Constraints between equations of state and mass-radius relations in general clusters of stellar systems

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(Received 28 September 2018; published 9 January 2019)

We prove three obstruction results on the existence of equations of state in clusters of stellar systems fulfilling mass-radius relations and some additional bound (on the mass, on the radius or a causal bound). The theorems are proved in great generality. We start with a motivating example of TOV systems and apply our results to stellar systems arising from experimental data.

DOI: 10.1103/PhysRevD.99.023007

I. INTRODUCTION

Since the works [1,2] of Chandrasekhar, the mathematical foundations of astrophysics have shown their importance and are being developed parallel to (and sometimes as part of) general relativity theory. In this article we focus on a specific topic of the mathematical foundations: the *axiomatization problem* (for a deep philosophical discussion see [3]). This means that we consider the question of which kind of objects can be used to model an astrophysical system. As a consequence, we obtain obstruction conditions constraining the existence of certain stellar systems.

In order to motivate our results, let us first see how this kind of obstruction appears in typical physical systems. Recall that the structure of general relativistic and spherically symmetric isotropic stars is modeled by the Tolman-Oppenheimer-Volkoff (TOV) equations [4], described in terms of its density ρ and pressure p (setting $G = 1$ and $c = 1$):

$$p'(r) = -\frac{(\rho(r) + p(r))(M(r) + 4\pi r^3 p(r))}{r^2(1 - \frac{2M(r)}{r})} \tag{1}$$

$$M'(r) = 4\pi r^2 \rho(r). \tag{2}$$

These equations are partially uncoupled and can be coupled through an equation of state. A typical example is the polytropic equation of state

$$f(p, \rho, k, k_0, \gamma) = p - k\rho^\gamma - k_0 = 0, \tag{3}$$

where $k, k_0 \in \mathbb{R}$ are the *polytropic constant* and the *stiffness constant*, respectively, and $\gamma = (n+1)/n \in \mathbb{Q}$ is the *polytropic exponent*. Since $f(p, \rho) = 0$ can obviously be solved for p and ρ, it can be used to couple a TOV system. Under this coupling, we say that the pair (p, ρ) corresponds to a *polytropic TOV system*.

In thermodynamic systems where pressure and density are related by an integrable equation of state, the speed of sound within the system can be defined as $v \coloneqq \sqrt{\partial p/\partial \rho}$. It follows from (3) that $v = \sqrt{k\gamma p^{\gamma-1}}$ in polytropic TOV systems. A TOV system is *causal* if the speed of sound is less than the speed of light, that is, $k\gamma\rho^{\gamma-1} < 1$. In particular,

$$k\gamma\rho_c^{\gamma-1} < 1, \tag{4}$$

where $\rho_c = \rho(0)$ is the density at the center.

Now, assume a star of radius R is modeled by a TOV system and let the *mass of the system* be $M = M(R)$, where $M(r)$ is the function given by (2). If this star is small, then it usually admits *mass-radius relations* $g(M, R, \delta) = 0$, relating M to R and other parameters [2]. For instance, it is known that in the Newtonian limit polytropic TOV systems satisfy

$$g(M, R, k, n, \rho_c) = M - A(k, n, \rho_c)R^2, \tag{5}$$

where

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$$A(k,n,\rho_c) = \left(\frac{4\pi}{(n+1)k}\right)^{3/2} \rho_c^{(n-3)/2n} \frac{|p'(R)|}{\rho_{\rm rel}}$$

and $\rho_{\rm rel} = \rho(R)/\rho_c$ is the *relative density* [2,4].

