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GUSTAVO HENRIQUE COSTA DE SOUZA

Relation between Lyapunov Exponents and Decoherence for Real Scalar Fields in de Sitter Spacetime

BELO HORIZONTE
2015

Gustavo Henrique Costa de Souza

Relaltion between Lyapunov Exponents and Decoerence for Real Scalar Fields in de Sitter Spacetime

Tese apresentada ao Programa de Pós-Graduação em Física da Universidade Federal de Minas Gerais como requisito parcial para obtenção do grau de Doutor(a) em Ciências. Área de concentração: Física.

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ATA DA SESSÃO DE ARGUIÇÃO DA 283ª TESE DE DOUTORADO DO PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA, DEFENDIDA POR GUSTAVO HENRIQUE COSTA DE SOUZA, orientado pelo professor Marcos Donizeti Rodrigues Sampaio e coorientado pela professora Karen Milena Fonseca Romero, para obtenção do grau de **DOUTOR EM CIÊNCIAS, área em concentração Física**. Às 9:00 horas de três de março de dois mil e quinze, na sala 4123A do Departamento de Física da UFMG, reuniu-se a Comissão Examinadora, composta pelos professores **Marcos Donizeti Rodrigues Sampaio** (Orientador - Departamento de Física/UFMG), **Pablo Lima Saldanha** (Departamento de Física/UFMG), **Giancarlo Queiroz Pellegrino** (Departamento de Física e Matemática/CEFET), **George Emanuel Avraam Matsas** (Instituto de Física Teórica/UNESP), **Sérgio Eduardo de Carvalho Eyer Jorás** (Instituto de Física/UFRJ) para dar cumprimento ao Artigo 37 do Regimento Geral da UFMG, submetendo a Mestre **Gustavo Henrique Costa de Souza** à arguição de seu trabalho de Tese de Doutorado, que recebeu o título de **"Relation between Lyapunov exponents and decoherence for real scalar fields in de Sitter spacetime"**. Às 13:00 horas do mesmo dia, o candidato fez uma exposição oral de seu trabalho durante aproximadamente 50 minutos, seguida de arguição pela Comissão Examinadora. A seguir a Comissão Examinadora se reuniu reservadamente e, após discussão, elaborou o parecer final que conclui pela aprovação da tese.

Belo Horizonte, 03 de março de 2015.

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Candidato

Dedico este trabalho à minha adorável esposa Joyce. Joyce, seu amor foi meu pouso nas minhas incontáveis idas e vindas morro acima e morro abaixo. Momentos inquietos, em que eu precisei saber que tinha uma casa. Onde quer que eu estivesse, *você* foi esta casa! Não tenho medo do asfalto: pra onde quer que nos leve a próxima estrada, sei que terei paz, na minha linda casa.

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Gustavo de Souza

“...youth is only being in a way like it might be an animal. No, it is not just like being an animal so much as being like one of these malenky toys you viddy being sold in the streets, like little chellovecks made out of tin and with a spring inside and then a winding handle on the outside and you wind it up grrr grrr grrr and off it itties, like walking, O my brothers. But it itties in a straight line and bangs straight into things bang bang and it cannot help what it is doing. Being young is like being one of these malenky machines.”

Anthony Burgess, in Nadsat

“The essence of the beautiful is unity in variety.”

Felix Mendelssohn

“Chega dessa história de dizer que o sonho acabou. A vida é sonho!”

Waly Salomão

Resumo

Investigamos neste trabalho a relação entre instabilidade orbital e decoerência no espaço-tempo de Sitter (dS). Consideramos um modelo quadrático simples proposto por Brandenberger, Laflamme and Mijić de dois campos escalares interagentes em um *background* dS. Ele admite uma separação modo-a-modo, sendo cada modo um par de osciladores harmônicos não-autônomos acoplados. Demonstramos que o expoente de Lyapunov maximal de cada modo é igual à taxa assintótica de produção de entropia de von Neumann de cada oscilador, assumindo o vácuo como estado inicial. Isto nos permite estabelecer uma divergência logarítmica da entropia modulada pela taxa de inflação do espaço-tempo e calcular a relação no regime super-Hubble entre a geração de entropia de um oscilador e a taxa exponencial de separação orbital do sistema. A conexão entre instabilidade orbital e a decoerência do estado de um oscilador também é examinada do ponto de vista da *nonclassical depth*, uma quantidade relacionada à existência para ele de uma representação de espaço-de-fase interpretável como uma distribuição estocástica clássica. Provamos que o comportamento desta medida no regime super-Hubble é determinada pelo balanço entre o *squeezing* de 1-modo e a entropia. Neste regime, a entropia de um dado modo e a taxa exponencial de separação orbital do sistema aumentam significativamente ao se passar de acoplamentos fracos para o limite de acoplamento forte. Se este aumento for grande o suficiente para que a entropia de um oscilador cresça mais rápido que o *squeezing*, por exemplo no limite de acoplamento forte para frequências não muito elevadas, então o ruído de qualquer quadratura do estado assintótico será maior que o ruído de vácuo. Os resultados obtidos sugerem a possibilidade de que processos interagentes não-lineares simples possuindo correspondentes clássicos instáveis (no sentido de Lyapunov) ou caóticos podem gerar contribuições significativas para a classicalização de campos escalares cosmológicos em um estágio de expansão de Sitter do espaço-tempo.

Palavras-chave: Campos em espaços-tempo curvos. Decoerência. Classicalização de estados quânticos.

Abstract

We investigate the relationship between orbital instability and decoherence in de Sitter (dS) spacetime. We consider a simple quadratic toy model proposed by Brandenberger, Laflamme and Mijić of two interacting scalar fields in a dS background. It admits a modewise separation, with each mode consisting of a pair of nonautonomous coupled harmonic oscillators. We show that the (classical) maximal Lyapunov exponent of every mode equals the asymptotic rate of (quantum) von Neumann entropy production of each oscillator, assuming an initial vacuum. This allows us to establish a logarithmic divergence of the entropy modulated by the spacetime inflation rate, and to calculate the late times superhorizon relationship between entropy generation of an oscillator and the system's exponential orbit separation rate. The connection between orbital instability and the decoherence of an oscillator's state is also examined from the point of view of the nonclassical depth, a quantity that is related to the existence of a phase-space representation for it interpretable as a classical stochastic distribution. We prove that its superhorizon behavior is determined by the balance between single-mode squeezing and entropy. In this regime, the entropy of a mode and the system's exponential orbit separation rate increase significantly as one moves from the weak- to the strong-coupling limit. If this increase is large enough for the entropy of an oscillator to grow more rapidly than squeezing, for example in the strong-coupling limit for not too high frequencies, the noise of every quadrature of the asymptotic state will be larger than the vacuum noise (zero nonclassical depth). The results suggest the possibility that simple, nonlinear interacting physical processes with unstable or chaotic classical counterparts may provide an important contribution to the effectiveness of the classicalization of cosmological scalar fields during a dS stage of spacetime expansion.

Keywords: Quantum fields in curved spacetimes. Decoherence. Classicalization of quantum states.

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1 Preliminaries

1.1 Introduction and Overview

The first studies of classicalization of primordial density fluctuations used quantum optics tools in the theory of cosmological perturbations. Such fluctuations were shown to unitarily evolve from an initial vacuum into a highly squeezed vacuum state in a purely de Sitter stage of spacetime expansion. In the time-asymptotic limit, where the squeezing is large, quantum expectation values calculated from the evolved state were found to become indistinguishable from classical averages calculated from a stochastic distribution [1, 2]. However, the evolution of isolated fluctuations is isoentropic. In order to evaluate the entropy of primordial fluctuations and describe their classicalization at quantum-state level, one needs to consider environment-induced decoherence, an irreversible process in which the system of interest loses quantum coherences and increases its von Neumann entropy. Because gravity has infinite range and couples to all sources of energy, interactions with some sort of environment are unavoidable. Therefore, environmentally induced decoherence that will certainly play a crucial role in the classicalization of primordial density fluctuations must be taken into account.

The entropy increase in usual models [3–5] occurs because of dynamically generated entanglement correlations between the system and the environment, which is assumed to consist of infinite degrees of freedom. The large environment is unaccessible in its entirety, and tracing it out leads to entropy generation at system level. On the other hand, it has been shown that in Minkowski spacetimes the coupling to a small environment (consisting of one or few degrees of freedom), in the presence of classical dynamical instabilities, can result in much stronger decoherence effects [7–9]. It has been found that for such interacting systems, displaying classical Lyapunov instability, von Neumann entropy generation rates at observed system level either coincide or are favored by the positive maximal Lyapunov exponents. This type of behavior is conjectured [6, 7] to hold up to a certain level of generality, but still not much is known beyond specific examples.

In the present work, we investigate such an example in the context of the quantum-to-classical transition of a massless scalar field state over de Sitter (dS) spacetime. We will consider for this purpose a simple, solvable model proposed in [10] of two interacting massless scalar fields coupled through a bilinear derivative interaction potential. One of the fields is taken to represent the system of interest, which can be thought as any massless real scalar field producing density fluctuations during a dS stage of inflationary spacetime expansion[1]. The other field represents an unobservable environment. The action for the

[1] For definitiveness, the reader may consider inflaton fluctuations in a first-order approximation, neglecting

model is quadratic and in reciprocal space it reduces to a modewise interaction between the system of interest and the unobservable field – that is, to a collection of pairs of interacting harmonic oscillators. Thus, for each mode of the system, the environment consists of a single degree of freedom. When considered over Minkowski spacetime the composite system's classical dynamics for such an action is of course stable, with a phase-space flow consisting of bounded periodic orbits. However, over dS spacetime the system becomes nonautonomous and unstable, with a positive maximal Lyapunov exponent μ that we will calculate to be equal to the background spacetime inflation rate (i.e., exponential expansion rate) given by the Hubble parameter H.

Assuming an initial vacuum state for a composite system mode, we will evaluate the asymptotic growth rate μ_S of its entanglement of formation (EoF) – that is, the von Neumann entropy of an observed system mode. We find that $\mu = \mu_S$, allowing us to calculate a logarithmic divergence for the EoF modulated by the inflation rate and to demonstrate the superhorizon relationship between the entropy generation at system level and the classical exponential orbit separation rate, after entanglement fluctuations cease.

We also examine the relationship between orbital instability and decoherence from the point of view of the nonclassical depth. This is a measure of the system's state decoherence that is also sensitive to its squeezing properties and focuses on the emergence of a phase-space representation for it interpretable as a classical probability distribution, which is a very important aspect of decoherence to take into account in cosmological contexts. We will show how orbital instability influences the nonclassical depth, and thus the effectiveness of the classicalization of the system in this sense. Although the maximal Lyapunov exponent/asymptotic entropy rate is independent of model details and is given here only by the spacetime inflation rate, the actual instantaneous exponential orbit separation rate for a given mode increases monotonically as function of the coupling strength. We shall find that it is proportional to the entropy for late times, in such a way that entropy values will get larger when we shift from the weak to the strong coupling extremes. On the other hand, because Gaussianity is preserved here in the course of evolution, the nonclassical depth of a system mode's state after horizon crossing will be given by the asymptotic balance between single-mode squeezing and von Neumann entropy. This indicates that orbital instability will influence the nonclassical depth asymptotics. We will quantify this influence, by evaluating the response of the nonclassical depth when we change between the weak- and strong-coupling regimes. We will prove that although in the weak-coupling regime every mode will evolve into a highly quadrature squeezed state as expected, in the strong-coupling limit all modes of the observed field evolve into a state with noise larger than the vacuum noise in every phase-space direction (zero nonclassical depth) except for the very high-frequency sector. As we will discuss, these results

backreaction effects (on the spacetime metric).

suggest the possibility that simple, nonlinear interacting physical processes with unstable or chaotic classical dynamical counterparts may provide an important contribution to the effectiveness of the classicalization of cosmological scalar fields during a dS stage of spacetime expansion.

This dissertation is organized as follows. First, in section 1.2 ahead, we will introduce the model, present its solution in the Heisenberg representation and calculate its classical maximal Lyapunov exponent. The Heisenberg picture solution makes it very easy to write the time evolution of the composite system's Robertson-Schrödinger covariance matrix. We continue by presenting some background material on how to compute the quantities relevant to our analysis of decoherence in terms of the covariance matrix in chapter 2. Our results will be presented in chapter 3 and will be finally discussed in chapter 4.

1.2 The BLM Model

The model we consider here was first proposed and used in investigations of decoherence of cosmological perturbations by Brandenberger, Laflamme and Mijić in [10], and for this reason we call it the BLM model. It describes a bipartite system of two coupled massless fields, the system of interest ϕ and the unobservable field ψ, over a curved spacetime with metric $g_{\mu\nu}$. The action of the BLM model reads (natural units will be used throughout the text)

$$S = \int d^4x \sqrt{g} \frac{1}{2} \left[\partial_\mu \phi \partial^\mu \phi + \partial_\mu \psi \partial^\mu \psi + 2\lambda \partial_\mu \phi \partial^\mu \psi\right] , \tag{1.1}$$

where $g = -\det(g_{\mu\nu})$ and λ is the dimensionless coupling parameter normalized such that $\lambda \neq 0$, $|\lambda| < 1$. The cases $\lambda = \pm 1$ are excluded because when $\lambda = \pm 1$, (1.1) reduces to the action of a single isolated field $\tilde{\phi}_\pm = \phi \pm \psi$.

