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From model to quasi-categories

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From model to quasi-categories

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RESUMO

Esta dissertação é uma humilde introdução a grandes ferramentas da teoria da homotopia abstrata: categorias modelo e ∞-categorias (através de quasicategorias). A primeira parte do texto apresenta estas estruturas, conectando com diversos tópicos de teoria das categorias, teoria da homotopia clássica e álgebra homológica. Na segunda parte, usamos este ferramental para estudar questões internas à própria teoria da homotopia, e para comparar essas duas linguagens que estudamos através da localização simplicial.

Palavras Chave: Teoria das Categorias. Teoria da Homotopia. Topologia Algébrica. Álgebra Homológica.

ABSTRACT

This dissertation is a humble introduction to important tools of abstract homotopy theory: model categories and ∞-categories (through quasicategories). The first part of the text presents these frameworks, connecting them with various topics from category theory, classical homotopy theory and homological algebra. In the second part, we use these tools to study issues internal to homotopy theory itself, and to compare these two languages that we study through simplicial localization.

Keywords: Category Theory. Homotopy Theory. Algebraic Topology. Homological Algebra.

CONTENTS

Introduction **7**

I Foundations **15**

1 Model Categories **16**

1.1 Localization 16

1.2 Model structures 18

1.3 Examples of model categories 20

1.4 The homotopy category 22

1.5 Quillen equivalences 25

2 Simplicial sets **27**

2.1 The simplicial category 27

2.2 Simplicial objects 31

2.3 Boundaries and horns 35

3 Examples of simplicial sets **40**

3.1 Categories 41

3.2 Topological spaces 47

3.3 Chain complexes 50

4 Quasicategories **53**

4.1 Definitions 53

4.2 Homotopy of morphisms 55

4.3 Functors 61

4.4 Trivial Kan fibrations 65

II Applications **69**

5 Homotopy hypothesis **70**

5.1 Spaces as ∞-groupoids 70

5.2 Simplicial categories 72

5.3 A nerve 74

5.4 Towards quasicategories 78

6 Simplicial localization **81**

6.1 Relative categories 82

6.2 Simplicial localizations 83

6.3 Towards ∞-categories 85

Appendix **88**

Monoidal categories 88

Enriched categories 90

Creating functors 91

INTRODUCTION

Although every category comes with a natural classification problem ("classify the objects up to isomorphism"), in most scenarios this endeavor is unreasonable. Homotopy theory studies the tools to deal with the situation where the category possesses a weaker notion of equivalence between objects.

For instance, in any introductory Topology course one is readily warned that the task of classifying spaces up to homeomorphism is simply impossible. While some invariants can be easily handed (compactness, connectedness, separability), these are far from enough to characterize all of **Top**. With this, *homotopy equivalence* is presented as a weaker notion of equivalence between spaces. We are given the much more robust *homotopical* invariants, the homology groups H_n and the homotopy groups π_n, and with these one can go as far as classifying, up to homotopy equivalence, all 2-dimensional surfaces [17].

Nevertheless, the niceness of homotopy equivalences is a double-edged sword: the category **HTop** of spaces and continuous functions up to homotopy is terrible. First it is not concrete, and also it loses both the completeness and cocompleteness of **Top** (see [45], pp. 135). Furthermore it's simply not a category where we got many more invariants.

To adress this issue, the paradigm adopted by homotopy theorists is the following: we work with an even weaker notion of equivalence, but between sufficiently "nice" spaces these will reproduce homotopy equivalences, and any space is weakly equivalent to a nice space. Then we can keep our nice, bicomplete category of spaces around, while still doing homotopical constructions in the homotopy category of nice spaces.

This strategy went a long way between the 30s and the 60s with the construction of powerful invariants, but it was only in 1967 [38] that a far reaching axiomatization of what was going on happened via *model categories*, which are categories with weak equivalences with more data. The additional bells and whistles from the model structure give an equivalent yet much more tractable construction of the homotopy theory at hand.

In this thesis we will explore many other scenarios where model categories clarify a homotopical situation, such as:

- chain complexes and chain homotopies;
- chain complexes and quasi-isomorphisms;
- categories and equivalences;
- $(\infty, 1)$-categories and categorical equivalences.

Another advantage of model categories is that they provide a clean way of comparing different homotopy theories through *Quillen equivalences*. In fact, this definition is so convenient that several theorems in homotopy theory are about Quillen equivalences, for instance the equivalence between the homotopy theories of simplicial sets and topological spaces, of chain complexes and simplicial abelian groups, and between all models of $(\infty, 1)$-categories. The first and third of these examples are the content of Chapters 2 and 4, respectively.

Because it's not all roses, model categories have their own tax. Most of the constructions are in the end done in the homotopy category, where we only consider homotopy *classes* of morphisms. This is a destructive procedure: by allowing ourselves to work up to homotopy, we lost the data of the homotopies themselves!

This is where $(\infty, 1)$-categories come into play: as we will see from Chapters 4 through 6, they provide a framework that contains the objects and morphisms at hand, with higher morphisms corresponding to homotopies and also higher coherence data. Moreover, we can recover the homotopy category by quotienting out by 2-morphisms.

Today $(\infty, 1)$-categories play a central to homotopy theory, with much of the research being given entirely in this new language. And in fact they are an enlargement of (simplicial) model categories, who constitute a smaller section of the theory; we will see this in Chapter 6.

This is not to say that model categories are now useless, quite the opposite. In fact, it is still standard to start a new direction by looking for a model category [4,33], even in the theory of $(\infty, 1)$-categories itself [6]; they are simply much more concrete, at this point of history.

OUTLINE

This thesis is divided in two parts. In the first part deals with foundational homotopical tools, and in the second we show how they can be used to enlighten our own understanding of homotopy theory itself.

Chapter 1 deals with model categories, the first abstract framework for homotopy theories. We start by discussing categories with weak equivalences and seeing how this concept broadens the usual notions of homotopy. Only then we define model structures, giving plenty of examples and developing the techniques that make model categories so useful. In particular, we construct the homotopy category of a model structure. We end this chapter by looking at Quillen equivalences, which give a way to compare different model categories.

In Chapter 2 we start to look at simplicial sets, basic instruments in a homotopy theorist's pocket. We do this by first defining and proving lemmas on the simplicial category $\mathbf{\Delta}$, and, because simplicial sets are defined as presheaves on $\mathbf{\Delta}$, we immediately get a mild understanding of these structures. There is also an emphasis on visualization of simplicial sets.

Examples of simplicial sets are deferred to Chapter 3, where we put spaces, categories and chain complexes (and their homotopy theories) in a simplicial context. Later, in Chapter 4, we see that the simplicial sets constructed in this section are actually quasicategories, i.e. $(\infty, 1)$-categories.

We finish Part I by devoting Chapter 4 to an introduction to quasicategories, the most developed model of $(\infty, 1)$-categories. We give basic definitions and exhibit how to retrieve the homotopy category of a quasicategory, which discards higher dimensional data. Afterwards we turn to (∞-)functors between quasicategories, focusing on intuition and examples, and we intuit a definition for ∞-equivalence between quasicategories. We close this section by using trivial Kan fibrations to formalize a popular slogan in ∞-category theory.

Part II has two chapters:

- In Chapter 5 we use model categories to describe precisely the slogan

  "$(\infty, 1)$-categories are categories enriched over spaces"

  Most precisely, we first discuss how the Quillen equivalence between the classical model structures of spaces and simplicial sets can be interpreted as an equivalence between sapces and ∞-groupoids - in the quasicategorical sense. Then we tour through simplicial categories, i.e. categories enriched over **sSet**, to formally describe "categories enriched over spaces" as we wanted. We close this section by formalizing the slogan as a Quillen equivalence.

- Chapter 6 is devoted to a short discussion on *simplicial localization*, a canonical way to compare the two languages we have for homotopy theory: model categories and quasicategories. We give two constructions for this localization, and discuss how exactly an ∞-category is retrieved from a model category.

INTRODUÇÃO

Embora cada categoria venha com um problema de classificação natural ("classifique os objetos a menos de isomorfismo"), na maioria dos cenários esta se mostra uma meta homérica. A teoria da homotopia traz então ferramentas para lidar com a situação em que a categoria possui uma noção mais fraca de equivalência entre objetos.

Por exemplo, em qualquer curso introdutório de Topologia se avisa como a tarefa de classificar espaços até o homeomorfismo é basicamente impossível. Enquanto alguns invariantes podem ser facilmente manipulados (compacidade, conectividade, separabilidade), estes estão longe de serem suficientes para caracterizar toda **Top**. Com isso, *equivalência de homotopia* se apresenta como uma noção mais fraca de equivalência entre espaços. Recebemos os invariantes *homotópicos*, muito mais robustos, os grupos de homologia H_n e os grupos de homotopia π_n, e com estes conseguimos até mesmo classificar, a menos de equivalência de homotopia, todas as superfícies bidimensionais [17].

No entanto, essa estratégia com equivalências de homotopia é uma faca de dois gumes: a categoria **HTop** de espaços e funções contínuas até homotopia é bem menos bem comportada que **Top**. Primeiramente essa categoria não é concreta, mas também ela perde tanto a completude quanto a cocompletude de **Top** (veja [45], pp. 135). Ainda por cima, essa simplesmente não é uma categoria onde temos muitos outros invariantes.

Para resolver esta questão, um paradigma adotado pelos pesquisadores homotópicos foi o seguinte: trabalhamos com uma noção ainda mais fraca de equivalência, mas entre espaços suficientemente "legais" estes irão reproduzir equivalências de homotopia, e qualquer espaço é fracamente equivalente a um espaço agradável . Então podemos manter nossa categoria bicompleta de espaços por perto enquanto ainda fazemos construções homotópicas na categoria de homotopia de espaços convenientes.

Essa estratégia percorreu um longo caminho entre os anos 30 e 60 com a construção de invariantes poderosos, mas foi em 1967 [38] que uma axiomatização de longo alcance do que estava acontecendo aconteceu via *categorias modelo*, que são categorias com equivalências fracas e outras definições técnicas. As ferramentas adicionais da estrutura do modelo fornecem uma construção muito equivalente, porém muito mais tratável, da teoria da homotopia em questão.

Nesta dissertação exploraremos muitos outros cenários em que as categorias modelo esclarecem um contexto em que se usa homotopia, como:

- espaços topológicos e homotopia;
- espaços e equivalência homotópicas fracas;
- complexos e quasi-isomorfismos;
- categorias e equivalências;
- $(\infty, 1)$-categorias e equivalências categóricas.

Outra vantagem das categorias de modelo é que elas fornecem uma maneira clara de comparar diferentes teorias de homotopia por meio de *equivalências de Quillen*. Na verdade, esta definição é tão conveniente que vários teoremas na teoria da homotopia são sobre equivalências de Quillen, por exemplo, a equivalência entre as teorias de homotopia de conjuntos simpliciais e espaços topológicos, de complexos de cadeia e de grupos abelianos simpliciais, e entre todos os modelos de $(\infty, 1)$-categorias. Discutimos alguns destes resultados ao longo do texto.

Como nem tudo são rosas, as categorias de modelos têm suas próprias limitações. Por exemplo, a maioria das construções são feitas na categoria de homotopia, onde consideramos apenas *classes de homotopia* de morfismos. Este é um procedimento destrutivo: ao nos permitimos trabalhar a menos homotopia, perdemos os dados das próprias homotopias!

É aqui que as $(\infty, 1)$-categorias entram em ação: como veremos nos Capítulos 4 a 6, elas fornecem uma estrutura que contém os objetos e morfismos em questão, com morfismos de dimensão maior correspondendo a homotopias e também a relações de coerência de dimensão superior. Além disso, podemos recuperar a categoria de homotopia quocientando essa $(\infty, 1)$-categoria a menos de 2-morfismos.

Hoje $(\infty, 1)$-categorias desempenham um papel central para a teoria da homotopia, com grande parte da pesquisa sendo dada inteiramente nesta nova linguagem. E, de fato, são uma ampliação de categorias de modelos (simpliciais), que constituem uma seção menor da teoria; veremos isso no Capítulo 6.

Isso não quer dizer que as categorias de modelos agora sejam inúteis, muito pelo contrário. Na verdade, ainda é padrão começar uma nova direção procurando uma categoria de modelo [4, 33], mesmo na teoria das próprias $(\infty, 1)$-categorias [6]. Neste momento da história, a concretude das categorias modelo ainda é muito bem vinda.

OUTLINE

Esta dissertação está dividida em duas partes. Na primeira tratamos de ferramentas homotópicas fundamentais, e na segunda mostramos como elas podem ser usadas para esclarecer nossa própria compreensão da própria teoria da homotopia.

O Capítulo 1 trata de categorias modelos, a primeira estrutura abstrata para teorias de homotopia. Começamos discutindo categorias com equivalências fracas e vendo como esse conceito amplia as noções usuais de homotopia. Só então definimos estruturas de modelo, dando muitos exemplos e desenvolvendo as técnicas que tornam as categorias de modelo tão úteis. Em particular, construímos a categoria de homotopia de uma estrutura modelo. Terminamos este capítulo examinando as equivalências de Quillen, que permitem comparar diferentes categorias de modelos.

No Capítulo 2 começamos a olhar para conjuntos simpliciais, instrumentos básicos em teoria da homotopia. Fazemos isso primeiro definindo e provando lemas na categoria simplicial $\mathbf{\Delta}$, e, como conjuntos simpliciais são definidos como pré-feixes em $\mathbf{\Delta}$, obtemos imediatamente alguma compreensão sobre estes. É dada ênfase na visualização de conjuntos simpliciais.

Exemplos de conjuntos simpliciais são adiados para o Capítulo 3, onde colocamos espaços, categorias e complexos de cadeia (e suas teorias de homotopia) em um contexto simplicial. Mais tarde, no Capítulo 4, vemos que os conjuntos simpliciais construídos nesta seção são na verdade quasicategorias, ou seja, $(\infty, 1)$-categorias.

Terminamos a Parte I dedicando o Capítulo 4 a uma introdução às quasicategorias, o modelo mais bem estudado de $(\infty, 1)$-categorias. Damos definições básicas e exibimos como recuperar a categoria de homotopia de uma quasicategoria, que descarta dados dimensionais mais altos. Depois nos voltamos para (∞-)funtores entre quasicategorias, focando em intuição e exemplos, e intuímos uma definição para ∞-equivalência entre quasicategorias. Fechamos esta seção usando fibrações de Kan triviais para formalizar um slogan popular na teoria de ∞-categorias.

A Parte II tem dois capítulos:

- No Capítulo 5 usamos categorias de modelo para descrever precisamente o slogan

  "$(\infty, 1)$-categorias são categorias enriquecidas sobre espaços"

  Mais precisamente, discutimos primeiro como a equivalência de Quillen entre as estruturas de modelos clássicos de espaços e conjuntos simpliciais pode ser interpretada como uma equivalência entre espaços e ∞-groupoides - no sentido quasicategórico. Em seguida, percorremos categorias simpliciais, ou seja, categorias enriquecidas sobre **sSet**, para descrever formalmente "categorias enriquecidas sobre espaços" como desejávamos. Fechamos esta seção formalizando o slogan como uma equivalência de Quillen.

- O Capítulo 6 é dedicado a uma breve discussão sobre *localização simplicial*, uma maneira canônica de comparar as duas linguagens que temos para a teoria da homotopia: categorias de modelo e quasicategorias. Damos duas construções para essa localização e discutimos como exatamente uma ∞-categoria é recuperada a partir de uma categoria modelo.

Part I

FOUNDATIONS

1

MODEL CATEGORIES

In commutative algebra, *localization* is a formal way to introduce the "inverses" to a given ring. That is, given a ring R and a *multiplicative subset* $S \subset R$ (a subset closed under multiplication and containing the identity), localization produces a ring $R \subset S^{-1}R$ such that any element of S has an inverse in $S^{-1}R$. Explicitly, the localization is defined as a quotient $S^{-1}R = R \times S / \sim$, where

$$(r_1, s_1) \sim (r_2, s_2) \iff \exists t \in R, t(r_1 s_2 - r_2 s_1) = 0.$$

To interpret this situation categorically, recall that a monoid M corresponds to a unique category BM with a single object $\bullet$ whose unique hom-set is M. In this picture the submonoid $S \subset M$ correponds simply to a subset of morphisms, and $S^{-1}M$ to the smallest category containing BM and such that each $s \in S$ is an isomorphism.

Now we can put the more general question: given a category $\mathcal{C}$ and a collection of morphisms $W \subset \mathrm{Mor}\mathcal{C}$, is there a *localization* $\mathcal{C}[W^{-1}]$ that "inverts" the morphisms in W? In the first section below, we will see that the answer in general is *yes*, but it is not too satisfying; the construction is too abstract and rather empty. In the subsequent sections we will explore *model categories*, an ubiquitous framework for understanding and working with the localization.

1.1 LOCALIZATION

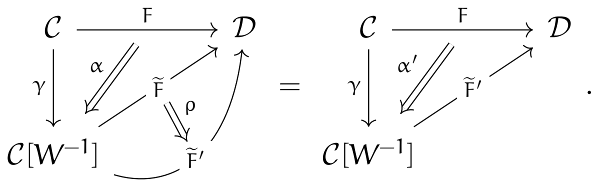
We start with a formal definition of localization.

1.1.1 Definition. The **localization** of a category $\mathcal{C}$ at a class of morphisms W is a category $\mathcal{C}[W^{-1}]$ equipped with a functor $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that

(i) γ takes W to isomorphisms in $\mathcal{C}[W^{-1}]$.

(ii) if $F : \mathcal{C} \to \mathcal{D}$ is another functor taking W to isomorphisms, then there is a functor $\widetilde{F} : \mathcal{C}[W^{-1}] \to \mathcal{D}$ and a natural isomorphism $\alpha : F \to \widetilde{F}\gamma$.

(iii) the pair $(\widetilde{F}, \alpha)$ above is unique in the following sense: if $(\widetilde{F}', \alpha')$ is another such pair, then there is a natural isomorphism $\rho : \widetilde{F} \to \widetilde{F}'$ such that $\alpha' = \alpha \circ (\rho \cdot \gamma)$:



1.1.2 Remark. The universal property of the localization guarantees its uniqueness up to equivalence. In fact, if $\gamma : \mathcal{C} \to \mathbf{L}$ and $\gamma : \mathcal{C} \to \mathbf{L}'$ both localize a class of morphisms W, then there is a pair of functors $\widetilde{\gamma} : \mathbf{L}' \leftrightarrows \mathbf{L} : \widetilde{\gamma}'$ and natural isomorphisms

$$\widetilde{\gamma}'\gamma \cong \gamma' \quad \text{and} \quad \widetilde{\gamma}\gamma' \cong \gamma,$$

1.1.3 Theorem (Gabriel-Zisman)**.** *The localization exists for any set of morphisms.*

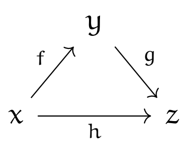
Proof. This is the content of Section 1.1 of the classic [16]; see Lemma 1.2 there. □

However in this general case it may be the case that $\mathcal{C}[W^{-1}]$ is large even if $\mathcal{C}$ is locally small. One anedoctal reasoning for this is that if $\mathcal{C}$ has a large set of objects,[1] then the zig-zags or spans may range over a large class of objects, and thus won't be small (see [1]). Among other things, model categories deal with this issue by ensuring that if $\mathcal{C}$ is locally small then so is $\mathcal{C}[W^{-1}]$.

1.1.4 Definition. A **category with weak equivalences** is a pair $(\mathcal{C}, W)$, where $\mathcal{C}$ is a category and W is a subset of morphisms of $\mathcal{C}$ such that

(i) W contains all isomorphisms;

(ii) (2-out-of-3) in any commutative triangle



if two morphisms are in W then so is the third.

1.1.5 Example. Any category $\mathcal{C}$ has a maximal class of weak equivalences given by *all* morphisms. Similarly, there is a minimal class of weak equivalences given by all isomorphisms.

1.1.6 Example. A functor $F : \mathcal{C} \to \mathcal{D}$ induces a class of weak equivalences in $\mathcal{C}$ by declaring that a morphism $f : x \to y$ in $\mathcal{C}$ is a weak equivalence iff $Ff : Fx \to Fy$ is an isomorphism in $\mathcal{D}$.

[1] As is the case for **Set**, **Cat**, **Top**, **Ab**5, $\mathbf{Ch}_R$ and **sSet**.

This appears frequently when we have a collection of functors $F_n : \mathcal{C} \to \mathcal{D}$ inducing a functor $\coprod_n F_n : \mathcal{C} \to \coprod_n \mathcal{D}$. Then a morphism $f : x \to y$ in $\mathcal{C}$ is a weak equivalence iff $(\coprod_n F_n)(f)$ is an isomorphism, or equivalently iff $F_n(f)$ is an isomorphism for each n. For example:

- In homological algebra, a **quasi-isomorphism** between chain complexes is a chain map $f : C_\bullet \to D_\bullet$ such that each group homomorphism $f_\sharp : H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism.
- In topology, two spaces have the same **homotopy type** if there is a continuous function $f : X \to Y$ such that each group homomorphism $f_\sharp : \pi_n(X) \to \pi_n(Y)$ is an isomorphism. Such an f is called a **weak homotopy equivalence**.

Notice that in these two particular examples the weak equivalences are defined in terms of *homotopical invariants* (the homotopy groups and homology groups). So, as the name suggests, being a weak homotopy equivalence is indeed a weaker notion than being a homotopy equivalence.

1.1.7 Example. If $\mathcal{C}$ is a 2-category (see the appendix), then a class of weak equivalences in its underlying category $\mathcal{C}_0$ is given by **equivalences**, which is a morphism that is invertible up to invertible 2-morphisms. To see this, first notice that the composition of equivalences is readily seen as an equivalence. On the other hand, suppose that $h = g \circ f$ and that h and f are equivalences, i.e. there are weak inverses h' and f' equipped with invertible 2-morphisms

$$hh' \sim 1, \quad h'h \sim 1, \quad ff' \sim 1 \quad \text{and} \quad f'f \sim 1,$$

Then fh' is a weak inverse for g. In fact,

$$gfh' = hh' \sim 1 \quad \text{and} \quad fh'g \sim fh'gff' = fh'hf' \sim ff' \sim 1.$$

Examples are categorical equivalences in **Cat** and homotopy equivalences in the 2-category **Top** of spaces, continuous functions and homotopy classes of homotopies.

1.2 MODEL STRUCTURES

We start by introducing model categories through weak factorization systems.[2]n the language of [38], what follows is actually the definition of a *closed* model category. We dropped the adjective "closed" because all examples we have ever encountered are closed. The definition is taken from [43]. Although we haven't yet defined these, we will give a definition right after.

[2] I

1.2.1 Definition. A **model structure** on a complete and cocomplete category $\mathcal{C}$ consists of three classes of morphisms W, Cof and Fib, such that

(i) $(\mathcal{C}, W)$ is a category with weak equivalences;

(ii) $(\mathrm{Cof} \cap W, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ are weak factorization systems.

We call the tuple $(\mathcal{C}, W, \mathrm{Cof}, \mathrm{Fib})$ a **model category**.

1.2.2 Definition. A morphism f in $\mathcal{C}$ has the **left lifting property (LLP)** against K if all commutative squares with f on the left and any $\mathsf{k} \in \mathsf{K}$ on the right have a lift:

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle k \in K} \\ \bullet & \longrightarrow & \bullet \end{array}$$

In this case, we write $\mathsf{f} \in \mathsf{LLP}(\mathsf{K})$. The **right lifting property (RLP)** is defined dually by putting f on the right and K on the left.

1.2.3 Definition. A **weak factorization system** on a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that

1. every morphism f in $\mathcal{C}$ can be factored as $\mathsf{f} = \mathsf{r}\mathsf{l}$, with $\mathsf{l} \in \mathcal{L}$ and $\mathsf{r} \in \mathcal{R}$;

2. $\mathcal{L} = \mathsf{LLP}(\mathcal{R})$ and $\mathcal{R} = \mathsf{RLP}(\mathcal{L})$.

