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Center Dynamics and Maximum Entropy Measures for Partially Hyperbolics Systems

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Center Dynamics and the Maximum Entropy Measure for Partially Hyperbolic Systems

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Resumo

Seja f um difeomorfismo dinamicalmente coherente sobre uma variedade Riemanniana fechada M com folheação central $\mathcal{W}^c$ de classe C^1 e $\lambda^c(f)$ =max $||Df|_{E^c} \leqslant 1$.

Conseguimos mostrar que a entropia topológica de f coincide com o crescimento exponencial de pseudo-órbitas periódicas respeitando a folheação central Per_n. Aplicando plaque expansividade da folheação central e a propriedade de center especificação, mostramos que $h(f) = \lim_{n\to\infty}\ frac1n \log \#\mathrm{Per}_n$. Além disso, quando f é um elemento regular de uma ação de grupo hiperbólica, mostramos a existência de uma única medida que maximiza a entropia.

Palavras Chave: Folheação central, plaque expansividade, entropia topológica, ações de grupo, medida de máxima entropia, placa central.

Abstract

Let f be a dynamically coherent partially hyperbolic diffeomorphism on a closed Riemannian manifold M with the central foliation $\mathcal{W}^c$ of C^1 class and $\lambda^c(f)$ =max $||Df|_{E^c} \leqslant 1$.

We managed to show that the topological entropy of f coincide with the growth exponential of periodic pseudo-orbits respecting the central foliation Per_n. Applying the plaque expansiveness of the central foliation and the center specification property, we show that $h(f) = \lim_{n\to\infty} \frac{1}{n} \log \#\mathrm{Per}_n$. Moreover, when f is an regular element of a hyperbolic action group, the existence of an unique measure that maximizes entropy is shown.

Key words: Central foliation, plaque expansiveness, topological entropy, group actions, measure of maximal entropy, center plaque.

Contents

Introduction

At the end of the 19th century, H. Poincaré began the study of chaotic dynamical systems while he was trying to understand the behavior of celestial bodies. He realized that small differences in initial conditions propagated during time, and produced widely different behaviors of the orbits. In fact, this allowed him to show the non-existence of analytic solutions of the 3-body problem, contrary to what was believed at the time (even by him). For the complete story, together with the digitalized documents of Poincaré the reader could check `http://www.mittag-leffler.se/library/henri-poincare`.

The problem stayed almost neglected (with perhaps the important work of Birkhoff) until around 1960, when S. Smale and his collaborators proposed to understand these "chaotic systems" from a qualitative point of view. Keeping the story short, this was the beginning of the concept of hyperbolicity, a cornerstone of modern mathematics.

Hyperbolic systems are by now very well (but not completely) understood, both from the geometrical and ergodic point of views. For example, if a diffeomorphism is Axiom A (see [28]), then its non-wandering set can be decomposed intro finitely many transitive pieces. Restricted to these pieces the map is expansive, has dense periodic points and satisfies the specification property. Regarding its ergodic properties, we can cite for example the existence of a unique entropy maximizing measure on each piece (see [7]).

Our goal in this work is to extend the above-cited results to a class of partially hyperbolic diffeomorphisms. Roughly speaking, a partially hyperbolic system has, in addition to contracting and expanding directions, an intermediate "center" direction whose behavior is dominated by the other two. The study of partially hyperbolic systems began independently with Hirsch, Pugh and Shub at [20], and with Brin and Pesin at [9] in the early 70's.

The existence of a center direction introduces a series of complications with respect to completely hyperbolic diffeomorphism. For instance, it may not be integrable, (see [19], [29]), it may fail to be smooth, and in general, since in principle the center direction is not dynamically defined (in contrast with the strong bundles), the analysis becomes much harder.

If the central direction is integrable to a smooth foliation then the system is milder: for example, a classical result due to Hirsch, Pugh and Shub guarantee that in this case the $\mathcal{C}^1$ perturbations of the original map have also integrable center bundles (although the resulting foliations fail to be smooth in general). To prove this they realized that a key concept shared by smooth center foliations and its perturbations is *plaque expansivity*, a generalization of the classical expansivity property. At the moment of writing there is no known example of partially hyperbolic system which does not satisfy plaque expansivity. In any case, in order to make initial progress, we restrict to systems with smooth center foliations. It is possible that with additional work the techniques can be extended to other situations.

We give here an overview of each chapter, pointing out the main results. In Chapter 1, we discuss a variety of definitions and results from foliation theory (inluding holonomy) and partially hyperbolic systems. The Chapter 2 is dedicated to the study of pseudo-orbits respecting to the central foliation. To this end, we denote by $\mathcal{P}^c(\delta)$ the space of such δ-pseudo-orbits, and establish some of its properties.

In [7] R. Bowen showed that, for an expansive system with the specification property, the topological entropy of the system coincides with the exponential growth of the periodic points. On the other hand, in [30], Wang and Zhu have showed that if the system is partially hyperbolic with a uniformly compact central foliation, then it admits the specification property, and its topological entropy is bounded by the sum of the growth rate of center periodic leaves with the entropy of the center foliation. In our case, we restrict ourselves to partially hyperbolic systems with $\lambda^c(f) = \max ||Df|_{E^c}|| \leqslant 1$. We show that this type of system is h-expansive and has center specification property. As a result, we obtained results analogous of those theorems.

To state it we give the following definition. First, we fix some appropriate $\delta > 0$ and consider $\overline{\mathcal{P}^c_{\text{per}}(\delta)}$ (the closure in the product topology) of the set of δ-pseudo-orbits. We say that a set $\underline{E} \subset \{\underline{x} \in \mathcal{P}^c_{\text{per}}(\delta);\ \sigma^n(\underline{x}) = \underline{x}\}$ is said to be n-centrally separated if for any $\underline{x}, \underline{y} \in \underline{E}$ there is a $j \in \{0, \cdots, n-1\}$ such that $d(x_j, y_j) \geqslant \delta$. Denote by $\#\text{Per}_n$ the cardinal of any n-centrally separated set of maximal size. We need to assume one of the following conditions.

Hypothesis 1. For any $\alpha > 0$ there exists $\rho = \rho(\alpha) > 0$ so that: for any $x \in M, z, w \in D^{su}(x), y \in W^c(x)$, it holds

$$d(z, w) \geqslant \alpha \Rightarrow d(h^c_{y,x}(z), h^c_{y,x}(w)) \geqslant \rho$$

(and in particular, $h^c(z), h^c(w)$ is well defined.)

Hypothesis 2. f is a regular element of an hyperbolic action.

Theorem A. *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism satisfying the following hypotheses*

- *foliations $\mathcal{W}^c, \mathcal{W}^{cs}$ are of class C^1;*
- *the strong unstable foliation $\mathcal{W}^u$ is minimal;*
- $\lambda^c(f) = \max ||Df|_{E^c}|| \leqslant 1$.

If f satisfies either Hypothesis 1 or 2, then

$$h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} \log \#Per_n.$$

Next we consider the ergodic theory part.

Theorem B. *Let $f : M \to M$ be a regular element of a hyperbolic action, and $\mathscr{E}$ be a family of maximally n-plaque periodic sets. Then*

$$\nu^{\mathscr{E}}_n = \frac{1}{n} \sum_{k=0}^{n-1} f^k_* \mu^{\mathscr{E}}_k \xrightarrow[n\to\infty]{\text{weakly}} \mu_{MME}$$

where μ_{MME} is the unique entropy maximizing measure of f.

Uniqueness of this measure can be deduced by the work of Climenhaga, Pesin and Zelerowicz in [14]. Our theorem gives a precise description, very much as in the case of the entropy maximizing measure for hyperbolic systems (Bowen measure).

Our contribution is given a precise description of this measure, very much as in the classical Anosov case.

We finish by giving an important application. We say that an hyperbolic action $\alpha\colon G \times M \to M$ is strongly Axiom A if there exists a regular element g in the center of G such that $f = \alpha(g, \cdot)$ is a regular element, and if the set of closed leaves is dense in M.

Theorem C. *If $\alpha : G \times M \to M$ is a strongly Axiom A action then there exists μ probability measure on M that is*

1. *G-invariant: for every measurable $A \subset M$ and $g \in G$, $\mu(\alpha(g, A)) = \mu(A)$.*
2. *μ is ergodic,*
3. $\operatorname{supp}(\mu) = M$.

This Theorem is proven by completely different methods in the recent paper of Y. Bonthonneau, C. Guillarmou and T. Weich [4]. It can be also obtained (again, with different methods) as consequence of the P.D. Carrasco and F. Rodriguez-Hertz [12]. Theorems A, B and C are proved in Chapter 3, which is the main part of the thesis.

Chapter 1

Preliminaries

In this chapter we present a brief review of the necessary definitions and properties that will be used throughout the work. We start with the concept of foliation. After this we introduce the notion of holonomy and list some of its properties that will be useful in the future. Lastly, we define partially hyperbolic systems, and review some of their properties. For a more detailed reading we suggest [10], [17], [20] and [24].

In this work we denote M as a closed (compact, without boundary) differentiable n-manifold.

1.1 Foliation

We recall some basic facts of foliation theory and related concepts. For more details see [10]. Take some $r \geqslant 1$, $s \geqslant 0$ and $1 \leqslant q \leqslant n-1$.

Definition 1.1.1. A *foliated atlas* of class $C^{r,s}$ and codimension q for M is an atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha);\ \varphi_\alpha \colon U_\alpha \to \mathbb{R}^{n-q} \times \mathbb{R}^q\}$ such that if $U_\alpha \cap U_\beta \neq \varnothing$, then the coordinate changes $\varphi_{\alpha\beta} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are of the form

$$\varphi_{\alpha\beta}(x, y) = (x_{\alpha\beta}(x, y), y_{\alpha\beta}(y)),$$

where $x_{\alpha\beta} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^{n-q}$ is of class C^r, and $y_{\alpha\beta} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^q$ is of class C^s.

Definition 1.1.2. A *foliation* of class $C^{r,s}$ and co-dimension q on M is a maximal foliated atlas of class $C^{r,s}$ and co-dimension q on M. In addition, a $C^{r,s}$-foliation

$\mathcal{W}$ is said to be a C^r-*foliation* if there is a foliated atlas for $\mathcal{W}$ whose coordinate changes are C^r.

Definition 1.1.3. A *fiber bundle* is a structure (K, B, π) that consists of differentiable manifolds K, B of dimension $n, n-k$, respectively, a differentiable map $\pi\colon K \to B$, an open neighborhood U_x of x in B and a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \phi\ } & U \times F \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle p} \\ U & \xrightarrow[\ Id\ ]{} & U \end{array}$$

where ϕ is a diffeomorphism, p is the canonical projection onto the first factor and F is a manifold of dimension k named fiber. In terms of nomenclature, K is the bundle space (or only, bundle), B is the base space.

It is important to remark that each subspace $\pi^{-1}(x)$ with $x \in B$ is an imbedded k-manifold diffeomorphic to F.

An example of fiber bundle is the tangent bundle TM of M defined by

$$TM = \bigcup_{x \in M} T_x M = \{(x, v) \colon x \in M \quad \text{and} \quad v \in T_x M\},$$

and $\pi\colon TM \to M$ is given by $\pi(x, v) = x$ for all $v \in T_x M$.

A sub-bundle TN of TM is a subset $TN \subset TM$ so that TN is a tangent bundle and for each $x \in M$, the fiber at x, $\pi^{-1}(x) \in T_x N$, is a subspace of $\pi^{-1}(x) = T_x M$

Definition 1.1.4. A foliation $\mathcal{W}$ of class $C^{r,0}$ on M is said to be of *class* $C^{r,0+}$ if each leaf W is C^1-immersed, and the inclusion $T\mathcal{W} \hookrightarrow TM$ embeds $T\mathcal{W}$ as a C^0 p-plane sub-bundle of TM.

A foliation $\mathcal{W} = \{(U_\alpha, \varphi_\alpha)\}$ is *nice* if every U_α is a cube[1] and if $U_\alpha \cap U_\beta \neq \emptyset$, then there is a cube in $\mathcal{W}$ containing $\overline{U_\alpha \cap U_\beta}$.

According to [10, Lemma 1.2.17, p. 30], every foliated atlas has a nice refinement, hence we can assume in this work that every foliation is nice.

[1] U_α is called a cube if $\varphi_\alpha(U_\alpha)$ is a cube in $\mathbb{R}^n$.

Definition 1.1.5. A *plaque* of U_α containing x is the set

$$P_x = \{y \in U_\alpha;\ \pi_q(\varphi_\alpha(y)) = \pi_q(\varphi_\alpha(x))\},$$

where $\pi_q\colon \mathbb{R}^{n-q} \times \mathbb{R}^q \to \mathbb{R}^q$ denotes the projection.

We say that two points $x, y \in M$ are *related* if there is a sequence $\{P_0, P_1, \ldots, P_m\}$ of plaques satisfying: $x \in P_0$, $y \in P_m$, and $P_i \cap P_{i-1} \neq \varnothing$ when $1 \leqslant i \leqslant m$. Such a sequence is called a *plaque chain*.

It is important to notice that being related is indeed an equivalence relation. Each equivalence class L is a union of plaques, and is called a *leaf of the foliation*; locally, it is a topologically immersed sub-manifold of M of dimension $n - q$.

1.2 Partially Hyperbolic Diffeomorphism

In this section we present the definition of partially hyperbolic systems. We will denote the continuous Riemannian metric on M by $\|\cdot\|$.

Definition 1.2.1. A C^1-diffeomorphism $f\colon M \to M$ is said to be *partially hyperbolic* if there is a nontrivial continuous splitting of the tangent bundle

$$TM = E^s \oplus E^c \oplus E^u,$$

that is df-invariant and it satisfies

$$\|d_x f(v^s)\| < \|d_x f(v^c)\| < \|d_x f(v^u)\|$$

for every $x \in M$ and all unitary vector $v^* \in E^*_x$ $(* = s, c, u)$. Moreover,

$$\lambda := \max\{||df|E^s||, ||df^{-1}|E^u||\} < 1.$$

The bundles E^s, E^c and E^u are called the *stable*, *center* and *unstable* bundle, respectively. We will also consider the following bundles: $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$. Besides, we can take a Lyapunov inner metric (see for instance [24, p.12]) such that, for every $x \in M$, the subspaces E^s_x, E^c_x and E^u_x are mutually orthogonal.