The weak- and strong-coupling limits are given by $\lambda \to 0$ and $|\lambda| \to 1$ respectively. Over the dS background, the metric is $ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2)$, $a(\eta) = -(H\eta)^{-1}$, where η is the conformal time $\eta(t) = \int_\infty^t \frac{ds}{a(s)}$. In this case we have $g = a^4$ and $a(t) = e^{Ht}$, where the Hubble parameter $H \equiv \frac{1}{a}\frac{da}{dt}$ is a constant.

Since the action is quadratic, the system is exactly solvable. In order to write its exact solution in the Heisenberg picture, we begin by expanding the fields in terms of their Fourier components, $\phi = \sum_{\vec{k}} \phi_{\vec{k}}(\eta) e^{i\vec{k}\cdot\vec{x}}$ and $\psi = \sum_{\vec{k}} \psi_{\vec{k}}(\eta) e^{i\vec{k}\cdot\vec{x}}$, in a large box of fixed comoving volume. Let $\Pi_{\phi,\vec{k}}(\eta)$ and $\Pi_{\psi,\vec{k}}(\eta)$ be the momenta conjugate to the Fourier field components. The Hamiltonian of the BLM model then reads $H = \sum_{\vec{k}} H_{\vec{k}}$, where

$$\begin{aligned} H_{\vec{k}} &= \frac{1}{2a^2(1-\lambda^2)} \left(\Pi^2_{\phi,\vec{k}} + \Pi^2_{\psi,\vec{k}} - 2\lambda \Pi_{\phi,\vec{k}} \Pi_{\psi,\vec{k}}\right) \\ &+ \frac{a^2 k^2}{2} \left(\phi^2_{\vec{k}} + \psi^2_{\vec{k}} + 2\lambda \phi_{\vec{k}} \psi_{\vec{k}}\right) . \end{aligned} \tag{1.2}$$

For simplicity, we have assumed that our comoving length units are such that the box volume in the Fourier expansion reduces to unity.

Let us define new field modes which diagonalize the Hamiltonian $H_{\vec{k}}(\eta)$, using the symplectic transformation (the subindex $\vec{k}$ was dropped to simplify the notation)

$$\begin{pmatrix}\phi_- \\ \phi_+\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix}, \begin{pmatrix}\pi_- \\ \pi_+\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}\Pi_\phi \\ \Pi_\psi\end{pmatrix}.$$

In terms of these new fields, the Hamiltonian (1.2) reads $H_{\vec{k}} = H_+ + H_-$, where

$$H_\pm = \frac{\pi_\pm^2}{2a^2(\eta)(1\pm\lambda)} + \frac{a^2(\eta)k^2(1\pm\lambda)}{2}\phi_\pm^2. \tag{1.3}$$

We can readily solve the equations of motion for these field modes:

$$\begin{aligned}\frac{d\hat{\phi}_\pm}{d\eta} &= \frac{1}{i\hbar}[\hat{\phi}_\pm, \hat{H}_\pm] = \frac{1}{a^2(1\pm\lambda)}\hat{\pi}_\pm, \\ \frac{d\hat{\pi}_\pm}{d\eta} &= \frac{1}{i\hbar}[\hat{\pi}_\pm, \hat{H}_\pm] = -a^2(1\pm\lambda)k^2\hat{\phi}_\pm.\end{aligned} \tag{1.4}$$

The general solution of the resulting second-order equation for these fields,

$$\frac{d}{d\eta}\left(a^2\frac{du}{d\eta}\right) + a^2k^2u = 0\ , \tag{1.5}$$

is a linear combination of Hankel functions, u and u^*, with

$$u(\eta) = \frac{1}{\sqrt{2}}\frac{H\eta}{k^{1/2}}e^{ik\eta}\left(1+\frac{1}{k\eta}\right)\ . \tag{1.6}$$

After some algebra, it can be shown that

$$\begin{pmatrix}\phi_\pm(\eta) \\ \pi_\pm(\eta)\end{pmatrix} = \begin{pmatrix}x(\eta) & \frac{y(\eta)}{1\pm\lambda} \\ (1\pm\lambda)z(\eta) & w(\eta)\end{pmatrix}\begin{pmatrix}\phi_\pm(\eta_0) \\ \pi_\pm(\eta_0)\end{pmatrix}, \tag{1.7}$$

is the solution of the equations of motion (1.4). The functions $x(\eta)$, $y(\eta)$, $z(\eta)$, and $w(\eta)$ are given by

$$\begin{aligned}x(\eta) = -i(u_\eta^* v_0 - u_\eta v_0^*) &= \frac{k\eta\cos k(\eta-\eta_0) - \sin k(\eta-\eta_0)}{k\eta_0}, \\ y(\eta) = i(u_\eta^* u_0 - u_\eta u_0^*) &= -\frac{H^2}{k^3}\Big(k(\eta-\eta_0)\cos k(\eta-\eta_0) - \\ &- (1+k^2\eta_0\eta)\sin k(\eta-\eta_0)\Big), \\ z(\eta) = -i(v_\eta^* v_0 - v_\eta v_0^*) &= -\frac{k}{H^2\eta_0\eta}\sin k(\eta-\eta_0), \\ w(\eta) = i(v_\eta^* u_0 - v_\eta u_0^*) &= \frac{k\eta_0\cos k(\eta-\eta_0) + \sin k(\eta-\eta_0)}{k\eta},\end{aligned}$$

where $v(\eta) \equiv a^2u'_\eta$ and the prime here stands for differentiation with respect to conformal time.

To obtain dynamically generated correlations between the system and auxiliary (unobservable) field parties at a given instant $\eta \geq \eta_0$, starting from a factorized initial condition $\hat{\rho}_T(\eta_0) = \rho_\phi(\eta_0) \otimes \rho_\psi(\eta_0)$, we have to express the nondiagonal fields at time η in terms of these diagonalized field coordinates at time η_0. The relation, in matrix form, is

$$\begin{pmatrix} \phi^\eta \\ \Pi_\phi^\eta \\ \psi^\eta \\ \Pi_\psi^\eta \end{pmatrix} = \underbrace{\begin{pmatrix} x & \frac{y}{1-\lambda^2} & 0 & \frac{\lambda y}{1-\lambda^2} \\ z & w & -\lambda z & 0 \\ 0 & \frac{\lambda y}{1-\lambda^2} & x & \frac{y}{1-\lambda^2} \\ -\lambda z & 0 & z & w \end{pmatrix}}_{=\mathbb{M}(\eta,\eta_0)} \begin{pmatrix} \phi^0 \\ \Pi_\phi^0 \\ \psi^0 \\ \Pi_\psi^0 \end{pmatrix}. \tag{1.8}$$

This equation can also be rewritten in a more compact form as

$$\boldsymbol{X}(\eta) = \mathbb{M}(\eta, \eta_0)\boldsymbol{X}(\eta_0), \tag{1.9}$$

where we have defined the vector

$$\boldsymbol{X}(\eta) = (\phi(\eta), \Pi_\phi(\eta), \psi(\eta), \Pi_\psi(\eta))^T. \tag{1.10}$$

The creation and annihilation operators for the reduced system and unobservable field are defined as

$$a_{1\vec{k}}(\eta) \;=\; \tfrac{1}{\sqrt{2}}\left(\phi_{\vec{k}}(\eta) + i\Pi_{\phi\vec{k}}(\eta)\right) = \left(a_{1\vec{k}}^\dagger(\eta)\right)^\dagger, \tag{1.11}$$

$$a_{2\vec{k}}(\eta) \;=\; \tfrac{1}{\sqrt{2}}\left(\psi_{\vec{k}}(\eta) + i\Pi_{\psi\vec{k}}(\eta)\right) = \left(a_{2\vec{k}}^\dagger(\eta)\right)^\dagger, \tag{1.12}$$

with the usual boson commutation relations $[a_{j\vec{k}}, a_{j'\vec{k}'}^\dagger] = \delta_{j,j'}\delta_{\vec{k},\vec{k}'}$, $j, j' = 1, 2$, being satisfied at any time $\eta \geq \eta_0$. Notice that (1.8) is the description of the quantum dynamics for a given mode in the Fock Space $\mathcal{F}_{\vec{k},\eta_0}$ determined by the reference vacua $|0_{\vec{k}}^\phi\rangle$ and $|0_{\vec{k}}^\psi\rangle$ annihilated respectively by $a_{1\vec{k}}(\eta_0)$ and $a_{2\vec{k}}(\eta_0)$ at time η_0.

Observe that the matrix relation (1.8) also describes the classical phase-space flow associated to the dynamics of a mode under the BLM model. That is, one just has to consider $(\phi^0\Pi_\phi^0\psi^0\Pi_\psi^0)^T$ as an initial condition in phase-space and $(\phi^\eta\Pi_\phi^\eta\psi^\eta\Pi_\psi^\eta)^T$ as the time-evolved generalized coordinates. The time evolution of the distance between two neighboring points, $d(\eta) = \|\mathbf{X}_1(\eta) - \mathbf{X}_2(\eta)\| = \|\delta\mathbf{X}(\eta)\| = \sqrt{\delta\mathbf{X}^T(\eta)\delta\mathbf{X}(\eta)}$, gives the maximal Lyapunov exponent

$$\mu = \lim_{\eta\to 0^-} \lim_{d(\eta_0)\to 0} -\frac{H}{2\ln(-H\eta)} \ln\frac{d^2(\eta)}{d^2(\eta_0)}.$$

Here, $\mathbf{X}(\eta)$ is the vector defined in Eq. (1.10). Taking into account that $\mathbf{r}(\eta)$ evolves according to Eq. (1.8), we see that the square of the distance varies as $d^2(\eta) = \delta\mathbf{X}^T(\eta_0)\mathbb{N}\delta\mathbf{X}(\eta_0)$. The matrix $\mathbb{N} = \mathbb{M}^T(\eta, \eta_0)\mathbb{M}(\eta, \eta_0)$ is a 4×4 matrix of the form

$$\mathbb{N} = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B} & \mathbb{A} \end{pmatrix},$$

where $\mathbb{A}$ and $\mathbb{B}$ are 2×2 symmetric matrices. The characteristic polynomial of $\mathbb{N}$ is the product of two quadratic polynomials; hence, the eigenvalues of $\mathbb{N}$ can be explicitly calculated. If we denote by μ_i, $i = \pm$, the roots with the largest real parts, the maximal Lyapunov exponent reads

$$\mu = \lim_{\eta \to 0^-} \max_{i=\pm} -\frac{H}{2\ln(-H\eta)} \ln \Re(\mu_i(\eta, \eta_0)),$$

where $\Re(\mu_i(\eta, \eta_0))$ stands for the real part of the eigenvalue μ_i, $i = \pm$. Making an expansion around $\eta = 0$, we obtain

$$\begin{aligned} \mu_\pm &= \frac{H^4\eta_0^2(k\eta_0 \cos k\eta_0 + \sin k\eta_0)^2 + k^4(1 \pm \lambda)^2 \sin^2 k\eta_0}{H^4 k^2 \eta_0^2\, \eta^2} \\ &= \frac{C}{\eta^2}, \quad C > 0. \end{aligned}$$

Finally, we find that the maximal Lyapunov exponent coincides with the Hubble parameter H giving the background spacetime inflation rate (i.e., exponential expansion rate),

$$\mu = \lim_{\eta \to 0^-} -\frac{H(\ln C - 2\ln(-\eta))}{2\ln(-H\eta)} = H. \tag{1.13}$$

2 Quantifying Decoherence

For the sake of completeness and to establish a notation for the sequence, we continue by briefly describing the quantities we will use to measure the decoherence process of the observed system. As mentioned in Chapter 1, we restrict our attention in this work to evaluate the time evolution of these quantities for an initial vacuum. It is a mode-wise factorized state of the form $\rho(\eta_0) = \prod_{\vec{k}} \rho_{\phi\vec{k}}(\eta_0) \otimes \rho_{\psi\vec{k}}(\eta_0)$, where the labels ϕ and ψ refer to the corresponding subsystem and where $\rho_{\phi\vec{k}}(\eta_0) = |0^{\phi}_{\vec{k}}\rangle \langle 0^{\phi}_{\vec{k}}|$, $\rho_{\psi\vec{k}}(\eta_0) = |0^{\psi}_{\vec{k}}\rangle \langle 0^{\psi}_{\vec{k}}|$. The initial state we refer to when we speak of a global vacuum initial condition for a given mode is $|0^{\phi}_{\vec{k}}\rangle \langle 0^{\phi}_{\vec{k}}| \otimes |0^{\psi}_{\vec{k}}\rangle \langle 0^{\psi}_{\vec{k}}|$. In the BLM model different modes do not interact; therefore, we can focus on a fixed $\vec{k}$. Unless it is absolutely necessary, we will omit from now on references to mode labels.

Even for an initial vacuum, it is a very difficult task to calculate the time evolution of information-theoretic quantities for an arbitrary interacting bipartite system. In the present case, however, our quadratic Hamiltonian will preserve the Gaussian character of the initial global vacuum in the course of evolution: the full composite system quantum state will be a generic two-mode squeezed vacuum at every instant. And for Gaussian states, these quantities can be written directly in terms of the Robertson-Schrödinger covariance matrix (CM), whose evolution can be easily found for the BLM model in terms of the Heisenberg picture solution to dynamics.