By means of isolating ρ_c in (5) and substituting the value found in (4), one can show that there are obstructions for a Newtonian polytropic TOV system to be causal. Indeed, the polytropic constant must satisfy

$$k < \frac{n}{n+1}\left(\frac{R^4}{n^3M^2}\beta\right)^{\frac{1}{n}}, \quad \text{where } \beta = 64\pi^3\left(\frac{|p'(R)|}{\rho_{\rm rel}}\right)^2.$$

The main aim of this article is to show that this kind of obstruction exists not only in TOV systems, but actually in generic clusters of stellar systems. In Sec. II we define what we mean by a *cluster* of stellar systems. Roughly speaking, for us a stellar system is a pair of functions describing the density and the pressure internal to some star, so that a cluster is just an arbitrary set of these systems (of which TOV systems are particular examples). In that same section we also formulate the equations of state and mass-radius relations in an axiomatic framework. Then, in Sec. III, we state and prove three different obstruction theorems, whose statements have the following common structure:

Theorem (Roughly). Consider a small radius stellar system with a mass-radius relation $g(M,R,\epsilon) = 0$. Each constraint on ϵ induces an obstruction on the possible equations of state, depending on ϵ, that can be introduced in that system.

The TOV systems were used as a motivating example. We notice that this claim (about generic systems) is reasonable. In general, mass-radius relations are closely related to the atomic nature of the star, as stellar systems with different atomic constitutions obey different relations [2,5]. Reciprocally, constraints on the parameters of a mass-radius relation provide information about the atomic structure of the system. For instance, Newtonian polytropic stars satisfying (5), but not the Chandrasekhar limit (resp. Oppenheimer-Volkoff limit) cannot be stable white dwarfs (resp. stable neutron stars); there are also bounds on n, of course [4]. On the other hand, equations of state arise from the statistical mechanics of the atomic structure. So, mass-radius relations with constraints restrict the atomic structure and, therefore, the possible equations of state, which is precisely the content of the above theorem.

Finally, in Sec. IV we apply our obstruction theorems to stellar systems arising from experimental data and in Sec. V we end this paper with some concluding remarks.

II. DEFINITIONS

A *stellar system* is defined as a pair (p,ρ) of real piecewise differentiable functions defined on an interval $I \subset \mathbb{R}$, possibly unbounded. It is natural to define a *cluster of stellar systems* of degree (k,l) as a vector subspace $\mathrm{Stellar}^{kl}(I)$ of $C^k_{pw}(I) \times C^l_{pw}(I)$, where $C^k_{pw}(I)$ denotes the vector space of piecewise C^k differentiable functions on I. It is important to notice that $C^k_{pw}(I)$ has the canonical generalized norm $\|f\|_k \coloneqq \sup_I |f^{(k)}(t)|$, which is a functional satisfying the norm axioms but possibly taking infinite values. Like classical norms, generalized norms induce a topology which can be characterized as the finest locally convex topology that makes sum and scalar multiplication continuous [6]. Therefore, $\mathrm{Stellar}^{kl}(I)$ has a canonical locally convex space structure.

We are interested in stellar systems having well-defined notions of mass and radius. This leads us to consider clusters of stellar systems that become endowed with maps $M,R\colon \mathrm{Stellar}^{kl}(I) \to \mathbb{R}$ assigning to each system (p,ρ) its mass and radius. In TOV systems, $M(p,\rho)$ is given by (2), and its inverse is $R(p,\rho)$. Looking at these expressions, we see that it is natural to assume M and R to be at least piecewise continuous, i.e., $M,R \in C^0_{pw}(\mathrm{Stellar}^{kl}(I))$. We say that a (locally convex) subspace $\mathrm{Bound}^{kl}(I) \subset \mathrm{Stellar}^{kl}(I)$ has *mass bounded from above* (resp. *radius bounded from above*) if when restricted to it the function M (resp. R) is bounded from above. Similarly, we define subspaces with mass and radius bounded from below.

A *function of state* for a cluster of stellar systems is a function $f\colon \mathrm{Stellar}^{kl}(I) \times E \to \mathbb{R}$, where E is a topological vector space of parameters. We say that a function of state is *locally integrable at* p if the corresponding equation of state $f(p,\rho,\epsilon) = 0$ can be locally solved for p. This means that there exists a neighborhood U of p, open sets $V \subset C^l_{pw}(I)$ and $W \subset E$, and a function $\xi\colon V \times W \to U$ such that $f(\xi(\rho,\epsilon),\rho,\epsilon) = 0$. Additionally, if ξ is monotone in both variables ρ, ϵ we say that f is a *monotonically locally integrable (MLI)* function of state.