Notice that, for each $x \in M$, the derivative of f contracts uniformly in the stable direction E^s_x (with contraction rate $\lambda_s = \lambda$), and expands uniformly in the unstable direction E^u_x (with expansion rate $\lambda_u = \lambda^{-1}$). In the central direction E^c_x, it can contract or expand but with smaller rates.

Next we present some of the main examples of partially hyperbolic diffeomorphisms. For a wider range of examples and properties, we suggest [17].

1.2.1 Examples

1. **Skew-products:** Let $f : M \to M$ be an Anosov diffeomorphism and $\phi : M \to \mathrm{Diff}^1(N)$[2] be a family of diffeomorphisms of a compact manifold N, satisfying

$$\|d_x f|_{E^s}\| < m(d_y \phi_x),$$

and

$$\|d_y \phi_x\| < m(d_x f|_{E^u}),$$

for all $x \in M$ and $y \in N$, where m denotes the conorm[3]. The skew-product $F\colon M \times N \to M \times N$ given by $F(x, y) = (f(x), \phi_x(y))$ is partially hyperbolic, where $E^s_F = E^s_f$, $E^u_F = E^u_f$ and $E^c_F = TN$. (we identify $T(M \times N) = TM \oplus TN$).

An example of skew-product is obtained by taking $N = S^1$ and $R_{\phi_x} : S^1 \to S^1$ to be the rotation by an angle ϕ_x. Hence, the skew-product

$$\begin{aligned} F\colon M \times S^1 &\to M \times S^1 \\ (x, y) &\mapsto (f(x), R_{\phi_x}(y)), \end{aligned}$$

is a partially hyperbolic diffeomorphism with central direction compact and one-dimensional.

2. **Time-one maps of Anosov flows:** A flow $\phi_t\colon M \to M$ is called Anosov if there is a $d\phi_t$-invariant decomposition $TM = E^s \oplus X \oplus E^u$, where X is the direction tangent to the flow, and E^s, E^u are the contracting and expanding directions.

The time-one map of the geodesic flow on negative-curvature surfaces is a partially hyperbolic diffeomorphism. In this example, the Anosov flow is topologically mixing. In addition, it is *accessible*, meaning that, given any

[2]$\mathrm{Diff}^1(N)$ denotes the set of all C^1-diffeomorphism of N

[3]The conorm of a matrix A is defined as $m(A) = \inf\{\|Av\|\colon \|v\| = 1\}$

two points in the manifold, one can take a piecewise C^1-curve connecting the two points, and whose derivative is always tangent to either the stable or the unstable bundle.

Another example of Anosov flow, called the suspension flow, can be seen as follows. Let f be a hyperbolic diffeomorphism on $\mathbb{T}^2$ and consider in $\mathbb{T}^2 \times \mathbb{R}$ the equivalence relation $(x, t+1) \sim (f(x), t)$. Then $M = \mathbb{T}^2 \times \mathbb{R}/\sim$ is a compact manifold and $F([x,t]) = [x, t+1]$ is a partially hyperbolic diffeomorphism. In this case, the distribution $E^s \oplus E^u$ is integrable (see [25]).

3. Linear Automorphisms on Tori: Consider a matrix $A \in \mathrm{SL}(n, \mathbb{Z})$, and denote by f_A the map induced by A on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Notice that f_A is differentiable. Indeed, the derivative $Df_A(x)$ at each point x is canonically identified with A. If A has no eigenvalues that are roots of the identity, then f_A is partially hyperbolic. In this case, the bundles E^s, E^c and E^u are the direct sum of the eigenspaces corresponding to the eigenvalues of norm less than, equal to and bigger than one, respectively.

The stable manifold theorem (see Theorem 7.3 in [29]) states that the bundles E^s and E^u are both integrable, which means that there are two f-invariant continuous foliations, denoted by $\mathcal{W}^s$ and $\mathcal{W}^u$, whose leaves are of class C^1, and satisfy $E^s = T\mathcal{W}^s$ and $E^u = T\mathcal{W}^u$. The transversal regularity of these foliations is only Hölder (see [27]).

On the other hand, the central foliation may not be integrable as can be seen in the example (see [17] and [29]). One of the main problem in the area is to establish necessary and sufficient conditions for integrability. Some results on this can be found in [3], [8] and [18].

However, for most of the known examples the central foliation is integrable. For this reason we are going to assume that E^c is integrable throughout this text. We actually ask that f be dynamically coherent, which the definition is the following.

Definition 1.2.2. A partially hyperbolic map f is *dynamically coherent* if:

- E^c, $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$ are integrable to continuous f-invariant foliations $\mathcal{W}^c$, $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$, respectively; and
- $\mathcal{W}^c$ sub-foliates $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$.

In [19]. F. Rodriguez Hertz and *et. al.* presented a example of a non-dynamically coherent partially hyperbolic diffeomorphism with one-dimensional center bundle. However, almost every known example satisfies this hypothesis, therefore we assume, in this work, that f is a partially hyperbolic diffeomorphism dynamically coherent.

A good characteristic of dynamically coherent partially hyperbolic diffeomorphisms is that they have geometric properties that improve the comprehension of its dynamics, *e.g.*, local product structure, that we are going to define at the end of this section.

For an invariant foliation $\mathcal{W}^*$ (where $* = s, c, u, cs, cu$), a point $x \in M$ and a constant $r > 0$, we denote

$$W^*(x,r) = \{y \in W^*(x) \colon d^*(x,y) \leqslant r\}$$

where d^* is the intrinsic distance in the corresponding leaf $W^*(x)$.

Hence, for a center leaf $L \in \mathcal{W}^c$ and a constant $r > 0$, we can define

$$W^s(L,r) = \bigcup_{x \in L} W^s(x,r)$$

and

$$W^u(L,r) = \bigcup_{x \in L} W^u(x,r).$$

Finally, the concept of local product structure is as follows.

Definition 1.2.3. We say that the map f has *local product structure* if there is a constant $c_{\text{lps}} > 0$, called the *constant of local product structure*, such that whenever $d(x,y) < c_{\text{lps}}$, then $W^s(W^c(x, c_{\text{lps}}), 2c_{\text{lps}})$ intersects $W^u(W^c(y, c_{\text{lps}}), 2c_{\text{lps}})$ along a plaque of $\mathcal{W}^c$ of radius at least $\frac{c_{\text{lps}}}{2}$.

Dynamically coherence is easily seen to imply local product structure (see for example Proposition 1.4 in [13]).

In summary, throughout this work f is a dynamically coherent partially hyperbolic diffeomorphism on a closed manifold M, and the central foliation $\mathcal{W}^c$ is of class C^1.

1.3 Holonomy

We are going to present in this section some basic notations, the concept of holonomy and how it will be used in this work.

Let M be a closed differentiable Riemannian n-manifold and $\mathcal{W}$ a continuous foliation of M. Moreover, we consider the bundle $E = T\mathcal{W}$ and the perpendicular bundle $F := (E)^{\perp}$.

We will also need the definition of vector bundle, which is given below.

Definition 1.3.1. A *vector bundle* is a fiber bundle (K, B, π) such that for every $x \in B$ the fiber $\pi^{-1}(x)$ is a vector space.

Given $\epsilon > 0$ we define the ϵ-disc sub-bundle of K as

$$K(\epsilon) := \bigsqcup_{x\in B} \{v \in K_x \colon \|v\| < \epsilon\},$$

where, $K_x = \pi^{-1}(x)$.

Also, we denote by

$$T_vK := \bigsqcup_{x\in M} TK_x,$$

where $v \in \pi^{-1}(x)$.

Now, let us to consider the vector bundle (TM, M, π).

Definition 1.3.2. Let $p \in M$, V a neighborhood of p in M and $\epsilon > 0$ small enough. We denote the open set $\mathcal{U} \in TM$ as

$$\mathcal{U} := \{(q, v) \in TM \colon q \in V,\ v \in T_qM \text{ and } \|v\| < \epsilon\}$$

and consider the map

$$\gamma\colon (-2, 2) \times \mathcal{U} \to M$$

then, the *exponential map* $\exp\colon \mathcal{U} \to M$ is defined by

$$\exp(q, v) = \gamma(1, q, v) = \gamma\big(\|v\|, q, \frac{v}{\|v\|}\big), \quad (q, v) \in \mathcal{U}.$$

Geometrically, $\exp_q(v) - \exp(q, v)$ is the point of M obtained by traversing a path of length equal to $\|v\|$, from q, on the geodesic passing through q with velocity $\frac{v}{\|v\|}$.

We will denote by R_{inj} the injectivity radius of $\exp : \mathcal{U} \to M$. It means that, given a point $q \in M$, then R_{inj} is the largest radius for which the exponential map applied to $B(0, R_{\text{inj}}) \subset T_qM$ is a diffeomorphism over its image.

Let W be a leaf of foliation $\mathcal{W}$. We define

$$N_W = \bigsqcup_{x \in W} F_x,$$

and the map $\pi_W : N_W \to L$ where $\pi_W(F_x) = x$ is the map projection of N_W in W.

Definition 1.3.3. N_W is the unwrapping bundle of the leaf W.

Then, N_W inherits the differentiable structure from F, and one can show that p_W is a submersion. Therefore, $\mathcal{W}$ lifts to a foliation $\mathcal{W}_W$ in N_W.

We would like to observe that $\mathcal{W}_W$ is transverse to the fibers of π_W, and also that N_W has a natural Riemannian metric (indeed, it is induced by the restriction of the Sasaki metric to F).

Now, we consider $\gamma \colon [0, k] \to W$ a path contained in W with $|\gamma'| = 1$, and define the map

$$\begin{aligned} h^\gamma : \Gamma(\gamma) \subset F_{\gamma(0)} &\to F_{\gamma(k)} \\ u &\mapsto h^\gamma(u), \end{aligned}$$

where $\Gamma(\gamma)$ is a neighborhood of $\gamma(0)$ in $F_{\gamma(0)}$ and $h^\gamma(u)$ is the terminal point of the unique curve $\gamma_u \colon [0, k] \to N_W$ satisfying:

(a) $\gamma_u(0) = u$;

(b) $\gamma_u' \in T\mathcal{W}_W$;

(c) $\pi_W(\gamma_u(t)) = \gamma(t)$ for every $t \in [0, k]$.

Since $\mathcal{W}_W$ intersects p_W transversely (denoted here by $\mathcal{W}_W \pitchfork p_W$), then γ_u is well defined, and so we can choose $\Gamma(\gamma)$ such that $\operatorname{Im} \gamma_u \subset F_{\gamma(k)}(R_{\text{inj}})$.

Therefore, it follows that the exponential map

$$\exp(h^\gamma(u)) := \text{hol}^\gamma_{x,y}(z)$$

denotes the holonomy relating to the foliation $\mathcal{W}$ in M defined by γ, from the disc $D(x) = \exp(F_x(R_{\text{inj}}))$ to $D(y) = \exp(F_y(R_{\text{inj}}))$, where $x = \gamma(0)$, $y = \gamma(k)$ and $z = \exp(u)$.

Remark 1. Since M is compact, the set $\Gamma(\gamma)$ can be chosen depending only on k, that is, $\Gamma(\gamma) = \Gamma(k)$.

By continuity of the tangent space $T\mathcal{W}$, given $\alpha > 0$, there is an $\epsilon > 0$ such that $|\measuredangle(T\mathcal{W}_W, T_vN_W) - \pi/2| < \alpha$ for points in $N_W(\epsilon)$ and for all W.

From now on, $\epsilon > 0$ is fixed and $\Gamma(k)$ is chosen so that every $u \in \Gamma(k) \subset F_{\gamma(t)}$ satisfies $\mathrm{Im}(\gamma_u) \subset B(\gamma(k), \epsilon) \subset F_{\gamma(k)}$.

Notice that each $w \in TN_W$ can be written uniquely as the direct sum $w = w^v \oplus w^H$, with $w^v \in T_vN_W$ and $w^H \in (T_vNW)^{\perp}$.

Let

$$m = \sup\{\|w^v\| \colon w \in T_vN_W, W \in \mathcal{W} \text{ and } \|D_{\pi_W}(w)\| = 1\}.$$

Given any $\gamma : [0,k] \to W$ and $u \in \Gamma(k) \subset F_{\gamma(0)}$, we define:

$$\begin{aligned} X_\gamma \colon [0,k] \times F_{\gamma(0)}(\Gamma(k)) &\to F_{\gamma(k)} \\ (t,u) &\mapsto \gamma_u'(t), \end{aligned}$$

where $h^\gamma(u) = X_\gamma(k,u)$.

Comparing X_γ with its vertical component we deduce that

$$\|h^\gamma(u)\| \geqslant \|u\| e^{-mk} \quad \text{if} \quad \|u\| < \Gamma(k).$$

More generally, the previous bound condition holds whenever $\mathrm{Im}(\gamma_u) \subset B(\gamma(k), \epsilon) \subset F_{\gamma(k)}$. This proves the following result.

Proposition 1.3.1. *For all curve γ tangent to a leaf in $\mathcal{W}$ it holds the following dichotomy: either*

1. *there are $y \in F(\Gamma(\gamma))$ and $z \in \mathrm{Im}(\gamma)$ such that $d(\mathrm{hol}^\gamma_{z,x}(y), z) \geqslant \epsilon$*

   or

2. *$d(\mathrm{hol}_{\gamma(t),x}(y), \gamma(t)) \geqslant d(x,y)e^{-mt}$ (assuming $|\gamma'| = 1$).*

Corollary 1.3.1. *If the foliation $\mathcal{W}$ is C^1, then it satisfies the Proposition 1.3.1. Furthermore, any C^1-perturbation in the foliation also satisfies.*

In this work, we will consider the center holonomy, that is, the holonomy map is defined in a disk contained in the foliation transversal to central foliation $\mathcal{W}^c$. More specifically:

We fix a leaf W^c in the center foliation $\mathcal{W}^c$ and consider a smooth family of discs $W^c \ni x \mapsto D^{su}(x, R_{\text{inj}})$ completely transverse to $\mathcal{W}^c$, here it is understood that the angle of $TD^{su}(x, R_{\text{inj}})$ with $E^s \oplus E^u$ is (uniformly) small. Given $x, y \in M$ such that $y \in W^c(x, \frac{c_{\text{lps}}}{2})$ we consider the holonomy transport

$$h^c_{x,y} : D^{su}(x, R_{\text{inj}}) \to D^{su}(y, c_{\text{lps}}).$$

- Case 1: If $y \in W^{cs}(x, \frac{c_{\text{lps}}}{2})$ we can consider

$$h^c_{x,y}|_{\mathcal{W}^s} : W^s(x, R_{\text{inj}}) \to W^s(y, c_{\text{lps}})$$

- Case 2: If $y \in W^{cu}(x, \frac{c_{\text{lps}}}{2})$ we can consider

$$h^c_{x,y}|_{\mathcal{W}^u} : W^u(x, R_{\text{inj}}) \to W^u(y, c_{\text{lps}})$$

(hopefully the abuse in notation won't cause any confusion).