Having this in mind, the present chapter consists of two parts. First, we will explain in section 2.1 how to write down the evolution of the CM under the BLM model and how it can be used to evaluate the dynamics of the Gaussian state parameters characterizing the reduced system's (single-mode squeezed thermal) state. This will be followed by a discussion in section 2.2 on the interpretation and calculation in the Gaussian state setting of the specific decoherence measures that we use: the von Neumann entropy generation at reduced system level, and the non-classical depth.

2.1 The Robertson-Schrödinger Covariance Matrix and the Reduced Density Operator

Let us first recall that the CM is the real symmetric matrix Σ given in terms of second-order correlation functions for the fields and their respective momenta as

$$\Sigma_{ij} = \mathrm{tr}\left(\frac{1}{2}\{X_i, X_j\}\,\rho\right) - \mathrm{tr}\,(X_i\rho)\,\mathrm{tr}\,(X_j\rho)\,, \tag{2.1}$$

where ρ denotes the full (two-mode) state and the X_i are entries of the 4-dimensional vector $\boldsymbol{X}^T = (\phi, \Pi_\phi, \psi, \Pi_\psi) = (X_1, X_2, X_3, X_4)$. Notice that the elements of this vector satisfy the canonical commutation relations $[X_i, X_j] = i\widetilde{\Lambda}_{ij}$, where

$$\tilde{\Lambda} = \text{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_2\right). \tag{2.2}$$

Its evolution can be calculated from our Heisenberg picture solution of the model, which gives us the dynamics of the second-order correlation functions in (2.1) in a form such that the CM at time η can be written as a linear transformation of Σ at the earlier time η_0:

$$\sigma_{ij}(\eta) = \sum_{mn} f_{ij}^{mn}(\eta, \eta_0)\sigma_{mn}(\eta_0). \tag{2.3}$$

Expressions for the relevant $f_{ij}^{mn}(\eta, \eta_0)$ are given in Appendix A.

Employing Williamson's theorem [11], we can diagonalize the CM 2.1 by a linear symplectic transformation S such that $\Sigma = S\Sigma_{th}S^T$, where $\Sigma_{th} = \text{diag}\left(\nu_1, \nu_1, \nu_2, \nu_2\right)$. In the previous canonical form Σ_{th} is the CM of the thermal state $\rho_{\text{th}}(\nu_1) \otimes \rho_{\text{th}}(\nu_2)$, where

$$\hat{\rho}_{\text{th}}(\nu_i) = \frac{1}{1+\nu_i} \exp\left(\ln\left(\frac{\nu_i}{\nu_i+1}\right)\hat{a}^\dagger\hat{a}\right) \tag{2.4}$$

and where ν_i is the number of thermal excitations of the state. In the limit $\nu_i \to 0$, the thermal state $\hat{\rho}_{\text{th}}(\nu_i)$ becomes the vacuum.

One interesting consequence of Williamson's Theorem is the definition of new coordinates, given by $\boldsymbol{X}' = S\boldsymbol{X}$, which correspond to the so-called **natural orbitals**. The Stone-von Neumann theorem [12, 13] implies that these natural orbitals are related to the old coordinates by a unitary transformation $\boldsymbol{X}' = U^\dagger \boldsymbol{X} U$, which can also be interpreted as a transformation of the thermal matrix ρ_{th} such that the two-mode state ρ is given by $\rho = U\rho_{\text{th}}U^\dagger$. It was demonstrated in [14] that this unitary operator U can be parametrized at any fixed instant as

$$U = \mathcal{D}(\boldsymbol{\alpha})\widehat{U}(\boldsymbol{z}, \zeta, \boldsymbol{\omega}, \Omega), \tag{2.5}$$

where the displacement operator $\mathcal{D}(\boldsymbol{\alpha})$ take care of non-vanishing first moments and corresponds to the product

$$\mathcal{D}(\boldsymbol{\alpha}) = \prod_i \mathcal{D}(\alpha_i) = \prod_i \exp\left(\alpha_i a_i^\dagger - \alpha^* a_i\right), \quad i = 1, 2, \tag{2.6}$$

and where $\widehat{U}$ corresponds to the exponentiation of a quadratic Hamiltonian stripped of the linear terms (which are absorbed by the displacement operators). The collective labels

$$\boldsymbol{z} = (z_1, z_2), \zeta, \boldsymbol{\omega} = (\omega_1, \omega_2), \Omega$$

in $\widehat{U}$ come from the following standard form for this operator, which was also established in [14]:

$$\widehat{U}(\boldsymbol{z},\zeta,\boldsymbol{\omega},\Omega) = \mathcal{S}_1(z_1)\mathcal{S}_2(z_2)\mathcal{S}_{12}(\zeta)\mathcal{R}_1(\omega_1)\mathcal{R}_2(\omega_2)\mathcal{R}_{12}(\Omega), \tag{2.7}$$

where the unitary transformations ($i = 1, 2$)

$$\mathcal{S}_i(z_i) = e^{-\frac{1}{2}z_i a_i^\dagger a_i^\dagger + \frac{1}{2}z_i^* a_i a_i}, \tag{2.8a}$$

$$\mathcal{S}_{12}(\zeta) = e^{-\zeta a_1^\dagger a_2^\dagger + \zeta^* a_1 a_2}, \tag{2.8b}$$

$$\mathcal{R}_i(\omega_i) = e^{-i\omega_i a_i^\dagger a_i}, \tag{2.8c}$$

$$\mathcal{R}_{12}(\Omega) = e^{\Omega a_1 a_2^\dagger - \Omega^* a_1^\dagger a_2}. \tag{2.8d}$$

describe the squeezing of one ($\mathcal{S}_i(z_i)$) and two modes ($\mathcal{S}_{12}(\zeta)$), the harmonic evolution of one mode ($\mathcal{R}_i(\omega_i)$) and rotation of two modes ($\mathcal{R}_{12}(\Omega)$). It is important to have in mind that the decomposition in 2.5 refers to a fixed instant, in such a way that the parameters above depend on (conformal) time.

With the representation $\rho = U\rho_{\text{th}}U^\dagger$ for the composite system state available, we can calculate the CM Σ at a given instant in terms of the Gaussian state parameters ν_i and $\boldsymbol{z}, \zeta, \boldsymbol{\omega}, \Omega$. Then, by inversion of the resulting expressions and by making use of 2.3, we can obtain the evolution of the Gaussian state ρ in a simple way in terms of the Heisenberg picture evolution of the CM. As we shall see in a minute, the dynamics of the information-theoretic quantities we are interested admit simple formulas in terms of these Gaussian state parameters.

However, since we are interested here only in the reduced system state obtained by tracing out the second (environmental, ψ) mode, it won't be necessary to perform such a calculation for the full system's state ρ and the full CM Σ. We can focus our attention exclusively on the reduced system state, which we shall denote from now on by ρ_ϕ. This state is of course represented by a general single-mode Gaussian density operator. But since the first and second moments of Gaussian states decouple, we can assume here that one always has $\alpha = 0$, in such a way that ρ_ϕ will be at any instant a single-mode squeezed thermal state (STS):

$$\hat{\rho}_\phi = \mathcal{S}(Z)\hat{\rho}_{\text{th}}(\nu)\mathcal{S}^\dagger(Z). \tag{2.9}$$

In the previous decomposition, we have used essentially the same notations as above, but we omitted for simplicity the index 1 indicating that we are dealing with operators and parameters for the first party ϕ. The only difference is that the single-mode squeezing parameter for ρ_ϕ is denoted for the sake of clarity by $Z = |Z|e^{i\theta}$.

Now, if we split the full CM Σ in blocks as

$$\Sigma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{2.10}$$

where A, B, and C are 2×2 matrices, then the submatrix A corresponds to the CM of the STS ρ_ϕ. To calculate it, we first notice that for the thermal density operator on the right-hand side of 2.9 the average values of squared creation and squared annihilation operators are zero, and thus we have

$$\mathrm{tr}\left(\hat{a}\hat{a}^\dagger \hat{\rho}_{\mathrm{th}}(\nu)\right) = \langle \hat{a}\hat{a}^\dagger \rangle_\nu = \langle \hat{a}^\dagger \hat{a} \rangle_\nu + 1 = \nu + 1 \tag{2.11}$$

(the index 1 in annihilation/creation operators for ϕ were suppressed). Next, the average value of a function of creation and annihilation operators in the STS ρ_ϕ is

$$\begin{aligned} \langle f(\hat{a}, \hat{a}^\dagger) \rangle_{\rho_\phi} &= \mathrm{tr}\left(f(\hat{a}, \hat{a}^\dagger)\mathcal{S}(Z)\hat{\rho}_\nu \mathcal{S}^\dagger(Z)\right) \\ &= \mathrm{tr}\left(\mathcal{S}^\dagger(Z) f(\hat{a}, \hat{a}^\dagger)\mathcal{S}(Z)\hat{\rho}_\nu\right) \\ &= \mathrm{tr}\left(f(\mathcal{S}^\dagger(Z)\hat{a}\mathcal{S}(Z), \mathcal{S}^\dagger(Z)\hat{a}^\dagger\mathcal{S}(Z))\hat{\rho}_\nu\right) \\ &= \mathrm{tr}\left(f(\tilde{a}, \tilde{a}^\dagger)\hat{\rho}_{\mathrm{th}}(\nu)\right), \end{aligned} \tag{2.12}$$

where the operator $\tilde{a}$ is calculated as

$$\tilde{a} = \mathcal{S}^\dagger(Z)\hat{a}\mathcal{S}(Z) = \cosh(|Z|)a - \frac{Z}{|Z|}\sinh(|Z|)a^\dagger. \tag{2.13}$$

Finally, having obtained the transformed quadratures $\tilde{\phi}$ and $\tilde{\Pi}_\phi$, given by

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\Pi}_\phi \end{pmatrix} = \mathcal{M} \begin{pmatrix} \phi \\ \Pi_\phi \end{pmatrix} \tag{2.14}$$

for

$$\mathcal{M} = \begin{pmatrix} \cosh|Z| - \cos\theta \sinh|Z| & -\sin\theta\sinh|Z| \\ -\sin\theta\sinh|Z| & \cosh|Z| + \cos\theta\sinh|Z| \end{pmatrix},$$

and $e^{i\theta} = Z/|Z|$, we can then readily obtain the covariance matrix for ρ_ϕ as desired:

$$\begin{aligned} A = &\left(\nu + \frac{1}{2}\right) \times \\ &\begin{pmatrix} \cosh 2|Z| - \cos\theta\sinh 2|Z| & -\sin\theta\sinh 2|Z| \\ -\sin\theta\sinh 2|Z| & \cosh 2|Z| + \cos\theta\sinh 2|Z| \end{pmatrix} \end{aligned} \tag{2.15}$$

with matrix elements $A_{11} = \mathrm{tr}(\tilde{\phi}\tilde{\phi}\hat{\rho}_{\mathrm{th}}(\nu))$, etc.

It is now straightforward to invert these expressions and write the Gaussian state parameters for ρ_ϕ in terms of the CM. First, observe that D_A =det $A = \left(\nu + \frac{1}{2}\right)^2$, whence $\nu = \sqrt{D_A} - \frac{1}{2}$. Moreover, $T_A = \mathrm{tr}A = A_{11} + A_{22} = 2\sqrt{D_A}\cosh(2|Z|)$. Thus,

$$|Z| = \frac{1}{2}\log\left(\frac{T_A}{2\sqrt{D_A}} + \sqrt{\frac{T_A^2}{4D_A} - 1}\right) . \tag{2.16}$$

Finally, since

$$\frac{2A_{12}}{T_A} = -\sin\theta\tanh(2|Z|) = -\sin\theta\frac{\sqrt{\frac{T_A^2}{4D_A} - 1}}{\frac{T_A}{2\sqrt{D_A}}},$$

we obtain

$$\theta = -\text{arcsin}\left(\frac{2A_{12}}{\sqrt{T_A^2 - 4D_A}}\right) . \tag{2.17}$$

2.2 Entropy and the Non-Classical Depth

It is possible to show [14, 15] that the von Neumann entropy of a single-mode squeezed thermal state such as ρ_ϕ, say $S(\rho_\phi)$, is given in the present notations as the following function of the determinant D_A of A: $S(\rho_\phi) = F(\sqrt{D_A})$, where

$$F(x) = \left(x + \frac{1}{2}\right)\ln\left(x + \frac{1}{2}\right) - \left(x - \frac{1}{2}\right)\ln\left(x - \frac{1}{2}\right) .$$

Since the full system evolves unitarily and we assume an initial global vacuum, the full quantum state ρ will be always pure in the course of evolution. It follows that the von Neumann entropy of the system or the auxiliary field is a direct measure of the quantum entanglement between the parties. More precisely, it coincides with the so-called Entanglement of Formation (EoF), defined by $\text{EoF}(\rho) = S(\text{tr}_\psi\rho)$. This explains how the reduced system's state von Neumann entropy can be considered here as a quantifier of the decoherence process: the generation of entropy at reduced system level is in direct correspondence to a developing degree of quantum entanglement between the parties, which in the present case is the only type of quantum mechanical correlation they can share. In this way, a process of entropy generation for the observed party is to be interpreted here as signaling an increasing amount of information loss (about the composite system) upon tracing out the environment degree-of-freedom, which is characteristic of environment-induced decoherence in open quantum systems. As we will establish in the next chapter, the entropy for an initial global vacuum will always (that is, irrespective of model details) enter a logarithmic divergence regime not long after a characteristic dynamical time set by the instant of horizon-crossing for the initial comoving length scale considered.

