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The Languages of Spacetime

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The Languages of Spacetime

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Orientador: Nelson de Oliveira Yokomizo

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ATA DE DEFESA DE DISSERTAÇÃO

ATA DA SESSÃO DE ARGUIÇÃO DA 688ª DISSERTAÇÃO DO PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA, DEFENDIDA POR FELIPE BRANDÃO GUISOLI, orientado pelo professor Nelson de Oliveira Yokomizo, para obtenção do grau de **MESTRE EM FÍSICA**. Às 14 horas de vinte de dezembro de 2022, reuniu-se a Comissão Examinadora, composta pelos professores **Nelson de Oliveira Yokomizo** (Orientador - Departamento de Física/UFMG), **Mario Sergio Carvalho Mazzoni** (Departamento de Física/UFMG) e **Gláuber Carvalho Dorsch** (Departamento de Física/UFMG), para dar cumprimento ao Artigo 37 do Regimento Geral da UFMG, submetendo o bacharel **FELIPE BRANDÃO GUISOLI** à arguição de seu trabalho de dissertação, que recebeu o título de **"The Languages of SpaceTime".** O candidato fez uma exposição oral de seu trabalho durante aproximadamente 50 minutos. Após esta, os membros da comissão prosseguiram com a sua arguição e apresentaram seus pareceres individuais sobre o trabalho, concluindo pela aprovação do candidato.

Belo Horizonte, 20 de dezembro de 2022.

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Todas as coisas cujos valores podem ser disputado no cuspe à distância servem para a poesia.

As coisas que não levam a nada têm grande importância.

(Manoel de Barros)

Resumo

A teoria da relatividade geral surge em 1915, e seu palco matemático é uma variedade quadridimensional Lorentziana. Nosso objetivo será explorar diferentes linguagens e formulações da teoria, com diferentes parâmetros atuando como variáveis dinâmicas. Iniciaremos com a formulação original desenvolvida por Einstein e passaremos então para a formulação lagrangeana da teoria, desenvolvida primeiramente por Hilbert. Desenvolveremos então duas formulações hamiltonianas da gravitação, baseadas em uma folheação (3+1) do espaço-tempo. A primeira será feita com o uso da métrica tridimensional como variável dinâmica. Tal formalismo é conhecido como formalismo ADM da relatividade geral e possibilita a construção de uma Hamiltoniana para a teoria em termos de vínculos e multiplicadores de Lagrange. Por fim, analisaremos a formulação hamiltoniana baseada na ação de Holst, em termos das variáveis de Ashtekar. Tal formulação é um dos caminhos possíveis para a quantização do campo gravitacional, na abordagem conhecida como gravitação de laços.

Palavras-chave: Relatividade Geral, Gravitação, Formalismo Hamiltoniano, Variáveis de Ashtekar, Gravitação Quântica de Laços

Abstract

The theory of general relativity emerges in 1915, and its mathematical stage is a four-dimensional Lorentzian manifold. Our goal will be to explore different languages and formulations of the theory, with different quantities playing the role of dynamical variables. We will start with the original formulation of the theory developed by Einstein, and then pass to the Lagrangian formulation, first developed by Hilbert. Then we will develop two Hamiltonian formulations of gravity, based on a (3+1) foliation of spacetime. The first will be done with the three-dimensional spatial metric as dynamical variable. Such a formalism is known as the ADM formalism of general relativity and allows for the construction of a Hamiltonian for the theory in terms of constraints and Lagrange multipliers. Finally, we will analyze the Hamiltonian formalism based on the Holst action, in terms of Ashtekar variables. This formulation provides a possible path for the canonical quantization of the gravitational field, in the approach known as loop quantum gravity.

Keywords: General Relativity, Gravitation, Hamiltonian Formalism, Ashtekar Variables, Loop Quantum Gravity

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Chapter 1

Introduction

Good physics is done *a priori*. Theory precedes the fact. The experience is useless because, before any experience, we already had the knowledge that we seek.

A. Koyré

Einstein's General theory of Relativity allowed us to explore the history of the universe in ways never done before. Just as Galileo did in the seventeenth century, Einstein had developed a new concept of space and time in his head, based on the equivalence principle and a lot of creativity. Subsequent experiments, such as the 1919 measurement of light deflection in Sobral and Principe, or the recent 2015 detection of black hole collision by the Laser Interferometer Gravitational-Wave Observatory (LIGO) showed the power, precision and revolutionary aspects of this new theory.

General Relativity (GR) considers a 4-dimensional manifold — the spacetime — as the center object of study. The geometrical properties of this manifold is given by the metric $g_{\mu\nu}$, and the evolution of the metric is how we study the history of spacetime in this paradigm.

One of the first results of the theory was the correct measurement of the precession of Mercury's perihelion, a fact known since Newton's theory of gravitation, but, since then, inexplicable using the same theory. Also, the measurement of the bending of light in gravitational fields in 1919 was a huge step towards the establishment of the theory and so were the measurements of the gravitational redshift done in 1925 by the american astronomer Walter Sidney Adams and by Popper in 1954 [12].

This is the first physical theory to use some very abstract concepts of mathematics, such as topology and differential geometry. The theory, built by Einstein using geometrical concepts and the Newtonian limit, or built by Hilbert using the Lagragian formalism, can be formulated in many different ways, using different objects playing dynamical roles. And the different languages that can be used to formulate the theory will allow us to explore different symmetries or concepts of spacetime.

So, for instance, we could see the high school relation $(a+b)^2 = a^2 + 2ab + b^2$ in a pure algebraic way, which may allow us to extract some information about algebra. On the other hand, one can choose to see this relation geometrically, as shown in figure 1.1, which will certainly enable him to see other aspects of the relation — the geometrical ones.

The best scenario would be to have the two points of view, or, if it is possible, all of them

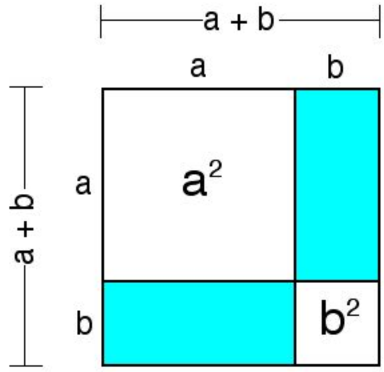

Figure 1.1: The square of the sum

that will lead us to build a broader view of the big picture.

This is our goal: to explore the different languages of spacetime. Therefore, in this work, we will explore some different ways to write the formalism of GR, with different variables playing the dynamical role. We will start with the first formalism developed by Einstein, then we will go to the Lagrangian formalism, done first by Hilbert, and finally we will end up in some different Hamiltonian formulations of the theory, one in terms of the metric, which is known as the ADM formalism, and another one in terms of some other variables — the Ashtekar variables — which is one possible path to quantize the theory of gravity.

The present work is organized as follows. In Chapter 2 we will set the mathematical background and conventions that will be used along the dissertation. The essence will be differential geometry, tensor calculus and the establishment of some notations. In Chapter 3 we will recover the main concepts of the formulation of GR done by Einstein and the main geometrical objects in this theory, like the metric, the Riemann curvature tensor and the connection. We will also discuss Einstein's equivalence principle, the base upon which GR lays. Furthermore, we are going to do here the Lagrangian formulation of GR, via Einstein-Hilbert action and also via the Palatini action, where we will let the connection play a dynamical role in the theory. In Chapter 4 we will review the formalism and structure of constrained Hamiltonian systems, and then we will apply it to the Hamiltonian formulation of GR — the ADM formalism. We will build this formalism in detail, via the foliation of spacetime, where it will be split in spatial slices evolving through time. This (3+1) split will allow us to tell the history of spacetime as the time evolution of these spatial slices. With these, we will be able to build an action for gravity in therms of a 3-metric h_{ab} in those slices — this will be the main dynamical variable of our formalism. We will end up with a constrained Hamiltonian system, and the symmetries of spacetime will be expressed as constraints in the Hamiltonian.

In chapter 5 we will construct the tetrad formalism, where we will trade the metric $g_{\mu\nu}$ for the local orthonormal frame e^I_μ as the dynamical variable. We will build all the formalism in terms of these new variables using Cartan's structural equations of differential geometry, ending with the Holst action in forms notation, which will be used for the next formalism. Finally, in chapter 6 we are going to mix up the two previous formalisms to build the Ashtekar formulation of GR, which is essentially the construction of the Hamiltonian formalism using triads as the dynamical variables in place of the metric. We will do the same (3+1) split, where the spatial part of the tetrad e^I_μ will be ε^I_μ — the triad, our main dynamical variable. This formalism, developed by Ashtekar in the mid 1980s, consists in rewriting the theory of GR in terms of some variables that made the theory resemble the theories of particle physics, which allowed the importation of techniques from particle physics to the quantization of gravity. This approach is known as *loop quantum gravity*.

Finally, we will introduce present our conclusions and discuss future developments in chapter 7.

Chapter 2

Manifolds, Topology and Differential Geometry

> Let no one ignorant of geometry enter.
>
> *Plato's Academy*

2.1 Manifolds

2.1.1 Introduction

The arena of differential geometry is a differentiable (or smooth) manifold. The kind of manifolds we are going to deal here will be spaces that locally look as $\mathbb{R}^n$, but not globally. Just as the Earth can be locally flat in a first approach where, locally, it looks just like Euclidean space.

One important thing is the fact that the differential geometry formalism will allow us to define objects in manifolds in a coordinate-independent manner, which is something that we seek, since the general laws of nature are to be expressed by equations that hold good for all systems of coordinates, i.e., in a generally covariant way. That is what general relativity is about: the generalization of the Galileo's principle of relativity for all coordinate systems, not only the inertial ones.

2.1.2 Topology

Definition of topological space

A topological space is a set X, together with its subsets (called the open sets), which satisfy:

- The empty set and X itself are open.
- If $U, V \in X$ are open, so is $U \cap V$.
- If the sets U_a are open, so is the union $\cup U_a$.

The collection of open sets is called the topology of X.

Continuous function and homeomorphism

The notion of continuos functions is allowed by the use of a topology. Suppose we have the two topological spaces X and Y, and the function $f : X \to Y$. This function is said to be continuous if, for any open set $U \in Y$, its inverse image $f^{-1}(U) \in X$ is also an open set in X, as is shown in figure 2.1.

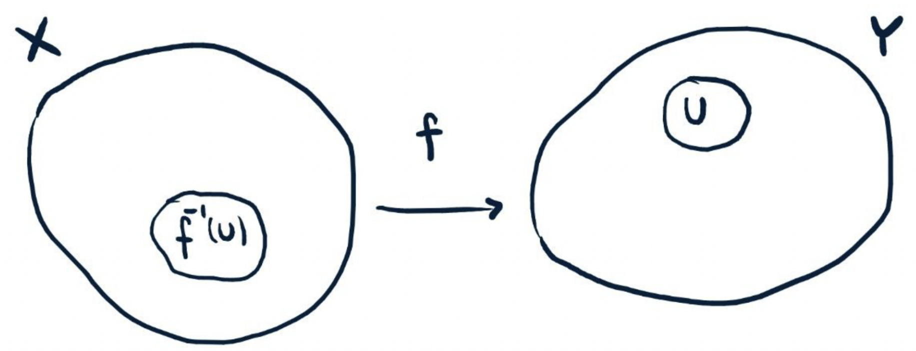

Figure 2.1: Continuous function from X to Y.

If the map f is a continuous and bijective between two topological spaces, whose invese is also continuous, then f is called a homeomorphism.

2.1.3 The manifold

Our idea here is to cover a space with patches that are locally just as $\mathbb{R}^n$.

We say that a collection of open sets U_a covers a topological space X if their union is all of X.

For an open set $U \in X$ we define a **chart** to be a continuous function $\varphi : U \to \mathbb{R}^n$ with a continuous inverse (where this inverse has its domain in $\varphi(U) \in \mathbb{R}^n$, just as figure 2.2 shows.)

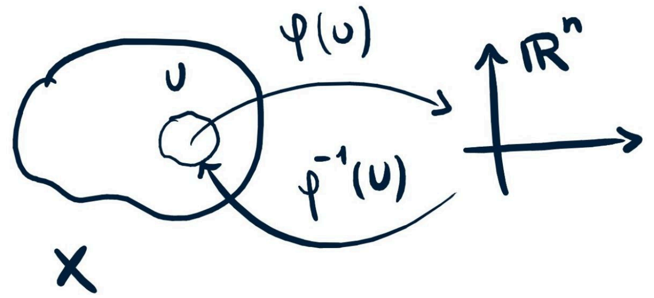

Figure 2.2: Charts.

The idea is that, as long as we work in the chart φ we can pretend we are in $\mathbb{R}^n$, just as the Earth looks perfectly flat if we do not go too far. Suppose, for example, we have a function $f : U \to \mathbb{R}$. We can turn it into a function from $\mathbb{R}^n \to \mathbb{R}$ using $f \circ \varphi^{-1}$, as figure 2.3 shows.

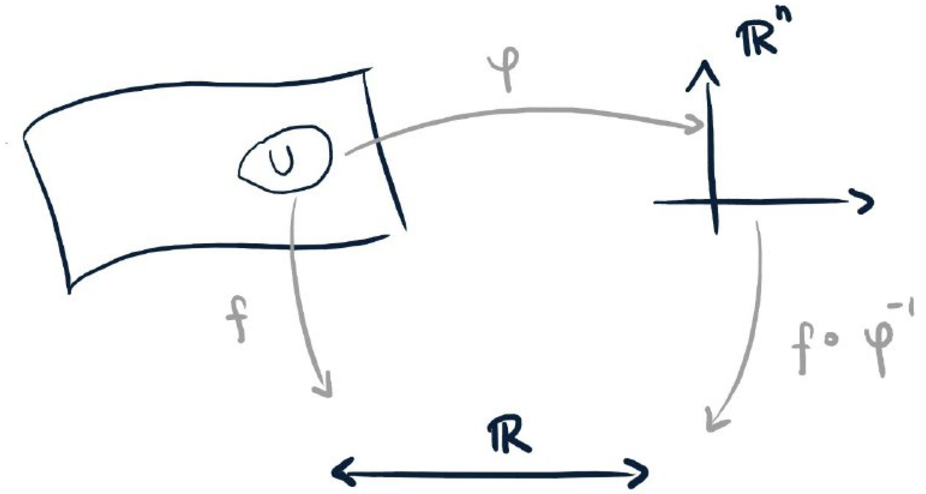

Figure 2.3: Turning functions in U in functions in $\mathbb{R}^n$.

Definition

A topological n-dimensional manifold M is a topological space such that every point has a neighbourhood U homeomorphic to an open subset in $\mathbb{R}^n$.

The manifold M is differentiable if the transition function $\varphi_\alpha^{-1} \circ \varphi_\beta$ is smooth where it is defined.

Some topological manifolds — the differentiable ones — M can be represented as a union of finite set of coordinate charts U, and the set of coordinate charts ψ_u that cover M is called an *atlas* on M.

The idea is that every point in a differentiable manifold lives in some open subset U_α that looks like $\mathbb{R}^n$, and that we can tell if any function on the manifold is smooth by looking at transition functions between charts. If there is a function $f : M \to \mathbb{R}$ and one uses a chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, then we say that f is smooth if

$$f \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$$

is smooth.

But one could instead use a chart $\varphi_\beta : U_\beta \to \mathbb{R}^n$. In this case, consider $V = U_\alpha \cup U_\beta$ the overlap of the two charts, the grey area represented in figure 2.4. The representation of f in this chart is

$$f \circ \varphi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}.$$

This function should also be smooth, for the smoothness of a function does not depend on the chart we use.

But for that to be true, we need

$$\varphi_\alpha \circ \varphi_\beta^{-1}$$

to be smooth, since

$$f \circ \varphi_\beta^{-1} = \left(f \circ \varphi_\alpha^{-1}\right) \circ \left(\varphi_\alpha \circ \varphi_\beta^{-1}\right).$$

From now on, when we mention any manifold, we will always be referring to a *smooth manifold*, as defined above.

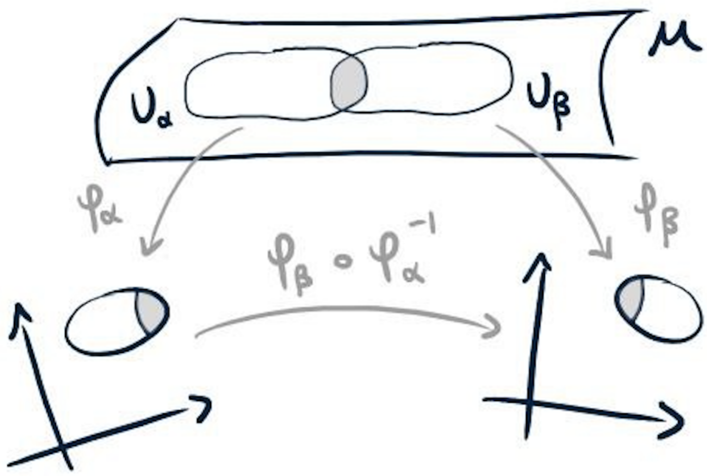

Figure 2.4: A manifold M.

