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On the phase transition for some percolation models in random environments

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On the phase transition for some percolation models in random environments

A transição de fase para alguns modelos de percolação em ambientes aleatórios

Tese apresentada ao Programa de Pós-Graduação da Universidade Federal de Minas Gerais, como requisito parcial à obtenção do título de Doutor em Matemática.

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Resumo

Nesta tese nós consideramos dois modelos de percolação em ambientes aleatórios e estamos interessados em seus fenômenos de transição de fase.

O primeiro modelo de percolação estudado é na rede cúbica apresentando desordem colunar. Este modelo é definido em dois passos: primeiro as colunas verticais de $\mathbb{Z}^3$ são removidas independentemente com probabilidade $1-\rho$ e, no segundo passo, os elos conectando sítios na sub-rede remanescente são declarados abertos com probabilidade p de modo independente. Nosso resultado mostra que existe $\delta > 0$ tal que o ponto crítico $p_c(\rho) < 1/2 - \delta$ para todo $\rho > \rho_c$, onde ρ_c denota o ponto crítico da percolação de sítios em $\mathbb{Z}^2$.

O segundo modelo é na rede quadrada esticada horizontalmente, que consiste de uma versão generalizada de $\mathbb{Z}_+^2$ obtida ao se esticar a distância entre suas colunas, segundo uma variável aleatória positiva ξ. Neste modelo a probabilidade de um elo ser declarado aberto decairá exponencialmente segundo seu comprimento. Nosso resultado mostra a existência da transição de fase quando $\mathbb{E}(\xi^\eta) < \infty$, para algum $\eta > 1$, e a ausência quando $\mathbb{E}(\xi^\eta) = \infty$, para algum $\eta < 1$.

Palavras-chave: Percolação, transição de fase, ambientes aleatórios, renormalização multi-escala.

Abstract

In this thesis we consider two percolation models in random environments and we are interested in their phase transition phenomenon.

The first percolation model we study is defined on the cubic lattice featuring columnar disorder. This model is defined in two steps: first the vertical columns of $\mathbb{Z}^3$ are removed independently with probability $1-\rho$ and, in the second step, the bonds connecting sites in the remaining sub-lattice are declared open with probability p, independently. Our result shows that there exists $\delta > 0$ such that $p_c(\rho) < 1/2 - \delta$ for any $\rho > \rho_c$, where ρ_c denotes the critical point of site percolation in $\mathbb{Z}^2$.

The second model is defined on a horizontally stretched square lattice, which is a generalized version of $\mathbb{Z}^2_+$ obtained by stretching the distances between its columns according to a positive random variable ξ. In this model the probability of a bond being declared open will decay exponentially according to its length. Our result shows the existence of a phase transition when $\mathbb{E}(\xi^\eta) < \infty$, for some $\eta > 1$, and the absence of phase transition when $\mathbb{E}(\xi^\eta) = \infty$ for some $\eta < 1$.

Keywords: Percolation, phase transition, random environments, renormalization, multiscale analysis.

Contents

Chapter 1

Introduction

In Section 1.1, the notation that will be used throughout the text is introduced and some important definitions are given. In Section 1.2, a brief summary on the history of percolation theory will be presented focusing on models that motivated this thesis. Such models present inhomogeneities which are introduced by means of a random environment. In Section 1.3 the models considered in this thesis will be introduced and the main results obtained will be stated.

1.1 Basic notation and definitions

In this section some of the notation and definitions used throughout this thesis will be introduced.

1.1.1 Graphs

A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G) \subseteq \binom{V(G)}{2}$ of *edges*, where

$$\binom{V(G)}{2} = \big\{\{v, w\} \subseteq V(G);\ v \neq w\big\}$$

is set of all (unordered) pairs of vertices in G. When there is no risk of ambiguity, we will abuse notation and not distinguish the graph G and its set of vertices $V(G)$. We say that G is an *infinite graph* if $|V(G)| = \infty$, where $|A|$ denote the *cardinality* of set A. Given G and $v, w \in G$, we write $v \sim w$ if v is a *neighbor* of w, i.e. $\{v, w\} \in E(G)$. In this case, v and w are called the *endpoints* of the edge $\{v, w\}$.

In specific situations throughout the text, other structures similar to graphs will be mentioned: *oriented graphs* and *multigraphs*. An oriented graph $\overrightarrow{G} = \big(V(G), \overrightarrow{E}(G)\big)$ is a graph in which an orientation is assigned to each one of its edges. A multigraph $\mathbf{G}$ is a graph in which the existence of *loops* (edges with only one endpoint) and *multiple edges* (more than one edge with the same endpoints) is permitted. From now on, all the definitions in this sections will be stated for graphs, however they can be easily extended to oriented graphs and multigraphs.

Given the graphs G and H, H is called a *subgraph* of G if

$$V(H) \subseteq V(G) \qquad \text{and} \qquad E(H) \subseteq E(G).$$

Now let $A \subseteq V(G)$. The *subgraph of G induced by A* is the subgraph whose set of vertices is A and whose set of edges consists of all edges in $E(G)$ with both endpoints in A. The boundary of A is

$$\partial A = \{w \in G \backslash A; \text{ there exists } v \in A \text{ such that } v \sim w\}.$$

For the sake of brevity, we do not distinguish the set $\{v\}$ from the element v, and denote ∂v the set of nearest neighbors of the vertex v. We say that the graph G is *locally finite* if $|\partial v| < \infty$ for every vertex v.

A *finite path* γ (in $A \subseteq G$) is a sequence of distinct vertices $(v_1, \ldots, v_n)$ (in A) such that $v_{i-1} \sim v_i$ for any $2 \leqslant i \leqslant n$. The vertices v_1 and v_n are called the *endpoints* of γ. The graph G is said to be *connected* if for every pair of vertices $v, w \in G$ there is a path with endpoints $\{v, w\}$. An *infinite path* $\gamma = (v_1, v_2, ...)$ is defined similarly with only one endpoint v_1. Here we regard paths in A as subsets of G.

Now for $d \in \{1, 2, ...\}$, consider the graph $\mathbb{Z}^d$ whose vertices are the d-tuples of integers, and where two vertices are neighbors if, and only if, only one of their coordinate disagree by one unit. It will be convenient to regard these lattices naturally embedded in $\mathbb{R}^d$. The two main lattices considered here will be $\mathbb{Z}^2$ and $\mathbb{Z}^3$ called square lattice and cubic lattice, respectively. Denote by o the origin of the lattice $\mathbb{Z}^d$, that is the d-tuple with zero on all entries.

The last definition in this section will be another frequent structure present in the next chapters. Let $a, b, c, d \in \mathbb{Z}$ with $a < b$ and $c < d$. Denote by

$$R = R\big([a, b) \times [c, d)\big) \tag{1.1}$$

the subgraph of $\mathbb{Z}^2$ defined by

$$\begin{aligned} V(R) &= [a,b] \times [c,d] \quad \text{and} \\ E(R) &= \big\{\{(x,y),(x+i,y+1-i)\};\ (x,y) \in [a,b-1] \times [c,d-1],\ i \in \{0,1\}\big\}, \end{aligned}$$

where $[a,b]$ denotes the set of all integers between a and b, including them both. R will be called a *rectangle*, and is the graph induced by $[a,b] \times [c,d]$ removed the edges in the right and top sides. See Figure 1.1.

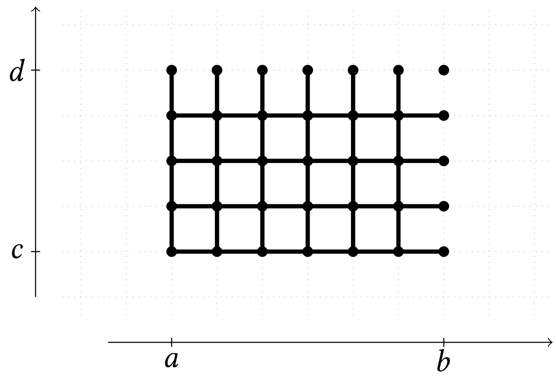

Figure 1.1: Illustration of the rectangle $R\big([a,b) \times [c,d)\big)$. Note that $R\big([a,b) \times [c,d)\big)$ is not equal to the graph induced by $[a,b-1] \times [c,d-1]$.

1.1.2 Percolation model

Consider G an infinite, locally finite, connected graph. A *bond percolation configuration* in G is an element

$$\omega = \big(\omega(e);\ e \in E(G)\big)$$

of the sample space $\{0,1\}^{E(G)}$. A *percolation event* is an element of the σ-algebra generated by the cylinder sets (events which only depend on the state of a finite number of edges). The edge $e \in E(G)$ is said to be *open* if $\omega(e) = 1$, otherwise it is said to be *closed*. Note that ω can be seen as a subgraph of G that contains only the edges which are open for ω, that is

$$V(\omega) := V(G) \quad \text{and} \quad E(\omega) := \{e \in E(G);\ \omega(e) = 1\}.$$

Now let $A, B, S \subseteq G$, with $A \cup B \subseteq S$. We denote $\{A \leftrightarrow B \text{ in } S\}$, the event that there is a path $(v_1, \dots, v_n)$ with $v_1 \in A$ and $v_n \in B$ such that $v_i \in S$ for any $1 \leqslant i \leqslant n$ and that $\omega\big(\{v_{i-1}, v_i\}\big) = 1$ for any $1 \leqslant i \leqslant n$. When $S = G$, we drop it from notation. Also we define the *cluster* of A in ω, by

$$C_A(\omega) = \big\{u \in V(G);\ \text{there is } v \in A \text{ such that } \omega \in \{v \leftrightarrow u\}\big\}.$$

For a vertex $v \in G$, let

$$\{v \leftrightarrow \infty\} = \big\{\omega \in \{0,1\}^{E(G)};\ |C_v(\omega)| = \infty\big\},$$

be the event that v is connected to infinity. A central question in percolation is whether $\{v \leftrightarrow \infty\}$ has positive probability, when ω is sampled at random.

In the Bernoulli bond percolation model on G, the $\omega(e)$'s are independent Bernoulli random variables with mean $p_e \in [0,1]$. When $p_e = p$ for some $p \in [0,1]$ and for every edge $e \in E(G)$, the model is said to be *homogeneous*. Otherwise it is said to be *inhomogeneous*. For the homogeneous percolation model, its law will be denoted by $\mathbb{P}_p(\cdot)$. The critical point of G is defined as

$$p_c(G) = \sup\big\{p \in [0,1];\ \mathbb{P}_p(v \leftrightarrow \infty) = 0\big\}.$$

Since G is connected, the FKG inequality implies that the critical point does not depend on the choice of vertex v. When $0 < p_c(G) < 1$, we say that the model undergoes a non-trivial phase transition, since the value of $\mathbb{P}_p(v \leftrightarrow \infty)$ changes from 0 when $p < p_c(G)$ to positive when $p > p_c(G)$.

Now consider the square lattice, $\mathbb{Z}^2$, and let $R = R\big([a,b) \times [c,d)\big)$ be as in (1.1). The horizontal and vertical crossing events in R are defined respectively as

$$\mathfrak{C}_h(R) = \big\{\{a\} \times [c,d] \leftrightarrow \{b\} \times [c,d] \text{ in } R\big\}, \tag{1.2}$$

$$\mathfrak{C}_v(R) = \big\{[a,b] \times \{c\} \leftrightarrow [a,b] \times \{d\} \text{ in } R\big\}. \tag{1.3}$$

Similarly, in order to define the Bernoulli site percolation model on G, we consider the *percolation configuration*

$$\omega = \big(\omega(v);\ v \in V(G)\big) \in \{0,1\}^{V(G)},$$

that can be seen as the subgraph of G induced by $\big\{v;\ \omega(v) = 1\big\}$. In the Bernoulli site percolation model on G, the $\omega(v)$'s are independent Bernoulli random variables with mean $p_v \in [0,1]$. The connectivity and crossing events are defined analogously. The horizontal and vertical crossing events in the rectangle R will be denoted respectively by $\mathfrak{C}_h^v(R)$ and $\mathfrak{C}_v^v(R)$. For the homogeneous Bernoulli site percolation model, its law will be denoted by $\mathbb{P}_p^{\text{site}}(\cdot)$. The critical point of G is defined by

$$p_c^{\text{site}}(G) = \sup\left\{p \in [0,1];\ \mathbb{P}_p^{\text{site}}(v \leftrightarrow \infty) = 0\right\}.$$

1.2 Motivation: a brief history of percolation theory in random environments

Percolation theory first appeared in mathematical literature in 1957 [6] when Broadbent and Hammersley modeled mathematically the flow of a fluid through a porous medium. They proved that for $d \geqslant 2$ the critical point of the (homogeneous) Bernoulli percolation model is non-trivial, i.e., $0 < p_c(\mathbb{Z}^d) < 1$. In 1960, Harris [14] showed that for the square lattice $\mathbb{P}_{1/2}(o \leftrightarrow \infty) = 0$, and twenty years later Kesten [18] showed that $p_c(\mathbb{Z}^2) = 1/2$. For a comprehensive list of references on percolation theory see Grimmett's book [12].

One of the main motivation for this work is the question:

How the introduction of inhomogeneities modifies the existence of the phase transition or shifts the critical points in percolation models?

This type of question was posed in a number of different situations, as mentioned below.

In several percolation models, inhomogeneties arise by introduction of an environment which specifies how to assign weights p_e for each edge e of the graph. For instance, for the square lattice, $\mathbb{Z}^2$, one way to introduce inhomogeneities is by fixing columns (the environment) whose edges will have a probability p of being open, while the other edges will be open with probability q. Formally let $\Lambda \subseteq \mathbb{Z}$, and set

$$E_{\text{vert}}(\Lambda) := \left\{\{(x,y),(x,y+1)\};\ x \in \Lambda, y \in \mathbb{Z}\right\},$$

which corresponds to the edges belonging to the columns that project to Λ. Let $\mathbb{P}_{p,q}^{\Lambda}(\cdot)$ be the probability law in $\{0,1\}^{E(\mathbb{Z}^2)}$ that governs the percolation model whose weights of edges are given by

$$p_e = \begin{cases} p, & \text{if } e \in E_{\text{vert}}(\Lambda), \\ q, & \text{if } e \notin E_{\text{vert}}(\Lambda). \end{cases}$$

In [28], Zhang nicely explores the ideas in [14] of constructing dual circuits around the origin, together with the Russo, Seymour and Welsh [25, 27] techniques, in order to prove that $\mathbb{P}^{\{0\}}_{p,q}(o \leftrightarrow \infty) = 0$ for any $p \in [0,1)$ and $q \leqslant p_c(\mathbb{Z}^2) = 1/2$. It follows from the results on percolation in half-spaces of Barsky, Grimmett and Newman [4] that $\mathbb{P}^{\{0\}}_{p,q}(o \leftrightarrow \infty) > 0$ whenever $q > p_c(\mathbb{Z}^2)$. It is also known that $\mathbb{P}^{\mathbb{Z}}_{p,q}(o \leftrightarrow \infty) > 0$ iff $p + q > 1$ (see page 54 in [19] or Section 11.9 in [12]).

Suppose there is k such that for every $l \in \mathbb{Z}$, $\Lambda \cap [l, l+k] \neq \varnothing$. A classical argument due to Aizenman and Grimmett [3] guarantees that for any $\varepsilon > 0$ there is $\delta = \delta(k, \varepsilon) > 0$ such that $\mathbb{P}^{\Lambda}_{p_c+\varepsilon, p_c-\delta}(o \leftrightarrow \infty) > 0$, where $p_c = p_c(\mathbb{Z}^2)$.

We now consider models for which the environment is taken randomly. Let υ_ρ be the probability measure on $\mathbb{Z}$ under which $\{i \in \Lambda\}$ are independent events having probability ρ. Recently Duminil-Copin, Hilário, Kozma and Sidoravicius [9] showed that for any $\varepsilon > 0$ and $\rho > 0$ there is $\delta = \delta(\rho, \varepsilon) > 0$ such that $\mathbb{P}^{\Lambda}_{p_c+\varepsilon, p_c-\delta}(o \leftrightarrow \infty) > 0$ for υ_ρ-almost everywhere environment Λ.

Using the arguments in Bramson, Durrett and Schonmann [5] one can prove that for any $\rho \in [0,1)$, there is $p < 1$ large enough such that $\mathbb{P}^{\Lambda}_{0,p}(o \leftrightarrow \infty) > 0$ for υ_ρ-almost every environment Λ. The last result says that if one deletes all edges of columns according to Bernoulli trials with mean ρ, the phase transition is present. In [16], Hoffman study the case where both rows and columns are deleted independently with the same probability ρ showing that the model still undergoes a non-trivial phase transition.

Another variation was studied by Kesten, Sidoravicius and Vares [20], who considered a site percolation model on the oriented square lattice, with the edges oriented from left to right and from bottom to top. Let Λ be taken in the same way as in [9]. The weights p_v of the vertices will be chosen according to whether or not they belong to diagonals given by Λ as follows

$$p_v = \begin{cases} p, & \text{if } v = (x - y, y) \text{ for some } x \in \Lambda \text{ and } y \in \mathbb{Z}, \\ q, & \text{otherwise.} \end{cases}$$

Let $\overrightarrow{\mathbb{P}}^{\Lambda}_{p,q}(\cdot)$ the probability law that governs this model. The main result in [20] shows that for any $p > p^{v}_c(\overrightarrow{\mathbb{Z}^2})$ and $q > 0$, there is $\rho < 1$ such that $\overrightarrow{\mathbb{P}}^{\Lambda}_{p,q}(o \leftrightarrow \infty) > 0$ for υ_ρ-almost everywhere environment Λ, where $p_c(\overrightarrow{\mathbb{Z}^2})$ denotes the critical point with respect to the measure $\overrightarrow{\mathbb{P}}^{\Lambda}_{p,p}(\cdot)$.

