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Classification of capillary CMC surfaces with symmetries in the unit ball

Brazil

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FOLHA DE APROVAÇÃO

Classificação de Hipersuperfícies CMC's Capilares com Simetrias na Bola Unitária

LUCAS CARVALHO SILVA

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Este trabalho é dedicado àqueles que buscam a Verdade.

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- Agora vamos, amadíssima esposa, pois há espera que também é caminhada.

"Não vos amoldeis às estruturas deste mundo,
mas transformai-vos pela renovação da mente,
a fim de distinguir qual é a vontade de Deus:
o que é bom, o que Lhe é agradável, o que é perfeito".
(Bíblia Sagrada, Romanos 12, 2)

"To those who do not know mathematics
it is difficult to get across a real feeling as to the beauty,
the deepest beauty, of nature.
If you want to learn about nature, to appreciate nature,
it is necessary to understand the language that she speaks in".
Richard Feynman

Resumo

Essa tese consiste de alguns resultados acerca de superfícies diferenciáveis, orientáveis, compactas, conexas, com bordo não vazio e curvatura média constante (CMC).

Na primeira parte, nós usamos o Método de Reflexão de Alexandrov para obter uma caracterização para anéis CMC, capilares, mergulhados na bola euclideana $\mathbb{B}^3$. Em especial, usando uma nova estratégia, nós apresentamos uma nova caracterização para o catenóide crítico. Precisamente, nós mostramos que sendo $\Sigma \subset \mathbb{B}^3$ um anel CMC, capilar, mergulhado em $\mathbb{B}^3$, tal que $\partial\Sigma$ é invariante sob reflexões através dos planos coordenados, então Σ deve ser rotacionalmente simétrico. Por fim, apresentamos uma nova demonstração para o Teorema de Pyo, no caso mergulhado.

Na segunda parte, nós estudamos imersões ϕ com curvatura média anisotrópica constante (CAMC), de uma variedade orientada, conexa, compacta, e com bordo não vazio, Σ, em uma região Ω cujo bordo é uma superfície de revolução. Percebemos que, diferentemente do caso clássico, as imersões CAMC, ϕ, não são necessariamente *free boundaries*. Assim, nos perguntamos quais seriam essas. Primeiramente, nós encontramos condições sobre o bordo, em seguida provamos que $\phi(\Sigma)$ deve ser um disco *flat* e, por fim, determinamos sob quais condições ele é estável.

Palavras-chave: Imersões capilares. Mergulhos capilares. Imersões anisotrópicas capilares.

Abstract

This thesis consists of some results about an orientable connected compact differentiable surface with boundary and constant mean curvature (CMC).

In the first part, we used the Alexandrov Reflection Method to obtain a characterization to embedded CMC capillary annulus in $\mathbb{B}^3$. In especial, using a new strategy, we present a new characterization to the critical catenoid. Precisely, we show that $\Sigma \subset \mathbb{B}^3$ being an embedded minimal free boundary annulus, such that $\partial\Sigma$ is invariant under reflection through of the coordinated planes, then Σ is the critical catenoid. Finally, in the case embedded, we presented a new proof for one Pyo's theorem.

In the second part we studied immersions with constant anisotropic mean curvature (CAMC) ϕ of a smooth oriented connected and compact surface Σ, such that $\partial\Sigma \neq \emptyset$, in a region Ω whose boundary is a revolution surface. Unlike the classic case, if ϕ is a CAMC immersion, it is not possible to state that ϕ is free boundary. Thus, we asked ourselves what should be the CAMC free boundaries immersions. First, we found one condition on the boundary, then we prove that $\phi(\Sigma)$ should be a flat disk and under what conditions it is stable.

Keywords: Capillary embedded. Capillary Immersion. Free boundary anisotropic Immersion.

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List of abbreviations and acronyms

ARM	Alexandrov Reflection Method.
CMC	Constant Mean Curvature.
CAMC	Constant Anisotropic Mean Curvature.

Contents

Introduction

In this work, we studied the behavior of surfaces, Σ, with constant mean curvature (CMC) and boundary $\partial\Sigma \neq \emptyset$, *properly immersed* in $\mathbb{B}^3$, i.e.,

$$\text{int}\Sigma \subset \text{int}\mathbb{B}^3 \tag{1}$$

$$\partial\Sigma \subset \partial\mathbb{B}^3 \tag{2}$$

and we suppose that $\partial\Sigma$ meet $\partial\mathbb{B}^3$ with a constant contact angle. For example, consider the disks

$$D_k = \{(x, y, z) \in \mathbb{R}^3 \; ; \; z = k\} \cap \mathbb{B}^3, \; k \in (-1, 1), \tag{3}$$

each one of this is an example from CMC surfaces, minimal surface when $k = 0$, properly immerse in $\mathbb{B}^3$ and such that ∂D_k meet $\partial\mathbb{B}^3$ with a constant contact angle $\theta_k = \arccos K$.

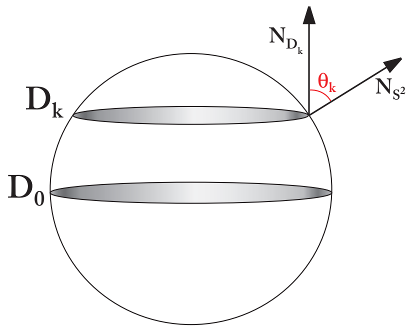

Figure 1 – To each flat disk D_k, ∂D_k meet $\partial\mathbb{B}^3$ with a constant contact angle θ_k.

Related issues to CMC surfaces, Σ, with $\partial\Sigma \neq \emptyset$, immersed in a ball, with others additional hypotheses, have been subject of study for a long time. In a classical result, due to Nitsche [1], he claims that the flat disk D_0 is the only immersed CMC disk, free boundary in a ball $\mathbb{B}^3$. Ros and Souam [2] considered an embedded CMC surface, capillary in an euclidean ball, such that $\partial\Sigma$ is contained in an open hemisphere of $\partial\mathbb{B}$ and they concluded that Σ must be of disk type.

Immersions $\phi : \Sigma^n \to B \subset \mathbb{R}^{n+1}$, such that $\partial\Sigma \neq \emptyset$, with constant mean curvature and constant contact angle $\theta = \frac{\pi}{2}$, called *free boundary CMC surface*, are known as solutions to the variational problem given by the area functional

$$\mathcal{A}(\epsilon) = \int_\Sigma d\Sigma_\epsilon. \tag{4}$$

Precisely, consider $\Phi : (-\epsilon_0, \epsilon_0) \times \Sigma^n \to B$ a *smooth variation of immersion* ϕ, i.e., $\Phi_\epsilon(p) := \Phi(\epsilon, p)$ is a smooth immersion of ϕ, $\forall\, \epsilon \in (-\epsilon_0, \epsilon_0)$, and $\Phi_0 = \phi$. In (4), $d\Sigma_\epsilon$ is the area element of Σ in the induced metric by Φ_ϵ.

A variation Φ is called *admissible* if

$$\Phi_\epsilon(\mathrm{int}\Sigma) \subset \mathrm{int}B \tag{5}$$
$$\Phi_\epsilon(\partial\Sigma) \subset \partial B \tag{6}$$

for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. Also consider the *volume function* given by

$$V(\epsilon) = \int_{[0,\epsilon]\times\Sigma} \Phi^* dV \tag{7}$$

where $\Phi^* dV$ is the pull back of canonical volume element dV. We called that the variation Φ is *volume preserving* if, $V(\epsilon) = V(0)$, $\forall\ \epsilon \in (-\epsilon_0, \epsilon_0)$.

Now, we can formalize what was said above about the relationship between free boundary CMC surfaces with non-empty boundary and the functional area shown in (4): these are critical points of the area functional, $\mathcal{A}$, for volume preserving admissible variations, is that,

$$\mathcal{A}'(0) := \left.\frac{d\mathcal{A}}{d\epsilon}\right|_{\epsilon=0} = 0, \tag{8}$$

for all variation, Φ, volume preserving admissible.

In [3], Ros and Vergasta partially classified, between the free boundary CMC immersions in a ball $\mathbb{B}^n$, those that are *stable*, i.e., the free boundary CMC immersions such that

$$\mathcal{A}''(0) \geq 0, \tag{9}$$

for all variation, Φ, normal volume preserving admissible, where *normal* means that the *variation vector*, $\left.\frac{d\Phi}{d\epsilon}\right|_{\epsilon=0} \in T_p^\perp\Sigma$, $\forall\, p \in \Sigma$. Their results were improved by [4], [5], and this problem was definitely solved by [6]. Other authors studied immersions in others ambient manifolds, as Pyo in [7] and Koiso in [8]. Below is a conjecture that motivated part of the our work and that recently received a response, but this response still needs confirmation.

Conjecture - Fraser and Li [9]: The critical catenoid is the unique properly embedded free boundary minimal annulus in $\mathbb{B}^3$, up to rotations.

There exist a parallel between the above conjecture and the

Conjecture - Lawson [9]: The Clifford Torus is the only embedded minimal torus in $\mathbb{S}^3$, up to rotations.

The Lawson's conjecture was definitively solved by Brendle in [10]. However, there was previously a partial demonstration due to Ros [11]:

Theorem 0.0.1 (Ros) *Let $\Sigma \subset \mathbb{S}^3$ be an embedded minimal torus, symmetric with respect to the coordinate hyperplanes of $\mathbb{R}^4$. Then Σ is the Clifford torus.*

In the case of the conjecture of Fraser and Li, there is an analogous result obtained by Ros, due to McGrath [9]:

Theorem 0.0.2 (McGrath) *Let $\Sigma \subset \mathbb{B}^n$, $n \geq 3$, be an embedded free boundary minimal annulus. If Σ is invariant under reflection through three orthogonal hyperplanes Π_i, $i = 1, 2, 3$, then Σ is the critical catenoid, up to rotation.*

It is well known that, if Σ is a minimal surface, free boundary in $\mathbb{B}^3$, then its coordinated functions are solutions for the *Steklov Problem with respect to* Σ,

$$\begin{cases} \Delta u = 0, & \text{on } \Sigma, \\ \dfrac{\partial u}{\partial \eta} = \sigma u, & \text{along } \partial\Sigma, \end{cases} \tag{10}$$

for $\sigma = 1$, where η is the unit normal vector outward of $\mathbb{S}^2$. In (10), $\sigma \geq 0$ is known as *eigenvalue for the Steklov Problem* and u is called *eigenfunction of Steklov associated with* σ. The smallest eigenvalue, $0 < \sigma_1 \leq 1$, is the *first eigenvalue of Steklov* and the functions associated with σ_1 are the *first eigenfunctions of Steklov*. Thus, when Σ is a free boundary minimal surface in $\mathbb{B}^3$, $\sigma = 1$ is a eigenvalue for Steklov Problem associated with Σ and $0 < \sigma_1 \leq 1$. McGrath used the assumptions of symmetry for prove that the coordinated functions are first eigenfunctions of Steklov problem and, to conclude the proof, he used the below result, that can be found in [12].

Theorem (Fraser and Schoen): Suppose Σ is a free boundary annulus in $\mathbb{B}^n$ such that the coordinated functions are first Steklov eigenfunctions. Then $n = 3$ and Σ is congruent to the critical catenoid.

In this work we presented, in the case $n = 2$, an improvement for the McGrath Theorem, as consequence of the following result:

Theorem 2.3.1 Let $\Sigma^2 \subset \mathbb{B}^3$ be an embedded CMC capillary annulus, such that $\partial\Sigma$ is symmetrical with respect to the coordinated planes, then Σ is rotationally symmetric.

This theorem improved significantly compared to those found in the literature. In their hypotheses, we consider cmc capillary surfaces instead of free boundary minimal surfaces. Compared to McGrath's results, we assume that $\partial\Sigma$ is invariant under reflection through three orthogonal hyperplanes, in contrast, he assumes such a propriety for Σ. Thus, when $n = 2$, the following corollary is an improvement that we give to McGrath's theorem, in addition to using another strategy.

Corollary 2.3.1 Let $\Sigma^2 \subset \mathbb{B}^3$ be an embedded annulus minimal free boundary. If $\partial\Sigma$ is symmetrical with respect to the coordinated planes, then Σ is the critical catenoid.

We also present a new version, in the embedded case, of the proof from following result, that can be found in [7], due to Juncheo Pyo.

Theorem 2.3.2 [Pyo] Let Σ^2 be an embedded minimal surface in $\mathbb{R}^3$ with two boundary components and let Γ be one component of $\partial\Sigma$. If Γ is a circle and Σ meets a plane along Γ at a constant angle, then Σ is part of the catenoid.

In chapter three, we considered a new concept of curvature, the anisotropic mean curvature, denoted by Λ, that will be defined below and is a generalization of the mean curvature H. Analogously to the classical case, associated to area functional, $\mathcal{A}$, given in (4), the immersion $\phi : \Sigma \to \Omega$ have constant anisotropic mean curvature (CAMC) if and only if ϕ is solution of variational problem with respect to functional

$$\mathcal{F}(\epsilon) = \int_{\Sigma} F(\nu(\epsilon, p))\, d\Sigma_{\epsilon} \tag{11}$$

where $F : \mathbb{S}^n \to \mathbb{R}^+$ is a smooth function and $\nu(\epsilon, p)$ is the unit normal vector from Σ_ϵ, at point p. The anisotropic mean curvature is defined by

$$\Lambda(p) := nH(p)F(\nu(0,p)) - \mathrm{div}_{\Sigma} DF(\nu(0,p)). \tag{12}$$

where $H(p)$ is the mean curvature from Σ at point p and DF is the gradient from F on $\mathbb{S}^2$. In this work, as well as in the consulted literature, we will consider the matrix $A := D^2F + F1$ positive definite.

In [13] it was proved that being Σ complete with respect to the induced metric and under other assumptions, a CAMC stable immersion is a Wulff Shape , up to a translation and homothety. In [14], they considered free boundary variations in a *slab* and classed the CAMC stable immersions.

In this work, we considered the generalized area functional $\mathcal{F}$, where F will have the form $F = f(\nu_3)$, and free boundary variations of immersions

$$\phi : (\Sigma^2, \partial\Sigma) \to (\Omega, \partial\Omega), \tag{13}$$

where $(\Sigma^2, \partial\Sigma)$ is an oriented connected compact surface, such that $\partial\Sigma \neq \emptyset$, as well as [3] and [14]. Besides that, we considered $\Omega \subset \mathbb{R}^3$ a region whose boundary is a revolution surface with profile curve α and axis e such that $\mathcal{J}_{\partial\Omega} := \{t \in I \mid \alpha'(t) /\!/ e\}$ is a discrete set.

Unlike the classic case, if ϕ is a critical immersion, it is not possible to say that ϕ is free boundary, that is, that $\phi(\partial\Sigma)$ meets $\partial\Omega$ orthogonally. In this work, classified the free boundary critical immersions ϕ, where Ω is a revolution surface.

In this context, we asked ourselves about the properties of $\phi(\Sigma)$. First, we found some features about $\phi(\partial\Sigma)$.

Proposition 3.2.2 Let $F = f(\nu_3)$ and f a smooth function such that $f' \neq 0$. Consider $\phi : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ a critical immersion to the functional $\mathcal{F}$, where $\mathcal{J}_{\partial\Omega}$ is a discrete set. Then $\partial\Sigma$ intersect $\partial\Omega$ orthogonally if, and only if, each connected component of $\partial\Sigma$ lies in a parallel of $\partial\Omega$, where $N \perp e_3$ along it.

So, we obtained the following characterization regarding the CAMC immersions:

Theorem 3.2.3 Let $F = f(\nu_3)$, where f is a smooth function such that $f' \neq 0$. Consider $\phi : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ a critical immersion to the $\mathcal{F}$ such that $\Lambda \leq 0$ and $\mathcal{J}_{\partial\Omega}$ a discrete set. Then, $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$ if, and only if, $\phi(\Sigma)$ is a totally geodesic disk whose boundary is a parallel of $\partial\Omega$, where $\bar{N} \perp e_3$ along it.

Then, of course, a question arises: as in the classic case, are the free boundary disks in the unit sphere stable, for any function F fixed? Or does it depend on the F function? The first step in answering that question was to obtain the second variation of F for all volume-preserving normal admissible variation:

$$\partial^2_{\epsilon\epsilon}\mathcal{F}\Big|_{\epsilon=0} = -\int_{\Sigma} u\dot{\Lambda}\, d\Sigma + \int_{\partial\Sigma} u\langle A\nabla u, \eta\rangle\, ds - \int_{\partial\Sigma} \mathrm{II}(\nu,\nu)u^2 F\, ds \tag{14}$$

where II is the second fundamental form of $\partial\Omega$ into $\mathbb{R}^{n+1}$, with respect to the inwards pointing unit normal direction and

$$\dot{\Lambda} = L[u] = \mathrm{div}_{\Sigma} A\nabla u + \langle A d\nu, d\nu\rangle u \tag{15}$$

is the *Jacobi Operator* of $\mathcal{F}$.

Throughout this text, the diagonal matrix associated with $A = D^2F + F1$ will be denoted by $\mathrm{diag}(\mu_1^{-1}, \mu_2^{-1})$ and its eigenvalues will help answer the question above.

Theorem 3.3.1 Let $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$ and $\Omega = \mathbb{B}^3$. The disk, $D = \phi(\Sigma)$, is stable with respect to F if, and only if, $f(1) \leq \mu_1^{-1} = \mu_2^{-1}$.

1 Preliminary

In this chapter, we present our objects of study and some results about these.

1.1 Capillary Immersions

Let Σ be a smooth manifold, $\bar{\Sigma}$ a Riemannian manifold, $\overline{\nabla}$ the Levi-Civita connection of $\bar{\Sigma}$ and $\phi : \Sigma^n \to \overline{\Sigma}^{n+m}$, $k := n + m$, an isometric immersion. The difference $k - n = m$ is called *co-dimension* of the immersion ϕ. This way, we have m normal directions to Σ, namely, $\nu_1, ..., \nu_m$. The connection

$$\nabla_X Y := (\overline{\nabla}_{\overline{X}} \overline{Y})^T \tag{1.1}$$

is the Levi-Civita connection of Σ in the induced metric by ϕ. Note that, $\nabla_X Y$ is the tangent part of the connection from ambient manifold $\overline{\nabla}_{\overline{X}} \overline{Y}$. On the other hand, being X, Y local fields in Σ, consider

$$B(X, Y) = \overline{\nabla}_{\overline{X}} \overline{Y} - \nabla_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^\perp \tag{1.2}$$

the normal part from $\overline{\nabla}_{\overline{X}} \overline{Y}$. For each point $p \in \Sigma$ and normal direction $\nu \in \{\nu_1, ..., \nu_m\}$, consider the symmetric bilinear form $H_\nu : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ given by

$$H_\nu(x, y) := \langle B(x, y), \nu\rangle, \ x, y \in T_p\Sigma. \tag{1.3}$$

The quadratic form, defined on $T_P\Sigma$, by

$$\mathrm{II}_\nu(x) := H_\nu(x, x) \tag{1.4}$$

is called *second fundamental form of ϕ, in the ν direction, at point p*. Now, we have conditions to define an important concept in geometry, the mean curvature. Consider $p \in \Sigma, x \in T_p\Sigma$, $\nu \in T_p^\perp\Sigma$ and N a local extension of ν, normal to Σ. We define the *Weingarten map*, $S_\nu : T_pM \to T_pM$, as

$$S_\nu(x) := -(\bar{\nabla}_x N)^T \tag{1.5}$$

Its trace is defined by

$$\operatorname{trace} S_\nu = \sum_{i=1}^{n} \langle S_\nu(e_i), e_i\rangle = \sum_{i=1}^{n} \langle B(e_i, e_i), \nu\rangle = \sum_{i=1}^{n} H_\nu(e_i, e_i) = \sum_{i=1}^{n} \mathrm{II}_\nu(e_i) \tag{1.6}$$

where $\{e_1, ..., e_n\}$ is an orthonormal base for $T_p\Sigma$. The *mean curvature vector of ϕ* is defined by

$$\overrightarrow{H} = \frac{1}{n}\sum_{i=1}^{m} (\operatorname{trace} S_i)\nu_i = \frac{1}{n}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} \mathrm{II}_\nu(e_j)\right)\nu_i \ \ ; \ \ S_i := S_{\nu_i}. \tag{1.7}$$

In the case $m = 1$,

$$n\overrightarrow{H} = (\operatorname{trace} S_i)\nu = \left(\sum_{j=1}^{n} \mathrm{II}_\nu(e_j)\right)\nu \ ; \ \langle\nu\rangle = T_p^\perp\Sigma, \tag{1.8}$$

and $H := |\overrightarrow{H}| = \frac{1}{n} \cdot (\operatorname{trace} S_\nu)$ is called the **mean curvature of** Σ. From now on, considered $m = 1$. In the especial case $\overline{\Sigma} = \mathbb{R}^{n+1}$,

$$S_\nu(x) = -d\nu(x) \tag{1.9}$$

where ν is the *Gauss Map of* Σ, for more details see [15].

