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On the Normal Sheaf of Gorenstein Curves

Belo Horizonte

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On the Normal Sheaf of Gorenstein Curves

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On the Normal Sheaf of Gorenstein Curves

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Resumo

Mostramos que qualquer curva integral Gorenstein tetragonal é uma interseção completa em seu respectivo scroll normal racional tridimensional S, implicando que o feixe normal de C mergulhada em S, e em $\mathbb{P}^{g-1}$ também, é instável para $g \geq 5$, a partir do fato em que S é suave. Nós também calculamos o grau do feixe normal de qualquer curva reduzida singular em termos dos números de Tjurina e Deligne, gerando a semicontinuidade do grau do feixe normal sobre certas deformações, revisitando resultados clássicos da teoria local de germes analíticos.

Palavras Chaves a Frases: Curvas Gorenstein, scrolls, estabilidade de feixes normais, semicontinuidade superior.

Abstract

We show that any tetragonal Gorenstein integral curve is a complete intersection in its respective 3-fold rational normal scroll S, implying that the normal sheaf on C embedded in S, and in $\mathbb{P}^{g-1}$ as well, is unstable for $g \geq 5$, provided that S is smooth. We also compute the degree of the normal sheaf of any singular reduced curve in terms of the Tjurina and Deligne numbers, providing a semicontinuity of the degree of the normal sheaf over suitable deformations, revisiting classical results of the local theory of analytic germs.

Key-words and Phrases: Gorenstein curves, scrolls, stability of normal sheafs, upper semicontinuity.

Sumário

1 Introduction

It is well known that a non-hyperelliptic Gorenstein curve C of arithmetical genus $g > 2$ can be embedded in the projective space $\mathbb{P}^{g-1}$ via its dualizing sheaf [Sto93-1]. Thus, C becomes a canonical Gorenstein curve, i.e. has genus g and degree $2g-2$. More recently, Kleimann & Martins in [KM09], followed by Lara, Martins & Souza in [LMS19], shown that if a singular curve has gonality k, then its *canonical model* lies on a $(k-1)$-fold rational normal scroll of degree $g-k+1$. In particular, assuming that C is Gorenstein, its canonical model coincides with the one given by the dualizing sheaf. Since the normal sheaf of a curve encodes a lot of geometrical information, it is only natural to ask about the stability of the normal sheaf of a canonical Gorenstein curve considered in its both natural ambient spaces, rational normal scrolls and $\mathbb{P}^{g-1}$.

The stability of the normal sheaf $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ of canonical curves is well known when C is a smooth curve of genus at most 8. The study of $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ starts with a remarkable work due to Aprodu, Farkas & Ortega [AFO16], where the authors show that the normal sheaf of any tetragonal smooth curve of genus 7 with maximal Clifford index is stable, showing in particular that Mercat conjecture fails for general curves of genus 7, for details see [AFO16, Thm. 0.3.]. With the same techniques, the authors also prove that the normal sheaf of a general tetragonal canonical smooth curve of genus $g \geq 6$ is unstable, in particular, the normal sheaf of a general canonical smooth curve of genus 6 is unstable. They also establish a conjecture.

Conjecture 1 (Aprodu–Farkas–Ortega)**.** The normal sheaf $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ of a general smooth canonical curve C of genus $g \geq 7$ is stable.

Later on, Bruns [Bru17] uses the fact that a smooth general canonical curve C of genus 8 is a transversal linear section of a Grassmannian $G(2,6)$ in its Plucker embedding, c.f. [MI03], to show the stability of the normal sheaf, in this case, confirming Aprodu–Farkas–Ortega conjecture for genus 8.

In the last decades many techniques and notions for singular curves have been carried out, such as gonality, canonical models, Petri's analysis, and Max Noether Theorem, see [LMS19], [CF18], [Sto93-2] and [CFV18]. In this thesis, we will study the stability of the normal sheaves on singular curves, starting from Gorenstein ones, and trying to avoid the normalization of these curves. One difficulty lies in finding the right notion of what a general Gorenstein curve is, without excluding the singular ones. We will do it here *genus by genus*.

This thesis is organized as follows:

The chapters 2 and 3 are devoted to present the main objects to be worked in Chapter 4. Definitions, results (which are not proven, only their demonstrations are indicated), and examples are presented to familiarize the reader with the theories worked.

In section 4.1 of this thesis, we recall the notions of linear system and gonality on singular curves. We also introduce the notion of (un)stability of the normal sheaf of a singular curve and provide some examples by showing that the normal sheaf of canonical Gorenstein curve in $\mathbb{P}^{g-1}$ of genus 4 and 5 is unstable, c.f. Example 4.1.2.

In section 4.2, we show that any tetragonal Gorenstein curve is a complete intersection in its corresponding 3-fold rational normal scroll S, c.f. Theorem 4.2.1, extending a result due Schreyer [Sch86, Sec. 4]. With the main result of Section 4.2 in hands, we are able to show that the normal sheaves $\mathcal{N}_{C/S}$ and $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ are unstable for $g \geq 5$, c.f. Theorem 4.3.1, provided that the 3-fold rational normal scroll S associated with C is smooth. We also provide a sufficient condition to the 3-fold scroll of a Gorenstein curve of genus 6 be smooth, see Definition 4.3.1 and Lemma 4.3.3, that concludes our section 4.3.

One important part of studying the normal sheaf on a curve lies in the computation of its degree. Since our curves are singular, the computation becomes more involved when the genus grows. Section 4.4 of this thesis, is addressed to computing the degree of the normal sheaf on any singular reduced projective curve using classical invariants of the local theory of singularities, namely Tjurina and Deligne numbers, c.f. Theorem 4.4.1. The provided formula can be very easy to manage, see Example 4.4.1 and Table 1. As an application of Theorem 4.4.1, we prove that the degree of the normal sheaf is an upper semicontinuous function over suitable deformations, c.f. Corollary 4.4.5.

This thesis resulted in a paper with the same title and this can be accessed at the address: https://doi.org/10.1016/j.bulsci.2022.103182.

2 Preliminaries

2.1 Gorenstein Rings

The reader can obtain more details of this section, including proofs of the results, in the following references [Tru18] and [BH98].

Let R be a ring and M an R-module, The *injective dimension* of M, denoted by $\mathrm{id}_R(M)$, is the smallest integer n for which there exists an injective resolution $I^\bullet$ of M with $I^m = 0$ for $m > n$. If there is no such n the injective dimension of M is infinite.

A Noetherian local ring R is a *Gorenstein ring* if, $\mathrm{id}_R(R) < \infty$. More generally R is a *Gorenstein ring* if $R_\mathfrak{m}$ is a Gorenstein ring for all maximal ideal $\mathfrak{m}$.

Proposition 2.1.1. *Let R be a ring and S be a multiplicatively closed subset of R. Then,*

1. *If R is Gorenstein then $S^{-1}R$ is Gorenstein;*
2. *R is Gorenstein if and only if $R_\mathfrak{m}$ is Gorenstein for every $\mathfrak{m} \in \mathrm{Specm}(R)$.*
3. *Let $\boldsymbol{x} = x_1, \ldots, x_t$ a regular sequence over R. We have, R is Gorenstein if and only if $R/(\boldsymbol{x})$ is Gorenstein.*

Proof. [BH98, Pro. 3.1.19]. □

The next proposition shows that the class of Gorenstein rings is a huge class of rings.

Proposition 2.1.2. *Let R be a Noetherian local ring. Then we have the following implications:*

R is regular $\Rightarrow$ R is a complete intersection $\Rightarrow$ R is Gorenstein $\Rightarrow$ R is Cohen-Macaulay.

Proof. [BH98, Pro. 3.1.20]. □

If $(R, \mathfrak{m}, \mathbf{k})$ is a Noetherian local ring with $\mathrm{depth}(R) = t$, we denote by $\mu^i(R) := \dim_\mathbf{k} \mathrm{Ext}^i_R(k, R)$, the ***i*-th Bass number** of R and by $r(R) := \dim_\mathbf{k} \mathrm{Ext}^t_R(k, R)$ the **type** of R.

Theorem 2.1.3. *Let R be a n-dimensional local ring. The following are equivalent:*

1. *R is Gorenstein.*
2. $\mu^i(R) = \begin{cases} 1, & \text{if } i = n \\ 0, & \text{otherwise} \end{cases}$
3. *R is Cohen-Macaulay of type 1.*

Proof. [Tru18, Thm. 3.2.6. and Thm. 3.4.5.] □

Remark 2.1.1. The item 2 of above theorem assures that, if $(R, \mathfrak{m}, \mathbf{k})$ is a n-dimensional Noetherian local ring, then R is Gorenstein if, only if, $\operatorname{Ext}^i_R(\mathbf{k}, R) = 0$ for $0 \leq i < n$ and $\operatorname{Ext}^n_R(\mathbf{k}, R) \cong \mathbf{k}$.

The next proposition gives a characterization of Gorenstein local rings of Krull dimension zero, which is quite useful for examples.

Proposition 2.1.4. *Let R be a Noetherian local ring with $\dim(R) = 0$. Then, R is Gorenstein if, only if, for any ideals $I, J \subseteq R$ such that $I \cap J = (0)$, then $I = (0)$ or $J = (0)$.*

Proof. [Tru18, Thm. 3.3.4.] □

Next, we provide three examples that show that the implications of the Proposition 2.1.2 do not admit its converses ($\mathbf{k}[\![x_1, \ldots, x_n]\!]$ is the ring of the formal series.).

Example 2.1.1. The ring $\mathbf{k}[\![x, y]\!]/(xy)$ is obviously a complete intersection ring but is not regular because it is not even a domain.

Example 2.1.2. Denote $R := \mathbf{k}[\![x, y]\!]/(x^2, xy, y^2)$. We have that R is Cohen-Macaulay ring, but is not Gorenstein ring. In fact, since $\sqrt{(x^2, xy, y^2)} = (x, y)$ then R is a Artinian ring. In particular, $\dim(R) = 0$ and so a Cohen-Macaulay ring. On other hand, we can see that $(\bar{x}^2, \bar{y}) \cap (\bar{x}, \bar{y}^2) = (\bar{0})$ and thus R is not Gorenstein by Proposition 2.1.4.

Example 2.1.3. The most difficult counterexample to find is a Gorenstein ring that is not a complete intersection. For instance the ring $R = \mathbf{k}[\![x, y, z]\!]/I$, where $I = (x^2, xy, yz, z^2, y^2 - xz)$ will work, but is not so easy to prove, because we need other sophisticated techniques c.f. [BH98, Exa. 3.2.11].

2.2 Divisors and Gonality on Singular Curves

Let C be a complete irreducible algebraic curve over an algebraically closed field $\mathbf{k}$ and $\upsilon : \overline{C} \to C$ its normalization map ($\overline{C}$ is called *non-singular model of* C). For each $P \in C$, the elements of the fiber $\upsilon^{-1}(P)$ (which is finite) are the *branches of* C *centered at* P. If $\#\upsilon^{-1}(P) = 1$ then P is called *unibranch point*. Automatically, smooth points are unibranch points. If P is a singular point and $\upsilon^{-1}(P) = \{Q_1, \dots, Q_m\}$ then

$$\overline{\mathcal{O}}_P = \mathcal{O}_{Q_1} \cap \ldots \cap \mathcal{O}_{Q_m},$$

and $\overline{\mathcal{O}}_P$ is a principal ideal domain. The Rosenlicht Theorem [Ros52, Thm. 1] gives us that the dimension

$$\delta_P := \dim_{\mathbf{k}} \overline{\mathcal{O}}_P / \mathcal{O}_P$$

is finite and this is called *singularity degree of* P. It is clear that P is smooth if only if $\delta_P = 0$. Since $\overline{\mathcal{O}}_P / \mathcal{O}_P$ has finite dimension, then δ_P is an *analitic invariant*, this is,

$$\delta_p = \dim_{\mathbf{k}} \widehat{\overline{\mathcal{O}}_P} / \widehat{\mathcal{O}_P}.$$

The sum $\delta := \sum_{P \in C} \delta_P$ is called *singularity degree of* C.

Example 2.2.1. Let C be *the projective nodal plane curve*, this is, the projective closure of $\mathrm{Spec}(R)$, where $R = \mathbf{k}[x, y]/(y^2 - x^2(x + 1))$. The unique singular point of C is the origin $(0 : 0 : 1)$ or yet $O = (0, 0)$ in the corresponding afine neighborhood. Note that $\dfrac{\bar{y}}{\bar{x}} \in \mathbf{k}(C)$ is a root of the monic polynomial $T^2 - \overline{x + 1} \in R_O[T]$ and moreover, this is the only integral element of R which is not in R. Thus,

$$\overline{\mathcal{O}}_O / \mathcal{O}_O \cong k.\frac{\bar{y}}{\bar{x}},$$

and $\delta = \delta_O = 1$. Similarly, if C is *the projective ordinary cuspidal plane curve*, this is, the projective closure of $\mathrm{Spec}(R)$, where $R = \mathbf{k}[x, y]/(y^2 - x^3)$, we also have $\delta = 1$. This also follows from the fact that $R \cong k[t^2, t^3]$ is a monomial curve of genus 1, see Section 2.5 of this thesis for the general case. Since the singularity degree is an analytic invariant, if P is a node or a cusp singularity (not necessarily planar) then $\delta_P = 1$.

Example 2.2.2. Let us take C the projective closure of the afine curve $\mathrm{Spec}(R)$, where $R = \mathbf{k}[t^3, t^4, t^5]$. We note that C has only singular point O, corresponding to the origin $t = 0$. On other hand, since t is integral over $R \subseteq \mathbf{k}[t]$, we get $\mathbf{k}[t]$ as its integral closure and so, the normalization of C is $\overline{C} = \mathbb{P}^1$. Moreover, O is an unibranch point and $\overline{\mathcal{O}}_O = \mathcal{O}_{\overline{O}} = \mathbf{k}[t]_{(t)}$. Thus,

$$\overline{\mathcal{O}}_O / \mathcal{O}_O = \mathbf{k}[t]_t / \mathbf{k}[t^3, t^4, t^5]_{(t^3, t^4, t^5)} = \mathbf{k}.t \oplus \mathbf{k}.t^2,$$

so $\delta = \delta_O = 2$. Again, this is a monomial example and the section 2.5 describes the general case.

A *divisor* on C is a formal product

$$\mathbf{a} = \prod_{P\in C} \mathbf{a}_P$$

where $\mathbf{a}_P$ is a non-zero fractional ideal of $\mathcal{O}_P$ and $\mathbf{a}_P = \mathcal{O}_P$, for almost all $P \in C$. We say that a divisor $\mathbf{a}$ is *locally principal* or a *Cartier divisor* if $\mathbf{a}_P$ is a principal ideal, for every $P \in C$. Given two divisors $\mathbf{a}$ and $\mathbf{b}$ we define the *product* $\mathbf{a}\cdot\mathbf{b}$ and the *quocient* $(\mathbf{a} : \mathbf{b})$ respectively by:

$$(\mathbf{a}\cdot\mathbf{b})_P := \mathbf{a}_P \cdot \mathbf{b}_P;$$

$$(\mathbf{a}_P : \mathbf{b}_P) = \{z \in \mathbf{k}(C) | z\mathbf{b}_P \subseteq \mathbf{a}_P\}.$$

In this way, the set of the locally principal $\mathrm{Div}(C)$ divisor together with the product defined above is a group with the identity element given by *structure divisor*

$$\mathcal{O} := \prod_{P\in C} \mathcal{O}_P.$$

We define a partial order in $\mathrm{Div}(C)$,

$$\mathbf{a} \geq \mathbf{b} \Longleftrightarrow \mathbf{a}_P \subseteq \mathbf{b}_P \ \text{ for each } P \in C$$

and a divisor $\mathbf{a}$ is *effective* if $\mathbf{a} \geq \mathcal{O}$. The effective divisors correspond one-to-one to the closed subschemes of dimension 0 of C. The *degree* of a divisor is defined by the proprieties: $\deg(\mathcal{O}) = 0$ and

$$\deg(\mathbf{a}) - \deg(\mathbf{b}) = \sum_{P\in C} \dim_{\mathbf{k}} \mathbf{a}_P/\mathbf{b}_P \ \text{ whenever } \mathbf{a} \geq \mathbf{b}.$$

We define the *principal divisor of a rational function* $z \in K(C)\backslash 0$ by:

$$\mathrm{div}(z) := \prod_{P\in C} z^{-1}\mathcal{O}_P$$

and for any divisor $\mathbf{a} \in \mathrm{Div}(C)$ its $\mathbf{k}$-vector space of global sections

$$H^0(C, \mathbf{a}) := \bigcap_{P\in C} \mathbf{a}_P = \{z \in K(C) | \mathrm{div}(z)\cdot\mathbf{a} \geq \mathcal{O}\} \cup \{0\},$$

which is also denoted by $L(\mathbf{a})$. Two divisors $\mathbf{a}, \mathbf{b} \in \mathrm{Div}(C)$ are said be *linearly equivalent*, $\mathbf{a} \sim \mathbf{b}$, if $\mathbf{a} = \mathrm{div}(z)\cdot\mathbf{b}$, for some rational function $z \in K(C)$. Therefore, the projectivization of $L(\mathbf{a})$ are exactly the effective divisors linearly equivalent to $\mathbf{a}$. It is possible to show that if $\mathbf{a} \sim \mathbf{b}$ then $\deg(\mathbf{a}) = \deg(\mathbf{b})$ and $\deg(\mathrm{div}(z)) = 0$, for any rational function $z \in K(C)\backslash 0$.

Making a connection with divisors on nonsingular curves, given $\mathbf{a} \in \mathrm{Div}(C)$, for each non-singular point $P \in C$ we have that the P-component can be written as $\mathbf{a}_P =$

$\mathfrak{m}_P^{-n_P}$, for some integer n_P ($\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_P$) and so we can associate the usual divisor $\sum_{P\in \mathrm{Reg}(C)} n_P P$ with the divisor $\mathbf{a} = \prod_{P\in C} \mathbf{a}_P$, where

$$\mathbf{a}_P = \begin{cases} \mathfrak{m}_P^{-n_P}, & \text{if } P \text{ is nonsingular;} \\ \mathcal{O}_P, & \text{if } P \text{ is singular.} \end{cases}$$

From a more global point of view, it is clear that we can rephrase the notion of divisors by coherent fractional ideal sheaves. In this way, the word *divisor* will be used for both notions.

A *linear system of dimension* r in C is a set of the form

$$\mathcal{L} = \mathcal{L}(\mathcal{F}, V) := \{x^{-1}\mathcal{F} \mid x \in V \setminus 0\},$$

where $\mathcal{F}$ is a coherent fractional ideal sheaf on C and V is a vector subspace of $H^0(C, \mathcal{F})$ of dimension $r+1$. The linear system is said to be *complete* if $V = H^0(C, \mathcal{F})$, in this case one simply writes $\mathcal{L} = |\mathcal{F}|$. A point $P \in C$ is called a *base point* of $\mathcal{L}$ if $x\mathcal{O}_P \subsetneq \mathcal{F}_P$ for all $x \in V$. Denote by $\mathcal{O}\langle V\rangle$ the subsheaf of the constant sheaf of rational functions $\mathcal{K}$ generated by all sections in $V \subseteq k(C)$. A base point is *removable* if it is not a base point of $\mathcal{L}(\mathcal{O}\langle V\rangle, V)$. Thus a point P is a non-removable base point of $\mathcal{L}$ if and only if $\mathcal{F}_P$ is not a free $\mathcal{O}_P$-module; In this case, P is singular. The *degree* of the linear system $\mathcal{L}$ is the integer

$$\deg(\mathcal{L}) := \chi(\mathcal{F}) - \chi(\mathcal{O}).$$

Particularly, if $\mathcal{F} \supseteq \mathcal{O}$ we have that the degree of $\mathcal{L}$ coincides with the degree of the effective divisor $\mathcal{F}$,

$$\deg(\mathcal{F}) = \sum_{P\in C} \dim_k(\mathcal{F}_P/\mathcal{O}_P).$$

The notation $\mathfrak{g}_k^r$ means a linear system of degree k and dimension r.

Example 2.2.3. Let C be the (projective) plane cuspidal cubic (Example 2.2.1). Now we consider $\mathcal{L}$ the linear system cut out on the plane cuspidal cubic by a net spanned by two double lines and a parabola, this is, taking the rational functions $\varphi_1 = \dfrac{x}{z}, \varphi_2 = \dfrac{y}{z} \in K(C)$, then

$$\mathcal{L} = \{\mathrm{div}(a_0\varphi_1^2 + a_1\varphi_2^2 + a_2(\varphi_2 - \varphi_1^2)) \prod(\varphi_1^2\mathcal{O}_P + \varphi_2^2\mathcal{O}_P + \mathcal{O}_P)|(a_0 : a_1 : a_2) \in \mathbb{P}^2\},$$

that is a $\mathfrak{g}_6^2$.

The smallest integer k for which there is a $\mathfrak{g}_k^1$ is called *gonality* of C. Equivalently, the gonality of C is the smallest integer k for which there is a torsion-free sheaf $\mathcal{F}$ of rank 1 on C of degree k and $h^0(\mathcal{F}) \geq 2$. If C has gonality k we say that C is *k-gonal*, or yet, *hyperelliptic*, *trigonal*, *tetragonal*, *pentagonal*, ... for $k = 2, 3, 4, 5, \ldots$, respectively. If C is smooth of genus at least 1, the gonality of C is the minimal degree of dominant morphisms $\lambda : C \to \mathbb{P}^1$. In this case, if C is k-gonal then any $\mathfrak{g}_k^1$ is a complete base point free linear system, this may not happen in the singular case.