Diffeomorphism

An isomorphism is a structure-preserving mapping between two structures of the same type that can be reversed by an inverse mapping. An homeomorphism, as previously defined, is an isomorphism of topological spaces. A diffeomorphism is a homeomorphism that preserves a differential structure.

Definition 2.1.1. Given two manifolds M and N, a **differentiable** map $f : M \to N$ is called a diffeomorphism if it is a bijection and if its inverse $f^{-1} : N \to M$ is also differentiable.

A map f from a manifold M to another manifold N can be built if one has the maps $g : M \mapsto \mathbb{R}$ and $h : N \mapsto \mathbb{R}$, as shown in figure 2.3, by the composition $h^{-1} \circ g$.

If there exists a diffeomorphism f between M and N we say that these two manifolds are diffeomorphic. We will consider the spacetime as a 4-dimensional differentiable manifold.

2.2 Vectors

2.2.1 Introduction

One can think of a vector field in a manifold as a field of arrows, tangent to the space in each point, as it is in $\mathbb{R}^n$. If we have a direction, we can differentiate a function f in that direction. The partial derivative of f in the direction of a vector v is, in $\mathbb{R}^n$:

$$vf = \mathbf{v} \cdot \nabla f = v^\mu \partial_\mu f,$$

where we are thinking of the vector v as something whose purpose is to get a function f and spit out another function, which is the partial derivative of f in the v direction, that's why we wrote it as vf (something like v is an operator acting on f).

If we look at the first and last member we have $vf = v^\mu \partial_\mu f$, which holds for every function f, so one may think that we can say that a vector field v can be written as

$$v = v^\mu \partial_\mu, \tag{2.1}$$

which says that the vector field v can be expanded in the basis ∂_μ.

Here, v is the vector field, while $v^\mu \partial_\mu$ is something that acts on a function and give its partial derivative. For us, a vector field on a manifold will be exactly that: entities whose main purpose is to differentiate functions.

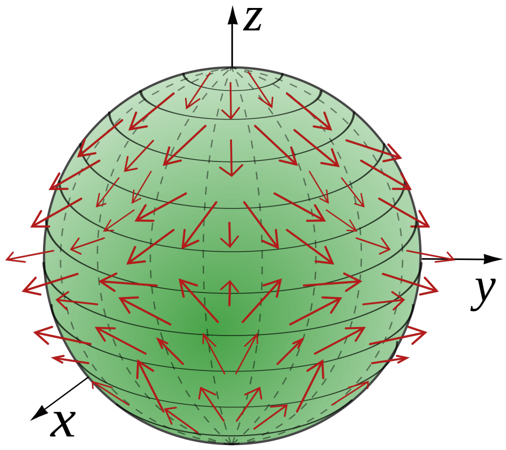

Figure 2.5: Vector Field.

Definition

A vector field v on a manifold M is a function from $C^\infty(M)$ to $C^\infty(M)$ satisfying:

- $v(f+g) = v(f) + v(g)$
- $v(\alpha f) = \alpha v(f)$
- $v(fg) = v(f)g + fv(g)$

for $\alpha \in \mathbb{R}$ and $f, g \in C^\infty(M)$. Here, $C^\infty(M)$ stands for the set of all complex functions infinitely differentiable in M, as usual.

So it is an object that acts linearly on functions and obeys the Leibniz rule. If we denote by $V(M)$ the set of all vector fields in a manifold M one can show that this is indeed a vector space, as expected.

2.2.2 Tangent Vectors

One can visualize a vector field v in a manifold M as assigning an arrow to each point P in the tangent space of the manifold. The tangent vector at each point P in M is the vector v_p living in the tangent plane at P, as showed in figure 2.6

We can differentiate the function f in the direction of the vector field v, represented by vf, and evaluate it in the point $p \in M$. We will call this $v_p f$ — the tangent vector in the point P.

So we have

$$v_p : C^\infty(M) \to \mathbb{R}\,, \qquad v_p(f) = v(f)(p)$$

where the last line means the partial derivative of f in the direction of the vector field v evaluated at point P.

It follows immediately that

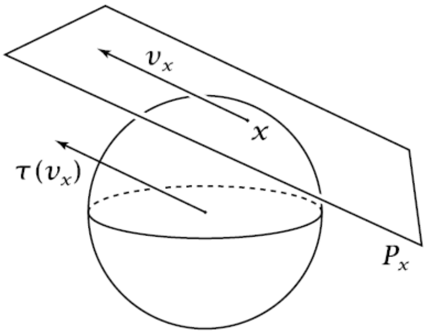

Figure 2.6: Tangent Space.

- $v_p(f+g) = v_p(f) + v_p(g)$
- $v_p(\alpha f) = \alpha v_p(f)$
- $v_p(fg) = v_p(f)g + f v_p(g)$

and we call the tangent vector v_p at point P the function $C^\infty \to \mathbb{R}$ that satisfies these 3 properties.

We call $T_p(M)$, the tangent space at P, the set of all tangent vectors at $p \in M$. The tangent space is indeed a vector space, with the sum of tangent vectors and the multiplication by a scalar defined in the natural way

- $(v_p + \omega_p)(f) = v_p(f) + \omega_p(f)$
- $(\alpha v_p)(f) = \alpha v_p(f)$

2.2.3 Lie Bracket

We define the Lie Bracket of two vectors v and w as

$$[v, w] = vw - wv, \tag{2.2}$$

which is just a short notation for

$$[v, w](f) = v(w(f)) - w(v(f)).$$

So, if v and w are vector fields, the Lie bracket $[v, w]$ is also a vector field, since its entry is a function $f \in C^\infty(M)$ and it spits out another function in $C^\infty(M)$.

For the basis vector ∂_μ and ∂_ν the Lie bracket is evidently zero, which follows from the commutation of partial derivatives:

$$\partial_\mu \partial_\nu = \partial_\nu \partial_\mu.$$

Geometrically this can be thought as flowing a little bit in the ∂_μ direction and then a little bit in the ∂_ν direction. If we invert the order we end up in the same place, at least in flat space.

For general vector fields this is not necessarily true, and the Lie bracket measures the difference between these who tracks, the failure of the two vector fields to commute, as shown in figure 2.7.

The Lie derivative $\mathcal{L}_w v$ of a vector v in the direction along w is defined as

$$\mathcal{L}_w v = [w, v], \tag{2.3}$$

which is the derivative of v along the flow [16] generated by w.

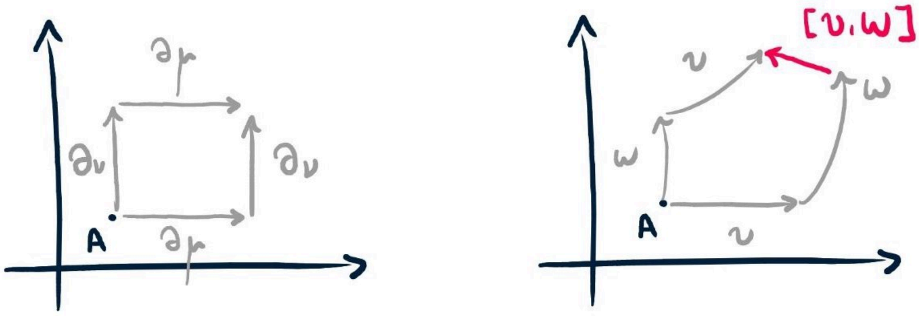

Figure 2.7: Lie Bracket.

2.3 Differential Forms

2.3.1 Introduction

The initial idea is to generalize the notion of the gradient of a function to functions on arbitrary manifolds.

For a function f on $\mathbb{R}^n$ we have its gradient expressed by ∇f. Here, we will define an operator d — the exterior derivative — and its action on a function f defined on an arbitrary manifold M will be expressed as df, and it will generalize the idea of the gradient in a first approach.

In $\mathbb{R}^n$, the directional derivative of f in the direction of the vector $\mathbf{v}$ is just the dot product between $\mathbf{v}$ and the gradient of f:

$$\nabla f \cdot v = vf, \tag{2.4}$$

where in the last step we have written the vector $\mathbf{v}$ in the vector basis ∂_μ:

$$v = v^\mu \partial_\mu,$$

hence

$$vf = v^\mu \partial_\mu f = v^\mu \frac{\partial f}{\partial x^\mu} = \nabla f \cdot v.$$

So, we are after an object df that keeps track of the derivative of f in all directions in the manifold M, just as the gradient does. In $\mathbb{R}^n$, the gradient of f is a vector field, and the directional derivative is calculated via a dot product, as shown before. But, taking dot products involves a choice of metric, and manifolds, in general, do not come pre-equipped with it. So, we will leave the choice of a metric to a further development. Hence it would be nice if, in a first approach, the df which will generalize the gradient was not a vector field, so that it would not be necessary to take a dot product in order to extract the directional derivative information.

We will call our df here a **1-form**, and it will have the same properties as the gradient does, so to speak, for each input vector v the operator $df \cdot v = vf$ spits out a scalar function, which is the directional derivative of f in the direction of v.

So, our df, when fed with a vector $v \in V(M)$ (the tangent vector space in a manifold M) will spit out a function $g \in C^\infty(M)$, and it will do it in a linear way, such as the gradient does:

$$\begin{aligned} df \cdot (v + u) &= df \cdot v + df \cdot u\,, \\ df \cdot (gv) &= g\,(df \cdot v)\ , \end{aligned} \tag{2.5}$$

for $g \in C^\infty(\mathbb{R}^n)$.

Definition 2.3.1. A **1-form** ω on a manifold M is a map from $V(M)$ to $C^\infty(M)$ that is linear over $C^\infty(M)$.

So, a **1-form** ω receives a vector v from $V(M)$ and spits out a function $\omega(v)$ such that:

$$\begin{aligned} \omega(v+u) &= \omega(v) + \omega(u)\,, \\ \omega(gv) &= g\omega(v)\,, \end{aligned} \tag{2.6}$$

where $g \in C^\infty(M)$. We represent the space of all 1-forms in a manifold M by $\Omega^1(M)$.

The exterior derivative as the generalization of the gradient

A simple example of a 1-form is, for any smooth function on M, the 1-form defined by:

$$df(v) = vf, \tag{2.7}$$

which is just a slick way to write the directional derivative, as observed in (2.4). We can see that this is really a 1-form by checking linearity:

$$df(v+u) = (v+u)f = vf + uf = df(v) + df(u),$$

and

$$df(gv) = (gv)f = g(vf) = gdf(v).$$

The 1-form df is called the differential of f, or the exterior derivative of f.

Composition

The addition of two 1-forms ω and μ and multiplication by a scalar (function) g is defined via

$$(\omega+\mu)(v) = \omega(v) + \mu(v) \tag{2.8}$$

and

$$(g\omega)(v) = g\omega(v)\,. \tag{2.9}$$

2.3.2 The Tangent and Cotanget spaces

Let us see what the exterior derivative is in any manifold, working in local coordinates. From equation (2.7) we can conclude that the 1-forms dx^μ form, at each point P, a local basis of 1-forms in $T_p^*(M)$ — the dual space of $T_p(M)$ — because, when we feed the 1-form dx^μ with a basis vector of the tangent space ∂_ν we get

$$dx^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu.$$

So, if ∂_μ is a basis of the tangent space on a manifold M and the action of the 1-forms dx^ν on that basis gives the Kronecker delta, then dx^ν is also a basis in the cotangent space. Therefore any 1-form $\omega \in \Omega^1(M)$ can be expanded and written in a unique form as

$$\omega = \omega_\mu dx^\mu, \tag{2.10}$$

with

$$\omega_\mu = \omega(\partial_\mu).$$

To see that this is the case, we just need to verify that the action of ω and $\omega_\mu dx^\mu$ on a vector v are the same:

1. $\omega(v) = \omega(v^\nu \partial_\nu) = v^\nu \omega_\nu$
2. $\omega_\mu dx^\mu(v) = \omega_\mu dx^\mu(v^\nu \partial_\nu) = v^\nu \omega_\mu(\partial_\nu dx^\mu) = v^\nu \omega_\mu(\delta^\mu_\nu) = \omega_\nu v^\nu$

which proves the statement.

One can then see the 1-forms as actually dual vectors. Just as a vector field v at M gives a tangent vector v_p at each point P of M, we can assign a cotangent vector ω_p at each point P of M. The space of all cotangent vectors at P, as mentioned before, is called $T^*_p M$. The cotangent vector ω at P is rigorously defined to be a linear map from the tangent space $T_p M$ to $\mathbb{R}$.

So, if we have a vector field v on M, we can define the cotangent vector field as

$$\omega(v) = \omega_\mu dx^\mu v^\mu \partial_\mu = \omega_\mu v^\mu, \tag{2.11}$$

which is indeed a map $\omega(v) : T_p M \mapsto \mathbb{R}$.

This really means that the 1-forms are the dual vectors of v. This is so since the dual vector space of V is the space V^* of all linear functionals $\omega : V \mapsto \mathbb{R}$. Hence, the cotangent space $T^*_p M$ is the dual vector space of $T_p M$.

It is important to note that, if we have a linear map f from one vector space V to another W

$$f : V \mapsto W,$$

we can automatically get a map f^*, the dual of f, from W^* to V^*

$$f^* : W^* \mapsto V^*,$$

that is defined by

$$(f^*\omega)(v) = \omega(f(v)). \tag{2.12}$$

For this we call the cotangent vectors covariant: linear maps between vector spaces gives rise to maps between their duals that go backwards. This is the convention used in [5], probably because this objects transforms with the same Jacobian matrix of the linear transformation itself, while tangent vectors transforms with its inverse, hence, they are called contravariant. We will develop more on this shortly.

So, if ϕ is a linear map between the tangent spaces at two different points P and Q in M

$$\phi : T_p M \mapsto T_q N,$$

the dual map goes the other way

$$\phi^* : T^*_q N \mapsto T^*_p M.$$

We call $\phi^*\omega$ the pullback of ω by ϕ.

In coordinates this means that the 1-forms, when we do a coordinate transformation, will transform with the inverse of the matrix that transform the coordinates of the vectors.

For instance, let the vector v be expressed in two different coordinate systems x^μ and x'^ν

$$v = v^\mu \partial_\mu = v'^\nu \partial'_\nu. \tag{2.13}$$

The object v is naturally the same, but its components v^μ or v'^ν are not, since they depend on the choice of basis ∂_μ or ∂'_ν where the components are written.

Since $\partial'_\nu = \dfrac{\partial x^\mu}{\partial x'^\nu}\partial_\mu$, then, in (2.13):

$$v'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} v^\mu, \tag{2.14}$$

and the components of the vector transforms with the inverse of the Jacobian matrix of the change of coordinates. Objects that behave this way live in the tangent bundle TM and are called contravariant.

However, for a 1-form ω, its components ω_μ and ω'_ν in the two coordinates systems are related via

$$\omega = \omega_\mu dx^\mu = \omega'_\nu dx'^\nu, \tag{2.15}$$

and since $dx'^\nu = \dfrac{\partial x'^\nu}{\partial x^\mu} dx^\mu$ we can see that the components of ω are related by

$$\omega'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} \omega_\mu, \tag{2.16}$$

which states that they transform with the Jacobian matrix of the change of coordinates. Objects that behave this way lives in the cotangent bundle T^*M and are called covariant.

A little more on the exterior derivative

We defined df in such a way that when fed with a vector v it spits out the directional derivative of v, if we are in $\mathbb{R}^n$. But we also know that $v = v^\mu \partial_\mu$, so:

$$df(v) = v^\mu \partial_\mu f,$$

but $df = f_\mu dx^\mu$, then

$$df(v) = f_\mu dx^\mu (v^\nu \partial_\nu) = v^\nu f_\mu \delta^\mu_\nu = v^\nu f_\nu,$$

hence, comparing with the first one, we have $f_\mu = \partial_\mu f$ and then

$$df = \partial_\mu f dx^\mu. \tag{2.17}$$

Therefore, the exterior derivative of scalar function is just its gradient in $\mathbb{R}^n$.

2.3.3 Wedge product and p-forms

In order to generalize the cross product in $\mathbb{R}^3$, which is anticommutative, we define the wedge product $\wedge$ of 1-forms ω and μ as

$$\omega \wedge \ \mu = -\mu \wedge \omega. \tag{2.18}$$

We can actually define the differential forms on M, denoted by $\Omega(M)$, to be the algebra generated by $\Omega^1(M)$ with the relations in equation (2.18).

The 0-forms, $\Omega^1(M)$, are the functions, and we define the wedge product of a a function with a differential form to be the ordinary product: $f \wedge \omega = f\omega$.

