as $[n] \to \mathrm{Func}(\mathscr{C}, \mathscr{D})$ and so we conclude that

$$\mathrm{Func}(\mathcal{N}\mathscr{C}, \mathcal{N}\mathscr{D}) = \mathcal{N}(\mathrm{Func}(\mathscr{C}, \mathscr{D})),$$

that is, the functor quasi-category of ordinary categories is their ordinary category of functors. More generally, by the same argument, for A a simplicial set and $\mathcal{N}\mathscr{C}$ a category, we have that

$$\mathrm{Func}(A, \mathcal{N}\mathscr{C}) = \mathcal{N}(\mathrm{Func}(\tau_1 A, \mathscr{C})).$$

Example 5.24. If A is a simplicial set and **Sing**(X) is a singular complex, then functors $F, G : A \rightrightarrows \mathbf{Sing}(X)$ are simply a continuous maps $F, G : |A| \rightrightarrows X$, in this case, a natural transformation from F to G is the same as a homotopy

$$\begin{tikzcd} {|A|} \arrow[r, hook] \arrow[dr, "F"'] & {|A| \times I} \arrow[d, "H"] & {|A|} \arrow[l, hook'] \arrow[dl, "G"] \\ & X & \end{tikzcd}$$

from F to G. This is the case because $|-|$ preserves products and $|\Delta^1| \cong I$.

Since we have a notion of natural isomorphisms between functors, we also have a notion of equivalence of quasi-categories:

Definition 5.25. Let $f : X \to Y$ be a functor between quasi-categories. Then f is an equivalence of quasi-categories if there exists $g : X \to Y$ such that gf is naturally isomorphic to id_X and fg naturally isomorphic to id_Y. □

This is all going to get tied together neatly once we have the Joyal Model Structure.

5.5 ∞-GROUPOIDS

A groupoid is a category in which all morphisms are invertible. Then seeing a quasi-category as an ∞-category, an ∞-groupoid would be a quasi-category in which all morphisms are invertible. The problem with this is that , in general, it doesn't make sense to say that two morphisms compose to the identity in a quasi-category, since composition is only defined up to homotopy. To solve this we think about what is the natural notion of "isomorphism" in a quasi-category, namely, homotopy equivalence.

Definition 5.26. Let X be a quasi category, then a morphism $f : x \to y$ is said to be an isomorphism if $[f]$ is an isomorphism in hX. □

Remark 5.27. By definition, to say that $f : x \to y$ is an isomorphism is equivalent to saying that there exists $g : y \to z$ such that $f \circ g \cong \mathrm{id}_y$ and $g \circ f \cong \mathrm{id}_x$. Since composition is defined up to homotopy, this is just saying that they indeed compose to identities.

Definition 5.28 (∞-Groupoids). An ∞-groupoid is a quasi-category in which all morphisms are isomorphisms. □

Example 5.29. Let X be a quasi-category and let X^{iso} be the simplicial set generated by the isomorphisms of X. More precisely, let X^{iso} be the simplicial subset such that $X^{\mathrm{iso}}_n \subset X_n$ is the set of n-simplices

$$\Delta^n \longrightarrow X$$

such that for any inclusion $\Delta^1 \hookrightarrow \Delta^n$ the 1-simplex

$$\Delta^1 \hookrightarrow \Delta^n \longrightarrow X$$

is an isomorphism. It can be shown that if a horn has as components elements of X^{iso}_n, then so does any filler, so that X^{iso} is also a quasi-category. By definition, the 1-simplices of X^{iso} are isomorphisms, so that X^{iso} is an ∞-groupoid, in fact, the largest ∞-groupoid contained in X.

As with quasi-categories, we can characterize this definition in terms of horn fillings:

Proposition 5.30. *X is an ∞-groupoid if, and only if, every 2-horn $\Lambda^2_k \to X$ admits a filler.*

Proof: We prove only the filling of the horns Λ^2_k with $k = 0$, the case $k = 1$ is immediate since X is quasi-category, and $k = 2$ is analogous. Suppose that $f : x \to y$ is a morphism, then it defines a 2-horn $\Lambda^2_0 \to X$

$$\begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur, "f"] \arrow[rr, equal, "\mathrm{id}_x"'] & & x. \end{tikzcd}$$

By hypothesis, there is a 2-simplex σ filling this horn, so that σ witnesses id_y as a composition of f and $d_0\sigma$, *i.e.*, $d_0\sigma$ is a left (homotopy)inverse for f. The same argument yields a right inverse for f.

Conversely, let X be an ∞-groupoid, and consider a horn

$$\begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & z. \end{tikzcd}$$

since f is invertible there is map f^{-1} such that $f^{-1} \circ f \cong \mathrm{id}_x$, then

$$[g \circ f^{-1}] \circ [f] = [g] \circ [f^{-1} \circ f] = [g],$$

that is, $[g]$ is a composition of $g \circ f^{-1}$ and f, so that a 2-simplex witnessing this is a filler for our horn. ■

Now that we have a definition, we may look at how this is a generalization of a groupoid.

Example 5.31. Let $\mathcal{N}\mathscr{C}$ be the nerve of a category, then for fixed a morphism f, the diagram

$$\begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & z. \end{tikzcd}$$

admits a filler for every g, if, and only if, f has a left inverse. Similarly, horns of type Λ^2_2 will have a filler if, and only if, f has a right inverse, thus we conclude that $\mathcal{N}\mathscr{C}$ is an ∞-groupoid if, and only if, $\mathscr{C}$ is a groupoid.

By recalling the definition of a Kan complex we get the following trivially:

Corollary 5.32. *Every Kan complex is an ∞-groupoid.*

This is the realization of the idea that a topological space is an ∞-groupoid. Not only that, by the next result, every ∞-groupoid comes from a topological space.

Theorem 5.33. *Let X be an ∞-groupoid, then X is a Kan complex.*

The proof of this theorem consists in showing that for horns of higher order, being inner or outer doesn't matter, similar to the case of the nerve, where for $n \geq 3$ every n-horn has a filler. A proof can be found in [16]. This theorem tells us that Kan complexes are indeed all ∞-groupoids and further develops the idea that quasi-categories are in fact a model for $(\infty, 1)$-categories rather than general (∞, n)-categories[2], since just by inverting 1-simplices all higher horns are guaranteed to have a filler.

By Theorem 4.36, we know that every topological space can be seen as an ∞-groupoid, so it is natural to ask if every ∞-groupoid comes from a topological space, and the answer is yes, up to homotopy, which is the correct notion of equivalence between quasi-categories. Note that it is impossible for every ∞-groupoid to be isomorphic as a simplicial set to some singular complex. To see this, consider the walking isomorphism[3] $\mathscr{J}$, and let $J = \mathcal{N}\mathscr{J}$ be its nerve. Clearly, $\mathscr{J}$ is an ∞-groupoid, and

$$J_1 = \{\mathrm{id}_x, f, f^{-1}, \mathrm{id}_y\}.$$

Now, if X is a topological space, and we have a path γ between the points $x, y \in X$, then the cardinality of $\mathbf{Sing}(X)_1$ is infinite, since any reparametrization of γ is also a 1-simplex. With that, it's clear J isn't isomorphic to any singular complex.

[2] Categories with higher morphisms such that all morphisms of order greater than n are isomorphisms.
[3] Recall that this is the groupoid with two objects and exactly one morphism in each hom-set.

6

SIMPLICIALLY ENRICHED CATEGORIES

A simplicial category is a category *enriched over* simplicial sets. An "enriched category" is, roughly, a category whose hom-sets are in fact objects of some category $\mathcal{V}$ (instead of objects of **Set**) and such that the compositions are morphisms in $\mathcal{V}$. A prime example of this is the category of vector spaces over a field $\mathbb{K}$, $\mathbf{Vect}_{\mathbb{K}}$, since for any vector spaces V, W, $\mathrm{Hom}(V, W)$ is a vector space. In this case, we may use the tensor product of vector spaces to get the composition of linear maps as a linear map. The map

$$\circ : \mathrm{Hom}(W, Z) \times \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(V, Z),$$

is bilinear, so it corresponds naturally with a morphism $\circ : \mathrm{Hom}(W, Z) \otimes \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, Z)$. What is happening here is that $(\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K})$ is a symmetric monoidal category.

More generally, given a monoidal category $(\mathcal{V}, \otimes, \mathrm{I})$, where I is the unit, it's possible to think of categories enriched over $\mathcal{V}$. In this scenario, a $\mathcal{V}$-enriched category $\mathscr{C}$ would consist of some collection $\mathrm{ob}(\mathscr{C})$ of objects, for each $x, y \in \mathrm{ob}(\mathscr{C})$ an object $\mathscr{C}(x, y) \in \mathcal{V}$ of morphisms, and for each $x, y, z \in \mathrm{ob}(\mathscr{C})$ a morphism

$$c_{z,y,x} : \mathscr{C}(y, z) \otimes \mathscr{C}(x, y) \longrightarrow \mathscr{C}(x, z)$$

in $\mathcal{V}$. Furthermore, the identities of each object are picked by morphisms $\mathrm{id}_x : \mathrm{I} \to \mathscr{C}(x, x)$. Of course this bunch of data must satisfy a series of axioms in order for this structure to be useful/workable. In this sense, ordinary categories are categories enriched over $(\mathbf{Set}, \times, *)$.

In this chapter we will only discuss simplicial categories, that is, categories enriched over the symmetric monoidal category $(\mathbf{sSet}, \times, \Delta^0)$, so we'll not go into more detail about the general theory of enriched categories, although later on we'll talk a bit more about it. For a complete treatment of enriched categories see [18].

6.1 SIMPLICIAL CATEGORIES

As discussed before, one of the main ideas of what should an $(\infty, 1)$-category be is a category enriched over the "category of ∞-grupoids". That is, each hom-set must be, in a way, a space in which we have a natural notion of homotopy. In this way, an ordinary category can be seen as a category enriched over discrete groupoids (sets), while $(2, 1)$-categories can be seen as (are) categories enriched over the ordinary category of groupoids.

We saw that Kan complexes are objects that can be used to represent ∞-groupoids, so it's natural to want to enrich categories over the category of Kan complexes, which is just a subcategory of **sSet**. This is the motivation of simplicial categories as $(\infty, 1)$-categories, or rather, Kan enriched categories. In the next part of this text we will show that Kan enriched categories are in fact the fibrant objects of a model structure that localizes equivalences that behave like the desired notion of equivalence between $(\infty, 1)$-categories.

Definition 6.1 (Simplicial Categories)**.** A simplicial category $\mathscr{C}$ consists of the following data

(i) a collection $\mathrm{ob}(\mathscr{C})$ of objects of $\mathscr{C}$;

(ii) for each $x, y \in \mathrm{ob}(\mathscr{C})$ a simplicial set $\mathscr{C}(x, y) \in$ **sSet**;

(iii) for each $x, y, z \in \mathrm{ob}(\mathscr{C})$ a natural transformation

$$c_{z,y,x} : \mathscr{C}(y, z) \times \mathscr{C}(x, y) \longrightarrow \mathscr{C}(x, z);$$

(iv) for each $x \in \mathrm{ob}(\mathscr{C})$, a 0-simplex $\mathrm{id}_x \in \mathscr{C}(x, x)_0$.

This data is required to satisfy the following:

$$\begin{tikzcd} \mathscr{C}(z,w) \times \mathscr{C}(y,z) \times \mathscr{C}(x,y) \arrow[r, "c_{wzy} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times c_{zyx}"'] & \mathscr{C}(y,w) \times \mathscr{C}(x,y) \arrow[d, "c_{wyx}"] \\ \mathscr{C}(z,w) \times \mathscr{C}(x,z) \arrow[r, "c_{wzx}"'] & \mathscr{C}(x,w) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathscr{C}(x,y) \times \Delta^0 \arrow[r, "(\mathrm{id}, \mathrm{id}_x)"] \arrow[dr, "\pi_1"'] & \mathscr{C}(x,y) \times \mathscr{C}(x,x) \arrow[d, "c_{yxx}"] \\ & \mathscr{C}(x,y) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}(y,y) \times \mathscr{C}(x,y) \arrow[d, "c_{yyx}"'] & \Delta^0 \times \mathscr{C}(x,y) \arrow[l, "(\mathrm{id}_y, \mathrm{id})"'] \arrow[dl, "\pi_2"] \\ \mathscr{C}(x,y) & \end{tikzcd}$$

commute for all $x, y, z, w \in \mathrm{ob}(\mathscr{C})$. The simplicial sets $\mathscr{C}(x, y)$ are called the morphism complexes, the natural transformations c_{zyx} are called the composition laws and id_x the identity of x. □

Remark 6.2. Above we used the cartesian product in **sSet** to define the composition of morphisms and used the unit Δ^0 to define the identities. This process is called the enrichment over the category of simplicial sets. This works in general for any monoidal category, and in this context the theory of enriched category theory is studied.

Example 6.3. Our first example of a simplicial category is the category of simplicial sets where the morphism complex from X to Y is the exponential object Y^X.

By definition, for x, y objects of a simplicial category $\mathscr{C}$, we have a simplicial set of morphisms $\mathscr{C}(x, y)$. Because of that, for each n we may form a category $\mathscr{C}_n$ as follows

- the objects of $\mathscr{C}_n$ are the objects of $\mathscr{C}$
- the morphisms from x to y are $\mathscr{C}_n(x, y) := (\mathscr{C}(x, y))_n$
- composition is given by restricting the composition map to the n-simplices
- the identity in x is given by

$$\Delta^n \to \Delta^0 \xrightarrow{\mathrm{id}_x} \mathscr{C}(x, y)$$

The case $n = 0$ receives a special name:

Definition 6.4 (Underlying Category)**.** Let $\mathscr{C}$ be a simplicial category, then the category $\mathscr{C}_0$ is called the underlying category[1] of $\mathscr{C}$. □

Example 6.5. Let $\mathscr{C}$ be an ordinary category, define $\underline{\mathscr{C}}$ to be the simplicial category whose objects are the same of $\mathscr{C}$, and for x, y objects of $\underline{\mathscr{C}}$, we let $\underline{\mathscr{C}}(x, y)$ be the constant simplicial set $\mathscr{C}(x, y)$, that is, the simplicial set such that

$$(\underline{\mathscr{C}}(x, y))_n = \mathscr{C}(x, y) \text{ for all } n,$$

and all faces and degeneracies maps are identities.

In the study of simplicial categories, many times we don't want the morphism complex to be any simplicial set, but rather simplicial sets with good properties, this what is called a *local property* of the simplicial category, this makes the following definition very useful.

Definition 6.6. Let $\mathcal{P}$ be a property of simplicial sets (*e.g.* being a Kan complex, quasi-category, etc.). A simplicial category $\mathscr{C}$ is said to have the property $\mathcal{P}$ locally if for every $x, y \in \mathscr{C}$, $\mathscr{C}(x, y)$ has the property $\mathcal{P}$. □

As mentioned before, locally Kan simplicial categories ($\mathscr{C}(x, y)$ is a Kan complex) will be very important. As an example of those, we consider **Top** as a simplicial category. For X, Y topological spaces, let $\mathrm{Maps}(X, Y)$ denote the space of continuous function, define **sSet-Top** to be the category whose objects are the same as **Top**, and for $X, Y \in$ **sSet-Top**, define

$$\textbf{sSet-Top}(X, Y) = \textbf{Sing}(\mathrm{Maps}(X, Y)).$$

Since the composition of functions is a continuous map, and since $\textbf{Sing}(-)$ preserves products, the compositions are given by their images under $\textbf{Sing}(-)$. Clearly this category is locally Kan, as the morphism complexes are singular complexes of topological spaces. Note that for any X, Y,

$$\textbf{sSet-Top}(X, Y)_0 = \mathrm{Maps}(X, Y)$$

so that $\textbf{sSet-Top}_0$ is **Top**.

6.2 SIMPLICIAL FUNCTORS

After we have given the definition of an enriched something, the definition of an enriched functor comes quite naturally. Recall that a functor is a function of objects, and a function for each hom-set. Now, our hom-sets are actually objects in some other category, thus it makes sense that if a functor sends objects of a category to objects in another category, to define the components of said functor to be morphisms of objects in the category which the hom-sets belong.

Definition 6.7 (Simplicial Functor)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be simplicial categories, then a simplicial functor $F : \mathscr{C} \to \mathscr{D}$ consists of:

(i) A function $F : \mathrm{ob}(\mathscr{C}) \to \mathrm{ob}(\mathscr{D})$;

(ii) For each $x, y \in \mathrm{ob}(\mathscr{C})$, a morphism $F_{x,y} : \mathscr{C}(x, y) \to \mathscr{D}(F(x), F(y))$,

which are required to satisfy the following properties:

1. $F_{x,x}(\mathrm{id}_x) = \mathrm{id}_{F(x)}$;

[1] This name comes from the general theory of enriched categories, and can be done in the general case as well. In it, the hom sets would be given by $\mathscr{C}_0(x, y) := \mathcal{V}(I, \mathscr{C}(x, y))$ where I is the unit of $\mathcal{V}$.