Example 2.2.4. Now let C be the nodal plane cubic curve and O its singular point (Example 2.2.1). The linear system of Cartier divisors $C \cap L$, where L is a line passing through O give us a $\mathfrak{g}_3^1$. Now, consider the divisor $\mathcal{F} := ((C \cap L) : O)$. The linear system $\mathcal{L} = |\mathcal{F}|$ gives us a $\mathfrak{g}_2^1$ and O is a base point for $\mathcal{L}$.

We also have the singular version of Riemann–Roch Theorem.

Theorem 2.2.1. *For any fractional ideal sheaf $\mathcal{F}$ on a curve C of arithmetic genus g we have that*

$$\chi(\mathcal{F}) = \deg(\mathcal{F}) + 1 - g.$$

Proof. [Sto93-1, Thm. 1.1.]. □

Now, since the geometric genus of C is $\tilde{g} := h^1(\overline{\mathcal{O}})$, $h^0(\overline{\mathcal{O}}) = 1$ and by definition $\deg(\overline{\mathcal{O}}) = \delta$, applying the Riemann-Roch Theorem for the divisor $\overline{\mathcal{O}}$ we get the following theorem.

Theorem 2.2.2. *Let C be a complete irreducible curve of arithmetic genus g, geometric genus $\tilde{g}$ and singularity degree δ, we have*

$$\tilde{g} = g - \delta.$$

2.3 The Dualizing Sheaf

Given a n-dimensional proprer scheme X, a *dualizing sheaf* for X is a coherent sheaf ω_X on X, together with a *trace* morphism $t : H^n(X, \omega_X) \to \mathbf{k}$ such that for each coherent sheaf $\mathcal{F}$ on X one has the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X)$$

and t induces an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^\vee.$$

In fact, for any proper scheme, the dualizing sheaf exists and is unique, up to isomorphism. For a projective scheme $X \subseteq \mathbb{P}^r$ of codimension c, the dualizing sheaf is given by

$$\omega_X = \mathcal{E}\mathrm{xt}^c_{\mathbb{P}^r}(\mathcal{O}_X, \omega_{\mathbb{P}^r}), \quad \text{c.f. [Har77, Chap.III, Pro. 7.5]}$$

In the case where X is nonsingular then the dualizing sheaf is a canonical sheaf c.f. [Har77, Chap.III, Cor. 7.12].

Proposition 2.3.1. *If X is a projective nonsingular variety, then the dualizing sheaf is isomorphic to the canonical sheaf.*

Let C be a complete integral curve. The dualizing sheaf is a torsion-free sheaf of rank 1 on C and is associated with a Cartier divisor $\mathbf{K}_C$ that is called of *canonial divisor*. If $\upsilon : \overline{C} \to C$ denote the normalization map of C, then the dualizing sheaf ω_C can be seen as follows (c.f. [HM06, 3.5]): each open set $U \subseteq C$ is associated with the space of the rational 1-forms λ on $\upsilon^{-1}(U) \subseteq \overline{C}$, such that for every $P \in U$ and $f \in \mathcal{O}_{C,P}$, we have

$$\sum_{Q \in \upsilon^{-1}(P)} \mathrm{Res}_Q(\upsilon^* f \cdot \lambda) = 0. \tag{2.1}$$

The dualizing sheaf can also be described as follows (c.f. [Sto93-1]): if $\eta \in \Omega^1_{\mathbf{k}(C)|\mathbf{k}}$ we denote by ω_η the fractional ideal sheaf such that for all point $P \in C$ the stalk $\omega_{\eta,P}$ is the largest fractional $\mathcal{O}_P$-ideal in the field of rational functions $\mathbf{k}(C)$ that satisfies the condition 2.1. Thus, the dualizing sheaf is given by $\omega_C = \omega_\eta \cdot \eta$, for any $\eta \in \Omega^1_{\mathbf{k}(C)|\mathbf{k}}$, since the vector space $\Omega^1_{\mathbf{k}(C)|\mathbf{k}}$ has dimension 1 over $\mathbf{k}(C)$. If P is a smooth point, then $\omega_{\eta,P} = t_P^{-v_P(\eta)} \cdot \mathcal{O}_P$, where v_P is the valuation at P.

Example 2.3.1. Let C be the curve described in the Example 2.2.2. By Theorem 2.2.2, the arithmetic genus of C is $g = 0 + 2 = 2$. Since the dualizing sheaf does not depend on the chosen 1-form, we can take $\eta := \frac{1}{t^3}dt$. Now we show that $\omega_C \cong \omega_{\eta,O} = \mathcal{O}_O + t\mathcal{O}_O$. The inclusion $\omega_{\eta,O} \subseteq \mathcal{O}_O + t\mathcal{O}_O$ it is clear, because $\mathrm{Res}_{\overline{O}}(f \cdot \eta) = 0$, for each $f \in \mathcal{O}_O + t\mathcal{O}_O$. Now, since $\mathrm{Res}_{\overline{O}}(t^2 \cdot \eta) = 1$, $t^2 \notin \omega_{\eta,O}$. We also have, $t^{-n} \notin \mathcal{O}_O, n \geq 1$, and $t^n \in \mathcal{O}_O, n \geq 3$, and follows the equality $\omega_{\eta,O} = \mathcal{O}_O + t\mathcal{O}_O$. Denoting by ∞ the point at the infinity of C, we have t^{-1} as local parameter and $\eta = -t^{-1}d(t^{-1})$ and we get $\omega_{\eta,\infty} = t\mathcal{O}_\infty$. For every nonsingular point P corresponding to $t = a \neq 0$, we have that $\frac{1}{t^3}$ is a unity in $\mathcal{O}_P$, hence $\omega_{\eta,P} = \mathcal{O}_P$. We can conclude then that $H^0(C, \omega_C) = \langle 1, t \rangle$ and the canonical divisor of C is

$$\mathbf{K}_C = (\mathcal{O}_O + t_O\mathcal{O}_O) \times (t_\infty \mathcal{O}_\infty) \times \prod_{P \in \mathrm{Reg}(C)} \mathcal{O}_P.$$

Note that $h^0(\omega_C) = 2 = \deg(\omega_C)$ which is in accordance with the corollary below, which follows as a consequence of the Riemann–Roch theorem.

Corollary 2.3.2. *Let C be a curve of arithmetic genus g and ω_C its dualizing sheaf. Then,*

1. $h^0(\omega_C) = g$.
2. $\deg(\omega_C) = 2g - 2$.

Proof. [Sto93-1, pg. 110]. □

Reciprocally, if $\mathbf{a} \in \mathrm{Div}(C)$ *such that* $\deg(\mathbf{a}) = 2g - 2$ *and* $h^0(\mathbf{a}) = g$ *then* $\mathbf{a} \sim \mathbf{K}_C$*, that is,* $\mathbf{a}$ *is also a canonical divisor.*

Remark 2.3.1. The class of the linearly equivalent divisors to canonical divisor is called *canonical class*, which is also denoted by $\mathbf{K}_C$.

2.4 Gorenstein Curves

Let $\mathfrak{C} = (\overline{\mathcal{O}} : \mathcal{O})$ be the condutor of $\overline{\mathcal{O}}$ into $\mathcal{O}$, and for each $P \in C$, denote by $\delta_P := \dim_k(\overline{\mathcal{O}}_P/\mathcal{O}_P)$ the *singularity degree* of C over P and $\eta_P := \delta_P - \dim_k(\mathcal{O}_P/\mathfrak{C}_P)$. Additionally, consider $\delta := \sum_{P\in C}\delta_P$ and $\eta := \sum_{P\in C}\eta_P$.

Remark 2.4.1. We already know that the singularity degree is finite. Rosenlicht also demonstrated that the number n_P is a non-negative integer c.f. [Ros52, Thm. 10].

Theorem 2.4.1. *Let C be a curve of arithmetic genus g and let ω_C be its dualizing sheaf. The following are equivalent:*

1. *C is Gorenstein.*
2. *ω_C is an invertible sheaf.*
3. *$n_p = 0$, for all $P \in C$.*
4. *The dualizing sheaf ω_C induces a morphism $\kappa : C \to \mathbb{P}^{g-1}$.*

Proof. The proof can be found in [Sto93-1] or [KM09]. □

Example 2.4.1. The projective curve of Example 2.3.1 is not Gorenstein, by the above theorem.

Remark 2.4.2. The image of C under κ, this is $C' := \kappa(C) \subseteq \mathbb{P}^{g-1}$, is called *canonical model* of C.

Remark 2.4.3. Every hyperelliptic curve is necessarily a Gorenstein curve.

Theorem 2.4.2. *Let C be a Gorenstein curve. Then $C \cong C'$ if only if C is non-hyperelliptic.*

Proof. The proof can be found in [Sto93-1] or [KM09]. □

Since for Gorenstein curves the dualizing sheaf ω_C is invertible, then the associated canonical divisor $\mathbf{K}_C$ is a Cartier divisor, defined up to linear equivalence. This provides the Serre duality c.f. [Har86, Thm. 1.4].

Theorem 2.4.3. *Let C be a Gorenstein complete irreducible integral curve. Then for any divisor $\mathbf{a} \in \mathrm{Div}(C)$ we have that $H^i(C, \mathbf{a})$ is dual to $H^{1-i}(C, (\mathbf{K}_C : \mathbf{a}))$ for $i = 0, 1$.*

2.5 Monomial Curves

A *numerical semigroup* $\mathcal{S}$ is a submonoid of the natural numbers $\mathbb{N}$ whose complement in $\mathbb{N}$, $L := \mathbb{N}\backslash\mathcal{S}$, is a finite set. Every element in L is called a *gap* and every element of $\mathcal{S}$ is a *nongap*. If $n_1 < \cdots < n_r$ are relatively prime positive integers, we set

$$\langle n_1, \cdots, n_r\rangle := \{a_1 n_1 + \cdots + a_r n_r | a_i \in \mathbb{N}\}.$$

Of course $\langle n_1, \cdots, n_r\rangle$ is a numerical semigroup (*semigroup generated for* $n_1, \ldots, n_r$) and all numerical semigroups are this way. Moreover, $\{n_1, \cdots, n_r\}$ is the unique minimal set of generators of $\mathcal{S} = \langle n_1, \cdots, n_r\rangle$. The cardinality of the minimal set of generators, denoted by $\mathrm{edim}(\mathcal{S})$, is the *embedding dimension* of $\mathcal{S}$ and the integer n_1 is the *multiplicity* of $\mathcal{S}$, which will be denoted by $m(\mathcal{S})$. The *genus* of $\mathcal{S}$, $g(\mathcal{S})$, is the cardinality of L, so if $g := g(\mathcal{S})$ we can write $L = \{l_1 < \cdots < l_g\}$. The last gap, l_g, is the *Frobenius number* denoted by $F(\mathcal{S})$. The least positive integer $c(\mathcal{S})$ such that $c(\mathcal{S}) + \mathbb{N} \subseteq \mathcal{S}$ is the *conductor* of $\mathcal{S}$ ($c(\mathcal{S})$ exists because L is finite).

Example 2.5.1. For the numerical semigroup $\mathcal{S} = \langle 3, 5\rangle$ we have: the set of nongaps is $\mathcal{S} = \{0, 3, 5, 6, 8, \rightarrow\}$, the minimal set of generators is $\{3, 5\}$, the embedding dimension $\mathrm{edim}(\mathcal{S}) = 2$, the multiplicity is $m(\mathcal{S}) = 3$ and the set of gaps is $L = \{1, 2, 4, 7\}$, genus $g(\mathcal{S}) = 4$, Frobenius number $F(\mathcal{S}) = 7$ and conductor $c(\mathcal{S}) = 8$.

Example 2.5.2. The unique numerical semigroup of genus 0 is $\mathbb{N}$. There is also only one numerical semigroup of genus 1, $\langle 2, 3\rangle$, and just two numerical semigroups of genus 2: $\langle 2, 5\rangle$ and $\langle 3, 4, 5\rangle$. For genus 3 exist four semigroups: $\langle 2, 7\rangle$, $\langle 3, 4\rangle$, $\langle 3, 5, 7\rangle$, $\langle 4, 5, 6, 7\rangle$.

Proposition 2.5.1. *Let $\mathcal{S}$ be a numerical semigroup. Then:*

1. $\mathrm{edim}(\mathcal{S}) \leq m(\mathcal{S})$;
2. $F(\mathcal{S}) = c(\mathcal{S}) - 1$;
3. $F(\mathcal{S}) \leq 2g(\mathcal{S}) - 1$.

Proof. [RG-S09, Pro. 2.10. and Lem. 2.14.] □

A numerical semigroup $\mathcal{S}$ is said be a *symmetric* if the bound of the item 3 is reached, this is, $F(\mathcal{S}) = 2g(\mathcal{S}) - 1$.

Example 2.5.3. The numerical semigroups: $\langle 2, 5\rangle$, $\langle 2, 7\rangle$ and $\langle 3, 4\rangle$ are symmetric, but all others of Example 2.5.2 are not symmetric. The numerical semigroup of Example 2.5.1 is also symmetric.

An integer x is called *a pseudo-Frobenius number* if $x \notin \mathcal{S}$ and $x+s \in \mathcal{S}$, for every $s \in \mathcal{S}^* := \mathcal{S}\backslash\{0\}$. We will denote by $PF(\mathcal{S})$ the set of pseudo-Frobenius numbers and will call $\lambda(\mathcal{S}) := \#PF(S)$ the *type* of $\mathcal{S}$. In other words, $\lambda(\mathcal{S})$ is the number of gaps l such that $l+n \in \mathcal{S}$ whenever n is a nongap.

Example 2.5.4. Let $\mathcal{S} = \langle 5,7,9\rangle = \{0,5,7,9,10,12,14,\rightarrow\}$. Thus, $PF(\mathcal{S}) = \{11,13\}$, $\lambda(\mathcal{S}) = 2$ and $\mathcal{S}$ is of type 2.

Proposition 2.5.2. *Let $\mathcal{S}$ be a numerical semigroup. Then,*

$$g(\mathcal{S}) \geq \frac{F(\mathcal{S})+\lambda(\mathcal{S})}{2}.$$

Proof. [Nar13, Lem. 2.0.16.] □

The numerical semigroups where the above inequality becomes an equality are called *almost symmetric*. An almost symmetric numerical semigroup of type 2 is called *pseudo-symmetric*.

Proposition 2.5.3. *Let $\mathcal{S}$ be a numerical semigroup. The following are equivalent:*

1. *$\mathcal{S}$ is symmetric;*
2. *$c(\mathcal{S}) = 2g(\mathcal{S})$;*
3. *$n \in \mathcal{S}$ if and only if $F(\mathcal{S}) - n \in L$;*
4. *$\mathcal{S}$ is almost symmetric of type* 1.

Proof. [RG-S09, Pro. 4.4. and Cor. 4.11.] □

Example 2.5.5. We have that $\langle 3,4,5\rangle$ is pseudo-symmetric and $\langle 5,8,9,12\rangle$ is almost symmetric of type 3 and the numerical semigroup of Example 2.5.4 is not even almost symmetric.

Now, for each numerical semigroup $\mathcal{S} = \langle n_1,\ldots,n_r\rangle$ we associate to the *rational affine monomial curve*

$$C_{\mathcal{S}} := \{(t^{n_1},\ldots,t^{n_r}) \in \mathbb{A}^r | t \in \mathbb{A}^1\}$$

with coordinate ring $k[\mathcal{S}] = k[t^{n_1},\ldots,t^{n_r}]$ and is called *semigroup ring* of $\mathcal{S}$. We can note that the morphism $\psi : \mathbf{k}[X_1,\ldots,X_r] \rightarrow \mathbf{k}[t]$, given by $X_i \mapsto t^{n_i}$ induces an isomorphism $\mathbf{k}[\mathcal{S}] \cong \mathbf{k}[X_1,\ldots,X_r]/I$, where $I = \left\langle \{X_i^{n_j} - X_j^{n_i}, i \neq j\}\right\rangle$. Since the rational functions fields $\mathbf{k}(\mathcal{S})$ and $\mathbf{k}(t)$ are isomorphic the map $\mathbb{A}^1 \rightarrow C_{\mathcal{S}}$ given by $c \mapsto (c^{n_1},\ldots,c^{n_r})$ is a birational morphism and this implies $\dim(C_{\mathcal{S}}) = 1$, $\overline{C_{\mathcal{S}}} = \mathbb{A}^1$, and so the name "rational affine monomial curve" it is justified. It is clear that the unique singular point of $C_{\mathcal{S}}$ is the origin $O = (0, \ldots, 0)$. For each $C_{\mathcal{S}}$ is associated with the *rational projective monomial curve*, which for abuse of notation will also be denoted by $C_{\mathcal{S}}$, and this is just its projective closure. By construction, $C_{\mathcal{S}}$ has only one point at the infinity $\infty = (0 : 0 : \cdots : 1)$, the origin $O = (1 : 0 : \cdots : 0)$ (again for abuse of notation) is its unique singular point and $\overline{C_{\mathcal{S}}} = \mathbb{P}^1$. In this way, its geometric genus is 0 and by Theorem 2.2.2, follows that $g = \delta$, where g is its arithmetic genus. On other hand, obviously, the singularity degree of $C_{\mathcal{S}}$ is the singularity degree at the origin. Since,

$$\overline{\mathbf{k}[\mathcal{S}]_O}/\mathbf{k}[\mathcal{S}]_O = \mathbf{k}[t]_t/\mathbf{k}[t^{n_1}, \ldots, t^{n_r}]_{(t^{n_1}, \ldots, t^{n_r})} = k.t^{l_1} \oplus \cdots \oplus k.t^{l_g},$$

follows that $\delta = g(\mathcal{S})$. So, we just showed you the following proposition.

Proposition 2.5.4. *Let $\mathcal{S}$ be a numerical semigroup and $C_{\mathcal{S}}$ be the rational projective monomial curve associated with arithmetic genus g and singularity degree δ. Then,*

$$g = \delta = g(\mathcal{S}).$$

Similarly to the above proposition, we can show the following property for projective monomial curves.

Proposition 2.5.5. *Let $\mathcal{S}$ be a numerical semigroup, $C_{\mathcal{S}}$ be the rational projective monomial curve associated with arithmetic genus g, singularity degree δ, $\mathfrak{C}$ its conductor and be $c = \dim_{\mathbf{k}}(\overline{K[\mathcal{S}]_P}/\mathfrak{C}_P)$. Then, $\dim_{\mathbf{k}}(K[\mathcal{S}]_P/\mathfrak{C}_P) = c - g$ and $\eta = 2\delta - c$. (P is its unique singular point.)*

As we have seen the vast majority of geometric properties of monomial curves can be described just by looking at its associated numerical semigroup. Thus, we list these property associations.

Proposition 2.5.6. *Let $\mathcal{S}$ be a numerical semigroup, $C_{\mathcal{S}}$ be the rational projective monomial curve associated with arithmetic genus g, singularity degree δ, Then,*

1. *$C_{\mathcal{S}}$ is hyperelliptic if and only if $\mathcal{S}$ is hyperelliptic, that is, $2 \in \mathcal{S}$.*
2. *$C_{\mathcal{S}}$ is Gorenstein if and only if $\mathcal{S}$ is symmetric.*
3. *$C_{\mathcal{S}}$ is almost Gorenstein (that is, $\eta_P = \mu^1(\mathbf{k}[\mathcal{S}]_P) - 1$, for all $P \in \mathbf{k}[\mathcal{S}]$) if and only if $\mathcal{S}$ is almost symmetric.*
4. *$C_{\mathcal{S}}$ is Kunz (that is, $\eta_P = 1$, for all $P \in \mathbf{k}[\mathcal{S}]$) if and only if $\mathcal{S}$ is pseudo-symmetric.*

2.6 Weierstrass Points

A point P on a smooth projective curve X of genus $g > 1$ is called a *Weierstrass point* if there is a meromorphic function on X which has a pole of the order less than or equal to g at P and is holomorphic on $X\backslash\{P\}$. For each smooth projective curve, there is a finite quantity of Weierstrass points. A point that is not a Weierstrass point is said to be a *ordinary point*. The Weierstrass Gap Theorem ensures that for every $P \in X$ there are exactly g orders $l_{i,P}$, such that

$$1 = l_{1,P} < \ldots < l_{g-1,P} < l_{g,P} = 2g - 1,$$

such that no meromorphic function exists having, as its only singularity, a pole of order $l_{i,P}$ at P c.f. [ACGH85]. The set $L_P = \{l_{1,P}, \ldots, l_{g-1,P}, l_{g,P}\}$ is called *Weierstrass gap sequence at* P and that way, $\mathcal{S}_P := \mathbb{N}\backslash L_P$ is a numerical semigroup, the *Weierstrass semigroup at* P. Note that $P \in X$ is an ordinary point if, only if, its Weierstrass gap sequence is $L_P = \{1, \ldots, g\}$. Now, let us consider C a complete integral Gorenstein curve of arithmetic genus $g > 1$ and let ω its dualizing sheaf. For each smooth point $P \in C$, the Weierstrass semigroup $\mathcal{S}_P$ is well defined. Thus, for every $n \in \mathcal{S}_P$ there is a rational function x_n on C with pole divisor nP, and this is equivalent to saying that n is a nongap if and only if $H^0(C, \mathcal{O}_C(n-1P)) \subsetneq H^0(C, \mathcal{O}_C(nP))$. Let $0 = n_0 < n_1 < \ldots < n_{g-1}$ the first g nongaps of $\mathcal{S}_P$. Since $\mathcal{S}_P$ is symmetric, $n_{g-1} = 2g-2$. Thus, the set $\{x_{n_0}, \ldots, x_{n_{g-1}}\}$ is a basis for the vector space $H^0(C, \omega)$ and $\omega \cong \mathcal{O}_C((2g-2)P)$. If C is non-hyperelliptic, the canonical morphism is given by

$$(x_{n_0} : \ldots : x_{n_{g-1}}) : C \to \mathbb{P}^{g-1}.$$

So, the canonical model C' is a curve of degree $2g - 2$ in $\mathbb{P}^{g-1}$ and the integers $l_i - 1$ are the contact orders of the curve with the hyperplanes at origin $O = (0 : 0 : \ldots : 1)$, where l_i form the Weierstrass gap sequence at P. Reciprocally, every non-hyperelliptic symmetric numerical semigroup can be realized as the Weierstrass semigroup of some curve at a smooth point: if $\mathcal{S} = \langle n_0, \ldots, n_{g-1} \rangle$ is a numerical semigroup in this way we can take the rational projective monomial curve

$$C^{(0)} := \{(s^{n_0}t^{l_g-1} : s^{n_1}t^{l_{g-1}-1} \ldots : s^{n_{g-1}}t^{l_1-1}) \in \mathbb{P}^{g-1} | (s:t) \in \mathbb{P}^1\},$$

then S is a Weierstrass semigroup of $C^{(0)}$ at infinity point $\infty = (0 : \ldots : 0 : 1)$.

