UNIVERSIDADE FEDERAL DE MINAS GERAIS
INSTITUTO DE CIÊNCIAS EXATAS
DEPARTAMENTO DE MATEMÁTICA

COMBINATORIAL RECONSTRUCTION PROBLEMS, HOPF ALGEBRAS AND GRAPH POSETS

Deisiane Lopes Gonçalves

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Orientador: Bhalchandra Digambar Thatte

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Combinatorial Reconstruction Problems, Hopf Algebras and Graph Posets

DEISIANE LOPES GONÇALVES

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Resumo

A conjectura de reconstrução de vértice afirma que todo grafo simples, finito e não direcionado com três ou mais vértices é determinado, via isomorfismo, pela coleção de subgrafos vértice deletados não rotulados. A conjectura de reconstrução de aresta afirma que todo grafo simples, finito e não direcionado com quatro ou mais arestas é determinado, via isomorfismo, por sua coleção de subgrafos aresta deletados não rotulados. Nós consideramos problemas análogos de reconstruir um grafo arbitrário G, via isomorfismo, de seu poset abstrato de subgrafo aresta, o poset abstrato de subgrafo induzido e o reticulado abstrato de ligação. Mostramos que, se um grafo não tem vértices isolados, então o reticulado abstrato de ligação e o poset abstrato de subgrafo induzido podem ser construídos a partir do poset abstrato de subgrafo aresta, exceto para as famílias de grafos que caracterizamos. Nós também estudamos outras estruturas relacionadas obtidas por considerar diferentes tipos de homomorfismos (ou seja, homomorfismos em geral, monomorfismos, epimorfismos, etc.) e questões sobre construções relacionando estas estruturas (por exemplo, quais estruturas podem ser construídas de quais outras estruturas). Estas questões são motivadas pela seguinte conjectura de Thatte. Seja $\mathscr{G}$ o conjunto de todos os grafos não rotulados. Seja $f : \mathscr{G} \to \mathscr{G}$ uma bijeção tal que para todo $G, H \in \mathscr{G}$, o número de homomorfismos de G para H é igual ao número de homomorfismos de $f(G)$ para $f(H)$. Então, f é o mapa de identidade. Esta conjectura é mais fraca do que a conjectura de reconstrução de aresta.

Em seguida, construímos uma subálgebra da álgebra UGQSym estudada por Borie. Os elementos desta álgebra são séries de potência formal que podem ser avaliadas em grafos, e conta o número de ocorrências de blocos. Nesta formulação, nós obtemos uma prova algébrica de um resultado de Whitney.

Dado o baralho de um grafo G, todos subgrafos vértice próprios de G podem ser contados usando um resultado básico em reconstrução de grafos, conhecido como lemma de Kelly. Consideramos o problema de refinar o lema para contar subgrafos enraizados de modo que o vértice raiz coincida com o vértice deletado. Mostramos que tal contagem não é possível em geral, a menos que a conjectura de reconstrução de vértice é verdadeira, mas um multiconjunto de subgrafos enraizados de altura fixa k pode ser construído de um baralho de G desde que G tem raio maior que k. Nós provamos um resultado análogo para o problema de reconstrução de aresta.

Palavras-chave: reconstrução de grafo, posets de grafo, homomorfismos de grafo, álgebras de Hopf, lema de Kelly.

Abstract

The vertex reconstruction conjecture asserts that every finite simple undirected graph on three or more vertices is determined, up to isomorphism, by its collection of unlabelled vertex-deleted subgraphs. The edge reconstruction conjecture asserts that every finite simple undirected graph with four or more edges is determined, up to isomorphism, by its collection of unlabelled edge-deleted subgraphs. We consider analogous problems of reconstructing an arbitrary graph G up to isomorphism from its abstract edge-subgraph poset, its abstract induced subgraph poset and its abstract bond lattice. We show that if a graph has no isolated vertices, then its abstract bond lattice and the abstract induced subgraph poset can be constructed from the abstract edge-subgraph poset except for the families of graphs that we characterise. We also study other relational structures obtained by considering different types of homomorphisms (e.g., general homomorphisms, monomorphisms, epimorphisms, etc.) and questions about constructions relating these structures, (for example, which structures can be constructed from which other structures). These questions are motivated by the following conjecture of Thatte. Let $\mathscr{G}$ be the set of all unlabelled graphs. Let $f : \mathscr{G} \to \mathscr{G}$ be a bijection such that for all $G, H \in \mathscr{G}$, the number of homomorphisms from G to H is equal to the number of homomorphisms from $f(G)$ to $f(H)$. Then, f is the identity map. The conjecture is weaker than the edge reconstruction conjecture.

Next we construct a subalgebra of the algebra UGQSym studied by Borie. The elements of this algebra are formal power series which can be evaluated on graphs, and count occurrences of blocks. In this formulation, we obtain an algebraic proof of a result of Whitney.

Given the vertex-deck of a graph G, all vertex-proper subgraphs of G can be counted using a basic result on graph reconstruction, known as Kelly's lemma. We consider the problem of refining the lemma to count rooted subgraphs such that the root vertex coincides the deleted vertex. We show that such counting is not possible in general unless the vertex reconstruction conjecture is true, but a multiset of rooted subgraphs of a fixed height k can be constructed from the vertex-deck of G provided G has radius more than k. We prove analogous result for the edge reconstruction problem.

Keywords: graph reconstruction, graph posets, graph homomorphisms, Hopf algebras, Kelly's lemma.

Nomenclature

Δ	maximum degree in G
$\delta(G)$	minimum degree in G
$\mathrm{Orbit}_G(x)$	set of vertices of G that are similar to x
Φ	null graph
$e(G)$	number of edges of G
$k(G)$	number of components of G
$u \approx v$	similar vertices
$v(G)$	number of vertices of G
$\mathrm{Aut}(G)$	set of all automorphisms of G
$\mathrm{c_v}(\mathscr{F}, G)$	number of ways to cover vertices of G by $\mathscr{F}$
$\mathrm{c}(\mathscr{F}, G)$	number of ways to cover G by $\mathscr{F}$
$\overline{\Omega}$	abstract connected partition lattice
$\mathrm{epi}(G, H)$	number of epimorphims from G to H
$\overline{Q}$	abstract edge subgraph poset
$\mathrm{hom}(G, H)$	number of homomorphisms from G to H
I	identity map
$\mathrm{ind}(G, H)$	number of monomorphisms f from G to H such that the image of f is an induced subgraph of H
$\overline{P}$	abstract induced subgraph poset
$\mathbb{K}$	field

$\mathscr{G}$	set of unlabelled simple graphs
$\mathscr{G}_B$	set of isomorphism classes of blocks with two or more vertices
$\mathrm{mon}(G, H)$	number of monomorphisms from G to H
$\overline{\mathscr{G}}$	set of unlabelled graphs without isolated vertices
$\Sigma(G, H)$	number of monomorphims f from G to H such that for each connected component B of G we have $H[f(V(B))]$ is isomorphic to B.
$\mathrm{surhom}(G, H)$	number of surjective homomorphisms from G to H
$\{H^{+e}\}$	set of graphs that can be obtained by adding a new edge to a copy of H
C_n	cycle on n vertices
$D_e(G)$	edge-deck of G
$D_v(G)$	deck of G
$d_v(G)$	degree of v in G.
G	graph
$G - X$	obtained by deleting vertices in X
$G[E]$	edge-subgraph
$G[X]$	subgraph induced by vertex set X
$G \uplus H$	disjoint union of G and H
G^c	complement of a graph G
G^x	graph with a root vertex x
G_k^v	subgraph of G rooted at v, induced by vertices at distance at most k from v
G_k^e	subgraph of G rooted at e induced by edges at distance at most k from e.
$K_2 + K_{1,3}$	disjoint union of K_2 and $K_{1,3}$
$K_4 \setminus e$	graph K_4 minus an edge

K_n	complete graph on n vertices
$K_{n,m}$	complete bipartite graph with n and m vertices in two partitions
$p(F, G^v)$	number of induced subgraphs of G that contain vertex v and are isomorphic to F
$p(F^x, G^v)$	number of induced rooted subgraphs of G^v that are isomorphic to F^x such that the root of the subgraph coincides with v
$p(H, G)$	number of induced subgraphs of G isomorphic to H
P_n	path on n vertices
$q(H, G)$	number of edge-subgraphs of G isomorphic to H
$r(G)$	radius of G
$s(F, G^v)$	number of subgraphs of G that contain vertex v and are isomorphic to F
$s(F^x, G)$	number of rooted subgraphs of G that are isomorphic to F^x
$s(F^x, G^v)$	number of rooted subgraphs of G^v that are isomorphic to F^x such that the root of the subgraph coincides with v
$s(H, G)$	number of subgraphs of G isomorphic to H

Contents

Chapter 1

Introduction

One of the beautiful conjectures in graph theory, which has been open for more than 70 years, is about vertex reconstruction of graphs [1]. It was proposed by Ulam and Kelly [15], and reformulated by Harary [10] in the more intuitive language of reconstruction. This conjecture asserts that every finite simple undirected graph on three or more vertices is determined, up to isomorphism, by its collection of vertex-deleted subgraphs (called the *deck*).

Harary [10] proposed an analogous conjecture, known as the edge reconstruction conjecture. It asserts that every finite simple undirected graph with four or more edges is determined, up to isomorphism, by its collection of edge-deleted subgraphs (called the *edge-deck*).

A graph G is reconstructible, if it is determined, up to isomorphism, by its deck. There are three principal types of results on the reconstruction conjecture: results that show that all graphs in a certain class are reconstructible; results that show that certain invariants of a graph can be calculated from the deck; and reductions of one conjecture to another. We survey some of these results below.

A class X of graphs is reconstructible if every member of X is reconstructible. The reconstruction conjecture has been proven true for several classes of graphs, such as trees, disconnected graphs and regular graphs [16], but the general case remains unsolved.

A graph invariant is reconstructible if it can be calculated from the deck (i.e. if it takes the same value on all reconstructions of the graph). Some reconstructible graph invariants are: number of subgraphs of a graph G isomorphic to a graph F when F has fewer vertices than G (Kelly's lemma, see [16]), number of coverings of a graph G by a graph family $\mathscr{F}$ (Kocay's lemma, see [17]), the number of disconnected spanning subgraphs of G having a specified number of components in each isomorphism class [34], the number of connected separable spanning subgraphs of

G having a specified number of blocks in each isomorphism class [34], the number of nonseparable spanning subgraphs with a given number of edges [34], characteristic polynomial and Tutte polynomial [34].

Whitney [38] showed that any invariant of graphs that counts subgraphs formed by a given collection of blocks can be expressed as a unique polynomial, with rational coefficients, in terms of the indeterminates which count blocks in a graph, and furthermore, these latter invariants are algebraically independent over the rationals. This result is known as Whitney's subgraph expansion theorem, which we refer to as Whitney's theorem.

One approach to the reconstruction conjecture is to reduce the conjecture to another. The edge reconstruction conjecture is weaker than the vertex reconstruction conjecture. This may be proved in two ways. We can construct the vertex deck of a graph from its edge deck [12]. Or we may prove the result via line graphs: a graph with at least 4 edges is edge reconstructible if its line graph is vertex reconstructible [11]. Another reduction was proved by Yang Yongzhi [39]. He proved that if all 2-connected graphs are vertex reconstructible then the vertex reconstruction conjecture is true.

An effective approach to the edge reconstruction conjecture was given by Lovász [19] who proved that a graph on n vertices with more that $\binom{n}{2}/2$ edges is edge reconstructible. Lovász's bound was improved to $n \log n$ by Müller [24]. The methods of Lovász and Müller were further refined by Nash-Williams [25] who characterised the hypothetical counter examples to the edge reconstruction conjecture.

Many variations of the original conjecture of Ulam and Kelly have been studied. For example, vertex reconstruction problems have been studied for directed graphs, hypergraphs, infinite graphs, necklaces, and so on, see [2]. There are non-reconstructible tournament families [29] and hypergraphs [18]. We consider problems of reconstructing an arbitrary graph G up to isomorphism from its *abstract edge-subgraph poset*, its *abstract induced subgraph poset* and its *abstract bond lattice*. These problems were proposed and studied by Thatte in [31–33].

Let G be a graph. The abstract edge-subgraph poset $\overline{Q}(G)$ of G is the isomorphism class of its partially ordered set of distinct unlabelled edge-subgraphs, that is, the subgraphs themselves are not required. The abstract induced subgraph poset $\overline{P}(G)$ of G is the isomorphism class of its partially ordered set of distinct unlabelled non-empty induced subgraphs. The abstract bond lattice $\overline{\Omega}(G)$ of G is the isomorphism class of the lattice of distinct unlabelled connected partitions of G. These posets are suitably weighted by subgraph counting numbers.

We say that a graph G is Q-reconstructible (P-reconstructible, or Ω- reconstructible, respectively) if it is determined, up to isomorphism, by its abstract edge-subgraph poset $\overline{Q}(G)$ (by its abstract induced subgraph poset $\overline{P}(G)$, or by its abstract bond lattice $\overline{\Omega}(G)$, respectively).

Thatte [31–33] proved that the edge reconstruction conjecture is equivalent to the Q-reconstruction problem, except for a family of graphs (which was fully characterised in [32]); the P-reconstruction problem is equivalent to the Ω- reconstruction problem, with a few exceptions, and the vertex reconstruction conjecture is equivalent to the P-reconstruction problem, except for empty graphs. Moreover, it was proved in [33] that many invariants of graphs (e.g., the chromatic polynomial, the symmetric Tutte polynomial, the number of spanning trees, the number of Hamiltonian cycles, and so on) are P-reconstructible.

Hopf algebras were introduced by Heinz Hopf in 1941. They have now been used in diverse fields, including physics, probability theory, combinatorics, etc. In the 1970s, Rota [14] found many combinatorial examples of Hopf algebras. He observed that the multiplication of combinatorial objects arises naturally as a disjoint union of two objects, and the comultiplication arises as linear combinations of decompositions of an object into pairs of objects.

The Milnor-Moore Theorem [6] says that if B is a linear basis in the space of primitives over a field of characteristic zero of a connected, commutative, cocommutative Hopf algebra, then such a Hopf algebra is the polynomial algebra in the elements of B (that is, isomorphic to the polynomial algebra in $|B|$ variables). Thus, we can translate some combinatorial questions to questions concerning polynomial algebras.

Schmitt [27] studied invariants of combinatorial objects by considering certain associated Hopf algebras. He proved that any invariant which counts subobjects of a particular type is given by a unique polynomial in invariants which count connected subobjects. Besides, he applied the above result for graphs, obtaining Whitney's theorem. For Hopf algebras of graphs, Iovanov and Jun [13] found a basis for the space of primitives, and proved that it satisfies a certain minimality property and a universal property, and applied it to known results on reconstruction of graphs. Borie [5] defined a Hopf algebra whose elements are formal power series which, when evaluated on graphs, count occurrences of subgraphs. He proved how this algebra is connected to invariants of graphs, and gave a sufficient criterion for two graphs to be isomorphic, and applied his result to the reconstruction problem of graphs.

The text of this thesis is divided into five chapters.

In Chapter 2, we develop the necessary background for the subsequent chapters. We define some concepts from graph theory, partially ordered sets, Hopf algebras, and the graph reconstruction theory. We state Ulam and Kelly's conjecture and some important lemmas on graph reconstruction. We also state some relevant results on Hopf algebras.

In Chapter 3, we define three subgraph posets: the abstract edge-subgraph poset, the abstract induced subgraph poset and the abstract bond lattice. We state some results about these posets given by Thatte [31–33]. We then prove the following new result [8].

Theorem. *Let G be a graph with no isolated vertices. Then*

1. *$\overline{\Omega}(G)$ can be constructed from $\overline{Q}(G)$ if and only if G does not belong to $\mathscr{M}$;*
2. *$\overline{P}(G)$ can be constructed from $\overline{Q}(G)$ if and only if G does not belong to $\mathscr{N}$;*

where $\mathscr{M}$ and $\mathscr{N}$ are certain families of graphs that we characterise completely.

The construction of $\overline{P}(G)$ from $\overline{Q}(G)$ generalises a well known result in reconstruction theory that states that the vertex deck of a graph with at least 4 edges and without isolated vertices can be constructed from its edge deck [12].

Let $\mathscr{G}$ be the set consisting of one representative element from each isomorphism class of simple graphs. Thatte [32] proposed the following conjecture.

Conjecture (Thatte). *Let $f : \mathscr{G} \to \mathscr{G}$ be a bijection such that for all $G, H \in \mathscr{G}$, the number of homomorphisms from G to H is equal to the number of homomorphisms from $f(G)$ to $f(H)$, then f is the identity map.*

He proved that this conjecture is weaker than the edge reconstruction conjecture. Let G^c be the complement of graph G, that is, a graph with the same vertex set but whose edge set consists of the edges not present in G. In Chapter 4, we propose a number of analogous conjectures.

Conjecture.

1. *Let $\sigma : \mathscr{G} \to \mathscr{G}$ be a bijection such that for all $G, H \in \mathscr{G}$, the number of monomorphisms from G to H is equal to the number of monomorphisms from $\sigma(G)$ to $\sigma(H)$, then σ is the identity map.*
2. *Let $\sigma : \mathscr{G} \to \mathscr{G}$ be a bijection such that for all $G, H \in \mathscr{G}$, the number of monomorphisms from G to H such that the image is an induced subgraph is equal to the number of monomorphisms from $\sigma(G)$ to $\sigma(H)$ whose image is an induced subgraph, then σ is the identity map or $\sigma(G) = G^c$ for all $G \in \mathscr{G}$.*

3. *Let G and H be two graphs. We define $\Sigma(H, G)$ as the number of monomorphims f from H to G such that for each connected component B of H, the subgraph of G induced by $f(V(B))$ is isomorphic to B. Let $\sigma : \mathscr{G} \to \mathscr{G}$ be a bijection such that $\Sigma(H, G) = \Sigma(\sigma(H), \sigma(G))$ for all $G, H \in \mathscr{G}$, then σ is the identity map.*

We prove relations among the above conjectures. We show that Conjecture 1 is weaker than Conjecture 2, and Thatte's Conjecture is weaker than Conjecture 1. Besides, we show a relation between Conjecture 2 and Conjecture 3.

In Chapter 5, we state Whitney's Theorem and present some results from [5]. We define a subalgebra of UGQSym and obtain a new proof of Whitney's Theorem. This chapter was motivated by another proof of Whitney's Theorem by Schmitt [27] and the Hopf algebra defined by Borie [5].

In Chapter 6, we consider the problem of refining Kelly's vertex and edge reconstruction lemmas for counting rooted subgraphs such that the root vertex coincides the deleted vertex, and for counting edge-rooted subgraphs such that the root edge coincides the deleted edge. We show that such counting is not possible in general unless the reconstruction conjecture is true, but a multiset of rooted subgraphs of a fixed height k can be counted if G has radius more than k. Let G_k^v be the subgraph of G rooted at v induced by the vertices at distance at most k from v. Let G_k^e denote the subgraph of G rooted at edge e (i.e., with a distinguished edge e) induced by edges at distance at most k from e. More specifically, we prove the following propositions.

Proposition. *If G is a connected graph with radius more than k, then the multiset $S_k(G) := \{G_k^v, v \in V(G)\}$ is reconstructible.*

Proposition. *If G is a connected graph with radius more that $k \geq 1$, then the multiset $T_k(G) := \{G_k^e, e \in E(G)\}$ is edge-reconstructible.*

Chapter 2

Preliminary concepts

In Section 2.1 we introduce basic graph theoretic terminology and notation. In Section 2.2 we give a background of graph reconstruction. We introduce the well known conjectures in this area, in particular, the conjectures of Ulam-Kelly and Harary, and summarise important known results. In Section 2.3 we define basic terminology and notation for posets. In Section 2.4, we give the necessary background of Hopf algebras. The material in this chapter is based on the books by Bondy and Murty [4], Stanley [28] and Grinberg and Reiner [9] and a survey by Bondy and Hemminger [3].

2.1 Graphs

In this section, notations and definitions are taken from Thatte's article [32] and Bondy and Murty's book [4].

A *graph* $G = (V(G), E(G))$ is an ordered pair, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges, together with an incidence function that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. Here the term graph means a finite graph (i.e., both its set of vertices and set of edges are finite).