Definition 1.1.1 *Let $\phi : \Sigma^n \to \overline{\Sigma}^{n+1}$ be an immersion, the number $H(p) = \frac{1}{n}\operatorname{trace}(d\nu_p)$ is called mean curvature of ϕ at p. An immersion is called constant mean curvature (CMC) if,*

$$H(p) = constant,\ \forall\, p \ \in\ \Sigma. \tag{1.10}$$

In the special case, $H(p) = 0,\ \forall\, p\ \in\ \Sigma$, the immersion ϕ is called minimal.

Planes and spheres, as well as its pieces, are trivial examples from hypersurfaces with constant mean curvature. A plane is a minimal surface and a sphere with radius r, S_r, have constant mean curvature $H(p) = \frac{1}{r}$, for all point p on S_r. A cylinder of radius r is an example also of CMC surface, in this case, $H = 2r^{-1}$.

Definition 1.1.2 *A hypersurface Σ^n is called symmetrical rotationally or axially symmetric if, there is a line, l (axis), such that any nonempty intersection of Σ with a hyperplane, orthogonal to l, is an open disk whose center lies on l. When $n = 2$, Σ also is called revolution surface.*

Example 1.1.1 (The first non-trivial example) *Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a curve parametrized by $\alpha(t) = (a\cosh\left(\frac{t}{a}\right), 0, t)$, where a is a non null constant, this curve is known by Catenary. Consider the Catenoid, which is a surface obtained by revolution from α around z-axis, whose parametrization is given by*

$$X(t,\theta) = (a\cos\theta\cosh\left(\frac{t}{a}\right), a\sin\theta\cosh\left(\frac{t}{a}\right), t),\ \theta \in (0, 2\pi). \tag{1.11}$$

The Catenoid is a symmetrical rotationally surface; α is called the profile curve and z the rotation axis of Catenoid. The Catenoid is an example of minimal surface, namely, the first non-trivial example.

The *genus* $g(\Sigma)$, of a surface Σ, corresponds to the number of torus present in it. The Euler characteristic of a surface Σ, with r connected components of the boundary, is the number given by

$$\chi(\Sigma) = 2 - 2g(\Sigma) - r. \tag{1.12}$$

Planes, Spheres, Catenoids and its pieces have genus zero. Follows below, an example of minimal surface with genus one.

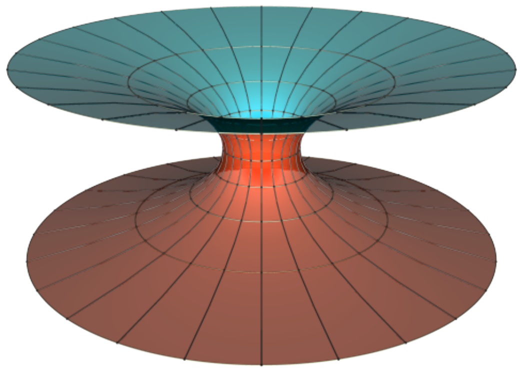

Figure 2 – The Catenoid. Credit: Matthias Weber, www.indiana.edu/$\sim$minimal.

Example 1.1.2 (The Costa-Hoffman-Meeks surface [16]) *Discovered in 1984 by Costa, this is a minimal surface with genus one and Euler characteristic zero.*

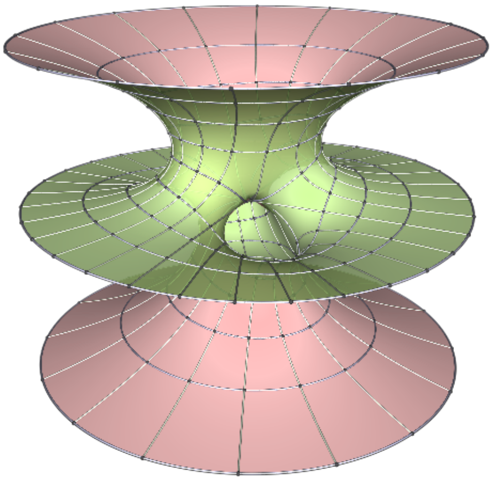

Figure 3 – The Costa-Hoffman-Meeks surface.
Credit: Matthias Weber, www.indiana.edu/$\sim$minimal.

In [16] there exist many examples of minimal surfaces. The Catenoid, it cited in the below example, is an example of *Delaunay surfaces*: revolution surfaces with constant mean curvature. This surfaces were classified by Delaunay in [17]:

Theorem 1.1.1 (Delaunay's Theorem) *A surface of revolution CMC is locally congruent (i.e., up to rotations and translations) to exactly one of these: plane, circular cylinder, sphere, catenoid, unduloid, or nodoid.*

Example 1.1.3 *Undulary and nodary are the curves obtained by roullete of ellipse and hyperbole, respectively. Unduloid and nodoid, revolution surface obtained from rotation of undulary and nodary, respectively, are examples of non-zero constant mean curvature surface. Figures 4 and 5 can be found in [18].*

Figure 4 – A cut Unduloid.

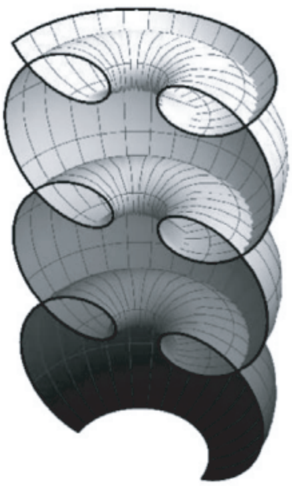

Figure 5 – A cut Nodoid.

Let B be a compact smooth region in $\bar{\Sigma}$ such that B is diffeomorphic to an euclidean ball and $\bar{N}$ the unit outward normal to ∂B. Consider also η the unit outward normal to $\partial\Sigma$ tangent to Σ and $\bar{\eta}$ be the unit normal to $\partial\Sigma$ tangent to ∂B such that the orthonormal frames to normal bundle $\{\eta, \nu\}$ and $\{\bar{\eta}, \bar{N}\}$ have the same orientation, see Figure 6.

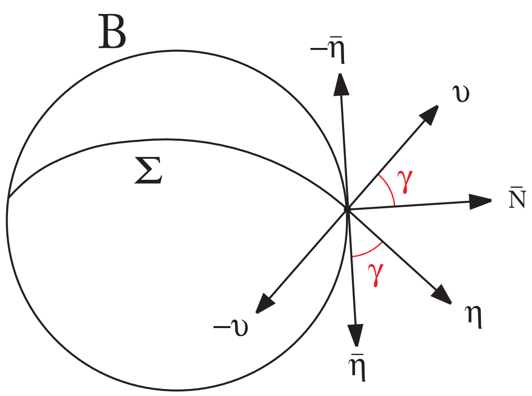

Figure 6 – The fields $\nu, \eta, \bar{\eta}$ and $\bar{N}$.

Thus, we called that $\partial\Sigma$ *meets* ∂B *in a constant contact angle* if,

$$\langle \nu, \bar{N} \rangle = \langle \eta, \bar{\eta} \rangle \equiv \text{constant, along } \partial\Sigma. \tag{1.13}$$

In especial, *this intersection is called orthogonal* when $\langle \nu, \bar{N} \rangle = \langle \eta, \bar{\eta} \rangle \equiv 0$, i.e., when $\gamma = \frac{\pi}{2}$. We are now able to enter an important definition.

Definition 1.1.3 *Let Σ be a manifold with boundary, $\partial\Sigma \neq \emptyset$, and let $\phi : \Sigma \to B$ be an immersion into $B \subset \overline{\Sigma}$. The immersion ϕ is called capillary if, $\phi(\partial\Sigma)$ meets ∂B with a constant contact angle. If $\phi(\partial\Sigma)$ intersects ∂B orthogonally, we called that the immersion is free boundary. Namely, the contact angle is those defined between the normal vectors of $\phi(\Sigma)$ and ∂B, respectively.*

Remark 1.1.1 *From now on, we will only say that a capillary immersion is one in which $\partial\Sigma$ meets ∂B at a constant angle.*

Remark 1.1.2 *It is common to find in the literature that an immersion $\phi : \Sigma \to B$, $\partial\Sigma \neq \emptyset$, is capillary if, ϕ is CMC and $\partial\Sigma$ meets ∂B in a constant angle γ and, in particular, also is found that an immersion ϕ is free boundary if, ϕ is minimal and $\gamma = \frac{\pi}{2}$. This is due to the existence of another characterization of the CMC immersions, $\phi : \Sigma \to B$, with constant contact angle. However, throughout this text we will consider the definition 1.1.3.*

When ϕ is capillary, we have the following equations in the normal bundle (see Figure 6):

$$\begin{aligned} \eta &= \cos\gamma\, \bar{\eta} + \sin\gamma\, \bar{N} \\ \nu &= \sin\gamma\, \bar{\eta} + \cos\gamma\, \bar{N} \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} \bar{\eta} &= \cos\gamma\, \eta + \sin\gamma\, \nu \\ \bar{N} &= \sin\gamma\, \eta + \cos\gamma\, \nu \end{aligned} \tag{1.15}$$

Example 1.1.4 (Trivial examples) *The disks $D_k = \{(x, y, z) \in \mathbb{R}^3\ ;\ z = k\} \cap \mathbb{B}^3$, $k \in (-1, 1)$, are trivial examples of capillary surfaces (minimal) in the ball $\mathbb{B}^3$, see Figure 7.*

The below lemma that, can be found in [19] and will be add here for completeness, will enable us to construction more one example of capillary surface.

Lemma 1.1.1 *Let $\beta(s) = (x(s), 0, z(s))$, $s \in [a, b]$, a smooth curve in $\mathbb{R}^3$. Consider the function $g : [a, b] \setminus \{s \in [a, b]; z'(s) = 0\} \to \mathbb{R}$ given by*

$$g(s) = x(s) - \frac{x'(s)}{z'(s)} z(s). \tag{1.16}$$

If, there exist $s_1 < s_2 \in [a, b]$ such that

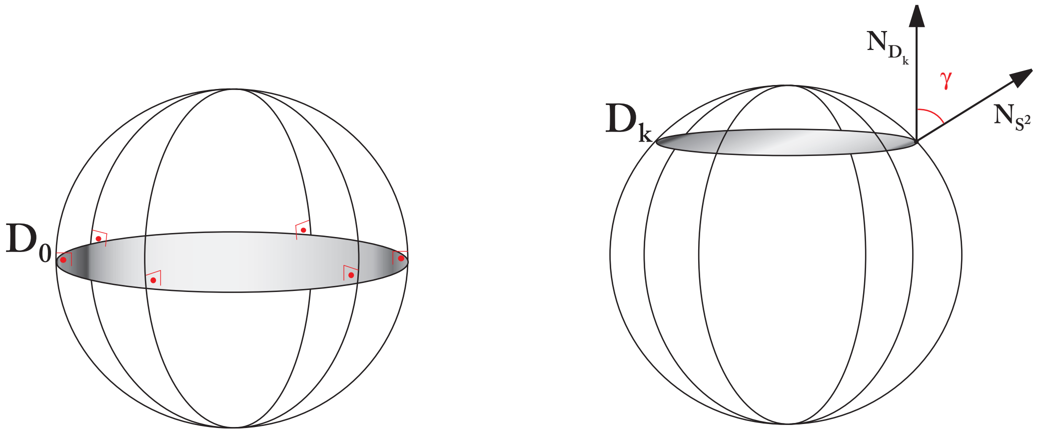

Figure 7 – In the left side, the disk D_0 minimal free boundary in $\mathbb{B}^3$; in the right side, the disk D_k minimal capillary in $\mathbb{B}^3$, with constant contact angle $\gamma = \arccos k$.

(i) $g(s_1) = g(s_2) = 0$;

(ii) $||\beta(s_1)|| = ||\beta(s_2)|| =: r$;

(iii) $||\beta(s)|| < r, \ \forall\, s \in (s_1, s_2)$;

Then, Σ, the surface obtained by rotation of $\beta([s_1, s_2])$ around the z-axis, satisfies: $\Sigma \subset B(r, 0)$ and $\partial\Sigma$ intersects $\partial B(r, 0)$ orthogonally, where $B(r, 0)$ is the origin-centered ball whose radius is r.

Proof of Lemma 1.1.1: Let Σ the surface obtained by rotation of $\beta([s_1, s_2])$ around the z-axis. Follows from **(ii)** and **(iii)** that, $\partial\Sigma \subset \partial B(r, 0)$ and $\mathrm{int}\Sigma \subset B(r, 0)$, respectively. As β is profile curve of Σ, $\beta'(s) \in T_{\beta(s)}\Sigma, \ \forall\, s \in [s_1, s_2]$. Once Σ is a revolution surface, and $\beta([s_1, s_2]) \subset \{y = 0\}$, so $t = (0, 1, 0) \in T_{\beta(s)}\Sigma$ and $t \perp \beta'(s), \ \forall\, s \in [s_1, s_2]$. Thus, $\nu(s) = \beta'(s) \wedge t = (-z'(s), 0, x'(s)), \ \forall\, s \in [s_1, s_2]$.

Follows from **(i)** that,

$$x(s_1) - \frac{x'(s_1)}{z'(s_1)} z(s_1) = x(s_2) - \frac{x'(s_2)}{z'(s_2)} z(s_2) = 0$$

i.e.

$$x'(s)z(s) - x(s)z'(s) = 0, \ s \in \{s_1, s_2\} \ \Leftrightarrow \tag{1.17}$$

$$\langle \beta(s), \nu(s) \rangle = 0, \ s \in \{s_1, s_2\} \ \Leftrightarrow \tag{1.18}$$

$$\langle r\bar{N}, \nu(s) \rangle = 0, \ s \in \{s_1, s_2\} \ \Leftrightarrow \tag{1.19}$$

$$\langle \bar{N}, \nu(s) \rangle = 0, \ s \in \{s_1, s_2\} \tag{1.20}$$

Therefore, $\partial\Sigma$ intersects $\partial B(r, 0)$ orthogonally, because Σ is a revolution surface contained in a ball.

∎

Example 1.1.5 *Consider the catenary given by $\beta(s) = (a\cosh(s/a), 0, s)$, $a > 0$, $s \in \mathbb{R}$. As* $\cosh$ *is an even function, the catenary is symmetric with respect to x axis. Note that, if $a \to 0$, $p = (a, 0, 0) = \beta(0) \to (0, 0, 0)$ and the catenary closes around x axis, see Figure 8.*

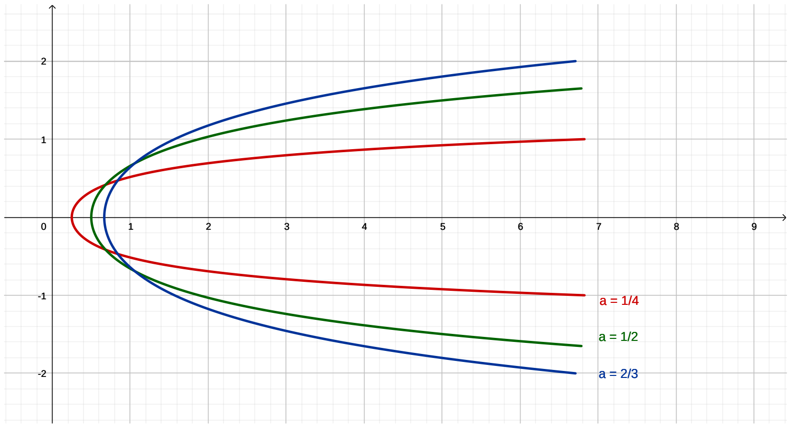

Figure 8 – Red catenary: $a = \frac{1}{4}$ and $-1 \le s \le 1$; green catenary: $a = \frac{1}{2}$ and $-1,65 \le s \le 1,65$; blue catenary: $a = \frac{2}{3}$ and $-2 \le s \le 2$.

Let's check for zeros of the g. Thus,

$$0 = x'(s)z(s) - x(s)z'(s) \iff \tag{1.21}$$
$$0 = s\sinh\left(\frac{s}{a}\right) - a\cosh\left(\frac{s}{a}\right) \iff \tag{1.22}$$
$$\frac{a}{s} = \tanh\left(\frac{s}{a}\right) \tag{1.23}$$

Then, with a suitable software we can calculate $|\frac{s}{a}| = \rho = 1,19968... \approx 1,2$ and define $s_1 := -a\rho$ and $s_2 := a\rho$. As $\cosh$ is an even function, $||\beta(s_1)|| = ||\beta(s_2)|| := r$; as $\cosh$ have a global minimal point at $s = 0$, see Figure 8, then $||\beta(s)|| < ||\beta(s_1)|| = ||\beta(s_2)||$, $s \in (s_1, s_2)$. Therefore, follows from Lemma 1.1.1 that, Σ, the surface obtained by rotation of $\beta([s_1, s_2])$ around the z axis, the catenoid, satisfies

$$\text{int}\Sigma \subset B(r, 0), \tag{1.24}$$
$$\partial\Sigma \subset \partial B(r, 0), \tag{1.25}$$
$$\partial\Sigma \perp \partial B(r, 0). \tag{1.26}$$

Therefore, this part of catenoid is an example of capillary surface in a ball.

Each $a > 0$, determines from **(ii)**, a single value for r. For $a = 1/4$, $a = 1/2$ and $a = 2/3$, we have $r \approx 0,54$, $r \approx 1,09$ and $r \approx 1,45$, respectively, see Figures 9, 10 and 11.

Fixing $r = 1$ in **(ii)** we obtain $a^* = 0,46048 \approx 0,46$. The surface Σ^*, obtained by rotation of $\beta^*(s) = (a^*\cosh(\frac{s}{a^*}), 0, s)$ around z axis, is called *Critical Catenoid*.

Example 1.1.6 *There exist a piece of sphere, called spherical cap, free boundary in $\mathbb{B}^3$. Indeed,* $\partial\mathbb{B}^3 = \mathbb{S}^2 = \{(x_1, x_2, x_3)\,;\, x_1^2 + x_2^2 + x_3^2 = 1\}$, $\Pi_\theta = \{(x_1, x_2, x_3)\,;\, (\cos\theta)x_1 + (\sin\theta)x_2 = 0\}$,

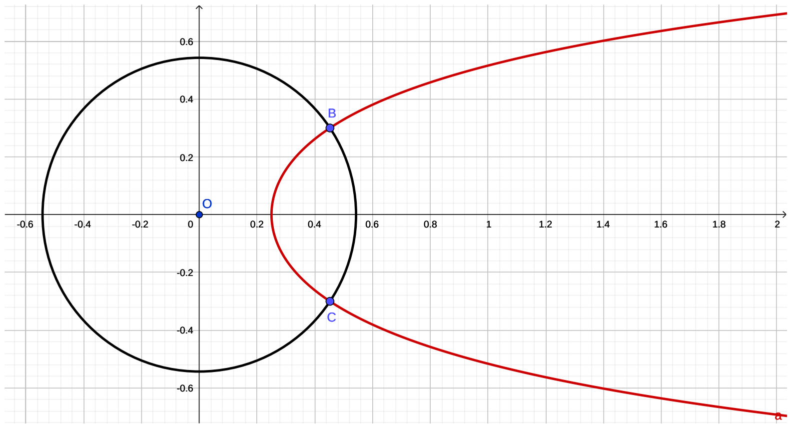

Figure 9 – Red catenary: $a = \frac{1}{4}$ and the ball $B(r, O)$, $r \approx 0,54$; the intersect point are $B = (0,45,0,0,3)$ and $C = (0,45,0,-0,3)$, approximately.

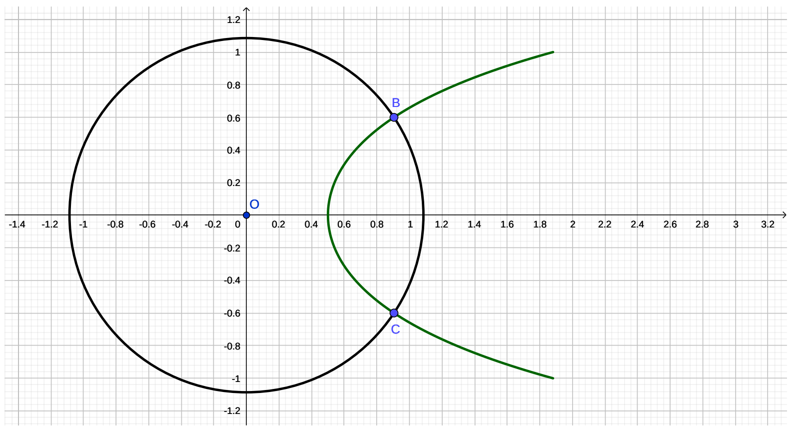

Figure 10 – Green catenary: $a = \frac{1}{2}$ and and the ball $B(r, O)$, $r \approx 1,09$; the intersect point are $B = (0,91,0,0,6)$ and $C = (0,91,0,-0,6)$, approximately.