Another related model is the Bernoulli line percolation introduced in the physics literature by Kantor [17] and studied mathematically by Hilário and Sidoravicius [15]. This site percolation model on $\mathbb{Z}^d$, $d \geqslant 3$, consists of selecting lines parallel to the coordinate axes at random, according to Bernoulli trials. A vertex is declared closed if it belongs to any of these lines, and open otherwise. In [15], the existence of a phase transition, the number of infinite clusters and the connectivity decay are studied. The presence of power-law decay contrasts sharply with the behavior of models with finite-range dependencies where the decay is exponential, see [1, 24]. This model was studied also from the numerical point of view in [10, 26].

1.3 Models and main results

In this section, we give a mathematical construction of the two models of percolation in random environments which will be studied in this thesis. We also provide the statements of the main results achieved. The proofs will be given in Chapter 2 and 3.

1.3.1 Percolation model on a dilute lattice with columnar disorder

In this section, we introduce a percolation model on the cubic lattice, $\mathbb{Z}^3$, featuring columnar disorder, consisting of a combination of the Bernoulli line percolation and the homogeneous Bernoulli bond percolation.

First consider the square lattice $\mathbb{Z}^2$ embedded in $\mathbb{Z}^3$ by identifying $\mathbb{Z}^2$ to the set of all the vertices of $\mathbb{Z}^3$ whose third coordinate vanishes, and call an environment every element of $\Lambda \in \{0,1\}^{\mathbb{Z}^2}$. Slightly abusing notation, Λ can be seen as the subset

$$\{v \in \mathbb{Z}^2;\ \Lambda(v) = 1\} \subseteq \mathbb{Z}^2.$$

Given an environment Λ and a vertex $v \in \mathbb{Z}^2$, the column $\{v\} \times \mathbb{Z} \subseteq \mathbb{Z}^3$ is said to be *occupied* if $\Lambda(v) = 1$ and *vacant* otherwise. Now define the set of excluded edges as

$$E_{\text{exc}}(\Lambda) = \big\{e \in E(\mathbb{Z}^3);\ e \text{ has at least one of its endpoints in a vacant column}\big\}.$$

Consider now the percolation model on $\mathbb{Z}^3$, as defined in Section 1.1.2, whose parameter $p_e = p_e(\Lambda)$ of the edge $e \in E(\mathbb{Z}^3)$ to be open is given by

$$p_e = \begin{cases} p, & \text{if } e \notin E_{\text{exc}}(\Lambda) \\ 0, & \text{if } e \in E_{\text{exc}}(\Lambda) \end{cases}.$$

Denote by $\mathbb{P}_p^\Lambda(\cdot)$ the respective probability measure on $\{0,1\}^{E(\mathbb{Z}^3)}$ under which $\{\omega(e)\}_{e\in E(\mathbb{Z}^3)}$ are independent Bernoulli random variables with mean p_e. The measure $\mathbb{P}_p^\Lambda$ is called the *percolation law in $\mathbb{Z}^3$ with quenched columnar disorder* Λ.

Note that this model is equivalent to the homogeneous percolation model in $\Lambda \times \mathbb{Z}$, viewed as a subgraph of $\mathbb{Z}^3$ induced by the occupied columns. For a fixed environment $\Lambda \in \{0,1\}^{\mathbb{Z}^2}$, we say that a path $\gamma = (v_0, \dots, v_n)$ in $\mathbb{Z}^2$ is occupied if all of its vertices v_i are occupied (that is, $\Lambda(v_i) = 1$ for all i).

The environments Λ will be taken at random. Denote by ν_ρ the probability measure on $\{0,1\}^{\mathbb{Z}^2}$ under which $\{\Lambda(v)\}_{v\in\mathbb{Z}^2}$ are independent Bernoulli random variables with parameter $\rho \in (0,1]$. Note that ν_ρ is the law of the homogeneous site percolation model on $\mathbb{Z}^2$, and regard an environment Λ as a percolation configuration in $\mathbb{Z}^2$. Set

$$\rho_c := p_c^{\text{site}}(\mathbb{Z}^2).$$

We will also consider the *annealed* probability measure on $\{0,1\}^{E(\mathbb{Z}^3)}$ given by

$$\mathbb{P}_p^\rho(\cdot) := \int_{\{0,1\}^{\mathbb{Z}^2}} \mathbb{P}_p^\Lambda(\cdot) d\nu_\rho(\Lambda).$$

Note that, given a typical Λ, under the quenched measure $\mathbb{P}_p^\Lambda$, the state of the non-excluded edges are independent of each other and the measure is invariant under vertical shift. Under the annealed measure $\mathbb{P}_p^\rho$, the state of the edges present in the same column are dependent, regardless of the distance between then, but this measure is invariant under lattice shifts.

Given Λ, we can define $p_c(\Lambda)$ the critical percolation threshold for the quenched law $\mathbb{P}_p^\Lambda$, that is, the value of p above which ω has an infinite cluster $\mathbb{P}_p^\Lambda$-almost surely but below which such a component does not exist, $\mathbb{P}_p^\Lambda$-almost surely. Similarly, define $p_c(\rho)$ the critical threshold for the annealed law $\mathbb{P}_p^\rho$. By standard ergodicity arguments one can show that $p_c(\rho) = p_c(\Lambda)$ for ν_ρ-almost any environment Λ.

This model was introduced in Grassberger [11] and was studied from both the numerical and the mathematical point of view in Grassberger-Hilário-Sidoravicius [10] where the authors show rigorously that the connectivity decays with a power-law when $\rho > \rho_c$ and $p \in \big(p_c(\mathbb{Z}^3), p_c(\rho)\big)$. Figure 1.2 shows the graph of $\rho \mapsto p_c(\rho)$ obtained numerically in [10].

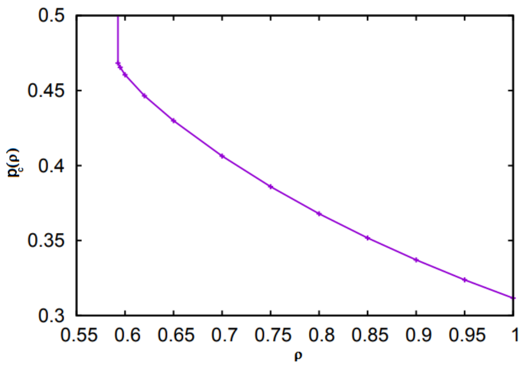

Figure 1.2: The critical curve $\rho \mapsto p_c(\rho)$. This graph was taken from [10].

Before we state our result, let us list some immediate properties fulfilled by $\rho \to p_c(\rho)$:

1. A standard coupling argument reveals that $p_c(\rho)$ is non-increasing. Furthermore, $p_c(1)$ equals the critical threshold for Bernoulli bond percolation on $\mathbb{Z}^3$.

2. For every $\rho \leqslant \rho_c$, $p_c(\rho) = 1$. In fact, ν_ρ-almost all Λ consists of a countable union of finite disjoint clusters, therefore the graph $\Lambda \times \mathbb{Z}$ is a countable union of disjoint cigar-shaped graphs extending infinitely in a single direction. Therefore, we will be only interested in the regime $\rho > \rho_c$.

3. For every $\rho > \rho_c$, $p_c(\rho) \leqslant \frac{1}{2}$. In fact, ν_ρ-almost all Λ contains an infinite path. Conditional on the existence of such a path, the subset of $\Lambda \times \mathbb{Z}$ which projects othogonally to this path resembles a cramped infinite sheet. Is is indeed isomorphic to $\mathbb{Z}_+ \times \mathbb{Z}$ whose critical point equals $1/2$ (see [18] and [4]).

As noted in Figure 1.2, the bound $p_c(\rho) \leqslant 1/2$ for $\rho > \rho_c$ should not be sharp: as one approaches ρ_c from the right the values of $p_c(\rho)$ fall strictly under $1/2$. A rigorous proof of this fact was still missing and is one of the main contributions of this thesis.

Theorem 1.1. *There exists $\delta > 0$ such that for every $\rho > \rho_c$,*

$$p_c(\rho) \leqslant \frac{1}{2} - \delta.$$

Roughly speaking, this result states that above ρ_c, the structure that remains after the columns are drilled are uniformly better to percolation than a simple two-dimensional sheet.

Furthermore, in our setting, Figure 1.2 suggests strongly that, apart from the continuity at $\rho_c = \rho_c^{\text{site}}(\mathbb{Z}^2)$, the results obtained in [7] should also hold. Note, however that the curve fails to be continuous at ρ_c which is the disconnection threshold for the dilute lattice. It is not very hard to show continuity on the other edge of the curve, that is at $\rho = 1$. We present a sketch of the argument in Section 2.3.

The fact that we can only obtain results on the extremes of the interval $[\rho_c, 1]$ is due to the lack of good tools to deal with strict monotonicity when the disorder presents long-range dependencies. In particular, the method of Aizenman and Grimmett [3] seems to be hard to apply in this context, since part of their arguments requires that one performs local modifications.

In our case, however, the application of such methods is hidden in the use of the so-called brochette percolation [9] as we will explain with more details below. In fact this result is based on a quantitative control of the prefactors appearing in the differential inequalities and require some knowledge of near-critical percolation in 2-dimensions. For this reason we are currently unable to prove similar results for higher dimensions. For instance, if we remove 2-d planes independently from the $\mathbb{Z}^4$, is it the case that the critical curve remains always below $p_c(\mathbb{Z}^3)$?

1.3.2 Percolation model on the stretched square lattice

Let $\Lambda = \{x_0, x_1, ...\} \subseteq \mathbb{R}$ be an increasing sequence which will be called *environment*. Given an environment Λ, consider the lattice $\mathfrak{L}_\Lambda = \big(V(\mathfrak{L}_\Lambda), E(\mathfrak{L}_\Lambda)\big)$ defined as

$$\begin{aligned} V(\mathfrak{L}_\Lambda) &= \Lambda \times \mathbb{Z}_+ = \big\{(x, y) \in \mathbb{R}^2; x \in \Lambda, y \in \mathbb{Z}_+)\big\} \quad \text{and} \\ E(\mathfrak{L}_\Lambda) &= \big\{\{(x_i, n), (x_j, m)\} \subseteq V(\mathfrak{L}_\Lambda); |i - j| + |n - m| = 1\big\}. \end{aligned}$$

Roughly speaking, $\mathfrak{L}_\Lambda$ is a version of $\mathbb{Z}_+^2$ horizontally stretched according to Λ.

Now let $\lambda \in (0, \infty)$ and consider the families $\{\mathcal{P}_m\}_{m\in\mathbb{Z}_+}$ and $\{\mathcal{Q}_m\}_{m\in\mathbb{Z}_+}$ of Poisson process with rate λ independent of each other. The percolation configurations $\omega \in \{0, 1\}^{E(\mathfrak{L}_\lambda)}$ will be defined according to the realization of these Poisson processes as follows: for the edge $e \in E(\mathfrak{L}_\Lambda)$

set $\omega(e) = 1$ iff

$$\begin{aligned} e &= \{(x_i, n), (x_{i+1}, n)\} \text{ and } \mathcal{P}_n\big((x_i, x_{i+1}]\big) = 0, \text{ or} \\ e &= \{(x_i, n), (x_i, n+1)\} \text{ and } Q_i\big((n, n+1]\big) = 0. \end{aligned}$$

Geometrically, points are marked on rows and columns of the lattice $\mathfrak{L}_\Lambda$ according to the above Poisson processes, and an edge is declared open iff it has not been marked. See Figure 1.3.

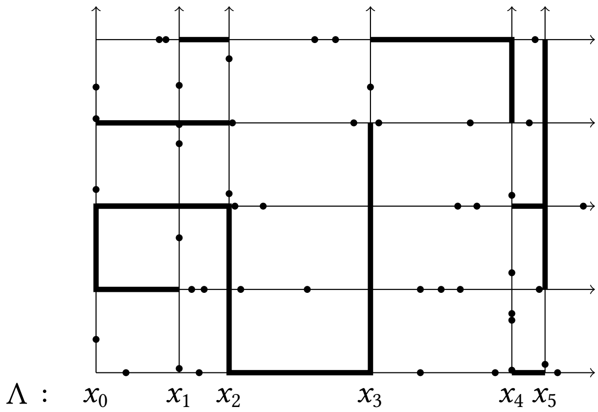

Figure 1.3: Illustration of the lattice $\mathfrak{L}_\Lambda$. The dots represent the arrivals times of the Poisson processes in the rows and columns of $\mathfrak{L}_\Lambda$. The open edges are highlighted by the thickness.

Denote by $\mathbb{P}_\lambda^\Lambda(\cdot)$ the probability law that governs this percolation model on $\mathfrak{L}_\Lambda$. Equivalently, $\mathbb{P}_\lambda^\Lambda$ is the measure on $\{0,1\}^{E(\mathfrak{L}_\Lambda)}$ under which the variables $\omega(e)$ (with $e \in E(\mathcal{L}_\Lambda)$) are independent Bernoulli random variables with mean

$$p_e = \exp(-\lambda|e|),$$

where $|e|$ denotes the length of the edge $e = \{v_1, v_2\} \in E(\mathfrak{L}_\Lambda)$, defined by $|e| = \|v_1 - v_2\|$, being $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^2$.

The goal is to study this model when the environment, Λ, is random and distributed according to a renewal process as we describe now. Let ξ be a positive random variable, and let $\{\xi_i\}_{i\in\mathbb{Z}_+^*}$ be copies of ξ independent of each other. Set

$$\Lambda = \left\{ x_k = \sum_{1\leqslant i\leqslant k} \xi_i;\ k \in \mathbb{Z}_+ \right\} = \{x_i \in \mathbb{R}; x_0 = 0 \text{ and } x_k = x_{k-1} + \xi_k \text{ for } k \in \mathbb{Z}_+^*\}.$$

And denote $\nu_\xi(\cdot)$ the probability measure that governs the renewal process Λ with interarrival time ξ. An overview of renewal processes will be provided in Section 3.2.

This model can be seen as an inhomogeneous percolation model on $\mathbb{Z}_+^2$, where each edge $e \in E(\mathbb{Z}_+^2)$ is open with probability

$$p_e = \begin{cases} \exp(-\lambda), & \text{if } e = \{(i,j),(i,j+1)\} \\ \exp(-\lambda\xi_{i+1}), & \text{if } e = \{(i,j),(i+1,j)\} \end{cases}. \tag{1.4}$$

The random variables ξ_i's indicate how far apart the columns of the stretched lattice lie from one another. Note that when $\xi = 1$ a.s. the model is homogeneous.

Our main results are

Theorem 1.2. *Let ξ be a positive random variable. If $\mathbb{E}(\xi^\eta) < \infty$ for some $\eta > 1$, then for any $\lambda > 0$ sufficiently small, we have*

$$\mathbb{P}_\lambda^\Lambda(o \leftrightarrow \infty) > 0, \textit{ for } \nu_\xi\textit{-almost every environment } \Lambda.$$

Theorem 1.3. *Let ξ be a positive random variable. If $\mathbb{E}(\xi^\eta) = \infty$ for some $\eta < 1$, then for any $\lambda \in (0,\infty)$,*

$$\mathbb{P}_\lambda^\Lambda(o \leftrightarrow \infty) = 0, \textit{ for } \nu_\xi\textit{-almost every environment } \Lambda.$$

Intuitively speaking, the absence of phase transition established in Theorem 1.3 follows from the fact that the columns are typically very far apart from one another. The proof relies on a simple application of Borel-Cantelli Lemma. The proof of Theorem 1.2 is more delicate and relies on the construction of a renormalization scheme. Theorem 1.2 can be viewed as a generalization of a discrete version of the results in [5]. Indeed the techniques there seem to apply only to the case when the ξ's are geometric random variables. We are currently unable to tackle the case when the lattice is stretched in both horizontal and vertical directions. This interesting case would generalize the results in [16].

Chapter 2

Strict inequality for bond percolation on a dilute lattice with columnar disorder

The chapter is devoted to the proof of Theorem 1.1. This proof has two main inputs: a construction of the so-called decorated path and an enhancing-type argument which are presented in Sections 2.1 and 2.2 respectively. Below we summarize these strategies.

Fix $\rho > \rho_c$. The first input consists of constructing an infinite sequence of crossing events taking place inside an infinite set of overlapping boxes in $\mathbb{Z}^2$. The side length of these boxes are chosen, depending on ρ, in such a way as to guarantee that they are all crossed simultaneously with positive probability. On the event that these crossings occur, we carefully pick one of them in each box in such a way that the box is split into two regions, being one of them unexplored.

Moreover, one can concatenate the crossings in order to obtain an infinite occupied path in $\mathbb{Z}^2$. The subgraph of $\mathbb{Z}^3$ that projects orthogonally to this infinite path is isomorphic to $\mathbb{Z}_+ \times \mathbb{Z}$ so the critical point for Bernoulli bond percolation restricted to it equals $1/2$. The idea now is to attach to this set a structure that will enhance the percolation process on it. Perhaps the most interesting point in our argument is how to have this accomplished uniformly on $\rho > \rho_c$. For that we use the so called brochette-percolation [9] as we explain below.

The key idea is that, as one follows the infinite path, the environment on the left-hand side is always fresh. This allows us to attach to the path an infinite sequence of evenly spaced threads of length one having one endpoint whose state is unexplored. Thus the states of the tip of these threads dominate an i.i.d. sequence of Bernoulli random variables with positive mean, uniformly bounded away from zero.

As their states are revealed, the path will become decorated with a sequence of randomly placed threads whose endpoints are occupied. The edges that project to these threads will serve to enhance the percolation on the cramped sheet. However, since their positions are random, the Aizenman and Grimmett enhancement-type arguments do not apply directly. In order to show that the decorated cramped sheet is strictly better for percolation than a simple cramped sheet we use a mild modified of the so-called brochette percolation studied in [9] (see Section 2.2.1).

2.1 Block argument

In this section we present the block argument leading to the construction of an infinite path decorated with open threads as mentioned above. But first some definitions. We write $[n] := \{1, \dots, n\} \subset \mathbb{Z}$ and $[n]_0 := \{0, \dots, n\}$ and we say that the subset $A \subset \mathbb{Z}$ is said 2-*spaced* if for every pair $i \neq j$ in A, one has $|i - j| \geqslant 2$. We denote Γ the set of all the paths in $\mathbb{Z}^2$. A subset of a path $\gamma \in \Gamma$ that is still a path is called a *subpath* of γ. A path γ is said *minimal* when its only subpath that has the same extremes as γ is itself.