1.2.4 Remark. The second condition above is saying that $\mathcal{L}$ and $\mathcal{R}$ determine each other. In a model category, this means that cofibrations and weak equivalences determine the fibrations and vice-versa, since

$$\mathrm{Fib} = \mathsf{LLP}(\mathrm{Cof} \cap W) \quad \text{and} \quad \mathrm{Cof} = \mathsf{RLP}(\mathrm{Fib} \cap W).$$

We end this section by describing some closure properties of sets defined by lifting properties, from which weak factorization systems are a particular example.

1.2.5 Lemma. *Let* K *be a class of morphisms in* $\mathcal{C}$ *and consider* $\mathsf{LLP}(\mathsf{K})$ *and* $\mathsf{RLP}(\mathsf{K})$. *Then*

1. *Both classes contain isomorphisms.*

2. *Both classes are closed under composition, and* $\mathsf{LLP}(\mathsf{K})$ *is closed under transfinite composition.*[3]

3. *Both classes are closed under forming retracts.*

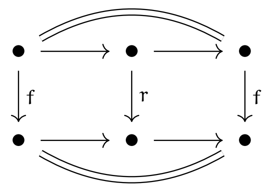
4. $\mathsf{LLP}(\mathsf{K})$ *is closed under pushouts,* $\mathsf{RLP}(\mathsf{K})$ *is closed under pullbacks.*

[3]The **transfinite composition** of a sequence of morphisms is the map from its limit to its colimit.

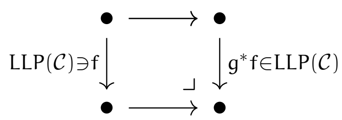
5. LLP(K) *is closed under coproducts*, RLP(K) *is closed under products.*

Proof. See [43], Proposition 2.10. □

1.2.6 Remark. In this lemma, by retract we mean a retract in the arrow category **Func**$(\mathbb{1}, \mathcal{C})$. Put more explicitly, a retract of a morphism f is a morphism r with a commutative square as below, with the horizontal arrows composing to identities.



Also, closure under pushouts (and dually, under pullbacks) means that if $f \in \mathrm{LLP}(K)$ then for any pushout square the morphism parallel to f is still in $\mathrm{LLP}(K)$:



1.2.7 Remark. Each pair of propositions in this lemma is formally dual to each other. This is because

$$f \in \mathrm{LLP}(K) \iff f \in \mathrm{RLP}(K^{op})$$

1.2.8 Lemma (Retract argument)**.** *Consider a composite morphism*

$$f : X \xrightarrow{i} A \xrightarrow{p} Y.$$

Then

1. *If* f *has LLP against* p*, then* f *is a retract of* i.

2. *If* f *has RLP against* i*, then* f *is a retract of* p.

Proof. See [43], Lemma 2.15. □

1.3 EXAMPLES OF MODEL CATEGORIES

1.3.1 Example. The **trivial model structure** in any category $\mathcal{C}$ is the model structure where the weak equivalences are the isomorphisms (see Example 1.1.5) and every morphism is both a fibration and a cofibration. This follows immediately by noticing that

1. any morphism can be written as itself followed by an identity (which is, in particular, an isomorphism);

2. any lifting problem such as

$$\begin{array}{ccc} \bullet & \xrightarrow{u} & \bullet \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \bullet & \xrightarrow[v]{} & \bullet \end{array}$$

   admits a solution if f is an isomorphism.

1.3.2 Example. The **classical model structure** on topological spaces is the first example given by Quillen in [38]. The weak equivalences are the weak homotopy equivalences (see Example 1.1.6) and the fibrations are Serre fibrations, i.e. continuous functions $f : X \to Y$ such that any lifting problem as the one below has a lift:

$$\begin{array}{ccc} \mathbb{D}^n & \longrightarrow & X \\ {\scriptstyle \delta^0}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ \mathbb{D}^n \times I & \longrightarrow & Y \end{array}$$

The cofibrations are determined, but an explicit description is in the appendix.

This model category is perhaps the most important, since it models the classical homotopy theory of spaces as in, for instance, homotopy types, and it clarifies the structure underlying highly abstract constructions, such as spectral sequences. For a proof that this is indeed a model category, see Chapter 3 of [43].

1.3.3 Example. Let $\mathbf{Ch}_R^{\geq 0}$ be the category of non-negatively graded chain complexes. The **projective model structure** on $\mathbf{Ch}_R^{\geq 0}$ is a model structure for quasi-isomorphisms (see Example 1.1.6) where the cofibrations are monomorphisms with projective cokernel; there is an analogous injective model structure. For a proof, see [13, Chapter 7].

1.3.4 Example. The **2-trivial model structure** of a 2-category $\mathcal{C}$ is the (unique) model structure on the category underlying $\mathcal{C}$ whose weak equivalences are equivalences (see Example 1.1.7).

A fibration in this model structure is a morphism $g : x \to y$ such that each induced functor $\mathcal{C}(z,x) \to \mathcal{C}(z,y)$ is an *isofibration* for all z. This means that

- for each $g : z \to x$ and each invertible 2-morphism $\eta : g \circ f \cong h$, there is an invertible 2-morphism $\varepsilon : f \cong h'$ such that $\varepsilon \cdot f = \eta$

The cofibrations are determined.

The proof of this model structure is rather non-trivial [26, Section 4], but it generalizes some canonical examples previously found in the literature:

- The **canonical model structure** in **Cat**, the unique model structure for equivalences.

- The **Hurewicz model structure** is the 2-trivial model structure for **Top**. This model category was originally defined [44] with
  - weak equivaleces given by homotopy equivalences;
  - fibrations as continuous functions with the RLP against cylinder inclusions $\delta^0 : A \hookrightarrow A \times I$.

  To see that this is in fact the aforementioned 2-trivial model category, notice that the existence of lifts

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow & \overset{K}{\nearrow} & \downarrow g \\ Z \times I & \xrightarrow[H]{} & Y \end{array}$$

  is the same as requiring that $g_* : \mathbf{Top}(Z, X) \to \mathbf{Top}(Z, Y)$ is an isofibration.

1.3.5 Example. There are exactly nine non-equivalent model structures on Set. For a proof, see [8].

1.4 THE HOMOTOPY CATEGORY

As promised, in this section we construct the localization of a model category as its *homotopy category*.

1.4.1 Definition. For any object $x \in \mathcal{C}$, a *cylinder object* is a factorization of the codiagonal

$$x \coprod x \xrightarrow{\in \mathrm{Cof}} \mathrm{Cyl}(x) \xrightarrow{\in W} x.$$

Dually, a path object is a factorization of the diagonal

$$x \xrightarrow{\in W} \mathrm{Path}(x) \xrightarrow{\in \mathrm{Fib}} x \times x.$$

1.4.2 Remark. In a model category, cylinder and path objects exist for any object. In fact, the weak factorization system $(\mathrm{Cof}, W \cap \mathrm{Fib})$ guarantees a factorization

$$x \coprod x \xrightarrow{\in \mathrm{Cof}} \hat{x} \xrightarrow{\in W \cap \mathrm{Fib}} x,$$

so $\tilde{x}$ is a cylinder object. The existence of path objects is obtained similarly with the weak factorization system $(W \cap \mathrm{Cof}, \mathrm{Fib})$.

1.4.3 Example. Any space X has a path object in $\mathbf{Top}_{\mathrm{Quillen}}$ given by $\mathbf{Top}(I, X)$ with the compact-open topology. If X is a CW complex, then $X \times I$ is a cylinder object (but not generally).

1.4.4 Definition. A **left homotopy** between a parallel pair $f, g : x \rightrightarrows y$ is a diagram

$$\begin{array}{ccc} x & & \\ {\scriptstyle \iota_0}\big\downarrow & \searrow^{f} & \\ \mathrm{Cyl}(x) & \xrightarrow{\ H\ } & y \\ {\scriptstyle \iota_1}\big\uparrow & \nearrow_{g} & \\ x & & \end{array} .$$

Dually, a **right homotopy** is a diagram

$$\begin{array}{ccc} & & y \\ & \nearrow & \big\downarrow \\ x & \longrightarrow & \mathrm{Path}(y) \\ & \searrow & \big\uparrow \\ & & y \end{array} .$$

1.4.5 Definition. Let $\mathcal{C}$ be a model category. An object $x \in \mathcal{C}$ is called

1. **cofibrant** if the initial morphism $\varnothing \to x$ is a cofibration.

2. **fibrant** is the terminal morphism $x \to *$ is a fibration.

A **cofibrant resolution** (resp. fibrant resolution) of an object x is a cofibrant (resp. fibrant) object $\hat{x}$ equipped with a weak equivalence $\tilde{x} \xrightarrow{\sim} x$.

1.4.6 Example. In the projective model structure of $\mathbf{Ch}_R^{\geq 0}$, an object is cofibrant if and only if it consists of projective modules. In homological algebra, a cofibrant resolution is then usually called a *projective resolution*. Similar remarks can be made about the injective model structure.

1.4.7 Remark. In a model category, any object is guaranteed to have a cofibrant resolution by factoring the initial morphism $\varnothing \to x$ through $(\mathrm{Cof}, W \cap \mathrm{Fib})$. Similarly, every object has a fibrant resolution.

1.4.8 Proposition. *Let x and y be, respectively, cofibrant and fibrant objects. Then*

1. *the left and right homotopy relations coincide;*

2. *the homotopy relation is an equivalence relation on $\mathcal{C}(x, y)$.*

Proof. See [13], Section 4. □

This proposition is crucial for the definition of the homotopy category Ho$\mathcal{C}$ below, as well to its proof.

1.4.9 Theorem (Whitehead theorem for model categories)**.** *A weak equivalence between fibrant-cofibrant objects is a homotopy equivalence.*

Proof. See [43], Lemma 2.27.' □

1.4.10 Definition. The **homotopy category** of a model category $\mathcal{C}$ is the category whose

- objects are fibrant-cofibrant objects in $\mathcal{C}$.
- morphisms are homotopy classes of morphisms between such objects:

$$\mathrm{Ho}\mathcal{C}(x, y) = \mathcal{C}(x, y)/\sim$$

1.4.11 Remark. The homotopy category could also be defined as having only cofibrant or fibrant or even all objects of $\mathcal{C}$. In this case, the morphisms would consider an appropriate resolution. In any case, the resulting categories are all equivalent.

1.4.12 Theorem. *Let $\mathcal{C}$ be a model category, and for each object $X \in \mathcal{C}$ fix a fibrant-cofibrant resolution $\hat{x}$, letting $\hat{x} = x$ if x is already fibrant-cofibrant. Then the mapping $x \mapsto \hat{x}$ defines a functor $\gamma : \mathcal{C} \to Ho\mathcal{C}$ exhibiting $Ho\mathcal{C}$ as the localization $\mathcal{C}[W^{-1}]$.*

Proof. See [43], from Definition 2.28 to Theorem 2.31. □

Since the localization is unique up to equivalence, we immediately have that γ is independent of choice of fibrant-cofibrant replacements.

1.4.13 Example. In the classical model structure of spaces, every object is fibrant, as the lift below always exists by setting $\tilde{f}(x, t) = f(x)$:

$$\begin{array}{ccc} D^n & \xrightarrow{f} & X \\ {\scriptstyle i_0}\downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow \\ D^n \times I & \longrightarrow & * \end{array}$$

On the other hand, a space is cofibrant if and only if it is a cell complex. The *CW approximation theorem* can be reinterpreted as saying that this fibrant-cofibrant replacement can always be taken to be a CW complex.[4]

The homotopy category $\mathrm{Ho}\mathbf{Top}_{\mathrm{Quillen}}$ describe the homotopy theory of spaces up to their homotopy types.

1.4.14 Example. The **derived category** from homological algebra is simply the homotopy category of $\mathbf{Ch}_R$ (with either the projective or injective structure).

[4] CW complexes are a very nice class of spaces large enough to include, for instance, topological manifolds.

1.4.15 Example. All objects in a 2-trivial model category $\mathcal{C}$ (Example 1.3.4) are both fibrant and cofibrant. From this it's easy to see that $\mathrm{Ho}\mathcal{C}$ is the standard homotopy category of $\mathcal{C}$, i.e. the category with the same objects of $\mathcal{C}$ and its morphisms only up to invertible 2-morphisms.

1.5 QUILLEN EQUIVALENCES

1.5.1 Definition. A **homotopical functor** between categories with weak equivalences is a functor preserving weak equivalences.

Given a homotopical functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}$, the universal property of the localization induces a functor $\mathcal{C}[W^{-1}] \to \mathcal{D}$, which can be composed with the localization of $\mathcal{D}$ to yield a functor between the localizations:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathsf{F}} & \mathcal{D} \\ \downarrow & \swarrow\!\!\!\sim & \downarrow \\ \mathrm{Ho}\mathcal{C} & \xrightarrow{\mathrm{Ho}\mathsf{F}} & \mathrm{Ho}\mathcal{D} \end{array}$$

The notion of homotopical functor is too strong, as many functors between model categories are not homotopical. However a functor $\mathrm{Ho}\mathsf{F} : \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$ already exists if

1. the restriction of F to the category of fibrant objects $\mathcal{C}_f$ is homotopical. In this case, the induced functor $\mathbb{R}\mathsf{F} : \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$ is called the *right derived functor* of F.

2. the restriction of F to the category of cofibrant objects $\mathcal{C}_c$ is homotopical. In this case, the induced functor $\mathbb{L}\mathsf{F} : \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$ is called the *left derived functor* of F.

This follows immediately from $\mathrm{Ho}\mathcal{C} = \mathrm{Ho}\mathcal{C}_c = \mathrm{Ho}\mathcal{C}_f$.

The reason we care about derived functors right now is that they provide a convenient language for talking about *Quillen adjunctions* (also known as *Quillen pairs*) and *Quillen equivalences*. The latter form a reasonable way of classification for model categories.

1.5.2 Definition. A **Quillen adjunction** between model categories $\mathcal{C}$ and $\mathcal{D}$ is an adjunction $\mathsf{F} : \mathcal{C} \leftrightarrows \mathcal{D} : \mathsf{G}$ such that $\mathbb{L}\mathsf{F}$ is left adjoint to $\mathbb{R}\mathsf{G}$.

1.5.3 Proposition. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and $\mathsf{F} : \mathcal{C} \leftrightarrows \mathcal{D} : \mathsf{G}$ an adjunction such that F preserves cofibrations and G preserves fibrations. Then F and G form a Quillen adjunction.*

Proof. See [13], Theorem 9.7. □

1.5.4 Definition. A **Quillen equivalence** is a Quillen adjunction $\mathsf{F} : \mathcal{C} \leftrightarrows \mathcal{D} : \mathsf{G}$ such that $\mathbb{L}\mathsf{F}$ is an equivalence of categories.

1.5.5 Proposition. *Let* $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ *be a Quillen adjunction such the following proposition holds:*

- *for every cofibrant object* $c \in \mathcal{C}$ *and fibrant object* $d \in \mathcal{D}$*, a morphism* $d \to Fc$ *is a weak equivalence in* $\mathcal{D}$ *iff its transpose* $Gd \to c$ *is a weak equivalence in* $\mathcal{C}$*.*

Then F *and* G *form a Quillen equivalence.*

Proof. Also see [13], Theorem 9.7. □

1.5.6 Example. In Chapter 4 we will see that there is a Quillen equivalence

$$|-| : \mathbf{sSet}_{\mathrm{Kan}} \leftrightarrows \mathbf{Top}_{\mathrm{Quillen}}$$

between the classical model structures on simplicial sets and on spaces. This Quillen equivalence may be interpreted as a way of formalizing that both of these homotopy theories are actually equivalent.

1.5.7 Example. The *Dold-Kan correspondence* asserts that there is a Quillen equivalence

$$N : \mathbf{sAb} \leftrightarrows \mathbf{Ch}_R^+ : \Gamma,$$

where **sAb** is the category of simplicial groups (with the model structure induced from $\mathbf{sSet}_{\mathrm{Kan}}$), and $\mathbf{Ch}_R^+$ is equipped with the projective model structure.

1.5.8 Example. There are several models for $(\infty, 1)$-categories: quasi-categories, simplicial categories, complete Segal spaces, Segal categories... As summarized by Bergner in [6], the canonical way to compare these models is to first construct a model category describing their homotopy theory, and then connect them through a zig-zag of Quillen equivalences; see the picture in page 12 of [6].

2

SIMPLICIAL SETS

> The definition looks a bit off-putting at first (what concerning simplicial sets doesn't?)
>
> — Greg Friedman

The first apparition of simpicial sets happened in [14] under the name *complete semi-simplicial complexes*. The authors were studying singular homology and, as the name suggests, simplicial sets originated as a generalization of simplicial complexes.

However, since then the importance and ubiquity of simplicial sets in homotopy theory is nearly unexplainable. For instance:

- the homotopy theory of spaces is equivalent to that of simplicial sets [38];
- the Waldhausen $S_\bullet$-construction in algebraic K-theory is completely (bi)simplicial [22];
- most modern definitions of $(\infty, 1)$-categories are of a simplicial nature [6,9];
- and a modern point of view on cohomology is through hom-spaces of $(\infty, 1)$-categories [34].

In this chapter we lay the foundations of this simplicial machinery.

2.1 THE SIMPLICIAL CATEGORY

In this section construct the simplicial category $\boldsymbol{\Delta}$ and prove a few lemmas about it. Simplicial sets will later be defined as simply presheaves on $\boldsymbol{\Delta}$.

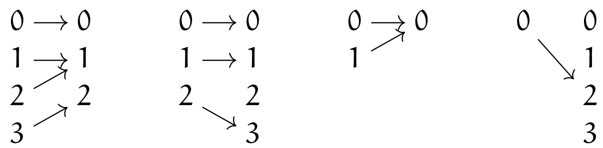
By $[n]$ we denote the ordered set $\{0 < 1 < 2 < \cdots < n\}$.

We also use the following notation throughout this section:

$$[n] = \begin{matrix} 0 \\ 1 \\ \vdots \\ n \end{matrix}$$

2.1.1 Definition. The **simplex category** is the category $\mathbf{\Delta}$ whose objects are the ordered sets $[0], [1], [2], \ldots$ and a morphism $f : [m] \to [n]$ is a *non-decreasing* map, that is, if $0 \le i < j \le m$ then $0 \le f(i) \le f(j) \le n$.

We may depict morphisms in $\mathbf{\Delta}$ as follows:



2.1.2 Notation. Whenever we write $f : [m] \to [n]$ we mean that f is a morphism in $\mathbf{\Delta}$, i.e. a non-decreasing map.

The following lemma is a particular instance of an *epi-mono factorization system*, which exists for any elementary topos [31, Chapter IV.6], such as the category of sets or as in a category of presheaves.

2.1.3 Lemma (Epi-mono factorization)**.** *Any morphism* $f : [m] \to [n]$ *in* $\mathbf{\Delta}$ *factors as* $f = \iota \circ \pi$*, where* π *is surjective and* ι *is injective.*

Proof. Let $i_0, \ldots, i_p$ be the image of $f : [m] \to [n]$.

We define $\pi : [m] \to [p]$ as follows: if $f(k) = i_\ell$, then $\pi(k) = \ell$. Then p is surjective because [p] has the size of the image of f. On the other hand, if we define $\iota : [p] \to [n]$ by $h(k) = i_k$, we get by construction an injective morphism. A quick calculation then shows that $\pi, \iota \in \mathbf{\Delta}$ and that $f = \iota \circ \pi$. □

2.1.4 Definition. For $n > 0$ and $0 \le i \le n$, the **i-th coface** is the unique injective morphism $\delta^i : [n-1] \to [n]$ in $\mathbf{\Delta}$ such that i is not in the image of δ^i.

Explicitly,

$$\delta^i(j) = \begin{cases} j & \text{se } j < i \\ j+1 & \text{se } j \ge i \end{cases}$$

Examples of cofaces:

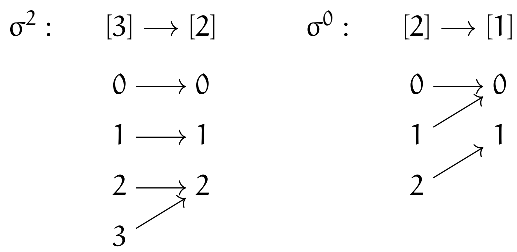
$\delta^2 : \quad [2] \longrightarrow [3] \qquad\qquad \delta^0 : \quad [0] \longrightarrow [1]$

2.1.5 Definition. For $n \geq 0$ and $0 \leq i \leq n$, the **i-th codegeneracy** is the unique surjective morphism $\sigma^i : [n+1] \to [n]$ which sends two elements of $[n+1]$ to i.

Explicitly,

$$\sigma^i(j) = \begin{cases} j & \text{se } j \leq i \\ j-1 & \text{se } j > i \end{cases}$$

Examples of codegeneracies:



2.1.6 Lemma (Decomposition in cofaces and codegeneracies)**.** *Any morphism* $f : [m] \to [n]$ *can be uniquely factored as*

$$f = \delta^{j_0} \cdots \delta^{j_k} \sigma^{i_0} \cdots \sigma^{i_l}.$$

where $j_0 < \cdots < j_k$ *and* $i_{n-k} < \cdots < i_l$.

Proof. From Lemma 2.1.3, it suffices to prove separately that surjective (resp. injective) morphisms factor as compositions of codegeneracies (resp. cofaces).

- Suppose that f is injective. In this case, there are $n-m$ elements of $[n]$ not in the image of f, which we denote by

  $$\{j_0 < j_1 < \cdots < j_{n-k}\} \subset [n].$$

  Notice that for all $0 \leq k \leq n$ we must have $j_k \leq n-k$, otherwise f couldn't be injective; so $\delta^{j_k} : [n-k] \to [n-k+1]$ is well-defined. Then a simple calculation checks that

  $$f = \delta^{j_0} \delta^{j_2} \cdots \delta^{j_{n-k}},$$

  as we wanted.

- Suppose that f is surjective. Then, since f is non-decreasing, for each $0 \leq i \leq n$ we must have either

  $$f(i+1) = f(i) \text{ or } f(i)+1,$$

  otherwise some element of the image would be skipped. In fact, there are exactly $n-m$ points in $[m]$ where $f(i+1) = f(i)$, which we denote by

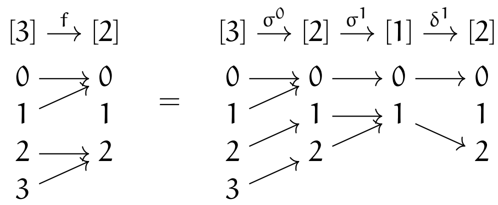
  $$\{i_0 < i_1 < \cdots < i_{n-k}\} \subset [m].$$

Then for all $k \leq n-k$ we must have $i_k \leq n-(n-p-k) = p+k$, otherwise f wouldn't be surjective; thus $\sigma^{i_k} : [p+k] \to [p+k-1]$ is well-defined. Thus again we could easily check that

$$f = \sigma^{i_{n-k}} \cdots \sigma^{i_1} \sigma^{i_0}.$$

□

2.1.7 Example. The proof of the lemma gives an algorithm for the factorization. We give an example below:



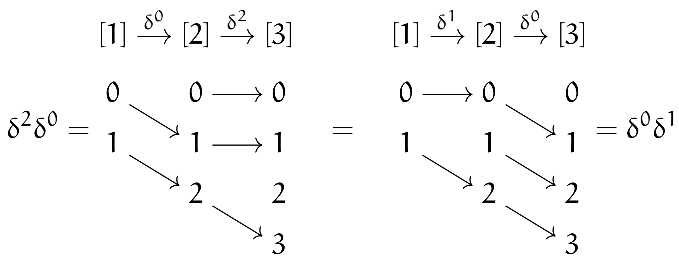
2.1.8 Lemma (Cosimplicial identities)**.** *The maps δ^i and σ^i satisfy:*

1. $\delta^j \delta^i = \delta^i \delta^{j-1}$ *if* $i < j$
2. $\sigma^j \sigma^i = \sigma^i \sigma^{j+1}$ *if* $i \leq j$
3. $\sigma^j \delta^i = \begin{cases} \delta^i \sigma^{j-1} & \textit{if } i < j \\ 1 & \textit{if } i = j \textit{ or } j+1 \\ \delta^{i-1} \sigma^j & \textit{if } i > j+1 \end{cases}$

Proof. Using the definitions, we calculate explicitly, for $i < j$,

$$\delta^j \delta^i(k) = \delta^i \delta^{j-1} = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } i \leq k < j \\ k+2 & \text{if } j \leq k \end{cases},$$

A concrete example:



The other identities are proved similarly by direct computations. We heavily encourage the reader to write down some concrete examples to get the hang of the equations. □

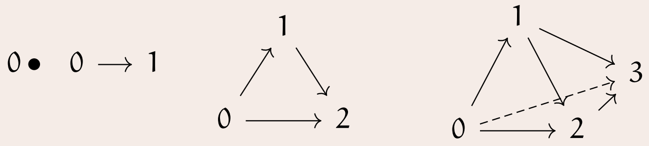
2.1.9 Corollary. *The coface-codegeneracy decomposition obtained in Lemma 2.1.6 is unique up to cosimplicial identities.*

Some pictures

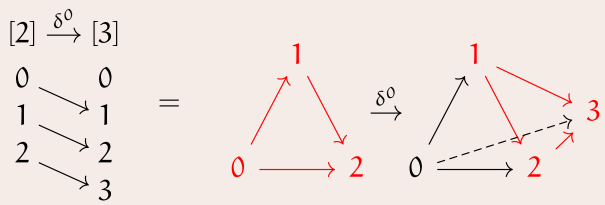
The ordered set $[n]$ can be regarded as the free category on the graph

$$0 \to 1 \to \cdots \to n.$$

For small n, we can draw all non-identity morphisms in this $[n]$ without getting a mess:



By freeness, a morphism $f : [m] \to [n]$ in $\boldsymbol{\Delta}$ corresponds to a unique "non-decreasing functor". In fact, the simplex category is a full subcategory $\boldsymbol{\Delta} \hookrightarrow \mathbf{Cat}$. In low dimensions we get a nice visualization of these morphisms. For instance, the coface $\delta^0 : [2] \to [3]$ becomes the 0-th face of the tetrahedron, i.e. the face opposite to the vertex 0:



2.2 SIMPLICIAL OBJECTS

We will be mainly concerned with simplicial sets, i.e. presheaves on $\boldsymbol{\Delta}$. However, there is an evident generalization of *simplicial object* defined as presheaves on $\boldsymbol{\Delta}$ but with values in any category $\mathcal{C}$. These also play an important role in the theory, so we develop this broader notion for the sake of generality.