When a locally integrable function of state f is such that ξ is differentiable (in some sense to be specified below), we can define the *squared speed of sound* as $v^2 = \partial_\rho \xi$. We are interested in situations in which both ξ and v are monotone with respect to both variables. A function of state satisfying these properties is called a *fully monotonically locally integrable* function of state.

The notion of differentiability of f depends on the topological nature of both $\mathrm{Stellar}^{kl}(I)$ and the space of parameters E, leading to different generalizations of the implicit function theorem (IFT). These versions of the IFT imply that any function whose derivative in the direction of p satisfies mild conditions is locally integrable.

We remark that some classical generalizations of IFT, such as the IFT for Banach spaces [7] and the Nash-Moser Theorem [8] for tame Fréchet spaces, cannot be used here, because $\mathrm{Stellar}^{kl}(I)$ is neither Banach nor Fréchet. General contexts that apply here are when E is an arbitrary topological vector space and, more concretely, when E is locally convex (see [9] and [10], respectively). We will work in an intermediate context: when E is Banach, so that by means of embedding $\mathbb{R}$ in E we can regard f as a map

$f\colon \mathrm{Stellar}^{kl}(I)\times E\to E$ and define the directional derivative $\partial_p f$ as usual for maps from a locally convex to a Banach space, without the technicalities needed in the more general situations. As proved in [11], the classical IFT holds for $f\colon X\times E\to E$ when X is locally convex and E is Banach, meaning that if $\partial_p f\neq 0$, then f is a locally integrable function of state in a neighborhood of that point and ξ is differentiable (in a certain generalized sense). This IFT also gives an expression for the derivative of ξ at each point, which depends on the derivative of f with respect to the other variables at points of the neighborhood where ξ is defined, and furthermore establishes that ξ is monotone on any neighborhood where both partial derivatives do not vanish.

A *mass-radius function* for a cluster of stellar systems is a function

$$g\colon\ C^0_{pw}(\mathrm{Stellar}^{kl}(I))\times C^0_{pw}(\mathrm{Stellar}^{kl}(I))\times F\to\mathbb{R},$$

where F is a topological vector space of parameters, such that the mass-radius relation $g(M,R,\delta)=0$ can be locally solved for some δ, so that we can locally write $\delta=\eta(M,R)$. If η is monotone, we say that g is a *locally monotone mass-radius function*. As in the previous case, if F is Banach, then a mass-radius function can be obtained by requiring $\partial_\delta g\neq 0$, and can be guaranteed to be locally monotone if the other partial derivatives also do not vanish.

III. STATEMENT AND PROOF

We can now state and prove our obstruction theorems.

Theorem 1. Let $\mathcal{C}=(\mathrm{Stellar}^{kl}(I),M,R)$ be a cluster of stellar systems of degree (k,l) endowed with a locally monotone mass-radius function $g(M,R,\epsilon)$. Then any upper (resp. lower) bound on the mass and on the radius induces a bound on each MLI function of state of $\mathcal{C}$ depending on ϵ (and possibly on other parameters). If a function of state is fully MLI at some p, then the bounds induce bounds on the speed of sound near p.

Proof. After the previous discussion, the proof becomes easy. We only work with upper bounds; the proof for lower bounds is essentially the same. By definition, the mass-radius relation $g(M,R,\epsilon)=0$ can be solved in a neighborhood of some ϵ, that is, locally $\epsilon=\eta(M,R)$. Since M and R are bounded, say $M\leq m$ and $R\leq r$, and g is locally monotone, we see that $\eta(M,R)\leq\eta(m,R)\leq\eta(m,r)$. Therefore, ϵ is also bounded, say by ϵ_0. Now, notice that if a parameter ϵ is bounded then any monotone function $f(x,\epsilon)$ depending on that parameter is bounded by the function $h(x)=f(x,\epsilon_0)$. In particular, any MLI function of state $f(p,\rho,\epsilon)$ for the cluster $\mathcal{C}$ depending on ϵ has an upper bound. Furthermore, if f is fully MLI at some p, then the squared speed of sound $v(\rho,\epsilon)^2$ near p is well defined and it also depends monotonically on ϵ, so that it also has an upper bound. □