We start the construction of a path decorated by defining the region of $\mathbb{Z}^2$ where we will attempt to find such a path.

Let $e_1 = (1, 0) \in \mathbb{R}^2$ and $\theta_{\pi/2}$ be the rotation by $\pi/2$ about the origin in $\mathbb{R}^2$. For some integer $L = L(\rho) > 1$, that will be chosen latter, consider the sequence of rectangles R_i defined recursively as follows (see Figure 2.1):

(i) $w_1 := (0, 0)$ and $R_1 := ([0,\ 4L] \times [0,\ L]) \cap \mathbb{Z}^2$,

(ii) $w_{n+1} := w_n + 2^{n+1}L \cdot \theta_{\pi/2}^{n-1}(e_1)$ and $R_{n+1} := w_{n+1} + 2^n \cdot \theta_{\pi/2}^n(R_1)$.

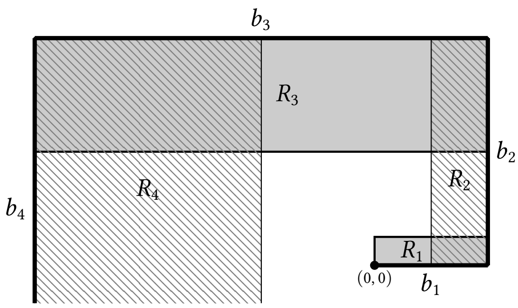

Figure 2.1: The sequence of overlapping rectangles R_i. Their bottom sides b_i are represented by the thick segments whose concatenation form the broken line spiral.

We also define $b_n = [w_n, w_{n+1}]$ where $[u, v]$ denotes the line segment between $u, v \in \mathbb{R}^2$, that is, $[u, v] = \{tv + (1 - t)u \in \mathbb{R}^2;\ t \in [0, 1]\}$. The concatenation of the segments $b_1, b_2, \dots$ form the infinite broken-line spiral as shown in Figure 2.1. The points w_i are the ones at which this spiral breaks. Notice that the rectangle R_i overlap with R_{i-1} and R_{i+1}. The union of all the rectangles R_i forms a spiral region inside which we will attempt to find an infinite minimal occupied path decorated with occupied threads.

It will be convenient to assign different reference frames to the successive rectangles R_i so that b_i is called the bottom side of R_i. That is, the bottom side of each rectangle is defined to be its outermost side as seem from the center of the spiral. Once the bottom side of R_i is fixed, we naturally define its right, top and left sides following the order of appearance as we travel along its the boundary counterclockwise. Note that for $i = 1, 5, 9, \ldots$ the orientation coincides with the natural one so that the right side r_i coincides with the natural. For $i = n \pmod 4$ the orientation of the bottom side is rotated by $(n-1)\pi/2$ with respect to the original one. With this convention, denote r_i the right side of R_i.

We say that a path γ crosses R_i in the hard (resp. easy) direction if γ is contained in R_i and one of its extremes belong to the right (resp. top) side of R_i and the other belongs to the left (resp. bottom) side of R_i.

Let us now assume $\rho > \rho_c$ and fix the value of $L = L(\rho)$ used in the definition of the rectangles R_i above. A straightforward adaptation of Lemma 4.12 in [2] or Lemma 2.3 in [8] guarantees the existence of an integer $L = L(\rho)$ such that for every $i \in \mathbb{Z}_+$

$$\nu_\rho\big(R_i \text{ is crossed in the hard direction}\big) \geqslant 1 - \frac{1}{2^{i+1}},$$

and therefore

$$\nu_\rho\Big(\bigcap_{i \in \mathbb{Z}_+} \{R_i \text{ is crossed in the hard direction}\}\Big) \geqslant \frac{1}{2}. \tag{2.1}$$

Suppose that a path γ in $\mathbb{Z}^2$ contains a subpath that crosses R_i in the easy direction and that $\partial\gamma \cap r_i = \emptyset$. We set $l_\gamma := \partial\gamma \cap R_i$ which does not need to be connected in the sense of $\mathbb{Z}^2$ but nevertheless, it spans $\mathbb{R}_i$ in the easy direction (see Figure 2.2 for an illustration of l_γ). Let $\Gamma(R_i, \gamma)$ be the set of all minimal paths γ' contained in R_i, such that $|\gamma' \cap l_\gamma| = 1$ and that has one extreme in l_γ and the other in r_i. Define $\{l_\gamma \overset{R_i}{\longleftrightarrow} r_i\}$ the event that l_γ and r_i are *connected* in R_i, i.e. the event that $\Gamma(R_i, \gamma)$ contains at least one occupied path. Given $\Lambda \in \{l_\gamma \overset{R_i}{\longleftrightarrow} r_i\}$, denote by $\mathcal{L}_{R_i,\gamma}$ the lowest occupied minimal path in $\Gamma(R_i, \gamma)$. For $\gamma' \in \Gamma(R_i, \gamma)$, denote by $\mathcal{D}(R_i, \gamma, \gamma')$ the region of R_i located below γ' and to the right of l_γ including γ' (see Figure 2.2). Note that, for

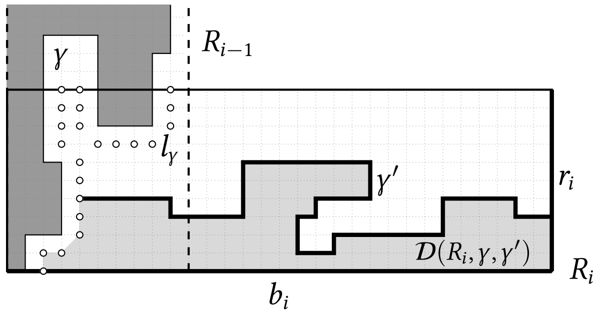

Figure 2.2: The bottom and right sides of R_i are represented as thick segments labeled b_i and r_i, respectively. The path γ has a subpath that crosses R_i in the easy way. The set l_γ, whose vertices are represented by $\circ$, belong to does not intersect the right r_i. The path γ' crosses R_i from l_γ to r_i. The light gray colored region corresponds to $\mathcal{D}(R_i,\gamma,\gamma')$.

all $\gamma' \in \Gamma(R_i,\gamma)$, the event $\{\mathcal{L}_{R_i,\gamma} = \gamma'\}$ is measurable with respect to the state of the vertices in $\mathcal{D}(R_i,\gamma,\gamma')$.

We are ready to construct the relevant crossing events to be used in our block argument. Let $\gamma_0 = \{-1\} \times [0,L]$ and assume that $\{l_{\gamma_0} \overset{R_1}{\longleftrightarrow} r_1\}$ occurs (note that $l_{\gamma_0} = \{0\} \times [0,L]$, the left side of R_1). We can extract $\mathcal{L}_{R_1,\gamma_0}$ the lowest minimal occupied path connecting l_{γ_0} to r_1. Now, once we condition on $\mathcal{L}_{R_1,\gamma_0} = \gamma_1$ and since such a path γ_1 necessarily has a subpath that crosses R_2 in the easy direction without intersecting r_2, we can consider the event $\{l_{\gamma_1} \overset{R_2}{\longleftrightarrow} r_2\}$. On this event we can extract the lowest minimal occupied path $\mathcal{L}_{R_2,\gamma_1}$. We continue progressively: Conditional on $\{\mathcal{L}_{R_i,\gamma_{i-1}} = \gamma_i\}$ and $\{l_{\gamma_i} \overset{R_{i+1}}{\longleftrightarrow} r_{i+1}\}$ we denote $\mathcal{L}_{R_{i+1},\gamma_i}$ the lowest minimal occupied path inside R_{i+1} crossing from l_{γ_i} to r_{i+1}.

We say that a sequence of paths $\{\gamma_i\}_{i\in[k]}$ in Γ is allowed if $\gamma_i \in \Gamma(R_i,\gamma_{i-1})$ for all $i \in [k]$. For each k we set

$$\Delta_k := \bigcup_{\{\gamma_i\}_{i\in[k]} \text{ allowed}} \Big\{\mathcal{L}_{R_i,\gamma_{i-1}} = \gamma_i \text{ for every } i \in [k]\Big\} \tag{2.2}$$

where the union is disjoint. Since the k first elements in an allowed sequence $\{\gamma_i\}_{i\in[k+1]}$ still form an allowed sequence, we have $\Delta_{k+1} \subset \Delta_k$.

Moreover,

$$\bigcap_{1\leqslant i\leqslant k} \Big\{R_i \text{ is crossed in the hard direction}\Big\} \subset \Delta_k,$$

which, in view of (2.1) implies

$$\nu_\rho\Big(\bigcap_{k\in\mathbb{Z}_+} \Delta_k\Big) \geqslant \frac{1}{2}. \tag{2.3}$$

The procedure is summarized in the following lemma. Recall that Γ denotes the set of all paths in $\mathbb{Z}^2$. Let $\mathcal{F}_k = \sigma(\Lambda(v) : v \in \cup_{j=1}^k R_j)$ be the σ-field generated by the random element Λ

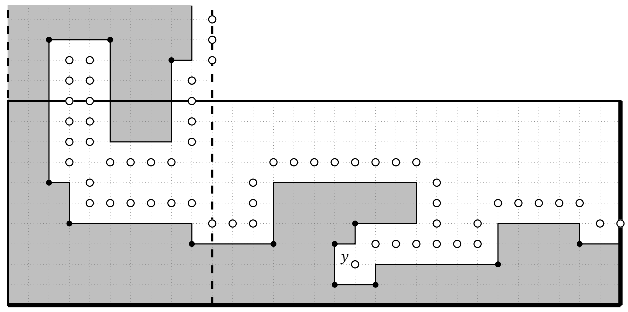

Figure 2.3: We illustrate part of the path $\gamma = \Phi_k$. The grey shaded area represents D_k. The black-filled dots are the sites v_i for $i \in \mathcal{B}_\gamma$. The sites in $\mathcal{H}_\gamma$ are represented as $\circ$ and their states are independent of the event $\{\Phi_k = \gamma\}$. They can neighbor up to 4 sites in γ, as for example, the site y illustrated in the picture.

restricted to the first k rectangles so that $(\mathcal{F}_k)_{k\in\mathbb{Z}_+}$ defines a filtration. Notice that the sequence $\{\Delta_k\}_{k\in\mathbb{Z}_+}$ is adapted, that is $\Delta_k \in \mathcal{F}_k$ for every k.

Lemma 2.1. *For every $\rho > \rho_c$ there exists an integer $L = L(\rho) > 0$ such that, the corresponding adapted sequence of crossing events $\{\Delta_k\}_{k\in\mathbb{Z}_+}$ defined in* (2.2) *satisfies*

(i) $\nu_\rho(\cap_k \Delta_k) \geqslant \frac{1}{2}$.

(ii) There for every k, there exists a function $\Phi_k : \Delta_k \to \Gamma$, such that, for every $\Lambda \in \Delta_k$ $\Phi_k(\Lambda)$ is an occupied minimal path having one extreme in $\partial\gamma_0$ and the other in r_k (the right side of the k-th rectangle in the spiral, R_k).

(iii) For every path $\gamma = (v_0, \dots, v_n) \in \Phi_k(\Delta_k)$, there is a 2-spaced set $\mathcal{B}_\gamma \subset [n]_0$ and a set $\mathcal{H}_\gamma \subset \partial\gamma$ such that for every $i \in [n]_0 \backslash \mathcal{B}_\gamma$, $\partial v_i \cap \mathcal{H}_\gamma \neq \varnothing$. Moreover, $\sigma(\Lambda(v) : v \in \mathcal{H}_\gamma)$, is independent of $\{\Phi_k = \gamma\}$.

Before we prove Lemma 2.1 let us clarify its statement. On the event Δ_k, the function Φ_k selects a minimal occupied path γ with one extreme in $\partial\gamma_0$ and the other in r_k by concatenating the crossings in successive rectangles. For each vertex $y \in \mathcal{H}_\gamma$, there exists $v_i \in \gamma$ such that $y \in \partial v_i \backslash \cup_{i=1}^{k} \mathcal{D}(R_i, \gamma_{i-1}, \gamma_i)$. We think of y as being an endpoint of a thread of length one that will be attached to v_i. Vertices of γ which are not attached to any thread are indexed by the set $\mathcal{B}_\gamma$. The fact that $\mathcal{B}_\gamma$ is 2-spaced implies that, for each pair of consecutive vertices in γ, at least one of these vertices has a thread attached. Finally, the ν_ρ-state of the vertices $y \in \mathcal{H}_\gamma$, is independent of the event that γ was the path selected.

Proof of Lemma 2.1. Item (i) is just (2.3).

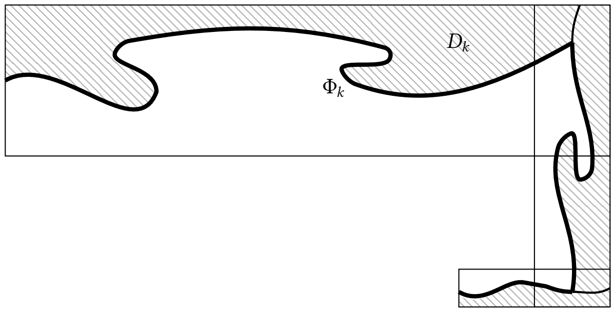

Figure 2.4: We illustrate the three first rectangles in the spiral region with the lowest minimal crossings within them. We highlight the path Φ_3 formed by the concatenation of parts of these crossings. The shaded region is D_3.

Fix $\Lambda \in \Delta_k$, and let $\{\gamma_i\}_{i\in[k]}$ be the unique allowed sequence such that we have $\mathcal{L}_{R_i,\gamma_{i-1}} = \gamma_i$ for every $i \in [k]$ and let $u_i := \gamma_i \cap \partial\gamma_{i-1}$ be the extreme γ_i that also lie in $l_{\gamma_{i-1}}$.

Now we define $\Phi_k(\Lambda)$ as being the path that starts at u_1, goes along γ_1 until hitting a neighbor of u_2, then follows along γ_2 until hitting a neighbor of the u_3 and so on until the last step when it starts at u_{k-1} and follows along γ_k until hitting r_k. It is clear that $\Phi_k(\Lambda)$ is an occupied minimal path contained in $\cup_{i=1}^k R_k$ and that it has one extreme on $\partial\gamma_0$ and the other on r_k.

Also, $\Phi_k(\Lambda)$ divides $\cup_{i=1}^k R_i$ into two regions being

$$D_k(\Lambda) \subset \cup_{i=1}^k \mathcal{D}(R_i, \gamma_i, \gamma_{i-1})$$

the one located between $\Phi_k(\Lambda)$ and the outermost part of the spiral. Moreover,

$$\{\Phi_k = \gamma\} \text{ is measurable with respect to } \sigma(\Lambda(v) : v \in D_k(\gamma)). \tag{2.4}$$

Now for some $\Lambda \in \Delta_k$ let $\Phi_k(\Lambda) = \gamma = (v_0, \dots, v_n)$ and define

$$B_\gamma := \{i \in [n]_0 : \partial v_i \subset D_k(\Lambda)\}. \tag{2.5}$$

We claim that B_γ is a 2-spaced set. Indeed, if ∂v_i and ∂v_{i+1} were both contained in $D_k(\Lambda)$, then we would have $v_{i-1} \sim v_{i+2}$ contradicting the minimality of γ.

Choose any ordering of $\mathbb{Z}^2$. For every $i \in [n] \backslash B_\gamma$ let y_i be the earliest vertex in $\partial v_i \backslash D_k(\Lambda)$ and define

$$\mathcal{H}_\gamma := \bigcup_{i\in[n]_0\backslash\mathcal{B}_\gamma} \{y_i\} \subset \partial\gamma \cap \left[\, \cup_{i=1}^k R_i \backslash D_k(\Lambda)\right]. \tag{2.6}$$

Notice that $\sigma(\Lambda(v) : v \notin D_k(\Lambda))$ is independent of $\{\Phi_k = \gamma\}$ since the latter is measurable with respect to the states of vertices in $D_k(\Lambda)$. □

We stress once more that, for the rest of the text we will fix $L = L(\rho)$ as given by Lemma 2.1. We will also abuse notation and write $\Delta_\gamma = \Phi_k^{-1}(\gamma)$, omitting the dependency on $k \in \mathbb{Z}_+$. It is the case that $\{\Delta_\gamma\}_{\gamma \in \Phi_k(\Delta_k)}$ form a partition of Δ_k.

2.2 Coupling with brochette percolation

In the previous section we showed that, with positive probability, under ν_ρ, we can find an infinite occupied path γ that is decorated with a set of threads $\mathcal{H}_\gamma$ whose states are unexplored. Roughly speaking, in order to prove Theorem 1.1 it remains to show that, uniformly on the realization of such pair $(\gamma, \mathcal{H}_\gamma)$, Bernoulli bond percolation on $(\gamma \cup \mathcal{H}_\gamma) \times \mathbb{Z} \subset \mathbb{Z}^3$ has a lower critical point than on $\mathbb{Z}_+ \times \mathbb{Z}$. While this results sound intuitively clear, it is not straightforward due to the random location of the occupied threads that should be used to enhance the percolation process. As already mentioned above, the comparison will be made by mapping to a bond percolation model on $\mathbb{Z}^2$ that resembles the so-called brochette percolation model [9]. In Section 2.2.1 we define the brochette percolation model precisely. Then, in Section 2.2.2 we will construct the desired coupling and prove Theorem 1.1.

2.2.1 Brochette percolation

We start this section presenting the so-called brochette percolation. Given $\Xi \in \{0,1\}^{\mathbb{Z}_+}$, for $i \in \mathbb{Z}_+$, the column $\{i\} \times \mathbb{Z}$ is said *strong* if $\Xi(i) = 1$ and *weak* otherwise. Fix two parameters $0 < p \leqslant q < 1$. Edges of the lattice $\mathbb{Z}_+ \times \mathbb{Z}$ will be declared open or closed independently as follows. Vertical edges $e \in E(\mathbb{Z}_+ \times \mathbb{Z})$ whose endpoints lie in strong columns, are declared open with probability q and closed with probability $1 - q$. Every other edge in $E(\mathbb{Z}_+ \times \mathbb{Z})$ is declared open with probability p and closed with probability $1 - p$. This gives rise to an inhomogeneous bond percolation on $\mathbb{Z}_+ \times \mathbb{Z}$ whose law we denote $P^{\Xi}_{p,q}$.