It is convenient to notice in the context of the discussion in the previous section that the average number of excitations ν for ρ_ϕ is related to $S(\rho_\phi) = \text{EoF}(\rho)$ by the formula

$$S = F(\nu + \frac{1}{2}),$$

where F is the same function above [16]. So, the von Neumann entropy of the reduced system state is also given in terms of CM-related quantities.

Concerning the non-classicality degree of the observed system, there are several quantifiers available besides the entropy focusing on different aspects of the decoherence process. We will also evaluate here a quantifier that is sensible to the squeezing properties of the observed system and focuses on the emergence of a phase-space representation for it interpretable as a classical probability distribution, which is a very important aspect of the quantum-to-classical transition in cosmological contexts. This is in agreement, for example, with the approach taken in [5]. More precisely, the idea is to consider that the reduced system is classical when it admits a positive, regular Glauber-Sudarshan P-representation[1] for its state. In this case, it is known [19] that the state's second order quantum coherence function $g^{(2)}$ will be ≥ 1, which for a single-mode state is a drastic restriction for detectable quantum effects to show up in its excitation statistics[2]. Taking this into account, a proper quantifier of the effectiveness of the decoherence process must measure how distant the system state is from having a positive and regular P-function.

There is a non-classicality measure which performs exactly this task. It is the *non-classical depth*, introduced independently by C.T. Lee [21, 22] and N. Lütkenhaus et al. [23]. Since its use and interpretation as a decoherence measure is not as widespread as the reduced party's von Neumann entropy, the remainder of this chapter will be dedicated to a quick review on this topic.

The first step in order to understand how the non-classical depth is defined (see subsection 2.2.2) and how it measures decoherence (see subsection 2.2.3 ahead) is to have a clear picture of the Fourier transformation properties of the Glauber-Sudarshan representation of a quantum state. For this reason, we shall begin in the next subsection with a refresher on the Glauber-Sudarshan P-function.

2.2.1 The Glauber-Sudarshan Representation of a Quantum State

Following the notation introduced in chapter 1, we consider the Fock space $\mathcal{F}_{1,\vec{k},\eta_0}$ constructed by cyclic operation of the creation operator $a^\dagger_{1\vec{k}}(\eta_0)$ at time η_0 on the reference vacuum state annihilated by the corresponding annihilation operator $a_{1\vec{k}}(\eta_0)$. Since η_0 and $\vec{k}$ will be fixed and since we will be working here with quantum states of the observed party ϕ, we will drop the indexes $1, \vec{k}, \eta_0$ from now on.

[1] Here, *regular* means no more singular than a Dirac delta, which describes the P-function for a coherent state [17, 18].

[2] To obtain second-order correlation effects one has to consider multi-mode field states, which would show up in a non-trivial interacting theory. This type of effect was investigated in the context of the statistics of inflaton quanta in [20].

Consider the coherent state

$$\alpha = \mathrm{D}(\alpha)\,|0\rangle = \exp\left(\alpha a^\dagger - \alpha^* a\right)|0\rangle \ .$$

Since we have the resolution of identity

$$\hat{1} = \int_{\mathbb{C}} d^2\,\alpha \ \ |\alpha\rangle\,\langle\alpha|$$

available (from now on, we will omit the suffix $\mathbb{C}$ when integration over the whole complex plane is subtended), any density operator $\hat{\rho}$ can be written as

$$\hat{\rho} = \frac{1}{\pi^2}\int d^2\alpha \int d^2\beta \ \rho(\alpha,\beta)\,|\alpha\rangle\,\langle\beta| \ , \tag{2.18}$$

where $\rho(\alpha,\beta) = \langle\alpha|\,\hat{\rho}\,|\beta\rangle$. If $\hat{f}$ is an operator, $\left\langle \hat{f}\right\rangle = \mathrm{Tr}[\hat{\rho}\hat{f}]$ reads as

$$\left\langle \hat{f}\right\rangle = \frac{1}{\pi^2}\int d^2\alpha \int d^2\beta \ \rho(\alpha,\beta) f(\beta,\alpha) \ , \tag{2.19}$$

where $f(\alpha,\beta) = \langle\alpha|\,\hat{f}\,|\beta\rangle$.

A natural question would be if (a) there exists a function $P(\alpha)$ associated to the state $\hat{\rho}$; and (b) there exists to each normally ordered operator function $F(a^\dagger, a)$ of $a^\dagger, a$ a function $F(\alpha^*, \alpha)$ (with possibly a non-zero antiholomorphic part), such that

$$\left\langle \hat{F}\right\rangle = \int d^2\alpha \ P(\alpha) F(\alpha,\alpha^*) \ . \tag{2.20}$$

It happens that under certain conditions (see the discussion ahead) the answer is positive, in which case the complex-variable function $P(\alpha)$ is called the **Glauber-Sudarshan phase-space representation (or distribution)**, or sometimes simply the **P-function**, for the state $\hat{\rho}$. The term "phase-space" refers of course to the 2-space spanned by $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$, that is, the two-dimensional complex plane. When specification of a possible non-trivial antiholomorphic part in $P(\alpha)$ is desired, one usually writes $P(\alpha^*, \alpha)$.

Formally, it is common to write

$$\hat{\rho} = \int d^2\alpha \ P(\alpha')\,|\alpha'\rangle\,\langle\alpha'| \ , \tag{2.21}$$

meaning that the matrix element between $|\alpha\rangle$ and $|\beta\rangle$ for $\hat{\rho}$ is given in terms of P by

$$\langle\alpha|\,\hat{\rho}\,|\beta\rangle = \int d^2 w \ P(w)\,\langle\alpha\,|\,w\rangle\,\langle w\,|\,\beta\rangle \ . \tag{2.22}$$

Using the coherent-state overlap formulas

$$
\begin{aligned}
\langle \alpha \,|\, w \rangle &= \exp\left(\frac{1}{2}(\alpha^* w - \alpha w^*)\right) \exp\left(-\frac{1}{2}|\alpha - w|^2\right) \\
\langle w \,|\, \beta \rangle &= \exp\left(\frac{1}{2}(w^* \beta - w \beta^*)\right) \exp\left(-\frac{1}{2}|w - \beta|^2\right) \ ,
\end{aligned}
$$

equation (2.22) reads

$$
\rho(\alpha,\beta) = \int d^2w \ \exp\left(-\frac{1}{2}[|\alpha - w|^2 + |w - \beta|^2]\right) \exp\left(\frac{1}{2}[w^*(\beta - \alpha) - w(\beta - \alpha)^*]\right) P(w) \ . \tag{2.23}
$$

At least on a formal level, the P-function is expected to be normalized to unit whenever it is well-defined, because

$$
\begin{aligned}
\int d^2\alpha \ P(\alpha) &= \int d^2\alpha \ P(\alpha) \langle \alpha \,|\, \alpha \rangle = \int d^2\alpha \ P(\alpha) \sum_0^{+\infty} \langle \alpha \,|\, n \rangle \langle n \,|\, \alpha \rangle \\
&= \mathrm{Tr}\left[\int d^2\alpha \ P(\alpha) \,|\alpha\rangle \langle\alpha|\right] = \mathrm{Tr}(\hat{\rho}) = 1 \ .
\end{aligned}
$$

Although this property certainly must hold for a phase-space distribution such as our $P(\alpha)$ to be interpretable as a classical stochastic distribution in phase-space, the P-function assumes negative values in general. It follows that it must be considered as a **quasi-probability distribution**, in the same sense in which this designation is applied to the well-known Wigner quasi-probability distribution for example. Notice that the normally ordered expectation value of any regular enough operator function $\hat{F}(a^\dagger, a)$ will in fact reduce to

$$
\left\langle : \hat{F}(a^\dagger, a) : \right\rangle = \mathrm{Tr}\left[\hat{\rho} : \hat{F}(a^\dagger, a) :\right] = \int d^2\alpha \ P(\alpha) F(\alpha^*, \alpha) \ . \tag{2.24}
$$

The immediate question to be addressed at this point is how $P(\alpha)$ is calculated for a given quantum state. In order to explain how this is done, we will first recollect a few facts on the complex-variable Fourier transform of Cahill and Glauber [17, 18]. Given a complex-variable function $g(u^*, u)$, its **Cahill-Glauber transform** is defined by

$$
f(\alpha^*, \alpha) = \frac{1}{\pi^2} \int d^2u \ g(u^*, u) e^{u^*\alpha - u\alpha^*} \ . \tag{2.25}
$$

It is not difficult to see that

$$
f(\alpha^*, \alpha) = \mathcal{F}\left[g(u^*, u) e^{2i \tan(\theta_e(u,\alpha))}\right](\alpha) \ ,
$$

where $\mathcal{F}[h(u^*,u)](\alpha^*,\alpha)$ denotes the standard two-dimensional Fourier transform of $h(u^*,u)$ and $\theta_e(u,\alpha)$ is the Euclidean angle between u and α in the complex plane. As a consequence, we see that (2.25) differs from the standard Fourier transform by a c-number factor. It also follows that (2.25) will be well-defined for a given complex-valued function $g(u^*,u)$ as long it also has a well-defined standard two-dimensional Fourier transform. The usual conditions that guarantee the existence of the Fourier transform may then be applied to (2.25) as well: the integral in (2.25) converges if (i) $|g(u^*,u)|$ is integrable, (ii) $g(u^*,u)$ has a finite number of discontinuities, and (iii) $g(u^*,u)$ has bounded variation.

For the same reason, the Cahill-Glauber transform is also applicable to distributions and tempered distributions. The relevant examples for our purposes will be the Dirac delta in the complex domain and its derivatives, which are defined for test functions $\varphi(u^*,u)$ in Schwartz space by the linear functionals

$$\langle \delta, \varphi \rangle \equiv \int d^2u \; \delta(u^*,u)\varphi(u^*,u) = \varphi(0,0) \tag{2.26}$$

and

$$\left\langle \delta^{(k,l)}, \varphi \right\rangle \equiv (-1)^{k+l} \left\langle \delta, \varphi^{(k,l)} \right\rangle \tag{2.27}$$

respectively. The superscript (k,l) in (2.27) indicates k-th order differentiation with respect to u^* and l-th order differentiation with respect to u. For a generic distribution $F(u^*,u)$, the Cahill-Glauber transform is defined through Parseval's identity; specifically, given the distribution $F(u^*,u)$ acting on test functions by

$$\langle F, \varphi \rangle \equiv \int d^2u \; F(u^*,u)\varphi(u^*,u) \; , \tag{2.28}$$

we define its Fourier transform as the distribution $\widetilde{F}(z^*,z)$ given by

$$\left\langle \widetilde{F}, \varphi \right\rangle \equiv \langle F, \widetilde{\varphi} \rangle \; , \tag{2.29}$$

where $\widetilde{\varphi}(z^*,z)$ is the Cahill-Glauber transform of the complex-valued function φ as defined above.