2. For any $x, y, z \in \mathrm{ob}(\mathscr{C})$ the diagram

$$\begin{tikzcd} \mathscr{C}(y,z) \times \mathscr{C}(x,y) \arrow[r, "c_{z,y,x}"] \arrow[d, "F_{y,z} \times F_{x,y}"'] & \mathscr{C}(x,z) \arrow[d, "F_{x,z}"] \\ \mathscr{D}(F(y),F(z)) \times \mathscr{D}(F(x),F(y)) \arrow[r, "c'_{F(z),F(y),F(x)}"'] & \mathscr{D}(F(x),F(z)) \end{tikzcd}$$

is commutative.

The morphisms $F_{x,y}$ are called the components of the functor. □

The above definition is merely mimicking the definition of an ordinary functor, namely, it must preserve identities and compositions, which are expressed in the diagrams. We now know what are morphisms of simplicial categories, hence we can define their category, which will be extremely important to us.

Definition 6.8. Let **sSet-Cat** denote the category whose objects are small simplicial categories and whose morphisms are simplicial functors between them. □

Let $\mathscr{C}, \mathscr{D}$ be categories and let $\underline{\mathscr{C}}$ and $\underline{\mathscr{D}}$ be their corresponding simplicial category according to Example 6.5. If $F : \mathscr{C} \to \mathscr{D}$ is a functor, we can define the simplicial functor

$$\underline{F} : \underline{\mathscr{C}} \longrightarrow \underline{\mathscr{D}}$$

by letting

$$\underline{(F)}_{x,y} = F_{x,y} : \mathscr{C}(x,y) \to \mathscr{D}(F(x),F(y)).$$

With this, we get a functor

$$\underline{(-)} : \mathbf{Cat} \hookrightarrow \mathbf{sSet\text{-}Cat}$$

which is fully faithful. As we did for quasi-categories, we have a natural way of seeing ordinary categories as simplicial categories, so in fact simplicial categories are a possible generalization for the concept of categories.

Let **sCat** denote the category of simplicial objects in **Cat** ($\mathrm{Func}(\Delta^{op}, \mathbf{Cat})$), which is complete and cocomplete and its (co)limits are computed component-wise. Then Example 6.3, gives a fully faithful inclusion

$$\mathbf{sSet\text{-}Cat} \hookrightarrow \mathbf{sCat}.$$

In fact, for a simplicial category $\mathscr{C}$, we have $[n] \mapsto \mathscr{C}_n$. This construction is such that $\mathrm{ob}(\mathscr{C}_n) = \mathrm{ob}(\mathscr{C}_m)$, so we let faces and cofaces be given by $s_i|_{\mathrm{ob}(\mathscr{C}_n)} = d_j|_{\mathrm{ob}(\mathscr{C}_n)} = \mathrm{id}_{\mathrm{ob}(\mathscr{C}_n)}$ on objects, and their action in hom-sets is given by the faces and cofaces existent on the morphism complexes (they are simplicial sets). In this way, we see that **sSet-Cat** is a full subcategory of **sCat** whose objects are functors $F : \Delta^{op} \to \mathbf{Cat}$ such that $\mathrm{ob}(F(n)) = \mathrm{ob}(F(m))$ for all m, n.

Let $\mathrm{ob} : \mathbf{Cat} \to \mathbf{Set}$ be the forgetful functor which takes a category into its set of objects, this functor admits left and right adjoints, so it preserves limits and colimits. For a diagram $F : J \to \mathrm{Func}(\Delta^{op}, \mathbf{Cat})$ whose image is in **sSet-Cat**, for any morphisms $f \in \mathrm{Mor}(J)$, the action of $(F(f))_n$ is the same in objects for all n. Thus, $\mathrm{ob}(F(f)_n) = \mathrm{ob}(F(f)_m)$ for all f in $\mathrm{Mor}(J)$ and all m, n. Since limits and colimits are computed component-wise in **sCat**, we conclude that **sSet-Cat** is closed under limits and colimits. In short, we proved the following result:

Proposition 6.9. **sSet-Cat** *is a complete and cocomplete category.*

Remark 6.10. Since we want to define a model structure on **sSet-Cat**, the above result is really important.

6.3 MORPHISMS AND THE HOMOTOPY CATEGORY FUNCTOR

The idea of using simplicial categories as models for higher categories comes from the naive idea that a higher category is a category with spaces for hom-sets. In an ordinary category, the hom-sets are in fact sets, which are discrete spaces, that is, we have no additional structure. When the hom-sets carry homotopical structure, we may think as morphisms as being connected by paths, and wish to think as morphisms in the same path component as doing more or less the "same thing", this becomes quite clear, and explicit, in the simplicial category **sSet-Top**. In this section we discuss how we may think of a simplicial category as a higher category.

Definition 6.11. Let $\mathscr{C}$ be a simplicial category, then the morphisms of $\mathscr{C}$ are the morphisms of $\mathscr{C}_0$. That is, for $x, y \in \mathscr{C}$, the morphisms from x to y are the 0-simplices of the simplicial set $\mathscr{C}(x, y)$. $\square$

Remark 6.12. Note that since composition is a map

$$c_{z,y,x} : \mathscr{C}(y, z) \times \mathscr{C}(x, y) \longrightarrow \mathscr{C}(x, z)$$

for any morphisms $f \in \mathscr{C}(x, y)_0$ and $g \in \mathscr{C}(y, z)$ we may define $g \circ f \in \mathscr{C}(x, z)_0$ simply by setting $g \circ f := (c_{z,y,x})_0(g, f)$.

From the point of view that $(\infty, 1)$-categories are categories enriched over ∞-groupoids, we see how Kan enriched categories become important.

Definition 6.13 (Homotopy). Let $f, g : x \rightrightarrows y$ be morphisms in $\mathscr{C}$. Then f is homotopic to g if there is a 1-simplex $\sigma \in \mathscr{C}(x, y)_1$, such that $d_1\sigma = f$ and $d_0\sigma = g$. $\square$

Remark 6.14. In general, being homotopic in a simplicial category is not an equivalence relation, since there's no need for symmetry, however, by the definition of Kan complexes, whenever the morphism complex is a Kan complex, being homotopic is an equivalence relation.

Note that saying that there is a homotopy from f to g is the same as saying that there is a morphism $\sigma : f \to g$ in the fundamental category of $\mathscr{C}(x, y)$, $\tau_1\mathscr{C}(x, y)$. Recall that for a category $\mathscr{C}$, $\pi_0\mathscr{C}$ is the set of connected components of $\mathscr{C}$, that is, for an on object $x \in \mathscr{C}$ its connected component is its equivalence class under the equivalence relation generated[2] by the following: $x \sim y$ if either $\mathscr{C}(x, y) \neq \emptyset$ or $\mathscr{C}(y, x) \neq \emptyset$. Put more simply, the path component of x is the set of all object that can be connected by to x by a "zig-zag" of morphism. For instance, all objects appearing below

$$x \to y \leftarrow z \to w \leftarrow u$$

are in the same path component.

Since functors preserve zig-zag's, this rule extends to a functor

$$\pi_0 : \mathbf{Cat} \longrightarrow \mathbf{Set}$$

which clearly preserves products. Thus, we may define a functor

$$\pi_0 : \mathbf{sSet\text{-}Cat} \longrightarrow \mathbf{Cat}$$

[2] The smallest equivalence relation that contains the given relation.

by setting for $x, y \in \mathscr{C}$,

$$(\pi_0\mathscr{C})(x, y) = \pi_0(\tau_1\mathscr{C}(x, y))).$$

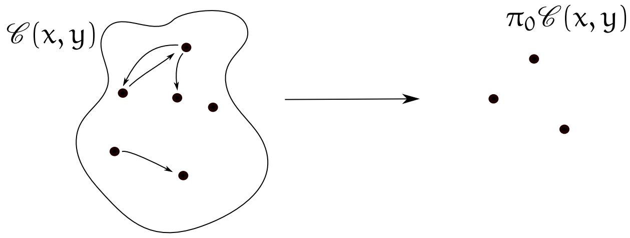


Figure 6.1: Illustration of the action of π_0 on hom-sets.

Example 6.15. For $\mathscr{C}$ a category, $\pi_0\underline{\mathscr{C}} = \mathscr{C}$. In this case, the space of morphisms is already discrete, so it is quite natural that π_0 doesn't contract any two morphisms to a single one.

This functor is called the homotopy category functor, and the category $\pi_0\mathscr{C}$. As this construction shows, we are "contracting" morphisms in the same path components to a point: we go from a space to a set!

However, this is a bit non natural, since we are saying that two morphisms, even non isomorphic ones, in the same path component become the same in the homotopy category. This problem would go away if all the 1-simplices of the morphism complexes were invertible. So, at least for Kan enriched categories, this definition is quite reasonable.

Definition 6.16. Let $e \in \mathscr{C}(x, y)_0$ be a morphism in $\mathscr{C}$, then e is said to be a homotopy equivalence if e becomes an isomorphism in $\pi_0\mathscr{C}$. $\square$

Example 6.17. In **sSet-Top** a homotopy equivalence is a homotopy equivalence. In fact, $\pi_0(\textbf{sSet-Top}) = \mathrm{H}\,\textbf{Top}$[3].

Since our hom-sets carry homotopical structure, we may change all concepts related to them to their homotopical counterpart. For instance, a functor $F : \mathscr{C} \to \mathscr{D}$ is full if for every $x, y \in \mathscr{C}$ the function $F_{x,y}$ is surjective. Since in general we are worried with things up to homotopy, this is too strong for simplicial categories. We can replace this concept by being the concept of "essential homotopy surjectivity". Let $F : \mathscr{C} \to \mathscr{D}$ be a simplicial functor, then F is homotopically full if $\pi_0 F$ is full. Later, this kind of construction will play a fundamental role in defining the equivalences of simplicial categories in Bergner's model structure.

[3] Here $\mathrm{H}\,\textbf{Top}$ denotes the homotopy category which localizes homotopy equivalences and not weak homotopy equivalences.

Part III

EQUIVALENCE

7 THE JOYAL MODEL STRUCTURE

In this chapter we begin the study of the homotopy theory of homotopy theories. Since quasi-categories form a category themselves, we could think to study the homotopy theory of said category, however that is not very convenient because we would like to use Quillen model structures, which can only be done in complete and cocomplete categories[1], and q**Cat** is neither complete nor cocomplete. As an example of this, let

$$\mathscr{C} = \{c_1 \to c_2\} \text{ and } \mathscr{D} = \{c_2' \to c_3\}$$

be categories. Let $*$ be the terminal category and consider the pushout

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \mathcal{N}\mathscr{D} \arrow[d] \\ \mathcal{N}\mathscr{C} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & P \end{tikzcd}$$

that identifies c_2 and c_2'. Then, without the identities,

$$P_1 = \{c_1 \to c_2, c_2 \to c_3\}$$

and there's no 1-simplex, $c_1 \to c_3$, thus this cannot be a quasi-category (they have compositions!).

To remedy this, we can simply work with **sSet**, and somehow describe q**Cat** inside of it. This is done by exhibiting quasi-categories as fibrant-cofibrant objects of some model structure on **sSet**.

Besides having a model structure which has as fibrant-cofibrant objects quasi-categories, we need weak equivalences to be meaningful, and not the same arbitrary class that makes our construction work. As the canonical model structure in **Cat**, we would like weak equivalences between quasi-categories to be equivalences of quasi-categories, in that way, we would truly capture the essence of equivalences on some category of categories.

In short, we need a model structure on the category **sSet** such that its fibrant-cofibrant objects are the quasi-categories and whose weak equivalences when restricted to quasi-categories, are equivalences of quasi-categories. This is precisely what the Joyal Model Structure does.

[1] At least a category with finite (co)limits, since the construction of homotopies is heavily dependent on them.

7.1 THE QUILLEN MODEL STRUCTURE

Although Quillen's Model Structure is not necessary to the definition of the Joyal Model Structure, it is nevertheless clarifying to study the topological inputs it provides, so we dedicate this short section to its definition.

Definition 7.1. Let $|-|: \mathbf{sSet} \to \mathbf{Top}$ be the Geometric Realization functor. Call $f: X \to Y$ a Kan weak equivalence[2] if the morphism

$$|f|: |X| \longrightarrow |Y|$$

is a weak homotopy equivalence. □

Since the collection of weak homotopy equivalences satisfies the 2-out-of-3 rule, so does the collection of the Kan weak equivalences.

Definition 7.2 (Kan Fibration). A morphism of simplicial set $f: X \to Y$ is said to be a Kan fibration if every diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

has a lift h. Equivalently, f is a Kan fibration if it has the right lifting property against all horn inclusions. □

By this definition, to say that X is a Kan complex, is equivalent to saying that $X \to *$ is a Kan fibration.

Note that fibrations were defined to be morphisms which lift against a very particular kind of morphism, indicating that the model structure will be cofibrantly generated. Denote by

$$J_k = \{\partial\Delta^n \hookrightarrow \Delta^n\} \text{ and } I_k = \{\Lambda^n_i \hookrightarrow \Delta^n\}$$

the collections of all boundary and horn inclusions. Clearly the morphisms in I_k are Kan weak equivalences. This next result is due to Quillen [26].

Theorem 7.3 (Quillen Model Structure). *Let $\mathfrak{C} \subset \mathrm{Mor}(\mathbf{sSet})$ denote the monomorphisms, $\mathfrak{F}_k \subset \mathrm{Mor}(\mathbf{sSet})$ the Kan fibrations and $\mathfrak{W}_k \subset \mathrm{Mor}(\mathbf{sSet})$ the Kan weak equivalences, then $(\mathbf{sSet}, \mathfrak{F}_k, \mathfrak{C}, \mathfrak{W}_k)$ is a model category. Moreover, said model structure is cofibrantly generated by I_k and J_k, where I_k is the set of trivial generating cofibrations.*

Definition 7.4. Write $\mathbf{sSet}_q$ for the category of simplicial sets equipped with the Quillen model structure. □

Remark 7.5. The trivial cofibrations, monomorphisms that are Kan weak equivalences, are often called anodyne maps.

Corollary 7.6. *The fibrant-cofibrant objects of $\mathbf{sSet}_q$ are the Kan complexes.*

Proof: The initial object of $\mathbf{sSet}$ is the empty simplicial set, therefore it is clear that $\emptyset \to X$ is a monomorphism for every X. By definition Kan complexes are the simplicial sets whose terminal morphism is a Kan fibration. ■

[2] Kan weak equivalences are also commonly known as weak homotopy equivalences, but we avoid this terminology so there is no confusion.

The motivation of the Quillen Model Structure was to show that the homotopy theory of simplicial sets and topological spaces were the same, this was attained in the following theorem, also due to Quillen:

Theorem 7.7. *The adjunction*

$$|-| : \mathbf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Top} : \mathbf{Sing}(-)$$

is a Quillen equivalence.

The proof of this result boils down to observing that $|-|$ preserves the generating cofibrations and that adjuncts of weak equivalences are weak equivalences. In fact, the procedure is the same as that of the proof of Theorem 9.7. Since CW-complexes form a full subcategory of $h\mathbf{Top}$ which is equivalent to it, we see that singular complexes of CW-complexes represent, up to homotopy, all possible Kan complexes, *i.e.*, ∞-groupoids are spaces. This is a realization of the Homotopy Hypothesis [3].

7.2 WEAK CATEGORICAL EQUIVALENCES

As we know from Section 5.4, **sSet** is a cartesian closed category. Let $\tau : \mathbf{sSet} \to \mathbf{Set}$ be any functor that preserves products, then we can define a category $\mathbf{sSet}^\tau$ with the same objects as **sSet**, and

$$\mathbf{sSet}^\tau(X, Y) := \tau(Y^X).$$

Composition is then given by the image under τ of the morphism adjunct of the morphism

$$Z^Y \times Y^X \times X \xrightarrow{(\mathrm{id}_{Z^Y}, ev)} Z^Y \times Y \xrightarrow{ev} Z \leftrightsquigarrow Z^Y \times Y^X \to Z^X$$

where ev is the evaluation morphism.

If $u : X \to Y$ is a morphism, then it corresponds naturally to a morphism in $X \to Y$ in $\mathbf{sSet}^\tau$. To see this, note that u corresponds to a morphism

$$\Delta^0 \xrightarrow{u} Y^X$$

so that we get an element

$$* \xrightarrow{\tau(u)} \tau(Y^X).$$

Define then, for any $Z \in \mathbf{sSet}^\tau$, the function

$$\mathbf{sSet}^\tau(u, X) : \mathbf{sSet}^\tau(Y, Z) \longrightarrow \mathbf{sSet}^\tau(X, Z)$$

by pre-composition with $\tau(u)$.

From the fundamental category functor $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ we can construct a **Set** valued product preserving functor as follows: let $G : \mathbf{Cat} \to \mathbf{Grpd}$ be the functor that takes a category into its groupoid by forgetting all morphisms which aren't isomorphisms, define

$$\tau_0 = \pi_0 \circ G \circ \tau_1 : \mathbf{sSet} \longrightarrow \mathbf{Set}.$$

In other words, τ_0 takes a simplicial set into the set of isomorphism classes of its fundamental category. Clearly, whenever X is a Kan complex, $\tau_0 X = \pi_0 X$.