For a measure μ on $\{0,1\}^{\mathbb{Z}_+}$ let us define

$$P^{\mu}_{p,q}(\cdot) := \int P^{\Xi}_{p,q}(\cdot) d\mu(\Xi). \tag{2.7}$$

In [9], $\{\Xi(i)\}_{i\in\mathbb{Z}_+}$ are assumed to be independent Bernoulli random variables with mean $u > 0$ that is, $\mu = \mu_u$, where

$$\mu_u := \otimes_{i\in\mathbb{Z}_+}[(1-u)\delta_0 + u\delta_1]. \tag{2.8}$$

The resulting measure $P^{\mu_u}_{p,q}$ is called the brochette percolation on $\mathbb{Z}_+ \times \mathbb{Z}$. The parameter u should be understood as a density of enhanced lines. The main result in [9] is:

Theorem 2.1 ([9], Theorem 1, p. 481)**.** *For every $u \in (0,1)$ and $\varepsilon \in (0,1/2)$ there exists $\delta = \delta(u,\varepsilon) > 0$ such that*

$$P^{\Xi}_{1/2-\delta,1/2+\varepsilon}\big(o \leftrightarrow \infty\big) > 0$$

for μ_u-almost all Ξ.

Strictly speaking, in [9] the model was defined on the $\mathbb{Z}^2$-lattice. However the exact same proof presented there works for the half-space $\mathbb{Z}_+ \times \mathbb{Z}$.

An immediate consequence of Theorem 2.1 is: For every $u \in (0,1)$ and every $\varepsilon \in (0,1/2)$, there exists $\delta = \delta(u,\varepsilon) > 0$ and $\alpha = \alpha(u,\varepsilon) > 0$ such that for every $p \geqslant 1/2 - \delta$, $P^{\mu_u}_{p,1/2+\varepsilon}(o \leftrightarrow \infty) > \alpha$.

For our purposes it will be useful to force $\Xi(i)$ to vanish at some indices i. For this reason, let us consider for every $B \subset \mathbb{Z}_+$, the measure

$$\mu_{u,B} := \otimes_{i\in\mathbb{Z}_+\setminus B}\big[(1-u)\delta_0 + u\delta_1\big] \times \otimes_{i\in B}\delta_0. \tag{2.9}$$

The following theorem is an adaptation of Theorem 2.1, the only difference being that instead of deciding the position of the strong columns independently we only allow for strong columns outside a 2-spaced set. This amounts to replace μ_u in (2.8) by $\mu_{u,B}$ in (2.9).

Theorem 2.2. *For every $\varepsilon \in (0,1/2)$ and $u \in (0,1)$, there exist $\delta = \delta(\varepsilon,u) > 0$ and $\alpha = \alpha(\varepsilon,u) > 0$, such that for every 2-spaced set $B \subset \mathbb{Z}_+$ and every $p \geqslant 1/2 - \delta$*

$$P^{\mu_{u,B}}_{p,\frac{1}{2}+\varepsilon}\big(o \leftrightarrow \infty\big) > \alpha.$$

We refrain from writing down a proof here since it would follow exactly the same lines as Theorem 2.1 with very straightforward modifications.

Corollary 2.3. *For every $\varepsilon \in (0, 1/2)$, $u \in (0,1)$ and $n \in \mathbb{Z}_+$, there exist $\delta = \delta(\varepsilon, u) > 0$ and $\alpha = \alpha(\varepsilon, u) > 0$, such that for every 2-spaced set $B \subset [n]_0$ and every $p \geqslant 1/2 - \delta$*

$$P^{\mu_{u,B}}_{p,\frac{1}{2}+\varepsilon}\big(o \leftrightarrow \{n\} \times \mathbb{Z} \text{ in } [n]_0 \times \mathbb{Z}\big) > \alpha. \tag{2.10}$$

2.2.2 Proof of Theorem 1.1

In this section we compare $\mathbb{P}^\rho_p$ restricted to a certain random subset of $\mathbb{Z}^3$ and the brochette percolation $P^{\mu_{u,B}}_{p,q}$ on $\mathbb{Z}_+ \times \mathbb{Z}$ with an appropriate choice of parameters u and q and of a random 2-spaced subset $B \subset \mathbb{Z}_+$. Then we use this comparison to prove Theorem 1.1.

Fix $\rho > \rho_c$ and $p \in (0,1)$. Let $L = L(\rho)$ be given as in Lemma 2.1. Define $u = u(\rho)$ and $q = q(p)$ as

$$u := 1 - (1-\rho)^{1/4}, \qquad q := p + \Big[(1-p)\big(1-(1-p)^{1/2}\big)\big)^2\big(1-(1-p)^{1/4}\big)\big)\Big]. \tag{2.11}$$

For $k \in \mathbb{Z}_+$ and a path $\gamma \in \Phi_k(\Delta_k)$, recall the definitions of $\mathcal{B}_\gamma$ as in Lemma 2.1 and $\Delta_\gamma = \Phi_k^{-1}(\gamma)$.

Lemma 2.2. *Let $\rho > \rho_c$ and $p \in (0,1)$. Fix $\gamma = (v_0, \dots, v_n) \in \Phi_k(\Delta_k)$ and let $u = u(\rho)$ and $q = q(p)$ be given as in* (2.11). *Then,*

$$\mathbb{E}_{\nu_\rho}\Big(\mathbb{P}^\Lambda_p[(v_0,0) \leftrightarrow \{v_n\} \times \mathbb{Z} \text{ in } (\gamma \cup \mathcal{H}_\gamma) \times \mathbb{Z} \mid \mathcal{F}_{\Lambda|_{D_k(\gamma)}}]\Big) \geqslant P^{\mu_{u,\mathcal{B}_\gamma}}_{p,q}\big(o \leftrightarrow \{n\} \times \mathbb{Z} \text{ in } [n]_0 \times \mathbb{Z}\big) \tag{2.12}$$

for ν_ρ-almost all Λ, where $\mathcal{F}_{\Lambda|_{D_k(\gamma)}} = \sigma\{\Lambda(v) : v \in D_k(\gamma)\}$.

The proof of the above lemma relies on a simple coupling of a percolation process $\mathbb{P}^\rho_p$ restricted to $(\gamma \cup \mathcal{H}_\gamma) \times \mathbb{Z}$ and the brochette percolation $P^{\mu_{u,\mathcal{B}_\gamma}}_{p,q}$ restricted to $[n]_0 \times \mathbb{Z}$ as we sketch now.

Fixed $\gamma = (v_0, \dots, v_n)$, we wish to associate each site v_i with $i \in [n]_0 \backslash \mathcal{B}_\gamma$ to an adjacent thread $y \in \mathcal{H}_\gamma$. Since each thread y may neighbor up to 4 sites in γ (see Figure 2.3), we split them into 4 quarter threads and color each one of them blue independently with probability u. Therefore, we can now assign to each v_i with $i \in [n]_0 \backslash \mathcal{B}_\gamma$ a neighboring quarter thread in an injective way. We now couple the environments Λ and Ξ in the two processes. Since we want to condition on Δ_γ, we fix $\Lambda(v) = 1$ for every $v \in \gamma$. Then declare each $y \in \mathcal{H}_\gamma$ occupied (that is, $\Lambda(y) = 1$) if at least one of the 4 quarter threads resulting from it is colored blue and unoccupied (that is, $\Lambda(y) = 0$) otherwise. In order to construct $\Xi \in \{0,1\}^{\mathbb{Z}_+}$, we force $\Xi(j) = 0$ if j belongs to $\mathcal{B}_\gamma$

and, for each j outside $\mathcal{B}_\gamma$ we let $\Xi(j) = 1$ if the quarter thread assigned to the site v_j is colored blue. One can check that $\{\Lambda(v); v \in \gamma \cup \mathcal{H}_\gamma\}$ is distributed as $\nu_\rho(\,\cdot \mid \Delta_\gamma)$ and that $\{\Xi(j)\}_{j\in[n]_0}$ is distributed as $\mu_{u,\mathcal{B}_\gamma}$.

Conditional on the above realizations of Λ and Ξ we now construct the desired bond percolation processes on $(\gamma \cup \mathcal{H}_\gamma) \times \mathbb{Z}$ and $[n]_0 \times \mathbb{Z}$. We start by defining the states of the edges in $E(\gamma \cup \mathcal{H}_\gamma) \times \mathbb{Z}$ independently as follows. All the edges in $E(\gamma \times \mathbb{Z})$ are declared open (resp. closed) with probability p (resp. $1 - p$). Now let us construct the state of edges that have at least one endpoint in $\mathcal{H}_\gamma \times \mathbb{Z}$. If this endpoint projects orthogonally into an unoccupied thread $y \in \mathcal{H}_\gamma$, then the edge is declared closed whereas, if it projects to an occupied thread, then we decide its state according to the following procedure:

1. If $f = \{(y, z), (y, z + 1)\}$ is a vertical edge, projecting into the vertex $y \in \mathcal{H}_\gamma$, then we divide f into 4 parallel edges (because y itself has been previously divided into 4 quarter threads) and color each one of these new edges green independently with probability $r = 1 - (1 - p)^{1/4}$. Now we declare f open if at least one of these 4 new edges is colored green.

2. If $e = \{(v_i, z), (y, z)\}$ is a horizontal edge so that $\Xi(i) = 1$ and y is the thread that had one of its quarters assigned to v_i, then divide it into 2 parallel horizontal edges. Color each one of these new edges red independently with probability $s = 1 - (1 - p)^{1/2}$ and declare e open if at least one of these new edges is colored red.

Before we construct the brochette percolation process on $\mathbb{Z}_+ \times \mathbb{Z}$, let us recall that vertical edges that project into occupied threads in $\mathcal{H}_\gamma$ have been divided into four whereas horizontal edges having one endpoint that project into occupied threads in $\mathcal{H}_\gamma$ have been divided into two. The reason for doing so is that we can now regard the handle-shaped detours around vertical edges $\{(v_i, z), (v_i, z + 1)\}$ for which $\Xi(i) = 1$ as illustrated in Figure 2.5.

Now, for a vertical edge, $f' = \{(i, z), (i, z + 1)\}$ with $\Xi(i) = 0$ or a horizontal edge, $e' = \{(i, z), (i + 1, z)\}$, declare it open if the edge $f = \{(v_i, z), (v_i, z + 1)\}$ or respectively $e = \{(v_i, z), (v_{i+1}, z)\}$, is open. Now for a vertical edge $f' = \{(i, z), (i, z + 1)\}$ with $\Xi(i) = 1$ declare it open if either the corresponding edge $f = \{(v_i, z), (v_i, z + 1)\}$ is open or if the handle-shaped detour around it has the bottom and top edges colored red and one of the 4 vertical edges green. This occurs with probability q. One can check that this defines a process in $\mathbb{Z}_+ \times \mathbb{Z}$ distributed as $P_{p,q}^{\mu_u \mathcal{B}_\gamma}$.

Moreover, we have coupled the processes in such a way that if $\big\{o \leftrightarrow \{n\} \times \mathbb{Z} \text{ in } [n]_0 \times \mathbb{Z}\big\}$ occurs, then also $\big\{(v_0, 0) \leftrightarrow \{v_n\} \times \mathbb{Z} \text{ in } (\gamma \cup \mathcal{H}_\gamma) \times \mathbb{Z}\big\}$ does. Thus (2.12) holds.

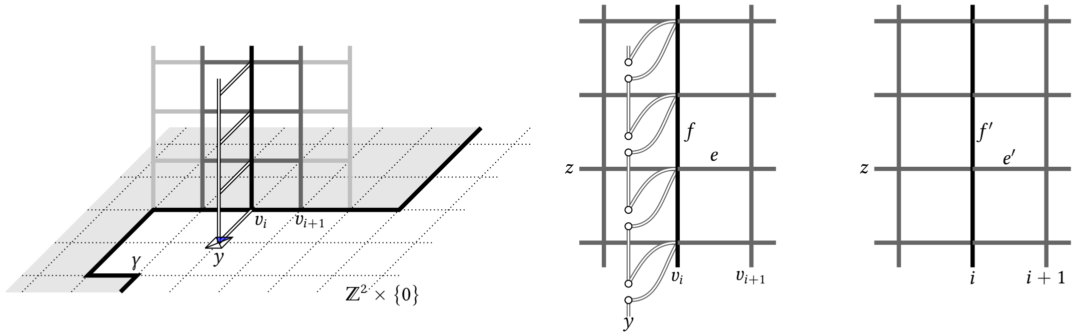

Figure 2.5: In the left we represent the path γ along with one of its sites v_i and the thread y whose quarter thread is assigned to v_i and is colored blue. In the middle we illustrate the division of the horizontal edges in $E\big((\gamma \cup \mathcal{H}_\gamma) \times \mathbb{Z}\big)$ of the type $\{(v_i, z), (y, z)\}$ into two parallel edges and the resulting handle-shaped detours. In the right we represent edges in $E([n]_0 \times \mathbb{Z})$. Horizontal edges e' are open if the corresponding e are open. Vertical edges f' along the i-th column are open if either the corresponding edge f is open or the handle-shaped detour around it has been colored green and red.

We are now ready to put together all the ingredients needed in order to prove our main result.

Proof of Theorem 1.1. Let $\varepsilon = (1 - 1/\sqrt{2})^2(1 - 1/\sqrt[4]{2})/4$ so that $q = 1/2 + 2\varepsilon$, when $p = 1/2$. Define also $u_c = 1-(1-\rho_c)^{1/4}$. For such $\varepsilon > 0$ and $u_c > 0$, let $\delta = \delta(u_c, \varepsilon) > 0$ and $\alpha = \alpha(u_c, \varepsilon)$ be given as in Theorem 2.2. By (2.11), q varies continuously as a function of p, therefore we can chose $0 < \delta' \leqslant \delta$ so that $q \geqslant 1/2 + \varepsilon$ when $p = 1/2 - \delta'$.

For each $k \in \mathbb{Z}_+$ and $\gamma = (v_0, \cdots, v_n) \in \Phi_k(\Delta_k)$, it is clear that $n \geqslant 2^k L$. Denote by C the largest cluster that touches the segment $\{0\} \times [0, L] \times \{0\}$ and note that $v_0 \in \{0\} \times [0, L] \times \{0\}$. Then we have

$$
\begin{aligned}
\mathbb{P}_p^\rho(|C| > 2^k L) &\geqslant \int_{\Delta_k} \mathbb{P}_p^\Lambda(|C| > 2^k L) d\nu_\rho(\Lambda) \\
&= \sum_{\gamma \in \Phi_k(\Delta_k)} \mathbb{E}_{\nu_\rho}\Big(\mathbb{E}_{\nu_\rho}[1_{\Delta_\gamma} \mathbb{P}_p^\Lambda(|C| > 2^k L) \mid \mathcal{F}_{\Lambda|_{D_k(\gamma)}}]\Big) \\
&\overset{(2.4)}{=} \sum_{\gamma \in \Phi_k(\Delta_k)} \mathbb{E}_{\nu_\rho}\Big(1_{\Delta_\gamma} \mathbb{E}_{\nu_\rho}[\mathbb{P}_p^\Lambda((v_0, 0) \leftrightarrow \{v_n\} \times \mathbb{Z} \text{ in } (\gamma \cup \mathcal{H}_\gamma) \times \mathbb{Z}) \mid \mathcal{F}_{\Lambda|_{D_k(\gamma)}}]\Big) \\
&\overset{(2.12)}{\geqslant} \sum_{\gamma \in \Phi_k(\Delta_k)} \mathbb{E}_{\nu_\rho}\Big(1_{\Delta_\gamma} P_{p,q}^{\mu_u, \mathcal{B}_\gamma}\big(o \leftrightarrow \{n\} \times \mathbb{Z} \text{ in } [n]_0 \times \mathbb{Z}\big)\Big) \\
&\overset{(2.10)}{\geqslant} \sum_{\gamma \in \Phi_k(\Delta_k)} \alpha \mathbb{E}_{\nu_\rho}(1_{\Delta_\gamma}) \geqslant \sum_{\gamma \in \Phi_k(\Delta_k)} \alpha \nu_\rho(\Delta_\gamma) \geqslant \alpha \nu_\rho(\Delta_k) \geqslant \alpha/2
\end{aligned}
$$

Since the lower bound above holds for every $k \in \mathbb{Z}_+$, this proves that $p_c(\rho) \leqslant \frac{1}{2} - \delta'$ uniformly for $\rho > \rho_c$ concluding the proof of Theorem 1.1. □

2.3 Concluding Remarks

In this work we have managed to compare the critical points of percolation on a dilute cubic lattice with columnar disorder in $\mathbb{Z}^3$ with the critical point of bond percolation in $\mathbb{Z}^2$ uniformly in the disorder intensity. Of course, one would expect to show that the critical curve $p_c(\rho)$ is non-decreasing throughout the interval $(\rho_c, 1]$. This seems to be an interesting and hard question.

Indeed, our argument goes like this: right above $\rho = \rho_c$ we can find copies of $\mathbb{Z}^2$ embedded into $\mathbb{Z}^3$ and independent percolation at $p = 1/2$ restricted to one of these copies is critical. By considering only the effect of the unit length threads, we actually have something substantially better for percolation these embedded copies of $\mathbb{Z}^2$. This last comparison, although intuitive is not at all trivial and relies on a coupling with brochette percolation.

Now assume that we have two densities $\rho' > \rho$ both in the interval $[\rho_c(\mathbb{Z}^2), 1]$ and consider the respective 2-*d* percolation processes in $\mathbb{Z}^2$ coupled in the usual monotone way. The removal of columns with density $1-\rho$ will of course leave a structure that is thinner than that with density $1 - \rho'$. It seem reasonable to state that the latter is strictly better for bond percolation then the former. However, the available techniques of differential inequalities do not seem to work in this context where enhancements are performed at random in the presence of correlations that do not decay with distance.

