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EXPLORING DIFFERENCE-ALGEBRAIC REPRESENTATION FOR DISCRETE-TIME NONLINEAR SYSTEMS: novel stabilization conditions based on alternative Lyapunov functions and nonlinear control laws

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FOLHA DE APROVAÇÃO

"EXPLORING DIFFERENCE-ALGEBRAIC REPRESENTATION FOR DISCRETE-TIME NONLINEAR SYSTEMS: NOVEL STABILIZATION CONDITIONS BASED ON ALTERNATIVE LYAPUNOV FUNCTIONS AND NONLINEAR CONTROL LAWS"

GABRIELA LÍGIA REIS

Tese de Doutorado submetida à Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em Engenharia Elétrica da Escola de Engenharia da Universidade Federal de Minas Gerais, como requisito para obtenção do grau de Doutor em Engenharia Elétrica.

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To my mother, Janete, and my fiancé, Caio.

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“What we know is a drop, what we don’t know is an ocean.”
(Isaac Newton)

RESUMO

Ao longo das últimas décadas, a estabilização robusta regional de sistemas não lineares tem sido estudada sob diferentes metodologias. Neste cenário, a teoria de Lyapunov tem sido utilizada com sucesso como ponto de partida para desenvolver condições de estabilização na forma de desigualdades matriciais lineares (LMIs, do inglês *Linear Matrix Inequalities*) para diferentes classes de sistemas não lineares. Essas abordagens consideram restrições nos estados do sistema devido a limitações físicas ou associadas à validade do modelo utilizado, além disso, restrições na entrada de controle decorrentes da saturação do atuador também têm sido estudadas de forma mais aprofundada. No contexto de controle baseado em modelos, uma alternativa pouco explorada na literatura, principalmente no domínio do tempo discreto, é o uso de Representações Algébrico-Diferenciais ou de Diferenças (DAR, do inglês *Difference-Algebraic Representation*). Neste sentido, esta tese aborda o problema de estabilização regional de sistemas não lineares em tempo discreto com parâmetros variantes no tempo modelados por DAR. Os resultados deste trabalho são apresentados em duas partes: (i) novas condições suficientes na forma de LMIs são desenvolvidas para projetar controladores por realimentação de estado com ganho escalonado, utilizando funções de Lyapunov dependentes de parâmetros. Dois problemas de otimização são propostos para obter o maior Domínio de Atração (DoA, do inglês, *Domain of Attraction*) estimado ou para minimizar o ganho ℓ_2 da entrada de perturbação limitada por energia para a saída de desempenho; e (ii) uma nova metodologia para projetar controladores por realimentação de estados e de saída baseada em funções de Lyapunov polinomiais é proposta. As condições LMIs obtidas garantem a estabilização do sistema e fornecem uma estimativa do DoA. Em ambos os casos, o uso de funções de Lyapunov alternativas e informações sobre as não linearidades do sistema para o projeto do controlador reduzem o conservadorismo das condições de síntese. Exemplos numéricos ilustram a eficácia da metodologia proposta, apresentando resultados favoráveis ao comparar com trabalhos recentes na literatura para controle de sistemas não lineares em tempo discreto.

Palavras-chave: Representação Algébrica e de Diferenças. Controle não linear regional. Funções de Lyapunov polinomiais. Funções de Lyapunov dependentes de parâmetros.

ABSTRACT

Over the past decades, the regional robust stabilization of nonlinear systems has been investigated under different viewpoints and methodologies. As a matter of fact, in this scenario, Lyapunov theory has been successfully used as a starting point to develop stabilization conditions in the form of Linear Matrix Inequality (LMI) for different classes of nonlinear systems. These approaches have considered state constraints generated by limitations in the physical system or associated with the model validity and, more recently, control input constraints arising from actuator saturation. In the context of model-based control, an alternative not widely explored in the literature, especially in the discrete-time domain, is the use of Differential or Difference-Algebraic Representation (DAR). In this sense, this thesis addresses the regional stabilization problem of discrete-time nonlinear systems with time-varying parameters modeled as DAR. The results of this investigation can be associated with two main contributions: (i) a novel set of sufficient LMI conditions is developed to design gain-scheduled state feedback controllers using parameter-dependent Lyapunov functions, and two optimization problems are proposed to either obtain the largest estimated Domain-of-Attraction (DoA) or to minimize the ℓ_2-gain from the energy-bounded disturbance input to the performance output; and (ii) a new methodology to design State Feedback (SF) and Static Output Feedback (SOF) controllers based on polynomial Lyapunov functions is provided. The proposed LMI conditions guarantee the system stabilization associated with an estimate of the DoA. In both cases, considering alternative Lyapunov functions and information on the system's nonlinearities to synthesize the control law reduces conservatism in the control design. Numerical examples illustrate the effectiveness of the proposed methodology, showing favorable comparisons with recently published results in the context of robust control of discrete-time uncertain nonlinear systems.

Keywords: Difference-Algebraic Representation. Regional nonlinear control. Polynomial Lyapunov functions. Parameter-dependent Lyapunov functions.

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LIST OF ABBREVIATIONS

DAR	Differential or Difference-Algebraic Representation
DoA	Domain-of-Attraction
LFR	Linear Fractional Representation
LMI	Linear Matrix Inequality
LPV	Linear Parameter Varying
LTI	Linear Time-Invariant
qLPV	quasi-Linear Parameter Varying
RAR	Recursive-Algebraic Representation
SF	State Feedback
SOF	Static Output Feedback
TS	Takagi-Sugeno

LIST OF SYMBOLS

$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{R}^{n \times m}$	The set of $n \times m$ real matrix
0	Null matrix of appropriate dimension
I	Identity matrix of appropriate dimension
I_n	Identity matrix of n-th order
diag$\{\ldots\}$	Block-diagonal matrix
A	Matrix $A = [a_{ij}]$
$A^\top$	Transpose of matrix A
He$\{A\}$	Shorthand notation for $A + A^\top$
$A_{(i)}$	i-th row vector of matrix A
$\mathcal{I}_n$	The set of natural numbers from 1 to n
$P > (<)\ 0$	The matrix P is positive (negative) definite
$P \geq (\leq)\ 0$	The matrix P is positive (negative) semidefinite
$\star$	Transpose element inside of a symmetric matrix
$\parallel w \parallel_2$	The 2-norm of vector w_k : $\parallel w \parallel_2 = \sqrt{\sum_{k=0}^{\infty} w_k^\top w_k}$
ℓ_2	The Lebesgue space of square-summable sequences over $[0, \infty)$, with norm $\parallel \cdot \parallel_2$
$\mathcal{X} \times \Delta \subset \mathbb{R}^{n_x + n_\delta}$	Cartesian product of $\mathcal{X} \subset \mathbb{R}^{n_x}$ and $\Delta \subset \mathbb{R}^{n_\delta}$
co $\{\cdot\}$	Convex hull of vectors
Λ_1	Unitary simplex $\Lambda_1 := \left\{\alpha_{\mathsf{p}_k} \in \mathbb{R}^N : \sum_{\mathsf{v}=1}^{N} \alpha_{\mathsf{p}(\mathsf{v})k} = 1,\ \alpha_{\mathsf{p}(\mathsf{v})k} \geq 0\right\}$ where p represents an index used to distinguish different polytopes, N is the number of vertices and $\alpha_{\mathsf{p}(\mathsf{v})k}$ is the vth entry in the vector at time k.

CONTENTS

Part I

Thesis Contextualization

1 INTRODUCTION

This chapter provides an overview of this thesis. Initially, some aspects related to analysis and control of nonlinear dynamical systems are introduced. Next, the motivations and objectives pursued in this research are stated. Finally, the outline of this thesis is presented and contributions of this investigation are highlighted.

1.1 Analysis and control of nonlinear dynamical systems

In real-world applications, most dynamical systems, such as electrical, electronic, mechanical, aerospace, thermal, chemical, and biological systems, present nonlinear behavior. In the context of control systems, the predominance of nonlinear systems is widely known [1, 2, 3, 4]. Generally, the stability analysis and the control design for nonlinear systems are a difficult task and require advanced and involved mathematical tools.

Due to the extensive knowledge and powerful methodologies available in the context of linear systems, a common engineering practice is to linearize the system model around an operating point for analysis and control purposes. Indeed linear analysis is a useful tool to understand the system behavior. However, according to Khalil [3], only the linearization may not be sufficient, since the results obtained may be guaranteed only in a small neighborhood of the operating point and the linear representations may be unable to model certain nonlinear phenomena.

Besides the fact that it is essential to consider the nonlinear characteristics of certain systems, in many cases the use of nonlinear control strategies, even for processes considered as Linear Time-Invariant (LTI) systems, can be important to achieve better performance for the closed-loop control system in comparison to what can be achieved relying only on linear techniques [1].

As a consequence of the benefits of addressing nonlinear behavior in control systems, the development of analysis and synthesis conditions for nonlinear control systems has received a lot of attention in the last few decades. For nonlinear systems, a general methodology is not known, but analysis and control techniques aimed at specific classes of systems, such as polynomial systems [5, 6, 7, 8, 9], rational systems [10, 11, 12, 13, 14, 15, 16, 17, 18], Takagi-Sugeno (TS) fuzzy models [19, 20, 21, 22, 23, 24, 25, 26, 27, 28], and quasi-Linear Parameter Varying (qLPV) systems [29, 30, 31, 32, 33, 34, 35, 36, 37] have been developed.

In this context, an alternative for nonlinear control systems that has received a lot of attention from researchers is to consider a compact region around the equilibrium point of interest, in which the stability of the equilibrium point is guaranteed [3]. Since constraints on system states and control inputs usually occur in practical applications due to physical limitations, these approaches are quite promising. In this investigation, we are particularly interested in the regional stabilization and input-output characterization of a specific class of nonlinear systems, while taking into account several relevant aspects of control theory.

1.2 Motivation

The development of synthesis conditions to robust stabilization of nonlinear systems with constraints is a fundamental problem in control systems theory, with several practical applications as, for instance, in spacecraft attitude control [38, 39], mobile robots [40], bioprocesses [41] and solar power generation [42]. These methods generally rely on semidefinite programming to solve stabilization conditions based on the Lyapunov stability theory. In the context of nonlinear systems, this approach can be very challenging, and therefore it is worth pursuing systematic closed-loop stability analysis and control design techniques.

In some real-world control applications, constraints on the control inputs and states should be fulfilled for physical, technical, or safety reasons [20, 24]. Input constraints usually arise from actuators saturation [43, 44]. On the other hand, besides the necessity to obey physical and safety requirements, system states are sometimes implicitly assumed to be bounded in order to guarantee the validity of underlying mathematical models used to solve analysis and/or synthesis control problems. In this sense, as it is not possible to guarantee global stability of the system, since the control law will be valid only in the region associated with the model validity, these constraints must be directly taken into consideration in the control problem formulation [11, 45]. Indeed, for many real-world applications, if input and state constraints are not taken into account in the control design, this may result in safety risks or even in closed-loop system instability.

In addition, real-world control systems are frequently subject to parameter variations and uncertainties arising from changes in the plant or in the environment, and from measurement noise. Considering time-varying parameters and system constraints at the same time can lead to conservative design results. However, it is possible to reduce the conservativeness by selecting non-quadratic Lyapunov functions, such as parameter-dependent Lyapunov functions or polynomial Lyapunov functions, to perform stability analysis and control synthesis [46, 47, 48, 49, 50, 15]. Alternatively, nonlinear controllers can be used instead of linear ones. In this context, due to the possibility of simple and intuitive control design, gain-scheduled control is one of the most used strategies for controlling several nonlinear systems [39, 51]. Besides that, by using gain-scheduled controllers it is sometimes possible to reduce conservativeness also by inserting information on the measured plant parameters variations explicitly in the control law.

At the same time, due to the recent advances in hardware, it is noteworthy that almost all controllers are implemented in a digital environment. Generally, digital controllers are more powerful, reliable, faster, and cheaper [52, 53]. These factors have motivated numerous contributions focused on control design for originally discrete-time systems or the counterparts of continuous-time systems obtained using discretization methods [54, 21, 29, 24].

Based on this discussion, this research aims to develop regional stabilization conditions for discrete-time nonlinear systems with time-varying parameters. The class of systems considered in this research covers all systems that can be modeled as a Difference-Algebraic Representation (DAR) [55], also called in the literature as Recursive-Algebraic Representation (RAR) [12, 50], the discrete-time counterpart of the Differential-Algebraic Representation [56]. From DAR, it is possible to obtain an exact representation of rational nonlinear systems as a set of algebraic and difference (or differential) equations. It is worth mentioning that even vector fields with elementary nonlinear functions, such as exponential, logarithmic, trigonometric, and hyperbolic functions, can be rewritten in rational form, as presented by Henrion & Garulli [57, p. 29]. In this sense, DAR can be used to model a wide range of phenomena in engineering, physical, chemical, biological and economic contexts [12]. Moreover, it is possible to apply the well-established Lyapunov theory [3] and LMI-based tools [58, 59] for the control design of parameter-varying systems.

The use of Differential and Difference-Algebraic Representations in analysis and control of nonlinear systems has been investigated in the last decades, under several aspects in the context of regional stability analysis [60, 50, 61, 62, 15], regional stabilization and DoA estimation [11, 45, 63, 64, 13], regional stabilization and output performance [11, 18], filters and observers analysis and design [65, 55, 66], anti-windup design [67, 68], event-triggered control [69], and more recently for the output regulation problem [14].

In the context of stability analysis, less conservative results to nonlinear systems described in DAR form were obtained by searching for polynomial and rational Lyapunov function candidates [60, 70, 71, 15]. However, due to inherent difficulties in the development of synthesis conditions, recent approaches to DAR use quadratic Lyapunov functions and/or linear State Feedback (SF) controllers to robust stabilization and DoA estimation [14, 12, 64, 13]. In this sense, the development of new stabilization conditions to obtain less conservative results is worth investigating.

Besides that, there are still relevant problems in the literature that have not been widely explored in the context of DAR, especially in the discrete-time domain as, for instance, the design of Static Output Feedback (SOF) controllers. The SOF control design problem has received a lot of attention in the past years because it is simple to be implemented in practical situations where only partial state information is available in real-time [24, 72, 44]. However, the design of SOF control schemes is considered to be harder to solve due to its non-convex characterization, even in the context of linear systems [46, 73, 44]. Most results are restrictive and conservative. For instance, some methodologies require a constant output matrix or particular similarity transformations [74, 75]. Besides that, it is possible to find in the literature methodologies based on iterative algorithms which increases the computational effort

[76, 77, 78]. There are also two-step approaches, where the first step consists of searching for an SF controller and then the SOF control is obtained from the initial results [18, 79]. More recently, in Peixoto, Coutinho & Palhares [72] and Peixoto *et al.* [44] an alternative one-step approach was proposed to compute output-feedback control gains for discrete-time nonlinear parameter-varying systems with time-varying delay in the state and also the case for fuzzy systems [21]. Thus, SOF control is a challenging stabilization problem and it is worth noting that there is still room for improvements, especially in the context of DAR systems.

1.3 Objectives

This thesis is concerned with the stabilization of discrete-time nonlinear systems. Therefore, based on what was previously discussed, the following specific objectives are pursued:

a) To explore the use of DAR in controller synthesis for discrete-time nonlinear systems with time-varying parameters;

b) To explore non-quadratic Lyapunov functions to obtain novel conditions for stabilization of discrete-time nonlinear systems;

c) To obtain new conditions for control design incorporating information about the system's nonlinearities in the control law;

d) To take into account state constraints in the synthesis of controllers for nonlinear discrete-time systems.

1.4 Thesis outline and contributions

This manuscript is divided into four parts organized as follows:

(i) **Part I - Chapters 1 and 2**:

This is the introductory part of the thesis. *Chapter* 1 provided an overview of our research. In the sequel, *Chapter* 2 presents the main concepts and definitions utilized throughout this investigation, concerning the use of DAR in the context of nonlinear control.

(ii) **Part II - Chapters 3 and 4**:

In this part, we explore the use of parameter-dependent Lyapunov functions to provide novel stabilization conditions for discrete-time nonlinear systems described in a DAR form. In *Chapter* 3, concepts, definitions, and the main problems addressed in this part of the research are discussed.

Chapter 4 presents the theorems based on parameter-dependent Lyapunov functions obtained to synthesize gain-scheduled SF controllers. Firstly a *non-iterative* method to regional stabilization applied to minimize the input-to-output performance index in the presence of energy-bounded disturbances is proposed, when considering zero initial conditions, together with a polytopic description of input saturation, as proposed by Hu & Lin [80], instead of the so-called generalized sector condition [81]. Secondly, the approach is adapted to the problem of estimating the largest DoA, disregarding the presence of exogenous disturbances. Our main contributions at this point can be summarized as follows:

– Proposition of new sufficient conditions to design gain-scheduled SF controllers for regional stabilization of nonlinear systems with state and input constraints, described in a DAR form, by using a novel set of LMIs obtained by considering parameter-dependent Lyapunov functions and incorporating information on the system's nonlinearities to synthesize the control law, reducing the design conservativeness;
– Development of LMI synthesis conditions to ensure input-to-state stability and worst-case input-to-output performance by minimizing the induced ℓ_2-gain from the disturbance input to the performance output, relating the system stabilization region to the admissible energy-bounded disturbance in the case of zero initial conditions;
– Modification of the synthesis conditions to investigate the problem of DoA estimation for the closed-loop system, for the case of no input disturbances.

It is worth emphasizing that input-to-output stability for nonlinear systems may hold only locally [3], and thus it is necessary to consider in the control design explicit bounds on the disturbance inputs and on the admissible initial conditions for the system. Otherwise, performance and stability cannot be effectively guaranteed. However, several available results do not consider the important practical aspects of regional characterization of nonlinear systems robust stabilization and performance with an estimation of the stabilization region [82, 83, 84], or the existence of limits on the control input [18, 19].

To the best of the author's knowledge, Oliveira, Gomes da Silva Jr. & Coutinho [61] and Oliveira, Gomes da Silva Jr. & Coutinho [85] proposed analysis conditions to constrained nonlinear systems described by DAR to investigate both problems, but without providing synthesis conditions. Klug, Castelan & Coutinho [19] also investigated similar problems for systems represented as Takagi-Sugeno fuzzy models with nonlinear consequent, but without taking into account time-varying parameters or control input saturation.

In the end of Part II, some numerical examples illustrate the effectiveness of the proposed approach.

The results presented in Chapter 4 have been published in Reis *et al.* [86] and Reis *et al.* [87].

(iii) **Part III - Chapters 5 and 6:**

This part of the investigation is focused on the use of polynomial Lyapunov functions. In *Chapter* 5, preliminary concepts and some relevant aspects considered in our methodology are discussed.

Chapter 6 presents new theorems to design gain-scheduled SF and SOF controllers for discrete-time nonlinear systems with time-varying parameters described in a DAR form. Convex optimization problems subject to LMI constraints are proposed to obtain the largest estimate of the closed-loop DoA. Our methodology consists in a one-step approach that requires no iterative algorithms, and auxiliary decision variables are introduced only aiming at less conservative results. More specifically, our main contributions can be summarized as follows:

– A novel sufficient condition to design nonlinear gain-scheduled SF controllers for regional stabilization of discrete-time nonlinear systems is provided. The novelty of this approach compared to that proposed by Oliveira, Gomes da Silva Jr & Coutinho [12] (based on quadratic Lyapunov functions) and our first proposal presented in Part II (based on parameter-dependent Lyapunov functions) is related not only to the fact that a polynomial Lyapunov function candidate is considered, but also to the methodology used to obtain the proposed conditions, in which no congruence transformations are required;

– A new sufficient condition based on polynomial Lyapunov functions for regional stabilization of discrete-time nonlinear systems by gain-scheduled SOF controllers, so far not explored in the context of DARs, is presented. The new control approach can be applied to systems with parameter-dependent and/or nonlinear output matrix. Besides that, no similarity transformation is required.

In the context of discrete-time nonlinear systems described in a DAR form, to the best of the author's knowledge, Coutinho & Souza [50] proposed analysis conditions based on polynomial Lyapunov functions, but without providing synthesis conditions.

Finally, *Chapter* 6 also brings some illustrative numerical examples to show how the use of polynomial Lyapunov functions can provide a larger and more accurate estimated DoA, reducing conservativeness.

Part of the results presented in Chapter 6 have been published in Reis *et al.* [88].

(iv) **Part IV - Chapter 7:**

To conclude, *Chapter* 7 points out conclusions and directions for the continuity of this study.

2 NONLINEAR SYSTEM REPRESENTATION

The purpose of this chapter is to introduce fundamental concepts for the general context of this thesis, concerning the stabilization of discrete-time nonlinear systems, described in a DAR form. The class of discrete-time nonlinear dynamical systems considered in this investigation is presented in Section 2.1. In the sequel, Section 2.2 shows how DAR can be used to describe this class of nonlinear systems and some numerical examples are given.

2.1 The class of nonlinear systems

Throughout this work, unless otherwise additional specific assumptions are made, the following class of discrete-time nonlinear dynamical systems is considered:

$$
\begin{aligned}
x_{k+1} &= f(x_k, \delta_k) + g(x_k, \delta_k)u_k + h(x_k, \delta_k)w_k, \\
z_k &= f_z(x_k, \delta_k) + g_z(x_k, \delta_k)u_k + h_z(x_k, \delta_k)w_k, \\
y_k &= f_y(x_k, \delta_k) = C_y(x_k, \delta_k)x_k,
\end{aligned}
\tag{2.1}
$$

where $x_k \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ is the state vector of the system, $\delta_k \in \Delta \subseteq \mathbb{R}^{n_\delta}$ is a time-varying parameter vector, which is supposedly known and available for measurement, $u_k \in \mathbb{R}^{n_u}$ is the control input, $w_k \in \mathbb{R}^{n_w}$ is the exogenous disturbance input which is supposed to be an arbitrary signal in the ℓ_2-space, $z_k \in \mathbb{R}^{n_z}$ is the performance output, $y_k \in \mathbb{R}^{n_y}$ is the measurement output, and $C_y(x_k, \delta_k) \in \mathbb{R}^{n_y \times n_x}$ is the output matrix.

In this investigation, the following assumptions are considered for system (2.1):

Assumption 2.1. *Functions $f(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_x}$, $f_z(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_z}$, $f_y(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_y}$, (with $f(0, \delta_k) = f_z(0, \delta_k) = f_y(0, \delta_k) = 0$), $g(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_x \times n_u}$, $g_z(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_z \times n_u}$, $h(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_x \times n_w}$ and $h_z(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_z \times n_w}$ are rational functions well-posed on $\mathcal{X} \times \Delta$.*

Assumption 2.1 regards the class of *rational systems* and guarantees existence and uniqueness of the solutions of the difference equation in the region $\mathcal{X} \times \Delta$ that contains the equilibrium point $f(0, \delta_k) = 0$, $\forall \delta_k \in \Delta$. It was already shown by Coutinho *et al.* [18] that the class of rational systems in the continuous-time domain can be represented in a Differential-Algebraic Representation form. Similarly, this representation could be used in the context of discrete-time systems, as also presented in the literature [45, 61, 85].

Assumption 2.2. *The disturbance input vector w_k belongs to the following class of square summable sequences:*

$$
\mathcal{W} := \left\{ w_k \in \mathbb{R}^{n_w} : \|w_k\|_2^2 \leq \lambda^{-1} \right\}, \quad \text{for some } \lambda > 0. \tag{2.2}
$$

2.2 Using a Difference-Algebraic Representation

From *Assumption 2.1*, the rational system (2.1) can be recast as a DAR given by

$$\begin{aligned} x_{k+1} &= A_1(x_k,\delta_k)x_k + A_2(x_k,\delta_k)\pi_k + A_3(x_k,\delta_k)u_k + A_4(x_k,\delta_k)w_k,\\ 0 &= \Omega_1(x_k,\delta_k)x_k + \Omega_2(x_k,\delta_k)\pi_k + \Omega_3(x_k,\delta_k)u_k + \Omega_4(x_k,\delta_k)w_k,\\ z_k &= C_{z_1}(x_k,\delta_k)x_k + C_{z_2}(x_k,\delta_k)\pi_k + C_{z_3}(x_k,\delta_k)u_k + C_{z_4}(x_k,\delta_k)w_k,\\ y_k &= C_{y_1}(x_k,\delta_k)x_k + C_{y_2}(x_k,\delta_k)\pi_k, \end{aligned} \tag{2.3}$$

where $\pi_k := \pi(x_k,\delta_k,u_k,w_k) \in \mathbb{R}^{n_\pi}$ is an auxiliary vector of rational functions with respect to (x_k,δ_k), and affine functions with respect to (u_k,w_k). The matrices $A_1(x_k,\delta_k) \in \mathbb{R}^{n_x\times n_x}$, $A_2(x_k,\delta_k) \in \mathbb{R}^{n_x\times n_\pi}$, $A_3(x_k,\delta_k) \in \mathbb{R}^{n_x\times n_u}$, $A_4(x_k,\delta_k) \in \mathbb{R}^{n_x\times n_w}$, $\Omega_1(x_k,\delta_k) \in \mathbb{R}^{n_\pi\times n_x}$, $\Omega_2(x_k,\delta_k) \in \mathbb{R}^{n_\pi\times n_\pi}$, $\Omega_3(x_k,\delta_k) \in \mathbb{R}^{n_\pi\times n_u}$, $\Omega_4(x_k,\delta_k) \in \mathbb{R}^{n_\pi\times n_w}$, $C_{z_1}(x_k,\delta_k) \in \mathbb{R}^{n_z\times n_x}$, $C_{z_2}(x_k,\delta_k) \in \mathbb{R}^{n_z\times n_\pi}$, $C_{z_3}(x_k,\delta_k) \in \mathbb{R}^{n_z\times n_u}$, $C_{z_4}(x_k,\delta_k) \in \mathbb{R}^{n_z\times n_w}$, $C_{y_1}(x_k,\delta_k) \in \mathbb{R}^{n_y\times n_x}$, and $C_{y_2}(x_k,\delta_k) \in \mathbb{R}^{n_y\times n_\pi}$, are affine functions of (x_k,δ_k), with $\Omega_2(x_k,\delta_k)$ a square full-rank matrix for all $(x_k,\delta_k) \in \mathcal{X}\times\Delta$.

The correctness of DARs can be verified by replacing the nonlinearity vector π_k given by the null algebraic equation in (2.3) with the corresponding expression:

$$\pi_k = -\Omega_2^{-1}(x_k,\delta_k)\left[\Omega_1(x_k,\delta_k)x_k + \Omega_3(x_k,\delta_k)u_k + \Omega_4(x_k,\delta_k)w_k\right] \tag{2.4}$$

in the equations related to x_{k+1}, z_k, and y_k in (2.3). This procedure must result in the same difference equation presented in (2.1). Figure 2.1 illustrate this idea presenting the block diagram of the DAR in (2.3).

Coutinho *et al.* [18] showed that the DAR includes the Linear Fractional Representation (LFR) of rational nonlinear systems discussed by Ghaoui & Scorletti [10]. Thus, it is possible to apply LFR modeling tools to derive a DAR model for rational nonlinear systems, which leads to constant matrices A_1, A_2, A_3, and A_4, as detailed in Coutinho *et al.* [18]. Furthermore, it is possible to obtain a DAR of a rational system directly from the state-space model (2.1), through a more intuitive procedure presented in Trofino & Dezuo [15], which was used in this research.

2.2.1 Numerical examples

In the sequel, we present some numerical examples to demonstrate how it is possible to obtain a DAR of a rational dynamical system, based on the ideas shown in Trofino & Dezuo [15].

Example 2.1 (Polynomial system)**.** *Consider the polynomial discrete-time nonlinear system without time-varying parameters*

$$x_{k+1} = a_0 x_k - a_1 x_k^4 + (a_2 + a_3 x_k^2)u_k + a_4 w_k. \tag{2.5}$$

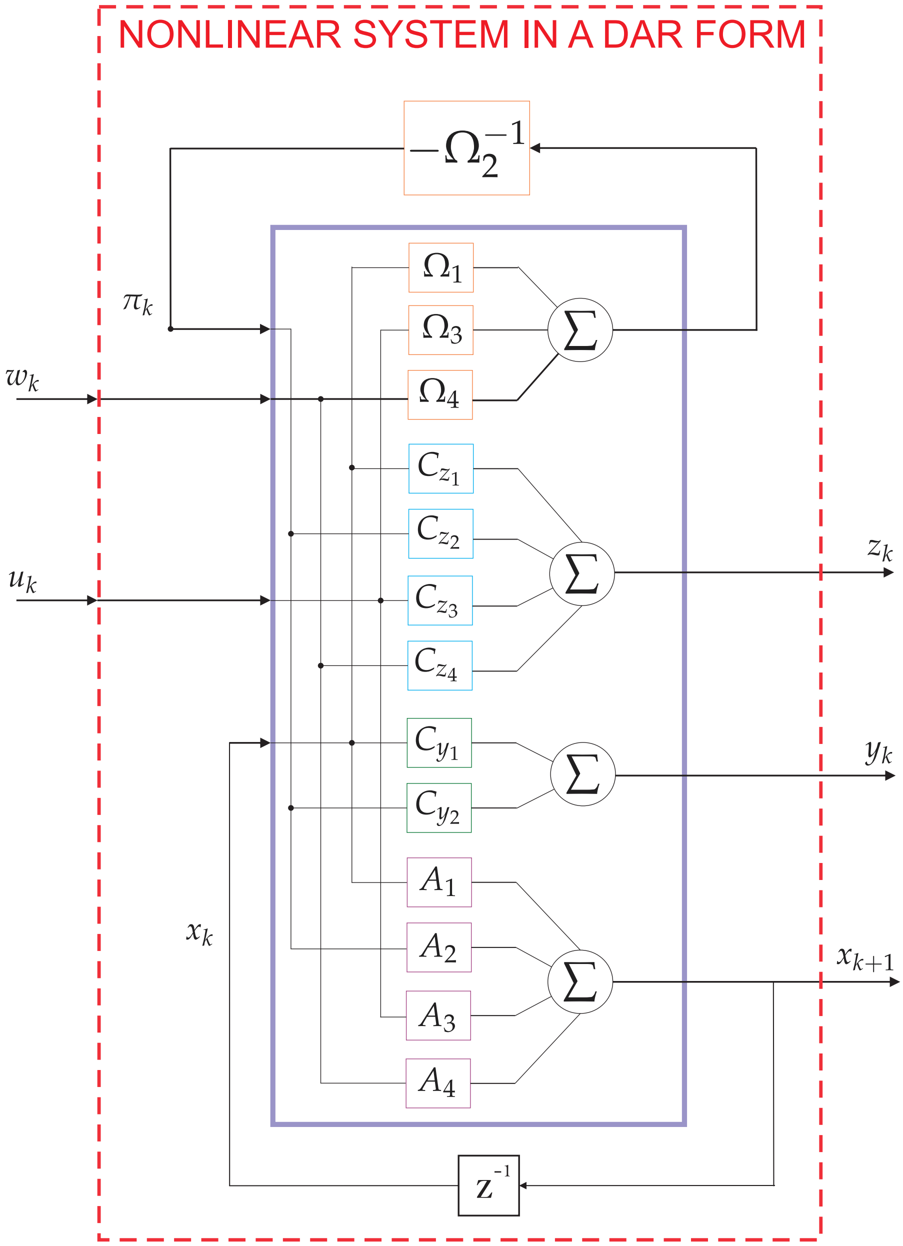


Figure 2.1 – Block diagram of DAR.

A DAR for this system can be obtained by the following three steps.

(i) ***Start by defining the nonlinearity vector*** π_k

In this case, a possible choice of π_k *which allows us to express the difference equation* (2.5) *as an affine combination of* x_k, π_k, u_k, *and* w_k *is:*

$$\pi_k = \begin{bmatrix} x_k^2 & x_k^3 & x_k u_k \end{bmatrix}^\top.$$

(ii) ***Now, it is possible to rewrite the system equation based on*** π_k

Considering the vector π_k *defined previously, the polynomial system in* (2.5) *can be* *recast as the difference equation in* (2.3), *with*

$$A_1 = a_0,\ A_2(x_k) = \begin{bmatrix} 0 & -a_1 x_k & a_3 x_k \end{bmatrix},\ A_3 = a_2,\ \text{and}\ A_4 = a_4.$$

(iii) ***Finally, one can obtain the algebraic equation based on each element of*** π_k

In this example, the entries of vector π_k *are such that:*

$$\begin{cases} \pi_{(1)k} = x_k^2 \\ \pi_{(2)k} = x_k \pi_{(1)k} \\ \pi_{(3)k} = x_k u_k \end{cases},\quad \text{or} \quad \begin{cases} x_k^2 - \pi_{(1)k} = 0 \\ x_k \pi_{(1)k} - \pi_{(2)k} = 0 \\ x_k u_k - \pi_{(3)k} = 0 \end{cases},$$

leading to the algebraic equation in (2.3) *with*

$$\Omega_1(x_k) = \begin{bmatrix} x_k \\ 0 \\ 0 \end{bmatrix},\ \Omega_2(x_k) = \begin{bmatrix} -1 & 0 & 0 \\ x_k & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ \Omega_3(x_k) = \begin{bmatrix} 0 \\ 0 \\ x_k \end{bmatrix},\ \text{and}\ \Omega_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, system (2.5) *can be recast in a DAR given by*

$$\begin{aligned} x_{k+1} &= a_0 x_k + \begin{bmatrix} 0 & -a_1 x_k & a_3 x_k \end{bmatrix} \pi_k + a_2 u_k + a_4 w_k, \\ 0 &= \begin{bmatrix} x_k \\ 0 \\ 0 \end{bmatrix} x_k + \begin{bmatrix} -1 & 0 & 0 \\ x_k & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \pi_k + \begin{bmatrix} 0 \\ 0 \\ x_k \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} w_k. \end{aligned} \tag{2.6}$$

Remark 2.1. *In this example, it is worth mentioning two important aspects:*

a) *Although the term* x_k^2 *is not necessary to describe the system in a difference equation form as in* (2.6), *the choice of* x_k^2 *as one of the entries of the nonlinearity vector* (π_k) *is essential to obtain the algebraic equation in* (2.6) *as an affine combination of* x_k, π_k, u_k, *and* w_k;

b) *Other possible choice for the nonlinearity vector is defining*

$$\pi_k = \begin{bmatrix} x_k^2 & x_k^3 & x_k^4 & x_k u_k & x_k^2 u_k \end{bmatrix}^\top.$$

In this case, following the steps described previously, system (2.5) *can be rewritten in a DAR form* (2.3) *such that*

$$A_1 = a_0,\ A_2 = \begin{bmatrix} 0 & 0 & -a_1 & 0 & a_3 \end{bmatrix},\ A_3 = a_2,\ A_4 = a_4,$$

$$\Omega_1(x_k) = \begin{bmatrix} x_k \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \Omega_2(x_k) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ x_k & -1 & 0 & 0 & 0 \\ 0 & x_k & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & x_k & -1 \end{bmatrix}, \Omega_3(x_k) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_k \\ 0 \end{bmatrix},\ \text{and}$$

$$\Omega_4 = 0_{5\times 1}.$$

Note that, by choosing this vector π_k *matrices* $A_i, i \in \mathcal{I}_4$ *are constant.*

Example 2.2 (Rational system). *Consider the following rational discrete-time nonlinear system*

$$x_{k+1} = a_0 x_k + f_n(x_k)x_k + a_2 u_k + a_3 w_k, \quad f_n(x_k) = \frac{a_1 x_k}{b_0 + b_1 x_k}. \tag{2.7}$$

Following the steps presented in Example 2.1, firstly, the vector π_k is defined. In this case, a possible choice that allows describing this rational system in a DAR form is

$$\pi_k = \begin{bmatrix} f_n(x_k) & f_n(x_k)x_k \end{bmatrix}^\top.$$

Then, the system equation (2.7) *can be decomposed as*

$$x_{k+1} = \underbrace{a_0}_{A_1} x_k + \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{A_2} \pi_k + \underbrace{a_2}_{A_3} u_k + \underbrace{a_3}_{A_4} w_k. \tag{2.8}$$

Finally, it is necessary to ensure the relation between π_k and x_k determined by the algebraic equation in the DAR. Since $b_0 + b_1 x_k \neq 0$, per Assumption 2.1 on $f_n(x_k)$, this rational function can be redefined as an augmented polynomial equation such that

$$a_1 x_k - f_n(x_k)(b_0 + b_1 x_k) = 0.$$

Thus, we have

$$\begin{cases} a_1 x_k - b_0 \pi_{(1)k} - b_1 \pi_{(2)k} = 0 \\ x_k \pi_{(1)k} - \pi_{(2)k} = 0 \end{cases},$$

leading to the following algebraic equation

$$0 = \underbrace{\begin{bmatrix} a_1 \\ 0 \end{bmatrix}}_{\Omega_1} x_k + \underbrace{\begin{bmatrix} -b_0 & -b_1 \\ x_k & -1 \end{bmatrix}}_{\Omega_2(x_k)} \pi_k + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\Omega_3} u_k + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\Omega_4} w_k. \tag{2.9}$$

Therefore, system (2.7) *can be recast in a DAR form* (2.3) *by equations* (2.8) *and* (2.9).

Example 2.3 (Rational system with time-varying parameters). *The following example is a more complex nonlinear system, including time-varying parameter and the output vectors z_k and y_k described previously*

$$\begin{aligned} x_{(1)k+1} &= x_{(2)k} + 0.5 w_k, \\ x_{(2)k+1} &= -x_{(2)k} + \frac{\delta_{(1)k} \left[x_{(1)k} x_{(2)k} + 1 \right] x_{(2)k}}{1 + \delta_{(2)k}} + u_k + w_k, \\ z_k &= x_{(2)k} + x_{(1)k}^2 u_k, \\ y_k &= x_{(1)k} + x_{(1)k}^3, \end{aligned} \tag{2.10}$$

where $1 + \delta_{(2)k} \neq 0$. A possible DAR (2.3) *for this system can be obtained with*

$$\pi_k = \begin{bmatrix} x_{(1)k}^2 & \frac{\delta_{(1)k} x_{(1)k} x_{(2)k}}{1+\delta_{(2)k}} & \frac{\delta_{(1)k} x_{(2)k}}{1+\delta_{(2)k}} & x_{(1)k} u_k \end{bmatrix}^\top,$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \; A_2(x_k) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x_{(2)k} & 1 & 0 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; A_4 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$

$$\Omega_1(x_k, \delta_k) = \begin{bmatrix} x_{(1)k} & 0 \\ 0 & 0 \\ 0 & \delta_{(1)k} \\ 0 & 0 \end{bmatrix}, \; \Omega_2(x_k, \delta_k) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & x_{(1)k} & 0 \\ 0 & 0 & -(\delta_{(2)k}+1) & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\Omega_3(x_k) = \begin{bmatrix} 0 & 0 & 0 & x_{(1)k} \end{bmatrix}^\top, \; \Omega_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top,$$

$$C_{z_1} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; C_{z_2}(x_k) = \begin{bmatrix} 0 & 0 & 0 & x_{(1)k} \end{bmatrix}, \; C_{z_3} = C_{z_4} = 0,$$
$$C_{y_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; C_{y_2}(x_k) = \begin{bmatrix} x_{(1)k} & 0 & 0 & 0 \end{bmatrix}.$$

Remark 2.2. *It is important to point out that the LFR of a rational nonlinear system is not unique. Besides that, using a procedure similar to the one presented in Trofino & Dezuo [15], it is possible to choose different nonlinearity vectors π_k, as illustrated in Example 2.1. Consequently, the composition of the nonlinear system in a DAR form is not unique, which can lead to conservative results, depending on the choice of the nonlinearity vector, or on the LFR being used to obtain the DAR.*

2.2.2 Polytopic description

As previously discussed, due to physical limitations and the validity region of the model, a domain of operation for the system must be considered. In this investigation, the state trajectories of system (2.3) will be constrained into the following polyhedral set:

$$\mathcal{X} := \left\{ x_k \in \mathbb{R}^{n_x} : a_v^\top x_k \leq 1, \quad v \in \mathcal{I}_{n_e} \right\}, \tag{2.11}$$

where a_v is a constant n_x-dimensional vector of parameters, and n_e is the number of affine constraints (represented as hyperplanes) which characterize the region $\mathcal{X}$. Figure 2.2 presents two examples of polytopic regions, considering 2D and 3D systems.

Taking this domain into account in the control design will be essential to ensure both suitable closed-loop performance and stability of nonlinear systems. The choice of the polyhedral set could be considered conservative. However, due to its characteristics of simplicity and convexity, this set is widely used [20, 12, 13, 14].

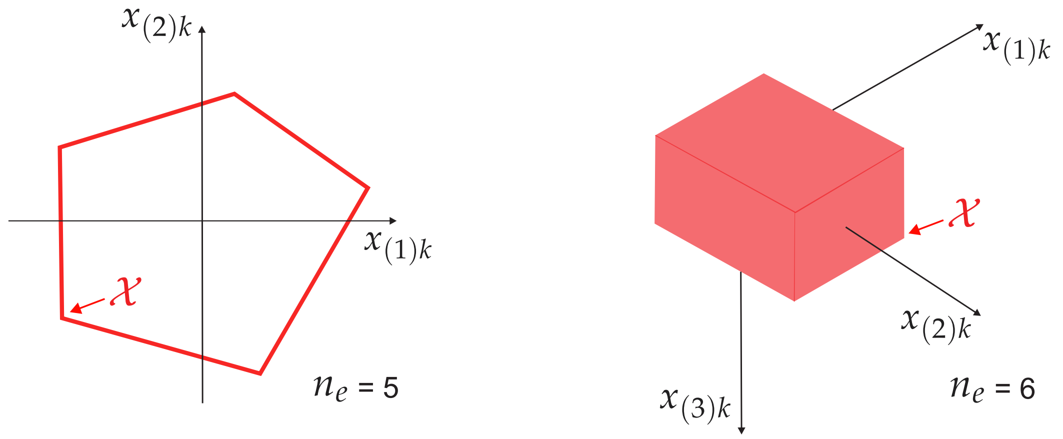


Figure 2.2 – Examples of polytopic regions $\mathcal{X}$ for 2D and 3D systems.

Similarly, the system time-varying parameters are considered to be confined to a polytopic region $\Delta \subset \mathbb{R}^{n_\delta}$. In this case, $\mathcal{X} \times \Delta$ is also a polytopic region, and since $A_{\text{v}}(\cdot), \Omega_{\text{v}}(\cdot)$, $C_{z_{\text{v}}}(\cdot)$, $\text{v} \in \mathcal{I}_4$, $C_{y_1}(\cdot)$, and $C_{y_2}(\cdot)$ are matrices of affine functions with respect to (x_k, δ_k), they belong to polytopes of matrices that can be compactly represented as convex combinations of N_x vertices in $\mathcal{X}$ and N_δ vertices in Δ, such that

$$M(x_k, \delta_k) = \sum_{i=1}^{N_x} \sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}} \alpha_{\delta_{(l)k}} M_{il}, \tag{2.12}$$

where $M(x_k, \delta_k)$ represents any matrix in (2.3), M_{il} represents the value of system matrices in each i-th, l-th vertex in $\mathcal{X}$ and Δ, respectively, and $\alpha_{x_k}, \alpha_{\delta_k} \in \Lambda_1$, with Λ_1 being the unitary simplex. Figure 2.3 illustrates this idea, for the case of an one-dimensional system with a scalar time-varying parameter.

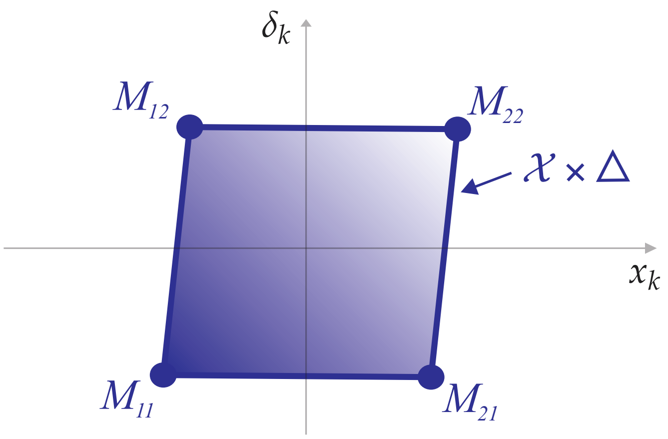


Figure 2.3 – Example of polytopic region $\mathcal{X} \times \Delta$ for an one-dimensional system with time-varying parameter.

As it is shown in the next example, the normalized vectors α_{x_k} and α_{δ_k} can be obtained using a polytopic decomposition that consists of multiplying among themselves the elements resulting from the Cartesian products

$$\Phi_x = \beta_{x_1k} \times \beta_{x_2k} \ldots \times \ldots \beta_{x_{n_x}k},$$
$$\Phi_\delta = \beta_{\delta_1k} \times \beta_{\delta_2k} \ldots \times \ldots \beta_{\delta_{n_\delta}k},$$

where $\beta_{x_jk} = \begin{bmatrix}\beta_{x_j(1)k} & \beta_{x_j(2)k}\end{bmatrix}^\top$ and $\beta_{\delta_mk} = \begin{bmatrix}\beta_{\delta_m(1)k} & \beta_{\delta_m(2)k}\end{bmatrix}^\top$, with

$$\beta_{x_j(1)k} = \frac{\overline{x}_{(j)k} - x_{(j)k}}{\overline{x}_{(j)k} - \underline{x}_{(j)k}}, \quad \beta_{x_j(2)k} = \frac{x_{(j)k} - \underline{x}_{(j)k}}{\overline{x}_{(j)k} - \underline{x}_{(j)k}}, \quad j \in \mathcal{I}_{n_x},$$

$$\beta_{\delta_m(1)k} = \frac{\overline{\delta}_{(m)k} - \delta_{(m)k}}{\overline{\delta}_{(m)k} - \underline{\delta}_{(m)k}}, \quad \beta_{\delta_m(2)k} = \frac{\delta_{(m)k} - \underline{\delta}_{(m)k}}{\overline{\delta}_{(m)k} - \underline{\delta}_{(m)k}}, \quad m \in \mathcal{I}_{n_\delta},$$

with $\overline{x}_{(j)k}$ and $\overline{\delta}_{(m)k}$ the maximum values, and $\underline{x}_{(j)k}$ and $\underline{\delta}_{(m)k}$ the minimum values, of $x_{(j)k}$ and $\delta_{(m)k}$, respectively.