A *subgraph* H of a graph G is another graph formed from a subset of the vertices and edges of G such that the incidence function of H is the restriction of the incidence function of G to $E(H)$.

An *induced subgraph* of a graph G is another graph formed from a subset of the vertices of G and all of the edges of G connecting pairs of vertices in that subset. For $X \subseteq V(G)$, we denote the subgraph induced by X by $G[X]$, the subgraph induced by $V(G) \setminus X$ by $G - X$, or $G - u$ when $X = \{u\}$. An *edge-subgraph* of G is a graph formed from a subset of the edges of G together with any vertices that are their endpoints. For $E \subseteq E(G)$, we denote the subgraph induced by E by $G[E]$. A *spanning subgraph* of a graph G is a subgraph of G with the same vertex set as G. We denote the spanning subgraph of G with edge set $E(G) \setminus E$ by $G - E$, or $G - e$ if $E = \{e\}$.

Two graphs G and H are *isomorphic* if there are bijections $f\colon V(G) \to V(H)$ and $g\colon E(G) \to E(H)$ such that the incidence function of G associates $e \in E(G)$ with vertices u and v of G if and only if the incidence function of H associates $g(e) \in E(H)$ with vertices $f(u)$ and $f(v)$ of H. We identify each isomorphism class of graphs (also called an *unlabelled graph*) with a unique representative of the class. If G is a graph, then $\overline{G}$ is the unique representative of the isomorphism class of G. A graph is *simple* if it has no loops or parallel edges. Let $\mathscr{G}$ be the set consisting of one representative element from each isomorphism class of simple graphs. We assume that all newly declared graphs are from $\mathscr{G}$. On the other hand, if a graph H is derived from a graph G (e.g., $H = G - e$), then H may not be in $\mathscr{G}$. If $H \in \mathscr{G}$ is isomorphic to a subgraph of G, we say that G contains H as a subgraph or G contains H, or H is contained in G.

We denote the number of subgraphs (induced subgraphs, edge-subgraphs) of G that are isomorphic to H by $\mathrm{s}(H, G)$ $(\mathrm{p}(H, G), \mathrm{q}(H, G)$, respectively). Let v be a vertex of G. We denote by $\mathrm{s}(H, G^v)$ $(\mathrm{p}(H, G^v))$ the number of subgraphs (induced subgraphs) of G that are isomorphic to H and that contain the vertex v.

Example 2.1.1. Let G be the graph represented by Figure 2.2. We have $\mathrm{s}(\Gamma, G) = 9$, $\mathrm{p}(\Gamma, G) = 6$ and $\mathrm{q}(\Gamma, G) = 9$, and $\mathrm{s}(\Gamma, G^b) = 3$ and $\mathrm{p}(\Gamma, G^b) = 2$.

An *empty graph* is a graph with empty edge set. A *null graph* Φ is a graph with no vertices.

Let G be a graph. We denote the number of vertices of G by $v(G)$, the number of edges of G by $e(G)$, and the number of components of G by $k(G)$.

The *degree* of a vertex v in a graph G is the number of edges of G incident with v. We denote the maximum degree in G by $\Delta(G)$, and the minimum degree in G by $\delta(G)$.

The path graph on n vertices is denoted by P_n and the cycle graph on n vertices is denoted by C_n. We denote the complete graph on n vertices by K_n, a complete bipartite graph with n and m vertices in the two partitions by $K_{n,m}$, and the graph K_4 minus an edge by $K_4 \setminus e$.

We refer to the graphs in Figure 2.1 in some proofs.

For terminology and notation about graphs not defined here, we refer to

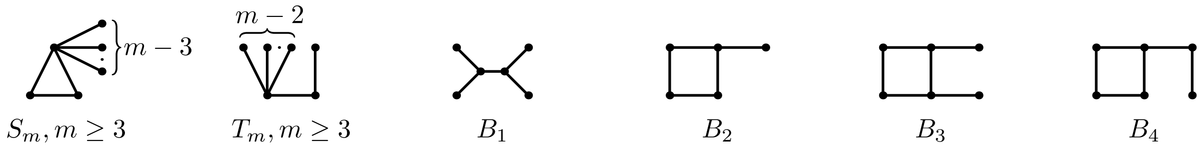


Figure 2.1: Some graphs referred to in proofs.

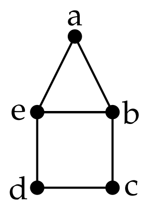


Figure 2.2: Graph G

Bondy and Murty [4].

2.2 Graph reconstruction problems

Let G and H be two graphs. We say that G and H are *vertex-hypomorphic* if there is a bijection $f\colon V(G) \to V(H)$ such that for all $v \in V(G)$, $G - v$ and $H - f(v)$ are isomorphic. The *vertex deck* or simply *deck* of G is the multiset $D_v(G) := \{G - u \mid u \in V(G)\}$ of unlabelled induced subgraphs of G. We say that G and H are *edge-hypomorphic* if there is a bijection $f\colon E(G) \to E(H)$ such that for all $e \in E(G)$, $G - e$ and $H - f(e)$ are isomorphic. The *edge-deck* of G is the multiset $D_e(G) := \{G - e \mid e \in E(G)\}$ of unlabelled spanning subgraphs of G.

Figure 2.3 shows the deck of G represented by Figure 2.2.

Two graphs G and H are vertex-hypomorphic if and only if they have the same vertex deck. A vertex reconstruction of a graph G is a graph H with the same deck as G. A graph G is *vertex reconstructible* if it is determined, up to isomorphism, by $D_v(G)$, equivalently, if every graph that is vertex-hypomorphic to G is also isomorphic to G.

Analogously, two graphs G and H are edge-hypomorphic if and only if they have the same edge-deck. A graph G is *edge reconstructible* if it is determined, up to isomorphism, by $D_e(G)$, equivalently, if every graph that is edge-hypomorphic to G is also isomorphic to G.

In the 1940s, Ulam [35] and Kelly [15] proposed the following conjecture, which is known as the vertex reconstruction conjecture.

Conjecture 2.2.1. *(Ulam and Kelly [15]) Every simple graph on at least three vertices is*

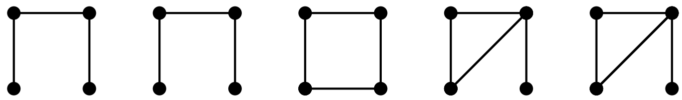


Figure 2.3: The deck of G represented by Figure 2.2.

vertex reconstructible.

The graphs $2K_1$ and K_2 have the same deck but they are not isomorphic. An analogous conjecture, known as the edge reconstruction conjecture, was proposed by Harary [10].

Conjecture 2.2.2. *(Harary [10]) Every simple graph on at least four edges is edge reconstructible.*

We denote by $K_3 + K_1$ the disjoint union of K_3 and K_1. Analogously, we denote by $K_{1,2} + K_1$ the disjoint union of $K_{1,2}$ and K_1. The pairs of graphs $(K_{1,3}, K_3 + K_1)$ and $(2K_2, K_{1,2} + K_1)$ have the same edge-deck but they are not isomorphic.

We can approach the reconstruction conjecture in two ways: reconstructing classes of graphs and reconstructing graph invariants. A class of graphs is reconstructible if every member of the class is vertex reconstructible. A graph invariant is reconstructible if the invariant takes the same value on all vertex reconstructions of a graph.

The classes of graphs such as regular graphs, disconnected graphs and tree are vertex reconstructible [2].

Next we state two fundamental lemmas, namely Kelly's Lemma and Kocay's Lemma, that are useful in many reconstruction proofs.

Lemma 2.2.3. *(Kelly's Lemma [16]) Let F and G be two graphs such that $v(F) < v(G)$. Then, $\mathrm{s}(F, G)$ is vertex reconstructible. Moreover, for each subgraph $G - v$ in the deck of G, the quantity $\mathrm{s}(F, G^v)$ is vertex reconstructible.*

For a tuple $\mathscr{F} := (F_1, \ldots, F_k)$ of graphs, define the number of ways to cover H by $\mathscr{F}$ as

$$\mathrm{c}(\mathscr{F}, H) := \left|\{(H_1, \ldots, H_k) \mid \forall i\ H_i \subseteq H,\ H_i \cong F_i, \text{ and } \cup_{i=1}^{k} H_i = H\}\right|.$$

Example 2.2.4. If $\mathscr{F} = (K_2, K_3)$, and $H = K_3$, then $\mathrm{c}(\mathscr{F}, H) = 3$. Here, we have one way to cover H by K_3 and three ways to put K_2 into K_3.

Lemma 2.2.5. *(Kocay's Lemma [17]) Let G be a graph. Let $\mathscr{F} := (F_1, \ldots, F_k) \in \mathscr{G}^k$. We have*

$$\prod_{i=1}^{k} \mathrm{s}(F_i, G) = \sum_{H} \mathrm{c}(\mathscr{F}, H)\, \mathrm{s}(H, G), \tag{2.1}$$

where the sum is over all distinct (mutually non-isomorphic) graphs H.

We also have an induced subgraph version of Kocay's lemma.

Definition 2.2.6. Let G be a graph and $\mathscr{F} := (F_1, \ldots, F_k) \in \mathscr{G}^k$. A vertex cover of G by $\mathscr{F}$ is a tuple $(H_1, \ldots, H_k)$ of induced subgraphs of G such that $H_i \cong F_i$ for all i, and $\cup_i V(H_i) = V(G)$. We denote by $\mathrm{c_v}(\mathscr{F}, G)$ the number of vertex covers of G by $\mathscr{F}$.

See Thatte [33] for a proof of the following lemma.

Lemma 2.2.7. *(Kocay's Lemma for induced subgraphs) Let $\mathscr{F} := (F_1, \ldots, F_k) \in \mathscr{G}^k$. We have*

$$\mathrm{c_v}(\mathscr{F}, G) = \prod_{i=1}^{k} \mathrm{p}(F_i, G) - \sum_{v(H)<v(G)} \mathrm{c_v}(\mathscr{F}, H)\,\mathrm{p}(H, G). \tag{2.2}$$

Furthermore, if $v(G_i) < v(G)$ for all i, then $\mathrm{c_v}(\mathscr{F}, G)$ is vertex reconstructible.

The following result, originally due to Hemminger [12], shows that the edge reconstruction conjecture is weaker than the vertex reconstruction conjecture.

Proposition 2.2.8. 1. *The number of isolated vertices is edge reconstructible.*

2. *If G is a graph without isolated vertices, then the deck of G can be constructed from its edge-deck.*

2.3 Partially ordered sets

For terminology and notation about partially ordered sets, we refer to Stanley [28]. Here we state the terms that we frequently use in this thesis.

A partially ordered set or *poset* $(S, \leq)$ is a set S with a partial order relation $\leq$, which is a reflexive, antisymmetric and transitive relation.

Let $(S, \leq)$ be a partially ordered set. If $x, y \in S$, $x \leq y$, $x \neq y$ and there is no $z \in S \setminus \{x, y\}$ such that $x \leq z \leq y$, then we say y covers x. We say that $\rho\colon S \to \mathbb{N}$ is a rank function if for all $x, y \in S$, y covers x implies $\rho(y) = \rho(x) + 1$.

A *weighted poset* is a poset $(S, \leq)$ with a compatible weight function $\omega\colon S \times S \to \mathbb{Z}$, where *compatible* means $\omega(x, y) = 0$ unless $x \leq y$. We say that weighted posets $(S, \leq, \omega)$ and $(S', \leq', \omega')$ are isomorphic if there is a bijection $f\colon S \to S'$ such that for all $x, y \in S$, we have $x \leq y$ if and only if $f(x) \leq' f(y)$ and $\omega(x, y) = \omega'(f(x), f(y))$.

Example 2.3.1. Let $n \in \mathbb{N}$. Let B_n be the set of all subsets of $\{1, \cdots, n\}$. We can consider B_n as a poset by defining $S \leq T$ in B_n if $S \subseteq T$ as sets. The *zeta function* $\zeta\colon B_n \times B_n \to \mathbb{Z}$ is defined by $\zeta(S, T) = 1$, if $S \leq T$ and $\zeta(S, T) = 0$, otherwise. Thus, $(B_n, \leq, \zeta)$ is a weighted poset.

2.4 Hopf algebras

In Chapter 5, we use the definitions and results presented in this section.

A Hopf algebra is a vector space with maps: product, coproduct, antipode, unit and counit. These maps have to satisfy certain compatibility conditions which can be defined in terms of commutative diagrams. The product and the unit constitute an *algebra*. The coproduct and the counit constitute a *coalgebra*. Next, we formally define these terms.

Let $\mathbb{K}$ be a field. Denote by $\otimes$ the tensor product over $\mathbb{K}$.

Definition 2.4.1 (Algebra)**.** An algebra is a triple (A, m, μ) consisting of a vector space A over $\mathbb{K}$ and $\mathbb{K}$-linear maps $m\colon A \otimes A \to A$ and $\mu\colon \mathbb{K} \to A$ that satisfy the following conditions.

1. The following diagram commutes:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{I} \otimes m"] \arrow[d, "m \otimes \mathrm{I}"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

   where the map I is the identity map.

2. The following diagram commutes:

$$\begin{tikzcd} A \otimes \mathbb{K} \arrow[r, "\mathrm{I} \otimes \mu"] \arrow[d, "s_2"'] & A \otimes A \arrow[dl, "m"] \\ A & \mathbb{K} \otimes A \arrow[l, "s_1"] \arrow[u, "\mu \otimes \mathrm{I}"'] \end{tikzcd}$$

   The map $s_1\colon \mathbb{K} \otimes A \to A$ is defined by $r \otimes a \to ra$ and the map $s_2\colon A \otimes \mathbb{K} \to A$ is defined by $a \otimes r \to ar$.

The map m is the *product* map and the map μ is the *unit* map.

Example 2.4.2. (Graphs) Let $\mathbb{K}$ be a field of characteristic zero. We define the following algebra on $\mathbb{K}[\mathscr{G}]$. The product map $m\colon \mathbb{K}[\mathscr{G}] \otimes \mathbb{K}[\mathscr{G}] \to \mathbb{K}[\mathscr{G}]$ is given by

$$m(G \otimes H) = G \uplus H$$

for all $G, H \in \mathscr{G}$, that is, the product is given by disjoint union of graphs, and the unit map $\mu\colon \mathbb{K} \to \mathbb{K}[\mathscr{G}]$ is given by

$$\mu(1) = \Phi.$$

The maps are extended linearly. This algebra is a *monoid algebra*.

Definition 2.4.3 (commutative algebra)**.** The algebra (A, m, μ) is commutative if $m\tau = m$, where τ denotes the twist map defined as $\tau(a \otimes b) = b \otimes a$ for $a, b \in A$.

Example 2.4.2 is a commutative algebra.

The objects called coalgebras that we are going to define now are (in some sense) the dual to algebras.

Definition 2.4.4 (coalgebra)**.** A coalgebra is a triple (C, Δ, ϵ) consisting of a vector space C over $\mathbb{K}$ and $\mathbb{K}$-linear maps $\Delta\colon C \to C \otimes C$ and $\epsilon\colon C \to \mathbb{K}$ that satisfy the following conditions.

1. The following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\mathrm{I} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{I}"] & C \otimes C \otimes C \end{tikzcd}$$

2. The following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "1 \otimes -"] \arrow[d, "- \otimes 1"] \arrow[dr, "\Delta"] & \mathbb{K} \otimes C \\ C \otimes \mathbb{K} & C \otimes C \arrow[l, "\mathrm{I} \otimes \epsilon"] \arrow[u, "\epsilon \otimes \mathrm{I}"] \end{tikzcd}$$

   Here the maps $- \otimes 1$ and $1 \otimes -$ are defined by $c \to c \otimes 1$ and $c \to 1 \otimes c$, respectively.

The maps Δ and ϵ are called *coproduct* and *counit* maps, respectively.

Example 2.4.5. (A coalgebra of graphs) The coproduct map $\Delta\colon \mathbb{K}[\mathscr{G}] \to \mathbb{K}[\mathscr{G}] \otimes \mathbb{K}[\mathscr{G}]$ is given by

$$\Delta(G) = \sum_{X \subseteq V(G)} G[X] \otimes G[\overline{X}]$$

for all $G \in \mathscr{G}$, where $\overline{X} = V(G) - X$, and the counit map $\mu\colon \mathbb{K}[\mathscr{G}] \to \mathbb{K}$ is given by

$$\epsilon(G) = \begin{cases} 1, & \text{if } G = \Phi; \\ 0, & \text{otherwise.} \end{cases} .$$

The maps are extended linearly.

Definition 2.4.6 (cocommutative coalgebra)**.** The coalgebra (C, Δ, ϵ) is cocommutative if

$$\tau(\Delta) = \Delta.$$

Example 2.4.5 is a cocommutative coalgebra.

Let C be a coalgebra and let C^* be its linear dual. If (C, Δ, ϵ) is a coalgebra, then (C^*, m, μ) is an algebra, where the maps m and μ are reduced from the transpose of Δ and ϵ, respectively. This result can be found in Underwood [36].

Definition 2.4.7 (bialgebra)**.** A bialgebra is a $\mathbb{K}$-vector space B together with maps m, μ, Δ, ϵ that satisfy the following conditions:

1. (B, m, μ) is an algebra and (B, Δ, ϵ) is a coalgebra,

2. Δ and ϵ are homomorphisms of algebras.

Example 2.4.8. (A bialgebra of graphs) The algebra and the coalgebra $\mathbb{K}[\mathscr{G}]$ given in Examples 2.4.2 and 2.4.5 form a bialgebra.

Definition 2.4.9 (primitive element)**.** Let B be a coalgebra. An element $b \in B$ is a primitive element of B if $\Delta(b) = 1 \otimes b + b \otimes 1$.

Definition 2.4.10 (Hopf algebra)**.** A Hopf algebra H is a bialgebra $(H, m, \mu, \Delta, \epsilon)$ over a field $\mathbb{K}$ together with a $\mathbb{K}$-linear map $S \colon H \to H$ that satisfies the coinverse property

$$m(\mathrm{I} \otimes S)\Delta(h) = \mu(\epsilon(h)) = m(S \otimes \mathrm{I})\Delta(h)$$

for all $h \in H$. The map S is called the coinverse (or antipode) map.

Next, we can see some properties of the coinverse map. See Grinberg [9] for a proof of the following proposition.

Proposition 2.4.11. *Let H be a Hopf algebra with coinverse S. Then the following properties hold.*

1. $S(ab) = S(b)S(a)$ *for all* $a, b \in H$,

2. $S(1) = 1$.

3. *If H is commutative, then $S^2 = \mathrm{I}$.*

4. *If x is a primitive element, then $S(x) = -x$.*

Definition 2.4.12 (graded module)**.** A graded $\mathbb{K}$-module V is one with a $\mathbb{K}$-module direct sum decomposition $V = \bigoplus_{n \geq 0} V_n$. Elements x in V_n are called homogeneous of degree n.

One endows tensor products $V \otimes W$ of graded $\mathbb{K}$-modules V, W with graded module structure in which $(V \otimes W)_n := \bigoplus_{i+j=n} V_i \otimes W_j$.

Definition 2.4.13 (graded map)**.** A $\mathbb{K}$-linear map $\psi\colon V \to W$ between two graded $\mathbb{K}$-modules is called graded if $\psi(V_n) \subset W_n$ for all n. We say that an algebra (coalgebra, bialgebra) is graded if it is a graded $\mathbb{K}$-module and all of the relevant structure maps $(\mu, \epsilon, m, \Delta)$ are graded. We say that a graded module V is connected if $V_0 \cong \mathbb{K}$.

We can see that in connected graded bialgebras, antipodes come for free.

Proposition 2.4.14. *A connected graded bialgebra H has a unique antipode S, which is a graded map $S\colon H \to H$, endowing it with a Hopf structure.*

See Grinberg [9] for a proof.

Chapter 3

Subgraph posets and the reconstruction conjectures

We define three subgraph posets of a graph G, namely, the abstract induced subgraph poset $\overline{P}(G)$, the abstract edge-subgraph poset $\overline{Q}(G)$, and the abstract bond lattice $\overline{\Omega}(G)$. We survey a number of results from [31–33].

Our main theorem states that $\overline{P}(G)$ can be constructed from $\overline{Q}(G)$, except when G belongs to a certain family $\mathscr{N}$ of graphs, and that $\overline{\Omega}(G)$ can be constructed from $\overline{Q}(G)$, except when G belongs to a certain family $\mathscr{M}$ of graphs [8]. We characterise the families $\mathscr{N}$ and $\mathscr{M}$.