2.7 Scrolls

Given $d > 1$ nonnegative integers $e_1 \leq \cdots \leq e_d$ $(e_d > 0)$, set $e = \sum e_i$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_d)$ which is a vector bundle over $\mathbb{P}^1$ of rank d generated by $e+d$ global sections. Let $\mathbb{P}(\mathcal{E}) := \mathbb{P}(\mathrm{Sym}(\mathcal{E})) \xrightarrow{\vartheta} \mathbb{P}^1$ be the projectivized vector bundle and let

$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be its tautological line bundle. Since $e_i \geq 0$, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is generated by its global sections and define a birational morphism $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N$, where $N = e + d - 1$. The image of this map is the *rational normal scroll* and will be denoted by $S := S(e_1, \ldots, e_d)$. More explicitly, $S := S(e_1, \ldots, e_d)$ is the image of the map $\pi : \mathbb{P}^1 \times \mathbb{P}^{d-1} \to \mathbb{P}^N$ given by

$$(x : y; t_1 : t_2 : \ldots : t_d) \mapsto (x^{e_1}t_1 : x^{e_1-1}yt_1 : \ldots : y^{e_1}t_1 : \ldots : x^{e_k}t_k : x^{e_k-1}yt_k : \ldots : y^{e_k}t_k).$$

Hence, the rational normal scroll is a d-dimensional projective variety with degree equal to e.

Remark 2.7.1. The scrolls are varieties of *minimal degree*, that is, its degree is equal to its codimension plus one. (c.f. [EH87-2])

Example 2.7.1. **(a)** The scroll $S(0, 0, 1)$ is just

$$S(0, 0, 1) = \{(t_1, t_2, xt_3, yt_3)|(x : y; t_1 : t_2 : t_3) \in \mathbb{P}^1 \times \mathbb{P}^2\} \subseteq \mathbb{P}^3,$$

and clearly $S(0, 0, 1) \cong \mathbb{P}^2$. Generally, $S(\underbrace{0, 0, \ldots, 0}_{r-zeros}, 1) \cong \mathbb{P}^r$.

(b) For each $a > 0$, the scroll $S(a)$ is given by the a-th Veronese embedding and so is a rational normal curve of degree a, in other words $S(a) \cong \mathbb{P}^1 \subseteq \mathbb{P}^a$ and $\deg(S(a)) = a$.

(c) $S(1, 1) = Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$ is the nonsingular quadric surface and $S(2, 1) \subseteq \mathbb{P}^4$ is a cubic scroll.

We can geometrically describe the scrolls $S(e_1, \ldots, e_d)$ in the following way: Take d complementary linear projective subspaces of dimension e_i, $L_i \cong \mathbb{P}^{e_i} \subseteq \mathbb{P}^N$. For each i, choose a rational normal curve $C_{e_i} \subseteq L_i$ and an isomorphism $\phi_i : \mathbb{P}^1 \to C_{e_i}$ (when $e_i = 0$, C_{e_i} is a point and ϕ_i is constant). Then,

$$S(e_1, \ldots, e_d) = \bigcup_{P \in \mathbb{P}^1} \langle \phi_1(P), \ldots, \phi_d(P) \rangle \subseteq \mathbb{P}^N,$$

and this means that we can see S as the disjoint union of all $(d-1)$-plane in $\mathbb{P}^N$ determinate by choosing a point in each one of the d rational normal curves. Each $(d-1)$-plane is called a *ruling* of S.

From this geometric description we can describe the homogeneous ideal of a scroll $S(e_1, \ldots, e_d)$ in $\mathbb{P}^N$: we can choose $x_0, \ldots, x_N$ in $\mathbb{P}^N$ such that $x_0, \ldots, x_{e_1}$ are homogeneous coordinates in L_{e_1}, $x_{e_1+1}, \ldots, x_{e_1+e_2}$ are homogeneous coordinates in L_{e_1}, and go on like this, $x_{N-e_d}, \ldots, x_N$ are homogeneous coordinates in L_{e_d} and so its homogeneous ideal is generated by 2×2 minors of the $2 \times e$ matrix,

$$M_{e_1,\ldots,e_d} := \begin{pmatrix} x_0 & \ldots & x_{e_1-1} & x_{e_1+1} & \ldots & x_{e_1+e_2} & \ldots & x_{N-e_d} & \ldots & x_{N-1} \\ x_1 & \ldots & x_{e_1} & x_{e_1+2} & \ldots & x_{e_1+e_2+1} & \ldots & x_{N-e_d+1} & \ldots & x_N \end{pmatrix}.$$

Example 2.7.2. We can choose $x_0, x_1, x_2, x_3, x_4, x_5$ the coordinates of $\mathbb{P}^5$ such that the scroll $S(1,1,1)$ is given by 2×2 minors of the matrix

$$M_{1,1,1} = \begin{pmatrix} x_0 & x_2 & x_4 \\ x_1 & x_3 & x_5 \end{pmatrix},$$

namely, $f_1 = x_0x_3 - x_1x_2, g_1 = x_0x_5 - x_1x_4, h_1 = x_2x_5 - x_3x_4$. Now, the scroll $S(2,1,0)$ is given by 2×2 minors of the matrix

$$M_{2,1,0} = \begin{pmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \end{pmatrix},$$

this is, $f_2 = x_0x_2 - x_1^2, g_2 = x_0x_4 - x_1x_3, h_2 = x_1x_4 - x_2x_3$. For the scroll $S(3,0,0)$ the matrix is given by

$$M_{3,0,0} = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix},$$

and, $f_3 = x_0x_2 - x_1^2, g_3 = x_0x_3 - x_1x_2, h_3 = x_1x_3 - x_2^2$ are its equations. Note that, by Jacobian criterion $S(1,1,1)$ is smooth, $S(2,1,0)$ is singular only at point $(0:0:0:0:0:1)$ and

$$\text{Sing}[S(3,0,0)] = \{(0:0:0:0:x_4:x_5) \in \mathbb{P}^5\} \cong \mathbb{P}^1.$$

We summarize the proprieties of scrolls in the following theorem (c.f. [ACGH85]).

Theorem 2.7.1. *Let $S := S(e_1, \ldots, e_d) \subseteq \mathbb{P}^N$, with $N = e + d - 1$ and $e = \sum e_i$. Then,*

1. $\dim(S) = d$ *and* $\deg(S) = e$*;*
2. *S is ACM (arithmetically Cohen-Macaulay) and its homogeneous ideal $I(S)$ is generated by $\binom{e}{2}$ quadrics;*
3. *S is smooth if and only if $e_i > 0$ for all $1 \leq i \leq d$ or $S = S(0,0,\ldots,0,1) \cong \mathbb{P}^r$;*
4. *If S is singular, it's a cone whose singular locus is a vertex V of dimension $\#\{i\,;\, e_i = 0\} - 1$;*
5. *If S is singular and $\mathcal{E} = \bigoplus \mathcal{O}_{\mathbb{P}^1}(e_i)$, then the birational morphism $\pi : \mathbb{P}(\mathcal{E}) \to S$ is a rational resolution of singularities of S.*
6. *This birational morphism $\pi : \mathbb{P}(\mathcal{E}) \to S$ is such that each fiber of $\vartheta : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ is mapped to a ruling.*

Moreover, one can show that $\text{Pic}(\mathbb{P}(\mathcal{E})) = \mathbb{Z}\tilde{H} \oplus \mathbb{Z}\tilde{R}$, where $[\tilde{H}] = [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)]$ is a hyperplane section while $[\tilde{R}] := [\vartheta^*(\mathcal{O}_{\mathbb{P}^1}(1))]$ is a fiber class, satisfying

$$\tilde{H}^d = e, \quad \tilde{H}^{d-1}\tilde{R} = 1, \quad \tilde{R}^2 = 0 \text{ and } K_{\mathbb{P}(\mathcal{E})} = (e-2)\tilde{R} - d\tilde{H}$$

where $K_{\mathbb{P}(\mathcal{E})}$ is the canonical class of $\mathbb{P}(\mathcal{E})$. We also fix the following notation

$$\mathcal{O}_S(aH + bR) := \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(a\tilde{H} + b\tilde{R}),$$

where $[H]$ is a hyperplane section of S and $[R]$ is a class of a ruling. Note that if S is singular with vertex V, then $\mathcal{O}_S(aH+bR)$ is a fractional ideal sheaf (or a divisorial sheaf) associated with a suitable Weil divisor of S, more precisely:

- if $\operatorname{codim}(V, S) > 2$ then $\operatorname{Pic}(\mathbb{P}(\mathcal{E}))$ is isomorphic to the group $\operatorname{Cl}(S)$ of Weil divisors of S, hence $\operatorname{Cl}(S) = \mathbb{Z}[H] \oplus \mathbb{Z}[R]$;
- if $\operatorname{codim}(V, S) = 2$ and E stands for the exceptional divisor of π, then sequence

  $$0 \to \mathbb{Z} \xrightarrow{\cdot E} \operatorname{Pic}(\mathbb{P}(\mathcal{E})) \xrightarrow{\pi_*} \operatorname{Cl}(S) \to 0,$$

  is exact, and in this case $E \sim \tilde{H} - e\tilde{R}$ and $\operatorname{Cl}(S) = \mathbb{Z}[R]$,

c.f. [Fer01, Proposition 2.1 and Corollary 2.2]. We also can conclude that the dualizing sheaf ω_S is $\pi_* K_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_S((e-2)R - dH)$.

3 Moduli Spaces and Deformation Theory

3.1 Moduli Spaces

A *moduli problem* is equipped with the following ingredients:

- A class of (geometric) *objects* $\mathcal{P}$.
- The notion of *family* of these objects.
- The notion of *equivalence* of these families.

Intuitively, a moduli space $\mathcal{M}_{\mathcal{P}}$ for a such problem is a geometric object such that:

- Its points are in bijective correspondence with equivalence classes of objects in $\mathcal{P}$.
- There is a natural bijective correspondence between families over a given base B and maps $B \to \mathcal{M}_{\mathcal{P}}$.

Example 3.1.1. An example of a moduli problem is given by the locally free sheaves (vector bundles) on a fixed scheme X, up to isomorphisms. The natural notion for a family over a base B is a locally free sheaf $\mathcal{F}$ over $X \times B$ flat over B with the equivalence relation being the relation of isomorphism.

Let us move on to the formal definitions.

Definition 3.1.1. Let $\mathcal{P}$ be a class of objects in some category $\mathbf{C}$ and $B \in \mathbf{C}$ any object. A *family of $\mathcal{P}$-objects over B* is an object $X \in \mathbf{C}$ together with a surjective morphism $\pi : X \twoheadrightarrow B$ such that the fiber at each point is a $\mathcal{P}$-object. Symbolically, for any $b \in B$ we have that $X_b := \pi^{-1}(b) \in \mathcal{P}$.

A family naturally defines a map $\varphi_\pi : B \to \mathcal{M}_{\mathcal{P}}$ given by $\varphi_\pi(b) := [X_b]$, this is, the image of a point b in the moduli space is the equivalence class of the fiber of that point. Strictly, in order to use the term *moduli space*, we need more ingredients:

- The map φ_π is a morphism.
- That two non-equivalent families give the two different maps.
- That every morphism from $f : B \to \mathcal{M}_{\mathcal{P}}$ arises as the map associated with a family.

If these three conditions are satisfied we say that there exists a full dictionary to translate the geometry of the moduli space in the geometry of the families of objects. This is the best scenario and they are formally called *fine moduli spaces*. The next definition reformulates these notions in categorical language.

Definition 3.1.2. Let X be a scheme. The contravariant functor which is described below is called *functor of points of* X (this name was motivated in reason of Example 3.1.2):

1. For each scheme $Y \in \mathbf{Sch}$, $\mathcal{H}om(Y, X) := \mathrm{Hom}(Y, X)$;
2. Every morphism $f : Y \to Z$ is sent in the map $\mathrm{H}_X(f) : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Y, X)$ given by $g \mapsto g \circ f$.

$$\begin{array}{ccc} & \mathcal{H}om(-, X) & \\ \mathbf{Sch} & \longrightarrow & \mathbf{Sets} \\ Y & \longmapsto & \mathrm{Hom}(Y, X) \\ \downarrow & & \uparrow \\ Z & \longmapsto & \mathrm{Hom}(Z, X) \end{array}$$

Example 3.1.2. For each scheme X, we have that $\mathcal{H}om(\mathrm{Spec}(\mathbf{k}), X)$ is the set of k-points of X and if Y is another scheme then $\mathcal{H}om(Y, X)$ is the set of Y-valued points of X.

Definition 3.1.3. A presheaf $\mathcal{F}$ on a category $\mathbf{C}$ is said be *representable* if there is an object $K \in \mathbf{C}$ and a natural isomorphism (a isomorphism of functors) $\mathcal{F} \cong \mathcal{H}om(-, C)$.

Definition 3.1.4. A *moduli problem* (or *moduli functor*) for a class $\mathcal{P}$ of objects in a category $\mathbf{C}$ which has fiber product is a contravariant functor $\mathcal{F}_{\mathcal{P}} : \mathbf{C} \to \mathbf{Sets}$ given by:

1. For each object $B \in \mathbf{C}$, $\mathcal{F}_{\mathcal{P}}(B)$ is the set of isomorphism classes of families of $\mathcal{P}$-objects.
2. For each morphism $f : B' \to B$, the map $\mathcal{F}_{\mathcal{P}}(f)$ represents a family $X \to B$ in the pullback family $f^*(X) \to B'$.

Definition 3.1.5. If there exists an object $\mathcal{M}_{\mathcal{P}} \in \mathbf{C}$ that represent the moduli problem $\mathcal{F}_{\mathcal{P}}$, it is called a *fine moduli space for* $\mathcal{F}_{\mathcal{P}}$.

Remark 3.1.1. If we have a fine moduli space, this gives us a natural bijection between the points of moduli space and the equivalence classes of objects that we want to parametrize.

Definition 3.1.6. Let $\mathcal{M}_{\mathcal{P}}$ be a fine moduli space and be $\mathbb{T}$ the natural transformation that identifies the moduli functor with the functor of points. The *universal family* is given by $\mathbb{T}(\mathrm{id}_{\mathcal{M}_{\mathcal{P}}})$.

Remark 3.1.2. The universal family is such that the fiber on each point $m \in \mathcal{M}_{\mathcal{P}}$ is exactly the object parametrize by the point m (up to equivalence classes), this is, for any $m \in \mathcal{M}_{\mathcal{P}}$ corresponding to an equivalence class $[X_m] \in \mathcal{P}$ is such that $[\pi^{-1}(m)] = [X_m]$.

Example 3.1.3. In this example, we show that $\mathbb{P}^n$ is a fine moduli space for the moduli problem of lines in $\mathbb{A}^{n+1}$ through the origin. For each scheme B we have that a family of lines in $\mathbb{A}^{n+1}$ through the origin is a line bundle $\mathcal{L}$ over B which is a subbundle of trivial vector bundle $\mathbb{A}^{n+1} \times B$. So, we can define the moduli functor $\mathcal{F} : \mathbf{Sch} \to \mathbf{Sets}$ as:

1. for each scheme B, we set $\mathcal{F}(B)$ as the equivalence class of

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{i} & \mathbb{A}^{n+1} \times B \\ \downarrow & \swarrow & \\ B & & \end{array}$$

   where i is the inclusion and the equivalence relation is given by isomorphism of line bundles over B.

2. for each morphism $f : B' \to B$ the pullback $\mathcal{F}(f) : \mathcal{F}(B) \to \mathcal{F}(B')$ is defined by

$$(\mathcal{L} \overset{i}{\hookrightarrow} \mathbb{A}^{n+1} \times B) \mapsto (f^*\mathcal{L} \overset{f^*i}{\hookrightarrow} f^*(\mathbb{A}^{n+1} \times B) = \mathbb{A}^{n+1} \times B').$$

We show that $\mathcal{F}$ is representable by $\mathbb{P}^n$. Now, for each scheme B, let $(\mathcal{L} \overset{i}{\hookrightarrow} \mathbb{A}^{n+1} \times B)$ be a representative of the class $\mathcal{F}(B)$. The injective map i gives us a short exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_B^{n+1} \to \mathcal{Q} \to 0$$

of locally free sheaves on B. Dualizing, we get a new exact sequence of locally free sheaves

$$0 \to \mathcal{Q}^\vee \to \mathcal{O}_B^{n+1} \to \mathcal{L}^\vee \to 0.$$

Thus, we have a surjection $i^\vee : \mathcal{O}_B^{n+1} \twoheadrightarrow \mathcal{L}^\vee$ which is determined by $n+1$ global sections $s_0, \ldots, s_n \in H^0(B, \mathcal{L}^\vee)$ without common zero. So, we can define $\mathcal{M} := \mathcal{L}^\vee$ and $\mathcal{F}'(B) := (\mathcal{M}, s_0, \ldots, s_n)$ up to isomorphism. In this way, we get a natural equivalence of the functors $\mathcal{F} \cong \mathcal{F}'$ and we have reduced the problem to show that the equivalence $\mathcal{F}' \cong \mathcal{H}om(-, \mathbb{P}^n)$. On the other hand, this is equivalent to saying that to give a line bundle on B together with $n+1$ global sections $s_0, \ldots, s_n$ without common zero (up to isomorphism) is the same that to give a morphism $B \to \mathbb{P}^n$ and this correspondence is already well known. In this case, the universal family is given by the tautological line bundle

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^n}(-1) = \{(P, \ell) \in \mathbb{A}^{n+1} \times \mathbb{P}^n | P \in \ell\} & \longrightarrow & \mathbb{A}^{n+1} \times \mathbb{P}^n \, . \\ \downarrow & \swarrow & \\ \mathbb{P}^n & & \end{array}$$

Example 3.1.4. As in the previous example, the Grassmannian variety $\mathrm{G}(d, n)$, in fact, is a fine moduli space for the moduli problem of d-dimensional linear subspaces in $\mathbb{A}^n$. In this case, a family over a base scheme B is a subbundle $\mathcal{E}$ of $\mathbb{A}^n \times B$ and the equivalence relation is given by isomorphism of vector bundles. See [Hos16] for details.