The elements that are a linear combination of a product of p 1-forms are called p-forms, and the space of all p-forms in M is $\Omega^p(M)$. Of course, the space of all differential forms in M is then the direct sum of the subspaces:

$$\Omega(M) = \bigoplus_p \Omega^p(M).$$

The 1-forms are given by $\omega_\mu dx^\mu$, with the coefficients ω_μ being functions.

2-forms look like

$$\frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu,$$

where the factor of 1/2 is inserted since $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. The $(dx^\mu \wedge dx^\nu)$ term is the basis of 2-forms.

In general, a p-form looks like

$$\frac{1}{p!}\omega_{\mu\nu\ldots\tau} dx^\mu \wedge dx^\nu \wedge \cdots \wedge dx^\tau,$$

where the product of p 1-forms $dx^\mu \wedge dx^\nu \wedge \cdots \wedge dx^\tau$ is the basis of all p-forms.

2.3.4 The exterior derivative

We can then extend the definition of the exterior derivate d to generalize the gradient, the divergence and the curl in any dimensions. The exterior derivative is defined to be the operator d

$$d : \Omega^p(M) \mapsto \Omega^{p+1}(M) \tag{2.19}$$

satisfying

- $d : \Omega^0(M) \mapsto \Omega^1(M)$ agrees with the previous definition
- $d(\omega + \mu) = d\omega + d\mu$ and $d(c\omega) = cd\omega$ for all $\omega, \mu \in \Omega(M)$ and $c \in \mathbb{R}$
- $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p \omega \wedge d\mu$ for all $\omega \in \Omega^p(M)$ and $\mu \in \Omega(M)$
- $d(d\omega) = 0$ for all $\omega \in \Omega(M)$

For instance, if we have a 1-form ω, its exterior derivative is

$$d\omega = d(\omega_\mu dx^\mu) = d\omega_\mu \wedge dx^\mu - \omega_\mu \wedge d(dx^\mu) = d(\omega_\mu) \wedge dx^\mu,$$

but $df = \partial_\nu f dx^\nu$, so

$$d\omega = (\partial_\nu \omega_\mu) dx^\nu \wedge dx^\mu, \tag{2.20}$$

which is a 2-form.

The third property is the Leibniz rule graded, which is necessary since the product of differential forms is anticommutative, and then, passing through p 1-forms we gain a sign of -1 at each step.

The last property can be demonstrated, as we now show. Recovering equation (2.20):

$$\begin{aligned} d(d\omega) &= d((\partial_\nu \omega_\mu) dx^\nu \wedge dx^\mu) \\ &= (\partial_\tau \partial_\nu \omega_\mu) dx^\tau \wedge dx^\nu \wedge dx^\mu \\ &= 0, \end{aligned} \tag{2.21}$$

since $\partial_\tau \partial_\nu$ is symmetric in $[\nu, \tau]$ but $dx^\tau \wedge dx^\nu$ is antisymmetric in the same indices, which means that $d(d\omega) = -d(d\omega)$ and hence it vanishes.

The exterior derivative generalizes all vector derivatives in 3D. For instance, one can easily show that

- Gradient: $d : \Omega^0(\mathbb{R}^3) \mapsto \Omega^1(\mathbb{R}^3)$
- Curl: $d : \Omega^1(\mathbb{R}^3) \mapsto \Omega^2(\mathbb{R}^3)$
- Divergence: $d : \Omega^2(\mathbb{R}^3) \mapsto \Omega^3(\mathbb{R}^3)$

The identity $d^2 = 0$ then contains the two identities of vector calculus

$$\nabla \times (\nabla f) = 0,$$

and

$$\nabla \cdot (\nabla \times v) = 0,$$

and has profound consequences in physics.

2.3.5 The Hodge Star operator

In the particular case of $\mathbb{R}^3$ there is something missing to really conclude that, for example, the exterior derivative reduces to the curl. In coordinates, take the two 1-forms $\omega = \omega_x dx + \omega_y dy + \omega_z dz$ and $\mu = \mu_x dx + \mu_y dy + \mu_z dz$ and their wedge product:

$$\omega \wedge \mu = (\omega_x \mu_y - \omega_y \mu_x) dx \wedge dy + (\omega_y \mu_z - \omega_z \mu_y) dy \wedge dz + (\omega_z \mu_x - \omega_x \mu_z) dz \wedge dx. \tag{2.22}$$

If we define a linear map $*$ to turn elements of $\Omega^2(M)$ in elements of $\Omega^1(M)$ such that

$$* : dx \wedge dy \mapsto dz$$

$$* : dy \wedge dz \mapsto dx$$

$$* : dz \wedge dx \mapsto dy$$

then we could really see that equation (2.22) would reduce to the curl, as expected. Note that defining this operator — the star or Hodge operator — in that way is incorporating the right-hand rule, since we could just as well have defined

$$* : dy \wedge dx \mapsto dz$$

$$* : dz \wedge dy \mapsto dx$$

$$* : dx \wedge dz \mapsto dy$$

which would imply in adopting a left-hand rule.

More generally, we define the Hodge star operator in a n-dimensional manifold M

$$* : \Omega^p(M) \mapsto \Omega^{n-p}(M), \tag{2.23}$$

to be the unique linear map from p-forms to $(n-p)$-forms such that, for all $\omega, \mu \in \Omega^p(M)$,

$$\omega \wedge *\mu = \langle \omega, \mu \rangle vol, \tag{2.24}$$

where $\langle \omega, \mu \rangle$ is the inner product of the forms, which is defined using the metric tensor as will be discussed shortly, and vol is the volume form:

$$vol = \sqrt{\det(g_{\mu\nu})} dx^1 \wedge dx^2 \cdots \wedge dx^n,$$

where $g_{\mu\nu}$ is the metric.

The definition in (2.24) simply implies a choice of orientation, since the existence of a volume form states that the manifold is orientable, and the choice of orientation — right-handed or left-handed — is what it is needed to make the map unique, as previously discussed in the 3D case.

2.4 Tensors

2.4.1 Definition

Having defined vectors—the geometrical objects living in TM whose basis in local coordinates are ∂_μ—we can define a new object constructed by composing those with the p-forms—the objects in the dual space T^*M with the dual basis dx^μ. Those objects are called tensors, and we define the bundle of (r,s) tensors to be the tensor product of r copies of TM and s copies of T^*M:

$$\underbrace{TM \otimes TM \otimes \cdots \otimes TM}_{\text{r}} \otimes \underbrace{T^*M \otimes T^*M \otimes \cdots \otimes T^*M}_{\text{s}}.$$

An object living in this space is an (r,s) tensor. The $(0,0)$ tensor are scalar fields. In local coordinates, any (r,s) tensor is just a linear combination of

$$\underbrace{\partial_\mu \otimes \partial_\nu \otimes \cdots \otimes \partial_\sigma}_{\text{r}} \otimes \underbrace{dx^\alpha \otimes dx^\beta \otimes \cdots \otimes dx^\gamma}_{\text{s}}.$$

Therefore, a tensor T can be written in components in this basis as

$$T = T^{\mu\nu\ldots\sigma}_{\alpha\beta\ldots\gamma} \underbrace{\partial_\mu \otimes \partial_\nu \otimes \cdots \otimes \partial_\sigma}_{\text{r}} \otimes \underbrace{dx^\alpha \otimes dx^\beta \otimes \cdots \otimes dx^\gamma}_{\text{s}},$$

where $T^{\mu\nu\ldots\sigma}_{\alpha\beta\ldots\gamma}$ are the components of the tensor in this basis, having r upper indices and s lower indices. This object, when we change coordinates, will transform r times in a covariant way and s times in a contravariant way. Hence, the components $\tilde{T}$ of the tensor T in a different coordinate system will be related to the components in the first coordinate system by

$$\tilde{T}^{\mu\nu\ldots\sigma}_{\alpha\beta\ldots\gamma} = T^{\tau\xi\ldots\delta}_{\theta\phi\ldots\omega} \underbrace{\Lambda^\mu{}_\tau \Lambda^\nu{}_\xi \ldots \Lambda^\sigma{}_\delta}_{\text{r}} \underbrace{(\Lambda^{-1})^\theta{}_\alpha (\Lambda^{-1})^\phi{}_\beta \ldots (\Lambda^{-1})^\omega{}_\gamma}_{\text{s}},$$

where Λ is the Jacobian matrix of the coordinate transformation and Λ^{-1} its inverse, as expected since vectors transform with Λ and 1-forms with its inverse. One can do the same thing using any basis e_μ of vector fields and its dual basis e^μ of 1-forms.

One way to think about the (r,s) tensor T is as a functional that accepts r 1-forms and s vector fields as inputs and outputs a function on M in a manner that is $C^\infty(M)$-linear in each input.

2.4.2 Metric tensor

A metric g is a $(0,2)$ tensor that is

- Symmetric: $g(v,w) = g(w,v)$
- Nondegenerate: if $g(v,w) = 0$ for all w then $v = 0$

The metric is the object that allows one to measure distances, angles and hence establishes the dot product in the manifold. For instance, in Minkoski spacetime the dot product of vectors v and w is

$$\eta(v,w) = v \cdot w = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3 = \eta_{\mu\nu} v^\mu w^\nu,$$

where $\eta_{\mu\nu}$ is the Minkowski metric given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.25}$$

and we will adopt this convetion where Minkowski spacetime has signature (3,1). The signature (m, n) of a metric tensor is the number of positive and negative eigenvalues of the symmetric tensor $\eta_{\mu\nu}$ written in a basis where it is diagonal. Hence, if there are m positive eingenvalues and n negative ones, one say that this metric has signature (m, n).

A metric g on a manifold M assigns to each point $P \in M$ a metric g_p on the tangent space T_pM. This is the object used to take inner products of tangent vectors v and w at P

$$g(v, w) = g_{\mu\nu} v^\mu w^\nu$$

and of 1-forms ω and μ in the dual space T_p^*M:

$$\langle \omega, \mu \rangle = g^{\alpha\beta} \omega_\alpha \mu_\beta.$$

In local coordinates the components $g_{\mu\nu}$ of the metric g are given by

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu), \tag{2.26}$$

and one can use the metric to calculate infinitesimal distances

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.27}$$

2.4.3 Covariant derivative

In curved spaces, as we change to a point of coordinates x^μ to a nearby point $x^\mu + dx^\mu$ not only the coordinates of a vector v change but also, in general, the basis vectors also change. So, when one take a derivative of a vector $v = v^\nu e_\nu$ written in the basis e_μ:

$$\begin{aligned} \nabla_\mu v &= \partial_\mu (v^\nu e_\nu) \\ &= (\partial_\mu v^\nu) e_\nu + v^\nu (\partial_\mu e_\nu) \\ &= (\partial_\mu v^\nu) e_\nu + v^\nu \Gamma^k_{\mu\nu} e_k \\ &= (\partial_\mu v^k + \Gamma^k_{\mu\nu} v^\nu) e_k, \end{aligned} \tag{2.28}$$

where we defined $\partial_\mu e_\nu \coloneqq \Gamma^k_{\mu\nu} e_k$. The symbol $\Gamma^k_{\mu\nu}$ tracks how the basis vectors e_μ changes from point to point and it is called the *connection*, since it allow one to connect a vector in one point to another.

There are a lot of ways to make this connection. There is, however, a unique connection that satisfies

- Metric compatibility: $\nabla g = 0$.
- Torsion free: for any vector fields v and w we have the Lie bracket $[v, w] = \nabla_v w - \nabla_w v = \mathcal{L}_v w$ vanishing.

This connection is called the Levi-Civita connection, and will allow us to take derivatives of any geometrical object in arbitrary spaces. For instance, for a 1-form $\omega = \omega_\mu dx^\mu$:

$$\nabla_\rho \omega = \partial_\rho \omega_\mu - \Gamma^k_{\rho\mu}\omega_k, \tag{2.29}$$

and, for a rank (r, s) tensor T we have

$$\nabla_\rho T = \partial_\rho T^{\mu\nu\ldots\sigma}_{\alpha\beta\ldots\gamma} + \underbrace{\Gamma^\mu_{\rho k} T^{k\nu\ldots\sigma}_{\alpha\beta\ldots\gamma} + \Gamma^\nu_{\rho k} T^{\mu k\ldots\sigma}_{\alpha\beta\ldots\gamma} + \cdots + \Gamma^\sigma_{\rho k} T^{\mu\nu\ldots k}_{\alpha\beta\ldots\gamma}}_{\text{r}} - \underbrace{\Gamma^k_{\rho\alpha} T^{\mu\nu\ldots\sigma}_{k\beta\ldots\gamma} - \Gamma^k_{\rho\beta} T^{\mu\nu\ldots\sigma}_{\alpha k\ldots\gamma} - \Gamma^k_{\rho\gamma} T^{\mu\nu\ldots\sigma}_{\alpha\beta\ldots k}}_{\text{s}}.$$

Chapter 3

The traditional formulation of GR

> I was sitting in a chair in the patent office in Bern when all of a sudden a thought occurred to me: "If a person falls freely he will not feel his own weight." I was stunned. This simple thought made a deep impression on me. It impelled me toward a theory of gravitation.
>
> *A. Einstein*

3.1 Introduction

The first disruptive article on relativity was the 1905 Einstein paper *On the Electrodynamics of Moving Bodies*. The paper introduced the now called special theory of relativity, a theory about space, time and the notion of simultaneity. It is essentially about the nature of light and the kinematic effects that it brings upon. The most famous of its effects are length contraction, time dilation and the relativity of simultaneity. This was all done in Einstein's free time after his work as an electrotechnology technical expert at the patent office, in Bern.

The subsequent generalization of the theory to the now called general theory of relativity was done just in 1915, and it happened in three major steps [12]:

- The formulation of the equivalence hypothesis, or the *equivalence principle*, in 1907.
- The incorporation of the metric tensor $g_{\mu\nu}$ as the main mathematical concept for a generally relativistic theory of gravitation in 1912.
- The construction of the generally covariant field equations of gravitation, in 1915.

3.2 Equivalence Principle

The general theory of relativity establishes a relation between gravity and geometry based on the equivalence principle, which is based upon the equality of the inertial and gravitational masses. Newton's law of gravitation states that the gravitational force between bodies of mass m and M is given by

$$F = G\frac{m_G M}{r^2}.$$

On the other hand, Newton's second law states that the dynamics of m is governed by the equation of motion

$$F = m_I \ddot{x}.$$

If the inertial mass m_I is equivalent to the gravitational mass m_G, then the dynamics of bodies due to a gravitational field will be independent of the body itself:

$$F = m_I \ddot{x} = G\frac{m_G M}{r^2},$$
$$\implies \ddot{x} = G\frac{M}{r^2}.$$

Hence, the acceleration of bodies due to the effect of gravity will be the same for all bodies and there are trajectories in spacetime that dictate how bodies will move if they are under the effect of gravity. These trajectories are a property of that region of spacetime and do not depend on the free falling body.

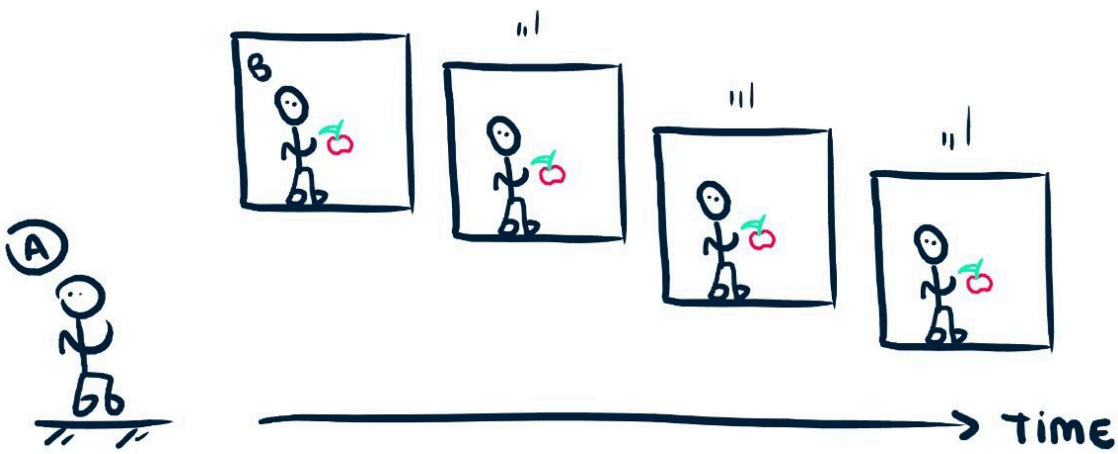

Figure 3.1: Observer A sees the apple free falling, but observer B, who is also in a free fall, does not feel the effect of gravity. For him, the apple is fluctuating over his hand.

This idea has huge consequences, such as the possibility of changing coordinates to cancel the effect of gravity, as we will briefly show. Consider observer A, which is in a uniform gravitational field g, studying the movement of particle C, of mass m. He then writes the equation of motion for that particle

$$m\ddot{x}_A = mg = F_A, \tag{3.1}$$

where F_A stands for the net force acting on particle C in the frame of reference A.