The main difficulty in the construction of $\overline{P}(G)$ from $\overline{Q}(G)$ is the following. The graphs K_3, $K_{1,3}$ and $3K_2$ have the same abstract edge-subgraph poset (see Figure 3.2). In fact we have an infinite family of pairs of nonisomorphic graphs with the same edge-subgraph poset. For these graphs, we cannot determine their number of vertices from their abstract edge-subgraph poset. On the other hand, given $\overline{P}(G)$, we trivially know the number of vertices of G. Hence a straightforward induction on the number of edges does not help in the construction of $\overline{P}(G)$ from $\overline{Q}(G)$. Our proof proceeds by showing that if a graph G does not belong to the family $\mathscr{N}$, then either G itself is Q-reconstructible, or the number of vertices and the number of components of G can be determined from $\overline{Q}(G)$.

3.1 The abstract edge-subgraph poset and the edge reconstruction

Recall that $\mathscr{G}$ denotes the set of unlabelled simple graphs.

Definition 3.1.1. (Thatte [32]) Let G and H be two graphs. Let $Q := (\mathscr{G}, \leq_e, q)$ be a weighted poset, where $H \leq_e G$ if H is isomorphic to an edge-subgraph of G, and $q\colon \mathscr{G} \times \mathscr{G} \to \mathbb{Z}$ is defined by: $q(G, H)$ equals the number of edge-subgraphs of G isomorphic to H, for all $G, H \in \mathscr{G}$. Let G be a graph and let $Q(G) := \{G_1, \ldots, G_m\}$, where $G_1 = K_2, G_2, \ldots, G_m = G$ are the distinct non-empty edge-subgraphs of G. The *concrete edge-subgraph poset* of G is the restriction of Q to $Q(G)$; it is denoted simply by $Q(G)$. The *abstract edge-subgraph poset* of G is the isomorphism class of $Q(G)$, and is written as $\overline{Q}(G) := (\{g_1, \ldots, g_m\}, \leq_e, q)$. Thus g_1 and g_m are the minimal and the maximal elements in $\overline{Q}(G)$, respectively. We define a rank function ρ on $\overline{Q}(G)$ such that $\rho(g_1) = 1$ and $\rho(g_i) = e(G_i)$ for all i.

Let $(S, \leq)$ be a poset. The *Hasse diagram* of S is the graph whose vertices are the elements of S, whose edges are the cover relations, and such that if $x, y \in S$, $x < y$ then y drawn above x. We use the Hasse diagram to illustrate the edge-subgraph poset.

Example 3.1.2. The edge-subgraph poset of $K_4 \setminus e$ (concrete and abstract) is shown in Figure 3.1. On the left side (a) is the concrete edge-subgraph poset and on the right side (b) is the abstract edge-subgraph poset. The Hasse diagram is only for illustration; it does not display the weights on all related pairs of graphs in $Q(G)$. For example, $q(g_3, g_8) = 5$. The weights not shown in the Hasse diagram may be calculated using Kelly's lemma.

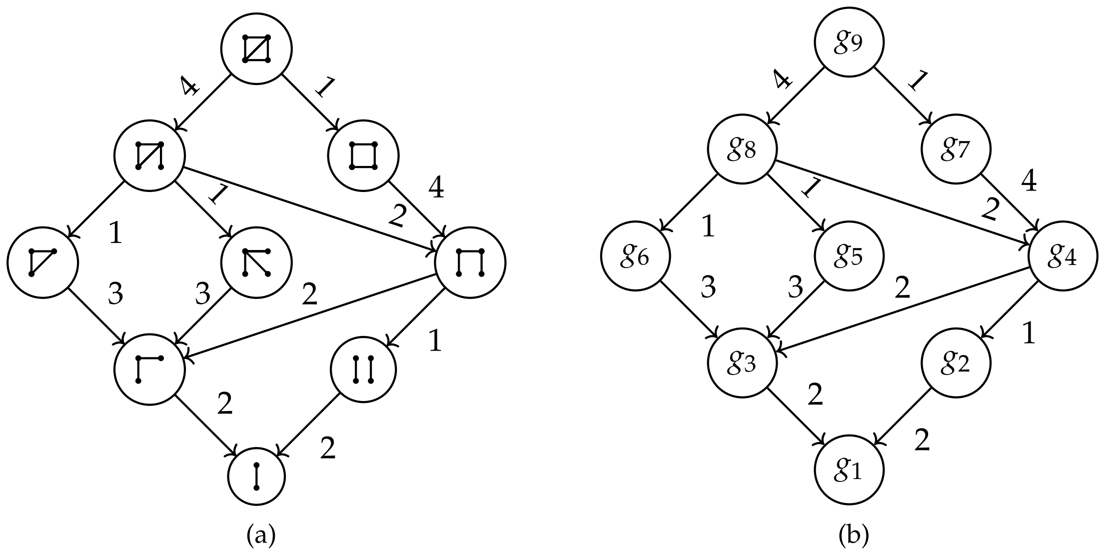


Figure 3.1: The concrete edge-subgraph poset of $K_4 \setminus e$ and the abstract edge-subgraph poset of $K_4 \setminus e$.

A graph G is *Q-reconstructible* if it is determined up to isomorphism by its abstract edge-subgraph poset. An invariant of G is said to be Q-reconstructible if it is determined by $\overline{Q}(G)$. A class of graphs is Q-reconstructible if each element of the class is Q-reconstructible. For all graph G, we have $\overline{Q}(G) = \overline{Q}(G + K_1)$. Therefore, we understand Q-reconstructibility to mean Q-reconstructibility modulo isolated vertices.

Example 3.1.3. Figure 3.2 shows the abstract edge-subgraph poset of K_3. Graphs K_3, $K_{1,3}$ and $3K_2$ are not Q-reconstructible, since they have the same abstract edge-subgraph poset. The graph $K_4 \setminus e$ is Q-reconstructible, i.e., if H is a graph with 5 edges, then we have $H \cong K_4 \setminus e$ if and only if $\overline{Q}(H) = \overline{Q}(K_4 \setminus e)$.

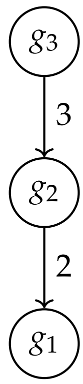


Figure 3.2: The abstract edge-subgraph poset of K_3.

We define the following families of graphs. Recall the definitions of graphs from Section 2.1.

Let

$$\begin{aligned}
\mathscr{F}_0 &:= \{3K_2, K_3, K_{1,3}\},\\
\mathscr{F}_1 &:= \{P_4, K_{1,2} + K_2, P_4 + K_2, T_4\},\\
\mathscr{F}_2 &:= \{C_4, 2K_{1,2}, C_4 + K_2, B_1, P_6, B_2, B_3, B_4\},\\
\mathscr{F}_3 &:= \{K_{1,m} \mid m > 1 \text{ and } m \neq 3\} \bigcup \{mK_2 \mid m > 1 \text{ and } m \neq 3\},\\
\mathscr{F}_4 &:= \{pK_3 + qK_{1,3} + F \mid p \neq q \text{ and } F \in \mathbb{N}^{\mathscr{X}}\} \setminus \{K_3, K_{1,3}\}, \text{ where}\\
\mathscr{X} &:= \{P_n \mid n \geq 2\} \bigcup \{C_n \mid n \geq 4\} \bigcup \{S_4, K_4 \setminus e, K_4\},
\end{aligned}$$

and $\mathbb{N}^{\mathscr{X}}$ is the set of all unlabelled finite graphs (including the null graph) with components from $\mathscr{X}$. We write $\mathscr{N} := \cup_{i=0}^{4} \mathscr{F}_i$.

The following result, showing a relationship between the edge reconstruction problem and the Q-reconstruction problem, was established in Thatte [32].

Theorem 3.1.4 (Theorem 2.1 [32])**.** *The graphs in $\mathscr{N}$ are not Q-reconstructible, and the edge reconstructible conjecture is true if and only if all graphs, except the graphs in $\mathscr{N}$, are Q-reconstructible.*

Graphs in each of the following sets have the same abstract edge-subgraph poset: $\{K_3, K_{1,3}, 3K_2\}$,$\{P_4, K_{1,2}+K_2\}$, $\{P_4+K_2, T_4\}$, $\{C_4, 2K_{1,2}\}$,$\{C_4+K_2, B_1\}$, $\{P_6, B_2\}$, $\{B_3, B_4\}$, $\{K_{1,m}, mK_2\}$, for all $m > 1$, and $\{pK_3+qK_{1,3}+F, qK_3+pK_{1,3}+F\}$, where $p \neq q$ and $F \in \mathbb{N}^{\mathscr{X}}$.

Although the Q-reconstruction problem and the edge reconstruction problem are equivalent (except for graphs in the family $\mathscr{N}$) when considered for all graphs, it is often difficult to show that a given graph G or a specific class of graphs is Q-reconstructible even when we know that it is edge reconstructible. On the other hand, if a class of edge reconstructible graphs is closed under edge deletion, it is Q-reconstructible. For example, consider the class of acyclic graphs. It is known that acyclic graphs with four or more edges are edge reconstructible and the class of acyclic graphs is closed under edge-deletion. Thus, we have the following corollary.

Corollary 3.1.5 (Corollary 2.11 [32]). *Acyclic graphs (trees and forests) that are not in $\mathscr{N}$ are Q-reconstructible.*

Let H be a graph. We define $\{H^{+e}\}$ as the set of graphs that can be obtained from H by adding a new edge having 0, 1, or 2 end-vertices in H. For example, $\{(3K_2)^{+e}\} = \{4K_2, P_4+K_2, 2K_2+K_{1,2}\}$.

Let G be a graph. A weight preserving isomorphism $\ell\colon \overline{Q}(G) \to Q(H)$ is called a *legitimate labelling* of $\overline{Q}(G)$, and H is called a Q-reconstruction of G. (Note that $Q(H)$ is a concrete poset.) Let $x \in \overline{Q}(G)$. We say that x is reconstructible or *uniquely labelled* if there is a graph F such that for all legitimate labelling maps ℓ from $\overline{Q}(G)$, we have $\ell(x) = F$. Similarly, a graph F is a distinguished subgraph of G (or simply F is distinguished) if there is an element x such that for all legitimate labelling maps ℓ from $\overline{Q}(G)$, we have $\ell(x) = F$.

We have already seen that for $m > 1$, the graphs $K_{1,m}$ and mK_2 have the same abstract edge-subgraph poset. We require the following lemmas from [32] in the proofs in this chapter.

Lemma 3.1.6 (Lemma 2.7 [32]). *For all $m \geq 4$, if G is a graph in the class $\{K_{1,m}^{+e} \setminus \{K_{1,m+1}\}\}$, then G is Q-reconstructible, and the label $K_{1,3}$ and the label K_3 (if K_3 is a subgraph of G) are uniquely assigned to elements of $\overline{Q}(G)$.*

Lemma 3.1.7 (Lemma 2.8 [32]). *For all $m \geq 4$, if G is a graph in the class $\{(mK_2)^{+e} \setminus \{(m+1)K_2\}\}$, then G is Q-reconstructible, and the label $3K_2$ is uniquely assigned to elements of $\overline{Q}(G)$.*

Lemma 3.1.8 (Proposition 2.9 [32]). *Graphs with at most 7 edges, except the ones in $\mathscr{N}$, are Q-reconstructible.*

3.2 The abstract induced subgraph poset and the abstract bond lattice

Definition 3.2.1. (Thatte [33]) Let $P := (\mathscr{G}, \leq_v, p)$ be a weighted poset, where $H \leq_v G$ if H is isomorphic to an induced subgraph of G, and $p \colon \mathscr{G} \times \mathscr{G} \to \mathbb{Z}$ is defined by: $p(H, G)$ equals the number of induced subgraphs of G isomorphic to H, for all $G, H \in \mathscr{G}$. Let G be a graph and let $P(G) := \{G_1, \ldots, G_m\}$, where $G_1 = K_1, G_2, \ldots, G_m = G$ are the distinct *non-empty* induced subgraphs of G. The *concrete induced subgraph poset* of G is the restrict of P to $P(G)$; it is denoted simply by $P(G)$. The *abstract induced subgraph poset* of G is the isomorphism class of $P(G)$, and is written as $\overline{P}(G) := (\{g_1, \ldots, g_m\}, \leq_v, p)$. Thus g_1 and g_m are the minimal and the maximal elements in $\overline{P}(G)$, respectively. We assume that g_2 covers g_1 (thus $G_2 \cong K_2$ and $e(G_i) = p(g_2, g_i)$). We define a rank function ν on $\overline{P}(G)$ such that $\nu(g_1) = 1$ and $\nu(g_i) = v(G_i)$ for all i.

Example 3.2.2. The induced subgraph poset of $K_4 \setminus e$ (concrete and abstract) is shown in Figure 3.3. On the left side (a) is the concrete induced subgraph poset and on the right side (b) is the abstract induced subgraph poset. The Hasse diagram is only for illustration; it does not display the weights on all related pairs of graphs in $P(G)$. As in the case of $Q(G)$, the weights not shown in the Hasse diagram can be calculated using Kelly's lemma.

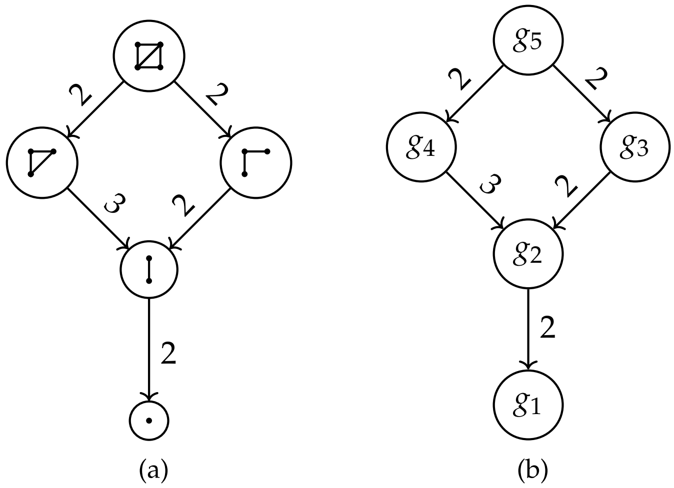


Figure 3.3: The concrete induced subgraph poset of $K_4 \setminus e$ and the abstract induced subgraph poset of $K_4 \setminus e$.

A graph G is *P-reconstructible* if it is determined up to isomorphism by its abstract induced subgraph poset $\overline{P}(G)$. An invariant of G is said to be P-

reconstructible if it is determined by $\overline{P}(G)$. A class of graphs is P-reconstructible if each element of class is P-reconstructible.

Thatte [31] proved the following theorem.

Theorem 3.2.3 (Proposition 2.7 [31])**.** *The vertex reconstruction conjecture is true if and only if every non-empty graph is P-reconstructible.*

Let G be a graph. We say that a partition $\pi := \{X_1, \ldots, X_n\}$ of $V(G)$ is a *connected partition* of $V(G)$, if the induced subgraphs $G[X_1], \ldots, G[X_n]$ are connected; we write $\pi \vdash_c V(G)$. We denote by $G[\pi]$ the subgraph $G[X_1] \uplus G[X_2] \uplus \ldots \uplus G[X_n]$ of G given by disjoint union.

Definition 3.2.4. (Thatte [33]) For $H_i, H_j \in \mathscr{G}$, if there exists $U \subseteq V(H_j)$ and $\pi \vdash_c U$ such that $H_j[\pi] \cong H_i$, we write $H_i \leq_\pi H_j$. Let $\Omega := (\mathscr{G}, \leq_\pi, \omega)$ be a weighted poset, where $\omega\colon \mathscr{G} \times \mathscr{G} \to \mathbb{Z}$ is defined as $\omega(H, G) := |\{(\pi, U) \mid U \subseteq V(G), \pi \vdash_c U \text{ and } G[\pi] \cong H\}|$. Let G be a graph and let $\Omega(G) := \{H_1, \ldots, H_m\}$, where $H_1 = v(G)K_1, H_2, \ldots, H_m = G$ are the distinct graphs induced by connected partitions of $V(G)$. The *folded bond lattice* of G is the restriction of Ω to $\Omega(G)$; it is denoted simply by $\Omega(G)$. The *abstract bond lattice* of G is the isomorphism class of $\Omega(G)$, written as $\overline{\Omega}(G) := (\{h_1, \ldots, h_m\}, \leq_\pi, \omega)$. Thus h_1 and h_m are the minimal and the maximal elements in $\overline{\Omega}(G)$, respectively. We define a rank function γ on $\overline{\Omega}(G)$ such that $\gamma(h_1) = 0$. The number of components in H_i is given by $k(H_i) = v(G) - \gamma(h_i)$.

Example 3.2.5. The bond lattice of $K_4 \setminus e$ (folded and abstract) is shown in Figure 3.4. On the left side (a) is the folded bond lattice and on the right side (b) is the abstract bond lattice. The Hasse diagram is only for illustration; it does not display the weights on all related pairs of graphs in $\Omega(G)$.

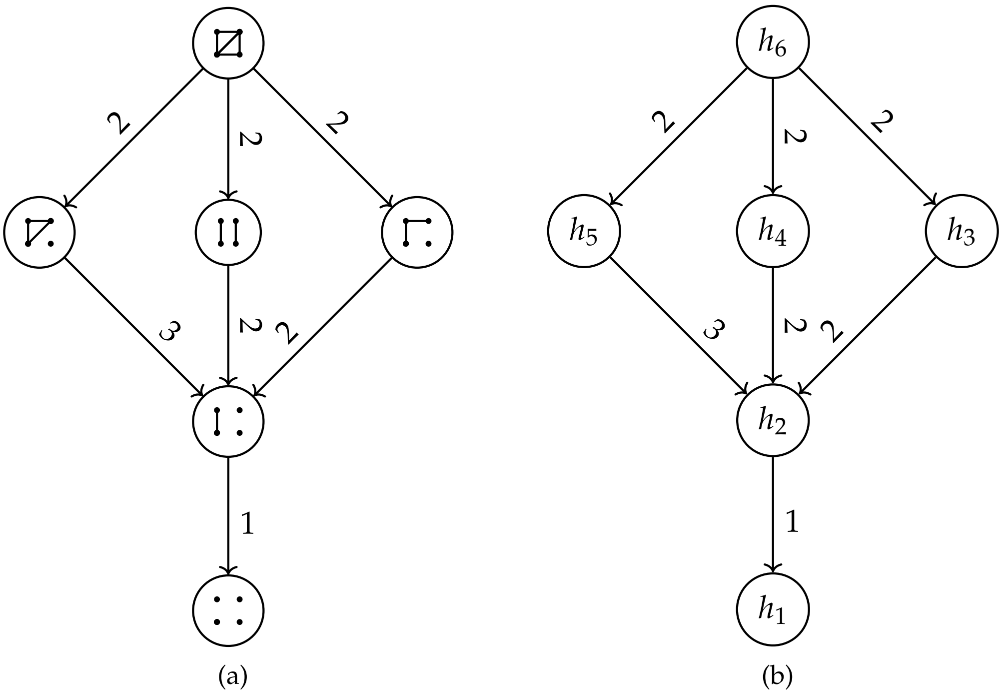


Figure 3.4: The folded bond lattice of $K_4 \setminus e$ and the abstract bond lattice of $K_4 \setminus e$.

Definition 3.2.6. (Thatte [33]) A graph G is Ω-*reconstructible* if it is determined up to isomorphism, modulo the number of isolated vertices, by its abstract bond lattice $\overline{\Omega}(G)$. An invariant of G is said to be Ω-reconstructible if it is determined by $\overline{\Omega}(G)$. A class of graphs is Ω-reconstructible if each element of class is Ω-reconstructible.

Theorem 3.2.7 shows that the P-reconstruction and Ω-reconstruction problems are equivalent, with a few simple exceptions.

Theorem 3.2.7 (Theorems 3.3 and 3.7 in [33])**.**

1. *The abstract bond lattice of G can be constructed from its abstract induced subgraph poset.*

2. *If a graph G has no isolated vertices and is not one of the graphs in* $\{K_{1,n}, nK_2 \mid n > 1\} \cup \{P_4, K_{1,2} + K_2, T_4, P_4 + K_2\}$*, then its abstract induced subgraphs poset can be constructed from its abstract bond lattice.*

For $n > 1$, the graphs $K_{1,n}$ and nK_2 have the same abstract bond lattice. Also, the pair of graphs P_4 and $K_{1,2} + K_2$, T_4 and $P_4 + K_2$ have the same abstract bond lattice.