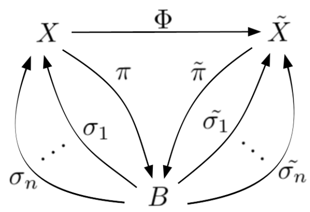
Example 3.1.5. Here, we will exhibit the fine moduli space $\mathcal{M}_{0,n}$ (the 0 indicates the genus of the curve $\mathbb{P}^1$) for isomorphism classes of n ordered distinct marked points $P_i \in \mathbb{P}^1$. Our object class is

$$\mathcal{P} = \{(P_1, \ldots, P_n) \mid P_i \in \mathbb{P}^1, P_i \neq P_j \text{ for } i \neq j\}$$

and the equivalence relation is given by

$$(P_1, \ldots, P_n) \sim (Q_1, \ldots, Q_n) \text{ if there is } \Phi \in \mathrm{Aut}(\mathbb{P}^1), \text{ such that } \Psi(P_i) = Q_i, \text{ for all } i.$$

A family of n distinct points on $\mathbb{P}^1$ over a base scheme is a flat proper morphism $\pi : X \to B$, such that each fiber is rational, $X_b \cong \mathbb{P}^1$ for each $b \in B$ together with n disjoint sections $\sigma_i : B \to X$. We say that two families $(\pi : X \to B, \sigma_1, \ldots, \sigma_n)$ and $(\tilde{\pi} : \tilde{X} \to B, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ are equivalent if there is an isomorphism $\Phi : X \to \tilde{X}$ making the following diagram commute:



Thus, we just defined a moduli functor $\mathcal{M}_{0,n} : \mathbf{Sch} \to \mathbf{Sets}$.
Since, $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2 := \mathbb{P}(\mathrm{GL}_2)$, where $\mathrm{GL}_2 := \{A \in \mathrm{Mat}_{2\times 2} = \mathbb{A}^4 : \det(A) \neq 0\}$ is the affine general linear group, we have a natural action of PGL_2 on $\mathbb{P}^1$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot [x : y] := [ax + by : cx + dx]$$

An important fact is that this action is 3-transitive, this is, for $P_1, P_2, P_3 \in \mathbb{P}^1$ are three distinct points there is a unique matrix in PGL_2 (up to equivalence class) sending them to $0 := [0 : 1], 1 := [1 : 1], \infty := [1 : 0] \in \mathbb{P}^1$ In fact, if $P_i := [z_i : 1] \neq \infty$, for $i = 1, 2, 3$ the matrix

$$\begin{bmatrix} z_2 - z_3 & -z_1(z_2 - z_3) \\ z_2 - z_1 & -z_3(z_2 - z_1) \end{bmatrix}$$

will work and this is unique up to multiplication by a scalar. In this case, $z_i = \infty$ for some i, the searched matrix is obtained from the above matrix dividing all entries by z_i and making it go to infinity.

Now, notice that this fact is equivalent to saying that the morphism

$$\mathrm{PGL}_2 \to (\mathbb{P}^1)^3 \backslash \Delta$$

$$A \mapsto (A \cdot 0, A \cdot 1, A \cdot \infty)$$

to the complement of the diagonal $\Delta := \{(Q_1, Q_2, Q_3) \,| Q_1 = Q_2 \text{ or } Q_1 = Q_3 \text{ or } Q_2 = Q_3\}$ is an isomorphism. In this way for $n = 3$, every triple (P_1, P_2, P_3) is equivalent the triple $(0, 1, \infty)$, then we expect that moduli space is $\mathcal{M}_{0,3}$ is a point. In fact, the moduli functor $\mathcal{M}_{0,3}$ is represented by $\mathrm{pt} := \mathrm{Spec}(\mathbf{k})$ c.f. [Sch20]. Now, for the cases of $n \geq 4$, given a n-tuple $(P_1, \dots, P_n)$ we still have a unique automorphism Φ such that $\Phi(P_1) = 0, \Phi(P_2) = 1$ and $\Phi(P_3) = \infty$ and so,

$$(P_1, \dots, P_n) \sim (0, 1, \infty, \Phi(P_4) \dots, \Phi(P_n)).$$

Hence, as before we can see that the moduli functor $\mathcal{M}_{0,n}$ is representable by the variety

$$\mathcal{M}_{0,n} = (\mathbb{P}^1 \backslash \{0, 1, \infty\})^{n-3} \backslash \Delta,$$

where $\Delta := \{(Q_i)_i \mid \exists i \neq j \text{ with } Q_i = Q_j\}$ is the diagonal.

Of course, not every moduli problem admits a fine moduli space (see Example 3.1.6). The next two pathologies prevent a moduli problem from admitting a fine moduli space:

- (*The Jump Phenomena:*) There exist a family $\mathcal{F}$ over $\mathbb{A}^1$ such that $\mathcal{F}_s \sim \mathcal{F}_t$ for all $s, t \in \mathbb{A}^1 \backslash \{0\}$, but $\mathcal{F}_0 \nsim \mathcal{F}_s$ for $s \in \mathbb{A}^1 \backslash \{0\}$.
- The moduli problem may be unbounded, in the sense that there is no family $\mathcal{F}$ over a scheme S, which parametrizes all objects in the moduli problem.

Example 3.1.6. Let us consider the moduli problem of classifying the endomorphism of a vector space of dimension n (obviously up to isomorphism), this is, the pairs (V, T), with V a n-dimensional vector space and T an endomorphism on V and the equivalence class is defined by $(V, T) \sim (V', T')$ if there is an isomorphism $\varphi : V \to V'$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V' \\ \downarrow T & & \downarrow T' \\ V & \xrightarrow{\varphi} & V' \end{array}$$

So, we define the moduli functor $\mathcal{E}\mathrm{nd}_n$ associating each scheme B to a vector bundle $\mathcal{V}$ over B together with an endomorphism $\Phi : \mathcal{V} \to \mathcal{V}$ and we the equivalence class is extended for vector bundles. However, this moduli problem does not admit a fine moduli space. In fact, for $n = 2$, consider the family $(\mathcal{O}_{\mathbb{A}^1}^{\oplus 2}, \Phi)$ over $\mathbb{A}^1$ given by:

$$\Phi_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \; s \in \mathbb{A}^1$$

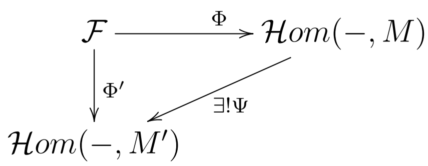
We can note that for $s, t \neq 0$, these matrices are similar and so $\Phi_s \sim \Phi_t$, but $\Phi_0 \nsim \Phi_1$ since its Jordan normal forms are distinct. Thus, we have the Jump Phenomena and so this moduli problem does not admit a fine moduli space.

Example 3.1.7. The moduli problem of vector bundles of rank 2 and degree 0 also doesn't admit a fine moduli space, because it is unbounded, c.f. [Hos16].

As we saw in the two above examples, in general, the moduli problems can fail to have fine moduli spaces, then comes the need to relax a little the definition for a scheme "almost represent" the moduli functor.

Definition 3.1.7. Let $\mathcal{F}$ be a moduli functor. A *coarse moduli space* for $\mathcal{F}$ is a pair (M, Φ) where M is a scheme and $\Phi : \mathcal{F} \to \mathcal{H}om(-, M)$ such that:

- $\Phi_{\text{Spec}(\mathbf{k})} : \mathcal{F}(\text{Spec}(\mathbf{k})) \to \mathcal{H}om(\text{Spec}(\mathbf{k}), M)$ is bijective
- For any other scheme M' and natural transformation $\Phi' : \mathcal{F} \to \mathcal{H}om(-, M')$ there is a unique natural transformation $\Psi : \mathcal{H}om(-, M) \to \mathcal{H}om(-, M')$ such that the bellow diagram commutes:



The coarse moduli spaces still parameterize the required objects and the natural transformation Ψ still matches a family of objects to a map to the corresponding parameter space. However, we may lose the universal family and the complete correspondence between maps and families.

Proposition 3.1.1. *A coarse moduli space (M, Φ) is a fine moduli space if and only if*

1. *there is a family $\mathcal{U}$ over M such that $\Phi_M(\mathcal{U}) = \text{id}_M$;*
2. *for any two families $\mathcal{F}$ and $\mathcal{G}$ over B, we have $\mathcal{F} \sim_B \mathcal{G} \Leftrightarrow \Phi_B(\mathcal{F}) = \Phi_B(\mathcal{G})$.*

It is also known that the two pathologies previously mentioned prevent the moduli problem from admitting coarse moduli space. In particular, the moduli problems of examples 3.1.6 and 3.1.7 also do not admit coarse moduli spaces.

The most classic coarse moduli space is the space that parametrizes the smooth curves with fixed genus g, $\mathcal{M}_g$, and its modifications as the space of smooth curves of genus g with n marked points $\mathcal{M}_{g,n}$ and the space of smooth marked curves with a Weierstrass point of semigroup $\mathcal{S}$ of genus g, $\mathcal{M}^{\mathcal{S}}_{g,1}$, and its compactifications: the space of stable curves of genus g, $\overline{\mathcal{M}_g}$, the space of stable curves of genus g with n marked point $\overline{\mathcal{M}_{g,n}}$ and the space of the smooth marked curves with a symmetric Weierstrass point of semigroup $\mathcal{S}$ of genus g $\overline{\mathcal{M}_{g,1}^{\mathcal{S}}}$. The third space is, actually, a compactification of the first space. For each of these moduli spaces, there is a vast theory and the reader will found in the following references: [HM06], [Cav16] and [Sto93-2].

3.2 The T^i Functors

The T^i Functors were introduced by Lichtenbaum and Schlessinger in the paper [LS67]. In this thesis, we will present most of its proprieties and this section will end with Buchweitz's method for the computation of the degree to T^1 functor on monomial curves. The proof of the results of this section can be found in [Har10].

Let us consider $A \to B$ a homomorphism of (commutative) rings and M be a B-module. We will construct the modules $T^i(B/A, M)$ for $i = 0, 1, 2$. Take a polynomial ring $R = A[\mathbf{x}]$, where $\mathbf{x}$ means a (possibly infinite) set of variables $\{x_i\}$ such that B is isomorphic to a quotient of R as A-algebra and let I be the kernel of the projection $R \to B$, so we have the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow B \longrightarrow 0$$

We also can choose a free R-module F and a surjection $j : F \twoheadrightarrow I$. Put $Q := \ker(j)$ and we get a new exact sequence

$$0 \longrightarrow Q \longrightarrow F \stackrel{j}{\longrightarrow} I \longrightarrow 0.$$

Now, let F_0 be the submodule of F generated by (Koszul) relations $j(a)b - j(b)a$, for $a, b \in F$. Thus, $j(F_0) = 0$ and $F_0 \subseteq Q$. So, we want to define the *cotangent complex*:

$$L_\bullet : L_2 \stackrel{d_2}{\longrightarrow} L_1 \stackrel{d_1}{\longrightarrow} L_0.$$

For this, we take $L_2 = Q/F_0$ as B-module structure inherited of R, $L_1 = F \otimes_R B = F/IF$ with d_2 the map induced from inclusion $Q \to F$ and $L_0 = \Omega_{R/A} \otimes_R B$, where $\Omega_{R/A}$ is the module of relative differentials. Finally, d_1 the composition of map $L_1 \to I/I^2$ and the map $I/I^2 \to L_0$ induced by derivation map $d : R \to \Omega_{R/A}$. We can see that $d_1 d_2 = 0$ and so $L_\bullet$ is a complex of B-modules. In this case, L_1 and L_0 are free B-modules, because F is a free module and $\Omega_{R/A}$ is free since R is a polynomial ring. Then, we can define for any B-module M the modules

$$T^i(B/A, M) = H^i(\mathrm{Hom}_B(L_\bullet, M)).$$

as the corresponding cohomologies modules. It is also shown that the above construction is independent of the choice of F and R and so the modules $T^i(B/A, M)$ are well-defined up to isomorphism, see [Har10, Lem. 3.2. and Lem. 3.3.].

Example 3.2.1. If B is a polymomial ring over A, we can take $R = B$ and the complex $L_\bullet$ is just the term L_0 and so, $T^1(B/A, M) = T^2(B/A, M) = 0$, for any B-module M.

By proper construction, from short exact sequences we get long exact sequences.

Theorem 3.2.1. *Let $A \to B$ be a homomorphism of rings. Then, for $i = 0, 1, 2$, $T^i(B/A, -)$ is a covariant, additive functor in the category of B-modules and:*

1. *If*
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$
is a short exact sequence of B-modules, then there is a long exact sequence
$$\begin{aligned} 0 \longrightarrow T^0(B/A, M') \longrightarrow T^0(B/A, M) \longrightarrow T^0(B/A, M'') \longrightarrow \\ T^1(B/A, M') \longrightarrow T^1(B/A, M) \longrightarrow T^1(B/A, M'') \longrightarrow \\ T^2(B/A, M') \longrightarrow T^2(B/A, M) \longrightarrow T^2(B/A, M''). \end{aligned}$$

2. *If $A \to B \to C$ be homomorphisms of rings, and M is a C-module, then there is an exact sequence of C-modules*
$$\begin{aligned} 0 \longrightarrow T^0(C/B, M) \longrightarrow T^0(C/A, M) \longrightarrow T^0(B/A, M) \longrightarrow \\ T^1(C/B, M) \longrightarrow T^1(C/A, M) \longrightarrow T^1(B/A, M) \longrightarrow \\ T^2(C/B, M) \longrightarrow T^2(C/A, M) \longrightarrow T^2(B/A, M). \end{aligned}$$

Proof. [Har10, Thm. 3.4. and Thm. 3.5.] □

The functor $T^0(B/A, -)$ is represented by the modules of derivations.

Proposition 3.2.2. *If $A \to B$ is a homomorphism of rings and M is a B-module then*
$$T^0(B/A, M) = \mathrm{Hom}_B(\Omega_{B/A}, M).$$
In particular, $T^0(B/A, B) = \mathrm{Hom}_B(\Omega_{B/A}, B)$ is the tangent module $T_{B/A}$ of B over A.

Proof. [Har10, Pro. 3.6.] □

The normal module also can be computed in terms of the cotangent complex.

Proposition 3.2.3. *If $A \twoheadrightarrow B$ is a surjective homomorphism and I is its kernel, then for any B-module M, we have that $T^0(B/A, M) = 0$ and*
$$T^1(B/A, M) = \mathrm{Hom}_B(I/I^2, M).$$
In particular, $T^1(B/A, B) = \mathrm{Hom}_B(I/I^2, B)$ is the normal module $N_{B/A}$ of $\mathrm{Spec}(B)$ over $\mathrm{Spec}(A)$.

Proof. [Har10, Pro. 3.8.] □

Proposition 3.2.4. *Let* $\mathbf{k}$ *be a base ring (not necessarily a field),* $A = \mathbf{k}[x_1, \dots, x_n]$*,* $I \subseteq A$ *be an ideal and* $B = A/I$*. Then, for any* B*-module there is a exact sequence*

$$0 \longrightarrow T^0(B/\mathbf{k}, M) \longrightarrow \operatorname{Hom}_B(\Omega_{A/\mathbf{k}}, M) \longrightarrow \operatorname{Hom}_B(I/I^2, M) \longrightarrow T^1(B/\mathbf{k}, M) \longrightarrow 0$$

and an isomorphism

$$T^2(B/A, M) \xrightarrow{\sim} T^2(B/\mathbf{k}, M).$$

Proof. [Har10, Pro. 3.10.] □

Since, the construction of the T^i functors is compatible with localization, given a morphism $f : X \to Y$ of schemes, we can extend them to functors $\mathrm{T}^i(X/Y, -)$ in the category of $\mathcal{O}_X$-modules: for any $\mathcal{O}_X$-module $\mathcal{F}$, an affine open $V \subseteq Y$ and an affine open $U \subseteq f^{-1}(V)$, we can write $\mathcal{F} = \tilde{M}$, and $\mathrm{T}^i(X/Y, \mathcal{F})(U) := T^i(U/V, M)$. All previous results are extend for the functors T^i.

Next, we will see that the vanishing of the functor T^1 characterizes smooth morphisms and the functor T^2 charaterizes the local complete intersections morphisms.

Proposition 3.2.5. *Let* $X = \operatorname{Spec}(B)$ *be an affine scheme over an algebraically closed field* $\mathbf{k}$*. Then* X *is nonsingular if and only if* $T^1(B/k, M) = 0$ *for all* B*-modules* M*. Moreover, if* X *is nonsingular, then also* $T^2(B/k, M) = 0$ *for all* B*-modules* M*.*

Proof. [Har10, Thm. 4.9.] □

Proposition 3.2.6. *Let* $\mathbf{k}$ *be an algebraically closed field,* A *be a regular local* $\mathbf{k}$*-algebra, and let* $B = A/I$ *be a quotient of* A*. Then* B *is a local complete intersection in* A *if and only if* $T^2(B/k, M) = 0$ *for all* B*-modules* M*.*

Proof. [Har10, Thm. 4.13.] □

For local complete intersections, the first cotangent module is just the quotient by the jacobian matrix.

Proposition 3.2.7. *Let* $\mathbf{k}$ *be an algebraically closed field,* $A = \mathbf{k}[x_1, \dots, x_n]$ *be a polynomial ring, let* $B = A/I$ *be a quotient of* A*. If,* $I = (f_1, \dots, f_t)$ *is generated by a regular sequence, then*

$$T^1(B/k, B) \cong \frac{A^t}{I + \mathbf{J}},$$

where $\mathbf{J}$ *is the ideal generated by entries of the Jacobian matrix*

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \dots & \vdots \\ \frac{\partial f_t}{\partial x_1} & \dots & \frac{\partial f_t}{\partial x_n} \end{bmatrix}.$$

In particular, if $B = A/(f)$ *is the ring of a hypersurface* $\mathbb{V}(f)$*, then*

$$T^1(B/k, B) \cong \frac{A}{(f) + \nabla},$$

where ∇ *is the ideal generated by gradient vector* $\left(\frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_1}{\partial x_n}\right)$.

Proof. [Ser07, Sec. 3.1.1.] □

Definition 3.2.1. Let $X = \mathrm{Spec}(B)$ be an affine scheme over an algebraically closed field $\mathbf{k}$ with isolated singularities. We call the (finite) number

$$\tau := \dim_k T^1(B/k, B),$$

as the **Tjurina number** [1] of X.

Example 3.2.2. With the above characterization we can compute the Tjurina number of some known singularities:

1. *Node:* By definition $\hat{B} \cong \mathbf{k}[\![x, y]\!]/(xy)$, so

   $$T^1(B/k, B) \cong \mathbf{k}[x, y]/(x, y) \cong \mathbf{k},$$

   and $\tau = 1$.

2. *Ordinary Cusp:* Now, $\hat{B} \cong \mathbf{k}[\![x, y]\!]/(x^2 + y^3)$, so

   $$T^1(B/k, B) \cong \mathbf{k}[x, y]/(x, y^2) \cong \mathbf{k} \oplus \mathbf{k} \cdot \bar{y}$$

   and $\tau = 2$.

3. *Tacnode:* In this case, $\hat{B} \cong \mathbf{k}[\![x, y]\!]/(y^2 + x^2 y)$, so

   $$T^1(B/k, B) \cong \mathbf{k}[x, y]/(xy, x^2 + 2y) \cong \mathbf{k} \oplus \mathbf{k} \cdot \bar{x} \oplus \mathbf{k} \cdot \bar{y}$$

   and $\tau = 3$.

[1] this name is due to Tjurina's work on analytical germs.

Actually, if we consider a curve embedded in a surface whose Tjurina number of its singularity is $\tau = 1, 2, 3$ then the singularity is exactly a node or an ordinary cusp or a tacnode, c.f. [Ser07, Pro. 3.1.5].

For the class of monomial curves $C_{\mathcal{S}}$, there is an implementable method to compute the Tjurina number τ. First, let us recall a result due to Herzog [Her70] assuring that the ideal of $C_{\mathcal{S}}$ can be generated by isobaric polynomials F_i that are differences between two monomials, namely

$$F_i := X_1^{\alpha_{i1}} \dots X_r^{\alpha_{ir}} - X_1^{\beta_{i1}} \dots X_r^{\beta_{ir}},$$

with $\alpha_i \cdot \beta_i = 0$. As usual, the weight of F_i is $d_i := \sum_j n_j \alpha_{ij} = \sum_j n_j \beta_{ij}$. For each i, let $v_i := (\alpha_{i1} - \beta_{i1}, \dots, \alpha_{ir} - \beta_{ir})$ be a vector in $\mathbf{k}^r$ induced by F_i. Next a result due to Buchweitz, c.f. [Buc80, Thm. 2.2.1], computes the Tjurina number for monomial curves.

Theorem 3.2.8 (Buchweitz)**.** $\tau = \sum_{s\in\mathbb{Z}} \dim_{\mathbf{k}} T^1_s$*, where for each* $\ell \notin \mathrm{End}(\mathcal{S}) := \{n \in \mathbb{N} | n \in \mathcal{S}, \forall s \in \mathcal{S} \backslash \{0\}\}$,

$$\dim T^1_\ell = \#\{i \in \{1, \dots, r\}\,;\, n_i + \ell \notin \mathcal{S}\} - \dim V_\ell - 1,$$

V_ℓ *is the subvector space of* $\mathbf{k}^r$ *generated by the vectors* v_i *such that* $d_i + \ell \notin \mathcal{S}$*. We also have that*

$$\dim T^1_s = 0,\ \forall\, s \in \mathrm{End}(\mathcal{S}).$$

Finally, by the very explicit and implementable method in [CF18] and [CS13], we know that the ideal of $C_{\mathcal{S}} \subseteq \mathbb{P}^{g-1}$ is given by suitable $\frac{1}{2}(g-3)(g-2)$ quadratic and isobaric forms, when the first non zero element n_1 of $\mathcal{S}$ is such that $3 < n_1 \leq g-1$ and $\mathcal{S} \neq \langle 4, 5 \rangle$, and by $\frac{1}{2}(g-3)(g-2)$ quadratic and isobaric forms added to $\binom{g+2}{3} - 5g + 5$ cubic and isobaric forms in the remaining cases. In this way, one may implement an algorithm to compute the Tjurina number τ in all these cases.

3.3 Extensions

The proof of the results and more details about the theory of this section and of the sections 3.4 and 3.5 are in the reference [Ser07].

Consider $A \to R$ a homomorphism of commutative rings. An exact sequence

$$(R', \varphi) : 0 \to I \to R' \to R \to 0,$$

such that R' is an A-algebra and φ is a homomorphism of A-algebras whose kernel I is an ideal of R' satisfying $I^2 = 0$ is called an A-*extension of* R.