Now consider the coordinate transformation

$$x_B = x_A - \frac{1}{2}gt^2$$
$$t_B = t_A = t$$

which, when plugged in equation (3.1), leads us to

$$\begin{aligned} F_A = mg &= m\ddot{x}_A \\ &= m\frac{d}{dt^2}\left(x_B + \frac{1}{2}gt^2\right) \\ &= m\ddot{x}_B + mg, \end{aligned}$$

and, finally, the dynamics of particle C in the non inertial reference frame B is given by the equation of motion

$$m\ddot{x}_B = F_A - mg = 0 = F_B. \tag{3.2}$$

Hence, the two observers write the same physical law, i.e. $F = m\ddot{x}$, the only difference is that A feels a uniform gravitational field and B does not. The observers do not agree on the forces acting on the body, but they agree on the physical law which describes its dynamics. We have the gravitational force being canceled by inertial forces. The B frame of reference represents a free falling observer: he does not feel the effect of gravity, although, using the equivalence principle, he can write the same physical law to describe the dynamics in his point of view.

We have seen that the equivalence of the inertial and gravitational masses leads us to the equivalence between gravity and acceleration: one can annihilate the effect of gravity using acceleration, or also create the effect of gravity by accelerating.

This will not be true for Earth's gravitational field, for instance, since it is not a uniform gravitational field. However, in a small enough region of space and for very small intervals of time, one can approximate the field of the Earth by a uniform gravitational field. Hence, the equivalence principle states that *in a small enough region of spacetime no experiment can tell us whether we are in a gravitational field or in an accelerated frame of reference.* Therefore, it is always possible to build a local inertial frame or reference, satisfying the laws of special relativity. We have then established the relation between metric and gravity: the absence of gravity corresponds to the flat spacetime metric, the Minkowski metric $\eta_{\mu\nu}$ such that $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$. However, in the presence of a non uniform gravitational field we need the metric $g_{\mu\nu}$ since here it is not possible to find coordinates such that the metric tensor reduces to the Minkowski metric, except in a infinitesimal neighborhood of a certain point, where $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Quoting Einstein [7]:

> For infinitely small four-dimensional regions the theory of relativity in the restricted sense is appropriate, if the coordinates are suitably chosen.

This connection between metric and gravity will lead us to Einstein's field equation very shortly.

3.3 The classical formulation of GR in four steps

3.3.1 Equation of motion

According to the ideas previously developed we can always find a local coordinate system ξ^α such that the equation of motion of a particle free falling reduces to

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0, \tag{3.3}$$

i.e. the effect of gravity is locally canceled via this coordinate transformation. Here, τ stands for proper time, which is the time elapsed in a reference frame where the space interval between the two events is zero, i.e., the two events have the same spatial coordinates.

We are trying to relate the equation of motion in the local inertial coordinates ξ^α to the reference frame in coordinates x^μ who is feeling the effects of gravity. Hence, we can rewrite equation (3.3) as

$$\begin{aligned}\frac{d^2\xi^\alpha}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial\tau}\right)\\ &= \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial\tau}\\ &= \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\\ &= \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2}\frac{\partial x^\rho}{\partial\xi^\alpha} + \frac{\partial x^\rho}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\\ &= \frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau},\end{aligned} \tag{3.4}$$

where we have defined the Christoffel symbol Γ as

$$\Gamma^\rho_{\mu\nu} := \frac{\partial x^\rho}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}, \tag{3.5}$$

and, in the fourth line we multiplied both sides of the equation by $\frac{\partial x^\rho}{\partial\xi^\alpha}$, which does not change the left-hand side since it is equal to zero. We have also used $\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\rho}{\partial\xi^\alpha} = \delta^\rho_\mu$.

Therefore, from (3.3) and (3.4) we get the geodesics equation

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{3.6}$$

which gives the curves in spacetime $x^\rho(\tau)$ that describe the trajectories of bodies moving under effect of gravity. These curves are called geodesics. They are a property of the geometry of spacetime and do not depend of the particle in motion, as previously discussed.

Taking equation (3.5) as the definition of the Christoffel symbol — the connection — one can show that its relation to the metric is given by

$$\Gamma^\rho_{\mu\lambda} = \frac{1}{2}g^{\rho\nu}\left(\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda}\right). \tag{3.7}$$

3.3.2 Newtonian limit

Our new theory needs to be reduced into Newtonian theory in non relativistic limits. This will set up some conditions that some components of the metric tensor must satisfy. The classical limit will imply the following conditions:

- The particle will be moving in low speed comparing with the speed of light:

$$\frac{dx}{d\tau} \ll \frac{dt}{d\tau}. \tag{3.8}$$

- The gravitational field will be stationary:

$$\partial_\tau g_{\mu\nu} = 0. \tag{3.9}$$

- The gravitational field is weak. Hence, we can introduce the tensor $h_{\mu\nu}$ which represents a low deviation of the spacetime metric $g_{\mu\nu}$ from the Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}. \tag{3.10}$$

The first condition allows us to reduce the equation of motion (3.6) by neglecting some of its components:

$$\frac{d^2x^\rho}{d\tau^2} = -\Gamma^\rho_{00}\left(\frac{dx^0}{d\tau}\right)^2,$$

and, from equation (3.7) we get

$$\begin{aligned}\Gamma^\rho_{00} &= \frac{1}{2}g^{\rho\nu}\left(\partial_0 g_{\nu 0} + \partial_0 g_{\lambda 0} - \partial_\nu g_{00}\right)\\ &= -\frac{1}{2}g^{\rho\nu}\partial_\nu g_{00},\end{aligned}$$

where we have used the third condition to annihilate the time derivatives. Then, we get, for the equation of motion:

$$\frac{d^2x^\rho}{d\tau^2} = \frac{1}{2}g^{\rho\nu}\partial_\nu g_{00}\left(\frac{dx^0}{d\tau}\right)^2,$$

however, using the weak gravitational field condition we are led to

$$\begin{aligned}\frac{d^2x^\rho}{d\tau^2} &= \frac{1}{2}(-h^{\rho\nu} + \eta^{\rho\nu})\partial_\nu(h_{00} + \eta_{00})\left(\frac{dx^0}{d\tau}\right)^2\\ &\approx -\frac{1}{2}\eta^{\rho\nu}\partial_\nu h_{00}\left(\frac{dx^0}{d\tau}\right)^2,\end{aligned}$$

and, hence, the equation of motion is

$$\frac{d^2x^\rho}{d\tau^2} - \frac{1}{2}\eta^{\rho\nu}\partial_\nu h_{00}\left(\frac{dt}{d\tau}\right)^2 = 0.$$

For $\rho = 0$ the the second term vanishes since $\eta^{0\nu}\partial_\nu h_{00} = -\delta^\nu_0\partial_\nu h_{00} = \partial_0 h_{00} = 0$, because of the stationary gravitational field condition. Hence, the equation states that $\dfrac{d^2x^0}{d\tau^2} = 0$. Therefore

$$\frac{dt}{d\tau} = \text{constant}.$$

Now, for $\rho = i = 1, 2, 3$ we get

$$\frac{d^2x^i}{d\tau^2} - \frac{1}{2}\eta^{i\nu}\partial_\nu h_{00}\left(\frac{dt}{d\tau}\right)^2 = 0,$$

which, dividing by $\left(\dfrac{dt}{d\tau}\right)^2$, which is just a constant, as we previously showed, leads us to

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}.$$

But, looking to Newton's second law for a gravitational force written in terms of the gravitational potential ϕ we have

$$\frac{d^2x^i}{dt^2} = -\nabla\phi,$$

which allow us to make the identification

$$h_{00} = -2\phi + c,$$

where c is a real constant. But, since at infinity the metric must become the Minkowski metric, we have, at infinity, $g_{00} = \eta_{00} + h_{00} = \eta_{00}$ and, therefore, the constant c must be zero, since the potential ϕ already vanishes at infinity.

Then we get an equation saying that the time-time component of the metric tensor must satisfy:

$$g_{00} = \eta_{00} + h_{00} = -\left(1 + 2\phi\right). \tag{3.11}$$

We are looking for an equation of motion that describes gravity, here represented by the metric $g_{\mu\nu}$. In the classical approach, gravity, or the gravity potential ϕ, is given by Poisson's equation

$$\nabla^2\phi = 4\pi G\rho, \tag{3.12}$$

where ρ is the mass density which is generating the gravitational field. In the language of differential geometry and tensors, the presence of mass will be carried in the energy momentum tensor $T_{\mu\nu}$. This tensor will represent the flux of the momentum p^μ through the surface where x^ν is constant. Hence, T^{00} will be the flux of energy — p^0 — through time – x^0 – which is the energy density in the reference frame where the system is at rest. The T^{0j} element is the density of momentum in j direction, and the T^{ij} component will be the flux of the momentum component in the i direction per unit of time (force) flowing through a surface oriented in the direction of j, and so on.

Conservation laws can then be written as

$$\partial_\mu T^{\mu\nu} = 0,$$

which will be the conservation of energy for $\nu = 0$ and the conservation of momentum in the i direction for $\nu = i$.

Therefore, we are looking here for an equation of the form

$$\nabla^2 g_{00} = -8\pi G T_{00}.$$

This is actually a special case, written in a reference frame where the particles are at low speed. We could write the equation in a more general way as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3.13}$$

where the tensor $G_{\mu\nu}$ must have, at most, second order derivatives of the metric tensor, since we need to recover Poissons's equation (3.12) in the classical limit. But then: who is this tensor $G_{\mu\nu}$ that we are looking for?

3.3.3 Riemann Curvature Tensor

One way to identify the presence of curvature in a certain surface is by the non-commutativity of the covariant derivatives. Of course, in flat space this is zero, i.e. $[\partial_\mu, \partial_\nu] = 0$.

However, in curved space $[\nabla_\mu, \nabla_\nu]$ is not necessarily zero, and the deviation of this relation from zero will be due to the curvature of space — the intrinsic curvature of a surface is actually defined as the failure of this equation to vanish. Let us then evaluate the expression $[\nabla_\mu, \nabla_\nu]$ acting in a certain vector V:

$$[\nabla_\mu, \nabla_\nu]V_\rho = [\partial_\mu(\nabla_\nu V_\rho) - \Gamma^\sigma_{\mu\nu}(\nabla_\sigma V_\rho) - \Gamma^k_{\mu\rho}(\nabla_\nu V_k)],$$

where the antisymmetrizator in the right handside is remembering us to antisymmetrize the expression in $[\mu, \nu]$ at the end.

The second term in the right handside will vanish since $\Gamma^\sigma_{..}$ is symmetric in its two lower indices. Developing the equation a bit more will lead us to

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]V_\rho &= [\partial_\mu(\partial_\nu V_\rho - \Gamma^\sigma_{\nu\rho}V_\sigma) - \Gamma^k_{\mu\rho}(\partial_\nu V_k - \Gamma^\sigma_{\nu k}V_\sigma)] \\ &= [\partial_\mu\partial_\nu V_\rho - \partial_\mu(\Gamma^\sigma_{\nu\rho}V_\sigma) - \Gamma^k_{\mu\rho}\partial_\nu V_k + \Gamma^k_{\mu\rho}\Gamma^\sigma_{\nu k}V_\sigma],\end{aligned}$$

and here the first term in the last line vanishes since it is symmetric in $[\mu, \nu]$. The second term will dismember in $-\partial_\mu(\Gamma^\sigma_{\nu\rho}V_\sigma) = -V_\sigma\partial_\mu\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\nu\rho}\partial_\mu V_\sigma = -V_\sigma\partial_\mu\Gamma^\sigma_{\nu\rho} - \Gamma^k_{\nu\rho}\partial_\mu V_k$, which, when plugged back in the expression above will give

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]V_\rho &= [-V_\sigma\partial_\mu\Gamma^\sigma_{\nu\rho} - \Gamma^k_{\nu\rho}\partial_\mu V_k - \Gamma^k_{\mu\rho}\partial_\nu V_k + \Gamma^k_{\mu\rho}\Gamma^\sigma_{\nu k}V_\sigma] \\ &= [-V_\sigma\partial_\mu\Gamma^\sigma_{\nu\rho} - \left(\Gamma^k_{\rho\nu}\partial_\mu V_k + \Gamma^k_{\mu\rho}\partial_\nu V_k\right) + \Gamma^k_{\mu\rho}\Gamma^\sigma_{\nu k}V_\sigma].\end{aligned}$$

Now note that the term in parenthesis is symmetric in $[\mu, \nu]$, therefore it vanishes due the antisymmetrization. Finally, the equation is reduced to

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]V_\rho &= [-V_\sigma\partial_\mu\Gamma^\sigma_{\nu\rho} + \Gamma^k_{\mu\rho}\Gamma^\sigma_{\nu k}V_\sigma] \\ &= [-\partial_\mu\Gamma^\sigma_{\nu\rho} + \Gamma^k_{\mu\rho}\Gamma^\sigma_{\nu k}]V_\sigma \\ &= [\partial_\mu\Gamma^\sigma_{\nu\rho} - \Gamma^k_{\mu\rho}\Gamma^\sigma_{\nu k}]V_\sigma \\ &= (\partial_\mu\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\mu\rho} + \Gamma^k_{\nu\rho}\Gamma^\sigma_{\mu k} - \Gamma^k_{\mu\rho}\Gamma^\sigma_{\nu k})V_\sigma. \qquad (3.14)\end{aligned}$$

The curvature is then given by

$$R^\sigma_{\ \rho\mu\nu} := \partial_\mu\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\mu\rho} + \Gamma^k_{\nu\rho}\Gamma^\sigma_{\mu k} - \Gamma^k_{\mu\rho}\Gamma^\sigma_{\nu k}. \qquad (3.15)$$

Equation (3.15) defines the Riemann tensor, a (1,3) tensor that carries the information about the curvature of the space. One can raise or lower indices of this tensor as with any other using the metric:

$$R_{\tau\rho\mu\nu} = g_{\tau\sigma}R^\sigma_{\ \rho\mu\nu}.$$

One can also define the Ricci tensor $R_{\rho\nu}$ by the contraction

$$R_{\rho\nu} = g^{\tau\mu}R_{\tau\rho\mu\nu} = g^{\tau\mu}(g_{\tau\sigma}R^\sigma_{\ \rho\mu\nu}) = \delta^\mu_\sigma R^\sigma_{\ \rho\mu\nu} = R^\mu_{\ \rho\mu\nu},$$

and also the curvature scalar R by the total contraction of indices

$$R = g^{\mu\nu}R_{\mu\nu}.$$

3.3.4 Bianchi's identity

Back to the discussion that ended the section *Newtonian limit*, to write the equation for gravity in a manifestly covariant way we were looking for a tensor $G_{\mu\nu}$ that had, at most, second order time derivatives of the metric tensor. One could think of the tensor $G_{\mu\nu}$ as being the curvature tensor, let us say the Ricci tensor $R_{\mu\nu}$, for instance. The idea is that the energy momentum tensor, on the right-hand side of (3.13), is the object that generates curvature, which must appear on the left-hand side of the equation.

However, from the conservation of energy and momentum, we must have

$$\nabla_\mu T^{\mu\nu} = 0, \tag{3.16}$$

where the conservation of energy is the equation for $\mu = 0$ and the conservation of the 3-momentum is satisfied for $\mu = 1, 2, 3$.

But the derivative of the Ricci tensor — and also of the Riemann tensor — is not zero, so $R_{\mu\nu} = 8\pi G T_{\mu\nu}$, although it has the element which generates curvature on one side and the curvature itself on the other, can not be the equation we are looking for. (Historically speaking, Einstein and Grossmann dismissed it because they were unable to recover Newtonian physics in the weak field limit[12], as discussed in section 3.2.2).

On the other hand, one can contract Bianchi's identity

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0,$$

to get to

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \right) = 0. \tag{3.17}$$

Now, we have built a symmetric tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{3.18}$$

which is related to the curvature, has its covariant derivative vanishing and it is of second order, since the curvature has at most second order derivatives of $g_{\mu\nu}$.

3.3.5 Einstein's field equations

Equation (3.13) was built in a covariant way from the Newtonian limit as a restriction. Our goal was to find the left-hand side that would make the physics hold. We could do some attempts of finding some tensors $G_{\mu\nu}$ that satisfies certain properties, however, we have already built the tensor $G_{\mu\nu}$ that we need.

In order to write an equation for gravity that will contain the equivalence principle (equation (3.4)), reduce to Newtonian gravity in the classical limit (equation (3.12), will be manifestly covariant and will, beyond satisfying the conservation laws (equation (3.17)), also contain the physical idea that the presence of mass — $T_{\mu\nu}$ — is responsible for the curvature — $R_{\mu\nu}$ — of spacetime, then our tensor $G_{\mu\nu}$ that does that is the Einstein tensor, given by (3.18). He have then built the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3.19}$$

Since the metric is compatible with the covariant derivative ∇_μ, a more general equation for gravity could be built by adding any term in the left-handside proportional do the metric, since this would not affect the conservation laws. So, a more general equation would then be

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3.20}$$

The parameter Λ is called the cosmological constant, and it was added by Einstein to give solutions for a static cosmological model. This was done before Hubble's work about the expansion of universe.