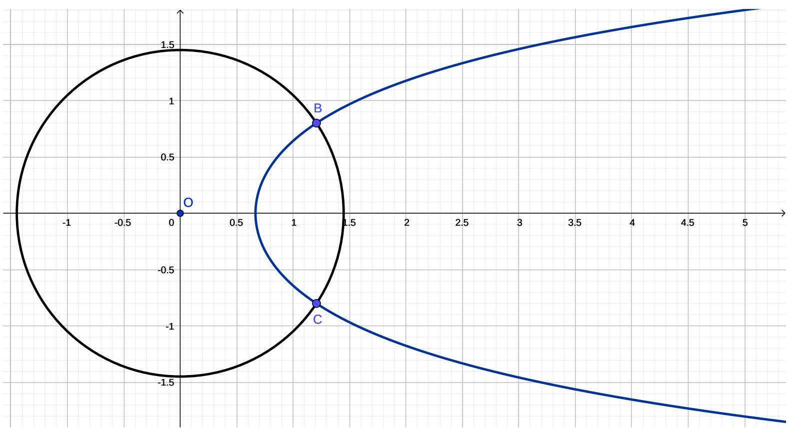

Figure 11 – Blue catenary: $a = \frac{2}{3}$ and and the ball $B(r, O)$, $r \approx 1,45$; the intersect point are $B = (1,21,0,0,8)$ and $C = (1,21,0,-0,8)$, approximately.

$\theta \in [0, \pi]$, and $\mathbb{S}^1_\theta := \mathbb{S}^3 \cap \Pi_\theta$. Consider the identification $\mathbb{S}^1_\theta \longleftrightarrow \mathbb{S}^1$ and $x, \nu : [0, \frac{\pi}{2}] \to \mathbb{S}^1_\theta$ given by $x(t) = (\cos t, \sin t)$, the position vector of $\mathbb{S}^3$ restricted to $\mathbb{S}^1_\theta$, and $\nu(t) = (-\cos t, \sin t)$, the unit normal vector to $\mathbb{S}^2$ restricted to $\mathbb{S}^1_\theta$, see Figure 13.

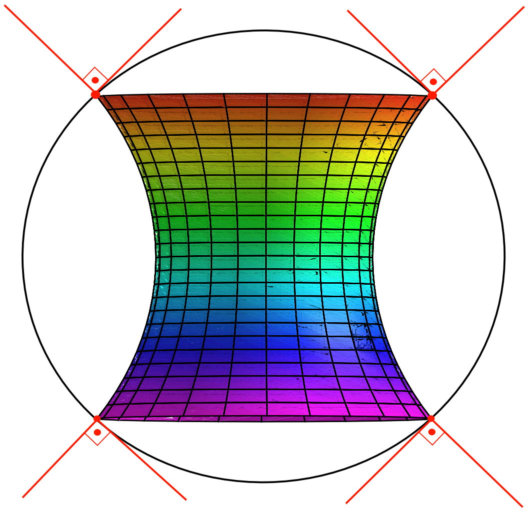

Figure 12 – The Critical Catenoid.

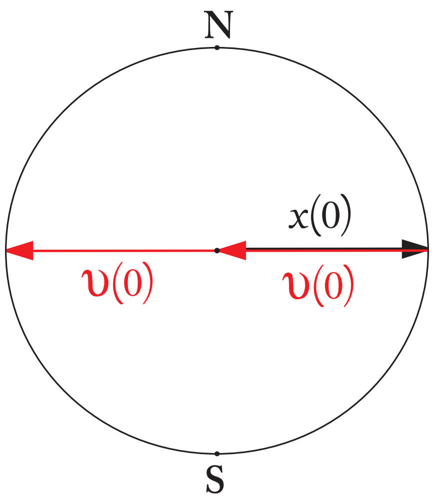

Figure 13 – The vectors $x(0)$ and $\nu(0)$.

Observe that, while t varies in $[0, \frac{\pi}{2}]$, $x(t)$ rotates, with its end point about $\mathbb{S}^1_\theta$, anticlockwise until its end point reaches N. In the other hand, $\nu(t)$ rotates in the clockwise until its end point reaches N. Consider the function $f : [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \to \mathbb{R}$ given by $f(s,t) := \langle x(s), \nu(t) \rangle$; note that $f(0,0) = -1$, $f(\frac{\pi}{2}, \frac{\pi}{2}) = 1$, and

$$0 = f(s,t) = \langle x(s), \nu(t) \rangle = -\cos s \cdot \cos t + \sin s \cdot \sin t \tag{1.27}$$

$$= -\cos(s+t), \tag{1.28}$$

so

$$0 \equiv f(s,t) \;\Leftrightarrow\; s + t = \frac{\pi}{2}, \tag{1.29}$$

i.e.,

$$x(s) \perp \nu\left(\frac{\pi}{2} - s\right), \ \forall\, s \in \left[0, \frac{\pi}{2}\right], \tag{1.30}$$

see Figure 14.

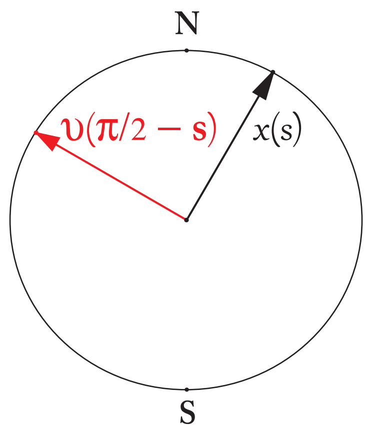

Figure 14 – The vectors $x(s)$ and $\nu(\frac{\pi}{2} - s)$ are orthogonal.

Consider $\nu(\frac{\pi}{2} - s)$ with end point in the center of $\mathbb{S}^1_\theta$; by the spherical geometry, the initial point $P = (p_1, p_2, p_3)$ of $\nu(\frac{\pi}{2} - s)$ will also belong to $\mathbb{S}^1_\theta$. The plane $\{x_3 = p_3\}$ divide $\mathbb{S}^2$ in two pieces, define by $\mathbb{S}_{\theta,s}$ the spherical cap below $\{x_3 = p_3\}$, i.e.,

$$\mathbb{S}_{\theta,s} = \mathbb{S}^2 \cap \{x_3 \leq p_3\}; \ -p_3 = \left\langle \nu\left(\frac{\pi}{2} - s\right), e_3 \right\rangle. \tag{1.31}$$

Note that, 1.30 does not depend of θ, thus it has validity on $\mathbb{S}_{\theta,s} = \cup_{\theta\in[0,\pi]}\mathbb{S}^1_\theta$. Hence, the unit normal vector to $\mathbb{S}_{\theta,s}$ along $\partial\mathbb{S}_{\theta,s}$ is constant and equal to $\nu(\frac{\pi}{2} - s)$ and the unit normal vector to $\mathbb{S}^2$ along $\mathbb{S}^2 \cap \{x_3 = \langle x(s), e_3\rangle\}$ is constant and equal to $x(s)$, for all $s \in [0, \frac{\pi}{2}]$. Thus, for the values of $s \in [0, \frac{\pi}{2}]$, such that $\partial\mathbb{S}_{\theta,s}$ has the same radius as $\mathbb{S}^2 \cap \{x_3 = \langle x(s), e_3\rangle\}$, we have $\mathbb{S}_{\theta,s}$ free boundary in $\mathbb{B}^3$ along $\mathbb{S}^2 \cap \{x_3 = \langle x(s), e_3\rangle\}$, and this happens only $s = \frac{\pi}{4} = t$, see Figure 15.

Example 1.1.7 *In [19] there are more examples of surfaces CMC free boundary in $\mathbb{B}^3$. Moreover, was proved that there exist a free boundary piece of unduloid.*

In definition of the Euler characteristic of a surface Σ, we mention the *torus* which is a topological space homeomorphic to the product of two circles.

Example 1.1.8 *There exist a minimal immersion $\phi : \mathbb{R}^2 \to \mathbb{R}^4$ such that $\phi(\mathbb{R}^2) \subset \mathbb{S}^3$ is a torus T, the Clifford Torus. Observe that,*

$$\phi(u, v) = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v) = (\cos u, \sin u, 0, 0) + (0, 0, \cos v, \sin v) \tag{1.32}$$

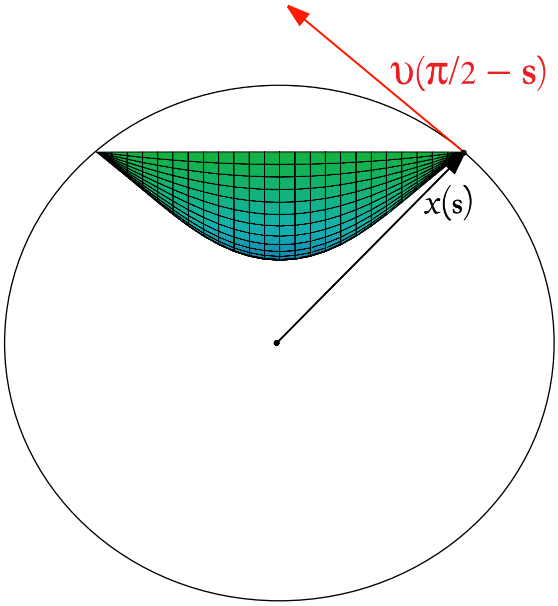

Figure 15 – The spherical cap CMC free boundary.

Figure 16 – A revolution torus.

is such that, $||\phi(u, v)|| = 1$ *and*

$$d\phi(u,v) = \begin{bmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & -\sin v \\ 0 & \cos v \end{bmatrix} \tag{1.33}$$

is injective, i.e., $\phi(\mathbb{R}^2)$ *is an immersed torus* $T^2 \subset \mathbb{S}^3$*. Also note that, considering for* T^2 *the induced metric from* $\mathbb{R}^4$*,* $\{e_1 = (-\sin u, \cos u, 0, 0), e_2 = (0, 0, -\sin v, \cos v)\}$ *is a orthonormal frame for* T_pT^2*,* $\forall\ (u, v) \in \mathbb{R}^2$*. From this frame,* $\{e_1, e_2\}$*, we can construct a frame for normal* *space at p, namely,*

$$\{\nu_1 = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v), \nu_2 = \frac{1}{\sqrt{2}}(-\cos u, -\sin u, \cos v, \sin v)\}. \tag{1.34}$$

The field $\nu = \nu_2$ is tangent to T_pT^2, unlike ν_1. Considering in $\mathbb{S}^3$ the Levi-Civita connection, associated to induced metric, we have

$$[S_\nu] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}. \tag{1.35}$$

Then, $\mathrm{trace}(S_\nu) = 0$ *and T^2 is a minimal immersion in $\mathbb{S}^3$.*

The generalization of this torus is the so-called *H-Torus*, a non-zero constant mean curvature torus, more precisely,

(i) consider the canonical immersions from $\mathbb{S}^{n-1}(r)$ in $\mathbb{R}^n$ and $\mathbb{S}^1(\sqrt{1-r^2})$ in $\mathbb{R}^2$, for $r \in (0,1)$;

(ii) and take the product manifold $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ immersed in $\mathbb{R}^n \times \mathbb{R}^2$.

The manifold obtained this way, is called an *H-Torus*, a n-manifold contained in $\mathbb{S}^{n+1}$ with non-zero constant mean curvature equal to r. More details can be found at [20].

It is possible to characterize free boundary CMC immersions ϕ in variational viewpoint as follows (for more details see [3] and [21]). Precisely, let $\Phi : (-\epsilon_0, \epsilon_0) \times \Sigma \to \overline{\Sigma}$ be a *smooth variation* of ϕ, i.e., for each $\epsilon \in (-\epsilon_0, \epsilon_0)$, $\Phi_\epsilon := \Phi(\epsilon, p)$ is a smooth immersion of Σ; $\Phi_0 = \phi$, see Figure 17, in the case $B = \mathbb{B}^{n+1}$.

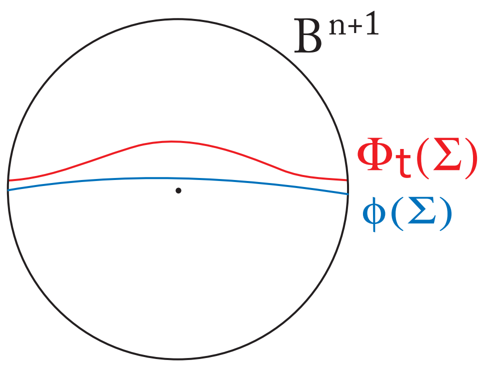

Figure 17 – Smooth variation $\Phi_\epsilon(\Sigma)$ of immersion $\phi(\Sigma)$ in the ball $B = \mathbb{B}^{n+1}$.

A smooth variation is called *admissible* if

$$\Phi_\epsilon(\mathrm{int}\Sigma) \subset \mathrm{int}B \quad \text{and} \quad \Phi_\epsilon(\partial\Sigma) \subset \partial B, \ \forall\, \epsilon \in (-\epsilon_0, \epsilon_0). \tag{1.36}$$

Being Φ a smooth variation, define an *area functional* $\mathcal{A} : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$, given by

$$\mathcal{A}(\epsilon) = \int_\Sigma d\Sigma_\epsilon \tag{1.37}$$

where $d\Sigma_\epsilon$ is the area element of Σ in the metric induced by Φ_ϵ and a *volume functional* $V : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$

$$V(\epsilon) = \int_{[0,\epsilon]\times\Sigma} \Phi^* dV \tag{1.38}$$

where $\Phi^* dV$ is the *pull back* from the volume element of $\bar{\Sigma}$. Geometrically, $V(\epsilon)$ is the volume (oriented) encompassed by B, $\Phi_\epsilon(\Sigma)$ and $\phi(\Sigma)$. The variation Φ is *volume-preserving* if, $V(\epsilon) = V(0) = 0,\ \forall\ \epsilon \in (-\epsilon_0, \epsilon_0)$, see Figure 18.

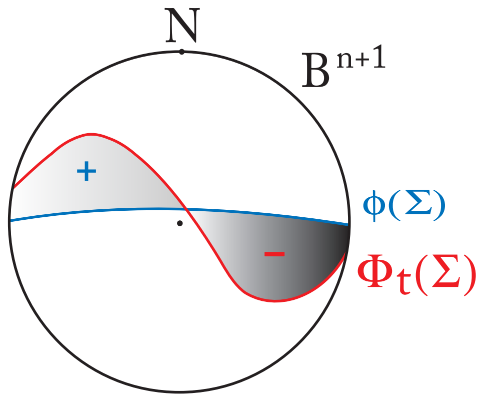

Figure 18 – Variation Φ volume-preserving, in the case $B = B^{n+1}$ and the unit normal vector of Σ pointing to for north pole N.

Consider

$$\dot{\Phi} = \frac{\partial \Phi}{\partial \epsilon}(p)\bigg|_{\epsilon=0} \tag{1.39}$$

the *variation vector* of Φ. Note that, if a variation is admissible then

$$\langle \dot{\Phi}, \bar{N} \rangle \equiv 0 \ \text{ sobre } \ \partial\Sigma. \tag{1.40}$$

where $\bar{N}$ is the unit outward normal of ∂B. It is said that Φ is *normal* if,

$$\frac{\partial \Phi}{\partial \epsilon}(p)\bigg|_{\epsilon=0} = f \cdot \nu \tag{1.41}$$

where $f = \langle \dot{\Phi}, \nu \rangle$.

Note that, $\phi(\Sigma)$ divide B in two connected regions B_1 and B_2. So, ∂B_i is the union of $\phi(\Sigma)$ and a domain D_i on ∂B, such that $\partial D_i = \phi(\partial\Sigma)$, $i \in \{1, 2\}$. Define D, between D_1 and D_2, those whose normal exterior is $\bar{\eta}$. Define, $\mathcal{T} : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ such that $\mathcal{T}(\epsilon)$ is the area of $D(\epsilon)$ in the metric induced by Φ_ϵ. Fix an angle $\gamma \in (0, \pi)$ and consider a variation Φ_ϵ, as defined above. Then, we can define the *energy functional*

$$\mathcal{E}(\epsilon) = \mathcal{A}(\epsilon) - \cos\gamma\, \mathcal{T}(\epsilon) \tag{1.42}$$

Below, we present the first variation for the functional $\mathcal{A}$, V, $\mathcal{T}$ and $\mathcal{E}$. However, before we state the following lemma:

Lemma 1.1.2 *Let $v \in C^\infty(\Sigma)$.*

(i) $\int_\Sigma uv\, d\Sigma = 0,\ \forall\, u \in C^\infty(\Sigma) \ \Rightarrow\ v \equiv 0;$

(ii) $\int_\Sigma uv\, d\Sigma = 0,\ \forall\, u \in C^\infty(\Sigma)\ ;\ \int_\Sigma u\, d\Sigma = 0 \ \Rightarrow\ v \equiv$ *constant;*

Proof of Lemma 1.1.2:

(i) If $v \not\equiv 0$, then, choosing $u = v \in C^\infty(\Sigma)$, we have $\int_\Sigma v^2\, d\Sigma \neq 0$.

(ii) Consider V, W open subsets of Σ and $g \in C^\infty(\Sigma)$ such that $\int_V g\, d\Sigma = \int_W g\, d\Sigma$ and $|V| = |W|$. Define

$$v(p) = \begin{cases} c, & on \quad \Sigma \setminus V, \\ g, & on \quad V, \end{cases} \tag{1.43}$$

where c is a constant such that $c < g$ on $\mathrm{int}V$ and $c = g$ along ∂V. Define also

$$\bar{u}(p) = \begin{cases} 0, & on \quad \Sigma \setminus (V \cup W), \\ g - c, & on \quad V, \\ -(g-c), & on \quad W, \end{cases} \tag{1.44}$$

and note that $\bar{u} > 0$ on V, $\bar{u} \equiv 0$ along ∂V, $\bar{u} < 0$ on W, $\bar{u} \equiv 0$ along ∂W, and

$$\int_\Sigma \bar{u}\, d\Sigma = \int_V g - c\, d\Sigma - \int_W g - c\, d\Sigma \tag{1.45}$$

$$= \int_V g\, d\Sigma - \int_V c\, d\Sigma - \int_W g\, d\Sigma + \int_W c\, d\Sigma \tag{1.46}$$

$$= -c|V| + c|W| \tag{1.47}$$

$$= 0. \tag{1.48}$$

Hereafter,

$$\int_\Sigma \bar{u}v\, d\Sigma = \int_V g(g-c)\, d\Sigma - \int_W c(g-c)\, d\Sigma \tag{1.49}$$

$$= \int_V g^2\, d\Sigma - c\int_V g\, d\Sigma - c\int_W g\, d\Sigma + c^2\int_W d\Sigma \tag{1.50}$$

$$= \int_V g^2\, d\Sigma - c\int_V g\, d\Sigma - c\int_V g\, d\Sigma + c^2\int_V d\Sigma \tag{1.51}$$

$$= \int_V g^2 - 2cg + c^2\, d\Sigma \tag{1.52}$$

$$= \int_V (g-c)^2\, d\Sigma \tag{1.53}$$

$$> 0. \tag{1.54}$$

■

Proposition 1.1.1 *The first variation of area functional $\mathcal{A}$ is given by*

$$\mathcal{A}'(0) = -\int_{\Sigma} nHf\ d\Sigma + \int_{\partial\Sigma} \langle \dot{\Phi},\ \eta \rangle\ ds$$

where η is the unit exterior normal along $\partial\Sigma$ and ds is the area element of $\partial\Sigma$.