Example 2.4. *Consider the following matrix and polytopic regions $\mathcal{X}$ and Δ*

$$M(x_k, \delta_k) = \begin{bmatrix} x_{(1)k} & \delta_{(1)k} \\ x_{(2)k} & 1 \end{bmatrix},$$

$$\mathcal{X} = \left\{ x_k \in \mathbb{R}^{n_x} : |x_{(1)k}| \leq 0.9 \text{ and } |x_{(2)k}| \leq 0.5 \right\},$$

and

$$\Delta = \left\{ \delta_k \in \mathbb{R}^{n_\delta} : 0 \leq \delta_{(1)k} \leq 0.8 \right\}.$$

This hypothetical matrix can be represented as in (2.12) *with*

$$M_{11} = \begin{bmatrix} -0.9 & 0 \\ -0.5 & 1 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} 0.9 & 0 \\ -0.5 & 1 \end{bmatrix}, \quad M_{31} = \begin{bmatrix} -0.9 & 0 \\ 0.5 & 1 \end{bmatrix}, \quad M_{41} = \begin{bmatrix} 0.9 & 0 \\ 0.5 & 1 \end{bmatrix},$$

$$M_{12} = \begin{bmatrix} -0.9 & 0.8 \\ -0.5 & 1 \end{bmatrix}, \; M_{22} = \begin{bmatrix} 0.9 & 0.8 \\ -0.5 & 1 \end{bmatrix}, \; M_{32} = \begin{bmatrix} -0.9 & 0.8 \\ 0.5 & 1 \end{bmatrix}, \; M_{42} = \begin{bmatrix} 0.9 & 0.8 \\ 0.5 & 1 \end{bmatrix}.$$

For $x_{(1)k} = 0.9$, $x_{(2)k} = 0.25$, and $\delta_{(1)k} = 0.64$, using the polytopic decomposition described previously, we have

$$\beta_{x_1k} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; \beta_{x_2k} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} \;\rightarrow\; \alpha_{x_k} = \begin{bmatrix} \beta_{x_1(1)k}\beta_{x_2(1)k} \\ \beta_{x_1(2)k}\beta_{x_2(1)k} \\ \beta_{x_1(1)k}\beta_{x_2(2)k} \\ \beta_{x_1(2)k}\beta_{x_2(2)k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.25 \\ 0 \\ 0.75 \end{bmatrix},$$

and

$$\alpha_{\delta_k} = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix}^\top .$$

Thus, by (2.12) *it is obtained the following result, which is the same as that obtained by applying the given matrix* $M(x_k, \delta_k)$ *directly*

$$M(x_k, \delta_k) = \sum_{i=1}^{4} \sum_{l=1}^{2} \alpha_{x_{(i)k}} \alpha_{\delta_{(l)k}} M_{il} = \begin{bmatrix} 0.9 & 0.64 \\ 0.25 & 1 \end{bmatrix} .$$

Part II

Novel stabilization conditions using parameter-dependent Lyapunov functions

3 PRELIMINARIES FOR PART II

This chapter presents relevant concepts and the main problems addressed in this part of the research. In Section 3.1, it is shown how Lyapunov theory can be applied in the development of synthesis conditions for regional stabilization, by using parameter-dependent Lyapunov functions. Afterwards, the main approaches used in the literature to describe input saturation are presented in Section 3.2. In Section 3.3, the problems addressed in this part of the investigation are summarized.

3.1 Regional stabilization and parameter-dependent Lyapunov function

As previously stated, this part of the research is concerned with the development of LMI based conditions to obtain nonlinear gain-scheduled controllers that will provide robust stabilization for nonlinear systems described in a DAR form, by using parameter-dependent Lyapunov functions.

To achieve the main purpose of this investigation, the following Lyapunov function candidate is considered:

$$V(x_k, \delta_k) = x_k^\top P^{-1}(\delta_k) x_k, \tag{3.1}$$

where

$$P(\delta_k) = \sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} P_l, \; P_l = P_l^\top > 0, \quad \alpha_{\delta_k} \in \Lambda_1. \tag{3.2}$$

Initially, two main problems are studied, which are described below.

3.1.1 Domain-of-Attraction estimation

As previously discussed, in the context of nonlinear systems, generally it is possible to guarantee only the regional stability of the system. Therefore, it is important to consider the concept of Domain-of-Attraction (DoA).

Definition 3.1 (Domain-of-Attraction (DoA))**.** *For a given autonomous system* (2.1)*, i.e. with $u_k = 0$ and $w_k = 0$, the DoA is a positively invariant region around an equilibrium point such that, for every x_0 inside it and $\delta_k \in \Delta$, the trajectory x_k asymptotically converges to the equilibrium point.*

Figure 3.1 illustrates the concept of DoA, using a 2D system as example. An important problem in control theory is to provide synthesis conditions to regional stabilization of the resulting autonomous closed-loop system while aiming to attain the largest estimated DoA.

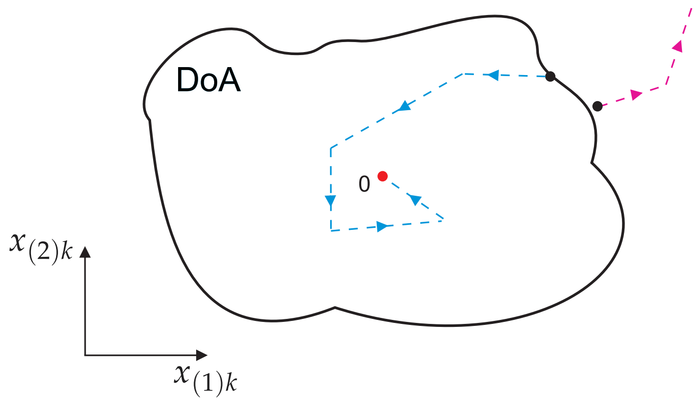


Figure 3.1 – Example of estimated DoA (black solid line) for a 2D system with two state trajectories. The trajectory in blue dashed line starts at the boundary of the DoA and converges to the origin. On the other hand, the trajectory in magenta dashed line starts outside DoA and diverges.

Finding the exact DoA is usually very challenging. Alternatively, from Lyapunov theory, it is possible to consider an estimate of the DoA using Lyapunov function level surfaces [3]. Without loss of generality, consider the state-space origin to be an equilibrium point of the autonomous ($u_k = 0$ and $w_k = 0$) system (2.1), which is inside the following normalized level set associated with the Lyapunov function candidate (3.1):

$$\mathcal{E}_{DoA} := \{x_k \in \mathbb{R}^{n_x} : V(x_k, \delta_k) \leq 1, \quad \forall \delta_k \in \Delta\}. \tag{3.3}$$

According to Jungers & Castelan [89], the level set (3.3) associated with the function (3.1) is defined as the intersection of the ellipsoids $\mathcal{E}\left(P_l^{-1}, 1\right)$, with

$$\mathcal{E}\left(P_l^{-1}, 1\right) := \left\{x_k \in \mathbb{R}^{n_x} : x_k^\top P_l^{-1} x_k \leq 1\right\}.$$

An alternative to find the largest DoA is to define an ellipsoid contained in this intersection region, i.e.,

$$\mathcal{E}\left(Q^{-1}, 1\right) \subseteq \mathcal{E}_{DoA}, \tag{3.4}$$

with $\mathcal{E}\left(Q^{-1}, 1\right) := \left\{x_k \in \mathbb{R}^{n_x} : x_k^\top Q^{-1} x_k \leq 1\right\}$, such that to maximize the volume of $\mathcal{E}\left(Q^{-1}, 1\right)$ corresponds to maximize the estimated DoA. Figures 3.2 and 3.3 depict a simple example to illustrate these ideas.

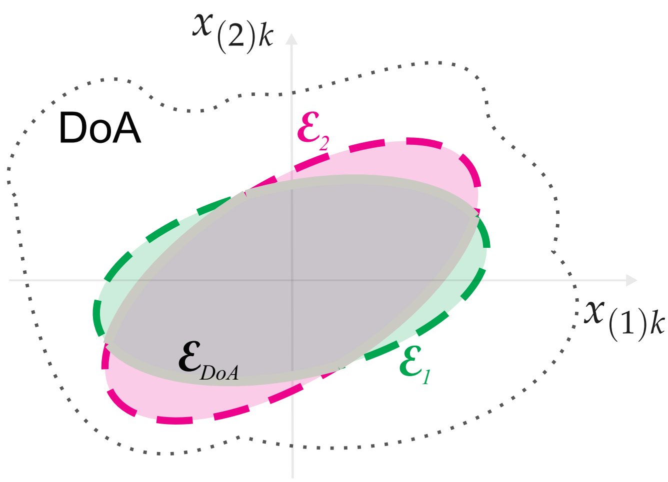


Figure 3.2 – Example of estimated DoA ($\mathcal{E}_{DoA} \subseteq$ DoA).

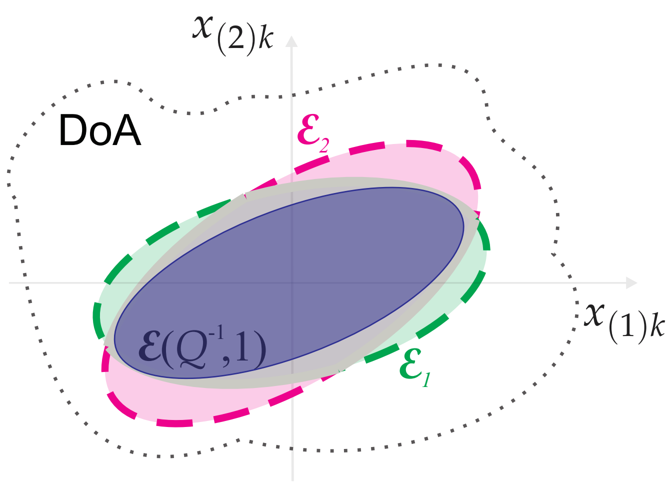


Figure 3.3 – Example of an ellipsoid in the intersection region $\mathcal{E}\left(Q^{-1},1\right) \subseteq \mathcal{E}_{DoA}$.

According to Boyd & Vandenberghe [90, Chapter 5], the largest volume of $\mathcal{E}\left(Q^{-1},1\right)$ can be obtained maximizing the objective function $log(det(Q))$, as it is a log-concave function, and the volume of the ellipsoid is proportional to $(det(Q))^{\frac{1}{2}}$. In the next chapter, we will present the conditions obtained in the proposed approaches, giving more details about this methodology.

3.1.2 ℓ_2-Performance

In Control Theory, another important problem is to provide synthesis conditions to ensure input-to-state stability and worst-case input-to-output performance by minimizing the induced ℓ_2-gain from the disturbance input to the performance output.

Definition 3.2 (ℓ_2-performance)**.** *Considering* $x_0 = 0$ *and* $w_k \in \mathcal{W}$*, an upper-bound for the* ℓ_2*-gain from the disturbance* w_k *to the output* z_k *is smaller or equal to* γ *when* $\|z\|_2 \leq \gamma \|w\|_2$*.*

For this case, consider the level set associated with the function (3.1) defined by

$$\mathcal{E}_{\mathcal{V}} := \left\{ x_k \in \mathbb{R}^{n_x} : V(x_k, \delta_k) \leq \lambda^{-1}, \quad \forall \delta_k \in \Delta \right\}, \tag{3.5}$$

where λ is the positive scalar defining the bound of $\mathcal{W}$ in (2.2).

Lemma 3.1 (adapted from Klug, Castelan & Coutinho [19])**.** *The unforced system* (2.1)*, with* $x_0 = 0$*, is locally input-to-state stable and there exists an upper bound* γ *on the* ℓ_2*-gain from* w_k *to* z_k *if*

$$\Delta V_k + \frac{1}{\gamma^2} z_k^\top z_k - w_k^\top w_k < 0 \tag{3.6}$$

holds $\forall x_k \in \mathcal{E}_{\mathcal{V}}, \forall \delta_k \in \Delta$*, and* $w_k \in \mathcal{W}$*, where* $\Delta V_k = V(x_{k+1}, \delta_{k+1}) - V(x_k, \delta_k)$*.*

Proof. If (3.6) holds for all $x_k \in \mathcal{E}_{\mathcal{V}}, \delta_k \in \Delta$ and $w_k \in \mathcal{W}$, then for $\bar{k} > 0$, since

$$\sum_{k=0}^{\bar{k}-1} \Delta V_k = \; V(x_{\bar{k}}, \delta_{\bar{k}}) - V(x_0, \delta_0),$$

the following inequality is satisfied

$$V(x_{\bar{k}}, \delta_{\bar{k}}) - V(x_0, \delta_0) + \frac{1}{\gamma^2} \sum_{k=0}^{\bar{k}-1} z_k' z_k - \sum_{k=0}^{\bar{k}-1} w_k' w_k < 0, \tag{3.7}$$

which implies that:

a) If $w_k = 0$, then $\Delta V_k < -\gamma^{-2} z_k' z_k \leq 0$, and therefore $\mathcal{E}_{\mathcal{V}}$ is a contractive positively invariant set which ensures that for $x_0 \in \mathcal{E}_{\mathcal{V}}$, $x_k \to 0$, when $k \to \infty$;

b) If $x_0 = 0$ and $w_k \in \mathcal{W}$, then $V(x_{\bar{k}}, \delta_{\bar{k}}) < \sum_{k=0}^{\bar{k}-1} \|w_k\|_2^2 \leq \lambda^{-1}, \forall \bar{k} > 0$, which guarantees that the trajectories of the system do not leave $\mathcal{E}_{\mathcal{V}}$; and $\|z\|_2 \leq \gamma \|w\|_2$, by taking $\bar{k} \to \infty$. Besides that, if $w_k = 0$, $\forall k \geq \tilde{k}$, from the previously analysis, $x_k \to 0$, when $k \to \infty$.

□

Figure 3.4 illustrates this idea. From Lemma 3.1 it is possible to obtain synthesis conditions, relating the system stabilization region to the admissible energy-bounded disturbance, in the case of zero initial conditions.

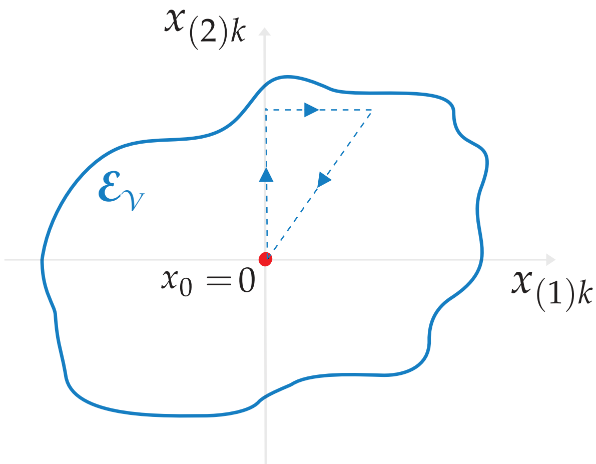


Figure 3.4 – Example of a region $\mathcal{E}_{\mathcal{V}}$ (blue solid line) for a 2D system with a state trajectory (blue dashed line), starting at the origin, changing over time inside the region $\mathcal{E}_{\mathcal{V}}$, and returning to the initial condition after the disturbance ends.

Similarly to the previous discussion, considering the Lyapunov function candidate in (3.1), one has

$$\mathcal{E}_{\mathcal{V}} = \bigcap_{l\in\{1,\ldots N_\delta\}} \mathcal{E}\left(P_l^{-1}, \lambda^{-1}\right), \tag{3.8}$$

with $\mathcal{E}\left(P_l^{-1}, \lambda^{-1}\right) := \left\{x_k \in \mathbb{R}^{n_x} : x_k^\top P_l^{-1} x_k \leq \lambda^{-1}\right\}$. In this case, the convex optimization problem for controller synthesis can be structured in a more direct way, aiming to minimize the upper-bound for the ℓ_2-gain (γ).

Remark 3.1. *Note that, in both cases, it is necessary to additionally consider that $\mathcal{E}_{\mathcal{V}}$ and $\mathcal{E}_{DoA}$ satisfy $\mathcal{E}(P_l^{-1}, \lambda^{-1}) \subset \mathcal{X}$ and $\mathcal{E}(P_l^{-1}, 1) \subset \mathcal{X}$, respectively, for $l = 1, \ldots, N_\delta$. Thus, as shown in Figure 3.5, the level sets* (3.3) *and* (3.5) *associated with the Lyapunov function* (3.1) *will be constrained in the region $\mathcal{X}$, which is defined by* (2.11).

3.2 Defining the control law and incorporating control input saturation

In addition to the problems described in Section 3.1, in this part of the research, we have included in the proposed stabilization conditions information about the control input saturation, while dealing with SF control design. Therefore, system (2.1) is rewritten as

$$\begin{aligned} x_{k+1} &= f(x_k, \delta_k) + g(x_k, \delta_k)\text{sat}(u_k) + h(x_k, \delta_k)w_k, \\ z_k &= f_z(x_k, \delta_k) + g_z(x_k, \delta_k)\text{sat}(u_k) + h_z(x_k, \delta_k)w_k, \end{aligned} \tag{3.9}$$

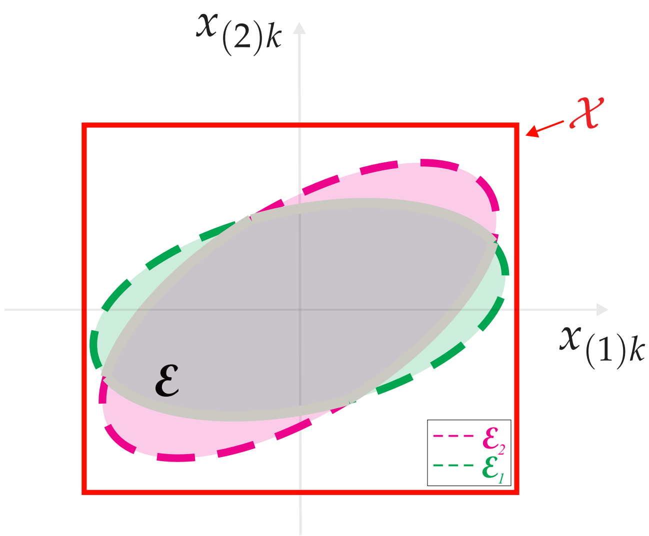


Figure 3.5 – Example of inclusion $\mathcal{E} \subset \mathcal{X}$. Depending on the context, $\mathcal{E}$ represents the regions $\mathcal{E}_{\mathcal{V}}$ or $\mathcal{E}_{DoA}$.

where the saturation function $\text{sat}(u_k) = \begin{bmatrix} \text{sat}(u_{(1)k}) & \text{sat}(u_{(2)k}) & \cdots & \text{sat}(u_{(n_u)k}) \end{bmatrix} \in \mathbb{R}^{n_u}$ corresponds to

$$\text{sat}\left(u_{(s)k}\right) := \text{sign}\left(u_{(s)k}\right) \min\left\{\left|u_{(s)k}\right|, u_{0(s)}\right\}, \quad s \in \mathcal{I}_{n_u},$$

such that $u_{0(s)}$ is the maximum absolute value of $u_{(s)k}$.

Considering Assumptions 2.1 and 2.2, we can also represent this system in the following DAR form:

$$\begin{aligned} x_{k+1} &= A_1(x_k,\delta_k)x_k + A_2(x_k,\delta_k)\pi_k + A_3(x_k,\delta_k)\,\text{sat}(u_k) + A_4(x_k,\delta_k)w_k, \\ 0 &= \Omega_1(x_k,\delta_k)x_k + \Omega_2(x_k,\delta_k)\pi_k + \Omega_3(x_k,\delta_k)\,\text{sat}(u_k) + \Omega_4(x_k,\delta_k)w_k, \\ z_k &= C_{z_1}(x_k,\delta_k)x_k + C_{z_2}(x_k,\delta_k)\pi_k + C_{z_3}(x_k,\delta_k)\,\text{sat}(u_k) + C_{z_4}(x_k,\delta_k)w_k, \end{aligned} \tag{3.10}$$

with the system matrices as described previously.

For the stabilization of the DAR model (3.10), the following nonlinear control law is proposed:

$$u_k = K(x_k,\delta_k)G^{-1}(x_k,\delta_k)x_k + R(x_k,\delta_k)N^{-1}(x_k,\delta_k)\pi_k, \tag{3.11}$$

with $K(x_k,\delta_k) \in \mathbb{R}^{n_u \times n_x}$, $G(x_k,\delta_k) \in \mathbb{R}^{n_x \times n_x}$, $R(x_k,\delta_k) \in \mathbb{R}^{n_u \times n_\pi}$ and $N(x_k,\delta_k) \in \mathbb{R}^{n_\pi \times n_\pi}$ matrices to be determined and represented in a polytopic form as in (2.12).

Remark 3.2. *It is worth mentioning that it is possible to implement* (3.11) *in practice only if* π_k *does not depend on* w_k*, since in many practical situations it is not possible to measure or estimate the disturbance input in real-time. In this case we must have* $\Omega_4(x_k,\delta_k) = 0$ *in* (2.3)*, as it is possible to infer by analyzing the block diagram presented in Figure 3.6.*

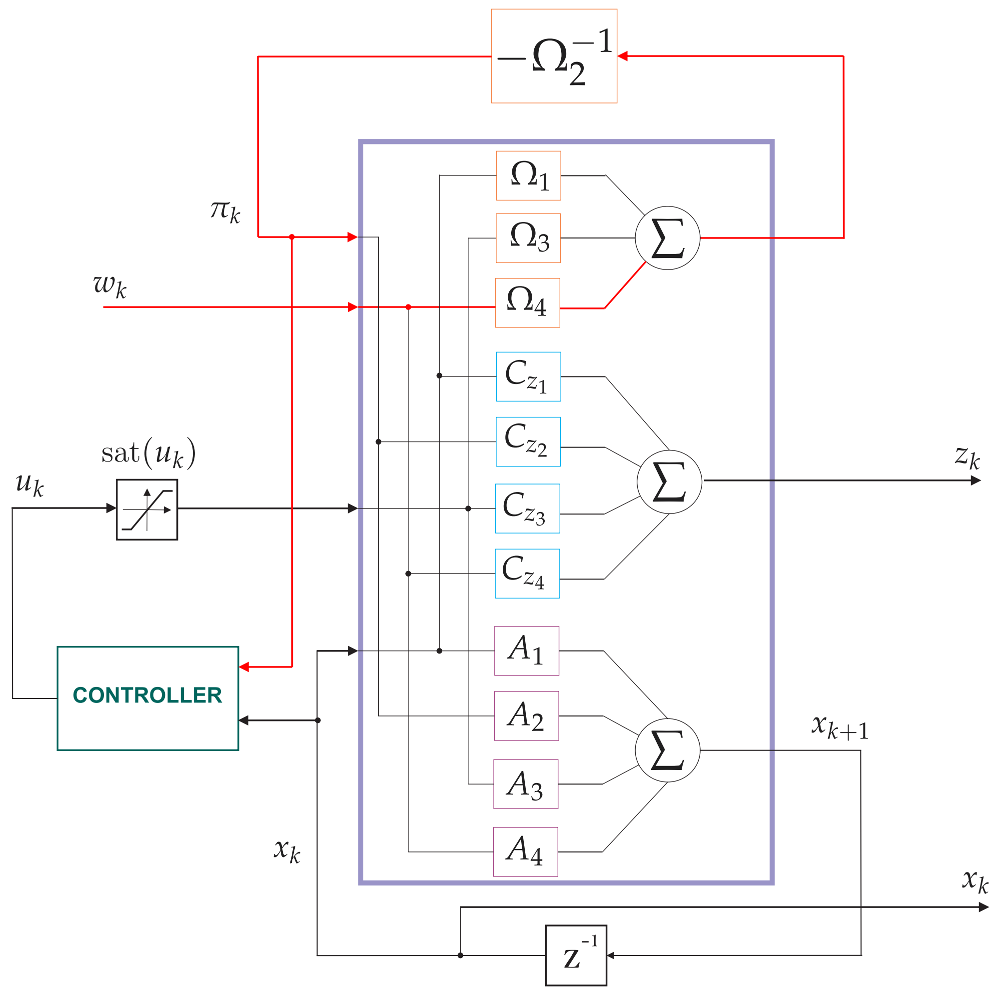


Figure 3.6 – Block diagram of a discrete-time nonlinear system in a DAR form including the controller dependent on π_k.

Remark 3.3. *Note that the proposed control law includes the particular case*

$$u_k = K(x_k, \delta_k)G^{-1}(x_k, \delta_k)x_k, \tag{3.12}$$

by considering $R(x_k, \delta_k) = 0$. *Although control strategy* (3.12) *is more conservative, it can be used instead of* (3.11) *in cases where* $\Omega_4(x_k, \delta_k) \neq 0$ *in* (3.10).

The input saturation can be described in many different forms [43]. In this investigation, we consider the polytopic representation of input saturation proposed by Hu & Lin [80].

Lemma 3.2 (Polytopic description of input saturation [80]). *Assume that the set*

$$\mathcal{D} := \left\{ D_r \in \mathbb{R}^{n_u \times n_u} : r \in \mathcal{I}_{N_u} \right\} \tag{3.13}$$

is a set of diagonal matrices D_r *whose diagonal elements are either 0 or 1, such that* $N_u = 2^{n_u}$. *Denoting* $D_r^- = I_{n_u} - D_r$, *one can see that* $D_r^- \in \mathcal{D}$. *Therefore, given any vector* $v_k \in \mathbb{R}^{n_u}$, *whose components satisfy* $|v_{(s)k}| \leq u_{0(s)}, \forall s \in \mathcal{I}_{n_u}$, *it is always possible to write*

$$\mathrm{sat}(u_k) \in co\left\{ D_r u_k + D_r{}^- v_k : r \in \mathcal{I}_{N_u} \right\}. \tag{3.14}$$

Note that, from (3.14), the saturated input vector is formed by the convex combination of u_k and v_k. The control input u_k is given by (3.11). In the same way, it is necessary to define the auxiliary vector v_k, which could be constructed in a similar form.

By considering

$$v_k = H(x_k,\delta_k)G^{-1}(x_k,\delta_k)x_k + S(x_k,\delta_k)N^{-1}(x_k,\delta_k)\pi_k,$$

with $H(x_k,\delta_k) \in \mathbb{R}^{n_u \times n_x}$ and $S(x_k,\delta_k) \in \mathbb{R}^{n_u \times n_\pi}$ to be determined, from (3.14) it is possible to represent the input saturation as

$$\begin{aligned}\text{sat}(u_k) = &\left[D(\alpha_{d_k})K(x_k,\delta_k)G^{-1}(x_k,\delta_k) + D^-(\alpha_{d_k})H(x_k,\delta_k)G^{-1}(x_k,\delta_k)\right] x_k \\ &+ \left[D(\alpha_{d_k})R(x_k,\delta_k)N^{-1}(x_k,\delta_k) + D^-(\alpha_{d_k})S(x_k,\delta_k)N^{-1}(x_k,\delta_k)\right] \pi_k,\end{aligned} \tag{3.15}$$

where

$$D(\alpha_{d_k}) = \sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r,\ D^-(\alpha_{d_k}) = \sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r^-,\ \alpha_{d_k} \in \Lambda_1. \tag{3.16}$$

Replacing (3.15) in (3.10), and considering $\Omega_4(x_k,\delta_k) = 0$, the closed-loop system is given by

$$\begin{aligned} x_{k+1} &= \mathbf{A_{1cl}} x_k + \mathbf{A_{2cl}} \pi_k + A_4(x_k,\delta_k) w_k, \\ 0 &= \mathbf{\Omega_{1cl}} x_k + \mathbf{\Omega_{2cl}} \pi_k, \\ z_k &= \mathbf{C_{z_1cl}} x_k + \mathbf{C_{z_2cl}} \pi_k + C_{z_4}(x_k,\delta_k) w_k, \end{aligned} \tag{3.17}$$

such that

$$\mathbf{M_{1cl}} = M_1(x_k,\delta_k) + M_3(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})K(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})H(x_k,\delta_k)\right] G^{-1}(x_k,\delta_k),$$
$$\mathbf{M_{2cl}} = M_2(x_k,\delta_k) + M_3(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})R(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})S(x_k,\delta_k)\right] N^{-1}(x_k,\delta_k),$$

where $\mathbf{M_{icl}}$ and M_j are placeholders for the matrices $\mathbf{A_{icl}}$, $\mathbf{\Omega_{icl}}$, $\mathbf{C_{z_icl}}$, $i \in \mathcal{I}_2$, and, A_j, Ω_j, C_{z_j}, $j \in \mathcal{I}_3$, respectively.

From this point, it is possible to develop LMI based conditions that provide the regional stabilization for system (3.9). Notice that, besides the inclusion $\mathcal{E} \subset \mathcal{X}$ discussed previously, it is necessary to ensure that $|v_{(s)k}| \leq u_{0(s)}, \forall s \in \mathcal{I}_{n_u}$, for all $\mathcal{E}_\mathcal{V}$ or $\mathcal{E}_{DoA}$, so that Lemma 3.2 can be applied. Figure 3.7 illustrates this inclusion, where $\mathcal{U} := \left\{x_k \in \mathbb{R}^{n_x} : |v_{(s)k}| \leq u_{0(s)},\ \forall \delta_k \in \Delta,\ s \in \mathcal{I}_{n_u}\right\}$.

Remark 3.4. *Due to the convex combination described in the Lemma 3.2, the use of polytopic representation generates 2^{n_u} LMIs to be considered in the control design, which could be prohibitive in some cases. Alternatively, one can use the approach based on a sector nonlinearity model presented in Tarbouriech et al. [43]. In this case, it is necessary to include in the system representation an extra auxiliary vector related to the dead-zone nonlinearity $\psi(u_k)$. As a result, the number of rows and columns of the LMIs increases proportionally with the number of control inputs.*

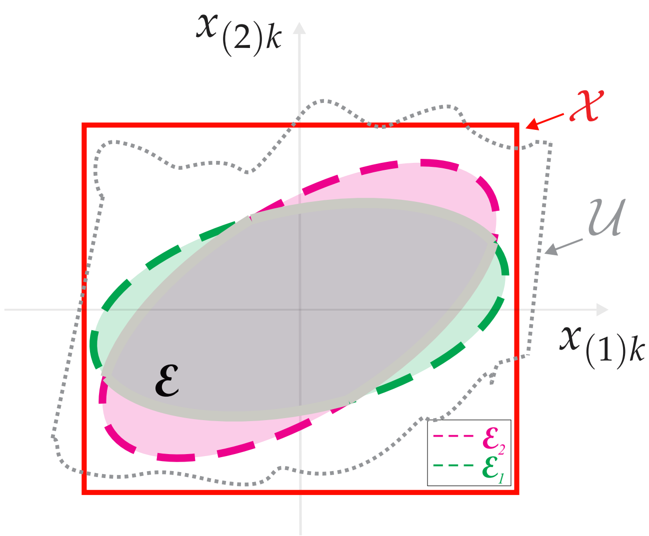


Figure 3.7 – Example of inclusions $\mathcal{E} \subset \mathcal{X}$ and $\mathcal{E} \subset \mathcal{U}$. Depending on the context, $\mathcal{E}$ represents the regions $\mathcal{E}_{\mathcal{V}}$ or $\mathcal{E}_{DoA}$.

3.3 Problem statement

This chapter has presented foundational elements that are relevant for this work. They are related to the use of parameter-dependent Lyapunov functions for the synthesis of SF nonlinear controllers based on DAR, while taking into consideration the saturation of the control input. In light of the previous discussions, the initial part of this research is particularly concerned with proposing sufficient conditions to solve the following control problems.

Problem 3.1 (Input-to-output performance)**.** *Consider system* (3.9) *in a DAR form* (3.10), *for* $w_k \in \mathcal{W}$. *Design a controller* (3.11) *that minimizes an upper-bound* γ *for the* ℓ_2*-gain from the disturbance* w_k *to the performance output* z_k, *for* $x_0 = 0$, *and also ensures that system states* x_k *remain bounded in* $\mathcal{E}_{\mathcal{V}}$ *for all* $k \geq 0$. *Moreover, if there exists* $\bar{k} > 0$ *such that* $w_k = 0, \forall k \geq \bar{k}$, *then* $x_k \to 0$, *when* $k \to \infty$.

Problem 3.2 (Domain-of-attraction estimation)**.** *Consider system* (3.9) *in a DAR form* (3.10), *for* $x_0 \in \mathcal{E}_{DoA}$ *and* $w_k = 0, \forall k \geq 0$. *Design a controller as in* (3.11) *such that* $\mathcal{E}_{DoA} \subset \mathcal{X}, \forall \delta_k \in \Delta$ *is as large as possible, and* $\mathcal{E}_{DoA}$ *is a positively invariant set for the closed-loop system comprised by* (3.10) *and* (3.11), *if* $\Omega_4(x_k, \delta_k) = 0$, *or* (3.10) *and* (3.12), *if* $\Omega_4(x_k, \delta_k) \neq 0$.

4 STABILIZATION CONDITIONS BASED ON PARAMETER-DEPENDENT LYAPUNOV FUNCTIONS

This chapter depicts the main results of this part of the research, regarding the proposed stabilization conditions based on parameter-dependent Lyapunov functions. An analysis of the computational complexity is provided and numerical examples illustrate the effectiveness of the proposed methodology.

4.1 Stabilization conditions

In this section, novel stabilization conditions for discrete-time nonlinear systems, considering all aspects stated previously are presented.

4.1.1 Control input depending on the nonlinearity vector

The first approach proposed in this research was developed using the control law in (3.11), in which the information about the nonlinearity vector π_k is taken into account to compute the control action to be applied to the system.

In *Theorem* 4.1 it is presented a sufficient condition to synthesize the proposed gain-scheduled controller (3.11) to stabilize the nonlinear system (3.9) with a guaranteed upper bound γ for the induced ℓ_2-gain from $w_k \in \mathcal{W}$ to z_k.

In the sequel, the two problems considered in this investigation are addressed in two corollaries:

a) *Corollary* 4.1 presents the optimization problem used to solve *Problem* 3.1, from the stabilization conditions presented in *Theorem* 4.1;

b) *Corollary* 4.2 proposes a modification in the conditions of *Theorem* 4.1 to solve *Problem* 3.2.

Theorem 4.1. *Consider system* (3.10), *on page 39, with* $\Omega_4(x_k, \delta_k) = 0$, *and* $w_k \in \mathcal{W}$ *for a given scalar* $\lambda > 0$, *following Assumption 2.2 on page 24. If there exist a positive scalar* μ, *a symmetric matrix* $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$, *and matrices* $G(x_k, \delta_k) \in \mathbb{R}^{n_x \times n_x}$, $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$, $H(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$, $R(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_\pi}$, $S(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_\pi}$ *and* $N(x_k, \delta_k) \in \mathbb{R}^{n_\pi \times n_\pi}$, *satisfying the following inequalities,* $\forall \delta_k \in \Delta$ *and* $\forall x_k \in \mathcal{X}$*:*

$$\Psi(x_k, \delta_k, \delta_{k+1}, \alpha_{d_k}) = \begin{bmatrix} \mathcal{A}_{11}^1 & \star & \star & \star & \star \\ \mathcal{A}_{21}^1 & -P(\delta_{k+1}) & \star & \star & \star \\ \mathcal{A}_{31}^1 & \mathcal{A}_{32}^1 & \mathcal{A}_{33}^1 & \star & \star \\ 0 & A_4^\top(x_k, \delta_k) & 0 & -I_{n_w} & \star \\ \mathcal{A}_{51}^1 & 0 & \mathcal{A}_{53}^1 & C_{z_4}(x_k, \delta_k) & -\mu I_{n_z} \end{bmatrix} < 0, \tag{4.1}$$

$$\text{He}\{\Omega_2(x_k,\delta_k)N(x_k,\delta_k)\} < 0, \tag{4.2}$$

$$\begin{bmatrix} \lambda u_{0(s)}^2 & \star & \star \\ H_{(s)}^\top(x_k,\delta_k) & \text{He}\{G(x_k,\delta_k)\} - P(\delta_k) & \star \\ S_{(s)}^\top(x_k,\delta_k) & \mathcal{B}_{32}^1 & \mathcal{B}_{33}^1 \end{bmatrix} \geq 0, \quad s \in \mathcal{I}_{n_u}, \tag{4.3}$$

and

$$\begin{bmatrix} \lambda & \star \\ G^\top(x_k,\delta_k)a_v & \text{He}\{G(x_k,\delta_k)\} - P(\delta_k) \end{bmatrix} \geq 0, \quad v \in \mathcal{I}_{n_e}, \tag{4.4}$$

where

$$\begin{aligned}
\mathcal{A}_{11}^1 &= -\text{He}\{G(x_k,\delta_k)\} + P(\delta_k),\\
\mathcal{A}_{21}^1 &= A_1(x_k,\delta_k)G(x_k,\delta_k) + A_3(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})K(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})H(x_k,\delta_k)\right],\\
\mathcal{A}_{31}^1 &= \Omega_1(x_k,\delta_k)G(x_k,\delta_k) + \Omega_3(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})K(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})H(x_k,\delta_k)\right],\\
\mathcal{A}_{51}^1 &= C_{z_1}(x_k,\delta_k)G(x_k,\delta_k) + C_{z_3}(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})K(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})H(x_k,\delta_k)\right],\\
\mathcal{A}_{32}^1 &= N^\top(x_k,\delta_k)A_2^\top(x_k,\delta_k) + \left[R^\top(x_k,\delta_k)D(\alpha_{d,k}) + S^\top(x_k,\delta_k)D^-(\alpha_{d,k})\right]A_3^\top(x_k,\delta_k),\\
\mathcal{A}_{33}^1 &= \text{He}\left\{\Omega_2(x_k,\delta_k)N(x_k,\delta_k) + \Omega_3(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})R(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})S(x_k,\delta_k)\right]\right\},\\
\mathcal{A}_{53}^1 &= C_{z_2}(x_k,\delta_k)N(x_k,\delta_k) + C_{z_3}(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})R(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})S(x_k,\delta_k)\right],\\
\mathcal{B}_{32}^1 &= -\Omega_1(x_k,\delta_k)G(x_k,\delta_k) - \Omega_3(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})K(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})H(x_k,\delta_k)\right],\\
\mathcal{B}_{33}^1 &= -He\left\{\Omega_2(x_k,\delta_k)N(x_k,\delta_k) + \Omega_3(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})R(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})S(x_k,\delta_k)\right]\right\},
\end{aligned}$$

then there exist a parameter-dependent Lyapunov function (3.1) *and a controller* (3.11) *such that, for zero initial conditions* ($x_0 = 0$), x_k *remains bounded in* $\mathcal{E}_\mathcal{V}$ *and* $\|z\|_2 \leq \gamma\|w\|_2, \forall w_k \in \mathcal{W}$, *with* $\gamma = \sqrt{\mu}$. *Moreover, if there exists* $\bar{k} > 0$ *such that* $w_k = 0, \forall k \geq \bar{k}$, *then* $x_k \to 0, k \to \infty$.

Proof. By using the property that

$$\begin{aligned}
&[G(x_k,\delta_k) - P(\delta_k)]^\top P^{-1}(\delta_k)[G(x_k,\delta_k) - P(\delta_k)] \geq 0 \Leftrightarrow\\
&G^\top(x_k,\delta_k)P^{-1}(\delta_k)G(x_k,\delta_k) \geq \text{He}\{G(x_k,\delta_k)\} - P(\delta_k),
\end{aligned} \tag{4.5}$$

if inequality (4.1) holds, then it can be rewritten with $\mathcal{A}_{11}^1 = -G^\top(x_k,\delta_k)P^{-1}(\delta_k)G(x_k,\delta_k)$.

From the feasibility of $\mathcal{A}_{11}^1 < 0$ in (4.1) and inequality (4.2), and since $\Omega_2(x_k,\delta_k)$ is a square full-rank matrix, matrices $G(x_k,\delta_k)$ and $N(x_k,\delta_k)$ must be invertible. Thus, one can apply a congruence transformation pre- and post-multiplying inequality (4.1) by diag$\{G^{-\top}(x_k,\delta_k), P^{-1}(\delta_{k+1}), N^{-\top}(x_k,\delta_k), I_{n_w}, I_{n_z}\}$ and its transpose, respectively, to obtain

$$\begin{bmatrix} \mathcal{A}_{11}^2 & \star & \star & \star & \star \\ \mathcal{A}_{21}^2 & -P^{-1}(\delta_{k+1}) & \star & \star & \star \\ \mathcal{A}_{31}^2 & \mathcal{A}_{32}^2 & \mathcal{A}_{33}^2 & \star & \star \\ 0 & \mathcal{A}_{42}^2 & 0 & -I_{n_w} & \star \\ \mathcal{A}_{51}^2 & 0 & \mathcal{A}_{53}^2 & C_{z_4}(x_k,\delta_k) & -\mu I_{n_z} \end{bmatrix} < 0,$$

$$\begin{aligned} \mathcal{A}_{11}^2 &= -P^{-1}(\delta_k), & \mathcal{A}_{33}^2 &= \text{He}\left\{N^{-\top}(x_k,\delta_k)\Omega_{\mathbf{2cl}}\right\}, \\ \mathcal{A}_{21}^2 &= P^{-1}(\delta_{k+1})\mathbf{A_{1cl}}, & \mathcal{A}_{42}^2 &= A_4^\top(x_k,\delta_k)P^{-1}(\delta_{k+1}), \\ \mathcal{A}_{31}^2 &= N^{-\top}(x_k,\delta_k)\Omega_{\mathbf{1cl}}, & \mathcal{A}_{51}^2 &= \mathbf{C_{z_1cl}}, \\ \mathcal{A}_{32}^2 &= \mathbf{A_{2cl}^\top}P^{-1}(\delta_{k+1}), & \mathcal{A}_{53}^2 &= \mathbf{C_{z_2cl}}. \end{aligned}$$

Applying the Schur complement [91] and choosing $\mu = \gamma^2$, the above inequality can be recast as:

$$\Xi_1 + L\Xi_2 + \Xi_2^\top L^\top + \gamma^{-2}\Xi_3^\top\Xi_3 < 0, \tag{4.6}$$

where

$$\Xi_1 = \text{diag}\left\{-P^{-1}(\delta_k), P^{-1}(\delta_{k+1}), 0_{n_\pi}, -I_{n_w}\right\}, \qquad L = \begin{bmatrix} 0 & P^{-1}(\delta_{k+1}) & 0 & 0 \\ 0 & 0 & N^{-1}(x_k,\delta_k) & 0 \end{bmatrix}^\top,$$

$$\Xi_2 = \begin{bmatrix} \mathbf{A_{1cl}} & -I_{n_x} & \mathbf{A_{2cl}} & A_4(x_k,\delta_k) \\ \Omega_{\mathbf{1cl}} & 0 & \Omega_{\mathbf{2cl}} & 0 \end{bmatrix}, \qquad \Xi_3 = \begin{bmatrix} \mathbf{C_{z_1cl}} & 0 & \mathbf{C_{z_2cl}} & C_{z_4}(x_k,\delta_k) \end{bmatrix}.$$

Pre- and post-multiplying (4.6) by $\begin{bmatrix} x_k^\top & x_{k+1}^\top & \pi_k^\top & w_k^\top \end{bmatrix}$ and its transpose results in (3.6), in Lemma 3.1 (page 37). This proves that if the condition (4.1) is feasible, then $V(x_k,\delta_k)$ is a Lyapunov function and the controller (3.11) ensures that for zero initial conditions, the origin of the closed-loop system is locally input-to-output stable with an upper bound γ on the ℓ_2-gain from w_k to z_k, $\forall x_k \in \mathcal{X}, \forall \delta_k \in \Delta$ and $w_k \in \mathcal{W}$.

Using the property (4.5) and multiplying (4.3) by diag$\{1, G^{-\top}(x_k,\delta_k), N^{-\top}(x_k,\delta_k)\}$ on the left and its transpose on the right, followed by the Schur complement, we obtain

$$\Upsilon^\top\sigma\Upsilon + \begin{bmatrix} -P^{-1}(\delta_k) & \Omega_{\mathbf{1cl}}^\top N^{-1}(x_k,\delta_k) \\ \star & \text{He}\left\{N^{-\top}(x_k,\delta_k)\Omega_{\mathbf{2cl}}\right\} \end{bmatrix} \le 0,$$

with $\sigma = 1/\lambda u_{0(s)}^2$ and $\Upsilon = \begin{bmatrix} H_{(s)}(x_k,\delta_k)G^{-1}(x_k,\delta_k) & S_{(s)}(x_k,\delta_k)N^{-1}(x_k,\delta_k) \end{bmatrix}$.