Remark 3.3.1. The condition I^2 assures that I inherits the structure of A-algebra of R'.

A *homomorphism of A-extensions of R*, (R', φ) and (R'', ψ), is a homomorphism of A-algebras $f : R' \to R''$ such that $\psi \circ f = \varphi$. Especially, an *isomorphism of extensions*, (R', φ) and (R'', ψ), is a isomomorphism of A-algebras $\xi : R' \xrightarrow{\sim} R''$ such that $\ker \varphi = \ker \psi = I$ and the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R' & \xrightarrow{\varphi} & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_I} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \mathrm{id}_R} & & \\ 0 & \longrightarrow & I & \longrightarrow & R'' & \xrightarrow{\psi} & R & \longrightarrow & 0 \end{array}$$

We say that an A-extension (R', φ) is *trivial* if it has a *section*, this is, if there is a homomorphism of A-algebras $\sigma : R \to R'$ such that $\varphi \circ \sigma = \mathrm{id}_R$. Given any R-module I, we can construct a trivial A-extension of R by I. We take the A-module $R \oplus I$ adding the operation of multiplication defined by:

$$(r, i)(s, j) := (rs, rj + si).$$

We denote this A-algebra by $R\tilde{\oplus}I$ and the first projection $p : R\tilde{\oplus}I \to R$ make $(R\tilde{\oplus}I, p)$ an trivial A-extension of R by I, because $q : R \to R\tilde{\oplus}I$, given by $q(r) = (r, 0)$ is a section of p. More generically, the sections of p can be identified with the A-derivations $d : R \to I$, once the sections of p are of the form $\sigma_d(r, d(r))$. Other important fact is that every trivial A-extension of R by I is isomorphic to $(R\tilde{\oplus}I, p)$. An A-extension (P, γ) of R is said be *versal* if always there exist a homomorphism $\alpha : (P, \gamma) \to (R', \varphi)$, for any other A-extension (R', φ) of R. If $P = A[\mathbf{x}]$ is a polynomial algebra, where $\mathbf{x}$ is a (possibly infinite) set of variables $\{x_i\}$ and $R = P/I$ then the sequence exact

$$0 \to I/I^2 \to P/I^2 \to R \to 0$$

is a versal A-extension of R. Moreover, since there is always a homomorphism from any A-algebra to a polynomial algebra then every A-algebra admits a versal extension. Note that this construction is very similar to the construction of the cotangent complex.

Example 3.3.1. Let us consider the A-algebra $R' = A[t]/(t^2)$. We can denote $R' = A[\varepsilon]$, where $\varepsilon = t \mod (t^2)$ and $\varepsilon^2 = 0$. Then, the exact sequence

$$0 \to (\varepsilon) \to A[\varepsilon] \to A \to 0$$

is called the *extension by dual numbers of A*.

Remark 3.3.2. Every A-extension of A is trivial once by definition it has a section. Particularly, the above A-extension is trivial.

Example 3.3.2. Consider K a field. Suppose R is a local K-algebra with residual field K. We call a K-extension of R by K of a *small extension of R*.

Now, suppose

$$0 \to (t) \to R' \xrightarrow{f} R \to 0$$

is a small K-extension, then $t \in \mathfrak{m}_{R'}$, $f(t) = 0$ and $(t) \cong K \cdot t$.
Fact: c.f. [Ser07, Exa. 1.1.2., (ii)] (R', f) is trivial if and only if the surjective map induced by f,

$$\bar{f} : \frac{\mathfrak{m}_{R'}}{\mathfrak{m}_{R'}^2} \to \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}$$

is not bijective.
By the above fact, it is clear that the small K-extension

$$0 \to \frac{(t^n)}{(t^{n+1})} \to \frac{K[t]}{(t^{n+1})} \to \frac{K[t]}{(t^n)} \to 0,$$

for each $n \geq 2$ is nontrivial.

Our main goal now is to describe the R-module structure to the set of isomorphisms classes of extension of an A-algebra R by a module I. For that, we need of the notions of *pullback* and *pushout*. Let (R', φ) be an A-extension of R by I and $f : S \to R$ a homomorphism of A-algebras. We can define the A-extension *pullback of* (R', φ) *by* f, denoted by $f^*(R', \varphi)$ as follows:

$$\begin{array}{lccccccccc} f^*(R',\varphi): & 0 & \longrightarrow & I & \longrightarrow & R' \times_R S & \longrightarrow & S & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\ (R',\varphi): & 0 & \longrightarrow & I & \longrightarrow & R' & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

where $R' \times_R S$ is the fiber product.

Example 3.3.3. Let K be a field. We can extend the K-algebra

$$K[\varepsilon, \epsilon] := K[t, t']/(t, t')^2$$

by two different ways.
First extension:

$$0 \to (\epsilon) \to K[\varepsilon, \epsilon] \xrightarrow{p_\varepsilon} K[\varepsilon] \to 0$$

which is a trivial extension, isomorphic to $p^*((K[\epsilon], p_\epsilon))$:

$$\begin{array}{lccccccccc} p^*((K[\epsilon],p_\epsilon)): & 0 & \longrightarrow & (\epsilon) & \longrightarrow & K[\varepsilon] \times_K K[\epsilon] & \longrightarrow & K[\varepsilon] & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} & & \\ (K[\epsilon],p_\epsilon): & 0 & \longrightarrow & (\epsilon) & \longrightarrow & K[\epsilon] & \xrightarrow{p'} & K & \longrightarrow & 0 \end{array}$$

The isomorphism is given by

$$\begin{array}{rcl} K[\varepsilon, \epsilon] & \longrightarrow & K[\varepsilon] \times_K K[\epsilon] \\ a + b\varepsilon + c\epsilon & \longmapsto & (a + b\varepsilon, a + c\epsilon). \end{array}$$

Second extension:

$$0 \to (\varepsilon - \epsilon) \to K[\varepsilon, \epsilon] \xrightarrow{s} K[\varepsilon] \to 0$$

where $s : K[\varepsilon, \epsilon] \to K[\varepsilon]$ is given by $s(a + b\varepsilon + c\epsilon) = a + (b + c)\varepsilon$. We also can show that $(K[\varepsilon, \epsilon], s)$ is isomorphic to $(K[\varepsilon, \epsilon], p_\varepsilon)$.

Now, let $\lambda : I \to J$ be a homomorphism of R-modules. We can define the A-extension of R by J *pushout of* (R', φ) *by* J, denoted by $\lambda_*(R', \varphi)$ as follows:

$$\begin{array}{lccccccccc}
(R', \varphi): & 0 & \longrightarrow & I & \xrightarrow{\alpha} & R' & \xrightarrow{\varphi} & R & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle\lambda} & & \downarrow & & \downarrow & & \\
\lambda_*(R', \varphi): & 0 & \longrightarrow & J & \longrightarrow & R' \coprod_I J & \longrightarrow & R & \longrightarrow & 0
\end{array}$$

where

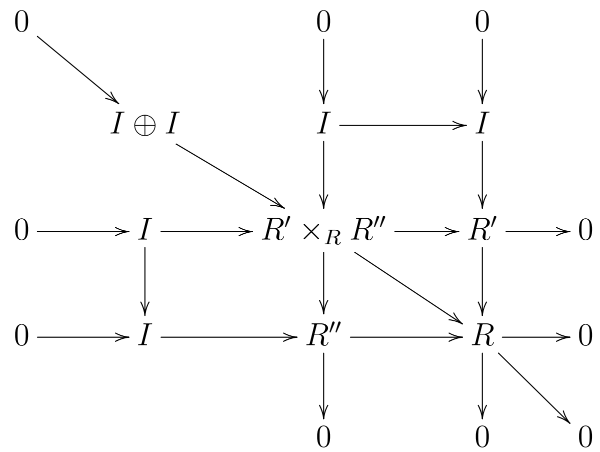
$$R' \coprod_I J = \frac{R' \tilde{\oplus} J}{\{(-\alpha(i), \lambda(i)), i \in I\}}.$$

Definition 3.3.1. For each A-algebra R and for each R-module I we define $\mathrm{Ex}_A(R, I)$ as being the set of isomorphism classes of A-extension of R by I. We will denote the class of a extension (R', φ) in $\mathrm{Ex}_A(R, I)$ by $[R', \varphi]$.

We will give an R-module structure on $\mathrm{Ex}_A(R, I)$:
Multiplication by scalar: If $r \in R$ and $[R', \varphi] \in \mathrm{Ex}_A(R, I)$ we can define

$$r[R', \varphi] := [r_*(R', \varphi)],$$

whereby abuse of notation $r : I \to I$ is the multiplication by r.
Addition: Given $[R', \varphi], [R'', \psi] \in \mathrm{Ex}_A(R, I)$, we use the following diagram:



to get an A-extension:

$$(R' \times_R R'', \zeta) : 0 \to I \oplus I \to R' \times_R R'' \xrightarrow{\zeta} R \to 0,$$

and then we define

$$[R', \varphi] + [R'', \psi] := [\delta_*(R' \times_R R'', \zeta)],$$

where $\delta : I \oplus I \to I$ is given by $\delta(i \oplus j) = i + j$.

Proposition 3.3.1. *Let $A \to R$ be a ring homomorphism and I an R-module. With the operations defined above $\mathrm{Ex}_A(R, I)$ is an R-module whose zero element is $[R\tilde{\oplus}I, p]$. This construction defines a covariant functor using the pushout:*

Proof. [Ser07, Pro. 1.1.4.] □

$$\begin{array}{ccc} & \mathrm{Ex}_A(R, -) & \\ R\mathbf{mod} & & R\mathbf{mod} \\ I & \longmapsto & \mathrm{Ex}_A(R, I) \\ \downarrow & & \downarrow \\ J & \longmapsto & \mathrm{Ex}_A(R, J) \end{array}$$

Remark 3.3.3. We can check that a homomorphism of A-algebras $f : R \to S$ where I is also a S-module, then the pullback induces a homomorphism of R-modules

$$f^* : \mathrm{Ex}_A(S, I) \to \mathrm{Ex}_A(R, I)$$

The following results allow us to link extension modules with cotangent complexes.

Proposition 3.3.2. *Let A be a ring, $f : S \to R$ a homomorphism of A-algebras and let I be an R-module. Then there is an exact sequence of R-modules:*

$$0 \longrightarrow \mathrm{Der}_S(R, I) \longrightarrow \mathrm{Der}_A(R, I) \longrightarrow \mathrm{Der}_A(S, I) \times_S R \xrightarrow{\rho}$$
$$\longrightarrow \mathrm{Ex}_S(R, I) \xrightarrow{\nu} \mathrm{Ex}_A(R, I) \xrightarrow{f^*} \mathrm{Ex}_A(S, I) \times_S R.$$

Proof. [Ser07, Pro. 1.1.5.] □

Proposition 3.3.3. *Let $A \to B$ be an e.f.t. ring homomorphism, this is, B is a localization of an A-algebra of finite type, and let $B = P/J$ where P is a smooth A-algebra. Then for every B-module M we have an exact sequence:*

$$\mathrm{Der}_A(P, M) \longrightarrow \mathrm{Hom}_B(J/J^2, M) \longrightarrow \mathrm{Ex}_A(B, M) \longrightarrow 0.$$

Proof. [Ser07, Pro. 1.1.7.] □

Corollary 3.3.4. *Let $A \to B$ is an e.f.t. ring homomorphism and M a B-module then $\mathrm{Ex}_A(B, M)$ is a finitely generated B-module and we have an exact sequence:*

$$0 \to \mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Hom}_B(\Omega_{P/A} \otimes_P B, M) \to \mathrm{Hom}_B(J/J^2, M) \to \mathrm{Ex}_A(B, M) \to 0,$$

if $B = P/J$ for a smooth A-algebra P amd an ideal $J \subseteq P$.

Proof. [Ser07, Cor. 1.1.8.] □

Remark 3.3.4. Using the Propositions 3.2.2, 3.2.4 and the Corollary 3.3.4 we obtain that the first cotangent module $T^1(B/A, M)$ and the module of extensions $\mathrm{Ex}_A(B, M)$ are isomorphic and essentially the same object in these specific cases. Actually, some authors define the first cotangent module as being exactly the module of extensions $\mathrm{Ex}_A(B, M)$, see [Ser07, Def. 1.1.6.].

Like the cotangent complex, given a morphism $f : X \to S$ of schemes we can extend the functor of extensions. An *extension* of X/S is a closed immersion $X \subseteq X'$, where X' is a S-scheme, defined by a sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_{X'}$ such that $\mathcal{I}^2 = 0$. It follows that $\mathcal{I}$ is, naturally, a sheaf of $\mathcal{O}_X$-modules, which coincides with the conormal sheaf of $X \subseteq X'$ and, clearly, gives an extension $X \subseteq X'$ of X/S is equivalent to giving an exact sequence on X:

$$\mathcal{E} : 0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{X'} \xrightarrow{\varphi} \mathcal{O}_X \longrightarrow 0$$

where $\mathcal{I}$ is a $\mathcal{O}_X$-module, φ is a homomorphism of O_S-algebras, $\mathcal{I}^2 = 0$ in $\mathcal{O}_{X'}$ and we call $\mathcal{E}$ *an extension of* X/S *by* $\mathcal{I}$. Two such entensions $\mathcal{O}_{X'}$ and $\mathcal{O}_{X''}$ are *isomorphic* if there is an $\mathcal{O}_S$-homomorphism $\alpha : \mathcal{O}_{X'} \to \mathcal{O}_{X''}$ inducing the identity on both $\mathcal{I}$ and $\mathcal{O}_X$ (α must necessarily be an S-isomorphism).

We denote by $\mathrm{Ex}(X/S, \mathcal{I})$ the set of isomorphism classes of extensions of X/S with kernel $\mathcal{I}$. If $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes and $\mathcal{I} = \tilde{M}$ we can identify

$$\mathrm{Ex}_A(B, M) = \mathrm{Ex}(X/S, \mathcal{I})$$

and if $S = \mathrm{Spec}(A)$ we will write $\mathrm{Ex}_A(X, \mathcal{I})$ instead of $\mathrm{Ex}(X/\mathrm{Spec}(A), \mathcal{I})$. Therefore, a theory of extensions can be expanded for schemes, and its properties are preserved. In particular, we can describe the first cotangent sheaf in terms of extensions of schemes.

We finalize this section by presenting a result that identifies the first cotangent module with the classic $\mathcal{E}\mathrm{xt}$ functors.

Theorem 3.3.5. *Let $X \to S$ be a morphism of finite type of algebraic schemes and $\mathcal{I}$ a coherent locally free sheaf on X. Assume that X is reduced and S-smooth on a dense open subset. Then there is a canonical identification*

$$\mathrm{Ex}(X/S, \mathcal{I}) = \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I}).$$

In particular, if X is a reduced algebraic scheme then

$$T^1_X \cong \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$$

and if, moreover, $X = \mathrm{Spec}(B_0)$ then

$$T^1_{B_0} \cong \mathrm{Ext}^1_{\mathbf{k}}(\Omega_{B_0/\mathbf{k}}, B_0).$$

Proof. [Ser07, Cor. 1.1.11.] □

3.4 Deformations

To simplify the notation we will consider the following categories of **k**-algebras:

- $\mathcal{A}$:= the category of local artinian **k**-algebras with residue field **k**;
- $\hat{\mathcal{A}}$:= the category of complete local noetherian **k**-algebras with residue field **k**;
- $\mathcal{A}^*$:= the category of local noetherian **k**-algebras with residue field **k**;

Let X be an algebraic scheme. We call of *deformation of X parametrized by S* or (*over S*) a cartesian diagram of morphism of schemes

$$\eta: \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

where π is flat and surjective, and S is connected. We say that S and $\mathcal{X}$ are, respectively, of *parameter scheme* (or *parameter space*) and *total scheme* (or *total space*) of the deformation. If S is algebraic, we also call of *deformation of X* the scheme-theoretic fibre $\mathcal{X}(t)$, for each **k**-rational point $t \in S$. In the case $S = \operatorname{Spec}(A)$, where $A \in \operatorname{ob}(\mathcal{A}^*)$, and $s \in S$ is a closed point we have a *local family of deformations* (or *local deformation*) of X over A. We also can denote the deformation η by (S, η) or (A, η), when $S = \operatorname{Spec}(A)$.

Example 3.4.1. Let $X = \operatorname{Spec}(\mathbf{k}[x,y]/(xy))$ be the affine scheme of a planar simple node and be $\mathcal{X}$ the family given by $\operatorname{Spec}(\mathbf{k}[x,y,t]/(xy-t))$ in $\mathbb{A}^3 = \operatorname{Spec}(\mathbf{k}[x,y,t])$. Via projection

$$\begin{array}{rcl} \pi : \mathbb{A}^3 & \longrightarrow & \mathbb{A}^1 \\ (x,y,t) & \longmapsto & t \end{array}$$

we get the following diagram

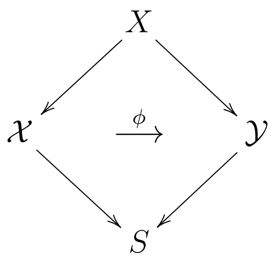
$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec}(\mathbf{k}) & \longrightarrow & \mathbb{A}^1 \end{array}$$

and we have a flat family $\mathcal{X} \to \mathbb{A}^1$ whose fibers are affine conics. Note that for $s = 0$ the fiber is the original nodal singularity and for $s \neq 0$ the fiber is a smooth hyperbola.

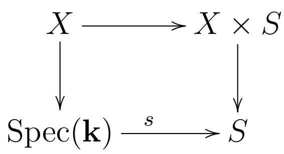
The local deformation (A, η) is *infinitesimal* (resp. *first-order*) if $A \in \operatorname{ob}(\mathcal{A})$ (resp. $A = \mathbf{k}[\varepsilon]$). If we have another deformation

$$\xi: \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathbf{k}) & \longrightarrow & S \end{array}$$

of X over S, an *isomorphism* of η with ξ is an S-isomorphism $\phi : \mathcal{X} \to \mathcal{Y}$ inducing the identity on X, this is, such that the following diagram commutes:



A *pointed scheme* is just a pair (S, s) where S is a scheme and $s \in S$. If K is a field we call (S, s) a K-pointed scheme if $K \cong \mathbf{k}(s)$. For every X and every $\mathbf{k}$-pointed scheme (S, s) there is at least one family of deformations of X over S, namely the *product family*:



and we say that a deformation is *trivial* if it is isomorphic to the product family. In this case, all fibers over $\mathbf{k}$-rational points are isomorphic to X (the deformation of Example 3.4.1 cannot be trivial). The converse is not true: some deformations are not trivial but have isomorphic fibers over all the $\mathbf{k}$-rational points.

Example 3.4.2. Given an integer $m \geq 0$, consider the rational ruled surface

$$F_m := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}).$$

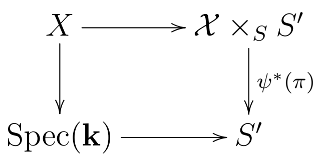
Then, the structural morphism $\pi : F_m \to \mathbb{P}^1$ defines a flat family whose fibers are all isomorphic to $\mathbb{P}^1$, but if $m > 0$ then π is not a trivial family because $F_m \ncong F_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

The scheme X is called *rigid* if every infinitesimal deformation of X over A is trivial for each $A \in \mathrm{ob}(\mathcal{A})$.

Theorem 3.4.1. *Every affine smooth algebraic variety is rigid.*

Proof. [Ser07, Thm. 1.2.4.] □

Given a deformation η of X over S as above and a morphism $\psi : (S', s') \to (S, s)$ of $\mathbf{k}$-rational pointed schemes there is induced a commutative diagram by base change



which, obviously, is a deformation of X over S' and this operation is functorial. A deformation (S, η) of X is called *miniversal* or *semi-universal* if every deformation (S', η') of X is isomorphic to a deformation $\psi^*(\pi)$, for some map $\psi : S' \to S$.

An infinitesimal deformation η of X is *locally trivial* if every point $x \in X$ has an open neighbourhood $U_x \subseteq X$ such that

$$\begin{array}{ccc} U_x & \longrightarrow & \mathcal{X}|_{U_x} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathbf{k}) & \longrightarrow & S \end{array}$$

is a trivial deformation of U_x.

Remark 3.4.1. Let

$$\eta : \qquad \begin{array}{ccc} X & \xrightarrow{j} & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

be a deformation of an algebraic scheme X parametrized by an algebraic scheme S and let $Z \subseteq X$ be a proper closed subset. Then

$$\begin{array}{ccc} X \backslash Z & \xrightarrow{j} & \mathcal{X} \backslash j(Z) \\ \downarrow & & \downarrow \pi' \\ \operatorname{Spec}(\mathbf{k}) & \xrightarrow{s} & S \end{array}$$

is a deformation of $X \backslash Z$ having the same fibers as π over $t \in S \backslash \{s\}$ and so such fibers are deformations both of X and of $X \backslash Z$. This shows that the definition of the family of deformations given above is somewhat ambiguous unless we assume that π is projective or that the deformation is infinitesimal. In what follows we will restrict to the consideration of deformations of projective schemes and/or of infinitesimal deformations when discussing the general theory so that such ambiguity will be removed; only occasionally will we consider non-infinitesimal deformations of affine schemes.