Definition 7.8 (Weak Categorical Equivalence). Let τ_0 denote the fundamental category functor. Then a morphism $u : X \to Y$ of simplicial sets is a categorical equivalence if u is an isomorphism in $\mathbf{sSet}^{\tau_0}$. u is a weak categorical equivalence if

$$\mathbf{sSet}^{\tau_0}(u, X) : \mathbf{sSet}^{\tau_0}(Y, Z) \longrightarrow \mathbf{sSet}^{\tau_0}(X, Z)$$

is a bijection for every quasi-category Z. □

Remark 7.9. By the Yoneda Lemma, every categorical equivalence is a weak categorical equivalence. Furthermore, we see that any weak categorical equivalence between quasi-categories is categorical equivalence.

This definition may seem a bit weird at first, but this definition is equivalent to Definition 5.25. We shall return to this issue once we have the model structure. For now, let's be reassured by the following proposition:

Proposition 7.10. *A functor $F : X \to Y$ is an equivalence of categories if and only if $\mathcal{N}(F)$ is a categorical equivalence.*

A proof of this result can be found at [27].

To continue the explanation of why this is a reasonable definition, we will define Kan weak equivalences in the same manner. Just as we defined $\mathbf{sSet}^{\tau_0}$ from $\tau_0 : \mathbf{sSet} \to \mathbf{Set}$ we can define $\mathbf{sSet}^{\pi_0}$ from $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$, since π_0 also preserve products, thus we can define:

Definition 7.11. A map $f : X \to Y$ of simplicial sets is called a Kan weak equivalence if

$$\mathbf{sSet}^{\pi_0}(u, X) : \mathbf{sSet}^{\pi_0}(Y, Z) \longrightarrow \mathbf{sSet}^{\pi_0}(X, Z)$$

is a bijection for every Kan complex Z. Similarly, a morphism is a Kan equivalence if it becomes an isomorphism in $\mathbf{sSet}^{\pi_0}$.□

This may seem a bit odd, to see that this definition is equivalent to Definition 7.1, remember that a continuous $f : X \to Y$ between cell complexes is a weak homotopy equivalence if and only if

$$\pi_0(f_*) : \pi_0\mathbf{Top}(Y, Z) \longrightarrow \pi_0\mathbf{Top}(X, Z)$$

where π_0 is the functor of path components, is a bijection for all cell complexes Z. So from the fact that for a Kan complex X, $\pi_0 X \cong \pi_0|X|$, we see that the definitions are equivalent.

Proposition 7.12. *Every weak categorical equivalence is a Kan weak equivalence.*

Proof: By Corollary 7.19, since τ_0 is the same as π_0 for Kan complexes, we get that $\mathbf{sSet}^{\tau_0}(A, X) = \mathbf{sSet}^{\pi_0}(A, X)$ for every Kan complex. Then it's clear that one definition implies the other. ■

Lastly, we show that some weak Kan equivalences are also weak categorical equivalences.

Proposition 7.13. *If f is a trivial Kan fibration then f is a categorical equivalence.*

Proof: Let f be a trivial Kan fibration. Then, the diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, "s"] & Y \end{tikzcd}$$

admits a lift s which is right inverse to f. So, if we show that sf is the identity in $\mathbf{sSet}^{\tau_0}$ we'll have shown that f is an isomorphism in $\mathbf{sSet}^{\tau_0}$.

Let J be the nerve of the walking isomorphism, and consider $\{0,1\} \overset{j}{\hookrightarrow} J$ the inclusion of a set onto each object of J. Then, again, the diagram

$$\begin{tikzcd} X \times \{0,1\} \arrow[r, "sf\circ\pi_1"] \arrow[d, "(id_X{,}j)"'] & X \arrow[d, "f"] \\ X \times \{0,1\} \arrow[r, "f\circ\pi_1"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

has a lift h. The map h is adjunct to a map $h^\sharp : J \to X^X$, which takes the isomorphism of J into a morphism in X^X which has as vertices id_X and sf, thus. sf becomes an isomorphism in $\mathbf{sSet}^{\tau_0}$, which implies that f is a categorical equivalence, and thus a weak categorical equivalence. ■

7.3 INNER FIBRATIONS

Before we proceed to Joyal's model structure, we need to say a few things about inner fibrations. We want quasi-categories to be fibrant objects in this new model structure, thus we need more fibrations, since Kan fibrations are too strict. One way to do this is simply ask fibrations to lift against inner horn inclusions, forcing quasi-categories to be fibrant. This is how the notion of an *inner fibration* arises. Such fibrations turn out to be a little too general, but they will be quite useful to understand the Joyal Model Structure.

Definition 7.14. Let $I_m := \{\Lambda^n_k \hookrightarrow \Delta^n \mid 0 < k < n\}$ be the set of inner horn inclusions. [3]. Then a map $f : X \to Y$ of simplicial sets is called an inner fibration if it has the right lifting property against every morphism in I_m. □

Remark 7.15. Note that, by definition, for any quasi-category X, $X \to *$ is an inner fibration.

The category **sSet** is a category of pre-sheaves over a small category, therefore it is locally presentable, which implies that all morphisms in I_m have small domains (See Theorem A.54), thus by Quillen's Small Object Argument, we have a factorization system $(\mathfrak{A}_m, \mathfrak{F}_m)$ in **sSet**, where $\mathfrak{F}_m$ is the collection of inner fibrations and $\mathfrak{A}_m$ is the collection of cell complexes generated by I_m.

Definition 7.16. A morphism in $\mathfrak{A}_m$ is called an inner anodyne. □

As we already know, **sSet** is a complete and cocomplete category, thus we may, for any morphism $u : A \to B$ of simplicial sets, construct an adjunction

$$u \boxtimes (-) : \mathrm{Func}(\Delta^1, \mathbf{sSet}) \rightleftarrows \mathrm{Func}(\Delta^1, \mathbf{sSet}) : \langle u, - \rangle$$

where $\mathrm{Func}(\Delta^1, \mathbf{sSet})$ is the arrow category of **sSet** and these functors are given as follows: for $f : X \to Y$, we let

$$\begin{tikzcd} A \times X \arrow[r, "(id_A{,}f)"] \arrow[d, "(u{,}id_X)"'] & A \times Y \arrow[d] \arrow[ddr, bend left, "(u{,}id_Y)"] & \\ B \times X \arrow[r] \arrow[drr, bend right, "(id_B{,}f)"'] & Po \arrow[dr, dashed, "u\boxtimes f"] & \\ & & B \times Y \end{tikzcd} \qquad \begin{tikzcd} X^B \arrow[dr, dashed, "\langle u{,}f\rangle"] \arrow[drr, bend left, "f_*"] \arrow[ddr, bend right, "u^*"'] & & \\ & Pb \arrow[r] \arrow[d] & Y^B \arrow[d, "u^*"] \\ & X^A \arrow[r, "f_*"'] & Y^A \end{tikzcd}$$

---

[3]The subscript "m" stands for "mid", as inner fibrations are also called mid fibrations.

where f_* is post composition with f and u^* is pre-composition. $u \boxtimes f$ is called the pushout product while $\langle u, f \rangle$ is called the pullback product.

This construction will be very useful for proving things relating to model structures on simplicial sets. This is the case because the pullback product preserves several lifting properties when it is taken with respect to monomorphisms, which will be the cofibrations of our model structure.

Remark 7.17. The pushout product is not particular to **sSet**, and can be done in general in any category with pushouts and pullbacks. Not only that, if we have a symmetric monoidal category, we could replace the product "$\times$" with the monoidal product "$\otimes$". This also justifies the use this weird "$\boxtimes$", since in a monoidal category both $f \otimes g$ and $f \boxtimes g$ make sense..

Theorem 7.18. *Let u be a monomorphism and let f be an inner anodyne (or anodyne), then $u \boxtimes f$ is also inner anodyne (anodyne). Besides that, let u be a monomorphism and let f be an inner fibration (or Kan fibration), then so is $\langle u, f \rangle$. Moreover, if u is anodyne or inner anodyne. Then $\langle u, f \rangle$ is a Kan fibration.*

Proof: We won't prove the first statement here, as it is too cumbersome, a proof of it can be found at [20], Corollary 2.3.2.4. For the second part, let u be monic and f be an inner fibration. Let v be an inner anodyne, then f, by first statement, has the right lifting property against $u \boxtimes v$, so by the adjunction $(u \boxtimes (-) \dashv \langle u, - \rangle)$, we get

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "u\boxtimes v"'] & \bullet \arrow[d, "f"] & & \bullet \arrow[r] \arrow[d, "v"'] & \bullet \arrow[d, "\langle u,f\rangle"] \\ \bullet \arrow[r] \arrow[ur, "h"] & \bullet \arrow[r, phantom, "\leftrightsquigarrow"] & {} & \bullet \arrow[r] \arrow[ur, "\tilde{h}"] & \bullet \end{tikzcd}$$

so that $\langle u, f \rangle$ has the right lifting property against every inner anodyne. From the fact that $(\mathfrak{A}_m, \mathfrak{F}_m)$ is a factorization system, we conclude that $\langle u, f \rangle$ is an inner fibration. ■

Corollary 7.19. *Let X be a quasi-category (Kan complex) and let A be a simplicial set, then X^A is a quasi-category (Kan complex).*

Proof: Let f be the terminal morphism $X \to *$ and u be the initial morphism $\varnothing \to A$. Since f is an inner fibration so is $\langle u, f \rangle$, but $\langle u, f \rangle$ is the terminal morphism $X^A \to *$, so that X^A is a quasi-category. ■

Corollary 7.20. *A morphism $u : A \to B$ is a weak categorical equivalence (Kan weak equivalence) if and only if $u^* : X^B \to X^A$ is a categorical (Kan) equivalence for every quasi-category (Kan complex) X.*

Proof: Let X be quasi-category and let $u : A \to B$ be a weak categorical equivalence, then for any simplicial set Z

$$\mathbf{sSet}^{\tau_0}(u, X^Z) : \mathbf{sSet}^{\tau_0}(B, X^Z) \longrightarrow \mathbf{sSet}^{\tau_0}(A, X^Z)$$

is a bijection, since X^Z is a quasi-category. Now, using the cartesian closed structure in **sSet**, we see that this map is the same as (isomorphic)

$$\mathbf{sSet}^{\tau_0}(Z, X^u) : \mathbf{sSet}^{\tau_0}(Z, X^B) \longrightarrow \mathbf{sSet}^{\tau_0}(Z, X^A).$$

This shows that u^* induces bijections on all hom-sets of $\mathbf{sSet}^{\tau_0}$, so it is an isomorphism. The converse is analogous. ■

Corollary 7.21. *Every inner anodyne map is a weak categorical equivalence.*

Proof: By Theorem 7.18, if u is inner anodyne and X is a quasi-category, then X^u is a trivial Kan fibration, and thus, by Proposition 7.13 it is a categorical equivalence. Thus, by the preceding corollary, u is a weak categorical equivalence. ■

7.4 THE JOYAL MODEL STRUCTURE

Since we know what are the weak equivalences (weak categorical equivalences) and cofibrations (monomorphisms) of the Joyal Model Structure, we are forced to define the fibrations as the morphisms with the right lifting property against trivial cofibrations, those are called *quasi-fibrations*.

Definition 7.22 (Quasi-fibrations). Call a map of simplicial sets a quasi-fibration if it has the right lifting property against every monomorphism which is also a categorical weak equivalence. Denote by $\mathfrak{F}_J$ the collection of such fibrations. □

Proposition 7.23. *Let f be a quasi-fibration. Then f is a categorical weak equivalence if and only if f is a trivial Kan fibration.*

Proof: By Proposition 7.13, every trivial Kan fibration is clearly both a weak categorical equivalence and a quasi-fibration. Now, suppose that f is a weak categorical equivalence, then we may factor it as factor is as

$$f : \bullet \xrightarrow{u} \bullet \xrightarrow{q} \bullet$$

where u is a monomorphism and q is a trivial Kan fibration. By the 2-out-of-3 rule, and again by Proposition 7.13, u is a weak categorical equivalence, thus f has the right lifting property against u and by the Retract Argument, f is a retract of q. Since monomorphisms and trivial Kan fibrations are a factorization system, trivial Kan fibrations are closed under retracts, and thus f is a trivial Kan fibration. ■

With this we get the first part we need for the Joyal Model Structure:

Corollary 7.24. $(\mathfrak{C}, \mathfrak{F}_J \cap \mathfrak{W}_J)$ *is a weak factorization system.*

Proof: By the previous proposition, $\mathfrak{F}_J \cap \mathfrak{W}_J = \mathfrak{F}_K \cap \mathfrak{W}_K$, thus this is the same factorization system of the Quillen Model Structure. ■

Although Definition 7.22 is the only possible one, it will be useful to characterize quasi-fibrations a little bit more, since we don't know exactly what $\mathfrak{C} \cap \mathfrak{W}_J$ is[4]. This is where inner fibrations come into play. More specifically, quasi-fibrations between quasi-categories will be inner fibrations which are also *isofibrations*.

Recall that in **Cat**, an isofibration is a functor that lifts isomorphisms. More precisely, let $\mathscr{J}$ be the walking isomorphism category, and let $* \to \mathscr{J}$ pick any of the two objects of $\mathscr{J}$. Then a functor $F : \mathscr{C} \to \mathscr{D}$ is an isofibration if every diagram of the form

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "F"] \\ \mathscr{J} \arrow[r] \arrow[ur, dashed, "H"] & \mathscr{D} \end{tikzcd}$$

admits a lift H.

To get an analogous definition for quasi-categories we use the Nerve Functor. Let J be the nerve of $\mathscr{J}$ – this is called the walking isomorphism in the context of quasi-categories –

[4] Unlike the Quillen Model structure, where we know a very simple generating set: the horn inclusions.

then a morphism of simplicial sets $f : X \to Y$ is called an isofibration if every diagram of the form

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ J \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

admits a lift h.

Lemma 7.25. *Let $f : X \to Y$ be an inner fibration and an isofibration between quasi-categories, then so is $\langle u, f\rangle$ for any monomorphism u. In addition, if u is also a weak categorical equivalence, then so is $\langle u, f\rangle$.*

The above result is analogous to 7.18, and it's just another consequence of the fact that the pushout product behaves well under lifting properties and that being an isofibration is the same as satisfying a lifting property. The important thing about it is that it allows us to prove the following:

Proposition 7.26. *Let $f : X \to Y$ be a morphism between quasi-categories, then f is a quasi-fibration if, and only if, f is an inner fibration and an isofibration.*

To see one side, note that inner horn inclusions are inner anodyne and that the morphism $* \to J$ is the image of the nerve functor of an equivalence of categories. Thus, a quasi-fibration has lifts against inner horn inclusions and is an isofibration.

The other side is more subtle so we omit it here. A proof can be found at [17], Theorem 5.22. ■

With this, we can prove that $\mathfrak{C} \cap \mathfrak{W}_J = \mathrm{LLP}(\mathfrak{F}_J)$.

Theorem 7.27. *$\mathfrak{C} \cap \mathfrak{W}_J$ is the collection of morphisms that have the left lifting property against every quasi-fibration between quasi-categories.*

Proof: Let $\mathfrak{F}_0 \subset \mathfrak{F}_J$ be the class of quasi-fibrations between quasi-categories. By definition, $\mathfrak{C} \cap \mathfrak{W}_J \subset \mathrm{LLP}(\mathfrak{F}_J) \subset \mathrm{LLP}(\mathfrak{F}_0)$, so we need only to show that $\mathrm{LLP}(\mathfrak{F}_0) \subset \mathfrak{C} \cap \mathfrak{W}_J$.

Let $u : A \to B \in \mathrm{LLP}(\mathfrak{F}_0)$, then we may factor $A \to *$ as an inner anodyne followed by an inner fibration:

$$A \xrightarrow{v} X \xrightarrow{f} *.$$

Assuming that $f \in \mathfrak{F}_0$, we get that the diagram

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

has a lift so that v factors through u. Since v is a monomorphism, so is u. Thus it only remains to show that u is also a weak categorical equivalence.

Since $u \in \mathrm{LLP}(\mathfrak{F}_0)$, for any monomorphism v, $\langle v, f\rangle$ is a quasi-fibration between quasi-categories by Lemma 7.25, so that u has the right left lifting property against $\langle v, f\rangle$ for every monomorphism and $f \in \mathfrak{F}_0$. From the adjunction $(u \boxtimes (-) \dashv \langle u, -\rangle)$, and the fact that $(-) \boxtimes (-)$ is symmetric, we conclude that $\langle u, f\rangle$ has the right lifting property against every monomorphism for every $f \in \mathfrak{F}_0$, *i.e.*, $\langle u, f\rangle$ is a trivial Kan fibration for every f. In particular, this implies that

$$X^u : X^B \longrightarrow X^A$$

is a trivial Kan fibration, so by Corollary 7.20 and Proposition 7.13, we conclude that u is a weak categorical equivalence. The proof that f can be taken to be in $\mathfrak{F}_0$ is below, in Proposition 7.30. ■

By recalling the string of inclusions $\mathfrak{C} \cap \mathfrak{W}_J \subset \mathrm{LLP}(\mathfrak{F}_J) \subset \mathrm{LLP}(\mathfrak{F}_0)$, we trivially conclude the following:

Corollary 7.28. $\mathfrak{C} \cap \mathfrak{W}_J = \mathrm{LLP}(\mathfrak{F}_J)$.