The process of inverse Fourier transformation in the sense of Cahill and Glauber is, just as for the standard Fourier transform, completely symmetrical. That is, if $g(u^*,u)$ has a Fourier transform $f(\alpha^*,\alpha)$ in the sense (2.25), then

$$g(u^*,u) = \int d^2\alpha \; f(\alpha^*,\alpha) e^{\alpha^* u - \alpha u^*} \; . \tag{2.30}$$

In particular, by noticing that the Fourier transform of the delta function as defined above is $\widetilde{\delta}(z^*, z) = \frac{1}{\pi^2}$, we see that we the following usual representation of the delta function will hold:

$$\delta(u^*, u) = \frac{1}{\pi^2} \int d^2z \; e^{z^* u - z u^*} \; . \tag{2.31}$$

One also writes

$$\delta(\alpha - \alpha') = \frac{1}{\pi^2} \int d^2u \; e^{u^*(\alpha - \alpha') - u(\alpha - \alpha')^*} \; . \tag{2.32}$$

The Fourier transforms of derivatives of the delta function are established as follows. By definition, we have

$$\begin{aligned} \left\langle \widetilde{\delta^{(k,l)}}, \varphi \right\rangle \equiv \left\langle \delta^{(k,l)}, \widetilde{\varphi} \right\rangle &= (-1)^{k+l} \left\langle \delta, \widetilde{\varphi}^{(k,l)} \right\rangle \\ &= (-1)^{k+l} \widetilde{\varphi}^{(k,l)}(0,0) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} \widetilde{\varphi^{(k,l)}}(z^*, z) &\equiv \frac{1}{\pi^2} \frac{\partial^{k+l}}{\partial z^{*k} \partial z^l} \int d^2u \; e^{u^* z - u z^*} \varphi(u^*, u) \\ &= \frac{1}{\pi^2} \int d^2u \; (-u^k) u^{*l} \exp\left(u^* z - u z^*\right) \varphi(u^*, u) \; . \end{aligned} \tag{2.34}$$

Combining these two equations leads to

$$\left\langle \widetilde{\delta^{(k,l)}}, \varphi \right\rangle = \frac{(-1)^{k+l}}{\pi^2} \int d^2u \; (-u)^k u^{*l} \varphi(u^*, u) \; , \tag{2.35}$$

which employing the representation

$$\left\langle \widetilde{\delta^{(k,l)}}, \varphi \right\rangle \equiv \int d^2u \; \widetilde{\delta^{(k,l)}}(u^*, u) \varphi(u^*, u) \tag{2.36}$$

gives

$$\widetilde{\delta^{(k,l)}}(u^*, u) = \frac{1}{\pi^2} (-u)^k u^{*l} \, , \forall k, l = 0, 1, 2, \cdots \; . \tag{2.37}$$

Now, having the previous facts on the complex-variable Fourier transform of Cahill and Glauber (2.25) in mind, we can go back to the question of how one evaluates the P-function for a given quantum state. The following method, put forward by Mehta (see [24,25] and the references therein), is standard: start by noticing that $\rho(-u,u) \equiv \langle -u|\,\hat{\rho}\,|u\rangle$ can be written after algebraic manipulation as

$$\rho(-u,u) = e^{-|u|^2} \int d^2\alpha \; P(\alpha) e^{-|\alpha|^2} e^{\alpha^* u - \alpha u^*} \;,$$

and conclude by taking the Fourier transform that the P-function is

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi^2} \int_{\mathbb{C}} d^2u \; \left[e^{|u|^2} \rho(-u,u) \right] \exp\left(u^*\alpha - u\alpha^*\right) \;. \tag{2.38}$$

This is a very simple and practical way to evaluate the P-function. As examples of use of formula (2.38), let us calculate the P-function for two notorious states. The results will be used in the sequence. First, consider the coherent state $\hat{\rho} = |\beta\rangle\langle\beta|$. In this case,

$$\rho(-u,u) = e^{-|\beta|^2} e^{-|u|^2} e^{-|\alpha|^2} e^{-u^*\beta + u\beta^*} \;,$$

from which (2.38) gives

$$P(\alpha) = e^{|\alpha|^2} e^{-|\beta|^2} \left\{ \int d^2u \; e^{u^*(\alpha-\beta) - u(\alpha-\beta)^*} \right\} \;.$$

Absorbing c-number factors, the result is

$$P(\alpha) = \delta(\alpha - \beta) \;. \tag{2.39}$$

The coherent states are often quoted (at least in the context of harmonic oscillators) as the most classical-like quantum states. At the extreme opposite, consider now the number state $\hat{\rho} = |n\rangle\langle n|$, $|n\rangle = (a^\dagger)^n |0\rangle$, a quantum state *par excellence*. In this case one has

$$\rho(-u,u) = e^{-|u|^2} \frac{(-u^*u)^n}{n!} \;,$$

from which (2.38) gives

$$P(\alpha) = \frac{e^{|\alpha|^2}}{n!} \left\{ \frac{\partial^{2n}}{\partial\alpha^n \partial\alpha^{*n}} \left(\frac{1}{\pi^2} \int d^2u \; e^{u^*\alpha - u\alpha^*} \right) \right\} \;.$$

In terms of derivatives of the delta function, the result is

$$P(\alpha^*, \alpha) = \frac{e^{|\alpha|^2}}{n!} \delta^{(n,n)}(\alpha^*, \alpha) \;. \tag{2.40}$$

Several other examples of P-function calculations can be found in [26–28]. For very detailed evaluations of P-functions for the classes of Gaussian states discussed in this work (single-mode squeezed thermal and generic two-mode Gaussian states), see [29].

2.2.2 The Cahill R-function and the Non-Classical Depth

Another very important relation for the P-function involving Fourier transformation is that it is the inverse Fourier transform of the normally ordered characteristic function [30, 31]:

$$P(\alpha) = \frac{1}{\pi^2} \int d^2\alpha \; e^{\alpha\beta^* - \alpha^*\beta} \Phi(\beta) \; .$$

Remember that the normally ordered characteristic function is defined as

$$\Phi(\beta) = \mathrm{Tr}\left[\hat{\rho} e^{\beta a^\dagger} e^{-\beta^* a}\right] = \langle : D(\beta) : \rangle \; .$$

The term "normally ordered" stems from the fact that normal ordering is applied above to the displacement operator. The usual characteristic function corresponds to the standard, Weyl-ordered displacement operator $D(\beta)$:

$$\Phi_W(\beta) = \mathrm{Tr}\left[\hat{\rho}\left(e^{\beta a^\dagger - \beta^* a}\right)\right] = \langle D(\beta) \rangle \; .$$

It completely characterizes the density operator $\hat{\rho}$, as it is the generating function for its statistical moments. The representation of the P-function as a integral transform of characteristic functions is therefore very important conceptually, from the point of view of probability theory. If the inverse Fourier transform of the standard characteristic function was taken instead of the normally ordered one, it is not difficult to show that the resulting phase-space function would be the Wigner quasi-probability distribution [24].

As we have mentioned above, the P-function is a very powerful computational tool when the target is calculating averages of normally-ordered operator functions of annihilation and creation operators. For anti-normally ordered operator functions of $a, a^\dagger$, another phase-space distribution function is more convenient: the **Husimi Q-function**. It is defined as the inverse Fourier transform of the anti-normally ordered characteristic function:

$$\Phi_Q(\beta) = \mathrm{Tr}\left[\hat{\rho} e^{-\beta^* a} e^{\beta a^\dagger}\right] = \langle \ddagger D(\beta) \ddagger \rangle \; .$$

It can be explicitly calculated to be equal to

$$Q(\alpha) = \frac{1}{\pi^2} \langle \alpha | \, \hat{\rho} \, | \alpha \rangle \; ,$$

in such a way that $Q(\alpha)$ is essentially given by the corresponding diagonal matrix element of the density operator. As a result, it is immediate that $Q(\alpha)$ is normalized to unit and that it is always positive and bounded. The Husimi Q-function exhibits then all the characteristics that are expected from an actual probability distribution (as opposed to a *quasi*-probability distribution) in phase-space.

It is possible to show [24] that the P-function and Q-function are related by the following equation:

$$Q(\alpha^*, \alpha) = \frac{1}{\pi^2} \int d^2u\, e^{-|\alpha-u|^2} P(u^*, u) \; . \tag{2.41}$$

As noticed by Lee [22], the reason why the Q-function behave better than the P-function can then be readily understood: it is because $Q(z)$ is the result of a convolution transformation (the convolution kernel being given by $\frac{1}{\pi}\exp(-|z-u|^2)$) applied to $P(u)$. As it is known from the classical theory of integral transforms and transfer functions, the convolution operation can be seen as a moving average, which has the effect of producing a smoother output function. The integral transform in equation (2.41) is a convolution transformation with a Gaussian-like mask which increases the regularity of the output function.

In [17,18], Cahill and Glauber used (2.41) as the starting point for the definition of a continuous 1-parameter family of phase-space distributions. Following a slight modification by Lee [21], what is done is to introduce a non-negative real parameter τ into (2.41) to define a general phase-space distribution as

$$R(\alpha^*, \alpha, \tau) = \tfrac{1}{\pi\tau} \int d^2u \; \exp\left(-\tfrac{1}{\tau}|\alpha-u|^2\right) P(u^*, u) \tag{2.42a}$$

$$R(\alpha^*, \alpha, 0) = P(\alpha^*, \alpha) \tag{2.42b}$$

The function $R(z^*, z, \tau)$ is called the **Cahill R-function**. The Q-function is the R-function corresponding to $\tau = 1$. For $\tau = 0$, equation (2.42a) breaks down. In this case it is defined by hand that the corresponding R-function is the original P-function, in order to ensure that the one-parameter family $\{R(z^*, z, \tau)\}_{\tau\geq 0}$ is indeed continuous.

Of course, equation (2.42a) can also be seen as a convolution transformation, generalizing (2.41). The generalized convolution mask $\frac{1}{\pi\tau}\exp(-\frac{1}{\tau}|z-u|^2)$ is broader for larger τ, in such a way that the resulting smoothing effect on the output function is enhanced for increasing τ. It was then a fundamental observation by Lee [21] that this suggests using the parameter τ as a measure of how non-classical quantum states are. This is the idea behind the non-classical depth.

The definition of the non-classical depth goes as follows. First, if a given value of τ is large enough so that the R-function corresponding to the P-function of a given quantum state becomes acceptable as a classical phase-space distribution – that is, it is a positive-definite ordinary function and normalizable – then we say that τ completes the smoothing operation (relative to the convolution transformation (2.42a)) for the considered state. Let $\Omega(\hat{\rho})$ denote the set of all τ that will complete the smoothing operation of the P-function for $\hat{\rho}$. This set is obviously bounded from below, and the **non-classical depth of** $\hat{\rho}$ is defined by

$$\tau_m(\hat{\rho}) \equiv \inf_{\tau \in \Omega(\hat{\rho})} \tau \ . \tag{2.43}$$

From this definition, we have $\tau_m = 0$ for an arbitrary coherent state $|\beta\rangle$. This is compatible with the result we have seen that the P-function for such a state is a delta function. This is desirable, since the quantum coherent state is considered to be at the borderline of classicality. On the other hand, $\tau = 1$ will give $R \equiv Q$, which is always acceptable as a classical phase-space distribution function for a given quantum state. This establishes an upper bound for τ_m, and we have $0 \leq \tau_m \leq 1$ for any quantum state. Using formula (2.40) for the P-function of a number state, a considerably more involved calculation (see for example [21, 22]) will show that $\tau_m(|n\rangle\langle n|) = 1$ for every n. So, the non-classical depth gives a pure, number state as being as non-classical as possible.

It follows from the convolution theorem that the Fourier transform of the R-function is given by

$$\widetilde{R}(u^*, u, \tau) = e^{-\tau|u|^2} \widetilde{P}(u^*, u) \ ,$$

where $\widetilde{P}$ denotes the Fourier transform of the P-function. If $R(z^*, z, \tau)$ is a non-negative ordinary function, by noticing that $uz^* - u^*z = 2i\mathrm{Im}(uz^*)$ we can write

$$\begin{aligned} |\widetilde{R}(u^*, u, \tau)| &\leq \frac{1}{\pi^2} \int d^2z \, |e^{uz^* - u^*z}| R(z^*, z, \tau) \\ &= \frac{1}{\pi^2} \int d^2z \, R(z^*, z, \tau) = \widetilde{R}(0, 0, \tau) = 1 \ . \end{aligned}$$

This is a necessary condition for the R-function to be an ordinary distribution function. The Fourier transform of the R-function is frequently used to establish τ_m.

Another useful technique in this direction is to employ the Husimi Q-function, which is sometimes more amenable to direct calculation. It can be written as a convolution transform of the R-function as

$$Q(z^*, z) = \frac{1}{\pi(1-\tau)} \int d^2u \, \exp\left(-\frac{|z-u|^2}{1-\tau}\right) R(u^*, u, \tau) \ ,$$

from which it follows that the Fourier transform of the R-function can be written as

$$\widetilde{R}(u^*, u, \tau) = e^{(1-\tau)|u|^2} \widetilde{Q}(u^*, u) \ ,$$

where $\widetilde{Q}$ is the Fourier transform of the Q-function.

2.2.3 Effectiveness of Decoherence

Although a process of von Neumann entropy production at system level indicates decoherence as explained above, this is not the full story. The traditional set-up to investigate environment-induced decoherence in open quantum systems also consists of starting with an initial reduced system state displaying quantum mechanical features (such as, for instance, a superposition state) and establishing the existence of a dynamical time-scale after which its Wigner phase-space representation becomes positive – and thus is interpretable as a classical stochastic distribution in phase-space. Of course, this will only make sense when the Wigner function for the initial system state considered assumes negative values, which is in turn recognized as corresponding to the quantum-mechanical character of the state. For such an initial system state, the entropy generation due to the establishment of correlations with the environment is considered to result in effective decoherence (that is, in the full elimination of quantum mechanical features) only if this so-called positivity threshold exists. This is the line of reasoning that leads, for example, to the use of the volume of the negative part of the system state's Wigner function (that is, the phase-space volume of the region where the Wigner function assumes negative values) as a decoherence measure [32], with an effective classicalization process meaning that this quantity evolves to zero.

Clearly, this approach won't work whenever the initial reduced state considered is Gaussian, being even worst in the case of quadratic evolutions. This is the case for the BLM model and the initial vacuum state for ρ_ϕ that we consider. However, the problem can be avoided by using the non-classical depth. From what we saw in the previous discussion, computing the non-classical depth of the reduced system state ρ_ϕ will give us a well-defined mathematical quantifier of how distant this quantum state is from admitting a phase-space representation interpretable as a classical probability distribution. This statement must be taken with a grain of salt, because the very definition of the non-classical depth shows that it is circumscribed to the so-called s-parametrized family of phase-space distributions – that is, to the family of phase-space distributions realized by the Cahill R-function[3]. Nevertheless, using the non-classical depth is already more general than looking solely at the Wigner function (which, by the way, does belong to the s-parametrized distribution family) and its positivity, since it takes into account a broader range of possible phase-space functions.