Next, we have a very important correspondence regarding first-order deformations: the *Koidara-Spencer correspondence.*

Theorem 3.4.2. *Let X be an algebraic variety. There exist a one-to-one correspondence:*

$$\kappa : \{\textit{isomorphism classes of first-order locally trivial deformations of } X\} \longrightarrow H^1(X, T_X)$$

where T_X is the tangent sheaf of X, such that $\kappa(\xi) = 0$ if and only if ξ is the trivial deformation class. In particular, if X is smooth then κ is a one-to-one correspondence

$$\kappa : \{\textit{isomorphism classes of first order deformations of } X\} \longrightarrow H^1(X, T_X).$$

Proof. [Ser07, Pro. 1.2.9.] □

Now, let X be a smooth algebraic variety. Consider a small extension

$$0 \to (t) \to \tilde{A} \to A \to 0$$

in $\mathcal{A}$ and let

$$\xi : \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

be an infinitesimal deformation of X. A *lifting of* ξ *to* $\tilde{A}$ is a pair consisting of a deformation

$$\tilde{\xi} : \quad \begin{array}{ccc} X & \longrightarrow & \tilde{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{k}) & \longrightarrow & \mathrm{Spec}(\tilde{A}) \end{array}$$

and an isomorphism of deformations

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & & \searrow & \\ \mathcal{X} & & \xrightarrow{\phi} & & \tilde{\mathcal{X}} \times_{\mathrm{Spec}(\tilde{A})} \mathrm{Spec}(A) \\ & \searrow & & \swarrow & \\ & & \mathrm{Spec}(A) & & \end{array}$$

Proposition 3.4.3. *Given* $A \in \mathrm{ob}(\mathcal{A})$ *and an infinitesimal deformation* ξ *of* X *over* A*:*

a) To every small extension e *of* A *there is associated an element* $o_\xi(e) \in H^2(X, T_X)$ *called the obstruction to lifting* ξ *to* $\tilde{A}$*, which is* 0 *if and only if a lifting of* ξ *to* $\tilde{A}$ *exists.*

b) The correspondence $e \mapsto o_\xi(e)$ *defines a* $\mathbf{k}$*-linear map*

$$o_\xi : \mathrm{Ex}_{\mathbf{k}}(A, \mathbf{k}) \to H^2(X, T_X).$$

Proof. [Ser07, Pro. 1.2.12.] □

Definition 3.4.1. The deformation ξ is called *unobstructed* if o_ξ is the zero map. Otherwise, ξ is called *obstructed*. Even more, X is said be *unobstructed* if every infinitesimal deformation of X is unobstructed and otherwise X is *obstructed*.

Corollary 3.4.4. *Let X be smooth algebraic variety. Then:*

a) X is unobstructed if $H^2(X, T_X) = 0$;

b) X is rigid if and only if $H^1(X, T_X) = 0$.

Proof. [Ser07, Cor. 1.2.14. and Cor. 1.2.15.] □

Example 3.4.3. If X is a projective smooth curve of genus g then from the Riemann-Roch theorem is follows that

$$h^1(X, T_X) = \begin{cases} 0, \text{ if } g = 0 \\ 1, \text{ if } g = 1 \\ 3g - 3, \text{ if } g \geq 2 \end{cases}$$

and $h^2(X, T_X) = 0$ and so projective nonsingular curves are unobstructed.

Example 3.4.4. It follows from the Euler sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to T_{\mathcal{O}_{\mathbb{P}^n}} \to 0$$

that $H^1(\mathbb{P}^n, T^n_{\mathbb{P}}) = 0$ and so the projective space $\mathbb{P}^n$ is rigid for each $n \geq 1$. Similarly, it is proved that finite products $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ of projective spaces are rigid.

3.5 Functors of Artin Rings and Obstructions

Given $\Lambda \in \text{ob}(\mathcal{A}^*)$ denote by:

- $\mathcal{A}_\Lambda :=$ the category of local artinian Λ-algebras with residue field $\mathbf{k}$;
- $\mathcal{A}^*_\Lambda :=$ the category of local noetherian Λ-algebras with residue field $\mathbf{k}$;

and if $\Lambda \in \text{ob}(\hat{\mathcal{A}})$ denote by $\hat{\mathcal{A}}_\Lambda$ the category of local artinian Λ-algebras with residue field $\mathbf{k}$.

A *functor of Artin Rings* is a covariant functor

$$F : \mathcal{A}_\Lambda \to (\text{sets})$$

where $\Lambda \in \text{ob}(\hat{\mathcal{A}})$.

The principal examples of such functors are obtained fixing an $R \in \text{ob}(\hat{\mathcal{A}}_\Lambda)$ and putting for each $A \in \text{ob}(\mathcal{A}_\Lambda)$:

$$h_{R/\Lambda}(A) = \text{Hom}_{\hat{\mathcal{A}}_\Lambda}(R, A).$$

Definition 3.5.1. Let F be a functor of Artin rings. Suppose that $\upsilon(F)$ is a $\mathbf{k}$-vector space such that for every $A \in \text{ob}(\mathcal{A}_\Lambda)$ and for every $\xi \in F(A)$ there is a $\mathbf{k}$-linear map

$$\xi_\upsilon : \text{Ex}_\Lambda(A, \mathbf{k}) \to \upsilon(F)$$

with the following property: the $\ker(\xi_\upsilon)$ consists of the isomorphism classes of extensions $(\tilde{A}, \varphi)$ such that

$$\xi \in \text{Im}[F(\tilde{A}) \to F(A))]$$

then $\upsilon(F)$ is called an *obstruction space* for the functor F. If F has (0) as an obstruction space then it is called *unobstructed*.

If X is an algebraic scheme then for each $A \in \text{ob}(\mathcal{A})$ put

$$\text{Def}_X(A) := \{\text{deformations of } X \text{ over } A\}/\{\text{isomorphisms}\}.$$

This defines a functor of Artin rings

$$\text{Def}_X : \mathcal{A} \to \{\text{sets}\}$$

and this is called of *local moduli functor* of X. We also define the subfunctor

$$\text{Def}'_X : \mathcal{A} \to \{\text{sets}\}$$

by

$$\text{Def}'_X(A) := \{\text{locally trivial deformations of } X \text{ over } A\}/\{\text{isomorphisms}\}$$

and we call of *locally trivial moduli functor* of X.

Theorem 3.5.1. *Let X be an algebraic scheme. Then:*

a) There is a canonical identification of $\mathbf{k}$*-vector spaces*

$$\text{Def}'_X(\mathbf{k}[\varepsilon]) = H^1(X, T_X).$$

In particular, if X is smooth then

$$\text{Def}_X(\mathbf{k}[\varepsilon]) = \text{Def}'_X(\mathbf{k}[\varepsilon]) = H^1(X, T_X).$$

b) If $X = \text{Spec}(B_0)$ is affine then

$$\text{Def}_{B_0}(\mathbf{k}[\varepsilon]) = T^1_{B_0}.$$

Proof. [Ser07, Thm. 2.4.1.] □

Proposition 3.5.2. *Let X be a smooth algebraic variety. Then $H^2(X, T_X)$ is an obstruction space for the functor* Def_X*. If X is an arbitrary algebraic scheme then $H^2(X, T_X)$ is an obstruction space for the functor* Def'_X*.*

Proof. [Ser07, Pro. 2.4.6.] □

Proposition 3.5.3. *Let X be a reduced l.c.i. algebraic scheme, and assume* $\mathrm{char}(k) = 0$*. Then* $\mathcal{E}\mathrm{xt}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ *is an obstruction space for the functor* Def_X*.*

Proof. [Ser07, Pro. 2.4.8.] □

Example 3.5.1. If X is a reduced l.c.i. curve then $\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ is a torsion sheaf and $\mathcal{E}\mathrm{xt}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 0$, therefore X is unobstructed by Proposition 3.5.3.

To finalizes this section, we will describe an obstruction space for the functor Def_{B_0}, where $B_0 = P/J$, P is a smooth $\mathbf{k}$-algebra and $J \subseteq P$ is an ideal. Here, we will recall the construction of the second cotangent module made in the section 3.2: we consider a presentation:

$$\eta : \; 0 \longrightarrow R \overset{\iota}{\longrightarrow} F \overset{j}{\longrightarrow} J \longrightarrow 0$$

where F is a finitely generated free P-module. Let $\lambda : \bigwedge^2 F \to F$ e defined by

$$\lambda(x \wedge y) = (jx)y - (jy)x$$

and $R^{tr} = \mathrm{Im}(\lambda)$. If $J(f_1, \ldots, f_n)$ then $F = P^n$ and

$$R/R^{tr} = H^1(K_\bullet(f_1, \ldots, f_n))$$

is the first homology module of the Koszul complex associated with $f_1, \ldots, f_n$ and in this way the second cotangent module $T^2_{B_0}$ is defined by the exact sequence

$$\mathrm{Hom}_{B_0}(J/J^2, B_0) \to \mathrm{Hom}_{B_0}(F \otimes_P B_0, B_0) \to \mathrm{Hom}_{B_0}(R/R^{tr}, B_0) \to T^2_{B_0} \to 0.$$

Proposition 3.5.4. *If P is a smooth k-algebra, $J \subseteq P$ is an ideal and $B_0 = P/J$ then $T^2_{B_0}$ is an obstruction space for the functor* Def_{B_0}*.*

Corollary 3.5.5. *Let* $X_0 = \mathrm{Spec}(B_0)$ *be an affine algebraic scheme such that* $\dim_{\mathbf{k}}(T^1_{B_0})$ *is finite, and let $(R, \{\eta_n\})$ be a semi-universal formal deformation of X_0. Then*

1. $\dim_{\mathbf{k}}(T^1_{B_0}) \geq \dim(R) \geq \dim_{\mathbf{k}}(T^1_{B_0}) - \dim_{\mathbf{k}}(T^2_{B_0})$ *and the first equality holds if and only if X_0 is unobstructed.*

2. *If B_0 is an e.f.t. local complete intersection* $\mathbf{k}$*-algebra then it is unobstructed. In particular, hypersurface singularities are unobstructed.*

4 The Normal Sheaf on Singular Curves

4.1 More Notation and background

Let C be an integral projective curve defined over an algebraically closed field $\mathbf{k}$. We recall that the *degree* of a coherent sheaf $\mathcal{F}$ on C is the integer

$$\deg(\mathcal{F}) = \chi(\mathcal{F}) - \operatorname{rank}(\mathcal{F})\chi(\mathcal{O}_C) \tag{4.1}$$

where χ stands for the usual Euler characteristic. By virtue of Riemann–Roch theorem for singular curves, we easily see that $\deg(\mathcal{F}) = \chi(\mathcal{F}) - \operatorname{rank}(\mathcal{F})(1-g)$, where g is the arithmetical genus of C. The *slope* of $\mathcal{F}$ is by definition

$$\mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}}.$$

For the sake of self-containedness, we include the proof of the following very naive result.

Lemma 4.1.1. *Let C be an integral projective curve and $\mathcal{F}$ a coherent sheaf over C. The following are true:*

(i) $\deg(\mathcal{F}) = \deg(\det(\mathcal{F}))$;

(ii) *If $\mathcal{F}$ is a torsion sheaf, then* $\deg(\mathcal{F}) = \sum_{P\in C} \dim_k(\mathcal{F}_P)$;

(iii) *Given an exact sequence,* $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$, *of coherent sheaves over C, we have* $\sum_{i=1}^{n}(-1)^i \deg(\mathcal{F}_i) = 0$.

Proof. Item **(i)** follows automatically from equation (4.1). Since the rank of $\mathcal{F}$ is zero, and $\mathcal{F}$ is a skyscraper sheaf, we get $H^1(C,\mathcal{F}) = 0$ and so $\deg(\mathcal{F}) = h^0(C,\mathcal{F})$, getting **(ii)**. The proof of **(iii)** follows by the additivity properties of Euler's characteristic and also of the rank on exact sequences. □

We also recall that the notion of linear systems on singular curves is characterized by interchanging line bundles by torsion-free sheaves of rank 1. A *linear system of dimension r and degree d* on a curve C, possibly singular is a set of the form

$$\mathfrak{g}^r_d := \{x^{-1}\mathcal{F} \mid x \in V \setminus 0\}$$

where $\mathcal{F}$ is a fractional ideal sheaf of degree d on C and V is a vector subspace of $H^0(C,\mathcal{F})$ of dimension $r+1$. Note that *non-removable* base points are allowed.

The *gonality of* C is the smallest k for which there exists a $\mathfrak{g}_k^1$ on C, or equivalently, a torsion-free sheaf $\mathcal{F}$ of rank 1 on C with degree k and $\dim \mathrm{H}^0(\mathcal{C}, \mathcal{F}) \geq 2$. We also note that singular curves may admit linear systems of degrees bigger than $\lfloor (g+3)/2 \rfloor$, c.f. [LMS19].

The next definition is just a simple adjustment of the notion of stability, and semi-stability, of a coherent sheaf on a smooth curve in a way to also include singular curves.

Definition 4.1.1. A coherent sheaf $\mathcal{F}$ on an integral projective curve, possibly singular, is *semistable* (resp. *stable*) if for each nontrivial coherent subsheaf $0 \neq \mathcal{G} \subsetneq \mathcal{F}$, with $\operatorname{rank} \mathcal{G} < \operatorname{rank} \mathcal{F}$, one has $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$ (resp. $\mu(\mathcal{G}) < \mu(\mathcal{F})$). Moreover, $\mathcal{F}$ is said to be *polistable* if it can be written as a sum of stables subsheaf, all of the same slope. As usual, a sheaf is *unstable* if it is not stable.

Example 4.1.1. If C is an integral plane curve of degree d, then $\mathcal{N}_{C/\mathbb{P}^2} \cong \mathcal{O}_C(d)$. Once $\mathcal{N}_{C/\mathbb{P}^2}$ is torsion-free, and every coherent sheaf of rank zero is a torsion sheaf, we get that $\mathcal{N}_{C/\mathbb{P}^2}$ is stable. In particular, the normal sheaf of a non-hyperelliptic canonical Gorenstein curve of genus 3 is stable.

The moduli space $\mathcal{M}_g$ of smooth curves of genus $g \geq 2$ admits a filtration

$$\mathcal{H}_g := \mathcal{M}_g(2) \subset \mathcal{M}_g(3) \subset \cdots \subset \mathcal{M}_g\left(\lfloor (g+3)/2 \rfloor\right) = \mathcal{M}_g,$$

where $\mathcal{M}_g(k) := \{[C] \in \mathcal{M}_g : C \text{ admits a } \mathfrak{g}_k^1\}$ is an irreducible closed subset of $\mathcal{M}_g$, c.f. [Ful69], of dimension $2g + 2k - 5$, see [AC81, eq. 2.3, pg. 346]. Here $\mathcal{H}_g$ stands for the space of hyperelliptic curves. Hence the locus of k-gonal smooth curves is just $\mathcal{M}_g^k := \mathcal{M}_g(k) \backslash \mathcal{M}_g(k-1)$. If $k \geq \lfloor (g+3)/2 \rfloor$, then $\mathcal{M}_g(k) = \mathcal{M}_g$, see [ACGH85]. In this way, for smooth curves of genus g, the number $\lfloor (g+3)/2 \rfloor$ is called *generic gonality*, and usually, a *general smooth curve* is defined in terms of the generic gonality.

In this thesis we also define a general possibly singular curve with a fixed arithmetic genus g in terms of the number $\lfloor (g+3)/2 \rfloor$. But it is also required, as the genus increases, to avoid also some specific cases, see for example the Definition 4.3.1 below.

Definition 4.1.2. A general curve of arithmetic genus 4 or 5 is just an integral Gorenstein curve whose gonality is 3 or 4, respectively.

Example 4.1.2. If C is a general curve of genus 4, then Petri's Theorem for Gorenstein curves, c.f. [Sto93-2, Section 3], assures that C is a complete intersection of a quadric with a cubic hypersurface in $\mathbb{P}^3$. Thus, the normal sheaf splits

$$\mathcal{N}_{C/\mathbb{P}^3} \cong \mathcal{O}_C(2) \oplus \mathcal{O}_C(3).$$

The nontrivial subsheaf $\mathcal{G} := \mathcal{O}_C(3)$ of $\mathcal{N}_{C/\mathbb{P}^3}$ has slope $\mu(\mathcal{G}) = 18$ while

$$\det(\mathcal{N}_{C/\mathbb{P}^3}) \cong \mathcal{O}_C(2) \otimes \mathcal{O}_C(3) \cong \mathcal{O}_C(5),$$

and so $\mu(\mathcal{N}_{C/\mathbb{P}^3}) = 30/2 = 15$. Hence $\mathcal{N}_{C/\mathbb{P}^3}$ is unstable.

Now, we consider C a general curve of genus 5. Again, by Petri's Theorem for Gorenstein curves, C is a complete intersection of three quadrics and so its normal sheaf can be written as the sum

$$\mathcal{N}_{C/\mathbb{P}^4} \cong \mathcal{O}_C(2) \oplus \mathcal{O}_C(2) \oplus \mathcal{O}_C(2).$$

By taking the subsheaf $\mathcal{G} := \mathcal{O}_C(2) \subsetneq \mathcal{N}_{C/\mathbb{P}^4}$ we get $\mu(\mathcal{G}) = 16 = 48/3 = \mu(\mathcal{N}_{C/\mathbb{P}^4})$, hence the normal sheaf $\mathcal{N}_{C/\mathbb{P}^4}$ is also unstable.

So we may conclude that the normal sheaf $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ of a general canonical curve of arithmetic genus 4 or 5 is unstable. In addition, the normal sheaf of a general canonical curve of genus 5 is polistable.

For further use, we finish this section presenting a naive generalization of a result present in Hartshorne's book [Har77, Thm. 7.11].

Let $X \subseteq \mathbb{P}^n$ be a regular scheme of codimension q, and $Y \subseteq X$ be a closed subscheme of codimension p in X. The dualizing sheaf ω_Y is given by $\omega_Y := \mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \omega_{\mathbb{P}^n})$. A result due to Grothendieck, c.f. [Gro59, Prop. 5], assures that for any coherent sheaf $\mathcal{F}$ on Y, the sheaf $\mathcal{E}^i(\mathcal{F}) := \mathcal{E}xt^{q+i}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \omega_{\mathbb{P}^n})$ does not depend on the regular space in which Y is considered. So we may interchange $\mathbb{P}^n$ with the regular projective scheme. The conclusion is that the dualizing sheaf of Y is also given by

$$\omega_Y = \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Y, \omega_X).$$

Following the steps in an entirely similar way of [Har77, Thm. 7.11], just replacing the projective space $\mathbb{P}^n$ by a regular variety X, one can establish the next result.

Lemma 4.1.2. *Let Y be a closed subscheme of a n-fold regular projective scheme X. Suppose that Y is a locally complete intersection of codimension p in X, we have*

$$\omega_Y = \omega_X \otimes \mathcal{O}_Y \otimes \det(\mathcal{N}_{Y/X}).$$

4.2 Tetragonal Gorenstein curves

A non-hyperelliptic Gorenstein curve C of genus $g > 2$ has two natural ambient spaces to embed it, namely the projective spaces $\mathbb{P}^{g-1}$ and suitable *rational normal scrolls*. While the projective space depends only on its genus g, it is also required to know the gonality of C to embedding it in a suitable scroll. So we briefly recall some basic facts on scrolls.

Given $d > 1$ non-negative integers $e_1 \leq \cdots \leq e_d$, set $e = \sum e_i$ and $N = e + d - 1$. The rational normal scroll $S := S(e_1, \ldots, e_d) \subseteq \mathbb{P}^N$ is the projective variety that, after a choice of coordinates, is the set of points $(x_0 : \ldots : x_N) \in \mathbb{P}^N$ such that

$$\operatorname{rank}\begin{pmatrix} x_0 & \ldots & x_{e_1-1} & x_{e_1+1} & \ldots & x_{e_1+e_2} & \ldots & x_{N-e_d} & \ldots & x_{N-1} \\ x_1 & \ldots & x_{e_1} & x_{e_1+2} & \ldots & x_{e_1+e_2+1} & \ldots & x_{N-e_d+1} & \ldots & x_N \end{pmatrix} \leq 1.$$

By taking d rational normal curves of degrees $e_1, \ldots, e_d$ lying on d complementary linear spaces in $\mathbb{P}^N$, one can see that S is the disjoint union of all $(d-1)$-plane in $\mathbb{P}^N$ determined by choosing a point in each one of the d rational normal curves. Hence, the dimension of S is d and its degree is e. Each $(d-1)$-plane is called a ruling of S. Additionally, S is a smooth variety if, and only if, $e_i > 0$ for all $1 \leq i \leq d$.