So it remains to show that $(\mathscr{C} \cap \mathfrak{W}_J, \mathfrak{F}_J)$ is a factorization system, but by the same argument as for anodynes, all morphisms in $\mathscr{C}$ have small domain, thus the Quillen Small Object Argument says that every morphism in factors as a cell complex of morphism in $\mathfrak{C} \cap \mathfrak{W}_J$ followed by a quasi-fibration, but since $\mathfrak{C} \cap \mathfrak{W}_J = \mathrm{LLP}(\mathfrak{F}_J)$, all cell complexes of it are monic categorical weak categorical equivalences. With this, we have finally proved the following theorem:

Theorem 7.29. *Let $\mathscr{C}$ denote the class of monomorphisms in* **sSet***, $\mathfrak{F}_J$ denote the class of quasi-fibrations and $\mathfrak{W}_J$ the class of weak categorical equivalences. Then $(\mathbf{sSet}, \mathfrak{F}_J, \mathfrak{C}, \mathfrak{W}_J)$ is a model category.*

Now that we have the model structure, we need to show that this is a model structure for quasi-categories: they are the fibrant-cofibrant objects of such a model structure. Since cofibrations are monomorphisms, every object is cofibrant, thus we need only to show that the quasi-categories are fibrant and are the only fibrant objects. The following result concludes this section.

Proposition 7.30. *X is a fibrant object in the Joyal Model Structure if and only if X is a quasi-category.*

Proof: Let X be a quasi-category, then $X \to *$ is an inner fibration by definition. If $x \in X_0$, then sending the objects of J into x and the isomorphism into $s_0 x$ determines a lift against the morphism $* \to J$. Thus $X \to *$ is an inner fibration and an isofibration.

Conversely, suppose that $X \to *$ is a quasi-fibration, then it has the right lifting property against monic weak categorical equivalences. Since every inner anodyne is a monic weak categorical equivalence, and since every inner horn inclusion is inner anodyne, X is a quasi-category. ■

Finally, we show that weak categorical equivalences are actually the same as equivalences of quasi-categories.

Theorem 7.31. *Let $f : X \to Y$ be a functor of quasi-categories. Then f is a weak categorical equivalence if and only if it is an equivalence of quasi-categories in the sense of Definition 5.25.*

Proof: Suppose that f is a weak categorical equivalence, so by Whitehead's Theorem, it is a homotopy equivalence in the Joyal model structure. So, let's find a cylinder object for X. Let J be the nerve of the walking isomorphism, we claim

$$X \bigsqcup X \hookrightarrow X \times J \xrightarrow{p} X$$

is a cylinder object. Clearly $X \bigsqcup X \hookrightarrow$ is a cofibration. Now, note that since $J \to *$ is a fibration since J is a quasi-category and $J \to *$ is also a weak categorical equivalence, since it is the nerve of an equivalence of categories. Now observe that p is given as the pullback

$$\begin{tikzcd} X \times J \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d, "\in \mathfrak{F}_J \cap \mathfrak{W}_J"] \\ X \arrow[r] & * \end{tikzcd}$$

so that p is also a trivial fibration. For the map

$$h : X \times J \to Y$$

we have a corresponding morphism $J \to \mathrm{Func}(X, Y)$. Since J is an ∞-groupoid, the 1-simplices in its image are isomorphisms so that the induced natural transformation

$$\Delta^1 \hookrightarrow J \to \mathrm{Func}(X, Y)$$

is a natural isomorphism. Thus, the definition of a homotopy equivalence in a model category immediately gives us Definition 5.25.

Now suppose that $f : X \to X$ is naturally isomorphic to the identity of X. We'll show that this implies that f is left homotopic to the identity on X, and from this the result will follow.

Let $h : X \times \Delta^1 \to X$ be a natural isomorphism, then $\Delta^1 \to \mathrm{Func}(X, X)$ factors through $\mathrm{Func}(X, X)^{\mathrm{iso}}$, which is an ∞-groupoid, that is, $\mathrm{Func}(X, X) \to *$ is a Kan fibration. Now, note that the inclusion

$$\Delta^1 \hookrightarrow J$$

is a trivial Kan cofibration, since their realizations are homotopic to a point. Thus, the diagram

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d, hook, "\in \mathfrak{C} \cap \mathfrak{W}_K"'] & \mathrm{Func}(X, X)^{\mathrm{iso}} \arrow[d, "\in \mathfrak{F}_K"] \\ J \arrow[r] \arrow[ur, dashed, "\tilde{h}"] & * \end{tikzcd}$$

has a lift $\tilde{h}$. This is a left homotopy between the same functors the natural isomorphism connected, since it is an extension of it. ■

7.5 QUASI-CATEGORIES AND ∞-GROUPOIDS

In this section we briefly discuss how Joyal's Model structure still preserves the Quillen Model Structure. First of all, note that this is a requirement, since the Quillen Model Structure serves as the homotopy theory of ∞-groupoids, and ∞-groupoids are quasi-categories.

We know that Quillen's structure has the same cofibrations as Joyal's, but we have less weak categorical equivalences than Kan weak equivalences (Proposition 7.12), so that $\mathfrak{F}_K \subset \mathfrak{F}_J$. This is called a *Left Bousfield Localization.*

Definition 7.32 (Left Bousfield Localizations). Let $\mathscr{C} = (\mathscr{C}, \mathfrak{F}, \mathfrak{C}, \mathfrak{W})$ be a model category, then a left Bousfield localization of $\mathscr{C}$ is a model category $\mathscr{C}_{\mathrm{loc}}$ such that:

(i) $\mathscr{C}_{\mathrm{loc}}$ has the same underlying category of $\mathscr{C}$, namely $\mathscr{C}$;

(ii) $\mathscr{C}_{\mathrm{loc}}$ has the same cofibrations as $\mathscr{C}$;

(iii) $\mathscr{C}_{\mathrm{loc}}$ has more weak equivalences than $\mathscr{C}$, that is, $\mathfrak{W} \subset \mathfrak{W}_{\mathrm{loc}}$.

Note that by its very definition, a left Bousfield localization is determined as soon as we know what $\mathfrak{W}_{\text{loc}}$ is. □

Now we show little of how the new fibrations are related with the original ones.

Proposition 7.33. *Let $\mathscr{C}_{\text{loc}}$ be a left Bousfield localization of $\mathscr{C}$ with weak equivalences $\mathfrak{W}_{\text{loc}}$, then:*

(i) $\mathfrak{F}_{\text{loc}} \subset \mathfrak{F}$;

(ii) $\mathfrak{F}_{\text{loc}} \cap \mathfrak{W}_{\text{loc}} = \mathfrak{F} \cap \mathfrak{W}$.

Proof: To show (i), note that $\mathfrak{C}_{\text{loc}} \cap \mathfrak{W}_{\text{loc}} = \mathfrak{C} \cap \mathfrak{W}_{\text{loc}} \supset \mathfrak{C} \cap \mathfrak{W}$, thus if a morphism has the right lifting property against all morphism in $\mathfrak{C}_{\text{loc}} \cap \mathfrak{W}_{\text{loc}}$ it has the right lifting property against all morphism in $\mathfrak{C} \cap \mathfrak{W}$.

For (ii), by the definition of a model category $\mathfrak{F} \cap \mathfrak{W} = \mathsf{RLP}(\mathfrak{C})$, thus $\mathfrak{F} \cap \mathfrak{W} = \mathsf{RLP}(\mathfrak{C}_{\text{loc}}) = \mathfrak{F}_{\text{loc}} \cap \mathfrak{F}_{\text{loc}}$. ■

Before we explain what this means for ∞-groupoids and quasi-categories, we develop the theory of Bousfield localizations just a bit more.

Definition 7.34. A full subcategory $\mathscr{D}$ of a category $\mathscr{C}$ is said to be reflective if the inclusion functor

$$\iota : \mathscr{D} \hookrightarrow \mathscr{C}$$

is right adjoint. □

Theorem 7.35. *Let $\mathscr{C}_{\text{loc}}$ be a left Bousfield localization of $\mathscr{C}$, then the adjunction*

$$\begin{tikzcd} \mathscr{C}_{\text{loc}} \arrow[r, shift right, "\mathrm{id}_{\mathscr{C}}"'] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, shift right, "\mathrm{id}_{\mathscr{C}}"'] \end{tikzcd}$$

is a Quillen adjunction. Furthermore, $\mathrm{h}\mathscr{C}_{\text{loc}}$ is a full subcategory of $\mathrm{h}\mathscr{C}$ and the right derived functor of the above adjunction can be taken to be the inclusion, thus exhibiting $\mathrm{h}\mathscr{C}_{\text{loc}}$ as a reflective subcategory of $\mathrm{h}\mathscr{C}$.

Proof: That this is a Quillen adjunction follows trivially: its preserves fibrations and acyclic fibrations since $\mathfrak{F}_{\text{loc}} \subset \mathfrak{F}$ and $\mathfrak{F}_{\text{loc}} \cap \mathfrak{W}_{\text{loc}} = \mathfrak{F} \cap \mathfrak{W}$.

That $\mathrm{h}\mathscr{C}_{\text{loc}}$ is a full subcategory follows from the fact that $\mathscr{C}_{\text{loc}}$ has the same cofibrations as $\mathscr{C}$: let X, Y be fibrant-cofibrant objects of $\mathscr{C}_{\text{loc}}$, then X, Y are also fibrant-cofibrant in $\mathscr{C}$, now we claim that f is left homotopic to g $(f, g : X \rightrightarrows Y)$ in $\mathscr{C}_{\text{loc}}$ if and only f is left homotopic to g in $\mathscr{C}$. This claim follows because left homotopy doesn't depend on the choice of cylinder object (for fibrant-cofibrant objects), thus we may always choose a left homotopy mediated by a cylinder object $\mathrm{Cyl}(X)$ that is cylinder object for X in $\mathscr{C}$ and $\mathscr{C}_{\text{loc}}$ simultaneously since they have the same cofibrations. Thus $[X, Y]_{\mathscr{C}_{\text{loc}}} = [X, Y]_{\mathscr{C}}$.

The last result is due to the fact that $\mathfrak{F}_{\text{loc}} \subset \mathfrak{F}$, since by the construction of the right derived and the fact that the functor is the identity, the right derived functor is simply an identity and can be taken to be the inclusion. ■

So, this above theorem is saying that the homotopy theory of the Bousfield localization includes itself in the homotopy theory of the original model category. Returning to our setting, we have that the homotopy theory of ∞-grupoid is included in the homotopy theory of quasi-categories! This is the realization of the Homotopy Hypothesis for quasi-categories, a must in any "wannabe" model of $(\infty, 1)$-category, for more details on this matter see [3].

8

THE BERGNER MODEL STRUCTURE

From what we know so far, simplicial categories categories are $(\infty, 1)$-categories when all their hom-sets are ∞-groupoids. For ∞-groupoids we already have a model structure, namely Quillen's model structure, which realizes ∞-grupoids as topological spaces up to homotopy. With this in mind, Bergner's model structure uses *local homotopical properties* to define its fibrations and weak equivalences in a natural manner. Although the proof of the model structure is rather technical, its definition is very simple and quite intuitive, so, we begin by exploring its weak equivalences and it is in fact a good definition.

The proof we'll give of the Bergner Model Structure is not the original one, given by Bergner in [4]. In it, we define the class of generating cofibrations and trivial cofibrations, and then follow the lines of Theorem 2.45. Here, we will use a much stronger result of Muro, [25], and conclude that our definitions indeed yield a model structure. The motive behind this is to explore Bergner's construction as general construction for categories enriched over model categories, and not something specific to simplicial categories. That being said, we won't go into much detail since we won't be able to go deeper into the theory of enriched categories.

8.1 DWYER-KAN EQUIVALENCES

We need a definition of equivalence of simplicial categories that captures the essence of what an equivalence between $(\infty, 1)$-category should be. We begin by looking at the equivalence of ordinary categories.

A functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories when two conditions are met. The first ones says that

$$F_{x,y} : \mathscr{C}(x, y) \longrightarrow \mathscr{D}(Fx, Fy)$$

must be a bijection. The second one is the requirement that F be essentially surjective. If $\pi_0\mathscr{C}$ denotes the set of isomorphism classes of $\mathscr{C}$, this is equivalent to saying that

$$\pi_0 F : \pi_0\mathscr{C} \longrightarrow \pi_0\mathscr{D}$$

is a surjective function.

This means that F must satisfy two properties: one *local* and one *global*. That is, one property for the action of F on hom-sets and one property for the action of F on objects. We use this fact as a precursor of the definition of equivalences of simplicial categories.

First, let's discuss what would be the desired local property. In **Cat**, the requirement that $F_{x,y}$ being a bijection translates to $F_{x,y}$ being an equivalence of sets. For simplicial categories, hom-sets are actually simplicial sets, and for those of our interest, Kan complexes. Thus, we could ask for $F_{x,y} : \mathscr{C}(x, y) \to \mathscr{D}(Fx, Fy)$ to be a natural isomorphism. However, this would be unnatural, since we are seeing simplicial categories as categories enriched over ∞-groupoids, for which natural isomorphisms are too strict. So, the natural definition would be to require $F_{x,y}$ to be an equivalence of ∞-groupoids: a Kan weak equivalence!

Now for the global property. For ordinary categories, essential surjectivity means that $\mathscr{C}$ "has all" objects of $\mathscr{D}$. Here, "has all" refers to the fact that objects of $\mathscr{C}$ represent all isomorphism classes of $\mathscr{D}$. Since simplicial categories are homotopical in nature, it is reasonable to replace "isomorphisms classes" for "homotopy equivalence classes". Thus, the second condition translates to F being "homotopically essentially surjective". Using the homotopy category functor

$$\pi_0 : \textbf{sSet-Cat} \longrightarrow \textbf{Cat}$$

these conditions can be expressed as $\pi_0 F$ being essentially surjective.

Having had this discussion, the following definition becomes extremely intuitive.

Definition 8.1 (Dwyer-Kan Equivalences)**.** Let $\mathscr{C}, \mathscr{D}$ be simplicial categories. Then a simplicial functor $F : \mathscr{C} \to \mathscr{D}$ is said to be a DK-equivalence[1] if

$$F_{x,y} : \mathscr{C}(x, y) \longrightarrow \mathscr{D}(Fx, Fy)$$

is a Kan weak equivalence for all x, y and

$$\pi_0 F : \pi_0 \mathscr{C} \longrightarrow \pi_0 \mathscr{D}$$

is an essentially surjective functor. □

Remark 8.2. For Kan enriched categories, the fact that $F_{x,y}$ is a Kan weak equivalence implies that $\pi_0 F_{x,y}$ is a bijection, thus weakly equivalent Kan enriched categories have equivalent homotopy categories.

8.2 FIBRATIONS

Now we discuss fibrations. Very much like DK-equivalences, fibrations will be defined via a local property combined with a global property.

Definition 8.3. Let $F : \mathscr{C} \to \mathscr{D}$ be a map of simplicial categories. Then F will be said to be a fibration if

$$F_{x,y} : \mathscr{C}(x, y) \longrightarrow \mathscr{D}(Fx, Fy)$$

is a Kan fibration for every x, y and

$$\pi_0 F : \pi_0 \mathscr{C} \longrightarrow \pi_0 \mathscr{D}$$

is an isofibration. □

Remark 8.4. By this definition, it's immediate that Kan enriched categories will be fibrant objects in Bergner's model structure, since for $\mathscr{C} \to *$, $F_{x,y} = (\mathscr{C}(x, y) \to *)$, which is a Kan fibration since $\mathscr{C}(x, y)$ is a Kan complex and $\pi_0 F = (\pi_0 \mathscr{C} \to *)$ is an isofibration, since every object is fibrant in **Cat**.

[1] The "DK" stands for "Dwyer-Kan", as they were the ones who introduced the concept in [8].

With this second definition we see a pattern forming. Simplicial categories are categories enriched over the model category **sSet**, that is, hom-sets are objects of **sSet**. Thus, the local condition in the above definition is related to the Quillen Model Structure in **sSet**. By Remark 8.2, the global conditions of DK-equivalences could be replaced by asking that $\pi_0 F$ be an equivalence of categories, thus, we get that the global conditions have to do with the Canonical Model Structure on **Cat**: weak equivalences are such that $\pi_0 F$ is weak equivalence in **Cat** and fibrations are such that $\pi_0 F$ is a fibration in **Cat**.

As another example of this, consider ordinary categories as categories enriched over **Set**. In **Set**, consider the trivial model structure: any morphism is a fibration and cofibrations and weak equivalences are bijections. Then, the Canonical Model Structure in **Cat** is obtained in the same way as the proposed one for simplicial categories.