One can note that the addition of this new term will make appear, in the classical limit, beyond the gravitational Newtonian force, a repulsive force proportional to Λ and to the distance. So, for small Λ, this repulsive term would be relevant only for great distances.

This term, although denied by Einstein and considered by him as an error, proved to be very important later on. It is responsible for the dark energy and it is the term that explained the accelerated expansion of the universe later detected in 1990.

However, since our work will not enter deep into cosmology, equation (3.19) will be our main subject.

3.4 The Lagrangian Formulation of GR

It is always possible to obtain the same evolution equation via the Lagrangian formalism, using variational calculus. Depending on what the dynamical variables in play are, one can write different Lagrangians for gravity. The idea of trying a variational approach came from Paul Bernays, a student of David Hilbert [12].

3.4.1 The Einstein-Hilbert action

Here we consider the metric $g_{\mu\nu}$ as our only dynamical variable. The field equations for gravity are extracted from the Einstein-Hilbert action:

$$\begin{aligned} S[g] &= \frac{1}{16\pi G}\int R\sqrt{-g}d^4x \\ &= \frac{1}{16\pi G}\int g^{\mu\nu}R_{\mu\nu}\sqrt{-g}d^4x. \end{aligned} \tag{3.21}$$

Since $g_{\mu\nu}$ is our only variable, the dynamics of gravity comes from setting $\frac{\delta S}{\delta g_{\mu\nu}} = 0$. From (3.21):

$$\delta S = \frac{1}{16\pi G}\int d^4x \left[R(\delta\sqrt{-g}) + \sqrt{-g}g^{\mu\nu}(\delta R_{\mu\nu}) + \sqrt{-g}R_{\mu\nu}(\delta g^{\mu\nu})\right].$$

We can break this integral in three terms (here we omited some constants)

- $\delta S_1 = \int d^4x R(\delta\sqrt{-g})$.
- $\delta S_2 = \int d^4x \sqrt{-g}g^{\mu\nu}(\delta R_{\mu\nu})$.
- $\delta S_3 = \int d^4x \sqrt{-g}R_{\mu\nu}(\delta g^{\mu\nu})$.

For the first one we use $\delta(\det M) = \det(M)\mathrm{M}_{ij}^{-1}\delta M_{ji}$, which leads us to

$$\begin{aligned}\delta(\sqrt{-g}) &= \frac{1}{2\sqrt{-g}} g g^{\mu\nu}\delta g_{\nu\mu} \\ &= \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \end{aligned} \tag{3.22}$$

where, in the last step, we used $\delta(g_{\mu\nu}g^{\mu\nu}) = g^{\mu\nu}\delta g_{\mu\nu} + g_{\mu\nu}\delta g^{\mu\nu} = 0$. Hence, we are left with

$$\delta S_1 = \int d^4x \left\{ -R\frac{1}{2}\sqrt{-g}g_{\mu\nu} \right\} \delta g^{\mu\nu}. \tag{3.23}$$

For the second term we will use the Palatini identity, which states that

$$\delta R^{\rho}_{\ \mu\sigma\nu} = \nabla_\sigma \delta\Gamma^{\rho}_{\ \mu\nu} - \nabla_\nu \delta\Gamma^{\rho}_{\ \mu\sigma}.$$

So, for the Ricci tensor we get

$$\delta R_{\mu\nu} = \delta R^{\rho}_{\ \mu\rho\nu} = \nabla_\rho \delta\Gamma^{\rho}_{\ \mu\nu} - \nabla_\nu \delta\Gamma^{\rho}_{\ \mu\rho}. \tag{3.24}$$

Therefore, the corresponding contribution to the variation of the action is written as

$$\begin{aligned}\delta S_2 &= \int d^4x\sqrt{-g}g^{\mu\nu}(\nabla_\rho \delta\Gamma^{\rho}_{\ \mu\nu} - \nabla_\nu \delta\Gamma^{\rho}_{\ \mu\rho}) \\ &= \int d^4x\sqrt{-g}(\nabla_\rho g^{\mu\nu} \delta\Gamma^{\rho}_{\ \mu\nu} - \nabla_\nu g^{\mu\nu} \delta\Gamma^{\rho}_{\ \mu\rho}) \\ &= \int d^4x\sqrt{-g}\nabla_\sigma(g^{\mu\nu} \delta\Gamma^{\sigma}_{\ \mu\nu} - g^{\mu\sigma} \delta\Gamma^{\rho}_{\ \mu\rho}) \\ &= \int d^4x\sqrt{-g}\nabla_\sigma \omega^\sigma, \end{aligned} \tag{3.25}$$

which is just a boundary term, that vanishes if $\delta\Gamma^{\rho}_{\mu\nu}$ vanishes at infinity.

The third term is already written in terms of the variation of the metric $\delta g^{\mu\nu}$. Hence, taking back the constants, we are left with

$$\begin{aligned}\delta S &= \frac{1}{16\pi G}(\delta S_1 + \delta S_2 + \delta S_3) \\ &= \frac{1}{16\pi G}\int \sqrt{-g}d^4x \left\{ -R\frac{1}{2}g_{\mu\nu} + R_{\mu\nu} \right\} \delta g^{\mu\nu}. \end{aligned}$$

Then, setting $\delta S/\delta g^{\mu\nu} = 0$ gives us Einstein equation in vacuum

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0.$$

If we consider matter, then the action would be, except for some constants, $S = S_{E.H.} + S_M$, where $S_{E.H.}$ stands for the Einstein-Hilbert action previously developed, and S_M for the action related to matter. Then, setting

$$\frac{\delta S}{\sqrt{-g}\delta g^{\mu\nu}} = 0,$$

we are led to

$$\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0, \tag{3.26}$$

which gives us the equation of motion

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu},$$

if one defines the energy momentum tensor as

$$T_{\mu\nu} := -\frac{1}{2\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{3.27}$$

3.4.2 The Palatini action

In the Palatini approach we consider that the connection Γ can also play a dynamical role. So we write the Palatini action

$$\begin{aligned} S[g, \Gamma] &= \int R\sqrt{-g} d^4x \\ &= \int g^{\mu\nu} R_{\mu\nu}(\Gamma)\sqrt{-g} d^4x. \end{aligned} \tag{3.28}$$

The curvature is completely determined by the connection, therefore it is not affected by variations of the metric. So, as we previously did, varying this action with respect to the metric and setting $\delta S/\delta g^{\mu\nu} = 0$ will lead us to

$$R_{\mu\nu}(\Gamma) - \frac{1}{2} R(\Gamma) g_{\mu\nu} = 0,$$

which is just the Einstein field equation.

However, for this method to be equivalent to the Einstein-Hilbert one we need the connection to be compatible with the metric.

So, for the second equation of motion, we vary the action with respect to the connection and set $\delta S/\delta \Gamma^{\rho}_{\mu\nu} = 0$. We will then get

$$\begin{aligned} \delta S_\Gamma &= \int d^4x \sqrt{-g} g^{\mu\nu} (\delta R_{\mu\nu}) \\ &= \int d^4x \sqrt{-g} g^{\mu\nu} (\nabla_\rho \delta\Gamma^{\rho}{}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\rho}{}_{\mu\rho}) \\ &= \int d^4x \sqrt{-g} (g^{\mu\nu} \nabla_\rho \delta\Gamma^{\rho}{}_{\mu\nu} - g^{\mu\sigma} \nabla_\sigma \delta\Gamma^{\rho}{}_{\mu\nu} \delta^{\nu}_{\rho}) \\ &= -\int d^4x \sqrt{-g} (\nabla_\rho g^{\mu\nu} - \delta^{\nu}_{\rho} \nabla_\sigma g^{\mu\sigma}) \delta\Gamma^{\rho}{}_{\mu\nu}, \end{aligned} \tag{3.29}$$

where we used the Palatini identity (3.24) in the second line and, in the last one, we did an integration by parts and neglected the boundary term.

Assuming that the connection is symmetric in $[\mu, \nu]$, the variation will vanish if the symmetrization of the integrand vanishes:

$$\nabla_\rho g^{\mu\nu} + \nabla_\rho g^{\nu\mu} - \delta^{\nu}_{\rho} \nabla_\sigma g^{\mu\sigma} - \delta^{\mu}_{\rho} \nabla_\sigma g^{\nu\sigma} = 0$$

$$2\nabla_\rho g^{\mu\nu} - \delta^{\nu}_{\rho} \nabla_\sigma g^{\mu\sigma} - \delta^{\mu}_{\rho} \nabla_\sigma g^{\nu\sigma} = 0, \tag{3.30}$$

and, contracting with δ^{ρ}_{μ} we get

$$\nabla_\mu g^{\mu\nu} - \nabla_\sigma g^{\nu\sigma} - \nabla_\sigma g^{\nu\sigma} = 0,$$

which, renaming dummy indices, leads us to $\nabla_\sigma g^{\nu\sigma} = 0$. When plugged in (3.30) we get

$$\nabla_\rho g^{\mu\nu} = 0, \tag{3.31}$$

which states that the covariant derivative ∇ with respect to the connection Γ gives a null derivative of the spacetime metric, i.e., the metric is compatible with the connection.

So, the first equation of motion gives us Einstein field equation and the second equation of motion states that our connection, previously placed as a dynamical variable, is fixed to be the Levi Civita connection.

Chapter 4

The Hamiltonian Formulation of GR

Lagrange has perhaps done more than any other analyst by showing that the most varied consequences respecting the motion of systems of bodies may be derived from one radical formula; the beauty of the method so suiting the dignity of the results, as to make of his great work a king of scientific poem.

Wiliam Rowan Hamilton

4.1 Introduction

In the canonical formulation of GR, the geometry of spacetime is given in terms of fields in spatial slices Σ, whose geometry is encoded in a 3-metric h_{ab}. Each spatial slice Σ corresponds to an instant of time, and, in this view, the 3-metric h_{ab} evolves in time through these slices.

Since there's no absolute time in GR, the decomposition of the metric g_{ab} in its spatial part h_{ab} will be necessary so one can define a time parameter in order to talk about the evolution of the system.

The spacetime geometry being generally covariant is expressed, in this formalism, by the presence of constraints in the fields. As said before, there is no absolute time in this description, neither a Hamiltonian generating evolution, there will be only the constraints: the *diffeomorphism* constraints and the *Hamiltonian* constraints, both of them will express the general covariance of spacetime.

4.2 Constrained Systems

We can see in Einstein's field equations some constraints regarding the temporal components of the Einstein tensor $G_{\mu\nu}$, defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{4.1}$$

The reduced Bianchi identity states that

$$\nabla_a G^a_b = 0 \,, \tag{4.2}$$

and, developing (4.2) we have

$$\nabla_\mu G^\mu_\nu = \partial_\mu G^\mu_\nu - \Gamma^\sigma_{\mu\nu} G^\mu_\sigma + \Gamma^\mu_{\mu k} G^k_\nu = 0.$$

Opening $\partial_\mu G^\mu_\nu$ as $\partial_0 G^0_\nu + \partial_i G^i_\nu$ for i spatial we get

$$\partial_0 G^0_\nu = -\partial_i G^i_\nu + \Gamma^\sigma_{\mu\nu} G^\mu_\sigma - \Gamma^\mu_{\mu k} G^k_\nu, \tag{4.3}$$

and, since

$$\Gamma^\sigma_{\mu\lambda} = \frac{1}{2} g^{\sigma\nu} \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda} \right),$$

the right-hand side of (4.3) has, at least, second order time derivatives, given that the Christoffel symbols Γ have at most first order time derivatives and the Einstein's tensor contains first derivatives of those symbols. Hence, from the left-hand side of the equation we can infer that G^0_ν has at most first order time derivatives. Therefore, since we have other equations with second order time derivatives, those four equations for G^0_ν are not evolution equations: they are constraints that the initial data must satisfy. From symmetry, the Einstein's field equations are a set of ten partial differential equations, of which only six are time evolution equations. The equations $G^0_\mu = 8\pi G T^0_\mu$ relate initial values of fields instead of determining how fields evolve.

If we proceed with the computation we can see that only spatial components of the metric g_{ab} appear with their second order time derivatives. The other components do not play the same dynamical role as g_{ab}. The g_{00} and g_{0a} equations will be the constraints — they will play the role of the lapse function and the shift vector, as we will see later.

4.2.1 The Lagrangian formalism

For a system with n (finite) degrees of freedom its action is

$$S[q^i(t)] = \int L(q^i, \dot{q}^i) dt, \tag{4.4}$$

for $i = 1, 2, 3, ..., n$. From the least action principle, one can get the Euler Lagrange equations by setting $\delta S = 0$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0. \tag{4.5}$$

By the chain rule, one can expand the time derivative as

$$\frac{d}{dt} = \frac{\partial}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial}{\partial \dot{q}^i}\frac{d\dot{q}^i}{dt}$$

and, plugging this in (4.5) one gets

$$\left(\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}\right) \ddot{q}^j + \left(\frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\right) \dot{q}^j - \frac{\partial L}{\partial q^i} = 0. \tag{4.6}$$

If we define the first term as

$$W_{ij} := \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}, \tag{4.7}$$

then, equation (4.6) is written as

$$W_{ij}\ddot{q}^j + \left(\frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\right)\dot{q}^j - \frac{\partial L}{\partial q^i} = 0. \tag{4.8}$$

If the matrix W_{ij} is non-degenerate, then one can invert (4.8) to obtain an explicit equation for $\ddot{q}^j$:

$$\ddot{q}^j = W_{ij}^{-1}\left(-\frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\dot{q}^j + \frac{\partial L}{\partial q^i}\right). \tag{4.9}$$

However, if W_{ij} is singular, then $\det(W_{ij}) = 0$ and equation (4.8) cannot be inverted. In that case, $\ddot{q}^j$ can not be uniquely determined by positions and velocities, and the system is said to be constrained, which we will detail better soon.

4.2.2 The Hamiltonian formulation

In this formulation constraints can arise in a similar way as happened in the Lagrangian formulation.

The starting point is to define the canonical momenta as

$$p_i := \frac{\partial L}{\partial \dot{q}^i}. \tag{4.10}$$

Equation (4.7) can then be rewritten as

$$W_{ij} = \frac{\partial p_i}{\partial \dot{q}^j}. \tag{4.11}$$

If W is nonsingular we can obtain in (4.11) the $\dot{q}'s$ in terms of $q's$ and $p's$, and then (4.10) will indeed provide n independent variables — the $p_i's$. However, if W is singular, there is no unique solution of the momenta definition equation expressing the velocities in terms of the canonical coordinates q^i and conjugate momenta p_j. In this case, there exists certain relations $\psi_s(q^i, p_j)$ connecting the momentum variables:

$$\psi_s(q^i, p_j) = 0. \tag{4.12}$$

The $q's$ and $p's$ — the dynamical variables of the system — are connected by the primary constraints, given by (4.12).

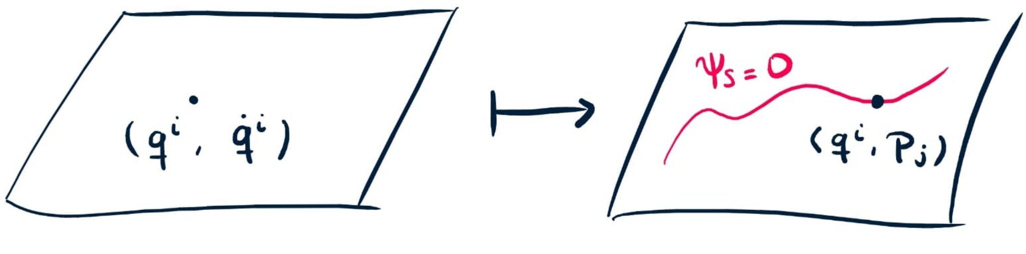

Figure 4.1: The constrained phase space

The map $(q^i, \dot{q}^i) \mapsto (q^i, p_j)$, when there are no constraints, is a one-to-one map. In the presence of constraints, it maps the unrestricted space $(q^i, \dot{q}^i)$ to the surface of primary constraint $\psi_s(q^i, p_j) = 0$ on the phase space, as shown in figure 4.1. We will name this constrained surface as $\mathcal{C}$ from now on.

Hamiltonian equations

Let us consider the usual Legendre transformation

$$H = \dot{q}^i p_i(q, \dot{q}) - L(q, \dot{q}), \tag{4.13}$$

on the unconstrained manifold $(q^i, \dot{q}^i)$.