Pre- and post-multiplying the above inequality, respectively, by $\begin{bmatrix} x_k^\top & \pi_k^\top \end{bmatrix}$ and its transpose lead to:

$$v_{(s)k}^{\top}(\lambda u_{0(s)}^{2})^{-1}v_{(s)k} - x_k^{\top}P^{-1}(\delta_k)x_k \leq 0.$$

Considering the S-procedure [58] we have $v_{(s)k}^{\top}v_{(s)k} \leq u_{0(s)}^{2}$, $\forall(x_k, \delta_k)$: $x_k^{\top}P^{-1}(\delta_k)x_k \leq \lambda^{-1}$, or $|v_{(s)k}| \leq u_{0(s)}$, $\forall x_k \in \mathcal{E}_{\mathcal{V}}$. Thus, $\mathcal{E}_{\mathcal{V}} \subset \mathcal{U}$ and Lemma 3.2 (page 40) is satisfied.

Using again the property (4.5), multiplying (4.4) by diag$\{1, G^{-\top}(x_k, \delta_k)\}$ on the left and its transpose on the right, and applying the Schur complement we have

$$a_v\lambda^{-1}a_v^{\top} - P^{-1}(\delta_k) \leq 0.$$

Then, multiplying the last inequality by $x_k^{\top}$ on the left and x_k on the right and considering the S-procedure leads to $x_k^{\top}a_v a_v^{\top}x_k \leq 1$, $\forall x_k : x_k^{\top}P^{-1}(\delta_k)x_k \leq \lambda^{-1}$. Thus, $|a_v^{\top}x_k| \leq 1$, $\forall v \in \mathcal{I}_{n_e}$, $\forall x_k \in \mathcal{E}_{\mathcal{V}}$. This proves the inclusion $\mathcal{E}_{\mathcal{V}} \subset \mathcal{X}$, which concludes the proof. □

Notice that the proposed control law involves the vector of nonlinearities π_k which can also be a function of the control input u_k, in the case where $\Omega_3(x_k, \delta_k) \neq 0$.

From the closed-loop system model in (3.17), one has that (dependency with (x_k, δ_k) was dropped for clarity purposes)

$$\pi_k = -\left[\Omega_2 + \Omega_3\left(DR + D^{-}S\right)N^{-1}\right]^{-1}\left[\Omega_1 + \Omega_3\left(DK + D^{-}H\right)G^{-1}\right]x_k. \tag{4.7}$$

Replacing (4.7) in the expression (3.11) for the control law results in

$$u_k = KG^{-1}x_k - RN^{-1}\left[\Omega_2 + \Omega_3\left(DR + D^{-}S\right)N^{-1}\right]^{-1}\left[\Omega_1 + \Omega_3\left(DK + D^{-}H\right)G^{-1}\right]x_k,$$

Thus, it is possible to compute u_k, if G, N, and $\Omega_2 + \Omega_3\left(DR + D^{-}S\right)N^{-1}$ are nonsingular.

It was pointed out that for the feasibility of conditions (4.1) and (4.2), matrices G and N must be invertible.

Moreover, from $\mathcal{A}_{33}^{1} < 0$ in (4.1), one has that

$$\mathrm{He}\left\{\Omega_2 N + \Omega_3\left(DR + D^{-}S\right)\right\} < 0.$$

So, one can apply a congruence transformation pre- and post-multiplying this inequality by N^{-T} and N^{-1}, respectively, to obtain

$$\mathrm{He}\left\{N^{-T}\left[\Omega_2 + \Omega_3\left(DR + D^{-}S\right)N^{-1}\right]\right\} < 0.$$

Since N is full rank, matrix $\Omega_2 + \Omega_3\left(DR + D^{-}S\right)N^{-1}$ must also be nonsingular so that the above condition is satisfied.

In this sense, if conditions in Theorem 4.1 are ensured, then it is possible to write u_k as a function of the states x_k and time-varying parameters δ_k only, even for the case where $\Omega_3(x_k, \delta_k) \neq 0$.

Remark 4.1. *In this investigation, the DAR* (3.10) *is considered to exhibit arbitrary variation rates for the time-varying parameters, as long as both* $\delta_k, \delta_{k+1} \in \Delta$. *Thus, by defining* $\alpha_{\delta_{(l)k+1}} = \alpha_{\Delta_{(n)k}}$ *one can write that*

$$P(\delta_{k+1}) = \sum_{n=1}^{N_\delta} \alpha_{\Delta_{(n)k}} P_n, \; P_n = P_n^\top > 0, \quad \alpha_{\Delta_k} \in \Lambda_1. \tag{4.8}$$

Alternatively, to obtain less conservative results, one can assume that the rates of variation of the parameters are bounded, which increases the complexity of developments (see for instance [32]).

Notice that, the use of a parameter-dependent Lyapunov function candidate can bring less conservative results, and the proposed approach includes the quadratic stabilizability conditions when $P(\delta_k) = P(\delta_{k+1}) = P$ is a special case. On the other hand, to further reduce the design conservativeness, it is also possible to consider the Lyapunov matrix $P(x_k, \delta_k)$ with affine functions on (x_k, δ_k). However, in this case, the polytope related to (x_{k+1}) must be considered, increasing the complexity and computational burden.

The following *Corollaries* 4.1 and 4.2 are used to solve *Problems* 3.1 and 3.2, stated at the end of *Chapter* 3. In the next Corollary, *Theorem* 4.1 can be used to solve *Problem* 3.1 as described previously.

Corollary 4.1. *For a given disturbance energy level* λ^{-1}*, the upper-bound* γ *for the* ℓ_2*-gain from* w_k *to* z_k *can be minimized solving the following optimization problem for all* $\delta_k \in \Delta$ *and* $x_k \in \mathcal{X}$*:*

$$\min_{\mu,P,G,K,H,R,S,N} \mu \quad \textit{subject to } (4.1)-(4.4). \tag{4.9}$$

In the next Corollary, *Theorem* 4.1 can also be adapted to solve *Problem* 3.2.

Corollary 4.2. *Consider system* (3.10)*, with* $w_k = 0$ *and disregard the influence of the disturbance input by removing the fourth and fifth rows and columns of* $\Psi(x_k, \delta_k, \delta_{k+1}, \alpha_{d_k})$ *(in Theorem 4.1), and consider* $\lambda = 1$ *in* (4.3) *and* (4.4)*.*

If there exist symmetric matrices $Q \in \mathbb{R}^{n_x \times n_x}$, $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$*, and matrices* $G(x_k, \delta_k) \in \mathbb{R}^{n_x \times n_x}$, $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$, $H(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$, $R(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_\pi}$, $S(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_\pi}$ *and* $N(x_k, \delta_k) \in \mathbb{R}^{n_\pi \times n_\pi}$*, satisfying the following optimization problem for all* $\delta_k \in \Delta$ *and* $x_k \in \mathcal{X}$*:*

$$\max_{Q,P,G,K,H,R,S,N} \log(\det(Q)) \quad \textit{subject to } (4.1)-(4.4), \tag{4.10}$$

with

$$Q - P_l < 0, \quad l \in \mathcal{I}_{N_\delta}, \tag{4.11}$$

then there exist a Lyapunov function (3.1) *and a controller* (3.11) *such that,* $\forall x_k(0)$ *inside* $\mathcal{E}_{DoA}$ *and* $\delta_k \in \Delta$*, the trajectory of* x_k *converge to the origin when* $k \to \infty$ *and* $\mathcal{E}_{DoA}$ *is an estimate of the DoA.*

Proof. The inequality (4.11) ensures that $\mathcal{E}\left(Q^{-1}, 1\right) \subseteq \mathcal{E}_{DoA}$ and the rest of the proof follows in a straightforward way as in the proof of *Theorem* 4.1. □

4.1.2 Control input not depending on the nonlinearity vector

The stabilization conditions for the synthesis of controller (3.12), on page 40, which do not incorporate the nonlinearity vector π_k, must be obtained in a similar way as in *Theorem* 4.1. In this case the auxiliary vector v_k used in the polytopic description of input saturation, presented in *Lemma* 3.2 (page 40), should be considered as

$$v_k = H(x_k, \delta_k) G^{-1}(x_k, \delta_k) x_k. \tag{4.12}$$

Theorem 4.2, in the sequel, presents a sufficient stabilization condition to stabilize the nonlinear system (3.9) with a guaranteed upper bound γ for the induced ℓ_2-gain from $w_k \in W$ to z_k, by relying on the use of controller (3.12), repeated here for convenience:

$$u_k = K(x_k, \delta_k) G^{-1}(x_k, \delta_k) x_k.$$

Theorem 4.2. *Consider system* (3.10)*, on page 39, with* $w_k \in \mathcal{W}$ *for a given scalar* $\lambda > 0$*, following Assumption 2.2 on page 24. If there exist a positive scalar* μ*, a symmetric matrix* $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$*, and matrices* $N(x_k, \delta_k) \in \mathbb{R}^{n_\pi \times n_\pi}$*,* $G(x_k, \delta_k) \in \mathbb{R}^{n_x \times n_x}$*,* $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$*,* $H(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$ *satisfying the following inequalities for all* $\delta_k \in \Delta$ *and* $x_k \in \mathcal{X}$*:*

$$\Gamma(x_k, \delta_k, \delta_{k+1}, \alpha_{d_k}) = \begin{bmatrix} \mathcal{A}_{11} & \star & \star & \star & \star \\ \mathcal{A}_{21} & -P(\delta_{k+1}) & \star & \star & \star \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} & \star & \star \\ 0 & A_4^\top(x_k, \delta_k) & \Omega_4^\top(x_k, \delta_k) & -I & \star \\ \mathcal{A}_{51} & 0 & \mathcal{A}_{53} & C_{z_4}(x_k, \delta_k) & -\mu I \end{bmatrix} < 0, \tag{4.13}$$

$$\begin{bmatrix} \lambda u_{0(s)}^2 & \star \\ H_{(s)}^\top(x_k, \delta_k) & \mathrm{He}\left\{G(x_k, \delta_k)\right\} - P(\delta_k) \end{bmatrix} \geq 0, \quad s \in \mathcal{I}_{n_u}, \tag{4.14}$$

and

$$\begin{bmatrix} \lambda & \star \\ G^\top(x_k,\delta_k)a_v & \text{He}\{G(x_k,\delta_k)\} - P(\delta_k) \end{bmatrix} \geq 0, \quad v \in \mathcal{I}_{n_e}, \tag{4.15}$$

where

$$\begin{aligned}
\mathcal{A}_{11} &= -\text{He}\{G(x_k,\delta_k)\} + P(\delta_k),\\
\mathcal{A}_{21} &= A_1(x_k,\delta_k)G(x_k,\delta_k) + A_3(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})K(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})H(x_k,\delta_k)\right],\\
\mathcal{A}_{31} &= \Omega_1(x_k,\delta_k)G(x_k,\delta_k) + \Omega_3(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})K(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})H(x_k,\delta_k)\right],\\
\mathcal{A}_{51} &= C_{z_1}(x_k,\delta_k)G(x_k,\delta_k) + C_{z_3}(x_k,\delta_k)\left[D(\alpha_{d_{(r)k}})K(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})H(x_k,\delta_k)\right],\\
\mathcal{A}_{32} &= N^\top(x_k,\delta_k)A_2^\top(x_k,\delta_k),\\
\mathcal{A}_{33} &= \text{He}\{\Omega_2(x_k,\delta_k)N(x_k,\delta_k)\},\\
\mathcal{A}_{53} &= C_{z_2}(x_k,\delta_k)N(x_k,\delta_k)
\end{aligned}$$

then there exist a parameter-dependent Lyapunov function (3.1) *and a controller* (3.12) *such that, for zero initial conditions* ($x_0 = 0$), x_k *remains bounded in* $\mathcal{E}_\mathcal{V}$ *and* $\|z\|_2 \leq \gamma\|w\|_2, \forall w_k \in \mathcal{W}$, *with* $\gamma = \sqrt{\mu}$. *Moreover, if there exists* $\bar{k} > 0$ *such that* $w_k = 0, \forall k \geq \bar{k}$, *then* $x_k \to 0$, $k \to \infty$.

Proof. The proof of *Theorem* 4.2 follows in a straightforward way as in the proof of *Theorem* 4.1, considering $R(x_k,\delta_k) = 0$, $S(x_k,\delta_k) = 0$, and $\Omega_4(x_k,\delta_k) \neq 0$. It is available in Reis *et al.* [86], and omitted here for brevity. □

The following *Corollaries* 4.3 and 4.4 are introduced to solve *Problems* 3.1 and 3.2, stated at the end of *Chapter* 3, considering the control input (3.12). In the next Corollary, *Theorem* 4.1 can be used to solve *Problem* 3.1 as described previously.

Corollary 4.3. *For a given disturbance energy level* λ^{-1}, *the upper-bound* γ *for the* ℓ_2*-gain from* w_k *to* z_k *can be minimized solving the following optimization problem for all* $\delta_k \in \Delta$ *and* $x_k \in \mathcal{X}$:

$$\min_{\mu,P,G,K,H,N} \mu \quad \textit{subject to}\ (4.13) - (4.15). \tag{4.16}$$

Similarly, the proposed approach can also be adapted to solve *Problem* 3.2, using the controller (3.12), as it is stated in the next Corollary.

Corollary 4.4. *Consider system* (3.10), *with* $w_k = 0$ *and disregard the influence of the disturbance input by removing the fourth and fifth rows and columns of* $\Gamma(x_k,\delta_k,\delta_{k+1},\alpha_{d_k})$ *(in Theorem 4.2) and consider* $\lambda = 1$ *in* (4.14) *and* (4.15).

If there exist symmetric matrices $Q \in \mathbb{R}^{n_x \times n_x}, P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$, *and matrices* $G(x_k,\delta_k) \in \mathbb{R}^{n_x \times n_x}, K(x_k,\delta_k) \in \mathbb{R}^{n_u \times n_x}$, $H(x_k,\delta_k) \in \mathbb{R}^{n_u \times n_x}$, *and* $N(x_k,\delta_k) \in \mathbb{R}^{n_\pi \times n_\pi}$, *satisfying the following optimization problem for all* $\delta_k \in \Delta$ *and* $x_k \in \mathcal{X}$:

$$\max_{Q,P,G,K,H,N} \log(\det(Q)) \quad \textit{subject to } (4.13) - (4.15), \tag{4.17}$$

with

$$Q - P_l < 0, \quad l \in \mathcal{I}_{N_\delta}, \tag{4.18}$$

then there exist a Lyapunov function (3.1) *and a controller* (3.12) *such that,* $\forall x_k(0)$ *inside* $\mathcal{E}_{DoA}$ *and* $\delta_k \in \Delta$*, the trajectory of* x_k *converge to the origin when* $k \to \infty$ *and* $\mathcal{E}_{DoA}$ *is an estimate of the DoA.*

The proofs of *Corollaries* 4.3 and 4.4 follow the proofs of *Theorems* 4.2 and *Corollary* 4.2, as described previously.

4.2 LMI relaxations

Note that the proposed stabilization conditions are supposed to be polynomially dependent on $(x_k, \delta_k, \delta_{k+1}, \alpha_{d_k})$. Thus, the problem is of infinite dimension such that its feasibility is not computationally tractable. However, since it is possible to use the multi-simplex framework based on (2.12), on page 30, (3.2), on page 34, (3.16), on page 41, and (4.8), on page 47, a finite set of LMIs in terms of the vertices of the polytopes $\mathcal{X}$, Δ, and $\mathcal{D}$ can be obtained, as follows.

Lemma 4.1 (LMI relaxation (adapted from Wang, Tanaka & Griffin [92])). *Suppose* Ψ_{ijlm}^{nr}*, with* $i, j \in \mathcal{I}_{N_x}, l, m, n \in \mathcal{I}_{N_\delta}$*, and* $r \in \mathcal{I}_{N_u}$*, are matrices of appropriate dimensions, such that*

$$\Psi(x_k, \delta_k, \delta_{k+1}, \alpha_{d_k}) = \sum_{r=1}^{N_u} \sum_{n=1}^{N_\delta} \sum_{i=1}^{N_x} \sum_{j=1}^{N_x} \sum_{l=1}^{N_\delta} \sum_{m=1}^{N_\delta} \alpha_k \Psi_{ijlm}^{nr} < 0. \tag{4.19}$$

with $\alpha_k = \alpha_{d_{(r)k}} \alpha_{\Delta_{(n)k}} \alpha_{x_{(i)k}}, \alpha_{x_{(j)k}} \alpha_{\delta_{(l)k}} \alpha_{\delta_{(m)k}}$.

If the following LMIs hold for all $i, j \in \mathcal{I}_{N_x}$*,* $l, m, n \in \mathcal{I}_{N_\delta}$ *and* $r \in \mathcal{I}_{N_u}$

$$\begin{aligned} &\Psi_{iill}^{nr} < 0, \; i = j, \; l = m, \\ &\Psi_{ijll}^{nr} + \Psi_{jill}^{nr} < 0, \; i < j, \; l = m, \\ &\Psi_{iilm}^{nr} + \Psi_{iiml}^{nr} < 0, \; i = j, \; l < m, \\ &\Psi_{ijlm}^{nr} + \Psi_{ijml}^{nr} + \Psi_{jilm}^{nr} + \Psi_{jiml}^{nr} < 0, \; i < j, \; l < m, \end{aligned} \tag{4.20}$$

then inequality (4.19) *is satisfied.*

Proof. The hypothetical matrix in (4.19) can be rewritten as

$$\Psi(x_k, \delta_k, \delta_{k+1}, \alpha_{d_k}) = \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha^2_{x_{(i)k}}\alpha^2_{\delta_{(l)k}}\Psi^{nr}_{iill}$$
$$+ \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x-1}\sum_{j=i+1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha_{x_{(i)k}}\alpha_{x_{(j)k}}\alpha^2_{\delta_{(l)k}}\left(\Psi^{nr}_{ijll}+\Psi^{nr}_{jill}\right)$$
$$+ \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta-1}\sum_{m=l+1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha^2_{x_{(i)k}}\alpha_{\delta_{(l)k}}\alpha_{\delta_{(m)k}}\left(\Psi^{nr}_{iilm}+\Psi^{nr}_{iiml}\right)$$
$$+ \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x-1}\sum_{j=i+1}^{N_x}\sum_{l=1}^{N_\delta-1}\sum_{m=l+1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha_{x_{(i)k}}\alpha_{x_{(j)k}}\alpha_{\delta_{(l)k}}\alpha_{\delta_{(m)k}}\Big(\Psi^{nr}_{ijlm}+\Psi^{nr}_{ijml}$$
$$+\Psi^{nr}_{jilm}+\Psi^{nr}_{jiml}\Big).$$

Since $\alpha_{\mathsf{p}_{(\mathsf{v})k}} \geq 0$, if inequalities (4.20) are satisfied, then condition (4.19) is guaranteed[1]. □

Appendix A depicts a guided proof of Lemma 4.1, exemplifying how the methodology presented to obtain LMI conditions can be used to computationally handle the proposed conditions.

4.3 Analysis of the computational complexity

The computational cost of interior-point algorithms used to solve LMI optimization problems can be estimated in terms of the number of scalar variables (S_v) and the number of LMI rows (L_r) [93]. According to [93], these methods have a polynomial-time complexity, which is proportional to ($L_r S_v^3$). Therefore, several recent works in the literature use these parameters to analyze the computational complexity of LMI conditions [94, 46].

Tables 4.1 and 4.2 show the computational complexity of the proposed approaches in terms of the number of scalar variables and the number of LMI rows, which are related to the dimension of the state, time-varying parameter, control input, performance output, and nonlinearity vector.

Remark 4.2. *Although the conditions seem to be involved in terms of the number of scalar variables and LMI rows, it is worth emphasizing that the proposed Theorems and Corollaries are associated with non-iterative algorithms, leading to less computational effort in solving the problems addressed. The computational burden can also be decreased if we consider the variable matrices of the optimization problems not dependent on* (x_k, δ_k). *However, it can provide conservative results. Besides that, it is worth emphasizing that the controller project is off-line, and the computational cost is not a problem in its practical implementation.*

[1] The term $\alpha_{\mathsf{p}_{(\mathsf{v})k}}$ is a generic remarktion where p represents an index $(x, \delta, \Delta,$ or, $d)$ used to distinguish different polytopes.

Table 4.1 – Computational complexity of LMI conditions to solve *Problem* 3.1

	Corollary 4.1
S_v	$\left[n_x\left(\frac{n_x+1}{2}\right)+\left(n_\pi^2+n_x^2+2n_un_x+2n_un_\pi\right)N_x\right]N_\delta+1$
L_r	$\left\{\left[(2n_x+n_\pi+n_w+n_z)N_\delta+n_\pi+(1+n_x+n_\pi)n_u\right]\left(\frac{(N_x+1)(N_\delta+1)N_u}{4}\right)+(1+n_x)n_e\right\}N_\delta N_x$
	Corollary 4.3
S_v	$\left[n_x\left(\frac{n_x+1}{2}\right)+\left(n_\pi^2+n_x^2+2n_un_x\right)N_x\right]N_\delta+1$
L_r	$\left[(2n_x+n_\pi+n_w+n_z)\left(\frac{(N_x+1)(N_\delta+1)N_uN_\delta}{4}\right)+(1+n_x)(n_u+n_e)\right]N_\delta N_x$

Table 4.2 – Computational complexity of LMI conditions to solve *Problem* 3.2

	Corollary 4.2
S_v	$n_x\left(\frac{n_x+1}{2}\right)(N_\delta+1)+\left(n_\pi^2+n_x^2+2n_un_x+2n_un_\pi\right)N_xN_\delta$
L_r	$\left\{\left[(2n_x+n_\pi)N_\delta+n_\pi+(1+n_x+n_\pi)n_u\right]\left(\frac{(N_x+1)(N_\delta+1)N_u}{4}\right)+(1+n_x)n_e\right\}N_\delta N_x+n_xN_\delta$
	Corollary 4.4
S_v	$n_x\left(\frac{n_x+1}{2}\right)(N_\delta+1)+\left(n_\pi^2+n_x^2+2n_un_x\right)N_xN_\delta$
L_r	$\left[(2n_x+n_\pi)\left(\frac{(N_x+1)(N_\delta+1)N_uN_\delta}{4}\right)+(1+n_x)(n_u+n_e)\right]N_\delta N_x+n_xN_\delta$

4.4 Extension to robust control design

The proposed approaches so far have considered, in the system model, the presence of time-varying parameters (δ_k), which are supposed to be exactly known and available for measurement, and this information is used in the gain-scheduled control strategy, aiming to achieve less conservative results.

In real-world applications, a more realistic situation is the case where the dynamical system presents physical parameters that are not precisely known, i.e., the nonlinear system model includes uncertain parameters whose bounds, in many cases, are known and can be taken into account in the stabilization conditions. This section shows how it is possible to extend the proposed methodology, by including *uncertain* time-varying parameters together with known time-varying parameters.

Consider the class of uncertain nonlinear systems described by the following DAR, similar to (3.10), on page 39, but incorporating uncertain parameters:

$$\begin{aligned}
x_{k+1} &= A_1(x_k,\delta_k,\overline{\delta}_k)x_k + A_2(x_k,\delta_k,\overline{\delta}_k)\pi_k + A_3(x_k,\delta_k,\overline{\delta}_k)\,\mathrm{sat}(u_k) + A_4(x_k,\delta_k,\overline{\delta}_k)w_k,\\
0 &= \Omega_1(x_k,\delta_k,\overline{\delta}_k)x_k + \Omega_2(x_k,\delta_k,\overline{\delta}_k)\pi_k + \Omega_3(x_k,\delta_k,\overline{\delta}_k)\,\mathrm{sat}(u_k) + \Omega_4(x_k,\delta_k,\overline{\delta}_k)w_k,\\
z_k &= C_{z_1}(x_k,\delta_k,\overline{\delta}_k)x_k + C_{z_2}(x_k,\delta_k,\overline{\delta}_k)\pi_k + C_{z_3}(x_k,\delta_k,\overline{\delta}_k)\,\mathrm{sat}(u_k) + C_{z_4}(x_k,\delta_k,\overline{\delta}_k)w_k,
\end{aligned} \tag{4.21}$$

where $\pi_k := \pi(x_k, \delta_k, \overline{\delta}_k, \text{sat}(u_k), w_k) \in \mathbb{R}^{n_\pi}$ is an auxiliary vector of nonlinear functions with respect to $(x_k, \delta_k, \overline{\delta}_k)$ and affine on $(\text{sat}(u_k))$ and (w_k), $\overline{\delta}_k \in \overline{\Delta} \subseteq \mathbb{R}^{n_{\overline{\delta}}}$ is the vector of parametric uncertainties, which is not precisely known. All system matrices are affine functions of $(x_k, \delta_k, \overline{\delta}_k)$ with appropriate dimensions, such that $\Omega_2(x_k, \delta_k, \overline{\delta}_k)$ is a square full-rank matrix for all $(x_k, \delta_k, \overline{\delta}_k) \in \mathcal{X} \times \Delta \times \overline{\Delta}$.

Since $\mathcal{X} \times \Delta \times \overline{\Delta}$ is a polytopic region and $A_{\text{v}}(\cdot), \Omega_{\text{v}}(\cdot)$, and $C_{z_{\text{v}}}(\cdot)$, $\text{v} \in \mathcal{I}_4$, are matrices of affine functions with respect to $(x_k, \delta_k, \overline{\delta}_k)$, they belong to polytopes of matrices, i.e., in the case where the polytopes are defined based on upper and lower bounds for each state and parameter, $N_x = 2^{n_x}$ number of vertices in $\mathcal{X}$, $N_\delta = 2^{n_\delta}$ number of vertices in δ_k and $N_{\overline{\delta}} = 2^{n_{\overline{\delta}}}$ number of vertices in $\overline{\delta}_k$:

$$M(x_k, \delta_k, \overline{\delta}_k) = \sum_{i=1}^{N_x} \sum_{l=1}^{N_\delta} \sum_{o=1}^{N_{\overline{\delta}}} \alpha_{x_{(i)k}} \alpha_{\delta_{(l)k}} \alpha_{\overline{\delta}_{(o)k}} M_{ilo}, \tag{4.22}$$

where $M(x_k, \delta_k, \overline{\delta}_k)$ represents any matrix in (4.21), M_{ilo} represents the value of system matrices in each i-th, l-th, o-th vertex in $\mathcal{X}, \Delta$ and $\overline{\Delta}$, respectively and $\alpha_{x_k}, \alpha_{\delta_k}, \alpha_{\overline{\delta}_k} \in \Lambda_1$.

In this case, the following Lyapunov function candidate is considered:

$$V(x_k, \delta_k, \overline{\delta}_k) = x_k^\top P^{-1}(\delta_k, \overline{\delta}_k) x_k, \tag{4.23}$$

with

$$P(\delta_k, \overline{\delta}_k) = \sum_{l=1}^{N_\delta} \sum_{o=1}^{N_{\overline{\delta}}} \alpha_{\delta_{(l)k}} \alpha_{\overline{\delta}_{(o)k}} P_{lo}, \; P_{lo} = P_{lo}^\top \geq 0. \tag{4.24}$$

Thus, the level set associated with (4.23) could be defined by

$$\mathcal{E}_{\mathcal{V}} := \left\{ x_k \in \mathbb{R}^{n_x} : V\left(x_k, \delta_k, \overline{\delta}_k\right) \leq \lambda^{-1}, \quad \forall \delta_k \in \Delta \text{ and } \forall \overline{\delta}_k \in \overline{\Delta} \right\}, \tag{4.25}$$

where λ is a positive scalar defining the bound of $\mathcal{W}$ in (2.2).

For simplicity, we will consider in the next development only the synthesis conditions for the control law (3.12), on page 40, that does not make use of the vector of nonlinearities π_k. Hence, the following Theorem provides sufficient conditions to robustly stabilize uncertain discrete-time nonlinear systems, described in the DAR form (4.21).

Theorem 4.3. *Consider system* (4.21) *with* $w_k \in \mathcal{W}$ *and a given scalar* $\lambda > 0$. *If there exist a positive scalar* μ, *a symmetric matrix* $P(\delta_k, \overline{\delta}_k) \in \mathbb{R}^{n_x \times n_x}$, *and matrices* $G(x_k, \delta_k) \in \mathbb{R}^{n_x \times n_x}$, $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$, $H(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$, *and* $N(x_k, \delta_k, \overline{\delta}_k) \in \mathbb{R}^{n_\pi \times n_\pi}$ *satisfying* *the following inequalities for all $\delta_k \in \Delta$, $\overline{\delta}_k \in \overline{\Delta}$, and $x_k \in \mathcal{X}$:*

$$\Upsilon(x_k, \delta_k, \delta_{k+1}, \overline{\delta}_k, \overline{\delta}_{k+1}, \alpha_{d_k}) = \begin{bmatrix} \mathcal{A}_{11} & \star & \star & \star & \star \\ \mathcal{A}_{21} & -P(\delta_{k+1}, \overline{\delta}_{k+1}) & \star & \star & \star \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} & \star & \star \\ 0 & A_4^\top(x_k, \delta_k, \overline{\delta}_k) & \Omega_4^\top(x_k, \delta_k, \overline{\delta}_k) & -I & \star \\ \mathcal{A}_{51} & 0 & \mathcal{A}_{53} & \mathcal{A}_{54} & -\mu I \end{bmatrix} < 0, \tag{4.26}$$

$$\begin{bmatrix} \lambda u_{0(s)}^2 & \star \\ H_{(s)}^\top(x_k, \delta_k) & \mathrm{He}\left\{G(x_k, \delta_k)\right\} - P(\delta_k, \overline{\delta}_k) \end{bmatrix} \geq 0, \quad s \in \mathcal{I}_{n_u}, \tag{4.27}$$

and

$$\begin{bmatrix} \lambda & \star \\ G^\top(x_k, \delta_k) a_v & \mathrm{He}\left\{G(x_k, \delta_k)\right\} - P(\delta_k, \overline{\delta}_k) \end{bmatrix} \geq 0, \quad v \in \mathcal{I}_{n_e}, \tag{4.28}$$

where

$$\begin{aligned}
\mathcal{A}_{11} &= -\mathrm{He}\left\{G(x_k, \delta_k)\right\} + P(\delta_k, \overline{\delta}_k), \\
\mathcal{A}_{21} &= A_1(x_k, \delta_k, \overline{\delta}_k) G(x_k, \delta_k) + A_3(x_k, \delta_k, \overline{\delta}_k) \left[D(\alpha_{d_{(r)k}}) K(x_k, \delta_k) + D^-(\alpha_{d_{(r)k}}) H(x_k, \delta_k) \right], \\
\mathcal{A}_{31} &= \Omega_1(x_k, \delta_k, \overline{\delta}_k) G(x_k, \delta_k) + \Omega_3(x_k, \delta_k, \overline{\delta}_k) \left[D(\alpha_{d_{(r)k}}) K(x_k, \delta_k) + D^-(\alpha_{d_{(r)k}}) H(x_k, \delta_k) \right], \\
\mathcal{A}_{51} &= C_{z_1}(x_k, \delta_k, \overline{\delta}_k) G(x_k, \delta_k) + C_{z_3}(x_k, \delta_k, \overline{\delta}_k) \left[D(\alpha_{d_{(r)k}}) K(x_k, \delta_k) + D^-(\alpha_{d_{(r)k}}) H(x_k, \delta_k) \right], \\
\mathcal{A}_{32} &= N^\top(x_k, \delta_k, \overline{\delta}_k) A_2^\top(x_k, \delta_k, \overline{\delta}_k), \qquad \mathcal{A}_{33} = \mathrm{He}\left\{\Omega_2(x_k, \delta_k, \overline{\delta}_k) N(x_k, \delta_k, \overline{\delta}_k)\right\}, \\
\mathcal{A}_{53} &= C_{z_2}(x_k, \delta_k, \overline{\delta}_k) N(x_k, \delta_k, \overline{\delta}_k), \qquad \mathcal{A}_{54} = C_{z_4}(x_k, \delta_k, \overline{\delta}_k).
\end{aligned}$$

then there exist a Lyapunov function (4.23) *and a controller* (3.12) *such that, for zero initial conditions* ($x_0 = 0$), x_k *remains bounded in* $\mathcal{E}_\mathcal{V}$ *and* $\|z\|_2 \leq \gamma \|w\|_2, \forall w_k \in \mathcal{W}$, *with* $\gamma = \sqrt{\mu}$. *Moreover, if there exists* $\overline{k} > 0$ *such that* $w_k = 0$, $\forall k \geq \overline{k}$, *then* $x_k \to 0$, $k \to \infty$.

Proof. Theorem 4.3 is obtained directly from *Theorem* 4.2 by incorporating the information about the parametric uncertainties in system matrices, Lyapunov function, and auxiliary decision variables. □

The following *Corollaries* 4.5 and 4.6 are used to solve *Problems* 3.1 and 3.2, stated at the end of *Chapter* 3, considering the uncertain nonlinear system described by (4.21).

In the next Corollary, *Theorem* 4.3 can be used to solve *Problem* 3.1 as described previously.

Corollary 4.5. *For a given disturbance energy level* λ^{-1}, *the upper-bound* γ *for the* ℓ_2-*gain from* w_k *to* z_k *can be minimized solving the following optimization problem for all* $\delta_k \in \Delta, \overline{\delta}_k \in \overline{\Delta}$ *and* $x_k \in \mathcal{X}$:

$$\min_{\mu, P, N, G, K, H} \mu \quad \textit{subject to } (4.26) - (4.28). \tag{4.29}$$

In the next Corollary, *Theorem* 4.3 can also be adapted to solve *Problem* 3.2.

Corollary 4.6. *Consider system* (4.21)*, with the disturbance input* $w_k = 0$*. Taking into account the normalized DoA:* $\mathcal{E}_{DoA} := \left\{x_k \in \mathbb{R}^{n_x} : V\left(x_k, \delta_k, \overline{\delta}_k\right) \leq 1,\ \forall \delta_k \in \Delta \text{ and } \forall \overline{\delta}_k \in \overline{\Delta}\right\}$*, an alternative to find the largest DoA is to consider the following subset of* $\mathcal{E}_{DoA}$*:*

$$\mathcal{E}\left(Q^{-1}, 1\right) \subseteq \bigcap_{l \,\in\, \mathcal{I}_{N_\delta},\ o \,\in\, \mathcal{I}_{N_{\overline{\delta}}}} \mathcal{E}\left(P_{lo}^{-1}, 1\right), \tag{4.30}$$

Now, disregard the influence of the disturbance input by removing the fourth and fifth rows and columns of $\Upsilon(x_k, \delta_k, \delta_{k+1}, \overline{\delta}_k, \overline{\delta}_{k+1}, \alpha_{d_k})$ *(in Theorem 4.3) and consider* $\lambda = 1$ *in* (4.27) *and* (4.28)*.*

If there exist symmetric matrices $Q \in \mathbb{R}^{n_x \times n_x}, P\left(\delta_k, \overline{\delta}_k\right) \in \mathbb{R}^{n_x \times n_x}$*, and matrices* $N\left(x_k, \delta_k, \overline{\delta}_k\right) \in \mathbb{R}^{n_\pi \times n_\pi}, G\left(x_k, \delta_k\right) \in \mathbb{R}^{n_x \times n_x}, K\left(x_k, \delta_k,\right) \in \mathbb{R}^{n_u \times n_x}, H\left(x_k, \delta_k\right) \in \mathbb{R}^{n_u \times n_x}$ *satisfying the following optimization problem for all* $\delta_k \in \Delta$*,* $\overline{\delta}_k \in \overline{\Delta}$*, and* $x_k \in \mathcal{X}$*:*

$$\max_{Q,P,N,G,K,H} \log(\det(Q)) \quad \textit{subject to } (4.26) - (4.28). \tag{4.31}$$

with

$$Q - P_{lo} < 0, \quad l \in \mathcal{I}_{N_\delta},\ o \in \mathcal{I}_{N_{\overline{\delta}}}, \tag{4.32}$$

then there exist a Lyapunov function (4.23) *and a controller* (3.12) *such that,* $\forall x_k(0)$ *inside* $\mathcal{E}_{DoA}$*,* $\delta_k \in \Delta$ *and* $\overline{\delta}_k \in \overline{\Delta}$*, the trajectory of* x_k *converge to the origin when* $k \to \infty$ *and* $\mathcal{E}_{DoA}$ *is an estimate of the DoA.*

Proof. The inequality (4.32) ensures that (4.30) holds. The rest of the proof follows in a straightforward way as in the proof of *Theorem* 4.3. □

Similarly to the case without uncertain time-varying parameters, despite the conditions in *Theorem* 4.3 are of infinite dimension, it is possible to use the multi-simplex framework based on (2.12), on page 30, (3.16), on page 41, (4.22), and (4.24) to obtain a finite set of LMIs in terms of the vertices of the polytopes $\mathcal{X}$, Δ, $\mathcal{D}$, and $\overline{\Delta}$, as described in Appendix A.2.

Remark 4.3. *This section showed how the proposed methodology could be adapted to provide robust stabilization conditions to design controller* (3.12)*. Similarly, it is possible to obtain robust stabilization conditions considering the control law* (3.11)*, on page 39, which include information about the nonlinearity vector. However, it is worth emphasizing that in this case, the nonlinearity vector must be independent of the uncertain parameters* $\overline{\delta}_k$*. Consequently, the matrices in the algebraic portion of* (4.21) *must be independent of* $\overline{\delta}_k$*.*

4.5 Numerical examples

In this section, numerical examples are presented to verify the effectiveness of the proposed methodology. The conditions proposed in this paper were implemented in MATLAB (R2019) using the parser Yalmip [95] and the solver Mosek [96].

4.5.1 DoA estimate for a polynomial system

Consider the following nonlinear system, without time-varying parameters, borrowed from Oliveira, Gomes da Silva Jr & Coutinho [12]:

$$\begin{aligned} x_{(1)k+1} &= x_{(2)k}, \\ x_{(2)k+1} &= x_{(1)k} + 3x_{(1)k}^3 + x_{(2)k} + \text{sat}(u_k), \end{aligned} \tag{4.33}$$

which can be recast in a DAR, as in (2.3), such that

$$\begin{gathered} \pi_k(x_k) = x_{(1)k}^2, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2(x_k) = \begin{bmatrix} 0 \\ 3x_{(1)k} \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ \Omega_1(x_k) = \begin{bmatrix} x_{(1)k} & 0 \end{bmatrix}, \quad \Omega_2 = -1, \quad \Omega_3 = 0. \end{gathered} \tag{4.34}$$

For $u_0 = 1$, as in Oliveira, Gomes da Silva Jr & Coutinho [12], the optimization problems (4.10) and (4.17) were solved to obtain the largest admissible polytope in state space and the largest estimated DoA. These results are compared with those from Oliveira, Gomes da Silva Jr & Coutinho [12] in Table 4.3, which presents the largest estimated DoA obtained and the computational effort in terms of the number of LMI rows (L_R) and scalar variables (S_v) required for each methodology.

Table 4.3 – Estimated DoA for system (4.33) with $u_0 = 1$.

Method	Polytopic Region $(\mathcal{X})$	$\log(\det(Q))$	S_v	L_R
Theorem 1 in [12]	$\lvert x_{(1)k}\rvert \leq 0.50$, $\lvert x_{(2)k}\rvert \leq 0.40$	-3.6952	12	35
Corollary 4.4	$\lvert x_{(1)k}\rvert \leq 0.66$, $\lvert x_{(2)k}\rvert \leq 0.65$	-1.7474	40	180
Corollary 4.2	$\lvert x_{(1)k}\rvert \leq 0.68$, $\lvert x_{(2)k}\rvert \leq 0.67$	-1.6616	48	248

The comparison shows that, from the methodologies proposed in this study, it is possible to obtain a larger estimated ellipsoidal DoA, in comparison with the results presented in Oliveira, Gomes da Silva Jr & Coutinho [12]. Besides that, a larger estimated DoA is achieved using the control law depending on π_k (3.11), synthesized by using *Corollary 4.2*, instead of using the control law not depending on π_k (3.12), from conditions stated in *Corollary 4.4*.

One can see that, using the proposed approaches, the computational complexity in terms of the number of scalar variables (S_v) and the number of LMI rows (L_r) increases. However, it is worth emphasizing that the control design is off-line and the computational cost is not a problem in its practical implementation. In general, reducing conservativeness is achieved considering more scalar variables.

Figure 4.1 depicts the largest estimated ellipsoidal DoA from *Corollaries 4.2* (blue solid line) and *4.4* (black dotted line), and the DoA obtained by Oliveira, Gomes da Silva Jr & Coutinho [12] (red dashed line). The largest value obtained for the objective function is $\log(\det(Q)) = -1.6616$, with $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : |x_{(1)k}| \leq 0.68, |x_{(2)k}| \leq 0.67\right\}$, as shown in Table 4.3.

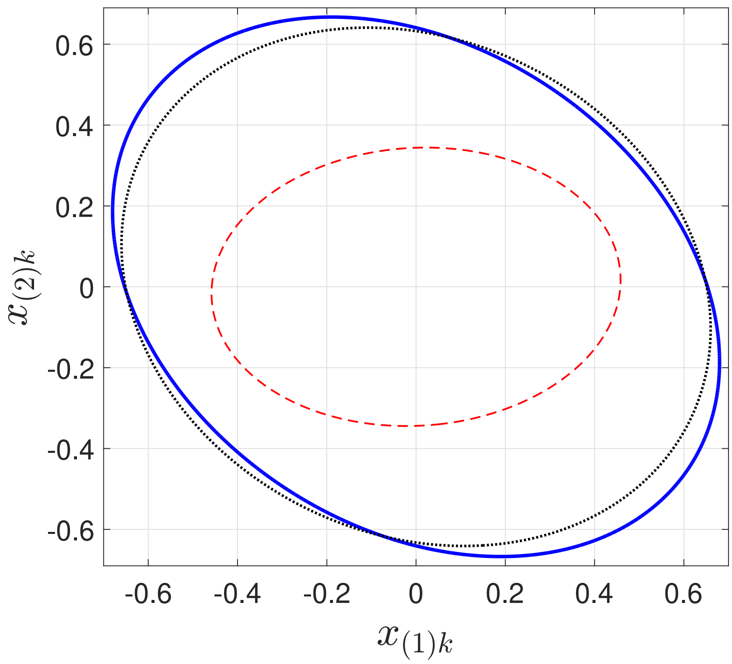
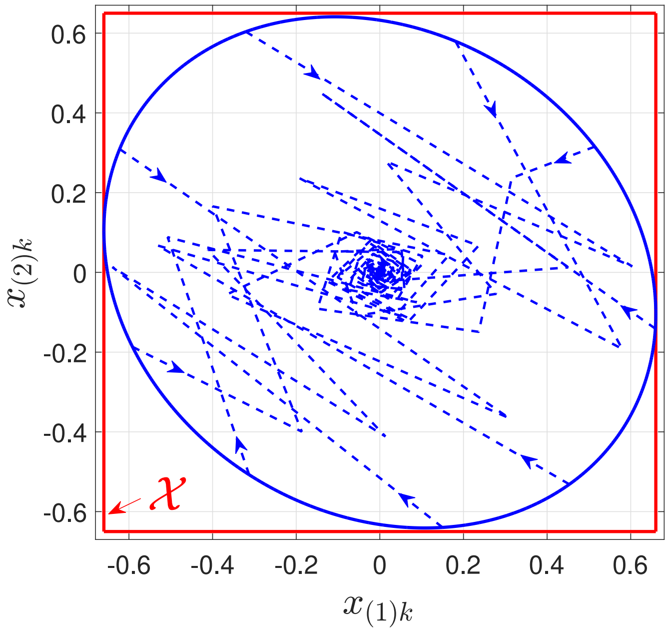
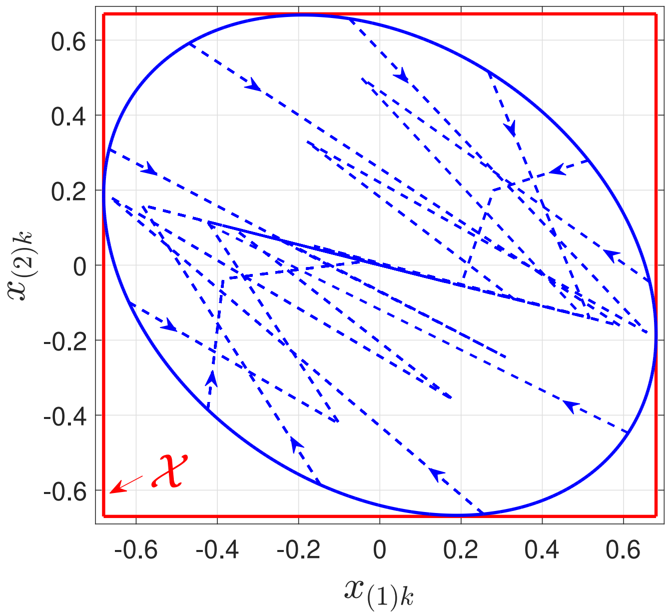


Figure 4.1 – Estimated DoAs for system (4.33) from *Corollary 4.2* (blue solid line), *Corollary 4.4* (black dotted line), and Theorem 1 in [12] (red dashed line).

From Table 4.3 and Figure 4.1 it is possible to verify that the proposed approaches provide less conservative results. In Figure 4.2, one can see the estimated DoAs from the proposed methodologies, with some trajectories initiating inside these regions. Note that all trajectories starting at the boundary of the DoAs converge to the origin.



(a)



(b)

Figure 4.2 – Estimated DoA (blue solid line) and some state trajectories (blue dash line) for system (4.33). (a) *Corollary* 4.4 (Controller not dependent on π_k) (b) *Corollary* 4.2 (Controller dependent on π_k).