The above theorem only applies for clusters whose function of state and mass-radius function depend on the same parameter. This hypothesis can be avoided by adding a continuity equation, defined as follows. Let $\mathcal{C}=(\mathrm{Stellar}^{kl}(I),M,R)$ be a cluster of stellar systems in which M is at least piecewise C^1. A *continuity equation* for $\mathcal{C}$ is an ordinary differential equation $M'(R)=F(R,\rho)$. We will only work with continuity equations which can be locally solved for ρ. The basic example is Eq. (2).

Theorem 2. Let $\mathcal{C}=(\mathrm{Stellar}^{kl}(I),M,R)$ be a cluster of stellar systems of degree (k,l) endowed with a locally monotone mass-radius function $g(M,R,\delta)$ and a continuity equation. Then any upper (resp. lower) bound on the derivative of the mass and on the radius induces a bound on each MLI function of state of $\mathcal{C}$. If a function of state is fully MLI at some p, then bounds are induced on the speed of sound near p.

Proof. The proof is similar. Because the mass-radius function $g(M,R,\delta)$ is locally monotone, it can be locally solved for M, allowing us to write $M(R,\delta)$. From the continuity equation, we have $\partial_R M(R,\delta)=F(R,\rho)$. On the other hand, the RHS can be solved for ρ as $\rho(R)=\partial_R M(R,\delta)$. It follows that any bound $\partial_R M(R,\delta)\leq M_0(\delta)$ induces a bound $\rho(R)\leq M_0(\delta)$. Consequently, if $f(p,\rho,\epsilon)$ is a MLI function of state, we have the desired bound $f(p,\rho,\epsilon)\leq M_0(\delta)$, which clearly induces a bound on the speed of sound when f is fully MLI. □

In the last two theorems we showed that bounds on the mass-radius function induce bounds on the functions of state, which can be taken to be on the speed of sound. On the other hand, additional conditions could be imposed *a priori* on the speed of sound, such as causal conditions. We will show that these conditions give rise to new bounds on the functions of state. In order to do this, given a fully MLI function of state $f(p,\rho,\delta)$, let us define a *causal condition* for f near p as an upper bound $v^2(\rho,\epsilon)\leq v_0^2(\rho,\epsilon)$ for the speed of sound near p. We assume that both v^2 and v_0^2 are *locally invertible*, meaning that we can locally invert $v^2(\rho,\epsilon)$ to write $\epsilon(v^2,\rho)$, and similarly for v_0^2. Restricting to the Banach space of parameters E, this can be formally described again through the IFT for Banach spaces.

Theorem 3. Let $\mathcal{C}=(\mathrm{Stellar}^{kl}(I),M,R)$ be a cluster of stellar systems of degree (k,l), with M piecewise C^2, endowed with a fully MLI function of state f and with a mass-radius function g satisfying a continuity equation. Then, any locally monotone invertible causal condition on f and any upper bound on the radius induce an upper bound on f depending only on the additional parameters of g.