In our view a better understanding on how the presence of lower-dimensional disorder affects the critical point is an interesting question to be addressed.

There is however another value of ρ at which we can say something related to the Figure 1.2, namely the other extreme of the interval $[\rho_c, 1]$. It is not hard to prove that the critical curve is continuous as ρ approaches 1. Here it follows a sketch of the argument which requires knowledge of the Grimmett and Marstrand arguments [13]. Consider the set of all paths in $\mathbb{Z}^2$ starting from the origin that are directed, that is, that only take up and right steps. For each such path γ let $F_\gamma = \gamma \times \mathbb{Z} \subset \mathbb{Z}^3$ be the set of sites in $\mathbb{Z}^3$ that project into γ. Theorem [13, Theorem A, page 447] states that there exists a positive integer $k = k(\varepsilon)$ such that $p_c^{\text{site}}(2kF_\gamma + B(k)) < p_c^{\text{site}}(\mathbb{Z}^3) + \varepsilon$. Although it does not follow from their statement, an inspection of the proof reveals that k can be taken uniformly in γ. That is, thickening F_γ by stretching $2k$ times and then filling with boxes of radius k results in a 'zigzagging slab' whose critical point falls below $p_c^{\text{site}}(\mathbb{Z}^3)+\varepsilon$ for every γ. Fixed ε and the corresponding $k = k(\varepsilon)$ we now pick $\rho \in (0,1)$ such that $\rho^{(2k+1)^2}$ exceeds the critical threshold for oriented percolation on $\mathbb{Z}^2$. Therefore, for such a value of ρ one can find, with positive probability, an infinite directed path of adjacent boxes of side length $2k$ whose sites are all occupied. Conditional on such a path, the structure of $\mathbb{Z}^3$ that projects to it is zigzag slab as above therefore, $p_c(\rho) < p_c^{\text{site}}(\mathbb{Z}^3) + \varepsilon$ which concludes the argument.

Our result leads naturally to the question whether an upper bound strictly smaller than $1/2$ still holds if the environment Λ is distributed like the incipient infinite cluster in $\mathbb{Z}^2$. We believe that this answer is positive but, unfortunately we fall short of proving so and leave it as an interesting open question.

Chapter 3

Percolation on a horizontally stretched square lattice

In this chapter we present the results on independent percolation on the lattice which is obtained from the square lattice by stretching by the same random amount all the edges that connect sites in two consecutive vertical columns. More precisely, every edge of the square lattice that links sites on the i-th and $(i+1)$-th columns is replaced by a segment of random length ξ_i. The ξ_i's are assumed to be i.i.d.. The edges of the resulting lattice are declared open independently with probability $p_e = p^{|e|}$ where $p \in [0,1]$ and $|e|$ is the length of edge e. The construction of this model and the notation can be found in Section 1.3.2. We relate the occurrence of nontrivial phase transition for this model to moment properties of ξ_1. More precisely, we prove that the model undergoes a nontrivial phase transition when $\mathbb{E}(\xi_1^\eta) < \infty$, for some $\eta > 1$ whereas, when $\mathbb{E}(\xi_1^\eta) = \infty$ for some $\eta < 1$, no phase transition occurs. This results were obtained in collaboration with Augusto Teixeira, Associate Professor of IMPA.

This chapter is organized as follows. In Section 3.1 we will prove Theorem 1.3 which rules out a non-trivial phase transition when the ξ_i's have sufficiently heavy tails. The rest of the chapter will be dedicated to the proof of Theorem 1.2 which establishes the existence of a non-trivial phase transition when the tails are sufficiently light. Section 3.2 contains a brief review on renewal processes and some results that will be useful for proving a decoupling inequality crucial in the proof of Theorem 1.2. Those who are familiar with the theory of renewal processes can perhaps skip reading this section.

In Section 3.3, we develop the multiscale scheme used to prove Theorem 1.2. First, in Section 3.3.1 we define a fast-growing sequence of numbers which correspond to the scales in which we analyze the model. Then we partition $\mathbb{Z}_+$ into the so-called k-blocks which are intervals whose length are related to the k-th scale. The k-blocks will be declared either bad or good hierarchically in such a way that being bad indicates that the renewal process within the block has arrivals that are close to each other. We will show that bad k-blocks are extremely rare under the appropriate moment condition. Section 3.3.2 will be dedicated to the construction of crossing events in rectangles whose bases are k-blocks and with large heights. Such crossings will have a very high probability on good k-blocks. In Section 3.4, we finish the proof of Theorem 1.2, using the results established throughout the chapter.

3.1 Proof of Theorem 1.3

Proof of Theorem 1.3. Let $\eta < 1$ be such that $\mathbb{E}(\xi^\eta) = \infty$. Then

$$\sum_{n=0} \mathbb{P}(\xi^\eta > n) = \infty. \tag{3.1}$$

Setting $\epsilon > 0$ such that $\eta^{-1} = 1 + 2\epsilon$, (3.1) reads

$$\sum_{n=0} \mathbb{P}(\xi > n^{1+2\epsilon}) = \infty.$$

Now consider the events $F_i = \{\xi_i \geqslant i^{1+2\epsilon}\}$, with $i \in \mathbb{Z}_+^*$ and recall that $\{\xi_i\}_{i\in\mathbb{Z}_+^*}$ are independent copies of ξ. Since the F_i are independent and $\sum_i \mathbb{P}(F_i) = \infty$, we have that $\nu_\xi(F_i \text{ i.o.}) = 1$.

Let us now fix $\Lambda \in \{F_i \text{ i.o.}\}$ together with an increasing subsequence $i_k = i_k(\Lambda)$, $k \in \mathbb{Z}_+$ such that F_{i_k} occurs for every k. Roughly speaking, the columns that project to $x_{i_{k-1}}$ and x_{i_k} are too distant from each other to allow paths to connect between them.

We now fix a $\lambda \in (0, \infty)$ and show that $\mathbb{P}^\Lambda_\lambda(o \leftrightarrow \infty) = 0$. Recall the definition of the model on $\mathbb{Z}_+^2$ whose inhomogeneities are given by p_e in (1.4) and also recall the crossing events introduced in (1.1), (1.2) and (1.3). For each $k \in \mathbb{Z}_+$, let

$$R_k = R\Big(\big[0, i_k\big) \times \big[0, \lceil \exp(i_k^{1+\epsilon}) \rceil\big]\Big),$$

and note that

$$\mathbb{P}^\Lambda_\lambda(o \leftrightarrow \infty) \leqslant \mathbb{P}^\Lambda_\lambda\big(\mathfrak{C}_h(R_k)\big) + \mathbb{P}^\Lambda_\lambda\big(\mathfrak{C}_v(R_k)\big). \tag{3.2}$$

The probability of $\mathfrak{C}_h(R_k)$ is bounded above by the probability that there is an open edge between the columns $\{i_k - 1\} \times \mathbb{Z}_+$ and $\{i_k\} \times \mathbb{Z}_+$. Since these columns are very far apart, the height of R_k is not large enough to ensure the existence of an open edge with good probability. In fact, let

$$J_k = \{0, 1, \dots, \lceil \exp(i_k^{1+\epsilon}) \rceil - 1\}$$

and note that

$$\begin{aligned} \mathbb{P}_\lambda^\Lambda\big(\mathfrak{C}_h(R_k)\big) &\leqslant \mathbb{P}_\lambda^\Lambda\Bigg(\bigcup_{j\in J_k}\big\{\{(i_k-1, j), (i_k, j)\} \text{ is open}\big\}\Bigg) \\ &\leqslant \lceil \exp(i_k^{1+\epsilon}) \rceil \exp(-\lambda \xi_{i_k}) \\ &\leqslant \lceil \exp(i_k^{1+\epsilon}) \rceil \exp(-\lambda i_k^{1+2\epsilon}) \overset{k\to\infty}{\longrightarrow} 0, \end{aligned} \tag{3.3}$$

where we used the definition of $i_k(\Lambda)$ in the last inequality.

In order to estimate the probability of $\mathfrak{C}_v(R_k)$, we note that, on this event there must be at least one vertical edge connecting the j-th and $(j+1)$-th layer in R_k, for $j \in J_k$. The height of R_k is large enough to guarantee that this event has vanishing probability as k grows. Let

$$\overline{\lambda} = \ln\big(1 - \exp(-\lambda)\big) < 0 \tag{3.4}$$

and note that

$$\begin{aligned} \mathbb{P}_\lambda^\Lambda\big(\mathfrak{C}_v(R_k)\big) &\leqslant \mathbb{P}_\lambda^\Lambda\Bigg(\bigcap_{j\in J_k}\bigcup_{l=0}^{i_k-1}\big\{\{(l, j), (l, j+1)\} \text{ is open}\big\}\Bigg) \\ &= \Big(1 - \big(1 - \exp(-\lambda)\big)^{i_k}\Big)^{|J_k|} \\ &\overset{(3.4)}{=} \big(1 - \exp\big(\overline{\lambda} i_k\big)\big)^{|J_k|} \\ &\leqslant \exp\big(-\exp\big(\overline{\lambda} i_k\big)|J_k|\big) \\ &\leqslant \exp\big(-\exp\big(\overline{\lambda} i_k + i_k^{1+\epsilon}\big)\big) \overset{k\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.5}$$

In the second inequality sign above we used $1 - x \leqslant \exp(-x)$.

Combining (3.2), (3.3) and (3.5),

$$\mathbb{P}_\lambda^\Lambda(o \leftrightarrow \infty) \overset{k\to\infty}{\longrightarrow} 0,$$

which concludes the proof. □

3.2 Renewal process

The purpose of this section is to prove a decoupling inequality (see Lemma 3.4), which will be used as a fundamental tool on our multiscale analysis in Section 3.3.1. For this a brief outline of some results on renewal processes, it taken from [23], will be presented.

3.2.1 Definition and notation

Let ξ be a positive integer-valued random variable, called *interarrival time*, and χ be a non-negative integer-valued random variable, called *delay*. As before, let $\{\xi_i\}_{i\in\mathbb{Z}_+^*}$ be i.i.d. copies of ξ which are also independent of χ. We define the *renewal process*

$$X = X(\xi,\chi) = \{X_i\}_{i\in\mathbb{Z}_+}$$

recursively as:

$$X_0 = \chi, \quad \text{and} \quad X_i = X_{i-1} + \xi_i \text{ for } i \in \mathbb{Z}_+^*.$$

We say that the i-th renewal takes place at time t if $X_{i-1} = t$.

It is convenient to define two other processes

$$Y = Y(\xi,\chi) = \{Y_n\}_{n\in\mathbb{Z}_+} \quad \text{and} \quad Z = Z(\xi,\chi) = \{Z_n\}_{n\in\mathbb{Z}_+},$$

as

$$Y_n = \begin{cases} 1, & \text{if a renewal of } X \text{ occurs at time } n, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

and the *residual life*

$$Z_n = \min\{X_i - n;\ i \in \mathbb{Z}_+ \text{ and } X_i - n \geqslant 0\}. \tag{3.7}$$

The processes Y and Z will also be called renewal process with interarrival time ξ and delay χ, since each one of X, Y and Z fully determines the two others (see Figure 3.1). The probability law that governs these renewal processes will be denoted by $\upsilon_\xi^\chi(\cdot)$. It is worth noting that Z is a

Markov chain with transition kernel given by

$$\mathbb{P}(Z_n = i | Z_{n-1} = j) = \begin{cases} \mathbb{P}(\xi = i+1), & \text{if } j = 0 \\ 1, & \text{if } i+1 = j > 0 \\ 0, & \text{otherwise} \end{cases}, \tag{3.8}$$

for all $n \in \mathbb{Z}_+^*$, and $i, j \in \mathbb{Z}_+$.

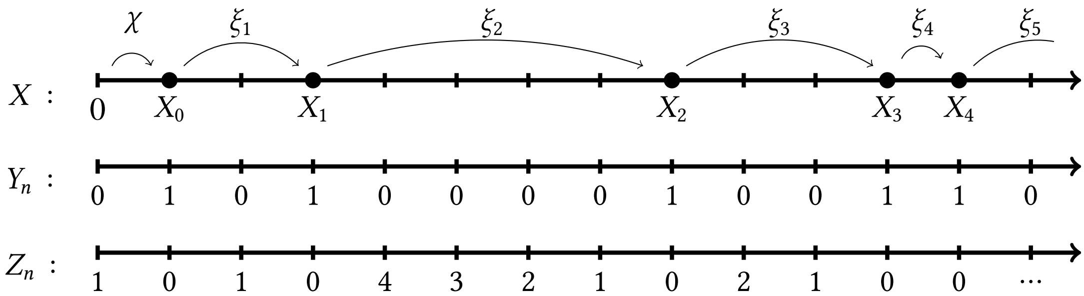

Figure 3.1: Illustration of the processes X, Y, Z. In this realization, $\chi = 1$, $\xi_1 = 2$, $\xi_2 = 5$, $\xi_3 = 3$ and $\xi_4 = 1$.

For $m \in \mathbb{Z}_+$ consider $\theta_m : \mathbb{Z}^\infty \mapsto \mathbb{Z}^\infty$, the shift operator given by

$$\theta_m(x_0, x_1, \ldots) = (x_m, x_{m+1}, \ldots).$$

It is desirable that the renewal process Z be invariant under shifts, i.e.

$$\theta_m Z \stackrel{d}{=} Z \quad \text{for any } m \in \mathbb{Z}_+^*, \tag{3.9}$$

where $\stackrel{d}{=}$ means equality in distribution. If $\mathbb{E}(\xi) < \infty$, we can define a random variable $\rho = \rho(\xi)$ with distribution

$$\rho_k = \mathbb{P}(\rho = k) := \frac{1}{\mathbb{E}(\xi)} \sum_{i=k+1} \mathbb{P}(\xi = i), \text{ for any } k \in \mathbb{Z}_+. \tag{3.10}$$

Using ρ as the delay, implies that the resulting renewal process $Z(\xi, \rho)$ satisfies (3.9). In fact, using (3.8) we have

$$\begin{aligned}
\mathbb{P}(Z_1 = k) &= \mathbb{P}(Z_1 = k | Z_0 = k+1)\mathbb{P}(Z_0 = k+1) + \mathbb{P}(Z_1 = k | Z_0 = 0)\mathbb{P}(Z_0 = 0) \\
&= 1 \cdot \rho_{k+1} + \mathbb{P}(\xi = k+1)\rho_0 \\
&= \frac{1}{\mathbb{E}(\xi)} \sum_{i=k+2} \mathbb{P}(\xi = i) + \frac{\mathbb{P}(\xi = k+1)}{\mathbb{E}(\xi)} \\
&= \frac{1}{\mathbb{E}(\xi)} \sum_{i=k+1} \mathbb{P}(\xi = i) = \rho_k.
\end{aligned}$$

Therefore

$$Z_1 \stackrel{d}{=} Z_0 \stackrel{d}{=} \rho.$$

Using the Markov property and induction, it follows that $\theta_m Z \stackrel{d}{=} Z$. Consequently it also follows that $\theta_m Y \stackrel{d}{=} Y$ for $Y = Y(\xi, \rho)$. For a fixed ξ, ρ is called *stationary delay*.

From same properties of expectation, one obtain

$$\text{if} \quad \mathbb{E}(\xi^{1+\varepsilon}) < \infty \quad \text{then} \quad \mathbb{E}(\rho^{\varepsilon}) < \infty. \tag{3.11}$$

3.2.2 Loss of memory

The purpose of this section is to show how the renewal process $Z = \{Z_n\}_{n \in \mathbb{Z}_+}$ forgets its initial state when $n \to \infty$. This fact will be important in what follows.

We say that a random variable ξ is *aperiodic* if

$$\gcd\left\{k \in \mathbb{Z}_+^*;\ \mathbb{P}(\xi = k) > 0\right\} = 1.$$

For the rest of this section we assume ξ is aperiodic and also $\mathbb{E}(\xi) < \infty$.

Let $Y = Y(\xi, \chi)$ and $Y' = Y(\xi, \chi')$ be two independent renewal processes with interarrival time ξ and delay χ and χ'. Set

$$T := \min\{k \in \mathbb{Z}_+^*;\ Y_k = Y'_k = 1\},$$

the *coupling time* of the renewal processes X and X'. And denote $v_\xi^{\chi,\chi'}(\cdot)$ the product measure $v_\xi^{\chi} \otimes v_\xi^{\chi'}$.

The next lemma will ensure that the processes X and X' meet a.s..

Lemma 3.1. *If ξ aperiodic and $\mathbb{E}(\xi) < \infty$, then*

$$v_{\xi}^{\chi,\chi'}(T < \infty) = 1. \tag{3.12}$$

Proof. Set $\widetilde{Y} = \widetilde{Y}(\xi, \chi, \chi') = \{\widetilde{Y}_n\}_{n \in \mathbb{Z}_+}$ as

$$\widetilde{Y}_n := Y_n \cdot Y'_n, \text{ for all } n \in \mathbb{Z}_+.$$

Notice that $\widetilde{Y}$ is also a renewal process, since ξ is aperiodic. Denote $\widetilde{\xi}$ its interarrival time. It is worth noting that $\widetilde{\xi}$ is nondefective, i.e.

$$\sum_{i \in \mathbb{Z}_+^*} \mathbb{P}\big(\widetilde{\xi} = i\big) = 1.$$

Otherwise we would have $\mathbb{P}(\widetilde{\xi} = \infty) > 0$ which would imply

$$\lim_{n \to \infty} v_{\xi}^{\rho,\rho}\big(\widetilde{Y}_n = 1\big) = 0,$$

where ρ is the stationary delay with respect to ξ. But this cannot happen, since

$$\begin{aligned} v_{\xi}^{\rho,\rho}\big(\widetilde{Y}_n = 1\big) &= v_{\xi}^{\rho}(Y_n = 1) v_{\xi}^{\rho}(Y'_n = 1) \\ &= v_{\xi}^{\rho}(Z_n = 0) v_{\xi}^{\rho}(Z'_n = 0) = \rho_0^2 = \left(\frac{1}{\mathbb{E}(\xi)}\right)^2 > 0, \end{aligned}$$

where we used the stationarity of ρ for equality between the lines. Hence

$$v_{\xi}^{\delta_0,\delta_0}\big(\widetilde{Y}_n = 1 \text{ i.o.}\big) = 1,$$

where δ_x denotes the delta distribution concentrated in $x \in \mathbb{Z}$.