Figure 4.3 depicts the trajectories for the closed-loop system from the synthesized controllers, with $x_0 = (0.15, -0.64)$ (chosen randomly), and Figure 4.4 shows the corresponding control input for each case.

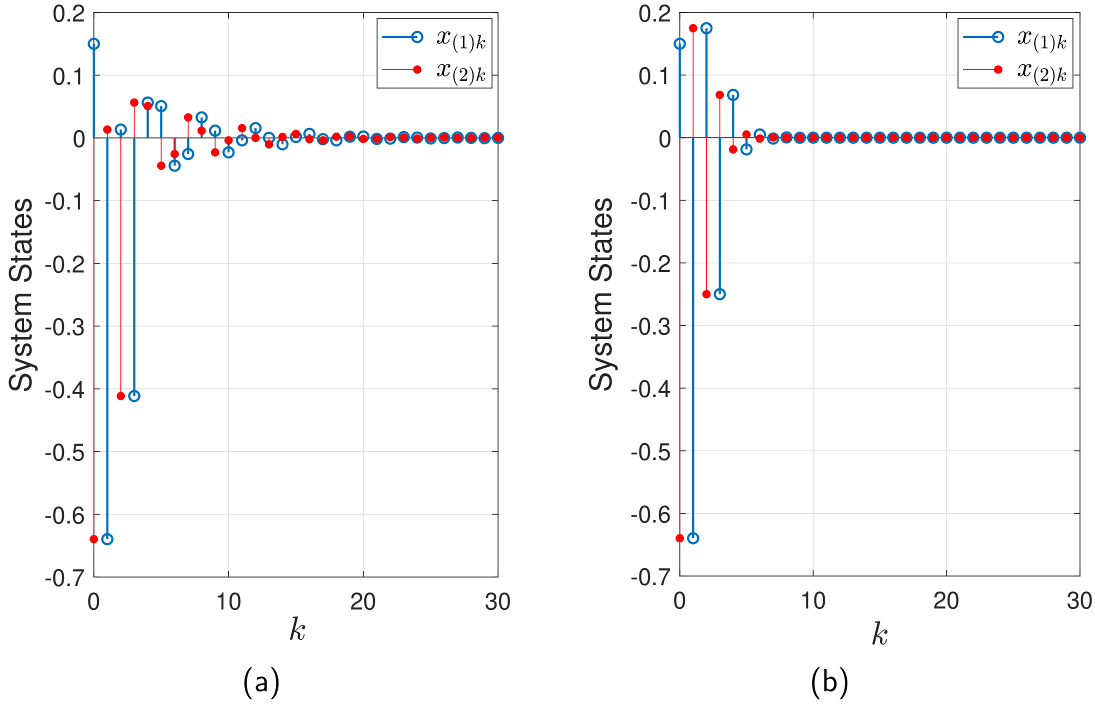


Figure 4.3 – Closed-loop behavior of system (4.33). System states trajectories (a) *Corollary* 4.4 (Controller not dependent on π_k) (b) *Corollary* 4.2 (Controller dependent on π_k).

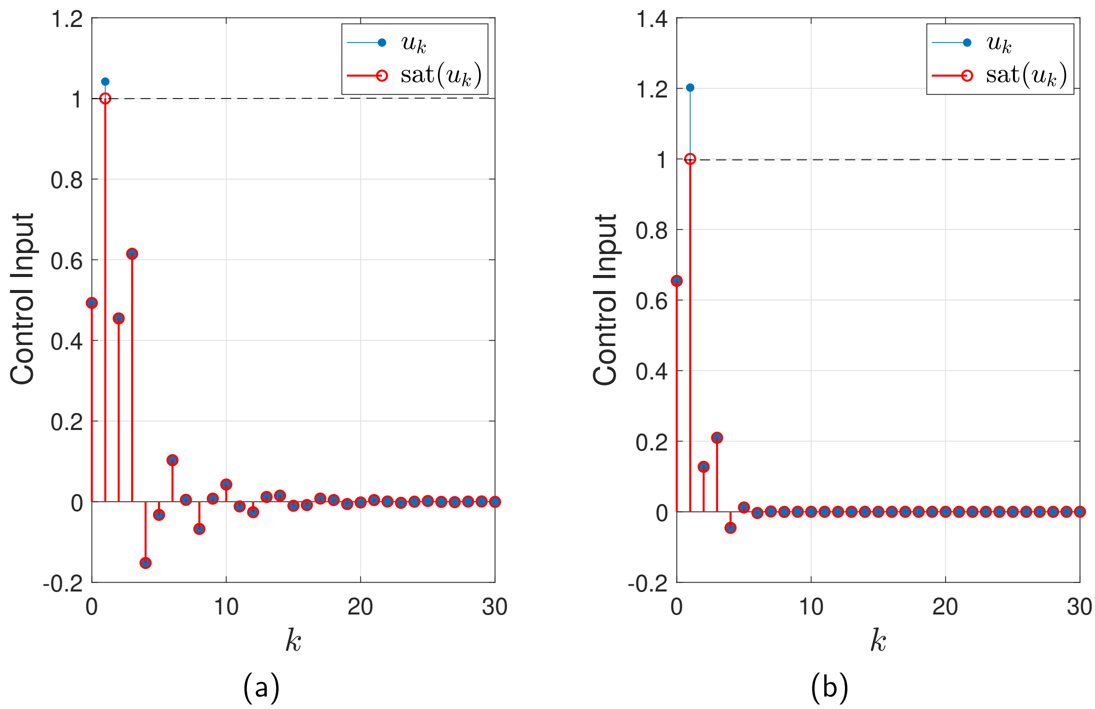


Figure 4.4 – Closed-loop behavior of system (4.33). Control input. (a) *Corollary* 4.4 (Controller not dependent on π_k) (b) *Corollary* 4.2 (Controller dependent on π_k).

Notice from Figure 4.4 that, although the input signal u_k exceeds its amplitude bound u_0, the actual control signal applied to the system is $\mathrm{sat}(u_k)$. Nevertheless, the proposed method guarantees system stability. It is worth emphasizing that the proposed conditions do not guarantee that input saturation will not occur. However, its effect is incorporated in the stability analysis by using the approach originally proposed by Hu & Lin [80].

To investigate the influence of the maximum absolute value of the control input on the results, tests were performed considering different values of u_0. Table 4.4 shows the largest DoA estimated in each case.

Table 4.4 – Estimated DoA for system (4.33) with different values of u_0.

	Corollary 4.4		*Corollary 4.2*	
u_0	Polytopic Region ($\mathcal{X}$)	Area ($\mathcal{E}_{DoA}$)	Polytopic Region ($\mathcal{X}$)	Area ($\mathcal{E}_{DoA}$)
1	$\lvert x_{(1)k}\rvert \leq 0.66,\ \lvert x_{(2)k}\rvert \leq 0.65$	1.3113	$\lvert x_{(1)k}\rvert \leq 0.68,\ \lvert x_{(2)k}\rvert \leq 0.67$	1.3687
1.5	$\lvert x_{(1)k}\rvert \leq 0.76,\ \lvert x_{(2)k}\rvert \leq 0.76$	1.8072	$\lvert x_{(1)k}\rvert \leq 0.79,\ \lvert x_{(2)k}\rvert \leq 0.78$	1.8666
2	$\lvert x_{(1)k}\rvert \leq 0.81,\ \lvert x_{(2)k}\rvert \leq 0.81$	2.0611	$\lvert x_{(1)k}\rvert \leq 0.87,\ \lvert x_{(2)k}\rvert \leq 0.87$	2.3037
3	$\lvert x_{(1)k}\rvert \leq 0.81,\ \lvert x_{(2)k}\rvert \leq 0.81$	2.0611	$\lvert x_{(1)k}\rvert \leq 1.00,\ \lvert x_{(2)k}\rvert \leq 1.00$	3.0640
5	$\lvert x_{(1)k}\rvert \leq 0.81,\ \lvert x_{(2)k}\rvert \leq 0.81$	2.0611	$\lvert x_{(1)k}\rvert \leq 1.18,\ \lvert x_{(2)k}\rvert \leq 1.18$	4.3609
10	$\lvert x_{(1)k}\rvert \leq 0.81,\ \lvert x_{(2)k}\rvert \leq 0.81$	2.0611	$\lvert x_{(1)k}\rvert \leq 1.49,\ \lvert x_{(2)k}\rvert \leq 1.49$	6.9703

Note that, as the maximum value for the control input increases, it is possible to obtain a larger estimated DoA. However, from *Corollary 4.4* (considering the control law not depending on π_k), when defining $u_0 = 2$ the maximum bound for the value of the cost function is reached, i.e., from this point on, even by increasing u_0, it was not possible to get a larger DoA. On the other hand, from *Corollary 4.2* (considering the control law depending on π_k), this limit was not achieved in the tests performed, where it was considered the maximum value for the control input up to $u_0 = 10$.

Figure 4.5 depicts the largest estimated ellipsoidal DoA for system (4.33) from *Corollaries 4.2* (blue solid line) and *4.4* (black dotted line), for $u_0 = 10$. The largest value obtained for the objective function from *Corollary 4.2* is $\log(\det(Q)) = 1.5939$, with the states' bounds defined by $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : \lvert x_{(1)k}\rvert \leq 1.49, \lvert x_{(2)k}\rvert \leq 1.49\right\}$. On the other hand, from *Corollary 4.4*, it was obtained $\log(\det(Q)) = -0.8429$, with the states' bounds defined by $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : \lvert x_{(1)k}\rvert \leq 0.81, \lvert x_{(2)k}\rvert \leq 0.81\right\}$, as shown in Table 4.4.

From these simulations, it is possible to conclude that incorporating the vector of nonlinearities into the control law can provide considerably less conservative results compared to results obtained using controllers that rely only on the system's state vector.

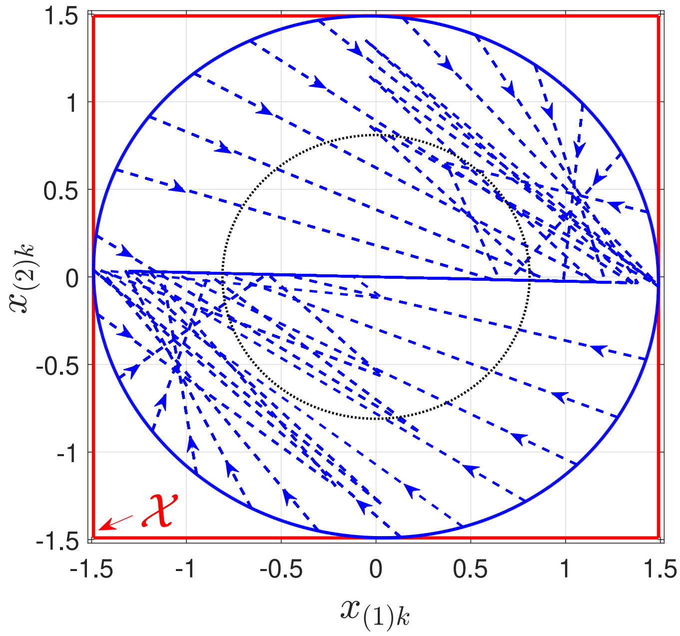


Figure 4.5 – Estimated DoAs for system (4.33) from *Corollary 4.2* (blue solid line) and *Corollary 4.4* (black dotted line), for $u_0 = 10$.

4.5.2 Input-to-Output performance for a polynomial system

Consider the discrete-time nonlinear system with disturbance input from Oliveira, Gomes da Silva Jr. & Coutinho [61]

$$\begin{aligned} x_{(1)k+1} &= x_{(2)k} + 0.22w_k, \\ x_{(2)k+1} &= (1 + x_{(1)k}^2)x_{(1)k} + x_{(2)k} + 0.3w_k + \text{sat}(u_k), \\ z_k &= x_{(1)k} + x_{(2)k}. \end{aligned} \tag{4.35}$$

It is worth emphasizing that in Oliveira, Gomes da Silva Jr. & Coutinho [61] only stability analysis conditions were considered. Despite this fact, this example is used to illustrate the effectiveness of the proposed method in the case of energy-bounded disturbance inputs, without performing any additional comparisons, since a controller design would be necessary and it cannot be done using the result in Oliveira, Gomes da Silva Jr. & Coutinho [61].

System (4.35) can be recast in a DAR (3.10) such that

$$\begin{aligned} \pi_k = x_{(1)k}^2,\ A_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ A_2(x_k) = \begin{bmatrix} 0 \\ x_{(1)k} \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ A_4 = \begin{bmatrix} 0.22 \\ 0.3 \end{bmatrix}, \\ \Omega_1(x_k) &= \begin{bmatrix} x_{(1)k} & 0 \end{bmatrix},\quad \Omega_2 = -1,\quad \Omega_3 = \Omega_4 = 0, \\ C_{z_1} &= \begin{bmatrix} 1 & 1 \end{bmatrix},\quad C_{z_2} = C_{z_3} = C_{z_4} = 0. \end{aligned} \tag{4.36}$$

Firstly, to verify the largest disturbance energy that the closed-loop system using the proposed controllers would be able to withstand, tests were carried out in which the optimization problems stated in *Corollaries 4.1* and *4.3* were solved, considering a search for the $\min(\lambda)$ (since the maximum energy of the disturbance signal, according to (2.2), is given by λ^{-1}) instead of searching for the $\min(\mu)$, with $\mu = \gamma^2$ and γ the induced ℓ_2-gain from w_k to z_k. Notice that in this case, we also have a LMI optimization problem.

Table 4.5 shows the results obtained, considering different values for u_0 (maximum absolute value for the control input) and $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : |x_{(1)k}| \leq 1.0, |x_{(2)k}| \leq 1.0\right\}$. Note that, by using the more general controller (3.11) it was possible to have considerably less conservative results in comparison with the use of controller (3.12).

Table 4.5 – Maximum guaranteed tolerated disturbance energy level (max(λ^{-1})) for different values of u_0.

u_0	1	1.5	2	3	5
Controller (3.12)	1.2198	2.7446	4.5735	4.5735	4.5735
Controller (3.11)	1.2239	2.7537	4.8883	7.1339	7.2055

Secondly, for the same polytopic region $\mathcal{X}$ and $u_0 = 5$, the optimization problems (4.9) and (4.16) were solved, using the proposed approaches. The results are depicted in Figure 4.6.

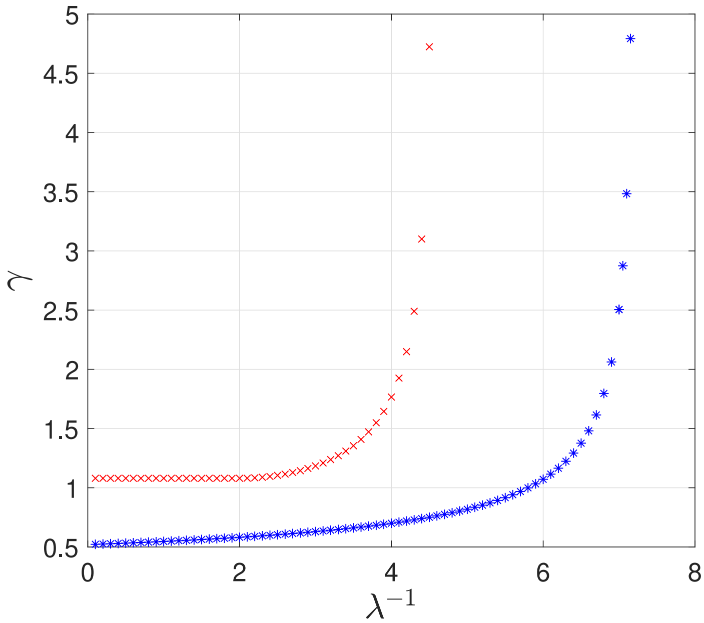
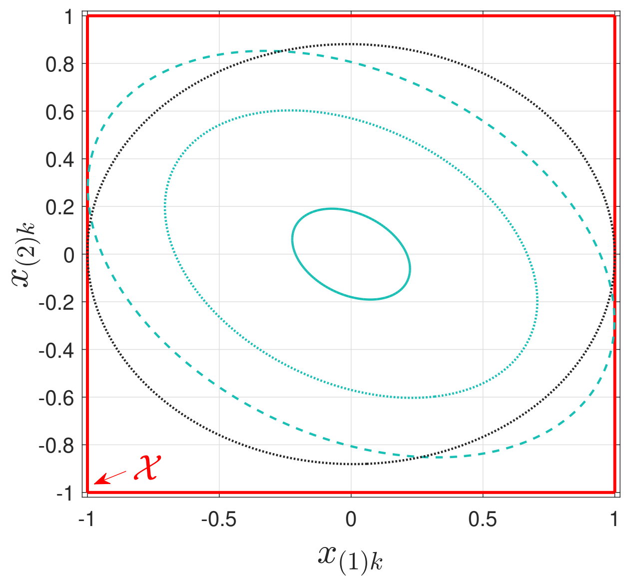


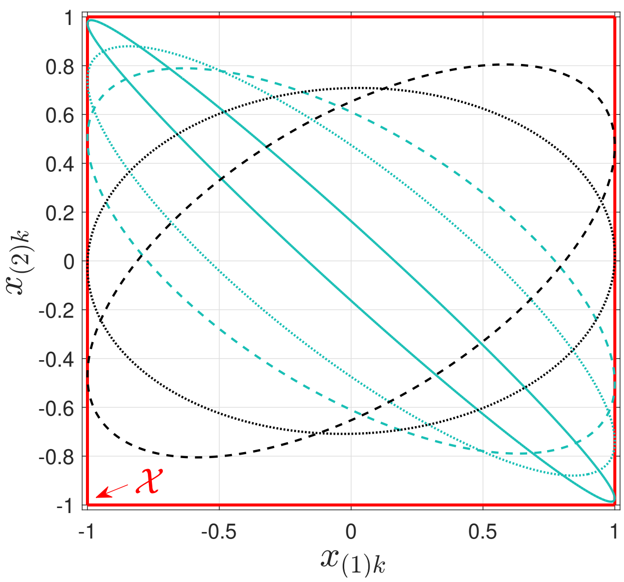
Figure 4.6 – The variation of γ (the upper-bound for the ℓ_2-gain from w_k to z_k) for different values of λ^{-1} (the admissible energy-bound for the disturbance input). × *Corollary 4.3* (Controller not dependent on π_k). * *Corollary 4.1* (Controller dependent on π_k).

The largest values of λ^{-1} for which it was possible to obtain feasible results from the proposed methods are $\lambda^{-1} = 4.5735$ using the controller (3.12), and $\lambda^{-1} = 7.2055$ for controller (3.11). From Figure 4.6, it is possible to note that less conservative results were obtained from controller (3.11) in this scenario, while considering the same values of λ^{-1} it was possible to obtain lower values for γ in comparison with the results obtained from (3.12).

Figure 4.7 depicts the level sets $\mathcal{E}(P^{-1}, \lambda^{-1})$ obtained to system (4.35) from the proposed approaches, considering $u_0 = 5$ and different values of λ^{-1}.



(a)



(b)

Figure 4.7 – Level sets $\mathcal{E}(P^{-1}, \lambda^{-1})$ obtained for $\lambda^{-1} = 0.1$ (green solid line), $\lambda^{-1} = 1.0$ (green dotted line), $\lambda^{-1} = 2.0$ (green dashed line), $\lambda^{-1} = 4.5$ (black dotted line), and $\lambda^{-1} = 7.2$ (black dashed line). (a) *Corollary 4.3* (Controller not dependent on π_k). (b) *Corollary 4.1* (Controller dependent on π_k).

Figure 4.8 shows three trajectories for the closed-loop system, with the controller (3.11) synthesized from *Corollary* 4.1, for zero initial conditions and different disturbances given by

$$w_k^{(1)} = \begin{cases} -0.7, & k \leq 5 \\ 0, & \text{elsewhere} \end{cases}, \quad w_k^{(2)} = \begin{cases} \sqrt{7.2}, & k = 3 \\ 0, & \text{elsewhere} \end{cases}, \quad w_k^{(3)} = \begin{cases} -10, & k = 2 \\ 0, & \text{elsewhere} \end{cases}$$

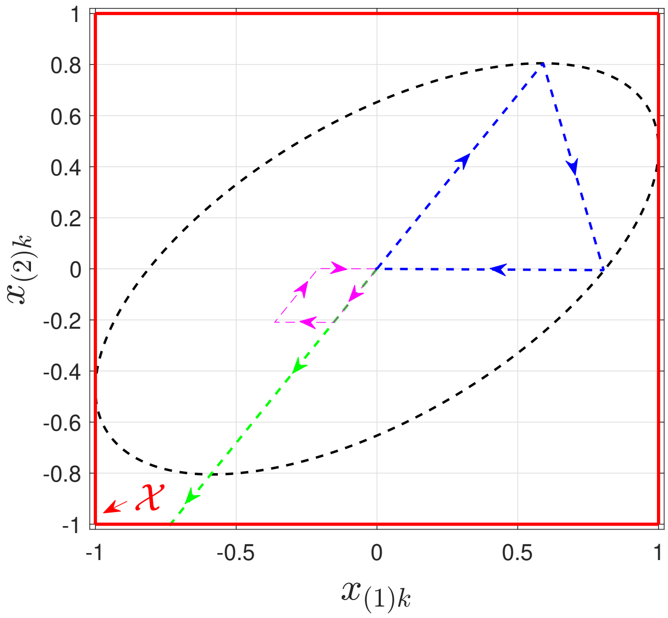


Figure 4.8 – Region $\mathcal{E}(P^{-1},\lambda^{-1})$ (black dashed line) obtained for $\lambda^{-1} = 7.2$ and the controller obtained from *Corollary* 4.1, with some trajectories for $x_0 = 0$, $w_k^{(1)}$ (magenta dashed line), $w_k^{(2)}$ (blue dashed line), and $w_k^{(3)}$ (green dashed line).

Note that, for $w_k^{(3)}$, the trajectory diverges because the condition (2.2) is not satisfied. For the other cases, the state trajectories do not leave the ellipsoidal region and, when the disturbance vanishes, the system states converge to the origin. It is worth emphasizing that, even when the largest admissible disturbance amplitude is applied in a single time instant (in the case of $w_k^{(2)}$), the saturated control input guarantees that the system states remain in the region $\mathcal{E}(P^{-1},\lambda^{-1})$.

4.5.3 DoA estimate for a 3-D polynomial system with time-varying parameter

In the previous examples, the synthesis conditions proposed in this investigation were applied for regional stabilization of nonlinear polynomial systems to the problems of estimating DoA and ensuring input-to-output performance. From the results obtained, it was possible to note that the controller using the nonlinearity vector provided less conservative results for the cases considered so far.

In this example, our goal is to illustrate the use of the proposed methodology for systems with time-varying parameters. For this purpose, the DoA estimation problem was taken into account, and the results obtained from *Corollary* 4.2 will be presented, as they are less conservative in comparison with *Corollary* 4.4.

Therefore, consider the following nonlinear system, with time-varying parameter:

$$\begin{aligned} x_{(1)k+1} &= -0.01x_{(1)k}^2 - (0.8+\delta_k)\,x_{(2)k} - (1.2+\delta_k)x_{(3)k}^2 \\ &\qquad + \delta_k \mathrm{sat}(u_{(1)k}) + \mathrm{sat}(u_{(2)k}) + f'(x_k), \\ x_{(2)k+1} &= 0.2(x_{(2)k} - x_{(3)k}^2) + \mathrm{sat}(u_{(2)k}) + f'(x_k), \\ x_{(3)k+1} &= 0.8x_{(3)k} + 0.1(x_{(1)k} - x_{(2)k} - x_{(3)k}^2), \end{aligned} \tag{4.37}$$

where $f'(x_k) = \left[0.8x_{(3)k} + 0.1(x_{(1)k} - x_{(2)k} - x_{(3)k}^2)\right]^2$. This system can be recast in a DAR form (3.10) such that

$$\pi_k(x_k) = \begin{bmatrix} x_{(3)k}^2 & x_{(1)k}x_{(3)k}^2 & x_{(2)k}x_{(3)k}^2 & x_{(3)k}^3 & x_{(3)k}^4 \end{bmatrix}^\top,$$

$$\begin{aligned} A_1(x_k,\delta_k) &= \begin{bmatrix} 0 & -0.8-\delta_k-0.02x_{(1)k}+0.01x_{(2)k} & 0.16x_{(1)k}-0.16x_{(2)k} \\ 0.01x_{(1)k} & 0.2-0.02x_{(1)k}+0.01x_{(2)k} & 0.16x_{(1)k}-0.16x_{(2)k} \\ 0.1 & -0.1 & 0.8 \end{bmatrix}, \\ A_2(\delta_k) &= \begin{bmatrix} -0.56-\delta_k & -0.02 & 0.02 & -0.16 & 0.01 \\ 0.44 & -0.02 & 0.02 & -0.16 & 0.01 \\ -0.1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3(\delta_k) = \begin{bmatrix} \delta_k & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{4.38}$$

$$\Omega_1(x_k) = \begin{bmatrix} 0 & 0 & x_{(3)k} \\ & 0_{4\times 3} & \end{bmatrix}, \quad \Omega_2(x_k) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ x_{(1)k} & -1 & 0 & 0 & 0 \\ x_{(2)k} & 0 & -1 & 0 & 0 \\ x_{(3)k} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & x_{(3)k} & -1 \end{bmatrix}, \quad \Omega_3 = 0_{5\times 2}.$$

For $u_0 = 1, \mathcal{X} := \{x_k \in \mathbb{R}^3 : |x_k| \le 1.0\}$, and $\Delta := \{\delta_k \in \mathbb{R} : 0 \le \delta_k \le 1\}$, the optimization problem (4.10) has been solved and the value obtained for the objective function is $\log(\det(Q)) = -1.6574$.

Figure 4.9 depicts the estimated DoA, with some trajectories initiating inside this region, which converges to the origin over time, and Figure 4.10 shows the projection of the ellipsoids onto the plans. This example illustrates the effectiveness of the proposed method when time-varying parameters are considered. In this case, note that the estimated DoA ($\mathcal{E}_{DoA}$)

consists of the intersection of the two ellipsoids (filled in magenta and green) associated with $\mathcal{E}(P_l^{-1},1)$.

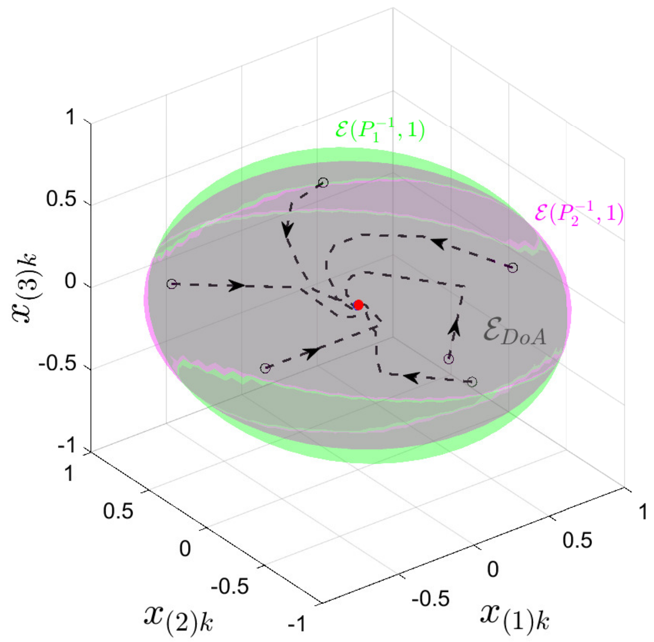


Figure 4.9 – Estimated DoA ($\mathcal{E}_{DoA}$) for system (4.37) obtained from *Corollary 4.2*, and some state trajectories (black dashed lines). $\mathcal{E}_{DoA}$ is the intersection of the two ellipsoids (in green and magenta) associated with $\mathcal{E}(P_l^{-1},1)$. The red point represents the origin.

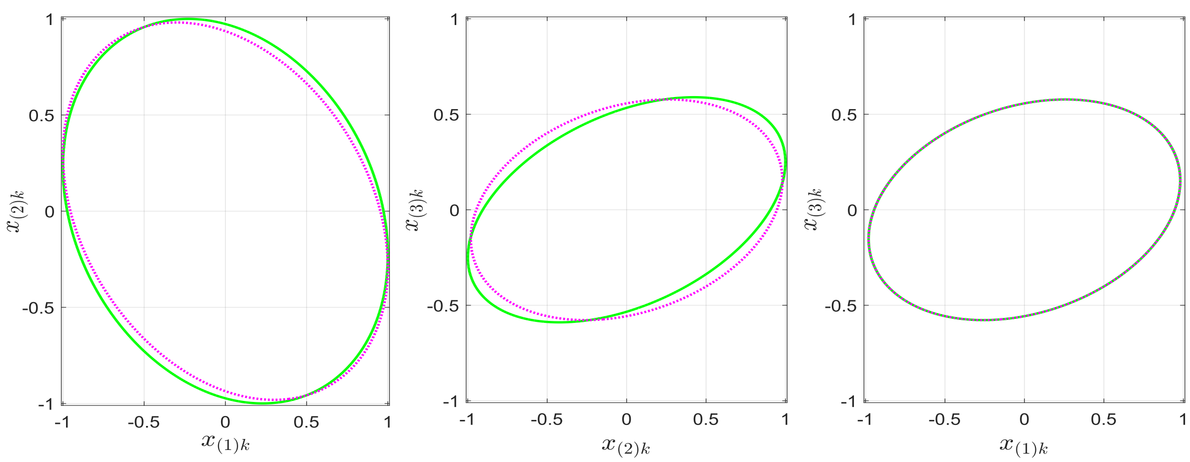


Figure 4.10 – Projection of the ellipsoids $\mathcal{E}(P_l^{-1},1)$ (in green and magenta) onto the plans $(x_{(1)k}, x_{(2)k})$, $(x_{(2)k}, x_{(3)k})$, and $(x_{(1)k}, x_{(3)k})$.

4.5.4 Input-to-Output performance for a rational system with time-varying parameter

In this example, our goal is to use *Corollary* 4.1, for a rational nonlinear system with time-varying parameters. Therefore, consider the following nonlinear system:

$$\begin{aligned} x_{(1)k+1} &= (1-\delta_k)x_{(2)k} + 0.5\frac{x_{(1)k}^2}{1+x_{(1)k}^2} + 0.5w_k, \\ x_{(2)k+1} &= -x_{(1)k} + x_{(2)k} + \frac{x_{(1)k}}{1+x_{(1)k}^2} + 0.1w_k + (1-\delta_k)\mathrm{sat}(u_k), \\ z_k &= x_{(1)k} + x_{(2)k}. \end{aligned} \tag{4.39}$$

which can be recast in a DAR form (2.3) such that

$$\pi_k = \begin{bmatrix} \dfrac{x_{(1)k}^2}{1+x_{(1)k}^2} & \dfrac{x_{(1)k}}{1+x_{(1)k}^2} \end{bmatrix}^\top,$$

$$C_{z_1} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad C_{z_2} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad C_{z_3} = C_{z_4} = 0,$$

$$A_1(\delta_k) = \begin{bmatrix} 0 & 1-\delta_k \\ -1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_3(\delta_k) = \begin{bmatrix} 0 \\ 1-\delta_k \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0.5 \\ 0.1 \end{bmatrix},$$

$$\Omega_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \Omega_2(x_k) = \begin{bmatrix} 1 & -x_{(1)k} \\ x_{(1)k} & 1 \end{bmatrix}, \quad \Omega_3 = \Omega_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Considering $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : |x_{(1)k}| \leq 2.0, |x_{(2)k}| \leq 1.0\right\}$, $\Delta := \{\delta_k \in \mathbb{R} : |\delta_k| \leq 0.5\}$, and $u_0 = 2$ the optimization problem (4.9) is solved and the obtained results are depicted in Table 4.6.

Table 4.6 – Disturbance attenuation ($\lambda^{-1} \times \gamma$) for $u_0 = 2$, where λ^{-1} is the largest admissible disturbance amplitude.

λ^{-1}	0.1	2	4	6	8
γ	0.8529	0.8587	1.0546	1.4886	3.1013

The largest value of λ^{-1} for which it was possible to obtain a feasible result from the proposed method is $\lambda^{-1} = 8.48$. Figure 4.11 depicts the level set $\mathcal{E}(P^{-1}, \lambda^{-1})$ considering $\lambda^{-1} = 8.48$, with two trajectories, for $x_0 = 0$, different time-varying sequences for $\delta_k \in \Delta$ (chosen randomly), and different input disturbances $w_k^{(1)}$ and $w_k^{(2)}$ as follows

$$w_k^{(1)} = \begin{cases} -e^{(0.22k)}, & 1 \leq k \leq 3 \\ 0, & \text{elsewhere} \end{cases}, \quad w_k^{(2)} = \begin{cases} \sqrt{8.48}, & k = 3 \\ 0, & \text{elsewhere} \end{cases}, \tag{4.40}$$

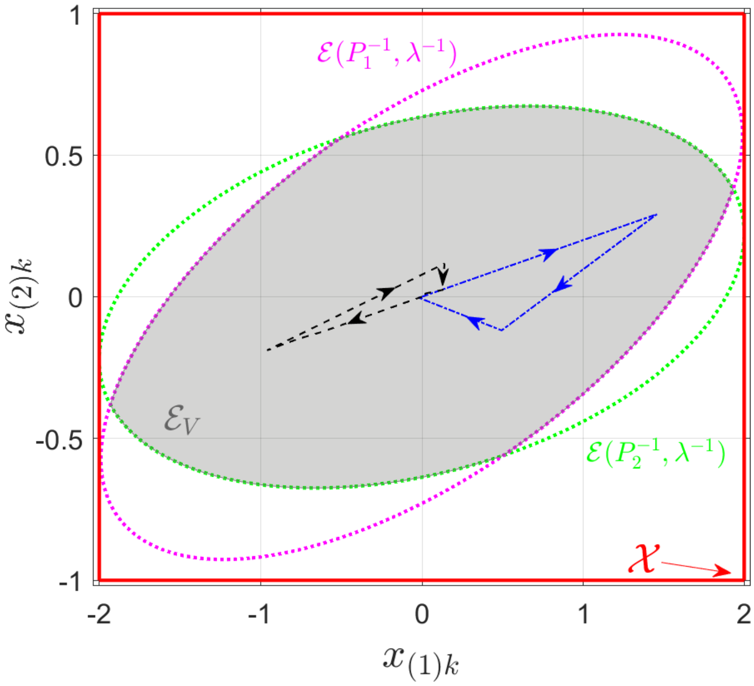


Figure 4.11 – Estimated region $(\mathcal{E}_V)$ for system (4.39) and the ellipsoidal regions $\mathcal{E}(P_1^{-1}, \lambda^{-1})$ (magenta dotted line) and $\mathcal{E}(P_2^{-1}, \lambda^{-1})$ (green dotted line) obtained from *Corollary 4.1* for $\lambda^{-1} = 8.48$. Two trajectories for $x_0 = 0$, $w_k^{(1)}$ (black dashed line) and $w_k^{(2)}$ (blue dash-dotted line) are also shown in this figure.

Note that the state trajectories do not leave the intersection region $(\mathcal{E}_V)$ and, when the disturbance vanishes, the system states converge to the origin, even when the largest admissible disturbance amplitude is applied in a single time instant (in the case of $w_k^{(2)}$).

4.5.5 DoA estimate for an uncertain rational system – The feedback linearised pendulum

In this example, our goal is to use *Corollary* 4.6, for a rational nonlinear system with both *known* and *uncertain* time-varying parameters. Consider the inverted pendulum model, from Azizi, Torres & Palhares [13]:

$$\ddot{\theta}(t) = \frac{g}{l}\sin(\theta(t)) - \frac{b\dot{\theta}(t)}{M} + \frac{\tau(t)}{Ml^2}, \tag{4.41}$$

where g is the gravitational acceleration, l is the length of the pendulum rod, M is the total mass and b is the damping coefficient. Besides that, $\theta(t)$ is the angle from the vertical direction and $\tau(t)$ is the control torque. Suppose that parameters b and M can change over time, such that $M = M_0(1+\delta(t))$ is a time-varying parameter, which is known, and $b = b_0(1+\bar{\delta}(t))$ is an uncertain parameter, with M_0 and b_0 being nominal values. Using the change of variables $r = \arctan(\theta)$, with $\sin(\theta) = (2r)/(1+r^2)$ and $\cos(\theta) = (1-r^2)/(1+r^2)$ proposed in Rohr, Pereira & Coutinho [62], and the input-to-state feedback linearization control proposed in Azizi, Torres & Palhares [13]

$$\tau(t) = \frac{2M_0 l^2}{1+x_1^2}\left[\text{sat}(u(t)) - \left(\frac{2x_1x_2^2}{1+x_1^2} + \frac{g}{l}x_1 - \frac{b_0}{M_0}x_2\right)\right], \tag{4.42}$$

it is possible to rewrite equation (4.41) as the following rational system:

$$\begin{aligned}\dot{x}_{(1)}(t) &= x_{(2)}(t),\\ \dot{x}_{(2)}(t) &= -\frac{b_0\bar{\delta}(t)}{M_0(1+\delta(t))}x_{(2)}(t) + \frac{\delta(t)}{1+\delta(t)}\left(\frac{2x_{(1)}(t)x_{(2)}^2(t)}{1+x_{(1)}^2(t)} + \frac{g}{l}x_{(1)}(t)\right) + \frac{1}{1+\delta(t)}\text{sat}(u(t)).\end{aligned} \tag{4.43}$$

Now, using Euler's first-order approximation, the following discrete-time model is obtained:

$$\begin{aligned}x_{(1)k+1} &= x_{(1)k} + Tx_{(2)k},\\ x_{(2)k+1} &= x_{(2)k} + T\left[-\frac{b_0\bar{\delta}_k}{M_0(1+\delta_k)}x_{(2)k} + \frac{\delta_k}{1+\delta_k}\left(\frac{2x_{(1)k}x_{(2)k}^2}{1+x_{(1)k}^2} + \frac{g}{l}x_{(1)k}\right) + \frac{1}{1+\delta_k}\text{sat}(u_k)\right].\end{aligned} \tag{4.44}$$

where T is the sampling period.

Thus, system (4.44) can be recast as an DAR, following expression (3.10) (page 39), with

$$\pi_k = \begin{bmatrix}\frac{x_{(1)k}}{1+\delta_k} & \frac{x_{(2)k}}{1+\delta_k} & \frac{x_{(1)k}x_{(2)k}\delta_k}{(1+\delta_k)\left(1+x_{(1)k}^2\right)} & \frac{x_{(1)k}\delta_k}{1+x_{(1)k}^2} & \frac{x_{(1)k}^2\delta_k}{1+x_{(1)k}^2} & \frac{\text{sat}(u_k)}{1+\delta_k}\end{bmatrix}^\top,$$

$$A_1 = \begin{bmatrix}1 & T\\ 0 & 1\end{bmatrix},\ A_2(x_k,\delta_k,\bar{\delta}_k) = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ \frac{Tg}{l}\delta_k & -\frac{Tb_o}{M_o}\bar{\delta}_k & 2Tx_{(2)k} & 0 & 0 & -T\delta_k\end{bmatrix},\ A_3 = \begin{bmatrix}0\\ T\end{bmatrix},$$

$$\Omega_1(\delta_k) = \begin{bmatrix}1 & 0 & 0 & \delta_k & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\end{bmatrix}^\top,\quad \Omega_3 = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}^\top,$$

$$\Omega_2(x_k,\delta_k) = \begin{bmatrix}-1-\delta_k & 0 & 0 & 0 & 0 & 0\\ 0 & -1-\delta_k & 0 & 0 & 0 & 0\\ 0 & 0 & -1-\delta_k & x_{(2)k} & 0 & 0\\ 0 & 0 & 0 & -1 & -x_{(1)_k} & 0\\ 0 & 0 & 0 & x_{(1)k} & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1-\delta_k\end{bmatrix}.$$

Choosing $T = 0.1$s and $u_0 = 1$, the optimization problem (4.31) was solved considering system (4.44) with $M_0 = 2$Kg, $l = 1$m, $g = 9.8$m/s^2, $b_0 = 0.1$Ns/m, $\Delta := \{\delta_k \in \mathbb{R} : |\delta_k| \leq 0.09\}$ and $\overline{\Delta} := \{\overline{\delta}_k \in \mathbb{R} : |\overline{\delta}_k| \leq 0.9\}$. In this scenario, the maximum value for the objective function is $\log(\det(Q)) = -0.21$, with the states' bounds defined by $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : |x_{(1)k}| \leq 1.2, |x_{(2)k}| \leq 1.1\right\}$.

The closed-loop results obtained by simulating equations (4.44) together with the control law (3.12) are shown in Figure 4.12, where the largest estimated DoA obtained and some trajectories initiating inside it are depicted for different time-varying sequences for $\delta_k \in \Delta$ and $\overline{\delta}_k \in \overline{\Delta}$.

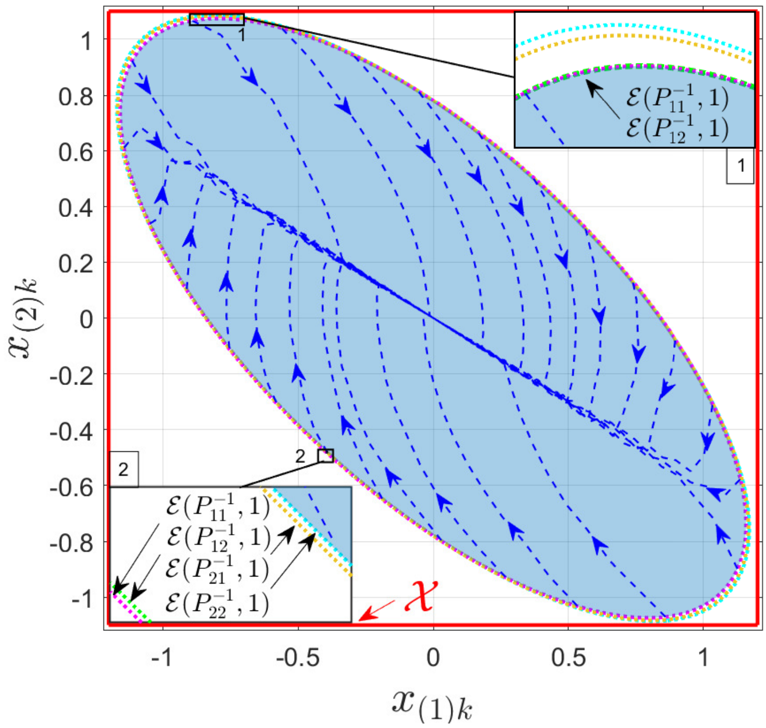


Figure 4.12 – Estimated DoA and some state trajectories (blue dashed lines) for system (4.44). $\mathcal{E}_{DoA}$ (region filled in blue) is the estimated DoA obtained from *Corollary 4.6*. The four ellipsoids (magenta, green, orange, and cyan dotted lines) associated with $\mathcal{E}(P_{lo}^{-1},1)$ are also shown in this figure.

This example illustrates the effectiveness of the proposed method for rational nonlinear systems when known and uncertain time-varying parameters are considered simultaneously. Notice that the estimated DoA (region filled in blue) in Figure 4.12 is not an ellipsoid in this case, but the intersection of the four ellipsoids (magenta, green, orange, and cyan dotted lines) associated with $\mathcal{E}(P_{lo}^{-1},1)$.

In addition, in Figure 4.12, zoom images are presented at different points. At point 1 (top right corner), taking the DoA as a reference, there are the overlapping ellipsoids $\mathcal{E}(P_{11}^{-1},1)$ (magenta) and $\mathcal{E}(P_{12}^{-1},1)$ (green), followed by the ellipsoid $\mathcal{E}(P_{21}^{-1},1)$ (orange) and the ellipsoid $\mathcal{E}(P_{22}^{-1},1)$ (cyan), which is the furthest from the DoA at this point. On the other hand, at point 2 (lower-left corner), this order is reversed, which shows that there are points of intersection between these regions and highlights the characteristic of the parameter-dependent Lyapunov function. Despite this, it is worth mentioning that in this example, the approach using a parameter-dependent Lyapunov function does not show a significant advantage compared to using a standard quadratic Lyapunov function.

4.6 Final remarks

This chapter has introduced new stabilization conditions to design gain-scheduled state feedback controllers for discrete-time nonlinear systems with time-varying parameters, described in a DAR form.

By taking advantage of the DAR, firstly it was considered the control strategy (3.11), which makes use of information on the nonlinearity vector π_k. Based on the Lyapunov theory and using a parameter-dependent Lyapunov function candidate, two optimization problems subjected to LMI constraints were stated to obtain the largest estimated Domain-of-Attraction (DoA), or to minimize the ℓ_2-gain from the energy-bounded disturbance input to the performance output for zero initial conditions. Furthermore, the proposed approach was adapted to the control law (3.12), where only the system's state vector information is used.

In the sequel, it was provided a preliminary analysis of the computational complexity for each case and it was shown how one can extend the proposed methodology to consider uncertain time-varying parameters.

Finally, it has presented numerical examples in which the proposed approaches were applied to provide the regional stabilization of polynomial and rational nonlinear systems.

The first example showed a favorable comparison between our methodology and that provided by Oliveira, Gomes da Silva Jr & Coutinho [12] to the problem of regional stabilization and DoA estimate, in the context of DAR. Moreover, from the first and second examples, it was possible to note that less conservative results can be obtained when the nonlinearity vector is taken into account in the control input.