Proof of Proposition 1.1.1: Initially, consider a local coordinate system φ in $p \in \Sigma$. As we are considering an isometric immersion,

$$\frac{\partial \varphi}{\partial x_i}(p) \leftrightarrow d\phi_p \frac{\partial \varphi}{\partial x_i} \tag{1.55}$$

and $\left\langle \frac{\partial \varphi}{\partial x_i}(p), \frac{\partial \varphi}{\partial x_j}(p) \right\rangle = \left\langle d\phi_p \frac{\partial \varphi}{\partial x_i}, d\phi_p \frac{\partial \varphi}{\partial x_j} \right\rangle$. Without loss of generality, consider φ a normal coordinate system at $p \in \Sigma$, i.e., $\left\{ \frac{\partial \varphi}{\partial x_1}, ..., \frac{\partial \varphi}{\partial x_n} \right\}_p$ is an orthonormal frame in T_pM. Deriving the area functional in t, follows

$$\left.\frac{d\mathcal{A}}{dt}\right|_{t=0} = \int_{\Sigma} \left.\frac{d}{dt}\right|_{t=0} d\Sigma_t \tag{1.56}$$

where $d\Sigma_t = \sqrt{\det(g_{ij}(t))}$. Thus,

$$\left.\frac{d}{dt}\right|_{t=0} d\Sigma_t = \frac{1}{2} \frac{1}{\sqrt{det(g_{ij}(0))}} \sum_{i=1}^{n} \det \begin{bmatrix} g_{11}(t) & \cdots & g_{1n}(t) \\ \vdots & \ddots & \vdots \\ g'_{i1}(t) & \cdots & g'_{in}(t) \\ \vdots & \ddots & \vdots \\ g_{n1}(t) & \cdots & g_{nn}(t) \end{bmatrix}_{t=0} . \tag{1.57}$$

So,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} d\Sigma_t &= \frac{1}{2} \sum_{i=1}^{n} g'_{ii}(t) = \frac{1}{2} \sum_{i=1}^{n} \frac{d}{dt} \left\langle \frac{\partial \varphi}{\partial x_i}, \frac{\partial \varphi}{\partial x_i} \right\rangle_p & (1.58) \\ &= \frac{1}{2} \sum_{i=1}^{n} 2 \left\langle \overline{\nabla}_{\dot{\Phi}} \frac{\partial \varphi}{\partial x_i}, \frac{\partial \varphi}{\partial x_i} \right\rangle_p & (1.59) \\ &= \frac{1}{2} \sum_{i=1}^{n} 2 \left\langle \overline{\nabla}_{\frac{\partial \varphi}{\partial x_i}} \dot{\Phi}, \frac{\partial \varphi}{\partial x_i} \right\rangle_p & (1.60) \\ &= \mathrm{div}_{\Sigma} \dot{\Phi}. & (1.61) \end{aligned}$$

On another hand,

$$\begin{aligned} \mathrm{div}_{\Sigma} \dot{\Phi} = \mathrm{div}_{\Sigma} \dot{\Phi}^{\top} + \mathrm{div}_{\Sigma} \dot{\Phi}^{\perp} &= \mathrm{div}_{\Sigma} \dot{\Phi}^{\top} - \langle \dot{\Phi}^{\perp}, n\overrightarrow{H} \rangle & (1.62) \\ &= -nH \langle \dot{\Phi}, \nu \rangle + \mathrm{div}_{\Sigma} \dot{\Phi}^{\top} & (1.63) \end{aligned}$$

and follows from divergence theorem that

$$\left.\frac{d\mathcal{A}}{dt}\right|_{t=0} = -\int_{\Sigma} nHf\ d\Sigma\ +\ \int_{\partial\Sigma} \langle \dot{\Phi}, \eta \rangle\ ds. \tag{1.64}$$

■

When $\bar{\Sigma} = \mathbb{R}^{n+1}$ and $B = \mathbb{B}^{n+1}$, the euclidean ball, analogously, we can to prove that $\mathcal{T}'(0) = \int_{\partial D} \langle \dot{\Phi}, \bar{\eta} \rangle \; ds$, where $\bar{\eta}$ is the unit normal exterior to D. Thus, the first variation of the energy functional is given by

$$\mathcal{E}'(\epsilon) = \mathcal{A}'(\epsilon) - \cos\gamma \cdot \mathcal{T}'(\epsilon) \tag{1.65}$$

$$= -\int_{\Sigma} nHf \; d\Sigma + \int_{\partial\Sigma} \langle \dot{\Phi}, \eta - (\cos\gamma)\bar{\eta} \rangle \; ds \tag{1.66}$$

Proposition 1.1.2 *The first variation of volume functional is given by*

$$V'(0) = \int_{\Sigma} f \; d\Sigma \tag{1.67}$$

Proof of Proposition 1.1.2: The pull back $\Phi^* dV$ is a $(n+1)$-differential form on $\tilde{M} := [0,t] \times M$, thus

$$\Phi^* dV = b \cdot dt \wedge dx_1 \wedge ... \wedge dx_n \; ; \; b : \tilde{M} \to \mathbb{R}, \tag{1.68}$$

where $dx_1 \wedge ... \wedge dx_n = dA$. Follows from duality between $\left\{ \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right\}$ and $\{dx_1, ..., dx_n\}$ that

$$b(s,p) = (\Phi^* dV)(s,p) \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right) \tag{1.69}$$

$$\tag{1.70}$$

and by pull back definition,

$$b(s,p) = Vol \left[\left.\frac{\partial \Phi}{\partial t}\right|_{t=s}, d\Phi_s \left(\frac{\partial}{\partial x_1} \right), ..., d\Phi_s \left(\frac{\partial}{\partial x_n} \right) \right]_p \tag{1.71}$$

$$= det \left[\left.\frac{\partial \Phi}{\partial t}\right|_{t=s}, d\Phi_s \left(\frac{\partial}{\partial x_1} \right), ..., d\Phi_s \left(\frac{\partial}{\partial x_n} \right) \right]_p \tag{1.72}$$

$$= \left\langle \left.\dot{\Phi}\right|_{t=s}, \nu(s,p) \right\rangle . \tag{1.73}$$

Then,

$$V'(t_0) = \frac{d}{dt} \int_0^{t_0} \int_{\Sigma} \left\langle \left.\dot{\Phi}\right|_{t=s}, \nu(s,p) \right\rangle \; d\Sigma_t = \int_{\Sigma} \left\langle \left.\dot{\Phi}\right|_{t=t_0}, \nu(t_0,p) \right\rangle \; d\Sigma_{t_0}. \tag{1.74}$$

Hence,

$$V'(0) = \int_{\Sigma} \left\langle \dot{\Phi}, \nu(0,p) \right\rangle \; d\Sigma = \int_{\Sigma} f \; d\Sigma. \tag{1.75}$$

■

Proposition 1.1.3 *An immersion $\phi : \Sigma^n \to B^{n+1} \subset \bar{\Sigma}$ is capillary if and only if ϕ is a critical point of energy functional $\mathcal{E}$. In particular, ϕ is called free boundary if and only if ϕ is a critical point of area functional $\mathcal{A}$.*

Proof of Proposition 1.1.3: Suppose ϕ capillary, i.e., ϕ is CMC and $\phi(\partial\Sigma)$ meets ∂B in a constant contact angle. Thus, follows from (1.13) that

$$\langle \nu, \bar{N}\rangle = \langle \eta, \bar{\eta}\rangle \equiv \text{constant, along } \partial\Sigma. \tag{1.76}$$

Consider Φ an admissible volume-preserving smooth variation, so

$$\begin{aligned}\langle \dot{\Phi}, \eta - \cos\gamma\, \bar{\eta}\rangle &= \langle \dot{\Phi}, \eta\rangle - \cos\gamma\, \langle \dot{\Phi}, \bar{\eta}\rangle \\ &= \langle \dot{\Phi}, \bar{\eta}\rangle\langle \eta, \bar{\eta}\rangle + \langle \dot{\Phi}, \bar{N}\rangle\langle \eta, \bar{N}\rangle - \cos\gamma\, \langle \dot{\Phi}, \bar{\eta}\rangle \\ &= \cos\gamma\langle \dot{\Phi}, \bar{\eta}\rangle + \langle \dot{\Phi}, \bar{N}\rangle\langle \eta, \bar{N}\rangle - \cos\gamma\, \langle \dot{\Phi}, \bar{\eta}\rangle \\ &= \langle \dot{\Phi}, \bar{N}\rangle\langle \eta, \bar{N}\rangle \\ &= 0,\end{aligned} \tag{1.77}$$

because of (1.14) and of the admissibility from Φ. Hereafter, follows from (1.65) and (1.77) that

$$\mathcal{E}'(0) = -\int_{\Sigma} nHf\, d\Sigma + \int_{\partial\Sigma} \langle \dot{\Phi}, \eta - (\cos\gamma)\bar{\eta}\rangle\, ds = -\int_{\Sigma} nHf\, d\Sigma. \tag{1.78}$$

As ϕ is CMC and Φ volume-preserving, then $\mathcal{E}'(0) = 0$.

Reciprocally, we suppose that ϕ is a critical point of energy functional $\mathcal{E}$.

Let Φ be an admissible volume-preserving variation with compact support, such that $f = \langle \dot{\Phi}, \nu\rangle$. Thus,

$$0 = \mathcal{E}'(0) = -\int_{\Sigma} nHf\, d\Sigma,\ \forall\, f \in H^1(\Sigma);\ \int_{\Sigma} f\, d\Sigma = 0, \tag{1.79}$$

where $H^1(\Sigma)$ denotes the Sobolev space of Σ, then, follows from Lemma 1.1.2 that

$$\phi \text{ is CMC }. \tag{1.80}$$

Now, consider the variation define by $\dot{\Phi} = g\langle \eta, \bar{\eta}\rangle\bar{\eta}$, where $g \in C^{\infty}(\partial\Sigma)$. Observe that, $0 = \langle \bar{\eta}, \bar{N}\rangle$, so Φ is admissible. And yet, $f = \langle \dot{\Phi}, \nu\rangle \equiv 0$, i.e., Φ is volume-preserving. Then, follows from (1.80) and (1.65) that

$$0 = \mathcal{E}'(0) = \int_{\partial\Sigma} g\langle \dot{\Phi}, \eta - \cos\gamma\, \bar{\eta}\rangle\, ds \tag{1.81}$$

$$= \int_{\partial\Sigma} g\langle \eta, \bar{\eta}\rangle^2 - g\cos\gamma\, \langle \eta, \bar{\eta}\rangle\, ds \tag{1.82}$$

$$= \int_{\partial\Sigma} g\langle \eta, \bar{\eta}\rangle(\langle \eta, \bar{\eta}\rangle - \cos\gamma)\, ds,\ \forall\, g \in C^{\infty}(\partial\Sigma). \tag{1.83}$$

Hence, follows from Lemma 1.1.2 that, $\langle \eta, \bar{\eta}\rangle(\langle \eta, \bar{\eta}\rangle - \cos\gamma) = 0$.

If $\gamma = \frac{\pi}{2}$, so we have $\langle \eta, \bar{\eta}\rangle = 0$ and $\eta = \bar{N}$, along $\partial\Sigma$, i.e., ϕ is free boundary. If $\gamma \neq \frac{\pi}{2}$, $\langle \eta, \bar{\eta}\rangle \neq 0$ and $\cos\gamma = \langle \eta, \bar{\eta}\rangle$, along $\partial\Sigma$, i.e., ϕ is capillary.

■

A CMC immersion is called CMC capillary *stable* if, $\mathcal{E}''(0) \geq 0$ for all volume-preserving admissible variations of ϕ. In [6], definitive classification was made about CMC capillary immersions in the Euclidean ball and in balls in other ambient manifolds.

In the chapter three, we studied an analogous variational problem with respect to functional

$$\mathcal{F}(\epsilon) = \int_{\Sigma} F(\nu(\epsilon, p))\, d\Sigma_{\epsilon}, \tag{1.84}$$

where $F : \mathbb{S}^n \to \mathbb{R}^+$ is a smooth function and $\nu(\epsilon, p)$ is the normal unit vector of Σ_ϵ, at point p. We consider immersions $\phi : \Sigma^2 \to \Omega \subset \mathbb{R}^3$, where $\partial\Omega$ is a revolution surface, and *admissible variation*, $\Phi_\epsilon := \Phi(\epsilon, p) = \phi(p) + \epsilon(u\nu + \xi)$, i.e.,

$$\Phi_\epsilon(\mathrm{int}(\Sigma)) \subset \mathrm{int}(\Omega), \tag{1.85}$$

$$\Phi_\epsilon(\partial\Sigma) \subset \partial\Omega, \tag{1.86}$$

where ξ is the tangent part of *variation vector*, $\dot{\Phi} = u\nu + \xi$. The variation Φ will also be considered *volume preserving* for

$$V(\epsilon) = \int_{[0,\epsilon]\times\Sigma} \Phi^* dV. \tag{1.87}$$

where $\Phi^* dV$ is the *pullback* of canonical volume element of $\mathbb{R}^3$.

The first variation of $\mathcal{F}$ is given by

$$\partial_\epsilon \mathcal{F}\big|_\epsilon = -\int_\Sigma u\Lambda\, d\Sigma_\epsilon + \int_{\partial\Sigma} \langle \chi \times \dot{\Phi}, d\Phi \rangle \tag{1.88}$$

where $\chi(\nu) = DF(\nu) + F(\nu)\nu$, $\dot{\Phi} = u\nu + \xi$, $d\Phi := t ds$, $t := t(\epsilon, p) = \nu(\epsilon, p) \times \eta(\epsilon, p)$ and $\eta_\epsilon := \eta(\epsilon, p)$ is the exterior normal along $\partial\Sigma_\epsilon$.

An immersion ϕ is called *critical* if and only if $\mathcal{F}'(0) = \partial_\epsilon \mathcal{F}\big|_{\epsilon=0} = 0$, for all admissible volume preserving variation, i.e., for all variation $\Phi = \phi + \epsilon(u \cdot \nu + \xi)$ such that

$$\int_\Sigma u\, d\Sigma = 0 \;\; and \;\; \langle \dot{\Phi}, N \rangle\Big|_{\partial\Sigma} \equiv 0 \tag{1.89}$$

where N is the exterior unit normal to $\partial\Omega$. In special, if Φ is compact supported, we conclude that

$$\Lambda(p) := n\, H(p)\, F(\nu(0, p)) - \mathrm{div}_\Sigma DF(\nu(0, p)) \equiv \text{constante}, \tag{1.90}$$

where H is the mean curvature of Σ, DF is the gradient of F on $\mathbb{S}^2$ and the number $\Lambda(p)$ is called *anisotropic mean curvature* of Σ, at point p.

Example 1.1.9 *A plane* Π *(or a piece of it) is a CAMC surface with* $\Lambda \equiv 0$*. Indeed, let* $p \in \Pi$*, so*

$$\Lambda(p) = n\,H(p)\,F(\nu(p)) - div_{\Sigma} DF(\nu(p)) = 0, \tag{1.91}$$

because $H \equiv 0$*, as well as*

$$div_{\Sigma} DF(\nu(p)) = 0, \tag{1.92}$$

once $d\nu = 0$*, because* ν *is constant on* Π*.*

Considering compact supported variations, in [13], Koiso and Palmer found, under some conditions, examples of CAMC surfaces.

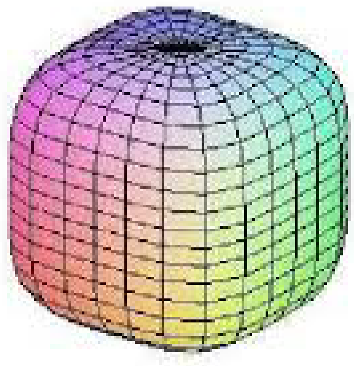

Figure 19 – Superellipsoid, an example of Wulff Shape [22].

Example 1.1.10 *The Wulff Shape of* F*, see Figure 19, defined by*

$$W_F := \partial \bigcap_{\nu \in \mathbb{S}^2} \Big\{ y \in \mathbb{R}^3 \mid \langle y, \nu \rangle \leq F(\nu) \Big\} \tag{1.93}$$

have constant anisotropic mean curvature $\Lambda < 0$ *(note that, when* $F \equiv 1$*,* W_F *is the* $\mathbb{S}^2$*). Associated to above Wulff Shape, we can construct the* ***Anisotropic Delaunay****: anisotropic catenoid, nodoid and unduloid, examples of CAMC surfaces. In Figure 20, the anisotropic catenoide associated to above superelipsoide.*

We will not be long on this subject due to the similarity with the classic case, for more details see [23], [13], [14], [22] and [24].

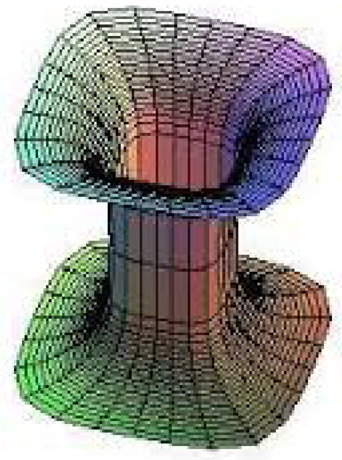

Figure 20 – Anisotropic catenoid with respect to superellipsoid presented above [22], $\Lambda \equiv 0$.

1.2 Maximum Principles

Let $A \subset \mathbb{R}^n$ an open set and

$$L(w) = \sum_{i,j} a_{ij}(x) w_{ij} + \sum_i b_i(x) w_i + c(x) w \tag{1.94}$$

where $w_i := \dfrac{\partial w}{\partial x_i}$, $w_{ij} := \dfrac{\partial w}{\partial x_i \partial x_j}$ and the functions a_{ij}, b_i and c are continuous on $\bar{A}$, a *differential elliptic operator on* A, i.e., the matrix $[a_{ij}(x)]$ is positive definite for all $x \in A$, that is,

$$0 < \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j, \ \forall\, x \in A, \ \forall\, \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.95}$$

We called L *uniformly elliptic on* A if, there exist a constant κ such that

$$\kappa|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j, \ \forall\, x \in A, \ \forall\, \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.96}$$

Now, we will present three maximum principles that we use during this work, especially in one step of the *Alexandrov Reflection Method* - ARM, and can be found in [25]. The first of them, for points $x \in \mathrm{int}A$:

Lemma 1.2.1 *Let L be an elliptic operator as in (1.94) and $w \in C^2(A)$ a function such that*

$$L(w) \geq 0, \ \ on\ A. \tag{1.97}$$

If exist $x_0 \in A$ such that $w(x_0) = 0$ and $w \leq 0$ on A, then $w \equiv 0$ on A.

The second, for points $x \in \partial A$ such that ∂A is of class C^1.

Lemma 1.2.2 *Let L be an uniformly elliptic operator as in (1.94), let A be a region in $\mathbb{R}^2$ and suppose that in a neighborhood of $x_0 \in \partial A$, ∂A is of class C^1. If*

$$L(w) \geq 0, \text{ on } A, \tag{1.98}$$

$w(x_0) = 0$, $w(x) \leq 0$, $\forall\, x \in \bar{A}$, and $\frac{\partial w}{\partial \nu} = 0$, where ν is the inward normal derivative, then $w \equiv 0$, on $\bar{A}$.

Finally, the third, for points $x \in \partial A$, at a corner.

Lemma 1.2.3 (Serrin's Boundary Point Lemma at a Corner [26]) *Let $A \subset \mathbb{R}^2$ be a bounded region which has a C^2 boundary in a neighborhood of $x_0 \in \partial A$. Consider T be a normal plane to ∂A at x_0 and A^+ be that component of A lying on one side of T which contains x_0 in its closure. Let L be an uniformly elliptic operator on A^+. Suppose also that*

$$|\sum_{i,j} a_{ij}(x)\xi_i\nu_j| \leq K \cdot [|(\xi \cdot \nu)| + |\xi| d] \tag{1.99}$$

for some constant $K > 0$, all $x \in \overline{A}^+$, any $\xi = (\xi_1, ..., \xi_n)$, where $\nu = (\nu_1, ..., \nu_n)$ is an unit normal to T, and where d is the distance from x to T.

Let $w \in C^2(\overline{A}^+)$ satisfy $L(w) \geq 0$ on $\overline{A}^+$ and suppose that $w(x_0) = 0$, $w(x) \leq 0$, for all $x \in \overline{A}^+$, and that $\partial w / \partial s = \partial^2 w / \partial s^2 = 0$, in any direction which enters A^+ non-tangentially at x_0.

2 A classification for the critical catenoid

In this chapter, we present results that classify surfaces with boundary by means of its mean curvature, topology and boundary properties. In the first section, we present some classification results of embedded surfaces; in the second section we show the path travelled by McGrath in [9] to obtain a classification for the embedded critical catenoid in an euclidean ball. Finally, in the third section we will present a classification for embedded annulus capillary in an euclidean ball. In special, as well as McGrath, we give our classification for the critical catenoid in an euclidean ball.

2.1 Classification of embedded CMC surfaces

Let Σ be an orientable compact connected immersed surface with constant mean curvature such that $\partial\Sigma \neq \emptyset$. In this section, we present the classical result due to Nitsche [1] and other similar results due to Ros and Souam [2], Koiso [8] and Wente [25]. They classified Σ from its topology and properties of the boundary.

Nitsche supposed that Σ is an immersed CMC topological disk such that $\partial\Sigma$ meets orthogonally $\mathbb{S}^2$ and proof the below result, see [1]:

Theorem 2.1.1 (Nitsche) *Let Σ^2 be an immersed minimal surface of disc type and free boundary in the euclidean ball $\mathbb{B}^3$. Then, Σ is the flat disk and $\partial\Sigma$ is an equator.*

Juncheol Pyo, in his work [7], considered an immersed minimal annulus and made different suppositions about $\partial\Sigma$. First, he supposed that $\partial\Sigma$ consists of two $C^{2,\alpha}$ planar Jordan curves Γ_1 and Γ_2, where Σ makes a constant contact angle with a plane Π_i along Γ_i, $i \in \{1, 2\}$, $\Pi_1 \neq \Pi_2$. More precisely, to prove the

Theorem 2.1.2 (Pyo) *Let Σ be an immersed minimal annulus such that $\partial\Sigma$ consists of two $C^{2,\alpha}$ planar Jordan curves Γ_1 and Γ_2. If Σ makes a constant contact angle with a plane Π_i along Γ_i, $i = 1, 2$, $\Pi_1 \neq \Pi_2$, then Σ is part of the catenoid.*

In a second moment, he supposed that Γ, one component of $\partial\Sigma$, is a circle and Σ meets a plane along Γ at a constant angle and he showed:

Theorem 2.1.3 (Pyo) *Let Σ be an immersed minimal surface with boundary and let Γ be one of the components of $\partial\Sigma$. If Γ is a circle and Σ meets a plane along Γ at a constant angle, then Σ is part of the catenoid.*

Two proofs can be found in [7] and will not be included here. Ros and Souam [2] considered the euclidean ball as ambient space and classed:

Proposition 2.1.1 *Let $\varphi : \Sigma^n \to \mathbb{B}^{n+1}$ be a capillary embedding in an euclidean ball. Assume that $\varphi(\partial\Sigma)$ is contained in an open hemisphere of $\partial\mathbb{B}$, then $\varphi(\Sigma)$ is a totally geodesic disk or a spherical cap.*

Ros and Souam used ARM (below, we explain this strategy better) to prove their proposition, as well as we will do. Other authors who also used the ARM were Miyuki Koiso [8] and Henry C. Wente [25].