If H is a Hamiltonian of the system we need to be able to express it in terms of q^i and p_j, and not only in terms of q^i and $\dot{q}^i$. However, in the constrained case, equation (4.10) cannot be inverted, so, we cannot express all of the $\dot{q}'s$ in terms of $p's$, which may lead us to conclude that it is not possible to write such a function as $H(q, p(q, \dot{q}))$ in the phase space.

Still, the function $H(q^i, p_j)$ is well defined, as we can easily see. From equation (4.13):

$$\begin{aligned}
\delta H &= \delta\dot{q}^i p_i + \dot{q}^i \delta p_i - \delta L(q^i, \dot{q}^i) \\
&= \delta\dot{q}^i p_i + \dot{q}^i \delta p_i - \frac{\partial L}{\partial q^i}\delta q^i - \frac{\partial L}{\partial \dot{q}^i}\delta\dot{q}^i \\
&= \delta\dot{q}^i p_i + \dot{q}^i \delta p_i - \frac{\partial L}{\partial q^i}\delta q^i - p_i \delta\dot{q}^i \\
&= -\frac{\partial L}{\partial q^i}\delta q^i + \dot{q}^i \delta p_i \\
&= -\dot{p}_i \delta q^i + \dot{q}^i \delta p_i \\
&= \frac{\partial H}{\partial q^i}\delta q^i + \frac{\partial H}{\partial p_i}\delta p_i .
\end{aligned}$$

In the fourth line one can easily see that the variation δH depends only on the variations of the momenta p_i and the position q^i, not on the velocities $\dot{q}^i$.

Equating the last two lines we get

$$\left(\frac{\partial H}{\partial q^i} + \dot{p}_i\right)\delta q^i + \left(\frac{\partial H}{\partial p_i} - \dot{q}^i\right)\delta p_i = 0. \tag{4.14}$$

For any variation $t^i = (\delta q^i, \delta p_i)$ tangent to the primary constraint surface, the equation above shows that the vector

$$V := \left(\frac{\partial H}{\partial q^i} + \dot{p}_i, \frac{\partial H}{\partial p_i} - \dot{q}^i\right) \tag{4.15}$$

is normal to the surface, since $t^i V_i = 0$

A basis of normal vectors to $\mathcal{C}$ is

$$v_s = \mathrm{grad}(\psi_s) = \left(\frac{\partial \psi_s}{\partial q^i}, \frac{\partial \psi_s}{\partial p_i}\right). \tag{4.16}$$

Then, for some functions λ on the surface of primary constraints, we have

$$V = \lambda^s v_s. \tag{4.17}$$

Finally, with equations (4.15), (4.16) and (4.17) one can get the equations of motion:

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + \lambda^s \frac{\partial \psi_s}{\partial q^i}, \tag{4.18}$$

$$\dot{q}^i = \frac{\partial H}{\partial p_i} - \lambda^s \frac{\partial \psi_s}{\partial p_i}. \tag{4.19}$$

Comparing the last equations with the Hamilton's equations of motion, those can be rewritten as

$$\dot{p}_i = -\frac{\partial(H - \lambda^s \psi_s)}{\partial q^i} - \psi_s \frac{\partial \lambda^s}{\partial q^i}, \tag{4.20}$$

$$\dot{q}^i = \frac{\partial(H - \lambda^s \psi_s)}{\partial p_i} + \psi_s \frac{\partial \lambda^s}{\partial p_i}, \tag{4.21}$$

where we can define the total Hamiltonian of the system as

$$H_{total} = H - \lambda^s \psi_s. \tag{4.22}$$

We can rewrite the Hamilton's equation in terms of the total Hamiltonian:

$$\dot{p}_i \approx -\frac{\partial H_{total}}{\partial q^i}, \tag{4.23}$$

$$\dot{q}^i \approx \frac{\partial H_{total}}{\partial p_i}. \tag{4.24}$$

Here, we introduce the weak equality symbol $\approx$, denoting an equality valid only in the constrained surface.

The value of the total Hamiltonian does not change on the surface of primary constraints by adding primary constraints and is independent of the λ^s. However, the evolution of the system depends on derivatives of the ψ_s, which might not be zero, and then the evolution depends on the λ^s. To see the role of the λ^s on the evolution the mathematical theory of constraints, described in terms of the Poisson structure, is very useful.

4.2.3 Poisson Brackets

In canonical coordinates (q^i, p_j) on the phase space, the Poisson bracket of the functions $f(q,p)$ and $g(q,p)$ is given by

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right). \tag{4.25}$$

It satisfies the following properties:

1. It is antisymmetric:
$$\{f, g\} = -\{g, f\}.$$

2. It is linear in both entries:
$$\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\},$$
$$\{g, f_1 + f_2\} = \{g, f_1\} + \{g, f_2\}.$$

3. It obeys the Leibniz law:
$$\{f_1 \cdot f_2, g\} = f_1 \{f_2, g\} + f_2 \{f_1, g\}.$$

4. It satisfies the Jacobi identity:
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Using the Poisson bracket we can rewrite the Hamilton's equations of motion 4.24 as:

$$\dot{p}_i \approx \{p_i, H_{total}\} \tag{4.26}$$
$$\dot{q}^i \approx \left\{q^i, H_{total}\right\}, \tag{4.27}$$

which actually is valid for any function $F(q,p)$ on the phase space, as is easily seen:

$$\begin{aligned}\dot{F} = \frac{dF}{dt} &= \frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial p_i}\dot{p}_i \\ &\approx \frac{\partial F}{\partial q^i}\left(\frac{\partial H_{total}}{\partial p_i}\right) + \frac{\partial F}{\partial p_i}\left(-\frac{\partial H_{total}}{\partial q^i}\right) \\ &= \{F, H_{total}\}\,. \end{aligned}\tag{4.28}$$

The total Hamiltonian then generates the dynamical flow of the variables of the phase space in time.

Because the primary constraints ψ_s are originated directly from the definition of the canonical momenta, they need to hold during all the evolution of the system. This means that the evolution of the system must be contained in the surface of primary constraint ψ_s. These are called consistency conditions, expressed by

$$\dot{\psi}_s \approx \{\psi_s, H_{total}\} = 0. \tag{4.29}$$

These conditions can add new constraints to the evolution of the system, known as secondary constraints. Those constraints must also satisfy the consistency conditions, which can lead to a new generation of constraints. This process goes on until no more constraints are generated.

Opening equation (4.29) we get:

$$\begin{aligned}\dot{\psi}_s \approx \{\psi_s, H_{total}\} &= \left\{\psi_s, H - \lambda^k\psi_k\right\} \\ &= \{\psi_s, H\} - \left\{\psi_s, \lambda^k\psi_k\right\} \\ &= \{\psi_s, H\} - \lambda^k\{\psi_s, \psi_k\} - \psi_k\left\{\psi_s, \lambda^k\right\} \\ &\approx \{\psi_s, H\} - \lambda^k\{\psi_s, \psi_k\} \\ &= \{\psi_s, H\} - \lambda^k C_{sk} = 0,\end{aligned}\tag{4.30}$$

where we have defined

$$C_{sk} \coloneqq \{\psi_s, \psi_k\}\,.$$

If C_{sk} is non singular the structure of the constraint system is uniquely determined: one can solve for the λ^k via

$$\lambda^k = C_{sk}^{-1}\{\psi_s, H\}\,.$$

In this case, no further constraints arise and we can fulfill the consistency condition. However, if the matrix C_{sk} is singular, we cannot determine all the λ^k. In that case, equation (4.30) implies the secondary constraints aforementioned. Those follow from the equations of motion, not from the definition of the momenta as the primary constraints.

4.2.4 Gauge Transformations

Since the Hamiltonian generates the evolution of the system, we can define, as stated in (4.28), the Hamiltonian vector field X_f associated to any function f as

$$X_f = \{\cdot, f\}\,.$$

We call a constraint ψ_k first class with respect to all constraints if its Hamiltonian vector field is everywhere tangent to the constraint surface $\mathcal{C}$. That is, for all constraints ψ_k on the constraint surface $\mathcal{C}$ we must have

$$\{\psi_s, \psi_k\} = 0,$$

and we call it second class if that Poisson bracket is nonvanishing on the constraint surface.

First class constraints generate gauge transformations, as we now show.

Consider all constraints, and consider also an arbitrary dynamical variable F, then define the transformation

$$F(q,p) \mapsto F(q,p) + \{F, \epsilon\psi_k\}\,, \tag{4.31}$$

where ϵ is a control parameter arbitrarily small. Due to the consistency conditions, this transformation does not affect the Hamiltonian

$$H(q,p) \mapsto H(q,p) + \{H, \epsilon\psi_k\} \approx H.$$

That is, the transformation takes solutions of the equations of motion and constraints into new solutions. This is a gauge transformation, and that is why constraints are generators of gauge transformations. Solutions that are related by gauge transformations are then treated as the same solution.

Any particular choice for the total Hamiltonian will result in equations of motion written in a specific gauge. But since the theory is invariant under gauge transformations generated by constraints, the choice of a total Hamiltonian does not matter, and all sets of equations of motion obtained for different gauges are equivalent.

4.3 Spacetime 3+1 decomposition

4.3.1 Introduction

When we are interested in studying the evolution of the spacetime, something strange immediately appears: it evolves with respect to what parameter?

GR treats space and time on the same footing, which is not what happens in Hamiltonian formulations. Spacetime does not evolve in time, it just is. However, we can interpret the spacetime as the evolution of the 3D space. For that, we will need to do a (3+1) decomposition, choosing an arbitrary parameter as time t, and considering that spacetime is the evolution of spatial slices Σ fixed for each t with respect to this time parameter.

This will be necessary because when we write down the Hamiltonian formalism it gives us the evolution of the system with respect to time, which is not absolute in GR. So, one needs to choose an arbitrary function to play the role of time and do this decomposition in order to write the Hamiltonian formalism for GR.

We assume the existence of a foliation of spacetime in terms of space-like 3 dimensional surfaces S of the spacetime manifold M. Thus, we consider the Lorentzian manifold M to be diffeomorphic to $\mathbb{R} \times S$.

There are lots of ways to build a diffeomorphism

$$\phi : M \mapsto \mathbb{R} \times S,$$

which means that time is not absolute in GR. There are different ways of defining a coordinate t on the manifold to play the role of time, which we will discuss later on. For now, assume that $\Sigma_t \in M$ is a slice of M for $t =$ constant for some time coordinate t. This can always be done in globally hyperbolic manifold [4].

4.3.2 Geometry of Hypersurfaces

Consider a spatial slice Σ_{t_0} in a foliated spacetime manifold $\mathbb{R} \times S$. This can be considered as a constraint surface such as $\psi_{t_0} = t - t_0 = 0$.

The spacetime is just the history of the space Σ_t with respect to t. In any instant t the spacetime is described as the immersion of Σ_t in the manifold M, as shown in figure 4.2, where the dashed line is the integral curve of the time vector field — which will be precisely defined later — joining the same point in the surface along its evolution.

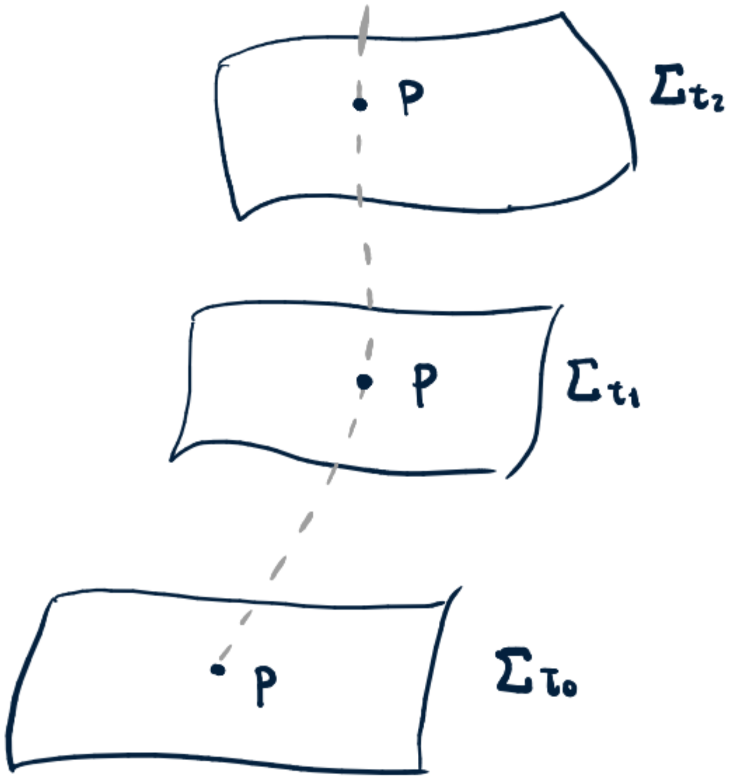

Figure 4.2: The 3+1 decomposition

The surface Σ is called a slice. The foliation is such that

1. $\Sigma_{t_1} \cap \Sigma_{t_2} = \varnothing$, if $t_1 \neq t_2$.
2. $\bigcup_t \Sigma_t = M$.

In this way, every point of spacetime belongs to a unique slice. Any embedding that satisfies this relations is a valid foliation, which reminds us that the foliation is not unique [4].

We can assign to each point of a slice Σ_t a time-like vector orthogonal to the surface at that point. That enables us to define, for a given foliation, a time-like normal vector field n^a, normalized such that

$$g(n, n) = n^a n_a = -1, \tag{4.32}$$

and the negative sign shows that this vector is time-like, as we wanted.

The foliation allows us to decompose all vectors in components parallel and perpendicular to the spatial slice Σ_t. This can be done via the projection operator [2]:

$$\begin{aligned} P_\parallel : TM &\mapsto T_\parallel M \\ x^a &\mapsto x^a + g(n, x) n^a = x^a + n_b x^b n^a, \end{aligned} \tag{4.33}$$

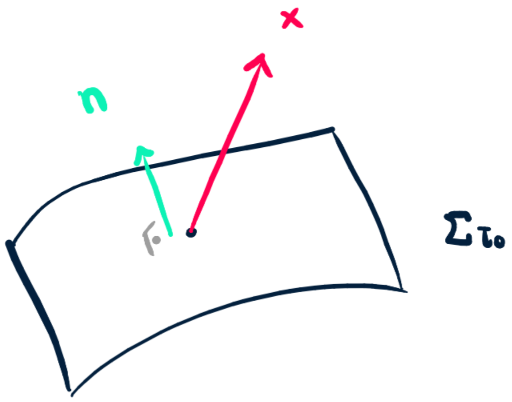

Figure 4.3: The spatial slice

and the orthogonal operator

$$\begin{aligned} P_\perp : TM &\mapsto T_\perp M \\ x^a &\mapsto -g(n,x)n^a = -n_b x^b n^a, \end{aligned} \tag{4.34}$$

as shown in the Figure 4.4.

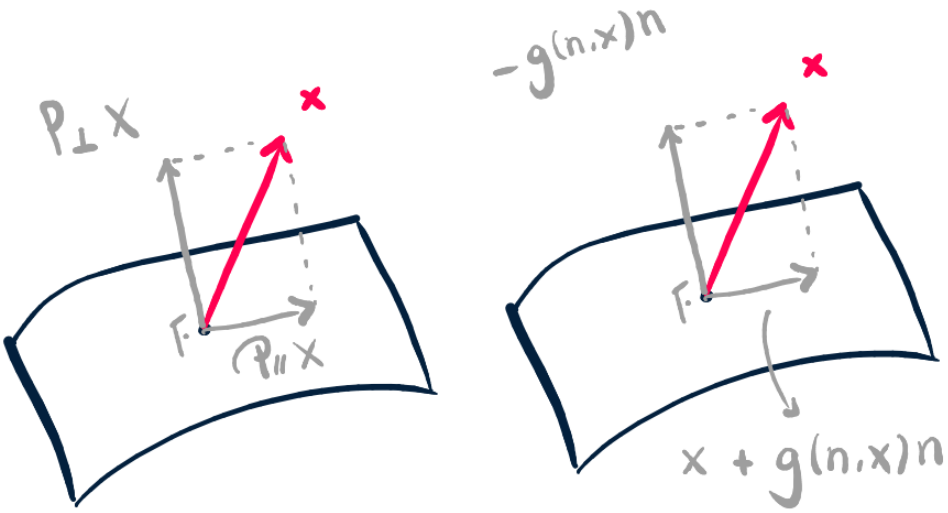

Figure 4.4: The projections on the slice

These projections allow us to break any geometrical object X (a vector or tensor) in its tangential ($\in T_\parallel M$) and perpendicular ($\in T_\perp M$) parts:

$$X = (P_\parallel X) + (P_\perp X).$$

For the dual space $T^*(M)$ the action of those operators is similar. The action of the projection operator on a 1-form ω , for instance, is:

$$\begin{aligned} P_\parallel : T^*M &\mapsto T^*_\parallel M \\ \omega_a &\mapsto \omega_a + g(n,\omega)n_a = \omega_a + n^b \omega_b n_a. \end{aligned} \tag{4.35}$$

For a rank (r, s) tensor T, the projection operator acts as follows

$$P_{\|}(T)(v^1, v^2, ..., v^r, \omega_1, \omega_2, ..., \omega_s) := T(P_{\|}v^1, P_{\|}v^2, ..., P_{\|}v^r, P_{\|}\omega_1, P_{\|}\omega_2, ..., P_{\|}\omega_s), \tag{4.36}$$

where $v^k \in T(M)$ and $\omega_k \in T^*M$. Therefore, the projection operator acting on a tensor is the same as the tensor acting on the projections of its entries. The same is true for the orthogonal operator.