Besides that, some numerical examples illustrated the use of the proposed methodology for nonlinear systems with simultaneously known and uncertain time-varying parameters, showing its effectiveness of the proposed control synthesis methodologies in this more general context.

Finally, it is worth mentioning that in this part of the research we did not find a solution to effectively include information about *linear annihilators*, as proposed by Trofino & Dezuo [97]. In the next part of the research, we present a new methodology based on polynomial Lyapunov functions. This new approach is developed without the need for congruence transformations and allows the use of *linear annihilators* to reduce conservatism, as shown by numerical examples in *Chapter* 6.

Part III

Novel stabilization conditions using polynomial Lyapunov functions

5 PRELIMINARIES FOR PART III

In this chapter, preliminary concepts regarding the use of polynomial Lyapunov functions to obtain regional stabilization conditions are presented. The Lyapunov function candidate is defined and some aspects are discussed. In the sequel, the concept of linear annihilator is introduced. Finally, the problems addressed in this part of the research are defined.

The following developments were greatly influenced by the discussion in Coutinho & Souza [50].

5.1 Lyapunov function candidate

This part of the research is concerned with developing LMI-based conditions that provide the stabilization of system (2.1) (page 24), using the DAR representation in (2.3) (page 25) and a polynomial Lyapunov function candidate, aiming to reduce conservativeness. In this sense, the following Lyapunov function candidate is considered

$$V(x_k) = \begin{bmatrix} x_k^\top & \Theta_k^\top \end{bmatrix} \begin{bmatrix} P & \star \\ Z^\top & W \end{bmatrix} \begin{bmatrix} x_k \\ \Theta_k \end{bmatrix}, \tag{5.1}$$

where $\Theta_k \in \mathbb{R}^{n_\Theta}$ is a vector of polynomial nonlinear functions of (x_k), P, W, Z are constant matrices to be determined, and P and W are symmetric matrices.

There is no systematic procedure for generating a less conservative polynomial Lyapunov function. Generally, increasing the complexity of vector Θ_k can lead to less conservative results at the cost of extra computations. An approach to deal with this problem is to start with a simple polynomial function and increase the complexity until some stopping criterion has been reached. Alternatively, we can use this procedure to include specific terms that are related to particular properties of the systems, such as its total mechanical energy, etc.

5.2 Regional stabilization and polynomial Lyapunov function

In this part of the research, for simplicity, we consider system (2.1) without the presence of disturbance inputs ($w_k = 0$), and focus on the problem of regional stabilization with DoA estimation, presented in the Subsection 3.1.1 (page 34).

The level set associated with the function (5.1) is defined by

$$\mathcal{R}_{DoA} := \{x_k \in \mathbb{R}^{n_x} : V(x_k) \leq 1\}. \tag{5.2}$$

If $V(x_k) > 0$ and $\Delta V = V(x_{k+1}) - V(x_k) < 0$ along the trajectories of the closed-loop system, $\forall x_k \in \mathcal{R}_{DoA}$ and $x_k \neq 0$, then (5.1) is said to be a Lyapunov function and

$\mathcal{R}_{DoA}$ is an invariant set with respect to the closed-loop system dynamics, which ensures that for $x_0 \in \mathcal{R}_{DoA}$, $x_k \to 0$, when $k \to \infty$.

Letting $\Theta_k := \Theta(x_k)$ and $\Theta_{k+1} := \Theta(x_{k+1})$, we have

$$\Delta V = \begin{bmatrix} x_{k+1}^\top & \Theta_{k+1}^\top \end{bmatrix} \begin{bmatrix} P & \star \\ Z^\top & W \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \Theta_{k+1} \end{bmatrix} - \begin{bmatrix} x_k^\top & \Theta_k^\top \end{bmatrix} \begin{bmatrix} P & \star \\ Z^\top & W \end{bmatrix} \begin{bmatrix} x_k \\ \Theta_k \end{bmatrix}.$$

As Θ_k is a polynomial function, similar to DAR (2.3), Θ_{k+1} and Θ_k can be related by the following difference and algebraic equations

$$\begin{aligned} &\begin{bmatrix} Y_{00}(x_k,\delta_k,x_{k+1}) \\ Y_{01}(x_k,\delta_k,x_{k+1}) \end{bmatrix} \Theta_{k+1} + \begin{bmatrix} Y_{10}(x_k,\delta_k,x_{k+1}) \\ Y_{11}(x_k,\delta_k,x_{k+1}) \end{bmatrix} x_{k+1} \\ &\qquad + \begin{bmatrix} Y_{20}(x_k,\delta_k,x_{k+1}) \\ Y_{21}(x_k,\delta_k,x_{k+1}) \end{bmatrix} \zeta_k + \begin{bmatrix} Y_{30}(x_k,\delta_k,x_{k+1}) \\ Y_{31}(x_k,\delta_k,x_{k+1}) \end{bmatrix} u_k = 0, \end{aligned} \tag{5.3}$$

$$X_0(x_k)\Theta_k + X_1(x_k)x_k = 0, \tag{5.4}$$

where $\zeta_k := \zeta_k(x_k,\delta_k,x_{k+1},u_k) \in \mathbb{R}^{n_\zeta}$ is an auxiliary vector of rational functions with respect to (x_k,δ_k,x_{k+1}) and affine with respect to (u_k). The matrices $Y_{00}(x_k,\delta_k,x_{k+1}) \in \mathbb{R}^{n_\Theta \times n_\Theta}$, $Y_{10}(x_k,\delta_k,x_{k+1}) \in \mathbb{R}^{n_\Theta \times n_x}$, $Y_{20}(x_k,\delta_k,x_{k+1}) \in \mathbb{R}^{n_\Theta \times n_\zeta}$, $Y_{30}(x_k,\delta_k,x_{k+1}) \in \mathbb{R}^{n_\Theta \times n_u}$, $Y_{01}(x_k,\delta_k,x_{k+1}) \in \mathbb{R}^{n_\zeta \times n_\Theta}$, $Y_{11}(x_k,\delta_k,x_{k+1}) \in \mathbb{R}^{n_\zeta \times n_z}$, $Y_{21}(x_k,\delta_k,x_{k+1}) \in \mathbb{R}^{n_\zeta \times n_\zeta}$, $Y_{31}(x_k,\delta_k,x_{k+1}) \in \mathbb{R}^{n_\zeta \times n_u}$ are affine functions on (x_k,δ_k,x_{k+1}), and matrices $X_0(x_k) \in \mathbb{R}^{n_\Theta \times n_\Theta}$ and $X_1(x_k) \in \mathbb{R}^{n_\Theta \times n_x}$ affine functions on (x_k).

In this case, the following assumption is considered.

Assumption 5.1. *Regarding equations* (5.3) *and* (5.4)*:*

a) $X_0(x_k)$ *has full rank for all* $x_k \in \mathcal{X}$*;*

b) $\begin{bmatrix} Y_{00}(x_k,\delta_k,x_{k+1}) & Y_{20}(x_k,\delta_k,x_{k+1}) \\ Y_{01}(x_k,\delta_k,x_{k+1}) & Y_{21}(x_k,\delta_k,x_{k+1}) \end{bmatrix}$ *has full rank for all* $(x_k,\delta_k) \in \mathcal{X} \times \Delta$.

Assumption 5.1 implies that matrices $X_0(x_k)$, $Y_{00}(x_k,\delta_k,x_{k+1})$, and $Y_{21}(x_k,\delta_k,x_{k+1})$ are nonsingular. As a result, it follows that

$$\begin{cases} \Theta_k = -X_0^{-1}(x_k)X_1(x_k)x_k, \\ \Theta_{k+1} = -Y_{00}^{-1}(x_k,\delta_k,x_{k+1})\left[Y_{10}(x_k,\delta_k,x_{k+1})x_{k+1} + Y_{20}(x_k,\delta_k,x_{k+1})\zeta_k + Y_{30}(x_k,\delta_k,x_{k+1})u_k\right], \\ \zeta_k = -Y_{21}^{-1}(x_k,\delta_k,x_{k+1})\left[Y_{01}(x_k,\delta_k,x_{k+1})\Theta_{k+1} + Y_{11}(x_k,\delta_k,x_{k+1})x_{k+1} + Y_{31}(x_k,\delta_k,x_{k+1})u_k\right]. \end{cases}$$

It is worth registering that the matrices of affine functions with respect to (x_k, δ_k, x_{k+1}) belong to polytopes of matrices that can be compactly represented as convex combinations of the vertices of the polytopic region $\mathcal{X} \times \Delta \times \mathcal{X}$, such that

$$O(x_k, \delta_k, x_{k+1}) = \sum_{i=1}^{N_x} \sum_{l=1}^{N_\delta} \sum_{t=1}^{N_x} \alpha_{x_{(i)k}} \alpha_{\delta_{(l)k}} \alpha_{x_{(t)k+1}} O_{ilt}, \tag{5.5}$$

In this investigation, we considered arbitrary variation rates for the system states. However, one can assume that these rates of variation are bounded, which can result in less conservativeness but increases the complexity of developments.

Equations (5.3) and (5.4) can be obtained using a similar procedure as described in Subsection 2.2.1 (page 25). In the sequel, an example is provided to illustrate some particularities in this case. More complex numerical examples are provided in Chapter 6.

Example 5.1. *Consider the following nonlinear system*

$$\begin{aligned} x_{(1)k+1} &= (1 + \delta_{(1)k}) x_{(2)k}, \\ x_{(2)k+1} &= x_{(1)k} + x_{(2)k}^3 + u_k. \end{aligned} \tag{5.6}$$

Following the next steps, it is possible to obtain the difference and algebraic equations for Θ_{k+1} and Θ_k, respectively.

(i) ***Start by defining vector Θ_k***

A possible choice of Θ_k which allows us to express the algebraic equation (5.4) *as an affine combination of x_k and Θ_k is*

$$\Theta_k = \begin{bmatrix} x_{(1)k}^2 & x_{(1)k} x_{(2)k} & x_{(2)k}^2 & x_{(1)k}^3 \end{bmatrix}^\top. \tag{5.7}$$

(ii) ***Now, it is possible to obtain the algebraic equation based on each element of Θ_k***

The entries of the chosen vector Θ_k are such that

$$\begin{cases} x_{(1)k}^2 - \Theta_{(1)k} = 0 \\ x_{(1)k} x_{(2)k} - \Theta_{(2)k} = 0 \\ x_{(2)k}^2 - \Theta_{(3)k} = 0 \\ x_{(1)k} \Theta_{(1)k} - \Theta_{(4)k} = 0 \end{cases},$$

Thus, it is possible to write equation (5.4) *as*

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ x_{(1)k} & 0 & 0 & -1 \end{bmatrix}}_{X_0(x_k)} \Theta_k + \underbrace{\begin{bmatrix} x_{(1)k} & 0 \\ 0 & x_{(1)k} \\ 0 & x_{(2)k} \\ 0 & 0 \end{bmatrix}}_{X_1(x_k)} x_k = 0.$$

(iii) ***Write vector Θ_{k+1}, considering the defined Θ_k and the system state-space model***

From (5.7)*, we have* $\Theta_{k+1} = \begin{bmatrix} x^2_{(1)k+1} & x_{(1)k+1}x_{(2)k+1} & x^2_{(2)k+1} & x^3_{(1)k+1} \end{bmatrix}^\top$*. Using the system equations in* (5.6)*, vector* Θ_{k+1} *can be recast as*

$$\Theta_{k+1} = \begin{bmatrix} (1+\delta_{(1)k})x_{(2)k}x_{(1)k+1} \\ (1+\delta_{(1)k})x_{(2)k}x_{(2)k+1} \\ (x_{(1)k} + x^3_{(2)k} + u_k)x_{(2)k+1} \\ (1+\delta_{(1)k})x_{(2)k}x^2_{(1)k+1} \end{bmatrix} \tag{5.8}$$

(iv) ***Then, choose a vector ζ_k so that the difference equation can be obtained as in* (5.3)*, where the matrices are affine functions on* (x_k, δ_k, x_{k+1})**

In this case, a possible choice of ζ_k *is*

$$\zeta_k = \begin{bmatrix} x_{(2)k}x_{(1)k+1} \\ x_{(2)k}x_{(2)k+1} \\ x^2_{(2)k}x_{(2)k+1} \end{bmatrix}. \tag{5.9}$$

(v) ***Finally, one can write the difference equation based on each element of Θ_{k+1} and ζ_k***

From (5.8) *and* (5.9) *it follows that*

$$\begin{cases} x_{(2)k}x_{(1)k+1} + \delta_{(1)k}\zeta_{(1)k} - \Theta_{(1)k+1} = 0, \\ x_{(2)k}x_{(2)k+1} + \delta_{(1)k}\zeta_{(2)k} - \Theta_{(2)k+1} = 0, \\ x_{(1)k}x_{(2)k+1} + x_{(2)k}\zeta_{(3)k} + x_{(2)k+1}u_k - \Theta_{(3)k+1} = 0, \\ x_{(1)k+1}\Theta_{(1)k+1} - \Theta_{(4)k+1} = 0, \end{cases}$$

$$\begin{cases} x_{(2)k}x_{(1)k+1} - \zeta_{(1)k} = 0, \\ x_{(2)k}x_{(2)k+1} - \zeta_{(2)k} = 0, \\ x_{(2)k}\zeta_{(1)k} - \zeta_{(3)k} = 0. \end{cases}$$

leading to the difference equation in (5.3) *with*

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ x_{(1)k+1} & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} \Upsilon_{00} \\ \Upsilon_{01} \end{bmatrix}} \Theta_{k+1} + \underbrace{\begin{bmatrix} x_{(2)k} & 0 \\ 0 & x_{(2)k} \\ 0 & x_{(1)k} \\ 0 & 0 \\ x_{(2)k} & 0 \\ 0 & x_{(2)k} \\ 0 & 0 \end{bmatrix}}_{\begin{bmatrix} \Upsilon_{10} \\ \Upsilon_{11} \end{bmatrix}} x_{k+1}$$

$$+ \underbrace{\begin{bmatrix} \delta_{(1)k} & 0 & 0 \\ 0 & \delta_{(1)k} & 0 \\ 0 & 0 & x_{(2)k} \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ x_{(2)k} & 0 & -1 \end{bmatrix}}_{\begin{bmatrix} \Upsilon_{20} \\ \Upsilon_{21} \end{bmatrix}} \zeta_k + \underbrace{\begin{bmatrix} 0 \\ 0 \\ x_{(2)k+1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\begin{bmatrix} \Upsilon_{30} \\ \Upsilon_{31} \end{bmatrix}} u_k = 0.$$

5.3 Linear annihilators

As previously discussed, the particular choice of system matrices in (2.3) *is not unique. As a result, a bad choice may lead to conservative results. In this part of the research, to reduce this potential conservativeness, we follow the procedure proposed by Trofino & Dezuo [97] for DAR models, which consists in using the concept of* linear annihilators.

Definition 5.1 (Linear annihilator [97]). *The matrix $\aleph_x(x_k) : \mathbb{R}^{n_x} \to \mathbb{R}^{n_q \times n_x}$ is a linear annihilator of the state vector x_k if $\aleph_x(x_k)x_k = 0$ and $\aleph_x(x_k)$ is linear with respect to x_k.*

Note that there is no single annihilator to a given system. In this thesis, the following annihilator, proposed by Trofino & Dezuo [97], that takes into account all possible product pairs $x_{(i)k}x_{(j)k}$, $\forall i,j \in \mathcal{I}_{n_x}$ and $i \neq j$, is used:

$$\aleph_x(x_k) = \begin{bmatrix} \mathcal{A}_1(x_k) & \mathcal{B}_1(x_k) \\ \vdots & \vdots \\ \mathcal{A}_{n_x-1}(x_k) & \mathcal{B}_{n_x-1}(x_k) \end{bmatrix}, \tag{5.10}$$

where

$$\mathcal{B}_i(x_k) = -x_{(i)k} I_{n_x - i}, \quad i \in \mathcal{I}_{n_x-1},$$

$$\mathcal{A}_1(x_k) = \begin{bmatrix} x_{(2)k} & \cdots & x_{(n_x)k} \end{bmatrix}^\top,$$

$$\mathcal{A}_i(x_k) = \begin{bmatrix} & x_{(i+1)k} \\ 0_{(n_x-i)\times(i-1)} & \vdots \\ & x_{(n_x)k} \end{bmatrix}, \quad i \in [2, n_x - 1],$$

with the number of rows $n_q = \sum_{j=1}^{n_x-1} j$.

Example 5.2. *For* $n_x = 3$*, the following expression is obtained from* (5.10)*:*

$$\aleph_x(x_k)x_k = \begin{bmatrix} x_{(2)k} & -x_{(1)k} & 0 \\ x_{(3)k} & 0 & -x_{(1)k} \\ 0 & x_{(3)k} & -x_{(2)k} \end{bmatrix} \begin{bmatrix} x_{(1)k} \\ x_{(2)k} \\ x_{(3)k} \end{bmatrix} = 0.$$

5.4 Problem statement

Based on the aforementioned, by considering system (2.1) *in a DAR form* (2.3) *and using polynomial Lyapunov functions as in* (5.1)*, this part of the research is particularly concerned with proposing sufficient conditions to design state and static output-feedback controllers such that* $\mathcal{R}_{DoA} \subset \mathcal{X}$*,* $\forall \delta_k \in \Delta$*, is an invariant set with respect to the closed-loop system dynamics such that,* $\forall x_0 \in \mathcal{R}_{DoA}$*,* $x_k \to 0$*, when* $k \to \infty$*, and* $\mathcal{R}_{DoA}$ *is as large as possible.*

It is worth mentioning that we do not incorporate information about the control input saturation in the following developments. However, this is an objective that we intend to pursue in future investigations.

6 STABILIZATION CONDITIONS BASED ON POLYNOMIAL LYAPUNOV FUNCTIONS

This chapter presents novel stabilization conditions based on polynomial Lyapunov functions. Firstly, a new strategy for gain-scheduled SF control is proposed. Secondly, a new gain-scheduled SOF control design solution is derived. Finally, numerical examples illustrate the proposed methodology's potential.

6.1 State Feedback control

In this section, novel stabilization conditions to design SF controllers are provided, considering polynomial Lyapunov functions, as defined in (5.1), on page 73, and the following control law

$$u_k = G^{-1}(x_k, \delta_k)K(x_k, \delta_k)\xi_k, \quad \xi_k = \begin{bmatrix} x_k^\top & \pi_k^\top \end{bmatrix}^\top, \tag{6.1}$$

with $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times (n_x + n_\pi)}$ and $G(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_u}$ matrices to be determined.

Remark 6.1. *Notice that we consider both the system model and the control input represented from the same basis function π_k. However, the elements of π_k that do not appear in the system representation can be removed by nulling the respective columns of the DAR matrix $A_2(x_k, \delta_k)$. On the other hand, it is possible to remove the elements of π_k that we do not want at the control input by nulling the respective columns of matrix $K(x_k, \delta_k)$ in the control input* (6.1)*. For instance, the proposed control law includes the particular case*

$$u_k = G^{-1}(x_k, \delta_k)\bar{K}(x_k, \delta_k)x_k, \tag{6.2}$$

by considering $K(x_k, \delta_k) = \begin{bmatrix} \bar{K}(x_k, \delta_k) & 0 \end{bmatrix}$ in (6.1)*, with $\bar{K}(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$.*

In the sequel, sufficient conditions to compute the SF control matrices that stabilize the nonlinear system (2.1) (page 24), for $w_k = 0$ are presented.

Theorem 6.1. *Consider the nonlinear system* (2.1) *in a DAR form* (2.3)*, on page 25, with $w_k = 0$, and a nonlinear vector Θ_k as described in Section 5.2. Let ϵ be a given positive scalar. If there exist positive scalars τ_v, symmetric matrices $P \in \mathbb{R}^{n_x \times n_x}$ and $W \in \mathbb{R}^{n_\Theta \times n_\Theta}$, and matrices $Z \in \mathbb{R}^{n_x \times n_\Theta}$, $G(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_u}$, $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times (n_x + n_\pi)}$, $L(x_k) \in \mathbb{R}^{(n_x + n_\Theta) \times (n_q + n_\Theta)}$, $R(x_k, \delta_k, x_{k+1}) \in \mathbb{R}^{(2n_x + n_\pi + n_u + 2n_\Theta + n_\zeta) \times (2n_\Theta + n_\zeta + n_x + n_\pi + n_q)}$, and $S_v \in \mathbb{R}^{(n_x + n_\Theta + 1) \times (n_x + n_\Theta + n_q)}$, $v \in \mathcal{I}_{n_e}$, such that the following inequalities hold*

$$\Xi_1 + L(x_k)N_1(x_k) + N_1^\top(x_k)L^\top(x_k) > 0, \tag{6.3}$$

$$\Xi_2(x_k, \delta_k) + R(x_k, \delta_k, x_{k+1})N_2(x_k, \delta_k, x_{k+1}) + N_2^\top(x_k, \delta_k, x_{k+1})R^\top(x_k, \delta_k, x_{k+1}) < 0, \tag{6.4}$$

$$\Xi_{3v} + S_v N_3(x_k) + N_3^\top(x_k)S_v^\top \geq 0, \; v \in \mathcal{I}_{n_e} \tag{6.5}$$

$$G^\top(x_k,\delta_k) + G(x_k,\delta_k) > 0, \tag{6.6}$$

where

$$\Xi_1 = \begin{bmatrix} P & \star \\ Z^\top & W \end{bmatrix}, \quad N_1(x_k) = \begin{bmatrix} X_1(x_k) & X_0(x_k) \\ \aleph(x_k) & 0 \end{bmatrix},$$

with the linear annihilator $\aleph(x_k)$ *given in* (5.10), *on page 77, and*

$$\Xi_2(x_k,\delta_k) = \begin{bmatrix} P & \star & \star & \star & \star \\ \epsilon K^\top(x_k,\delta_k)A_3^\top(x_k,\delta_k) & -P_a & \star & \star & \star \\ -\epsilon G^\top(x_k,\delta_k)A_3^\top(x_k,\delta_k) & 0 & 0 & \star & \star \\ Z^\top & 0 & 0 & W & \star \\ 0 & -Z_a^\top & 0 & 0 & -W_a \end{bmatrix},$$

$$P_a = \begin{bmatrix} P & \star \\ 0 & 0 \end{bmatrix}, \quad W_a = \begin{bmatrix} W & \star \\ 0 & 0 \end{bmatrix}, \quad Z_a = \begin{bmatrix} Z & \star \\ 0 & 0 \end{bmatrix},$$

$$N_2(x_k,x_{k+1}) = \begin{bmatrix} \mathcal{Y}_{10} & 0 & 0 & \mathcal{Y}_{30} & \mathcal{Y}_{00} & 0 & \mathcal{Y}_{20} \\ \mathcal{Y}_{11} & 0 & 0 & \mathcal{Y}_{31} & \mathcal{Y}_{01} & 0 & \mathcal{Y}_{21} \\ 0 & X_1(x_k) & 0 & 0 & 0 & X_0(x_k) & 0 \\ -I & A_1(x_k,\delta_k) & A_2(x_k,\delta_k) & A_3(x_k,\delta_k) & 0 & 0 & 0 \\ 0 & \Omega_1(x_k,\delta_k) & \Omega_2(x_k,\delta_k) & \Omega_3(x_k,\delta_k) & 0 & 0 & 0 \\ 0 & \aleph(x_k) & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\mathcal{Y}_{ab} = Y_{ab}(x_k,\delta_k,x_{k+1}), a = 0,1,2,3, b = 0,1$ *are given in* (5.3),

$$\Xi_{3v} = \begin{bmatrix} 2\tau_v - 1 & \star & \star \\ -\tau_v a_v & P & \star \\ 0 & Z^\top & W \end{bmatrix}, \text{ and } N_3(x_k) = \begin{bmatrix} x_k & -I & 0 \\ 0 & X_1(x_k) & X_0(x_k) \\ 0 & \aleph(x_k) & 0 \end{bmatrix},$$

then there exist a Lyapunov function (5.1) *(on page 73) and a controller given in* (6.1) *such that,* $\forall x_0$ *inside* $\mathcal{R}_{DoA}$*, the trajectory of* x_k *converge to the origin when* $k \to \infty$ *and* $\mathcal{R}_{DoA}$ *is an estimate of the DoA.*

Proof. Consider inequality (6.3) and the vector $\vartheta_1 = \begin{bmatrix} x_k^\top & \Theta_k^\top \end{bmatrix}^\top$. Since $N_1\vartheta_1 = 0$, pre- and post-multiplying (6.3) by $\vartheta_1^\top$ and ϑ_1, respectively, leads to $V(x_k) > 0, \forall x_k \in \mathcal{X}$ and $x_k \neq 0$.

On the other hand, inequality (6.4) can be recast as

$$\underbrace{\Phi_1 + J\Phi_2 + \Phi_2^\top J^\top}_{\Xi_2} + RN_2 + N_2^\top R^\top < 0, \tag{6.7}$$

with

$$\Phi_1 = \begin{bmatrix} P & \star & \star & \star & \star & \star \\ 0 & -P_a & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ Z^\top & 0 & 0 & W & \star & \star \\ 0 & -Z_b^\top & 0 & 0 & -W & \star \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; J = \begin{bmatrix} \epsilon A_3(x_k,\delta_k)G(x_k,\delta_k) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; Z_b = \begin{bmatrix} Z \\ 0 \end{bmatrix},$$

and

$$\Phi_2 = \begin{bmatrix} 0 & G^{-1}(x_k, \delta_k)K(x_k, \delta_k) & -I & 0 & 0 & 0 \end{bmatrix}.$$

Defining $\vartheta_2 = \begin{bmatrix} x_{k+1}^\top & \xi_k^\top & u_k^\top & \Theta_{k+1}^\top & \Theta_k^\top & \zeta_k^\top \end{bmatrix}^\top$, with $\xi_k = \begin{bmatrix} x_k^\top & \pi_k^\top \end{bmatrix}^\top$, as stated previously in (6.1), one has that $\Phi_2\vartheta_2 = 0$ and $N_2\vartheta_2 = 0$. By pre- and post-multiplying (6.7) by $\vartheta_2^\top$ and ϑ_2, respectively, results in $\Delta V_k < 0$.

Therefore, if the conditions (6.3) and (6.4) are feasible, then $V(x_k)$ in (5.1) is a Lyapunov function and the controller (6.1) ensures that the origin of the closed-loop system is asymptotically stable.

Multiplying (6.5) with $\begin{bmatrix} 1 & x_k^\top & \Theta_k^\top \end{bmatrix}$ on the left and its transpose on the right, yields

$$\begin{bmatrix} 1 & x_k^\top & \Theta_k^\top \end{bmatrix} \begin{bmatrix} 2\tau_v - 1 & \star & \star \\ -\tau_v a_v & P & \star \\ 0 & Z^\top & W \end{bmatrix} \begin{bmatrix} 1 \\ x_k \\ \Theta_k \end{bmatrix} \geq 0$$

Applying the S-Procedure, if the previous inequality is satisfied, then the following condition holds

$$2 - x_k^\top a_v - a_v^\top x_k \geq 0,\ v \in \mathcal{I}_{n_e}, \quad \forall x_k \in \mathcal{X} : V(x_k) = \begin{bmatrix} x_k^\top & \Theta_k^\top \end{bmatrix} \begin{bmatrix} P & \star \\ Z^\top & W \end{bmatrix} \begin{bmatrix} x_k \\ \Theta_k \end{bmatrix} \leq 1.$$

This guarantees the inclusion $\mathcal{R}_{DoA} \subseteq \mathcal{X}$.

Finally, condition (6.6) ensures the existence of the inverse of matrix $G(x_k, \delta_k)$, $\forall x_k \in \mathcal{X}$ and $\delta_k \in \Delta$, which is necessary to guarantee the computation of the control law in (6.1). □

Remark 6.2. *Notice that since matrix N_2 is given, it is possible to define $R = R(x_k, \delta_k)$ in* (6.7) *as a variable matrix affinely dependent on (x_k, δ_k). On the other hand, matrix Φ_2 in* (6.7) *is a variable of the problem. Thus, we defined J aiming to obtain numerically tractable conditions, where ϵ is a given positive scalar, $G(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_u}$ is a matrix to be determined, and $A_3(x_k, \delta_k) \in \mathbb{R}^{n_x \times n_u}$ is given by the DAR in* (2.3).

In this case, the positive scalar ϵ is introduced only to likely yield a less conservative result and the system matrix $A_3(x_k, \delta_k)$ is used in the defined matrix J to ensure the appropriate dimension of the element $J_{11} \in \mathbb{R}^{n_x \times n_u}$.

Therefore, inequality (6.7) *can be recast as in* (6.4)*, in Theorem 6.1. This inequality is in an infinite-dimensional form, but it can be converted into a finite set of LMIs, by using an LMI relaxation similar to that given in Appendix A.*

It is worth mentioning that from the control law defined, there are other possibilities for the choice of matrix J which allow us to obtain LMI conditions, as for instance

$$J = \begin{bmatrix} \epsilon_1 A_3(x_k, \delta_k) G(x_k, \delta_k) \\ \epsilon_2 A_3(x_k, \delta_k) G(x_k, \delta_k) \\ \epsilon_3 \Omega_3(x_k, \delta_k) G(x_k, \delta_k) \\ \epsilon_4 G(x_k, \delta_k) \\ \epsilon_5 Y_{30}(x_k, \delta_k, x_{k+1}) G(x_k, \delta_k) \\ \epsilon_6 Y_{30}(x_k, \delta_k, x_{k+1}) G(x_k, \delta_k) \\ \epsilon_7 Y_{31}(x_k, \delta_k, x_{k+1}) G(x_k, \delta_k) \end{bmatrix},$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6$, and ϵ_7 are given positive scalars. However, the chosen matrix J provided less conservative results among the alternatives we tested in numerical examples without the necessity of using more auxiliary decision variables.

By using the following Corollary, based on the idea presented by [50] for stability analysis applying polynomial Lyapunov functions in the context of DAR, it is possible to obtain the largest estimated DoA from *Theorem* 6.1.

Corollary 6.1. *The estimated DoA for the closed-loop system* (2.1)-(6.1)*, with $w_k = 0$, can be maximized over $\mathcal{X}$, solving the following optimization problem for all $x_k \in \mathcal{X}$, $\delta_k \in \Delta$.*

$$\min\ \sigma \quad \textit{subject to } (6.3)-(6.6), \tag{6.8}$$

with

$$\sigma - \operatorname{trace}\left(\Xi_1 + L(x_k)N_1(x_k) + N_1^\top(x_k)L^\top(x_k)\right) \geq 0. \tag{6.9}$$

The optimization problem presented in *Corollary* 6.1 is motivated by the fact that in the particular case of a quadratic Lyapunov function $V(x_k) = x_k^\top P x_k$, the minimization of the trace of P has the effect of maximizing the sum of the squared semi-axis lengths of the ellipsoid $\mathcal{E}(P,1) := \{x_k \in \mathbb{R}^{n_x} : x_k^\top P x_k \leq 1\}$, see the references [10, 90, 12, 14].

Remark 6.3. *Note that the novelty of this approach compared to that presented in Part II is related not only to the fact that a polynomial Lyapunov function candidate is considered, but also to the proposed control law and the methodology used to obtain the stabilization conditions, in which no congruence transformations are required. In our approach, the fact that congruence transformations are not applied is essential to consider a polynomial Lyapunov function candidate, as in* (5.1)*, and to incorporate the use of linear annihilators, effectively.*

The polynomial Lyapunov function candidate defined in (5.1) encompasses the quadratic Lyapunov function, by setting $\Theta_k = 0$, and the proposed methodology can also be adapted to the use of parameter-dependent Lyapunov functions, as

$$V(x_k, \delta_k) = x_k^\top P(\delta_k) x_k, \quad P(\delta_k) = \sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} P_l, \quad P_l = P_l^\top > 0, \quad \alpha_{\delta_k} \in \Lambda_1. \tag{6.10}$$

The level set associated with the function (6.10) is defined by

$$\mathcal{L}_{DoA} := \{x_k \in \mathbb{R}^{n_x} : V(x_k, \delta_k) \leq 1, \quad \forall \delta_k \in \Delta\}. \tag{6.11}$$

Lemma 6.1 (adapted from [89]). *The level set* (6.11) *associated with the function* (3.1) *verifies that*

$$\mathcal{L}_{DoA} = \bigcap_{l \in \{1,\ldots N_\delta\}} \mathcal{E}(P_l, 1), \tag{6.12}$$

with $\mathcal{E}(P_l, 1) := \left\{x_k \in \mathbb{R}^{n_x} : x_k^T P_l x_k \leq 1\right\}$.

Proof. $x_k \in \mathcal{L}_{DoA} \Leftrightarrow \forall \delta_k \in \Delta, V(x_k, \delta_k) < 1 \Leftrightarrow x_k \in \bigcap_{\delta_k \in \Delta} \mathcal{E}(P(\delta_k), 1)$. Moreover,

$$\bigcap_{\delta_k \in \Delta} \mathcal{E}(P(\delta_k), 1) \subset \bigcap_{l \in \{1,\ldots N_\delta\}} \mathcal{E}(P_l, 1)$$

To prove that

$$\bigcap_{l \in \{1,\ldots N_\delta\}} \mathcal{E}(P_l, 1) \subset \bigcap_{\delta_k \in \Delta} \mathcal{E}(P(\delta_k), 1),$$

consider $x_k \in \bigcap_{l \in \{1,\ldots N_\delta\}} \mathcal{E}(P_l, 1)$, then $x_k^T P_l x_k < 1$, $l \in \mathcal{I}_{N_\delta}$.

Since $\sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} = 1$, the above inequality can be recast as

$$x_k^T \left(\sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} P_l\right) x_k < 1, \quad l \in \mathcal{I}_{N_\delta}.$$

Thus,

$$x_k^T P(\delta_k) x_k < 1, \quad \delta_k \in \Delta.$$

This implies that $x_k \in \mathcal{E}(P(\delta_k), 1)$, or $x_k \in \bigcap_{\delta_k \in \Delta} \mathcal{E}(P(\delta_k), 1)$. □

Theorem 6.2, in the sequel, presents sufficient conditions adapted from *Theorem* 6.1 to compute the SF control matrices in (6.1) that stabilize the nonlinear system (2.1) (on page 24), with $w_k = 0$, by using parameter-dependent Lyapunov functions (6.10).

Theorem 6.2. *Consider the nonlinear system* (2.1) *and its DAR* (2.3)*, with* $w_k = 0$*. Let* ϵ *be a given positive scalar. If there exist positive scalars* τ_v*, matrices* $P(\delta_k) = P^\top(\delta_k) > 0$*,* $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$*,* $R(x_k, \delta_k) \in \mathbb{R}^{(2n_x + n_u + n_\pi) \times (n_x + n_\pi + n_q)}$*,* $G(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_u}$*, and* $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times (n_x + n_\pi)}$*, such that the following inequalities hold*

$$\Xi_2(x_k, \delta_k, \delta_{k+1}) + R(x_k, \delta_k) N_2(x_k, \delta_k) + N_2^\top(x_k, \delta_k) R^\top(x_k, \delta_k) < 0, \tag{6.13}$$

$$\begin{bmatrix} 2\tau_v - 1 & \star \\ -\tau_v a_v & P(\delta_k) \end{bmatrix} \geq 0, \quad v \in \mathcal{I}_{n_e}, \tag{6.14}$$

and

$$G^\top(x_k,\delta_k) + G(x_k,\delta_k) > 0, \tag{6.15}$$

where

$$\Xi_2(x_k,\delta_k,\delta_{k+1}) = \begin{bmatrix} P(\delta_{k+1}) & \star & \star \\ \epsilon K^\top(x_k,\delta_k) A_3^\top(x_k,\delta_k) & -P_a(\delta_k) & \star \\ -\epsilon G^\top(x_k,\delta_k) A_3^\top(x_k,\delta_k) & 0 & 0 \end{bmatrix}, \quad P_a(\delta_k) = \begin{bmatrix} P(\delta_k) & \star \\ 0 & 0 \end{bmatrix},$$

$$N_2(x_k,\delta_k) = \begin{bmatrix} -I & A_1(x_k,\delta_k) & A_2(x_k,\delta_k) & A_3(x_k,\delta_k) \\ 0 & \Omega_1(x_k,\delta_k) & \Omega_2(x_k,\delta_k) & \Omega_3(x_k,\delta_k) \\ 0 & \aleph_x(x_k) & 0 & 0 \end{bmatrix},$$

then there exist a Lyapunov function (6.10) *and a controller* (6.1) *such that,* $\forall x_0$ *inside* $\mathcal{L}_{DoA}$ *and* $\delta_k \in \Delta$*, the trajectory of* x_k *converge to the origin when* $k \to \infty$ *and* $\mathcal{L}_{DoA}$ *is an estimate of the DoA.*

Proof. Theorem 6.2 was obtained from *Theorem* 6.1, disregarding the influence of the nonlinearity vectors Θ_k and ζ_k, and incorporating information about time-varying parameters in the Lyapunov matrix P. Thus, following similar steps as in the proof of *Theorem* 6.1, if condition (6.13) is satisfied, then (6.10) is a Lyapunov function and the controller (6.1) ensures that the origin of the closed-loop system is asymptotically stable. Condition (6.14) guarantees the inclusion $\mathcal{L}_{DoA} \subseteq \mathcal{X}$ and condition (6.15) ensures that $G(x_k,\delta_k)$ is invertible, $\forall x_k \in \mathcal{X}$ and $\delta_k \in \Delta$. □

Similarly as presented in Subsection 3.1.1, one can consider the following subset of $\mathcal{L}_{DoA}$ to find the largest estimated DoA

$$\mathcal{E}(Q,1) \subseteq \bigcap_{l \in \{1,\ldots N_\delta\}} \mathcal{E}(P_l,1). \tag{6.16}$$

Thus, by solving the optimization problem in the next Corollary, it is possible to maximize the estimated DoA from *Theorem* 6.2

Corollary 6.2. *Given a positive scalar* $\epsilon > 0$*. If there exist positive scalars* τ_v*, symmetric matrices* $Q \in \mathbb{R}^{n_x \times n_x}$ *and* $P(\delta_k) > 0$*,* $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$*, and any matrices* $R(x_k,\delta_k) \in \mathbb{R}^{(2n_x+n_u+n_\pi)\times(n_x+n_\pi+n_q)}$*,* $G(x_k,\delta_k) \in \mathbb{R}^{n_u \times n_u}$*, and* $K(x_k,\delta_k) \in \mathbb{R}^{n_u \times (n_x+n_\pi)}$*, satisfying the following optimization problem for all* $\delta_k \in \Delta$ *and* $x_k \in \mathcal{X}$*:*

$$\min\{\text{trace}(Q)\} \quad \textit{subject to } (6.13)-(6.15), \tag{6.17}$$

with

$$Q - P_l > 0, \tag{6.18}$$

then the SF controller (6.1) *asymptotically stabilizes the closed-loop system, composed by* (2.3)*, on page 25, with* $w_k = 0$*, and* (6.1)*, around the origin, and* $\mathcal{L}_{DoA}$ *is an estimate of the DoA.*

Proof. Condition (6.18) ensures that $\mathcal{E}(Q,1) \subseteq \mathcal{L}_{DoA}$, which is defined in (6.12), and the rest of the proof follows in a straightforward way as in the proof of *Theorem* 6.2.

□

6.2 Static Output Feedback control

In this section, considering practical applications in which only the system output is measured in real-time, our goal is to design an SOF controller in the form

$$u_k = F^{-1}(\delta_k)H(\delta_k)y_k, \tag{6.19}$$

with $H(\delta_k) \in \mathbb{R}^{n_u \times n_y}$ and $F(\delta_k) \in \mathbb{R}^{n_u \times n_u}$ matrices to be determined.

Remark 6.4. *It is worth mentioning that in this case, the information about the system states vector is not taken into account in the gain matrices $F(\cdot)$ and $H(\cdot)$ which are only dependent on parameters (δ_k).*

Theorem 6.3 in the sequel presents a new SOF control design for the nonlinear system (2.3) (page 25), disregarding the influence of disturbance inputs by having $w_k = 0$.

Theorem 6.3. *Consider the nonlinear system* (2.1) *in a DAR form* (2.3)*, with $w_k = 0$, and a nonlinear vector Θ_k as described previously in Section 5.2 (page 73). Let ϵ be a given positive scalar. If there exist positive scalars τ_v, symmetric matrices $P \in \mathbb{R}^{n_x \times n_x}$ and $W \in \mathbb{R}^{n_\Theta \times n_\Theta}$, and matrices $Z \in \mathbb{R}^{n_x \times n_\Theta}$, $B(x_k) \in \mathbb{R}^{(n_x+n_\Theta)\times(n_\Theta+n_q)}$, $E(x_k,\delta_k,x_{k+1}) \in \mathbb{R}^{(2n_x+2n_\Theta+n_y+n_u+n_\pi+n_\zeta)\times(2n_\Theta+n_\zeta+n_x+n_y+n_\pi+n_q)}$, $T_v \in \mathbb{R}^{(n_x+n_\Theta+1)\times(n_x+n_\Theta+n_q)}$, $v \in \mathcal{I}_{n_e}$, $F(\delta_k) \in \mathbb{R}^{n_u \times n_u}$, and $H(\delta_k) \in \mathbb{R}^{n_u \times n_y}$, such that the following inequalities hold*

$$\Gamma_1 + B(x_k)M_1(x_k) + M_1^\top(x_k)B^\top(x_k) > 0, \tag{6.20}$$

$$\Gamma_2(x_k,\delta_k) + E(x_k,\delta_k,x_{k+1})M_2(x_k,\delta_k,x_{k+1}) + M_2^\top(x_k,\delta_k,x_{k+1})E^\top(x_k,\delta_k,x_{k+1}) < 0, \tag{6.21}$$

$$\Gamma_{3v} + T_v M_3(x_k) + M_3^\top(x_k)T_v^\top \geq 0,\ v \in \mathcal{I}_{n_e} \tag{6.22}$$

$$F(\delta_k)^\top + F(\delta_k) > 0, \tag{6.23}$$

where

$$\Gamma_1 = \begin{bmatrix} P & \star \\ Z^\top & W \end{bmatrix}, \quad M_1(x_k) = \begin{bmatrix} X_1(x_k) & X_0(x_k) \\ \aleph(x_k) & 0 \end{bmatrix},$$

with the linear annihilator $\aleph(x_k)$ given in (5.10) *(page 77),*

$$\Gamma_2(x_k,\delta_k) =$$
$$\begin{bmatrix} -P & \star & \star & \star & \star & \star & \star & \star \\ 0 & P & \star & \star & \star & \star & \star & \\ -Z^\top & 0 & -W & \star & \star & \star & \star & \star \\ 0 & Z^\top & 0 & W & \star & \star & \star & \star \\ \epsilon H^\top(\delta_k)A_3^\top(x_k,\delta_k) & H^\top(\delta_k)A_3^\top(x_k,\delta_k) & 0 & 0 & 0 & \star & \star & \star \\ -\epsilon F^\top(\delta_k)A_3^\top(x_k,\delta_k) & -F^\top(\delta_k)A_3^\top(x_k,\delta_k) & 0 & 0 & \epsilon H(\delta_k) & -\epsilon\,\mathrm{He}\{F(\delta_k)\} & \star & \star \\ 0 & 0 & 0 & 0 & \Omega_3(x_k,\delta_k)H(\delta_k) & -\Omega_3(x_k,\delta_k)F(\delta_k) & 0 & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_2(x_k,\delta_k,x_{k+1}) = \begin{bmatrix} X_1(x_k) & 0 & X_0(x_k) & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{Y}_{10} & 0 & \mathcal{Y}_{00} & 0 & \mathcal{Y}_{30} & 0 & \mathcal{Y}_{20} \\ 0 & \mathcal{Y}_{11} & 0 & \mathcal{Y}_{01} & 0 & \mathcal{Y}_{31} & 0 & \mathcal{Y}_{21} \\ A_1(x_k,\delta_k) & -I & 0 & 0 & 0 & A_3(x_k,\delta_k) & A_2(x_k,\delta_k) & 0 \\ C_{y_1}(x_k,\delta_k) & 0 & 0 & 0 & -I & 0 & C_{y_2}(x_k,\delta_k) & 0 \\ \Omega_1(x_k,\delta_k) & 0 & 0 & 0 & 0 & \Omega_3(x_k,\delta_k) & \Omega_2(x_k,\delta_k) & 0 \\ \aleph(x_k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\mathcal{Y}_{ab} = Y_{ab}(x_k,\delta_k,x_{k+1}), a = 0,1,2,3, b = 0,1$ *are given in* (5.3),

$$\Gamma_{3v} = \begin{bmatrix} 2\tau_v - 1 & \star & \star \\ -\tau_v a_v & P & \star \\ 0 & Z^\top & W \end{bmatrix}, \text{ and } M_3(x_k) = \begin{bmatrix} x_k & -I & 0 \\ 0 & X_1(x_k) & X_0(x_k) \\ 0 & \aleph(x_k) & 0 \end{bmatrix},$$

then there exist a Lyapunov function (5.1) *(page 73) and a controller given in* (6.19) *such that,* $\forall x_0$ *inside* $\mathcal{R}_{DoA}$*, the trajectory of* x_k *converge to the origin when* $k \to \infty$ *and* $\mathcal{R}_{DoA}$ *is an estimate of the DoA.*

Proof. Notice that conditions (6.20) and (6.22) are the same as (6.3) and (6.5) in *Theorem* 6.1, respectively. Thus, we have demonstrated that if condition (6.20) is satisfied, then $V(x_k) > 0$, $\forall x_k \in \mathcal{X}$ and $x_k \neq 0$.