The calculation of the non-classical depth for ρ_ϕ will be addressed in the next subsection, where the statement that it takes into account its squeezing properties, in addition

[3] Although we used τ in equation 2.42a as the parameter whose variation covers the 1-parameter family of phase-space distributions described by the R-function, the standard convention is to dub these anyway as the s-parametrized family of phase-space distributions.

to thermalization, will be justified. Before that, let us mention another interpretation of the non-classical depth that provides further clarification of its meaning as a quantifier of decoherence. According to Lee [21], the non-classical depth of a quantum state can also be seen as quantifying the robustness of its non-classical features with respect to quantum-mechanical superpositions. More precisely: it is equal to the minimum average number of excitations in a single-mode thermal state that are necessary to destroy all the non-classical effects present, by taking a quantum superposition. So, it measures the robustness of a state's non-classical features with respect to the addition of thermal noise.

The reason is the following. Given two states with P-functions $P_1(z)$ and $P_2(z)$, it was shown by Glauber in [26] that the P-function $P(z)$ of their quantum-mechanical superposition is given by the convolution product of P_1 and P_2,

$$P(z) = \frac{1}{\pi^2} \int d^2u \; P_1(z-u) P_2(u) \; .$$

Now, the P-function for a single-mode thermal state $\hat{\rho}_{\text{th}}$ whose mean number of excitations is $\langle n_{\text{th}} \rangle$ is given by [29]:

$$P_{\text{th}}(z) = \frac{1}{\langle n_{\text{th}} \rangle} \exp\left(-\frac{|z|^2}{\langle n_{\text{th}} \rangle} \right) \; .$$

So, if we consider the superposition of $\hat{\rho}_{\text{th}}$ with an arbitrary $\hat{\rho}$, we see that the P-function of this quantum state with thermal noise will be given by

$$P(z) = \frac{1}{\langle n_{\text{th}} \rangle} \frac{1}{\pi^2} \int d^2u \; \exp\left(-\frac{|z-u|^2}{\langle n_{\text{th}} \rangle} \right) P(u) \; . \tag{2.44}$$

From (2.41), notice that this is identical to the Q-function when $\langle n_{\text{th}} \rangle = 1$. This is interpreted as saying that noise with one thermal-like excitation is always enough to destroy whatever non-classical effects the state $\hat{\rho}$ might have.

Next, the R-function for the superposition state in (2.44) can be calculated to be [21]

$$R(z^*, z, \tau) = \frac{1}{\tau + \langle n_{\text{th}} \rangle} \frac{1}{\pi^2} \int d^2u \; \exp\left(-\frac{|z-u|^2}{\tau + \langle n_{\text{th}} \rangle} \right) P(u) \; , \tag{2.45}$$

in such a way that the non-classical depth of the state $\hat{\rho}$ with and without the addition of thermal noise are related by

$$\tau_m^{\text{th}} = \tau_m - \langle n_{\text{th}} \rangle \; .$$

This means that the non-classical depth of a quantum state with thermal noise decreases by an amount exactly equal to the average number of excitations present. This is then the reason for the physical meaning of the non-classical depth according to Lee mentioned above: the non-classical depth of a quantum state is equal to the minimum average number of excitations in a single-mode thermal state that are necessary to eliminate non-classical effects, by taking a quantum superposition.

2.2.4 The Non-Classical Depth of a STS and the Generalized Squeeze Variance

For a STS, the non-classical depth can be easily evaluated in terms of the CM. If $\epsilon_<$ denotes the smallest eigenvalue of the CM – called the **generalized squeeze variance** (GSV) for the corresponding state – then it can be shown [22] that its non-classical depth is

$$\tau_m = \max\left(\frac{1-2\epsilon_<}{2}, 0\right) \ . \tag{2.46}$$

For the present purposes, this formula reduces the evaluation of the non-classical depth of the observed system state ρ_ϕ in the course of evolution to keeping track of the GSV.

There is a simple expression for the GSV of a STS such as ρ_ϕ. In the previous notations, it is given by the dispersion [32]

$$\epsilon_< = \left(\nu + \tfrac{1}{2}\right) e^{-2|Z|} \ , \tag{2.47}$$

where $|Z|$ is the single-mode squeezing strength. This formula, albeit simple, will be important in our analysis in the next chapter of the effectiveness of the decoherence process for ρ_ϕ. It shows that it is the relative ratio between the quantities $\nu + \frac{1}{2}$ and $e^{-2|Z|}$, related respectively to entropy/thermalization and single-mode squeezing, which determines the non-classical depth. We must be able then to write it in terms of the CM. This is done by remembering that $|Z|$ is given by (see 2.16)

$$|Z| = \frac{1}{2}\log\left(\frac{T_A}{2\sqrt{D_A}} + \sqrt{\frac{T_A^2}{4D_A} - 1}\right) \ ,$$

which taking that

$$\nu = \sqrt{D_A} + \tfrac{1}{2}.$$

into account gives

$$\epsilon_< = \frac{2D_A}{T_A + \sqrt{T_A^2 - 4D_A}} = \frac{\sqrt{D_A}}{V_A}, \tag{2.48}$$

where as before $T_A = \mathrm{tr}A$, $D_A = \det\ A$, and where we define V_A through $2\sqrt{D_A}V_A = T_A + \sqrt{T_A^2 - 4D_A}$. The formula also shows that in our context the limiting values of the non-classical depth have the following interpretation: a value of τ_m close to the limit $\frac{1}{2}$ means that the STS ρ_ϕ is a highly-quadrature squeezed state, while $\tau_m = 0$ means that its noise is larger than the vacuum noise in every phase-space direction and no squeezed generalized quadrature exists.

For further details on the aspects of Gaussian information theory discussed in this section, we refer the reader to [32–36].

3 Results

We are now in position to present our results. We begin by the study of the entropy production at system level, followed by the analysis of the nonclassical depth/GSV.

3.1 Von Neumann Entropy Generation

As we discussed in the previous chapter, for a global vacuum initial condition the von Neumann entropy $\mathcal{S}(\rho_\phi)$ of the reduced system state (or, equivalently, the EoF of the full system state) accounts for both its degree of mixing and the amount of bipartite entanglement. We have seen that it is given by the determinant D_A of the CM A for ρ_ϕ. The calculation of $A(\eta)$ for an initial vacuum is relatively easy because, at time η_0, the only nonzero elements of the two-mode CM are those of the diagonal, all equal to 1/2. The determinant of the CM at time η is given by

$$D_{\text{vacua}}(\eta) = \frac{1 + d_2(\eta)\lambda^2 + d_4(\eta)\lambda^4 + x^2(\eta)z^2(\eta)\lambda^6}{4(1-\lambda^2)^2}, \tag{3.1}$$

where $d_2(\eta) = x^2z^2 - 2x^2w^2 + 2xyzw + 2y^2z^2 + y^2w^2$ and $d_4(\eta) = x^2w^2 + y^2z^2 - 2x^2z^2$. It is worth noticing that the entropy of entanglement increases monotonically with D_A in the interval $D_A \in [1/4, \infty)$. By performing an asymptotic expansion for $\eta \to 0^-$, we find $D_A \approx A(\eta_0; \lambda; k; H)\eta^{-2}$, where $A(\eta_0; \lambda; k; H) > 0$ is a function only of η_0, λ, k and H.

We define the asymptotic entropy generation rate as

$$\mu_S = \lim_{\eta \to 0^-} -\frac{H}{\ln(-H\eta)} S(\eta).$$

Taking into account the previous discussion we find $\mu_S = H$. In fact, for large values of D_A and $S(\rho_\phi)$, the approximations

$$S(\rho_\phi) \approx 1 + \ln\left(\sqrt{D_A} + \tfrac{1}{2}\right) \approx \tfrac{1}{2} \ln D_A$$

are good and we can write

$$\mu_S = \lim_{\eta \to 0^-} -\frac{H}{2\ln(-H\eta)} \ln D_A(\eta).$$

Taking into account the asymptotic expansion for D_A this gives us $\mu_S = H = \mu$, as claimed. Thus, the maximal Lyapunov exponent and the asymptotic entropy generation rate at reduced system level are equal.

From this equality, we see that for late times in the superhorizon regime the von Neumann entropy of the reduced system state relates to the maximal Lyapunov exponent as

$$S(\rho_\phi;\eta) \approx -\frac{\mu}{H}\ln(-H\eta)\ . \tag{3.2}$$

As the μ here is also equal to H, this establishes a logarithmic divergence for the late time entropy modulated by the Hubble parameter, $S(\rho_\phi;\eta) \approx -\ln(-H\eta)$. An exception for this behavior will be found only if $k = 0$, because $A(\eta_0;\lambda;k;H)$ vanishes in this case and we get

$$D_A(k \to 0^+) = \frac{\lambda^2\left(H^4\left(\eta^3 - \eta_0^3\right)^2 - 18\right) + 9\lambda^4 + 9}{36\left(\lambda^2 - 1\right)^2}. \tag{3.3}$$

But of course, this limit is not physical.

We thus see that although the general dynamical behavior of the EoF is modulated by the absolute value of the wavevector for earlier times, it diverges logarithmically for late times ($\eta \to 0^-$) for any $k \neq 0$, in a way that is completely frequency independent (also of any other model detail) and is modulated only by the spacetime inflation rate. Its early time behavior is markedly different for small and large values of k. In the former case, the EoF remains small and corresponds to the determinant D_A given by (3.3); in the latter case, entanglement oscillates with an amplitude which grows with k. Entanglement oscillations, which are characteristic of the subhorizon regime $k\eta > 1$, cease in the superhorizon regime $k\eta \ll 1$, when the EoF diverges asymptotically.

The result in equation (3.2) also shows that the late time entropy relates to the actual instantaneous exponential orbit separation rate $\Gamma(\eta)$ by

$$S(\rho_\phi;\eta) \approx \tfrac{1}{2}\Gamma(\eta), \tag{3.4}$$

with $\Gamma(\eta) = \ln \Re(\max_{i=\pm}\mu_i(\eta,\eta_0))$ in the notation of section 1.2. This, on the other hand, displays dependence on the model details, and for a given mode it is larger for stronger couplings. In fact, a numerical study, summarized in figure 1, shows that it increases monotonically as a function of $\lambda \in (0,1)$. The dependence in k only changes its early time oscillatory behavior. As a consequence of this relation, we see that the superhorizon regime values of the von Neumann entropy of the system mode will then be favored by the stronger orbital instability when we shift from the weak- to the strong-coupling extremes. A numerical study of the EoF showed that this is indeed the case: the EoF of course vanishes in the limit of no interaction, $\lambda = 0$, and the numerical investigation summarized in figure 2 demonstrated that it also increases with an increasing coupling constant in the interval $\lambda \in (0,1)$.

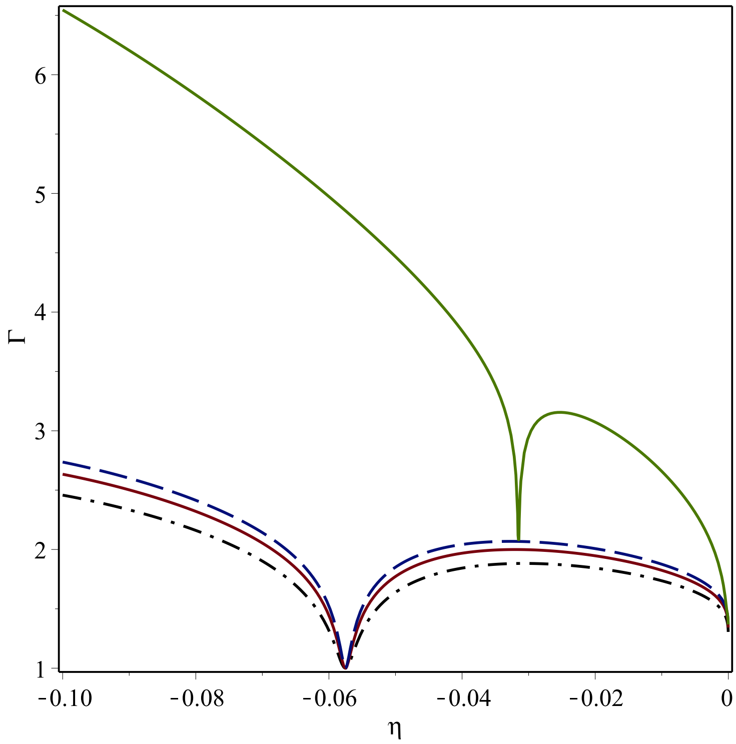

Figure 1 – Instantaneous orbit separation rate $\Gamma(\eta)$ for an initial vacuum. We scale conformal time into units such that the Hubble parameter is $H = 1$ and choose initial time $\eta_0 = -1$. Here, $k = 30$. The dot-dashed black line corresponds to $\lambda = 0.1$, the solid red line to $\lambda = 0.3$, the dashed blue line to $\lambda = 0.5$, and the solid green line to $\lambda = 0.999$.