2.2.1 Definition. A **simplicial object** in a category $\mathcal{C}$ is a functor $X : \boldsymbol{\Delta}^{op} \to \mathcal{C}$. A **simplicial set** is a simplicial object in **Set**.

The category of simplicial objects in $\mathcal{C}$ is the presheaf category

$$s\mathcal{C} := \mathbf{Func}(\boldsymbol{\Delta}^{op}, \mathcal{C})$$

Notice that a morphism between simplicial objects is a natural transformation between them.

In particular, there is the category of simplicial sets denoted by **sSet**. By a *simplicial map* we will mean a morphism between simplicial sets.

2.2.2 Notation. Let $X : \boldsymbol{\Delta}^{op} \to \mathcal{C}$ be a simplicial object. The action of X on objects will be written as $X_n := X[n]$.

Furthermore, we denote the action on the generating morphisms δ^i e σ^i of $\mathbf{\Delta}$ by

$$d_i := X(\delta^i) \text{ and } s_i := X(\sigma^i).$$

These are respectively called the **faces** and *degeneracies* of the simplicial object.

In the context of simplicial objects, lemmas 2.1.6 and 2.1.8 translate to the following corollaries, which are basically the same statement but with the indices swapped because of contravariance:

2.2.3 Lemma (Face-degeneracy decomposition)**.** *Given a simplicial object* X *and a morphism* $f : [m] \to [n]$ *in* $\mathbf{\Delta}$*, there is a factorization*

$$Xf = s_{i_l} \cdots s_{i_0} d_{j_k} \cdots d_{j_0},$$

where $j_0 < \cdots < j_k$ *and* $i_0 < \cdots < i_l$*. Moreover this factorization is unique up to simplicial identities.*

2.2.4 Lemma (Simplicial identities)**.** *The faces and degeneracies of a simplicial object satisfy:*

1. $d_i d_j = d_{j-1} d_i$ *if* $i < j$*;*
2. $s_i s_j = s_{j+1} s_i$ *if* $i \leq j$*;*
3. $d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } i = j \text{ ou } j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}.$

2.2.5 Corollary. *The data described by a simplicial object is a collection of objects* $(X_n)_{n \in \mathbb{N}}$ *equipped with morphisms* $d_i : X_{n+1} \to X_n$ *e* $s_i : X_n \to X_{n-1}$, $0 \leq i \leq n$ *satisfying the simplicial identities.*

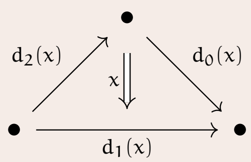
For the rest of this section we retain our attention to simplicial *sets*.

2.2.6 Definition. An **n-simplex** of a simplicial set X is an element $x \in X_n$.

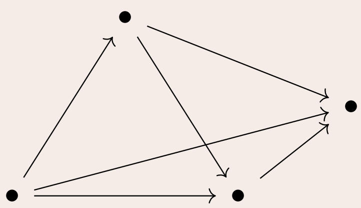
Visualization of simplices

We usually think of an n-simplex $x \in X_n$ as a tetrahedron in $\mathbb{R}^{n+1}$, whose faces are the $(n-1)$-simplices $d_0(x), \ldots, d_n(x)$. For instance,

$$x \in X_1 \implies d_1(x) \xrightarrow{x} d_0(x) \ , \qquad x \in X_2 \implies$$



$x \in X_3 \implies$



2.2.7 Definition. For each $[n] \in \mathbf{\Delta}$, the **standard n-simplex** is the simplicial set $\Delta^n \in \mathbf{sSet}$ represented by $[n]$, that is,

$$\Delta^n_k := \mathbf{\Delta}([k],[n]).$$

The faces and degeneracies are given by precomposition with the cofaces and codegeneracies, respectively.

2.2.8 Example. The standard 0-simplex Δ^0 is the terminal object in **sSet**: for each $n \geq 0$, the only element in Δ^0_n is the map $[n] \to [0]$ constant in $[0]$.

Another point of view

Notice that by definition Δ^n is the image of $[n]$ by the Yoneda embedding $\mathbf{\Delta} \hookrightarrow$ **sSet**. Then the Yoneda lemma gives a natural bijection

$$\Phi : X_n \cong \mathbf{sSet}(\Delta^n, X).$$

In other words, an n-simplex $x \in X_n$ corresponds to a simplicial map $x : \Delta^n \to X$:

$$x \in X_n \leftrightsquigarrow \Delta^n \xrightarrow{x} X.$$

The naturality of the Yoneda lemma says that given $x \in X_n$ and $f : [m] \to [n]$, the m-simplex $(Xf)(x) \in X_m$ is mapped to xf through Φ:

$$\begin{array}{ccc} \mathrm{id} & & x \\ \Delta^n_n & \xrightarrow{x} & X_n \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle Xf} \\ \Delta^n_m & \xrightarrow{x} & X_m \\ f & & xf = Xf(x) \end{array}$$

From this point of view, a simplicial set can be regarded as a sequence of sets $(X_n)_{n \geq 0}$ acted by the morphisms of $\mathbf{\Delta}$:

$$Xf(x) \in X_m \leftrightsquigarrow \Delta^m \xrightarrow{f} \Delta^n \xrightarrow{x} X.$$

2.2.9 Definition. An n-simplex $x \in X_n$ is called **degenerate** if it's in the image of any degeneracy map $s_i : X_{n-1} \to X_n$, that is $x = s_i(y)$ for some $y \in X_{n-1}$.

Equivalently, $x : \Delta^n \to X$ is degenerate if $x = y \circ \sigma^i$ for some $y : \Delta^{n+1} \to X$.

The tautological observations below may shed some light on the nature of degenerate morphisms.

2.2.10 Lemma. *An* n*-simplex* $\Phi(f) \in \Delta^m_n$ *is non-degenerate if and only if* $f : [n] \to [m]$ *is injective.*

2.2.11 Lemma. *An* n*-simplex* $x \in X_n$ *is non-degenerate if and only if it can be written as a composition of face maps.*

Proof. Both results follow directly from the decomposition of a morphism in face and degeneracy simplices. □

The following is a technical lemma.

2.2.12 Proposition. *Let* $x : \Delta^n \to X$ *be an* n*-simplex. Then there is a unique* $m \leq n$ *and a unique factorization*

$$\Delta^n \xrightarrow{f} \Delta^m \xrightarrow{u} X, \qquad x = u \circ f,$$

such that u *is non-degenerate and* $f : [n] \to [m]$ *is surjective.*

Proof. For uniqueness, suppose that there are two such factorizations $x = uf = u'f'$. By surjectivity of $f : [n] \to [m]$ and $f' : [n] \to [m']$, there are left inverses g and g'. Thus

$$u = xg = u'f'g.$$

Since u and u' are non-degenerate, the map $f'g : [m] \to [m']$ is injective. Analogously, $fg' : [m'] \to [m]$ is injective, thus $m = m'$ and as the only non-decreasing map $[m] \to [m]$ is the identity, we conclude that

$$fg' = f'g = id_{[m]}.$$

Thus the factorization of x is unique, because

$$f = (fg')f' = f' \text{ and } u = u'f'g = u'.$$

For existence, notice that the set

$$S = \{m' \in \mathbb{N} \mid \text{ there is a factorization } x = \Delta^n \xrightarrow{f} \Delta^m \xrightarrow{u} X\}$$

is non-empty, because $n \in S$. Thus, by the well-ordering principle, S contains a minimal element m. We argue that minimality implies that the associated factorization $x = uf$ satisfies the statement:

- f is surjective, otherwise we could replace m with the size of the image of f;
- u is non-degenerate, otherwise we would have

$$x = uf = (u'\sigma_i)f$$

where $\sigma_i f : [n] \to [m-1]$ is surjective, contradicting the minimality of f.

□

2.2.13 Corollary. *Let $\varphi : X \to Y$ be a simplicial map inducing a bijection between non-degenerate simplices. Then f is an isomorphism.*

Proof. Let $y \in Y_n$ and apply Proposition 2.2.12 to write $y = g(v)$, where v is non-degenerate and g is surjective.

Since $v \in Y_m$ is non-degenerate, by hypothesis we have $v = \varphi(u)$ for a unique $u \in X_m$. But, by naturality, we have a commuting square

$$\begin{array}{ccc} Y_m & \xrightarrow{\varphi} & X_m \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ Y_n & \xrightarrow[\varphi]{} & X_n \end{array} \implies \varphi(g(u)) = g(\varphi(u)) = y.$$

Thus φ is surjective. The proof that φ is injective is analogous, and that suffices. □

2.3 BOUNDARIES AND HORNS

If X is a simplicial set and $T_n \subset X_n$ is a sequence of subsets such that, for each $0 \le i \le n$,

$$d_i(T_n) \subset T_{n-1} \quad \text{and} \quad s_i(T_n) \subset T_{n+1},$$

then $d_i|_{T_n}$ and $s_i|_{T_n}$ define a simplicial set structure on $(T_n)_{n\ge 0}$. This data is what is referred to as a **simplicial subset**.

2.3.1 Definition. The **boundary** of Δ^n is the simplicial subset $\partial\Delta^n \subset \Delta^n$ defined by

$$\partial\Delta^n_k := (\partial\Delta^n)_k = \{f : [k] \to [n] \mid f \text{ non-surjective}\}$$

Notice that if f is not surjective then there is no way either $d_i f = f \circ \delta^i$ or $s_i f = f \circ \sigma^i$ are surjective, thus $\partial\Delta^n \subset \Delta^n$ is indeed a simplicial subset.

2.3.2 Definition. A **boundary** in a simplicial set X is a sequence $(x_0, \dots, x_n)$ of $(n-1)$-simplices $x_i \in X_{n-1}$ such that $d_i(x_j) = d_{j-1}(x_i)$ for $i < j$.

2.3.3 Proposition (Characterization of boundaries)**.** *For any simplicial set* X*, the map below is injective:*

$$\Phi : \mathbf{sSet}(\partial\Delta^n, X) \to \prod_{0\le i\le n} X_{n-1}$$
$$x \mapsto (x \circ \delta^0, \dots, x \circ \delta^n)$$

The image of Φ *consists of lists* $(x_0, \dots, x_n)$ *of* $(n-1)$*-simplices such that* $d_i(\sigma_j) = d_{j-1}(\sigma_i)$ *for* $i < j$*.*

Proof. First notice that $\delta^i(\Delta^{n-1}) \subset \partial\Delta^n$. Thus if $x : \partial\Delta^n \to X$ is a simplicial map then $x \circ \delta^i \in X_{n-1}$.

Let $f, g : \partial\Delta^n \to X$ and suppose $\Phi(f) = \Phi(g)$, i.e. that $f \circ \delta^i = g \circ \delta^i$ for each $0 \le i \le n$. Also, notice that any $u : [m] \to [n]$ in $\partial\Delta^n_m$ can be written as $u = \delta^i \circ v$. Then for each i we have

$$f(u) = (f \circ \delta^i) \circ v = (g \circ \delta^i) \circ v = g(u).$$

Since u was arbitrary, we conclude that $f = g$ and so Φ is injective.

For the second claim, lists in the image of Φ trivially satisfy the equation in the statement. In fact, that equation corresponds to

$$x_j \circ \delta^i = x_i \circ \delta^{j-1} \text{ if } i < j.$$

Then, given a list $(x_0, \dots, x_n) \subset \prod X_{n-1}$ satisfying the equation above we can define $\hat{\varphi} : \partial\Delta^n \to X$ by

$$\hat{\varphi} : u = \delta^i \circ v \mapsto x_i \circ v$$

where $u = g\delta^i$ is the representation of k-simplices we used above.

For well-definedness, if we have two decompositions of $u \in \partial\Delta^n$, e.g. $\delta^i \circ v$ and $\delta^j \circ v'$, then neither i nor j are in the image of u. Assuming without loss of generality that $i < j$, this means that we can write $u = \delta^j \circ \delta^i \circ \tilde{w}$. Then we can use the simplicial identities and the uniqueness part of 2.1.9 to conclude that $v = \delta^{j-1} w$ and $v' = \delta^i w$.

Thus, for $i < j$,

$$\begin{aligned}
\varphi(\delta^i \circ v) &= x_i \circ v \\
&= x_i \circ \delta^{j-1} w \\
&= x_j \circ \delta^i \circ w \\
&= x_j \circ v' \\
&= \varphi(u).
\end{aligned}$$

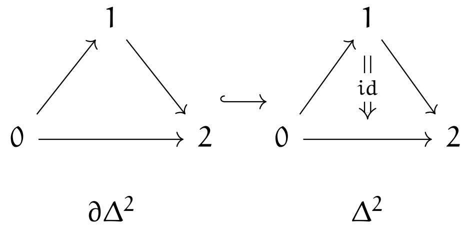
□

2.3.4 Example. Proposition 2.3.3 identifies the inclusion $\partial\Delta^n \hookrightarrow \times\Delta^n$ with the sequence

$$(\delta^0, \dots, \delta^n) \in \Delta^n_{n-1} \times \cdots \Delta^n_{n-1}.$$

The boundary $\partial\Delta^n \hookrightarrow \Delta^n$ fits into this picture as the faces of this tetrahedron, i.e. Δ^n without its interior.

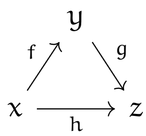
For instance, for $n = 2$:



2.3.5 Example. A boundary $\partial\Delta^2 \to X$ is a sequence (g, h, f) of 1-simplices such that

$$x := d_1 h = d_0 f,$$
$$y := d_1 f = d_1 g,$$
$$z := d_0 g = d_0 h.$$

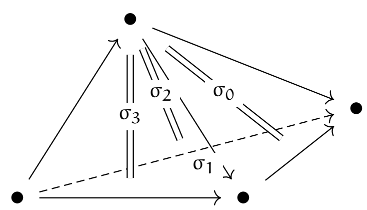
If we depict 0-simplices as points and 1-simplices as line segments, what these equations are saying is that the edges of the simplices f, g and h are tied together as in the following picture:



2.3.6 Example. A boundary $\partial\Delta^3 \to X$ is a sequence $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ of 3-simplices whose faces satisfy

$$d_0\sigma_1 = d_0\sigma_0 \quad d_1\sigma_2 = d_1\sigma_1$$
$$d_0\sigma_2 = d_1\sigma_0 \quad d_1\sigma_3 = d_2\sigma_1$$
$$d_0\sigma_3 = d_2\sigma_0 \quad d_2\sigma_3 = d_2\sigma_2$$

If we depict 1-simplices as line segments and 2-simplices as triangle, what these equations are saying is that the sides of the simplices σ_0, σ_1, σ_2 and σ_3 are tied together as in the following pictures:



2.3.7 Example. The composition of an n-simplex $X : \Delta^n \to X$ with the inclusion $\partial\Delta^n \overset{i}{\hookrightarrow} \Delta^n$ is the list of faces of x:

$$(d_0(x), \dots, d_n(x)).$$

2.3.8 Definition. For $0 \leq k \leq n$, the **k-th horn** in Δ^n is its simplicial subset Λ^n_k defined by

$$(\Lambda^n_k)_m = \{f : [m] \to [n] \mid [n] \not\subset f[m] \cup k\}.$$

2.3.9 Proposition. *For each* X*, the map below is injective:*

$$\mathbf{sSet}(\Lambda^n_k, X) \to \prod_{i \neq k} X_{n-1}$$
$$x \mapsto (x \circ \delta^0, \dots, x \circ \delta^{k-1}, \bullet, x \circ \delta^{k+1}, \dots, \delta^n)$$

Its image consists of lists

$$(x_0, \dots, x_{k-1}, \bullet, x_{k+1}, \dots, x_n) \in \prod_{i \neq k} X_{n-1}$$

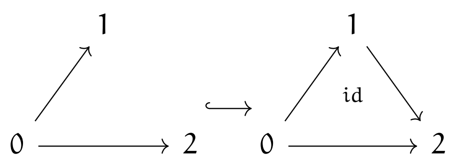
such that $d_i(\sigma_j) = d_{j-1}(\sigma_i)$ *for* $i < j$ *and* $i, j \neq k$.

Proof. Analogous to the proof of 2.3.3. □

2.3.10 Example. The proposition identifies the inclusion : $\Lambda^n_k \hookrightarrow \Delta^n$ with the list

$$(\delta^0, \dots, \delta^{k-1}, \bullet, \delta^{k+1}, \dots, \delta^n) \in \prod_{i \neq k} \Delta^n_{n-1}.$$

In other words, Λ^n_k corresponds to all faces of Δ^n, except the k-th one. For instance, for Λ^2_0:



Λ^2_0 Δ^2

2.3.11 Definition. 2.3.8 A **horn** $\Lambda^n_k \to X$ in a simplicial set X is a list

$$(x_0, \dots, x_{k-1}, \bullet, x_{k+1}, \dots, x_n) \ni \prod_{i \neq k} X_{n-1}$$

such that $d_i(x_j) = d_{j-1}(x_i)$ for $i < j$ and $i, j \neq k$.

2.3.12 Example. A horn $\Lambda^2_1 \to X$ is a sequence of 1-simplices $(g, \bullet, f)$ such that

$$y := d_1 g = d_0 f.$$

Defining $x := d_1 f$ and $z := d_0 g$, we can draw a picture for this equation:

$$\begin{array}{ccccc} & & y & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ x & & & & z \end{array}$$

Analogously, horns $\Lambda^2_0 \to X$ and $\Lambda^2_2 \to X$ may be depicted as follows:

$$\begin{array}{ccc} & y & \\ {\scriptstyle f}\nearrow & & \\ x & \xrightarrow[h]{} & z \end{array} \qquad \begin{array}{ccc} & y & \\ & & \searrow{\scriptstyle g} \\ x & \xrightarrow[h]{} & z \end{array}$$

2.3.13 Example. A horn $\Lambda^3_1 \to X$ is a sequence of 2-simplices $(\sigma_0, \bullet, \sigma_1, \sigma_2)$ whose faces satisfy

$$d_0\sigma_2 = d_1\sigma_0, \quad d_0\sigma_3 = d_2\sigma_0, \quad \text{and } d_2\sigma_3 = d_2\sigma_2.$$

We can depict this visually as tetrahedron without its interior *and* also without the face opposite to the its second vertex. (see also Example 2.3.6)

2.3.14 Example. The following question will repeat throughout this text and much of the homotopy theory literature:

> Is it possible to extend a horn $\Lambda^n_k \to X$ to a simplicial set Δ^n?

We depict this extension problem with a diagram

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{f} & X \\ \downarrow & \overset{\sigma}{\nearrow} & \\ \Delta^n. & & \end{array}$$

The proof of Proposition 2.3.9 shows that a solution to this problem is given by $\sigma \in X_n$ satifying

$$\sigma = (d_0(\sigma), \ldots, d_{k-1}(\sigma), \bullet, d_{k+1}(\sigma), \ldots, d_n(\sigma)).$$

3

EXAMPLES OF SIMPLICIAL SETS

> Simplicial sets, like complex numbers, are magic.
> We don't completely understand why they are so great.
>
> André Joyal

Ultimately, we want to show how simplicial sets are a good context to do homotopy theory. In this chapter, we start doing so by reinterpreting *categories, spaces and chain complexes as simplicial sets.*

It will turn out, however, that these are very special examples of simplicial sets - in fact, they are all $(\infty, 1)$-*categories*! In fact, they are very special examples of $(\infty, 1)$-categories. We will return to these matters in the next chapter.

Notation

- *Inner horn:* a horn $\Lambda^n_k \to X$ with $0 < k < n$.
- *Outer horn:* a horn $\Lambda^n_k \to X$ with $k = 0$ or n.
- *Filler:* given a horn $\Lambda^n_k \to X$, it's an extension to Δ^n:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow_{\exists} & \\ \Delta^n & & \end{array}$$

- *Kan complex:* a simplicial set such that every horn has a filler.
- *Quasi-category:* a simplicial set such that every inner horn has a filler.

3.1 CATEGORIES

3.1.1 Definition. The **nerve** of a category $\mathcal{C}$ is the simplicial set $N\mathcal{C}$ whose n-simplices are

$$N\mathcal{C}_n := \mathbf{Func}([n], \mathcal{C}).$$

A morphism $f : [m] \to [n]$ is taken to the map $N(f) : N\mathcal{C}_n \to N\mathcal{C}_m$ defined by precomposition.

In practice, an n-simplex $\sigma \in N\mathcal{C}_n$ is a chain of n consecutive morphisms in $\mathcal{C}$:

$$\sigma = \quad x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_n$$

The face maps $d_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ omits x_i from this string:

$$d_i\sigma = \quad x_0 \longrightarrow \cdots \longrightarrow x_{i-1} \xrightarrow{f_i \circ f_{i-1}} x_{i+1} \longrightarrow \cdots \longrightarrow x_n$$

The degeneracy maps $s_i : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ insert an identity after x_i:

$$s_i\sigma = \quad x_0 \longrightarrow \cdots \longrightarrow x_i = x_i \longrightarrow \cdots \longrightarrow x_n$$

So a degenerate simplex of $N\mathcal{C}$ is a string of morphisms which has at least one identity in its entries.

3.1.2 Remark. From the nerve of $\mathcal{C}$ we can extract all of its categorical structure:

- The set of objects is $N\mathcal{C}_0$ and the set of morphisms is $N\mathcal{C}_1$.
- The identity of $x \in N\mathcal{C}_0$ is $s_0(x) \in N\mathcal{C}_1$.
- The domain of $f \in N\mathcal{C}_1$ is $d_0(f) \in N\mathcal{C}_0$, and its codomain is $d_1(f) \in N\mathcal{C}_0$.
- A pair of composable morphisms $f, g \in N\mathcal{C}_1$ define a 2-simplex $\sigma \in N\mathcal{C}_2$:

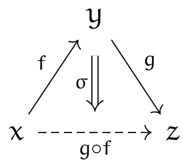
$$\sigma = \quad x \xrightarrow[f]{} y \xrightarrow[g]{} z \ .$$

  Their composite $g \circ f$ is $d_1(\sigma)$.