On the other hand, inequality (6.21) can be recast as

$$\underbrace{\Phi_1 + J\Phi_2 + \Phi_2^\top J^\top}_{\Upsilon_2} + EM_2 + M_2^\top E^\top < 0, \tag{6.24}$$

with

$$\Phi_1 = \begin{bmatrix} -P & \star & \star & \star & \star & \star & \star & \star \\ 0 & P & \star & \star & \star & \star & \star & \star \\ -Z^\top & 0 & -W & \star & \star & \star & \star & \star \\ 0 & Z^\top & 0 & W & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} \epsilon A_3(x_k,\delta_k)F(\delta_k) \\ A_3(x_k,\delta_k)F(\delta_k) \\ 0 \\ 0 \\ 0 \\ \epsilon F(\delta_k) \\ \Omega_3(x_k,\delta_k)F(\delta_k) \\ 0 \end{bmatrix}$$

and

$$\Phi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & F^{-1}(\delta_k)H(\delta_k) & -I & 0 & 0 \end{bmatrix}.$$

Defining $\vartheta = \begin{bmatrix} x_k^\top & x_{k+1}^\top & \Theta_k^\top & \Theta_{k+1}^\top & y_k^\top & u_k^\top & \pi_k^\top & \zeta_k^\top \end{bmatrix}^\top$, implies that $\Phi_2\vartheta = 0$ and $M_2\vartheta = 0$. By pre- and post-multiplying (6.24) by $\vartheta^\top$ and ϑ, respectively, results in $\Delta V_k < 0$.

Therefore, if the conditions (6.20) and (6.21) are feasible, then $V(x_k)$ in (5.1) is a Lyapunov function and the controller (6.19) ensures that the origin of the closed-loop system is asymptotically stable.

Similarly to the previous case for SF control (see Remark 6.2), the chosen matrix J provided less conservative results among other alternatives in the numerical examples tested without the necessity of using more auxiliary decision variables.

The inclusion $\mathcal{R}_{DoA} \subseteq \mathcal{X}$ is ensured by condition (6.22), as we have demonstrated in the proof of *Theorem* 6.1.

Besides that, condition (6.23) guarantees the existence of the inverse of matrix $F(\delta_k)$, $\forall x_k \in \mathcal{X}$ and $\delta_k \in \Delta$, which is necessary to the computation of the control law in (6.19). $\square$

The next Corollary can be used to obtain the largest estimated DoA from *Theorem* 6.3, similarly as presented for the case of SF control design.

Corollary 6.3. *The estimated DoA for the closed-loop system* (2.3)-(6.19)*, with* $w_k = 0$*, can be maximized over* $\mathcal{X}$*, by solving the following optimization problem for all* $x_k \in \mathcal{X}$*,* $\delta_k \in \Delta$*.*

$$\min \sigma \quad \textit{subject to } (6.20) - (6.23), \tag{6.25}$$

with

$$\sigma - \text{trace}\left(\Gamma_1 + B(x_k)M_1(x_k) + M_1^\top(x_k)B^\top(x_k)\right) \geq 0. \tag{6.26}$$

As for the case of SF control design, presented in Section 6.1, *Theorem* 6.3, can also be adapted to consider the use of parameter-dependent Lyapunov functions. In this sense, *Theorem* 6.4, in the sequel, presents sufficient conditions to compute the SF control matrices in (6.19) that stabilize the nonlinear system (2.3), with $w_k = 0$, by considering parameter-dependent Lyapunov functions.

Theorem 6.4. *Consider the nonlinear system* (2.1) *and its DAR* (2.3)*. Let* ϵ *be a given positive scalar. If there exist positive scalars* τ_v *and matrices* $P(\delta_k) = P^\top(\delta_k) > 0$, $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$, $E(x_k, \delta_k) \in \mathbb{R}^{(2n_x+n_y+n_u+n_\pi)\times(n_x+n_y+n_\pi+n_q)}$, $F(\delta_k) \in \mathbb{R}^{n_u \times n_u}$*, and* $H(\delta_k) \in \mathbb{R}^{n_u \times n_y}$*, such that the following inequalities hold*

$$\Gamma_2(x_k, \delta_k, \delta_{k+1}) + E(x_k, \delta_k)M_2(x_k, \delta_k) + M_2^\top(x_k, \delta_k)E^\top(x_k, \delta_k) < 0, \tag{6.27}$$

$$\begin{bmatrix} 2\tau_v - 1 & \star \\ -\tau_v a_v & P(\delta_k) \end{bmatrix} \geq 0, \quad v \in \mathcal{I}_{n_e}, \tag{6.28}$$

$$F^\top(\delta_k) + F(\delta_k) > 0, \tag{6.29}$$

where

$$\Gamma_2(x_k, \delta_k, \delta_{k+1}) =$$
$$\begin{bmatrix} -P(\delta_k) & \star & \star & \star & \star \\ 0 & P(\delta_{k+1}) & \star & \star & \star \\ \epsilon H^\top(\delta_k)A_3^\top(x_k,\delta_k) & H^\top(\delta_k)A_3^\top(x_k,\delta_k) & 0 & \star & \star \\ -\epsilon F^\top(\delta_k)A_3^\top(x_k,\delta_k) & -F^\top(\delta_k)A_3^\top(x_k,\delta_k) & \epsilon H(\delta_k) & -\epsilon\,\text{He}\{F(\delta_k)\} & \star \\ 0 & 0 & \Omega_3(x_k,\delta_k)H(\delta_k) & -\Omega_3(x_k,\delta_k)F(\delta_k) & 0 \end{bmatrix},$$

and

$$M_2(x_k,\delta_k) = \begin{bmatrix} A_1(x_k,\delta_k) & -I & 0 & A_3(x_k,\delta_k) & A_2(x_k,\delta_k) \\ C_{y_1}(x_k,\delta_k) & 0 & -I & 0 & C_{y_2}(x_k,\delta_k) \\ \Omega_1(x_k,\delta_k) & 0 & 0 & \Omega_3(x_k,\delta_k) & \Omega_2(x_k,\delta_k) \\ \aleph_x(x_k) & 0 & 0 & 0 & 0 \end{bmatrix},$$

then there exist a Lyapunov function (6.10) *and a controller* (6.19) *such that,* $\forall x_0$ *inside* $\mathcal{L}_{DoA}$ *and* $\delta_k \in \Delta$*, the trajectory of* x_k *converge to the origin when* $k \to \infty$*.*

Proof. Theorem 6.4 was obtained from *Theorem* 6.3, disregarding the influence of the non-linearity vectors Θ_k and ζ_k, and incorporating information about time-varying parameters in the Lyapunov matrix P. Thus, following similar steps as in the proof of *Theorem* 6.3, since $P(\delta_k) > 0$, if condition (6.27) is satisfied, then (6.10) is a Lyapunov function and the controller (6.19) ensures that the origin of the closed-loop system is asymptotically stable. Condition (6.28) guarantees the inclusion $\mathcal{L}_{DoA} \subseteq \mathcal{X}$ and condition (6.29) ensures that $F(\delta_k)$ is invertible, $\forall \delta_k \in \Delta$. □

By solving the optimization problem in the next Corollary, it is possible to maximize the estimated DoA from *Theorem* 6.4.

Corollary 6.4. *Given a positive scalar* $\epsilon > 0$*. If there exist positive scalars* τ_v*, symmetric matrices* $Q \in \mathbb{R}^{n_x \times n_x}$ *and* $P(\delta_k) > 0$*,* $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$*, and any matrices* $E(x_k,\delta_k) \in \mathbb{R}^{(2n_x+n_y+n_u+n_\pi)\times(n_x+n_y+n_\pi+n_q)}$*,* $F(\delta_k) \in \mathbb{R}^{n_u \times n_u}$*, and* $H(\delta_k) \in \mathbb{R}^{n_u \times n_y}$*, satisfying the following optimization problem for all* $\delta_k \in \Delta$ *and* $x_k \in \mathcal{X}$*:*

$$\min\{\text{trace}(Q)\} \quad \textit{subject to } (6.27)-(6.29), \tag{6.30}$$

with

$$Q - P_l > 0, \tag{6.31}$$

then the SF controller (6.19) *asymptotically stabilizes the closed-loop system, composed by* (2.3)*, with* $w_k = 0$*, and* (6.19)*, around the origin, and* $\mathcal{L}_{DoA}$ *is an estimate of the DoA.*

Remark 6.5. *Notice that our methodology can be applied to rational systems with nonlinear and/or parameter-dependent output matrix. Besides that, no structural constraint is imposed on the output matrix, and the SOF control design problem is solved directly, without the necessity to obtain an SF controller in the first step or to use iterative algorithms, unlike other approaches [74, 75, 76, 77, 18, 79].*

The proposed stabilization conditions for SF and SOF control design based on polynomial Lyapunov functions, presented in *Theorems* 6.1 and 6.3, respectively, are supposed to be polynomially dependent on (x_k, δ_k, x_{k+1}). However, since it is possible to use a multi-simplex framework based on (5.5) a finite set of LMIs in terms of the vertices of the polytopes $\mathcal{X}$ and Δ can be obtained, using the same ideas presented in Appendix A. This procedure can also be applied to the conditions in *Theorems* 6.2 and 6.4, which are supposed to be polynomially dependent on $(x_k, \delta_k, \delta_{k+1})$.

6.3 Analysis of the computational complexity

In the sequel, Tables 6.1 and 6.2 show the computational complexity of the proposed approaches in terms of the number of scalar variables (S_v) and the number of LMI rows (L_r), which are related to the dimension of the state, time-varying parameter, control input, measurement output, and nonlinearity vectors.

Table 6.1 – Computational complexity of LMI conditions to SF control design.

	Corollary 6.1
S_v	$1+n_e+n_x\left(\frac{n_x+1}{2}+n_\Theta\right)+n_\Theta\left(\frac{n_\Theta+1}{2}\right)+\left[n_u(n_u+n_x+n_\pi)N_\delta+(n_x+n_\Theta)(n_q+n_\Theta)\right]N_x \ldots$ $+(2n_x+n_\pi+n_u+2n_\Theta+n_\zeta)(2n_\Theta+n_\zeta+n_x+n_\pi+n_q)N_x^2N_\delta+n_e(n_x+n_\Theta+1)(n_x+n_\Theta+n_q)$
L_r	$\left[(n_x+n_\Theta+1)\left(\frac{N_x+1}{2}+n_e\right)+(2n_x+n_\pi+n_u+2n_\Theta+n_\zeta)\frac{(N_x+1)^2(N_\delta+1)}{8}N_xN_\delta+n_uN_\delta\right]N_x$
	Corollary 6.2
S_v	$n_e+n_x(1+N_\delta)\left(\frac{n_x+1}{2}\right)+\left[n_u(n_u+n_x+n_\pi)+(2n_x+n_u+n_\pi)(n_x+n_\pi+n_q)\right]N_xN_\delta$
L_r	$\left[2n_x+(2n_x+n_\pi+n_u)\frac{(N_x+1)(N_\delta+1)}{4}N_xN_\delta+(1+n_x)n_e+n_uN_x\right]N_\delta$

Table 6.2 – Computational complexity of LMI conditions to SOF control design.

	Corollary 6.3
S_v	$1+n_e+n_x\left(\frac{n_x+1}{2}+n_\Theta\right)+n_\Theta\left(\frac{n_\Theta+1}{2}\right)+n_u(n_u+n_y)N_\delta+n_e(n_x+n_\Theta+1)(n_x+n_\Theta+n_q)\ldots$ $+(2n_x+2n_\Theta+n_y+n_u+n_\pi+n_\zeta)(2n_\Theta+n_\zeta+n_x+n_y+n_\pi+n_q)N_x^2N_\delta+(n_x+n_\Theta)(n_\Theta+n_q)N_x$
L_r	$\left[(n_x+n_\Theta+1)\left(\frac{N_x+1}{2}+n_e\right)+(2n_x+2n_\Theta+n_y+n_u+n_\pi+n_\zeta)\frac{(N_x+1)^2(N_\delta+1)}{8}N_xN_\delta\right]N_x+n_uN_\delta$
	Corollary 6.4
S_v	$n_e+n_x(1+N_\delta)\left(\frac{n_x+1}{2}\right)+\left[n_u(n_u+n_y)+(2n_x+n_y+n_u+n_\pi)\times(n_x+n_y+n_\pi+n_q)N_x\right]N_\delta$
L_r	$\left[2n_x+(2n_x+n_\pi+n_y+n_u)\frac{(N_x+1)(N_\delta+1)}{4}N_xN_\delta+(1+n_x)n_e+n_uN_x\right]N_\delta$

It is noteworthy that the use of polynomial Lyapunov functions significantly increases the computational burden, especially in terms of the number of scalar variables. However, as discussed previously the proposed Theorems and Corollaries are associated with *non-iterative* algorithms and a one-step approach, which can lead to less computational effort in solving the problems addressed. Besides that, the control design is off-line and the computational cost is not a problem in its practical implementation.

6.4 On the implementation of the control law

The proposed approaches have considered, in the system model, the presence of time-varying parameters (δ_k), which are supposed to be exactly known, and this information is used in the gain-scheduled control strategy, aiming to achieve less conservative results. However, as discussed in Part II, the proposed control laws can be adapted to deal with parametric uncertainties associated with unknown parameters.

For SF robust control, *Theorems* 6.1 and 6.2 can be applied by considering the matrices $G(\cdot)$ and $K(\cdot)$ only affine with respect to states (x_k). In this case, if the nonlinearity vector, π_k, depends on part of the parametric uncertainties, we must null the respective columns of matrix $K(\cdot)$, as discussed in Remark 6.1, on page 79. For SOF robust control, it is possible to apply *Theorems* 6.3 and 6.4, defining F and H as constant matrices.

Another situation that requires attention in practical applications is when $\Omega_3(x_k, \delta_k) \neq 0$, i.e., vector π_k is dependent on (u_k). Concerning the design of SF controllers, as for the case of π_k dependent on uncertain parameters, it is possible to deal with this problem by nulling the respective columns of matrix $K(\cdot)$ and solving the stabilization conditions in *Theorems* 6.1 and 6.2. Alternatively, the following Corollaries can be used to synthesize a SF controller incorporating the complete information of vector π_k, in which the final implementation of the control law is guaranteed.

By using the next Corollary, it is possible to apply the conditions proposed in *Theorem* 6.1 for SF control design considering a polynomial Lyapunov function, in the specific case where $\Omega_3(x_k, \delta_k) \neq 0$.

Corollary 6.5. *Consider a given positive scalar $\epsilon > 0$. If there exist positive scalars τ_v, symmetric matrices $P \in \mathbb{R}^{n_x \times n_x}$ and $W \in \mathbb{R}^{n_\Theta \times n_\Theta}$, and matrices $Z \in \mathbb{R}^{n_x \times n_\Theta}$, $L(x_k) \in \mathbb{R}^{(n_x+n_\Theta)\times(n_q+n_\Theta)}$, $R(x_k, \delta_k, x_{k+1}) \in \mathbb{R}^{(2n_x+n_\pi+n_u+2n_\Theta+n_\zeta)\times(2n_\Theta+n_\zeta+n_x+n_\pi+n_q)}$, $G(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_u}$, $K(x_k, \delta_k) = \begin{bmatrix} \bar{K}(x_k, \delta_k) & \hat{K}(x_k, \delta_k) \end{bmatrix}$, with $\bar{K}(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$, $\hat{K}(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_\pi}$, and $S_v \in \mathbb{R}^{(n_x+n_\Theta+1)\times(n_x+n_\Theta+n_q)}$, $v \in \mathcal{I}_{n_e}$ satisfying the following optimization problem for all $\delta_k \in \Delta$ and $x_k \in \mathcal{X}$:*

$$\begin{cases} \min \sigma \\ \textit{subject to } (6.3)-(6.6), (6.9) \textit{ and} \end{cases} \tag{6.32}$$

$$\begin{bmatrix} G^T(x_k, \delta_k) + G(x_k, \delta_k) & \hat{K}(x_k, \delta_k) + \Omega_3^T(x_k, \delta_k) \\ \hat{K}^T(x_k, \delta_k) + \Omega_3(x_k, \delta_k) & -(\Omega_2^T(x_k, \delta_k) + \Omega_2(x_k, \delta_k)) \end{bmatrix} > 0, \tag{6.33}$$

then the SF controller (6.1) *asymptotically stabilizes the closed-loop system composed by* (2.3), *with $w_k = 0$, and* (6.1) *around the origin, and $\mathcal{R}_{DoA}$ is an estimate of the DoA.*

Proof. Note that, from (6.1), we have

$$u_k = G^{-1}(x_k, \delta_k)\left[\bar{K}(x_k, \delta_k)x_k + \hat{K}(x_k, \delta_k)\pi_k\right],$$

where $K(x_k, \delta_k) = \begin{bmatrix}\bar{K}(x_k, \delta_k) & \hat{K}(x_k, \delta_k)\end{bmatrix}$ in (6.1). At the same time, from (2.4), if $\Omega_3(x_k, \delta_k) \neq 0$ and $w_k = 0$, using the assumption that $\exists \Omega_2^{-1}(x_k, \delta_k)$, one has that (dependency with (x_k, δ_k) was dropped for clarity purposes)

$$u_k = G^{-1}\left[\bar{K}x_k - \hat{K}\Omega_2^{-1}\left(\Omega_1 x_k + \Omega_3 u_k\right)\right],$$

which can be rewritten as

$$\left[G + \hat{K}\Omega_2^{-1}\Omega_3\right]u_k = \bar{K}x_k - \hat{K}\Omega_2^{-1}\Omega_1 x_k.$$

Therefore, the final implementation of the control law (6.1) is given by

$$u_k = \left[G + \hat{K}\Omega_2^{-1}\Omega_3\right]^{-1}\left[\bar{K} - \hat{K}\Omega_2^{-1}\Omega_1\right]x_k, \tag{6.34}$$

as long as the matrix $M(x_k,\delta_k) = \left[G + \hat{K}\Omega_2^{-1}\Omega_3\right]$ is invertible.

If $\Omega_3 \equiv 0$ or $\hat{K} \equiv 0$, $M(x_k, \delta_k)$ is nonsingular from the satisfaction of (6.6) in *Theorem* 6.1. On the other hand, if $\Omega_3 \neq 0$ and $\hat{K} \neq 0$, from (6.33) one has

$$\Phi^T(x_k, \delta_k) + \Phi(x_k, \delta_k) > 0, \quad \Phi(x_k, \delta_k) = \begin{bmatrix} G & \hat{K} \\ \Omega_3 & -\Omega_2 \end{bmatrix}$$

From the feasibility of the above inequality, $\Phi(x_k, \delta_k)$ must be invertible. Notice that matrix $\Phi(x_k, \delta_k)$ can be recast as

$$\Phi(x_k, \delta_k) = \begin{bmatrix} I & -\hat{K}\Omega_2^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} G + \hat{K}\Omega_2^{-1}\Omega_3 & 0 \\ 0 & -\Omega_2 \end{bmatrix}\begin{bmatrix} I & 0 \\ -\Omega_2^{-1}\Omega_3 & I \end{bmatrix}.$$

Thus, $\det(\Phi(x_k,\delta_k)) = \det(G + \hat{K}\Omega_2^{-1}\Omega_3)\det(-\Omega_2) = \det(M)\det(-\Omega_2)$.

Therefore, if condition (6.33) is satisfied, then $\det(\Phi(x_k, \delta_k)) \neq 0$ and $M(x_k, \delta_k)$ is invertible. The rest of the proof follows in a straightforward way as in the proofs of *Theorem* 6.1 and *Corollary* 6.1.

□

Similarly, from the following Corollary, it is possible to apply the conditions proposed in *Theorem* 6.2 for SF control design considering a parameter-dependent Lyapunov function, when $\Omega_3(x_k, \delta_k) \neq 0$.

Corollary 6.6. *Consider a given positive scalar $\epsilon > 0$. If there exist positive scalars τ_v, symmetric matrices $Q \in \mathbb{R}^{n_x \times n_x}$ and $P(\delta_k) > 0$, $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$, and matrices $R(x_k, \delta_k) \in \mathbb{R}^{(2n_x+n_u+n_\pi)\times(n_x+n_\pi+n_q)}$, $G(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_u}$, and $K(x_k, \delta_k) = \begin{bmatrix} \bar{K}(x_k, \delta_k) & \hat{K}(x_k, \delta_k) \end{bmatrix}$, $\bar{K}(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$, $\hat{K}(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_\pi}$ satisfying the following optimization problem for all $\delta_k \in \Delta$ and $x_k \in \mathcal{X}$:*

$$\begin{cases} \min \{\text{trace}(Q)\} \\ \textit{subject to } (6.13) - (6.15), (6.33) \textit{ and } Q - P(\delta_k) > 0, \end{cases} \tag{6.35}$$

then the SF controller (6.1) *asymptotically stabilizes the closed-loop system comprised by* (2.3)*, with $w_k = 0$, and* (6.1) *around the origin, and $\mathcal{L}_{DoA}$ is an estimate of the DoA.*

Proof. The proof uses steps similar to those presented in *Theorem* 6.2 and *Corollaries* 6.2 and 6.5. □

Remark 6.6. *Notice that condition* (6.33) *in Corollary 6.5 requires the additional restriction that $\Omega_2^\top(\cdot) + \Omega_2(\cdot)$ must be a negative definite matrix in the DAR model. Despite this fact, considerably less conservative results can be obtained using the complete information of the nonlinearity vector (π_k), as shown in Example 6.3 below.*

6.5 Numerical examples

In this section, numerical examples are presented to demonstrate the effectiveness of the methodology provided in this part of the research. As for the *Part* II, the stabilization conditions in a LMI form were implemented in MATLAB (R2019) using the parser Yalmip and the solver Mosek.

Example 6.1. *Consider the following nonlinear system that does not have time-varying parameters, adapted from Oliveira, Gomes da Silva Jr & Coutinho [12]:*

$$\begin{aligned} x_{(1)k+1} &= x_{(2)k}, \\ x_{(2)k+1} &= x_{(1)k} + 3x_{(1)k}^3 + x_{(2)k} + u_k, \end{aligned} \tag{6.36}$$

which can be recast in a DAR, such that

$$\begin{aligned} &\pi_k = x_{(1)k}^2,\ A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 \\ 3x_{(1)k} \end{bmatrix},\ A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ &\Omega_1 = \begin{bmatrix} x_{(1)k} & 0 \end{bmatrix},\ \Omega_2 = -1, \quad \Omega_3 = 0. \end{aligned} \tag{6.37}$$

Let's define the Lyapunov function candidate in (5.1) *with*

$$\Theta_k = \begin{bmatrix} x_{(1)k}^2 & x_{(1)k}x_{(2)k} & x_{(2)k}^2 & x_{(1)k}^3 & x_{(1)k}x_{(2)k}^2 & x_{(1)k}^2x_{(2)k} & x_{(2)k}^3 \end{bmatrix}^\top. \tag{6.38}$$

By using the first equation in (6.36), *we have*

$$\Theta_{k+1} = \begin{bmatrix} x_{(2)k}x_{(1)k+1} \\ x_{(2)k}x_{(2)k+1} \\ x_{(2)k+1}^2 \\ x_{(2)k}^2 x_{(1)k+1} \\ x_{(2)k}x_{(2)k+1}^2 \\ x_{(2)k}^2 x_{(2)k+1} \\ x_{(2)k+1}^3 \end{bmatrix}.$$

Choosing $\zeta_k = \begin{bmatrix} x_{(2)k}x_{(1)k+1} & x_{(2)k}x_{(2)k+1} \end{bmatrix}^\top$, *leads to the following matrices for equations* (5.3) *and* (5.4)

$$Y_{00} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & x_{(2)k} & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & x_{(2)k+1} & 0 & 0 & 0 & -1 \end{bmatrix}, Y_{10} = \begin{bmatrix} 0 & 0 \\ 0 & x_{(2)k} \\ 0 & x_{(2)k+1} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, Y_{20} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ x_{(2)k} & 0 \\ 0 & 0 \\ 0 & x_{(2)k} \\ 0 & 0 \end{bmatrix},$$

$$Y_{01} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y_{11} = \begin{bmatrix} x_{(2)k} & 0 \\ 0 & x_{(2)k} \end{bmatrix}, Y_{21} = -I_2, Y_{30} = 0_{7\times 1}, Y_{31} = 0_{2\times 1},$$

$$X_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ x_{(1)k} & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & x_{(1)k} & 0 & -1 & 0 & 0 \\ x_{(2)k} & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & x_{(2)k} & 0 & 0 & 0 & -1 \end{bmatrix}, X_1 = \begin{bmatrix} x_{(1)k} & 0 \\ x_{(2)k} & 0 \\ 0 & x_{(2)k} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Two situations are taken into account in this example. Firstly, we consider that the whole information about system states is available in real-time. In this case, Corollaries 6.1 and 6.2 are used to synthesize SF controllers, and the results are compared with those obtained using the approach proposed in Oliveira, Gomes da Silva Jr & Coutinho [12]. Secondly, we suppose that only partial state information is measured, and SOF controllers are designed by applying Corollary 6.3 and 6.4.

- ***Case 1: SF Control Design***

The problem, in this case, is to determine the largest admissible polytope in state space and the largest estimated DoA, such that system (6.36) *can be stabilized. Thus, the optimization problems presented in Corollaries 6.1 (by considering the polynomial Lyapunov function in* (5.1) *with* Θ_k *defined in* (6.38)*) and 6.2 (by considering a quadratic Lyapunov function) were solved and the results are compared with those from Oliveira, Gomes da Silva Jr & Coutinho [12] in Table 6.3. For a fair comparison, we applied the DAR-based methodologies proposed in Oliveira, Gomes da Silva Jr & Coutinho [12], disregarding the effect of input saturation.*

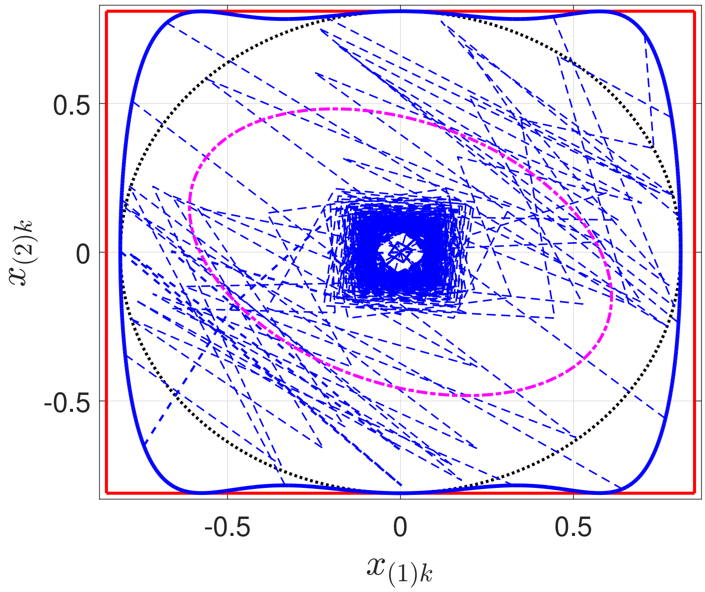
Table 6.3 – Estimated DoA by using different synthesis conditions.

Synthesis Conditions	Control Law	Polytopic Region ($\mathcal{X}$)	Estimated Area
Theorem 1 in [12]	not dependent on π_k	$\lvert x_{(1)k}\rvert \leq 0.61,\ \lvert x_{(2)k}\rvert \leq 0.61$	0.8766
Corollary 6.2	not dependent on π_k	$\lvert x_{(1)k}\rvert \leq 0.81,\ \lvert x_{(2)k}\rvert \leq 0.81$	2.0612
Corollary 6.1	not dependent on π_k	$\lvert x_{(1)k}\rvert \leq 0.85,\ \lvert x_{(2)k}\rvert \leq 0.81$	2.4717
Corollary 6.2	dependent on π_k	$\lvert x_{(1)k}\rvert \leq 10.0,\ \lvert x_{(2)k}\rvert \leq 10.0$	314.1583
Corollary 6.1	dependent on π_k	$\lvert x_{(1)k}\rvert \leq 10.0,\ \lvert x_{(2)k}\rvert \leq 10.0$	377.9260

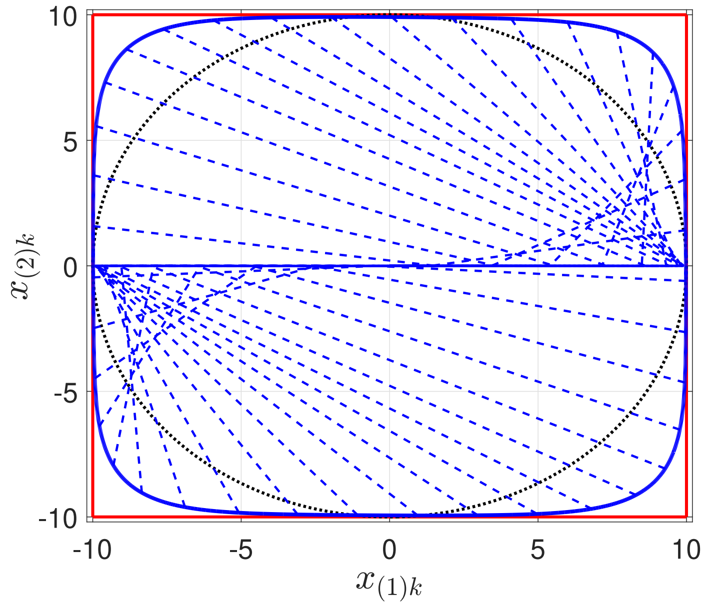
The comparison shows that from the methodology proposed in this study, it is possible to obtain feasible results for a considerably larger polytopic region, using the control law dependent on π_k*. The largest estimated DoA is obtained from Corollary 6.1, with the area computed of* 377.9260*, for* $\mathcal{X} := \left\{ x_k \in \mathbb{R}^2 : |x_{(1)k}| \leq 10.0, |x_{(2)k}| \leq 10.0 \right\}$*, and* $\epsilon = 1$*. In this case, incorporating information about vector* π_k *in the control law might be providing the cancellation of the system's nonlinearity, significantly increasing the estimated DoA.*

Figures 6.1(a) and 6.1(b) depicts the estimated DoAs obtained from the methodologies presented in Table 6.3 considering control law not dependent on π_k *and dependent on* π_k*, respectively. The figures also show some trajectories initiating inside the largest estimated DoA, which start at the boundary region and converge to the origin.*

From Figure 6.1 and Table 6.3, one can see the proposed approach's potential in reducing conservativeness, when the system's nonlinearities are taken into account in the control law. Besides that, it is also possible to note the effect of using polynomial Lyapunov functions in the estimated DoA that contributes to obtain a larger and more accurate estimate.



(a)



(b)

Figure 6.1 – Estimated DoAs and some state trajectories (blue dashed line). (a) Estimate DoAs for the control law not dependent on π_k, by using the stabilization conditions in *Corollary* 6.1 (blue solid line), *Corollary* 6.2 (black dotted line), and in Oliveira, Gomes da Silva Jr & Coutinho [12] (magenta dash-dotted line). (b) Estimate DoAs for the control law dependent on π_k, by using the stabilization conditions in *Corollary* 6.1 (blue solid line) and *Corollary* 6.2 (black dotted line).

- ***Case 2: SOF Control Design***

Now, suppose that only the information about $x_{(1)k}$ is available, given by the measurement output y_k. Therefore, it is possible to synthesize SOF controllers from Corollaries 6.3 *and 6.4. In this scenario, two situations are taken into account in our study: a linear and a nonlinear measurement output.*

– *Linear measurement output*

Consider the following linear measurement output and the respective matrices C_{y_1} *and* C_{y_2} *in the DAR* (2.3)*:*

$$y_k = x_{(1)k}, \quad C_{y_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_{y_2} = 0.$$

Based on this information and considering the same polynomial Lyapunov function candidate as in the previous case, the optimization problems in Corollary 6.3 and Corollary 6.4 were solved to obtain the largest admissible polytope in state space and the largest estimated DoA. Table 6.4 presents a comparison between the results obtained and Figure 6.2 depicts the estimated DoAs in each case. In addition, Figure 6.2 also shows some state trajectories starting at the boundary of the largest estimated DoA that converge to the origin.

Table 6.4 – Largest estimated DoA for the closed-loop system (6.36)-(6.19) with $y_k = x_{(1)k}$.

Method	Lyapunov function	Polytopic region ($\mathcal{X}$)	Estimated Area
Corollary 6.4	quadratic	$\lvert x_{(1)k}\rvert \leq 0.55,\ \lvert x_{(2)k}\rvert \leq 0.50$	0.7533
Corollary 6.3	polynomial	$\lvert x_{(1)k}\rvert \leq 0.68,\ \lvert x_{(2)k}\rvert \leq 0.54$	0.9497

In this case, the largest estimated DoA was obtained by Corollary 6.3, for $\epsilon = 0.1$ *and* $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : |x_{(1)k}| \leq 0.68 \text{ and } |x_{(2)k}| \leq 0.54\right\}$. *By applying Corollary 6.4 the better result was achieved for* $\epsilon = 0.2$ *and* $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : |x_{(1)k}| \leq 0.55 \text{ and } |x_{(2)k}| \leq 0.50\right\}$.

From Table 6.4 and Figure 6.2 it is possible to verify that the proposed approach based on polynomial Lyapunov functions can provide not only a larger estimated ellipsoidal DoA, but also feasible results for a larger polytopic region $\mathcal{X}$*, in comparison with the results obtained by considering quadratic Lyapunov functions.*

Figure 6.3 shows the same estimated DoA in Figure 6.2 superimposed to initial conditions that were numerically evaluated in simulation for the closed-loop system. Points denoted by '$+$' correspond to initial conditions leading to asymptotically stable behavior, and points indicated by '$\times$' correspond to initial conditions associated with unstable behavior. Note that, in this case, the non-convex characteristic of region $\mathcal{R}_{DoA}$ *contributes to obtain a more accurate estimated DoA.*

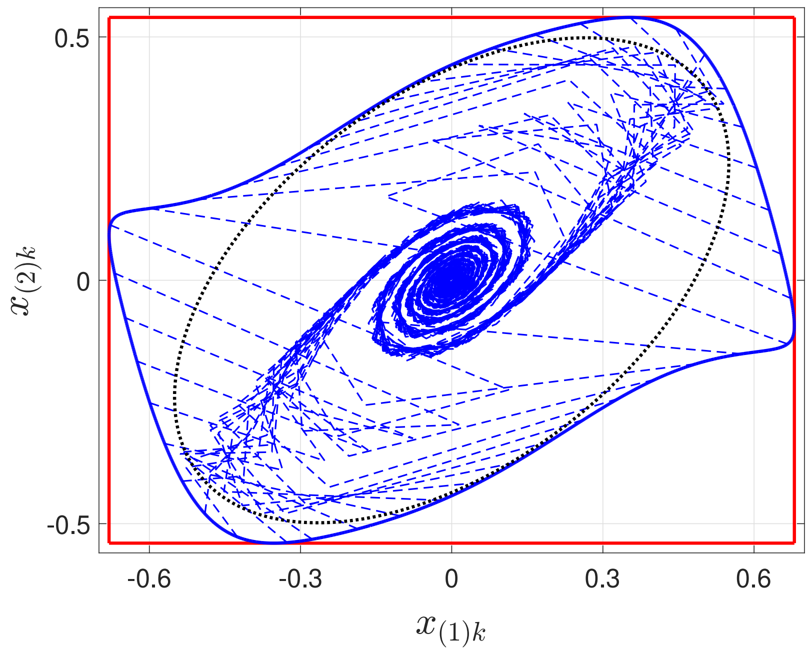


Figure 6.2 – Largest estimated DoA and some state trajectories for the closed-loop system (6.36)-(6.19) with $y_k = x_{(1)k}$, by using *Corollary* 6.3 (blue solid line) and *Corollary* 6.4 (black dotted line).

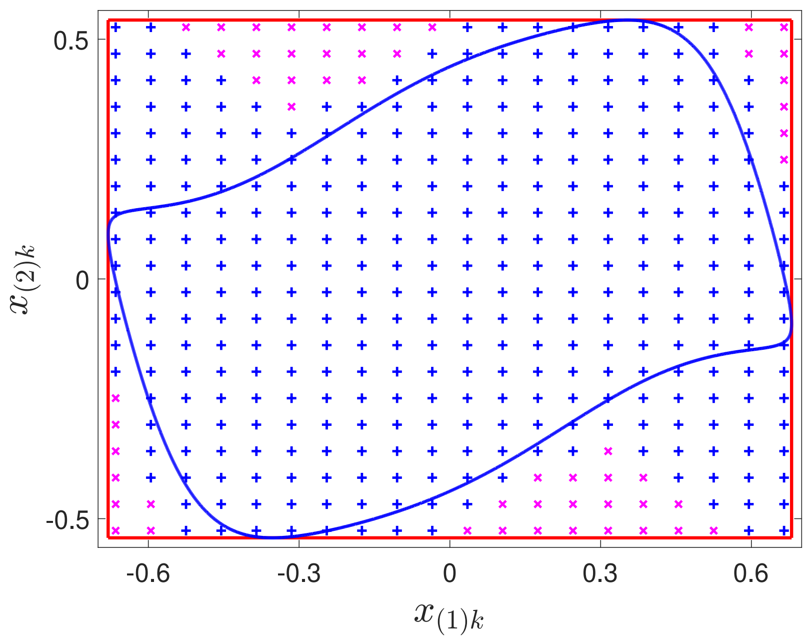


Figure 6.3 – Estimated DoA (blue solid line), stability region ($+$), and instability region ($\times$) for the closed-loop system (6.36)-(6.19) with $y_k = x_{(1)k}$, using *Corollary* 6.3 and the polynomial Lyapunov function candidate in (5.1) with Θ_k defined in (6.38).

– *Nonlinear measurement output*

Now, consider the following nonlinear measurement output and the respective matrices C_{y_1} *and* C_{y_2} *for the DAR* (2.3)*:*

$$y_k = x_{(1)k} + 1.2x_{(1)k}^3, \quad C_{y_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_{y_2} = 1.2x_{(1)k}.$$

Table 6.5 summarizes the results obtained by using the proposed approaches to synthesize SOF controllers, concerning the maximization of the estimated DoA. In this case, the largest estimated DoA was obtained for $\mathcal{X} := \left\{ x_k \in \mathbb{R}^2 : |x_{(1)k}| \leq 1.24 \text{ and } |x_{(2)k}| \leq 1.08 \right\}$*, by solving the optimization problem in Corollary 6.3 with* $\epsilon = 0.1$*.*

Table 6.5 – Largest estimated DoA for the closed-loop system (6.36)-(6.19) with $y_k = x_{(1)k} + 1.2x_{(1)k}^3$.

Method	Lyapunov function	Polytopic region ($\mathcal{X}$)	Estimated Area
Corollary 6.4	quadratic	$\lvert x_{(1)k}\rvert \leq 1.09, \lvert x_{(2)k}\rvert \leq 0.97$	2.8639
Corollary 6.3	polynomial	$\lvert x_{(1)k}\rvert \leq 1.24, \lvert x_{(2)k}\rvert \leq 1.08$	3.8726

Figure 6.4 presents the estimated DoAs in each case and some trajectories initiating inside the largest estimated DoA, which converge to the origin over time. Figure 6.5 also shows the largest estimated DoA with the stability region for the closed-loop system denoted by '+' and the instability region indicated by using '×'. From Figures 6.4 and 6.5 one can see the potential of our methodology based on polynomial Lyapunov functions to provide a larger and more accurate estimated DoA in comparison with the use of quadratic Lyapunov functions.

This example illustrates the effectiveness of the proposed method when the complete information about system states is not available, and the measured output presents information about the system's nonlinearities. Note that, when considering a nonlinear measurement output, from the SOF controller designed by applying the proposed methodology, it was possible to obtain an even larger estimated DoA than the regions found using SF controllers which do not take into account the system's nonlinear behavior in the control law.

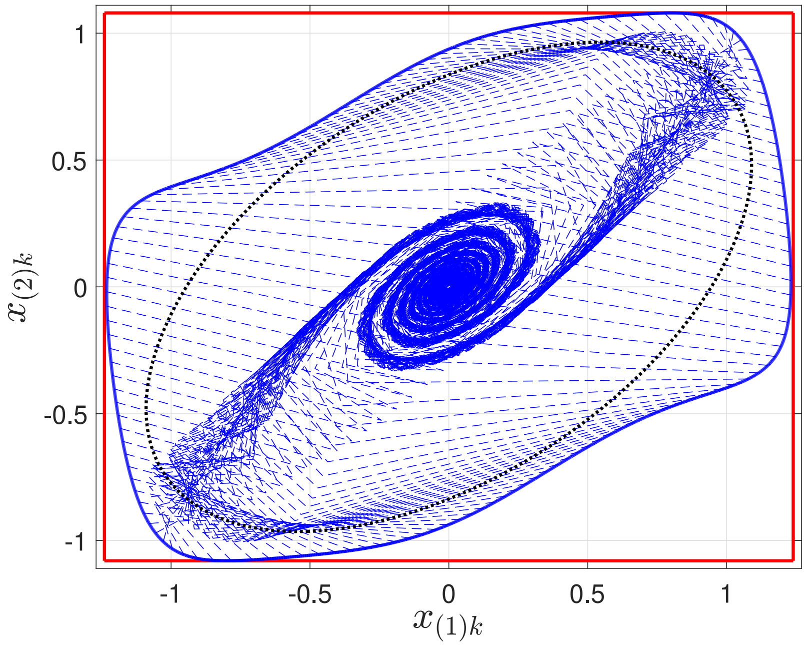


Figure 6.4 – Largest estimated DoA and some state trajectories for the closed-loop system (6.36)-(6.19) with $y_k = x_{(1)k} + 1.2x^3_{(1)k}$, by using *Corollary* 6.3 (blue solid line) and *Corollary* 6.4 (black dotted line).

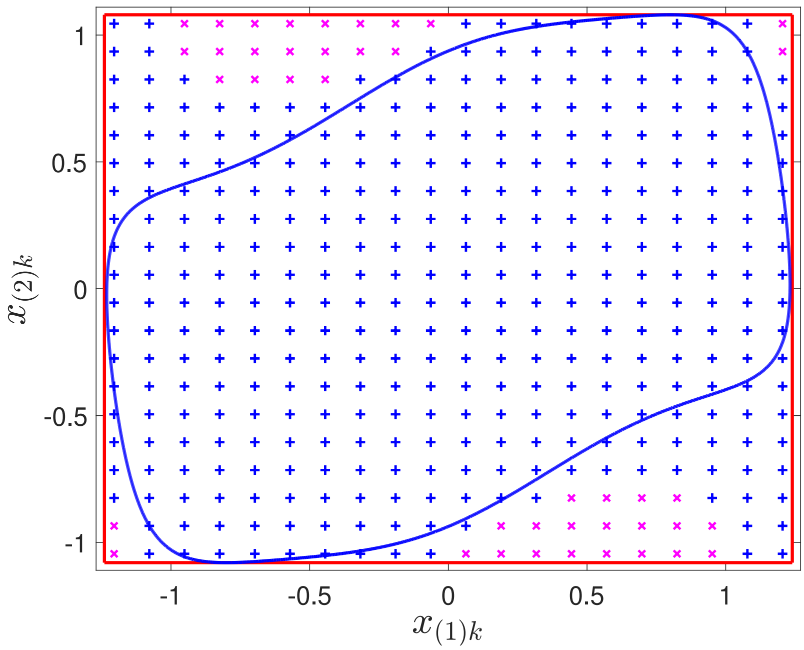


Figure 6.5 – Estimated DoA (blue solid line), stability region (+), and instability region (×) for the closed-loop system (6.36)-(6.19) with $y_k = x_{(1)k}$, using *Corollary* 6.3 and the polynomial Lyapunov function candidate in (5.1) with Θ_k defined in (6.38).