8.3 CATEGORIES ENRICHED OVER MODEL CATEGORIES

In this section, $\mathcal{V}$ will denote a symmetric monoidal category $(\mathcal{V}, \otimes, I)$. Without loss to this text, you can think of $\mathcal{V}$ as $(\mathbf{sSet}, \times, \Delta^0)$.

Suppose that $\mathcal{V}$ is also a model category, and consider $\mathcal{V}$-**Cat** as the category of categories enriched over $\mathcal{V}$. We'll try to put a model structure on $\mathcal{V}$-**Cat**, to do this, we will mimic and we've done so far for simplicial categories. First of all, we need to define what would be the π_0 functor in this case.

In **sSet**, the definition of π_0 is quite simple, since its objects already have a notion of "elements". In the general case, an object in $\mathcal{V}$ is completely abstract, thus we need a more intrinsic definition. We do this by using the unit of its symmetric monoidal structure. Since $\mathcal{V}$ is a model category, it has a localization

$$\gamma : \mathcal{V} \longrightarrow h\mathcal{V}$$

and with it we can define

$$\pi_0 : \mathcal{V} \longrightarrow \mathbf{Set}$$

as

$$\pi_0 X = h\mathcal{V}(\gamma(I), \gamma(X)).$$

Note that this in fact returns our definition for **sSet**: $I = *$, and a morphism $x : * \to X$ is homotopic to another $y : * \to X$ if, and only if, there exists a path from x to y. If you wish to make this even more concrete, consider $\mathcal{V} = (\mathbf{Top}, \times, *)$ with the model structure of your choice.

With this we hand, we define a functor

$$\pi_0 : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}$$

by setting $ob(\pi_0 \mathscr{C}) = ob(\mathscr{C})$ and $(\pi_0 \mathscr{C})(x, y) = \pi_0 \mathscr{C}(x, y)$. With this in hand we can make the following definition:

Definition 8.5. Let $F : \mathscr{C} \to \mathscr{D}$ be a $\mathcal{V}$-enriched functor. Call F a DK-equivalence if $F_{x,y}$ is a weak equivalence in $\mathcal{V}$ for all x, y and $\pi_0 F$ is essentially surjective. Call F a DK-fibration if $F_{x,y}$ is a fibration in $\mathcal{V}$ for all x, y and $\pi_0 F$ is an isofibration. □

So, we have a prototype of a model structure in $\mathcal{V}$-**Cat**, however, there's no reason for this to actually be a model structure. However, given certain conditions in $\mathcal{V}$, this can be shown to be a model structure. Because of that, we now make definitions which will become this condition.

First of all, note that for the enrichment over $\mathcal{V}$, we only used the monoidal structure, thus, it is reasonable to ask for some sort of compatibility of said monoidal structure and the model structure.

Definition 8.6 (Monoidal Model Category)**.** Let $(\mathscr{C}, \otimes, I)$ be a symmetric monoidal model category which is also a model category. Then $\mathscr{C}$ is a monoidal model category if the following conditions are satisfied

(i) if i and j are cofibrations, then so is $i \boxtimes j$, their pushout product;

(ii) for any cofibrant object X and for any cofibrant resolution of the unity $\varnothing \to QI \xrightarrow{p} I$, the morphism

$$QI \otimes X \xrightarrow{p \otimes X} I \times X$$

is a weak equivalence.

Here, the pushout product is defined as in Section 7.3. □

This may seem a bit strange, but this definition makes a great connection between the model structure and the monoidal structure, for more details see [14].

Definition 8.7 (Monoid Axiom)**.** Let $\mathscr{C}$ be a monoidal model category, then $\mathscr{C}$ satisfies the monoid axiom if every morphism that is obtained as a transfinite composition of pushouts of tensor products of trivial cofibrations with any object is a weak equivalence. □

Lastly, we need our category to be well behaved as a plain category, so we have the following:

Definition 8.8 (Combinatorial Model Category)**.** Let $\mathscr{C}$ be a model category, then $\mathscr{C}$ is said to be combinatorial if $\mathscr{C}$ is locally presentable and cofibrantly generated.

Finally, we can state the main result of [25]:

Theorem 8.9. *Let $\mathcal{V}$ be a combinatorial monoidal model category which satisfies the monoid axiom. Then* DK*-equivalences and* DK*-fibrations form a combinatorial model structure in $\mathcal{V}$-**Cat**.*

So, if we want to prove that Bergner's model structure is indeed a model structure, it suffices to check that these axioms hold for **sSet** with the Quillen Model Structure.

8.4 THE BERGNER MODEL STRUCTURE

As mentioned above, to define Bergner's model structure we need only to check the hypothesis of Theorem 8.9. Since **sSet** is the presheaf category of a small category we know that it is locally presentable. Furthermore, we know that Quillen's model structure is cofibrantly generated by boundaries and horn inclusions, thus $\mathbf{sSet}_q$ is a combinatorial model category. So, we need to show that $\mathbf{sSet}_q$ is a monoidal model category, when considering the symmetric monoidal structure given by the product and that it satisfies the monoid axiom.

Lemma 8.10. $\mathbf{sSet}_q$ *is a monoidal model category.*

Proof: We need to show conditions (i) and (ii) of Definition 8.6. To see (i), recall that monomorphisms in **sSet** are level wise injections, and since **sSet** is a presheaf category its colimits are computed component wise. Thus, we must show if $u : A \to B$ and $j : X \to Y$

are injective functions, then so is the induced map in the diagram below

$$\begin{tikzcd}
A \times X \arrow[r, "{(id,j)}"] \arrow[d, "{(u,id)}"'] & A \times Y \arrow[d] \arrow[ddr, bend left, "{(u,id)}"] & \\
B \times X \arrow[r] \arrow[drr, bend right, "{(id,j)}"'] & Po \arrow[dr, dashed] & \\
& & B \times Y
\end{tikzcd}$$

The pushout of the diagram is the set $(B \times X \coprod A \times Y)/\sim$, where $(u(a), x) \sim (a, j(x))$. The induced map is then

$$(a, y) \xmapsto{f_1} (u(a), y)$$

$$(b, x) \xmapsto{f_2} (b, j(a))$$

factored through the quotient. Now suppose that $f_1(a, y) = f_2(b, x)$, that is, $(u(a), y) = (b, j(x))$, then $y = j(x)$ and $b = u(a)$, thus (a, y) and (b, x) belong to the same equivalence class, and thus the induced map is injective.

For the second condition we need to show that

$$Q * \times X \xrightarrow{(p,X)} X$$

is a weak equivalence. This is simply the projection onto X. Now, since the geometric realization preserves products (it is left adjoint), the image of this morphism by it is a weak homotopy equivalence. ■

Lemma 8.11. $\mathbf{sSet}_q$ *satisfies the monoid axiom.*

Proof: The monoid axiom asks that transfinite compositions of morphisms of the form (u, id_X) where $X \in \mathbf{sSet}$ and $u \in \mathfrak{C} \cap \mathfrak{W}_k$ be weak equivalences. Now, note that, since the geometric realization functor preserves products, the realization of (u, id_X)

$$|A| \times |X| \xrightarrow{(|u|, |id_X|)} |B| \times |X|$$

is clearly a weak homotopy equivalence, thus (u, id_X) is always a Kan weak equivalence. Furthermore, it is obvious that (u, id_X) is also monomorphism, so that $(u, id_X) \in \mathfrak{C} \cap \mathfrak{W}_k$ for all $u \in \mathfrak{C} \cap \mathfrak{W}_k$ and $X \in \mathbf{sSet}$. Now, since $\mathfrak{C} \cap \mathfrak{W}_k$ is given by the class of morphisms that have the left lifting property against another class, it is closed under transfinite composition, and thus $\mathbf{sSet}_q$ satisfies the Monoid Axiom. ■

So, we conclude what we wanted:

Theorem 8.12. $(\mathbf{sSet\text{-}Cat}, \mathfrak{F}_B, \mathfrak{C}_B, \mathfrak{W}_B)$*, where* $\mathfrak{F}_B$ *and* $\mathfrak{W}_B$ *are DK-fibrations and DK-fibrations and* $\mathfrak{C}_B = \mathrm{LLP}(\mathfrak{W}_B)$ *is a model category. In this category, the fibrant objects are the simplicial categories enriched over Kan complexes.*

Note that, although we succeeded in making categories enriched over Kan complexes the fibrant objects of the model structure, it is not clear what the cofibrant objects are. It turns out that the cofibrant objects are not as easy to compute as the fibrant ones and we will not need to know them explicitly, so we'll not discuss them here. A detailed account of the cofibrant objects look like can be found in Bergner's original paper.

9 THE QUILLEN EQUIVALENCE

Now that we have two model structures, one for each model of $(\infty, 1)$-categories, we may compare them, and of course, this comparison is done by means of Quillen adjunctions. More specifically, we'll define a functor

$$\mathcal{N}^{hc} : \mathbf{sSet\text{-}Cat} \longrightarrow \mathbf{sSet}$$

and show that it has a left-adjoint, and then, we'll show that this adjunction is in fact a Quillen equivalence. The functor $\mathcal{N}^{hc}$ is called the Homotopy Coherent Nerve, and it is very similar to the nerve of a category in many ways, with the exception that it is *coherent up to homotopy*.

9.1 THE HOMOTOPY COHERENT NERVE

As the name implies, the Homotopy Coherent Nerve is a kind of nerve. One way to get a nerve functor

$$\mathcal{N} : \mathbf{sSet\text{-}Cat} \longrightarrow \mathbf{sSet}$$

would be to simply set

$$(\mathcal{N}\mathscr{C})_n = \mathbf{sSet\text{-}Cat}([n], \mathscr{C})$$

where we regard $[n]$ as a simplicial category with constant hom-sets. However, this construction completely ignores the higher information encoded in the morphism complexes of simplicial categories. Our solution to this will be to replace $[n]$ with a more suitable version, one which can capture information on other levels of the morphism complexes in a "coherent" manner.

Regarding $[n]$ as a category, we see that for each object $i, j \in [n]$ with $i \leq j$, there is a single morphism $q_{ij} : i \to j$, in that way, if $q_{jk} : j \to k$, we get that

$$q_{ij} \circ q_{jk} = q_{ik}.$$

This says that the *path* you take between any two objects in $[n]$ doesn't matter. For instance, going from 1 to 3 is the same as going from 1 to 2 and going to 3. This is too strict, since strict composition is not really well defined in $(\infty, 1)$-categories, so we'll construct another object to serve the same purpose as $[n]$ in a way that the path you take between two objects of $[n]$ matters.

Definition 9.1. Let $[n] \in \Delta$. Define the simplicial category Path$[n]$ as follows:

(i) $\mathrm{ob}(\mathrm{Path}[n]) = \{0, 1, \ldots, n\}$

(ii) $\mathrm{Path}[n](i, j) = \mathcal{N}P_{ij}$, where P_{ij} is the linearly ordered set given by

$$P_{ij} = \{I \subset [n] \mid i, j \in I \text{ and } k \in I \implies i \leq k \leq j\}$$

if $i \leq j$ ordered by reverse inclusion[1] and the empty set otherwise;

(iii) for $i, j, k \in \mathrm{Path}[n]$, the composition is the nerve of the functor

$$\circ : P_{jk} \times P_{ij} \longrightarrow P_{ik}$$

$$(J, I) \mapsto J \cup I \in P_{ik};$$

(iv) identities are the singleton sets $\{i\} \in P_{ii}$.

The category $\mathrm{Path}[n]$ is called the simplicial path category of $[n]$. □

Note that the difference between $[n]$ and $\mathrm{Path}[n]$ arises when we consider different paths in $[n]$, that is, if all the paths were the same ($P_{ij} = \{*\}$), then the above construction would return $[n]$ when considered as a simplicial category.

This rule clearly extends to a functor

$$\mathrm{Path} : \Delta \longrightarrow \mathbf{sSet\text{-}Cat}$$

whose action on a morphism $f : [n] \to [m]$ is given by the nerve of

$$\mathrm{Path}(f) : P_{ij} \longrightarrow P_{f(i)f(j)}$$

$$I \mapsto f(I).$$

With this in hand, we may define the Homotopy Coherent Nerve:

Definition 9.2. Let $\mathscr{C}$ be a simplicial category, then let $\mathcal{N}^{hc}\mathscr{C}$ denote the simplicial set whose simplices are given by

$$(\mathcal{N}^{hc}\mathscr{C})_n := \mathbf{sSet\text{-}Cat}(\mathrm{Path}[n], \mathscr{C}).$$

Faces and degeneracy maps are given by pre composition with the induced maps $\mathrm{Path}[n] \to \mathrm{Path}[m]$. This in tuns defines a functor

$$\mathcal{N}^{hc} : \mathbf{sSet\text{-}Cat} \longrightarrow \mathbf{sSet}$$

where its action on simplicial functors is simply post-composition. The functor is called the Homotopy Coherent Nerve. □

This functor is not called nerve for nothing: it is the right adjoint of a Kan extension along a Yoneda embedding. Below we go deeper into this fact, but first, let's have a look at what it does to an ordinary category.

For a category $\mathscr{C} \in \mathbf{Cat}$, let $\underline{\mathscr{C}}$ denote the simplicial category obtained from $\mathscr{C}$ as in Example 6.5. Let $F : \mathrm{Path}[n] \to \underline{\mathscr{C}}$ be a simplicial functor, then F is, among other things, a function

$$F : \{0, 1, 2 \ldots, n\} \longrightarrow \mathrm{ob}(\mathscr{C}).$$

[1] $A \leq B \iff B \subset A$.

So, just as the nerve, F picks $n+1$ objects in $\underline{\mathscr{C}}$. Consider $x_i = F(i)$ and $x_j = F(j)$, then F gives a natural transformation

$$F : \mathrm{Path}[n](i,j) \longrightarrow \underline{\mathscr{C}}(x_i, x_j).$$

Thus, since all the faces and degeneracies of $\underline{\mathscr{C}}(x_i, x_j)$ are identities, we get that the diagram

$$\begin{array}{ccc} (\mathcal{N}P_{ij})_1 & \xrightarrow{F_1} & \mathscr{C}(x_i, x_j) \\ {\scriptstyle d}\big\downarrow & & \big\| \\ (\mathcal{N}P_{ij})_0 & \xrightarrow[F_0]{} & \mathscr{C}(x_i, x_j) \end{array}$$

commutes for every face map d. Let $I \subset P_{ij}$, since $\{x,y\}$ is a maximum of the ordered set P_{ij}, we can compute

$$F_0(I) = F_0(d_1(I \to \{i,j\})) = F_1(I \to \{i,j\}) = F_0(d_0(I \to \{i,j\})) = F_0(\{i,j\}).$$

That is, F_0 (therefore F_i) is constant and depends only on its action on $\{i,j\}$: it selects exactly one morphism in $\mathscr{C}$! So we see that it doesn't matter if we consider $[n]$ or Path$[n]$ in this case, or better yet, in this case the Homotopy Coherent Nerve doesn't discriminate along different paths (as one should expect since composition in categories is strict)! As a conclusion to this discussion, we get that

$$\mathcal{N}^{hc}\underline{\mathscr{C}} \cong \mathcal{N}\mathscr{C}.$$

Now, let's have a look at what the Homotopy Nerve of a simplicial category looks like in the general case. Let $\mathscr{C}$ be a simplicial category. Then is quite clear that

$$(\mathcal{N}^{hc}\mathscr{C})_0 = \mathrm{ob}(\mathscr{C})$$

since Path[0] is the terminal category. Also, for $n = 1$, $P_{01} = \{\{x,y\}\}$, so that $\mathcal{N}P_{0,1}$ is the terminal simplicial set, Δ^0. Thus, we conclude that $(\mathcal{N}^{hc}\mathscr{C})_1$ are the morphisms of $\mathscr{C}$.

Note that this is very convenient (as it was made to be), since the objects of $\mathcal{N}^{hc}\mathscr{C}$ are the objects of $\mathscr{C}$ and its morphisms are the morphisms of $\mathscr{C}$. Now we have to see what happens for $n = 2$. Since Path[2] has three objects, we get that a simplicial functor

$$F : \mathrm{Path}[2] \longrightarrow \mathscr{C}$$

picks three objects in $\mathscr{C}$: x_1, x_2 and x_3. We have three non trivial morphisms complexes, P_{01}, P_{02} and P_{12}. Of these, P_{01} and P_{12} are the final category, and thus the action of F on these is simply to pick morphisms $f_{01} \in \mathscr{C}(x_0, x_1)$ and $f_{12} \in \mathscr{C}(x_1, x_2)$. Now let's see what is going on with P_{02}. $P_{02} = \{\{0,1,2\} \to \{0,2\}\}$ and we have a non trivial composition $P_{01} \times P_{12} \to P_{01}$, so that

$$F : \mathcal{N}(P_{02}) \longrightarrow \mathscr{C}(x,y)$$

is such that $F(\{0,1,2\}) = f_{12} \circ f_{01}$ (simplicial functors preserve compositions). Let $f_{0,2} \in \mathscr{C}(x_0, x_2)_0$ be the morphism $F(\{0,2\})$. Since we have a unique morphism $\{\{0,1,2\} \to \{0,1\}\}$, the action of F in P_{02} is to pick a homotopy in $\mathscr{C}$ that goes from $f_{12} \circ f_{01}$ to f_{02}.