Since $\mathcal{W}^c$ is $\mathcal{C}^1$ the holonomy maps are Lipschitz: there exist $L > 0$ so that

$$z, z' \in D^{su}(x, R_{\text{inj}}) \Rightarrow d(h^c_{x,y}(z), h^c_{x,y}(z')) \leqslant Ld(z, z').$$

By reducing R_{inj}, and by eventually modifying the discs $D^{su}(x, R_{\text{inj}})$ we can take L arbitrarily close to 1. In particular we can assume that

$$\lambda_s L < 1$$
$$\lambda_u^{-1} L < 1.$$

1.4 Entropy

We will recall the definition of topological entropy.

Let $f\colon M \to M$ be a continuous function on a compact metric space M.

Definition 1.4.1 (Separated set)**.** Given $\epsilon > 0$ and $n \in \mathbb{N}$ we say that $E \subset M$ is (n, ϵ)-*separated for* f if: for any $x, y \in E$, we can find a $j \in \{0, \ldots, n-1\}$ such that $d(f^j(x), f^j(y)) > \epsilon$. We denote by $s(n, \epsilon)$ the largest cardinality of the (n, ϵ)-separated sets.

Definition 1.4.2 (Generator set)**.** Given $\epsilon > 0$ and $n \in \mathbb{N}$ we say that $G \subset M$ is a (n, ϵ)-*generator for* f if: for any $x \in M$, we can find an $a \in G$ such that $d(f^j(x), f^j(a)) < \epsilon$ for every $j \in \{0, \ldots, n-1\}$. We denote by $g(n, \epsilon, f)$ the minimum cardinality of the (n, ϵ)-generator sets.

Denote

$$s(\epsilon, f) = \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon)$$
$$g(\epsilon, f) = \limsup_{n \to \infty} \frac{1}{n} \log g(n, \epsilon).$$

Definition 1.4.3. The *topological entropy* of f is given by

$$h_{top}(f) = \lim_{\epsilon \to 0} s(\epsilon, f) = \lim_{\epsilon \to 0} g(\epsilon, f).$$

For the proof that the two limits coincide see for instance [23, Proposition 10.1.6, p. 307]).

Definition 1.4.4. Given a closed set $K \subset M$, the *topological entropy of* K is given by

$$h_{top}(f|_K) = \lim_{\epsilon \to 0} g_K(\epsilon)$$

where

$$g_K(\epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log g_K(n, \epsilon),$$

and $g_K(n, \epsilon)$ denotes the smallest cardinality of a (n, ϵ)-generator with respect to for $f|_K$.

If $x \in M$ and $\epsilon > 0$, we denote

$$B^n(x, \epsilon) = \bigcap_{i=0}^{n-1} f^{-i}(B(f^i(x), \epsilon)),$$

the n-dynamical ball cantered at x and of radius ϵ; here $B(x, \epsilon)$ is the closed ball centered at x of radius ϵ. We extend the definition to

$$B^\infty(x, \epsilon) := \bigcap_{i \in \mathbb{Z}} f^{-i}(B(f^i(x), \epsilon))$$

Under our working hypotheses it holds that our map is h-expansive.

Definition 1.4.5. We say that a map f is *h-expansive* if there exists $\epsilon > 0$ such that

$$h_{top}\left(f, B^{\infty}(x, \epsilon)\right) = 0 \quad \text{for all} \quad x \in M.$$

This will be used in the proof Proposition 3.1.1.

Remark 2. In [6] it is showed that if f is h-expansive, then there exists some ϵ_0 so that for $0 \leqslant \epsilon \leqslant \epsilon_0$ it holds

$$h_{top}(f) = g(f, \epsilon) = s(f, \epsilon). \tag{1.1}$$

Chapter 2

Dynamics of Center Plaques

In this chapter we will introduce our object of study, which is the space $\mathcal{P}^c(\delta)$ of δ-pseudo-orbits respecting the central foliation. We will discuss some basic results about it, such as compactness and some type of shadowing property. Additionally, we present some important subsets of $\mathcal{P}^c(\delta)$, like the space $\mathcal{P}^c_{\text{per}}(\delta)$ periodic δ-pseudo orbits, and $\mathcal{P}^c_{\text{rec}}(\delta)$ of recurrent δ-pseudo-orbits respecting the central foliation. Finally, we define the stable and unstable sets of a pseudo-orbit respecting the central foliation.

The work in this Chapter was done in collaboration with Catalina Freijó.

Recall that M is a closed Riemannian manifold and d denotes the corresponding induced metric on it.

Definition 2.0.1. Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism. Consider a sequence $\underline{x} = (x_n)_{n\in\mathbb{Z}} \in M^{\mathbb{Z}} = \prod_{i=-\infty}^{\infty} M$ and $\delta > 0$. We say that $\underline{x}$ is a *δ-pseudo-orbit* for f if

$$d(f(x_n), x_{n+1}) \leqslant \delta \text{ for every } n \in \mathbb{Z}.$$

We say that a δ-pseudo-orbit $\underline{x}$ *respects the central foliation* or that is a *center pseudo-orbit* if

$$f(x_n) \in W^c(x_{n+1}, \delta) \text{ for every } n \in \mathbb{Z}.$$

Whenever the size δ of the pseudo-orbit is not relevant for the discussion we will omit the explicit reference.

Let

$$\mathcal{P}(\delta) = \{\underline{x} \in M^{\mathbb{Z}} \colon \underline{x} \text{ is a } \delta\text{-pseudo-orbit}\}$$

and

$$\mathcal{P}^c(\delta) = \{\underline{x} \in \mathcal{P}(\delta) \colon \underline{x} \text{ respects the central foliation}\}.$$

We remark that if $0 < \hat{\delta} < \delta$, then $\mathcal{P}^c(\hat{\delta}) \subset \mathcal{P}^c(\delta)$. We fix some $0 < \delta_* < \frac{c_{\text{lps}}}{2}$, whose precise size will be given later and denote

$$\mathcal{P}^c = \mathcal{P}^c(\delta_*).$$

We denote by $\pi : \mathcal{P}^c \to M$ the natural projection, that is,

$$\mathcal{P}^c \ni \underline{x} \quad \longmapsto \quad \pi(\underline{x}) = x_0.$$

Clearly π is continuous.

On $M^{\mathbb{Z}}$ we consider the distance

$$d_{\text{pro}}(\underline{x}, \underline{y}) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \frac{d(x_i, y_i)}{1 + d(x_i, y_i)}.$$

It is well known that d_{pro} is compatible with the product topology induced from M, hence by Tychonoff's theorem $(M^{\mathbb{Z}}, d_{\text{pro}})$ is a compact metric space. The subsets $\mathcal{P}(\delta) \subset M^{\mathbb{Z}}$ will we endowed with the subspace metric.

Before we state the first property of the space $\mathcal{P}^c$, let us define the following.

Definition 2.0.2. The *center plaque* centered at $x \in M$ with radius $\delta > 0$ is defined to be

$$P_\delta(x) = W^c(x, \delta).$$

In the next result, we show that the set of pseudo-orbits respecting the central foliation is compact in $M^{\mathbb{Z}}$ with respect to the metric d_{pro}. By the discussion above it suffices to show that $\mathcal{P}^c(\delta)$ is a closed set, for $0 < \delta \leqslant \delta_*$

Lemma 2.0.1. *The space $\mathcal{P}^c(\delta)$ is closed in $M^{\mathbb{Z}}$.*

Proof. Let $\{\underline{x}^k\}_{k \in \mathbb{N}}$ be a sequence in $\mathcal{P}^c(\delta)$ converging to some $\underline{x}$. Since the limit of δ-pseudo-orbits is still a δ-pseudo-orbit, it is enough to show that $\underline{x}$ respects the central foliation.

The map $x \mapsto W^c(x,\delta)$, that associates each $x \in M$ to the plaque $W^c(x,\delta)$ containing x, is continuous. Hence, the sequence $W^c(x_n^k,\delta)$ converges to $W^c(x_n,\delta)$ for every $n \in \mathbb{Z}$. Analogously, $W^c(f(x_n^k),\delta)$ converges to $W^c(f(x_n),\delta)$ and, since $f(x_n^k) \in W^c(x_{n+1}^k,\delta)$, we conclude that $f(x_n) \in W^c(x_{n+1},\delta)$. □

2.1 Shadowing

Since the idea is to extend some of the theory of Anosov maps to partially hyperbolic diffeomorphism, we will need an analogue of the classical Shadowing Lemma (see, for instance, [28, Proposition 8.20, p.109]). To explain this we need some definitions.

Definition 2.1.1. Let $\underline{x}$ and $\underline{y}$ be two pseudo-orbits. Given an $\epsilon > 0$, we say that $\underline{y}$ *ϵ-shadows* $\underline{x}$ if $d(x_n, y_n) < \epsilon$ for all $n \in \mathbb{Z}$.

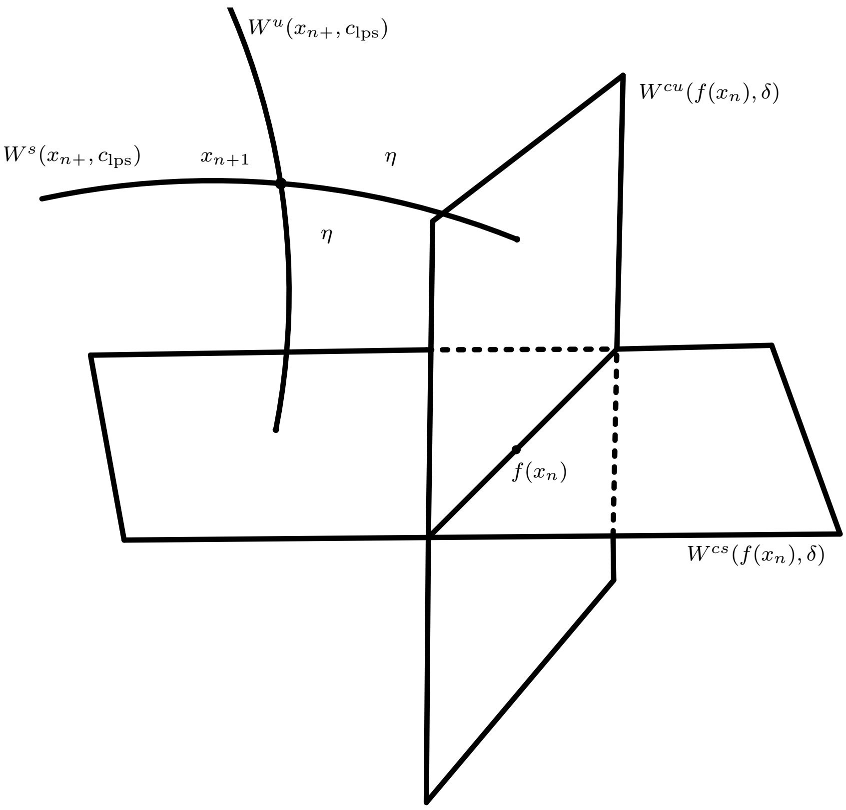
Now we can state the above-mentioned lemma involving pseudo-orbits.

Theorem 1 (Shadowing Lemma)**.** *Let $f\colon M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism. Then there exists $C > 0$ so that for any $\delta > 0$ such that $C\delta < c_{\mathrm{lps}}$ it holds: any δ-pseudo-orbit $\underline{x}$ for f can be $C\delta$-shadowed by a $C\delta$-pseudo orbit $\underline{y}$ respecting the central foliation. Moreover, if $\underline{x}$ is periodic then $\underline{y}$ can be taken periodic as well.*

This is a variation of [20, Lemma 7A.2, p.133]. In this work however, we will require a finer control in the size of the shadowing pseudo-orbit, and on its distance to the original one. This was pointed out by Javier Correa, whom we would like to thank.

Definition 2.1.2. Let $0 < \eta \leqslant \delta$. We say that the δ-pseudo-orbit $\underline{x}$ is a (δ,η)-quasi-center pseudo-orbit if for all $n \in \mathbb{Z}$

$$d(x_{n+1}, W^s(x_{n+1}, c_{\mathrm{lps}}) \cap W^{cu}(f(x_n),\delta)) < \eta$$
$$d(x_{n+1}, W^u(x_{n+1}, c_{\mathrm{lps}}) \cap W^{cs}(f(x_n),\delta)) < \eta.$$



Then we have the following version of the Shadowing theorem.

Theorem 2 (Shadowing Lemma'). *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism with smooth center foliation. Then there exist $0 < \delta_0 < c_{\text{lps}}$, $C > 0$ such that if $0 < \delta < \delta_0$ then there exist $\eta_\delta > 0$ and $D_\delta : (0, \eta_\delta) \to [1, 2]$ verifying*

1. *D_δ is continuous and $\lim_{\eta \to 0} D_\delta(\eta) = 1$.*
2. *If $\underline{x}$ is a (δ, η)-quasi-center pseudo-orbit then it can be shadowed by a center pseudo-orbit $\underline{y}$ and verifying for all $n \in \mathbb{Z}$:*
   (a) *$d(f(y_n), y_{n+1}) < D_\delta(\eta)\delta$, and*
   (b) *$d(x_n, y_n) < C\eta$.*

If $\underline{x}$ is periodic then $\underline{y}$ is periodic.

For the proof see [11].

Naturally, one could ask about the uniqueness of the shadowing pseudo-orbit in Lemma 1. The concept of plaque expansivity (that we introduce next) will provide an answer to this question.

Definition 2.1.3. The central foliation $\mathcal{W}^c$ is *plaque expansive* if there is a constant $c_{\exp} > 0$ such that for any $\underline{x}, \underline{y} \in \mathcal{P}^c(\delta)$ satisfying $d(x_n, y_n) < c_{\exp}$ for every $n \in \mathbb{Z}$, we have that x_n and y_n are always in the same center plaque.