Example 6.2. *Consider the following nonlinear system:*

$$
\begin{aligned}
x_{(1)k+1} &= (1-\delta_{(2)k})x_{(2)k} + \frac{x_{(1)k}^2}{1+x_{(1)k}^2}, \\
x_{(2)k+1} &= -x_{(1)k} + x_{(2)k} + 0.5\frac{\delta_{(1)k}x_{(1)k}}{1+x_{(1)k}^2} + (1+2\delta_{(1)k})u_k, \\
y_k &= x_{(1)k} + 2\delta_{(1)k}x_{(2)k}.
\end{aligned} \tag{6.39}
$$

with a corresponding DAR such that

$$
\pi = \begin{bmatrix} \frac{x_{(1)k}^2}{1+x_{(1)k}^2} & \frac{x_{(1)k}}{1+x_{(1)k}^2} \end{bmatrix}^\top, \; C_{y_1} = \begin{bmatrix} 1 & 2\delta_{(1)k} \end{bmatrix}, \; C_{y_2} = \begin{bmatrix} 0 & 0 \end{bmatrix},
$$

$$
A_1 = \begin{bmatrix} 0 & 1-\delta_{(2)k} \\ -1 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5\delta_{(1)k} \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 \\ 1+2\delta_{(1)k} \end{bmatrix},
$$

$$
\Omega_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \; \Omega_2 = \begin{bmatrix} 1 & -x_{(1)k} \\ x_{(1)k} & 1 \end{bmatrix}, \; \Omega_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

The SOF control design and an estimated DoA for the closed-loop system are derived considering the following Lyapunov functions:

(a) Quadratic LF: $V_1 = x_k^\top P x_k$;

(b) Parameter-dependent LF: $V_2 = x_k^\top P(\delta_k) x_k$;

(c) Polynomial LF: $V_3 = \theta_k^\top \mathcal{P} \theta_k$, *with*

$$
\theta_k = \begin{bmatrix} x_{(1)k} & x_{(2)k} & x_{(1)k}^2 & x_{(1)k}x_{(2)k} & x_{(2)k}^2 \end{bmatrix}^\top \text{ and } \mathcal{P} = \begin{bmatrix} P & \star \\ Z^\top & W \end{bmatrix}.
$$

From the polynomial Lyapunov function given, we have

$$
\Theta_k = \begin{bmatrix} x_{(1)k}^2 & x_{(1)k}x_{(2)k} & x_{(2)k}^2 \end{bmatrix}^\top.
$$

By using the first equation in (6.39) *we have*

$$
\Theta_{k+1} = \begin{bmatrix} \left(1-\delta_{(2)k}\right) x_{(2)k}x_{(1)k+1} + \frac{x_{(1)k}^2 x_{(1)k+1}}{1+x_{(1)k}^2} \\ \left(1-\delta_{(2)k}\right) x_{(2)k}x_{(2)k+1} + \frac{x_{(1)k}^2 x_{(2)k+1}}{1+x_{(1)k}^2} \\ x_{(2)k+1}^2 \end{bmatrix}.
$$

Defining

$$\zeta_k = \begin{bmatrix} \frac{x_{(1)k}^2 x_{(1)k+1}}{1+x_{(1)k}^2} & \frac{x_{(1)k} x_{(1)k+1}}{1+x_{(1)k}^2} & \frac{x_{(1)k}^2 x_{(2)k+1}}{1+x_{(1)k}^2} & \frac{x_{(1)k} x_{(2)k+1}}{1+x_{(1)k}^2} & x_{(2)k}x_{(1)k+1} & x_{(2)k}x_{(2)k+1} \end{bmatrix}^\top,$$

we get additional equations in (5.4) *and* (5.3) *with*

$$Y_{00} = -I_3, Y_{10} = \begin{bmatrix} x_{(2)k} & 0 \\ 0 & x_{(2)k} \\ 0 & x_{(2)k+1} \end{bmatrix}, Y_{20} = \begin{bmatrix} 1 & 0 & 0 & 0 & -\delta_{(2)k} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\delta_{(2)k} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Y_{30} = 0_{3\times 1},$$

$$Y_{01} = 0_{6\times 3}, Y_{11} = \begin{bmatrix} 0 & 0 \\ x_{(1)k} & 0 \\ 0 & 0 \\ 0 & x_{(1)k} \\ x_{(2)k} & 0 \\ 0 & x_{(2)k} \end{bmatrix}, Y_{21} = \begin{bmatrix} -1 & x_{(1)k} & 0 & 0 & 0 & 0 \\ -x_{(1)k} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & x_{(1)k} & 0 & 0 \\ 0 & 0 & -x_{(1)k} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, Y_{31} = 0_{6\times 1},$$

$$X_0 = -I_3, \quad X_1 = \begin{bmatrix} x_{(1)k} & 0 \\ x_{(2)k} & 0 \\ 0 & x_{(2)k} \end{bmatrix}.$$

Defining $\Delta := \left\{\delta_k \in \mathbb{R} : |\delta_{(1)k}| \le 0.1 \text{ and } |\delta_{(2)k}| \le 0.19\right\}$ *and considering the Lyapunov functions described previously, the optimization problems presented in Corollaries 6.3 and 6.4 were solved to determine the largest admissible polytope in state space and the largest estimated DoA, such that system* (6.39) *can be stabilized. The results obtained are summarized in Table 6.6.*

Table 6.6 – Largest estimated DoA for the closed-loop system (6.39)-(6.19), by considering different Lyapunov functions.

Method	Lyapunov function	Polytopic region ($\mathcal{X}$)	Estimated Area
Corollary 6.4	quadratic	$\|x_{(1)k}\| \le 0.55, \|x_{(2)k}\| \le 0.30$	0.3678
Corollary 6.4	parameter-dependent	$\|x_{(1)k}\| \le 0.60, \|x_{(2)k}\| \le 0.30$	0.4147
Corollary 6.3	polynomial	$\|x_{(1)k}\| \le 0.99, \|x_{(2)k}\| \le 0.42$	1.0316

In this scenario, the less conservative result was obtained by defining $\epsilon = 1 \times 10^{-8}$ *and* $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : |x_{(1)k}| \le 0.99 \text{ and } |x_{(2)k}| \le 0.42\right\}$, *to solve the optimization problem in Corollary 6.3, which results in an estimated DoA with an area of 1.0316. Note that although the use of parameter-dependent Lyapunov function provides a largest estimated DoA compared to* *that obtained by using quadratic Lyapunov function, this result is still conservative in comparison with the estimated DoA from the synthesis conditions based on polynomial Lyapunov function, as shown in Figure 6.6.*

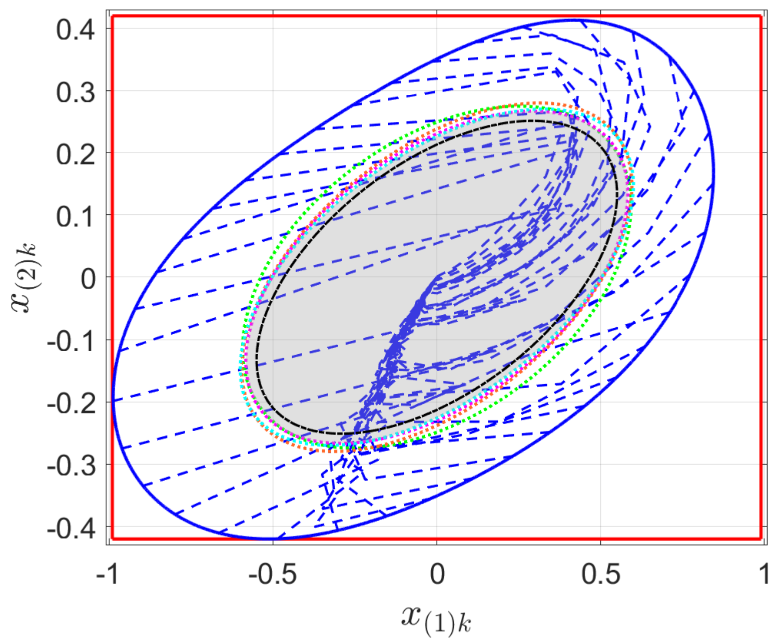


Figure 6.6 – Estimated DoAs based on polynomial Lyapunov fuction (blue solid line), parameter-dependent Lyapunov function (region filled in gray), and quadratic Lyapunov function (black dash-dotted line). Some state trajectories initiating in the border of the largest estimated DoA and the four ellipsoids (orange, magenta, green, and cyan dotted line) associated with the parameter-dependent Lyapunov function are also shown in this figure.

In Figure 6.6 one can see the largest estimated DoA obtained (blue solid line) with some trajectories initiating inside it (blue dashed line) considering different time-varying sequences of $\delta_k \in \Delta$. Note that, in this case, the estimated DoA is a non-symmetric region. In addition, Figure 6.6 also shows the estimated DoA by using parameter-dependent Lyapunov function (region filled in gray), which corresponds to the intersection of four ellipsoids (magenta, green, orange, and cyan dotted lines), and the estimated DoA based on quadratic Lyapunov function (black dash-dotted line).

This example illustrates the effectiveness of the proposed method for rational nonlinear systems with time-varying parameters.

Example 6.3. *Consider the following nonlinear system, without time-varying parameters, borrowed from Guerra & Vermeiren [98]:*

$$\begin{aligned} x_{(1)k+1} &= x_{(1)k} - x_{(1)k}x_{(2)k} + (5 + x_{(1)k})u_k, \\ x_{(2)k+1} &= -x_{(1)k} - 0.5x_{(2)k} + 2x_{(1)k}u_k, \end{aligned} \tag{6.40}$$

which can be recast in a DAR, such that

$$\pi_k = \begin{bmatrix} x_{(1)k}x_{(2)k} \\ x_{(1)k}u_k \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ -1 & -0.5 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \; A_3 = \begin{bmatrix} 5 \\ 0 \end{bmatrix},$$
$$\Omega_1 = \begin{bmatrix} 0 & x_{(1)k} \\ 0 & 0 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Omega_3 = \begin{bmatrix} 0 \\ x_{(1)k} \end{bmatrix}. \tag{6.41}$$

By defining the following Lyapunov function

$$V_1(x_k) = \theta_{1_k}^\top \begin{bmatrix} P_1 & \star \\ Z_1^\top & W_1 \end{bmatrix} \theta_{1_k}, \quad \theta_{1_k} = \begin{bmatrix} x_{(1)k} & x_{(2)k} & x_{(1)k}x_{(2)k} \end{bmatrix}^\top, \tag{6.42}$$

from (5.1)*, we have* $\Theta_k = x_{(1)k}x_{(2)k}$*, and using the first equation in* (6.40) *it follows that*

$$\Theta_{k+1} = x_{(1)k}x_{(2)k+1} - x_{(1)k}x_{(2)k}x_{(2)k+1} + 5u_kx_{(2)k+1} + x_{(1)k}u_kx_{(2)k+1}.$$

Letting $\zeta_k = \begin{bmatrix} x_{(2)k}x_{(2)k+1} & x_{(2)k+1}u_k \end{bmatrix}^\top$*, we get the additional equations in* (5.3) *and* (5.4) *with*

$$\begin{aligned}
&Y_{00} = -1, \quad Y_{10} = \begin{bmatrix} 0 & x_{(1)k} \end{bmatrix}, \quad Y_{20} = \begin{bmatrix} -x_{(1)k} & 5 + x_{(1)k} \end{bmatrix}, \quad Y_{30} = 0, \\
&Y_{01} = 0_{2\times 1}, \quad Y_{11} = \begin{bmatrix} 0 & x_{(2)k} \\ 0 & 0 \end{bmatrix}, \quad Y_{21} = -I_2, \quad Y_{31} = \begin{bmatrix} 0 \\ x_{(2)k+1} \end{bmatrix}, \\
&X_0 = -1, \quad X_1 = \begin{bmatrix} 0 & x_{(1)k} \end{bmatrix},
\end{aligned}$$

Considering $|x_{(1)k}| \leq b$*, the goal is to obtain the maximum variation for the value* b *such that there still exists a feasible solution, that is, there is a SF control guaranteeing the asymptotic stability. Table 6.7 presents the largest* b *obtained from the proposed methodology for SF control design and other control design conditions existing in the literature in the context of fuzzy discrete-time Takagi-Sugeno fuzzy models [28]. It is worth pointing out that the results of our methodology presented in Table 6.7 were obtained considering information about the nonlinearity vector in the control law and the states' bounds defined by* $\mathcal{X} := \left\{ x_k \in \mathbb{R}^2 : |x_{(1)k}| \leq b, |x_{(2)k}| \leq 40 \right\}$*. Since system* (6.40) *do not present time-varying parameters, Corollary 4.2 in Part II and Corollary 6.6 in Part III are applied for a quadratic Lyapunov function candidate. On the other hand, for Corollary 6.5, we consider the Lyapunov function candidate in* (6.42).

Table 6.7 – Maximum variations of parameter b obtained using existing conditions and the proposed approach.

Synthesis condition	**Maximum b**
Theorem 5 in [98]	1.539
Theorem 2 in [99]	1.757
Corollary 4.2	1.765
Theorem 4 in [25]	2.041
Corollary 6.6	4.800
Corollary 6.5	**4.900**

Initially, in Guerra & Vermeiren [98], feasible solutions were obtained for $b \leq 1.539$. This result is improved using the methodology proposed by Lendek, Guerra & Lauber [99], which allowed to solve the problem for $b \leq 1.757$. Recently, a better result was obtained by Coutinho et al. *[25] using delayed non-quadratic Lyapunov functions, with $b \leq 2.041$. The comparison shows that, from the proposed methodology based on polynomial Lyapunov functions, it is possible to obtain feasible results for a larger value of b, given by $b = 4.900$, with $\epsilon = 1 \times 10^2$.*

It is worth registering that we notice that incorporating the vector of nonlinearities into the control law together with the use of linear annihilator contributed significantly to reduce conservativeness in this example. Note that from Corollary 4.2, where the linear annihilator is not considered, the results obtained were conservative. Besides that, simulations of stabilization conditions in Corollaries 6.5 and 6.6 with the control law not dependent on π_k or without using the linear annihilator also provided more conservative results.

Another important point to be highlighted is that, in the other methodologies presented in Table 6.7, the regional stabilization with an estimated DoA was not took into account. In this sense, Figure 6.7 shows a comparison between the stability regions for the closed-loop system with the control gains designed using Corollary 6.5 ($\circ$) and that proposed by Coutinho et al. *[25] ($\cdot$), considering $b = 2.041$. Note that the proposed conditions provide the largest stability region encompassing that obtained the methodology in Coutinho* et al. *[25]. Figure 6.7 also shows the estimated DoAs from Corollary 6.5 (blue dash-dotted line) and Corollary 6.6 (green dotted line). It is worth mentioning that the stability regions obtained from these Corollaries are almost the same for $b = 2.041$. However, as one can see in Figure 6.7, from Corollary 6.5 based on polynomial Lyapunov functions, it was possible to obtain a considerably less conservative DoA estimate.*

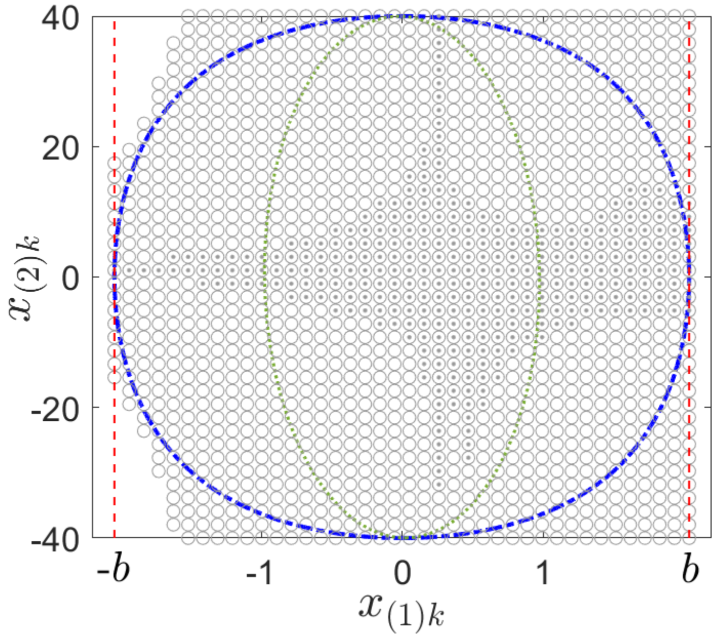


Figure 6.7 – Stability regions for the closed-loop system with the control gains designed using *Corollary* 6.5 (○) and Theorem 4 in Coutinho *et al.* [25] (·) with $b = 2.041$. Figure 6.7 also shows the estimated DoAs from *Corollary* 6.5 (blue dash-dotted line) and *Corollary* 6.6 (green dotted line).

The control gains obtained for $b = 4.900$ *using Corollary 6.5 are:*

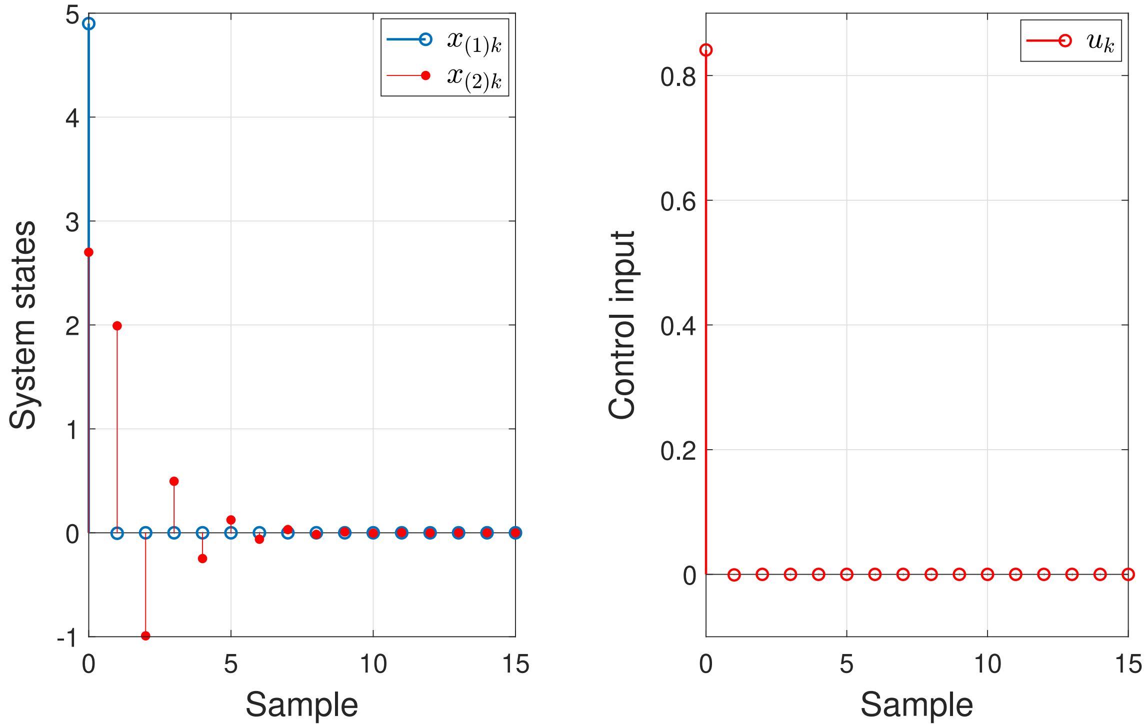
$$G_1 = 20.1196, \quad G_2 = 0.2551, \quad G_3 = 20.0784, \quad G_4 = 0.2550,$$

$$\bar{K}_1 = \begin{bmatrix} -2.3746 & -10.0720 \end{bmatrix}, \qquad \bar{K}_2 = \begin{bmatrix} -2.4247 & 10.0721 \end{bmatrix},$$
$$\bar{K}_3 = \begin{bmatrix} -1.6299 & -10.0721 \end{bmatrix}, \qquad \bar{K}_4 = \begin{bmatrix} -1.6885 & 10.0720 \end{bmatrix}.$$

$$\hat{K}_1 = \begin{bmatrix} -0.0446 & -4.0653 \end{bmatrix}, \qquad \hat{K}_2 = \begin{bmatrix} -0.0096 & -4.1003 \end{bmatrix},$$
$$\hat{K}_3 = \begin{bmatrix} -0.0532 & -4.0559 \end{bmatrix}, \qquad \hat{K}_4 = \begin{bmatrix} -0.0099 & -4.0997 \end{bmatrix}.$$

Figure 6.8 presents state trajectories and the control input sequence for $b = 4.900$. *Notice that for this value of* b, *no feasible solution is found for the other stabilization conditions as described in Table 6.7.*

In this example, we demonstrate that the proposed approach based on polynomial Lyapunov functions can provide considerably less conservative results in terms of DoA estimate, even using a simple polynomial function. Now, we show that increasing the complexity of vector Θ_k *can provide a more accurate DoA estimate at the price of increasing the computational burden.*



(a) (b)

Figure 6.8 – Time series of the state trajectories and the control input sequence of the closed-loop system with the controller designed using *Corollary* 6.5 for b = 4.900.

For this purpose, let us consider the following Lyapunov function:

$$V_2(x_k) = \theta_{2_k}^\top \begin{bmatrix} P_2 & \star \\ Z_2^\top & W_2 \end{bmatrix} \theta_{2_k},$$

$$\theta_{2_k} = \begin{bmatrix} x_{(1)k} \\ x_{(2)k} \\ x_{(1)k}^2 \\ x_{(1)k}x_{(2)k} \\ x_{(2)k}^2 \end{bmatrix}. \tag{6.43}$$

Thus, we have $\Theta_k = \begin{bmatrix} x_{(1)k}^2 & x_{(1)k}x_{(2)k} & x_{(2)k}^2 \end{bmatrix}^\top$*, and using the first equation in* (6.40) *it follows that*

$$\Theta_{k+1} = \begin{bmatrix} x_{(1)k}x_{(1)k+1} - x_{(1)k}x_{(2)k}x_{(1)k+1} + 5u_kx_{(1)k+1} + x_{(1)k}u_kx_{(1)k+1} \\ x_{(1)k}x_{(2)k+1} - x_{(1)k}x_{(2)k}x_{(2)k+1} + 5u_kx_{(2)k+1} + x_{(1)k}u_kx_{(2)k+1} \\ x_{(2)k+1}^2 \end{bmatrix}.$$

Choosing

$$\zeta = \begin{bmatrix} x_{(1)k}x_{(1)k+1} & x_{(2)k}x_{(2)k+1} & u_k x_{(1)k+1} & u_k x_{(2)k+1} \end{bmatrix}^\top,$$

the additional equations in (5.4) *and* (5.3) *can be written with*

$$Y_{00} = -I_3,\ Y_{10} = \begin{bmatrix} x_{(1)k} & 0 \\ 0 & x_{(1)k} \\ 0 & x_{(2)k+1} \end{bmatrix},\ Y_{20} = \begin{bmatrix} -x_{(2)k} & 0 & x_{(1)k} & 0 \\ 0 & -x_{(1)k} & 0 & x_{(1)k} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_{30} = \begin{bmatrix} 5x_{(1)k+1} \\ 5x_{(2)k+1} \\ 0 \end{bmatrix}, Y_{01} = 0_{4\times 3},\ Y_{11} = \begin{bmatrix} x_{(1)k} & 0 \\ 0 & x_{(2)k} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ Y_{21} = -I_4,\ Y_{31} = \begin{bmatrix} 0 \\ 0 \\ x_{(1)k+1} \\ x_{(2)k+1} \end{bmatrix},$$

$$X_0 = -I_3, \quad X_1 = \begin{bmatrix} x_{(1)k} & 0 \\ x_{(2)k} & 0 \\ 0 & x_{(2)k} \end{bmatrix}.$$

Figure 6.9 depicts the results obtained by solving the optimization problem in Corollary 6.5 for $b = 2.041$*, considering the Lyapunov function candidate* (6.43)*. Figure 6.9(a) shows the estimated DoA with the stability region for the closed-loop system denoted by '*$+$*' and the instability region indicated by using '*$\times$*'. In Figure 6.9(b) there are some trajectories starting at the boundary of the DoA, which converge to the origin. Notice that, in this case, a non-symmetrical region* $\mathcal{R}_{DoA}$ *was obtained, resulting in a more accurate estimated DoA compared to those shown in Figure 6.7.*

In addition, Table 6.8 shows a comparison between the results obtained from the stabilization conditions proposed in this part of the research, by considering different Lyapunov functions.

From Table 6.8, one can see that using the proposed approaches, the computational effort in terms of the number of scalar variables (S_v) *and the number of LMI rows* (L_r) *can considerably increase, depending on the complexity of the polynomial vector function chosen* Θ_k*. However, the area of the estimated DoA increases significantly in comparison with the use of quadratic Lyapunov functions.*

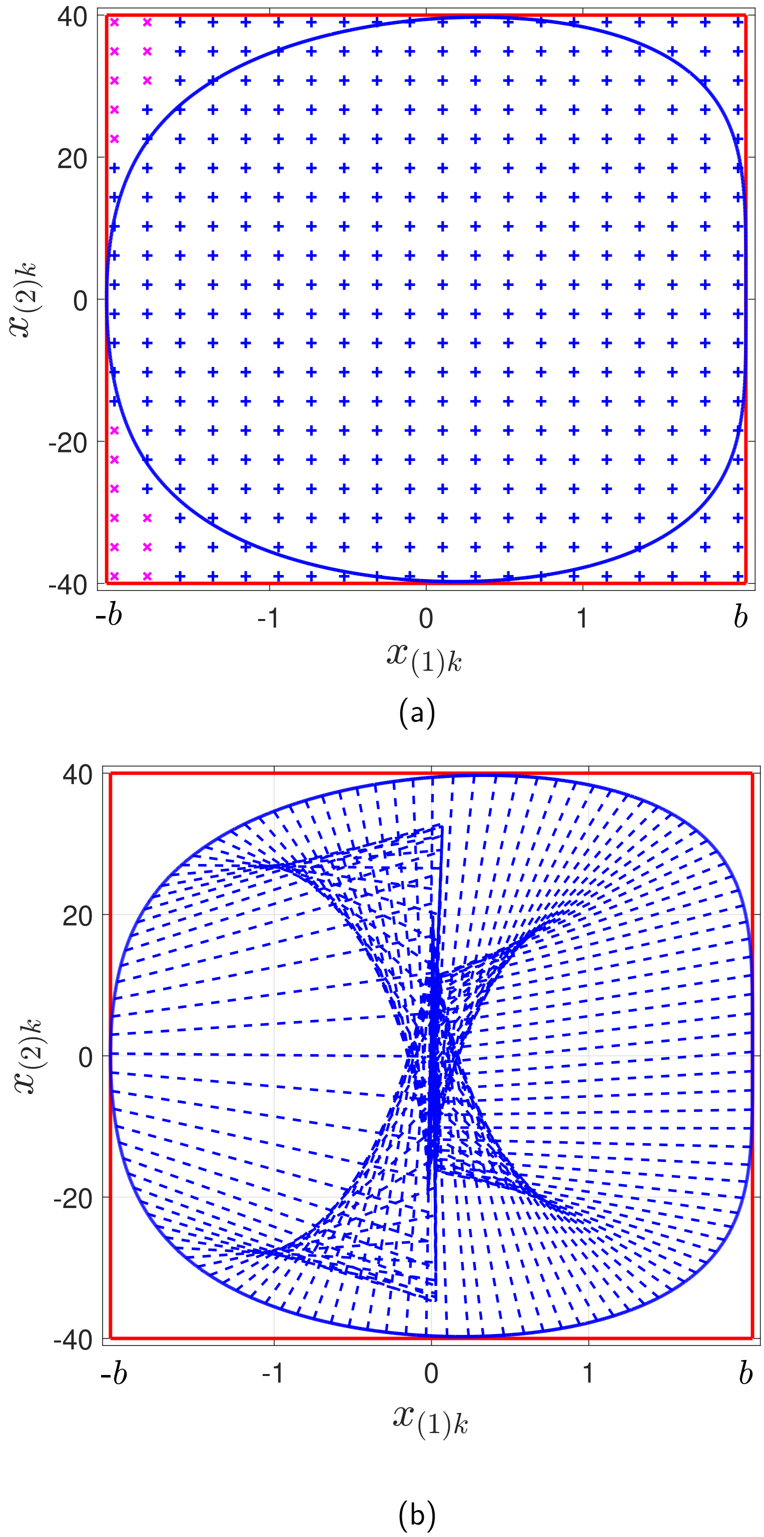


Figure 6.9 – Estimated DoA (blue solid line) for system (6.40) with $b = 2.041$, using *Corollary* 6.5 and the Lyapunov function candidate in (6.43). (a) Stability region ($+$) and instability region ($\times$). (b) State trajectories (blue dashed line).

Table 6.8 – Area of the estimated DoA and computational burden from the proposed methodology, by considering different Lyapunov functions.

Synthesis condition	Lyapunov Function (LF)	Area of Estimated DoA	S_v	L_R
Corollary 6.6	quadratic LF	1.2247×10^2	167	88
Corollary 6.5	polynomial LF in (6.42)	2.6800×10^2	1703	1208
Corollary 6.5	polynomial LF in (6.43)	2.7659×10^2	4344	1860

6.6 Final remarks

This chapter has introduced new stabilization conditions to design gain-scheduled SF and SOF controllers for discrete-time nonlinear systems with time-varying parameters, described in a DAR form.

Firstly, based on the Lyapunov theory and using a polynomial Lyapunov function candidate, synthesis conditions to design SF controllers were presented. By taking advantage of the DAR, information on the nonlinearity vector π_k was taken into account in the proposed control law (6.1).

Secondly, a new approach based on polynomial Lyapunov function candidate to design SOF controller (6.19) was introduced. Our methodology can be applied to discrete-time systems with nonlinear and/or parameter-dependent output matrix and no structural constraint is imposed on the output matrix.

As stated previously, in the context of discrete-time nonlinear systems described in a DAR form, Coutinho & Souza [50] proposed analysis conditions based on polynomial Lyapunov functions, but without providing synthesis conditions. Therefore, this is an important contribution of this doctoral work.

Besides that, for each case, it was shown how one can adapt the proposed methodology to consider the simpler case of using parameter-dependent Lyapunov functions and a preliminary analysis of the computational complexity was provided.

Numerical examples were presented in which the proposed approaches were applied to provide the regional stabilization of polynomial and rational nonlinear systems, by SF and SOF control design.

In the first numerical example, adapted from Oliveira, Gomes da Silva Jr & Coutinho [12], the design of SF and SOF controllers was considered. This example showed the efficiency of the proposed approach in both scenarios and how the information about the system's nonlinearities in the control law can provide less conservative results.

In the sequel, our methodology was applied to control a rational nonlinear system with time-varying parameters, taking into account three different Lyapunov functions. This example illustrates that the use of polynomial Lyapunov functions can provide considerably less conservative results in terms of a largest estimated DoA, than the use of quadratic or parameter-dependent Lyapunov functions.

Finally, the third example showed a favorable comparison between our methodology and those provided by Guerra & Vermeiren [98], Lendek, Guerra & Lauber [99] and Coutinho *et al.* [25] in the context of discrete-time Takagi-Sugeno fuzzy models. It has also been shown that the methodology presented in this part of the research can provide less conservative results, for the examples considered in this chapter, in comparison with that proposed in *Part* II, taking into account the use of *linear annihilators*.

Part IV

Concluding remarks

7 CONCLUSIONS AND FUTURE DIRECTIONS

7.1 Conclusions

This thesis has addressed the regional stabilization of discrete-time nonlinear systems with time-varying parameters described in a Difference-Algebraic Representation (DAR) form. The results obtained were divided into two parts that correspond to *Part* II and III of this thesis.

Part II focused on the use of parameter-dependent Lyapunov functions to obtain new stabilization conditions. In this part, *Chapter* 4 has presented novel LMI conditions to design gain-scheduled state feedback controllers. This kind of control approach offers the possibility of using information on time-varying parameters in the control law. Two state-feedback nonlinear control laws were proposed: one that does not explicitly incorporate the vector of nonlinearities used in the description of the system in its DAR form, and another one that incorporates this nonlinearity vector. In addition, this investigation considered control input saturation and two optimization problems have been stated, namely: (*i*) minimization of the upper-bound on the induced ℓ_2-gain for the case of zero initial conditions, and (*ii*) maximization of the estimated DoA in the absence of exogenous disturbances.

Part III focused on a new methodology for controller synthesis based on polynomial Lyapunov functions. In this context, *Chapter* 6 has presented novel stabilization conditions in terms of LMIs with a linear search parameter to design gain-scheduled SF and SOF controllers. The new solution for SOF control, not explored in the context of DARs for discrete-time systems, requires neither specific matrix-rank constraints in the system state-space model nor iterative algorithms. In addition, it can be applied to rational nonlinear systems with nonlinear and/or parameter-dependent output matrix.

Numerical examples have illustrated the effectiveness of the proposed methodologies and have shown favorable comparisons with recently published approaches. It has been also verified that using alternative Lyapunov functions and incorporating the nonlinearity vector in the control law can provide considerably less conservative results in comparison with the use of quadratic Lyapunov functions and the case when only the state vector is used to compute the control action. Furthermore, it was possible to verify the potential of the proposed methodology based on polynomial Lyapunov functions to provide a larger and more accurate estimated DoA in comparison with the use of quadratic or parameter-dependent Lyapunov functions.

7.2 Further steps

The next steps of this research will be discussed in this section, based on the further development of the main objective of this proposal, which is to provide efficient stabilization conditions for discrete-time uncertain nonlinear systems represented in a DAR form.

In this sense, topics for future investigations are:

a) To extend the proposed method to investigate the problem of amplitude-bounded disturbances:

One topic of interest in the continuity of this research is to consider exogenous signals that can present an upper and a lower bound of its values, following the approach reported by klug2017local applied to the context of systems represented as Takagi-Sugeno fuzzy models. In this case, the disturbance input vector w_k lies inside the following set:

$$\mathcal{W} := \left\{ w_k \in \mathbb{R}^{n_w} : w_k^\top L w_k \leq \lambda^{-1} \right\}, \tag{7.1}$$

where L is a positive definite diagonal matrix and λ is a positive scalar. Considering $L = \text{diag}\{l_1, l_2, \ldots, l_{n_w}\}$, $l_i \in \mathbb{R}$, thus $|w_{(i)k}| \leq \sqrt{\dfrac{1}{\lambda l_i}}$.

Note that in this case, the concept of ultimately bounded systems must be used.

This approach could provide an interesting application problem related to the use of discretized models of nonlinear systems. The idea is based on the fact that for state constrained nonlinear systems, it could be possible to estimate the worst-case numerical integration error in the region $\mathcal{X}$. This value could be considered in the amplitude bound of the exogenous disturbance when establishing stabilization conditions.

b) To extend the proposed method to address the problem of multiobjective controller design:

The development of synthesis conditions for multiobjective controllers is a challenge for control engineering with several practical applications [100, 101, 102]. In this case, the optimization problem aims at minimizing or maximizing two or more objective functions simultaneously, and the concept of Pareto optimality could be used [103]. Over the past few decades, several works aiming to synthesize controllers using different norms simultaneously as performance criteria have been presented in the literature [104, 105, 106, 107, 103]. As this thesis pursues different optimization problems related to the control theory, the study of multiobjective control for discrete-time nonlinear systems could be interesting for future steps.

c) To address other control problems that stand out in the literature and that were not explored in depth in the context of DARs:

As previously discussed, the use of DAR has been investigated in the last decades over several aspects. However, there are still many possibilities to be explored, especially in the discrete-time domain. Some examples are Robust Model Predictive

Control [108], Fault Tolerant Control [109, 49, 54], Time-Delay System Analysis and Control [110, 111, 112, 113], and Event-triggered control [114, 29, 115, 116, 30]. These problems have been studied in the D!FCOM Laboratory in the context of systems represented as Linear Parameter Varying (LPV) systems and TS fuzzy models.

7.3 Publications

The publications related to the specific topic of this thesis are listed below:

- **Reis, G. L.**; R. F. Araújo; L. A. B Torres; and R. M. Palhares. Gain-scheduled control design for discrete-time nonlinear systems using difference-algebraic representations. In: **International Journal of Robust and Nonlinear Control.** 31.5, pp. 1542 – 1563. 2021. DOI: 10.1002/rnc.5362;
- **Reis, G. L.**; R. F. Araújo; L. A. B Torres; and R. M. Palhares. Stabilization of rational nonlinear discrete-time systems by State Feedback and Static Output Feedback. In: **European Journal of Control**. 67, Article no. 100718. 2022. DOI: 10.1016/j.ejcon.2022.100718;
- **Reis, G. L.**; R. F. Araújo; L. A. B Torres; and R. M. Palhares. Improved Gain-Scheduled Control Design for Rational Nonlinear Discrete-Time Systems with Input Saturation. 2022. **European Control Conference (ECC), 2022**, pp. 560-565, 10.23919/ECC55457.2022.9838200.

In addition, following we have a submission involving a topic closely related with the main subject of this doctoral research and which can be a starting point for future works:

- Peixoto, M. L. C.; **Reis, G. L.**; Coutinho, P. H. S.; L. A. B Torres; and R. M. Palhares. Stability analysis of uncertain discrete-time systems with time-varying delays using Difference-Algebraic Representation. 2022. **European Control Conference (ECC), 2022**, pp. 2069-2074, 10.23919/ECC55457.2022.9838021.

BIBLIOGRAPHY

[1] SLOTINE, Jean-Jacques E; LI, Weiping *et al.* **Applied Nonlinear Control**. Englewood Cliffs, NJ, USA: Prentice hall, 1991. v. 199. Page 18.

[2] SASTRY, Shankar. **Nonlinear Systems: Analysis, Stability, and Control**. New York, NY, USA: Springer-Verlag, 1999. Page 18.

[3] KHALIL, H. K. **Nonlinear Systems**. 3rd. ed. Upper Saddle River, NJ, USA: Prentice Hall, 2002. Pages 18, 20, 22, and 35.

[4] VIDYASAGAR, Mathukumalli. **Nonlinear Systems Analysis**. 2nd. ed. [S.l.]: Society for Industrial and Applied Mathematics (SIAM), 2002. Page 18.

[5] TIBKEN, Bernd. Estimation of the domain of attraction for polynomial systems via LMIs. In: IEEE. **Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No. 00CH37187)**. [S.l.], 2000. v. 4, p. 3860–3864. Page 18.

[6] OSOWSKY, Jefferson; SOUZA, Carlos E de; COUTINHO, Daniel. Regional stability of two-dimensional nonlinear polynomial fornasini-marchesini systems. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 28, n. 6, p. 2318–2339, 2018. Page 18.

[7] COUTINHO, Daniel; SOUZA, Carlos E de; JR., João Manoel Gomes da Silva; CALDEIRA, André F; PRIEUR, Christophe. Regional stabilization of input-delayed uncertain nonlinear polynomial systems. **IEEE Transactions on Automatic Control**, IEEE, v. 65, n. 5, p. 2300–2307, 2020. Page 18.

[8] CHESI, Graziano. On the estimation of the domain of attraction for uncertain polynomial systems via LMIs. In: IEEE. **2004 43rd IEEE Conference on Decision and Control (CDC)(IEEE Cat. No. 04CH37601)**. [S.l.], 2004. v. 1, p. 881–886. Page 18.

[9] PITARCH, Jose Luis; SALA, Antonio; LAUBER, Jimmy; GUERRA, Thierry-Marie. Control synthesis for polynomial discrete-time systems under input constraints via delayed-state Lyapunov functions. **International Journal of Systems Science**, Taylor & Francis, v. 47, n. 5, p. 1176–1184, 2016. Page 18.

[10] GHAOUI, Laurent El; SCORLETTI, Gérard. Control of rational systems using linear-fractional representations and linear matrix inequalities. **Automatica**, Elsevier, v. 32, n. 9, p. 1273–1284, 1996. Pages 18, 25, and 82.

[11] COUTINHO, Daniel; TROFINO, Alexandre; FU, Minyue. Guaranteed cost control of uncertain nonlinear systems via polynomial Lyapunov functions. **IEEE Transactions on Automatic control**, IEEE, v. 47, n. 9, p. 1575–1580, 2002. Pages 18, 19, and 20.

[12] OLIVEIRA, M. Z.; Gomes da Silva Jr, J. M.; COUTINHO, D. Regional stabilization of rational discrete-time systems with magnitude control constraints. In: PROCEEDINGS OF IEEE AMERICAN CONTROL CONFERENCE. [S.l.], 2013. p. 241–246. Pages 9, 10, 18, 20, 23, 29, 56, 57, 71, 82, 92, 93, 94, 95, and 109.

[13] AZIZI, S; TORRES, Leonardo A B; PALHARES, Reinaldo M. Regional robust stabilisation and domain-of-attraction estimation for MIMO uncertain nonlinear systems with input saturation. **International Journal of Control**, Taylor & Francis, v. 91, n. 1, p. 215–229, 2018. Pages 18, 20, 29, 68, and 69.

[14] CASTRO, Rafael Silveira; FLORES, Jeferson Vieira; SALTON, Aurelio Tergolina; CHEN, Zhiyong; COUTINHO, Daniel F. A stabilization framework for the output regulation of rational nonlinear systems. **IEEE Transactions on Automatic Control**, IEEE, v. 65, n. 11, p. 4860–4865, 2020. Pages 18, 20, 29, and 82.

[15] TROFINO, A; DEZUO, T J M. LMI stability conditions for uncertain rational nonlinear systems. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 24, n. 18, p. 3124–3169, 2014. Pages 18, 19, 20, 25, and 29.

[16] POLCZ, Péter; PÉNI, Tamás; SZEDERKÉNYI, Gábor. Improved algorithm for computing the domain of attraction of rational nonlinear systems. **European Journal of Control**, Elsevier, v. 39, p. 53–67, 2018. Page 18.

[17] POLCZ, Péter; PÉNI, Tamás; SZEDERKÉNYI, Gábor. Reduced linear fractional representation of nonlinear systems for stability analysis. **IFAC–PapersOnLine**, Elsevier, v. 51, n. 2, p. 37–42, 2018. Page 18.

[18] COUTINHO, D F; FU, M; TROFINO, A; DANES, P. $\mathcal{L}_2$-gain analysis and control of uncertain nonlinear systems with bounded disturbance inputs. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 18, n. 1, p. 88–110, 2008. Pages 18, 20, 21, 22, 24, 25, and 88.

[19] KLUG, Michael; CASTELAN, Eugênio B; COUTINHO, Daniel. A T–S fuzzy approach to the local stabilization of nonlinear discrete-time systems subject to energy-bounded disturbances. **Journal of Control, Automation and Electrical Systems**, Springer, v. 26, n. 3, p. 191–200, 2015. Pages 18, 22, and 37.

[20] COUTINHO, Pedro H S; ARAÚJO, Rodrigo F; NGUYEN, Anh-Tu; PALHARES, Reinaldo M. A multiple-parameterization approach for local stabilization of constrained Takagi-Sugeno fuzzy systems with nonlinear consequents. **Information Sciences**, Elsevier, v. 506, p. 295–307, 2020. Pages 18, 19, and 29.

[21] PEIXOTO, Márcia L. C.; COUTINHO, P. H. S.; LACERDA, M. J.; PALHARES, R. M. Guaranteed region of attraction estimation for time-delayed fuzzy systems via static output-feedback control. **Automatica**, Elsevier, v. 143, p. 110438, 2022. Pages 18, 20, and 21.

[22] CAMPOS, Victor C S; SOUZA, Fernando O; TORRES, Leonardo A B; PALHARES, Reinaldo M. New stability conditions based on piecewise fuzzy Lyapunov functions and tensor product transformations. **IEEE Transactions on Fuzzy Systems**, IEEE, v. 21, n. 4, p. 748–760, 2013. Page 18.

[23] LEE, Dong Hwan; PARK, Jin Bae; JOO, Young Hoon. Improvement on nonquadratic stabilization of discrete-time Takagi–Sugeno fuzzy systems: Multiple-parameterization approach. **IEEE Transactions on Fuzzy Systems**, IEEE, v. 18, n. 2, p. 425–429, 2010. Page 18.

[24] NGUYEN, Anh-Tu; COUTINHO, Pedro; GUERRA, Thierry-Marie; PALHARES, Reinaldo; PAN, Juntao. Constrained output-feedback control for discrete-time fuzzy systems with local

nonlinear models subject to state and input constraints. **IEEE Transactions on Cybernetics**, IEEE, v. 51, n. 9, p. 4673–4684, 2021. Pages 18, 19, and 20.

[25] COUTINHO, Pedro Henrique S; LAUBER, Jimmy; BERNAL, Miguel; PALHARES, Reinaldo M. Efficient LMI conditions for enhanced stabilization of discrete-time Takagi–Sugeno models via delayed nonquadratic Lyapunov functions. **IEEE Transactions on Fuzzy Systems**, IEEE, v. 27, n. 9, p. 1833–1843, 2019. Pages 11, 18, 104, 105, and 110.

[26] COUTINHO, Pedro Henrique S; PEIXOTO, Márcia L C; LACERDA, Márcio J; BERNAL, Miguel; PALHARES, Reinaldo M. Generalized non-monotonic Lyapunov functions for analysis and synthesis of Takagi-Sugeno fuzzy systems. **Journal of Intelligent & Fuzzy Systems**, IOS Press, v. 39, n. 3, p. 4147–4158, 2020. Page 18.

[27] BESSA, Iury; PUIG, Vicenç; PALHARES, Reinaldo Martinez. TS fuzzy reconfiguration blocks for fault tolerant control of nonlinear systems. **Journal of the Franklin Institute**, Elsevier, v. 357, n. 8, p. 4592–4623, 2020. Page 18.

[28] NGUYEN, Anh-Tu; TANIGUCHI, T.; ECIOLAZA, L.; CAMPOS, V.; PALHARES, R.M.; SUGENO, M. Fuzzy control systems: Past, present and future. **IEEE Computational Intelligence Magazine**, v. 14, n. 1, p. 56–68, 2019. Pages 18 and 103.

[29] COUTINHO, Pedro Henrique S; PEIXOTO, Márcia L C; BESSA, Iury; PALHARES, Reinaldo M. Dynamic event-triggered gain-scheduling control of discrete-time quasi-LPV system. **Automatica**, v. 141, p. Article no. 110292, 2022. Pages 18, 20, and 114.

[30] COUTINHO, Pedro Henrique Silva; PALHARES, Reinaldo Martínez. Dynamic periodic event-triggered gain-scheduling control co-design for quasi-LPV systems. **Nonlinear Analysis: Hybrid Systems**, Elsevier, v. 41, p. 101044, 2021. Pages 18 and 114.

[31] COUTINHO, Pedro Henrique S; BERNAL, Miguel; PALHARES, Reinaldo M. Robust sampled-data controller design for uncertain nonlinear systems via Euler discretization. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 30, n. 18, p. 8244–8258, 2020. Page 18.

[32] SADEGHZADEH, Arash. LMI relaxations for robust gain-scheduled control of uncertain linear parameter varying systems. **IET Control Theory & Applications**, IET, v. 13, n. 4, p. 486–495, 2019. Pages 18 and 47.