3.3 The connected induced subgraphs of a graph and the vertex reconstruction conjecture

We prove in Proposition 3.3.3 that reconstruction from the deck is equivalent to reconstruction from the collection of all connected induced subgraphs. This result is implicit in Lemma 2.4 in [33]. Before proving it, we prove the following lemmas.

Lemma 3.3.1. *Let G be a graph. Then,*

$$p(H,G) = \omega(H,G) - \sum_{F} \omega(H,F)p(F,G), \tag{3.1}$$

where summation is over all graphs F not isomorphic to H such that H is a spanning subgraph of F.

Proof. For each $U \subseteq V(G)$ and a connected partition π of U such that $G[\pi] \cong H$, we have an induced subgraph F of G on vertex set U such that $F[\pi] \cong H$. Hence by summing over induced subgraphs F of G such that H is a spanning subgraph of F, we obtain

$$\omega(H,G) = \sum_{F} \omega(H,F)p(F,G), \tag{3.2}$$

Now we rearrange Equation 3.2 to obtain Equation 3.1. □

The following lemma is obtained by applying Kocay's Lemma for induced subgraphs (see Equation 2.2).

Lemma 3.3.2 (Lemma 2.7 [33])**.** *Let $H \cong \sum_{i=1}^{n} k_i H_i$. Then,*

$$\omega(H,G) = \frac{\prod_{i=1}^{n}(p(H_i,G))^{k_i} - \sum_{F<G} \mathrm{c_v}(H,F)p(F,G)}{\prod_{i=1}^{n} k_i!}, \tag{3.3}$$

where the summation is over all graph F with $v(F) < v(G)$.

Let $D_c(G)$ be the multiset of connected graphs in $D_v(G)$. It is known that in general $D_c(G)$ is not sufficient to reconstruct G.

Proposition 3.3.3. *Let G be a graph. Let $D_c^*(G)$ be the multiset of all connected proper induced subgraphs of G. The graph G is vertex reconstructible if and only if it is reconstructible from $D_c^*(G)$.*

Proof. By Kelly's lemma, we can construct $D_c^*(G)$ from $D_v(G)$. Hence if G is reconstructible from $D_c^*(G)$, then it is also reconstructible from $D_v(G)$.

Now, suppose that G is vertex reconstructible. Given $D_c^*(G)$, we will construct $D_v(G)$ as follows. First, note that the number of vertices is known by multiplicity of K_1 in $D_c^*(G)$. We have $|D_v(G)| = v(G)$ and all graphs in $D_v(G)$ have $v(G) - 1$ vertices. Thus, the graphs with number of vertices equal to $v(G) - 1$ in $D_c^*(G)$ belong to $D_v(G)$. Now, we need to construct disconnected graphs in $D_v(G)$ that have $v(G) - 1$ vertices. Let $\{G_1, \ldots, G_m\}$ be the set of unlabelled distinct disconnected graphs with components in $D_c^*(G)$, with $v(G) - 1$ vertices each. Assume that for all $1 \leq i < j \leq m$, we have $e(G_i) > e(G_j)$. For each $1 \leq i \leq m$, we can use inductively Equations (3.1) and (3.3) to calculate $p(G_i, G)$, since on the right-hand side in Equation 3.3, we have only connected induced subgraphs in the first coordinate in $p(.,.)$ and if the graph H is a component of G_i then $p(H, G)$ is known from $D_c^*(G)$. Thus, we can decide if $G_i \in D_v(G)$, and we can calculate its multiplicity in $D_v(G)$. Therefore, G is reconstructible from $D_c^*(G)$. $\square$

Lemma 3.3.4 expresses the number of induced subgraphs of a graph in terms of induced subgraphs of its components.

Lemma 3.3.4 (Lemma 2.3 [33]). *Let $\mathscr{S} := (S_1, \ldots, S_m)$ and $\mathscr{T} := (T_1, \ldots, T_n)$ be tuples of connected graphs. Then*

$$p(G_{\mathscr{S}}, G_{\mathscr{T}}) = \sum_{(B_1,\ldots,B_n)} \prod_{k=1}^{n} p(G_{B_k}, T_k), \tag{3.4}$$

where the summation is over all ordered partitions $(B_1, \ldots, B_n)$ of the multiset $\{S_1, \ldots, S_m\}$ in n parts (where some parts are possibly empty), $G_{\mathscr{S}}$ is the graph with components $S_1, \ldots, S_m$ and G_{B_k} is the graph with components B_k.

Lemma 3.3.5. *Let G, H be graphs. Then,*

$$p(H, G) = P(p(F_i, F_j)) \tag{3.5}$$

where P is a polynomial evaluated at $p(F_i, F_j)$, where (F_i, F_j) are pairs from the collection of connected induced subgraphs of G.

Proof. Equation 3.5 is obtained as follows: first, we apply Equation 3.4, and for each term, we apply recursively Equation 3.1, substituting $\omega(.,.)$. $\square$

Example 3.3.6. By Equation 3.4,

$$p(K_2 + K_1, S_4 + K_2) = p(K_2 + K_1, S_4) + p(K_2, S_4)p(K_1, K_2) + p(K_1, S_4)p(K_2, K_2).$$

We apply Equation 3.1,

$$p(K_2 + K_1, S_4) = 8 - 2p(K_{1,2}, S_4) - 3p(K_3, S_4).$$

So, we obtain

$$p(K_2 + K_1, S_4 + K_2) = 8 - 2p(K_{1,2}, S_4) - 3p(K_3, S_4) + p(K_2, S_4)p(K_1, K_2) + p(K_1, S_4)p(K_2, K_2).$$

3.4 The abstract induced subgraph poset and the abstract edge-subgraph poset

In this section, we prove the following result which is the main theorem of this chapter.

Theorem 3.4.1. *Let G be a graph with no isolated vertices. Then*

1. *$\overline{\Omega}(G)$ can be constructed from $\overline{Q}(G)$ if and only if G does not belong to $\mathscr{M} := \mathscr{F}_0 \cup \mathscr{F}_2 \cup \mathscr{F}_4$.*
2. *$\overline{P}(G)$ can be constructed from $\overline{Q}(G)$ if and only if G does not belong to $\mathscr{N} := \bigcup_{i=0}^{4} \mathscr{F}_i$.*

Throughout this section we assume that G is a graph without isolated vertices, and that we are given $\overline{Q}(G)$. The main idea in proving Theorem 3.4.1 is the following Lemma 3.4.2. When it is applicable, it allows us to recognise which elements of $\overline{Q}(G)$ must be in $\overline{\Omega}(G)$ (since any element in $\overline{\Omega}(G)$ is in $\overline{Q}(G)$, except for isolated vertices) and to calculate the weight function in the definition of $\overline{\Omega}(G)$.

Lemma 3.4.2. *Let $g_i, g_k \in \overline{Q}(G)$. Then $\omega(g_i, g_k)$ may be expressed as a polynomial in $q(g_r, g_s)$, where $g_i \leq_e g_r <_e g_s \leq_e g_k$ and, for all g_r, we have $v(g_i) = v(g_r)$ and $k(g_i) = k(g_r)$, and for all $g_s \neq g_k$, we have $v(g_i) = v(g_s)$ and $k(g_i) = k(g_s)$.*

Proof. If $(v(g_i), k(g_i)) = (v(g_k), k(g_k))$, then

$$\omega(g_i, g_k) = \begin{cases} 1 & \text{if } g_i = g_k, \\ 0 & \text{otherwise.} \end{cases}$$

Also, if $g_i \not\leq_e g_k$, then $\omega(g_i, g_k) = 0$. Hence in the following calculation we assume that $(v(g_i), k(g_i)) \neq (v(g_k), k(g_k))$ and $g_i <_e g_k$. We have

$$q(g_i, g_k) = \sum_{\substack{g_j | v(g_j) = v(g_i), \\ k(g_j) = k(g_i)}} q(g_i, g_j)\omega(g_j, g_k). \tag{3.6}$$

We rewrite Equation (3.6) as

$$\omega(g_i, g_k) = q(g_i, g_k) - \sum_{\substack{g_j | g_i <_e g_j \\ v(g_j)=v(g_i), \\ k(g_j)=k(g_i)}} q(g_i, g_j)\omega(g_j, g_k), \tag{3.7}$$

and repeatedly expand the factors $\omega(g_j, g_k)$ in each term on the right hand side, with the condition that $\omega(g_j, g_k) = q(g_j, g_k)$ if there is no g_r such that $g_j <_e g_r <_e g_k$ and $v(g_r) = v(g_j)$ and $k(g_r) = k(g_j)$. Thus we obtain the required polynomial. $\square$

Lemma 3.4.2 can be used to calculate $\omega(g_i, g_k)$ only if we know the number of vertices and the number of components of all elements that appear in the computation. Hence most of the following lemmas are meant to either Q-reconstruct G or to show that the number of vertices and the number of components of all elements that appear in the computation of $\omega(g_i, g_k)$ can be reconstructed.

Next we prove several propositions and lemmas that eventually imply the main theorem. We will first prove part (1) of the theorem, and then use Theorem 3.2.7 to prove the part (2).

The following proposition shows the "only if" parts of the main theorem.

Proposition 3.4.3.

1. *If G belongs to $\mathscr{M}$, then $\overline{\Omega}(G)$ cannot be constructed from $\overline{Q}(G)$.*
2. *If G belongs to $\mathscr{N}$, then $\overline{P}(G)$ cannot be constructed from $\overline{Q}(G)$.*

Proof. 1. The graphs $3K_2, K_{1,3}, K_3$ have $\overline{Q}(3K_2) = \overline{Q}(K_{1,3}) = \overline{Q}(K_3)$, and the graphs $3K_2, K_{1,3}$ have $\overline{\Omega}(3K_2) = \overline{\Omega}(K_{1,3})$, but $\overline{\Omega}(K_3)$ is different.

If $G \in \mathscr{F}_4$, then suppose that $G = rK_3 + sK_{1,3} + F$ and $H = sK_3 + rK_{1,3} + F$, where $r \neq s$. Now $\overline{\Omega}(G) \neq \overline{\Omega}(H)$: this follows from the fact that the rank of the maximal element in $\overline{\Omega}(.)$ is $v(.) - k(.)$, hence the two posets have different heights.

Graphs in $\mathscr{F}_2$ form pairs so that graphs in each pair have the same abstract edge-subgraph poset but different abstract bond lattice.

2. If $G, H \in \mathscr{N}$ such that $\overline{Q}(G) \cong \overline{Q}(H)$ and $G \ncong H$, then there are only two cases in which $v(G) = v(H)$, which are $\{G, H\} = \{C_4 + K_2, B_1\}$ and $\{G, H\} = \{B_2, B_3\}$. In these cases, we verify that $\overline{P}(G) \ncong \overline{P}(H)$. In all other cases, $v(G) \neq v(H)$, hence G and H cannot have the same abstract induced subgraph poset. Hence if $G \in \mathscr{N}$, then $\overline{P}(G)$ cannot be constructed from $\overline{Q}(G)$. $\square$

Proposition 3.4.4. *If $G \in \mathscr{F}_1 \cup \mathscr{F}_3$, then $\overline{\Omega}(G)$ can be constructed from $\overline{Q}(G)$.*

Proof. Graphs in each pair $\{K_{1,m}, mK_2\}, m \neq 3$ have the same abstract edge-subgraph poset and the same abstract bond lattice. We prove that no other graph has the same abstract edge-subgraph poset as one of these graphs by induction on the number of edges of $G \in \{K_{1,m}, mK_2\}$. If $e(G) = 2$ or $e(G) = 4$ then the result holds. Suppose that the result holds for all $G \in \{K_{1,m}, mK_2\}$ such that $m \geq 4$. If $e(G) = m+1$, then by induction hypothesis, there is an element $x \in \overline{Q}(G)$ such that x represents $K_{1,m}$ or mK_2. By Lemmas 3.1.6 and 3.1.7, we have $G \in \{K_{1,m+1}, (m+1)K_2\}$.

Graphs in each pair $\{P_4, K_{1,2} + K_2\}$, $\{P_4 + K_2, T_4\}$ have the same abstract edge-subgraph poset and the same abstract bond lattice. By Lemma 3.1.8, no other graph has the same abstract edge-subgraph poset as one of these graphs. □

Lemma 3.4.5. *If $G \notin \mathscr{N}$, then $v(G)$ and $k(G)$ are Q-reconstructible.*

Proof. We prove the result by induction on the number of edges of G. All graphs on at most seven edges that are not in $\mathscr{N}$ are Q-reconstructible (Lemma 3.1.8). Hence we take $e(G) = 7$ as the base case, for which the result is true. Suppose now that the result is true for all $G \notin \mathscr{N}$ such that $7 \leq e(G) \leq m$. Let $G \notin \mathscr{N}$ be a graph on $m+1$ edges.

By Lemma 3.1.5, if G is an acyclic graph, then G is Q-reconstructible, thus $k(G)$, $v(G)$ are known. So assume that G contains a cycle. Hence each edge e in G that is on a cycle is such that $k(G-e) = k(G)$, and $v(G-e) = v(G)$, and $G-e$ has no isolated vertices (since we have assumed that G has no isolated vertices).

Now consider an arbitrary element x of rank m in $\overline{Q}(G)$. Suppose that $x := \overline{G-e}$ (minus the resulting isolated vertices). We claim that x cannot be in $\mathscr{F}_0 \cup \mathscr{F}_1 \cup \mathscr{F}_2$ since $m \geq 7$, and all graphs in $\mathscr{F}_0 \cup \mathscr{F}_1 \cup \mathscr{F}_2$ have at most six edges. By Lemmas 3.1.6 and 3.1.7, we also assume that x cannot be in $\mathscr{F}_3$ since in that case G would be Q-reconstructible . If $x \notin \mathscr{N}$, then we know $v(x)$ and $k(x)$ by induction hypothesis; and we annotate x with this information. If it is known from $\overline{Q}(G)$ that G has an edge e such that e is on a cycle and $\overline{G-e}$ is not in $\mathscr{F}_4$, then $v(G)$ is the maximum $v(H)$ among all graphs of rank m in $\overline{Q}(G)$ that are not in $\mathscr{N}$. Next we show that such an edge must exist.

Consider a graph $H \in \mathscr{F}_4$. Suppose that $G = \overline{H+uv}$, where u and v are non-adjacent vertices in the same component of H (so that uv is on a cycle in $H+uv$). Each component of H is in $\mathscr{X} \cup \{K_{1,3}, K_3\}$. We verify that in all possible ways of adding an edge uv between two vertices of the same component of H, we either obtain a graph in $\mathscr{N}$ or obtain a graph G which contains an edge e on a cycle such that $\overline{G-e}$ is not in $\mathscr{N}$. Since $G \notin \mathscr{N}$, it is the latter case. Thus $v(G)$ is reconstructible.

Now, $k(G)$ is equal to the minimum number of components among graphs not in $\mathscr{N}$ that correspond to elements of rank m and that have the same number of vertices as G. □

Lemma 3.4.6 (Lemma 2.5 [32]). *The graph $K_{1,2} + 2K_2$ is Q-reconstructible, and all elements of $\overline{Q}(K_{1,2} + 2K_2)$ are uniquely labelled.*

Lemma 3.4.7. *If $G \notin \mathscr{N}$ and contains both $K_{1,2} + 2K_2$ and T_4 as subgraphs, then $v(x)$ and $k(x)$ are reconstructible for all x in $\overline{Q}(G)$.*

Proof. Let x be an element of $\overline{Q}(G)$. If $x \notin \mathscr{N}$, then $v(x)$ and $k(x)$ are known by Lemma 3.4.5. Hence assume that $x \in \mathscr{N}$.

We claim that all elements of $\overline{Q}(G)$ of rank 3 are uniquely labelled. Indeed, all elements of $\overline{Q}(K_{1,2} + 2K_2)$ are uniquely labelled (by Lemma 3.4.6), $\overline{Q}(T_4) \cong \overline{Q}(P_4 + K_2)$, and there is no other graph H such that $Q(H) \cong Q(T_4)$, hence we distinguish T_4. Now $K_{1,3}$ is distinguished since T_4 contains $K_{1,3}$, but does not contain K_3 or $3K_2$. Furthermore, $3K_2$ is distinguished. Thus K_3 is distinguished also. Other subgraphs with 3 edges are distinguished since they are edge reconstructible, and graphs with 2 edges are distinguished. Thus, the graphs in $\mathscr{F}_0$, $P_4 + K_2$ and T_4 are distinguished.

Graphs P_4 and $K_{1,2} + K_2$ are distinguished; since $K_{1,2} + 2K_2$ contains $K_{1,2} + K_2$, but does not contain P_4.

Pairs of graphs in $\mathscr{F}_2$ are distinguished; since P_4 and $K_{1,3}$ are distinguished.

Graphs $K_{1,m}$ and mK_2 are distinguished; since $K_{1,m}$ contains $K_{1,3}$ as a subgraph, while mK_2 does not.

Graphs $rK_3 + sK_{1,3} + F$ and $sK_3 + rK_{1,3} + F$, where $r \neq s$, are distinguished since $K_{1,3}$ and K_3 are distinguished. □

Lemma 3.4.8. *If $G \notin \mathscr{N}$, $e(G) \geq 5$, and $\Delta(G) \geq 4$, then $v(x)$ and $k(x)$ are reconstructible for all x in $\overline{Q}(G)$.*

Proof. Let $x \in \overline{Q}(G)$. If $x \notin \mathscr{N}$, then $v(x)$ and $k(x)$ are known by Lemma 3.4.5. Hence assume that $x \in \mathscr{N}$. By Lemma 3.1.6, $K_{1,3}$ is distinguished. This allows distinguishing subgraphs isomorphic to $P_4 + K_2, T_4, B_1, C_4 + K_2, B_2, P_6, B_3, B_4$ and all subgraphs in $\mathscr{F}_3 \cup \mathscr{F}_4$. Hence for such elements, $v(x)$ and $k(x)$ are known. The only elements in $\mathscr{N}$ that are not yet distinguished are $P_4, K_{1,2} + K_2, C_4, 2K_{1,2}$.

We have $\Delta(G) \geq 4$ and G itself is not $K_{1,\Delta(G)}$, hence G contains one of the graphs in $\{K_{1,4}^{+e}\} \setminus \{K_{1,5}\}$ (i.e., one of $K_{1,4} + K_2, S_5, T_5$) as a subgraph. We use such a subgraph to distinguish P_4 and $K_{1,2} + K_2$.

If G contains $K_{1,4} + K_2$, then $K_{1,4} + K_2$ is distinguished (by Lemma 3.1.6). Now $K_{1,2} + K_2$ and P_4 are distinguished since $K_{1,4} + K_2$ does not contain P_4 but contains $K_{1,2} + K_2$.

If G contains S_5, then it also contains S_4, and S_4 is Q-reconstructible. Hence S_4 is distinguished. Now $K_{1,2} + K_2$ and P_4 are distinguished since S_4 does not contain $K_{1,2} + K_2$ but contains 2 subgraphs isomorphic to P_4.

If G does not contain S_4 but contains T_5, then G contains T_4, which is distinguished (as noted above). Now P_4 and $K_{1,2} + K_2$ are distinguished since T_4 contains one subgraph isomorphic to $K_{1,2} + K_2$ and two subgraphs isomorphic to P_4.

Once P_4 is distinguished, we also distinguish C_4 and $2K_{1,2}$ since only C_4 contains P_4 as a subgraph.

Now $v(x)$ and $k(x)$ are known for all x in $\overline{Q}(G)$. □

Lemma 3.4.9. *If $G \notin \mathscr{N}$, $\Delta(G) \leq 3$, and G contains $K_{1,2} + 2K_2$ but does not contain T_4, then G is Q-reconstructible.*

Proof. Let $\mathscr{I}$ be the class of graphs satisfying the conditions in the statement of the lemma.