Under the hypothesis of $\mathcal{W}^c$ being C^1, the condition of plaque expansivity is satisfied (see [20, Theorem 7.2, p.119] for a proof). Actually, there are not known counterexamples to this fact, although a variation of the examples in [2] is suspected to give such a counterexample.

Putting everything together, by eventually reducing δ_0 in Theorem 2, we get the following.

Corollary 2.1.1. *Assume that f is a dynamically coherent partially hyperbolic diffeomorphism with smooth center foliation. Then the shadowing pseudo-orbit given in Theorem 2 are unique in the following sense: if $\underline{x}$ is a (δ, η)-quasi-center pseudo-orbit which is $C\eta$-shadowed by the $D_\delta(\eta)\delta$-center pseudo-orbits $\underline{y}, \underline{z}$, then y_n and z_n are in the same center plaque for every $n \in \mathbb{Z}$.*

Bookeeping of constants. From now on we redefine δ_* so that $\delta_* < \frac{\delta_0}{100}$.

2.2 Recurrence

Next we investigate the concept of recurrence in $\mathcal{P}^c_{\mathrm{per}}$.

Definition 2.2.1.

1. A pseudo-orbit $\underline{x}$ is said to be *periodic* if there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{Z}$, $x_{k+n} = x_n$. The smallest of such k is called the period of $\underline{x}$.

   We denote $\mathcal{P}_{\mathrm{per}}(\delta)$ the set of all δ-periodic pseudo-orbits, and by

   $$\mathcal{P}^c_{\mathrm{per}}(\delta) = \mathcal{P}_{\mathrm{per}}(\delta) \cap \mathcal{P}^c(\delta)$$

   the set of all δ-periodic pseudo-orbits respecting the central foliation.

2. A point $x \in M$ is called *chain-recurrent* if for every $\epsilon > 0$ there exists an ϵ-periodic pseudo-orbit $\underline{x}$ with $x_0 = x$. The set of all chain-recurrent points of M is the chain-recurrent set of f and is denoted $\mathrm{CR}(f)$.

According to Conley's theory [15], the chain-recurrent set is the most general (invariant) set relevant for the dynamics. Still, even when dealing with true orbits of a map $f : M \to M$ it is important to consider other sets with good recurrent properties, like

- the set of periodic points of f, $\mathrm{Per}(f) = \{x \in M : \exists k \in \mathbb{N} \text{ with } f^k(x) = x\}$,
- the set of non-wandering points of f, $\mathrm{NW}(f) = \{x \in M : \forall \text{ open } U \ni x \;\; \exists k > 0;\;\; f^k(U) \cap U \neq \varnothing\}$, or
- the set of bi-recurrent points of f, $\mathrm{Rec}^{\pm}(f) = \{x : x \in \omega(x) \cap \alpha(x)\}$, where

  $$\omega(x) = \{y \in M : \;\exists \text{ a sequence } n_i \to \infty;\;\; f^{n_i}(x) \to y\}$$

  and

  $$\alpha(x) = \{y \in M : \;\exists \text{ a sequence } n_i \to -\infty;\;\; f^{n_i}(x) \to y\}$$

It is a basic fact of topological dynamics that the sets $\mathrm{NW}(f), \mathrm{CR}(f)$ are closed, and the following sequence of inclusions hold:

$$\overline{\mathrm{Per}(f)} \subset \overline{\mathrm{Rec}^{\pm}(f)} \cap \subset \mathrm{NW}(f) \subset \mathrm{CR}(f).$$

A more subtle fact is that C^1 generically, $\overline{\mathrm{Per}(f)} = \mathrm{CR}(f)$: see [1]. For pseudo-orbits however, analogous notions to non-wandering and chain-recurrence seem to lead to very restricted situations. Because of this, we will opt to work with the sets

$$\overline{\mathcal{P}^c_{\mathrm{per}}(\delta)}.$$

Let us show $\overline{\mathcal{P}^c_{\mathrm{per}}(\delta)}$ contains sets with good recurrence properties.

Definition 2.2.2. Let $\underline{x}$ be a δ-pseudo-orbit. We say that $\underline{x}$ is recurrent if for every $N > 0$ there exist $k, k' > N$ so that $x_{-k'} = x_0 = x_k$.

We denote by $\mathcal{P}_{\mathrm{rec}}(\delta)$ the set of δ-recurrent pseudo-orbits and by

$$\mathcal{P}^c_{\mathrm{rec}}(\delta) = \mathcal{P}_{\mathrm{rec}}(\delta) \cap \mathcal{P}^c(\delta)$$

the set of all δ-recurrent pseudo-orbits respecting the central foliation.

Example 1. For $x \in M$ let $\underline{x} = (f^n(x))_{n \in \mathbb{Z}}$ and $\delta > 0$. It is direct to check that

- if x is periodic then $\underline{x} \in \mathcal{P}^c_{\mathrm{per}}(\delta)$;

- if x is bi-recurrent then $\underline{x} \in \mathcal{P}^c_{\text{rec}}(\delta)$.

The following inclusion is direct

$$\mathcal{P}_{\text{per}}(\delta) \subset \mathcal{P}_{\text{rec}}(\delta) \Rightarrow \overline{\mathcal{P}^c_{\text{per}}(\delta)} \subset \overline{\mathcal{P}^c_{\text{rec}}(\delta)} \subset \overline{\mathcal{P}_{\text{rec}}(\delta)}.$$

Lemma 2.2.1. *For all $\delta > 0$ there exists δ', with $\delta' > \delta$ such that*

$$\overline{\mathcal{P}^c_{\text{rec}}(\delta)} \subset \overline{\mathcal{P}^c_{\text{per}}(\delta')}.$$

Proof. Note that we are taking $\delta > 0$ much smaller than the constant δ_0 given in Theorem 2 and consider $\delta' > \delta$ (which we could assume close to δ). Consider η_δ, C, D_δ as given in that Theorem 2.

Now, fix $\underline{x} \in \mathcal{P}^c_{\text{rec}}(\delta)$ and $\epsilon > 0$. Take $n_0 \in \mathbb{N}$ so that

$$\sum_{|i|>n_0} \frac{1}{2^{|i|}} < \frac{\epsilon}{2}.$$

Since $\underline{x}$ is recurrent we can find $k, k' > n_0$ so that $x_k = x_0 = x_{-k'}$. We define

$$y_n = x_n \qquad -k' \leqslant n \leqslant k$$

and complete to a bi-infinite periodic sequence $\underline{y} = (y_n)_{n\in\mathbb{Z}}$. Note that $d_{\text{pro}}(\underline{x}, \underline{y}) < \frac{\epsilon}{2}$.

On the other hand, since $\underline{x}$ preserves the center foliation, it is in particular a (δ, η)-quasi-center pseudo-orbit, for every $0 < \eta \leqslant \eta_\delta$. We reduce η even further so it satisfies

- $D_\delta(\eta)\delta \leqslant \delta'$.
- $C\eta < \frac{\epsilon}{2^{n_0+1}}$

Due to Theorem 2 it follows that we can find $\underline{z}^\eta$ satisfying

1. $\underline{z}^\eta \in \mathcal{P}^c_{\text{per}}(\delta')$.
2. $d_{\text{pro}}(\underline{y}, \underline{z}^\eta) < \frac{\epsilon}{2}$.

In particular $d_{\text{pro}}(\underline{x}, \underline{z}^\eta) < \epsilon$. This shows that $\underline{x} \in \overline{\mathcal{P}^c_{\text{per}}(\delta')}$. □

Corollary 2.2.1. *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism with smooth center foliation. Then for every δ sufficiently small,*

$$\overline{\mathrm{Rec}^{\pm}(f)} \subset \pi(\overline{\mathcal{P}^c_{\mathrm{per}}(\delta)}).$$

Proof. Indeed, for any $\delta > 0$ it holds $\mathrm{Rec}^{\pm}(f) \subset \pi(\mathcal{P}^c_{\mathrm{rec}}(\delta/2))$ (Example 1), hence by continuity of π and the lemma above,

$$\mathrm{Rec}^{\pm}(f) \subset \pi\big(\mathcal{P}^c_{\mathrm{rec}}\big(\delta/2\big)\big) \subset \overline{\pi(\mathcal{P}^c_{\mathrm{rec}}(\delta/2))} \subset \pi(\overline{\mathcal{P}^c_{\mathrm{rec}}(\delta/2)}) \subset \pi(\overline{\mathcal{P}^c_{\mathrm{per}}(\delta)}).$$

From here follows. □

Remark 3. It is not hard to construct partially hyperbolic systems without periodic points. In this regard, the corollary above gives a mechanism to obtain recurrent points from periodic structures inherent to the system.

2.3 Stable and Unstable sets of plaques

It will be convenient to introduce the natural (shift) dynamics on $\mathcal{P}^c(\delta)$, namely

$$\begin{aligned} \sigma\colon \mathcal{P}^c(\delta) &\to \mathcal{P}^c(\delta) \\ \underline{x} &\mapsto \underline{y} \text{ with } y_i = x_{i+1} \text{ for every } i \in \mathbb{Z}. \end{aligned}$$

For two δ-pseudo orbits $\underline{x}$ and $\underline{y}$ in $\mathcal{P}^c(\delta)$ we define the quantities d^+ and d^-,

$$\begin{aligned} d^+(\underline{x}, \underline{y}) &= \limsup_{n\to\infty} d_{\mathrm{Haus}}(W^c(x_n, \delta), W^c(y_n, \delta)) \\ d^-(\underline{x}, \underline{y}) &= \limsup_{n\to-\infty} d_{\mathrm{Haus}}(W^c(x_n, \delta), W^c(y_n, \delta)). \end{aligned}$$

In the definition above we are considering the Hausdorff distance[1].

Remark 4. Since the map $M \ni x \mapsto W^c(x, \delta)$ is continuous, we have that if, for any $\underline{x}, \underline{y} \in \mathcal{P}^c(\delta)$,

$$d(x_n, y_n) \to 0 \text{ when } n \to \infty \text{ then } d^+(\underline{x}, \underline{y}) = 0$$

and

$$d(x_n, y_n) \to 0 \text{ when } n \to -\infty \text{ then } d^-(\underline{x}, \underline{y}) = 0.$$

[1]For any sets $X, Y \subset (M, d)$, their Hausdorff distance is $d_{\mathrm{Haus}}(X, Y) = \inf\{\epsilon \geqslant 0\colon X \subset Y_\epsilon \text{ and } Y \subset X_\epsilon\}$, where $X_\epsilon := \bigcup_{x\in X} \{z \in M\colon d(z, x) \leqslant \epsilon\}$.

Definition 2.3.1. If $\underline{x} \in \mathcal{P}^c(\delta)$ its stable set is

$$S_\delta(\underline{x}) = \{\underline{y} \in \mathcal{P}^c(\delta) \colon d^+(\underline{x}, \underline{y}) = 0\},$$

while its unstable set is

$$U_\delta(\underline{x}) = \{\underline{y} \in \mathcal{P}^c(\delta) \colon d^-(\underline{x}, \underline{y}) = 0\}.$$

Remark 5. When $\underline{x} \in \mathcal{P}^c_{\text{per}}(\delta)$ with period k we have that if $\underline{y} \in S_\delta(\underline{x})$ then $\sigma^{n \cdot k}(\underline{y})$ also belongs to $S_\delta(\underline{x})$ for every $n > 0$. Analogously, if $\underline{y} \in U_\delta(\underline{x})$ then $\sigma^{n \cdot k}(\underline{y}) \in U_\delta(\underline{x})$ for every $n < 0$.

The next result assures us that stable and unstable sets of pseudo-orbit $\underline{x} \in \mathcal{P}^c(\delta)$ are nonempty.

Lemma 2.3.1. *For* $0 < \delta < \frac{c_{\text{lps}}}{2}$ *it holds: for any* $\delta < \delta' < \frac{c_{\text{lps}}}{2}$ *there exists* $\zeta > 0$ *so that for every* $\underline{x} \in \mathcal{P}^c(\delta)$,

$$\forall\ y \in W^{cs}(x_0, \zeta), \quad \exists\ \underline{y} \in S_{\delta'}(\underline{x});\ y_0 = W^c(y) \cap W^s(x_0, \zeta),$$

and

$$\forall\ y \in W^{cu}(x_0, \zeta), \quad \exists\ \underline{y} \in U_{\delta'}(\underline{x});\ y_0 = W^c(y) \cap W^u(x_0, \zeta).$$

Proof. Fix δ' with $0 < \delta < \delta' < \frac{c_{\text{lps}}}{2}$ and consider $0 < \zeta < \delta$ to be determined. For $y \in W^{cs}(x_0, \zeta)$, define $y_0 = W^c(y) \cap W^s(x_0, \zeta)$, we are going to generate a pseudo-orbit $\underline{y} \in S_{\delta'}(\underline{x})$ recursively:

$$\begin{aligned} y_j &= f^j(y) \ \text{ for every } \ j \leqslant 0 \\ y_j &= h^c_{x_j, f(x_{j-1})}(f(y_{j-1})) \ \text{ for every } \ j > 0. \end{aligned}$$

We will prove simultaneously that y_j is well defined for $j \geqslant 0$, $\lim_{j \to \infty} d(x_j, y_j) = 0$, and that $\underline{y} \in \mathcal{P}^c(\delta')$. This in turn will show that $\underline{y} \in S_{\delta'}(\underline{x})$.

First we use the (uniform) continuity of the stable foliation: for $z, w \in M$, $z \in W^c(w, \delta)$ there exists $\zeta > 0$ so that $z' \in W^s(z, \zeta), w' \in W^s(w, \zeta) \cap W^c(z')$ then $d(z', w') < \delta'$.

We now proceed by induction, remember that $\lambda L < 1$, and assume that,

- $y_j \in W^s(x_j, \zeta(\lambda_s \cdot L)^j)$, where L is Lipschitz constant of center holonomy.
- $d(f(y_{j-1}), y_j) < \delta'$.