Proof. Once more we start by writing $\rho(R)=\partial_R M(R,\delta)$. The difference is that, instead of using bounds on the right-hand side in order to get bounds on f, we consider the speed of sound $v^2(\rho,\epsilon)$, which becomes $v^2(R,\delta,\epsilon)$. If we have a bound for v^2 we can write $v^2(R,\delta,\epsilon)\leq v_0^2(R,\delta,\epsilon)$, translating to a bound on ϵ via the invertibility hypothesis. If $R \leq R_0$ is a bound on the radius, from the monotone property of v_0^2 we obtain $\epsilon(R, v^2\delta) \leq \epsilon_0(\delta)$, where $\epsilon_0(\delta) = \epsilon(R_0, v_0^2\delta)$. This gives the desired bound on f, due to the MLI hypothesis. □

IV. APPLICATIONS

Here, we apply our obstruction theorems to stellar systems inspired by experimental mass-radius relations. By this we mean relations found in the astrophysical literature as optimal approximations to experimental data describing zero age main sequence (ZAMS) stars and terminal age main sequence (TAMS) stars. These stars satisfy polytropic equations of state and the causal condition, so that the previous theorems can be applied.

A. Monomial mass-radius relations

A simple model for mass-radius relations of ZAMS and TAMS stars was developed in [12] by considering monomial polynomials. The model was also applied recently to neutron stars in [5]. This means that in the cluster $\mathcal{C} = (\mathrm{Stellar}^{kl}(I), M, R)$ of ZAMS and TAMS stars we have a mass-radius function given by

$$g(M, R, a, b) = M - aR^b, \tag{6}$$

where a, b, M, R are real functions on $\mathrm{Stellar}^{kl}(I)$. By definition, a stellar system belongs to the main sequence (MS) when the temperature at its nucleus is high enough to enable hydrogen fusion in such a way that the system becomes stable. This generally happens if the mass is at least $0.1\ M_\odot$ [2], so that these systems are naturally endowed with a lower bound on the mass. On the other hand, the mass of a MS-star determines many of its properties, such as its luminosity and MS lifetime. Thus, different classes of MS-stars are characterized by upper bounds $M \leq M_0$.

Additionally, stars at the beginning of the MS have smaller radius than those near MS's end. Therefore, ZAMS and TAMS have intrinsic bounds $R \leq R_0$. Furthermore, each partial derivative of g in (6) does not vanish except at $R = 0$ and $b = 0$; the point $R = 0$ is excluded by the bound $M \geq 0.1\ M_\odot$, and $b = 0$ is excluded from experimental data. Therefore, Theorem 1 applies, giving the following corollary:

Corollary 1. Let $\mathcal{C} = (\mathrm{Stellar}^{kl}(I), M, R)$ be a cluster of ZAMS or TAMS stars fulfilling natural upper bounds $M \leq M_0$ and $R \leq R_0$. Then any function of state of $\mathcal{C}$ depending monotonically on a and b is bounded from above. □

Because ZAMS and TAMS stars are MS-stars, they follow Eddington's standard model, which means that polytropic function of states $f = p - K\rho^\gamma$ are good models to be chosen. In order to use Corollary 1 on f, we need some dependence on a and b. If the dependence is on K, i.e., if $K(a, b)$ is a monotone function, Corollary 1 leads to bounds on γ in terms of b, $\rho(R_0)$ and $p(R_0)$. If the dependence is on γ, we find bounds on K in terms of the same parameters. Finally, if both K and γ depend of a, b, we get bounds on b in terms of $\rho(R_0)$ and $p(R_0)$.

In Eddington's standard model, the MS-stars fulfill the continuity equation (2), allowing us to apply Theorem 2 and find bounds even when K and γ do not depend on a, b. Explicitly, Eq. (2) combined with the mass-radius function [12] allows us to write the density as

$$\rho(R, a, b) = \frac{abR^{b-3}}{4\pi}. \tag{7}$$

The speed of the sound within the star becomes

$$v(R, a, b) = k\left(\frac{ab}{4\pi}\right)^\beta R^{\beta-1} \quad \text{with} \quad \beta = \gamma(b-3). \tag{8}$$