Now since ξ is aperiodic, it follows that there is $n_0 \in \mathbb{Z}_+$ such that

$$v_{\xi}^{\delta_0}(Y_n = 1) > 0 \text{ for all } n \geqslant n_0.$$

(This result can be seen in [21], Lemma 1.30) Therefore, for $m \in \mathbb{Z}$,

$$\begin{aligned} 1 &= \upsilon_{\xi}^{\delta_0,\delta_0}\big(\widetilde{Y} = 1 \text{ i.o.}\big) \\ &= \upsilon_{\xi}^{\delta_0,\delta_0}\big(\widetilde{Y} = 1 \text{ i.o.}|(Y_{n_0}, Y'_{n_0+m}) = (1,1)\big)\upsilon_{\xi}^{\delta_0,\delta_0}\big((Y_{n_0}, Y'_{n_0+m}) = (1,1)\big) + \\ &\quad +\upsilon_{\xi}^{\delta_0,\delta_0}\big(\widetilde{Y} = 1 \text{ i.o.}|(Y_{n_0}, Y'_{n_0+m}) \neq (1,1)\big)\upsilon_{\xi}^{\delta_0,\delta_0}\big((Y_{n_0}, Y'_{n_0+m}) \neq (1,1)\big). \end{aligned} \tag{3.13}$$

Note that if

$$xz + y(1-z) = 1 \text{ for } x, y \in [0,1] \text{ and } z \in (0,1),$$

then $x = 1$. Therefore, it follows from (3.13) that

$$\upsilon_{\xi}^{\delta_0,\delta_m}\big(\widetilde{Y} = 1 \text{ i.o.}\big) = \upsilon_{\xi}^{\delta_0,\delta_0}\big(\widetilde{Y} = 1 \text{ i.o.}|(Y_{n_0}, Y'_{n_0+m}) = (1,1)\big) = 1.$$

Therefore for any $i, j \in \mathbb{Z}_+$,

$$\upsilon_{\xi}^{\delta_i,\delta_j}\big(\widetilde{Y} = 1 \text{ i.o.}\big) = \upsilon_{\xi}^{\delta_0,\delta_{|i-j|}}\big(\widetilde{Y} = 1 \text{ i.o.}\big) = 1.$$

The result can finally be established

$$\begin{aligned} \upsilon_{\xi}^{\chi,\chi'}(T < \infty) &\geqslant \upsilon_{\xi}^{\chi,\chi'}\big(\widetilde{Y} = 1 \text{ i.o.}\big) \\ &= \sum_{i\in\mathbb{Z}_+}\sum_{j\in\mathbb{Z}_+} \upsilon_{\xi}^{\delta_i,\delta_j}\big(\widetilde{Y} = 1 \text{ i.o.}\big)\mathbb{P}(\chi = i)\mathbb{P}(\chi' = j) = 1, \end{aligned}$$

where the above equality follows the conditioning in the first renewal (delay). □

The lemma below shows that renewal processes forget the delay.

Lemma 3.2 (Renewal theorem)**.** *If ξ aperiodic and $\mathbb{E}(\xi) < \infty$, then for all event A,*

$$\lim_{n\to\infty}\Big|\upsilon_{\xi}^{\chi}(\theta_n Y \in A) - \upsilon_{\xi}^{\chi'}(\theta_n Y' \in A)\Big| = 0 \tag{3.14}$$

for any delays χ and χ'.

Moreover,

$$\lim_{n\to\infty}\upsilon_{\xi}^{\chi}(Z_n = k) = \rho_k \tag{3.15}$$

for all $k \in \mathbb{Z}_+$ and any delay χ.

Proof. For every event A, we have

$$\left| v_\xi^{\chi}(\theta_n Y \in A) - v_\xi^{\chi'}(\theta_n Y' \in A) \right| \leqslant v_\xi^{\chi,\chi'}(T > n). \tag{3.16}$$

Taking the limit as $n \to \infty$ on both sides of (3.16) and using (3.12) we have

$$\lim_{n\to\infty} v_\xi^{\chi,\chi'}(T > n) = v_\xi^{\chi,\chi'}(T = \infty) = 0.$$

This establishes (3.14). In order to prove (3.15), plug $\chi' = \rho$ in (3.14) and note that $\{Z_n = k\} = \{Y_{n+k} = 1\}$,

$$\lim_{n\to\infty} v_\xi^{\chi}(Z_n = k) = \lim_{n\to\infty} v_\xi^{\rho}(Z_n = k) = v_\xi^{\rho}(Z_0 = k) = \rho_k.$$

□

3.2.3 Decoupling

The goal of this section is to prove a decoupling for stationary renewals. To prove decoupling inequality we will use (3.16). We will also need to bound the value of $v_\xi^{\chi,\chi'}(T > n)$. The upper bound for $v_\xi^{\chi,\chi'}(T > n)$ will be obtained by the Markov inequality applied to the random variable T^ε, where $\varepsilon > 0$. The lemma below guarantees that T^ε has finite expected value and will be crucial to prove the decoupling inequality. Its proof was taken from [23] (Theorem 4.2, page 27) and/or [22] (Proposition 1).

Lemma 3.3. *Let ξ be an aperiodic positive integer-valued random variable. Suppose that for some $\varepsilon \in (0, 1)$, $\mathbb{E}(\xi^{1+\varepsilon}) < \infty$, and that χ, χ' are non-negative integer-valued random variables with $\mathbb{E}(\chi^\varepsilon)$ and $\mathbb{E}(\chi'^\varepsilon)$ finite. Then $\mathbb{E}(T^\varepsilon) < \infty$.*

Proof. Denote by $\mathbb{E}_\xi^{\chi,\chi'}(\cdot)$ the expected value with respect to measure $v_\xi^{\chi,\chi'}$, and note that

$$\begin{aligned}
\mathbb{E}_\xi^{\chi',\chi'}(T^\varepsilon) &= \mathbb{E}_\xi^{\chi,\chi'}\Big(\big(\min(\chi,\chi') + T - \min(\chi,\chi')\big)^\varepsilon\Big) \\
&\leqslant \mathbb{E}_\xi^{\chi,\chi'}\big(\min(\chi,\chi')^\varepsilon\big) + \mathbb{E}_\xi^{\chi,\chi'}\Big(\big(T - \min(\chi,\chi')\big)^\varepsilon\Big) \\
&\leqslant \mathbb{E}_\xi^{\chi,\chi'}\big(\min(\chi,\chi')^\varepsilon\big) + \mathbb{E}_\xi^{\delta_0,\varphi}(T^\varepsilon),
\end{aligned}$$

where $\varphi = \max(\chi, \chi') - \min(\chi, \chi')$. By hypothesis $\mathbb{E}_{\xi}^{\chi,\chi'}\left(\min(\chi, \chi')^{\varepsilon}\right) < \infty$ and $\mathbb{E}_{\xi}^{\chi,\chi'}(\varphi^{\varepsilon}) < \infty$. Therefore without loss of generality, we will assume that $\chi = \delta_0$. To simplify the notation we will omit the variables ξ, χ and χ' in $\mathbb{E}_{\xi}^{\chi,\chi'}$ when there is no risk of ambiguity.

Let

$$Y = Y(\xi, \delta_0) \text{ and } Y' = Y(\xi, \chi')$$

mutually independent (as well as X and X'). The idea behind the proof is to make a series of attempts for processes Y and Y' find each other, where each attempt will have positive probability of occurring. The uniform lower bound for the success of each attempt is obtained by taking $n_0 \in \mathbb{Z}_+$ and $\sigma > 0$ such that

$$v_{\xi}^{\delta_0}(Z_n = 0) \geqslant \sigma \text{ for all } n \geqslant n_0.$$

This follows from the Lemma 3.2, because

$$\lim_{n \to \infty} v_{\xi}^{\delta_0}(Z_n = 0) = \frac{1}{\mathbb{E}(\xi)}.$$

Define the random variables F_n, H_n and v_n for $n \in \mathbb{Z}_+^*$ inductively by $F_0 = 0$ and,

$$\begin{aligned}
H_{2n} &= \min\{X_i' - F_{2n};\ X_i' - F_{2n} \geqslant 0\} = X_{v_{2n}}' - F_{2n}, \\
F_{2n+1} &= X_{v_{2n}+n_0}', \\
H_{2n+1} &= \min\{X_i - F_{2n+1};\ X_i - F_{2n+1} \geqslant 0\} = X_{v_{2n+1}} - F_{2n+1}, \\
F_{2n+2} &= = X_{v_{2n+1}+n_0}.
\end{aligned}$$

Note that $H_n = 0$ corresponds to a success. See Figure 3.2.The reason for adding n_0 steps to the variables $X_{v_1}, X_{v_2}', \ldots$ is that the probability of success is at least equal to σ. Now define

$$\tau = \min\{k \in \mathbb{Z}_+;\ H_k = 0\}$$

and

$$U_1 = X_{v_1} - X_{v_0}', \quad U_2 = X_{v_2}' - X_{v_1}, \quad \ldots \quad .$$

Consider also the filtration $\{\mathcal{N}_i\}_{i\in\mathbb{Z}_+}$, where $\mathcal{N}_i$ is generated by the variables

$$X_j, X'_k, \text{ where } j \leqslant v_i \text{ and } k \leqslant v_{i-1}+n_0, \quad \text{when } i \text{ is odd, or}$$
$$X_j, X'_k, \text{ where } k \leqslant v_i \text{ and } j \leqslant v_{i-1}+n_0, \quad \text{when } i \text{ is even.}$$

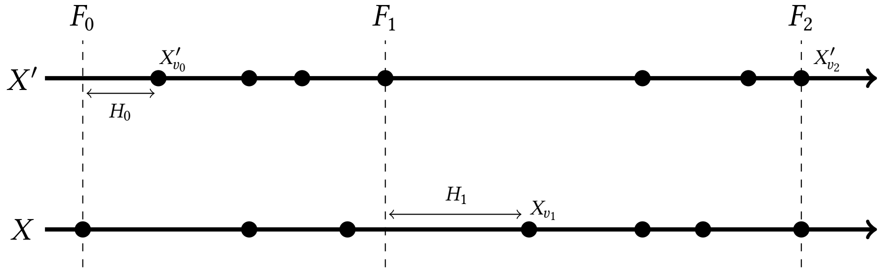

Figure 3.2: In this example we have $n_0 = 3$, $H_0 > 0$, $H_1 > 0$ and $H_2 = 0$. Hence $\tau = 2$ and we have a successful coupling.

Note that,

$$T \leqslant X'_0 + \sum_{i=1}^{\tau} U_i = \chi' + \sum_{i=1} U_i \mathbb{I}_{\{\tau \geqslant i\}}, \tag{3.17}$$

where $\mathbb{I}_S$ denotes the indicator function of the set S. Since

$$\Big(\sum_{i=1} x_i\Big)^{\varepsilon} \leqslant \sum_{i=1} x_i^{\varepsilon} \tag{3.18}$$

when $x_i \geqslant 0$ and $\varepsilon \in (0,1)$, the inequality (3.17) implies that

$$\mathbb{E}^{\delta 0,\chi'}\big(T^{\varepsilon}\big) \leqslant \mathbb{E}(\chi'^{\varepsilon}) + \sum_{i=1} \mathbb{E}(U_i^{\varepsilon} \mathbb{I}_{(\tau \geqslant i)}). \tag{3.19}$$

For each term of the above sum,

$$\begin{aligned} \mathbb{E}(U_i^{\varepsilon}\mathbb{I}_{(\tau\geqslant i)}) &= \mathbb{E}\Big(\mathbb{E}(U_i^{\varepsilon}\mathbb{I}_{(\tau\geqslant i)}|\mathcal{N}_{i-1})\Big) \\ &\leqslant \mathbb{E}\Big(\mathbb{E}(U_i^{\varepsilon}|\mathcal{N}_{i-1})\mathbb{I}_{(\tau\geqslant i)}\Big). \end{aligned} \tag{3.20}$$

To conclude the proof it suffices to show that $\mathbb{E}(U_i^\varepsilon|\mathcal{N}_{i-1}) < K$, for some constant K, because combining (3.19), (3.20), $\mathbb{E}(\chi'^\varepsilon) < \infty$ and the choice of n_0

$$\begin{aligned}\mathbb{E}^{\delta 0,\chi'}\left(T^\varepsilon\right) &\leqslant \mathbb{E}(\chi'^\varepsilon) + \sum_{i=1} K\mathbb{P}(\tau \geqslant i)\\ &\leqslant \mathbb{E}(\chi'^\varepsilon) + \sum_{i=1} K(1-\sigma)^i < \infty.\end{aligned}$$

Applying one more time (3.18)

$$\begin{aligned}\mathbb{E}(U_i^\varepsilon|\mathcal{N}_{i-1}) &= \mathbb{E}\big((X_{v_{i-1}+n_0} - X_{v_{i-1}} + H_i)^\varepsilon|\mathcal{N}_{i-1}\big)\\ &= n_0\mathbb{E}(\xi^\varepsilon) + \mathbb{E}(H_i^\varepsilon|H_{i-1}).\end{aligned} \tag{3.21}$$

Now think that $F_{i-1} = 0$, and note that $H_i = Z_k^*$, where $Z^* = Z(\delta_0, \xi)$ as defined in (3.7) and $k = H_{i-1} + X_{v_{i-1}+n_0} - X_{v_{i-1}}$. Note that

$$\begin{aligned}\mathbb{E}\big((Z_k^*)^\varepsilon\big) &= \sum_{i=1} i^\varepsilon \mathbb{P}(Z_k^* = i)\\ &= \sum_{i=1} i^\varepsilon \sum_{m=0}^{k-1} \mathbb{P}(Z_m^* = 0)\mathbb{P}(\xi = k+i-m)\\ &\leqslant \sum_{i=1} i^\varepsilon \sum_{m=i} \mathbb{P}(\xi = m)\\ &= \sum_{i=1} (1^\varepsilon + 2^\varepsilon + \ldots + i^\varepsilon)\mathbb{P}(\xi = i)\\ &\leqslant \sum_{i=1} (i+1)^{1+\varepsilon}\mathbb{P}(\xi = i) < \infty,\end{aligned} \tag{3.22}$$

where the last conclusion follows from $\mathbb{E}(\xi^{1+\varepsilon}) < \infty$. The lemma follows combining (3.21) and (3.22). □

We can now prove the decoupling.

Lemma 3.4 (Decoupling). *Let ξ be a positive integer-valued, aperiodic random variable with $\mathbb{E}(\xi^{1+\varepsilon}) < \infty$, for some $\varepsilon > 0$, and consider the renewal process $Y = Y\big(\xi, \rho(\xi)\big)$ defined in (3.6).*

Then there is $c_1 = c_1(\xi, \varepsilon) \in (0, \infty)$ such that for all $n, m \in \mathbb{Z}_+$ and for all events A and B, where

$$A \in \sigma(Y_i;\ 0 \leqslant i \leqslant m) \quad \textit{and} \quad B \in \sigma(Y_i,\ i \geqslant m+n)$$

we have

$$v_\xi^\rho(A \cap B) \leqslant v_\xi^\rho(A) v_\xi^\rho(B) + c_1 n^{-\varepsilon}.$$

Proof. For simplicity the indices ξ will be omitted in this proof. If $v^\rho(A) = 0$ there is nothing to be proved. Suppose then that $v^\rho(A) > 0$. Recall the definition of Z in (3.7), and note that

$$\begin{aligned}
v^\rho(A \cap B) &= v^\rho(A \cap B \cap \{Z_m > n/2\}) + v^\rho(A \cap B \cap \{Z_m \leqslant n/2\}) \\
&\leqslant v^\rho(Z_m > n/2) + v^\rho(A) v^\rho(B \cap \{Z_m \leqslant n/2\} | A) \\
&\leqslant v^\rho(Z_m > n/2) + v^\rho(A) \sum_{\substack{0 \leqslant i \leqslant \lfloor n/2 \rfloor \\ v^\rho(Z_m = i|A) > 0}} v^\rho(B|A, Z_m = i) v^\rho(Z_m = i|A) \\
&\leqslant v^\rho(Z_m > n/2) + v^\rho(A) \max_{0 \leqslant j \leqslant \lfloor n/2 \rfloor} v^{\delta_{m+j}}(B) \sum_{0 \leqslant i \leqslant \lfloor n/2 \rfloor} v^\rho(Z_m = i|A) \\
&\leqslant v^\rho(Z_m > n/2) + v^\rho(A) \max_{0 \leqslant j \leqslant \lfloor n/2 \rfloor} v^{\delta_{m+j}}(B). \qquad (3.23)
\end{aligned}$$

Now we compare $v^{\delta_{m+j}}(B)$ with $v^\rho(B)$, when $0 \leqslant j \leqslant \lfloor n/2 \rfloor$. Using that $v^{\delta_{m+j}}(B) = v^{\delta_0}\big(\theta_{m+j}(B)\big)$ and by the stationarity of ρ

$$\begin{aligned}
\left| v^{\delta_{m+j}}(B) - v^{\delta_\rho}(B) \right| &= \left| v^{\delta_0}\big(\theta_{m+j}(B)\big) - v^\rho\big(\theta_{m+j}(B)\big) \right| \\
&\overset{(3.16)}{\leqslant} v_\xi^{\delta_0,\rho}(T > n - j) \\
&\leqslant v_\xi^{\delta_0,\rho}(T > n/2). \qquad (3.24)
\end{aligned}$$

By (3.23), (3.24) and by the fact $Z_m \overset{d}{=} Z_0 \overset{d}{=} \rho$

$$\begin{aligned}
v^\rho(A \cap B) &\leqslant v^\rho(A) v^\rho(B) + v^\rho(\rho > n/2) + v_\xi^{\delta_0,\rho}(T > n/2) \\
&\leqslant v^\rho(A) v^\rho(B) + 2^\varepsilon \mathbb{E}(\rho^\varepsilon) n^{-\varepsilon} + 2^\varepsilon \mathbb{E}_\xi^{\delta_0,\rho}(T^\varepsilon) n^{-\varepsilon},
\end{aligned}$$

where the last inequality follows from the Markov inequality for ρ^ε and T^ε. Finally take $c_1 = 2^\varepsilon \mathbb{E}(\rho^\varepsilon) + 2^\varepsilon \mathbb{E}_\xi^{\delta_0,\rho}(T^\varepsilon)$ which is finite by (3.11) and Lemma 3.3. □

3.3 The multiscale scheme

Throughout this section we fix ξ positive, integer-valued and aperiodic. We also assume $\mathbb{E}(\xi^{1+\varepsilon}) < \infty$ for a some $\varepsilon > 0$ and denote $\rho = \rho(\xi)$ the respective stationary delay given in (3.10). These conditions on ξ allow us to redefine the percolation model on horizontally stretched square lattice to obtain an equivalent model on $\mathbb{Z}_+^2$ as follows:

Consider the environment $\Lambda \subseteq \mathbb{Z}_+$ distributed as υ_ξ and set

$$E_{\text{vert}}(\Lambda^c) := \big\{\{(x,y),(x,y+1)\} \in E(\mathbb{Z}_+^2); x \notin \Lambda, y \in \mathbb{Z}_+\big\}.$$

Let each edge $e \in E(\mathbb{Z}_+^2)$ be open independently with probability

$$p_e = \begin{cases} 0, & \text{if } e \in E_{\text{vert}}(\Lambda^c), \\ p, & \text{if } e \notin E_{\text{vert}}(\Lambda^c). \end{cases}$$

Edges which are not open are called closed.