Then $d(f(x_j), f(y_j)) < \zeta(\lambda_s \cdot L)^j \lambda_s$ implies

$$\begin{aligned} d(x_{j+1}, y_{j+1}) &= d(h^c_{x_{j+1}, f(x_j)}(f(x_j)), h^c_{x_{j+1}, f(x_j)}(f(y_j))) \\ &\leqslant L d(f(x_j), f(y_j)) \\ &< L\zeta(\lambda_s \cdot L)^j \lambda_s \\ &= \zeta(\lambda_s \cdot L)^{j+1} \end{aligned}$$

and $y_{j+1} = h^c_{x_{j+1}, f(x_j)}(f(y_j)) \in W^s(x_{j+1}, \zeta(\lambda_s \cdot L)^{j+1})$. By choice of ζ it follows that $\underline{y}$ is a δ'-pseudo-orbit.

The case when $y \in W^{cu}(x_0, \zeta)$ is analogous. □

Remark 6. What the lemma above says is that the stable and unstable sets of any $\underline{x} \in \mathcal{P}^c(\delta)$ are non-empty, if we allow to slightly increase δ.

Corollary 2.3.1. *Given* $0 < \delta < \delta' < \frac{c_{\text{lps}}}{2}$ *there exists* $\zeta > 0$ *so that for every* $\underline{x}, \underline{y} \in \mathcal{P}^c(\delta)$ *with* $d(x_i, y_j) < \zeta$ *for some* $i, j \in \mathbb{Z}$ *we have*

$$S_{\delta'}(\underline{x}) \cap U_{\delta'}(\underline{y}) \neq \varnothing$$

and

$$U_{\delta'}(\underline{x}) \cap S_{\delta'}(\underline{y}) \neq \varnothing.$$

Proof. For $0 < \zeta < c_{\text{lps}}$ and any $x, y \in M$ with $d(x, y) < \zeta$ both $W^{cs}(x, \zeta) \cap W^{cu}(y, \zeta)$ and $W^{cs}(y, \zeta) \cap W^{cu}(x, \zeta)$ are nonempty. Therefore, by Lemma 2.3.1 is possible to obtain $\underline{z} \in S_{\delta'}(\underline{x}) \cap U_{\delta'}(\underline{y})$ and $\underline{w} \in U_{\delta'}(\underline{x}) \cap S_{\delta'}(\underline{y})$. □

Given $x, y \in M$ such that $d(x, y) < \delta$, where $0 < \delta \leqslant c_{\text{lps}}$, we define the bracket between x and y as follows

$$[x, y] = W^s(x, 2c_{\text{lps}}) \cap W^{cu}(y, 2c_{\text{lps}}).$$

We finish this part by noting that $\overline{\mathcal{P}^c_{\text{per}}(\delta)}$ satisfies a weak "local product structure".

Proposition 2.3.1. *Given* $0 < \delta < \delta' < \frac{c_{\text{lps}}}{2}$ *it holds: for every* $\underline{x}, \underline{y} \in \overline{\mathcal{P}^c_{\text{per}}(\frac{\delta}{2})}$ *such that* $d(x_0, y_0) < c_{\text{lps}}$ *the point* $z = [x_0, y_0]$ *is in* $\pi(\overline{\mathcal{P}^c_{\text{per}}(\delta')})$.

Proof. First, suppose $\underline{x}$ and $\underline{y}$ are $\frac{\delta}{2}$-periodicals with period k_1, k_2 respectively. Let $z = [x_0, y_0]$ and $z' = [y_0, x_0]$, by Lemma 2.3.1 there is

$$\underline{z} \in S_{\frac{\delta'}{2}}(\underline{x}) \cap U_{\frac{\delta'}{2}}(\underline{y})$$

and

$$\underline{z'} \in S_{\frac{\delta'}{2}}(\underline{y}) \cap U_{\frac{\delta'}{2}}(\underline{x}).$$

Consider η_δ, C, D_δ as given in Theorem 2. Since $\underline{x}$ and $\underline{y}$ respecting the central foliation, they are $(\frac{\delta}{2}, \eta)$-quasi-center, for every $0 < \eta \leqslant \eta_\delta$. We fix $\eta > 0$ small enough to satisfy $\eta < \delta' - \delta$.

By uniform continuity of f, there is an ϵ_0 so that $d(f(x), f(y)) < \eta$ whenever $d(x, y) < \epsilon_0$. Moreover, by periodicity of $\underline{x}$ and $\underline{y}$ there is $N_0 \in \mathbb{N}$ large enough such that, if $n > N_0$ then

$$d(z_{n\cdot k_1,}, x_0) < \frac{\epsilon_0}{2}, \quad d(z_{-n\cdot k_2,}, y_0) < \frac{\epsilon_0}{2}$$

and

$$d(z'_{-n\cdot k_1,}, x_0) < \frac{\epsilon_0}{2}, \quad d(z'_{n\cdot k_2,}, y_0) < \frac{\epsilon_0}{2}$$

Now, we define $\underline{q}$ as follows:

$$\begin{array}{rclcrcl} q_0 &=& z_0 & \ldots & q_{nk_1} &=& z_{nk_1}, \\ q_{nk_1+1} &=& z'_{-nk_1+1} & \ldots & q_{(2k_1+k_2)n} &=& z'_{nk_2}, \\ q_{n(2k_1+k_2)+1} &=& z_{-k_2 n+1} & \ldots & q_{2n(k_1+k_2)} &=& z_0. \end{array}$$

Notice that $\underline{q}$ is $2(k_1+k_2)n$-periodic. We would like to prove that $d(f(q_n), q_{n+1}) \leqslant \delta'$ for every $n \in \mathbb{Z}$. For this, it is enough to show that $d(f(z_{nk_1}), z'_{-nk_1+1}) \leqslant \delta'$ and $d(f(z'_{nk_2}), z_{-nk_2+1}) \leqslant \delta'$.

Indeed, the triangular inequality gives us that

$$d(f(z_{nk_1}), z'_{-nk_1+1}) \leqslant d(f(z_{nk_1}), f(z'_{-nk_1})) + d(f(z'_{-nk_1}), z'_{-nk_1+1}) < \delta'.$$

The other case is analogous.

Furthermore, $\underline{q}$ é (δ', η)-quasi-center. Indeed, observe that,

$$d(z_{n\cdot k_1}, x_0) < \frac{\epsilon_0}{2} \Longrightarrow d(f(q_{n\cdot k_1}), f(x_0)) < \frac{\eta}{2}$$

and

$$
\begin{aligned}
d(q_{n\cdot k_1+1}, x_1) = d(z'_{-n\cdot k_1+1}, x_1) &\leqslant d(z'_{-n\cdot k_1+1}, f(z'_{-n\cdot k_1})) + d(f(z'_{-n\cdot k_1}), f(x_0)) + d(f(x_0), x_1) \\
&\leqslant d(q_{n\cdot k_1+1}, f(q_{n\cdot k_1})) + d(f(q_{n\cdot k_1}), f(x_0)) + d(f(x_0), x_1) \\
&< \frac{\delta'}{2} + \frac{\eta}{2} + \frac{\delta}{2}
\end{aligned}
$$

$$
d(q_{nk_1+1}, W^s(q_{nk_1+1}, c_{\text{lps}}) \cap W^{cu}(f(q_{nk_1}), \delta') < \eta.
$$

Then, by Theorem 2, there is a periodic $D'_\delta(\eta)\delta'$ pseudo-orbit $\underline{p}$ respecting the central foliation and shadowing $\underline{q}$.

□

Chapter 3

Entropy of f

We recall the following classical Theorem due to Bowen [7].

Theorem 3. *Let $f : X \to X$ be a homeomorphism of a compact metric space. If f is expansive and satisfies the specification property then it holds*

1. $h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} \log \#\{x : f^n(x) = x\}$.
2. *There exists a unique entropy maximizing measure μ_{MME}. That is,*
   (a) $h_{top}(f) = h_{\mu_{MME}}(f)$.
   (b) *If $\mu \neq \mu_{MME}$ is any other invariant measure for f, then $h_{\mu_{MME}}(f) > h_{\mu(f)}$.*
3. *The system (f, μ_{MME}) is isomorphic to a Bernoulli Shift.*

In this Chapter we will consider an analogous Theorem for some class of partially hyperbolic diffeomorphism. Namely, we will assume that

$f : M \to M$ is a dynamically coherent partially hyperbolic diffeomorphism with $\mathcal{C}^1$ bundles E^c, E^{cs}, minimal unstable foliation and $\lambda^c = \|Df|_{E^c}\| \leqslant 1$.

In this setting it is a result of Climenhaga, Pesin and Zelerowicz that f has a unique entropy maximizing measure [14], but our methods are different and seem to give a more concrete representation of this measure.The fact that (f, μ_{MME}) is isomorphic to a Bernoulli scheme is proven in [12].

Bookeeping of constants. The constants $c_{\text{lps}}, c_{\text{exp}}, \eta = \eta_\delta$ keep the same meaning as in Chapter 2. We fix $0 < \delta < \frac{\delta_0}{2}$ and write

$$\mathcal{P}^c = \mathcal{P}^c(\delta).$$

3.1 h-expansivity and Center specification property

In the following proposition we will show that plaque expansivity and $\lambda^c \leqslant 1$ imply h-expansivity.

Proposition 3.1.1. *If $f\colon M \to M$ is plaque expansive and satisfies that $\lambda^c(f) \leqslant 1$, then f is h-expansive.*

Proof. Fix $0 < \epsilon < \frac{c_{\text{exp}}}{2}$ (where c_{exp} is a constant of plaque expansivity of f) and $x \in M$. For any y, z in $B^\infty(x, \epsilon)$ we have,

$$d(f^n(y), f^n(z)) \leqslant d(f^n(y), f^n(x)) + d(f^n(x), f^n(z)) < 2\epsilon < c_{\text{exp}} \quad \forall\, n \in \mathbb{Z}.$$

Thus, by plaque expansivity, $y \in W^c(z, c_{\text{exp}})$. Therefore, the entropy of the set $B^\infty(x, \epsilon)$ is dominated by the c-topological entropy of f, which is zero since $\lambda^c \leqslant 1$. □

Bookeeping of constants. From now on, and by reducing ϵ_0 we assume that $\epsilon_0 \leqslant c_{\text{exp}}$.

Minimality of the unstable foliation gives us the next consequence.

Proposition 3.1.2. *If the unstable foliation $\mathcal{W}^u$ is minimal then f is mixing.*

Proof. Let U, V be two non-empty open sets in M and $x \in V$. Take $\epsilon > 0$ small enough such that $W^u(x, \epsilon) \subset V$. We have $\overline{W^u(x)} = M$ and $W^u(x) = \bigcup_{n \geqslant 0} f^n(W^u(f^{-n}(x), \epsilon))$.

Therefore,

$$M = \overline{W^u(x)} = \overline{\bigcup_{n\geqslant 0} f^n(W^u(f^{-n}(x), \epsilon))} \subset \overline{\bigcup_{n\geqslant 0} f^n(V)} \subset M,$$

then $M = \overline{\bigcup_{n\geqslant 0} f^n(V)}$ and since M is compact there is $N \in \mathbb{N}$ large enough such that for every $n > N, f^n(V) \cap U \neq \varnothing$. □

The fact that f is mixing implies that f has the center specification property, which is define below.

Definition 3.1.1. We say that f satisfies the *center specification property (at scale δ)* if for every $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ satisfying the following property: given $k \geqslant 2$ points

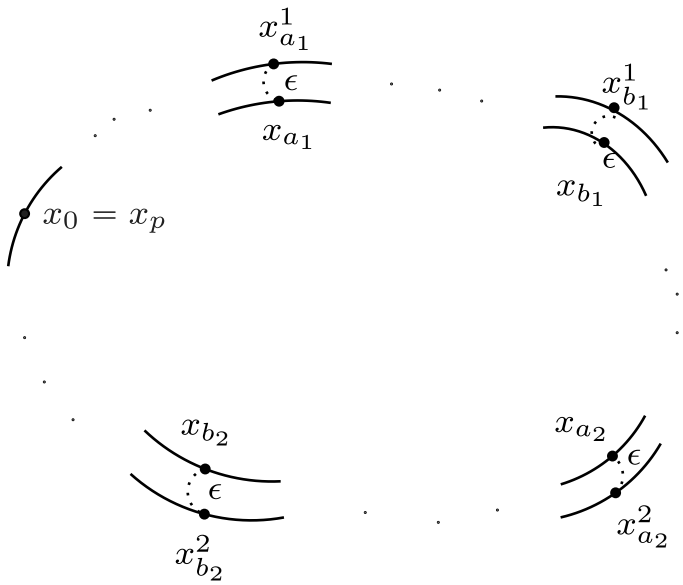
$$x^1, \ldots, x^k \in M,$$

integers

$$a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_k \leqslant b_k$$

with $a_i - b_{i-1} \geqslant N(\epsilon)$ $\forall 2 \leqslant i \leqslant k$, and $p \geqslant N(\epsilon) + b_k - a_1$, then one can find a p-periodic pseudo-orbit $\underline{x} \in \mathcal{P}^c(\delta)$ satisfying

$$d(x^i_n, x_n) \leqslant \epsilon \;\text{ for }\; a_i \leqslant n \leqslant b_i, \;\; 1 \leqslant i \leqslant k, \;\; x^i_n = f^n(x^i).$$



Lemma 3.1.1. *Under the hypotheses considered, the f has the center specification property at scale δ_0.*

Proof. Consider $\delta = \frac{\delta_0}{2}$ and let $\eta = \eta_\delta, D = D_\delta(\eta), C$ as given in Theorem 2. Without loss of generality, assume that $D\delta < \delta_0$. Fix $0 < \epsilon < C\eta$.

Let $\mathcal{U} = \{U_1, \cdots U_l\}$ be a finite cover of M with mesh$(\mathcal{U}) < \epsilon$. Since f is topologically mixing, for any $U_i, U_j \in \mathcal{U}$, there is an $N_{ij} > 0$ such that $f^n(U_i) \cap U_j \neq \varnothing$

for all $n > N_{ij}$. Define $N = \max\{N_{ij} : i, j = 1, \ldots, l\}$, and denote by $U(x)$ the set $U \in \mathcal{U}$ containing the point x.

To prove the specification property, take $x^1, \ldots, x^k$ points in M, and integers $a_1 \leqslant b_1 < \cdots < a_k \leqslant b_k$ with $a_j - b_{j-1} > N$, for $j = 2, \ldots, k$. Now, let p be a natural number with $p - (b_k - a_1) > N$. Consider $a_{k+1} = p + a_1$ and $x^{k+1} = x^1$.