On the other hand, taking the smooth variety $\mathbb{P}(\mathcal{E})$ where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_d)$, one can see that there is a birational morphism $\pi : \mathbb{P}(\mathcal{E}) \to S$ induced by $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and π is a rational resolution of singularities of S. When S is singular, it is a cone whose singular locus is a vertex V of dimension $\#\{i\,;\, e_i = 0\} - 1$. The birational morphism π is such that each fiber of $\vartheta : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ is mapped to a ruling. Moreover, one can show that $\operatorname{Pic}(\mathbb{P}(\mathcal{E})) = \mathbb{Z}\tilde{H} \oplus \mathbb{Z}\tilde{R}$, where $[\tilde{H}] = [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)]$ is a hyperplane section while $[\tilde{R}] := [\vartheta^*(\mathcal{O}_{\mathbb{P}^1}(1))]$ is a fiber class, satisfying

$$\tilde{H}^d = e, \quad \tilde{H}^{d-1}\tilde{R} = 1, \quad \tilde{R}^2 = 0 \text{ and } K_{\mathbb{P}(\mathcal{E})} = (e-2)\tilde{R} - d\tilde{H}$$

where $K_{\mathbb{P}(\mathcal{E})}$ is the canonical class of $\mathbb{P}(\mathcal{E})$. We also fix the following notation

$$\mathcal{O}_S(aH + bR) := \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(a\tilde{H} + b\tilde{R}),$$

where $[H]$ is a hyperplane class of S and $[R]$ is a class of a ruling. Note if S is singular with vertex V then $\mathcal{O}_S(aH + bR)$ is a fractional ideal sheaf (or a divisorial sheaf) associated with a suitable Weil divisor of S, more precisely:

- if $\operatorname{codim}(V, S) > 2$ then $\operatorname{Pic}(\mathbb{P}(\mathcal{E}))$ is isomorphic to the group $\operatorname{Cl}(S)$ of Weil divisors of S, hence $\operatorname{Cl}(S) = \mathbb{Z}[H] \oplus \mathbb{Z}[R]$;
- if $\operatorname{codim}(V, S) = 2$ and E stands the exceptional divisor of π, then sequence

  $$0 \to \mathbb{Z} \xrightarrow{\cdot E} \operatorname{Pic}(\mathbb{P}(\mathcal{E})) \xrightarrow{\pi_*} \operatorname{Cl}(S) \to 0,$$

  is exact, and in this case $E \sim \tilde{H} - e\tilde{R}$ and $\operatorname{Cl}(S) = \mathbb{Z}[R]$,

c.f. [Fer01, Proposition 2.1 and Corollary 2.2]. We also can conclude that the dualizing sheaf ω_S is $\pi_*K_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_S((e-2)R - dH)$.

In [LMS19] the authors show that the *canonical model* of any k-gonal singular curve C can be embedded in a $(k-1)$-fold scroll S. In addition, they also show that the possible base points of a $\mathfrak{g}^1_k$ are exactly those lying on the vertex of S. The *canonical model* is defined in [KM09] and it coincides with the canonical curve in $\mathbb{P}^{g-1}$ when C is

Gorenstein. Thus, it is only natural to ask about the stability of the normal sheaves of canonical curves in $(k-1)$-fold scrolls.

It is very easy to see that a canonical smooth (or even a Gorenstein) curve is not a complete intersection in $\mathbb{P}^{g-1}$, provided that its genus is bigger than 5. On the other hand, it is a hard problem to realize canonical (smooth) curves as a complete intersection in known projective varieties. For instance, Mukai [Muk95, Muk10] and Mukai & Ide [MI03] realize the canonical model of smooth curves of small genus (≤ 9) as complete intersections in suitable known projective varieties, such as the product of projective spaces, weighted projective spaces, and Grassmannians. Moreover, in [Sch86] Schreyer proves that any tetragonal smooth curve is a complete intersection in its corresponding 3-fold smooth Scroll $\mathbb{P}(\mathcal{E})$. The next theorem extends the above-cited result due to Schreyer by allowing Gorenstein curves too.

Theorem 4.2.1. *A tetragonal Gorenstein curve is a complete intersection of $Y_1 \sim 2H - b_1R$ and $Y_2 \sim 2H - b_2R$, with $b_1 + b_2 = e - 2$, in the corresponding 3-fold scroll $S = S(e_1, e_2, e_3)$.*

Proof. Let $C \subset \mathbb{P}^{g-1}$ be a tetragonal canonical Gorenstein curve and S be the 3-fold scroll where C lies on it. Let us also take the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E})$ over $\mathbb{P}^1$, that is a resolution of singularities of S by the birational morphism $\pi : \mathbb{P}(\mathcal{E}) \to S$ with exceptional divisor F.

Considering the pushforward π_* , we have that $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_S$. Then we may consider π_* as a functor from the category of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-modules to the category of $\mathcal{O}_S$-modules. Since the scroll S has only rational singularities, one has

$$\pi_*\omega_{\mathbb{P}(\mathcal{E})} = \omega_S, \quad R^j\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} = 0 \text{ and } \quad R^j\pi_*\omega_{\mathbb{P}(\mathcal{E})} = 0 \quad \text{for } j > 0,$$

where $R^j\pi_*$ denote the derived functors, c.f. [Vie77, Kov22]. Hence the pushforward of an exact sequence of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-modules is also an exact sequence of $\mathcal{O}_S$-modules.

Let us first suppose that the $\mathfrak{g}^1_4$ on C has no base points, then we assume that C lies on $\mathbb{P}(\mathcal{E})$ just because C does not pass through the possible singular locus of S. In this way, C is isomorphic to its lifting C' to $\mathbb{P}(\mathcal{E})$. By abuse of notation, we use C to denote the lifting of C as well. Since $\mathbb{P}(\mathcal{E})$ is smooth, the result follows just like in [Sch86], in the following way. The pencil $\mathfrak{g}^1_4$ has no base points. So, if D is a canonical divisor given by the intersection of C with a ruling, then the linear span of any subscheme of D with degree 3 is not a straight line, otherwise, we get a $\mathfrak{g}^1_3$. Hence we are in the range of [Sch86, Lemma, item 3, pg. 119]. The geometric version of Riemann–Roch Theorem for Gorenstein curves assures that $C \subset \mathbb{P}(\mathcal{E})$ has constant Betti numbers over $\mathbb{P}^1$, c.f. [Sch86, Prop. 4.3]. Finally, like in [Sch86, Cor 4.4], $C \subset \mathbb{P}(\mathcal{E})$ admits the following free resolution

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}((e-2)\tilde{R} - 4\tilde{H}) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(b_1\tilde{R} - 2\tilde{H}) \oplus O_{\mathbb{P}(\mathcal{E})}(b_2\tilde{R} - 2\tilde{H}) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_C \to 0, \tag{4.2}$$

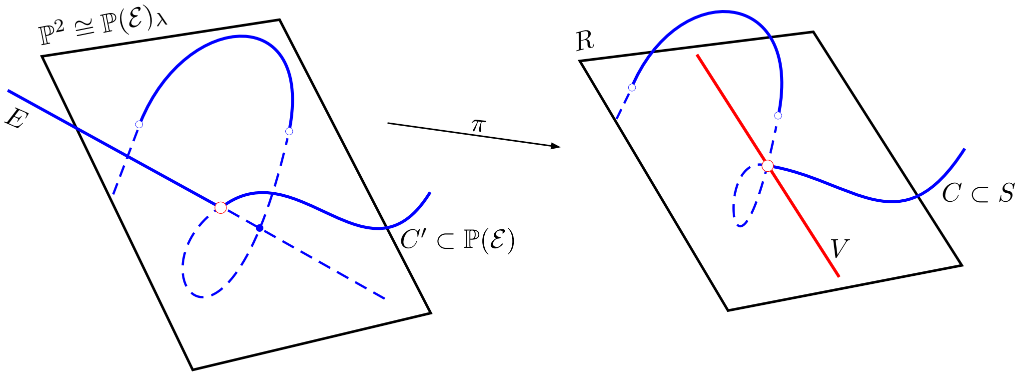
where $b_1 + b_2 = e - 2$ with $b_1, b_2 \geq -1$. Taking the pushforward π_* on the above exact sequence (4.2), we get a free resolution of C on S, implying that C is a complete intersection in S determined by the two mentioned divisors in the statement.

Now let us assume that the pencil $\mathfrak{g}^1_4$ has base points, thus C meets the vertex of S. We also may assume that the base points of $\mathfrak{g}^1_4$ are singular points of C, otherwise, each such base point is removable and so C is no longer tetragonal. The vertex V is either a straight line or a point, i.e. $\text{codim}(V, S) \geq 2$.

We first show that $C \subset S$ has constant Betti numbers over the rulings. So, let D be a subcanonical divisor of degree 4 that is the intersection of C with a ruling. By considering D as a subscheme of dimension 0, by the construction of S we get that the linear space spanned by D in $\mathbb{P}^{g-1}$, say $\overline{D}$, is isomorphic to a $\mathbb{P}^2$. Let $E \subset D$ be a zero-dimensional subscheme of degree 3. If E is contained in a straight line of $\mathbb{P}^2 \cong \overline{D}$, then the geometric version of the Riemann-Roch Theorem for Gorenstein curves implies that E compounds a $\mathfrak{g}^1_3$, that is a contradiction because C is tetragonal. Hence E is not contained in a hyperplane of $\mathbb{P}^2$. Now, by virtue of [Sch86, Lemma 4.2], the Betti numbers of C are constants over the rulings, depending only on the gonality of C.

Next, we consider the lifting of C to $\mathbb{P}(\mathcal{E})$ by the birational morphism π. Since the degree of the divisors given by the intersection of the lifting with the fibers of $\mathbb{P}(\mathcal{E})$ may decreases, i.e. could be smaller than 4, we then take a reducible curve C' that is the lifting of C union with suitable exceptional divisors $\mathbb{P}^1 \cong E_i \subset E$ such that the intersection of C' with the fibers is still 4, see Figure 1.

Figura 1 – Construction of C'



By construction the fibers of $\mathbb{P}(\mathcal{E})$ meet C' in four points, and they also satisfy [Sch86, Lemma 4.2 (3)]. So C' has constant Betti numbers over the fibers depending only on the number of intersections of C' with the fibers, then they are equal to the Betti numbers of the $C \subset S$ over the rulings. Hence $\mathcal{O}_{C'}$ admits a free resolution, c.f. [Sch86, Theorem 3.2 and Corollary 4.4], of the form

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}((e-2)\tilde{R} - 4\tilde{H}) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(b_1\tilde{R} - 2\tilde{H}) \oplus O_{\mathbb{P}(\mathcal{E})}(b_2\tilde{R} - 2\tilde{H}) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_{C'} \to 0,$$

where $b_1 + b_2 = e - 2$ with $b_1, b_2 \geq -1$. Taking the pushforward to S we get that

$$0 \to \mathcal{O}_S((e-2)R - 4H) \to \mathcal{O}_S(b_1 R - 2H) \oplus O_S(b_2 R - 2H) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

is a free resolution of $\mathcal{O}_C$ and we are done. □

4.3 The normal sheaf of a tetragonal curve

If a Gorenstein curve C has gonality 3, then it is determined by a single divisor on the associated 2-fold scroll, automatically implying that $\mathcal{N}_{C/S}$ is stable. The next step is then to consider the normal sheaf of tetragonal Gorenstein curves.

Theorem 4.3.1. *Let C be a tetragonal Gorenstein curve of genus $g \geq 5$ and S its associated 3-fold scroll. If S is smooth, then $\mathcal{N}_{C/S}$ and $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ are unstable.*

Proof. By virtue of Theorem 4.2.1, C is a complete intersection in S, say that C is determined by the divisors Y_1 and Y_2. So the normal sheaf $\mathcal{N}_{C/S}$ splits as a sum of two subsheafs, namely $\mathcal{N}_{C/S} = \mathcal{N}_{Y_1/S} \oplus \mathcal{N}_{Y_2/S}$. Since $\operatorname{rank}(\mathcal{N}_{Y_i/S}) = 1$, one of these two rank one subsheafs destabilize $\mathcal{N}_{C/S}$.

Now we move our attention to the normal sheaf $\mathcal{N}_{C/\mathbb{P}^{g-1}}$. In this case C is no longer a complete intersection whenever $g \geq 6$. The way we choose to show that $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ is also unstable, requires to compute the degree of $\mathcal{N}_{C/S}$.

Let H and R be the hyperplane and the ruling sections of S. Since C is a tetragonal canonical curve, $R \cdot C = 4$ and $\deg(\omega_C) = 2g - 2 = H \cdot C$. Thus, by Lemma 4.1.2 we get

$$\deg(\mathcal{N}_{C/S}) = \deg(\omega_C) - \deg(K_S \otimes \mathcal{O}_C) = 4H \cdot C - (g-5)R \cdot C = 4g + 12,$$

and so $\mu(\mathcal{N}_{C/S}) = 2g + 6$. Since C is a complete intersection in S and S is smooth, the following sequence is exact,

$$0 \longrightarrow \mathcal{N}_{C/S} \longrightarrow \mathcal{N}_{C/\mathbb{P}^{g-1}} \longrightarrow \mathcal{N}_{S/\mathbb{P}^{g-1}} \otimes \mathcal{O}_C \longrightarrow 0.$$

Taking degrees we get $\deg(\mathcal{N}_{C/\mathbb{P}^{g-1}}) = \deg(\mathcal{N}_{C/S}) + \deg(\mathcal{N}_{S/\mathbb{P}^{g-1}} \otimes \mathcal{O}_C)$. On the other hand, by the flatness of $\mathcal{N}_{S/\mathbb{P}^{g-1}}$, we get the exact sequence

$$0 \longrightarrow \mathcal{T}_S \otimes \mathcal{O}_C \longrightarrow \mathcal{T}_{\mathbb{P}^{g-1}} \otimes \mathcal{O}_C \longrightarrow \mathcal{N}_{S/\mathbb{P}^{g-1}} \otimes \mathcal{O}_C \longrightarrow 0.$$

So we may conclude that

$$\deg(\mathcal{N}_{S/\mathbb{P}^{g-1}} \otimes \mathcal{O}_C) = \deg(\mathcal{T}_{\mathbb{P}^{g-1}} \otimes \mathcal{O}_C) - \deg(\mathcal{T}_S \otimes \mathcal{O}_C) =$$
$$- \deg(K_{\mathbb{P}^{g-1}} \otimes \mathcal{O}_C) + \deg(K_S \otimes \mathcal{O}_C) = g(2g-2) - 2g - 14 = 2(g^2 - 2g - 7).$$

Hence $\deg(\mathcal{N}_{C/\mathbb{P}^{g-1}}) = 2(g-1)(g+1)$ and $\mu(\mathcal{N}_{C/\mathbb{P}^{g-1}}) = 2(g-1)(g+1)/(g-2)$, and then

$$2(g-1)(g+1)/(g-2) = \mu(\mathcal{N}_{C/\mathbb{P}^{g-1}}) \leq \mu(\mathcal{N}_{C/S}) = 2(g+3) \text{ for } g \geq 5,$$

finishing the proof. □

We now move our attention to providing sufficient conditions to assure that the 3-fold scroll of a tetragonal Gorenstein curve of genus 6 is smooth. It is known that the ideal of any non-hyperelliptic canonical Gorenstein curve is generated by quadratic forms, except for a quintic plane curve of genus 6, c.f. [CF18, Section 3]. Hence the divisors $Y_i \in \mathrm{Div}(S)$ in the Theorem 4.2.1 are also given by quadratic forms on S, say

$$\mathfrak{Q}_i = \sum_{1 \le j \le k \le 3} P_{ijk}\sigma_j\sigma_k, \quad (\text{with } i = 1, 2), \tag{4.3}$$

where we pick up suitable sections $\sigma_i \in H^0(S, \mathcal{O}_S(H - e_jR))$ corresponding to the rational normal curves, and each P_{ijk} is a homogeneous polynomial in $\mathbf{k}[s,t]$ of degree $e_j + e_k - b_i$.

Lemma 4.3.2. *Let C be a tetragonal Gorenstein curve canonically embedded in $\mathbb{P}^{g-1}$. If $S := S(e_1, e_2, e_3)$ is the 3-fold scroll where C is a complete intersection of the quadratic forms Y_1 and Y_2 as above, then*

(a) *$b_1 \le 2e_2$ and $b_2 \le 2e_3$;*

(b) *$e_3 = b_2 = 0$ if, and only if, C is bi-elliptic;*

(c) *If $e_3 = 0$ and $b_2 = -1$, then C has a $\mathfrak{g}^2_5$.*

Proof. Item **(a)** was proved by Brawner in [Bra97, Proposition 3.1] just applying Riemann–Roch theorem and the irreducibility of C, so it works if C is singular as well. Item **(b)** was also provided by Brawner, [Bra97, Proposition 3.2], provided that C is smooth. Here we just have to make minor adjustments to reach the Gorenstein case.

Assuming that $e_3 = 0$, S is a cone over a scroll of dimension 2, and from $b_2 = 0$, the intersection of Y_1 with a ruling is a conic. Hence the linear system $\mathfrak{g}^1_4$ is given by the composition of two double covers $C \to E \to \mathbb{P}^1$ and Y_1 is a birational ruled surface over E with a rational curve $\tilde{E}$ of double points. Thus, intersecting $\tilde{E}$ with Y_2 we have that

$$2m = \tilde{E} \cdot (2H - b_2R) = 2\deg(\tilde{E}),$$

and $m = 0$ if, and only if, C is smooth. The intersection of H with Y_1 is a curve birational to E, and its arithmetic genus is $g(E) = 1$, because

$$2g(E) - 2 = H(2H - b_1R)(f - 2 - b_1)R = 0.$$

Hence C is bi-elliptic. The proof of the converse of item **(b)** follows exactly as in [Bra97, Proposition 3.2]. It remains to prove item **(c)**. Since $b_1 = e - 1$, C admits a $\mathfrak{g}^1_3$ or a $\mathfrak{g}^2_5$. To see this, we just have to adapt the arguments in [Sch86, pg. 128] using that each Y_i is given by a quadratic form in S, the intersection theory of S and Riemann–Roch theorem for singular curves. By the hypothesis, C does not admit a $\mathfrak{g}^1_3$, hence C has a $\mathfrak{g}^2_5$. $\square$

Definition 4.3.1. A general curve of arithmetic genus 6 is a non-bielliptic tetragonal integral Gorenstein curve not admitting a $\mathfrak{g}_5^2$.

Lemma 4.3.3. *The corresponding scroll of a general curve C of arithmetic genus 6 is smooth.*

Proof. The scroll S can be of three types, namely $S(3,0,0)$, $S(2,1,0)$ or $S(1,1,1)$. If C is in a scroll with $e_3 = 0$ (resp. $e_2 = 0$) then by item **(a)** of Lemma 4.3.2 we get $b_2 \leq 0$. Hence $b_2 = 0$ or $b_2 = -1$. If $b_2 = 0$ (resp. $b_2 = -1$), then by **(b)** (resp. **(c)**) of the Lemma 4.3.2 follows that the curve C is bi-elliptic (resp. C has a $\mathfrak{g}_5^2$). In both cases C is not a general curve as in Definition 4.3.1. Therefore, $S = S(1,1,1)$. □

We finish this section by pointing out two naive declinations of the above results.

Corollary 4.3.4. *The normal sheaf $\mathcal{N}_{C/S}$ of a general curve C embedded in the associated scroll S is stable for arithmetic genera 3 and 4, and unstable for 5 and 6.*

Corollary 4.3.5. *The normal sheaf $\mathcal{N}_{C/\mathbb{P}^5}$ of a general curve C of arithmetic genus 6 is unstable.*

4.4 Local-global method

An important part of studying the stability of the normal sheaf lies in the computation of its degree. Since here we assume Gorenstein singularities, this computation tends to become more involved when the genus grows. So local methods can be useful, addressing the study of the cotangent complex and suitable local invariants, like Deligne and Tjurina numbers. At the very beginning of this section, we just fix some required notation for the remaining of this thesis.

Let $C \subset \mathbb{P}^r$ be a reduced projective curve defined over an algebraically closed field $\mathbf{k}$ of characteristic zero. When C is not smooth, the fundamental sequence involving the tangent and the normal sheafs is no longer a short exact sequence, but it extends to an exact sequence with four terms

$$0 \to \mathcal{T}_C \to \mathcal{T}_{C/\mathbb{P}^r} \to \mathcal{N}_{C/\mathbb{P}^r} \to \mathrm{T}^1 \to 0\,, \tag{4.4}$$

where $\mathrm{T}^1 := \mathrm{T}^1_{C/\mathbf{k}}$ stands for the first cohomology module of the cotangent complex of C, see 3.2.4. Furthermore, [LS67, Lemma 3.1.2] assures that the $\mathcal{O}_C$-module T^1 can be taken as the cokernel of the map $\mathcal{T}_{C/\mathbb{P}^r} \to \mathcal{N}_{C/\mathbb{P}^r}$.