First we show that we can recognise if G is in $\mathscr{I}$. Graphs not in $\mathscr{N}$ with at most 7 edges are Q-reconstructible (Lemma 3.1.8), so we assume that $e(G) > 7$. Since G is not $K_{1,m}$ for any m, we can assume that $\Delta(G) < e(G)$. By Lemma 3.1.6, we can recognise if $\Delta(G) \geq 4$. Since $K_{1,2} + 2K_2$ is Q-reconstructible, we can recognise if G contains $K_{1,2} + 2K_2$ as a subgraph. We can recognise if G contains T_4 as a subgraph as follows: if G contains T_4, then G has a 7-edge-subgraph F that contains T_4, and thus cannot be in $\mathscr{N}$. Hence F is Q-reconstructible (by Lemma 3.1.8). Thus we know that G contains T_4 as a subgraph. Hence we now assume that G and all its Q-reconstructions are in $\mathscr{I}$.

Next we make some observations about the structure of any graph G in class $\mathscr{I}$. Graphs $S_4, K_4 \setminus e, K_4$ are in $\mathscr{I}$, and since G does not contain T_4, these graphs can only occur as subgraphs of 4-vertex components. Also, we can verify that any subgraph isomorphic to K_3 or $K_{1,3}$ is either a component of G or a subgraph of a component isomorphic to S_4, $K_4 \setminus e$ or K_4. All other components of G are paths or cycles. Hence G is of the form $rK_3 + sK_{1,3} + F$, where components of F are in $\mathscr{X}$. But G is not in $\mathscr{N}$, hence $r = s$. Since $q(K_{1,3}, H) = q(K_3, H)$ for all $H \in \{S_4, K_4 \setminus e, K_4\}$, we have $k(K_3, G) = k(K_{1,3}, G)$ and $q(K_3, G) = q(K_{1,3}, G)$ for all graphs in $\mathscr{I}$.

Graphs $S_4, K_4 \setminus e, K_4$ are all Q-reconstructible, hence $q(S_4, G)$, $q(K_4 \setminus e, G)$ and $q(K_4, G)$ are known. Therefore, $k(S_4, G)$, $k(K_4 \setminus e, G)$, $k(K_4, G)$, $k(K_3, G)$ and $k(K_{1,3}, G)$ are all reconstructible by the following sequence of calculations:

$$\begin{aligned}
k(K_4, G) &= q(K_4, G) \\
k(K_4 \setminus e, G) &= q(K_4 \setminus e, G) - q(K_4 \setminus e, K_4)k(K_4, G) \\
k(S_4, G) &= q(S_4, G) - q(S_4, K_4 \setminus e)k(K_4 \setminus e, G) - q(S_4, K_4)k(K_4, G), \\
k(K_3, G) = k(K_{1,3}, G) &= q(K_3, G) - k(S_4, G) - 2k(K_4 \setminus e, G) - 4k(K_4, G).
\end{aligned}$$

The graph $K_{1,2} + 2K_2$ is contained in G, and by Lemma 3.4.6, the graphs K_2, P_3, $2K_2$, $K_{1,2} + K_2$, $3K_2$, and $K_{1,2} + 2K_2$ are distinguished. Now P_4 is distinguished because it is edge reconstructible and all edge-deleted subgraphs of P_4 are distinguished. The argument extends to C_4, P_5, C_5, P_6 and C_6 - they are all distinguished. (Of these graphs, C_5, C_6 and P_5 are Q-reconstructible.) Thus we know $q(P_i, G), 2 \leq i \leq 6$ and $q(C_i, G), 4 \leq i \leq 6$.

Graphs with at most 7 edges that are not in $\mathscr{N}$ are Q-reconstructible, hence we prove the lemma by induction on $e(G)$, with $e(g) = 7$ as the base case. Suppose that the result is true for all $G \in \mathscr{I}$ such that $7 \leq e(G) \leq m$. Let G be a graph with $e(G) = m + 1$, and suppose that we are given $\overline{Q}(G)$.

Paths and cycles with i edges, for $7 \leq i \leq m$ are distinguished by induction hypothesis. Hence if G itself is a cycle or path, then its edge-deck is known, and by edge reconstructibility of cycles and paths, G is Q-reconstructible. Hence we assume that G is not a path or cycle.

Cycles of length 5 or more can only be components (since G does not contain T_4). Now, we calculate $k(P_i, G), 2 \leq i \leq m$ and $k(C_i, G), 4 \leq i \leq m$ by solving the following equations (in that order):

$$\begin{aligned}
&k(C_4, G) = q(C_4, G) - q(C_4, K_4 \setminus e)k(K_4 \setminus e, G) - q(C_4, K_4)k(K_4, G), \\
&k(C_i, G) = q(C_i, G) \text{ for } 5 \leq i \leq m, \\
&k(P_i, G) = q(P_i, G) - \sum_H q(P_i, H)k(H, G) \text{ for } i = m, m-1, \ldots, 2,
\end{aligned}$$

where the summation in the last equation is over $H \in \{K_{1,3}, K_3, S_4, K_4 \setminus e, K_4, P_r, r > i, C_s, s \geq 4\}$.

Now all components of G are known along with their multiplicities, completing the induction step and the proof. $\square$

Lemma 3.4.10. *If $G \notin \mathscr{N}$, $\Delta(G) \leq 3$, and G contains T_4 but does not contain $K_{1,2} + 2K_2$, then G is Q-reconstructible.*

Proof. Graphs with at most 7 edges that are not in $\mathscr{N}$ are Q-reconstructible, hence we assume that $e(G) > 7$.

If a graph contains T_4, and has 2 or more non-trivial components, then it also contains $K_{1,2} + 2K_2$. Therefore we assume that G is connected. The conditions on G also imply that $v(G) \in \{5, 6\}$. If $v(G) = 5, \Delta(G) \leq 3$, then $e(G) \leq 7$ and the result is true. Therefore, we can assume that $v(G) = 6$. Now $\Delta(G) \leq 3$ and $e(G) > 7$ imply that have $e(G) \in \{8, 9\}$.

For any graph H, if $4 \leq e(H) \leq 9$, then $v(H) \leq 10$ or H is disconnected. In either case, H is vertex reconstructible, and hence edge reconstructible (see [12, 16, 21]). If $e(G) = 8$, the degree sequence of G is $3, 3, 3, 3, 2, 2$ or $3, 3, 3, 3, 3, 1$. Now a 7-edge-subgraph of G cannot have a component isomorphic to K_3 or $K_{1,3}$ (since the degree sequence of K_3 and $K_{1,3}$ is $2, 2, 2$ and $3, 1, 1, 1$, respectively), hence G does not have a 7-edge-subgraph in $\mathscr{N}$. Hence all edge-deleted subgraphs of G are Q-reconstructible, and since G is edge reconstructible, it is also Q-reconstructible.

If $e(G) = 9$, then the degree sequence of G is $3, 3, 3, 3, 3, 3$, by a similar argument as in the case $e(G) = 8$, we note that no edge deleted subgraph of G is in $\mathscr{N}$, hence using the case $e(G) = 8$, we can construct the edge deck of G, and then reconstruct G since G is edge reconstructible. □

Proof of Theorem 3.4.1 (the 'if' part). Let G be a graph such that $G \notin \mathscr{M}$.

If $G \notin \mathscr{N}$ and $e(G) \leq 7$, then G is Q-reconstructible by Lemma 3.1.8. Hence $\overline{\Omega}(G)$ can be constructed from $\overline{Q}(G)$. If $G \notin \mathscr{N}$, and contains $K_{1,2} + 2K_2$ or T_4, and $\Delta(G) \leq 3$, then G is Q-reconstructible by Lemmas 3.4.9 and 3.4.10. Hence $\overline{\Omega}(G)$ can be constructed from $\overline{Q}(G)$. If G contains neither T_4 nor $K_{1,2} + 2K_2$ as a subgraph, then $e(G) \leq 7$. If $G \in \mathscr{F}_3 \cup \{P_4, K_{1,2} + K_2, P_4 + K_2, T_4\}$, then $\overline{\Omega}(G)$ can be constructed from $\overline{Q}(G)$ by Propositions 3.4.4. In all other cases, by Lemmas 3.4.7 and 3.4.8, we can reconstruct $v(x)$ and $k(x)$ for all $x \in \overline{Q}(G)$.

Now we apply Lemmas 3.4.2 to g_i and g_k, with g_k as the maximal element, to recognise if g_i must be $\overline{\Omega}(G)$. Then we again apply Lemma 3.4.2 for all g_i, g_k that are marked as elements of $\overline{\Omega}(G)$.

Let G be a graph such that $G \notin \mathscr{N}$. Given $\overline{Q}(G)$, we construct $\overline{\Omega}(G)$. Now, by Theorem 3.2.7, we construct $\overline{P}(G)$ from $\overline{\Omega}(G)$. □

The following corollary generalises the result that edge reconstruction conjecture is weaker than the vertex reconstruction conjecture.

Corollary 3.4.11. *If a graph G not in $\mathscr{N}$ is P-reconstructible, then it is Q-reconstructible.*

Remark 3.4.12. Consider the infinite posets $\overline{Q}$ and $\overline{\Omega}$. We can construct $\overline{\Omega}$ from $\overline{Q}$, since the elements $G \in \mathscr{M}$ can be distinguished.

3.5 Conclusions and outlook

We have the following result of Lovász on edge reconstruction.

Theorem 3.5.1. *(Lovász [19]) Graphs on n vertices having more than $\binom{n}{2}/2$ edges are edge reconstructible.*

We tried to prove the following generalisation of Lovász's result.

Conjecture 3.5.2. *Let G be a graph (without isolated vertices) with n vertices and m edges such that $m > 7$. If $m > \frac{1}{2}\binom{n}{2}$, then G is Q-reconstructible.*

Remark 3.5.3.

1. By Lemma 3.1.8, we consider $m > 7$.
2. We cannot apply Theorem 3.1.4, since we cannot use the induction on number of edges.

A proof of Theorem 3.5.1 uses Nash-William's Lemma (Lemma 2.26 in [4]). We cannot use the same technique to prove Conjecture 3.5.2, since each element in $\overline{Q}$ is abstract. We studied other proofs of Theorem 3.5.1 in Thatte [30] and Mnukhin [22].

In the language of Hopf algebras, Schmitt [27] proved that any invariant which counts subobjects of a particular type is given by a unique polynomial in invariants which count connected subobjects. Besides, he applied the above result for graphs, obtaining Whitney's subgraph expansion theorem. For a Hopf algebra of graphs, Iovanov and Jun [13] found a basis of the space of primitives and applied it to known results on reconstruction of graphs. We investigated (in collaboration with Monique Müller Lopes Rocha (UFSJ)) whether constructions such as $\overline{P}(G)$ to $\overline{\Omega}(G)$ and $\overline{\Omega}(G)$ to $\overline{P}(G)$ may be described in the language of Hopf algebras. It appears that the enumerative methods used in these constructions are similar to the counting in Whitney's subgraph expansion theorem.

Chapter 4

Other relational structures on graphs and associated reconstruction problems

The contents of this chapter are motivated by the following conjecture proposed in [32]: If f is a permutation of the set $\mathscr{G}$ of all graphs such that f preserves the number of homomorphisms from G to H, for all graphs G and H in $\mathscr{G}$, then f is the identity permutation. In other words, the relational structure on $\mathscr{G}$ arising from graph homomorphisms has no non-trivial automorphisms, or is *rigid*. It was proved in [32] that this conjecture is weaker than the edge reconstruction conjecture. We consider a variety of relational structures on $\mathscr{G}$ arising from general graph homomorphisms, monomorphisms, epimorphisms, and so on. We then formulate a class of reconstruction conjectures for these relational structures. All these conjectures say that the structure under consideration is rigid. We prove relationships among the various conjectures.

4.1 Some definitions and conjectures

Let $G, H \in \mathscr{G}$. We say a map $f\colon V(H) \to V(G)$ is a *homomorphism* from H to G if for each $\{x, y\} \in E(H)$ we have $\{f(x), f(y)\} \in E(G)$. A one-to-one homomorphism is called a *monomorphism*. An edge-surjective homomorphism is called an *epimorphism*. A vertex-surjective homomorphism is called a *surjective homomorphism*.

We denote the number of homomorphisms from H to G by $\hom(H, G)$; the number of monomorphisms from H to G by $\mathrm{mon}(H, G)$; the number of monomorphisms f from H to G such that the image of f is an induced subgraph of G by $\mathrm{ind}(H,G)$; the number of epimorphisms from H to G by $\mathrm{epi}(H,G)$; and the number of surjective homomorphisms from H to G by $\mathrm{surhom}(H,G)$.

Let G and H be graphs. We define $\Sigma(H,G)$ to be the number of monomorphisms f from H to G such that for each component B of H, the subgraph of G induced by $f(V(B))$ is isomorphic to B.

Example 4.1.1. Consider the bipartite graph $K_{1,2}$ and the cycle graph C_4. We have $\mathrm{hom}(K_{1,2},C_4)=16$, $\mathrm{mon}(K_{1,2},C_4)=8$ and $\Sigma(K_{1,2},C_4)=8$.

Now, consider K_3. We have $\mathrm{hom}(K_{1,2},K_3)=12$, $\mathrm{mon}(K_{1,2},K_3)=6$ and $\mathrm{surhom}(K_{1,2},K_3)=6$, however $\mathrm{epi}(K_{1,2},K_3)=0$ and $\Sigma(K_{1,2},K_3)=0$.

Definition 4.1.2. We define $\mathscr{H}$, $\mathscr{M}$, $\mathscr{I}$ and $\mathscr{L}$ as weighted directed complete infinite graphs, where the vertex set is $\mathscr{G}$, and the weights are, respectively, $\mathrm{hom}(H,G)$, $\mathrm{mon}(H,G)$, $\mathrm{ind}(H,G)$ and $\Sigma(H,G)$ for all G,H in $\mathscr{G}$.

Observe that we can ignore the arcs with zero weights and consider $\mathscr{M}$, $\mathscr{I}$ and $\mathscr{L}$ as posets. Note that we do not obtain a poset from homomorphisms since it is not antisymmetric. For example, if $G \cong K_{1,2}$ to $H \cong K_2$, then there are homomorphisms from G to H as well as from H to G. We propose the following class of conjectures.

Conjecture 4.1.3. *1. (HOM) Let $\sigma\colon \mathscr{G} \to \mathscr{G}$ be a bijection such that* $\mathrm{hom}(H,G) = \mathrm{hom}(\sigma(H),\sigma(G))$ *for all $G,H \in \mathscr{G}$, then σ is the identity map. That is, $\mathscr{H}$ has no non-trivial automorphisms.*

2. *(MON) Let $\sigma\colon \mathscr{G} \to \mathscr{G}$ be a bijection such that* $\mathrm{mon}(H,G) = \mathrm{mon}(\sigma(H),\sigma(G))$ *for all $G,H \in \mathscr{G}$, then σ is the identity map. That is, $\mathscr{M}$ has no non-trivial automorphisms.*

3. *(IND) Let $\sigma\colon \mathscr{G} \to \mathscr{G}$ be a bijection such that* $\mathrm{ind}(H,G) = \mathrm{ind}(\sigma(H),\sigma(G))$ *for all $G,H \in \mathscr{G}$, then σ is the identity map or $\sigma(G) = G^c$ for all $G \in \mathscr{G}$. That is, $\mathscr{I}$ has only two automorphisms.*

4. *(LRC) Let $\sigma\colon \mathscr{G} \to \mathscr{G}$ be a bijection such that $\Sigma(H,G) = \Sigma(\sigma(H),\sigma(G))$ for all $G,H \in \mathscr{G}$, then σ is the identity map. That is, $\mathscr{L}$ has no non-trivial automorphisms.*

In the next section, we will prove relations among the above conjectures.

4.2 Some results on Conjectures 4.1.3

In this section we show that Conjecture 4.1.3(3) is weaker than the vertex reconstruction conjecture (Proposition 4.2.1) and that the Conjecture 4.1.3(2) is weaker than edge reconstruction conjecture (Proposition 4.2.2).

Proposition 4.2.1. *(VRC $\Rightarrow$ IND) Let σ be an automorphism of $\mathscr{I}$. If the vertex reconstruction conjecture is true, then σ is the identity map or $\sigma(G) = G^c$ for all $G \in \mathscr{G}$.*

Proof. We claim that $\sigma(K_1) = K_1$. Indeed, if $\sigma(K_1) = G$ and $\sigma(H) = K_1$, then

$$\mathrm{ind}(K_1, H) = \mathrm{ind}(\sigma(K_1), \sigma(H)) = \mathrm{ind}(G, K_1).$$

But $\mathrm{ind}(K_1, H)$ is the number of vertices of H and $\mathrm{ind}(G, K_1) = 0$ unless $G = K_1$.

Note that the number of vertices is preserved by the automorphism σ, thus $\sigma(K_2) = K_2$ or $\sigma(K_2) = 2K_1$. First, we will prove that if $\sigma(K_2) = K_2$, then $\sigma(H) = H$ for all H. Suppose that $\sigma(H) = H$ for all H such that $2 \leq v(H) \leq k$. Let H be a graph with $k+1$ vertices. We have, by the induction hypothesis,

$$\mathrm{ind}(G, H) = \mathrm{ind}(G, \sigma(H))$$

for all G such that $v(G) \leq k$. In the other words, H and $\sigma(H)$ have the same vertex-deck. The vertex reconstruction conjecture implies that $\sigma(H) = H$.

Now we will show that if $\sigma(K_2) = 2K_1$, then $\sigma(H) = H^c$ for all H. Suppose that $\sigma(H) = H^c$ for all H such that $2 \leq v(H) \leq k$. Let H be a graph with $k+1$ vertices. We have, by induction hypothesis,

$$\mathrm{ind}(G, H) = \mathrm{ind}(G^c, \sigma(H))$$

for all G such that $v(G) \leq k$. Since $\mathrm{ind}(G, H) = \mathrm{ind}(G^c, H^c)$ for all $G, H \in \mathscr{G}$, H^c and $\sigma(H)$ have the same vertex-deck. The vertex reconstruction conjecture implies that $\sigma(H) = H^c$, completing the result. $\square$

Proposition 4.2.2. *(ERC $\Rightarrow$ MON) Let σ be an automorphism of $\mathscr{M}$. If the edge reconstruction conjecture is true, then σ is the identity map.*

Proof. We can see $\sigma(H) = H$ for all H such that $e(H) \leq 3$.

Suppose that $\sigma(H) = H$ for all H such that $3 \leq e(H) \leq k$. Let H be a graph with $k+1$ edges. We have, by induction hypothesis,

$$\mathrm{mon}(G, H) = \mathrm{mon}(G, \sigma(H))$$

for all G such that $e(G) \leq k$. In the other words, H and $\sigma(H)$ have the same edge-deck. Now, the edge reconstruction conjecture implies that $\sigma(H) = H$ and we have the result. $\square$

We show in Theorem 4.2.5 that Conjecture 4.1.3 (2) is weaker than Conjecture 4.1.3 (3), and Conjecture 4.1.3 (1) is weaker than Conjecture 4.1.3 (2). First, we need the following lemma.

Lemma 4.2.3. *Let G and H be graphs. Then, the following identities hold*

$$\mathrm{hom}(H,G) = \mathrm{surhom}(H,G) + \sum_{F \mid v(F)<v(G)} \mathrm{surhom}(H,F)p(F,G) \tag{4.1}$$

$$\mathrm{hom}(H,G) = \mathrm{mon}(H,G) + \sum_{F \mid v(F)<v(H)} \mathrm{surhom}(H,F)p(F,G) \tag{4.2}$$

$$\mathrm{mon}(H,G) = \mathrm{ind}(H,G) + \sum_{\substack{H' \mid e(H')>e(H),\\ v(H')=v(H)}} \mathrm{mon}(H,H')p(H',G) \tag{4.3}$$

$$\mathrm{hom}(H,G) = \mathrm{epi}(H,G) + \sum_{F \mid e(F)<e(G)} \mathrm{epi}(H,F)s(F,G) \tag{4.4}$$

$$p(H,G) = \frac{\mathrm{ind}(H,G)}{\mathrm{aut}(H)} \tag{4.5}$$

$$s(H,G) = \frac{\mathrm{mon}(H,G)}{\mathrm{aut}(H)}. \tag{4.6}$$

Proof. For Equation (4.1) each homomorphism f from H to G, the image of f is a surjective homomorphism from H to $f(H)$. The summation is over all the induced subgraphs of G, since a surjective homomorphism is a vertex-surjective homomorphism. Thus, Equation (4.1) holds. We use a similar argument to prove Equation (4.4), but now the summation is over all the subgraphs of G, since an epimorphism is an edge-surjective homomorphism.