Observe that, for each $j = 1, \ldots, k$, there is an y^j satisfying $y^j \in U(f^{b_j}(x^j))$ and $f^{a_{j+1}-b_j}(y^j) \in U(f^{a_{j+1}}(x^{j+1}))$. Now define $\underline{z} = \{z_i\}$ as follows:

1. $z_i = f^i(x^j)$ when $a_j \leqslant i < b_j$;
2. $z_i = f^{i-b_j}(y^j)$ when $b_j \leqslant i < a_{j+1}$; and
3. $z_{i+p} = z_i$ when $i \in \mathbb{Z}$.

We claim that $\underline{z}$ is a (δ, η)-quasi-center pseudo-orbit. In fact, it is sufficient to check $d(f(z_{b_j-1}), z_{b_j}) < \eta$ and $d(f(z_{a_{j+1}-1}), z_{a_{j+1}}) < \eta$. This follows by direct computation:

$$\begin{aligned} d(f(z_{b_j-1}), z_{b_j}) &= d(f(f^{b_j-1}(x^j)), f^{b_j-b_j}(y^j)) \\ &= d(f^{b_j}(x^j), y^j) \\ &< C\eta < \eta \end{aligned}$$

and

$$\begin{aligned} d(f(z_{a_{j+1}-1}), z_{a_{j+1}}) &= d(f(f^{a_{j+1}-1-b_j}(y^{a_{j+1}-1})), f^{a_{j+1}}(x^{j+1})) \\ &= d(f^{a_{j+1}-b_j}(y^{a_{j+1}-1}), f^{a_{j+1}}(x^{j+1})) \\ &< C\eta < \eta. \end{aligned}$$

The conclusion is obtained by Theorem 2 (Shadowing Lemma'): there exists a pseudo-orbit $\underline{x}$ such that

$$d(f(x_n), x_{n+1}) < D\delta < \delta_0$$

and

$$d(z_n, x_n) < C\eta, \quad \forall n \in \mathbb{Z}.$$

and therefore

$$d(f^n(x^i), x_n) \leqslant \epsilon < C\eta \ \text{ for } \ a_i \leqslant n \leqslant b_i, \quad 1 \leqslant i \leqslant k.$$

□

The idea now is based in [7], where the expansive case is considered.

Definition 3.1.2. For $p \in M$, $\Gamma(p) = \overline{W^u(p, \frac{c_{\text{lps}}}{2})}$.

Our interest is computing $h_{\text{top}}(f|_{\Gamma(p)})$. We will add the superscript "u" to denote that we are working with sets inside the unstable foliation, and extend the notations in the natural way. For example

$$s^u_K(n, \epsilon)$$

denotes the cardinal of a maximally (n, ϵ)-separated set inside K, where K is relatively compact inside a leaf of $\mathcal{W}^u$. To simplify we also write

$$s^u(n, \epsilon, p) = s^u_{\Gamma(p)}(n, \epsilon).$$

Since $f|_{\mathcal{W}^u}$ is uniformly expanding one sees directly that for every $n \geqslant 0$, $E \subset f^{-n}(\Gamma(p))$ is (n, ϵ) separated if and only if $f^n(E) \subset \Gamma(p)$ is separated (meaning, $d(x, y) \geqslant \epsilon$ for all $x \neq y \in f^n(E)$).

Lemma 3.1.2. *There exists $N_0 > 0$ and $k > 0$ constants (not depending on p) such that for every $p \in M$ the set $f^{N_0}(\Gamma(p))$ can be covered with at most k sets $\Gamma(q_1(p)), \cdots, \Gamma(q_k(p))$, where q_i is a point in $f^{N_0}(\Gamma(p))$ for every $i = 1, \cdots, k$.*

This is direct. From here it follows that for any given p, $s^u(n, \epsilon, p)$ is uniformly comparable with $s^u(n, \epsilon, q_i(p))$. Pushing a little more the same argument, this implies the following.

Lemma 3.1.3. *For $N_1 \geqslant N_0$ there exists $C_{N_1} > 0$ so that for every $p \in M$ and $q \in f^{N_1}(\Gamma(p))$ it holds*

$$\Gamma(q) \subset f^{N_1}(\Gamma(p)) \Rightarrow s^u(n, \epsilon, f^{-N_1}\Gamma(q)) \leqslant C_{N_1} s^u(n, \epsilon, p).$$

There exists $p_1, \cdots, p_l$ so that

$$M = \bigcup_{i=1}^{l} \overline{W^{cs}(\Gamma(p_i), \frac{c_{\text{lps}}}{2})} = \bigcup_{i=1}^{l} B_i. \tag{3.1}$$

By minimality of $\mathcal{W}^u$ there exists $N_2 \geqslant N_1$ so that for every $p \in M$, there is a connected component $E_i(p)$ of $f^N(\Gamma(p)) \cap B_i$ whose cs-projection on $\Gamma(p_i)$ is surjective. Using that the center stable holonomy is Lipschitz, and since $\lambda^c \leqslant 1$ we can compare

$$s^u(n,\epsilon,p) \sim s^u_{E_i(p)}(n,\epsilon) \sim s^u(n,\epsilon,p_i)$$

where $a \sim b$ means that $\frac{a}{b}$ is bounded above and below by some constant (independent of n and ϵ).

Corollary 3.1.1. *There exists $C_1 > 0$ so that for every $p, q \in M$, $n \in \mathbb{N}$ and $\epsilon > 0$,*

$$s^u(n,\epsilon,p) \leqslant C_1 s^u(n,\epsilon,q).$$

In particular, $h_{\text{top}}(f|_{\Gamma(p)}) = h_{\text{top}}(f|_{\Gamma(q)})$.

By the work of Hu, Hua and Wu [21] it is known that (in our working conditions, with $\lambda^c \leqslant 1$)

$$h_{\text{top}}(f) = \sup_{p\in M} h_{\text{top}}(f|_{\Gamma(p)})$$

hence we deduce that for every $p \in M$,

$$h_{\text{top}}(f) = h_{\text{top}}(f|_{\Gamma(p)}).$$

Nonetheless, in our particular case we can obtain a refinement. Indeed, considering the decomposition (3.1), and by an analogous argument as the one written above we get that

$$s(n,\epsilon) \sim s_{B_i}(n,\epsilon) \sim s^u(n,\epsilon,p_i).$$

Corollary 3.1.2. *There exists constants $C_2, C_3 > 0$ so that for any $p \in M$, $n \in \mathbb{N}$ and $\epsilon > 0$,*

$$s(n,\epsilon) \leqslant C_2 s^u(n,\epsilon,p) \leqslant C_3 s(n,\epsilon)$$

To study the behavior of $\{s(n,\epsilon)\}$ we can instead look at $\{s^u(n,\epsilon,p)\}$ (for some fixed p). Note in particular that, since $f|_{\mathcal{W}^u}$ is uniformly expanding, the analysis is much simpler.

Lemma 3.1.4. *Given $0 < \epsilon < \epsilon^* < \frac{c_{\text{lps}}}{4}$ there exists $C(\epsilon,\epsilon^*) > 0$ so that for every $n \geqslant 0$*

$$s(n,\epsilon) \leqslant C(\epsilon,\epsilon^*) s(n,\epsilon^*)$$

Proof. The existence of $C'(\epsilon,\epsilon^*)$ satisfying

$$s^u(n,\epsilon,p) \leqslant C(\epsilon,\epsilon^*) s^u(n,\epsilon^*,p)$$

is simple to check. From here and the previous corollary we get the result. □

Exactly the same argument (working with $\Gamma(p)$ instead of the whole manifold) allow us to deduce the following two lemmas.

Lemma 3.1.5. *For any $\epsilon > 0$ like in Lemma 3.1.4, there is a constant D_ϵ such that*

$$s(n_1 + \cdots + n_k, \epsilon) \leqslant \prod_{i=1}^{k} D_\epsilon s(n_i, \epsilon),$$

whenever $n_1, \ldots, n_k \geqslant 1$.

Lemma 3.1.6. *For any $\epsilon > 0$ like in Lemma 3.1.4, there is a constant E_ϵ such that*

$$s(n_1 + \cdots + n_k, \epsilon) \geqslant \prod_{i=1}^{k} E_\epsilon s(n_i, \epsilon)$$

whenever $n_1, \ldots, n_k \geqslant 1$.

With the above we can prove the following precise estimate for the numbers $s(n, \epsilon)$.

Proposition 3.1.3. *Consider $h = h_{\text{top}}(f)$. For a sufficiently small $\epsilon > 0$, the constants given by Lemmas 3.1.5 and 3.1.6 satisfy:*

$$D_\epsilon^{-1} e^{nh} \leqslant s(n, \epsilon, f) \leqslant E_\epsilon^{-1} e^{nh}$$

for every $n \geqslant 0$.

Proof. Suppose, by contradiction, that $s(n, \epsilon, f) < D_\epsilon^{-1} e^{nh}$ for some $n \in \mathbb{N}$. By Lemma 3.1.5, we have that $s(kn, \epsilon, f) \leqslant (D_\epsilon s(n, \epsilon, f))^k$ and hence

$$\begin{aligned} \frac{1}{kn} \log s(kn, \epsilon, f) &\leqslant \frac{1}{kn} \log(D_\epsilon s(n, \epsilon, f))^k \\ &\leqslant \frac{1}{n} (\log D_\epsilon + \log s(n, \epsilon, f)). \end{aligned}$$

Using h-expansivity, we know that $h = \lim_{k \to \infty} \frac{1}{kn} \log s(kn, \epsilon, f)$, if ϵ is sufficiently small. Thus,

$$h \leqslant \frac{1}{n} (\log D_\epsilon + \log s(n, \epsilon, f)) < \frac{1}{n} (\log D_\epsilon + \log(D_\epsilon^{-1} e^{nh})) = h$$

that is a contradiction. The other inequality is obtained in a similar way. □

3.2 Topological entropy

Our goal in this section is to compute the entropy of a partially hyperbolic diffeomorphism in terms of the growth of the periodic pseudo-orbits.

Theorem A. *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism satisfying the following hypotheses*

- *foliations $\mathcal{W}^c, \mathcal{W}^{cs}$ are of class C^1;*
- *the strong unstable foliation $\mathcal{W}^u$ is minimal;*
- $\lambda^c = \max ||df|_{E^c}|| \leqslant 1$.

Suppose that either

- *the central foliation $\mathcal{W}^c$ satisfies* ***Hypothesis 1***

  or

- ***Hypothesis 2****: f is a regular element of an C^1 hyperbolic action.*

Then,

$$h_{top}(f) = \lim_{n\to\infty} \frac{1}{n} \log \#Per_n.$$

The quantity $\#\mathrm{Per}_n$ is, roughly speaking, the number of different orbits of n-periodic plaques. Since formalizing this concept precisely seems difficult, we will opt to work with n-periodic δ-pseudo-orbits. We explain this later.

Now let us discuss the meaning of **Hypothesis 1** and **Hypothesis 2**, and their role in the proof of Theorem A. Recall that **Hypothesis 1** mentioned in above theorem is given by

Hypothesis 1. For any $\alpha > 0$ there exists $\rho = \rho(\alpha) > 0$ so that: for any $x \in M, z, w \in D^{su}(x), y \in W^c(x)$, it holds

$$d(z, w) \geqslant \alpha \Rightarrow d(h^c_{y,x}(z), h^c_{y,x}(w)) \geqslant \rho$$

(and in particular, $h^c(z), h^c(w)$ is well defined.)

This is equivalent to what we wrote in the introduction. It essentially means that the center foliation is "almost parallel", in the sense that its leaves do not stray away too much (when lifted to its unwrapping bundle as in the introduction). We will use this to estimate, given periodic pseudo-orbits $\underline{x}, \underline{y} \in \mathcal{P}^c_{\text{per}}(\delta)$, the distance between $f^n(x_0), f^n(y_0)$ in terms of the distance between x_n and y_n. Observe that in principle the center (intrinsic) distance $d^c(x_n, f^n(x_0))$ could be large. **Hypothesis 1** gives us control in the transverse direction, even for far away points.

To elaborate on this observe that simply by continuity of the center foliation one gets the following.

Lemma 3.2.1. *Given $\beta > 0$ consider $\gamma > 0$ so that for every $x, y \in M$ with $y \in W^c(x, \beta)$ the holonomy map $h^c_{y,x} : D^{su}(x, \gamma) \to D^{su}(y)$ is well defined. Then for every $\alpha > 0$ there exists $\rho(\alpha, \beta)$ so that $z, w \in D^{su}(x, \gamma)$,*

$$d(z, w) \geqslant \alpha \Rightarrow d(h^c_{y,x}(z), h^c_{y,x}(w)) \geqslant \rho(\alpha, \beta).$$

What we are requiring with **Hypothesis 1** is independence of ρ with β.

Note the following simple lemma.

Lemma 3.2.2. *Assume that $\lambda^c < 1$. Then there exists $\gamma > 0$ so that: for every $0 < \alpha < \gamma$ there exists $\rho = \rho(\alpha)$ so that if $\underline{x}, \underline{y} \in \mathcal{P}^c_{\text{per}}(\delta)$ are of the same period with $d(x_k, y_k) \leqslant \gamma$ then*

$$d(x_k, y_k) \geqslant \alpha \Rightarrow d(f^k(x_0), f^k(y_0)) \geqslant \rho.$$

Proof. Indeed, note that for every $\underline{x} \in \mathcal{P}^c_{\text{per}}(\delta)$ we have

$$d^c(f^k(x_0), x_k) \leqslant \delta \sum_{l=1}^{k} (\lambda^c)^l \leqslant \frac{\delta}{1 - \lambda^c}$$

where d^c is the intrinsic distance inside $W^c(x_k)$. Now we just apply Lemma 3.2.1. □

Remark 7. The case when $\lambda^c < 1$ corresponds to the situation when f is Anosov. Under these hypotheses it is known that

$$h_{\text{top}}(f) = \lim_{n \to \infty} \frac{1}{n} \log \#\{x : f^n(x) = x\}.$$

When applied to this case, our methods permit to obtain the entropy of (transitive) Anosov maps in terms of the number pseudo-orbits that preserve the center foliation. This means that in some sense we are allowed to make small "mistakes" when looking at periodic orbits, and count pseudo-periodic orbits instead. This fact may be suitable to be implemented in software in order to obtain estimates for the entropy of an Anosov map.

Example 2 (Rigid skew-products)**.** We say that two dynamically coherent partially hyperbolic systems $f\colon M \to M$ and $g\colon M \to M$ are centrally conjugated if there exists an homeomorphism $h\colon M \to M$ such that $h(W^c(x; f)) = W^c(h(x); g)$, for every $x \in M$.