3.2 Effectiveness of Decoherence

Let us examine the decoherence process from the point of view of the nonclassical depth. In the previous chapter, we saw that the GSV $\epsilon_<$ determines the nonclassical depth and is given by the ratio of $\sqrt{D_A}$ to $e^{2|Z|}$, which measure thermalization (entropy) and squeezing of the system. As $\epsilon_<$ ranges from 0 to $\frac{1}{2}$ the nonclassical depth varies from its maximum Gaussian state value of $\frac{1}{2}$ to its minimum, 0; for $\epsilon_<$ larger than $\frac{1}{2}$ the nonclassical depth is zero. We also saw in equation (3.4) that the values of the entropy (and thus of $\sqrt{D_A}$) increase in the superhorizon regime when the orbital instability measured by $\Gamma(\eta)$ gains in importance, as we shift from the weak- to the strong-coupling limit. This indicates that orbital instability will influence the nonclassical depth asymptotics. We will quantify this influence, by evaluating the response of the nonclassical depth when we change between the weak- and strong-coupling regimes.

From equation (2.48), we have to analyze the late time behavior of $\sqrt{D_A}$ and V_A. Their asymptotic expansions can be calculated to be of the form $\sqrt{D_A}(\eta \to 0) = -\frac{1}{\eta}P + O(1)$ and $V_A(\eta \to 0) = -\frac{1}{\eta}Q + O(1)$, where the coefficients P and Q are functions of λ, k, H. We see then that the balance between these quantities is going to be determined by P, Q.

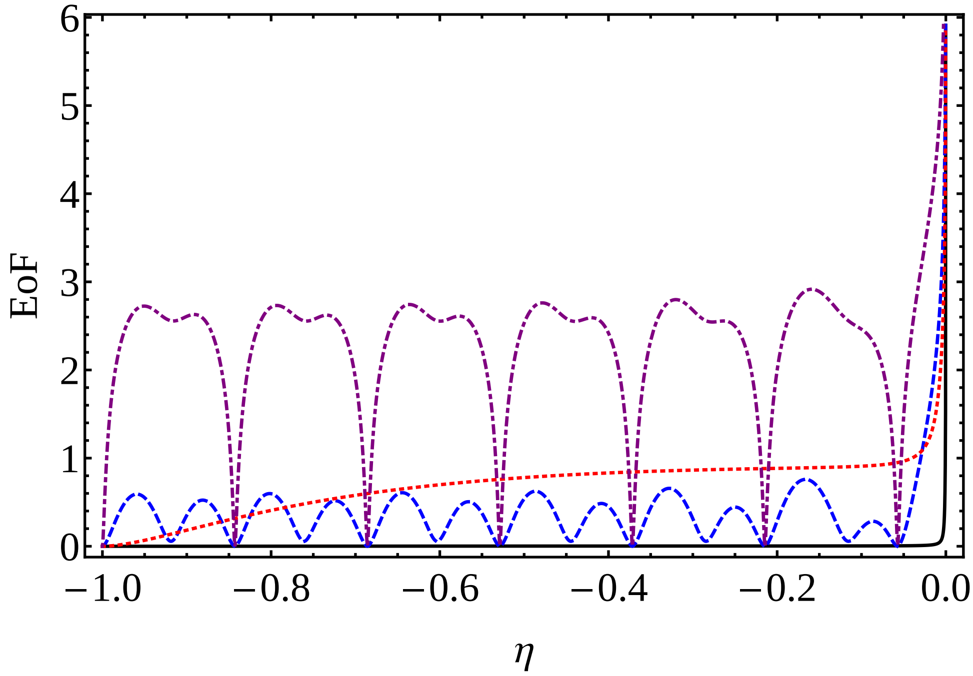

Figure 2 – Entanglement of formation (EoF) as a function of the η for an initial vacuum. We scale conformal time into units such that the Hubble parameter is $H = 1$ and choose initial time $\eta_0 = -1$. Entanglement oscillations are seen for initial states in the subhorizon regime ($k = 20$, the dot-dashed purple line corresponds to $\lambda = \frac{9}{10}$, the dashed blue line to $\lambda = \frac{1}{10}$), but not for initial states in the superhorizon regime ($k = \frac{1}{10}$, the solid black line corresponds to $\lambda = \frac{1}{10}$, the dotted red line to $\lambda = \frac{9}{10}$). The Entanglement of Formation diverges asymptotically in all cases.

The detailed form of these coefficients is very cumbersome and we will omit the details. For simplicity, we will assume henceforth that conformal time has been scaled into units such that the Hubble parameter is $H = 1$ and choose initial time $\eta_0 = -1$, corresponding to the standard cosmic time instant $t = 0$. The resulting asymptotic expansion for $\epsilon_< = \frac{\sqrt{D_A(\eta)}}{V_A(\eta)}$ (eq. (2.48)) is

$$\epsilon_<(\eta \to 0) = \frac{\lambda^2}{2k^6(1-\lambda^2)^2}\frac{G_1 + G_2}{G} , \tag{3.5}$$

where

$$G_1 = G \times (k\cos k - \sin k)^2, \tag{3.6}$$

$$G_2 = \frac{1}{2}k^4\sin^2(k)[(1-\lambda^2)^2k^4\sin^2(k) + (1+\lambda^2)(k\cos k - \sin k)^2] . \tag{3.7}$$

and

$$G = \frac{1}{2}\left[(1+\lambda^2)k^4\sin^2(k) + (k\cos k - \sin k)^2\right] . \tag{3.8}$$

With this formula available, let us begin by examining the conditions for asymptotic quadrature squeezing. From eq. (3.5), the condition $\epsilon_<(\eta \to 0) < \frac{1}{2}$ reduces to a constraint on λ and k,

$$\frac{\lambda^2}{(1-\lambda^2)^2} < k^6 \frac{G}{G_1 + G_2} \,. \tag{3.9}$$

In the weak-coupling regime $\lambda \to 0$, the right-hand side of eq. (3.9) has a positive finite limit which never exceeds $\frac{9}{10}$. Since the left-hand side tends to zero as $\lambda \to 0$, it follows that (3.9) will be satisfied for every k as long as λ is small enough. Furthermore, it is not difficult to see from the expressions for G, G_1, and G_2 that the right-hand side of (3.9) diverges with increasing k, so that taking initial length scales on the particle horizon, $k = 1$, or deeper, $k > 1$, will lead to a larger asymptotic quadrature squeezing. Thus, the typical output reduced system state in the weak-coupling limit is a highly quadrature-squeezed state. This is consistent with what is expected, for example, for isolated massless inflaton fluctuations [2].

On the other hand, the condition for absence of quadrature squeezing (zero non-classical depth) is $\epsilon_< \geq \frac{1}{2}$, which leads to

$$\frac{\lambda^2}{(1-\lambda^2)^2} \geq k^6 \frac{G}{G_1 + G_2} \,. \tag{3.10}$$

Since the left-hand side of (3.10) diverges as $\lambda \to 1$, this inequality can hold in the strong-coupling limit as long as we place a restriction in k. In fact, the limit of $k^6 G/(G_1 + G_2)$ as $\lambda \to 1$ is a positive function of k which diverges when k is equal to one of the roots of the equation $k\cos(k) = \sin(k)$ and when $k \to \infty$. Nonetheless, experimenting numerically with the coupling parameter shows that taking λ close enough to 1 guarantees (3.10) to hold up to considerably large values of k. Thus, the asymptotic reduced system state $\rho_{\phi,\infty}$ here is qualitatively very different from the weak-coupling case. As we already observed, taking the $|\lambda| \to 1$ limit increases the instantaneous exponential orbit separation rate of the model, reflecting in larger entropy values. In fact, remember that we showed that although the EoF presents a logarithmic divergence for nonzero coupling, its growth increases as λ varies from $\lambda \approx 0$ to $\lambda \approx 1$. The numerical analysis, summarized in figure 3 (see the next page), shows that for $|\lambda| \to 1$ this orbit separation rate will become strong enough to render the asymptotic state $\rho_{\phi,\infty}$ quadrature-squeezing free except for very small subhorizon length scales. In conclusion, we see that in the strong-coupling cases when $\tau_m(\rho_{\phi,\infty}) = 0$, decoherence is sufficiently effective to make the noise for the output state exceed the vacuum noise in all phase-space directions. This should be compared with the results described in [5].

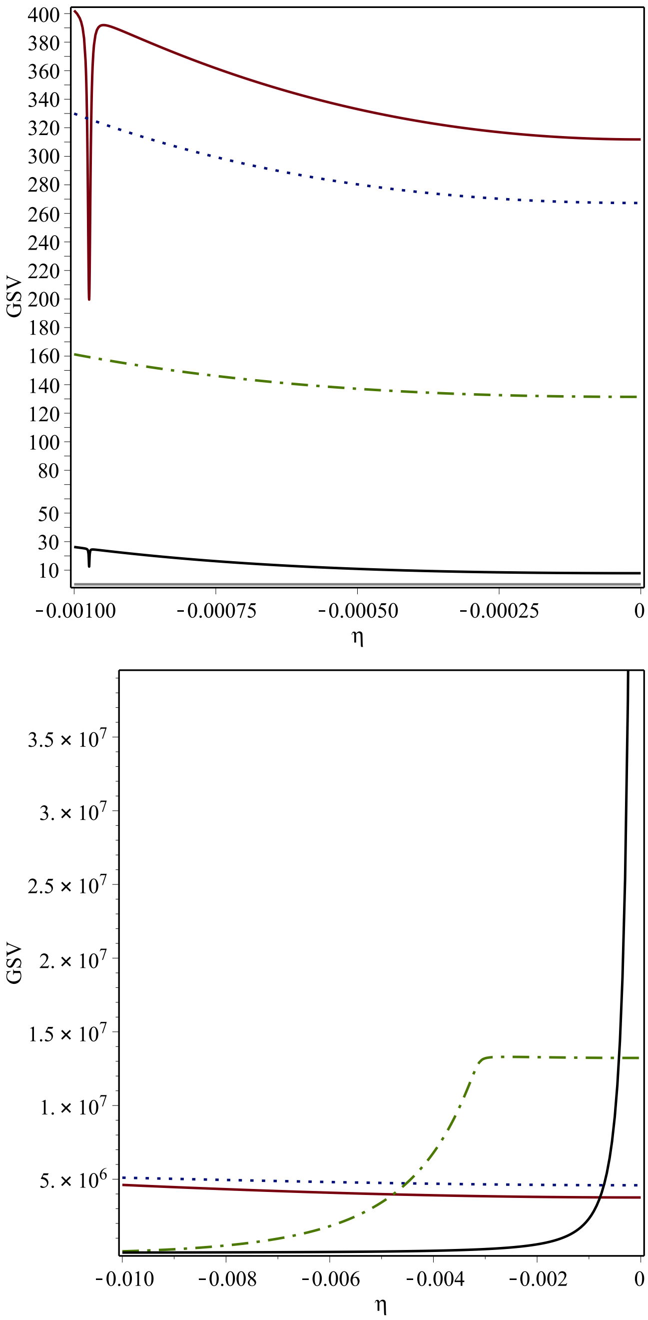

Figure 3 – GSV Asymptotics. The coupling is $\lambda = 0.999999$. (a) Top: the solid black line corresponds to $k = 1000$, with associated initial length scale being 0.1% of the Hubble radius and $\epsilon_<(\eta \to 0) > 10$. The solid red line corresponds to $k = 500$, the dotted blue line to $k = 350$, and the dot-dashed green line to $k = 350$. (b) Bottom: for initial length scales of order $\approx 1\% H$, $\epsilon_<(\eta \to 0)$ is huge, $\approx 10^6$. The dot-dashed green line corresponds to $k = 35$, the dotted blue line to $k = 40$ and the solid red line to $k = 50$. The solid black line is practically the Hubble radius, $k = 1.5$; in this case $\epsilon_<(\eta \to 0) \approx 10^{11}$.

4 Discussion

We examined in this work the relation between Lyapunov exponents and decoherence in de Sitter spacetime with spatially flat simultaneity hypersurfaces. We considered a quadratic example model, the BLM model, which reduces modewise to a model of nonautonomous coupled harmonic oscillators. Assuming an initial vacuum, we demonstrated that there is a relationship here between classical orbital instability as measured by the maximal Lyapunov exponent μ and the von Neumann entropy generation rate for the reduced subsystem, μ_S. We found that these are equal, leading to a relation between the entropy and the exponential orbit separation in the late times superhorizon regime of the form

$$S(\rho_\phi;\eta) \approx -\frac{\mu}{H}\ln(-H\eta)\ .$$

Thus, the von Neumann entropy presents in this regime a logarithmic divergence modulated by the background spacetime inflation rate given by the Hubble parameter H and proportional to the maximal Lyapunov exponent. In the present case, the above relation reduces to a logarithmic divergence of $S(\rho_\phi)$ depending only on H. But if such a relationship between entropy and μ holds in greater generality, then other simple interacting processes for the system presenting nonlinearities or other more complicated unstable classical counterparts can lead to greater entropy generation rates. We believe that the example here is then instructive in the sense that it gives an idea of how we can expect the linear relationship between entropy and the maximal Lyapunov exponent of Zurek-Paz type in Minkowski spacetime [6] to be altered in de Sitter spacetime.