4.3.3 Metric decomposition

Since we can apply the projection operator to any geometrical object, let us do this for the metric. The part of the metric that is tangential to the slice Σ_t is called the induced metric, and we will denote it by h, so:

$$h = P_{\|}g, \tag{4.37}$$

or, in components:

$$\begin{aligned} P_{\|}g(X, Y) &= g(P_{\|}X, P_{\|}Y) \\ &= g_{ab}(X^a + n^d X_d n^a, Y^b + n^d Y_d n^b) \\ &= g_{ab}X^a Y^b + g_{ab}X^a n^d Y_d n^b + g_{ab}Y^b n^d X_d n^a + g_{ab}n^c X_c n^a n^d Y_d n^b \\ &= g_{ab}X^a Y^b + X_b n^b n^d Y_d + Y_a n^a n^d X_d - n^c X_c n^d Y_d \\ &= g_{ab}X^a Y^b + X^b n_b Y^d n_d + Y^a n_a X^d n_d - n^c X_c n^d Y_d \\ &= (g_{ab} + n_a n_b)X^a Y^b \\ &:= h_{ab}X^a Y^b. \end{aligned} \tag{4.38}$$

One can note that

1. The metric h_{ab} lives in Σ_t:

$$n^a h_{ab} = n^a(g_{ab} + n_a n_b) = n^a g_{ab} + n^a n_a n_b = n_b - n_b = 0. \tag{4.39}$$

2. Let s^a be a vector tangent to Σ_t, then

$$h_{ab}s^a = (g_{ab} + n_a n_b)s^a = g_{ab}s^a + s^a n_a n_b.$$

But $s^a n_a = 0$ since they are orthogonal, hence:

$$h_{ab}s^a = g_{ab}s^a. \tag{4.40}$$

So, when applied to vectors tangent to Σ_t, the induced metric h_{ab} gives the same geometry as g_{ab}.

One can then use the induced metric h_{ab} to describe projections of any geometrical object. In coordinates, for a rank (m, n) tensor, one gets:

$$(P_{\|}T)^{a_1 \dots a_m}_{b_1 \dots b_n} = h^{a_1}_{c_1} \dots h^{a_m}_{c_m} h^{d_1}_{b_1} \dots h^{d_n}_{b_n} T^{c_1 \dots c_m}_{d_1 \dots d_n}. \tag{4.41}$$

To study the dynamics of the canonical formulation, we consider the induced 3-metric h_{ab} as a time-dependent 3-dimensional tensor field evolving on a family of manifolds Σ_t. Then, the time dependent field h_{ab} will be the configuration variables of canonical gravity.

However, in order to do this, we have to define a time evolution vector field t^a that specifies the directions of time derivatives, since one will need to take time derivatives of the induced metric or any other vector fields.

4.3.4 Time derivatives

If the spacetime is the history of the evolution of the slices Σ_t, how can one say how a field in Σ_t, let us say, h_{ab}, evolves?

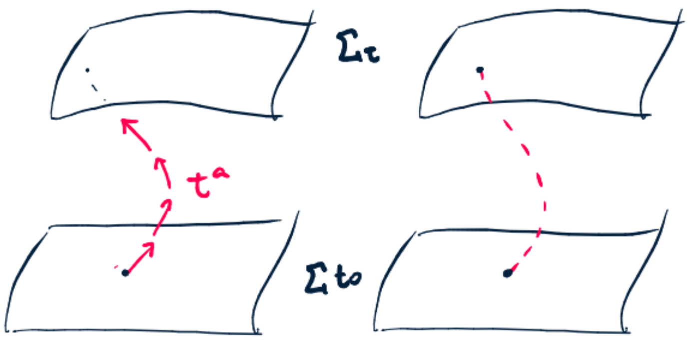

Figure 4.5: The time vector field

If one has just two slices in the foliation, it is impossible to say how a field defined on them changes, unless we can uniquely associate a point on one slice to a point on the other one. The vector field that connects a point in one slice to its correspondent point in another one is the time evolution vector field t^a, whose integral curves are shown in the right part of figure (4.5).

To ensure that this vector field agrees with the concept of time it is required that

$$t^a \nabla_a t = 1, \tag{4.42}$$

which states that the change of t in the direction of the time evolution vector field t^a is just the unity.

It is assumed that the spatial coordinates x^b are held fixed:

$$t^a \nabla_a x^b = 0, \tag{4.43}$$

so that

$$t^a \nabla_a := \frac{\partial}{\partial t}. \tag{4.44}$$

By introducing the **shift vector** N^a

$$N^a := P_\parallel t^a = h^{ab} t_b \tag{4.45}$$

and the **lapse function** N, which is the amount of the vector field t^a in the direction orthogonal to Σ_t:

$$N n^a := t^a - h^{ab} t_b, \tag{4.46}$$

and by acting with n_a on both sides of equation (4.46) one gets

$$N n^a n_a = n_a t^a - n_a h^{ab} t_b = n_a t^a.$$

Since $n^a n_a = -1$, we get

$$N = -n_a t^a. \tag{4.47}$$

The time evolution vector field t^a can then be written in its normal and tangential parts with respect to the surface Σ_t:

$$t^a = N n^a + N^a. \tag{4.48}$$

And now, with the projection operators, the definition of a time derivative of any tensor field is also possible:

$$\dot{T}^{a_1 \ldots a_m}_{b_1 \ldots b_n} := P_\parallel \left(\mathcal{L}_t T^{a_1 \ldots a_m}_{b_1 \ldots b_n} \right) = \left(h^{a_1}_{c_1} \ldots h^{a_m}_{c_m} h^{d_1}_{b_1} \ldots h^{d_n}_{b_n} \right) \mathcal{L}_t T^{c_1 \ldots c_m}_{d_1 \ldots d_n}. \tag{4.49}$$

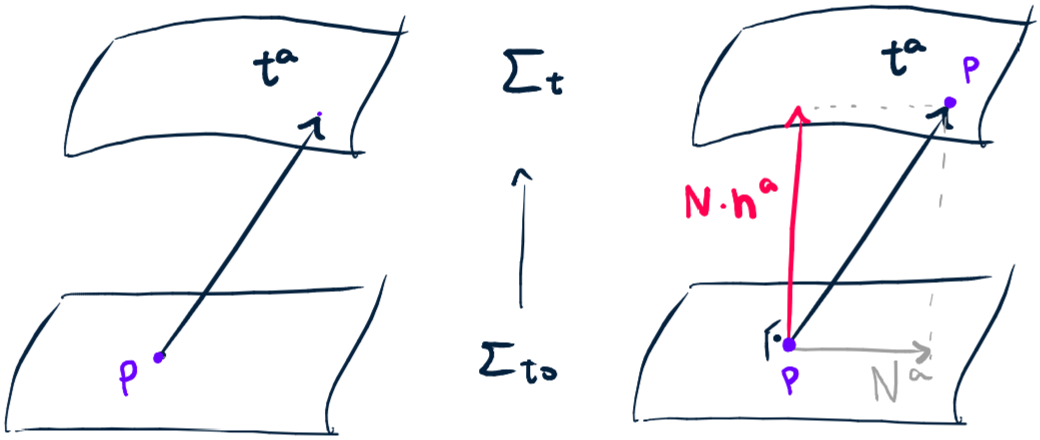

Figure 4.6: The components of the time vector field

4.3.5 Metric decomposition

From (4.48), the normal vector field can be written as

$$n^a = \frac{1}{N}(t^a - N^a) \,, \tag{4.50}$$

which allows us to write the inverse spacetime metric as

$$g^{ab} = h^{ab} - n^a n^b = h^{ab} - \frac{1}{N^2}(t^a - N^a)(t^b - N^b). \tag{4.51}$$

We can then invert this matrix and get the line element

$$ds^2 = g_{ab}dx^a dx^b = -N^2 dt^2 + h_{ab}(dx^a + N^a dt)(dx^b + N^b dt). \tag{4.52}$$

We have then decomposed the metric in ten independent terms: the lapse function N and the three components of the shift vector N^a; and six independent terms h_{ab} . The idea is to express any geometrical property in terms of these variables: N, N^a and h_{ab}.

It will be also useful to express the determinant g of the metric g_{ab} in terms of the determinant h of the induced metric h_{ab}, since it appears in the Einstein-Hilbert action. One can do this as follows: from equation (4.51) we can see that $g^{00} = -1/N^2$. Then we can use the relation

$$(A^{-1})^{ij} = \frac{C_{ij}}{\det A} \,,$$

where $(A^{-1})^{ij}$ is the element of the i-th row and j-th column of the inverse matrix of A, and C_{ij} is the correspondent cofactor matrix, i.e. the determinant of the minor matrix obtaining by eliminating the i-th row and j-th column from the matrix A. Then we have

$$\begin{aligned} g^{00} &= -\frac{1}{N^2} \\ &= \frac{C_{00}}{\det(g_{ab})} \\ &= \frac{\det(h_{cd})}{\det(g_{ab})} \,, \end{aligned}$$

from which we conclude that

$$g = \det(g_{ab}) = -N^2 \det(h_{cd}) = -N^2\, h \,. \tag{4.53}$$

4.3.6 Intrinsic and Extrinsic Geometry

The induced metric h_{ab} allows us to define a unique covariant derivative metric compatible in Σ_t. If we represent it by D_a, the metric compatibility requires — beyond the torsion free condition — that

$$D_a h_{bc} = 0. \tag{4.54}$$

One can show that this covariant derivative D_a compatible with the induced metric h_{ab} is just the parallel part of ∇_a, i.e.

$$D_a := P_{\|} \nabla_a. \tag{4.55}$$

This is proven as follows:

$$\begin{aligned} D_a h_{bc} = P_{\|} [\nabla_a h_{bc}] &= P_{\|} [\nabla_a (g_{bc} + n_b n_c)] \\ &= P_{\|} [\nabla_a (n_b n_c)] \\ &= P_{\|} [n_c \nabla_a n_b + n_b \nabla_a n_c)] \\ &= P_{\|}(n_c) P_{\|}(\nabla_a n_b) + P_{\|}(n_b) P_{\|}(\nabla_a n_c) \\ &= 0 \,, \end{aligned} \tag{4.56}$$

where in the second line we used the compatibility of ∇_a with the metric g_{ab} and in the last line the fact that $P_{\|}(n_b) = 0$.

This covariant derivative D_a can be seen as the projection in Σ_t of the derivative ∇_a by the induced metric h_{ab}:

$$D_c T^{a_1 \ldots a_m}_{b_1 \ldots b_n} := (h^{a_1}_{c_1} \ldots h^{a_m}_{c_m} h^{d_1}_{b_1} \ldots h^{d_n}_{b_n}) h^f_c \nabla_f T^{c_1 \ldots c_m}_{d_1 \ldots d_n}. \tag{4.57}$$

Definition 4.3.1 (Intrinsic Curvature)**.** Given the three dimensional covariant derivative D_a, we can define the intrinsic-curvature tensor ${}^3R_{abc}{}^d$ as for any other covariant derivative:

$${}^3R_{abc}{}^d \omega_d = D_a D_b \omega_c - D_b D_a \omega_c \tag{4.58}$$

for any spatial 1-form ω_c, i.e., $\omega_a n^a = 0$.

With this definition, one can obtain the Ricci tensor ${}^3R_{ab}$ and the Ricci scalar ${}^3R$ by the usual contractions.

The intrinsic geometry refers only to (Σ, h_{ab}). But because Σ is spatial, we cannot talk about the evolution of the system using only parameters intrinsic to the manifold.

A geometrical object — the extrinsic curvature K_{ab} — will naturally arise when we try to make the induced metric evolve:

$$\begin{aligned} \mathcal{L}_n h_{ab} &= n^c \nabla_c h_{ab} + h_{ac} \nabla_b n^c + h_{bc} \nabla_a n^c \\ &= n^c \nabla_c (g_{ab} + n_a n_b) + \nabla_b n_a + \nabla_a n_b \\ &= n^c \nabla_c (n_a n_b) + \nabla_b n_a + \nabla_a n_b \\ &= n^c n_a \nabla_c n_b + n^c n_b \nabla_c n_a + \nabla_b n_a + \nabla_a n_b \\ &= (g^c_a + n_a n^c) \nabla_c n_b + (g^c_b + n_b n^c) \nabla_c n_a \\ &= h^c_a \nabla_c n_b + h^c_b \nabla_c n_a \\ &= K_{ab} + K_{ba}, \end{aligned} \tag{4.59}$$

where the object K_{ab} appears in the context of the evolution of the induced metric h_{ab}. Also, in the third line we developed $h_{ac} \nabla_b n^c = (g_{ac} + n_a n_c) \nabla_b n^c = \nabla_b n_a + n_a n_c \nabla_b n^c = \nabla_b n_a$, since $n_c \nabla_b n^c = 0$ as shown in equation (4.61).

Definition 4.3.2 (Extrinsic Curvature)**.** Given any normal vector n^a to the surface Σ, the extrinsic-curvature tensor is a spatial tensor on Σ defined by

$$K_{ab} \coloneqq D_a n_b = h_a^c \, h_b^d \nabla_c n_d. \tag{4.60}$$

We could also omit the first projector h_b^d on the definition because

$$\begin{aligned} K_{ab} \coloneqq D_a n_b &= h_a^c \, h_b^d \nabla_c n_d \\ &= h_a^c \, (g_b^d + n^d n_b) \nabla_c n_d \\ &= h_a^c \, g_b^d \nabla_c n_d + h_a^c n_b n^d \nabla_c n_d \\ &= h_a^c \, \nabla_c n_b, \end{aligned}$$

since, in the third line, $n^d \nabla_c n_d = 0$. This is easy to see, since

$$n^d \nabla_c n_d = \frac{1}{2}(n^d \nabla_c n_d + n_d \nabla_c n^d) = \frac{1}{2} \nabla_c (n_d n^d) = 0. \tag{4.61}$$

Another way of thinking about the extrinsic curvature tensor is as the normal component of the derivative of v with respect to u, for u and v spatial:

$$K(u, v) = -g(\nabla_u v, n). \tag{4.62}$$

This notion is captured when one splits the derivative $\nabla_u v$ in its normal and tangential parts

$$\nabla_u v = -g(\nabla_u v, n) n + (\nabla_u v + g(\nabla_u v, n) n),$$

where the first term represents the normal part of it and the second one the tangential part. So, when we parallel transport v, who lives in Σ, in the direction of u, which also lives in Σ, the emergence of a normal component in this parallel transport measures exactly the curvature in that region.

This way of thinking agrees with our previous definition of K_{ab}, since, from the that definition, we had

$$\begin{aligned} K_{ab} u^a v^b &= (D_a n_b) u^a v^b \\ &= h_a^c (\nabla_c n_b) u^a v^b \\ &= (\nabla_c n_b) u^c v^b, \end{aligned} \tag{4.63}$$

and, from the notion now placed, we have

$$\begin{aligned} K(u, v) &= -g(\nabla_u v, n) \\ &= -g_{ab} (\nabla_u v^a) n^b \\ &= -(u^c \nabla_c v^a) n_a \\ &= u^c (\nabla_c n_a) v^a \\ &= (\nabla_c n_b) u^c v^b, \end{aligned} \tag{4.64}$$

where in the third line we used the metric to lower the index of n^b and in the fourth line we used the fact that both v and u are spatial, then $\nabla_c (v^a n_a) = 0$, then $n_a \nabla_c v^a = -v^a \nabla_c n_a$. In the last line we only renamed a dummy index so it agrees with equation (4.63).

With this view, the tensor K measures how much the surface Σ is curved in the way it sits in M, because it says how much a vector tangent to Σ will fail to be tangent if parallel transported using the Levi-Civita connection ∇ on M.