Suppose that Γ^n is a compact connected topological sub-manifold without boundary of $\mathbb{R}^{n+1}$. Then, see [8], $\mathbb{R}^{n+1} \setminus \Gamma$ is the union of two regions one bounded and another unbounded which we will define *inside of* Γ and *outside of* Γ, respectively. In [8] there are the following results:

Theorem 2.1.4 (Koiso) *Let Γ_0 be a (n-1)-dimensional sphere in some hyperplane $\pi \subset \mathbb{R}^{n+1}$ and $\Sigma \subset \mathbb{R}^{n+1}$ a compact C^2-hypersurface with Γ_0 as its boundary. Assume that Σ does not intersect the outside of Γ_0 in π. Then, if Σ is of non-zero constant mean curvature, Σ must be a spherical cap.*

And

Theorem 2.1.5 (Koiso) *Let E be a $(n-1)$-dimensional linear subspace of $\mathbb{R}^{n+1}$, and let e_1, e_2 be two unit vectors which are both perpendicular to E and perpendicular to each other. Suppose that Γ is an $(n-1)$-dimensional C^2 sub-manifold of $\mathbb{R}^{n+1}$ which satisfies the following condition: There exist a bounded domain $A \subset E$ and a real-valued C^1 function f on $\bar{A}$ such that f is positive in A, identically zero on ∂A, and*

$$\Gamma = \{x + f(x) \cdot e_1 \; ; \; x \in A\} \cup \{x - f(x) \cdot e_1 \; ; \; x \in A\} \tag{2.1}$$

Suppose that Σ is a compact C^2 hypersurface with Γ as its boundary and that Σ does not intersect with the outside of Γ in the hyperplane $\pi_1 = E + \mathbb{R} \cdot e_1$. Then Σ is orientable, and if Σ is of non-zero constant mean curvature, then Σ is symmetric with respect to the hyperplane $\pi_2 = E + \mathbb{R} \cdot e_2$.

Note that, in Theorem 2.1.5, by assuming the symmetry of Γ we got a gift: the symmetry for Σ. Talking about symmetries, in [25], in the case $n = 2$, for example, under some assumptions, it is proved that a liquid-air interface Σ, is an axially symmetric surface, in the sense that there is an axis such that any nonempty intersection of Σ with an horizontal hyperplane is an open disk whose center lies on the vertical axis. Besides that, [25] as well as [8], are nice references for the ARM.

2.2 Classification of embedded CMC annulus capillary in a ball.

In [27], we can find the question that motivates this section.

Conjecture 2.2.1 (Fraser and Li) *The critical catenoid is the unique properly embedded free boundary minimal annulus in $\mathbb{B}^3$, up to rotations.*

Philosophically, there exists a parallel between the conjecture 2.2.1 and

Conjecture 2.2.2 (Lawson) *The Clifford Torus is the only embedded minimal torus in $\mathbb{S}^3$, up to rotations.*

The Lawson's conjecture was definitively resolved by Brendle in [10]. However, there was previously a partial demonstration due to Ros, in [11]:

Theorem 2.2.1 (Ros) *Let $\Sigma \subset \mathbb{S}^3$ be an embedded minimal torus, symmetric with respect to the coordinated hyperplanes of $\mathbb{R}^4$. Then Σ is the Clifford torus.*

In the case of the conjecture (2.2.1), there is an analogous result to that obtained by Ros, due to McGrath [9]:

Theorem 2.2.2 (McGrath) *Let $\Sigma^2 \subset \mathbb{B}^n$, $n \geq 3$, be an embedded free boundary minimal annulus. If Σ is invariant under reflection through three orthogonal hyperplanes Π_i, $i = 1, 2, 3$, then Σ is the critical catenoid, up to rotation.*

Is well known that, if Σ is a minimal free boundary surface in $\mathbb{B}^3$, then its coordinated functions are solutions for the Steklov Problem

$$\begin{cases} \Delta u = 0, & \text{on } \Sigma. \\ \dfrac{\partial u}{\partial \eta} = u, & \text{along } \partial\Sigma. \end{cases} \tag{2.2}$$

where η is the unit normal vector outward of $\mathbb{S}^2$, see [16]. In his proof, McGrath uses the result below, that can be found in [12].

Theorem 2.2.3 (Fraser and Schoen) *Suppose Σ is a free boundary annulus in $\mathbb{B}^n$ such that the coordinated functions are first Steklov eigenfunctions. Then $n = 3$ and Σ is congruent to the critical catenoid.*

Still needing confirmation, we have the recent result [28]:

Theorem 2.2.4 (Liu and Yu) *An immersed minimal free boundary annulus in unit ball $\mathbb{B}^3 \subset \mathbb{R}^3$ is congruent to the critical catenoid.*

In this work, we take another path - ARM.

2.3 ARM and our results

Let $\Sigma \subset \mathbb{B}^3$ an orientable compact connected embedded annulus minimal free-boundary, such that

$$\operatorname{int}(\Sigma) \subset \operatorname{int}(\mathbb{B}^3) \tag{2.3}$$

$$\partial\Sigma = \Gamma \cup \Gamma' \subset \partial\mathbb{B}^3 \tag{2.4}$$

where Γ and Γ' are the connected components of the boundary of Σ.

Consider a hyperplane $\Pi \subset \mathbb{R}^3$ and R_Π the map such that $R_\Pi(x)$ is the orthogonal reflection of x through Π. If $R_\Pi(\Sigma) = \Sigma$, we say that Σ is Π-*invariant*. Note that, the map $R_\Pi : \Sigma \to \Sigma$ is an isometry such that $\partial\Sigma \mapsto \partial\Sigma$ and $\operatorname{int}(\Sigma) \mapsto \operatorname{int}(\Sigma)$. From now on, consider

$$\Pi_i := \{(x_1, x_2, x_3) \mid x_i = 0\}, \tag{2.5}$$

Let $G = \{R_{\Pi_1}, R_{\Pi_2}, R_{\Pi_3}\}$ be the group of reflection with respect to the coordinated planes. We say that Σ is $G-$ *invariant* if, $R_{\Pi_i}(\Sigma) = \Sigma$, for all $i \in \{1, 2, 3\}$. In this work, we considered $\partial\Sigma = \Gamma \cup \Gamma'$ G-invariant.

Then, we will prove that the property G-invariant of $\partial\Sigma$ implies that it intersects each of the eight octants and there exists a plane such that Γ' is the reflection of Γ through it.

Lemma 2.3.1 *Let $\Sigma^2 \subset \mathbb{B}^3$ an embedded annulus such that $\partial\Sigma$ is G-invariant. Then,*

(i) *there exist $i, j \in \{1, 2, 3\}$, $i \neq j$, such that*

$$\Gamma = R_{\Pi_i}(\Gamma) = R_{\Pi_j}(\Gamma), \tag{2.6}$$

$$\Gamma' = R_{\Pi_i}(\Gamma') = R_{\Pi_j}(\Gamma'), \tag{2.7}$$

and

(ii) *there exists $k \in \{1, 2, 3\}$, $k \notin \{i, j\}$, such that*

$$\Gamma' = R_{\Pi_k}(\Gamma) \ \textit{and} \ \Gamma = R_{\Pi_k}(\Gamma') \tag{2.8}$$

Proof of Lemma 2.3.1: Let $\partial\Sigma = \Gamma \cup \Gamma'$, where Γ and Γ' are the connected components of the boundary of Σ, and $\Gamma \cap \mathcal{O} =: \gamma : [0, 1] \to \partial\Sigma$, where $\mathcal{O} = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1, x_2, x_3 \geq 0\}$. As $\partial\Sigma$ is G-invariant, we can find it by joining the possible reflections of γ, i.e., there are $i, j, k \in \{1, 2, 3\}$, different from each other, such that

$$\partial\Sigma = \gamma \cup \tilde{\gamma} \cup R_{\Pi_k}(\tilde{\gamma}) \tag{2.9}$$

where

$$\tilde{\gamma} := \gamma \cup R_{\Pi_i}(\gamma) \cup R_{\Pi_j}(\gamma) \cup (R_{\Pi_i} \circ R_{\Pi_j})(\gamma) \subset \partial\Sigma \tag{2.10}$$

As Σ is embedded, $\gamma(0,1)$ does not intersect Π_i, $i \in \{1,2,3\}$, since $\partial\Sigma$ is G-invariant, otherwise Γ would have self intersections. Once Γ and Γ' are closed curves, $\gamma(0) \in \Pi_i$ and $\gamma(1) \in \Pi_j$, $i \neq j$, because $\partial\Sigma$ is G-invariant. Indeed, if $\gamma(1) \notin \Pi_j$, then there exist $p_j \in \Pi_j$ such that $d(\gamma(1), p_j) = d > 0$ and follows from (2.9) and (2.10) that, $\partial\Sigma$ would not be the union of closed curves and this would be a contradiction, see Figure 21.

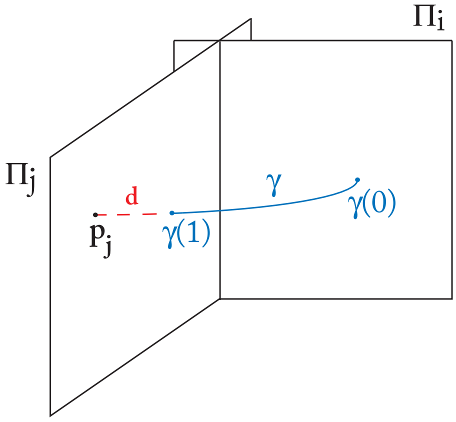

Figure 21 – $\gamma(1) \notin \Pi_j$.

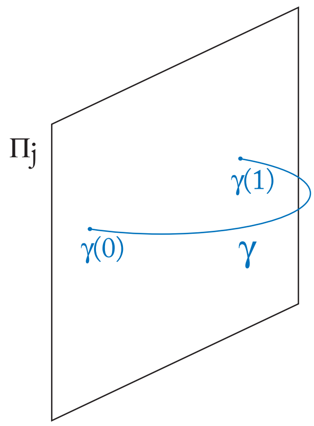

Figure 22 – $\gamma(0), \gamma(1) \in \Pi_i = \Pi_j$.

In the other hand, if $\gamma(0), \gamma(1) \in \Pi_i = \Pi_j$, see Figure 22, then the curve $\gamma \cup R_{\Pi_i}(\gamma) := \beta : [a,b] \to \partial\Sigma$ would be a closed curve contained in $\partial\Sigma$. Thus, the curves β, $R_{\Pi_j}(\beta)$, $R_{\Pi_k}(\beta)$ and $(R_{\Pi_j} \circ R_{\Pi_k})(\beta)$ would be closed curves contained in $\partial\Sigma$, but this is also a contradiction, since Σ is a topological annulus.

Then, if we define $\Gamma := \tilde{\gamma}$, we have **(i)** and **(ii)**.

■

Let P_λ, $\lambda \in \mathbb{R}$, be a family of planes parallel to each other. We call *moving planes*, the process of changing the parameter λ, from the geometric view point, we have a movement between this parallel planes.

Soon, we will present the main result of this chapter, obtained from ARM. This method can be divided into the following steps:

(1) Consider a subsidiary plane P and an arbitrary family, P_λ, $\lambda \in \mathbb{R}$, of parallel planes each other and orthogonal to P (in our case, P will be the plane Π_k provided by Lemma 2.3.1).

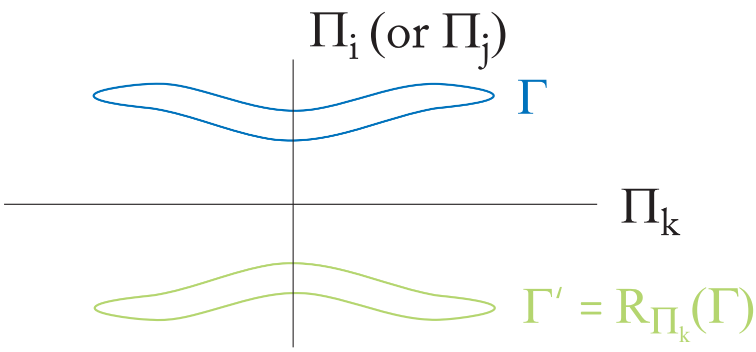

Figure 23 – $\Gamma' = R_{\Pi_k}(\Gamma);\ k \in \{1,2,3\} \setminus \{i,j\}$.

(2) Varying the parameter λ, a *moving planes* process is started by means of family P_λ. For some $\lambda \in \mathbb{R}$, $P_\lambda \cap \Sigma \neq \emptyset$ and can be considered the reflection, through P_λ, of the part of Σ surpassed by P_λ.

(3) For a critical parameter, λ^*, it is considered the reflection through P_{λ^*} of the part of Σ surpassed by P_{λ^*}, see Figure 24.

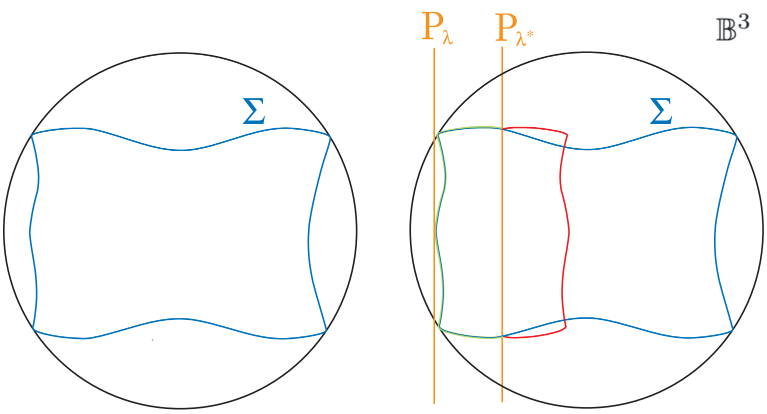

Figure 24 – In red, the reflection, through P_{λ^*}, of the part of Σ surpassed by P_{λ^*}.

(4) Considering an appropriate coordinated system, we use a suitable maximum principle and it is concluded that the reflection, through P_{λ^*}, of the part of Σ surpassed by P_{λ^*} coincide, locally, with the part of Σ non surpassed by P_{λ^*}.

(5) The single continuation principle is used and it is concluded that the reflection, through P_{λ^*}, of the part of Σ surpassed by P_{λ^*} coincide with the part of Σ non surpassed by P_{λ^*}.

(6) Finally, from arbitrariness of P_λ, it is concluded that Σ is symmetrical rotationally.

A natural question around the steps above:

Question 2.3.1 *How to determine the critical parameter λ^*?*

Consider

$$\Lambda \text{ the region bounded by } C_+ \cup \Sigma \cup C_- \ \subset \ \mathbb{B}^3, \tag{2.11}$$

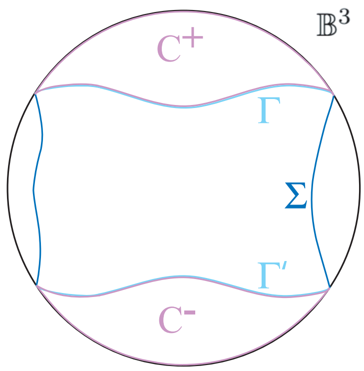

Figure 25 – Λ is the connected region bounded by $C_+ \cup \Sigma \cup C_- \subset \mathbb{B}^3$.

where C_+ is the upper portion of $\mathbb{S}^2$ such that $\partial C_+ = \Gamma$; C_- is the lower portion of $\mathbb{S}^2$ such that $\partial C_- = \Gamma'$. As Σ is embedded, Λ is connected (Figure 25).

Define

$$\lambda^- = \min\{\lambda \in \mathbb{R}\,;\; P_\lambda \cap \Sigma \neq \emptyset\}, \tag{2.12}$$

$$\lambda^+ = \max\{\lambda \in \mathbb{R}\,;\; P_\lambda \cap \Sigma \neq \emptyset\}, \tag{2.13}$$

and observe that, as $\Sigma \subset \mathbb{B}^3$, so $-1 \leq \lambda^- < \lambda^+ \leq 1$. To better organize the text, consider the following definition:

(i) Σ_λ being *the part of Σ between P_{λ^-} and P_λ*, $\lambda \in (\lambda^-, \lambda^+)$, is that, the part of Σ surpassed by P_λ;

(ii) $\widetilde{\Sigma}_\lambda$ being *the reflection of Σ_λ through P_λ*;

(iii) $\Sigma \backslash \Sigma_\lambda$ being *the part of Σ between P_λ and P_{λ^+}*, $\lambda \in (\lambda^-, \lambda^+)$, is that, the part of Σ non surpassed by P_λ.

For some value of parameter λ, called λ^*, we say that the reflected part, $\widetilde{\Sigma}_\lambda$, *definitely extrapolates* Λ if,

$$\exists\, x^* \in \widetilde{\Sigma}_\lambda\,;\; x^* + \mu \cdot N_P \notin \Lambda,\ \forall\, \mu > 0, \tag{2.14}$$

where N_P is the unit normal vector to family P_λ, pointing in the sense of increasing λ. Thus, it is defined the critical parameter of moving planes process with respect to family P_λ, see Figure 26. Note that, we should not worry with the possibility of $\widetilde{\Sigma}_{\theta,\lambda}$ definitively extrapolates Λ by C^+ or C^- and also in the possibility of a point along $\partial\widetilde{\Sigma}_{\theta,\lambda}$ definitively extrapolates Λ, at a point $p \in \mathrm{int}\Lambda$, because $\partial\Sigma$ is G-invariant and due to spherical geometry (this is other relevance of $\partial\Sigma$ to be G-invariant).

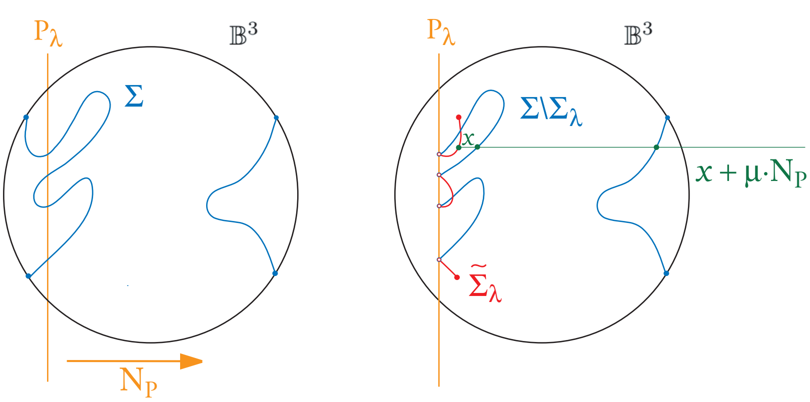

Figure 26 – The moving planes process in two moments, λ and λ^*.

Observation 2.3.1 *We created and adopted the concept definitively extrapolates instead of touching, the latter already existing in the literature, to avoid the possibility of a boundary point of $\widetilde{\Sigma}_\lambda$ intersects a interior point of $\Sigma\backslash\Sigma_\lambda$.*

Note that, being $P^\perp$ be the plane containing the origin and orthogonal to Π_3 and to family P_λ, there exist $p, q \in P^\perp \cap \Gamma$ such that

$$<p, e_3> = <q, e_3>, \tag{2.15}$$

because $\partial\Sigma$ is G-invariant, where $\{e_1, e_2, e_3\}$ is the canonical base in $\mathbb{R}^3$. Define σ^p, σ^q the connected component of $P^\perp \cap \Sigma$ that contain p and q, respectively. As Σ is embedded, either $\sigma^p = \sigma^q$ or $\sigma^p \cap \sigma^q = \emptyset$, for an illustration see Figure 27.

This extrapolation may occur in the following ways:

(P1) At a point x^* on $\mathrm{int}(\widetilde{\Sigma}_{\lambda^*}) \cap \mathrm{int}(\Sigma\backslash\Sigma_{\lambda^*})$.

(P2) At a point $x^* \in \partial\widetilde{\Sigma}_{\lambda^*} \cap \partial(\Sigma\backslash\Sigma_{\lambda^*})$.

(P3) At a point x^* such that $T_{x^*}\Sigma \perp P_{\lambda^*}$.

(P4) At a point x^* such that $T_p\partial\Sigma \perp P_{\lambda^*}$.

Now that the ARM was presented, we can state our main result:

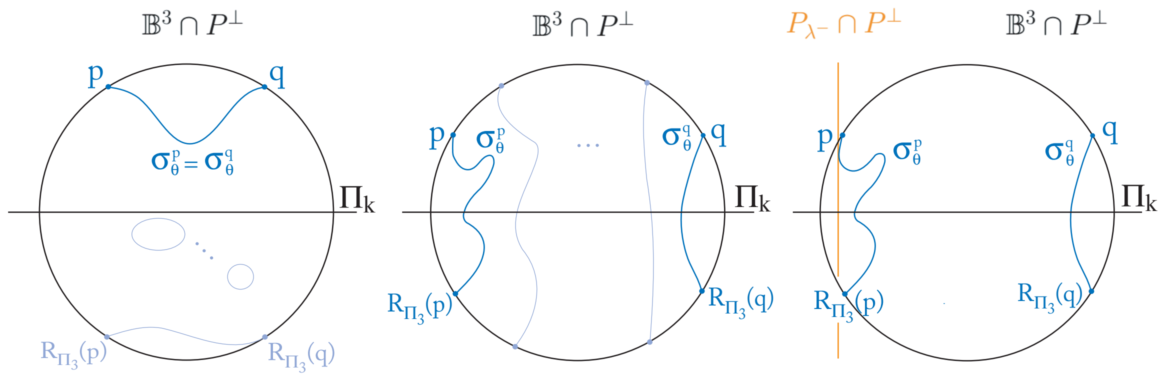

Figure 27 – σ^p, σ^q and the other connected components of $P^\perp \cap \Sigma$. On the right side figure $\sigma^p \cap \sigma^q = \emptyset$.