With this discussion, we conclude that a 2-simplex of $\mathcal{N}^{hc}\mathscr{C}$ corresponds to a not necessarily commutative diagram

$$\begin{tikzcd} & x_1 \arrow[dr, "f_{12}"] & \\ x_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & x_2 \end{tikzcd}$$

together with a chosen homotopy $h \in \mathscr{C}(x_0, x_1)$ that goes from $f_{12} \circ f_{01}$ to f_{02}. Hence the name Homotopy Coherent Nerve: we chose diagrams that commute up to a homotopy such that this homotopy is given by the functor, and does not depend on arbitrary choice.

9.2 THE ADJUNCTION

As mentioned before, the Homotopy Coherent Nerve is a right adjoint to some functor **sSet** $\to$ **sSet-Cat**. We get this nerve by considering the same construction as in Section 4.4. We defined a functor

$$\mathrm{Path} : \Delta \longrightarrow \mathbf{sSet}$$

so that we have a diagram

$$\begin{tikzcd} \Delta \arrow[r, "\mathrm{Path}"] \arrow[d, hook, "y"'] & \mathbf{sSet\text{-}Cat} \\ \mathbf{sSet} & \end{tikzcd}$$

where y is the Yoneda embedding. Thus, we get a Kan extension

$$\begin{tikzcd} \Delta \arrow[rr, "\mathrm{Path}"] \arrow[dr, hook, "y"'] & \arrow[d, phantom, "\Downarrow"] & \mathbf{sSet\text{-}Cat} \arrow[dl, bend right, "R"'] \\ & \mathbf{sSet} \arrow[ur, bend right, "\mathrm{Path}"'] \arrow[ur, phantom, "\top"] & \end{tikzcd}$$

where $R(\mathscr{C}) = \mathbf{sSet\text{-}Cat}(\mathrm{Path}(-), \mathscr{C})$, *i.e.* $R = \mathcal{N}^{hc}$. For a simplicial set X, Path(X) is called the simplicial path category of X.

With this discussion, we conclude that there exists an adjunction

$$\mathrm{Path} : \mathbf{sSet} \underset{\perp}{\rightleftarrows} \mathbf{sSet\text{-}Cat} : \mathcal{N}^{hc}.$$

Below we show that this adjunction is in fact a Quillen equivalence.

9.3 THE QUILLEN EQUIVALENCE

Before we can start, we need to tweak a definition we used for the Joyal Model Structure, namely, the definition of weak equivalences.

Definition 9.3. Call a morphism $u : X \to Y$ a categorical weak equivalence if $\mathrm{Path}(u) : \mathrm{Path}(X) \to \mathrm{Path}(Y)$ is a DK-equivalence. □

In the literature, this is often the chosen definition of weak categorical equivalences, for instance in [20], Lurie constructs the Joyal model structure for **sSet** with the same fibrations and cofibrations and the maps defined above as the weak equivalences. Fortunately, if two model structures have the same fibrations and cofibrations, then they have the same weak equivalences:

Proposition 9.4. *Let* $(\mathfrak{F}, \mathfrak{C}, \mathfrak{W})$ *and* $(\mathfrak{F}', \mathfrak{C}', \mathfrak{W}')$ *be two model structures on a category* $\mathscr{C}$. *If* $\mathfrak{F} = \mathfrak{F}'$ *and* $\mathfrak{C} = \mathfrak{C}'$, *then* $\mathfrak{W} = \mathfrak{W}'$.

Proof: Since fibrations and calibrations are equal we get that so are trivial cofibrations and cofibrations since they are given by lifting properties against fibrations and cofibrations. Now, observe that

$$\mathfrak{W} = \{p \circ i \mid i \in \mathfrak{C} \cap \mathfrak{W}, p \in \mathfrak{F} \cap \mathfrak{W} \text{ and } \operatorname{cod}(i) = \operatorname{dom}(p)\}.$$

In fact, $w \in \mathfrak{W}$, then w factors as

$$w : \bullet \xrightarrow{i \in \mathfrak{C}} \bullet \xrightarrow{p \in \mathfrak{F} \cap \mathfrak{W}} \bullet$$

and by the 2-out-of-3 rule, $i \in \mathfrak{C} \cap \mathfrak{W}$. This concludes the proof, since, as mentioned, $\mathfrak{F} \cap \mathfrak{W} = \mathfrak{F}' \cap \mathfrak{W}'$ and $\mathfrak{C} \cap \mathfrak{W} = \mathfrak{C}' \cap \mathfrak{W}'$. ∎

Now, by definition, we see that

$$\mathrm{Path} : \mathbf{sSet} \longrightarrow \mathbf{sSet\text{-}Cat}$$

carries weak equivalences into weak equivalences, so if we show that Path also carries cofibrations to cofibrations, then we'll have shown that the $(\mathrm{Path} \dashv \mathcal{N}^{hc})$ is a Quillen equivalence. That's what we will do now:

Theorem 9.5. *The adjunction*

$$\mathrm{Path} : \mathbf{sSet} \underset{\perp}{\rightleftarrows} \mathbf{sSet\text{-}Cat} : \mathcal{N}^{hc}.$$

is a Quillen adjunction.

Proof: We know that the cofibrations of **sSet** are the monomorphisms, and the monomorphisms are generated by the boundary inclusions

$$\{\partial\Delta^n \hookrightarrow \Delta^n\}.$$

Since $\mathrm{Path} : \mathbf{sSet} \to \mathbf{sSet\text{-}Cat}$ is left adjoint, it preserves colimits, and so, it carries relative cell complexes into relative cell complexes. Thus, if we show that Path carries the boundary inclusions into cofibrations, we'll have shown that it preserves cofibrations, since the class of cofibrations of any model category is closed under forming relative cell complexes.

Let's compute what $\mathrm{Path}(\partial\Delta^n) \to \mathrm{Path}(\Delta^n)$ is. Note that $\mathrm{Path}(\Delta^n) = \mathrm{Path}[n]$ by definition. Now, for $\mathrm{Path}(\partial\Delta^n)$ we get almost the same thing:

$$\mathrm{Path}(\partial\Delta^n)(i,j) = \mathrm{Path}(\Delta^n)(i,j) \text{ if } i \neq 0 \text{ or } j \neq n.$$

So, let's look at $\mathrm{Path}(\partial\Delta^n)(0,n)$. This can be identified with the boundary of the $n-1$-cube, $\partial(\Delta^1)^{n-1}$. Now, observe that $(\Delta^1)^{n-1} = \mathcal{N}([1]^{n-1})$, since the nerve preserves products. We claim that there is a natural identification

$$\mathrm{Path}[n](0,n) \cong (\Delta^1)^{n-1}.$$

To prove this it suffices to show that we have an identification of P_{0n} and $[1]^{n-1}$. The elements of $[1]^{n-1}$ are of the form

$$(x_1, \dots, x_{n-1}), \quad x_i \in \{0,1\}.$$

In the product order, $(x_1, \dots, x_{n-1}) \leq (y_1, \dots, y_{n-1}) \iff x_i \leq y_i$ for all i. With this, it is easy to see the identification: interpret the sequence $(x_1, \dots, x_{n-1})$ as the path from 0 to n that passes the numbers x_i when $x_i = 0$. That is, define the morphism $[1]^{n-1} \to P_{0n}$ and

$$(x_1, \dots, x_{n-1}) \mapsto (\{1,n\} \cup \{x_i \mid x_i = 0\}) \in P_{0n}.$$

This is clearly an isomorphism of partially ordered sets. Thus, we conclude that $\mathrm{Path}(\partial\Delta^n) \to \mathrm{Path}(\Delta^n)$ is the identity on objects and morphism complexes except for $\mathrm{Path}(\partial\Delta^n)(0,n) \to \mathrm{Path}(\Delta^n)(0,n)$, which is an inclusion of simplicial sets.

Now, suppose that we have a diagram

$$\begin{tikzcd} \mathrm{Path}(\partial\Delta^n) \arrow[r, "F"] \arrow[d] & \mathscr{C} \arrow[d, "\in \mathfrak{F} \cap \mathfrak{W}"] \\ \mathrm{Path}(\partial\Delta^n) \arrow[r] & \mathscr{D} \end{tikzcd}$$

Clearly we have a lift at the level of objects, since we have an identity on objects at the left. For i, j, with $i \neq 0$ or $j \neq 0$, we have

$$\begin{tikzcd} \mathrm{Path}(\partial\Delta^n)(i,j) \arrow[r, "F_{i,j}"] \arrow[d, equal] & \mathscr{C}(x_i, x_j) \arrow[d, "\in \mathfrak{F}\cap\mathfrak{W}"] \\ \mathrm{Path}(\partial\Delta^n)(i,j) \arrow[r] & \mathscr{D}(y_i, y_j) \end{tikzcd}$$

Which clearly has a unique lift, namely $F_{i,j}$, which of course fit into a simplicial lift functor (satisfies the required diagrams) $\mathrm{Path}(\partial\Delta^n) \to \mathscr{C}$, with maybe the exception of $F_{0,n}$, which we are yet to define. Now, the diagram

$$\begin{tikzcd} \mathrm{Path}(\partial\Delta^n)(0,n) \arrow[r, "F"] \arrow[d, hook] & \mathscr{C}(x_0, x_n) \arrow[d, "\in \mathfrak{F}\cap\mathfrak{W}"] \\ \mathrm{Path}(\partial\Delta^n)(0,n) \arrow[r] \arrow[ur, "h"] & \mathscr{D}(y_0, y_n) \end{tikzcd}$$

admits a lift h, since the right vertical morphism is a trivial Kan fibration. Thus, setting

$$H : \mathrm{Path}(\partial\Delta^n) \longrightarrow \mathscr{C}$$

$$H_{i,j} = \begin{cases} F_{i,j}, & \text{if } i \neq 0 \text{ or } j \neq n \\ h, & \text{if } i = 0 \text{ and } j = n \end{cases}$$

define a simplicial functor that is a lift for the initial diagram, and thus $\mathrm{Path}(\partial\Delta^n) \longrightarrow \mathrm{Path}(\Delta^n)$ is a cofibration in the Bergner model structure. ■

With this result in hand, we are almost ready to show that this is a Quillen equivalence, but before we need a lemma:

Lemma 9.6. *For the adjunction*

$$\mathrm{Path} : \mathbf{sSet} \rightleftarrows_{\perp} \mathbf{sSet\text{-}Cat} : \mathcal{N}^{hc}$$

all components of the counit

$$\beta : FG \Rightarrow \mathrm{id}_{\mathbf{sSet\text{-}Cat}}$$

of this adjunction are DK*-equivalences.*

The proof of the above lemma is not trivial and is quite cumbersome, so it won't be given here. The proof can be found in Section 2.2 of [20]. With this result we can finally prove that the homotopy coherent nerve establishes a Quillen equivalence between **sSet** with the Joyal Model Structure and **sSet-Cat** with the Bergner Model Structure.

Theorem 9.7. *Let* **sSet** *be equipped with the Joyal Model Structure and* **sSet-Cat** *with the Bergner Model Structure. Then the adjunction*

$$\mathrm{Path} : \mathbf{sSet} \rightleftarrows_{\perp} \mathbf{sSet\text{-}Cat} : \mathcal{N}^{hc}$$

is a Quillen equivalence.

Proof: We must show that a map $u : X \to \mathcal{N}^{hc}(\mathscr{C})$ is a weak categorical equivalence if and only if its transpose $u^\sharp : \mathrm{Path}(X) \to \mathscr{C}$ is a DK-equivalence. As with any adjunction [29], we can factor $u^\sharp$ as

$$u^\sharp : \mathrm{Path}(X) \xrightarrow{\mathrm{Path}(u)} \mathrm{Path}(\mathcal{N}^{hc}(\mathscr{C})) \xrightarrow{\beta} \mathscr{C}$$

where β is the counit map. By the previous lemma, β is DK-equivalence, and thus, by the 2-out-of-3 rule, we conclude that $u^\sharp$ is a DK-equivalence if and only if $\mathrm{Path}(u)$ is also a weak equivalence. But, by definition, this happens if and only if u is a categorical weak equivalence, and this concludes the proof. ■

In the next chapter we'll discuss some of the consequences of this result, but for now remember that the weak equivalences in these model structures are the "correct" notion of equivalence between $(\infty, 1)$-categories, clearly, so we conclude that the property of being equivalent as $(\infty, 1)$-categories is invariant under these models.

CONCLUSION

We now have two ways of defining what is an $(\infty, 1)$-category, and we know that these definitions are connected by a Quillen Equivalence. Now what? Here we will try to give further motivations of why $(\infty, 1)$-categories matter and why this equivalence is good and where to follow from here.

First of all, $(\infty, 1)$-categories are a *unifying* concept. That is, every flavour of homotopy theory we have so far can be described in terms of $(\infty, 1)$-categories. As we mentioned in the introduction, the simplest kind of homotopy theory is that of a category with weak equivalences $(\mathscr{C}, \mathfrak{W})$, which can be made into a simplicial category by means of Simplicial Localizations, developed by Dwyer and Kan in [9]. In this process, the homotopy category of the simplicial localization of a category is a localization for the category with weak equivalences. Of course, by using the Homotopy Coherent Nerve, we could also think of any category with weak equivalences as a quasi-category. Also, as we saw, any ordinary category can be seen as a quasi-category, by means of the Nerve. Moreover, there also a nerve functor [7] which carries $(2, 1)$-categories (2-categories where every 2-morphism is an isomorphism) into quasi-categories preserving the "homotopy" category of the 2-category. Also, in this sense of being a unifying concept, every cohomology theory to this day can be formalized inside some $(\infty, 1)$-category, as the set of path components of some hom-set in it. More details on this can be found in Chapter 7 of [20].

Besides being an unifying concept, $(\infty, 1)$-categories are the natural setting to doing anything up to homotopy. We didn't mention it here, but one of the most important tools of homotopy theory are the homotopy (co)limits. These can be thought as universal properties which hold up to homotopy, instead of up to unique isomorphism. Examples of these are mapping cones and cocones of continuous functions or chain maps, and they are essential to computing homotopy groups. Also, homotopy limits play an essential role in stable homotopy theory and cohomology theories. There are several good references on these, for instance [6, 33, 32].

In a model category, the construction of such limits depends on the arbitrary choice of homotopies between morphisms and thus this construction are often not well behaved and hard to work with. This happens because a functor taking values on a model category is just an ordinary functor and does not have any way of "interacting" with the homotopical aspects of the category. This is where $(\infty, 1)$-categories come into play. A functor between $(\infty, 1)$-categories already carries homotopical information, since it acts on every level of morphisms, thus, diagrams in $(\infty, 1)$ categories are already commutative up to homotopy so that universal properties are naturally homotopical. This is the theory of ∞-limits and it is already a very well developed theory and very much in use. A complete discussion of their construction and uses can be found at Lurie's [23], which serves as a more "user friendly" version of the classic "Higher Topos Theory", also by Lurie.

$(\infty, 1)$-categories have found their way into Physics, Logic, and other areas. Today, Differential Cohomology plays an essential role in modern Physics, and this is well understood in terms of $(\infty, 1)$-categories. Some references of this topic are [2, 31, 24]. In Logic, we have the rapidly growing field of Homotopy Type Theory, which uses concepts of ∞-category theory such as homotopy and ∞-groupoids, a fairly complete account of this subject can be found here [36]. For more on point results we have the classification of extended Topological Quantum Field Theories [21] and the existence of smash products in any stable ∞-category, which isn't the case for the homotopy category of spectra [22, 19, 24].

A

CATEGORY THEORY

The following serves as a brief discussion of all categorical tools assumed to be known during this text. This chapter is based of Riehl's "*Category Theory in Context*", [29], which contains a much more detailed account of the subjects studied here.

A.1 CATEGORIES

Definition A.1. A category $\mathscr{C}$ consists of the following data

(i) A collection [1] of objects $\mathrm{ob}(\mathscr{C})$.

(ii) A collection $\mathrm{Mor}(\mathscr{C})$, called the morphisms of $\mathscr{C}$, which satisfies the following properties:

- Each morphism has a distinguished domain and codomain, both of which are objects of $\mathscr{C}$. We denote by $\mathscr{C}(x, y)$ the collection of morphisms with domain x and codomain y.

- For every $x, y, z, w \in \mathrm{ob}(\mathscr{C})$ there is a function

$$\circ : \mathscr{C}(x, y) \times \mathscr{C}(y, z) \longrightarrow \mathscr{C}(x, z)$$

$$(f, g) \mapsto g \circ f$$

called the composition of morphisms, such that for every $h \in \mathscr{C}(z, w)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Moreover, for each x there is a morphism $\mathrm{id}_x \in \mathscr{C}(x, x)$, the identity of x, such that for every $b \in \mathrm{ob}(\mathscr{C})$, $f \in \mathscr{C}(x, y)$ and $g \in \mathscr{C}(y, x)$, we have

$$\mathrm{id}_x \circ g = g, \quad f \circ \mathrm{id}_x = f.$$

Most of the time we write $x \in \mathscr{C}$ to denote an object of $\mathscr{C}$ instead of $x \in \mathrm{ob}(\mathscr{C})$. □

Remark A.2. If f is a morphism with domain x and codomain y, we denote it by $f : x \to y$. Be warned that this is merely a notation, as morphisms are not necessarily functions!