Geometrically, this formulation consists in preserving the columns of the $\mathbb{Z}_+^2$ lattice that project to Λ while deleting the ones that project to Λ^c. The resulting graph is similar to the stretched lattice $\mathcal{L}_\Lambda$ defined in Section 1.3.2, however, the edges are now split into unit length segments. Each one of these edges is open independently with probability p.

Note that, we can recover the original formulation on $\mathcal{L}_\Lambda$ by declaring an edge open if all the corresponding unitary edges in $\mathbb{Z}_+^2$ are open in the new formulation. Therefore, these two formulations are equivalent and we slightly abusing notation, denoting $\mathbb{P}_p^\Lambda(\cdot)$ the law of this new model.

Since the model is now defined on $\mathbb{Z}_+^2$, one can define the rectangle $R = R\big([a,b) \times [c,d)\big)$ for any $a,b,c,d \in \mathbb{Z}_+$ and the corresponding (horizontal and vertical) crossing events as in (1.1), (1.2) and (1.3). In the remainder of this section only this new definition of the model will be adopted.

3.3.1 Environments

Let us fix constants

$$\alpha \in \left(0, \frac{\varepsilon}{2}\right] \quad \text{and} \quad \gamma \in \left(1, 1 + \frac{\alpha}{\alpha+2}\right). \tag{3.25}$$

which will appear as exponents in several expressions below. The exponent γ will give the rate of growth for the scales in which we study the environment while α will give the rate of decay of probability that bad events occur in each scale (see (3.26) and (3.30)).

Let us also fix $L_0 = L_0(\xi, \varepsilon, \alpha, \gamma) \in \mathbb{Z}_+$ sufficiently large so that

(i) $L_0^{\gamma-1} \geqslant 3$,

(ii) $L_0^{\varepsilon-\alpha} \geqslant \mathbb{E}(\rho^{\varepsilon})$ and

(iii) $L_0^{c_2} \geqslant c_1 + 1$, where c_1 is given by the Lemma 3.4 and

$$c_2 = 2 + 2\alpha - \gamma\alpha - 2\gamma.$$

Note that $c_2 > 0$ by the choice of γ in (3.25).

Once L_0 is fixed, we can define recursively the sequence of scales $(L_k)_{k\in\mathbb{Z}_+}$ by putting

$$L_k = L_{k-1}\lfloor L_{k-1}^{\gamma-1}\rfloor, \text{ for any } k \geqslant 1. \tag{3.26}$$

Item (i) in the definition of L_0 together with (3.25) and (3.26) implies that the scales grow superexponentially. In fact,

$$\left(\frac{2}{3}\right)^k L_0^{\gamma^k} \leqslant \ldots \leqslant \frac{2}{3}L_{k-1}^{\gamma} \leqslant L_k \leqslant L_{k-1}^{\gamma} \leqslant \ldots \leqslant L_0^{\gamma^k}. \tag{3.27}$$

The items (ii) and (iii) are necessary to prove the Lemma 3.5 below.

For $k \in \mathbb{Z}_+$, consider the partition of $\mathbb{R}_+$ into the intervals

$$I_j^k = \big[jL_k, (j+1)L_k\big), \text{ with } j \in \mathbb{Z}_+.$$

The interval I_j^k is said to be the *j-th k-block*. Note that the k-blocks have size L_k and are formed by $\lfloor L_{k-1}^{\gamma-1}\rfloor$ $(k-1)$-blocks. More precisely, for $k \geqslant 1$

$$I_j^k = \bigcup_i I_i^{k-1}, \text{ with } i \in \big\{j\lfloor L_{k-1}^{\gamma-1}\rfloor, \ldots, (j+1)\lfloor L_{k-1}^{\gamma-1}\rfloor - 1\big\}. \tag{3.28}$$

Now fix an environment $\Lambda \subseteq \mathbb{Z}_+$. Blocks will be labeled good or bad depending on Λ in a recursive way. For $k = 0$ declare the j-th 0-block, I_j^0, good if $\Lambda \cap I_j^0 \neq \varnothing$, and bad otherwise. Once labeled the $(k-1)$-blocks, declare a k-block bad, if there are at least two non-consecutive bad $(k-1)$-blocks contained in it. Otherwise, we declare the k-block good. More precisely, for

$j, k \in \mathbb{Z}_+$ consider the events A_j^k defined recursively by

$$\begin{aligned} A_j^0 &= \{\Lambda \subseteq \mathbb{Z}_+; \Lambda \cap I_j^0 = \varnothing\} = \{I_j^0 \text{ is bad}\}, \text{ and} \\ A_j^k &= \bigcup_{\substack{j_k^- \leqslant i_1, i_2 \leqslant j_k^+ \\ |i_1 - i_2| \geqslant 2}} \left(A_{i_1}^{k-1} \cap A_{i_2}^{k-1}\right) = \{I_j^k \text{ is bad}\}, \text{ for } k \geqslant 1, \end{aligned} \tag{3.29}$$

where j_k^- and j_k^+ correspond respectively to the smallest and greatest among the indices of the $(k-1)$-blocks that form I_j^k. By (3.28) we know that

$$j_k^- = j\lfloor L_{k-1}^{\gamma-1} \rfloor \quad \text{and} \quad j_k^+ = (j+1)\lfloor L_{k-1}^{\gamma-1} \rfloor - 1.$$

It follows from (i) that every block is formed by at least three blocks from the previous scale, which results that the events A_j^k (for $k \geqslant 1$) are non triviality defined.

We now define

$$p_k := \upsilon^\rho(A_0^k) = \upsilon^\rho(A_j^k)$$

where the equality follows from the stationarity of ρ.

The next lemma establishes an upper bound for the p_k's, which is a power law in L_k with exponent α.

Lemma 3.5. *For every $k \in \mathbb{Z}_+$ we have*

$$p_k \leqslant L_k^{-\alpha}. \tag{3.30}$$

Proof. We proceed by induction on k. First observe that

$$p_0 = \upsilon^\rho(A_0^0) \overset{(3.7)}{=} \upsilon^\rho(Z_0 > L_0) \overset{(3.10)}{=} \mathbb{P}(\rho > L_0) \leqslant \frac{\mathbb{E}(\rho^\varepsilon)}{L_0^\varepsilon},$$

where the last inequality follows from the Markov's inequality for ρ^ε. Therefore (ii) implies $p_0 \leqslant L_0^{-\alpha}$.

Now suppose that for some $k \in \mathbb{Z}_+$, $p_k \leqslant L_k^{-\alpha}$. Set

$$\mathfrak{I} := \left\{(i_1, i_2) \in \mathbb{Z}_+^2;\ i_1, i_2 \leqslant \lfloor L_k^{\gamma-1} \rfloor - 1 \text{ and } |i_1 - i_2| \geqslant 2\right\},$$

and note that

$$\begin{aligned} p_{k+1} &= \nu^{\rho}(A_0^{k+1}) \\ &\overset{(3.29)}{\leqslant} \sum_{(i_1,i_2)\in\mathfrak{I}} \nu^{\rho}(A_{i_1}^k \cap A_{i_2}^k) \\ &\leqslant \lfloor L_k^{\gamma-1} \rfloor^2 (p_k^2 + c_1 L_k^{-\varepsilon}) \\ &\leqslant L_k^{2\gamma-2}(L_k^{-2\alpha} + c_1 L_k^{-\varepsilon}) \\ &\leqslant (1+c_1) L_k^{2\gamma-2-2\alpha}, \end{aligned}$$

where the second inequality follows from the Lemma 3.4 and the third inequality follows from the induction hypothesis.

Therefore

$$\frac{p_{k+1}}{L_{k+1}^{-\alpha}} \leqslant (1+c_1) L_k^{2\gamma-2-2\alpha} L_{k+1}^{\alpha} \leqslant (1+c_1) L_k^{2\gamma-2-2\alpha+\gamma\alpha} \overset{(iii)}{\leqslant} 1.$$

Finishing the proof. □

3.3.2 Crossings

In this section we will define crossing events in certain rectangles of $\mathbb{Z}_+^2$. The base of such an rectangle will be a k-block, for some k. Also, the rectangles will be very elongated on the vertical direction, meaning that the height is a stretched exponential function of the length of the base. First fix

$$\mu \in \left(\frac{1}{\gamma}, 1\right) \tag{3.31}$$

and define recursively the sequence of heights $(H_k)_{k\in\mathbb{Z}_+}$ by

$$H_0 = 100 \text{ and } H_k = 2\lceil \exp(L_k^{\mu}) \rceil H_{k-1}, \text{ for } k \geqslant 1.$$

The choice $H_0 = 100$ is arbitrary and we could have used any other positive integer.

Recall of the crossing events defined in (1.2) and (1.3). For $i, j, k \in \mathbb{Z}_+$ consider the events (see Figure 3.3).

$$C_{i,j}^k = \mathfrak{C}_h\Big((I_i^k \cup I_{i+1}^k) \times \big[jH_k, (j+1)H_k \big) \Big) \text{ and} \tag{3.32}$$

$$D_{i,j}^k = \mathfrak{C}_v\Big(I_i^k \times \big[jH_k, (j+2)H_k\big)\Big). \tag{3.33}$$

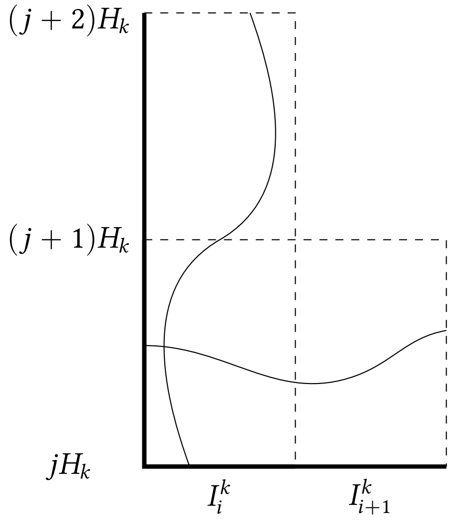

Figure 3.3: Illustration of the occurrences of events $C_{i,j}^k$ and $D_{i,j}^k$.

Now set

$$q_k(\lambda, i, j) = \max\left\{ \max_{\substack{\Lambda;\, I_i^k \text{ and} \\ I_{i+1}^k \text{are good}}} \mathbb{P}_\lambda^\Lambda(\{C_{i,j}^k\}^c), \max_{\substack{\Lambda;\, I_i^k \text{ is} \\ \text{good}}} \mathbb{P}_\lambda^\Lambda(\{D_{i,j}^k\}^c) \right\}.$$

And note that for any $k \in \mathbb{Z}_+$,

$$q_k(\lambda) := q_k(\lambda, 0, 0) = q_k(\lambda, i, j), \text{ for any } i, j \in \mathbb{Z}_+. \tag{3.34}$$

The next lemma will guarantee that the crossing events above have high probability, since the bases of these rectangles are good blocks. Using this lemma, we will construct an infinite cluster with positive probability, by taking intersections of these rectangles. But before stating it, note that by (3.31), we have $\gamma\mu - \gamma < 0$ and so we can choose

$$\beta \in (\gamma\mu - \gamma + 1, 1). \tag{3.35}$$

Lemma 3.6. *There are $c_3 = c_3(\gamma, L_0, \mu, \beta) \in \mathbb{Z}_+$ sufficiently large and $\lambda = \lambda(\gamma, L_0, \mu, \beta, c_3) > 0$ sufficiently small such that*

$$q_k(\lambda) \leqslant \exp\Big(-L_k^\beta\Big), \textit{ for any } k \geqslant c_3.$$

To prove the previous lemma it suffices to prove the following auxiliary lemmas:

Lemma 3.7. *Let $\lambda < 1$, then there is $c_4 = c_4(\gamma, L_0, \mu, \beta) \in \mathbb{Z}_+$, such that for all $k \geqslant c_4$ we have the implication*

$$q_k(\lambda) \leqslant \exp\left(-L_k^\beta\right) \Rightarrow \mathbb{P}_\lambda^\Lambda(C_{0,0}^{k+1}) \leqslant \exp\left(-L_{k+1}^\beta\right),$$

for all environment Λ satisfying that the blocks I_0^{k+1} and I_1^{k+1} are good.

Lemma 3.8. *There is $c_5 = c_5(\gamma, L_0, \mu, \beta) \in \mathbb{Z}_+$, such that for all $k \geqslant c_5$ we have the implication*

$$q_k(\lambda) \leqslant \exp\left(-L_k^\beta\right) \Rightarrow \mathbb{P}_\lambda^\Lambda(D_{0,0}^{k+1}) \leqslant \exp\left(-L_{k+1}^\beta\right),$$

for all environment Λ satisfying that the block I_0^{k+1} is good.

Now we give the proofs of the above lemmas.

Proof of Lemma 3.6. Lemmas 3.7 and 3.8 imply that it is enough to set $c_3 = \max\{c_4, c_5\}$ and choose $\lambda = \lambda(\gamma, L_0, \mu, \beta, c_3) > 0$ small enough so that $q_{c_3}(\lambda) \leqslant \exp\left(-L_{c_3}^\beta\right)$. □

Proof of Lemma 3.7. Fix an environment Λ such that I_0^{k+1} and I_1^{k+1} are good blocks. Both $(k+1)$-blocks can be formed by up to two (consecutive) bad k-blocks. Although the probability of crossing a bad k-block is small, this will be compensated for giving a lot of opportunities. For this, divide the height H_{k+1} into bands of size $2H_k$, and see if there are crossings in these bands (see Figure 3.4). Recall the notations (1.1), (1.2) and consider the events

$$G_j = \mathfrak{C}_h\Big(R\big([0, 2L_{k+1}) \times [jH_k, (j+2)H_k)\big)\Big)$$

and note that $\{C_{0,0}^{k+1}\}^c \subseteq \cap_j G_{2j}^c$, where $0 \leqslant j \leqslant \lceil \exp\left(L_{k+1}^\mu\right) \rceil - 1$. This along with the independence and invariance of events G's implies that

$$\mathbb{P}_\lambda^\Lambda\left(\{C_{0,0}^{k+1}\}^c\right) \leqslant \mathbb{P}_\lambda^\Lambda(G_0^c)^{\lceil \exp\left(L_{k+1}^\mu\right) \rceil} \leqslant \mathbb{P}_\lambda^\Lambda(G_0^c)^{\exp\left(L_{k+1}^\mu\right)}. \tag{3.36}$$

To obtain the crossing G_0 we will intercept the crossing events C's and D's whose bases are good k-blocks that form I_0^{k+1} and I_1^{k+1}, plus the crossing of the bad k-blocks, which we shall denote by B_0 and B_1 (with respect to I_0^{k+1} and I_1^{k+1} respectively), see Figure 3.5.