Equation (4.2) is true, since for each homomorphism f from H to G, if f is not a monomorphism, then it is a surjective homomorphism from H to F, where F is an induced proper subgraph of G.

Equations (4.5) and (4.6) are true by definition. We have,

$$\begin{aligned}
\mathrm{mon}(H,G) &= \mathrm{aut}(H)s(H,G)\\
&= \mathrm{aut}(H)\left(p(H,G) + \sum_{\substack{H' \mid e(H')>e(H),\\ v(H')=v(H)}} s(H,H')p(H',G)\right)\\
&= \mathrm{ind}(H,G) + \sum_{\substack{H' \mid e(H')>e(H),\\ v(H')=v(H)}} \mathrm{mon}(H,H')p(H',G),
\end{aligned}$$

hence Equation (4.3) is true. □

Lemma 4.2.4. *[Thatte [32]] Let σ be an automorphism of $\mathscr{H}$. Then for all G, we have $v(G) = v(\sigma(G))$ and $e(G) = e(\sigma(G))$. Moreover, $\sigma(G) = G$ for all G such that $e(G) \leq 3$.*

Theorem 4.2.5. *(IND $\Rightarrow$ MON $\Rightarrow$ HOM) Let σ be an automorphism of $\mathscr{H}$. Then, for all graphs $G, H \in \mathscr{G}$,* $\mathrm{mon}(G,H)$*,* $\mathrm{ind}(G,H)$*, and* $\mathrm{surhom}(H,G)$ *are preserved by σ. Furthermore, if σ is an automorphism of $\mathscr{M}$, then σ is also an automorphism of $\mathscr{I}$.*

Proof. We will prove the result by induction on $v(G)$. For $v(G) = 1$, $\sigma(K_1) = K_1$, and we have

$$\begin{aligned}
\mathrm{mon}(K_1, H) &= \mathrm{hom}(K_1, H) = \mathrm{hom}(\sigma(K_1), \sigma(H)) = \mathrm{mon}(\sigma(K_1), \sigma(H)) \\
\mathrm{surhom}(H, K_1) &= \mathrm{hom}(H, K_1) = \mathrm{hom}(\sigma(H), \sigma(K_1)) = \mathrm{surhom}(\sigma(H), \sigma(K_1)) \\
\mathrm{ind}(K_1, H) &= \mathrm{mon}(K_1, H) = \mathrm{mon}(\sigma(K_1), \sigma(H)) = \mathrm{ind}(\sigma(K_1), \sigma(H))
\end{aligned}$$

for each graph H.

Suppose that for all graphs H and for all graphs G with $v(G) < n$, we have $\mathrm{surhom}(H,G) = \mathrm{surhom}(\sigma(H), \sigma(G))$, $\mathrm{mon}(G,H) = \mathrm{mon}(\sigma(G), \sigma(H))$, and $\mathrm{ind}(G,H) = \mathrm{ind}(\sigma(G), \sigma(H))$. First we prove that for all graphs H and for all graphs G with $v(G) = n$, we have $\mathrm{surhom}(H,G) = \mathrm{surhom}(\sigma(H), \sigma(G))$.

We have

$$\begin{aligned}
\mathrm{surhom}(H,G) &= \mathrm{hom}(H,G) - \sum_{F \mid v(F) < v(G)} \mathrm{surhom}(H,F) p(F,G) \\
&= \mathrm{hom}(\sigma(H), \sigma(G)) - \sum_{F \mid v(F) < v(G)} \mathrm{surhom}(\sigma(H), \sigma(F)) p(\sigma(F), \sigma(G)), \\
&= \mathrm{hom}(\sigma(H), \sigma(G)) - \sum_{F \mid v(F) < v(\sigma(G))} \mathrm{surhom}(\sigma(H), F) p(F, \sigma(G)) \\
&= \mathrm{surhom}(\sigma(H), \sigma(G))
\end{aligned}$$

The first line is given by Lemma 4.2.3. In the second line, we use the fact that σ preserves the number of homomorphism, and we use the induction hypothesis, since

$$p(F,G) = \frac{\mathrm{ind}(F,G)}{\mathrm{aut}(F)} = \frac{\mathrm{ind}(F,G)}{\mathrm{mon}(F,F)}.$$

In the third line, we use Lemma 4.2.4.

Next we prove that $\mathrm{mon}(G,H) = \mathrm{mon}(\sigma(G), \sigma(H))$, for all graphs G with $v(G) = n$.

We have, by the similar reasoning,

$$\begin{aligned}
\operatorname{mon}(G,H) &= \hom(G,H) - \sum_{F \mid v(F)<v(G)} \operatorname{surhom}(G,F)p(F,H) \\
&= \hom(\sigma(G),\sigma(H)) - \sum_{F \mid v(F)<v(G)} \operatorname{surhom}(\sigma(G),\sigma(F))p(\sigma(F),\sigma(H)) \\
&= \hom(\sigma(G),\sigma(H)) - \sum_{F \mid v(F)<v(\sigma(G))} \operatorname{surhom}(\sigma(G),F)p(F,\sigma(H)) \\
&= \operatorname{mon}(\sigma(G),\sigma(H)).
\end{aligned}$$

Now, we prove that $\operatorname{ind}(G,H) = \operatorname{ind}(\sigma(G),\sigma(H))$ for all graphs G with $v(G) = n$. By Lemma 4.2.3,

$$\operatorname{ind}(G,H) = \operatorname{mon}(G,H) - \sum_{\substack{G' \mid e(G')>e(G), \\ v(G')=v(G)}} \operatorname{mon}(G,G')p(G',H)$$

Since we have shown that $\operatorname{mon}(G,H) = \operatorname{mon}(\sigma(G),\sigma(H))$ for all graphs H and for all graphs G with $v(G) = n$, we obtain $\operatorname{ind}(G,H) = \operatorname{ind}(\sigma(G),\sigma(H))$ by applying the above equation recursively in decreasing order on the number of edges of G.

Thus, we conclude that for all graphs G, H, $\operatorname{surhom}(H,G)$, $\operatorname{mon}(G,H)$ and $\operatorname{ind}(G,H)$ are preserved by σ. $\square$

Proposition 4.2.6. *Let σ be an automorphism of $\mathscr{H}$. Then, for all graphs $G, H \in \mathscr{G}$, the quantity* $\operatorname{epi}(H,G)$ *is preserved by σ.*

Proof. Let σ be an automorphism of $\mathscr{H}$. Let $G, H \in \mathscr{G}$. We prove by induction on $e(G)$ that $\operatorname{epi}(H,G) = \operatorname{epi}(\sigma(H),\sigma(G))$. For $e(G) = 0$, we use Lemma 4.2.4 and the result is true. Now, suppose that for all graphs H and for all graphs G with $e(G) < n$, we have $\operatorname{epi}(H,G) = \operatorname{epi}(\sigma(H),\sigma(G))$. Let G be a graph with $e(G) = n$. We have

$$\begin{aligned}
\operatorname{epi}(H,G) &= \hom(H,G) - \sum_{F \mid e(F)<e(G)} \operatorname{epi}(H,F)s(F,G) \\
&= \hom(\sigma(H),\sigma(G)) - \sum_{F \mid e(F)<e(G)} \operatorname{epi}(\sigma(H),\sigma(F))s(\sigma(F),\sigma(G)) \\
&= \hom(\sigma(H),\sigma(G)) - \sum_{F \mid e(F)<e(\sigma(G))} \operatorname{epi}(\sigma(H),F)s(F,\sigma(G)) \\
&= \operatorname{epi}(\sigma(H),\sigma(G)),
\end{aligned}$$

where
$$s(H,G) = \frac{\mathrm{mon}(H,G)}{\mathrm{aut}(H)} = \frac{\mathrm{mon}(H,G)}{\mathrm{mon}(H,H)}.$$
The first line is given by Lemma 4.2.3. In the second line, we use the fact that σ preserves the number of homomorphism, the induction hypothesis, and that σ preserves the number of monomorphisms, by Theorem 4.2.5. In the third line, we use Lemma 4.2.4.

Hence, $\mathrm{epi}(H,G) = \mathrm{epi}(\sigma(H),\sigma(G))$ for all graphs G and H. □

We prove the following lemmas.

Lemma 4.2.7. *Let σ be an automorphism of $\mathscr{I}$ such that $\sigma(K_2) = K_2$. If G is a connected graph, then $\sigma(G)$ is also a connected graph.*

Proof. We prove the result by induction on number of vertices of G. We have $\sigma(G) = G$ for all graph G such that $v(G) \leq 3$.

Suppose that the result is true for all G such that $3 \leq v(G) \leq k$. Let G be a connected graph such that $v(G) = k+1$. We use the fact that a graph on two or more vertices is connected if and only if at least two of its vertex-deleted subgraphs are connected. Consider two not necessarily distinct connected graphs G_1 and G_2 such that the number of vertices of G_i is equal to k and $\mathrm{ind}(G_i,G) \neq 0$, for $i = 1,2$. We have,
$$\mathrm{ind}(G_i,G) = \mathrm{ind}(\sigma(G_i),\sigma(G)),$$
and since by hypothesis $\sigma(G_1)$ and $\sigma(G_2)$ are connected graphs, $\sigma(G)$ is a connected graph. □

Lemma 4.2.8. *Let σ be an automorphism of $\mathscr{I}$ such that $\sigma(K_2) = K_2$. Then, for any graph $G = \sum_{i=1}^{k} m_i G_i$, where for all $i = 1,\cdots,k$, G_i is a connected graph and m_i is a positive integer, we have*
$$\sigma(G) = \sum_{i=1}^{k} m_i \sigma(G_i).$$

Proof. Assume that $v(G_1) \leq v(G_2) \leq \cdots \leq v(G_k)$. We will prove, for all $i = 1,\cdots,k$, that $\sigma(G)$ has exactly m_i components isomorphic to $\sigma(G_i)$, and since $v(G) = v(\sigma(G))$, the graph $\sigma(G)$ does not have any other components.

We prove the result for $k-j$ by induction on j. For $j = 0$, we have
$$m_k = p(G_k,G) = p(\sigma(G_k),\sigma(G)).$$

If there exists a connected graph H such that $v(H) > v(G_k)$, and $p(H,\sigma(G)) \neq 0$, then $p(\sigma^{-1}(H),G) \neq 0$. This is not possible since G_k is the largest component of G, therefore $\sigma(G_k)$ is the largest component of $\sigma(G)$, and $\sigma(G)$ has exactly m_k components isomorphic to $\sigma(G_k)$.

We suppose that, for $0 \le j < l$, $\sigma(G)$ has exactly m_{k-j} components isomorphic to $\sigma(G_{k-j})$, and we prove the result for $j = l$. We have

$$\begin{aligned} m_{k-l} &= p(G_{k-l}, G) - \sum_{r=0}^{l-1} p(G_{k-l}, G_{k-r}) m_{k-r} \\ &= p(\sigma(G_{k-l}), \sigma(G)) - \sum_{r=0}^{l-1} p(\sigma(G_{k-l}), \sigma(G_{k-r})) m_{k-r}. \end{aligned}$$

The last line is equal to the number of components of $\sigma(G)$ isomorphic to $\sigma(G_{k-l})$, since, by induction hypothesis, m_{k-r} is the number of components of $\sigma(G)$ isomorphic to $\sigma(G_{k-r})$. □

In Theorem 4.2.9, we show a relation between the Conjecture 4.1.3 (3) and the Conjecture 4.1.3 (4).

Theorem 4.2.9. *If σ is an automorphism of $\mathscr{L}$, then σ is an automorphism of $\mathscr{I}$. Furthermore, if σ is an autormorphism of $\mathscr{I}$ and $\sigma(K_2) = K_2$, then σ is an automorphism of $\mathscr{L}$.*

Proof. Let $\sigma\colon \mathscr{G} \to \mathscr{G}$ be a bijection such that $\Sigma(G, H) = \Sigma(\sigma(G), \sigma(H))$ for all $G, H \in \mathscr{G}$ (see Definition 4.1.2). We have $v(G) = v(\sigma(G))$ and $e(G) = e(\sigma(G))$. Thus, $\sigma(K_n) = K_n$ for all $n \ge 1$.

We have

$$\operatorname{ind}(G, H) = \Sigma(G, H) - \sum_{\substack{G' \mid e(G') > e(G), \\ v(G') = v(G)}} \Sigma(G, G') p(G', H) \tag{4.7}$$

for all graphs G and H.

Let H be a graph and G be a graph with n vertices. If $e(G) = \binom{n}{2}$, then $G = K_n$ and $\operatorname{ind}(G, H) = \Sigma(G, H) = \Sigma(\sigma(G), \sigma(H)) = \operatorname{ind}(\sigma(G), \sigma(H))$. Suppose that $\operatorname{ind}(G, H) = \operatorname{ind}(\sigma(G), \sigma(H))$ for all graphs G with $e(G) > \binom{n}{2} - k$ where $k \ge 1$. We use Equation (4.7) to prove that $\operatorname{ind}(G, H) = \operatorname{ind}(\sigma(G), \sigma(H))$ for all graphs G with $e(G) = \binom{n}{2} - k$ so that the result follows by induction.

$$\begin{aligned}
\operatorname{ind}(G,H) &= \Sigma(G,H) - \sum_{\substack{G' \mid e(G')>e(G),\\ v(G')=v(G)}} \Sigma(G,G')p(G',H) \\
&= \Sigma(\sigma(G),\sigma(H)) - \sum_{\substack{G' \mid e(G')>e(G),\\ v(G')=v(G)}} \Sigma(\sigma(G),\sigma(G'))p(\sigma(G'),\sigma(H)) \\
&= \Sigma(\sigma(G),\sigma(H)) - \sum_{\substack{G' \mid e(G')>e(\sigma(G)),\\ v(G')=v(\sigma(G))}} \Sigma(\sigma(G),G')p(G',\sigma(H)) \\
&= \operatorname{ind}(\sigma(G),\sigma(H)),
\end{aligned}$$

where in the third line, we use the fact that the number of vertices and edges are preserved by σ.

On the other hand, suppose now that $\sigma\colon \mathscr{G} \to \mathscr{G}$ is a bijection such that $\operatorname{ind}(G,H) = \operatorname{ind}(\sigma(G),\sigma(H))$ for all $G,H \in \mathscr{G}$, and $\sigma(K_2) = K_2$. So, σ preserves the number of vertices and edges, and $\sigma(G) = G$ for all graph G such that $v(G) \leq 3$.

We have, for any graphs G and H,

$$\begin{aligned}
\Sigma(G,H) &= \operatorname{aut}(G)\omega(G,H) \\
&= \operatorname{ind}(G,G)\omega(G,H)
\end{aligned}$$

But $\omega(G,H)$ can be written as a polynomial in the variables $p(F_i,F_j)$'s, where the graphs F_i's and F_j's are connected graphs (see Equation (3.3)). Thus, using the fact that σ only depends on connected graphs, by Lemma 4.2.7 and Lemma 4.2.8, we have

$$\begin{aligned}
\Sigma(G,H) &= \operatorname{ind}(\sigma(G),\sigma(G))P(p(\sigma(F_i),\sigma(F_j))) \\
&= \Sigma(\sigma(G),\sigma(H)),
\end{aligned}$$

where $P(\dots)$ denotes a polynomial that can be explicitly obtained. □

4.3 Conclusions and outlook

In this section, we formulate some open problems.

Consider the weighted posets $Q = (\mathscr{G}, \leq_e, q)$ and $P = (\mathscr{G}, \leq_v, p)$. We state the following conjecture.

Conjecture 4.3.1. *1. (QRC) Q has no non-trivial automorphisms.*

2. *(PRC) P has only two automorphisms - the identity automorphism and the automorphism defined by the map $G \mapsto G^c$ for all G.*

Verify that ERC implies QRC and VRC implies PRC. Verify that PRC and IND are equivalent, and QRC and MON are equivalent. Then we can only talk about IND and MON so that we can formulate everything in terms of morphisms.

We know that ERC implies MON. Are they equivalent? Also, we know that VRC implies IND. Are they equivalent?

Every tree T can be constructed from $P(T)$ and also from $Q(T)$, with possibly a few exceptions. But similar exceptions should not occur in $\mathscr{H}$, $\mathscr{M}$, $\mathscr{I}$, etc). Analogous problem may be formulated as follows for the automorphisms of the various relational structures defined in this chapter. Prove that certain subsets of $\mathscr{H}$,$\mathscr{M}$, $\mathscr{I}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{L}$ are fixed by every automorphism (i.e., they are uniquely marked). In particular, prove that trees and unicyclic graphs are uniquely marked in all the above relational structures.

Prove that various invariants can be uniquely constructed for some points in these structures. This should be possible since we have similar results for $P(G)$.

Chapter 5

Hopf algebras of graph invariants and Whitney's Theorem

Whitney [38] showed that any graph invariant which counts subgraphs formed by a given collection of blocks can be expressed as a unique polynomial, with rational coefficients, in the invariants which count blocks in a graph, and furthermore, these latter invariants are algebraically independent over the rationals.

We construct a subalgebra of the algebra UGQSym given by Borie [5], and show that the subalgebra is generated by elements that are formal power series which can be evaluated on graphs and count occurrences of blocks. This result is an algebraic proof of Whitney's Theorem.

In this Chapter, we state Whitney's Theorem, define the algebra UGQSym and a specific subalgebra of UGQSym.

5.1 Whitney's theorem

Definition 5.1.1. (Mohar and Thomassen [23]) Let G be a graph. Define a relation $\sim$ on $E(G)$ as follows: for all $e \in E(G)$, $e \sim e$; for all distinct $e_1, e_2 \in E(G)$, $e_1 \sim e_2$ if there is a cycle in G that contains the edges e_1 and e_2. The relation $\sim$ is an equivalence relation. A *block* of G is either a subgraph consisting of an isolated vertex in G or a subgraph consisting of the edges in an equivalence class of $\sim$ and their incidence vertices.

Example 5.1.2. The blocks of a nontrivial tree are the copies of K_2 induced by its edges.

A connected graph G is called *2-connected*, if it contains at least three vertices and for every vertex $v \in V(G)$, the induced subgraph $G - v$ is connected. We note the following basic facts about blocks.

Proposition 5.1.3. *(Mohar and Thomassen [23]) Let G be a graph.*

1. *If G is the null graph, then it has no blocks.*
2. *If G is a single-vertex graph, then it has exactly one block.*
3. *G is a union of its blocks.*
4. *If a block of G contains at least two edges with distinct sets of incident vertices, then it is a 2-connected subgraph of G.*

A graph B is a block graph or just a *block* if the only block in the graph is the graph itself. Let $\mathscr{G}_B := \{B_1, B_2, B_3, \ldots\}$ be the set of isomorphism classes of blocks with two or more vertices. We associate each $B_i \in \mathscr{G}_B$ with an indeterminate x_i, and assume that the indeterminates commute, and let $X := \{x_1, x_2, \ldots\}$. The block structure of a graph G is the multiset of isomorphism classes of the blocks of G.

Theorem 5.1.4 (Whitney [38]). *Let $S := \{B_{n_1}, B_{n_2}, \ldots, B_{n_k}\}$ be a finite nonempty multiset of elements in $\mathscr{G}_B$. Let $\mathscr{H}_S := \{G_1, G_2, \ldots, G_m\}$ be the set of all unlabelled graphs with block structure S. We say that the graphs in $\mathscr{H}_S$ are graphs of type S. Then there is a polynomial $P_S \in \mathbb{Q}[X]$ such that, for all graphs G, the number of subgraphs of G in $\mathscr{H}_S$ is the evaluation of P_S at $x_j = s(B_j, G)$, where $s(B_j, G)$ counts the number of blocks of G isomorphic to B_j.*

An enumerative combinatorial proof of Theorem 5.1.4 using Kocay's lemma was given by Thatte [31] and an algebraic proof using the language of Hopf algebras was given by Schmitt [27].

Example 5.1.5. Let $S = \{K_2, K_2, K_3\}$. For all graphs G, the number of subgraphs of type S in G is obtained by evaluating the polynomial

$$6x_2 - \frac{7}{2}x_1x_2 - 2x_3 + \frac{1}{2}x_1^2x_2$$

at $x_1 = s(K_2, G)$, $x_2 = s(K_3, G)$ and $x_3 = s(K_4 \setminus e, G)$.