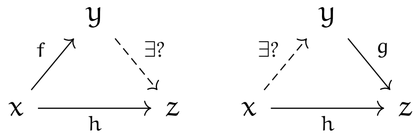
In fact, not only the structure but also the *properties* of unitality and associativity are encoded as *structure* on the nerve. This will be spelled out in Proposition 3.1.5.

It turns out that nerves of categories have *unique* fillers for all inner horns (Proposition 3.1.5). In fact, this property actually characterizes the simplicial sets which arise as nerves (Proposition 3.1.7).

3.1.3 Example. Every horn $(g, \bullet, f) : \Lambda^2_1 \to N\mathcal{C}$ has a unique extension to $\sigma \in \Delta^2$, characterized by $d_1(\sigma) = g \circ f$:



In general, the statement is no longer true if we replace Λ^2_1 with Λ^2_0 or Λ^2_2, as there isn't necessarily a morphism closing the triangle:



3.1.4 Lemma. *A simplicial map* $f : X \to N\mathcal{C}$ *is determined by* f_0 *and* f_1*.*

Proof. Let $n \geq 2$ and $0 \leq i \leq j \leq n$. For a simplicial set Y we define

$$d_{ij} := d_0 \cdots \hat{d}_i \cdots \hat{d}_j \cdots d_n : Y_n \to Y_1.$$

In particular, for $Y = N\mathcal{C}$ the map sends $g_0 \to \cdots \to g_n \in N\mathcal{C}_n$ to its composition. Then, by naturality,

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & N\mathcal{C}_n \\ {\scriptstyle d_{ij}}\downarrow & & \downarrow{\scriptstyle d_{ij}} \\ X_1 & \xrightarrow{f_1} & N\mathcal{C}_1 \end{array} \implies d_{ij}(f_n\sigma) = f_1(d_{ij}\sigma)$$

So the elements in the string $f_n(\sigma)$ are defined by f_1 (and d_{ij}). $\square$

3.1.5 Proposition. *Every inner horn* $f : \Lambda^n_k \to N\mathcal{C}$ *has a unique filler.*

Proof. The proposition is empty for $n = 0, 1$, and we treated the case $n = 2$ in Example 3.1.3 above. So we assume that $n \geq 3$.

We will do the proof by constructing $\tilde{f} : [n] \to \mathcal{C}$ explicitly. Let $(i) : [0] \to [n]$ and $(i, j) : [1] \to [n]$ be the maps defined by be defined by $0 \mapsto i$ and $(0, 1) \mapsto (i, j)$, respectively. Since $n \geq 3$, both (i) and (i, j) are, by definition, simplices in Λ^n_k.

We define the functor $\tilde{f} : [n] \to \mathcal{C}$ by

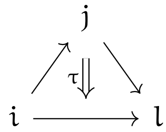
$$i \leq j \mapsto f(i, j) : f(i) \xrightarrow{f} (j)$$

In this case, by construction the restriction of $\tilde{f}$ to $(\Lambda^n_k)_0$ and $(\Lambda^n_k)_1$ is respectively f_0 and f_1. So, by Lemma 3.1.4, $\tilde{f}$ will be an extension of f.

It remains to check that $\tilde{f}$ is a functor. First notice that $(i, i) = (i) \circ \sigma^0$, from which we can conclude, by naturality of f, that $\tilde{f}$ preserves identities:

$$\tilde{f}(1_i) = f(i, i) = f((i) \circ \sigma^0) = s_0(f(i)) = 1_{\tilde{f}(i)}.$$

Now we show that $\tilde{f}$ preserves compositions, i.e. $f(j,l) \circ f(i,j) = f(i,l)$ for $0 \leq i \leq j \leq l \leq n$. Notice that $\tau := (i,j,l) \in \Delta^n_2$:



We separate in two cases:

- If $\tau \in (\Lambda^n_k)_2$, then $f\tau \in N\mathcal{C}_2$ is such that

  $$\begin{aligned} d_0 f\tau &= f d_0 \tau = f(j,l) \\ d_1 f\tau &= f d_1 \tau = f(i,l) \\ d_2 f\tau &= f d_2 \tau = f(i,j) \end{aligned}$$

  However the inner horn $(d_2 f\tau, \bullet, d_0\tau) : \Lambda^2_1 \to N\mathcal{C}$ has a unique extensionto $\tilde{\tau} \in \Delta^2$ (see Example 2.3.12) given by

  $$d_1 \tilde{\tau} = f(j,l) \circ f(i,j).$$

  Then, by the uniqueness of the extension,

  $$d_1 \tau = d_1 \tilde{\tau} \implies f(i,l) = f_{j,l} \circ f_{i,j},$$

  as we wanted.

- If $\tau \notin (\Lambda^n_k)_2$, then, by definition, necessarily $n = 3$ and then $\tau = \delta^k : [2] \to [3]$. We work with the case $k = 1$; the case $k = 2$ is analogous.

  Notice that $\delta^0, \delta^2, \delta^3 : [1] \to [3]$ are 2-simplices in Λ^3_1. Applying f to these simplices, the same uniqueness argument displayed above shows that

  $$f(0,2) = f(1,2) \circ f(0,1), \quad f(0,3) = f(1,3) \circ f(0,1), f(1,3) = f(2,3) \circ f(1,2).$$

  Hence

  $$\begin{aligned} f(0,3) &= f(1,3) \circ f(0,1) \\ &= (f(2,3) \circ f(1,2)) \circ f(0,1) \\ &= f(2,3) \circ f(0,2), \end{aligned} \tag{3.1}$$

  as we wanted to show.

The uniqueness of $\tilde{f}$ is immediate because any other extension g must satisfy $g_{i,j} = f_{i,j}$ and $g_i = f_i$, so that $g = \tilde{f}$ by Lemma 3.1.4 □

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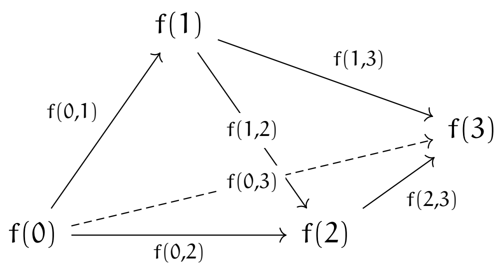
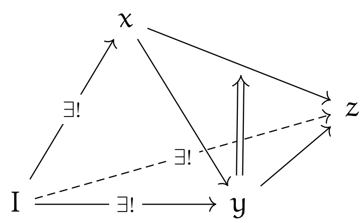


Figure 3.1: Visual interpretation of Equation 3.1: having shown that the faces opposite to the vertices $f(0)$, $f(1)$ and $f(3)$ commute, with some diagram chasing we see that the face opposite to $f(2)$ also commutes and then it defines a 3-simplex in $N\mathcal{C}$.

3.1.6 Remark. (This example is due to Eric Wofsey.) The previous theorem is indeed only true for inner horns. To see this, let I be an initial object in a category $\mathcal{C}$. If every outer horn $\Lambda^3_0 \to N\mathcal{C}$ had an extension to Δ^3, then any horn with I in the 0-th vertex could be filled; we would conclude that every triangle in $\mathcal{C}$ commutes, which of course is not always the case.



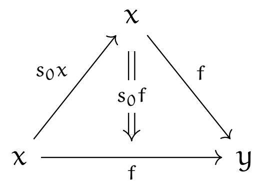
3.1.7 Proposition. *Let X be a simplicial set such that every inner horn has a unique filler. Then there is a cateogry $\mathcal{C}$ such that $X \cong N\mathcal{C}$.*

Proof. We define $\mathcal{C}$ explicitly:

- The set of objects is X_0.
- The set of morphisms is X_1. A morphism $f : x \to y$ is a 1-simplex such that $d_1 f = x$ and $d_0 f = y$.
- The identity $x \in X_0$ is $s_0 x \in X_1$.
- A pair of morphisms $f, g \in X_1$ is composable if $d_1 g = d_0 f$. In this case, we have the horn $(g, \bullet, f) : \Lambda^2_1 \to X$, which, by hypothesis, has a unique extension to $\tau \in \Delta^2$. With this, we set $g \circ f = d_1 \tau$, which is well-defined by uniqueness of the extension.

Now we verify that $\mathcal{C}$ is indeeed a category:

- *Unitality:* given a morphism $f : x \to y$, to verify that $f \circ 1_x = f$ corresponds to finding a 2-simplex $\tau \in X_2$ such that $d_0 \tau d_1 \tau = f$ and $d_2 \tau = s_0(x)$. This can be achieved by taking $\tau = s_0 f$ and using the simplicial identities:



The case $1_y \circ f = f$ is analogous by replacing $\tau = s_1 f$.

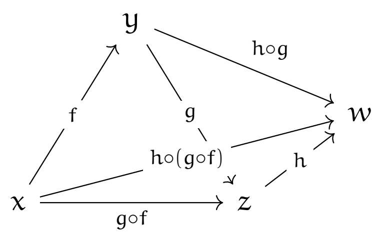
- As observed above, composable morphisms $f : x \to y$, $g : y \to z$ and $h : z \to w$ define the following 2-simplices:

$$\sigma_0 = (h, h \circ g, g)$$
$$\sigma_1 = (h, h \circ (g \circ f), g \circ f)$$
$$\sigma_3 = (g, g \circ f, f)$$

We can notice that

$$d_0\sigma_1 = d_0\sigma_0, \quad d_0\sigma_3 = d_2\sigma_0 \quad \text{and} \quad d_1\sigma_3 = d_2\sigma_1,$$

so that $(\sigma_0, \sigma_1, \bullet, \sigma_3)$ is a horn $\Lambda^3_2 \to X$, as depicted below.



This horn has a unique extension to $\tau \in \Delta^3$. Defining $\sigma_2 := d_2\tau$, a simple calculation with simplicial identities shows that

$$d_0\sigma_2 = h \circ g, \quad d_1\sigma_2 = h \circ (g \circ f) \text{ and } d_2\sigma_2 = f.$$

But then we also have $d_1\sigma_2 = (h \circ g) \circ f$, by definition, so $h \circ (g \circ f) = (h \circ g) \circ f$, as we wanted.

The map $f : X \to N\mathcal{C}$ is defined trivially in X_0 and X_1. Its definition on higher simplices can be obtained by the simplices of X (see [39, Definition 5.1]). The extension condition on X guarantees that the f obtained is a bijection on each degree, and thus an isomorphism. □

The two previous propositions characterize completely nerves of categories, as summarized in the corollary below.

3.1.8 Corollary. *A simplicial set X is the nerve of a category if, and only if, every inner horn in X has a unique filler.*

3.1.9 Proposition. *The nerve functor* $\mathrm{N} : \mathit{Cat} \to \mathit{sSet}$ *is fully faithful.*

Proof. We must exhibit that the map

$$\mathrm{N} : \mathrm{Func}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Func}(\mathrm{N}\mathcal{C}, \mathrm{N}\mathcal{D})$$

is a bijection. We will achieve that by building the inverse map M explicitly.

For each simplicial map $\phi : \mathrm{N}\mathcal{C} \to \mathrm{N}\mathcal{D}$, consider the functor $M(\phi) := F : \mathcal{C} \to \mathcal{D}$ defined as follows:

- For each $x \in \mathrm{N}\mathcal{C}_0$, let $F(x) = \phi_0(x)$.
- For each $f : x \to y \in \mathrm{N}\mathcal{C}_1$, let $Fu = \phi_1 u$.

With this definition, the domain of Fu is Fx by naturality:

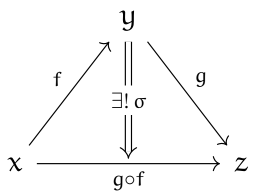
$$d_1(Fu) = d_1 f_0(f) = f_0 d_1 f = Fx.$$

Analogously, the domain of Ff is Fy, thus indeed $Ff : Fx \to Fy$. It remains to check for functoriality:

- F preserves identities:

$$F(1_x) = F(s_0 x) = f_1 s_0 x = s_0 f_0 x = 1_{Fx}.$$

- A pair of morphisms $f : x \to y$ and $g : y \to z$ in $\mathcal{C}$, defines a horn $(f, \bullet, g) : \Lambda^2_1 \to \mathrm{N}\mathcal{C}$. By Proposition 3.1.5, this horn has a unique extension σ, which is in this case characterized by $d_1\sigma = g \circ f$:



  Analogously, $Ff : Fx \to Fy$ e $Fg : Fy \to Fz$ defines a horn $(Ff, \bullet, Fg) : \Lambda^2_1 \to \mathrm{N}\mathcal{D}$ with a unique extension $\tilde{\sigma}$, which is characterized by $d_1\tilde{\sigma} = Fg \circ Ff$.

  On the other hand, ntice that $f_2\sigma \in \mathrm{N}\mathcal{D}_2$ is another extension of $(Ff, \bullet, Fg)$, because, by naturality,

$$d_0 f_2 \sigma = f_1 d_0 \sigma = Fg$$
$$d_2 f_2 \sigma = f_1 d_2 \sigma = Ff.$$

  Thus, by uniqueness, $\tilde{\sigma} = f_2\sigma$. So

$$d_1\tilde{\sigma} = d_1 f_2 \sigma = f_1 d_1 \sigma = F(g \circ f),$$

  and then $F(g \circ f) = Fg \circ Ff$.

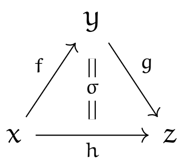
It follows by functoriality that $NM(\phi)_0 = \phi_0$ e $NM(\phi)_1 = \phi_1$, thus $NM(\phi) = \phi$ by Lemma 3.1.4. Also for any functor $G : \mathcal{C} \to \mathcal{D}$ we trivially have $MN(G) = G$, so we're done. □

Notice that a simplicial set X determines a (directed) graph $\mathrm{Graph}(X)$ with X_0 and X_1 as the sets of vertices and edges, respectively. Explicitly, an edge from x to y is a simplex $f \in X_1$ such that $d_1 f = x$ and $d_0 f = y$.

3.1.10 Definition. The **homotopy category** of X is the category hX freely generated by $\mathrm{Graph}(X)$ *modulo* the following equivalence relation:

$$h \sim gf \iff \text{there exists } \sigma \in X_2 \text{ such that } \begin{cases} d_0\sigma = f \\ d_1\sigma = h \\ d_2\sigma = g \end{cases}$$

In other words, the objects of hX are 0-simplices and morphisms are words generated by X_1. The relation used implies that the composition $g \circ f$ is exhibited by a 2-simplex:[1]



$$\implies h = g \circ f$$

3.1.11 Remark. When X is a quasicategory, this hX is the category whose objects are the obejcts of X and whose morphisms are 1-morphisms of X "modulo higher morphisms".

The construction of the homotopy category is functorial because it can be shown to be a composition of simpler functors. Let $h : \mathbf{sSet} \to \mathbf{Cat}$ be the functor it defines.

3.1.12 Proposition. *The pair $h \dashv N$ is an adjunction. In other words, there is a natural bijection*

$$\mathbf{Cat}(hX, \mathcal{C}) \xrightarrow{\sim} \mathbf{sSet}(X, N\mathcal{C}).$$

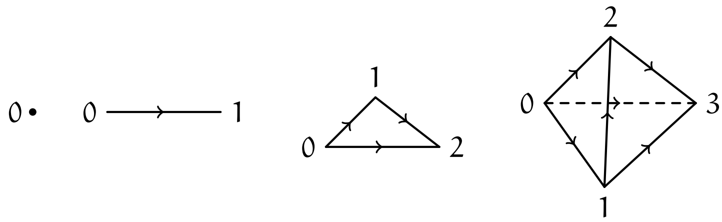
3.1.13 Corollary. *The adjunction above exhibits* **Cat** *as a reflexive subcategory of* **sSet**. *In particular, the counit of the adjunction provides natural isomorphisms $hN\mathcal{C} \xrightarrow{\sim} \mathcal{C}$.*

Proof. Immediate from 3.1.9. □

3.2 TOPOLOGICAL SPACES

For each $[n]$, the **topological n-simplex** is the n-dimensional tetrahedron in $\mathbb{R}^{n+1}$. Precisely, $|\Delta^n| \subset \mathbb{R}^{n+1}$ is defined by

$$|\Delta^n| = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1\}$$



3.2.1 Remark.

- Each $\delta^i : [n-1] \to [n]$ defines a map $|\Delta^{n-1}| \to |\Delta^n|$ which inserts $|\Delta^n|$ in the i-th face of $|\Delta^{n+1}|$:

$$(x_0, \dots, x_{i-1}, x_i, , \dots, x_{n-1}) \mapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_n).$$

- Each $\sigma^i : [n+1] \to [n]$ defines a map $|\Delta^{n+1}| \to |\Delta^n|$ that collapses $|\Delta^{n+1}|$ in its i-th face:

$$(x_0, \dots, x_i, x_{i+1}, \dots, x_n) \mapsto (x_0, \dots, x_i + x_{i+1}, \dots, x_n).$$

A simple calculation shows that these define a functor $\mathbf{\Delta} \to \mathbf{Top}$.

3.2.2 Definition. The **singular complex** of a topological space X is the simplicial set **Sing**(X) given by

$$\mathbf{Sing}(X)_n = \mathbf{Top}(|\Delta^n|, X),$$

with its action on morphisms defined by precomposition maps defined in Remark 3.2.1.

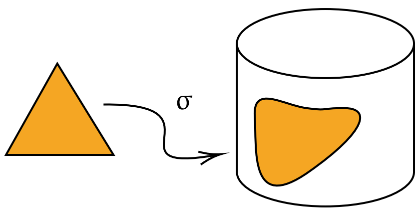


Figure 3.2: A 2-simplex σ in $S^1 \times I$.

In fact, the singular complexes assemble into a functor $\mathbf{Sing} : \mathbf{Top} \to \mathbf{sSet}$.

Geometric realization defines a functor $|-| : \mathbf{sSet} \to \mathbf{Top}$.

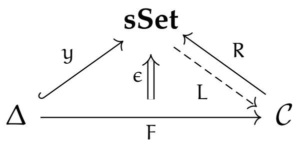
3.2.3 Lemma. *Given a functor $F : \mathbf{\Delta} \to \mathcal{C}$, define $R : \mathcal{C} \to \mathbf{sSet}$ representably by*

$$Rc_n = \mathcal{C}(F[n], c),$$

with face and degeneracy maps given by pre-composing. Then R is a right adjoint.

[1]"The 2-simplex σ *witnesses* the composition of f and g."

Proof. The left adjoint of $L : \mathbf{sSet} \to \mathcal{C}$ is given by a left Kan extension along the Yoneda embedding $y : \mathbf{\Delta} \hookrightarrow \mathbf{sSet}$:



For an explicit construction of L through coends and a proof of adjunction, see [40, Section 4]. □

3.2.4 Proposition. *The functor* **Sing** *is a right adjoint.*

Proof. In the language of Lemma 3.2.3, notice that **Sing** is the right adjoint defined by the functor $\mathbf{\Delta} \to \mathbf{Top}$ defined by $[n] \mapsto |\Delta^n|$. □

3.2.5 Definition. The **geometric realization** is the functor $|-| : \mathbf{sSet} \to \mathbf{Top}$ left adjoint to **Sing**.

3.2.6 Remark. The geometric realization of a simplicial sets admits the explicit description by

$$|X| = \coprod \hat{X} \times |\Delta^n| / \sim,$$

where $\hat{X} = \cup X_i$ with the discrete topology, and the equivalence relation is generated by $(d_i(x), t) \sim (x, \delta^i(t))$ and $(s_i(x), t) \sim (x, \sigma^i(t))$.

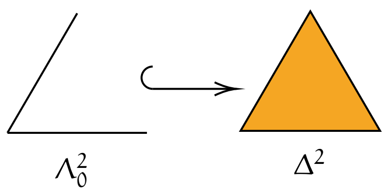
For further discussion about the geometric realization and its meaning, see [15, Section 4].

We collect the following results:

3.2.7 Remark. The geometric realization of Δ^n is $|\Delta^n|$, and $|\partial\Delta^n| = \partial|\Delta^n|$.

The geometric realization of $\Lambda^n_k \subset \Delta^n$ is $|\Delta^n|$ with its k-th face removed.

For instance:



3.2.8 Proposition. *For each space* X*, every horn* $\Lambda^n_k \to \mathbf{Sing}(X)$ *has a filler.*

Proof. Notice that $|\Lambda^n_k|$ is a retract of $|\Delta^n|$ i.e. the inclusion $|\Lambda^n_i| \hookrightarrow |\Delta^n|$ has an left inverse r. This can be seen by noticing that $|\Delta^n|$ and Λ^n_k are both homeomorphic to I^n and

$I^{n-1} \times \partial I$, respectively, and that the latter is a retract of the former; see the drawing in 3.2.7

With the adjunction $|-| \dashv \mathbf{Sing}$, we can transpose this extension problem to **Top**, where r provides a solution Transposing back to **sSet**, we obtain the desired extension:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathbf{Sing}(Y) \\ \big\downarrow & & \\ \Delta^n & & \end{array} \quad \to \quad \begin{array}{ccc} |\Lambda^n_k| & \longrightarrow & Y \\ \big\downarrow & \nearrow & \\ |\Delta^n| & & \end{array} \quad \to \quad \begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathbf{Sing}(Y) \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

□

Kan complexes as spaces

We will see in Chapter 4 that the adjunction $|-| \dashv \mathbf{Sing}$ is in fact a Quillen equivalence, where **Top** has the classical model structure (Example 1.3.2) and **sSet** has its classical model structure defined by Quillen. The fibrant-cofibrant objects in the latter are Kan complexes, so we have the frequent slogan:

> Kan complexes model the homotopy type of spaces.

Which, using the tools from Chapter 1, actually means:

> Given a topological space X there is a Kan complex K and a weak equivalence $|K| \xrightarrow{\in W} X$.

Moreover, it can be shown that $|K|$ is a CW complex and that every CW complex arises in this manner. For this reason homotopy theorists frequently swap the words *space*, *Kan complex*, *CW complex* and *∞-groupoid*.

3.3 CHAIN COMPLEXES

In this section we give a very brief survey of the *Dold-Kan correspondence*, the statement that, up to a well-defined notion of homotopy, chain complexes of abelian groups comprise a very peculiar class of spaces.

This is perhaps surprising, as the foundations of Algebraic Topology is itself intertwined with homological algebra. For instance, (co)homology is literally the (co)homology of a chain complex, and basic lemmas such as the long exact sequences and the Universal Coefficient Theorem are nothing but applied homological algebra.

3.3.1 Theorem (Dold-Kan)**.** *There is an equivalence of categories*

$$N : \mathbf{sAb} \leftrightarrows Ch^{\geq 0}_{\mathbb{Z}} : \Gamma.$$

The following proposition explains why we are referring to **sAb** as a particular class of spaces.

3.3.2 Proposition. *The simplicial set underlying a simplicial group* G *is a Kan complex.*

Proof. Considering a horn

$$(\sigma_0, \dots, \sigma_{i-1}, \bullet, \sigma_{i+1}, \dots, \sigma_n) : \Lambda_i^n \to G.$$

The filler can be calculated explicitly by using the group structure of G; we do this below for $i = 0$ and refer the reader to [36] for the other cases.

- $i = 0$: let $\tau_n = s_{n-1}(\sigma_n)$ and define inductively $\tau_i = \tau_{i+1} \cdot (s_{i-1} d_i \tau_{i+1})^{-1} \cdot s_{i-1}(\sigma_i)$, where $\cdot$ is the group structure in G_n. Then w_1 satisfies $d_i w_1 = y_i$ for $i \neq 0$, i.e. w_1 fills the horn.

□

For a proof of Theorem 3.3.1, see [29, Section 2.5.6] or the survey [32].

3.3.3 Definition. The **Moore complex** of a simplicial abelian group A is the chain complex $(C(A)_\bullet, \partial)$, where $C(A)_n = A_n$ and the differential is given by

$$\partial : C(A)_n \to C(A)_{n-1} \quad \partial(\sigma) = \sum_{i=1}^{n} (-1)^i d_i(\sigma).$$

The construction can be extended to any simplicial set X by considering the free simplicial abelian group defined by X.