Now we formally define the events B_l for $l \in \{0, 1\}$. If all the k-blocks that form I_l^{k+1} are good, define $B_l = \varnothing$. Otherwise denote by j_l the lowest index of a bad k-block that forms I_l^{k+1}

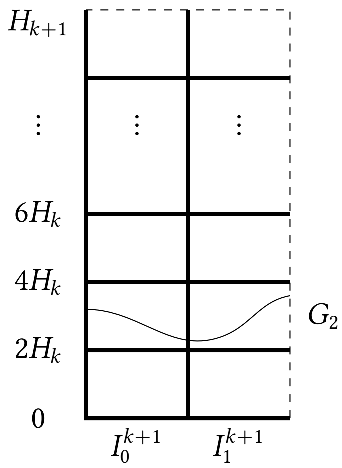

Figure 3.4: Illustration of the event G_2, implying in the occurrence of $\{C_{0,0}^{k+1}\}^c$.

and set the interval

$$I_l^* = (I_{j_l-1}^k \cup I_{j_l}^k \cup I_{j_l+1}^k \cup I_{j_l+2}^k) \cap (I_0^{k+1} \cup I_1^{k+1}) \subseteq \mathbb{Z}_+.$$

Note that the interval I_l^* contains all the bad k-blocks that form I_l^{k+1}, plus the previous k-blocks and the posterior one (as long as they are contained in $I_0^{k+1} \cup I_1^{k+1}$). And set

$$B_l = \Big\{ \text{ all edges of the form} \big\{(m,0),(m+1,0)\big\} \text{ with } m \in I_l^* \text{ are open } \Big\}.$$

Since $\lambda < 1$, and by the definition of B_0 and B_1, we have

$$\mathbb{P}_\lambda^\Lambda(B_0 \cap B_1) \geqslant \exp(-8\lambda L_k) \geqslant \exp(-8L_k). \tag{3.37}$$

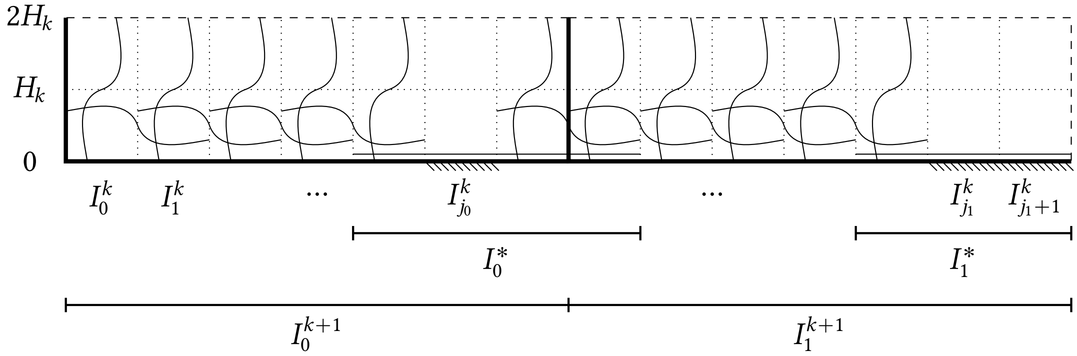

Figure 3.5: Illustration of the occurrence of the events C's, D's, B_0 and B_1 that imply in the occurrence of G_0. $I_{j_0}^k$, $I_{j_1}^k$ and $I_{j_1+1}^k$ correspond to bad k-blocks.

Note that

$$\Bigg(\bigcap_{\substack{i;\ I_i^k, I_{i+1}^k \\ \text{are good}}} C_{i,0}^k\Bigg) \cap \Bigg(\bigcap_{\substack{j;\ I_j^k \text{ is} \\ \text{good}}} D_{j,0}^k\Bigg) \cap B_0 \cap B_1 \subseteq G_0 \tag{3.38}$$

where $0 \leqslant i, j \leqslant 2\lfloor L_k^{\gamma-1} \rfloor - 1$. Since the events B's, C's and D's are increasing events, it follows from the FKG inequality, (3.34), (3.37) and (3.38) that

$$\begin{aligned} \mathbb{P}_\lambda^\Lambda(G_0) &\geqslant (1 - q_k(\lambda))^{4\lfloor L_k^{\gamma-1} \rfloor} \exp(-8L_k) \\ &\geqslant (1 - 4L_k^{\gamma-1} q_k(\lambda)) \exp(-8L_k). \end{aligned} \tag{3.39}$$

Now suppose $q_k(\lambda) \leqslant \exp(-L_k^\beta)$ and let $c_6 = c_6(L_0, \gamma, \beta) \in \mathbb{Z}_+$ sufficiently large such that for $k \geqslant c_6$ we have

$$1 - 4L_k^{\gamma-1} q_k(\lambda) \geqslant 1 - 4L_k^{\gamma-1} \exp\left(-L_k^\beta\right) \geqslant 1/2. \tag{3.40}$$

By (3.36), (3.39) and (3.40)

$$\begin{aligned} \frac{\mathbb{P}_\lambda^\Lambda\left(\{C_{0,0}^{k+1}\}^c\right)}{\exp\left(-L_{k+1}^\beta\right)} &\leqslant \exp\left(L_{k+1}^\beta\right)\left(1 - \frac{\exp(-8L_k)}{2}\right)^{\exp\left(L_{k+1}^\mu\right)} \\ &\leqslant \exp\left(L_{k+1}^\beta - \exp\left(-8L_k + L_{k+1}^\mu - \log 2\right)\right) \\ &\overset{(3.27)}{\leqslant} \exp\left(L_k^{\gamma\beta} - \exp(-8L_k + 0.66^\mu L_k^{\gamma\mu} - \log 2)\right), \end{aligned}$$

where in the second inequality we use that $1 - x \leqslant \exp(-x)$.

Since $\gamma\mu > 1$, by (3.31), we can take $c_4 = c_4(\gamma, L_0, \mu, \beta, c_6) \geqslant c_6$ sufficiently large such that for any $k \geqslant c_4$ the above inequality becomes at most 1. Thus proving the lemma. □

Proof of Lemma 3.8. Fix an environment Λ such that I_0^{k+1} is good. We will estimate $\mathbb{P}_p^\Lambda(D_{0,0}^{k+1})$ using a Peierls-type argument in a renormalized lattice. Each rectangle $I_i^k \times \left[jH_k, (j+1)H_k\right)$ will correspond to a vertex (i, j) in this renormalized lattice. This renormalized lattice is then just the $\mathbb{Z}_+^2$ lattice and the vertex $(i, j) \in \mathbb{Z}_+^2$ is declared open if the event $C_{i,j}^k \cap D_{i,j}^k$ occurs, see Figure 3.6. This gives rise to a dependent percolation process in the the renormalized lattice.

Since I_0^{k+1} is good, either I_i^k is good for every $i \in \left\{0, 1, \dots, \left\lfloor \frac{1}{2}\lfloor L_k^{\gamma-1} \rfloor \right\rfloor - 1\right\}$ or I_i^k is good for every $i \in \left\{\left\lfloor \frac{1}{2}\lfloor L_k^{\gamma-1} \rfloor \right\rfloor + 1, \dots, \lfloor L_k^{\gamma-1} \rfloor - 1\right\}$. Assume without loss of generality that the former holds and define

$$L := \left\lfloor \tfrac{1}{2} \lfloor L_k^{\gamma-1} \rfloor \right\rfloor - 1. \tag{3.41}$$

Consider the rectangle

$$R = R\Big(\big[0, L\big) \times \big[0, 4\lceil \exp\big(L_{k+1}^{\mu}\big) \rceil \big] \Big),$$

and the event $C_v(R)$ that this rectangle is crossed vertically. Note that

$$\mathbb{P}_p^{\Lambda}(D_{0,0}^{k+1}) \geqslant \mathbb{P}\big(C_v(R)\big).$$

Now we use the Peierls argument: suppose that the event $C_v(R)$ does not occur. Then there exists a sequence of distinct vertices $(i_0, j_0), (i_1, j_1), \ldots, (i_n, j_n)$ in R such that

1. $\max\big\{ |i_l - i_{l-1}|, |j_l - j_{l-1}| \big\} = 1,$
2. $(i_0, j_0) \in \{0\} \times \big[0, 4\lceil \exp\big(L_{k+1}^{\mu}\big) \rceil \big]$ and $(i_n, j_n) \in \{L\} \times \big[0, 4\lceil \exp\big(L_{k+1}^{\mu}\big) \rceil \big]$ and
3. (i_k, j_k) is closed for every $k = 0, \ldots, n$.

Note that there are at most $4\lceil \exp\big(L_{k+1}^{\mu}\big) \rceil 8^n$ sequences with $n+1$ vertices that satisfy 1. and 2. Also, the probability that a vertex of R be open is at least

$$1 - 2q_k(\lambda) \geqslant 1 - 2\exp\Big(-L_k^{\beta}\Big).$$

By the geometry of the crossing events in the original lattice, for any $(i, j) \in \mathbb{Z}_+^2$, the event $\{(i, j) \text{ is open}\}$ in the renormalized lattice depends on $\{(i', j') \text{ is open}\}$ for, at most 7 distinct vertices (i', j') (see Figure 3.6). Therefore, for every set containing $n+1$ vertices, there are at least $\lfloor n/7 \rfloor$ vertices whose states are mutually independent.

Therefore,

$$\begin{aligned}
\mathbb{P}\big(C_v(R)^c\big) &\leqslant \sum_n \mathbb{P}(\text{there is a sequence of } n+1 \text{ vertices satisfying 1., 2. and 3.}) \\
&\leqslant \sum_{n \geqslant L} 4\lceil \exp\big(L_{k+1}^{\mu}\big) \rceil 8^n \Big(2\exp\Big(-L_k^{\beta}\Big)\Big)^{\lfloor n/7 \rfloor} \\
&\leqslant 4\lceil \exp\big(L_{k+1}^{\mu}\big) \rceil \sum_{n \geqslant L} \exp\Big(n \ln 8 + \lfloor n/7 \rfloor \ln 2 - \lfloor n/7 \rfloor L_k^{\beta}\Big) \\
&\overset{(3.41)}{\leqslant} c_7 \exp\Big(L_{k+1}^{\mu} - c_8 \cdot L_k^{\beta+\gamma-1}\Big),
\end{aligned}$$

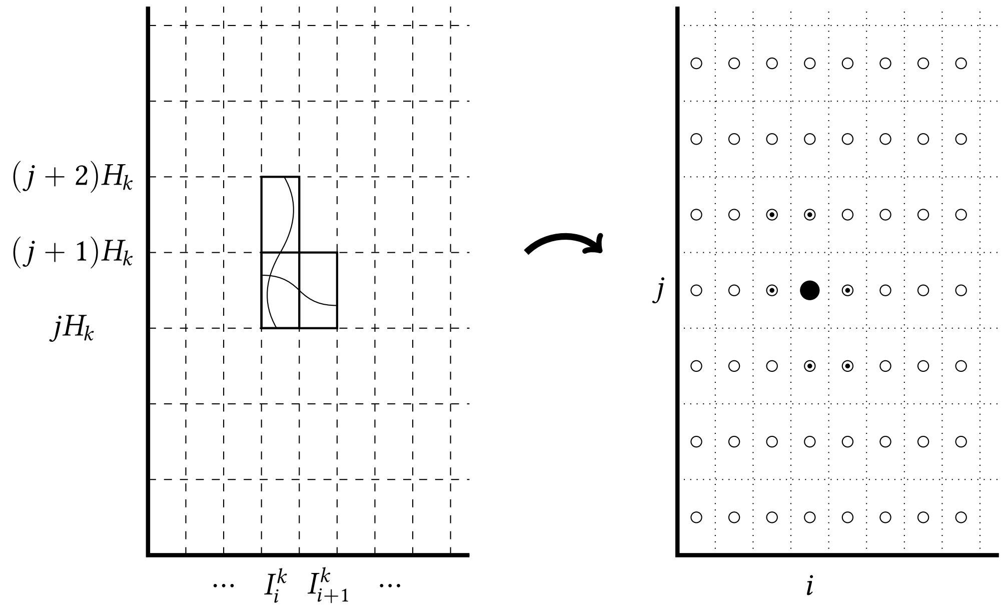

Figure 3.6: On the left, we illustrate the occurrence of the event $C_{i,j}^k \cap D_{i,j}^k$ on the original lattice. On the right, we depict the renormalized square lattice where the circles represent the sites. The occurrence of $C_{i,j}^k \cap D_{i,j}^k$ in the original lattice implies that the site (i, j) (represented as a black circle) is open in the renormalized lattice. The state of the site (i, j) only depends on the state of the other six sites represented as circles with a dot inside.

for some $c_7 = c_7(\gamma, L_0, \beta) > 0$ and $c_8 = c_8(\gamma, L_0, \beta) > 0$ sufficiently large.

Therefore,

$$\frac{\mathbb{P}_p^\Lambda(\{D_{0,0}^{k+1}\}^c)}{\exp\left(-L_{k+1}^\beta\right)} \leqslant c_7 \exp\left(L_k^{\gamma\mu} + L_k^{\gamma\beta} - c_8 L_k^{\beta+\gamma-1}\right) \tag{3.42}$$

It follows from the choice of β in (3.35), that

$$\beta + \gamma - 1 > \max\{\gamma\beta, \gamma\mu\}.$$

The proof now follows by choosing $c_5 = c_5(\gamma, L_0, \mu, \beta)$ sufficiently large so that the right-hand side of the (3.42) is less than 1 whenever $k \geqslant c_5$. $\square$

3.4 Proof of Theorem 1.2

Proof of Theorem 1.2. In the hypotheses of Theorem 1.2, the random variable ξ assumes values in $\mathbb{R}$. It turns out that, it is enough to consider the case where ξ is a positive integer-valued, aperiodic random variable. In fact, suppose that Theorem 1.2 holds whenever ξ has support on

$\mathbb{Z}$ and be aperiodic. Now let ξ be any positive random variable such that $\mathbb{E}(\xi^\eta) < \infty$ for a given $\eta > 1$. Let

$$m = \text{gcd}\left\{k \in \mathbb{Z}_+^*;\ \mathbb{P}(\lceil \xi \rceil = k) \neq 0\right\}$$

and consider the positive integer-valued, aperiodic random variable $\xi' = \lceil \xi \rceil / m$. Note that $\mathbb{E}\big((\xi')^\eta\big) < \infty$ so that for every λ sufficiently large, $\mathbb{P}_\lambda^{\Lambda'}(o \leftrightarrow \infty) > 0$ for $\upsilon_{\xi'}$- almost every environment Λ'. Now if Λ is distributed according to υ_ξ, $\mathbb{P}_{\lambda/m}^{\Lambda}$ dominates stochastically $\mathbb{P}_\lambda^{\Lambda'}$ as it can be seen by a simple coupling argument. Therefore, $\mathbb{P}_{\lambda/m}^{\Lambda}(o \leftrightarrow \infty)$ is also strictly positive. In view of this fact we assume from now on that ξ is positive integer-valued and aperiodic. So we can use all the results developed in Section 3.3.

By Lemma 3.5

$$\upsilon_\xi^\rho\Bigg(\bigcup_{0 \leqslant i \leqslant \lfloor L_k^{\gamma-1} \rfloor - 1} \{I_i^k \text{ is bad}\}\Bigg) \leqslant L_k^{\gamma-1} L_k^{-\alpha} \overset{(3.31)}{\leqslant} L_k^{-\frac{\alpha}{2}}.$$

Since $\sum_k L_k^{-\alpha/2} < \infty$, it follows from Borel-Cantelli Lemma that for (υ_ξ^ρ)-almost every environment Λ there is $c_9 = c_9(\Lambda, c_3) > c_3$ such that, for every $k \geqslant c_9$, all the k-blocks that form the first $(k+1)$-block are good. Now fix Λ one of these environments.

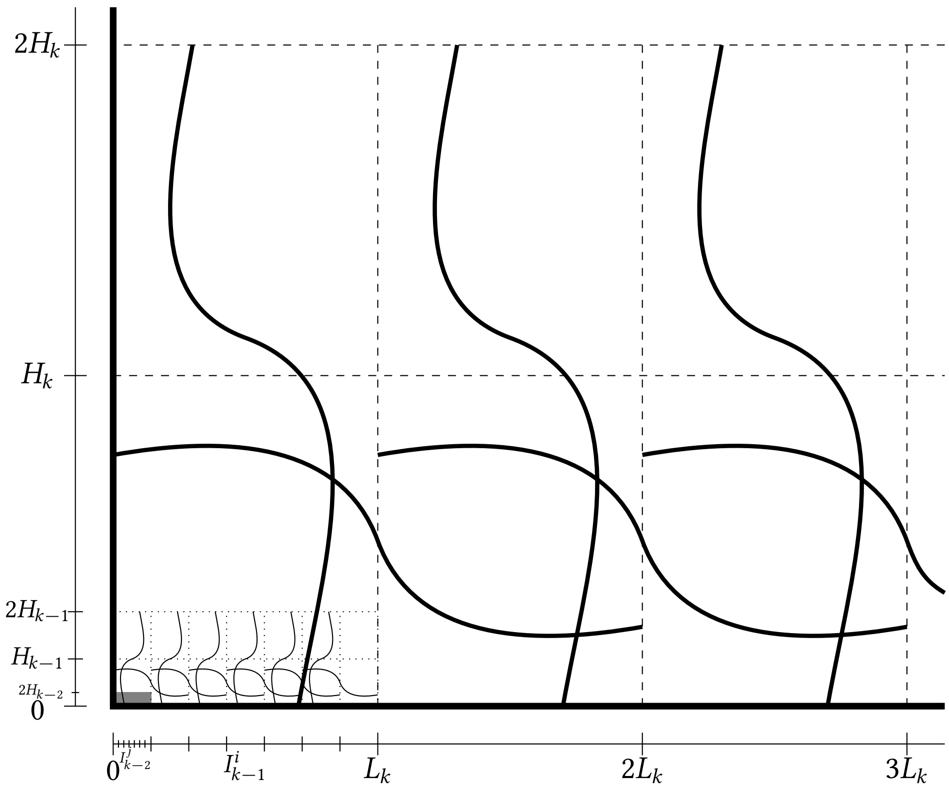

Figure 3.7: Representation of the intersections of events C's e D's, showing the existence of the infinite cluster.

Recall the definitions (3.32) and (3.33) and note that

$$\bigcap_{k\geqslant c_{10}}\left(\bigcap_{i}(C_{i,0}^{k}\cap D_{i,0}^{k})\right)\subseteq\{\text{there is an infinite cluster}\},$$

where $i\in\left\{0,1,\ldots,\lfloor L_k^{\gamma-1}\rfloor-2\right\}$ and c_{10} is any positive integer (see Figure 3.7). It follows from FKG inequality and from (3.34) that

$$\begin{aligned}\mathbb{P}_{\lambda}^{\Lambda}(\text{there is an infinite cluster}) &\geqslant \prod_{k\geqslant c_{10}}(1-2q_k(\lambda))^{\lfloor L_k^{\gamma-1}\rfloor-1}\\ &\geqslant 1-\sum_{k\geqslant c_{10}}2L_k^{\gamma-1}q_k(\lambda)\\ &\geqslant 1-\sum_{k\geqslant c_{10}}2L_k^{\gamma-1}\exp\left(-L_k^{\beta}\right),\end{aligned}\tag{3.43}$$

where the last inequality follows from Lemma 3.6.

The theorem is proved by choosing $c_{10}=c_{10}(\gamma,L_0,\beta,c_9)>c_9$ sufficiently large so that the summation in (3.43) is less that 1. $\square$

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