Given a singular point $P \in C$, $\tau_P := \dim_k(\mathrm{T}^1_P)$ stands for the *Tjurina number* at $P \in C$, see [Gre20] for the connection between the Tjurina algebra and the cotangent complex of an analytic germ. As an immediate consequence of Lemma 4.1.1, we get

$$\deg(\mathrm{T}^1) = \sum_{P \in \mathrm{Sing}(C)} \tau_P =: \tau,$$

that is the so-called *Tjurina number of* C. We also recall that the *singularity degree of* C *at* P is defined by $\delta_P := \dim_{\mathbf{k}} \overline{\mathcal{O}}_{C,P}/\mathcal{O}_{C,P}$, where $\overline{\mathcal{O}}_{C,P}$ is the normalization of $\mathcal{O}_{C,P}$, and then we set $\delta := \sum_{P\in C} \delta_P$, the singularity degree of C. We also consider the skyscraper sheaf $\mathfrak{D}_C$ such that its stalk at any $P \in C$ is

$$\mathfrak{D}_{C,P} = \mathrm{coker}(\mathrm{Der}_{\mathbf{k}}(\mathcal{O}_{C,P}, \mathcal{O}_{C,P}) \to \mathrm{Der}_{\mathbf{k}}(\overline{\mathcal{O}}_{C,P}, \overline{\mathcal{O}}_{C,P})). \tag{4.5}$$

Here we assume that the ground field has characteristic zero in order that the above map is an inclusion between the derivations modules, and so we fix[1]

$$\theta_P := \dim_{\mathbf{k}} \mathfrak{D}_{C,P} \quad \text{and} \quad \theta := \sum_{P \in \mathrm{Sing}(C)} \theta_P.$$

Finally, we also consider the *Deligne numbers*:

$$e_P := 3\delta_P - \theta_P \quad \text{and} \quad e := \sum_{P \in \mathrm{Sing}(C)} e_p.$$

Theorem 4.4.1. *Let $C \subset \mathbb{P}^r$ be an integral projective curve of degree d whose arithmetic genus is g. Then the degree of the normal sheaf of C in $\mathbb{P}^r$ is given by*

$$\deg(\mathcal{N}_{C/\mathbb{P}^r}) = 2g - 2 + (r+1)d + \tau - e. \tag{4.6}$$

Proof. We start by taking the Euler characteristic in the sequence (4.4),

$$\chi(\mathcal{T}_C) - \chi(\mathcal{T}_{C/\mathbb{P}^r}) + \chi(\mathcal{N}_{C/\mathbb{P}^r}) - \chi(\mathrm{T}^1) = 0.$$

The sheaf of differentials $\Omega_{C|\mathbf{k}}$ of C may not be torsion-free, but the tangent sheaf $\mathcal{T}_C$ is a coherent fractional ideal sheaf, so the singular version of the Riemann–Roch Theorem ensures that

$$\chi(\mathcal{T}_C) = \deg(\mathcal{T}_C) + 1 - g.$$

Let $\widetilde{C}$ be the non-singular model of C and $\widetilde{g}$ its geometric genus. The bundle of differentials $\Omega_{\widetilde{C}|\mathbf{k}}$ is invertible of degree $2\widetilde{g} - 2$, where $\widetilde{g} = g - \delta$. Thus, $\mathcal{T}_{\widetilde{C}}$ is an invertible sheaf of degree $2 - 2\widetilde{g}$ and so

$$\deg(\mathcal{T}_C) = \deg(\mathcal{T}_{\widetilde{C}}) - \theta + \delta = 2 - 2g - e.$$

Since $\mathcal{N}_{C/\mathbb{P}^r}$ and $\mathcal{T}_{C/\mathbb{P}^r}$ have rank $r-1$ and r, respectively, the sequence (4.4) and Lemma 4.1.1 imply that $\deg(\mathcal{T}_C) - \deg(\mathcal{T}_{C/\mathbb{P}^r}) + \deg(\mathcal{N}_{C/\mathbb{P}^r}) - \tau = 0$, and so

$$\deg(\mathcal{N}_{C/\mathbb{P}^r}) - \deg(\mathcal{T}_{C/\mathbb{P}^r}) = 2g - 2 + \tau - e.$$

[1]The number θ_P is usually detonated by m_1, it was introduced by Deligne in [Del73] when dealing with only one singular point, we changed the notation just to avoid multiples sub-indexes.

Now we just have to compute the degree of the relative tangent sheaf $\mathcal{T}_{C/\mathbb{P}^r}$, that is given by $\det(\mathcal{T}_{\mathbb{P}^r} \otimes \mathcal{O}_C) = \det(\mathcal{T}_{\mathbb{P}^r}) \otimes \mathcal{O}_C = \omega_{\mathbb{P}^r}^{\vee} \otimes \mathcal{O}_C = \mathcal{O}_{\mathbb{P}^r}(r+1) \otimes \mathcal{O}_C$, where i is embedding $i : C \hookrightarrow \mathbb{P}^r$. Hence

$$\deg(\mathcal{O}_{\mathbb{P}^r}(r+1) \otimes \mathcal{O}_C) = \deg(i^*(\mathcal{O}_{\mathbb{P}^r}(r+1))) = (\deg i)\deg(\mathcal{O}_{\mathbb{P}^r}(r+1)) = d(r+1),$$

that concludes the proof. □

Example 4.4.1. The above formula for the degree of the normal sheaf can be very manageable for suitable classes of curves. For example, let $\mathcal{S} \subset \mathbb{N}$ be a numerical semigroup of genus $g := \#(\mathbb{N} \setminus \mathcal{S}) > 1$ generated by $a_1, \ldots, a_r$, and

$$C_{\mathcal{S}} := \{(t^{a_1}, \ldots, t^{a_r})\,;\, t \in \mathbf{k}\} = \operatorname{Spec} \mathbf{k}[\mathcal{S}] \subset \mathbb{A}^r$$

be the affine monomial curve associated to $\mathcal{S}$. It is very known that $C_{\mathcal{S}}$ has a unique unibranch singular point $Q = (0, \ldots, 0)$. Equivalently, the semigroup algebra $\mathbf{k}[\mathcal{S}]$ corresponds uniquely to an equisingular class of an (analytic germ when $\mathbf{k} = \mathbb{C}$) algebra of a branch whose singular semigroup is $\mathcal{S}$, see [Zar06, Corollary 1.2.4 pg. 117]. The affine curve $C_{\mathcal{S}}$ is Gorenstein if, and only if, $\mathcal{S}$ is symmetric, i.e. the largest gap of $\mathcal{S}$ is the biggest possible, $\ell_g = 2g - 1$.

We associated to $C_{\mathcal{S}}$ a projective curve in $\mathbb{P}^r$ by adding just one smooth point at the infinite. Here, we use the same notation for this projective curve and the affine one. Assuming that $\mathcal{S}$ is symmetric and $2 \notin \mathcal{S}$, $C_{\mathcal{S}}$ can be considered as a canonical Gorenstein curve, i.e. a curve of genus g and degree $2g-2$ in $\mathbb{P}^{g-1}$. So all the invariants that appear in formula (4.6) of Theorem 4.4.1 are known, with the possible exception of the Tjurina number, namely $3\delta = 3g$, $(r+1)d = g(2g-2)$ and $\theta = \theta_Q = 1 + g - \lambda(\mathcal{S}) = g$, where $\lambda(\mathcal{S}) := [\operatorname{End}(\mathcal{S}) : \mathcal{S}] = 1$ because $\mathcal{S}$ is symmetric, and by the very definition, $\operatorname{End}(\mathcal{S})$ is the set of all $n \in \mathbb{N}$ such that $n + s \in \mathcal{S}\, \forall\, s \in S \setminus \{0\}$. Hence

$$\deg(\mathcal{N}_{C_{\mathcal{S}}/\mathbb{P}^{g-1}}) = g(2g-2) + \tau - 2.$$

Unfortunately, we do not know a general formula for the Tjurina number τ depending only on the genus of $C_{\mathcal{S}}$, but there is an implementable method to compute it. We first recall a result due to Herzog [Her70] assuring that the ideal of $C_{\mathcal{S}}$ can be generated by isobaric polynomials F_i that are differences of two monomials, namely

$$F_i := X_1^{\alpha_{i1}} \ldots X_r^{\alpha_{ir}} - X_1^{\beta_{i1}} \ldots X_r^{\beta_{ir}},$$

with $\alpha_i \cdot \beta_i = 0$. As usual, the weight of F_i is $d_i := \sum_j n_j \alpha_{ij} = \sum_j n_j \beta_{ij}$. For each i, let $v_i := (\alpha_{i1} - \beta_{i1}, \ldots, \alpha_{ir} - \beta_{ir})$ be a vector in $\mathbf{k}^r$ induced by F_i. Next a result due to Buchweitz, c.f. [Buc80, Thm. 2.2.1], computes the Tjurina number for monomial curves.

Theorem 4.4.2 (Buchweitz)**.** $\tau = \sum_{s\in\mathbb{Z}} \dim_{\mathbf{k}} T^1_s$, *where for each* $\ell \notin \mathrm{End}(\mathcal{S})$,

$$\dim T^1_\ell = \#\{i \in \{1,\ldots,r\}\,;\, n_i + \ell \notin \mathcal{S}\} - \dim V_\ell - 1,$$

V_ℓ *is the subvector space of* $\mathbf{k}^r$ *generated by the vectors* v_i *such that* $d_i + \ell \notin \mathcal{S}$*. We also have that*

$$\dim T^1_s = 0,\ \forall\, s \in \mathrm{End}(\mathcal{S}).$$

Finally, by the very explicit and implementable method in [CF18] and [CS13], we know that the ideal of $C_{\mathcal{S}} \subseteq \mathbb{P}^{g-1}$ is given by suitable $\frac{1}{2}(g-3)(g-2)$ quadratic and isobaric forms, when the first non zero element n_1 of $\mathcal{S}$ is such that $3 < n_1 \le g-1$ and $\mathcal{S} \ne <4,5>$, and by $\frac{1}{2}(g-3)(g-2)$ quadratic and isobaric forms added to $\binom{g+2}{3} - 5g + 5$ cubic and isobaric forms in the remaining cases. In this way, one may implement an algorithm to compute the Tjurina number τ in all these cases. To conclude this example, we collect in Table 1 some examples in low genus.

Tabela 1 – degree of the normal sheaf $\mathcal{N}_{C_{\mathcal{S}}/\mathbb{P}^{g-1}}$ when $\mathcal{S}$ is symmetric and $4 \le g \le 7$

$\mathcal{S}$	g	τ	$\deg(\mathcal{N}_{C_{\mathcal{S}}/\mathbb{P}^{g-1}})$	$\mathcal{S}$	g	τ	$\deg(\mathcal{N}_{C_{\mathcal{S}}/\mathbb{P}^{g-1}})$
$<3,5>$	4	8	30	$<6,7,8,9,10>$	6	15	73
$<4,5,6>$	4	8	30	$<3,8>$	7	14	96
$<4,6,7>$	5	10	48	$<4,7,10>$	7	14	96
$<5,6,7,8>$	5	10	48	$<4,6,11>$	7	14	96
$<3,7>$	6	12	70	$<5,7,9,11>$	7	14	96
$<4,6,9>$	6	12	70	$<5,6,9>$	7	14	96
$<4,5>$	6	12	70	$<6,8,9,10,11>$	7	17	99
$<5,7,8,9>$	6	12	70	$<7,8,9,10,11,12>$	7	21	103

An immediate consequence of the above global results Theorem 4.4.1 and Lemma 4.1.2 is the following known local result, c.f. [Gre20, Section 2.6].

Corollary 4.4.3. *If* $C \subset \mathbb{P}^r$ *is a locally complete intersection integral curve, then* $\tau = e$*.*

Now let $\mathcal{Z}$ be a reduced projective scheme just admitting isolated singularities and assume that $\eta : \mathcal{Z} \to S$ is a flat morphism whose fibers $\mathcal{Z}_s$ are reduced projective curves. So, under these conditions, we are able to consider the sheaves $\mathrm{T}^1_{\mathcal{Z}}, \mathcal{D}_{\mathcal{Z}}$ and $\overline{\mathcal{O}}_{\mathcal{Z}}/\mathcal{O}_{\mathcal{Z}}$ as sheaves whose stalks on a fiber Z_s are just $\mathrm{T}^1_{Z_s}$, $\mathcal{D}_{Z_s}$ and $\overline{\mathcal{O}}_{\mathcal{Z}_s}/\mathcal{O}_{\mathcal{Z}_s}$, see [Zar06, pg. 148] for further details. Moreover, these sheaves are flat at over S. Therefore, we can apply the upper semicontinuity theorem, c.f. [Har77, Thm. 12.8], to $s \mapsto \dim H^0(\mathcal{Z}_s, \mathrm{T}^1_{\mathcal{Z}_s})$, $s \mapsto \dim H^0(\mathcal{Z}_s, \mathcal{D}_{\mathcal{Z}_s})$ and $s \mapsto \dim H^0(\mathcal{Z}_s, \overline{\mathcal{O}}_{\mathcal{Z}_s}/\mathcal{O}_{\mathcal{Z}_s})$, one can easily check the following result.

Proposition 4.4.4. *Under the above conditions, the invariants* τ*,* θ *and* δ *are upper semicontinuous functions.*

Complete intersections are stable under flat deformations with a connected base, provided that the ground field has characteristic zero, and once the degree of the normal sheaf of a complete intersection curve only depends on the genus and the degree of the curve, c.f. Corollary 4.4.3 and Theorem 4.4.1, it follows that the degree of the normal sheaf of a complete intersection curve is stable under flat deformations.

On the other hand, it is easy to see that the degree of the normal sheaf cannot be an upper semi-continuous function, just because smooth curves can degenerate to singular ones. But if one keeps the singular degree under deformations, it is easy to see that the degree of the normal sheaf is a semicontinuous function as well.

The notion of a deformation preserving the singularity degree appears in the literature when we (formally) deform an analytic germ of a singularity. In a survey, [Gre20] by Greuel, the author shows that the singularity degree of an analytic germ is preserved if, and only if, it is *contractible*, cf. [Gre20, Corollary 2.46]. We also should mention a result due to Teissier, cf. [Zar06, 2.10 Theorem 3], showing that for a given numerical semigroup $\mathcal{S}$, the positively graded part $\mathrm{T}^{1,+}(\mathbf{k}[\mathcal{S}])$ of the cotangent complex is a miniversal space for deformations of the affine monomial curve $C_{\mathcal{S}}$ with a reducible base such that each fiber has a singular point whose singular semigroup is $\mathcal{S}$.

An immediate consequence of Theorem 4.4.1 and Proposition 4.4.4 is that, over suitable conditions, the degree of the normal sheaf is an upper semicontinuous function.

Corollary 4.4.5. *Let $C \subseteq \mathbb{P}^r$ be a reduced projective curve of degree d and arithmetic genus g. We have that $\deg(\mathcal{N}_{C/\mathbb{P}^n})$ is an upper semicontinuous function for embedded deformations preserving the singularity degree δ. In other words, if $\eta : \mathcal{X} \to S$ is a deformation of $C = \mathcal{X}_{\mathbf{0}}$, with $\mathcal{X} \subseteq S \times \mathbb{P}^r$ just admitting isolated singularities, such that each fiber has constant singularity degree, i.e. $\delta(\mathcal{X}_s) = \delta(C)$ for each $s \in S$, then*

$$\deg(\mathcal{N}_{\mathcal{X}_s/\mathbb{P}^r}) \leq \deg(\mathcal{N}_{C/\mathbb{P}^r}) = 2g - 2 + (r+1)d + \tau - e.$$

References

[AFO16] M. Aprodu, G. Farkas, and A. Ortega. Restricted Lazarsfeld-Mukai bundles and canonical curves, *Development of moduli theory in: Advanced studies in pure mathematics. Mathematical Society of Japan*, 96 (2016) pp. 303–322.

[AC81] E. Arbarello and M. Cornalba, Footnotes to a paper of Beniamino Segre, *Math. Ann.*, 256, (1981), 341–362.

[ACGH85] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, Geometry of algebraic curves, Vol I, *Grundlehren der Mathematischen Wissenschaften* 267, Springer-Verlag, New York (1985).

[Bra97] J. Brawner, Tetragonal Curves, Scrolls and $K3$ Surfaces. *Trans. Am. Math. Soc.*, 349 (1997) 3075-3091.

[Bru17] G. Bruns, The normal bundle of canonical genus 8 curves. *arXiv preprint:* https://arxiv.org/abs/1703.06213 (2017).

[BH98] W. Bruns and H. J. Herzog, Cohen-Macaulay Rings. *Cambridge university press*, 1998.

[Buc80] R-O. Buchweitz, On deformations of monomial curves, *Lecture notes in Mathematics* 777 (1980) 205–220.

[Cav16] R. Cavalieri, Moduli spaces of pointed rational curves. Lecture Notes for the Graduate Student School in the Combinatorial Algebraic Geometry program at the Fields Institute July, p. 18-22, 2016.

[CS13] A. Contiero and K. O. Stöhr, Upper bounds for the dimension of moduli spaces of curves with symmetric Weierstrass semigroups, *J. London Math. Soc.* 88 (2013) 580–598.

[CFV18] A. Contiero, L. Feital and R. V. Martins, Max Noether's Theorem for integral curves. *Journal of Algebra*, v. 494, p. 111-136, 2018.

[CF18] A. Contiero and A. Fontes, On the locus of curves with an odd subcanonical marked point, *arXiv preprint:* https://arxiv.org/abs/1804.09797 (2018).

[Del73] P. Deligne, Intersections sur les surfaces régulières *SGA 7, Exposé X, Lectures notes in Mathematics* 340 (1973) 1–37.

[EH87-1] D. Eisenbud and J. Harris,‘Existence, decomposition, and limits of certain Weierstrass points’, *Invent. Math.* 87 (1987) 495–515.

[EH87-2] D. Eisenbud and J. Harris. On varieties of minimal degree. *In Proc. Sympos. Pure Math* (Vol. 46, No. 1, pp. 3-13) (1987).

[Fer01] R. Ferraro, Weil divisors on rational normal scrolls, *Geometric and combinatorial aspects of Commutative Algebra, Lectures Notes in Applied and Pure Mathematics*, Vol. 217, Marcel Dekker Ser., New York (2001) 183–198.

[Ful69] W. Fulton, Hurwitz schemes and irreducibility of moduli of algebraic curves, *Ann. Math.* 90 (1969) 542–575.

[Gre20] GM Greuel, Deformation and Smoothing of Singularities. *Handbook of Geometry and Topology of Singularities I*, Springer Verlage, Cham. (2020) 389–448.

[Gro59] A. Grothendieck, Théorèmes de dualité pour les faisceaux algebraic cohérents. *Bourbaki*, 1959.

[Har77] R. Hartshorne, Algebraic geometry. *Graduate Texts in Mathematics*, Springer Verlage, 1977.

[Har86] R. Hartshorne, Generalized divisors on Gorenstein curves and a theorem of Noether. *Journal of Mathematics of Kyoto University*, v. 26, n. 3, p. 375-386, 1986.

[Har10] R. Hartshorne, Deformation theory. *New York: Springer* Vol. 257, 2010.

[HM06] J. Harris and I. Morisson, Moduli of Curves. *Graduate Texts in Mathematics*, Springer Verlage, 2006.

[Her70] J. Herzog, Generators and relations of abelian semigroups and semigroup rings. *Manuscripta Math* 3 (1970) 175–193.

[Hos16] V. Hoskins, Moduli problems and geometric invariant theory. Lecture Notes (2016).

[KM09] S. Kleiman and R.V. Martins, The canonical model of a singular curve. *Geometriae Dedicata*, 139 (2009) 139–166.

[Kov22] S. Kovacs, Rational Singularites, *arXiv preprint:* https://arxiv.org/abs/1703.02269 (2022).

[LMS19] D. Lara, Martins, R. and J. Souza, M. On gonality, scrolls, and canonical models of non-Gorenstein curves. *Geometriae Dedicata*, (2019) 1–23.

[LS67] S. Lichtenbaum and M. Schlessinger, The Cotangent Complex of a Morphism, *Trans. Am. Math. Soc.* 128 (1967) 41–70.

[Muk95] S. Mukai. Curves and symmetric spaces, I. *American Journal of Mathematics*, 117 (1995) 1627–1644.

[Muk10] S. Mukai. Curves and symmetric spaces, II. *Ann. of Math.* 172 (2010)

[MI03] S. Mukai and M. Ide, Canonical curves of genus eight, *Proceedings of the Japan Academy* 79 (2003) 59-64.

[Nar13] H. Nari, Symmetries on almost symmetric numerical semigroups, Semigroup Forum, 86(1), (2013), 140-154.

[Pin74] H. Pinkham, Deformations of algebraic varieties with G_m-action', *Astérisque 20*, (1974).

[RG-S09] J.C. Rosales e P.A. García-Sánchez, Numerical Semigroups, Springer-Verlag, New York, 2009.

[Ros52] M.Rosenlicht, Equivalence relations on algebraic curves, *Ann.of math* (1952), 228-240.

[Sch86] F. O. Schreyer. Syzygies of canonical curves and special linear series. *Math. Ann.*, 275 (1986) 105–137.

[Sch20] J. Schmitt, The moduli space of curves. Genus, 1, 46 (2020).

[Ser07] E. Sernesi, Deformations of algebraic schemes. Springer Science & Business Media, 2007.

[Sto93-1] K. O. Stöhr, On the Poles of Regular Differentials of Singular Curves, *Bol. Soc. Bras. Mat*, 24, (1993), 105–136.

[Sto93-2] K. O. Stöhr, On the moduli spaces of Gorenstein curves with symmetric Weierstrass semigroups, *J. reine angew. Math.* 441 (1993) 189–213.

[Tru18] C. C. Trung, Gorenstein Rings, *Bachelor Thesis*, Ho Chi Minh City, 66 (2018)

[Vie77] E. Viehweg, Rational singularities of higher dimensional schemes, *Proc. Amer. Math. Soc.* 63 (1977) 6–8.

[Zar06] O. Zariski, with an appendix by B. Teissier, The moduli problem for plane branches, *American Mathematical Society*, University Lecture Series, 39 (2006) 111–151.