Remark 5.1.6. For the indeterminate x_j in P_S, we have that B_j is a nonempty finite union of blocks in S (i.e., if we order S and consider S as a tuple, then $\mathrm{c}(S, B_j)$ is nonzero). In Example 5.1.5, observe that $\mathrm{c}(\{K_3, K_2, K_2\}, K_4 \setminus e) \neq 0$ and we have the term $x_3 = s(K_4 \setminus e, G)$.

Corollary 5.1.7 is an application of Whitney's Theorem to the graph reconstruction theory.

Corollary 5.1.7. *Let G be a graph on at least three vertices. Let S be a finite multiset of elements in $\mathscr{G}_B$ such that each block in S has fewer than $v(G)$ vertices. Then the number of spanning subgraphs of G with block structure S is determined by vertex deck of G.*

Since there may be many mutually non-isomorphic graphs with a given block structure, Whitney's theorem is not sufficient to conclude that if H has block structure S and each block in S has fewer than $v(G)$ vertices then $s(H, G)$ is determined by the vertex deck of G.

5.2 The Hopf algebra UGQSym

In this section we present some results given by Borie [5]. We define and investigate the Hopf structure of the algebra UGQSym. First, we realise functions counting occurrences of subgraphs, as power series in an infinite number of variables.

Let $\overline{\mathscr{G}}$ be a subset of $\mathscr{G}$ such that if $G \in \overline{\mathscr{G}}$, then G has no isolated vertices. The null graph is in $\overline{\mathscr{G}}$ and the single vertex graph is not in $\overline{\mathscr{G}}$.

Definition 5.2.1. Let $G \in \overline{\mathscr{G}}$. Let g be a labelling of the vertices of G using positive integers. We define the following monomial:

$$m(g) := \prod_{\substack{0<i<j \\ i \text{ and } j \text{ are adjacent}}} x_{ij}.$$

and an invariant power series $\mathscr{M}_G$ as:

$$\mathscr{M}_G := \sum_{g\text{: labelling of } G} m(g).$$

We set $\mathscr{M}_\Phi := 1$.

Example 5.2.2. 1. $\mathscr{M}_{K_2} = \sum\limits_{0<i<j} x_{ij} = x_{12} + x_{13} + x_{23} + \dots$

2. $\mathscr{M}_{K_{1,2}} = \sum\limits_{0<i<j<k} x_{ij}x_{ik} + x_{ij}x_{jk} + x_{ik}x_{jk}$

We define the map $\mathscr{M}_G : \mathscr{G} \to \mathbb{N}$ as follows. For each $H \in \mathscr{G}$ with n vertices, let a bijection $h\colon V(H) \to \{1, \dots, n\}$ be a labelling of the vertices of H. For each variable x_{ij} in $\mathscr{M}_G$, we evaluate $x_{ij} = 1$, if i and j are adjacent in H and $x_{ij} = 0$, otherwise. The main motivation justifying the definition of $\mathscr{M}_G$ is given by the following result.

Theorem 5.2.3. *(Borie [5]) $\mathscr{M}_G(H)$ counts the number of subgraphs of H that are isomorphic to G.*

Proof. Each term of $\mathscr{M}_G(H)$ that is equal to 1 counts a distinct subgraph of H isomorphic to G. This number of subgraphs isomorphic to G is independent of the choice of the labelling of the graph H since the functions $\mathscr{M}_G$ are invariant under the relabelling action. □

Example 5.2.4. Let H be the cycle graph with 5 vertices. We consider the labelling of H shown in Figure 5.1.

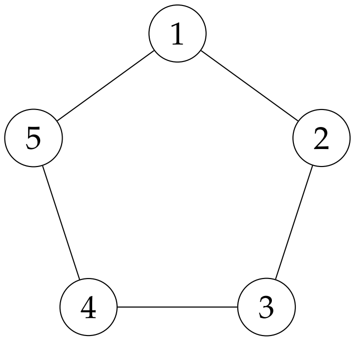


Figure 5.1: A labelling of H

We have

$$\begin{aligned}\mathscr{M}_{2K_2} = \sum_{0<i<j<k<l} x_{ij}x_{kl} + x_{ik}x_{jl} + x_{il}x_{jk} &= x_{12}x_{34} + x_{13}x_{24} + x_{14}x_{23} \\ &+ x_{12}x_{35} + x_{13}x_{25} + x_{15}x_{23} + x_{23}x_{45} + x_{24}x_{35} + x_{25}x_{34} \\ &+ x_{12}x_{45} + x_{14}x_{25} + x_{15}x_{24} + x_{13}x_{45} + x_{14}x_{35} + x_{15}x_{34} + \dots\end{aligned}$$

Thus,

$$\begin{aligned}\mathscr{M}_{2K_2}(H) = 1\cdot 1 + 0\cdot 0 + 0\cdot 1 + 1\cdot 0 + 0\cdot 0 + 1\cdot 1 + 1\cdot 1 + 0\cdot 0 + 0\cdot 1 + \\ 1\cdot 1 + 0\cdot 0 + 1\cdot 0 + 0\cdot 1 + 0\cdot 0 + 1\cdot 1 + \dots = 5.\end{aligned}$$

Observe that all the remaining terms in the series evaluate to 0.

Now, we define the algebra UGQSym and investigate its Hopf structure. Let $\mathbb{K}$ be a field of characteristic zero. Let $\mathscr{A}$ be the subspace of $\mathbb{K}[\![x_{ij} \mid i<j]\!]/(x_{ij}^2 - x_{ij})$ generated by $\mathscr{M}_G$'s, that is,

$$\mathscr{A} := \{\sum_{i=1}^{k} \lambda_i \mathscr{M}_{G_i} \mid k \in \mathbb{N}, \lambda_i \in \mathbb{K}, G_i \in \overline{\mathscr{G}}\}.$$

The subspace $\mathscr{A}$ is called the *Unlabelled Graph Quasi Symmetric functions* (UGQSym).

Theorem 5.2.5 (Borie [5]). *1. $\mathscr{A}$ is a subalgebra of $\mathbb{K}[\![x_{ij} \mid i < j]\!]/(x_{ij}^2 - x_{ij})$.*

2. *The set $\{\mathscr{M}_G \mid G \in \overline{\mathscr{G}}\}$ forms a linear basis of the algebra $\mathscr{A}$.*

3. *The set $\{\mathscr{M}_G \mid G \in \overline{\mathscr{G}}$ and G is a connected graph$\}$ generates the algebra $\mathscr{A}$.*

Let $G_1, G_2 \in \overline{\mathscr{G}}$. The product is given by

$$\mathscr{M}_{G_1} \cdot \mathscr{M}_{G_2} = \sum_{G \in \overline{\mathscr{G}}} c^{G}_{G_1,G_2} \mathscr{M}_G, \tag{5.1}$$

where $c^{G}_{G_1,G_2}$ counts the number of ways to cover G by G_1 and G_2 and the unit map u given by $u(1) = \mathscr{M}_\Phi$. Equation 5.1 is equivalent to Kocay's equation 2.1 which also gives the coefficients $c^{G}_{G_1,G_2}$.

We do not want to create multi-edges when calculating the product of two terms. For this reason, we define $\mathscr{A}$ as a subspace of $\mathbb{K}[\![x_{ij} \mid i < j]\!]/(x_{ij}^2 - x_{ij})$.

Example 5.2.6. 1. $\mathscr{M}_{K_2} \cdot \mathscr{M}_{K_2} = \mathscr{M}_{K_2} + 2\mathscr{M}_{K_{1,2}} + 2\mathscr{M}_{2K_2}$

2. $\mathscr{M}_{K_2} \cdot \mathscr{M}_{K_{1,2}} = \mathscr{M}_{K_2+K_{1,2}} + 2\mathscr{M}_{P_3} + 2\mathscr{M}_{K_{1,2}} + 3\mathscr{M}_{K_3} + 3K_{1,3}$

We define the coproduct as follows:

Definition 5.2.7. Let $G \in \overline{\mathscr{G}}$. The *coproduct* of $\mathscr{A}$ is given by

$$\Delta(\mathscr{M}_G) := \sum_{G_1 \uplus G_2 = G} \mathscr{M}_{G_1} \otimes \mathscr{M}_{G_2},$$

where the summation runs over ordered pairs (G_1, G_2) of graphs in $\overline{\mathscr{G}}$ such that G is the disjoint union of G_1 and G_2. That is, the primitive elements are connected graphs.

The *counit* of $\mathscr{A}$ is defined by

$$\epsilon(\mathscr{M}_G) := \begin{cases} 1, & \text{if } G = \Phi; \\ 0, & \text{otherwise.} \end{cases}$$

The coproduct and counit of $\mathscr{A}$ are extended linearly to all elements.

Example 5.2.8. 1. $\Delta(\mathscr{M}_{K_3}) = \mathscr{M}_{K_3} \otimes 1 + 1 \otimes \mathscr{M}_{K_3}$

2. $\Delta(\mathscr{M}_{2K_2}) = \mathscr{M}_{2K_2} \otimes 1 + \mathscr{M}_{K_2} \otimes \mathscr{M}_{K_2} + 1 \otimes \mathscr{M}_{2K_2}$

Theorem 5.2.9. *(Borie [5]) $(\mathscr{A}, \cdot, \Delta, u, \epsilon)$ is a Hopf algebra.*

Proof. By Theorem 5.2.5, $\mathscr{A}$ is a subalgebra of $\mathbb{K}[\![x_{ij} \mid i < j]\!]/(x_{ij}^2 - x_{ij})$, thus we can see that $\mathscr{A}$ is a commutative algebra. Besides, $\mathscr{A}$ is a cocommutative coalgebra with maps Δ and ϵ.

The product and coproduct are compatible by induction on the number of connected components of the operands. Thus, $\mathscr{A}$ is a bialgebra.

Finally, since $\mathscr{A}$ is a connected graded bialgebra by the number of vertices, and we conclude that $\mathscr{A}$ is a Hopf algebra (see Proposition 2.4.14). □

If S is the antipode of $\mathscr{A}$, we have $S(\mathscr{M}_G) = -\mathscr{M}_G$, for all connected graphs $G \in \overline{\mathscr{G}}$, since the primitive elements of the Hopf algebra $\mathscr{A}$ are the connected graphs.

5.3 A subalgebra of UGQSym

We now define another algebra such that it is a subalgebra of UGQSym.

Definition 5.3.1. Let

$$\mathscr{T} := \{\mathscr{H}_S \mid S \text{ is a finite multiset of blocks in } \mathscr{G}_B\},$$

where $\mathscr{H}_S$ is the class of unlabelled graphs with a given block structure S.

Example 5.3.2. 1. $\{B\} \in \mathscr{T}$, where $B \in \mathscr{G}_B$

2. $\{S_4, K_3 + K_2\} \in \mathscr{T}$

Let $\mathscr{B}$ be the subspace of $\mathscr{A}$ generated by the set

$$\{\sum_{G \in \mathscr{H}_S} \mathscr{M}_G \mid \mathscr{H}_S \in \mathscr{T}\} \cup \{\mathscr{M}_\Phi\}.$$

Proposition 5.3.3. *$\mathscr{B}$ is a subalgebra of $\mathscr{A}$.*

Proof. Let $\mathscr{H}_{S_1}, \mathscr{H}_{S_2} \in \mathscr{T}$. We have

$$(\sum_{G \in \mathscr{H}_{S_1}} \mathscr{M}_G) \cdot (\sum_{H \in \mathscr{H}_{S_2}} \mathscr{M}_H) = \sum_S c_S (\sum_{F \in \mathscr{H}_S} \mathscr{M}_F), \tag{5.2}$$

where the outer summation on the right-hand side is over all multisets S of blocks, and c_S counts the number of ways to cover a graph $F \in \mathscr{H}_S$ by graphs $G \in \mathscr{H}_{S_1}$ and $H \in \mathscr{H}_{S_2}$. Note that c_S does not depend on the choice of F, G and H, but only on their block structure. □

Example 5.3.4. Let $S_1 = \{K_3, K_2\}$ and $S_2 = \{K_3\}$.

$$(\mathscr{M}_{\triangledown\,\vert} + \mathscr{M}_{\boxslash}) \cdot (\mathscr{M}_{\triangledown}) = 2\left(\sum_{G \in \mathscr{H}_{\{K_3,K_3,K_2\}}} \mathscr{M}_G\right) + \sum_{G \in \mathscr{H}_{\{K_3,K_2\}}} \mathscr{M}_G + 6\left(\sum_{G \in \mathscr{H}_{\{K_3,K_3\}}} \mathscr{M}_G\right)$$
$$+2\left(\sum_{G \in \mathscr{H}_{\{K_4 \backslash e,K_2\}}} \mathscr{M}_G\right) + 4\left(\mathscr{M}_{K_4 \backslash e}\right) + 12\left(\mathscr{M}_{K_4}\right).$$

Proposition 5.3.5. *The set* $\{\sum_{G \in \mathscr{H}_S} \mathscr{M}_G \mid \mathscr{H}_S \in \mathscr{T}\} \cup \{\mathscr{M}_\Phi\}$ *forms a linear basis of the algebra* $\mathscr{B}$.

Proof. By definition, the set generates $\mathscr{B}$. Since for all H, there is a unique $\mathscr{H}_S \in \mathscr{T}$ such that $\mathscr{M}_H$ is a term of the sum $\sum_{G \in \mathscr{H}_S} \mathscr{M}_G$ and by Theorem 5.2.5, the elements of the set $\{\mathscr{M}_G \mid G \in \overline{\mathscr{G}}\}$ are linearly independent, and the set $\{\sum_{G \in \mathscr{H}_S} \mathscr{M}_G \mid \mathscr{H}_S \in \mathscr{T}\} \cup \{\mathscr{M}_\Phi\}$ is also linearly independent. $\square$

Theorem 5.3.6 is Whitney's Theorem in terms of our language.

Theorem 5.3.6. *The algebra* $\mathscr{B}$ *is generated by* $\Gamma := \{\mathscr{M}_B \mid B \in \mathscr{G}_B\} \cup \{\mathscr{M}_\Phi\}$.

Proof. We will prove that for any $\mathscr{H}_S \in \mathscr{T}$, the sum $\sum_{G \in \mathscr{H}_S} \mathscr{M}_G$ can be written as a polynomial in the elements of Γ as indeterminates, with rational coefficients.

Let B_1 and B_2 be blocks. By Equation 5.2,

$$\mathscr{M}_{B_1} \cdot \mathscr{M}_{B_2} = c\left(\sum_{G \in \mathscr{H}_{\{B_1,B_2\}}} \mathscr{M}_G\right) + \sum_{B \in \mathscr{G}_B} c^B_{B_1,B_2} \mathscr{M}_B, \tag{5.3}$$

where c is a constant. (Note that $c = 1$ if B_1 and B_2 are non-isomorphic, and $c = 2$ if they are isomorphic). Thus,

$$\sum_{G \in \mathscr{H}_{\{B_1,B_2\}}} \mathscr{M}_G = \frac{1}{c}\left(\mathscr{M}_{B_1} \cdot \mathscr{M}_{B_2} - \sum_{B \in \mathscr{G}_B} c^B_{B_1,B_2} \mathscr{M}_B\right). \tag{5.4}$$

Therefore, by induction on the cardinality of S, we have the result. $\square$

Remark 5.3.7. Let H be a graph. For each multiset of blocks S, the sum $\sum_{G \in \mathscr{H}_S} \mathscr{M}_G(H)$ counts the number of subgraphs X of H with block structure S. Thus, Theorem 5.3.6 is equivalent to Whitney's Theorem.

Proposition 5.3.8. *The subalgebra* $\mathscr{B}$ *is a Hopf subalgebra of* $\mathscr{A}$.

Proof. Since the algebra $\mathscr{B}$ is generated by $\mathscr{M}_B$'s and the blocks are primitive elements, we can see that $\Delta(\mathscr{B}) \subseteq \mathscr{B} \otimes \mathscr{B}$ and thus $\mathscr{B}$ is a subcoalgebra . Now, we restrict the antipode S to $\mathscr{B}$ and we use Theorem 5.3.6. Since the blocks are primitive elements, $S(\mathscr{M}_B) = -\mathscr{M}_B$, for all $B \in \mathscr{G}_B$ and $S(\mathscr{M}_{B_1} \cdot \mathscr{M}_{B_2}) = S(\mathscr{M}_{B_2}) \cdot S(\mathscr{M}_{B_1})$ (a property of the antipode), we can conclude that S maps $\mathscr{B}$ into $\mathscr{B}$. $\square$

Example 5.3.9. Let B_1 and B_2 be blocks.

$$\begin{aligned} S(\sum_{G \in \mathscr{H}_{\{B_1, B_2\}}} \mathscr{M}_G) &= S(\frac{1}{c}(\mathscr{M}_{B_1} \cdot \mathscr{M}_{B_2} - \sum_{B \text{ is a block}} c^B_{B_1,B_2} \mathscr{M}_B)) \\ &= \frac{1}{c}(S(\mathscr{M}_{B_2}) \cdot S(\mathscr{M}_{B_1}) - \sum_{B \text{ is a block}} c^B_{B_1,B_2} S(\mathscr{M}_B)) \\ &= \frac{1}{c}(\mathscr{M}_{B_2} \cdot \mathscr{M}_{B_1} + \sum_{B \text{ is a block}} c^B_{B_1,B_2} \mathscr{M}_B). \end{aligned}$$

5.4 Conclusions and outlook

Theorem 5.3.6 is not a new result but it is an interesting algebraic proof of Whitney's theorem. Our proof is much simpler that a Hopf algebra proof of the same result given by Schmitt [27]. Further, we observe that Equation 5.4 and the Equation 7 in Schmitt's paper are very similar to Equation 2.1.

Chapter 6

Refining Kelly's reconstruction lemma for counting rooted subgraphs

Kelly's lemma (Lemma 6.1.1) is a basic result on graph reconstruction. It states that given the deck of a graph G on n vertices, and a graph F on fewer than n vertices, we can count the number of subgraphs of G that are isomorphic to F. Moreover, for a given card $G - v$ in the deck, we can count the number of subgraphs of G that are isomorphic to F and that contain v. We consider the problem of refining the lemma to count rooted subgraphs such that the root vertex coincides the deleted vertex. We show that such counting is not possible in general, but a multiset of rooted subgraphs of a fixed height k can be counted if G has radius more than k.

6.1 Vertex reconstruction and counting rooted subgraph

We recall Kelly's Lemma. For graphs F and G, and a vertex v, we denote by $\mathrm{s}(F, G^v)$ the number of subgraphs of G that contain v and are isomorphic to F. Analogously, we denote by $\mathrm{p}(F, G^v)$ the number of induced subgraphs of G that contain v and are isomorphic to F.

Lemma 6.1.1. *(Kelly's Lemma) If a graph F is such that $v(F) < v(G)$, then*

i) $\mathrm{s}(F, G)$ *and* $\mathrm{p}(F, G)$ *are reconstructible;*

ii) $\mathrm{s}(F, G^v)$ *and* $\mathrm{p}(F, G^v)$ *are reconstructible.*

Observe that part (ii) of Kelly's lemma is not sufficient to determine the contributions to $\mathrm{s}(F, G^v)$ from different *configurations* in which copies of F may appear at v. For example, if F is the 4-vertex path P_4, then we do not immediately know how many subgraphs isomorphic to P_4 have their end-vertex at v and how many have their internal vertex at v. We make this observation precise.

First we define isomorphism of rooted graphs. Let F_1^x and F_2^y be graphs F_1 and F_2, considered to be rooted at x and y, respectively. We say that they are *isomorphic* if there are isomorphisms $f\colon V(F_1) \to V(F_2)$ and $g\colon E(F_1) \to E(F_2)$ such that the incidence function of F_1 associates $e \in E(F_1)$ with vertices u and v of F_1 if and only if the incidence function of F_2 associates $g(e) \in E(F_2)$ with vertices $f(u)$ and $f(v)$ of F_2 and $f(x) = y$. For a graph F rooted at x, we denote by $\mathrm{s}(F^x, G^v)$ the number of rooted subgraphs of G^v that are isomorphic to F^x such that the root x of the subgraph coincides with v, and by $\mathrm{p}(F^x, G^v)$ the number of induced rooted subgraphs of G^v that are isomorphic to F^x such that the root x of the subgraph coincides with v.

We say that two vertices u and v in a graph F are *similar*, written as $u \approx v$, if there exists $g \in \mathrm{Aut}(F)$ such that $v = gu$. Note that $\approx$ is an equivalence relation on $V(F)$.