The calculation of $\partial^2 = 0$ is the same as in singular homology [20, Lemma 2.1]. It can also be checked that the Moore complex assembles into a functor $M : \mathbf{sAb} \to \mathbf{Ch}_{\mathbb{Z}}$.

3.3.4 Example. This definition encapsulates the singular homology of spaces as the composite functors

$$\mathbf{Top} \xrightarrow{\mathbf{Sing}} \mathbf{sSet} \xrightarrow{F} \mathbf{sAb} \xrightarrow{M} \mathbf{Ch}_{\mathbb{Z}} \xrightarrow{H_n} \mathbf{Ab},$$

where the last arrow is the standard homology of chain complexes, i.e.

$$H_n(C_\bullet) = \ker\, \partial_n / \partial_{n+1}(C_{n+1}).$$

3.3.5 Example. The simplicial abelian group freely generated by Δ^0 is the abelian group constant in $\mathbb{Z}$. The boundaries of its Moore complex are

$$\partial = \sum (-1)^i = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

We can calculate its homology directly

$$H_n(\Delta^0) = \begin{cases} \mathbb{Z} \text{ if } n = 0 \\ 0 & n > 0 \end{cases}$$

Since $\Delta^0 = \mathbf{Sing}(*)$, this recovers the most basic calculation in algebraic topology.

3.3.6 Remark. The Moore complex isn't *de facto* the functor N in Theorem 3.3.1. Instead, it is the *normalized* Moore complex $NA_\bullet$, defined by considering the vanishing set of the face maps d_i, for $0 < i$, with the n-th face map $d_0 : A_n \to A_{n-1}$ serving as the differential.

However the inclusion $NA_\bullet \hookrightarrow CA_\bullet$ is a quasi-isomorphism [29, Proposition 2.5.5.11].

3.3.7 Definition. For a chain complex of abelian groups $C_\bullet$, let $\Gamma(C_\bullet)$ be the simplicial group given by

$$\Gamma(C_\bullet)_n = \bigoplus_{\substack{[n]\to[k] \\ \text{surjective}}} C_k.$$

For each $f : [m] \to [n]$ in $\mathbf{\Delta}$, we define $f_* : \Gamma(C_\bullet)_n \to \Gamma(C_\bullet)$ as follows:

- if $\nu \in \Gamma(C_\bullet)_n$ is indexed by $g : [n] \to [k]$, then consider the factorization

  $$g \circ f = [m] \overset{\pi}{\twoheadrightarrow} [p] \overset{\iota}{\hookrightarrow} [k]$$

  as in Lemma 2.1.3. Since π is surjective, we can define $f_*(\nu) = \nu$.

We finish this section this section with a few facts relating Γ to homological algebra. We won't give any details at all, hoping that the sheer statements will foster the curiosity of the reader.

3.3.8 Proposition. *Let A be an abelian group and $A[n]_\bullet \in \mathbf{Ch}_\mathbb{Z}$ be the chain complex concentrated in A in degree n. Then $\Gamma(A[n]_\bullet)$ is an Eilenberg-MacLane space $K(A, n)$.*

3.3.9 Lemma. *Let $C_\bullet$ be a chain complex of abelian groups. Then the simplicial homotopy groups of $\Gamma(C_\bullet)$ are the homology groups of $C_\bullet$.*

3.3.10 Corollary. *A delooping of $\Gamma(C_\bullet)$ is given by $\Gamma(C_{\bullet+1})$.*

4

QUASICATEGORIES

Quasicategories model ∞-categories. The definition dates back to 1972 by Boardman and Vogt [7], but their intensive use as models for ∞-categories was first done by Joyal [23, 24], and later much extended by Lurie in the remarkable [27]. In this book, many standard theorems of Category Theory find their quasicategorical counterparts, for instance: equivalences as fully faithful and essentially surjective functors, the Yoneda lemma and the Adjoint Functor Theorem.

There are other models for ∞-categories, but we settle with quasicategories aiming for agility towards concrete results. For brief (and excellent) introductions to this larger web of models for ∞-category theory, we refer the reader to [9] and [6].

4.1 DEFINITIONS

We start with the necessary definition.

4.1.1 Definition. A **quasicategory** is a simplicial set $\mathcal{C}$ such that every inner horn has a filler.

In other words, in a quasicategory the extension below always exists, for $0 < k < n$:

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{f} & \mathcal{C} \\ \downarrow & \nearrow_{\exists \tilde{f}} & \\ \Delta^n & & \end{array}$$

4.1.2 Example. The initial and terminal quasicategories are respectively $\varnothing$ and Δ^0.

4.1.3 Example. Corollary 3.1.8 shows that nerves of categories are precisely those quasicategories whose fillers are unique. In particular, $\Delta^n = N[n]$ are quasicategories. Furthermore, the fully faithfulness of the nerve (Proposition 3.1.9) guarantees that we can exchange freely between a category and its nerve.

4.1.4 Example. The three examples of the last chapter give very particular examples of $(\infty, 1)$-categories, i.e. quasicategories:

- Categories are simply categories. (Corollary 3.1.8)
- Topological spaces correspond to ∞-groupoids. (Theorem 3.2.8; see Definition)
- Chain complexes are a special case of ∞-abelian groups. (Theorem 3.3.1)

4.1.5 Example (Products)**.** Given a collection $\{X_\alpha\}_{\alpha \in \mathscr{A}}$ of simplicial sets and another simplicial set Y, standard category theory shows that

$$\mathbf{sSet}(Y, \prod_\alpha X_\alpha) \cong \prod_\alpha \mathbf{sSet}(Y, X_\alpha).$$

This observation makes it clear that is each X_α is a quasicategory then its product $\prod_\alpha X_\alpha$ is also a quasicategory. The converse hold if each X_α is non-empty (otherwise the product is also empty).

The opposite quasicategory

Given a quasicategory $\mathcal{C}$, we wish to define its opposite $\mathcal{C}^{op}$. To do that, first we define a functor $\mathrm{Op} : \mathbf{\Delta} \to \mathbf{\Delta}$ by $[n] \mapsto [n]$ and

$$\delta_i \mapsto \delta_{n-i}, \quad \sigma_i \mapsto \sigma_{n-i}.$$

Now we can define $\mathcal{C}^{op}$ as the composite

$$\mathbf{\Delta}^{op} \xrightarrow{\mathrm{Op}} \mathbf{\Delta}^{op} \xrightarrow{\mathcal{C}} \mathrm{Set}.$$

Explicitly, $\mathcal{C}^{op}$ has the same vertices as $\mathcal{C}$ and

$$d_i^{op} \mapsto d_{n-i}, \quad s_i^{op} \mapsto s_{n-i}.$$

Furthermore, from the definition of Op we have isomorphisms $(\Delta^n)^{op} \cong \Delta^n$ and $(\Lambda^n_i)^{op} \cong \Lambda^n_{n-i}$. Therefore, there is a correspondence between the extension problems

$$\Lambda^n_i \to \mathcal{C}^{op} \leftrightsquigarrow \Lambda^n_{n-i} \to \mathcal{C}.$$

In particular, the opposite of a quasicategory is also a quasicategory.

4.1.6 Remark. A quasicategory $\mathcal{C}$ possesses

- a collection of **objects** $x, y, z, \dots$, the elements of X_0.
- a collection of **morphisms** $f, g, h, \dots$, the elements of X_1.

so that:

- each morphism $f \in X_1$ has specified **domain** and **codomain** objects given by $d_1(f)$ and $d_0(f)$, respectively; the notation $f : x \to y$ signifies that f is a morphism with domain x and codomain y.
- Each object has a designated **identity morphism** $1_x := s_0(x) : x \to x$.
- For any pair of morphisms f, g with the codomain of f equal to the codomain of g, the filler of the horn
$$(g, \bullet, f) : \Lambda^2_1 \to \mathcal{C}$$
provides a **composite morphisms** gf whose domain equal to the domain of f and whose codomain is equal to the codomain of g, i.e.:
$$f : x \to y, \quad g : y \to z \quad \rightsquigarrow gf : y \to z$$

4.2 HOMOTOPY OF MORPHISMS

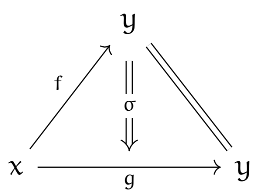
Most difficulties of higher category theory are due to the follow paradigm:

> In an n-category, commutative diagrams of k-morphisms commute up to $(k+1)$-morphisms.

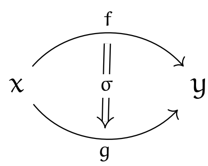
We devote this section to developing this paradigm for quasicategories in a rather succint manner.[1]

4.2.1 Definition. A **homotopy** between a pair of parallel morphisms $f, g : x \to y$ in $\mathcal{C}$ is a 2-simplex σ satisfying
$$d_0(\sigma) = 1_y, \quad d_1(\sigma) = g, \quad d_2(\sigma) = f,$$

We may depict a homotopy $\sigma : f \Rightarrow g$ visually as follows:



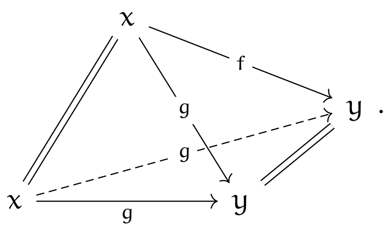
4.2.2 Remark. A homotopy may be interpreted as a 2-morphism $\sigma : f \Rightarrow g$. At first, this sounds foreign because we're more used to the *globular representation* of 2-morphisms:



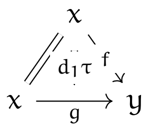
[1] Actually, in this section we show that the composition of morphisms is well-defined up to 2-morphisms. The general case will be discussed in the next section, and we also refer the reader to [29].

Notice however that a homotopy is like a 2-morphism, except that we've opened an identity at y. In the literature, this usually goes by the name of *simplicial representation* of n-morphisms.

4.2.3 Remark. There is an evident choice when we open an identity at y instead of x. This choice is, however, irrelevant. To see this, given a homotopy $\sigma : f \Rightarrow g$ we define the horn $(\sigma, s_1(g), \bullet, s_0(g)) : \Lambda^3_2 \to \mathcal{C}$, depicted visually below,



Since $\mathcal{C}$ is a quasicategory, we can fill this horn to a 3-simplex τ. This is such that

$d_2(\tau) =$


exhibiting a "left homotopy" from f to g.

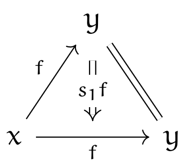
4.2.4 Example. Let $f, g : x \to y$ be morphisms in a ordinary category $\mathcal{C}$. Since $\mathrm{id}_y \circ f = f$ on the nose, two morphisms in $N\mathcal{C}$ are homotopic if, and only if, $f = g$.

4.2.5 Example. Since $|\Delta^1| \cong [0, 1]$, morphisms $f, g : x \to y$ in **Sing**(X) may be identified with paths $f, g : [0, 1] \to X$ such that $f(0) = x = g(0)$ and $f(1) = y = g(1)$. Then a simple but nasty calculation shows that f and g are homotopic in the quasicategorical sense if, and only if, the corresponding paths are homotopic in the topological sense [29].

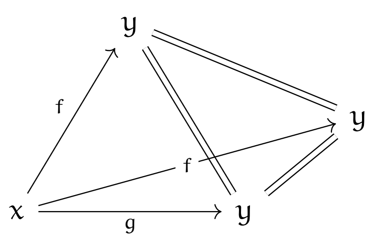
4.2.6 Proposition. *Homotopy is an equivalence relation.*

Proof.

- *Symmetry:* simply take $s_1(f) \in \mathcal{C}_2$:

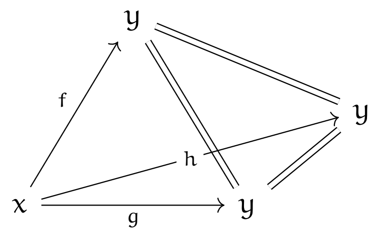


- *Reflexitivity:* let σ be a homotopy from f to g. Then, consider the horn $(s_0 s_0 y, \bullet, s_1(f), \sigma) : \Lambda^3_1 \to \mathcal{C}$, as depicted below:

$\tau =$


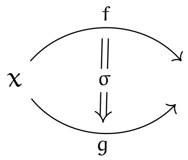
Since $\mathcal{C}$ is a quasicategory, this horn extends to a 3-simplex τ; a homotopy from g to f is given by $d_1(\tau)$.

Transitivity: let σ be a homotopy from f to g, and η a homotopy from g to h. Then we can build the horn $(s_0 s_0 y, \eta, \bullet, \sigma) : \Lambda^3_2 \to \mathcal{C}$, as depicted below:



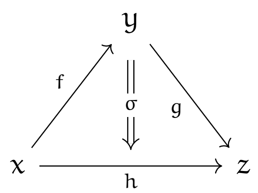
Since $\mathcal{C}$ is a quasicategory, this horn extends to a 3-simplex τ; a homotopy from g to h is given by $d_2(\tau)$.

□

4.2.7 Remark. The symmetry of the homotopy relation indicates that in a quasicategory every 2-morphism 

is invertible. This is the first indication that quasicategories actually are a model for $(\infty, 1)$-categories, rather than general ∞-categories.

4.2.8 Definition. Let $f : x \to y$, $g : y \to z$ and $h : x \to z$ be morphisms in a quasicategory $\mathcal{C}$. We say that h is **a composition** of f and g if there is a 2-simplex $\sigma \in \mathcal{C}_2$ such that $d_0(\sigma) = g$, $d_1(\sigma) = h$ and $d_2(\sigma) = f$:



In this case, we say that σ *witnesses* h as a composition of g with f.

Notice that compositions in a quasicategory are far from uniquely defined, They are, however, unique up to homotopy:

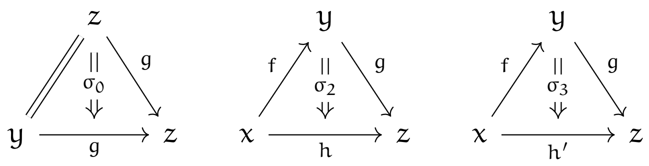
4.2.9 Proposition. *Let $f : x \to y$ and $g : y \to z$ be morphisms in a quasicategory $\mathcal{C}$.*

(i) There is a composition $h : x \to z$ of f and g.

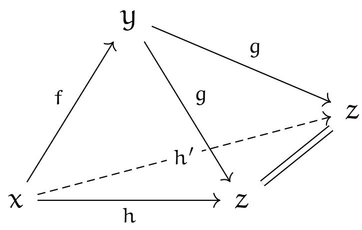
(ii) Another morphism $h' : x \to z$ is a composition of f and g if, and only if, there is a homotopy between h and h'.

Proof. Since $\mathcal{C}$ is a quasicategory, we can fill the horn $(g, \bullet, f) : \Lambda^2_1 \to \mathcal{C}$ to a 2-simplex σ such that $h := d_1(\sigma)$ is a composition of f and g.

For the other assertion, we build the following 2-simplices:



Then we define the horn $(\sigma_0, \bullet, \sigma_2, \sigma_3) : \Lambda^3_1 \to \mathcal{C}$, as depicted below:

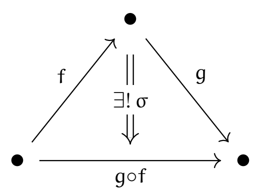


Since $\mathcal{C}$ is a quasicategory, this horn may be extended to a 3-simplex τ. The face $d_1(\tau)$ gives the desired homotopy from h to h'.

The converse is similar: if σ_1 is a homotopy from h to h', then the 1-th face of the filler of $(\sigma_0, \sigma_1, \bullet, \sigma_3) : \Lambda^3_1 \to \mathcal{C}$ exhibits h' as a composition of f and g. ☐

4.2.10 Notation. We write $h \in g \circ f$ as a shorthand for "h is a composition of f and g".

4.2.11 Example. Example 3.1.3 shows that, given a category $\mathcal{C}$, the quasicategorical composition in $N\mathcal{C}$ is unique and given by the composition of $\mathcal{C}$:



4.2.12 Example. Two morphisms $f, g : [0,1] \to X$ in $\mathrm{Sing}(X)$ are composable iff $f(1) = g(0)$. One possible composition of these morphisms is given by path concatenation $g \star f : [0,1] \to X$, defined explicitly by

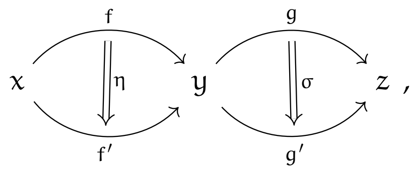
$$(g \star f)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq 1/2 \\ g(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}.$$

The following 2-simplex $\sigma : |\Delta^2| \to X$ witnesses $g \star f$ as a composition of f and g:

$$\sigma(t_0, t_1, t_2) = \begin{cases} f(t_1 + 2t_2) & \text{if } t_0 \geq t_2 \\ g(t_2 - t_0) & \text{if } t_0 \leq t_2 \end{cases}$$

In this case, composition is clearly not uniquely defined. In fact, combining Example 4.2.5 with Theorem 4.2.9 we see that the compositions of g and f are given precisely by the paths homotopic to $g \star f$.

The next proposition provides an exchange rule for homotopies. In globular notation, this means that given a diagram

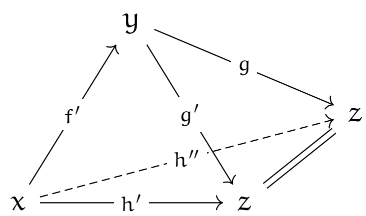


the compositions of f and g are all homotopic to all compositions of f' and g'.

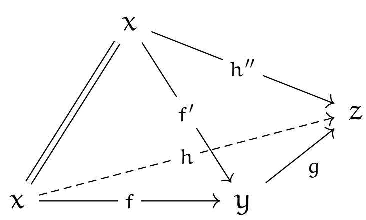
4.2.13 Proposition. *Let $f, f' : x \to y$ and $g, g' : y \to z$ be pairs of homotopic morphisms in a quasicategory $\mathcal{C}$. Let h be a composition of f and g, and h' be a composition of f' and g'. Then h and h' are homotopic.*

Proof. This proof is some sort of "quasicategorical pasting". Let h'' be a composition of f' and g. We will show that h and h' are homotopic to h'', from which the result follows by transitivity.

- To see that h' and h'' are homotopic, we extend the horn $\Lambda^3_1 \to \mathcal{C}$ depicted below:



- To see that h and h'' are homotopic, we extend the horn $\Lambda^3_2 \to \mathcal{C}$ depicted below:



□

Propositions 4.2.9 and 4.2.13 indicate that the composition of homotopy *classes* is well-defined by $[g] \circ [f] = [h]$, where h is any composition of f and g.

Letting $h\mathcal{C}(x, y)$ be set of homotopy classes of morphisms between two objects $x, y \in \mathcal{C}$, this means that composition gives a map

$$\circ : h\mathcal{C}(x, y) \times h\mathcal{C}(y, z) \to h\mathcal{C}(x, z).$$

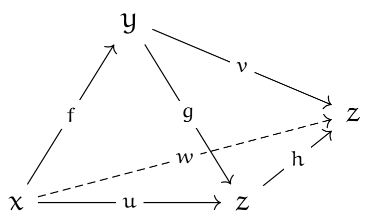
4.2.14 Proposition. *Let $\mathcal{C}$ be a quasicategory. Then composition of homotopy classes*

(i) is associative.

(ii) has a neutral elements given by the classes of the identities $id_x = s_0(x)$.

Proof.

(i) *Associativity:* given morphisms $f : x \to y$, $g : y \to z$ and $h : z \to w$, let $u \in g \circ f$, $v \in h \circ g$ and $w \in h \circ u$. Then consider the horn $\Lambda^3_2 \to \mathcal{C}$ depicted below:



Since $\mathcal{C}$ is a quasicategory, this horn has a filler τ. Its face $d_2\tau$ is a homotopy $v \circ f \Rightarrow w$, i.e. $(h \circ g) \circ f \sim h \circ (g \circ f)$.

(ii) *Unitality:* given a morphism $f : x \to y$, the 2-simplices $s_0(f)$ and $s_1(f)$ show that $f \in 1_y \circ f$ and that $f \in f \circ 1_x$.

□

The propositions above are all summarized by the following, crucial definition.

4.2.15 Definition. The **homotopy category** of a quasicategory $\mathcal{C}$ is the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are homotopy classes of morphisms in $\mathcal{C}$. The composition is defined by $[g] \circ [f] = [h]$, where $h \in g \circ f$, and the identities are $id_x := s_0(x)$.

4.2.16 Example. The homotopy category of $N\mathcal{C}$ is isomorphic to $\mathcal{C}$.

4.2.17 Example. The homotopy category of **Sing**(X) has as objects points of X and as morphisms homotopy classes of paths in X. Well, actually h**Sing**(X) is a very famous category: it's the fundamental groupoid $\Pi(X)$.

4.2.18 Definition. The set of **connected components** of a simplicial sets X is defined by

$$\pi_0 X = X_0 / \sim, \quad x \sim y \iff \exists f : x \to y.$$

If X is a Kan complex, then

$$\pi_0 \mathcal{C} = \mathrm{ob}(h\mathcal{C})/\{\text{isomorphisms}\}.$$

4.2.19 Example. The connected components of $\mathbf{Sing}(X)$ corresponds to the path connected components of X.

The hom ∞-groupoid

Let $\mathcal{C}$ be a quasicategory with objects x and y. The **hom ∞-grupoid**, or the **mapping space**, is the simplicial set $\mathcal{C}(x,y)$ defined by the pullback

$$\begin{array}{ccc} \mathcal{C}(x,y) & \longrightarrow & \mathbf{Func}(\Delta^1,\mathcal{C}) \\ \big\downarrow & \lrcorner & \big\downarrow p \\ \Delta^0 & \xrightarrow[(y,x)]{} & \mathbf{Func}(\partial\Delta^1,\mathcal{C}) \end{array},$$

For a proof, we refer the reader to Proposition 4.6.1.18 of [29].

4.2.20 Example. One of the most striking features of ∞-categories is the powerful definition of *cohomology theory* [34] it provides. Given an ∞-category $\mathbf{H}$ and an object X, the **0-th degree cohomology** of X with values in another object A is defined to be the set of connected components of the mapping space $\mathbf{H}(X,A)$:

$$H^0(X;A) = \pi_0\mathbf{H}(X,A).$$

Cohomology in n-th degree is defined if A possesses a delooping of degree n, i.e., an object A_n such that $A \cong \Omega^n A_n$.[2] In this case,

$$H^n(X;A) = \pi_0\mathbf{H}(X,A_n).$$

All different kinds of cohomology theories fit into this picture. We heavily encourage the reader to skim through the nLab page.

4.3 FUNCTORS

As models for ∞-categories, there is a natural notion of $(\infty-)$functor between quasicategories that generalizes ordinary functors.

4.3.1 Definition. A **functor** between quasicategories $\mathcal{C},\mathcal{D}$ is a simplicial map $F:\mathcal{C}\to\mathcal{D}$.

4.3.2 Example. By naturality, a functor between quasicategories takes objects to objects and morphisms to morphisms. Also, identities are preserved:

$$F(1_x) = F(s_0x) = s_0(Fx) = 1_{Fx}.$$

[2] The loop space ΩA is defined as the ∞-pullback of $* \to A \leftarrow *$.

Furthermore, if $\sigma \in \mathcal{C}_2$ is a 2-simplex witnessing a composition of f and g, then $F(\sigma)$ exhibits a composition of Ff and Fg:

$$d_0 F(\sigma) = F(d_0 \sigma) = Fg$$
$$d_2 F(\sigma) = F(d_2 \sigma) = Ff$$

In other words, a functor between quasicategories preserves not only compositions, but also the 2-simplices witnessing these compositions.