[33] CALDERÓN, Horacio M; CISNEROS, Pablo S G; WERNER, Herbert. qLPV predictive control - a benchmark study on state space vs input-output approach. **IFAC–PapersOnLine**, Elsevier, v. 52, n. 28, p. 146–151, 2019. Page 18.

[34] SHAMMA, Jeff S; ATHANS, M. Analysis of gain scheduled control for nonlinear plants. **IEEE Transactions on Automatic Control**, v. 35, n. 8, p. 898–907, 1990. Page 18.

[35] ROTONDO, Damiano; NEJJARI, Fatiha; PUIG, Vicenç. Quasi-LPV modeling, identification and control of a twin rotor MIMO system. **Control Engineering Practice**, Elsevier, v. 21, n. 6, p. 829–846, 2013. Page 18.

[36] CISNEROS, Pablo S G; WERNER, Herbert. Parameter-dependent stability conditions for quasi-LPV model predictive control. In: IEEE. **2017 American Control Conference (ACC)**. [S.l.], 2017. p. 5032–5037. Page 18.

[37] ROBLES, Ruben; SALA, Antonio; BERNAL, Miguel. Performance-oriented quasi-LPV modeling of nonlinear systems. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 29, n. 5, p. 1230–1248, 2019. Page 18.

[38] SOFYALI, Ahmet; JAFAROV, Elbrous M. Robust stabilization of spacecraft attitude motion under magnetic control through time-varying integral sliding mode. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 29, n. 11, p. 3446–3468, 2019. Page 19.

[39] WANG, Qian; ZHOU, Bin; DUAN, Guang-Ren. Robust gain scheduled control of spacecraft rendezvous system subject to input saturation. **Aerospace Science and Technology**, Elsevier, v. 42, p. 442–450, 2015. Page 19.

[40] CHEN, Hua. Robust stabilization for a class of dynamic feedback uncertain nonholonomic mobile robots with input saturation. **International Journal of Control, Automation and Systems**, Springer, v. 12, n. 6, p. 1216–1224, 2014. Page 19.

[41] COUTINHO, Daniel; WOUWER, Alain Vande. A robust non-linear feedback control strategy for a class of bioprocesses. **IET Control Theory & Applications**, IET, v. 7, n. 6, p. 829–841, 2013. Page 19.

[42] NASRI, Mohamed; SAIFIA, Dounia; CHADLI, Mohammed; LABIOD, Salim. H_∞ switching fuzzy control of solar power generation systems with asymmetric input constraint. **Asian Journal of Control**, Wiley Online Library, v. 21, n. 4, p. 1869–1880, 2019. Page 19.

[43] TARBOURIECH, Sophie; GARCIA, Germain; Gomes da Silva Jr., João Manoel; QUEINNEC, Isabelle. **Stability and Stabilization of Linear Systems with Saturating Actuators**. London, UK: Springer, 2011. Pages 19, 40, and 41.

[44] PEIXOTO, Márcia Luciana Costa; COUTINHO, Pedro Henrique Silva; BESSA, Iury; PALHARES, Reinaldo Martinez. Static output-feedback stabilization of discrete-time LPV systems under actuator saturation. **International Journal of Robust and Nonlinear Control**, v. 32, n. 9, p. 5799–5809, 2022. Pages 19, 20, and 21.

[45] COUTINHO, Daniel Ferreira; FU, Minyue; TROFINO, Alexandre. Robust analysis and control for a class of uncertain nonlinear discrete-time systems. **Systems & Control Letters**, Elsevier, v. 53, n. 5, p. 377–393, 2004. Pages 19, 20, and 24.

[46] PEIXOTO, Márcia Luciana Costa; PESSIM, Paulo Sérgio Pereira; LACERDA, Márcio Júnior; PALHARES, Reinaldo Martinez. Stability and stabilization for LPV systems based on Lyapunov functions with non-monotonic terms. **Journal of the Franklin Institute**, v. 357, n. 11, p. 6595–6614, 2020. Pages 19, 20, and 51.

[47] PEIXOTO, Márcia Luciana Costa; LACERDA, Márcio Júnior; PALHARES, Reinaldo Martinez. On discrete-time LPV control using delayed Lyapunov functions. **Asian Journal of Control**, v. 23, p. 2359–2369, 2021. Page 19.

[48] MOZELLI, Leonardo A.; ADRIANO, Ricardo L. S. On computational issues for stability analysis of LPV systems using parameter-dependent Lyapunov functions and LMIs. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 29, n. 10, p. 3267–3277, 2019. Page 19.

[49] QUADROS, Mariella Maia; BESSA, Iury Valente; LEITE, Valter J. S.; PALHARES, Reinaldo Martinez. Fault tolerant control for linear parameter varying systems: An improved robust virtual actuator and sensor approach. **ISA Transactions**, Elsevier, v. 104, p. 356–369, 2020. Pages 19 and 114.

[50] COUTINHO, Daniel; SOUZA, Carlos E. Local stability analysis and domain of attraction estimation for a class of uncertain nonlinear discrete-time systems. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 23, n. 13, p. 1456–1471, 2013. Pages 19, 20, 23, 73, 82, and 109.

[51] RUGH, Wilson J; SHAMMA, Jeff S. Research on gain scheduling. **Automatica**, Elsevier, v. 36, n. 10, p. 1401–1425, 2000. Page 19.

[52] SAAT, Mohd Shakir Md; NGUANG, Sing Kiong; NASIRI, Alireza. **Analysis and synthesis of polynomial discrete-time systems: an SOS approach**. [S.l.]: Butterworth-Heinemann, 2017. Page 19.

[53] JORDÁN, Mario A.; BUSTAMANTE, Jorge L. **Discrete Time Systems**. [S.l.]: InTech, 2011. Page 19.

[54] QUADROS, Mariella Maia; LEITE, Valter J S; PALHARES, Reinaldo Martínez. Robust fault hiding approach for T–S fuzzy systems with unmeasured premise variables. **Information Sciences**, Elsevier, v. 589, p. 690–715, 2022. Pages 20 and 114.

[55] COUTINHO, Daniel Ferreira; SOUZA, Carlos E de; BARBOSA, Karina A. Robust h_∞ filter design for a class of discrete-time parameter varying systems. **Automatica**, Elsevier, v. 45, n. 12, p. 2946–2954, 2009. Page 20.

[56] TROFINO, Alexandre. Robust stability and domain of attraction of uncertain nonlinear systems. In: PROCEEDINGS OF IEEE AMERICAN CONTROL CONFERENCE. **Proceedings of the 2000 American Control Conference. ACC (IEEE Cat. No. 00CH36334)**. [S.l.], 2000. v. 5, p. 3707–3711. Page 20.

[57] HENRION, Didier; GARULLI, Andrea. **Positive polynomials in control**. [S.l.]: Springer Science & Business Media, 2005. v. 312. Page 20.

[58] SCHERER, Carsten; WEILAND, Siep. Linear matrix inequalities in control. **Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands**, v. 3, n. 2, 2000. Pages 20 and 46.

[59] BOYD, Stephen; GHAOUI, Laurent El; FERON, Eric; BALAKRISHNAN, Venkataramanan. **Linear matrix inequalities in system and control theory**. [S.l.]: SIAM, 1994. Page 20.

[60] COUTINHO, D F; Gomes da Silva Jr., J M. Computing estimates of the region of attraction for rational control systems with saturating actuators. **IET Control Theory & Applications**, IET, v. 4, n. 3, p. 315–325, 2010. Page 20.

[61] OLIVEIRA, M Z; Gomes da Silva Jr., J. M.; COUTINHO, Daniel. Stability analysis for a class of nonlinear discrete-time control systems subject to disturbances and to actuator saturation. **International Journal of Control**, Taylor & Francis, v. 86, n. 5, p. 869–882, 2013. Pages 20, 22, 24, and 61.

[62] ROHR, Eduardo Rath; PEREIRA, Luís Fernando Alves; COUTINHO, Daniel Ferreira. Robustness analysis of nonlinear systems subject to state feedback linearization. **SBA: Controle & Automação Sociedade Brasileira de Automatica**, SciELO Brasil, v. 20, n. 4, p. 482–489, 2009. Pages 20 and 69.

[63] COUTINHO, Daniel Ferreira; BAZANELLA, Alexandre Sanfelice; TROFINO, Alexandre; SILVA, Aguinaldo Silveira e. Stability analysis and control of a class of differential-algebraic nonlinear systems. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 14, n. 16, p. 1301–1326, 2004. Page 20.

[64] OLIVEIRA, M Z; Gomes da Silva Jr., J M; COUTINHO, Daniel. State feedback design for rational nonlinear control systems with saturating inputs. In: PROCEEDINGS OF IEEE AMERICAN CONTROL CONFERENCE. **2012 American Control Conference (ACC)**. [S.l.], 2012. p. 2331–2336. Page 20.

[65] DEZUO, Tiago; TROFINO, Alexandre. LMI conditions for designing rational nonlinear observers with guaranteed cost. **IFAC Proceedings Volumes**, Elsevier, v. 47, n. 3, p. 67–72, 2014. Page 20.

[66] SOUZA, Carlos E De; XIE, Lihua; COUTINHO, Daniel F. Robust filtering for 2-d discrete-time linear systems with convex-bounded parameter uncertainty. **Automatica**, Elsevier, v. 46, n. 4, p. 673–681, 2010. Page 20.

[67] Gomes da Silva Jr., J M; OLIVEIRA, M Z; COUTINHO, D; TARBOURIECH, Sophie. Static anti-windup design for a class of nonlinear systems. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 24, n. 5, p. 793–810, 2014. Page 20.

[68] Gomes da Silva Jr., J M; LONGHI, Mauricio Borges; OLIVEIRA, Mauricio Zardo. Dynamic anti-windup design for a class of nonlinear systems. **International Journal of Control**, Taylor & Francis, v. 89, n. 12, p. 2406–2419, 2016. Page 20.

[69] MOREIRA, L G; GROFF, L B; Gomes da Silva Jr., J M; COUTINHO, D. Event-triggered control for nonlinear rational systems. **IFAC–PapersOnLine**, Elsevier, v. 50, n. 1, p. 15307 –15312, 2017. Page 20.

[70] COUTINHO, Daniel F; PAGANO, Daniel J; TROFINO, Alexandre. On the estimation of robust stability regions for nonlinear systems with saturation. **Sba: Controle & Automação Sociedade Brasileira de Automatica**, SciELO Brasil, v. 15, n. 3, p. 269–278, 2004. Page 20.

[71] COUTINHO, Daniel F; Gomes da Silva Jr., J M. Estimating the region of attraction of nonlinear control systems with saturating actuators. In: PROCEEDINGS OF IEEE AMERICAN CONTROL CONFERENCE. **2007 American Control Conference**. [S.l.], 2007. p. 4715–4720. Page 20.

[72] PEIXOTO, Márcia L C; COUTINHO, Pedro H S; PALHARES, Reinaldo M. Improved robust gain-scheduling static output-feedback control for discrete-time LPV systems. **European Journal of Control**, Elsevier, v. 58, p. 11–16, 2021. Pages 20 and 21.

[73] PESSIM, Paulo S. P.; PEIXOTO, Márcia L. C.; PALHARES, Reinaldo M.; LACERDA, Márcio J. Static output-feedback control for cyber-physical LPV systems under DoS attacks. **Information Sciences**, Elsevier, v. 563, p. 241–255, 2021. Page 20.

[74] CRUSIUS, Cesar A R; TROFINO, Alexandre. Sufficient LMI conditions for output feedback control problems. **IEEE Transactions on Automatic Control**, IEEE, v. 44, n. 5, p. 1053–1057, 1999. Pages 20 and 88.

[75] DONG, Jiuxiang; YANG, Guang-Hong. Robust static output feedback control for linear discrete-time systems with time-varying uncertainties. **Systems & Control Letters**, Elsevier, v. 57, n. 2, p. 123–131, 2008. Pages 20 and 88.

[76] WU, Huai-Ning. An ILMI approach to robust H_2 static output feedback fuzzy control for uncertain discrete-time nonlinear systems. **Automatica**, Elsevier, v. 44, n. 9, p. 2333–2339, 2008. Pages 21 and 88.

[77] NGUANG, Sing Kiong; HUANG, Dan; SHI, Peng. Robust H_∞ static output feedback control of fuzzy systems: an ILMI approach. **IFAC Proceedings Volumes**, Elsevier, v. 38, n. 1, p. 826–831, 2005. Pages 21 and 88.

[78] ROSA, Tabitha E; MORAIS, Cecilia F; OLIVEIRA, Ricardo C L F. New robust LMI synthesis conditions for mixed H_2/H_∞ gain-scheduled reduced-order DOF control of discrete-time LPV systems. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 28, n. 18, p. 6122–6145, 2018. Page 21.

[79] OLIVEIRA, Maurıcio Carvalho De; GEROMEL, José Claudio; BERNUSSOU, Jacques. Design of dynamic output feedback decentralized controllers via a separation procedure. **International Journal of Control**, Taylor & Francis, v. 73, n. 5, p. 371–381, 2000. Pages 21 and 88.

[80] HU, Tingshu; LIN, Zongli. **Control Systems with Actuator Saturation: Analysis and Design**. Boston, MA, USA: Birkhäuser Boston, 2001. Pages 22, 40, and 60.

[81] HU, Tingshu; HUANG, Bin; LIN, Zongli. Absolute stability with a generalized sector condition. **IEEE Transactions on Automatic Control**, IEEE, v. 49, n. 4, p. 535–548, 2004. Page 22.

[82] FIGUEREDO, Luis F C; ISHIHARA, João Y; BORGES, Geovany A; BAUCHSPIESS, Adolfo. Robust h_∞ output tracking control for a class of nonlinear systems with time-varying delays. **Circuits, Systems, and Signal Processing**, Springer, v. 33, n. 5, p. 1451–1471, 2014. Page 22.

[83] LIEN, Chang-Hua; YU, Ker-Wei; LIN, Yen-Feng; CHUNG, Yeong-Jay; CHUNG, Long-Yeu. Robust reliable h_∞ control for uncertain nonlinear systems via LMI approach. **Applied Mathematics and Computation**, Elsevier, v. 198, n. 1, p. 453–462, 2008. Page 22.

[84] ZHANG, Xiaoyu; HAN, Yuntao; BAI, Tao; WEI, Yanhui; MA, Kemao. h_∞ controller design using LMIs for high-speed underwater vehicles in presence of uncertainties and disturbances. **Ocean engineering**, Elsevier, v. 104, p. 359–369, 2015. Page 22.

[85] OLIVEIRA, M Z; Gomes da Silva Jr., J M; COUTINHO, D F. Asymptotic and $\mathcal{L}_2$ stability analysis for a class of nonlinear discrete-time control systems subject to actuator saturation. In: PROCEEDINGS OF IEEE AMERICAN CONTROL CONFERENCE. **Proceedings of the 2010 American Control Conference**. [S.l.], 2010. p. 5179–5184. Pages 22 and 24.

[86] REIS, Gabriela L.; ARAÚJO, Rodrigo F.; TORRES, Leonardo A. B.; PALHARES, Reinaldo M. Gain-scheduled control design for discrete-time nonlinear systems using difference-algebraic representations. **International Journal of Robust and Nonlinear Control**, v. 31, n. 5, p. 1542–1563, 2021. Pages 22 and 49.

[87] REIS, Gabriela L.; ARAÚJO, Rodrigo F.; TORRES, Leonardo A. B.; PALHARES, Reinaldo M. Improved gain-scheduled control design for rational nonlinear discrete-time systems with input saturation. In: **20th European Control Conference (ECC 2022), London**. [S.l.: s.n.], 2022. p. 560–565. Page 22.

[88] REIS, Gabriela L.; ARAÚJO, Rodrigo F.; TORRES, Leonardo A. B.; PALHARES, Reinaldo M. Stabilization of rational nonlinear discrete-time systems by state feedback and static output feedback. **European Journal of Control**, v. 67, p. 100718, 2022. Page 23.

[89] JUNGERS, M; CASTELAN, E B. Gain-scheduled output control design for a class of discrete-time nonlinear systems with saturating actuators. **Systems & Control Letters**, Elsevier, 60, n. 3, p. 169–173, 2011. Pages 35 and 83.

[90] BOYD, Stephen; VANDENBERGHE, Lieven. **Convex Optimization**. [S.l.]: Cambridge University Press, 2004. Pages 36 and 82.

[91] ZHANG, Fuzhen. **The Schur complement and its applications**. [S.l.]: Springer Science & Business Media, 2006. v. 4. Page 45.

[92] WANG, Hua O; TANAKA, Kazuo; GRIFFIN, Michael F. An approach to fuzzy control of nonlinear systems: Stability and design issues. **IEEE transactions on Fuzzy Systems**, IEEE, v. 4, n. 1, p. 14–23, 1996. Page 50.

[93] GAHINET, P; NEMIROVSKI, A; LAUB, AJ; CHILALI, M. **The LMI Control Toolbox. For Use with Matlab. User's Guide. Natick, MA: The MathWorks**. [S.l.]: Inc, 1995. Page 51.

[94] NGUYEN, Anh-Tu; LAURAIN, Thomas; PALHARES, Reinaldo; LAUBER, Jimmy; SENTOUH, Chouki; POPIEUL, Jean-Christophe. LMI-based control synthesis of constrained Takagi–Sugeno fuzzy systems subject to $\mathcal{L}_2$ or $\mathcal{L}_\infty$ disturbances. **Neurocomputing**, Elsevier, v. 207, p. 793–804, 2016. Page 51.

[95] LÖFBERG, J. YALMIP: A toolbox for modeling and optimization in MATLAB. In: **2004 IEEE International Symposium on Computer Aided Control Systems Design**. Taipei, Taiwan: [s.n.], 2004. p. 284–289. Page 56.

[96] ANDERSEN, Erling D; ANDERSEN, Knud D. The mosek interior point optimizer for linear programming: an implementation of the homogeneous algorithm. In: **High Performance Optimization**. US: Springer, 2000. v. 33, p. 197–232. Disponível em: <http://www.mosek.com>. Page 56.

[97] TROFINO, Alexandre; DEZUO, Tiago J M. Global stability of uncertain rational nonlinear systems with some positive states. In: PROCEEDINGS OF 50TH IEEE CONFERENCE ON DECISION AND CONTROL AND EUROPEAN CONTROL CONFERENCE. **2011 50th IEEE Conference on Decision and Control and European Control Conference**. [S.l.], 2011. p. 7337–7342. Pages 71 and 77.

[98] GUERRA, Thierry Marie; VERMEIREN, Laurent. LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi–Sugeno's form. **Automatica**, Elsevier, v. 40, n. 5, p. 823–829, 2004. Pages 102, 104, and 110.

[99] LENDEK, Zsófia; GUERRA, Thierry-Marie; LAUBER, Jimmy. Controller design for TS models using delayed nonquadratic Lyapunov functions. **IEEE Transactions on Cybernetics**, IEEE, v. 45, n. 3, p. 439–450, 2015. Pages 104 and 110.

[100] ALLISON, John. Robust multi-objective control of hybrid renewable microgeneration systems with energy storage. **Applied Thermal Engineering**, Elsevier, v. 114, p. 1498–1506, 2017. Page 113.

[101] GAO, Huijun; LAM, James; WANG, Changhong. Multi-objective control of vehicle active suspension systems via load-dependent controllers. **Journal of Sound and Vibration**, Elsevier, v. 290, n. 3-5, p. 654–675, 2006. Page 113.

[102] SHEN, Chao; SHI, Yang; BUCKHAM, Brad. Path-following control of an AUV: A multiobjective model predictive control approach. **IEEE Transactions on Control Systems Technology**, IEEE, v. 27, n. 3, p. 1334–1342, 2019. Page 113.

[103] TAKAHASHI, Ricardo H C; PALHARES, Reinaldo M; DUTRA, Daniel A; GONÇALVES, Leila P S. Estimation of Pareto sets in the mixed control problem. **International Journal of Systems Science**, Taylor & Francis, v. 35, n. 1, p. 55–67, 2004. Page 113.

[104] SCHERER, Carsten; GAHINET, Pascal; CHILALI, Mahmoud. Multiobjective output-feedback control via LMI optimization. **IEEE Transactions on Automatic Control**, IEEE, v. 42, n. 7, p. 896–911, 1997. Page 113.

[105] BALANDIN, Dmitry V; KOGAN, Mark M. Multi-objective generalized H_2 control. **Automatica**, Elsevier, v. 99, p. 317–322, 2019. Page 113.

[106] LI, Xiao-Jian; WANG, Ning. Data-driven fault estimation and control for unknown discrete-time systems via multiobjective optimization method. **IEEE Transactions on Cybernetics**, IEEE, v. 52, n. 5, p. 3289–3301, 2020. Page 113.

[107] CAIGNY, Jan De; CAMINO, Juan F; OLIVEIRA, Ricardo C L F; PERES, Pedro Luis Dias; SWEVERS, Jan. Gain-scheduled H_2 and H_∞ control of discrete-time polytopic time-varying systems. **IET Control Theory & Applications**, IET, v. 4, n. 3, p. 362–380, 2010. Page 113.

[108] LOPES, Rosileide O; MENDES, Eduardo M A M; TORRES, Leonardo A B; PALHARES, Reinaldo M. Constrained robust model predicted control of discrete-time markov jump linear systems. **IET Control Theory & Applications**, IET, v. 13, n. 4, p. 517–525, 2019. Page 114.

[109] BESSA, Iury; PUIG, Vicenç; PALHARES, Reinaldo Martínez. Passivation blocks for fault tolerant control of nonlinear systems. **Automatica**, Elsevier, v. 125, p. 109450, 2021. Page 114.

[110] OLIVEIRA, Maurício Zardo; COUTINHO, Daniel Ferreira. Estabilidade robusta de sistemas lineares em tempo discreto sujeitos a atraso no estado. **Sba: Controle & Automação Sociedade Brasileira de Automatica**, SciELO Brasil, v. 19, n. 3, p. 270–280, 2008. Page 114.

[111] COUTINHO, Daniel F; SOUZA, Carlos E de. Delay-dependent robust stability and $\mathcal{L}_2$-gain analysis of a class of nonlinear time-delay systems. **Automatica**, Elsevier, v. 44, n. 8, p. 2006–2018, 2008. Page 114.

[112] SOUZA, Lucas T F De; PEIXOTO, Márcia L C; PALHARES, Reinaldo M. New gain-scheduling control conditions for time-varying delayed LPV systems. **Journal of the Franklin Institute**, Elsevier, v. 359, n. 2, p. 719–742, 2022. Page 114.

[113] PEIXOTO, Márcia L C; BRAGA, Márcio F; PALHARES, Reinaldo M. Gain-scheduled control for discrete-time non-linear parameter-varying systems with time-varying delays. **IET Control Theory & Applications**, IET, v. 14, n. 19, p. 3217–3229, 2020. Page 114.

[114] COUTINHO, Pedro Henrique S; MOREIRA, Luciano G.; PALHARES, Reinaldo M. Event-triggered control of quasi-LPV systems with communication delays. **International Journal of Robust and Nonlinear Control**, v. 32, n. 15, p. 8689–8710, 2022. Page 114.

[115] COUTINHO, Pedro Henrique Silva; PALHARES, Reinaldo Martinez. Co-design of dynamic event-triggered gain-scheduling control for a class of nonlinear systems. **IEEE Transactions on Automatic Control**, IEEE, v. 67, n. 8, p. 4186–4193, 2022. Page 114.

[116] JR, Luiz A Q Cordovil; COUTINHO, Pedro H S; BESSA, Iury; PEIXOTO, Márcia L C; PALHARES, Reinaldo Martínez. Learning event-triggered control based on evolving data-driven fuzzy granular models. **International Journal of Robust and Nonlinear Control**, Wiley Online Library, v. 32, n. 5, p. 2805–2827, 2022. Page 114.

Appendix

APPENDIX A – FINITE-DIMENSIONAL LMI RELAXATIONS

A.1 Guided proof of *Lemma* 4.1

The idea in this Section is to give more details to support the previous proof of *Lemma* 4.1 and demonstrate how the concept behind this lemma can be used to obtain a finite set of LMIs. For this purpose, we use *Theorem* 4.1 as an application example. Consider the inequality (4.1) from *Theorem* 4.1. By (2.12), (3.1), (3.15), and (4.8), this inequality can be recast as

$$\begin{bmatrix} \mathcal{M}_{11}^1 & \star & \star & \star & \star \\ \mathcal{M}_{21}^1 & \mathcal{M}_{22}^1 & \star & \star & \star \\ \mathcal{M}_{31}^1 & \mathcal{M}_{32}^1 & \mathcal{M}_{33}^1 & \star & \star \\ 0 & \mathcal{M}_{42}^1 & 0 & -I & \star \\ \mathcal{M}_{51}^1 & 0 & \mathcal{M}_{53}^1 & \mathcal{M}_{54}^1 & -\mu I \end{bmatrix} < 0, \tag{A.1}$$

$$\mathcal{M}_{11}^1 = -\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} G_{il}^\top - \sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} G_{il} + \sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} P_l,$$

$$\begin{aligned}\mathcal{M}_{21}^1 = {} & \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} A_{1jm}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} G_{il}\right) + \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} A_{3jm}\right) \\ & \times\left[\left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} K_{il}\right) + \left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r^-\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} H_{il}\right)\right],\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{31}^1 = {} & \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} \Omega_{1jm}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} G_{il}\right) + \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} \Omega_{3jm}\right) \\ & \times\left[\left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} K_{il}\right) + \left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r^-\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} H_{il}\right)\right],\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{51}^1 = {} & \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} C_{z_1jm}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} G_{il}\right) + \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} C_{z_3jm}\right) \\ & \times\left[\left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} K_{il}\right) + \left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r^-\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} H_{il}\right)\right],\end{aligned}$$

$$\mathcal{M}_{22}^{1} = -\sum_{n=1}^{N_\delta} \alpha_{\Delta_{(n)k}} P_n,$$

$$\mathcal{M}_{32}^{1} = \left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} N_{il}^{\top}\right)\left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} A_{2\ jm}^{\top}\right) + \left\{\left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} A_{3jm}\right)\right.$$
$$\left.\times\left[\left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} R_{il}\right) + \left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r^{-}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} S_{il}\right)\right]\right\}^{\top},$$

$$\mathcal{M}_{42}^{1} = \sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} A_{4\ jm}^{\top},$$

$$\mathcal{M}_{33}^{1} = \mathrm{He}\left\{\left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} \Omega_{2jm}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} N_{il}\right) + \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} \Omega_{3jm}\right)\right.$$
$$\left.\times\left[\left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} R_{il}\right) + \left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r^{-}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} S_{il}\right)\right]\right\},$$

$$\mathcal{M}_{53}^{1} = \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} C_{z_2jm}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} N_{il}\right) + \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} C_{z_3jm}\right)$$
$$\times\left[\left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} R_{il}\right) + \left(\sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r^{-}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} S_{il}\right)\right],$$

$$\mathcal{M}_{54}^{1} = \sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} C_{z_4jm}.$$

Note that, the system matrix indices have been chosen as "jm", without loss of generality. To obtain a finite set of LMIs in terms of the vertices of the polytopes $\mathcal{X}$, Δ, and $\mathcal{D}$, it is necessary to homogenize the terms of the previous inequality, as follows

$$\begin{bmatrix} \mathcal{M}_{11}^{2} & \star & \star & \star & \star \\ \mathcal{M}_{21}^{2} & \mathcal{M}_{22}^{1} & \star & \star & \star \\ \mathcal{M}_{31}^{2} & \mathcal{M}_{32}^{2} & \mathcal{M}_{33}^{2} & \star & \star \\ 0 & \mathcal{M}_{42}^{2} & 0 & \mathcal{M}_{44}^{2} & \star \\ \mathcal{M}_{51}^{2} & 0 & \mathcal{M}_{53}^{2} & \mathcal{M}_{54}^{2} & \mathcal{M}_{55}^{2} \end{bmatrix} < 0, \tag{A.2}$$

$$\mathcal{M}_{11}^{2} = \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} \left[-\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} G_{il}^{\top} - \sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} G_{il}\right.$$
$$\left. + \sum_{i=1}^{N_x} \alpha_{x_{(i)k}} \left(\sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} P_l\right)\right],$$

$$\mathcal{M}^2_{21} = \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} A_{1jm}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} G_{il}\right) + \sum_{n=1}^{N_\delta}\alpha_{\Delta_{(n)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}} \right.$$
$$\left. \times\ \alpha_{\delta_{(m)k}} A_{3jm}\right)\left[\left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} K_{il}\right) + \left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r^-\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} H_{il}\right)\right],$$

$$\mathcal{M}^2_{31} = \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} \Omega_{1jm}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} G_{il}\right) + \sum_{n=1}^{N_\delta}\alpha_{\Delta_{(n)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}} \right.$$
$$\left. \times\ \alpha_{\delta_{(m)k}} \Omega_{3jm}\right)\left[\left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} K_{il}\right) + \left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r^-\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} H_{il}\right)\right],$$

$$\mathcal{M}^2_{51} = \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} C_{z_1jm}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} G_{il}\right) + \sum_{n=1}^{N_\delta}\alpha_{\Delta_{(n)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}} \right.$$
$$\left. \times\ \alpha_{\delta_{(m)k}} C_{z_3jm}\right)\left[\left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} K_{il}\right) + \left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r^-\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} H_{il}\right)\right],$$

$$\mathcal{M}^2_{22} = -\sum_{r=1}^{N_u}\sum_{i=1}^{N_x}\sum_{j=1}^{N_x}\sum_{l=1}^{N_\delta}\sum_{m=1}^{N_x} \alpha_{d_{(r)k}}\alpha_{x_{(i)k}}\alpha_{x_{(j)k}}\alpha_{\delta_{(l)k}}\alpha_{\delta_{(m)k}} \left(-\sum_{n=1}^{N_\delta}\alpha_{\Delta_{(n)k}} P_n\right),$$

$$\mathcal{M}^2_{32} = \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}} \left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} N_{il}^\top\right)\left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} A_2^\top{}_{jm}\right) + \sum_{n=1}^{N_\delta}\alpha_{\Delta_{(n)k}} \left\{\left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}} \right.\right.$$
$$\left.\left. \times\ \alpha_{\delta_{(m)k}} A_{3jm}\right)\left[\left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} R_{il}\right) + \left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r^-\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} S_{il}\right)\right]\right\}^\top,$$

$$\mathcal{M}^2_{42} = \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} A_4^\top{}_{jm}\right),$$

$$\mathcal{M}^2_{33} = \sum_{n=1}^{N_\delta}\alpha_{\Delta_{(n)k}} \left\{\mathrm{He}\left\{\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} \Omega_{2jm}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} N_{il}\right) + \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}} \right.\right.\right.$$
$$\left.\left.\left. \times\ \alpha_{\delta_{(m)k}} \Omega_{3jm}\right)\left[\left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} R_{il}\right) + \left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r^-\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} S_{il}\right)\right]\right\}\right\},$$

$$\mathcal{M}^2_{53} = \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} C_{z_2jm}\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} N_{il}\right) + \sum_{n=1}^{N_\delta}\alpha_{\Delta_{(n)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}} \right.$$
$$\left. \times\ \alpha_{\delta_{(m)k}} C_{z_3jm}\right)\left[\left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} R_{il}\right) + \left(\sum_{r=1}^{N_u}\alpha_{d_{(r)k}} D_r^-\right)\left(\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} S_{il}\right)\right],$$

$$\mathcal{M}^2_{54} = \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} \left(\sum_{j=1}^{N_x}\sum_{m=1}^{N_\delta} \alpha_{x_{(j)k}}\alpha_{\delta_{(m)k}} C_{z_4jm}\right),$$

with the terms $\mathcal{M}_{44}^2$ and $\mathcal{M}_{55}^2$, given by the respective constant terms in (A.1) multiplied by $\sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x}\sum_{j=1}^{N_x}\sum_{l=1}^{N_\delta}\sum_{m=1}^{N_x}\alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha_{x_{(i)k}}\alpha_{x_{(j)k}}\alpha_{\delta_{(l)k}}\alpha_{\delta_{(m)k}}$.

It is worth emphasizing that, as defined in *Notation* section, $\sum_{\mathsf{v}=1}^{N}\alpha_{\mathsf{p}_{(\mathsf{v})k}} = 1$, for this reason, the previous procedure does not modify the inequality in (A.1). The next step is to group the terms according to the polynomials in $(\alpha_{d_{(r)k}}, \alpha_{\Delta_{(n)k}}, \alpha_{x_{(i)k}}, \alpha_{x_{(j)k}}\alpha_{\delta_{(l)k}}, \alpha_{\delta_{(m)k}})$. Let us consider the matrix Ψ_{ijlm}^{nr} defined as

$$\Psi_{ijlm}^{nr} = \begin{bmatrix} -G_{il}^\top - G_{il} + P_l & \star & \star & \star & \star \\ A_{1jm}G_{il} + A_{3jm}\left(D_rK_{il} + D_r^- H_{il}\right) & -P_n & \star & \star & \star \\ \Omega_{1jm}G_{il} + \Omega_{3jm}\left(D_rK_{il} + D_r^- H_{il}\right) & \mathcal{M}_{32}^3 & \mathcal{M}_{33}^3 & \star & \star \\ 0 & A_{4\ jm}^\top & 0 & -I & \star \\ C_{z_1jm}G_{il} + C_{z_3jm}\left(D_rK_{il} + D_r^- H_{il}\right) & 0 & \mathcal{M}_{53}^3 & C_{z_4jm} & -\mu I \end{bmatrix}, \tag{A.3}$$

$$\begin{aligned} \mathcal{M}_{32}^3 &= N_{il}^\top A_{2\ jm}^\top + \left[R_{il}^\top D_r + S_{il}^\top D_r^-\right] A_{3\ jm}^\top, \\ \mathcal{M}_{33}^3 &= \mathrm{He}\left\{\Omega_{2jm}N_{il} + \Omega_{3jm}\left[D_rR_{il} + D_r^- S_{il}\right]\right\}, \\ \mathcal{M}_{53}^3 &= C_{z_2jm}N_{il} + C_{z_3jm}\left[D_rR_{il} + D_r^- S_{il}\right]. \end{aligned}$$

Thus inequality (A.2) is recasted as

$$\sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta}\alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha_{x_{(i)k}}^2\alpha_{\delta_{(l)k}}^2\Psi_{iill}^{nr}$$

$$+\sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x-1}\sum_{j=i+1}^{N_x}\sum_{l=1}^{N_\delta}\alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha_{x_{(i)k}}\alpha_{x_{(j)k}}\alpha_{\delta_{(l)k}}^2\left(\Psi_{ijll}^{nr} + \Psi_{jill}^{nr}\right)$$

$$+\sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta-1}\sum_{m=l+1}^{N_\delta}\alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha_{x_{(i)k}}^2\alpha_{\delta_{(l)k}}\alpha_{\delta_{(m)k}}\left(\Psi_{iilm}^{nr} + \Psi_{iiml}^{nr}\right)$$

$$+\sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x-1}\sum_{j=i+1}^{N_x}\sum_{l=1}^{N_\delta-1}\sum_{m=l+1}^{N_\delta}\alpha_{d_{(r)k}}\alpha_{\Delta_{(n)k}}\alpha_{x_{(i)k}}\alpha_{x_{(j)k}}\alpha_{\delta_{(l)k}}\alpha_{\delta_{(m)k}}\left(\Psi_{ijlm}^{nr} + \Psi_{ijml}^{nr} + \Psi_{jilm}^{nr} + \Psi_{jiml}^{nr}\right) < 0.$$

As $\alpha_{\mathsf{p}_{(\mathsf{v})k}} \geq 0$, if the LMIs in (4.20) are satisfied, with Ψ_{ijlm}^{nr} defined in (A.3), then condition (4.1) in *Theorem* 4.1 is guaranteed. Similar procedure can be performed to condition (4.3), disregarding the variable n, since this inequality does not present the term $P(\delta_{k+1})$. Finally, considering the polytopic description of matrices $P(\delta_k)$ and $G(x_k, \delta_k)$ it is possible to obtain the finite-dimensional LMIs to computationally handle condition (4.4)

$$\begin{bmatrix} \lambda & \star \\ G_{il}^\top a_v^\top & G_{il}^\top + G_{il} - P_l \end{bmatrix} \geq 0, \quad v \in \mathcal{I}_{n_e},\ l \in \mathcal{I}_{N_\delta},\ i \in \mathcal{I}_{N_x}$$

A.2 Extension to include uncertain time-varying parameters

As discussed previously, similarly to the case shown in Lemma 4.1, without uncertain time-varying parameters, it is possible to use the multi-simplex framework based on (2.12), (4.23), (3.16), and (4.22) to obtain a finite set of LMIs in terms of the vertices of the polytopes $\mathcal{X}$, Δ, $\mathcal{D}$, and $\overline{\Delta}$, as follows.

Lemma A.1 (Extension of Lemma 4.1 to include uncertain time-varying parameters). *Suppose* Y^{nqr}_{ijlmop}, *with* $i,j \in \mathcal{I}_{N_x}$, $l,m,n \in \mathcal{I}_{N_\delta}$, $o,p,q \in \mathcal{I}_{N_{\overline{\delta}}}$, *and* $r \in \mathcal{I}_{N_u}$, *are matrices of appropriate dimensions, such that*

$$Y(x_k, \delta_k, \delta_{k+1}, \overline{\delta}_k, \overline{\delta}_{k+1}, \alpha_{d_k}) = \sum_{r=1}^{N_u}\sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x}\sum_{j=1}^{N_x}\sum_{l=1}^{N_\delta}\sum_{m=1}^{N_\delta} \alpha_k Y^{nqr}_{ijlmop} < 0. \tag{A.4}$$

with $\alpha_k = \alpha_{d_{(r)k}} \alpha_{\overline{\Delta}_{(q)k}} \alpha_{\Delta_{(n)k}} \alpha_{x_{(i)k}} \alpha_{x_{(j)k}} \alpha_{\delta_{(l)k}} \alpha_{\delta_{(m)k}} \alpha_{\overline{\delta}_{(o)k}} \alpha_{\overline{\delta}_{(p)k}}$.

If the following LMIs hold for all $i,j \in \mathcal{I}_{N_x}$, $l,m,n \in \mathcal{I}_{N_\delta}$, $o,p,q \in \mathcal{I}_{N_{\overline{\delta}}}$, *and* $r \in \mathcal{I}_{N_u}$

$$\begin{aligned}
&Y^{nqr}_{iilloo} < 0, \quad i=j,\ l=m,\ o=p,\\
&Y^{nqr}_{ijlloo} + Y^{nqr}_{jilloo} < 0, \quad i<j,\ l=m,\ o=p,\\
&Y^{nqr}_{iilmoo} + Y^{nqr}_{iimloo} < 0, \quad i=j,\ l<m,\ o=p,\\
&Y^{nqr}_{iillop} + Y^{nqr}_{iillpo} < 0, \quad i=j,\ l=m,\ o<p,\\
&Y^{nqr}_{ijlmoo} + Y^{nqr}_{ijmloo} + Y^{nqr}_{jilmoo} + Y^{nqr}_{jimloo} < 0, \quad i<j,\ l<m,\ o=p,\\
&Y^{nqr}_{ijllop} + Y^{nqr}_{ijllpo} + Y^{nqr}_{jillop} + Y^{nqr}_{jillpo} < 0, \quad i<j,\ l=m,\ o<p,\\
&Y^{nqr}_{iilmop} + Y^{nqr}_{iilmpo} + Y^{nqr}_{iimlop} + Y^{nqr}_{iimlpo} < 0, \quad i=j,\ l<m, o<p,\\
&Y^{nqr}_{ijlmop} + Y^{nqr}_{ijlmpo} + Y^{nqr}_{ijmlop} + Y^{nqr}_{ijmlpo} + Y^{nqr}_{jilmop} + Y^{nqr}_{jilmpo} + Y^{nqr}_{jimlop} + Y^{nqr}_{jimlpo} < 0,\\
&\qquad\qquad i<j,\ l<m,\ o<p,
\end{aligned} \tag{A.5}$$

then inequality (A.4) *is satisfied.*

Proof. The hypothetical matrix in (A.4) can be rewritten as

$$
\begin{aligned}
&\Upsilon(x_k, \delta_k, \delta_{k+1}, \overline{\delta}_k, \overline{\delta}_{k+1}, \alpha_{d_k}) = \\
&\quad \sum_{r=1}^{N_u} \sum_{q=1}^{N_{\overline{\delta}}} \sum_{n=1}^{N_\delta} \alpha_{d_{(r)k}} \alpha_{\overline{\Delta}_{(q)k}} \alpha_{\Delta_{(n)k}} \Bigg[\sum_{i=1}^{N_x} \sum_{l=1}^{N_\delta} \sum_{o=1}^{N_{\overline{\delta}}} \alpha^2_{x_{(i)k}} \alpha^2_{\delta_{(l)k}} \alpha^2_{\overline{\delta}_{(o)k}} \Upsilon^{nqr}_{iilloo} \\
&\quad + \sum_{i=1}^{N_x-1} \sum_{j=i+1}^{N_x} \sum_{l=1}^{N_\delta} \sum_{o=1}^{N_{\overline{\delta}}} \alpha_{x_{(i)k}} \alpha_{x_{(j)k}} \alpha^2_{\delta_{(l)k}} \alpha^2_{\overline{\delta}_{(o)k}} \left(\Psi^{nqr}_{ijlloo} + \Psi^{nqr}_{jilloo} \right) \\
&\quad + \sum_{i=1}^{N_x} \sum_{l=1}^{N_\delta-1} \sum_{m=l+1}^{N_\delta} \sum_{o=1}^{N_{\overline{\delta}}} \alpha^2_{x_{(i)k}} \alpha_{\delta_{(l)k}} \alpha_{\delta_{(m)k}} \alpha^2_{\overline{\delta}_{(o)k}} \left(\Psi^{nqr}_{iilmoo} + \Psi^{nqr}_{iimloo} \right) \\
&\quad + \sum_{i=1}^{N_x} \sum_{l=1}^{N_\delta} \sum_{o=1}^{N_{\overline{\delta}-1}} \sum_{p=o+1}^{N_{\overline{\delta}}} \alpha^2_{x_{(i)k}} \alpha^2_{\delta_{(l)k}} \alpha_{\overline{\delta}_{(o)k}} \alpha_{\overline{\delta}_{(p)k}} \left(\Psi^{nqr}_{iillop} + \Psi^{nqr}_{iillpo} \right) \\
&\quad + \sum_{i=1}^{N_x-1} \sum_{j=i+1}^{N_x} \sum_{l=1}^{N_\delta-1} \sum_{m=l+1}^{N_\delta} \sum_{o=1}^{N_{\overline{\delta}}} \alpha_{x_{(i)k}} \alpha_{x_{(j)k}} \alpha_{\delta_{(l)k}} \alpha_{\delta_{(m)k}} \alpha^2_{\overline{\delta}_{(o)k}} \left(\Psi^{nqr}_{ijlmoo} + \Psi^{nqr}_{ijmloo} + \Psi^{nqr}_{jilmoo} + \Psi^{nqr}_{jimloo} \right) \\
&\quad + \sum_{i=1}^{N_x-1} \sum_{j=i+1}^{N_x} \sum_{l=1}^{N_\delta} \sum_{o=1}^{N_{\overline{\delta}-1}} \sum_{p=o+1}^{N_{\overline{\delta}}} \alpha_{x_{(i)k}} \alpha_{x_{(j)k}} \alpha^2_{\delta_{(l)k}} \alpha_{\overline{\delta}_{(o)k}} \alpha_{\overline{\delta}_{(p)k}} \left(\Psi^{nqr}_{ijllop} + \Psi^{nqr}_{ijllpo} + \Psi^{nqr}_{jillop} + \Psi^{nqr}_{jillpo} \right) \\
&\quad + \sum_{i=1}^{N_x} \sum_{l=1}^{N_\delta-1} \sum_{m=l+1}^{N_\delta} \sum_{o=1}^{N_{\overline{\delta}-1}} \sum_{p=o+1}^{N_{\overline{\delta}}} \alpha^2_{x_{(i)k}} \alpha_{\delta_{(l)k}} \alpha_{\delta_{(m)k}} \alpha_{\overline{\delta}_{(o)k}} \alpha_{\overline{\delta}_{(p)k}} \left(\Psi^{nqr}_{iilmop} + \Psi^{nqr}_{iilmpo} + \Psi^{nqr}_{iimlop} + \Psi^{nqr}_{iimlpo} \right) \\
&\quad + \sum_{i=1}^{N_x-1} \sum_{j=i+1}^{N_x} \sum_{l=1}^{N_\delta-1} \sum_{m=l+1}^{N_\delta} \sum_{o=1}^{N_{\overline{\delta}-1}} \sum_{p=o+1}^{N_{\overline{\delta}}} \alpha_{x_{(i)k}} \alpha_{x_{(j)k}} \alpha_{\delta_{(l)k}} \alpha_{\delta_{(m)k}} \alpha_{\overline{\delta}_{(o)k}} \alpha_{\overline{\delta}_{(p)k}} \Big(\Psi^{nqr}_{ijlmop} + \Psi^{nqr}_{ijlmpo} \\
&\qquad\qquad + \Psi^{nqr}_{ijmlop} + \Psi^{nqr}_{ijmlpo} + \Psi^{nqr}_{jilmop} + \Psi^{nqr}_{jilmpo} + \Psi^{nqr}_{jimlop} + \Psi^{nqr}_{jimlpo} \Big) .
\end{aligned}
$$

Since $\alpha_{\mathsf{p}_{(\mathsf{v})k}} \geq 0$, if (4.20) are satisfied, then condition (4.19) is guaranteed. □