Theorem 2.3.1 *Let $\Sigma^2 \subset \mathbb{B}^3$ be an embedded CMC capillary annulus, such that $\partial\Sigma$ is symmetrical with respect to the coordinated planes, then Σ is rotationally symmetric.*

Proof of Theorem 2.3.1: Let $\Sigma \subset \mathbb{B}^3$ an embedded annulus capillary such that $\partial\Sigma = \Gamma \cup \Gamma'$ is G-invariant and

$$\mathrm{int}(\Sigma) \subset \mathrm{int}(\mathbb{B}^3) \tag{2.16}$$

$$\partial\Sigma = \Gamma \cup \Gamma' \subset \partial\mathbb{B}^3 \tag{2.17}$$

As $\partial\Sigma$ is G-invariant, follows from Lemma 2.3.1 that there exists a coordinated plane, without loss of generality, let's say Π_3, such that

$$\Gamma' = R_{\Pi_3}(\Gamma) \ \text{ and } \ \Gamma = R_{\Pi_3}(\Gamma') \tag{2.18}$$

and

$$\Gamma = R_{\Pi_1}(\Gamma) = R_{\Pi_2}(\Gamma) \ \text{ and } \ \Gamma' = R_{\Pi_1}(\Gamma') = R_{\Pi_2}(\Gamma'). \tag{2.19}$$

Consider the family $\mathcal{T}_\theta$ of orthogonal planes to Π_3 and parallels to each other, such that

$$T_{\theta,\lambda} = \{(x_1, x_2, x_3) \in \mathbb{R}^3; \cos\theta \cdot x_1 + \sin\theta \cdot x_2 = \lambda\} \in \mathcal{T}_\theta, \tag{2.20}$$

where $\theta \in [0, \pi)$ and $\lambda \in \mathbb{R}$. Let

$$\lambda_\theta^- = \min\{\lambda \in \mathbb{R} \ ; \ T_{\theta,\lambda} \cap \Sigma \neq \emptyset\}, \tag{2.21}$$

$$\lambda_\theta^+ = \max\{\lambda \in \mathbb{R} \ ; \ T_{\theta,\lambda} \cap \Sigma \neq \emptyset\}, \tag{2.22}$$

Define $\Sigma_{\theta,\lambda}$ as the part of Σ between $T_{\theta,\lambda_\theta^-}$ and $T_{\theta,\lambda}$, $\lambda \in (\lambda_\theta^-, \lambda_\theta^+)$, precisely

$$\Sigma_{\theta,\lambda} := \{x \in \Sigma \ ; \ \lambda_\theta^- \leq \cos\theta \cdot x_1 + \sin\theta \cdot x_2 \leq \lambda\}. \tag{2.23}$$

We will call $\widetilde{\Sigma}_{\theta,\lambda}$ the reflection of $\Sigma_{\theta,\lambda}$ through $T_{\theta,\lambda}$, i.e.,

$$\widetilde{\Sigma}_{\theta,\lambda} := \{x \in \Sigma \, ; \; \lambda \leq \cos\theta \cdot x_1 + \sin\theta \cdot x_2 \leq \lambda + \lambda_\theta^-\}, \tag{2.24}$$

and $\Sigma \backslash \Sigma_{\theta,\lambda}$ the part of Σ between $T_{\theta,\lambda}$ and $T_{\theta,\lambda_\theta^+}$, $\lambda \in (\lambda_\theta^-, \lambda_\theta^+)$, see Figure 28.

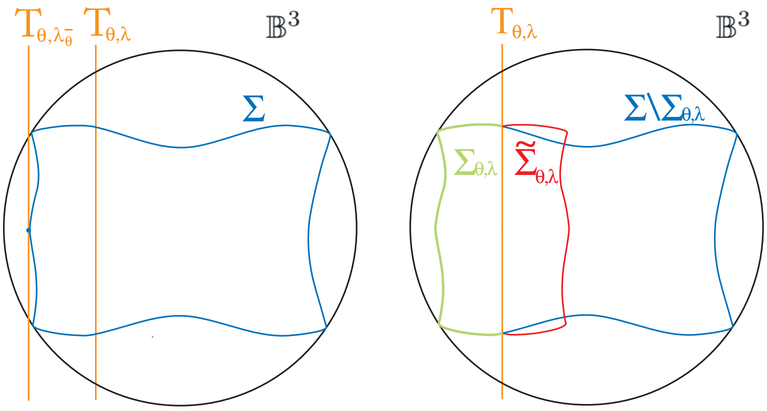

Figure 28 – $\Sigma_{\theta,\lambda}$, $\widetilde{\Sigma}_{\theta,\lambda}$ and $\Sigma\backslash\Sigma_{\theta,\lambda}$.

For example, note that for $\theta = \lambda = 0$, $T_{0,0} = \Pi_1$ and for $\theta = \frac{\pi}{2}$ and $\lambda = 0$, $T_{\frac{\pi}{2},0} = \Pi_2$. In these cases, $\partial\widetilde{\Sigma}_{\theta,\lambda} = \partial\,(\Sigma \setminus \Sigma_{\theta,\lambda})$, see Figure 29.

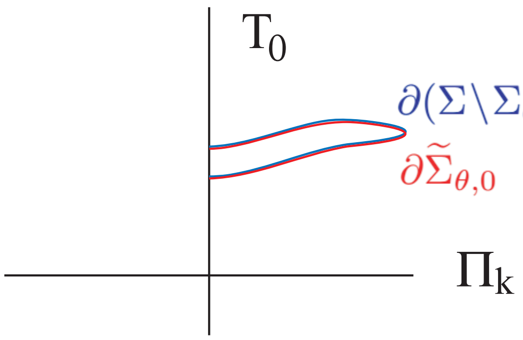

Figure 29 – $\partial\widetilde{\Sigma}_{\theta,0} = \partial(\Sigma\backslash\Sigma_{\theta,0})$, for $\theta \in \{0, \frac{\pi}{2}\}$.

Now, we start the moving planes process from $\lambda = \lambda_\theta^-$ until $\lambda = \lambda_\theta^*$, by means the family $\mathcal{T}_\theta$.

Let x^* the extrapolation point, for some of the possibilities **(P1)** $\sim$ **(P2)**. Consider a coordinate system such that $x^* = (0,0,0)$ and smooth functions $u, v : \bar{A} \to \mathbb{R}$, where A is an open in $\mathbb{R}^2$ such that $(0,0) \in \bar{A}$, such that

$$u(0,0) = v(0,0) = 0 \tag{2.25}$$

and $\Sigma\backslash\Sigma_{\theta,\lambda_\theta^*} = \text{graph}(u)$ and $\widetilde{\Sigma}_{\theta,\lambda_\theta^*} = \text{graph}(v)$ in a neighborhood of $(0,0)$. Note that, as $\Sigma\backslash\Sigma_{\theta,\lambda_\theta^*}$ and $\widetilde{\Sigma}_{\theta,\lambda_\theta^*}$ are CMC (for the same constant), u and v satisfy the same CMC equation. Hence, the function $w = v - u$ satisfy an homogeneous linear elliptic pde, see [25].

In the possibility **(P1)**, define a coordinate system such that $T_{x^*}\Sigma = \{z = 0\}$, where the axis z pointing to $T_{\theta,\lambda_\theta^*}$ and use the Lemma 1.2.1 to conclude that $w = 0$ in a neighborhood of $(0,0)$, i.e., $\widetilde{\Sigma}_{\theta,\lambda_\theta^*} = \Sigma\backslash\Sigma_{\theta,\lambda_\theta^*}$ in a neighborhood of x^*.

In **(P2)**, define a coordinate system such that $T_{x^*}\partial\Sigma = \{x = z = 0\}$ and $T_{x^*}\Sigma = \{z = 0\}$, where the axis z pointing to $T_{\theta,\lambda_\theta^*}$ and axis x point to $\text{int}A$. So, use the Lemma 1.2.2 to conclude that $w = 0$ in a neighborhood of $(0,0)$, i.e., $\widetilde{\Sigma}_{\theta,\lambda_\theta^*} = \Sigma\backslash\Sigma_{\theta,\lambda_\theta^*}$ in a neighborhood of x^*.

For the case **(P3)**, define a coordinate system such that $T_{x^*}\Sigma = \{z = 0\}$, the plane $T_{\theta,\lambda_\theta^*}$ coincide with $\{x = 0\}$, where the axis z pointing to $T_{\theta,\lambda_\theta^*}$ and axis x point to $\text{int}A$. So, use the Lemma 1.2.2 to conclude that $w = 0$ in a neighborhood of $(0,0)$, i.e., $\widetilde{\Sigma}_{\theta,\lambda_\theta^*} = \Sigma\backslash\Sigma_{\theta,\lambda_\theta^*}$ in a neighborhood of x^*.

In **(P4)**, define a coordinate system such that x^* is the origin of a coordinate system (x, y, z), $T_{x^*}\partial\Sigma = \{x = z = 0\}$ and $T_{x^*}\Sigma = \{z = 0\}$, where the axis z points into Σ and axis x pointing to $\widetilde{\Sigma}_{\theta,\lambda_\theta^*}$. So, use the Lemma 1.2.3 and the capillarity of Σ for conclude that $w = 0$ in a neighborhood of $(0,0)$, i.e., $\widetilde{\Sigma}_{\theta,\lambda_\theta^*} = \Sigma\backslash\Sigma_{\theta,\lambda_\theta^*}$ in a neighborhood of x^*.

Using the unique continuation we concluded that $\widetilde{\Sigma}_{\theta,\lambda_\theta^*} = \Sigma\backslash\Sigma_{\theta,\lambda_\theta^*}$.

Affirmation: $\lambda_\theta^* = 0, \ \forall\, \theta \in [0, \pi)$.

Indeed, suppose absurdly, that $\lambda_\theta^* < 0$.

Hereafter, as $\partial\Sigma_{\theta,\lambda_\theta^*} \subset \mathbb{S}^2$, then Σ is a surface whose boundary satisfy

$$\partial\Sigma = \partial\Sigma_{\theta,\lambda_\theta^*} \cup \partial\widetilde{\Sigma}_{\theta,\lambda_\theta^*}, \tag{2.26}$$

i.e., $\partial\Sigma$ does not contained in $\mathbb{S}^2$. Contradiction, because ϕ is admissible! Then, the affirmation is true.

Finally, as $\lambda_\theta^* = 0$ and $\widetilde{\Sigma}_{\theta,0} = \Sigma\backslash\Sigma_{\theta,0}$, $\forall\, \theta \in [0, \pi)$, because θ was taken arbitrarily, if Π is a plane parallel to Π_3, the straight line $r_\theta := \Pi \cap T_{\theta,0}$ intersects $\Sigma \cap \Pi$ orthogonally, for all $\theta \in [0, \pi)$. Besides that, as $\lambda_\theta^* = 0$, $\forall\, \theta \in [0, \pi)$, all these straight lines intersects each other at point $p_0 \in \Pi \cap$ eixo x_3, $\forall\, \theta \in [0, \pi)$, i.e., $\Sigma \cap \Pi$ is a circle.

Therefore, as θ was taken arbitrarily, Σ is symmetrical rotationally.

■

Observation 2.3.2 *Follows from above affirmation that, for example,* **(P3)** *does not occur for $\lambda < 0$. So, the curve defined by intersection $T_\theta^\perp \cap \Sigma$, where $T_\theta^\perp$ is the plane containing the origin and orthogonal to Π_3 and $\mathcal{T}_\theta$, can be represented, globally, as the graphic of a smooth function $f(z)$, where $z \in I \subset T_\theta^\perp \cap T_{\theta,0}$, see Figure 30. Thus, not exist the possibility of a boundary point of $\widetilde{\Sigma}_\lambda$ intersects $\Sigma\backslash\Sigma_\lambda$, i.e., we could considered the concept touching from the start.*

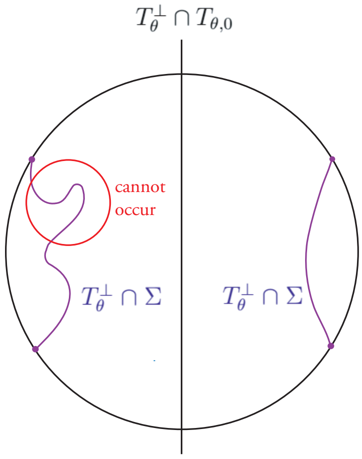

Figure 30 – As **(P3)** does not occur for $\lambda < 0$, the circled part does not occur either.

In Theorem 2.2.2, McGrath assume that an embedded annulus minimal free boundary, $\Sigma \subset \mathbb{B}^n$, is G-invariant, to prove that Σ is the critical catenoid. From Theorem 2.3.1, we improved the result of McGrath [9], to $n = 2$, because we assume only that $\partial\Sigma$ is G-invariant.

Corollary 2.3.1 *Let $\Sigma^2 \subset \mathbb{B}^3$ be an embedded annulus minimal free boundary. If $\partial\Sigma$ is G-invariant, then Σ is the critical catenoid.*

Proof of Corollary 2.3.1: Follows directly of proof from Theorem (2.3.1) and of fact that the critical catenoid is the only minimal surface rotationally symmetric free boundary in $\mathbb{B}^3$.

$\blacksquare$

With this methodology, we also get a new demonstration for

Theorem 2.3.2 (Pyo) *Let Σ^2 be an embedded minimal surface in $\mathbb{R}^3$ with two boundary components and let Γ be one component of $\partial\Sigma$. If Γ is a circle and Σ meets a plane along Γ at a constant angle, then Σ is part of the catenoid.*

Proof of Theorem 2.3.2: Let Π the plane that contain Γ and $\mathcal{T}_\theta$ be a family of parallel planes with each other and orthogonal to Π.

As Γ is a circle, during the moving plane process, for some value of the parameter λ, $\widetilde{\Sigma}_{\theta,\lambda}$ definitively extrapolates $\Sigma\backslash\widetilde{\Sigma}_{\theta,\lambda}$, of some of the forms **(P1)** $\sim$ **(P4)**. As we have no information about Γ', another component connected of $\partial\Sigma$, we cannot say anything about the occurrence of the cases **(P2)** and **(P4)**. If the cases **(P1)** or **(P3)** occur for some $\lambda_\theta^* \leq 0$, we have by ARM that

$$\widetilde{\Sigma}_{\theta,\lambda_\theta^*} \text{ coincide to } \Sigma\backslash\widetilde{\Sigma}_{\theta,\lambda_\theta^*} \tag{2.27}$$

in the neighborhood of the some point $p \in \text{int}\Sigma$, because Σ is capillary. Repeatedly applying the maximum principle until we get to Γ, we get

$$\Gamma = \Gamma_{\theta,\lambda_\theta^*} \cup \tilde{\Gamma}_{\theta,\lambda_\theta^*}, \tag{2.28}$$

But since Γ is a circle, it follows of the circular geometry of Γ, which $\lambda_\theta^* = 0$.

Consider r the orthogonal straight line to $T_{\theta,0}$ passing through the center of Γ and let p the point given by the intersection between Γ, r and $\Sigma\backslash\widetilde{\Sigma}_{\theta,0}$. Once Γ is a circle and **(P1)** and **(P3)** do not occur for $\lambda < 0$, we have $\widetilde{\Sigma}_{\theta,0}$ stays above $\Sigma\backslash\widetilde{\Sigma}_{\theta,0}$, relative to $\tilde{\nu}$, normal vector for $\widetilde{\Sigma}_{\theta,0}$ at the point p.

Consider a coordinated system such that $\{x_3 = 0\} = T_{x^*}\widetilde{\Sigma}_{\theta,0}$. Thus, using the ARM, the unique continuation principle, the arbitrariness in choosing θ, as well as in the proof of Theorem 2.3.1, we conclude that Σ is the critical catenoid.

■

3 Surfaces of constant anisotropic mean curvature with free-boundary in revolution surfaces

In this chapter we consider immersions ϕ, with constant anisotropic mean curvature (CAMC), of a smooth oriented connected and compact surface Σ, such that $\partial\Sigma \neq \emptyset$, in a region Ω whose boundary is a revolution surface. First, we find one condition on the boundary of CAMC free boundaries immersions, then we prove that $\phi(\Sigma)$ should be a flat disk and under what conditions it is stable.

3.1 Anisotropic Introduction

Let Σ be a smooth oriented connected and compact surface, $\Omega \subset \mathbb{R}^3$ a region such that $\partial\Omega$ is a revolution surface and $\phi : \Sigma \to \Omega$ a smooth immersion. Consider $\Phi : (-\epsilon_0, \epsilon_0)\times\Sigma \to \Omega$ a smooth variation of ϕ, i.e., $\Phi_\epsilon : \Sigma \to \Omega$, defined by $\Phi_\epsilon(p) := \Phi(\epsilon, p)$, $\epsilon \in (-\epsilon_0, \epsilon_0)$, is a smooth immersion, and $\Phi_0 = \phi$.

The study of CAMC immersions is a generalization of classical case about CMC immersions. Is well known that CMC hypersurfaces are solutions of a variation problem associated to area functional

$$\mathcal{A}(\epsilon) = \int_\Sigma d\Sigma_\epsilon \tag{3.1}$$

and we say that Σ is *CMC stable*, roughly, when it minimizes area up to second order. Important contribuitions for the classical case were made in [21] and [3], where they classified the stable immersions in cases $\partial\Sigma = \emptyset$ and free boundary in an euclidean ball, respectively:

Theorem 3.1.1 (Barbosa and Do Carmo) *Let Σ^n be compact orientable manifold and let $\phi : \Sigma \to \mathbb{R}^{n+1}$ be an immersion with nonzero constant mean curvature. Then ϕ is stable if and only if $\phi(\Sigma) \subset \mathbb{R}^{n+1}$ is a (round) sphere $S^n \subset \mathbb{R}^{n+1}$.*

Theorem 3.1.2 (Ros and Vergasta) *Let $\phi : \Sigma^2 \to B \subset \mathbb{R}^3$, where B is an euclidean ball, an immersion CMC stable. Then the only possibilities are*

i *$\phi(\Sigma)$ is a totally geodesic disk,*

ii *$\phi(\Sigma)$ is a spherical cap,*

iii *$g = 1$ and $r = 1$ or 2.*

However, this result of Ros and Vergasta is just partial and it was complemented by the work of Nunes [4] and Barbosa [5], discarding the third possibility. Recently, Wang and Xia [6] definitely classified the CMC stable immersions free boundary in the euclidean ball B^{n+1}.

The study of CAMC immersions is a generalization of CMC case, adding a function F in the area functional:

$$\mathcal{F}(\epsilon) = \int_{\Sigma} F(\nu(\epsilon, p))\, d\Sigma_{\epsilon} \tag{3.2}$$

where $F : \mathbb{S}^n \to \mathbb{R}^+$ is a smooth function and $\nu(\epsilon, p)$ is the unit normal vector from Σ_ϵ, at point p. In [13], to compactly supported variations, they proved that Euler-Lagrange equation of $\mathcal{F}$ is

$$\Lambda(p) := nH(p)F(\nu(0,p)) - \mathrm{div}_{\Sigma} DF(\nu(0,p)) \equiv \text{constant}, \tag{3.3}$$

where $H(p)$ is the mean curvature from Σ at point p, DF is the gradient from F on $\mathbb{S}^2$ and the number $\Lambda(p)$ is called *anisotropic mean curvature* of Σ at point p. They also proved that, under some assumptions, a CAMC stable immersion $\phi : \Sigma^2 \to \mathbb{R}^3$ is an Wulff Shape, up to a translation and homothety. In [14], they considered free boundary variations and classified the CAMC stable immersions in a *slab*.

In this work, we consider the generalized area functional $\mathcal{F}$ with some conditions under F and free boundary variations of immersions $\phi : (\Sigma^2, \partial\Sigma) \to (\Omega, \partial\Omega)$, where $(\Sigma^2, \partial\Sigma)$ is an oriented connected compact surface, such that $\partial\Sigma \neq \emptyset$, as well as [3] and [14], and $\Omega \subset \mathbb{R}^3$ is a region whose boundary is a revolution surface with profile curve α and axis e such that $\mathcal{J}_{\partial\Omega} := \{t \in I \mid \alpha'(t) /\!/ e\}$ is a discrete set. In this context we ask ourselves about the properties of $\phi(\Sigma)$. First, we found some features about $\phi(\partial\Sigma)$.

3.2 About the orthogonality of $\partial\Sigma$ in relation to $\partial\Omega$

Let $(\Sigma^2, \partial\Sigma)$ be an oriented connected compact hypersurface with boundary $\partial\Sigma \neq \emptyset$. Consider a free boundary admissible variation $\Phi = \Phi(\epsilon, p)$ of a smooth immersion $\phi : (\Sigma^2, \partial\Sigma) \to (\Omega, \partial\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a region such that $\partial\Omega$ is a revolution surface. Consider the *generalized area functional*

$$\mathcal{F}(\epsilon) = \int_{\Sigma} F(\nu(\epsilon))\, d\Sigma_{\epsilon} \tag{3.4}$$

and the *volume function*

$$V(\epsilon) = \int_{[0,\epsilon]\times\Sigma} \Phi^* dV. \tag{3.5}$$

where $\Phi^* dV$ is the *pullback* of the canonical volume element of $\mathbb{R}^3$.