4.3.3 Example. A functor $F : N\mathcal{C} \to N\mathcal{D}$ contains all the data defining a functor $\tilde{F} : \mathcal{C} \to \mathcal{D}$: it takes objects to objects and morphisms to morphisms, whereas identities and compositions are respectively taken to identities and compositions (see the proof of Proposition 3.1.5). Moreover, Proposition 3.1.9 gives a natural bijection

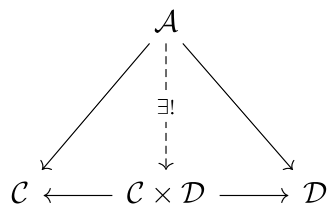
$$\text{functors } \mathcal{C} \to \mathcal{D} \leftrightsquigarrow \text{functors } N\mathcal{C} \to N\mathcal{D}.$$

4.3.4 Example. Let $\mathcal{C}$ be a quasicategory and $\mathcal{D}$ be an ordinary category. The adjunction $N : \mathrm{Cat} \leftrightarrows \mathbf{sSet} : h$ gives a natural bijection

$$\text{functors } \mathcal{C} \to N\mathcal{D} \leftrightsquigarrow \text{functors } h\mathcal{C} \to \mathcal{D}.$$

4.3.5 Example. Recall that the product of quasicategories is a quasicategory, as seen in Example 4.1.5.

Thus given a pair of functors $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{D}$ between quasicategories, there is a unique functor $F \times G : \mathcal{A} \to \mathcal{C} \times \mathcal{D}$ such that the following diagram of functors commutes:



Diagrams in quasicategories

A commutatitve diagram in a category $\mathcal{C}$ is a functor $J \to \mathcal{C}$, where J is a small category. We casually describe a diagram as a collection of objects $\{x_i\}$ and morphisms $\{f_{ij} : x_i \to x_j\}$ between them. Commutativity of a diagram is captured by the following property (which follows from functoriality):

- Any two chains $f_{i_n} \circ \cdots \circ f_{i_1}$ and $f_{j_n} \circ \cdots \circ f_{j_1}$ with the same domain and codomain are equal.

Analogously, a **commutative diagram** in a quasicategory $\mathcal{C}$ is a simplicial map $F : NJ \to \mathcal{C}$, where J is a small category. This time the diagram will select not only a collection of objects and morphisms, but also the specific 2-simplices witnessing the commutativity of each triangle in the diagram. The name, however, is a stretch, because this time "commutativity" only makes sense in the homotopical sense, i.e.

- Any two chains $f_{i_n} \circ \cdots \circ f_{i_1}$ and $f_{j_n} \circ \cdots \circ f_{j_1}$ with the same domain and codomain are homotopic.

For instance, while a commutative square in a category is simply a diagram $\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ \bullet & \xrightarrow[g']{} & \bullet \end{array}$ such that $g \circ f = g' \circ f'$, a commutative diagram is a quasicategory comes with specified 2-simplices witnessing these compositions:

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ {\scriptstyle f'}\downarrow & \overset{\sigma}{\searrow}_{\sigma'} & \downarrow{\scriptstyle g} \\ \bullet & \xrightarrow[g']{} & \bullet \end{array}$$

(i.e. a commutative square is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$)

In the rest of this section, we develop the quasicategory of functors $\mathbf{Func}(\mathcal{C}, \mathcal{D})$.

4.3.6 Definition. Let X and Y be simplicial sets. The **internal hom of simplicial sets** is the simplicial set given by

$$\mathbf{Func}(X, Y)_n := \mathbf{sSet}(X \times \Delta^n, Y),$$

and whose faces and degeneracies are induced by precomposition with (id, δ^i) and (id, σ^i), respectively.

4.3.7 Proposition. *The internal hom defines a closed monoidal structure on* ***sSet****. In other words, for each triple of simplicial sets $X, Y, Z \in$* ***sSet*** *there is a natural bijection*

$$\boldsymbol{sSet}(X \times Y, Z) \cong \boldsymbol{sSet}(X, \mathbf{Func}(Y, Z)).$$

Proof. First we define the evaluation map $\mathrm{ev} : \mathbf{Func}(Y, Z) \times Y \to Z$ at $(g, y) \in \mathbf{Func}(Y, Z) \times Y$ as the composite

$$\Delta^n \xrightarrow{\Delta} \Delta^n \times \Delta^n \xrightarrow{(\sigma, \mathrm{id})} \Delta^n \times X \xrightarrow{g} Y.$$

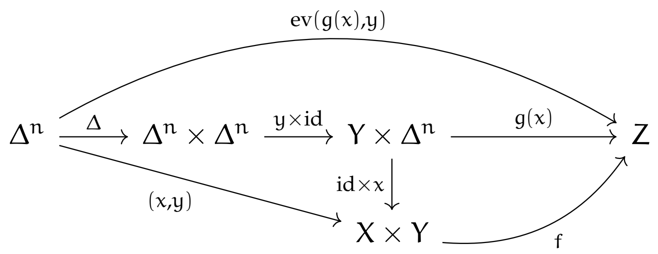
The bijection in the statement then follows from

$$\mathbf{sSet}(X, \mathbf{Func}(Y, Z)) \to \mathbf{sSet}(X \times Y, \mathbf{Func}(Y, Z) \times Y) \xrightarrow{\mathrm{ev}_*} \mathbf{sSet}(X \times Y, Z).$$

Indeed, we may define the inverse $f : X \times Y \to Z$ at $x \in X$ as the composite

$$g(x) = \Delta^n \times Y \xrightarrow{x \times \mathrm{id}} X \times Y \xrightarrow{f} Z.$$

It remains to check that $\mathrm{ev}(g(x), y) = f(x, y)$. This follows from the commutative diagram below:



$\square$

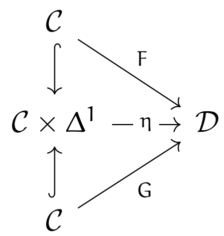
4.3.8 Remark. To show that **sSet** is cartesian closed, without invoking the internal hom explicitly, we could have proceeded by abstract nonsense by fixing $X \in$ **sSet** and defining the functor $\mathbf{\Delta} \to$ **sSet** given by $[n] \mapsto X \times \Delta^n$. The adjunction is then obtained by taking the left Kan extension along the Yoneda lemma.

4.3.9 Theorem. *Let X be a simplicial set and $\mathcal{C}$ a quasicategory. Then* **Func**$(X, \mathcal{C})$ *is a quasicategory.*

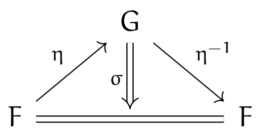
We will glance over the proof of this theorem in the next section.

4.3.10 Remark. When $\mathcal{C}$ and $\mathcal{D}$ are quasicategories, the objects of **Func**$(\mathcal{C}, \mathcal{D})$ are simplicial maps $\mathcal{C} \times \Delta^0 \to \mathcal{D}$ which may be identified as functors $\mathcal{C} \to \mathcal{D}$.

In this case, a **natural transformation** between functors $F, G : \mathcal{C}, \mathcal{D}$ is simply a morphism $\eta : F \Rightarrow G$, i.e., a morphism $\eta : \mathcal{C} \times \Delta^1 \to \mathcal{D}$ such that $\eta\ |_{\mathcal{C}\times\{0\}} = F$ and $\eta\ |_{\mathcal{C}\times\{1\}} = F$:



4.3.11 Definition. A **natural equivalence** is a natural transformation $\eta : \mathcal{C} \times \Delta^1 \to \mathcal{D}$ which is an equivalence in **Func**$(\mathcal{C}, \mathcal{D})$:



A functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence of quasicategories** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural equivalences $\eta : GF \cong \mathrm{id}_{\mathcal{C}}$ and $\epsilon : \mathrm{id}_{\mathcal{D}} \cong FG$.

By definition, two functors are equivalent in **Func**$(\mathcal{C}, \mathcal{D})$ iff they are isomorphic in h**Func**$(\mathcal{C}, \mathcal{D})$. Thus we get

4.3.12 Lemma. *An equivalence of quasicategories* $F : \mathcal{C} \to \mathcal{D}$ *induces an equivalence of categories* $hF : h\mathcal{C} \to h\mathcal{D}$.

Proof. Applying h to the natural isomorphisms $\eta : GF \cong \mathrm{id}_{\mathcal{C}}$ and $\epsilon : \mathrm{id}_{\mathcal{D}} \cong FG$ we see that hF is a categorical equivalence. □

4.3.13 Example. The quotient $\Delta^1/\partial\Delta^1$ is the simplicial set generated by the graph below:

Now consider $B\mathbb{N}$, the category freely generated by the same graph. There is an equivalence of quasicategories

$$\Delta^1/\partial\Delta^1 N(B\mathbb{N}).$$

4.3.14 Theorem. *Let* X *be a simplicial set such that the terminal map* $p : X \to \Delta^0$ *is a trivial fibration. Then* X *is a Kan complex and* p *is a quasicategorical equivalence.*

Proof. See [39, Proposition 23.3]. □

4.3.15 Remark. If $\mathcal{C}$ is a quasicategory, then the converse holds: if $\mathcal{C}$ is equivalent to Δ^0 then $\mathcal{C} \to \Delta^0$ is a trivial Kan fibration.

4.3.16 Remark. Quasicategorical equivalences may be generalized to simplicial sets as follows:

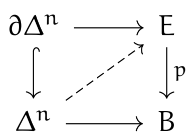
- a simplicial map $f : X \to Y$ is an **equivalence** if the induced functor $f^* : \mathbf{Func}(Y, \mathcal{C}) \to \mathbf{Func}(X, \mathcal{C})$ is an equivalence of quasicategories for all quasicategories $\mathcal{C}$.

They are the weak equivalences of Joyal's model structure, whose fibrant objects are precisely the quasicategories. Unravelling the definitions, we notice that for quasicategories this is the same as Definition 4.3.11.

4.4 TRIVIAL KAN FIBRATIONS

In order to show that $\mathbf{Func}(X, \mathcal{C})$ is a quasicategory for any quasicategory $\mathcal{C}$ (Theorem 4.4.9), we will use some definitions and results involving *trivial Kan fibrations*. These are actually tools borrowed from the homotopy theory of quasicategories.

4.4.1 Definition. Let $J = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \in \mathbb{N}\}$. A **trivial Kan fibration** is a simplicial map $p : E \to B$ with the left lifting property against the set $J = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \in \mathbb{N}\}$. In other words, p is a Kan fibration iff every lifting problem as below has a solution:



4.4.2 Corollary. *Trivial Kan fibrations are closed under pullback.*

Proof. This is a general closure result for sets defined by left lifting properties. □

4.4.3 Proposition. *A simplicial map* $p : E \to B$ *is a trivial Kan fibration if, and only if,* p *has the LLP against monomorphisms.*

Proof. See the Section 1.4.4 of [29]. □

The following corollary shows how Kan fibrations may resemble topological bundles.

4.4.4 Corollary. *Let* $p : E \to B$ *be a trivial Kan fibration.*

(i) There exists a section of p*, i.e. there is a simplicial map* $s : B \to E$ *such that* $p \circ s = id_B$*.*

(ii) Given any section s *of* p*, the composite* $sp : E \to E$ *is homotopic to the identity map. This means that there is a simplicial map* $H : E \times \Delta^1 \to E$ *such that* $H|_{E\times\{0\}} = s \circ p$ *and* $H|_{E\times\{0\}} = id_E$*.*

Proof.

(i) As in any category, the initial morphism $\varnothing \to B$ is a monomorphism. Then the section $s : B \to E$ is given by the lift provided by Corollary 4.4.3:

$$\begin{array}{ccc} \varnothing & \longrightarrow & E \\ \downarrow & \overset{s}{\nearrow} & \downarrow p \\ B & = & B \end{array}$$

(ii) Consider the lift problem depicted below:

$$\begin{array}{ccc} E \coprod E & \xrightarrow{(s\circ p, id)} & E \\ i_0 \coprod i_1 \downarrow & \overset{H}{\nearrow} & \downarrow p \\ E \times \Delta^1 & \xrightarrow[p \circ p_1]{} & B \end{array}$$

The map $i_0 \coprod i_1 \to E \times E$ is a monomorphism because $i_0, i_1 : E \cong E \times \Delta^0 \hookrightarrow E \times \Delta^1$ are monomorphisms and these are closed under coproducts. Then we may rewrite the triangle above as

$$\begin{array}{ccc} E & & \\ i_0 \downarrow & \overset{s\circ p}{\searrow} & \\ E \times \Delta^1 & \xrightarrow{H} & E \ , \\ i_1 \uparrow & \nearrow\!\!\!\!= & \\ E & & \end{array}$$

where $H : s \circ p \Rightarrow id_E$ is the homotopy we wanted.

□

4.4.5 Lemma. *A trivial Kan fibration* $p : E \to B$ *between quasicategories is an equivalence of quasicategories.*

Proof. Corollary 4.4.4 gives a section $s : B \to E$ of p such that $sp \cong \mathrm{id}_E$ in $\mathbf{Func}(E, E)$. Moreover, because s is a section, we already have $p \circ s = \mathrm{id}_B$, so p is indeed an equivalence of quasicategories. □

4.4.6 Corollary. *Let* $E \to B$ *be a trivial Kan fibration and* $i : X \hookrightarrow Y$ *be a monomorphism of simplicial sets. Then the canonical morphism*

$$\theta : \mathbf{Func}(Y, E) \to \mathbf{Func}(Y, B) \times_{\mathbf{Func}(X,B)} \mathbf{Func}(X, E)$$

is a trivial Kan fibration.

Proof. See [29, Corollary 1.4.5.6]. □

4.4.7 Corollary. *Let* $p : E \to B$ *be a trivial Kan fibration and* X *be a simplicial set. Then the induced map*

$$\mathbf{Func}(Y, E) \to \mathbf{Func}(Y, B) \times_{\mathbf{Func}(X,B)} \mathbf{Func}(X, B)$$

is a trivial Kan fibration.

Proof. Let $X = \emptyset$ in Corollary 4.4.6. □

4.4.8 Theorem (Joyal)**.** *A simplicial set* X *is a quasicategory if, and only if, the restriction morphism*

$$\mathbf{Func}(\Delta^2, \mathcal{C}) \to \mathbf{Func}(\Lambda^2_1, \mathcal{C})$$

is a trivial Kan fibration.

Proof. See Section 1.4.6 at [29]. □

4.4.9 Theorem. *Let* X *be a simplicial set and* $\mathcal{C}$ *be an* ∞*-category. Then* $\mathbf{Func}(X, \mathcal{C})$ *is an* ∞*-category.*

Proof. By Theorem 4.4.8, it remains to show that

$$\mathbf{Func}(\Delta^2, \mathbf{Func}(X, \mathcal{C})) \to \mathbf{Func}(\Lambda^2_1, \mathbf{Func}(X, \mathcal{C}))$$

is a trivial Kan fibration. Since products commute, by cartesian closedness this corresponds to a map

$$\mathbf{Func}(X, \mathbf{Func}(\Delta^2, \mathcal{C})) \to \mathbf{Func}(X, \mathbf{Func}(\Lambda^2_1, \mathcal{C})),$$

Now, since $\mathcal{C}$ is a quasicategory, Theorem 4.4.8 applied to Corollary 4.4.7 gives the desired result. □

4.4.10 Definition. A simplicial set X is a **contractible Kan complex**, or simply **contractible**, if the terminal morphism $X \to \Delta^0$ is a trivial Kan fibration.[3]

4.4.11 Remark. Theorem 4.4.3 asserts that contractible Kan complexes are indeed Kan complexes.

4.4.12 Example. A singular complex $\mathbf{Sing}(X)$ is contractible if, and only if, X is weakly contractible.[4] This follow from the Quillen equivalence $|-| \dashv \mathbf{Sing}$, but we can also directly. Indeed, saying that $\pi_n(X) \cong 0$ means that every continuous function $f : S^n \to X$ may be extended to D^{n+1},[5] implying that Sing(X)is contractible as $|\partial\Delta^n| \cong S^n$ and $|\Delta^n| \cong D^{n+1}$.

4.4.13 Example. The fiber of a trivial Kan fibration $p : E \to B$ at the vertex $b \in B_0$ is defined by the pullback

$$\begin{array}{ccc} E \times_B b & \longrightarrow & E \\ \big\downarrow & & \big\downarrow p \\ \Delta^0 & \xrightarrow[b]{} & B \end{array}$$

Closure under pullbacks shows that $E \times_B b$ is contractible.

The space of compositions is contractible

With Proposition 4.2.9 we saw that, in quasicategory, composition of morphisms is defined up to homotopy.
For two morphisms $f : x \to y$ $g : y \to z$, the set of all possible choices for their composition is the fiber defined by the pullback below:

$$\begin{array}{ccc} \mathbf{Func}(\Delta^2, \mathcal{C}) \times_{\mathbf{Func}(\Lambda^2_1, \mathcal{C})} (g, \bullet, f) & \longrightarrow & \mathbf{Func}(\Delta^2, \mathcal{C}) \\ \big\downarrow & & \big\downarrow p \\ \Delta^0 & \longrightarrow & \mathbf{Func}(\Lambda^2_1, \mathcal{C}) \end{array} .$$

Thus Theorem 4.4.8 implies that not only the composition is defined up to homotopy, but also that these are defined only up to a contractible space of choices. For more discussion on this point of view, see Section 3.2 at [9].

[3] In other words, the contractible simplicial sets are the fibrant objects weakly homotopic to Δ^0.

[4] A space is weakly contractible iff $X \to *$ is a weak equivalence.

[5] Given an extension $\tilde{f} : D^{n+1} \to X$, take a homotopy H of $\tilde{f}$ with a point (which exists because D^{n+1} is contractible). Then a homotopy between f and a point is given by $H \circ (i, \mathrm{id}) : S^n \times I \to D^{n+1} \times I$.

Part II

APPLICATIONS

5

HOMOTOPY HYPOTHESIS

A general paradigm in homotopy theory is that

$$\text{spaces} \leftrightsquigarrow \infty\text{-groupoids.} \tag{5.1}$$

This idea was first laid by Grothendieck in his famous letter *Pursuing Stacks* [19], and later got nicknamed as *homotopy hypothesis* by John Baez [3].

From the quasicategorical point of view, ∞-groupoids are naturally defined to be Kan complexes. Moreover, the singular complex functor already maps spaces to Kan complexes. Intuitively, the ∞-groupoids associated to a space X has the points of X as objects, paths in it as morphisms, path homotopies as 2-morphisms, and so on. The bold statement present in the homotopy hypothesis then asserts that, in a suitable sense, these singular complexes describe *all* ∞-groupoids.

Moreover, thinking of $(\infty, 1)$-categories as categories enriched over ∞-groupoids, the homotopy hypothesis leads to an equivalence

$$\text{categories enriched over spaces} \leftrightsquigarrow (\infty, 1)\text{-categories} \tag{5.2}$$

At this point, these are all but ideas. Our goal in this chapter will be to lay them down in a very concrete form, using the machinery developed in Part I of this dissertation

5.1 SPACES AS ∞-GROUPOIDS

As the title suggests, in this section we will explore the mantra

$$\text{spaces} \leftrightsquigarrow \infty\text{-groupoids.} \tag{5.3}$$

To get a feel of how a formalization for this idea will look like, we will first look at a lower-dimensional example.[1]

5.1.1 Theorem (Homotopy hypothesis in dimension 1)**.** *Let* **1Type** *be the category of 1-types and continuous functions. There is an equivalence of categories*

$$\pi_1 : \mathbf{1Type} \cong \mathbf{Grpd} : |-|$$

Proof. See [35]. □

This trend suggests that, in the end, Equation 5.3 will be incarnated as an ∞-categorical theorem:

5.1.2 Guess. *There is an ∞-equivalence of ∞-categories*

$$\infty\mathbf{Type} \cong \infty\mathbf{Grpd} \tag{5.4}$$

Two questions are left open:

i) What are the ∞-categories ∞**Type** and ∞**Grpd**?

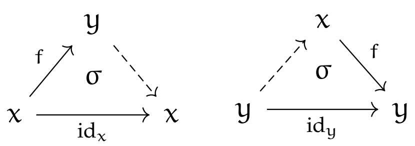
ii) How do we get the ∞-equivalence in Equation 5.4?

We start by defining ∞-groupoids in the most natural way:

5.1.3 Definition. An **∞-groupoid** is a quasicategory in which every morphism is invertible.

5.1.4 Theorem. *Kan complexes correspond precisely to ∞-groupoids.*

Proof. If **K** is a Kan complex, then for any morphism $f : x \to y$ we can fill the horns $(\bullet, \mathrm{id}_x, f), (f, \mathrm{id}_y, \bullet) : \Lambda^2_1 \to \mathbf{K}$ to find the right and left inverses of f (which coincide by basic algebra):



The converse follows from Joyal's Extension Theorem; see [39], Theorems 32.2 and 33.2. □

Recall that, by Proposition 3.2.8, the functor **Sing** : **Top** → **sSet** already takes spaces to Kan complexes. It remains to understand *how* is **Sing** giving *all* Kan complexes.

[1]Recall that a **homotopy type** is a topological space regarded up to weak homotopy equivalences, and a **homotopy** n**-type** is an n-connected homotopy type ($\pi_i = 0$ for $i > n$).

5.1.5 Remark. Singular complexes don't describe all Kan complexes. For instance, no singular complex is finite but there are finite Kan complexes.

Conversely, the geometric realization of a Kan complex is always a CW complex [18, Proposition 2.3], so not all spaces arise in this manner.

5.1.6 Theorem. *There is a model category* $\mathbf{sSet}_{\mathrm{Kan}}$ *describing the homotopy theory of Kan complexes. Its weak equivalences are the simplicial maps whose geometric realizations are weak equivalences, and its fibrant-cofibrant objects are precisely Kan complexes.*

5.1.7 Theorem. *There is a model category* $\mathbf{Top}_{\mathrm{Quillen}}$ *describing the homotopy theory of weak equivalences. In this model structure, the fibrant resolution of any space can be chosen to be a CW complex.*

5.1.8 Theorem. *The adjunction* $|-| \dashv \mathbf{Sing}$ *forms a Quillen equivalence*

$$|-| : \mathbf{sSet}_{\mathrm{Kan}} \leftrightarrows \mathbf{Top}_{\mathrm{Quillen}} : \mathbf{Sing}$$

We can finally understand the homotopy hypothesis: ∞-groupoids correspond to spaces up to weak equivalences.

5.1.9 Corollary. *If* $\mathbf{K}$ *is a Kan complex, then the derived unit map* $\mathbf{K} \to \mathbf{Sing}(|\mathbf{K}|)$ *is a weak equivalence.*

It will be a little anticlimatic to end the chapter with this result, because above we promised to write $\infty\mathbf{Grpd} \cong \mathbf{Top}$ as an equivalence of ∞-categories. Worry not: in the next chapter, we will study *simplicial localization*, a construction that takes a model category and spits a corresponding quasicategory, and that maps a Quillen equivalence to a quasicategorical equivalence.

5.2 SIMPLICIAL CATEGORIES

We have seen at 4.3.7 that **sSet** is carterian closed, and as such we may consider *simplicially enriched categories*, or simply *simplicial categories*. Their definition is written down below, in detail, for the sake of notation.

5.2.1 Definition. A **simplicial category** is described by the following data:

(i) A collection of objects $x, y, z, \dots \in \mathrm{ob}\mathcal{C}$

(ii) For each pair of objects $x, y \in \mathrm{ob}(\mathcal{C})$, a *hom-simplicial set* $\mathcal{C}(x, y)$.

(iii) For each triple of objects $x, y, z \in \mathrm{ob}(\mathcal{C})$, a simplicial map, the *composition*,

$$c : \mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z).$$

(iv) For each object $x \in \mathrm{ob}(\mathcal{C})$, a 0-simplex $\mathrm{id}_x \in \mathcal{C}(x,x)_0$, the identity.

Moreover the composition is associative and unital with respect to the identities.

5.2.2 Example (Simplicial sets)**.** Any cartesian closed category can be enriched over itself in a canonical manner (see [41, Section 3.3]), and **sSet** is cartesian closed. Thus we may define a simplicial category whose objects are simplicial sets and whose hom-simplicial sets are the inner hom-simplicial sets defined at Definition 4.3.6.