The results above consider the exponential orbit separation rate only in the limit and are seen to be independent of the model parameters. But when we consider the actual instantaneous orbit separation rate, we see that it is favored for any given mode by stronger couplings. Thus, we can actually illustrate the effect of stronger orbital instability in the decoherence process of the system even within the present model. We have seen that the late times von Neumann entropy is proportional to the late times instantaneous orbit separation rate. So, is the corresponding entropy generation enough to result in classicality? In this direction, we have also analyzed the superhorizon behavior of the nonclassical depth, which measures the emergence of a phase-space representation of the system oscillator quantum state corresponding to a stochastic distribution. In the present case, it depends upon a competition between the effect of single-mode squeezing and thermalization, as in equation (2.47). This indicates that an influence on the nonclassical depth asymptotics will be present when the orbital instability measured by $\Gamma(\eta)$ gains in importance, as we shift from the weak- to the strong-coupling limit. We verified this in quantitative terms, by evaluating the response of the nonclassical depth when we change between the weak- and strong-coupling regimes. We showed that in the strong-coupling limit all modes of the observed field evolve into a state with noise larger than the vacuum noise in every phase-space direction (zero nonclassical depth) except for the very high-frequency sector, corresponding to very large k.

This analysis offers then more supporting evidence that increasing the classical orbit instability will increase the effectiveness of classicalization. It suggests that if cosmological perturbations participate in simple but realistic physical processes during a dS stage of expansion, the nonlinear interactions involved could lead to a very significant contribution to their quantum-to-classical transition. These nonlinearities can lead to very complicated dynamics, and increase the rate in which the system explores its phase-space. From what we have learned, this can have a sensitive impact on thermalization at observed system level and make quantum correlation effects very difficult to show up on the statistics of the classicalized output state (this has been the subject of several studies in the Minkowski case; see [8]). It is remarkable from an information-theoretic point of view that this can already be seen for a system of coupled harmonic oscillators over expanding spacetimes.

This type of contribution to entanglement entropy generation for cosmological fields has also been discussed, in the different context of isolated self-interacting scalar field perturbations, in [37]. The issue of the quantum-to-classical transition of cosmological perturbations is indeed a very subtle one. If quantum-mechanical features in the correlation structure of fields are to survive a period of inflationary spacetime expansion and help us understand through the cosmic microwave background sky the early history of the Universe, it is determinant that we understand in a clear way the underlying decoherence mechanisms.

Bibliography

[1] L.P. Grishchuk and Y.V. Sidorov, Phys. Rev. D **42**, 3413 (1990). Cited on page 11.

[2] C. Kiefer and D. Polarski, Adv. Sci. Lett. **2**, 164 (2009). Cited 2 times on pages 11 and 39.

[3] P. Martineau, Classical Quantum Gravity **24**, 5817 (2007). Cited on page 11.

[4] C.P. Burgess, R. Holman and D. Hoover, Phys. Rev. D **77**, 063534 (2008). Cited on page 11.

[5] C. Kiefer, I. Lohmar, D. Polarski and A. Starobinsky, Classical Quantum Gravity **24**, 1699 (2007). Cited 3 times on pages 11, 22, and 39.

[6] W.H. Zurek and J.P. Paz, Phys. Rev. Lett. **72**, 2508 (1994). Cited 2 times on pages 11 and 41.

[7] R. Blume-Kohout and W. H. Zurek, Phys. Rev. A **68**, 032104 (2003). Cited on page 11.

[8] K.M. Fonseca-Romero, J.E. Parreira, L.A.M. Souza, M.C. Nemes, and W. Wreszinski, J. Phys. A **41**, 115303 (2008). Cited 2 times on pages 11 and 42.

[9] H.R. Jauslin, O. Sapin, S. Guérin, and W. Wreszinski, J. Math. Phys. (N.Y.) **45**, 4377 (2004). Cited on page 11.

[10] R. Brandenberger, R. Laflamme and M. Mijic, Mod. Phys. Lett. A **05**, 2311 (1990). Cited 2 times on pages 11 and 13.

[11] J. Williamson, Am. J. Math. **5**8, 141 (1936); **5**9, 599 (1936); **6**1, 897 (1936). Cited on page 18.

[12] M.H.Stone, Proc. Nat. Acad. Sci. USA **16**, 172 (1930). Cited on page 18.

[13] J. von Neumann, Math. Ann. **104** (1931). Cited on page 18.

[14] G. Agarwal and H. Huang, Phys. Rev. A **49**, 52 (1994). Cited 3 times on pages 18, 19, and 21.

[15] G. Agarwal, Phys. Rev. A **3**, 828 (1971). Cited on page 21.

[16] J. Parreira, K.M. Fonseca-Romero and M.C. Nemes, Phys. Rev. D **83**, 064033 (2011). Cited on page 22.

[17] K.E. Cahill and R.J. Glauber, Phys. Rev. **177**, 1857 (1969). Cited 3 times on pages 22, 24, and 29.

[18] K.E. Cahill and R.J. Glauber, Phys. Rev. **177**, 1882 (1969). Cited 3 times on pages 22, 24, and 29.

[19] L. Davidovitch, Rev. Mod. Phys. **68**, 127 (1996). Cited on page 22.

[20] M. Giovannini, Phys. Rev. D **83**, 023515 (2011). Cited on page 22.

[21] C. Lee, Phys. Rev. A **44**, R2775 (1991). Cited 4 times on pages 22, 29, 30, and 32.

[22] C. Lee, Phys. Rev. A **45**, 6586 (1992). Cited 4 times on pages 22, 29, 30, and 33.

[23] N. Lütkenhaus and S.M. Barnnett, Phys. Rev. A **51**, 3340 (1995). Cited on page 22.

[24] W.P. Schleich, Quantum Optics in Phase-Space. Wiley, 2001. Cited 3 times on pages 26, 28, and 29.

[25] J. Garrison and R. Chiao, Quantum Optics. Oxford University Press, 2014. Cited on page 26.

[26] R.J. Glauber, Phys. Rev. A **131**, 2766 (1963). Cited 2 times on pages 27 and 32.

[27] E.C.G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963). Cited on page 27.

[28] D.F. Walls, Nature **306**, 141 (1983). Cited on page 27.

[29] M.S. Kim, F.A.M de Oliveira, P.L. Knight, Phys. Rev. A **40**, 2494 (1989). Cited 2 times on pages 27 and 32.

[30] C. Gehrke, Quantitative Characterization of Nonclassicality. Diplomarbeit, Universität Rostock, 2008. Cited on page 28.

[31] T. Kiesel, Nonclassicality in terms of the Glauber-Sudarshan representation. Diplomarbeit, Universität Rostock, 2008. Cited on page 28.

[32] A. Serafini, F. Illuminati, and S. De Siena, J. Phys. B: At. Mol. Opt. Phys. **37**, L21 (2004). Cited 3 times on pages 31, 33, and 34.

[33] B. Yurke and M. Potasek, Phys. Rev. A **36**, 3464 (1987). Cited on page 34.

[34] D. Buono, G. Nocerino, V.D'Auria, A. Porzio, S. Olivares, and M.G.A. Paris, J. Opt. Soc. Am. B **27**, A110 (2010). Cited on page 34.

[35] J. Laurat, G. Keller, J.A. Oliveira-Huguenin, C. Fabre, T. Coudreau, A. Serafini, G. Adesso, and F. Illuminati, J. Opt. Soc. Am. B **7**, S577 (2005). Cited on page 34.

[36] A. Serafini, F. Illuminati and S. de Siena, J. Phys. B **37**, L21 (2004). Cited on page 34.

[37] D. Mazur, J. Heyl. Phys. Rev. D **80**, 023523 (2009). Cited on page 42.

Appendix

APPENDIX A – Covariance Matrix Dynamics

We collect here the expressions of the functions $f^{mn}_{\vec{k}ij}(\eta, \eta_0)$ in (2.3). They have been obtained by using the Heisenberg picture solution for the BLM model expressed in equation (1.8), section 1.2. The expressions are (we drop the index $\vec{k}$ and the time argument):

$f^{11}_{11} = x^2$	$f^{12}_{11} = 2\frac{xy}{1-\lambda^2}$	$f^{14}_{11} = 2\frac{\lambda xy}{1-\lambda^2}$
$f^{22}_{11} = \left(\frac{y}{1-\lambda^2}\right)^2$	$f^{24}_{11} = 2\frac{\lambda y^2}{(1-\lambda^2)^2}$	$f^{44}_{11} = \left(\frac{\lambda y}{1-\lambda^2}\right)^2$
$f^{11}_{12} = xz$	$f^{22}_{12} = \frac{yw}{1-\lambda^2}$	$f^{34}_{12} = -\frac{\lambda^2 yz}{1-\lambda^2}$
$f^{12}_{12} = xw + \frac{yz}{1-\lambda^2}$	$f^{13}_{12} = -\lambda xz$	$f^{23}_{12} = -\frac{\lambda yz}{1-\lambda^2}$
$f^{14}_{12} = \frac{\lambda yz}{1-\lambda^2}$	$f^{24}_{12} = \frac{\lambda yw}{1-\lambda^2}$	
$f^{11}_{22} = z^2$	$f^{22}_{22} = w^2$	$f^{33}_{22} = \lambda^2 z^2$
$f^{12}_{22} = 2zw$	$f^{13}_{22} = -2\lambda z^2$	$f^{23}_{22} = -2\lambda zw$
$f^{12}_{13} = \frac{\lambda xy}{1-\lambda^2}$	$f^{13}_{13} = x^2$	$f^{14}_{13} = \frac{xy}{1-\lambda^2}$
$f^{22}_{13} = \frac{\lambda y^2}{(1-\lambda^2)^2}$	$f^{23}_{13} = \frac{xy}{1-\lambda^2}$	$f^{24}_{13} = \left(\frac{y}{1-\lambda^2}\right)^2 + \left(\frac{\lambda y}{1-\lambda^2}\right)^2$
$f^{34}_{13} = \frac{\lambda xy}{1-\lambda^2}$	$f^{44}_{13} = \frac{\lambda y^2}{(1-\lambda^2)^2}$	
$f^{11}_{14} = -\lambda xz$	$f^{13}_{14} = xz$	$f^{14}_{14} = xw - \frac{\lambda^2 yz}{1-\lambda^2}$
$f^{12}_{14} = -\frac{\lambda yz}{1-\lambda^2}$	$f^{23}_{14} = \frac{yz}{1-\lambda^2}$	$f^{24}_{14} = \frac{yw}{1-\lambda^2}$
$f^{34}_{14} = \frac{\lambda yz}{1-\lambda^2}$	$f^{44}_{14} = \frac{\lambda yw}{1-\lambda^2}$	
$f^{12}_{23} = \frac{\lambda yz}{1-\lambda^2}$	$f^{13}_{23} = xz$	$f^{14}_{23} = \frac{yz}{1-\lambda^2}$
$f^{22}_{23} = \frac{yw}{1-\lambda^2}$	$f^{23}_{23} = xw - \frac{\lambda^2 yz}{1-\lambda^2}$	$f^{24}_{23} = \frac{yw}{1-\lambda^2}$
$f^{33}_{23} = -\lambda xz$	$f^{34}_{23} = -\frac{\lambda yz}{1-\lambda^2}$	
$f^{11}_{24} = -\lambda z^2$	$f^{13}_{24} = z^2 + \lambda^2 z^2$	$f^{14}_{24} = zw$
$f^{12}_{24} = -\lambda zw$	$f^{23}_{24} = zw$	$f^{24}_{24} = w^2$
$f^{33}_{24} = -\lambda z^2$	$f^{34}_{24} = -\lambda zw$	
$f^{22}_{33} = \left(\frac{\lambda y}{1-\lambda^2}\right)^2$	$f^{23}_{33} = 2\frac{\lambda xy}{1-\lambda^2}$	$f^{24}_{33} = 2\frac{\lambda y^2}{(1-\lambda^2)^2}$
$f^{33}_{33} = x^2$	$f^{34}_{33} = \frac{xy}{1-\lambda^2}$	$f^{44}_{33} = \left(\frac{y}{1-\lambda^2}\right)^2$
$f^{12}_{34} = -\frac{\lambda^2 yz}{1-\lambda^2}$	$f^{23}_{34} = \frac{\lambda yz}{1-\lambda^2}$	$f^{24}_{34} = \frac{\lambda yw}{1-\lambda^2}$
$f^{13}_{34} = -\lambda xz$	$f^{33}_{34} = xz$	$f^{34}_{34} = xw + \frac{yz}{1-\lambda^2}$
$f^{14}_{34} = -\frac{\lambda yz}{1-\lambda^2}$	$f^{44}_{34} = \frac{yw}{1-\lambda^2}$	
$f^{11}_{44} = \lambda^2 z^2$	$f^{13}_{44} = -2\lambda z^2$	$f^{14}_{44} = -2\lambda zw$
$f^{33}_{44} = z^2$	$f^{34}_{44} = 2zw$	$f^{44}_{44} = w^2$

We observe that $f^{mn}_{21} = f^{mn}_{12}$, $f^{mn}_{43} = f^{mn}_{34}$, $f^{mn}_{31} = f^{mn}_{13}$, $f^{mn}_{32} = f^{mn}_{14}$, $f^{mn}_{41} = f^{mn}_{23}$, $f^{mn}_{42} = f^{mn}_{24}$.

Otherwise, functions f_{ij}^{mn} not appearing above are null.