The first variation of functional $\mathcal{F}$ can be find in [14] and is given by

$$\partial_\epsilon \mathcal{F}|_\epsilon = -\int_\Sigma u\Lambda \, d\Sigma_\epsilon + \int_{\partial\Sigma} \langle \chi \times \dot{\Phi}, d\Phi \rangle \tag{3.6}$$

where $\chi(\nu) = DF(\nu) + F(\nu)\nu$, $\dot{\Phi} = u\nu + \xi$, $d\Phi := tds$ and

$$t := t(\epsilon, p) = \nu(\epsilon, p) \times \eta(\epsilon, p) \tag{3.7}$$

where $\eta_\epsilon := \eta(\epsilon, p)$ is the unit exterior normal along $\partial\Sigma_\epsilon$ and $\eta := \eta(0, p)$ is the unit exterior normal along $\partial\Sigma$.

An immersion ϕ is called *critical* if and only if $\mathcal{F}'(0) = \partial_\epsilon \mathcal{F}|_{\epsilon=0} = 0$, for all volume-preserving admissible variation, i.e., for all variation $\Phi = \phi + \epsilon(u \cdot \nu + \xi)$ such that

$$\int_\Sigma u \, d\Sigma = 0 \tag{3.8}$$

and

$$\langle \dot{\Phi}, N \rangle \equiv 0 \text{ along } \partial\Sigma, \tag{3.9}$$

where ξ is a field tangent to Σ and N is the unit outward normal along $\partial\Omega$.

Example 3.2.1 *A plane Π (or a piece of it) is a CAMC surface with $\Lambda \equiv 0$. Indeed, let p on Π, so*

$$\Lambda(p) = nH(p)F(\nu(p)) - div_\Sigma DF(\nu(p)) = 0,$$

since the mean curvature $H(p) = 0$, as well as

$$div_\Sigma DF(\nu(p)) = 0,$$

once that $d\nu = 0$, since ν is constant along Π.

For more examples of CAMC surfaces see [13], and [14] to find the following

Proposition 3.2.1 *An immersion ϕ is critical if and only if*

$$\Lambda \equiv \Lambda_0, \quad on \ \ \Sigma, \tag{3.10}$$

for some constant Λ_0 *and*

$$\langle \chi(\nu), N \rangle = 0, \quad along \ \ \partial\Sigma. \tag{3.11}$$

Remark 3.2.1 *Although* ϕ *is critical, we do not obtain the orthogonality of* $\partial\Sigma$ *in relation to* $\partial\Omega$*, unlike the classical case (CMC), where* $F \equiv 1$ *and the above proposition reduces to* $H \equiv constant$ *on* Σ *and* $\langle \nu, N \rangle = 0$ *along* $\partial\Sigma$*.*

In this work, as well as in [13] and [14], we will consider $F = f(\nu_3)$, i.e.,

$$F = f \circ x_3$$

where $\nu = (\nu_1, \nu_2, \nu_3)$, x_3 is the projection in the third component and $f : [-1, 1] \to \mathbb{R}^+$ is a smooth function such that $f' \neq 0$. In addition, we will demand that the matrix $A = D^2F + F1$ is positive definite, where D^2F is the Hessian matrix of F and 1 is the identity on $T_\nu\mathbb{S}^2$, whose eigenvalues, that were determined in [14], are given by

$$\frac{1}{\mu_1} = (1 - \nu_3^2)f'' + \frac{1}{\mu_2} \quad \text{and} \quad \frac{1}{\mu_2} = f - \nu_3 f' \tag{3.12}$$

Example 3.2.2 *Consider the following family of functions*

$$\begin{aligned} f : [-1, 1] &\to \mathbb{R}^+ \\ \nu_3 &\mapsto a\nu_3^2 + b\nu_3 + c \end{aligned} \tag{3.13}$$

where $a > 0$ *and* $\Delta = b^2 - 4ac < 0$*, is such that,* $c > b^2/4a$*. In this case,* $f > 0$ *and* $f' = 2a\nu_3 + b$*.*

$$0 \neq f' \Leftrightarrow \frac{-b}{2a} < -1 \ \ or \ \ \frac{-b}{2a} > 1 \tag{3.14}$$

$$\Leftrightarrow 0 < 2a < b \ \ or \ \ b < 0 < 2a < -b. \tag{3.15}$$

The matrix $A = D^2F + F1$ *must be positive definite, i.e., we should have* $\frac{1}{\mu_2} > 0$*, because* $f''(\nu_3) = 2a > 0$*. So,*

$$0 < \frac{1}{\mu_2} = f - \nu_3 f' = a\nu_3^2 + b\nu_3 + c - 2a\nu_3^2 - b\nu_3 = c - a\nu_3^2, \ \forall\, \nu_3 \in [-1, 1].$$

Since the right side of the above inequality reaches its lower value when $\nu_3 = \pm 1$, we should request $a < c$ in (3.13). However, as $f' \neq 0$, $b^2/4a > a$. Hence, we should only have $c > b^2/4a$.

Therefore, the functions of the family (3.13) such that

$$0 < 2a < b \ \textit{and} \ c > \frac{b^2}{4a} \ \textit{or} \ b < 0 < 2a < -b \ \textit{and} \ c > \frac{b^2}{4a} \tag{3.16}$$

meets the conditions requested. In addition, when $a = b = 0$ we return to the classical case.

The divergent of F on $\mathbb{S}^2$ is

$$DF(\nu) = f'(\nu_3)(e_3 - \nu_3\nu), \ \forall\, \nu \text{ on } \mathbb{S}^2, \tag{3.17}$$

where $e_3 = (0, 0, 1)$. In possession of the previous proposition we will classify the critical immersions $\phi : (\Sigma^2, \partial\Sigma) \to (\Omega, \partial\Omega)$ that intersect orthogonally $\partial\Omega$, i.e., the immersions such that

$$\eta = N, \text{ along } \partial\Sigma. \tag{3.18}$$

Let $\alpha : I \to \mathbb{R}^3$ and e, be a profile curve of and the axis of rotation of $\partial\Omega$, respectively (note that all parallel of $\partial\Omega$ is a circle in a plane orthogonal to the axis e). Let's also define $\mathcal{J}_{\partial\Omega} := \{t \in I \mid \alpha'(t) \,/\!/\, e\}$.

Unlike the classic case, as seen in Proposition 3.2.1, if ϕ is a critical immersion for the functional $\mathcal{F}$, it is not possible to say that ϕ is free boundary, that is, that $\phi(\partial\Sigma)$ meets $\partial\Omega$ orthogonally. In this work, we asked ourselves about the properties of CAMC free boundaries immersions. Below, our first result.

Proposition 3.2.2 *Let $F = f(\nu_3)$ and f a smooth function such that $f' \neq 0$. Consider $\phi : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ a critical immersion to the functional $\mathcal{F}$, where $\mathcal{J}_{\partial\Omega}$ is a discrete set. Then $\partial\Sigma$ intersect $\partial\Omega$ orthogonally if and only if each connected component of $\partial\Sigma$ lies in a parallel of $\partial\Omega$, where $N \perp e_3$ along it.*

Proof of Proposition 3.2.2: Consider an orthonormal frame $\{e_1, e_2, e_3\} \in \mathbb{R}^3$ such that e_3 be the director vector of axis e of $\partial\Omega$. As ϕ is a critical immersion, is a consequence of Proposition 3.2.1 and equation (3.17) that

$$\begin{aligned} \langle \chi(\nu(p)), N(p) \rangle &= 0, \quad \forall\, p \in \partial\Sigma. \\ \langle DF + F\nu, N \rangle &= 0, \quad \forall\, p \in \partial\Sigma. \\ f'\langle e_3, N \rangle - \nu_3 f' \langle \nu, N \rangle + f\langle \nu, N \rangle &= 0, \quad \forall\, p \in \partial\Sigma. \end{aligned} \tag{3.19}$$

Suppose $\phi(\Sigma)$ intersect $\partial\Omega$ orthogonally, then $\langle \nu, N \rangle \mid_{\partial\Sigma} \equiv 0$, and follows from (3.19) that

$$f'\langle e_3, N \rangle = 0, \ \ \forall\, p \in \partial\Sigma. \tag{3.20}$$

that is,

$$\langle e_3, N\rangle = 0, \ \ \forall\, p \in \partial\Sigma. \tag{3.21}$$

Because $\mathcal{J}_{\partial\Omega}$ is a discrete set, so each component of $\partial\Sigma$ lies in a parallel of $\partial\Omega$ and $N \perp e_3$ along $\partial\Sigma$.

In return, let $\nu = \nu(p), p \in \partial\Sigma$, and suppose $\partial\Sigma$ lies in a parallel of $\partial\Omega$ such that $N \perp e_3$ along this. Then, follows from (3.19) that

$$(f - \nu_3 f')\langle \nu, N\rangle = 0, \text{ along } \partial\Sigma. \tag{3.22}$$

However, if $\nu \not\parallel e_3$, then $(f - \nu_3 f') = \mu_2^{-1} > 0$, because μ_2^{-1} is an eigenvalue of positive definite matrix $A = D^2F + F1$. So, $\langle \nu, N\rangle = 0$, along $\partial\Sigma$. If $\nu(p) /\!/ e_3$, follows from (3.17) that $DF(\nu) = 0$ and using (3.19) follows the result. Therefore,

$$\langle \nu, N\rangle \equiv 0, \ \text{ on } \ \partial\Sigma. \tag{3.23}$$

■

Henceforth, we will denote by $\mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$ the family of critical immersions of $(\Sigma, \partial\Sigma)$, with respect to the function $F = f \circ x_3$, $f' \neq 0$, such that $\partial\Sigma \neq \emptyset$ intersect $\partial\Omega$ orthogonally.

Example 3.2.3 *Consider $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \mathbb{S}^2)$, where $f(\nu_3) = \nu_3^2 + 1$ and note that $f'(0) = 0$. Then, in this case, follows from (3.20) that each connected component of $\partial\Sigma$ can lies in a plane containing the the axis e.*

Corollary 3.2.1 *Under the assumptions of Proposition 3.2.2, we have:*

(i) *If $\mathcal{J}_{\partial\Omega} = \emptyset$, then $\mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega) = \emptyset$.*

(ii) *If $\mathcal{J}_{\partial\Omega}$ is an unit set, then $\phi(\Sigma)$ has an unique connected boundary component.*

Corollary 3.2.2 *Towards the above assumptions, if $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$, then $\nu /\!/ e_3$ along $\partial\Sigma$.*

Example 3.2.4 *If $\partial\Omega$ is a right circular cone (with one leaf), then does not exist critical immersion such that $\partial\Sigma$ intersect $\partial\Omega$ orthogonally, because $\mathcal{J}_{\partial\Omega} = \emptyset$.*

Example 3.2.5 *If $\partial\Omega$ is a catenoid and ϕ a critical immersion such that $\partial\Sigma$ intersect $\partial\Omega$ orthogonally, then $\partial\Sigma$ should lies in catenoid neck, see Figure 31.*

If $p = \alpha(t_0)$ on $\partial\Omega$ and $\alpha'(t_0) /\!/ e$, then the parallel, generated by the revolution of p, is a geodesic of $\partial\Omega$, see [29], so

Corollary 3.2.3 *If $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$ and $\mathcal{J}_{\partial\Omega}$ is a discrete set, then $\partial\Sigma$ is the union of closed geodesics of $\partial\Omega$.*

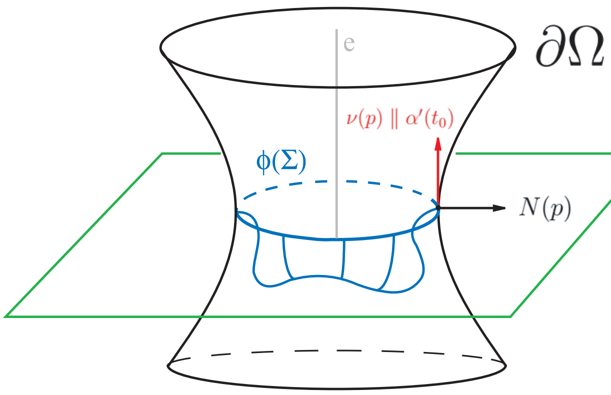

Figure 31 – Critical immersion ϕ in a catenoid, whose boundary $\partial\Sigma$ intersect the catenoid orthogonally.

In the classical case, proven by M. Koiso [8] that, being Σ^n a compact non-zero CMC hypersurface such that $\partial\Sigma$ is a $(n-1)$-sphere Γ_0 in some hyperplane $\Pi \subset \mathbb{R}^{n+1}$ and such that $\text{int}(\Sigma)$ does not intersect the outside of Γ_0 in Π, then Σ must be a spherical cap. In this work, precisely when $\mathcal{J}_{\partial\Omega}$ is an unit set, we already know that if Σ is CAMC, then $\partial\Sigma$ is a 1-sphere lying in a plane orthogonal to axis of rotation of $\partial\Omega$ and that $\text{int}(\Sigma)$ does not intersect the outside of this 1-sphere. Will this also allow us to conclude something about Σ?

Example 3.2.6 *Let $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$ and $\mathcal{J}_{\partial\Omega}$ an unit set, so follows from Proposition 3.2.2 that $\partial\Sigma$ lies in the unique parallel of $\partial\Omega$ such that $N \perp e_3$ and $\nu = e_3 = (0,0,1)$ (without loss of generality) along of $\partial\Sigma$. Consider $F = f \circ x_3$, where $f(\nu_3) = a\nu_3 + b$ and $0 < a < b$. In this case,*

$$div_\Sigma DF = f''(\nu_3)\langle \nabla\nu_3, \nabla x_3\rangle + 2H\nu_3 f' \tag{3.24}$$

and

$$\Lambda = 2HF - div_\Sigma DF = 2Hb \tag{3.25}$$

i.e.,

$$H = \frac{\Lambda}{2b} \equiv constant. \tag{3.26}$$

Then, follows from Theorem 2.1.4 that, Σ is a flat disk.

A natural question: will we have the same result for other functions f?

Adapting the tangency principle for boundary points, theorem 1.2 found in [30], for anisotropic case and when $r = 1$, we have:

Theorem 3.2.1 (Tangency Principle) *Let Σ^n and $\overline{\Sigma}^n$ be hypersurfaces of $\mathbb{R}^{n+1}$ with not empty boundaries $\partial\Sigma^n$ and $\partial\overline{\Sigma}^n$, respectively. Suppose Σ^n and $\overline{\Sigma}^n$ are tangent at $p \in \partial\Sigma^n \cap \partial\overline{\Sigma}^n$ and let $\nu(p)$ be normal to Σ^n at p. Suppose Σ^n remains above $\overline{\Sigma}^n$ and $\Lambda \leq \overline{\Lambda}$ in a neighborhood of p with respect to $\nu(p)$. Then, Σ^n and $\overline{\Sigma}^n$ coincide in a neighborhood of p.*

Proof of Theorem 3.2.1: Only remember that A is positive defined and follow the proof in [30] analogously.

■

In [31] we find the following result.

Theorem 3.2.2 (Unique Continuation for Elliptic Equations) *Let L be a second-order elliptic linear operator. If $Lv = 0$ on a convex domain $\mathcal{D}$ and v vanish on an open subset of $\mathcal{D}$, then $v \equiv 0$ on $\mathcal{D}$.*

Now we can present our next result and classify the critical immersions of $\mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$. Without loss of generality, we consider the unit outwards normal vector ν on Σ.

Theorem 3.2.3 *Let $F = f(\nu_3)$, where f is a smooth function such that $f' \neq 0$. Consider $\phi : (\Sigma, \partial\Sigma) \to (\Omega, \partial\Omega)$ a critical immersion to the $\mathcal{F}$ such that $\Lambda \leq 0$ and $\mathcal{J}_{\partial\Omega}$ a discrete set. Then, $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$ if and only if $\phi(\Sigma)$ is a totally geodesic disk whose boundary is a parallel of $\partial\Omega$, where $\bar{N} \perp e_3$ along it.*

Proof of Theorem 3.2.3: If $\phi(\Sigma)$ is a totally geodesic disk whose boundary is a closed geodesic in $\partial\Omega$, nothing to do. That being said, suppose $\partial\Sigma$ intersect $\partial\Omega$ orthogonally. Follows from Proposition 3.2.2, with each connected component of $\partial\Sigma$ lies in a plane perpendicular the rotation axis, which means, $\partial\Sigma = \partial\mathcal{D}_1 \cup ... \cup \partial\mathcal{D}_n$, where $\mathcal{D}_i$ is a totally geodesic disk, $i = 1, ..., n$, and still, $\nu /\!/ e_3$ along $\partial\Sigma$.

Let $p \in \partial\Sigma \cap \partial\mathcal{D}_i$, for some $i \in \{1, ..., n\}$, and consider, no loss generality, $\partial\mathcal{D}_i \subset \Pi_3$, where $\Pi_3 = \{(x_1, x_2, x_3) \; ; \; x_3 = 0\}$.

Affirmation: There exist a point $p \in \partial\Sigma$ and a neighborhood $V_p \subset \Sigma$, at p, such that $g(x) = \langle x, e_3 \rangle$ does not change signal ($g \geq 0$).

Absurdly, suppose that:

$$\forall \, p \in \partial\Sigma \;\text{ and }\; \forall \, V_p \ni p \in \partial\Sigma, V_p \subset \Sigma, \; g^+(V_p) \neq \emptyset \neq g^-(V_p), \tag{3.27}$$

where $g^+(V_p) = \{x \in V_p \mid g(x) > 0\}$ and $g^-(V_p) = \{x \in V_p \mid g(x) < 0\}$.

Henceforth, for all $p \in \partial\Sigma$ there are sequences in Σ,

$$x_n^+ \to p; \; g(x_n^+) > 0, \; \forall \, n \in \mathbb{N}; \tag{3.28}$$

$$x_n^- \to p; \; g(x_n^-) < 0, \; \forall \, n \in \mathbb{N}; \tag{3.29}$$

By continuity of g, also there exist a sequence such that

$$x_n \to p;\ g(x_n) = 0,\ \forall\, n \in \mathbb{N}. \tag{3.30}$$

Let $\alpha_p : [a,b] \to \Sigma$ be a curve such that $\alpha_p(a) = p$ and $x_n \in \alpha_p([a,b])$, $n \geq n_0$, that is, the curve α_p connect the points of sequence (x_n), for $n \geq n_0$. So, follows from (3.30) that, there exist $\delta_p > 0$ such that $(g \circ \alpha_p)([a,\delta_p)) = \{0\}$, because $(g \circ \alpha_p)^{-1}(0)$ is a closed subset of the compact set $[a,b]$. Thus, $\alpha_p([a,\delta_p)) \subset V_p \cap D_i$. Analogously, we can define the curves $\alpha_p^+, \alpha_p^- : [a,b] \to \Sigma$, i.e., $\alpha_p^\pm([a,\delta_p)) \subset g^\pm(V_p)$ (Figure 32).

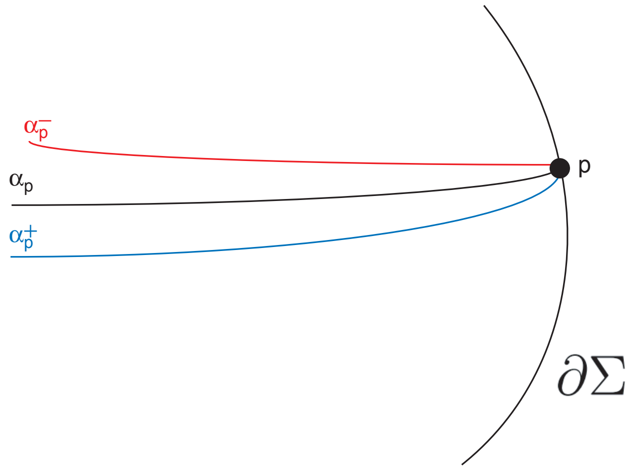

Figure 32 – We have $g = 0$, $g > 0$ and $g < 0$ along α_p, α_p^+ and α_p^-, respectively.

Let $p_1, p_2 \in V_p \cap \partial\Sigma$. Consider arc$(p_1,p_2)$ the smallest arc of circle containing $p, p_1, p_2 \in V_p \cap \partial\Sigma$ and $W_p \subset V_p$ a neighborhood in $p \in \partial\Sigma$ (Figure 33), such that

(w1) $W_p \cap \partial\Sigma = \text{arc}(p_1,p_2)$;

(w2) $\alpha_{p_1}^\pm([a,b]) \cap W_p = \alpha_{p_1}([a,b]) \cap W_p = \{p_1\}$; $\alpha_{p_2}^\pm([a,b]) \cap W_p = \alpha_{p_2}([a,b]) \cap W_p = \{p_2\}$;

(w3) $\forall\, p \in \text{arc}(p_1,p_2)$, $\exists\, t, t^\pm \in [a,\delta_p)$; $\alpha_p(t) \notin W_p$ and $\alpha_p^\pm(t^\pm) \notin W_p$.