[1] We will take the naive point of view on set theory. Often these "collections" do not take the form of sets in the "traditional" (ZFC) sense, but rather the form of *proper classes*. There are ways to deal with this, such as class formalism or the use of Grothendieck universes. For more details on these issues see [34].

Example A.3 (Some Examples)**.** Here are examples of categories most people have encountered

- **Set** is the category of sets. In it, objects are sets, morphisms are functions and the composition of morphisms is the composition of functions.
- **Grp** is the category of groups. Its objects are groups, and its morphisms are group homomorphisms.
- Let X be a set, and let $\mathscr{P}(X)$ be the power set of X. We define the category X, in which the $\mathrm{ob}(X) = \mathscr{P}(X)$ and the morphisms are the inclusions. This works because "containment" is a partial order, given a (non strict) partial order $(X, \leq)$ it's possible to define a category whose collection of objects in the set X and there is a morphisms $x \to y$ if and only if $x \leq y$.
- Let G be a group and $*$ any any set. We define the category BG as the category that has a single object $*$ and morphisms $\mathrm{Mor}(BG) = G$. This is a good example of a category whose morphisms are not functions.
- **Top** is the category of topological spaces and its morphisms are continuous functions.

Definition A.4. Let x, y be objects of $\mathscr{C}$. x is said to be isomorphic to y in $\mathscr{C}$ of there exists morphisms $f : x \to y$ and $g : y \to x$ such that

$$fg = \mathrm{id}_y, \quad gf = \mathrm{id}_x.$$

In this case we write $x \cong y$. □

Definition A.5 (Locally Small Category)**.** A small category is a category such that its collection of morphisms forms a set [2]. A category is said to be locally small if for any two objects x, y we have that $\mathscr{C}(x, y)$ is a set. □

A.2 DUALITY

Whenever we have a category, we automatically get another one for free, namely its dual category.

Definition A.6. Let $\mathscr{C}$ be a category, we define its dual category $\mathscr{C}^{op}$ in the following manner

- $\mathrm{ob}(\mathscr{C}^{op}) = \mathrm{ob}(\mathscr{C})$;
- $\mathscr{C}^{op}(x, y) = \mathscr{C}(y, x)$, in the sense that for every morphism $f : x \to y$ in $\mathscr{C}$, we have a morphism $f^{op} : y \to x$.

Note that the morphisms of $\mathscr{C}^{op}$ "are" the same ones from $\mathscr{C}$ but with their domains and codomains switched. □

Here is a definition in which a concept is clearly dual to another.

Definition A.7 (Terminal and Initial Objects)**.** An object $* \in \mathscr{C}$ is said to be terminal if for any object $x \in \mathscr{C}$ there exists exactly one morphisms

$$x \longrightarrow *.$$

[2] as opposed to a proper class.

Dually, an object $\varnothing \in \mathscr{C}$ is said to be initial if for every object $x \in \mathscr{C}$ there exists exactly one morphism

$$\varnothing \longrightarrow x.$$

These notations are due to the fact that in **Set** a set with a single element (which is final) is typically denoted by $*$ and the empty set is an initial object (via the empty function). □

To illustrate why duality is useful we prove a simple Lemma, but first a definition.

Definition A.8. Let $f : x \to y$ be a morphism, and z a object, then we define

- The post-composition $f_* : \mathscr{C}(z, x) \to \mathscr{C}(z, y)$ as the function

$$g \mapsto f \circ g$$

- Dually, the pre-composition $f^* : \mathscr{C}(y, z) \to \mathscr{C}(x, z)$ as the function

$$h \mapsto h \circ f$$

Lemma A.9. *Let $f : x \to y$ be a morphism, the following are equivalent:*

(i) $f : x \to y$ *is an isomorphism in* $\mathscr{C}$.

(ii) $f_* : \mathscr{C}(z, x) \to \mathscr{C}(z, y)$ *is a bijection for every* $z \in \mathscr{C}$.

(iii) $f^* : \mathscr{C}(y, z) \to \mathscr{C}(x, z)$ *is a bijection for every* $z \in \mathscr{C}$.

Proof: $(\mathrm{i}) \iff (\mathrm{ii})$

If f is an isomorphism, then it admits an inverse g, then the function $g_* : \mathscr{C}(z, y) \to \mathscr{C}(z, x)$ is the inverse f^*. Conversely, taking $z = y$, we have $f_* : \mathscr{C}(y, x) \to \mathscr{C}(y, y)$ is surjective, then for some morphism this composition yields the identity in y. The rest of the argument is analogous.
$(\mathrm{i}) \iff (\mathrm{iii})$
Taking the category $\mathscr{C}^{op}$ and the morphism $f^{op} : y \to x$ we have

$$f^{op}_* : \mathscr{C}^{op}(z, y) \longrightarrow \mathscr{C}^{op}(z, x)$$

now observe that f^{op}_* and f^* are equal, then the statement is reduced to the last one. Since we proved it for any category. ■

Definition A.10. Let $f : x \to y$ be a morphism, then

- f is a monomorphism if for any object z and morphisms $h, k : z \rightrightarrows x$, $f \circ h = f \circ k \implies h = k$.
- f is an epimorphism if for any object z and morphisms $h, k : y \rightrightarrows z$, $h \circ f = k \circ f \implies h = k$.

Observe that an (mono)epimorphism is a (epi)monomorphism in the dual category. □

Example A.11. In **Set**, the monomorphisms are injections and the epimorphisms are surjections.

A.3 FUNCTORS

Definition A.12 (Functors). A functor between $\mathscr{C}$ and $\mathscr{D}$ is rule which assigns

- to each object x of $\mathscr{C}$ an object $F(x)$ in $\mathscr{D}$.
- to each morphism $f : x \to y$ of $\mathscr{C}$ a morphism $F(f) : F(x) \to F(y)$, such that:

$$F(fg) = F(f)F(g) \quad \text{and} \quad F(\mathrm{id}_x) = \mathrm{id}_{F(x)}.$$

We write $F : \mathscr{C} \to \mathscr{D}$ to denote that F is a functor from $\mathscr{C}$ to $\mathscr{D}$. □

Sometimes functors "invert the arrows", so we define the following.

Definition A.13. A contravriant functor from $\mathscr{C}$ to $\mathscr{D}$ is a functor from $\mathscr{C}^{op}$ to $\mathscr{D}$, an ordinary functor is also called a covariant functor. □

Lemma A.14. *A functor preserves isomorphisms.*

Proof: Let $f : x \to y$ be an isomorphism and $g : y \to x$ be its inverse. Then

$$F(f)F(g) = F(fg) = F(\mathrm{id}_y) = \mathrm{id}_{F(y)}.$$

The converse is analogous. ■

The following are extremely important classes of functors.

Definition A.15 (Hom Functors). Let $\mathscr{C}$ be a locally small category and x an object of $\mathscr{C}$, then we define the covariant and contravariant functors

$$\mathscr{C}(x,-) : \mathscr{C} \longrightarrow \mathbf{Set}$$

$$\mathscr{C}(-,x) : \mathscr{C} \longrightarrow \mathbf{Set}$$

where for an object y, $\mathscr{C}(x,-)(y) := \mathscr{C}(x,y)$ and $\mathscr{C}(-,x)(y) = \mathscr{C}(y,x)$, and a morphism $f : y \to z$ is taken to its post-composition $f_* : \mathscr{C}(x,y) \to \mathscr{C}(x,z)$ or its pre-composition $f^* : \mathscr{C}(z,x) \to \mathscr{C}(y,x)$. In terms of diagrams

$$\begin{tikzcd} y \arrow[r, "\mathscr{C}(x,-)"] \arrow[d, "f"'] & \mathscr{C}(x,y) \arrow[d, "f_*"] \\ z \arrow[r, "\mathscr{C}(x,-)"'] & \mathscr{C}(x,z) \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "\mathscr{C}(-,x)"] \arrow[d, "f"'] & \mathscr{C}(y,x) \\ z \arrow[r, "\mathscr{C}(-,x)"'] & \mathscr{C}(z,x). \arrow[u, "f^*"'] \end{tikzcd}$$

These functors are called the covariant and contravariant, respectively, functors represented by x. □

A.4 NATURAL TRANSFORMATIONS

Definition A.16 (Natural Transformations). Let $F, G : \mathscr{C} \to \mathscr{D}$ functors, then a natural transformation from F to G is a family of morphisms $\{\alpha_x : F(x) \to G(x)\}_{x \in \mathscr{C}} \subset \mathrm{Mor}(\mathscr{D})$, such that

$$\begin{tikzcd} F(x) \arrow[r, "\alpha_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\alpha_y"'] & G(y) \end{tikzcd}$$

commutes for every morphism $f : x \to y$. The morphisms α_x are called the components of the natural transformation. We write $\alpha : F \Rightarrow G$ to indicate that α is a natural transformation from F to G. □

Example A.17.

- Let $F, G : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ be the cartesian product functor and the product inverting the order of the cartesian product, then we have natural transformation with components given by the map $x \times y \to y \times x$, $(a, b) \mapsto (b, a)$.
- Let G be a group, a functor from BG to $\mathbf{Set}$ is just a group action, then a natural transformation from functors with domain BG is just a G-equivariant function.

Definition A.18. A natural isomorphism is a natural transformation whose components are isomorphisms. If two functors F, G are naturally isomorphic we write $F \cong G$. □

Definition A.19. Let $\mathscr{C}, \mathscr{D}$ be categories, given $F, G, H : \mathscr{C} \to \mathscr{D}$ functors and natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ we define $\beta \circ \alpha : F \Rightarrow H$ to be the natural transformation whose components are $\beta_x \circ \alpha_x : F(x) \to H(x)$.

With this composition, we can turn the collection of functors from $\mathscr{C}$ to $\mathscr{D}$ into a category whose morphisms are natural transformations (it is easy to check associativity and existence of identities). This category is denoted by $\mathrm{Func}(\mathscr{C}, \mathscr{D})$. □

Remark A.20. Let **Cat** denote the category of small categories, then we see that, as a set, $\mathbf{Cat}(\mathscr{C}, \mathscr{D}) = \mathrm{Func}(\mathscr{C}, \mathscr{D})$, thus we sometimes also use $\mathbf{Cat}(\mathscr{C}, \mathscr{D})$ to denote the category of functors from $\mathscr{C}$ to $\mathscr{D}$, at least when they are small categories.

Definition A.21. A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be an equivalence of categories if there exists a functor $G : \mathscr{D} \to \mathscr{C}$ such that

$$F \circ G \cong \mathrm{id}_{\mathscr{D}} \quad \text{and} \quad G \circ F \cong \mathrm{id}_{\mathscr{C}}.$$

In this case we say that $\mathscr{C}$ is equivalent to $\mathscr{D}$. □

Definition A.22 (Full and Faithful Functors). Given a functor $F : \mathscr{C} \to \mathscr{D}$ and objects $x, y \in \mathscr{C}$ we have a function

$$F : \mathscr{C}(x, y) \longrightarrow \mathscr{D}(F(x), F(y)).$$

If this function is surjective for every x and y the functor is said to be full and if it's injective for every x and y the functor is said to be faithful. □

Definition A.23. A functor

$$F : \mathscr{C} \longrightarrow \mathscr{D}$$

is essentially surjective if for every $y \in \mathscr{D}$ there exists $x \in \mathscr{C}$ such that $F(x) \cong y$. □

Theorem A.24. *Every fully faithful essentially surjective functor establishes an equivalence of categories.*

Proof: Let $F : \mathscr{C} \to \mathscr{D}$ be a fully faithful essentially surjective functor. For $d \in \mathscr{D}$ chose Gd such that $FGd \cong d$ and an isomorphism $\epsilon_d : FGd \cong d$. For each $l : d \to d'$ it can be shown that there exists an unique morphism making the following diagram commute

$$\begin{tikzcd} FGd \arrow[r, "\epsilon_d"] \arrow[d, dotted] & d \arrow[d, "l"] \\ FGd' \arrow[r, "\epsilon_{d'}"'] & d' \end{tikzcd}$$

this unique morphism is defined to be Gl. This is set up so that each isomorphism ϵ_d becomes the components of a natural isomorphism $\epsilon : FG \Rightarrow \mathrm{id}_{\mathscr{D}}$. It is readily checked that the assignment $l \mapsto Gl$ is functorial. Now it remains to define the natural isomorphism $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow GF$. We may define the components of η by specifying morphisms $F\eta_c : Fc \to FGFc$ because F is full and faithful. So we define $F\eta_c$ to be ϵ_{Fc}^{-1}, for any $f : c \to c'$ we have that the outer square of

$$\begin{tikzcd} Fc \arrow[r, "F\eta_c"] \arrow[d, "f"'] & FGFc \arrow[r, "\epsilon_{Fc}"] \arrow[d, "FGFf"'] & Fc \arrow[d, "Ff"] \\ Fc' \arrow[r, "F\eta_c"'] & FGFc' \arrow[r, "\epsilon_{Fc'}"'] & Fc' \end{tikzcd}$$

commutes. The right square commutes by naturality while the left square commutes because ϵ_{Fc} is an isomorphism. Therefore, the faithfulness of F implies that $\eta_{c'} \cdot f = GFf \cdot \eta_c$ and we conclude that η is a natural isomorphism. ■

A.5 YONEDA'S LEMMA

Yoneda's Lemma is a powerful tool to study representable functors.

Definition A.25 (Representable Functors)**.** Let $\mathscr{C}$ be a locally small category and $F : \mathscr{C} \to \mathbf{Set}$ a functor from $\mathscr{C}$ to **Set**, then we say that F is representable if there exists $x \in \mathscr{C}$ such that

$$F \cong \mathscr{C}(x, -)$$

Dually a contravariant functor is representable if there exists $x \in \mathscr{C}$ such that $F \cong \mathscr{C}(-, x)$. □

One thing one might ask is when do we know that a functor is representable, so we need to know when there are natural transformations between $\mathscr{C}(x, -)$ for given $x \in \mathscr{C}$ and a functor $F : \mathscr{C} \to \mathbf{Set}$, this is exactly what Yoneda's Lemma tells us.

Theorem A.26 (Yoneda's Lemma)**.** *Let $\mathscr{C}$ be a locally small category. Then for any functor $F : \mathscr{C} \to \mathbf{Set}$ and any $x \in \mathscr{C}$ there is a natural bijection*

$$\mathrm{Nat}(\mathscr{C}(x, -), F) \cong F(x)$$[3]

which is given by

$$(\alpha : \mathscr{C}(x, -) \Rightarrow F) \mapsto \alpha_C(\mathrm{id}_x) \in F(x)$$

Proof: We already specified map, now we need to show that it is indeed a bijection. We have the function

$$\Phi : \mathrm{Nat}(\mathscr{C}(x, -), F) \longrightarrow F(x)$$
$$\Phi(\alpha) := \alpha_x(\mathrm{id}_x)$$

for which we will construct an inverse function

$$\Psi : F(x) \longrightarrow \mathrm{Nat}(\mathscr{C}(x, -), F).$$

[3] Here $\mathrm{Nat}(\mathscr{C}(x, -), F)$ denotes the collection of natural transformations from $\mathscr{C}(x, -)$ to F, of course we could've used $\mathrm{Func}(\mathscr{C}, \mathbf{Set})(\mathscr{C}(x, -), F))$, but this would make the notation too cumbersome.

We will do this by specifying the components of Ψ for given $a \in F(x)$. For $f : x \to y$ The diagram below

$$\begin{tikzcd} \mathscr{C}(x,x) \arrow[r, "\Psi(a)_x"] \arrow[d, "f_*"'] & F(x) \arrow[d, "F(f)"] \\ \mathscr{C}(x,y) \arrow[r, "\Psi(a)_y"'] & F(y) \end{tikzcd}$$

forces us to define

$$\Psi(a)_y(f) := F(f)(a).$$

Indeed, the image of id_x under the bottom left composition is $\Psi(a)_y(f) \in F(x)$ while its image by the top right composition is $F(f)(\Psi(a)_x(id_x))$, so if we want Ψ to be an inverse for Φ we need to set $\Psi(a)_x(id_x) = a$, and the naturality of the square takes care of the rest. So by construction, we have that $\Psi(a)_y$ commutes when it is in a square with $\Psi(a)_x$, now we need to show that

$$\begin{tikzcd} \mathscr{C}(x,y) \arrow[r, "\Psi(a)_y"] \arrow[d, "g_*"'] & F(y) \arrow[d, "F(g)"] \\ \mathscr{C}(x,z) \arrow[r, "\Psi(a)_z"'] & F(z) \end{tikzcd}$$

commutes for every $g : y \to z$. The image of $f : x \to y$ along the bottom composition is $\Psi(a)_y(gf) := F(gf)(a)$ while the top composition yields $F(g)(\Psi(a)_y(f)) = F(g)(F(f)(a))$, by the functionality of F ($F(gf) = F(g)F(f)$) we have that the diagram commutes. Now it remains to show that these functions really are the inverses of one another. To show that $\Psi\Phi(\alpha) = \alpha$ we just need to show that their components are equal. By definition, $\Psi(\alpha_x(id_x))_y(f) = F(f)(\alpha_x(id_x))$, the diagram

$$\begin{tikzcd} \mathscr{C}(x,x) \arrow[r, "\alpha_x"] \arrow[d, "f_*"'] & F(x) \arrow[d, "F(f)"] \\ \mathscr{C}(x,y) \arrow[r, "\alpha_y"'] & F(y) \end{tikzcd}$$

implies that $F(f)(\alpha_x(id_x)) = \alpha_y(id_x)$, showing that Ψ is a left inverse for Φ. Ψ is a right inverse to Φ by construction, so the result follows. ■

One great thing about Yoneda's Lemma is that it captures the idea that an object is determined by the morphisms into it or out of it, this is the content of Yoneda's Embedding.