Note that if f is centrally conjugated to a system g with linear center foliation, then it satisfies **Hypothesis 1**. Assuming further differentiability of $\mathcal{W}^c, \mathcal{W}^{cs}$, minimality of $\mathcal{W}^u$ and $\lambda^c \leqslant 1$ then we are in the hypothesis of Theorem A. In spite of appearing two restricted (which it is), this case contains interesting examples.

As a concrete one, we can consider an Anosov extension $f_0 : \mathbb{T}^3 \to \mathbb{T}^3$ as described in the first example of the Preliminaries (Chapter 1). Clearly f_0 satisfies the hypotheses. On the other hand, it is a classical result of Hirsch, Pugh and Shub (Corollary 8.3 in [20]) that small $\mathcal{C}^1$ perturbations f of f_0 are centrally conjugated to f_0; considering the subset of these that have

- differentiable bundles,
- $\lambda^c \leqslant 1$,
- minimal $\mathcal{W}^u$,

we get examples where Theorem A applies. The first two conditions are not difficult to get (albeit being serious restrictions), while the latter can be controlled with a result of Katok (Theorem 1 in [22]).

Example 3 (Rigid absolute partially hyperbolic diffeomorphisms)**.** If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is absolutely partially hyperbolic diffeomorphism, meaning

$$\sup\{\|D_x f|_{E^s} : x \in \mathbb{T}^3\|\} < \inf\{\|D_x f|_{E^c} : x \in \mathbb{T}^3\|\} \leqslant \lambda^c \leqslant 1 < \inf\{\|D_x f|_{E^u} : x \in \mathbb{T}^3\|\}$$

then f is centrally conjugated to a linear matrix (cf. [16]), and a similar argument as the previous example can be carried.

The **Hypothesis 2**, that is, when f is a regular element of an Anosov action, requires introducing some well known notions.

Definition 3.2.1. Let G be a Lie group and $\alpha : G \times M \to M$ a $\mathcal{C}^1$ action. We say that α is hyperbolic if the following holds.

1. The action is foliated, meaning that its orbits form a foliation whose leaves have the same dimension as G.
2. There exists $g_0 \in G$ such that $f = \alpha(g_0, \cdot) \in \mathrm{Diff}^1(M)$ is partially hyperbolic, with center foliation given by the orbit foliation of G. In this case f is called a regular element of the action.

If f is a regular element of an Anosov action, then its center leaves are homogeneous spaces, and (modulo changing the metric to an equivalent one) f acts isometrically on each one of them (in particular $\lambda^c = 1$). See for example the introduction of [12] for a quick review on this type of maps, and references.

Bookeeping of constants. Recall that $c_{\exp}$ denotes the size of a plaque expansivity constant, and we assume (with no loss of generality) that $c_{\exp} \leqslant \delta$. Note that any $\epsilon \leqslant c_{\exp}$ is also a plaque expansivity constant for δ-pseudo orbits.

Definition 3.2.2. A set $\underline{E} \subset \mathcal{P}^c(\delta)$ is called

1. n-plaque periodic if every $\underline{x} \in \underline{E}$ is n-periodic.
2. Separated if it is n-plaque periodic for some n and satisfies $\underline{x} \neq \underline{y} \in \underline{E}$ implies that for some i, $x_i \notin P_\delta(y_i)$ or $y_i \notin P_\delta(x_i)$ with $i \in \{0, \cdots, n-1\}$.

The following proposition guarantees that every separated set in $\mathcal{P}^c(\delta)$ is finite.

Proposition 3.2.1. *If $\underline{E} \subset \mathcal{P}^c(\delta)$ is a separated set, then $\#\underline{E} < \infty$. Moreover, for a given period this cardinality is uniformly bounded from above.*

Proof. Cover $\prod_{n=1}^{\infty} M$ with l sets such that their diameter in the maximum distance is less than $\frac{c_{\exp}}{2}$. If $\underline{E} \subset \mathcal{P}^c(\delta)$ is separated then necessarily $\#\underline{E} \leqslant l + 1$.

□

Definition 3.2.3. We denote by $\#\mathrm{Per}_n$ the cardinal of any n-plaque periodic separated set $\underline{E_n}$ of maximal size. Such a set $\underline{E_n}$ is said to be *maximally n-plaque periodic.*

Remark 8. Let $\mathcal{P}^c_{\text{per},n} = \{\underline{x} \in \mathcal{P}^c_{\text{per}} : \underline{x} \text{ is } n-\text{periodic}\}$. Note that the cyclic group $\mathbb{Z}_n$ acts naturally on the set $\mathcal{P}^c_{\text{per},n}$,

$$[m] \in \mathbb{Z}_n \Rightarrow [m] \cdot \underline{x} = \sigma^m(\underline{x}).$$

It follows that if $\underline{E}$ is n-plaque periodic, separated and it does not contain fixed pseudo-orbits (meaning pseudo-orbits with $x_n = x_0 \forall n$) then $\mathbb{Z}_n \cdot \underline{E}$ is also n-plaque periodic and separated.

Fixed pseudo-orbits do not contribute to entropy. It is thus safe to assume (and we will do so from now on) that any maximally n-plaque periodic set does not contain fixed pseudo-orbits, and in particular it is saturated by cyclic permutations ($\underline{x} \in \underline{E} \Rightarrow \sigma(\underline{x}) \in \underline{E}$).

Lemma 3.2.3. *Let $\underline{E_n}$ be a maximally n-plaque periodic set. Then for every $\underline{x} \neq \underline{y} \in \underline{E_n}$, there is a $k = k(\underline{x}, \underline{y})$ with $0 \leqslant k \leqslant n-1$ and such that $d(x_k, y_k) > c_{\exp}$.*

Proof. Otherwise the bi-infinite pseudo-orbits $\underline{x}^i, \underline{x}^j$ stay closer than the plaque expansitivity constant at all times, which implies that x^i_0, y^j_0 are in the same plaque, contradicting the fact that $\underline{E_n}$ is separated. □

The next lemma will give us a relationship between the topological entropy of f and $\#\text{Per}_n$.

Proposition 3.2.2. *There are constants $0 < D < E$ such that*

$$D \cdot e^{hn} \leqslant \#Per_n \leqslant E \cdot e^{hn}$$

for any sufficiently large n.

Proof. Fix $\underline{E_n}$ a maximally n-plaque periodic set. First we will show the following inequality:

$$\#\underline{E_n} \leqslant s(n, \epsilon).$$

Given two distinct pseudo-orbits $\underline{x}, \underline{y} \in \underline{E_n}$, Lemma 3.2.3 gives us a constant k with $0 \leqslant k \leqslant n-1$ and such that

$$d(x_k, y_k) \geqslant \epsilon.$$

Thus, denoting $x = x_0$ and $y = x_0$, we obtain:

$$\begin{cases} f^k(x) \in W^c(x_k); \text{and} \\ f^k(y) \in W^c(y_k). \end{cases}$$

We split the proof into two cases, depending on whether **Hypothesis 1** or **Hypothesis 2** hold.

Case. **Hypothesis 1**. Since all considered quantities are small, and due to our assumption, we deduce

$$d(f^k(x), f^k(y)) \geqslant \rho$$

which in turn implies that $\underline{E}$ is $\rho(\epsilon)$ separated, therefore

$$\#\underline{E} \leqslant s(n, \rho(\epsilon)) \leqslant E_{\rho(\epsilon)} e^{nh}$$

due to Proposition 3.1.3. It suffices thus to take any (fixed) ϵ sufficiently small and consider E as the corresponding constant for such ϵ.

Case. **Hypothesis 2**. Now $f = \alpha(g_0, \cdot)$ is a regular element of a hyperbolic action $\alpha : G \times M \to M$. We recall that f acts as an isometry on each center leaf (which it fixes). The typical way to obtain this is to consider a left invariant metric on G, induce the metric on E^c_f, and complete to an adapted metric on the remaining bundles. With this we get that for every x, $\alpha \colon G \times \{x\} \to W^c(x)$ is a Riemannian isometry (and a covering map). Going back to our case, observe that we can write

$$x_k = \alpha(g_{k,x}, x)$$

for some $g_{k,x} \in G$. Using the referred metrics, it follows that

$$k|g_0| - k\delta \leqslant |g_{k,x}| \leqslant k|g_0| + k\delta \tag{3.2}$$

where $|g|$ denotes the distance in G from g to the identity. Needless to say, we are assuming that $|g_0| >> \delta$.

Fix ϵ small (we'll be explicit shortly), and for $\underline{x} \neq \underline{y} \in \underline{E}$ consider k to be the first index such that $d(x_k, y_k) \geqslant \epsilon$. We can assume that ϵ is sufficiently close so that for $z \in W^c(w, |g_0| + \delta)$ the holonomy $h^c_{z,w}$ is well defined in a transverse disc of size ϵ. It follows that, maybe sliding along $\mathcal{W}^c$ the points x, y so they lie in the same transverse disc (which is no loss of generality), the segments $W^c(x, k(|g_0| + \delta))$ and $W^c(y, k(|g_0| + \delta))$ do no stray away, when looked in the unwrapped bundle $N_{W^c(x)}$

(cf. introduction), and in particular the holonomy transport $h^c_{x_k,x}$ is defined on y. From this we deduce the existence of k_0 (uniform) such that if $y \in W^{cs}(x, \frac{c_{\text{lps}}}{2})$ then necessarily $k > k_0$: indeed, if $y \in W^{cs}(x)$ we have that f contracts exponentially the distance between nearby plaques, and obtain that for sufficiently "long" holonomy transports the image of x, y will be closer than ϵ, provided that their orbits did the did not separate along the center direction. In this case,

$$d(x,y) \geqslant \rho(\epsilon, k_0).$$

Cover the manifold with sets similarly as in (3.1)

$$M = \bigcup_{i=1}^{l} \overline{W^{cs}(\Gamma(p_i), \frac{C_{\text{lps}}}{2})} = \bigcup_{i=1}^{l} B_i.$$

and consider $n >> k_0$. Fix B_i and look at $\underline{F^i_n}$, the set of pseudo-orbits of $\underline{E_n}$ whose initial points lie in B_i, and such that the first time that they separate more than ϵ occurs at time $\geqslant k_0$. Then

$$\#\{\underline{x} \in \underline{E_n} : x_0 \in B_i\} \leqslant \#\underline{F^i_n} \cdot s(k_0, \rho), \tag{3.3}$$

and by our previous discussion, for each $\underline{x} \in \underline{F^i_n}$ we can project x_0 onto $\Gamma(p_i)$ and obtain x'_0, in such a way that the assignment $x_0 \to x'_0$ is injective. Now the rest is simple: the set $\{f^{k_0}(x'_0) : \underline{x} \in \underline{F^i_n}\}$ is $(n - k_0, \rho(\epsilon))$-separated (same argument as before), hence

$$\#\underline{E_n} \leqslant l \cdot \max\{\#\underline{F^i_n} : 1 \leqslant i \leqslant l\} \cdot s(k_0, \rho) \leqslant l \cdot s^u(n - k_0, \rho, f^{k_0}(p_i))s(k_0, \rho) \leqslant e^{nh}E(\epsilon)$$

by Corollary 3.1.2 and Proposition 3.1.3.

Next we prove the other inequality:

$$De^{hn} \leqslant \#\underline{E_n}.$$

For it, we will use strongly the center specification property of f proved in Lemma 3.1.1.

Take ϵ small so that f has the specification property at size δ and let $N = N(\epsilon)$ be as in the definition of that property. We fix $B_p = \overline{W^{cs}(\Gamma(p), \frac{c_{\text{lps}}}{2})}$. Take any set $E \subset \Gamma(p)$ which is $(n - N, 3\epsilon)$-separated for f..

By center specification, for each $z \in E$, there is an n-periodic δ-pseudo-orbit $\underline{x}(z)$ respecting the central foliation such that

$$d(f^i(z), x_i(z)) \leqslant \epsilon \quad \text{for all } 0 \leqslant i \leqslant n - N - 1.$$

We claim that the set $\#\text{Per}_n \geqslant \#E$. Note first that the map $E \ni z \mapsto \underline{x}(z)$ is injective. Indeed, if we assume that this does not happen, then we could find points $z \neq z'$ satisfying $\underline{x}(z) = \underline{x}(z')$. Thus,

$$\begin{aligned} d(f^j(z), f^j(z')) &\leqslant d(f^j(z), x_j(z)) + d(x_j(z'), f^j(z)) \\ &< 2\epsilon \\ &< c_{\text{exp}} \end{aligned}$$

for every $0 \leqslant j \leqslant n - N - 1$, which contradicts our initial hypothesis. In principle the set $\{\underline{x}(z) : z \in E\}$ is not separated; on the other hand, arguing as in the previous part we could guarantee that for $z \neq z'$, either $\underline{x}(z), \underline{x}(z')$ start in the same center stable plaque, or they have to separate for $n \geqslant n_0$, for some uniform n_0.

Therefore, for some uniform C,

$$\#\underline{E_n} \geqslant C \cdot s(n - N, 3\epsilon)$$

and, by Lemma 3.1.3,

$$\#\text{Per}_n \geqslant De^{hn}$$

where $D = D_{3\epsilon}^{-1} e^{-Nh} C$. □

Remark 9. Observe that in the previous proof we are able to change the size of adjust the determine the size of ϵ first, and then determine the size δ of the pseudo-orbits needed to shadow/specify at this scale. The non-trivial argument showing independence of these choices is given by Lemma 3.1.3.

Theorem B is direct consequence from the Proposition above.

3.3 Entropy maximizing measure

In this section we present the second most important theorem of this chapter. It gives a concrete construction of the unique measure of maximal entropy. The proof of this result is based in Bowen's work for the hyperbolic case [5], [7].

In this part we will assume **Hypothesis 2**, namely that $f = \alpha(g_0, \cdot)$ is a regular element of an hyperbolic action $\alpha : G \times M \to M$. The methods used seem to be applicable also in the other case, but the bookkeeping of constants becomes cumbersome and hinders the clarity. In any case, the case considered is of more interest, and we will thus focus on it.