Let $\gamma : [c,d] \to W_p$ be a curve such that $\gamma(d) = p_1$ and $\gamma([c,d]) \cap \partial\Sigma = \{p_1\}$. In a neighborhood of p_1, the trace of curve γ intersect $\alpha_p([a,\delta_p))$ to an infinite $p \in \partial\Sigma$ (the same happens with respect to curves $\alpha_p^\pm([a,\delta_p))$, as a consequence of **(w2)** and **(w3)**). Then, the closed subset $g^{-1}(0) \subset [c,d]$ have infinity points. Contradiction! Therefore, the affirmation is true.

Henceforth, there exist a point $p \in \partial\Sigma$ and a neighborhood $V_p \subset \Sigma$, such that $g(x) = \langle x, e_3\rangle$ does not change signal ($g \geq 0$). So, as $\nu /\!/ e_3$ along $\partial\Sigma$, Σ and $\mathcal{D}_i$ are tangents at $p \in \partial\Sigma$; as $g \geq 0$, $\Sigma \cap V_p$ remains above D_i with respect to ν_Σ; and $\Lambda_\Sigma \leq 0 = \Lambda_{D_i}$ with respect to ν_Σ. So, follows from Theorem 3.2.1 that, there exist a neighborhood $V_p \subset \Sigma$, $p \in \partial\Sigma$, such that $V_p \subset D_i$. This way, being $\nu = (\nu_1,\nu_2,\nu_3)$, we have

$$\nu_1 \equiv \nu_2 \equiv 0,\ \ and\ \ \nu_3 \equiv 1\ \ on\ \ \Sigma \cap V_p. \tag{3.31}$$

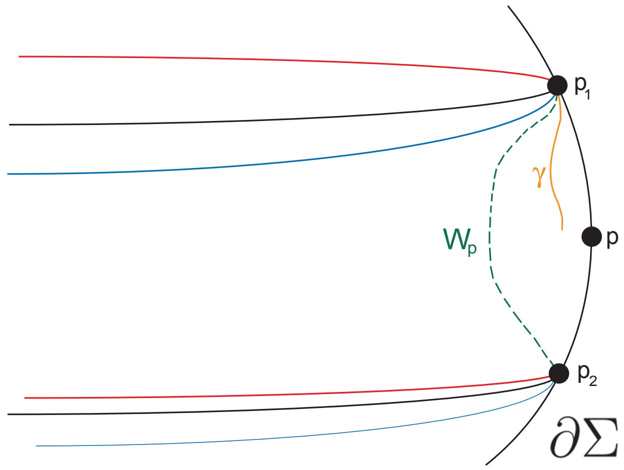

Figure 33 – The neighborhood W_p and the curve γ.

On the other hand, see [13], to $j = 1, 2$ and 3, ν_j verify

$$L[\nu_j] = 0, \quad \text{on } \Sigma, \tag{3.32}$$

where L is the second-order elliptic differential operator

$$L[u] = \mathrm{div}_\Sigma\,(A\nabla u) + \langle Ad\nu, d\nu\rangle u. \tag{3.33}$$

Then, follows from (3.32) and Theorem 3.2.2 that $\nu_1 \equiv \nu_2 \equiv 0$, in Σ. Therefore, $\phi(\Sigma)$ is a totally geodesic disk.

■

Example 3.2.7 *If $\partial\Omega = \mathbb{S}^2$ and ϕ is a critical immersion such that $\partial\Sigma$ intersect $\partial\Omega$ orthogonally, then Σ is the totally geodesic disk whose boundary must be an equator contained in an orthogonal plane to the axis e of $\mathbb{S}^2$ [Figura 34].*

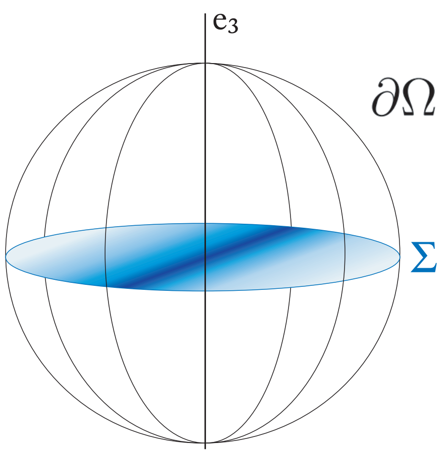

Figure 34 – Critical immersion $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \mathbb{S}^2)$.

In the next example, we show the relevance of hypothesis $f' \neq 0$ in the Theorem 3.2.3.

Example 3.2.8 *Let $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \mathbb{S}^2)$ such that $f'(k) = 0$, where $-1 < k < 0$ (for example, $f(\nu_3) = \nu_3^2 + \nu_3 + 3$). Fixed the function f, we can define the Wulff Shape of F*

$$W_F = \partial \bigcap_{\nu \in \mathbb{S}^2} \{y \in \mathbb{R}^3 | \langle y, \nu \rangle \leq f(\nu_3)\} \tag{3.34}$$

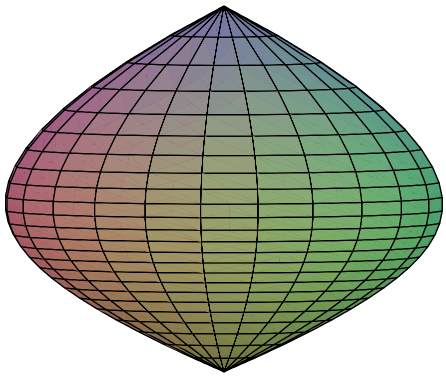

Figure 35 – The Wulff Shape with respect to the function $f(\nu_3) = \nu_3^2 + \nu_3 + 3$.

whose normal vector $\nu_W = (\nu_1, \nu_2, \nu_3)$, without loss of generality, we choose outward pointing. As $F = f(\nu_3)$, so W is a rotation surface around the axis $e /\!/ e_3$.

Consider the following steps:

1. *Define W_- the lower hemisphere of W_F.*

2. *Let W_k be the Wulff Shape Cap of W_F such that it normal vector $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3)$ satisfy $\bar{\nu}_3 = \nu_3 \equiv k$ along ∂W_k; and consider $p, q \in \partial W_k \cap$ plano $e_2 e_3$.*

3. *Move W_k to the upper half-space of $\mathbb{R}^3$, along of the axis e and in the sense of e_3, until the straight containing the origin and $\tilde{p}$ ($\tilde{p}$ being a translation of p) become orthogonal to $\bar{\nu}(\tilde{p})$ (the same will occur with respect to the straight containing the origin and $\tilde{q}$, where $\tilde{q}$ is a translation of q, by symmetry), see Figure 38.*

4. *Finally, consider the sphere $\mathbb{S}_r^2$, where $r = d(\tilde{p}, 0) = d(\tilde{q}, 0)$. So, this translation of W_k is free boundary in $\mathbb{S}_r^2$.*

Then, this Wulff Shape Cap W_k is critical free boundary, with respect to the functional defined by f, in $\mathbb{S}_r^2$. That is, if we with draw the assumption $f' \neq 0$, so appear other immersions CAMC free boundary in $\mathbb{B}^3$.

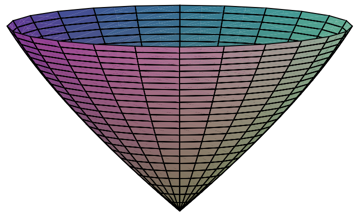

Figure 36 – An Wulff Shape Cap with respect to the function $f(\nu_3) = \nu_3^2 + \nu_3 + 3$.

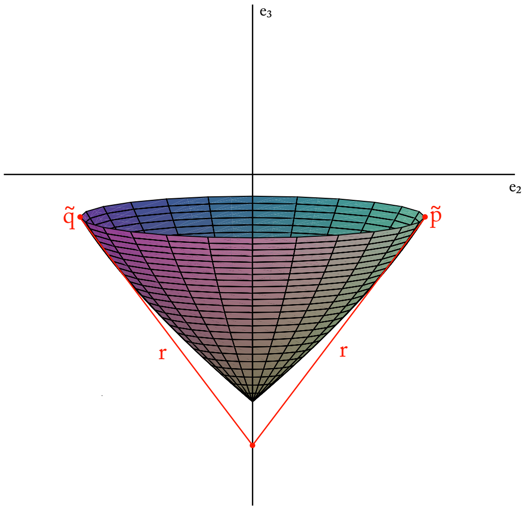

Figure 37 – The Wulff Shape Cap before translation.

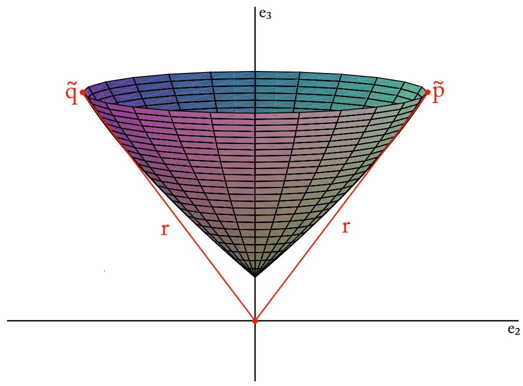

Figure 38 – The Wulff Shape Cap after translation.

3.3 About the stability of totally geodesic disk

Following the same steps of the classical case, we must verify the stability of the critical immersions $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$. However, is necessary that $\phi(\Sigma)$ be CAMC to be stable. So, for each $\mathcal{F}$, we must just verify the stability of the totally geodesic disk whose boundary is a parallel of $\partial\Omega$ contained in a plane Π that intersect the axis e orthogonally.

Proposition 3.3.1 *Let $F = f(\nu_3)$, where $f' \neq 0$. Consider $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$ and $\mathcal{J}_{\partial\Omega}$ a discrete set. Then, for all volume-preserving normal admissible variation, the second variation of $\mathcal{F}$ is*

$$\partial^2_{\epsilon\epsilon}\mathcal{F}\Big|_{\epsilon=0} = -\int_\Sigma u\dot{\Lambda}\, d\Sigma + \int_{\partial\Sigma} u\langle A\nabla u, \eta\rangle\, ds - \int_{\partial\Sigma} \mathrm{II}(\nu,\nu)u^2 F\, ds \tag{3.35}$$

where II is the second fundamental form of $\partial\Omega$ into $\mathbb{R}^{n+1}$, with respect to the inwards pointing unit normal direction and

$$\dot{\Lambda} = L[u] = div_\Sigma A\nabla u + \langle Ad\nu, d\nu\rangle u \tag{3.36}$$

is the Jacobi Operator of $\mathcal{F}$.

Proof of Proposition 3.3.1: Follows from (3.6)

$$\partial_\epsilon \mathcal{F}\big|_\epsilon = -\int_\Sigma u\Lambda \, d\Sigma_\epsilon + \int_{\partial\Sigma} \langle \chi \times \dot{\Phi}, d\Phi \rangle \tag{3.37}$$

so

$$\partial^2_{\epsilon\epsilon} \mathcal{F}\Big|_{\epsilon=0} = -\int_\Sigma u\dot{\Lambda} \, d\Sigma - \int_\Sigma u\Lambda_0 \, \partial_\epsilon d\Sigma_\epsilon|_{\epsilon=0} + \int_{\partial\Sigma} \partial_\epsilon \langle \chi \times \dot{\Phi}, d\Phi \rangle\Big|_{\epsilon=0} \tag{3.38}$$

In [13] and [14] we find,

$$\dot{\Lambda} = L[u] = \mathrm{div}_\Sigma A \nabla \, u + \langle A d\nu, d\nu \rangle u, \text{ on } \Sigma. \tag{3.39}$$

As ϕ is critical, Λ_0 is constant, hence

$$\int_\Sigma u\Lambda_0 \, \partial_\epsilon d\Sigma_\epsilon|_{\epsilon=0} = \Lambda_0 \partial_\epsilon \left(\left(\int_\Sigma u \, d\Sigma_\epsilon \right) \right)\Bigg|_{\epsilon=0} = \Lambda_0 \, \partial_\epsilon V'(\epsilon)|_{\epsilon=0} = 0, \tag{3.40}$$

because V is constant. The integral along $\partial\Sigma$ takes the following form

$$\int_{\partial\Sigma} \langle \dot{\chi} \times \dot{\Phi}, t \rangle \, ds + \int_{\partial\Sigma} \langle \chi \times \ddot{\Phi}, t \rangle \, ds + \int_{\partial\Sigma} \langle \chi \times \dot{\Phi}, (\dot{d\Phi}) \rangle, \tag{3.41}$$

where

$$\chi = \chi(\nu(p)) = DF(\nu(p)) + F(\nu(p))\nu(p) = F(\nu(p))\nu(p). \tag{3.42}$$

As $\phi \in \mathcal{I}^\perp_F(\partial\Sigma, \partial\Omega)$, so $\nu \,/\!/\, e$ along $\partial\Sigma$. Then $DF \equiv 0$ along $\partial\Sigma$ (3.17). We also have

$$\dot{\chi} = d\chi|_{\nu(p)} \left(-\nabla u \right)_p = -A|_{\nu(p)} \left(\nabla u \right)_p \tag{3.43}$$

hence

$$\langle \dot{\chi} \times \dot{\Phi}, t \rangle ds = -\langle t \times u\nu, \dot{\chi} \rangle ds = -u \langle \eta, \dot{\chi} \rangle ds = u \langle \eta, A\nabla u \rangle, \text{ on } \partial\Sigma. \tag{3.44}$$

Follows from second integral, using (3.42), that

$$\langle \chi \times \ddot{\Phi}, t \rangle ds = F \langle \eta, \ddot{\Phi} \rangle ds, \text{ along } \partial\Sigma. \tag{3.45}$$

On the other hand,

$$\begin{aligned} 0 = \partial_\epsilon \langle \eta, \dot{\Phi} \rangle &= \langle \dot{\eta}, \dot{\Phi} \rangle + \langle \eta, \ddot{\Phi} \rangle = \langle \bar{\nabla}_{\dot{\Phi}} \eta, \dot{\Phi} \rangle + \langle \eta, \ddot{\Phi} \rangle = u^2 \langle \bar{\nabla}_\nu \eta, \nu \rangle + \langle \eta, \ddot{\Phi} \rangle & (3.46) \\ &= u^2 \langle d\eta(\nu), \nu \rangle + \langle \eta, \ddot{\Phi} \rangle = -u^2 \langle d\eta(-\nu), \nu \rangle + \langle \eta, \ddot{\Phi} \rangle = u^2 \mathrm{II}^{\partial\Omega}_{-\eta}(\nu, \nu) + \langle \eta, \ddot{\Phi} \rangle & (3.47) \\ &=: u^2 \mathrm{II}(\nu, \nu) + \langle \eta, \ddot{\Phi} \rangle, \text{ along } \partial\Sigma, & (3.48) \end{aligned}$$

so

$$\langle \eta, \ddot{\Phi} \rangle = -u^2 \mathrm{II}(\nu, \nu), \text{ along } \partial\Sigma. \tag{3.49}$$

Then,

$$\langle \chi \times \ddot{\Phi}, t\rangle ds = -u^2 F\,\mathrm{II}(\nu,\nu)ds, \text{ along } \partial\Sigma. \tag{3.50}$$

Once more, as $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$ so $DF \equiv 0$ along $\partial\Sigma$. Hence,

$$\langle \chi \times \dot{\Phi}, (\dot{d\Phi})\rangle = 0, \text{ along } \partial\Sigma. \tag{3.51}$$

Therefore,

$$\partial^2_{\epsilon\epsilon}\Big|_{\epsilon=0} = -\int_\Sigma u\dot{\Lambda}\, d\Sigma + \int_{\partial\Sigma} u\langle A\nabla u, \eta\rangle\, ds - \int_{\partial\Sigma} \mathrm{II}(\nu,\nu)u^2 F\, ds \tag{3.52}$$

■

With the second variation of $\mathcal{F}$, we can discuss the stability of the immersions of $\mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$. As in the classical case, we define the stability of the follows form:

Definition 3.3.1 *An immersion $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$ is stable with respect to $\mathcal{F}$ if and only if*

$$0 \le \partial^2_{\epsilon\epsilon}\mathcal{F}\,|_{\epsilon=0}\ , \textit{ for all volume-preserving normal admissible variation.} \tag{3.53}$$

or, equivalently,

$$0 \le \partial^2_{\epsilon\epsilon}\mathcal{F}\,|_{\epsilon=0}\ ,\ \forall\, u \in C^\infty(\Sigma)\ ;\ \int_\Sigma u\, d\Sigma = 0\ . \tag{3.54}$$

We can associate to $\partial^2_{\epsilon\epsilon}\mathcal{F}\,|_{\epsilon=0}$ the symmetrical bilinear form on $\mathcal{H} = \{g \in H^1(\Sigma);$ $\int_\Sigma g\, d\Sigma = 0\}$,

$$\mathcal{I}_F(f,g) = -\int_\Sigma gL[f]\, d\Sigma + \int_{\partial\Sigma} g\langle A\nabla f, \eta\rangle\, ds - \int_{\partial\Sigma} \mathrm{II}(\nu,\nu)fgF\, ds \tag{3.55}$$

$$= \int_\Sigma \left(\langle A\nabla f, \nabla g\rangle - \langle Ad\nu, d\nu\rangle fg\right)\, d\Sigma - \int_{\partial\Sigma} \mathrm{II}(\nu,\nu)fgF\, ds \tag{3.56}$$

where ∇ is the gradient of the metric induced by ϕ. Note that, ϕ is stable with respect to F if and only if $\mathcal{I}(f,f) \ge 0,\ \forall\, f \in \mathcal{H}$.

Example 3.3.1 *The totally geodesic disk D is stable with respect to functions of the family (3.13) such that $b < 0 < 2a < -b$ and $c > \frac{b^2}{4a}$, indeed*

$$\mathcal{I}_F(u,u) = (c-a)\int_D |\nabla u|^2\, dA - (a+b+c)\int_{\partial D} u^2\, ds \tag{3.57}$$

$$= (c-a)\mathcal{I}(u,u) - (2a+b)\int_{\partial D} u^2\, ds \ge 0,\ \forall\, u \in \mathcal{H}, \tag{3.58}$$

since D is CMC stable and then $\mathcal{I}(u,u) \ge 0$.

Then, of course, a question arises: as in the classic case, are the free boundary disks in the unit sphere stable, for any function F fixed? Or does it depend on the F function? Our next result answers that question.

Theorem 3.3.1 *Let $\phi \in \mathcal{I}_F^{\perp}(\partial\Sigma, \partial\Omega)$ and $\Omega = \mathbb{B}^3$. The disk $D = \phi(\Sigma)$ is stable with respect to F if and only if $f(1) \leq \mu_1^{-1} = \mu_2^{-1}$.*

Proof of Theorem 3.3.1: Suppose than $f(1) \leq \mu_1^{-1} = \mu_2^{-1}$ and let $u \in \mathcal{H}$.

$$\mathcal{I}_F(u,u) = \int_D (\langle A\nabla u, \nabla u\rangle - \langle Ad\nu, d\nu\rangle u)\ d\Sigma - \int_{\partial D} \mathrm{II}(\nu,\nu)u^2 F\ ds \tag{3.59}$$

$$= \mu^{-1}\int_D |\nabla u|^2\ d\Sigma - f(1)\cdot \int_{\partial D} u^2\ ds \tag{3.60}$$

As D is a totally geodesic disk, then $d\nu \equiv 0$, $\nu_3 \equiv 1$, $A = \mu^{-1}\cdot 1$ and $\mathrm{II}(\nu,\nu) = 1$, where $\mu^{-1} = \mu_1^{-1} = \mu_2^{-1}$. Henceforth,

$$\mathcal{I}_F(u,u) = \mu^{-1}\mathcal{I}(u,u) + (\mu^{-1} - f(1))\int_{\partial D} u^2\ ds \tag{3.61}$$

Then, as A is positive definite and D is CMC stable, follows the result. Reciprocally, suppose $f(1) > \mu_1^{-1} = \mu_2^{-1}$.

$$\mathcal{I}_F(u,u) = \mu^{-1}\cdot\mathcal{I}(u,u) + (\mu^{-1} - f(1))\cdot\int_{\partial D} u^2\ ds \tag{3.62}$$

where $\mu^{-1} := \mu_1^{-1} = \mu_2^{-1}$. Consider the function $x_2 : \Sigma \to \mathbb{R}$; $x_2(x) = \langle x, e_2\rangle$.

$$D \text{ is minimal } \Rightarrow x_2\Delta x_2 = 0 \tag{3.63}$$

$$\Rightarrow \int_{\partial D} x_2\frac{\partial x_2}{\partial\eta}\ ds - \int_D |\nabla x_2|^2\ dA = 0 \tag{3.64}$$

However,

$$\frac{\partial x_2}{\partial\eta} = \eta\langle x, e_2\rangle = \langle\bar{\nabla}_\eta x, e_2\rangle = \langle dx\cdot\eta, e_2\rangle \tag{3.65}$$

$$= \langle 1\cdot\eta, e_2\rangle = \langle\eta, e_2\rangle = \langle x, e_2\rangle = x_2 \tag{3.66}$$

because ϕ is free boundary. Is that, $\mathcal{I}(x_2,x_2) = 0$. As

$$\int_{\partial D} x_2^2\ ds > 0 \tag{3.67}$$

follows from (3.62) that D is no stable.

■

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