If we assume bounds on M' we get bounds on (7) and, therefore, on the parameters of the polytropic equation of state, as well as on the speed of sound (8), as ensured by Theorem 2. For instance, if $M' \leq M_0$ and we are working only with stars of radius $R \leq R_0$ we find that γ must satisfy $M_0^{1/\gamma} \geq abR_0^{b-1}$. This can also be understood as an upper bound on the radius that a polytropic MS-star of mass M_0 fulfilling the mass-radius relation defined by (7) may have: $R_0 \leq (\frac{M_0^{1/\gamma}}{ab})^{1/(b-1)}$. Just to illustrate, for a massive ZAMS stars, say with $M_0 \approx 120\ M_\odot$, we have $\gamma = 3$, $a = 0.85$ and $b = 0.67$ [12], so that $b - 1 < 0$, implying that R_0 is very small, agreeing with the fact that massive MS-stars stay only a short time as ZAMS stars. On the other hand, $b = 1.78$ for $M_0 \approx 120\ M_\odot$ TAMS stars, meaning that in the terminal stage massive MS-stars may have a large radius.

We could also go in the direction of Theorem 3 and use causal conditions (instead of conditions on M') to get bounds. In natural units, the canonical choice of causal condition is $v < 1$. Assuming this and working in the same regime $R \leq R_0$, we obtain

$$k < \left[\left(\frac{ab}{4\pi}\right)^\beta R_0^{\beta-1}\right]^{-1} \quad \text{with} \quad \beta = \gamma(b-3). \tag{9}$$

B. Rational mass-radius relations

In the last subsection, we studied monomial mass-radius relations for ZAMS and TAMS stars, found in [12] following analysis of experimental data. However, for ZAMS stars with luminosity $Z = 0.02$, the monomial relation can be replaced by a rational one, as pointed out in [13]. So, let us consider mass-radius functions

$$g(M, R, a_i, c_i, b_i, d_i) = p(M, a_i, c_i) - Rq(M, b_i, d_i), \tag{10}$$

on a cluster of stellar systems, where $p(M) = \sum_i a_i M^{b_i}$ and $q(M) = \sum_j c_j M^{d_j}$ are polynomials and a_i, b_i, c_i, d_i are real functions on the cluster. We can apply Theorems 1, 2 and 3 to find constraints on the possible functions of state. Since we are interested in MS-stars, we study the distinguished polytropic function of state. Let us see how Theorems 2 and 3 work in this case. Notice that $g(M,R,a_i,c_i,b_i,d_i)=0$ can be globally solved for R as

$$R(M,a_i,c_i,b_i,d_i)=\frac{p(M,a_i,c_i)}{q(M,b_i,d_i)}. \tag{11}$$

This function is clearly piecewise C^1 and, for each fixed $A\equiv(a_i,b_i,c_i,d_i)$, it is singular in a finite number of points. In the neighborhood of each regular point, $R(M,a_i,b_i,c_i,d_i)$ can be inverted for M, so that we have $M(R,A)$. Using the continuity equation (2), we can write $\rho(R,A,M')=M'/4^2$. Setting bounds $M'\le M_0$ and working with $R=R_0$, we get a bound $\rho\le M^0/piR_0^2$ and, therefore, a bound on the polytropic function of state, as in Theorem 2. If instead of $M'\le M_0$ we impose the canonical causal condition $v^2=k\gamma\rho^{\gamma-1}<1$, recalling that $\rho=\rho(R,A,M'(R,A))$, we get the constraint

$$k<\frac{1}{\gamma\rho_0^{\gamma-1}(A)},\quad \text{where } \rho_0(A)=\rho(R_0,A), \tag{12}$$

exactly as in Theorem 3.

V. CONCLUDING REMARKS

We established, in an axiomatic way and in great generality, that generic clusters with finite mass and finite radius can only simultaneously accommodate mass-radius relations and equations of state when certain bounds are satisfied.

Mass-radius relations can be found experimentally, while equations of state arise from theoretical modeling. In this context, our results emphasize that experimental data constrain *a priori* the possible theoretical models. We believe that these general results point towards an axiomatic formulation of Astrophysics, a problem pointed out and extensively studied by Chandrasekhar.

ACKNOWLEDGMENTS

The authors thank the referee for helpful suggestions. Y. X. M. and L. F. A. C. were supported by CAPES and CNPq, respectively.

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