Corollary A.27 (Yoneda's Embedding)**.** *The functors*

$$\mathscr{C} \hookrightarrow \mathrm{Func}(\mathscr{C}^{op}, \mathbf{Set}) \qquad \mathscr{C}^{op} \hookrightarrow \mathrm{Func}(\mathscr{C}, \mathbf{Set})$$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & \mathscr{C}(-,x) \arrow[d, "f_*"'] \\ y \arrow[r] & \mathscr{C}(-,y) \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & \mathscr{C}(x,-) \\ y \arrow[r] & \mathscr{C}(y,-) \arrow[u, "f^*"'] \end{tikzcd}$$

define full and faithful embeddings.

Proof: We need to show that there is a bijection

$$\mathscr{C}(x,y) \cong \mathrm{Nat}(\mathscr{C}(-,x), \mathscr{C}(-,y)).$$

Is easy to see that different morphisms $x \to y$ induce different natural transformations. By Yoneda's Lemma, natural transformations

$$\alpha : \mathscr{C}(y, -) \Rightarrow \mathscr{C}(x, -)$$

correspond to elements of $\mathscr{C}(x, y)$, that is, to morphisms $f : x \to y$, where f is $\alpha_y(\mathrm{id}_y)$. The natural transformationf^* defined by pre-composition by f sends id_y to f. Thus, the bijection implies that $\alpha = f^*$. ■

As a result of Yoneda's Lemma we also have a nice way of defining universal properties properly.

Definition A.28 (Universal Properties)**.** A universal property of an object x is a functor F represented by x together with an appropriate universal element $a \in F(x)$ that specifies the natural isomorphism. □

Even though we will not use the above definition explicitly, it is very important, especially to define what limits/colimits are in terms or cones rigorously.

A.6 LIMITS AND COLIMITS

One special case of representable functors are the so called limits and colimits. A huge amount of mathematical concepts can be seen as particular instances of such functors, so they serve as a great tool in translating mathematical concepts from one setting to another. Throughout $\mathscr{C}$ will be a locally small category.

Definition A.29. Let J be a category. A diagram of shape J in $\mathscr{C}$ is a functor $F : J \to \mathscr{C}$. A diagram is said to be small if the indexing category J is small. □

Definition A.30 (Cones and Cocones)**.** Let $x \in \mathscr{C}$ be an object in a category $\mathscr{C}$ and let $F : J \to \mathscr{C}$ be a (small) diagram. A cone over F with apex x is a is a set of morphisms $\{\lambda_i : x \to F(i)\}_{i \in J}$ such that for every morphism $f : i \to j \in \mathrm{Mor}(J)$ the diagram

$$\begin{tikzcd} x \arrow[d, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ F(i) \arrow[r, "Ff"'] & F(j) \end{tikzcd}$$

commutes. Dually, a cocone under F with nadir (or bottom) C is a set of morphisms $\{\lambda_i : F(i) \to x\}_{i \in J}$ such that for every morphism $f : i \to j$ the diagram

$$\begin{tikzcd} F(i) \arrow[r, "Ff"] \arrow[d, "\lambda_i"'] & F(j) \arrow[dl, "\lambda_j"] \\ x & \end{tikzcd}$$

commutes. Observe that since J is small, such collections are indeed sets. □

Remark A.31. Note that for any object x of $\mathscr{C}$ and any category J there is a functor $x : J \to \mathscr{C}$ defined as the constant functor at x, that sends every object to x and every morphism to id_x. Thus, for a functor $F : J \to \mathscr{C}$, a cone over F with apex x is simply a natural transformation $\lambda : x \Rightarrow F$ while a cocone with nadir x is a natural transformation $\lambda : F \Rightarrow x$.

Definition A.32. For every small diagram $F : J \to \mathscr{C}$ there is a functor

$$\mathrm{Cone}(-, F) : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

that sends an object x to the set of cones over F with apex x and sends a morphism $f : x \to y$ to the function that takes a natural transformation $\lambda : x \Rightarrow F$ to the natural transformation $\lambda \circ f : y \Rightarrow F$. Dually, there is a functor

$$\mathrm{Cocone}(F, -) : \mathscr{C} \longrightarrow \mathbf{Set}$$

that sends an object x to the set of cocones under F with nadir x and sends a morphism $f : x \to y$ to the function that takes a natural transformation $\lambda : F \Rightarrow x$ to the natural transformation $f \circ \lambda : F \Rightarrow y$. □

Remark A.33. Note that the requirement that $\mathscr{C}$ is locally small and J is small is important, because they guarantee that the above functors indeed take value in **Set**.

Definition A.34 (Limits and Colimits). Let $F : J \to \mathscr{C}$ be a small diagram, then a limit for F is representation for $\mathrm{Cone}(-, F)$. The chosen object for the representation is denoted $\lim F$ and, by Yoneda's Lemma, the chosen natural isomorphism

$$\mathrm{Cone}(-, F) \cong \mathscr{C}(-, \lim F)$$

has a universal cone $\lambda : \lim F \Rightarrow F$ specifying it. Dually, a colimit for F is representation for $\mathrm{Cocone}(F, -)$. The chosen object for the representation is denoted $\mathrm{colim}\, F$ and, by Yoneda's Lemma, the chosen natural isomorphism

$$\mathrm{Cocone}(F, -) \cong \mathscr{C}(\mathrm{colim}\, F, -)$$

has a universal cocone $\lambda : F \Rightarrow \mathrm{colim}\, F$ specifying it. □

As we already know, a representation is the same as a universal property. So a limit for a diagram $F : J \to \mathscr{C}$ is an object $\lim F \in \mathscr{C}$ together with a set of morphisms $\{\lambda_i : \lim F \to F(i)\}_{i \in J}$ (its universal cone) with the commuting property that satisfy the following universal property: for any object $x \in \mathscr{C}$ and a set of morphisms $\{\xi_i : \lim F \to F(i)\}_{i \in J}$ satisfying the commuting property there exists a unique morphism $x \to \lim F$ such that the diagram

$$\begin{tikzcd}
 & x \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "\xi_i"'] \arrow[ddr, bend left, "\xi_j"] & \\
 & \lim F \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
F(i) \arrow[rr] & & F(j)
\end{tikzcd}$$

commutes. Dually, the diagram below describes the universal property of the colimit

$$\begin{tikzcd}
F(i) \arrow[rr] \arrow[dr, "\xi_i"] \arrow[ddr, bend right, "\lambda_i"'] & & F(j) \arrow[dl, "\xi_j"'] \arrow[ddl, bend left, "\lambda_j"] \\
 & x & \\
 & \mathrm{colim}\, F. \arrow[u, phantom] &
\end{tikzcd}$$

Example A.35. Let $\varnothing$ be the empty category, and $F : \varnothing \to \mathscr{C}$ the empty diagram, then a limit of F is simply an initial object in $\mathscr{C}$ while a colimit of F is a terminal object in $\mathscr{C}$.

The first thing one might ask is when do limits/colimits exist, and the answer is not always straightforward, fortunately we will see that they always exist in **Set**.

Definition A.36. A category $\mathscr{C}$ is said to be complete if every small limit exists. Dually, it is cocomplete if every small colimit exists. □

Remark A.37. By the last example, if a category is complete then it has an initial object, dually, it it has cocomplete , it has a terminal object.

Theorem A.38. *The category* **Set** *is a complete and cocomplete category.*

Proof: See [29], theorems 3.2.6 and 3.4.12. ■

An even stronger, yet standard result, is that for any small category $\mathscr{C}$, the category $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set})$ is complete and cocomplete.

A.7 ADJUNCTIONS

Now we see the last definition we will need from the theory of categories.

Definition A.39 (Adjunctions). Let $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ be functors. These pair of functors is called an adjunction if for every $y \in \mathscr{D}$ and $x \in \mathscr{C}$ there is a natural bijection

$$\mathscr{C}(F(x), y) \cong \mathscr{D}(x, G(y)).$$

In this case, we say that F is left adjoint to G and that G is right adjoint to F. □

The naturality condition in the definition is in the sense these bijections become a natural isomorphism of functors as in the diagram below:

$$\begin{tikzcd} \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \arrow[r, bend left, "\mathscr{D}(F(-){,}-)"] \arrow[r, bend right, "\mathscr{C}(-{,}G(-))"'] \arrow[r, phantom, "\Downarrow\cong"] & \mathbf{Set}. \end{tikzcd}$$

Of course this just makes sense for locally small categories, but we could reformulate it in terms that didn't need the definition of this functor.

Remark A.40. Let $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ be an adjunction with F left adjoint to G. We write $F \dashv G$ to indicate that F is a left adjoint to G, or that G is right adjoint to F. We also implement this notation in diagrams:

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

Definition A.41. Given an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

we have the bijections

$$\mathscr{C}(F(x), y) \cong \mathscr{D}(x, G(y)).$$

The image of a morphism $f^\sharp \in \mathscr{C}(F(x), y)$ under this bijection is denoted $f^\flat$. $f^\sharp$ and $f^\flat$ are called the transpose of each other. □

Lemma A.42. *The naturality of the bijections of an adjuration can be formulated as follows: for any pair diagrams of the form*

$$\begin{tikzcd} F(x) \arrow[r, "f^\sharp"] \arrow[d, "Fh"'] & y \arrow[d, "k"] \\ F(x') \arrow[r, "g^\sharp"'] & y' \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f^\flat"] \arrow[d, "h"'] & G(y) \arrow[d, "Gk"] \\ y' \arrow[r, "g^\flat"'] & G(y') \end{tikzcd}$$

the right square commutes if and only if the left one does as well.

Proof: Note that naturality in Definition A.39 amounts in saying that for any $f^\sharp : F(x) \to y$ and $k : y \to y'$, the transpose of $k \circ f^\sharp : F(x) \to y'$ is the composite of $f^\flat : x \to G(y)$ with $Gk : G(y) \to G(y')$ which is one side of the proof, the other side is dual to this. ∎

Definition A.43 (Units and Counits)**.** For an adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

we can fix $x \in \mathscr{C}$ and by the definition of an adjunction the object $F(x) \in \mathscr{D}$ represents the functor $\mathscr{C}(x, G(-)) : \mathscr{D} \to \mathbf{Set}$ via the bijection

$$\mathscr{D}(F(x), -) \cong \mathscr{C}(x, G(-))$$

so by Yoneda's Lemma this bijection is determined by an element of $\mathscr{C}(x, GF(x))$, the transpose of $\mathrm{id}_{F(x)}$, denoted α_x. So for each $x \in \mathscr{C}$, we have a morphism α_x representing the adjunction. The collection $\{\alpha_x : x \to x\}$ assembles into a natural transformation

$$\alpha : \mathrm{id}_{\mathscr{C}} \Rightarrow GF.$$

The naturality of the above construction is guaranteed by the naturality of the adjunction. Dually, we could've fixed $y \in \mathscr{D}$ and gotten a natural transformation

$$\beta : FG \Rightarrow \mathrm{id}_{\mathscr{D}}.$$

These transformations are called the unit and counit of the adjunction. □

Remark A.44. It is possible to define adjunctions equivalently in terms of units and counits, but for our purposes this won't be necessary, a complete discussion of this concept can be found at [29].

Now we move on to the last result we need about adjunctions, an extremely useful one.

Theorem A.45. *Let $F : \mathscr{C} \to \mathscr{D}$ be right adjoint function, then F preserves limits, in the sense that for a diagram*

$$H : J \longrightarrow \mathscr{C}$$

with limit $\lim H$*, then* $F(\lim H)$ *is a limit for the diagram*

$$FH : J \longrightarrow \mathscr{D}.$$

Dually, if F is left adjoint then F preserves colimits.

Proof: See [29] Theorem 4.5.2, p.136. ∎

Remark A.46. The above result is an extremely useful result, so much so that they receive acronyms : RAPL and LAPC.

Functors that preserve limits are said to be continuous while colimit preserving functors are called cocontinuous. That being said, the above theorem states that continuity is a necessary condition for a functor to be adjoint.

A.8 SMALL OBJECTS

Here we give a very brief introduction on some constructions related to the "size" of objects. This discussion is very technical and its goal is to state the Quillen Small Object Argument, a powerful tool for constructing factorization systems.

Definition A.47. A cardinal κ is said to be regular if k is infinite and can't be written as a union of fewer than κ sets with cardinality less than κ. □

Example A.48. $\mathbb{N}$ is a regular cardinal, since a finite union of finite sets is finite.

Definition A.49. A poset P is said to be κ-filtered if every subset $P' \subset P$ with cardinality less than κ has an upper bound in P. □

Example A.50. $\mathbb{N}$ is an $\mathbb{N}$-filtered poset.

Definition A.51. Let $\mathscr{C}$ be a locally small cocomplete category and κ a regular cardinal, then an object $x \in \mathscr{C}$ is said to be κ-small if there for any k-filtered poset P and any functor $F : P \to \mathscr{C}$ the induced map

$$\operatorname{colim} \mathscr{C}(x, F(-)) \longrightarrow \mathscr{C}(x, \operatorname{colim} F)$$

is a bijection. Call an object x small if it is κ-small for some regular cardinal κ. □

Above, the map is induced by the universal property of the colimit

$$\begin{tikzcd}
\dots \arrow[r] & \mathscr{C}(x,F(j)) \arrow[r] \arrow[dr] \arrow[ddr, bend right] & \mathscr{C}(x,F(l)) \arrow[r] \arrow[d] \arrow[dd, bend right] & \mathscr{C}(x,F(m)) \arrow[r] \arrow[dl] \arrow[ddl, bend left] & \dots \\
& & \operatorname{colim} \mathscr{C}(x,F(-)) \arrow[d, dashed, "\exists!"] & & \\
& & \mathscr{C}(x, \operatorname{colim} F) & &
\end{tikzcd}$$

The requirement of the map $\operatorname{colim} \mathscr{C}(x, F(-)) \to \mathscr{C}(x, \operatorname{colim} F)$ being surjective is equivalent to asking that every morphism $f : x \to \operatorname{colim} F$ factors through the colimit diagram:

$$\begin{tikzcd}
\dots \arrow[r] & F(j) \arrow[r] & F(m) \arrow[r] \arrow[d] & \dots \\
& x \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & \operatorname{colim} F &
\end{tikzcd}$$

With this in mind, and for our needs, we make a more general definition:

Definition A.52. Let $\mathscr{C}$ be a cocomplete category and let $I \subset \operatorname{Mor}(\mathscr{C})$ be any collection of morphisms from $\mathscr{C}$, then we say that an object x is small with relative to I if there is an ordinal α such that for every morphism $x \to Y$, where $X \to Y$ is an I-relative cell complex factors as

$$\begin{tikzcd}
X = Y_0 \arrow[r] & \dots \arrow[r] & Y_\beta \arrow[r] & \dots \arrow[r] & Y \\
& & x \arrow[u, dashed, "\tilde{f}"] \arrow[urr, "f"'] & &
\end{tikzcd}$$

at a stage $\beta < \alpha$. □

Theorem A.53 (Small Object Argument)**.** *Let $\mathscr{C}$ be a cocomplete category and I be class of morphisms of $\mathscr{C}$. Suppose that each domain of the morphisms in I are small relative to I.Then every morphism $f : x \to y$ has a factorization of the form*

$$f : x \xrightarrow{\in \mathtt{cell}(I)} \tilde{x} \xrightarrow{\in \mathsf{RLP}(I)} y$$

where $\mathtt{cell}(I)$ is the collection of I-relative cell complexes and $\mathsf{RLP}(I)$ is the collection of morphisms that have the right lifting property against I.

This is an extremely useful tool for model category theory, since it helps us construct model structures, as shown in Section 2.5 of Chapter 2. A proof of this result can be found on [13].

Another quite useful result is the following:

Theorem A.54. *Let $\mathscr{C}$ be a small category and let $\mathrm{Func}(\mathscr{C}^{op}, \mathbf{Set})$ be its category of presheaves, then all objects in $\mathrm{Func}(\mathscr{C}^{op}, \mathbf{Set})$ are small.*

Actually, the category $\mathrm{Func}(\mathscr{C}^{op}, \mathbf{Set})$ is always a *locally presentable* category, which is a stronger result statement. For details see [1].

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