5.2.3 Example. A general fact about monoidal categories is that

$$\{\text{simplicial monoids}\} \cong \{\text{simplicial categories with one object}\}$$

This equivalence is explicitly given by the map

$$(M,\mu,\varepsilon) \mapsto \begin{cases} \mathrm{ob}(\mathcal{C}) = * \\ \mathcal{C}(*,*) = M \\ c = \mu : M \times M \to M \\ \varepsilon : \Delta^0 \to M \end{cases}$$

5.2.4 Example (Spaces)**.** The category of topological spaces Top becomes a simplicial category with the hom-simplicial sets roughly given by

$$\mathcal{S}_n(X,Y) := \mathrm{Top}(|\Delta^n| \times X, Y).$$

In particular, when $X = *$ we have $S_n(*,Y) \cong \mathbf{Sing}(Y)$.

5.2.5 Example. By the main construction in the Appendix, a lax monoidal functor $F : \mathcal{D} \to \mathbf{sSet}$ induces a functor between enriched categories

$$F_* : \mathbf{Cat}_{\mathcal{D}} \to \mathbf{Cat}_{\mathbf{sSet}}$$

With this we may generate several examples:

(a) Let $\underline{(-)} : \mathrm{Set} \to \mathbf{sSet}$ be the functor taking a set X to the constant simplicial set

$$\underline{X} = \quad X = X = X = \cdots$$

This functor is monoidal since $\underline{(-)}$ trivially preserves products and terminal objects. Then, identifying $\mathbf{Cat}_{\mathrm{Set}} \cong \mathbf{Cat}$, the induced functor becomes $\underline{(-)}_* : \mathbf{Cat} \to \mathbf{Cat}_{\mathbf{sSet}}$. It takes a category to the simplicial category with the same objects, and the same hom-sets, but seen as simplicial sets.

(b) As a left adjoint, the nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$ preserves limits and, in particular, products. Moreover $N* \cong \Delta^0$, so N is monoidal.

Identifying $\mathbf{Cat}_{\mathbf{Cat}} \cong 2\mathbf{Cat}_{\mathrm{str}}$, the induced functor becomes $N_* : 2\mathbf{Cat}_{\mathrm{str}} \to \mathbf{Cat}_{\mathbf{sSet}}$, the functor taking a 2-category to the simplicial category with the same objects, and the same categories of morphisms, but regarded as simplicial sets through their nerves.

(c) Analogously, the singular complex functor $\mathbf{Sing} : \mathbf{Top} \to \mathbf{sSet}$ is monoidal, inducing a functor

$$\mathbf{Sing}_* : \mathbf{Cat}_{\mathbf{Top}} \to \mathbf{Cat}_{\mathbf{sSet}}.$$

A non-trivial result also shows that the geometric realization also preserves finite limits [18, Proposition 2.4] With that, we also have the induced functor $|-|_* : \mathbf{Cat}_{\mathbf{sSet}} \to \mathbf{Cat}_{\mathbf{Top}}$, and a direct calculation shows that we actually have an adjunction

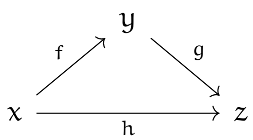
$$|-|_* : \mathbf{Cat}_{\mathbf{sSet}} \rightleftarrows \mathbf{Cat}_{\mathbf{Top}} : \mathbf{Sing}_*.$$

5.2.6 Remark. On a convenient category of spaces we may define the structure of a simplicial category by

$$\mathrm{Top}(|\Delta^n| \times X, Y) \cong \mathrm{Top}(|\Delta^n|, \mathrm{Map}(X, Y)) =: \mathbf{Sing}(\mathrm{Map}(X, Y))_n.$$

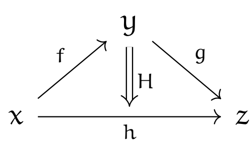
5.3 A NERVE

The 2-simplices of a category $\mathcal{C}$ are given by commutative triangles:



In a simplicial category, we can relax this condition up to higher simplices:

$H : g \circ f \to h \rightsquigarrow$



5.3.1 Definition. For each $[n] \in \mathbf{\Delta}$, we shall define a simplicial category $S[n]$ as follows

(i) the objects of $S[n]$ are the numbers $0, 1, 2 \ldots$.

(ii) the hom-simplicial set between $i, j \in S[n]$ is the nerve

$$S[n](i, j) := NP_{i,j}$$

where $P_{i,j}$ is the poset of paths $i \rightsquigarrow j$, equivalently defined by

- the poset of paths $\gamma : i \rightsquigarrow j$ in the category $[n]$, where $\gamma \leq \gamma'$ if γ' visits all vertices of γ.
- the poset of subsets of $\{i, i+1, \ldots, j\}$ that contain both i and j, ordered by inclusion.

(iii) the composite

$$c : S[n](j,k) \times S[n](i,j) \to S[n](i,k)$$

is given by path concatenation, or equivalently by subset union.

5.3.2 Example.

- $S[0]$ has a single object and a trivial hom-simplicial set.
- $S[1]$ has two objects and we have $S[1](0,1) \cong \Delta^0$:

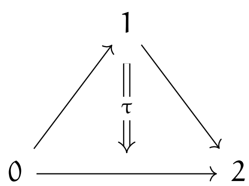
$$S[1] = \quad 0 \longrightarrow 1$$

- $S[2]$ has three objects; there are unique paths $0 \rightsquigarrow 1$ and $1 \rightsquigarrow 2$, so that $S[1](0,1) \cong S[1](1,2) \cong \Delta^0$.

  On the other hand, there are two paths $0 \rightsquigarrow 2$:

$$P_{0,2} = \{\{0,2\} \geq \{0,1,2\}\}.$$

  So that $S[1](0,2) \cong \Delta^1$:



$\in S[1](0,2)$

- the new feature brought by $S[3]$ is $S[3](0,3)$, whose simplices can be pictured as below:

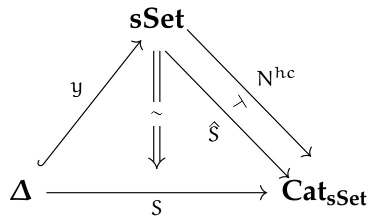
$$P_{0,3} = \begin{array}{ccc} 03 & \longrightarrow & 013 \\ \downarrow & \searrow & \downarrow \\ 023 & \longrightarrow & 0123 \end{array} \cong \Delta^1 \times \Delta^1$$

A quick calculation shows that $[n] \mapsto S[n]$ defines a functor $S : \mathbf{\Delta} \to \mathbf{Cat_{sSet}}$.

5.3.3 Definition. The **homotopy coherent nerve** $N^{hc}(\mathcal{C})$ of a simplicial category $\mathcal{C}$ is the simplicial set

$$N^{hc}(\mathcal{C})_\bullet = \mathbf{Func}(S[\bullet], \mathcal{C}).$$

The homotopy coherent nerve assembles into a functor $N^{hc} : \mathbf{Cat}_{\mathbf{sSet}} \to \mathbf{sSet}$. This kind of functor, defined "representably" with the aid of a functor $\mathbf{\Delta} \to \mathcal{C}$, always admits a left adjoint given by the left Kan extension of S:



See [40] for more details.

5.3.4 Example. A 0-simplex of $N^{hc}(\mathcal{C})$ is a functor $S[0] \to \mathcal{C}$, corresponding to an object of $\mathcal{C}$.

5.3.5 Example. A 1-simplex in $N^{hc}(\mathcal{C})$ is a functor $F : S[1] \to \mathcal{C}$. It consists of two objects $x = F(0)$ and $y = F(1)$, and a simplicial map

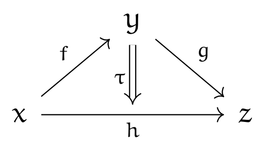
$$F_{01} : S[1](0,1) \to \mathcal{C}(x,y).$$

Since $S[1](0,1) \cong \Delta^0$, we can identify F_{01} with a morphism $f : x \to y$.

5.3.6 Example. A 2-simplex in $N^{hc}(\mathcal{C})$ is a functor $F : S[2] \to \mathcal{C}$. Functoriality means that F takes the objects $0, 1, 2 \in S[2]$ to objects $x, y, z \in \mathcal{C}$, and that there are simplicial maps

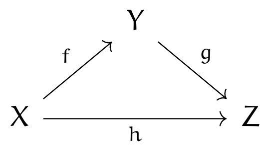
$$F_{01} : S[2](0,1) \to \mathcal{C}(x,y)$$
$$F_{12} : S[2](1,2) \to \mathcal{C}(y,z)$$
$$F_{02} : S[2](0,2) \to \mathcal{C}(x,z)$$

These can be informally denoted by F

$F =$ 

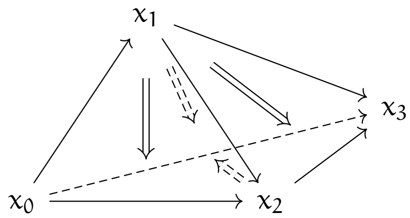
Where $f = F(01)$, $g = F(12)$ e $h = F(02)$. In this case, $h \neq g \circ f = F(012)$, but the image of $(12) \circ (01) \to (012)$ defines a homotopy $g \circ f \Rightarrow h \in \mathcal{C}(x,y)_1$.

5.3.7 Example. A 2-simplex $\sigma \in N^{hc}(\mathbf{Top})$ (Example 5.2.4) consists of a *possibly non-commutative* triangle of continuous functions:

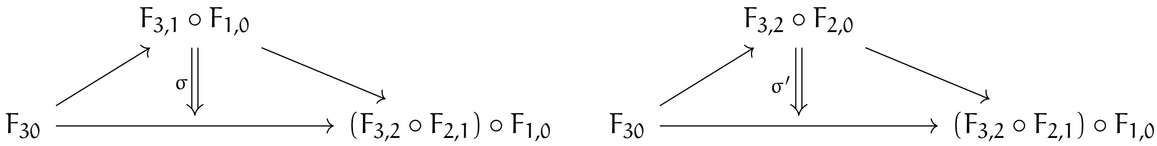


Moreover, the image of the non-degenerate 1-simplex $(12) \circ (01) \to (012)$ is a 1-simplex $\tau \in \mathbf{Top}(x,z)_1$ such that $d_1(\tau) = g \circ f$ and $d_0(\tau) = h$. Unwinding the definitions, we see that $\tau : X \times I \to Y$ is a homotopy $g \circ f \Rightarrow h$.

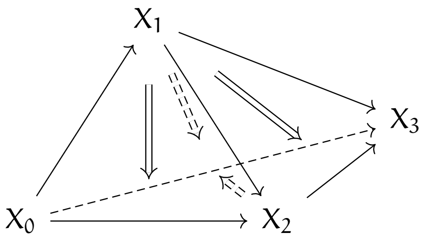
5.3.8 Example. A 3-simplex in $N^{hc}(\mathcal{C})$ is a functor $F : S[3] \to \mathcal{C}$. Analyzing each face separately, we denote F as tetrahedron with homotopies $H_{kji} : F(jk) \circ F(ij) \Rightarrow F(ik)$ on each face:



Besides, the non-degenerate 2-simplices in $NP_{0,3}$ give a compatibility condition in $\mathcal{C}(x_0, x_3)$:



5.3.9 Example. A 3-simplex τ in $N^{hc}(\mathbf{Top})$ is a tetrahedron of continuous functions



Example 5.3.7 shows that the faces of τ correspond to homotopies $X_i \times I \to X_k$ between $F_{jk} \circ F_{ij}$ and F_{ik}.

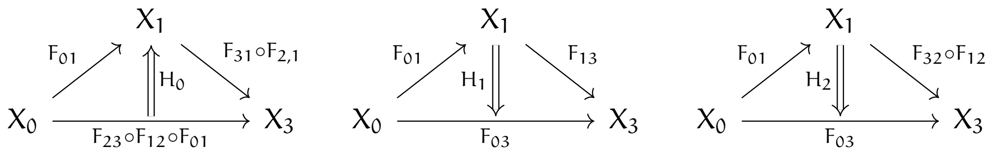
Furthermore, the image of the non-degenerate 2-simplices provide *homotopies between homotopies.* For instance, the image of $F(\sigma)$ is the map

$$F(\sigma) : X_0 \times |\Delta^2| \to X_3$$

such that the restriction to each edge of $|\Delta^2|$ gives a homotopy

$$d_0F(\sigma) : X_0 \times I \to X_3$$
$$d_1F(\sigma) : X_0 \times I \to X_3$$
$$d_2F(\sigma) : X_0 \times I \to X_3$$

corresponding to the faces



Identifying $|\Delta^2| \cong I \times I$, we see that τ itself is a homotopy

$$\tau : (X_0 \times I) \times I \to X_3, \quad H_1 \circ H_0 \Rightarrow H_2.$$

,

5.4 TOWARDS QUASICATEGORIES

The construction of connected components defines a functor

$$\pi_0 : \mathbf{sSet} \to \mathrm{Set}.$$

This functor preserves products (see [29, Corollary 1.1.6.26]), thus, from the appendix, there is an induced functor

$$(\pi_0)_* : \mathbf{Cat}_{\mathbf{sSet}} \to \mathbf{Cat}.$$

The **homotopy category** of a simplicial category $\mathcal{C}$ is its image through $(\pi_0)_*$. We will drop the $*$ and write $\pi_0(\mathcal{C})$.

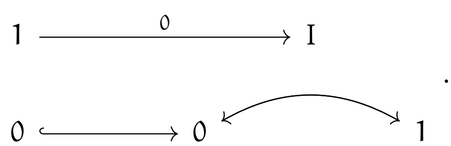
5.4.1 Definition. A **Dwyer-Kan equivalence** is a simplicial functor $F : \mathcal{C} \to \mathcal{D}$ such that

(i) the functor $(\pi_0)_* F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is essentially surjective.

(ii) for each $x, y \in \mathcal{C}$, the map $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is a weak equivalence of simplicial sets.

5.4.2 Theorem (Bergner). *There is a model structure* $\mathbf{Cat}_{\mathbf{sSet}}^{\mathrm{Bergner}}$ *whose*

- *weak equivalences are Dwyer-Kan equivalences.*
- *fibrations are simplicial functors* $F : \mathcal{C} \to \mathcal{D}$ *such that*
  - *the map* $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ *is a Kan fibration.*
  - *the functor* $(\pi_0)_* F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ *is an isofibration, i.e.* $(\pi_0)_* F$ *has the RLP against the walking isomorphism*



Every object in this model category is cofibrant, and an object is fibrant if its hom-simplicial sets are all Kan complexes; such a simplicial category is called ***locally Kan****.*

5.4.3 Definition. A **categorical equivalence** between simplicial sets is a map $f : X \to Y$ such that

$$\hat{S}f : \hat{S}X \to \hat{S}Y$$

is a Dwyer-Kan equivalence.

5.4.4 Theorem (Joyal)**.** *There is a model structure* $\mathbf{sSet}^{Joyal}$

- *the weak equivalences are categorical equivalences.*
- *the cofibrations are trivial Kan fibrations.*

Every object in this model category is cofibrant, and its fibrant objects are precisely the quasicategories.

5.4.5 Theorem (Lurie)**.** *The adjunction*

$$\hat{S} : \mathbf{sSet}^{Joyal} \leftrightarrows \mathbf{Cat}_{\mathbf{sSet}}^{Bergner} : N^{hc}$$

is a Quillen equivalence.

5.4.6 Corollary. *The homotopy coherent nerve of a locally Kan simplicial category is a quasicategory.*

5.4.7 Example. The **Dold-Kan equivalence** establishes an equivalence of categories

$$\Gamma : \mathrm{Ch}_{\mathbb{Z}} \leftrightarrows \mathrm{sAb} : N$$

The right adjoint N takes a simplicial abelian group $A_\bullet$ to the chain complex $NA = (A_\bullet, \partial)$, where

$$\partial_n = \sum_{i=1}^{n} (-1)^i d_i : A_n \to A_{n-1}.$$

It's possible to show that, N is colax monoidal and then, by Kelly's doctrinal adjunction [25], Γ is lax monoidal, inducing a functor

$$\Gamma_* : \mathbf{Cat}_{\mathrm{Ch}_{\mathbb{Z}}} \to \mathbf{Cat}_{\mathrm{sAb}} \hookrightarrow \mathbf{Cat}_{\mathbf{sSet}}.$$

In particular, the category of chain complexes $\mathbf{Ch}(\mathcal{A})$ of any abelian category $\mathcal{A}$ can be enriched over $\mathbf{Ch}_{\mathbb{Z}}$ [29, Section 2.5.1]. As such, we obtain a simplicial category $\Gamma_*(\mathbf{Ch}(\mathcal{A}))$, which is[2] a model for the ∞-*category of chain complexes*.

[2] Maybe after taking a fibrant replacement.

x'

6

SIMPLICIAL LOCALIZATION

Given a category with equivalences $(\mathcal{C}, W)$, we can formally "add inverses" to the morphisms in W by constructing its localization, the category $\mathcal{C}[W^{-1}]$ [16]. The localization serves as a *derived category* in a precise manner: if a functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ is such that $\mathsf{F}(W) \subset \mathrm{Iso}(\mathcal{D})$, then it extends through $\mathcal{C}[W^{-1}]$ (in a certain unique sense):

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathsf{F}} & \mathcal{D} \\ \downarrow & \Nearrow \quad \nearrow_{\widetilde{\mathsf{F}}} & \\ \mathcal{C}[W^{-1}] & & \end{array}$$

This idea of localization is spread throughout mathematics:

- **Homological algebra:** inverting quasi-isomorphisms between chain complexes.
- **Topology:** inverting weak homotopy equivalences between spaces/simplicial sets.
- **Spectra of spaces:** inverting weak equivalences between spectra.
- **(Bi)category theory:** inverting (bi)equivalences between (bi)categories.

However, the reader may notice that these are all examples of model categories. In this case, $\mathsf{C}[W^{-1}]$ is isomorphic to the homotopy category of the model structure (see Chapter 1), but take by granted some properties of this localization. For instance,

- this localization is automatically locally small.[1]
- the objects of the localization are objects of the base category (rather than abstract zig-zags of morphisms).
- there are categorical equivalences

$$\mathcal{C}[W^{-1}] \cong \mathcal{C}_{\mathrm{f}}[W^{-1}] \cong \mathcal{C}_{\mathrm{c}}[W^{-1}] \cong \mathcal{C}_{\mathrm{cf}}[W^{-1}].$$

[1] In general, the localization of a locally small large category may have large hom-sets [1].

In particular, due to Whitehead's theorem (see Theorem 1.4.9), weak equivalences in $\mathcal{C}_{\mathrm{cf}}[W^{-1}]$ behave much more like "usual homotopies", and as such they are much more familiar: its morphisms are *equivalence classes*. However, the *data* of the homotopies is discarded because we only keep the equivalence classes. Would it be possible to construct from $(C, W, \mathrm{Fib}, \mathrm{Cof})$ some structure where we can localize W, but without destroying this homotopical data?

The answer is *yes*: in [11, 12], the authors construct the **simplicial localization** of a pair $(\mathcal{C}, W)$. This is a *simplicial category* $\mathrm{L}\mathcal{C}$ which is compatible with $\mathcal{C}[W^{-1}]$ in the sense that

$$\pi_0(\mathrm{L}\mathcal{C}) = \mathcal{C}[W^{-1}].$$

If (C, W) is a model category, we again have stronger results, such as DK equivalences

$$\begin{array}{ccc} \mathrm{L}\mathcal{C}_{\mathrm{cf}} & \xrightarrow{\in W} & \mathrm{L}\mathcal{C}_{\mathrm{c}} \\ \downarrow & & \downarrow{\scriptstyle \in W} \\ \mathrm{L}\mathcal{C}_{\mathrm{f}} & \xrightarrow{\in W} & \mathrm{L}\mathcal{C}_{\mathrm{cf}} \end{array}$$

However it may be the case that $\mathrm{L}\mathcal{C}_{\mathrm{cf}}$ is not immediately an $(\infty, 1)$-category (the hom-simplicial sets might not be ∞-groupoids). This requires a resolution!

6.1 RELATIVE CATEGORIES

Simplicial localization is performed in the context of *relative categories*, which are a slight generalization of weak equivalences.

6.1.1 Definition. A **relative category** is a pair $(\mathcal{C}, W)$, where $\mathcal{C}$ is a category and W is a *wide subcategory* of $\mathcal{C}$, i.e. a subcategory containing all objects of $\mathcal{C}$.

Notice that the data of a relative category $(\mathcal{C}, W)$ can be equivalently described by a subset of morphisms of $\mathcal{C}$ that contains identities and is closed under composition.

6.1.2 Example. To any category $\mathcal{C}$ we associate its

- **maximal** relative category $\widehat{\mathrm{C}}$ by taking W to contain *all* morphisms.
- **minimal** relative category $\check{\mathrm{C}}$ by taking W to contain only identities.

6.1.3 Example. A category with weak equivalences (Definition 1.1.4) is a relative subcategory, since it contains identities and 2-out-of-3 implies close under composition. In particular model categories may be regarded as relative categories with extra structure. Furthermore, homotopical functors between categories of with weak equivalences (Definition 1.5.1) are morphisms in **RelCat**.

6.2 SIMPLICIAL LOCALIZATIONS

The following result is true for any model category.

6.2.1 Lemma. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a morphism between simplicial categories. Considering the Bergner model structure on* $\mathbf{Cat_{sSet}}$*, we may choose resolutions of* $\mathcal{C}$ *and* $\mathcal{D}$ *such that* F *factors through these resolutions, i.e.*

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \longrightarrow & \mathcal{C} \\ {\scriptstyle \tilde{F}}\downarrow & & \downarrow{\scriptstyle F} \\ \tilde{\mathcal{D}} & \longrightarrow & \mathcal{D} \end{array}$$

Proof. We first find the resolutions by factoring $\varnothing \to \mathcal{C}$ (resp. $\mathcal{D}$) as a cofibration followed by a trivial fibration:

$$\begin{array}{ccccc} \varnothing & \xrightarrow{\in \mathrm{Cof}} & \tilde{\mathcal{C}} & \xrightarrow{W \cap \mathrm{Fib}} & \mathcal{C} \\ \| & & & & \downarrow{\scriptstyle f} \\ \varnothing & \xrightarrow[\in \mathrm{Cof}]{} & \tilde{\mathcal{D}} & \xrightarrow[W \cap \mathrm{Fib}]{} & \mathcal{D} \end{array}$$

We may rearrange this diagram as follows:

$$\begin{array}{ccc} \varnothing & \longrightarrow & \widetilde{\mathcal{D}} \\ {\scriptstyle \mathrm{Cof} \ni}\downarrow & \nearrow & \downarrow{\scriptstyle \in W \cap \mathrm{Fib}} \\ \widetilde{\mathcal{C}} & \longrightarrow & \mathcal{D} \end{array},$$

from which we have the depicted lift, which is the map $\tilde{F}$ that we were looking for. □

6.2.2 Remark. In the Bergner model strucuture, the resolutions may be chosen in a way that the resolution of a monomorphism is a monomorphism. This is the approach originally taken by Dwyer and Kan in [11]. They construct the resolution through the *free simplicial category* of a category $\mathcal{C}$. This is a simplicial category roughly described by having

- the same objects a $\mathcal{C}$;
- the morphisms of $\mathcal{C}$ as $\mathcal{C}(x, y)_0$;
- *words* of morphisms as $\mathcal{C}(x, y)_1$;
- inductively, *words of words* of morphisms as $\mathcal{C}(x, y)_2$, and so on for higher n.

6.2.3 Definition. The **standard simplicial localization** of a relative category $(\mathcal{C}, W)$ is the simplicial category LC defined as follows:

- view the relative category as a morphism $W \hookrightarrow \mathcal{C}$ in **SCat** (see Example 5.2.5.i);
- consider the resolution $\widetilde{W} \to \widetilde{C}$ as in the previous lemma;

- define $LC := \tilde{C}[\tilde{W}^{-1}]$.

A functor $\varphi : F\mathcal{C} \to \mathcal{C}$ is obtained by composing the words of morphisms. This is in fact a DK equivalence, in fact a deformation retract.

6.2.4 Proposition. *The components of* $L\mathcal{C}$ *correspond to morphisms in* $\mathcal{C}[W^{-1}]$*, i.e.* $\pi_0 L\mathcal{C} \cong \mathcal{C}[W^{-1}]$.