Recall that for any x the center plaque $P_{2\delta}(x)$ is obtained as $\alpha(B^G(2\delta), x)$, where $B^G(2\delta)$ is the ball in G of radius 2δ centered at the identity. Now we cheat: by re-scaling the metric we assume that the measure of $B^G(2\delta)$ is equal to 1, and in particular (since the metric on center leaves is induced from the one on G) we have that

$$\mu^c(P_{2\delta}(x)) = 1, \forall x \in M.$$

Above μ^c denotes the corresponding (Riemannian) measure on $W^c(x)$. These type of simplifications are lacking in the non-homogeneous case, and the reason why we are focusing in the setting of group actions.

Given $\underline{x} \in \mathcal{P}^c_{\mathrm{per},n}(\delta)$ we consider the measure $\omega_{\underline{x}}$ on M given by

$$\omega_{\underline{x}}(A) = \frac{1}{n}\sum_{i=0}^{n-1} \mu^c(P_{2\delta}(x_i) \cap A).$$

Consider a family $\mathscr{E} = \{\underline{E_k} : k \geqslant 1\}$ where each $\underline{E_k}$ is maximally k-plaque periodic,and define the measure

$$\mu_k^{\mathscr{E}} = \frac{1}{\#\mathrm{Per}_k} \sum_{\underline{x} \in \underline{E_k}} \omega_{\underline{x}}. \tag{3.4}$$

Clearly $\mu_k^{\mathscr{E}}$ is a probability measure on M.

Definition 3.3.1. We say that $\{\mu_k^{\mathscr{E}}\}$ is a family of empirical plaque-periodic measures.

Remark 10. If $\underline{x} \in \underline{E_n}$ then $\sigma(\underline{x}), \cdots, \sigma^{n-1}(\underline{x})$ are in $\underline{E_n}$ as well, and

$$\omega_{\underline{x}} = \omega_{\sigma(\underline{x})} = \cdots = \omega_{\sigma^{n-1}(\underline{x})}.$$

In the sum (3.4) we are counting each one of these measures separately.

Given such a family we define

$$\nu_n^{\mathscr{E}} = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \mu_k^{\mathscr{E}}$$

Note that by the classical Krillov-Bogolyubov argument, any weak accumulation point of $\{\nu_n^{\mathscr{E}}\}$ is f-invariant.

Theorem B. *Let $f : M \to M$ be a regular element of a hyperbolic action, and $\mathscr{E}$ be a family of maximally n-plaque periodic sets. Then*

$$\nu_n^{\mathscr{E}} \xrightarrow[n\to\infty]{weakly} \mu_{MME}$$

where μ_{MME} is the unique entropy maximizing measure of f.

This tells us that we can detect the entropy maximizing measure by information on periodic pseudo-orbits.

Fix $\epsilon > 0$ small (whose size will be elucidated as we go on): the size δ of the pseudo-orbits is chosen so f has the central specification property at scale δ.

We will be interested in computing the measure of (n, ϵ)-Bowen balls. Since f is isometric on its center, we see that for every $n \geqslant n_0$ and every $x \in M$

$$W^s(f^{-n}W^{cu}(f^n(x), \epsilon), \epsilon) \subset B^n(x, 2\epsilon) \subset W^s(f^{-n}W^{cu}(f^n(x), 3\epsilon), 3\epsilon).$$

In view of this, it's typical to work with the sets $W^s(f^{-n}W^{cu}(f^n(x), \epsilon), \epsilon)$ instead of Bowen balls, and it is useful then to think

$$B^n(x, \epsilon) \text{``} = \text{''} W^s(f^{-n}W^{cu}(f^n(x), \epsilon), \epsilon).$$

Next observe the following.

Lemma 3.3.1. *There exists a constant $B_\epsilon > 0$ so that for every family $\mathscr{E} = \{\underline{E_k} : k \geqslant 1\}$ of maximal k-plaque periodic sets the induced family of measures $\{\mu_k^{\mathscr{E}} : k \geqslant 1\}$ satisfies:*

$$\mu_k^{\mathscr{E}}(B^n(x, \epsilon)) \leqslant B_\epsilon \cdot e^{-nh},$$

for every $x \in M$ and $n \geqslant 1$, provided that $k > n$.

Proof. The proof is essentially contained in Proposition 3.2.2. Indeed, by a minimal adaptation of the argument of the last part of Case 2, we deduce that for any given family $\mathscr{E} = \{\underline{E_k} : k \geqslant 1\}$ of maximally k-plaque periodic sets,

$$\#\{\underline{x} \in \underline{E_k} : x_i \in B^n(x,\delta)\} \leqslant B_0 \cdot s^u(k-n, \rho(\delta))$$

for some uniform B_0, provided that n is sufficiently large (say, larger than n_0); here we emphasize that we are assuming δ small. Hence, since $\epsilon < \delta$,

$$\mu_k^{\mathscr{E}}(B^n(x,\epsilon)) = \frac{\#\{\underline{x} \in \underline{E_k} : x_i \in B^n(x,\delta)\}}{\#\mathrm{Per}_k} \mu^c(W^c(x_i,\epsilon)) \leqslant \frac{B_0 B_1 e^{(k-n)h}}{B_2 e^{kh}} = B_\epsilon e^{-nh}$$

due to Lemma 3.1.4 and Proposition 3.1.3. □

We are interested in obtaining the opposite inequality.

Lemma 3.3.2. *There is an $A_\epsilon > 0$ such that for every family $\mathscr{E} = \{\underline{E_k} : k \geqslant 1\}$ of maximal k-plaque periodic sets the induced family of measures $\{\mu_k^{\mathscr{E}} : k \geqslant 1\}$ satisfies:*

$$\mu_k^{\mathscr{E}}(B^n(x,\epsilon)) \geqslant A_\epsilon e^{-nh}$$

for all $x \in M$ and $n \geqslant 1$.

Proof. The idea is again given Proposition 3.2.2, and consists of using the center specification property. We fix $p \in M$ and denote $B_p = W^{cs}(\Gamma(p, \frac{c_{\mathrm{lps}}}{2}), c_{\mathrm{lps}})$.

It is no loss of generality to assume that $0 < \epsilon < \frac{\epsilon^*}{3} < \delta$, and f has the center specification property at scale ϵ^*. Let $N = N(\epsilon)$ as in the definition of that property

Given $k = n + 2N + m$ we choose E an $(m, 3\epsilon)$ separated set inside B_p with the property that no two points are in the same center stable of the other, as we did in Proposition 3.2.2. Take $x \in M$.

Note that for each $y \in E$ there is a k-periodic ϵ^*-pseudo-orbit $\underline{z}(y)$ respecting the central foliation such that

$$d(z_j(y)), f^j(x)) \leqslant \epsilon \text{ for all } 0 \leqslant j < n; \text{ and}$$
$$d(z_{n+N+j}(y), f^j(y)) \leqslant \epsilon \text{ for all } 0 \leqslant j < m,$$

Observe that if $\underline{E_k}$ is a maximally k-plaque periodic then for any $\underline{z}(y)$ we can find one (and only one) element in $\underline{E_k}$ so that they share the same δ-plaque. Since $3\epsilon < \delta$, we get

$$\mu_k^{\mathscr{E}}(B^n(x,\epsilon)) \geqslant \frac{s(m,3\epsilon)}{\#\mathrm{Per}_k}\mu^c(W^c(x,3\epsilon)) \geqslant A_0 \cdot \frac{e^{mh}}{e^{(n+2N+m)h}} = Ae^{-nh}$$

□

Remark 11. In the lemmas above, the constants A, B do not depend on the family $\mathscr{E}$, and depend only on ϵ (and δ), of course.

For a $\underline{x} \in \mathcal{P}^c$ denote $f\underline{x} := \{f(x_n) : n \in \mathbb{Z}\}$; since f preserves the center foliation we get that $\underline{x} \in \mathcal{P}^c_{\mathrm{per},n} \Rightarrow f\underline{x} \in \mathcal{P}^c_{\mathrm{per},n}$. If $\underline{E}$ is maximally n-plaque periodic, then $f(\underline{E})$ is maximally n-plaque periodic as well. With this we see that given a family $\mathscr{E} = \{\underline{E_n} : n \geqslant 1\}$ of maximally n-plaque periodic sets, then $f\mathscr{E} := \{\underline{E_n} : n \geqslant 1\}$ is of the same type.

On the other hand, f acts as an isometry on its center and therefore of any $\underline{x} \in \mathcal{P}^c_{\mathrm{per}}$

$$f_*\omega_{\underline{x}} = \omega_{f\underline{x}},$$

which in turn implies that

$$f_*\mu_k^{\mathscr{E}} = \frac{1}{\#\mathrm{Per}_k}\sum_{\underline{x}\in\underline{E_n}_k} f_*\omega_{\underline{x}} = \frac{1}{\#\mathrm{Per}_k}\sum_{\underline{y}\in f\underline{E_n}_k} \omega_{f\underline{x}} = \mu_k^{f\mathscr{E}}.$$

Lemma 3.3.3. *Let A, B be as given in Lemmas 3.3.1,3.3.2, and consider a family of plaque-empirical measures $\{\mu_k^{\mathscr{E}}\}$. Then any accumulation point ν of the corresponding measures $\{\nu_k^{\mathscr{E}}\}$ satisfy, for every $x \in M$ and $n \geqslant 1$*

$$Ae^{-nh} \leqslant \nu(B^n(x,\epsilon)) \leqslant Be^{-nh}.$$

Proof. Fix n, $k > n$ and $x \in M$. Then

$$\nu_k^{\mathscr{E}}(B^n(x,\epsilon)) = \frac{1}{k}\sum_{i=0}^{n-1}\mu_i^{f^i\mathscr{E}}(B^n(x,\epsilon)) + \frac{1}{k}\sum_{i=n}^{k-1}\mu_i^{f^i\mathscr{E}}(B^n(x,\epsilon)) \leqslant \frac{n}{k} + Be^{-nh}$$

where we have used Remark 11. Now comes a small subtle point: given ν an accumulation point of $\{\nu_k^{\mathscr{E}}\}$, we want to use Alexandrov's theorem to pass to the limit in the previous inequality, but since $B^n(x,\epsilon)$ is closed we cannot argue directly. This is by-passed by slightly increasing ϵ so that $B^n(x,\epsilon) \subset \text{int}(B^n(x,\epsilon^*))$, and noticing that in the proof of Lemma 3.3.1, the value of B is unaffected by such small change. Then

$$\nu_k^{\mathscr{E}}(B^n(x,\epsilon)) \leqslant \nu(\text{int}(B^n(x,\epsilon^*))) \leqslant \liminf_{k\to\infty} \nu_k^{\mathscr{E}}(\text{int}(B^n(x,\epsilon^*))) \leqslant Be^{-nh}$$

The other inequality is analogous, and direct from Alexandrov's theorem. □

Given any f-invariant measure μ, the function

$$h_{\mu,loc}(f,\cdot) = \lim_{\epsilon\to 0} \limsup_n -\frac{1}{n} \log \mu(B^n(\cdot,\epsilon))$$

is the local entropy of f with respect to μ at the point x. Due to the Brin-Katok formula this function is in $L^1(\mu)$ and

$$h_\mu(f) = \int h_\mu(f,x) d\mu(x).$$

In our case it follows, by the previous Lemma, that if ν is any accumulation point $\{\nu_k^{\mathscr{E}}\}$ then

$$h_\nu(f) = h = h_{top}(f) :$$

observe that even though A, B depend on ϵ we are taking lim in n first, and already $\limsup_n -\frac{1}{n} \log \mu(B^n(\cdot,\epsilon)) = h$.

This shows that ν is an entropy maximizing measure: its metric entropy coincides with the topological entropy of the system. Due to work of [14] we know that in our setting the map f has a unique entropy maximizing measure μ_{MME}, hence:

Corollary 3.3.1 (Theorem B)**.** *Let $\mathscr{E}$ be a family of maximally n-plaque periodic sets, and let $\{\mu_k^{\mathscr{E}} : k \geqslant 1\}$ be the corresponding plaque empirical measures. Then $\{\frac{1}{n}\sum_{k=0}^{n-1} f_*^k \mu_k^{\mathscr{E}}\}_n$ converges weakly to μ_{MME}.*

Remark 12. For a probability measure μ its support supp(μ) is defined as the set of all points $x \in M$ such that $\mu(U_x) > 0$, for all open neighborhood U_x of x. From the corollary above it follows directly that supp(μ) is δ-dense. Observe however that the same arguments can be carried by reducing δ, hence supp(μ) is δ-dense for every $\delta > 0$ small, and therefore dense.

We finish with the proof of Theorem C.

Theorem C. *If $\alpha : G \times M \to M$ is a strongly Axiom A action then there exists μ probability measure on M that is*

1. *G-invariant: for every measurable $A \subset M$ and $g \in G$, $\mu(\alpha(g, A)) = \mu(A)$.*
2. *μ is ergodic,*
3. $\operatorname{supp}(\mu) = M$.

Proof. Let f be a regular element of the action. Due to [26] we know that the unstable foliation of f is minimal; let $\mu = \mu_{MME}$ its entropy maximizing measure. We already know that μ has full support.

Take any $\mathscr{E} = \{\underline{E_n} : n \geqslant 1\}$ of maximally n-centrally periodic sets. Now, since f commutes with every other $\tilde{f} = \alpha(g, \cdot)$, we get if $\underline{x} \in \mathcal{P}^c_{\text{per},n}$, then $\tilde{f}\underline{x} \in \mathcal{P}^c_{\text{per},n}$: indeed

$$d(f(\tilde{f}x_n), \tilde{f}x_{n+1}) = d(\tilde{f}(fx_n), \tilde{f}x_{n+1}) = d(fx_n, x_{n+1}) < \delta.$$

In the last part we've used that $\tilde{f}$ acts isometrically on center leaves of f. It follows also that $\tilde{f}\mathscr{E}$ is a family of maximally n-plaque periodic sets (for the dynamics of f).

Again using that $\tilde{f}|_{\mathcal{W}^c}$ acts isometrically, by arguing as in the case of f we get that

$$\tilde{f}\mu_k^{\mathscr{E}} = \mu_k^{\tilde{f}\mathscr{E}}.$$

From this and Theorem B (in particular, the uniqueness part) we obtain that

$$\tilde{f}\mu = \lim_{n\to\infty} \tilde{f}\frac{1}{n}\sum_{i=0}^{n-1} f^i_*\mu_k^{\mathscr{E}} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f^i_*\mu_k^{\tilde{f}\mathscr{E}} = \mu.$$

This concludes the result.

□

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