Lemma 6.1.2. *Suppose that $V_i, \cdots, V_s$ are the equivalence classes of $V(F)$ under the relation of similarity. That is, they are the orbits of the action of* $\mathrm{Aut}(F)$ *on $V(F)$. Let $U \subseteq V(F)$ consist of one representative vertex from each V_i. Then we have*

$$\mathrm{s}(F, G^v) = \sum_{u \in U} \mathrm{s}(F^u, G^v) \tag{6.1}$$

and

$$\mathrm{p}(F, G^v) = \sum_{u \in U} \mathrm{p}(F^u, G^v). \tag{6.2}$$

Ideally, given the deck of a graph G, we would like to count the individual terms in the above summation, which would significantly refine Kelly's lemma. We show that this is not possible in general. An example is provided by graphs with pseudo-similar vertices. Two vertices u and v in a graph G are *pseudo-similar* if the vertex-deleted subgraphs $G - v$ and $G - u$ are isomorphic but u and v are not similar.

Example 6.1.3. Figure 6.1 shows a graph G with pseudo-similar vertices u and v; this example is from [7].

Now let F be the graph . Observe that the vertex set of F partitions in three orbits, namely, the vertex x of degree 1, the two vertices of degree 2 (which are similar, so let one of them be y), and the vertex z of degree 3. Now we have

$$\begin{aligned}
\mathrm{s}(F^x, G^v) = 0 \text{ and } \mathrm{s}(F^x, G^u) = 1 \\
\mathrm{s}(F^y, G^v) = 1 \text{ and } \mathrm{s}(F^y, G^u) = 0 \\
\mathrm{s}(F^z, G^v) = 1 \text{ and } \mathrm{s}(F^z, G^u) = 1
\end{aligned}$$

These observations are consistent with our calculations in Section 6.3 that show that $\mathrm{s}(F^x, G^w) + \mathrm{s}(F^y, G^w)$ is reconstructible for each $w \in V(G)$. Similarly, if $F = P_4$ and x is a vertex of degree 1 in F, then $\mathrm{s}(F^x, G^v) = 3$ and $\mathrm{s}(F^x, G^u) = 3$, but $\mathrm{p}(F^x, G^v) = 2$ and $\mathrm{p}(F^x, G^u) = 1$.

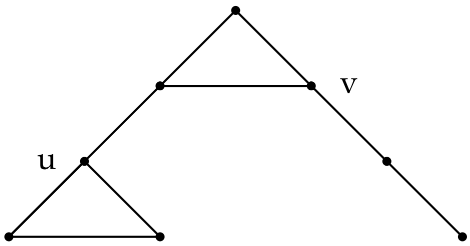


Figure 6.1: An example from [7] of a graph G with pseudo-similar vertices u and v.

The limitation described here is similar to the degree-pair and degree-pair sequence reconstruction for digraphs. We know that in general the degree-pair of a deleted vertex cannot be reconstructed, but the degree-pair sequence of the graph can be reconstructed. See Manvel [20] and Stockmeyer [29]. So we ask if the multiset $\{G_k^v, v \in V(G)\}$ could be reconstructed, where G_k^v is the subgraph of G rooted at v induced by the vertices at distance at most k from v. Here, we partially answer this question. We will prove a similar result for edge reconstruction in Section 6.2.

Lemma 6.1.4. *Let $x \in V(G)$. Define* $\mathrm{Orbit}_G(x)$ *the set of vertices of G that are similar to x. We denote by* $\mathrm{s}(F^x, G)$ *the number of rooted subgraphs of G that are isomorphic to F^x. We have*

$$\mathrm{s}(F^x, G) = |\mathrm{Orbit}_F(x)|\, \mathrm{s}(F, G).$$

The *distance* between two vertices is the number of edges in a shortest path between the two vertices. The minimum among all the maximum distances between a vertex and all other vertices is called the *radius* of the graph G and we denote by $r(G)$.

Let G be a connected graph. Let $v \in V(G)$. Let G_k^v denote the subgraph of G rooted at v, induced by vertices at distance at most k from v. Let $S_k(G)$ denote the multiset $\{G_k^v, v \in V(G)\}$. It is known that if $v(G) > 2$, then the degree sequence of G is reconstructible; also for each card $G - v$ in the deck, the degree of v and the neighbourhood degree sequence of v are reconstructible. But we do not know if $S_1(G)$ is reconstructible, and for a given card $G - v$, we cannot in general reconstruct G_1^v, G_2^v as shown by the examples of graphs containing pseudo-similar vertices. The following proposition partially answers the question of constructing $S_k(G)$.

Proposition 6.1.5. *If G is a connected graph with radius more that k, then $S_k(G)$ is reconstructible.*

Proof. If the radius $r(G)$ of G is more than k then for all $v \in V(G)$, the graph G_k^v has fewer than $v(G)$ vertices. Hence we claim that $S_k(G)$ is a subset of $\bigcup_{v \in V(G)} S_k(G - v)$ (taken as a multiset union), and the latter is reconstructible. Let S be the set of distinct rooted graphs in $\bigcup_{v \in V(G)} S_k(G - v)$. In the following, we determine which members of S are in $S_k(G)$ along with their multiplicities.

Let $A^u \in S$. Let $n(A^u)$ be the number of vertices $v \in V(G)$ such that $G_k^v \cong A^u$. We want to prove that $n(A^u)$ is reconstructible.

We have

$$\mathrm{s}(A^u, G) = \sum_{B^w \in S} \mathrm{s}(A^u, B^w) n(B^w). \tag{6.3}$$

Since $\mathrm{s}(A^u, G) = |\mathrm{Orbit}_A(u)|\, \mathrm{s}(A, G)$ and $v(A) < v(G)$, we can reconstruct $\mathrm{s}(A^u, G)$. If A^u is a maximal element in S (i.e., $\mathrm{s}(A^u, B^w) = 0$ for all $B^w \ncong A^u$), then $\mathrm{s}(A^u, G) = n(A^u)$. Now we order graphs in S as $A_1^{u_1}, A_2^{u_2}, \ldots$ so that $|E(A_i)| \geq |E(A_j)|$ for $i < j$. We can then solve Equation 6.3 recursively for each $A_i^{u_i}$ in the order $i = 1, 2, \ldots$. Thus, we can reconstruct $S_k(G)$. $\square$

We do not know if radius or diameter are in general reconstructible parameters. But the result can be applied to bounded degree graphs, graphs containing a vertex of degree 1, and possibly some other classes of graphs for which estimates for the radius can be made from the deck.

Example 6.1.6. Let G be the graph in Figure 6.1. The rooted graphs G_2^u and G_2^v are not isomorphic. We reconstruct the multiset $S_2(G)$ by the Equation 6.3 counting $n(G_2^w)$ where $w \in V(G)$ (see Section 6.3).

Now, let F be the graph  with a root vertex y of degree 2. We have seen that the parameter $\mathrm{s}(F^y, G^v)$ is not reconstructible. But, since $S_2(G)$ is reconstructible, the multiset $\{\mathrm{s}(F^y, G^v) \mid v \in V(G)\}$ is reconstructible.

6.2 Edge reconstruction and counting edge-rooted subgraphs

In this section we show a result of Proposition 6.1.5 for edge reconstruction.

Lemma 6.2.1. *(Kelly's Lemma- edge version) If a graph F is such that $e(F) < e(G)$, then* $\mathrm{s}(F, G)$ *and* $\mathrm{p}(F, G)$ *are edge-reconstructible.*

Let G be a graph. Let e_1 and e_2 be two edges in G. We define the *distance* between e_1 and e_2 to be the number of edges on a minimal path that contains e_1 and e_2. If e_1 and e_2 are in different components, then we define the distance between them to be infinity.

Let $e \in E(G)$. Let G_k^e denote the subgraph of G rooted at e (i.e., with a distinguished edge e) induced by edges at distance at most k from e. Let $T_k(G)$ denote the multiset $\{G_k^e, e \in E(G)\}$.

Edges a and b of a graph G are *similar* if there exists an automorphism of G that maps the ends vertices of a to the ends vertices of b, and are *pseudo-similar* if $G - a \cong G - b$, but a is not similar to b in G.

Example 6.2.2. Let G be the graph in Figure 6.2. The edges a and b are pseudo-similar. Let G_4^a and G_4^b be two elements in $T_4(G)$, we have $G_4^a \ncong G_4^b$.

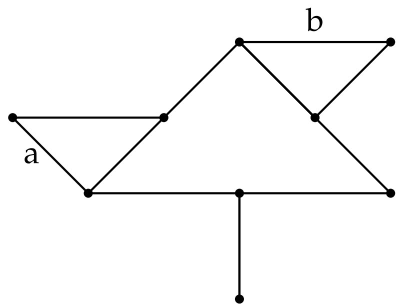


Figure 6.2: An example of a pair of pseudo-similar edges a and b in a graph from Poirier [26].

Proposition 6.2.3. *If G is a connected graph with radius more that $k \geq 1$, then $T_k(G)$ is edge-reconstructible.*

Proof. Let $e \in E(G)$, and let v be an end vertex of e. The distance between v and any other vertex in G_k^e is at most k. Thus, if the radius of G is more than k then every element in $T_k(G)$ has fewer than $v(G)$ vertices. Since G is connected, we have $e(G_k^e) < e(G)$. Hence, $T_k(G)$ is a subset of $\bigcup_{e \in E(G)} T_k(G - e)$. Let T be the set of distinct elements in $\bigcup_{e \in E(G)} T_k(G-e)$. Now, we will determine which members of T are in $T_k(G)$ along with their multiplicities.

Let $A^e \in T$. Let $\mathrm{s}(A^e, G)$ denote the number of edge-rooted subgraphs of G that are isomorphic to A^e. We have $\mathrm{s}(A^e, G) = |\mathrm{Orbit}_A(e)|\, \mathrm{s}(A, G)$, where $\mathrm{Orbit}_A(e)$ is the set of edges of A that are similar to e. Let $m(A^e)$ be the number of edges $f \in E(G)$ such that $G_k^f \cong A^e$. We have

$$\mathrm{s}(A^e, G) = \sum_{B^f \in T} \mathrm{s}(A^e, B^f) m(B^f). \tag{6.4}$$

The term on the left-hand side of Equation (6.4) is edge reconstructible. If A^e is a maximal element in T, then $\mathrm{s}(A^e, G) = m(A^e)$. We order graphs in T as $A_1^{e_1}, A_2^{e_2}, \ldots$ so that $e(A_i) \geq e(A_j)$ for $i < j$. We can then solve Equation 6.4 recursively for each $A_i^{e_i}$ in the order $i = 1, 2, \ldots$. □

Remark 6.2.4. Given $e \in E(G)$, we have $r(G-e) \geq r(G)$. We say that G is *radius-minimal* if for every edge e of G we have $r(G-e) > r(G)$. A connected graph G is radius-minimal if and only if G is a tree (see Walikar [37]). Thus, $r(G) = \min_{e \in E(G)} r(G-e)$ when G is not a tree. Hence, $r(G)$ is edge reconstructible.

6.3 Calculations for rooted graphs with a small number of vertices

The following calculations for small graphs illustrate the difficulties in counting the number of (induced) rooted subgraphs.

Let F^x be a graph with a root vertex x. Let G^v be a graph with a root vertex v. If $F^x \in \{ , , , \}$ (where the root vertex is marked by a bold dot), then the parameters $\mathrm{p}(F^x, G^v)$ and $\mathrm{s}(F^x, G^v)$ are reconstructible (since the underlying unrooted graph F is vertex transitive). We obtain the following equations for rooted graphs with a small number of vertices. We denote $d_v(G)$ by the degree of v in G.

$$\mathrm{p}(\ , G^v) = \mathrm{p}(\ , G^v) = d_v(G) \tag{6.5}$$

$$\mathrm{p}(\ , G^v) + \mathrm{p}(\ , G^v) = \mathrm{s}(\ , G^v) = \binom{d_v(G)}{2} \tag{6.6}$$

$$\mathrm{s}(\ , G^v) = \binom{d_v(G)}{3} = \mathrm{p}(\ , G^v) + \mathrm{p}(\ , G^v) + \mathrm{p}(\ , G^v) + \mathrm{p}(\ , G^v) \tag{6.7}$$

$$\mathrm{s}(\;, G^v) = \mathrm{p}(\;, G^v)(d_v(G) - 2) = \mathrm{p}(\;, G^v) + 2\,\mathrm{p}(\;, G^v) + 3\,\mathrm{p}(\;, G^v) \tag{6.8}$$

$$\mathrm{s}(\;, G^v) = \mathrm{p}(\;, G^v) + \mathrm{p}(\;, G^v) + \mathrm{p}(\;, G^v) + \mathrm{p}(\;, G^v) \tag{6.9}$$

$$\mathrm{s}(\;, G^v) = \mathrm{p}(\;, G^v) + \mathrm{p}(\;, G^v) + \mathrm{p}(\;, G^v) + \mathrm{p}(\;, G^v) + 2\,\mathrm{p}(\;, G^v) + 3\,\mathrm{p}(\;, G^v) \tag{6.10}$$

$$\begin{aligned} \mathrm{s}(\;, G^v) &= \mathrm{s}(\;, G^v)(d_v(G) - 1) - 2\,\mathrm{p}(\;, G^v) \\ &= \mathrm{p}(\;, G^v) + 2\,\mathrm{p}(\;, G^v) + \mathrm{p}(\;, G^v) + 2\,\mathrm{p}(\;, G^v) \\ &\quad + 4\,\mathrm{p}(\;, G^v) + 6\,\mathrm{p}(\;, G^v) + 2\,\mathrm{p}(\;, G^v) \end{aligned} \tag{6.11}$$

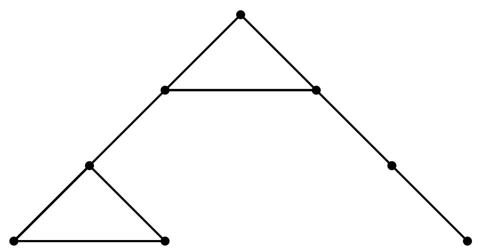
$$\begin{aligned} \mathrm{s}(\;, G^v) &= \mathrm{p}(\;, G^v) + 2\,\mathrm{p}(\;, G^v) + 2\,\mathrm{p}(\;, G^v) \\ &\quad + \mathrm{p}(\;, G^v) + 2\,\mathrm{p}(\;, G^v) + 4\,\mathrm{p}(\;, G^v) + 6\,\mathrm{p}(\;, G^v) \end{aligned} \tag{6.12}$$

By Equation (6.6), $\mathrm{p}(\;, G^v)$ and $\mathrm{s}(\;, G^v)$ are reconstructible, hence by Equation (6.1), $\mathrm{p}(\;, G^v)$ and $\mathrm{s}(\;, G^v)$ are reconstructible. Thus, for all F^x with $v(F) \leq 3$ and G^v with $v(G) \geq 4$, we can calculate $\mathrm{p}(F^x, G^v)$ and $\mathrm{s}(F^x, G^v)$ from the deck of G.

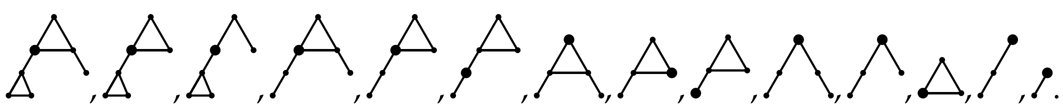
For rooted graphs with four vertices, we have already shown an example where the parameters $\mathrm{p}(\;, G^v)$ and $\mathrm{p}(\;, G^v)$ are not reconstructible. By Equation (6.8), we can calculate $\mathrm{s}(\;, G^v)$, thus $\mathrm{s}(\;, G^v) + \mathrm{s}(\;, G^v)$ is reconstructible. But we do not know how to calculate $\mathrm{p}(\;, G^v)$. Also, by Equations (6.12) and (6.1), $\mathrm{s}(\;, G^v)$ and $\mathrm{s}(\;, G^v)$ are reconstructible. But, we have already shown that $\mathrm{p}(\;, G^v)$ is not reconstructible.

The next example illustrates Proposition 6.1.5.

Example 6.3.1. Consider the following graph G:



We calculate $S_2(G)$ using Proposition 6.1.5. We have $S = \{A_1^{u_1}, A_2^{u_2}, \ldots, A_{14}^{u_{14}}\}$, where the elements are represented by the following rooted graphs, respectively,



Now, we use recursively Equation 6.3.

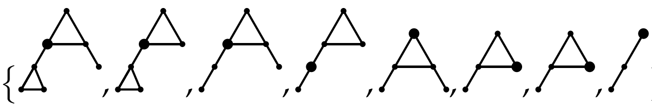
$$n(A_1^{u_1}) = \mathrm{s}(A_1^{u_1}, G) = 1$$

$$n(A_2^{u_2}) = \mathrm{s}(A_2^{u_2}, G) - \mathrm{s}(A_2^{u_2}, A_1^{u_1})n(A_1^{u_1}) = 2 - 1 \cdot 1 = 1$$

$$n(A_3^{u_3}) = \mathrm{s}(A_3^{u_3}, G) - \mathrm{s}(A_3^{u_3}, A_1^{u_1})n(A_1^{u_1}) - \mathrm{s}(A_3^{u_3}, A_2^{u_2})n(A_2^{u_2}) = 5 - 3 \cdot 1 - 2 \cdot 1 = 0.$$

$$\vdots$$

$$n(A_{14}^{u_{14}}) = \mathrm{s}(A_{14}^{u_{14}}, G) - \sum_{i=1}^{13} \mathrm{s}(A_{14}^{u_{14}}, A_i^{u_i})n(A_i^{u_i}) = 0.$$

Hence, $S_2(G) = \{$  $\}$.

6.4 Conclusions and outlook

We formulate a few open problems related to a further refinement of Kelly's lemma and the local structure of graphs.

Let $f\colon V(G) \to V(H)$ be a hypomorphism from G to H.

1. Define an equivalence relation $\sim$ on $V(G)$ by $u \sim v$ if $G - u \cong G - v$. Let $V_i, i = 1, 2, \ldots$ be the equivalence classes of $\sim$. Suppose that G has radius more than k. Show that for each equivalence class V_i, the set $\{G_k^v \mid v \in V_i\}$ is vertex-reconstructible. Equivalently, show that if G and H are of radius more than k, then there exists a hypomorphism $g\colon V(G) \to V(H)$ such that $G_k^v \cong H_k^{g(v)}$.

   Similar problem may be formulated for edge reconstruction by defining an equivalence relation on the set of edges.

   Analogously, in the case of directed graphs, either show that the degree pair sequence restricted to each equivalence class of vertices is reconstructible, or find examples of non-reconstructible graphs (e.g., in the families of tournaments constructed by Stockmeyer and others) that show that it is not possible.

2. Let $\mathscr{C}$ be the class of graphs with no pseudo-similar vertices. Let G and H be hypomorphic graphs in $\mathscr{C}$, of radius more than k, with a hypomorphism $f\colon V(G) \to V(H)$. Show that $G_k^v \cong H_k^{f(v)}$. We are not aware if existence of pseudo-similar vertices can be recognised from the deck.

3. Let G and H be hypomorphic graphs. Prove that there exists a hypomorphism $g\colon V(G) \to V(H)$ such that vertices u and v in G are similar if and only if vertices $g(u)$ and $g(v)$ in H are similar.

4. Let $\mathscr{F}_k$ be the family of distinct rooted k-vertex induced subgraphs of G. The family $\mathscr{F}_k$ is clearly reconstructible from the deck when $k < n$. Define the $\mathscr{F}_k$-degree of a vertex v to be the multiset of induced k-vertex subgraphs rooted at v.

   (a) Similar to Proposition 6.1.5: Is the $\mathscr{F}_k$-degree sequence of G reconstructible for some $k < n$?

   (b) Similar to Problem 1: Is the $\mathscr{F}_k$-degree sequence of G restricted to each V_i reconstructible?

5. Manvel has shown that the degree pair sequence of any digraph with five or more points is reconstructible. Can Manvel's reconstruction of degree-pair sequence be extended to multisets of rooted (not necessarily induced) subgraphs of height k for some k, independent of the radius? Manvel's construction works for tournaments, hence amounts to reconstructing certain spanning subgraphs. Which other spanning subgraphs can be reconstructed?

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