Proof. Regarding the inclusion $W \hookrightarrow \mathcal{C}$ as a map in **SCat**, applying π_0 to the diagram obtained by Lemma 6.2.1 we get

$$\begin{array}{ccc} \pi_0\widetilde{W} & \xrightarrow{\sim} & W \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{C}} & \xrightarrow[\sim]{} & \mathcal{C} \\ \downarrow & & \downarrow \\ \pi_0 LC & \overset{\leftarrow}{\underset{\rightarrow}{\dashrightarrow}} & \mathcal{C}[W^{-1}] \end{array},$$

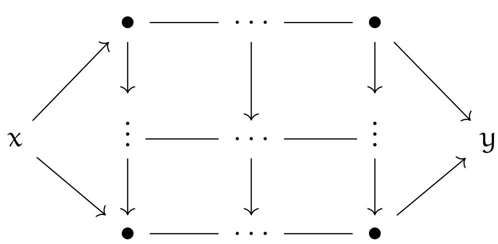
where

- we identified $\pi_0\mathcal{C} = \mathcal{C}$ and $\pi_0 W = W$ as that follows by definition;
- the horizontal arrows are categorical equivalences from the definition of DK equivalences.
- the dashed arrows exist because commutativity of the square is implying that both the equivalence $\pi_0\widetilde{\mathcal{C}} \cong \mathcal{C}$ and its inverse are homotopical functors.

Moreover, the dashed arrows form a categorical equivalence because of the uniqueness of the localization. This is what we were looking for. □

An alternative to the standard simplicial localization is the *hammock localization*. This approach is more concrete, and it's equivalent to the standard simplicial localization in the sense that there is a zig-zag of DK equivalences $L^H\mathcal{C} \longleftrightarrow L\mathcal{C}$ [12, Proposition 2.2].

6.2.5 Definition. The **hammock localization** of a relative category (C, W) is the simplicial category $L^H\mathcal{C}$ with the same objects as $\mathcal{C}$, and whose hom-simplicial sets have as n-simplices diagrams of the form



where

- there are $n + 1$ rows and any number of columns;
- morphisms in the same column have the same direction;
- each row is a zig-zag;
- vertical and left directed morphisms are in W;
- no column has only identities.

6.2.6 Proposition. *There is an equality $\pi_0 L^H \mathcal{C} = \mathcal{C}[W^{-1}]$.*

Proof. The morphisms in $\mathcal{C}[W^{-1}]$ are zig-zags of morphisms whose left pointing morphisms are in W [16, Chapter 1]. That is precisely the quotient of 0-hammocks by 1-hammocks. □

As promised [12, Proposition 8.4]:

6.2.7 Proposition. *If the context $(\mathcal{C}, W)$ is a model category, then there is a diagram of DK equivalences*

$$\begin{array}{ccc} L^H\mathcal{C}_{cf} & \xrightarrow{\in W} & L^H\mathcal{C}_c \\ \downarrow & & \downarrow{\scriptstyle \in W} \\ L^H\mathcal{C}_f & \xrightarrow{\in W} & L^H\mathcal{C}_{cf} \end{array}$$

6.3 TOWARDS ∞-CATEGORIES

For us, ∞-categories are presented by either quasicategories or Kan-enriched categories. Constratingly, both $L\mathcal{C}$ and $L^H\mathcal{C}$ aren't necessarily ∞-categories, so how will we fulfill the promise of giving the ∞-category underlying a model category?

Recall that $N^{hc} : \mathbf{Cat}_{\mathbf{sSet}} \to \mathbf{sSet}$ is a right Quillen equivalence, so it takes KanCat to **qCat** (fibrant-cofibrant preservation). Since N^{hc} is right Quillen, we may take the right derived functor $\mathbb{R}N^{hc} : \mathrm{Ho}(\mathbf{Cat}_{\mathbf{sSet}}) \to \mathbf{sSet}$, which will take values in quasicategories.

We're done! The underlying quasicategory is

$$\mathbb{R}N^{hc}(L^H(\mathcal{C}, W)) = N^{hc}(\widetilde{L^H(\mathcal{C}, W)})$$

6.3.1 Remark. It turns out that relative categories are an equally reasonable model for $(\infty, 1)$-categories as quasicategories or simplicial categories. This idea is developed in [5], where the authors construct a model structure on **RelCat** Quillen equivalent to Rezk's model structure for complete Segal spaces. The equivalence to $\mathbf{sSet}_{\mathsf{Joyal}}$ or $\mathbf{Cat}_{\mathbf{sSet}}^{\mathsf{Bergner}}$ is then obtained by composing with the appropriate zig-zag of Quillen equivalences.

This result is particularly interesting because it imposes a notion of weak equivalence between relative categories that gets sent to equivalences between the corresponding quasicategories. In particular, we get that every quasicategory (in fact, every simplicial set) may be presented by a relative category.

6.3.2 Remark. As expected, a Quillen equivalence induces an equivalence between the respective underlying quasicategories. This isn't however a trivial fact; for a modern account, see [21, Proposition 1.5.1].

6.3.3 Remark. While there isn't a complete classification for quasicategories that arise as localizations of model categories, a partial result is found at [28, A.3.7.6]:

6.3.4 Proposition. *A quasicategory $\mathcal{C}$ is **presentable** if and only if there is a simplicial model category $\mathcal{M}$ and an equivalence $\mathcal{C} \cong \mathbb{R}\mathrm{N}^{\mathrm{hc}}(\mathcal{M})$.*

A related result shows that the relative category of combinatorial model categories (and left Quillen functors) is weakly equivalent to the relative category of presentable quasicategories (and left adjoint functors) [37].

f

APPENDIX: ENRICHED CATEGORIES

In this appendix we develop the minimal theory of monoidal and enriched categories sufficient to understand the constructions in Chapter 5. In particular, we wish to prove the following:

Theorem. *A lax functor* $F : \mathcal{V} \to \mathcal{W}$ *induces a functor* $F_* : \mathbf{Cat}_{\mathcal{V}} \to \mathbf{Cat}_{\mathcal{W}}$*.*

MONOIDAL CATEGORIES

A *monoidal category* is essentially a category which roughly is a monoid, i.e. it has a product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an indentity $1 \to \mathcal{C}$ satisfying associativity and unitality diagrams.

Definition. A **symmetric monoidal category** $(\mathcal{M}, \otimes, 1)$ consists of

- a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, the *tensor product*.
- an object $I \in \mathcal{V}$, the *unit*.
- specified natural isomorphisms

  $$(x \otimes y) \otimes z \to x \otimes (y \otimes z) \quad \lambda_x : 1 \otimes x \to x \quad \rho_x : x \otimes 1 \to x,$$

  expressing associativity and unit conditions on the tensor product.
- natural isomorphisms

  $$b_{xy} : x \otimes y \to y \otimes z$$

  such that $b_{yx} \circ b_{xy} = \mathrm{id}_{x \otimes y}$.

This data is required to satisfy certain coherence diagrams, which we won't pursue here.

Example. A category with products and terminal objects can be canonically equipped with a symmetric monoidal structure $(\mathcal{C}, \times, I)$ whose natural isomorphisms are induced by the universal properties of these limits. Examples are the complete categories **Set**, $\mathbf{Ch}_R$, **Top**, **Cat**, **sSet** and $\mathbf{Mod}_R$.

Example. The classical tensor product of R-modules [30, pp. 24–27] defines a monoidal structure on $\mathbf{Mod}_R$. The unit is R, because $M \otimes R \cong M$. The tensor product of vector spaces (modules over fields) and abelian groups (modules over $\mathbb{Z}$) are particular cases.

Example. The tensor of product in $\mathbf{Ch}_R$ is defined degreewise by

$$(V \otimes W)_k = \bigoplus_{i+j=k} (V_i \otimes W_j).$$

The boundary map is defined by

$$\partial_{ij} := \partial|_{V_i \otimes W_j} = d_V \otimes \mathrm{id} + (-1)^i \mathrm{id} \otimes d_W.$$

The minus sign is here to guarantee that $\partial^2 = 0$. Indeed,

$$\begin{aligned}\partial^2 &= \partial_{i+1,j} \circ (d_V \otimes \mathrm{id}) + (-1)^i \partial_{i,j+1} \circ (\mathrm{id} \otimes d_W) \\ &= d_V^2 \otimes \mathrm{id} + (-1)^{i+1} d_V \otimes d_W + (-1)^i d_V \otimes d_W + (-1)^i \mathrm{id} \otimes d_W^2 \\ &= 0\end{aligned}$$

We conclude this section by proving that the tensor products in $\mathbf{Mod}_R$ and $\mathbf{Ch}_R$ are symmetric.

Example. One way to define the tensor products is as a representation for the functor $\mathrm{Bilin}(V, W; -)$ [42, Example 2.3.7]. Furthermore, a natural isomorphism $\mathrm{Bilin}(V, W; -) \Rightarrow \mathrm{Bilin}(W, V; -)$ is defined by precomposing bilinear maps with the map $f : W \times V \to V \times W$ defined by $f(w, v) = (v, w)$. Then we get a sequence of natural isomorphisms,

$$\begin{aligned}\hom(M \otimes N, -) &\cong \mathrm{Bilin}(M, N; -) \\ &\cong \mathrm{Bilin}(N, M; -) \\ &\cong \hom(N \otimes M, -),\end{aligned}$$

and a corollary of the Yoneda lemma [42, Proposition 2.3.1] implies that this natural isomorphism comes from an isomorphism $b_{MN} : M \otimes N \xrightarrow{\sim} N \otimes M$. This isomorphism defines a symmetric structure on $\mathbf{Mod}_R$.

Example. The trivial way to define the braiding for $\mathbf{Ch}_R$ would be given degreewise by

$$b(v_i \otimes w_j) = w_j \otimes v_i.$$

However, this isn't a chain map! In fact, the signs of

$$\begin{aligned}(\partial \circ b)(v_i \otimes w_j) &= \partial(w_j \otimes v_i) \\ &= dw_j \otimes v_i + (-1)^j w_j \otimes dv_i,\end{aligned}$$

disagree with

$$\begin{aligned}(b \circ \partial)(v_i \otimes w_j) &= b(dv_i \otimes w_j + (-1)^i v_i \otimes dw_j) \\ &= (-1)^i dw_j \otimes v_i + w_j \otimes dv_i.\end{aligned}$$

To solve this sign problem, we add a sign to the braiding:

$$b(v_i \otimes w_j) := (-1)^{ij} w_j \otimes v_i.$$

This time b is a chain map since

$$\begin{aligned}(\partial \circ b_g)(v_i \otimes w_j) &= \partial((-1)^{ij} w_j \otimes v_i) \\ &= (-1)^{ij}(dw_j \otimes v_i + (-1)^j w_j \otimes dv_i),\end{aligned}$$

coincides with

$$\begin{aligned}(b \circ \partial)(v_i \otimes w_j) &= b(dv_i \otimes w_j + (-1)^i v_i \otimes dw_j) \\ &= (-1)^{(i+1)j} w_j \otimes dv_i + (-1)^{i+i(j+1)} dw_j \otimes v_i \\ &= (-1)^{ij+j} w_j \otimes dv_i + (-1)^{2i+ij} dw_j \otimes v_i \\ &= (-1)^{ij}(dw_j \otimes v_i + (-1)^j w_j \otimes dv_i).\end{aligned}$$

ENRICHED CATEGORIES

Frequently, the hom-sets of a category can be regarded as objects of some category. For instance, for linear transformations form a vector space, and for topological spaces there sets of continuous functions admit several topologies, such as the compact-open topology. These are examples of **enriched categories**.

Definition. An **enriched category** $\mathcal{C}$ over a symmetric monoidal category $(\mathcal{V}, \otimes, I)$ consists of

- a collection of objects $x, y, z \in \mathcal{C}$
- for each $x, y \in \mathcal{C}$, a **hom-object** $\mathcal{C}(x, y) \in \mathcal{V}$
- for each $x \in \mathcal{C}$, $1_x \in \mathcal{C}(x, x)$
- for each $x, y, z \in \mathcal{C}$, a morphism

$$\circ : \mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \to \mathcal{C}(x, z)$$

such that the following diagrams commute for all $x, y, z, w \in \mathcal{C}$:

- *Associativity:*

$$\begin{array}{ccc} \mathcal{C}(x,y)\times\mathcal{C}(y,z))\times\mathcal{C}(z,w) & \xrightarrow[\circ\times\mathrm{id}]{} & \mathcal{C}(x,z)\times\mathcal{C}(x,w) \\ \downarrow{\scriptstyle \mathrm{id}\times\circ} & & \downarrow{\scriptstyle\circ} \\ \mathcal{C}(x,y)\times\mathcal{C}(y,w) & \xrightarrow{\circ} & \mathcal{C}(y,z) \end{array}$$

- *Unitality:*

$$\begin{array}{ccccc} 1\times\mathcal{C}(x,y) & \xrightarrow{1_x\times\mathrm{id}} & \mathcal{C}(x,y)\times\mathcal{C}(x,x) & \xleftarrow{1_x\times\mathrm{id}} & 1\times\mathcal{C}(x,y) \\ & \searrow{\scriptstyle p_2} & \downarrow{\scriptstyle\circ} & \swarrow{\scriptstyle p_2} & \\ & & \mathcal{C}(x,y) & & \end{array}$$

Remark. While this definition is independent of the braiding, the symmetric structure is convenient when dealing with compositions.

Example. A one object category enriched over $(\mathbf{Set},\times)$ is a monoid. More generally, an enriched category with one object is described entirely by its hom-object $M\in\mathcal{V}$ and morphisms $\circ : M\otimes M\to M$ and $\mathrm{id} : I\to M$. The diagrams in the definition of enriched categories imply that this is just a monoid in $\mathcal{V}$.

Definition. A **closed monoidal category** is a monoidal category $\mathcal{C}$ such that the functor $-\otimes y:\mathcal{C}\to\mathcal{C}$ has a right adjoint for each $y\in\mathcal{C}$:

$$\mathcal{V}(x\otimes y,z)=\mathcal{V}(x,[y,z])\text{ for all }y,z\in\mathcal{C}.$$

In this case, the objects $[y,z]$ are called **internal homs**.

A closed monoidal category can be enriched over itself using the internal homs as the hom-objects. The composition is defined by transposing the composite

$$[y,z]\otimes[x,y]\otimes x\xrightarrow{\mathrm{id}\otimes\varepsilon_y}[y,z]\otimes y\xrightarrow{\mathrm{id}\otimes\varepsilon_x}z.$$

through the adjunction $-\otimes y\dashv[y,-]$.

Example. The monoidal category $(\mathbf{Mod}_R,\otimes)$ is closed. In fact, that's how the tensor product was defined in the first place, through the bijection

$$\text{bilinear maps }x\times y\to z\leftrightsquigarrow\text{linear maps }x\to\hom(y,z)$$

Thus the right adjoint to $-\otimes y$ is $\hom(y,-)$:

$$\hom(x\otimes y,z)\cong\mathrm{Bilin}(x,v;z)\cong\hom(x,\hom(y,z))$$

Example. The monoidal category $(\mathbf{Top},\times)$ is *not* closed. This can be proven by showing that the functor $(-)\times X$ doesn't preserves colimits; see [10, Proposition 7.1.2]. A *convenient category of spaces* is a cartesian closed subcategory of **Top** that is large enough to contain CW complexes; this includes the homotopy theorists' category of compactly generated Hausdorff spaces.

CREATING FUNCTORS

Our goal in this section is to give the main theorem of this appendix. We must define lax functors and $\mathbf{Cat}_{\mathcal{V}}$ before proceeding.

Definition. A **lax functor** between monoidal categories $(\mathcal{C}, \otimes_C, \_C)$ and $(\mathcal{D}, \otimes_D, I_D)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ with natural morphisms

$$\mu_{xy} : Fx \otimes Fy \xrightarrow{\sim} F(x \otimes y) \quad \varepsilon : I_{\mathcal{D}} \xrightarrow{\sim} FI_{\mathcal{C}}$$

satisfying the following compatibility diagrams:

1. **Associators:**

$$\begin{array}{ccc} (Fx \otimes Fy) \otimes Fz & \xrightarrow{a_{FxFyFz}} & (Fx \otimes Fx) \otimes Fx \\ {\scriptstyle \mu_{xy} \otimes id} \downarrow & & \downarrow {\scriptstyle id \otimes \mu_{yz}} \\ F(x \otimes y) \otimes Fz & & Fx \otimes F(y \otimes z) \\ {\scriptstyle \mu_{x \otimes y, z}} \downarrow & & \downarrow {\scriptstyle \mu_{x, y \otimes z}} \\ F((x \otimes y) \otimes z) & \xrightarrow{Fa_{xyz}} & F((x \otimes (y \otimes z)) \end{array}$$

2. **Unitors:**

$$\begin{array}{ccc} I_{\mathcal{D}} \otimes Fx & \xrightarrow{\varepsilon \otimes id} & FI_{\mathcal{C}} \otimes Fx \\ {\scriptstyle \lambda_{\mathcal{D}}} \downarrow & & \downarrow {\scriptstyle \mu_{I_{\mathcal{C}}, x}} \\ Fx & \xleftarrow{F\lambda_{\mathcal{C}}} & F(I_{\mathcal{C}} \otimes x) \end{array} \qquad \begin{array}{ccc} Fx \otimes I_{\mathcal{D}} & \xrightarrow{id \otimes \varepsilon} & Fx \otimes FI_{\mathcal{C}} \\ {\scriptstyle \rho_{\mathcal{D}}} \downarrow & & \downarrow {\scriptstyle \mu_{x, I_{\mathcal{C}}}} \\ Fx & \xleftarrow{F\rho_{\mathcal{C}}} & F(x \otimes I_{\mathcal{C}}) \end{array}$$

Proposition. *The composition of lax monoidal functors is lax monoidal.*

Proof. If $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathbf{E}$ are lax monoidal functors, then their composite $GF : \mathcal{C} \to \mathbf{E}$ is a lax monoidal functor.

The natural transformations μ^{GF} and ε_{GF} are defined by the following diagrams:

$$\begin{array}{ccc} GF - \otimes GF - & \xRightarrow{\mu^G \circ F} & G(F - \otimes F -) \\ & {\scriptstyle \mu^{GF}} \searrow & \Downarrow {\scriptstyle G \circ \mu^F} \\ & & GF(- \otimes -) \end{array} \qquad \begin{array}{ccc} I_E & \xrightarrow{\varepsilon_G} & GI_D \\ & {\scriptstyle \varepsilon_{GF}} \searrow & \downarrow {\scriptstyle G\varepsilon_F} \\ & & GFI_C \end{array}$$

It must be checked that $(GF, \mu GF, \varepsilon GF)$ is again lax monoidal. The proof is lengthy but straightforward so we defer it to[2, Theorem 3.21]. □

Remark. Monoidal categories and lax monoidal functors form define a category.

monoidal functors can be found at Examples 5.2.5 and 5.4.7.

Definition. An **enriched functor** $F : \mathcal{C} \to \mathcal{D}$ between categories enriched over $\mathcal{V}$ consists of

- an assignment of an object $Fc \in \mathcal{D}$ for each $c \in \mathcal{C}$.
- a morphism $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ in $\mathcal{V}$.

This data is preserves identities and composites in the following sense:

- *Identities:*

$$\begin{array}{ccc} & I & \\ {\scriptstyle 1_x}\swarrow & & \searrow{\scriptstyle 1_{Fx}} \\ \mathcal{C}(x,x) & \xrightarrow[F_{xx}]{} & \mathcal{D}(y,y) \end{array}$$

- *Composition:*

$$\begin{array}{ccc} \mathcal{C}(y,z) \otimes \mathcal{C}(x,y) & \xrightarrow{\circ} & \mathcal{C}(x,z) \\ {\scriptstyle F_{yz}\otimes F_{xy}}\downarrow & & \downarrow{\scriptstyle F_{xz}} \\ \mathcal{D}(Fy,Fz) \otimes \mathcal{D}(Fx,Fy) & \xrightarrow[\circ]{} & \mathcal{D}(Fx,Fz) \end{array}$$

Definition. We define $\mathbf{Cat}_{\mathcal{V}}$ to be the category of categories enriched over $\mathcal{V}$ and enriched functors between them.

Theorem. *Let $\mathcal{V} \to \mathcal{W}$ be a lax functor between symmetric monoidal categories. Then there is an induced functor $F_* : \mathbf{Cat}_{\mathcal{V}} \to \mathbf{Cat}_{\mathcal{W}}$.*

Proof. Given a category $\mathcal{C}$ enriched over $\mathcal{V}$, we first define $F_*\mathcal{C}$ to be the category enriched over $\mathcal{W}$ with the same objects as $\mathcal{C}$ and whose with hom-objects are $F\mathcal{C}(x, y)$. Then we use the lax structure of F to define composites and identities:

- *Composition:*

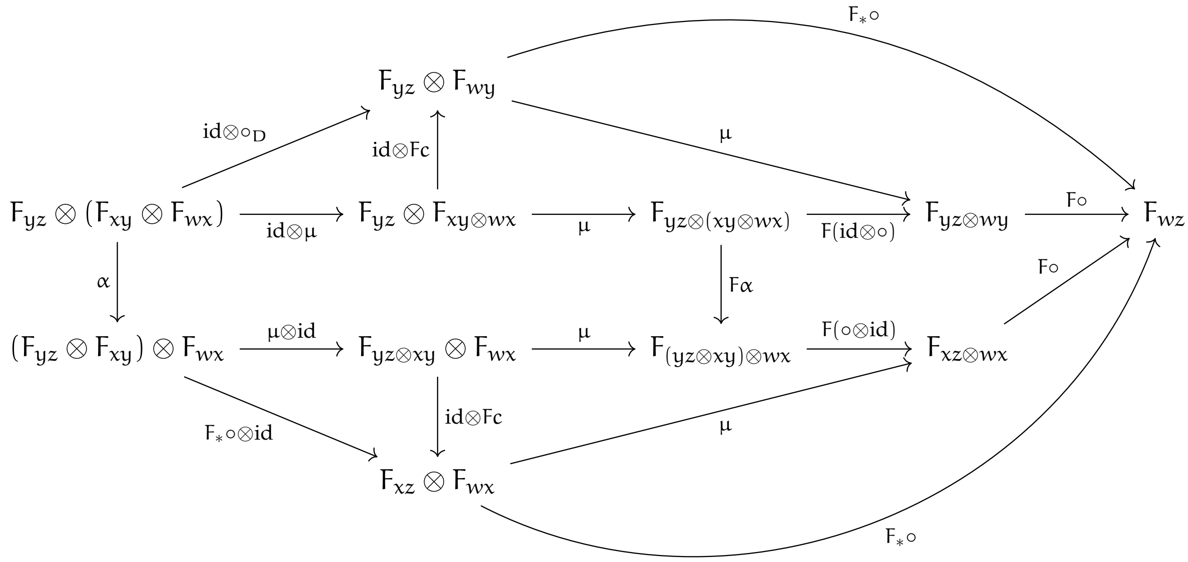
$$F\mathcal{C}(y,z) \otimes F\mathcal{C}(x,y) \xrightarrow{\mu} F(\mathcal{C}(y,z) \otimes \mathcal{C}(x,y)) \xrightarrow{F(\circ)} F\mathcal{C}(x,z)$$

- *Identities:*

$$I_{\mathcal{D}} \xrightarrow{\varepsilon} FI_{\mathcal{C}} \xrightarrow{F(1_x)} F\mathcal{C}(x,x)$$

We still have to check that $F_*\mathcal{C}$ is indeed a category and that F_* is a functor. This is a haunting yet straightforward task, and we try to convince the reader by proving that the composition in $F_*\mathcal{C}$ is associative.

Denote by $(-)_{xy}$ the hom-object $\mathcal{C}(x, y)$ and consider the diagram below: 1



Associativity in $F_*\mathcal{C}$ corresponds to the commutativity of the outer pentagon above. This is true, because because the smaller bits commute:

- The rectangle in the middle comes from the lax structure from F.
- The triangles are either the triangles defining $F_*\circ$ or the tensoring of those with id.
- The right triangles with four morphisms are both naturality squares.
- The uneven pentagon is the pentagon coherence diagram.

Thus we are done. Unitality is proven in a similar fashion, and so is the functoriality of F_*.

□

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