In components, we have

$$K(u,v) = K_{ij}u^i v^j$$

in local coordinates, where

$$K_{ij} = K(\partial_i, \partial_j).$$

From this point of view it is easy to see that this tensor is symmetric, since

$$\begin{aligned} K_{ij} - K_{ji} &= K(\partial_i, \partial_j) - K(\partial_j, \partial_i) \\ &= -g(\nabla_i \partial_j, n) + g(\nabla_j \partial_i, n) \\ &= -g(\nabla_i \partial_j - \nabla_j \partial_i, n) \\ &= -g([\partial_i, \partial_j], n) \\ &= -g(0, n) \\ &= 0. \end{aligned} \tag{4.65}$$

The extrinsic-curvature tensor has some important properties:

1. It is symmetric:
$$K_{ab} = K_{ba}, \tag{4.66}$$
as shown right above.

2. As developed in equation (4.59) and using the property above, we get that the extrinsic curvature tensor is half of the Lie derivative of the intrinsic metric along the unit normal:
$$K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}. \tag{4.67}$$

3. The extrinsic curvature tensor can be related to the intrinsic curvature h_{ab}, the shift vector N^a and the lapse function N via
$$K_{ab} = \frac{1}{2N}\left(\dot{h}_{ab} - D_a N_b - D_b N_a\right), \tag{4.68}$$
which can be proven as follows:

$$\begin{aligned} K_{ab} &= \frac{1}{2}\mathcal{L}_n h_{ab} \\ &= \frac{1}{2}[n^c \nabla_c h_{ab} + h_{ac}\nabla_b n^c + h_{bc}\nabla_a n^c] \\ &= \frac{1}{2N}[N n^c \nabla_c h_{ab} + h_{ac}\nabla_b (N n^c) + h_{bc}\nabla_a (N n^c)] \\ &= \frac{1}{2N}[(t^c - N^c)\nabla_c h_{ab} + h_{ac}\nabla_b (t^c - N^c) + h_{bc}\nabla_a (t^c - N^c)] \\ &= \frac{1}{2N}\mathcal{L}_{t-N} h_{ab} \\ &= \frac{1}{2N} h_a^d h_b^c \mathcal{L}_{t-N} h_{cd} \\ &= \frac{1}{2N} h_a^d h_b^c [\mathcal{L}_t h_{cd} - \mathcal{L}_N h_{cd}] \\ &= \frac{1}{2N}\left(h_a^d h_b^c \mathcal{L}_t h_{cd} - h_a^d h_b^c \mathcal{L}_N h_{cd}\right) \\ &= \frac{1}{2N}\left(hab - D_a \dot{N}_b - D_b N_a\right), \end{aligned}$$

where, from the third to the fourth line we used equation (4.48), and in the sixth line we just smuggled in the induced metric to get the spatial part of the calculation, since K_{ab} is purely spatial. In the last line we just used the definition of the time derivative of a tensor given by (4.49) and used the fact that the shift vector is spatial, then

$$\begin{aligned}\mathcal{L}_N h_{ab} &= P_{\|}[N^c\nabla_c h_{ab} + h_{ac}\nabla_b N^c + h_{bc}\nabla_a N^c] \\ &= P_{\|}[N^c\nabla_c h_{ab}] + P_{\|}[h_{ac}\nabla_b N^c + h_{bc}\nabla_a N^c] \\ &= N^c P_{\|}[\nabla_c h_{ab}] + P_{\|}[(g_{ac} + n_a n_c)\nabla_b N^c + (g_{bc} + n_b n_c)\nabla_a N^c] \\ &= N^c D_c h_{ab} + P_{\|}[\nabla_b N_a + \nabla_a N_b] \\ &= 0 + D_a N_b + D_b N_a \\ &= D_a N_b + D_b N_a\,.\end{aligned}$$

4.3.7 Curvature relations

Using the definitions and properties previously mentioned we can prove the following relations among the curvature tensors [4].

The Gauss equation

This relation comes from computing the Riemann curvature tensor $R_{efg}{}^h$ in terms of the intrinsic curvature ${}^3R_{abc}{}^d$ and the extrinsic curvature K_{ab}:

$$h_a^e\ h_b^f\ h_c^g\ R_{efg}{}^h = {}^3R_{abc}{}^d + K_{ac}K_b^d - K_{bc}K_a^d \tag{4.69}$$

The Codazzi equation

This relation comes from computing the parallel part of the Riemann curvature tensor contracted with the unitary normal vector $P_{\|}(R_{abcd}n^d)$ which equals

$$P_{\|}(R_{abcd}n^d) = h_e^a\ h_f^b\ h_g^c\ R_{abcd}n^d = D_e K_{fg} - D_f K_{eg}. \tag{4.70}$$

The Ricci equation

This last equation comes from taking the lie derivative $\mathcal{L}_n$ along the unit normal n^a if the extrinsic curvature K_{ab}:

$$R_{abcd}n^c n^d = -\mathcal{L}_n K_{ab} - K_{ac}K_b^c + D_{(a}a_{b)} + a_a a_b, \tag{4.71}$$

where a_a is the normal acceleration $a_a := n^c\nabla_c n_a$ (with $a_a n^a = 0$).

We could also use the Ricci equation (4.71) with the relation $R_{ab}n^a n^b = R_{acd}{}^d n^a n^b$ to get

$$R_{ab}n^a n^b = (K_a^a)^2 - K_a^b\ K_b^a + \nabla_a v^a, \tag{4.72}$$

where the vector field v^a is defined as

$$v^a := -n^a\nabla_c n^c + n^c\nabla_c n^a.$$

Using the Gauss-Codazzi equations with the Ricci equations one can read the Ricci scalar R:

$$R = {}^3R + K_{ab}K^{ab} - (K_a^a)^2 - 2\nabla_a v^a. \tag{4.73}$$

Hence, up to a divergence term, we can decompose the Ricci scalar into a potential term ${}^3R$ and a kinetic term — quadratic in extrinsic curvature. Then, the extrinsic curvature, as shown in equation (4.68), plays the role of a velocity of the spatial metric h_{ab} and is, thus, a candidate for its momentum when we formulate the GR in terms of canonical variables, as we do next.

4.4 The ADM formalism

The action of general relativity in metric variables is given, as already presented, by the Einstein-Hilbert action

$$S_{E.H.}[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-\det g}R := \int dt L_{grav}.$$

Using equations (4.53) and (4.73) one can write the Lagragian for gravity as

$$L_{grav} = \frac{1}{16\pi G}\int d^3x N\sqrt{\det h}\left({}^3R + K_{ab}K^{ab} - (K^a_{\ a})^2\right), \tag{4.74}$$

where the term proportional to $\nabla_a v^a$ was left out once it is a boundary term which does not affect the equations of motion.

From equation (4.68) we can see that the action depends on $\dot{h}_{ab}$ because of the K_{ab} term, but it is independent of time derivatives of the remaining space-time metric components, as expected, and also of time derivatives of N and N^a.

So we may already extract the primary constraints:

$$p_N(x) = \frac{\delta L_{grav}}{\delta \dot{N}(x)} = 0, \tag{4.75}$$

and

$$p_a(x) = \frac{\delta L_{grav}}{\delta \dot{N}^a(x)} = 0. \tag{4.76}$$

The conjugate momenta of the induced metric h_{ab} is

$$\begin{aligned}\pi^{ab}(x) &= \frac{\delta L_{grav}}{\delta \dot{h}_{ab}(x)} \\ &= \frac{\delta L_{grav}}{\delta K_{ab}}\frac{\delta K_{ab}}{\dot{h}_{ab}(x)} \qquad (4.77)\\ &= \frac{\delta L_{grav}}{\delta K_{ab}}\frac{1}{2N}, \qquad (4.78)\end{aligned}$$

where the last line comes from equation (4.68).

So we get

$$\pi^{ab}(x) = \frac{\sqrt{\det h}}{16\pi G}\left(K^{ab} - K^c_{\ c}\,h^{ab}\right). \tag{4.79}$$

Contracting this relation with h_{ab} we get

$$\begin{aligned}\frac{16\pi G}{\sqrt{\det h}}\pi^{ab}h_{ab} &= h_{ab}K^{ab} - K^c_{\ c}\,h^{ab}h_{ab} \\ &= K^a_{\ a} - 3K^c_{\ c} \\ &= -2K^a_{\ a},\end{aligned}$$

and it follows that

$$K^a_{\ a} = -\frac{8\pi G}{\sqrt{\det h}}\pi^a_{\ a}, \tag{4.80}$$

which allows us to isolate K^{ab} in equation 4.79:

$$K^{ab} = \frac{8\pi G}{\sqrt{\det h}}(2\pi^{ab} - \pi^c_{\ c} h^{ab}). \tag{4.81}$$

With this last relation we can express $\dot{h}_{ab}$ in (4.68) in terms of its conjugate momenta π^{ab}:

$$\dot{h}_{ab} = \frac{16\pi GN}{\sqrt{\det h}}(2\pi^{ab} - \pi^c_{\ c} h^{ab}) + 2D_{(a}N_{b)}. \tag{4.82}$$

Then we can obtain the Hamiltonian through

$$H(t) = \int d^3x \left([\pi^{ab}\dot{h_{ab}}] + \lambda p_N + \mu^a p_a\right) - L(t), \tag{4.83}$$

where the λ and $\mu' s$ are the Lagrange multipliers of the constraints.

Using equation (4.82) to write $\dot{h}_{ab}$ in terms of its conjugate momenta π^{ab} we can write (4.83) as

$$H = \int d^3x \left[\frac{16\pi GN}{\sqrt{\det h}}\left(\pi_{ab}\pi^{ab} - \frac{1}{2}(\pi^a_{\ a})^2\right) + 2\pi^{ab}D_a N_b - \frac{N\sqrt{\det h}}{16\pi G}\,{}^3R + \lambda p_N + \mu^a p_a\right]. \tag{4.84}$$

Applying the consistency conditions (4.29) to the constraints we get secondary constraints:

$$0 = \dot{p_N} = \{p_N, H_{total}\} := -C_{grav}(h_{ab}, \pi^{ab}), \tag{4.85}$$
$$0 = \dot{p_a} = \{p_a, H_{total}\} := -C^{grav}_a(h_{ab}, \pi^{ab}). \tag{4.86}$$

And if we work out the Poisson's brackets above we get [4]

$$C_{grav} = \frac{16\pi GN}{\sqrt{\det h}}\left(\pi_{ab}\pi^{ab} - \frac{1}{2}(\pi^a_{\ a})^2\right) - \frac{N\sqrt{\det h}}{16\pi G}\,{}^3R \approx 0, \tag{4.87}$$

which is called the *Hamiltonian constraint*.
Working out the second Poisson bracket [4] we get

$$C^{grav}_a = -2D_b\pi^b_{\ a} \approx 0, \tag{4.88}$$

which is called the *diffeomorphism constraint*.

We can now see that, putting these in (4.84), the lapse function N and the shift vector N^a play the role of Lagrange multipliers of the secondary constraints:

$$H = \int d^3x\,[NC_{grav} + N^a C^{grav}_a + \lambda p_N + \mu^a p_a] + H_{\partial\Sigma}, \tag{4.89}$$

where the last term refers to the Hamiltonian of the boundary term.

We have finally built a Hamiltoninan representation of the dynamics of the spacetime geometry. The canonical variables here are the induced metric h_{ab} and its conjugate momenta π^{ab}. With this Hamiltonian it is now possible to study the spacetime dynamics in a canonical way, using every tool of the Hamiltonian formalism.

4.5 The equations of motion

Let us now obtain the evolutionary part of Einstein's equations canonically.

The Hamilton equations give $\dot{N}(x) = \lambda(x)$ and $\dot{N}^a(x) = \mu^a(x)$, which means that these functions can change arbitrally due to reparametrizations. We have also the equations

$$\dot{h}_{ab} = \{h_{ab}, H_{grav}\},$$

which gives us back equation (4.82). Finally, we have also the equation of motion

$$\dot{\pi}^{ab} = \left\{\pi^{ab}, H_{grav}\right\},$$

which, when developed, gives us

$$\begin{aligned}\dot{\pi}^{ab} = & -\frac{N\sqrt{\det h}}{16\pi G}\left({}^3R^{ab} - \frac{1}{2}\,{}^3Rh^{ab}\right) + \frac{8\pi GN}{\sqrt{\det h}}h^{ab}\left(\pi^{cd}\pi_{cd} - \frac{1}{2}(\pi^c_{\ c})^2\right) + \\ & -\frac{32\pi GN}{\sqrt{\det h}}\left(\pi^{ac}\pi^b_{\ c} - \frac{1}{2}\pi^{ab}\pi^c_{\ c}\right) + \frac{\sqrt{\det h}}{16\pi G}\left(D^aD^bN - h^{ab}D_cD^cN\right) \\ & + \sqrt{\det h}D_c\left(\frac{\pi^{ab}N^c}{\sqrt{\det h}}\right) - 2\pi^{c(a}D_cN^{b)}. \end{aligned} \tag{4.90}$$

Chapter 5

Tetrads Formalism and Palatini Action

Geometry is the archetype of the beauty of the world.

Johannes Kepler

5.1 Introduction

Since Riemannian manifolds are locally flat, one can choose an orthonormal basis of vectors $\{e_0, e_1, e_2, e_3\}$ for each point P on the manifold M, as shown in figure (5.1).

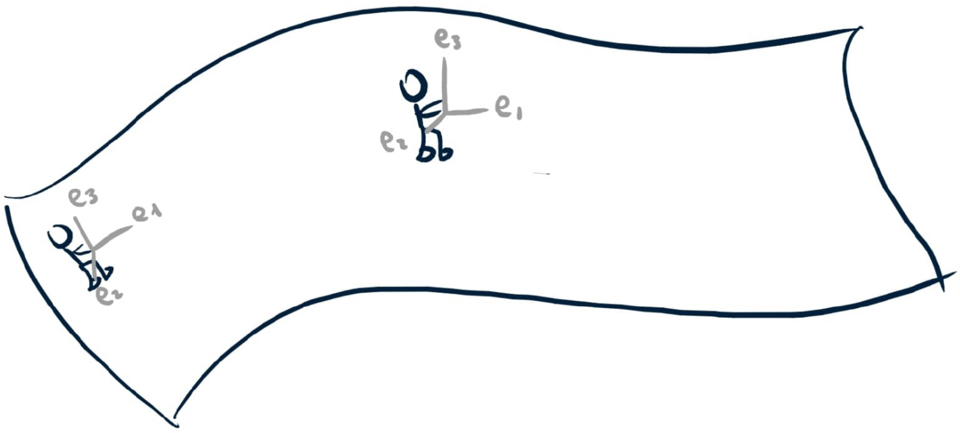

Figure 5.1: The local tetrad basis.

One can write the basis vector of T_pM as a linear combination of the orthornormal local tetrad basis at P:

$$\partial_\mu = e^I_\mu e_I, \tag{5.1}$$

where the matrix e^I_μ contain the coefficients of the linear combination. The greek letters represent the world indices in the manifold — it refers to the coordinalization of the manifold. The latin letters represent an internal index, referring to the components of the local orthonormal basis at P. This is shown in figure (5.2).

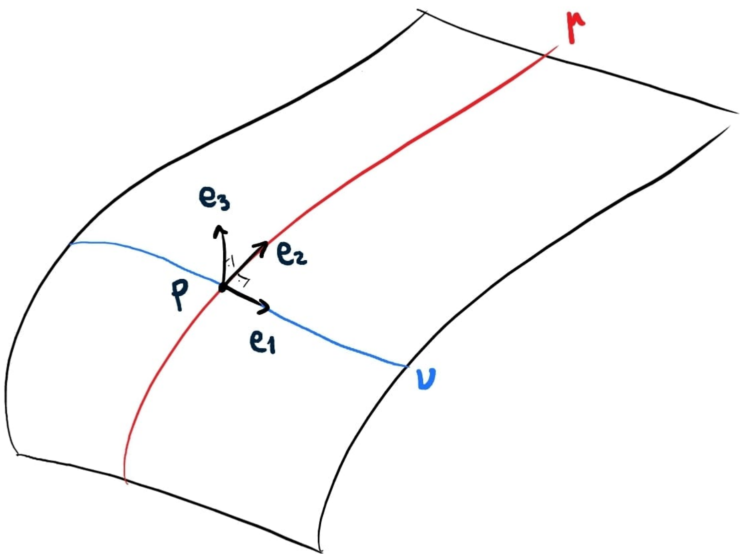

Figure 5.2: Internal vs World indices.

5.2 Tetrad formalism

Since the tetrads represent an orthonormal basis, it is required that $e_I \cdot e_J = \eta_{IJ}$, then, using (5.1) we get

$$\begin{aligned} g_{\mu\nu} &= g(\partial_\mu, \partial_\nu) \\ &= \partial_\mu \cdot \partial_\nu \\ &= (e_\mu^I e_I) \cdot (e_\nu^J e_J) \\ &= e_\mu^I e_\nu^J \eta_{IJ}, \end{aligned} \tag{5.2}$$

and we can also rewrite this relation in index free notation as

$$g = e^T \eta \, e.$$

In this way, we can see the tetrad as a similarity transformation that diagonalizes the metric $g_{\mu\nu}$ and scales it to the unit. The η matrix is the euclidean metric if we are talking about 3D space (then the basis e should be called a *triad*) or the Minkowski metric if we are talking about spacetime — where the name *tetrad* makes more sense.

Taking the determinant of this equation we get

$$g = -e^2, \tag{5.3}$$

where g stands for the determinant of the spacetime metric $g_{\mu\nu}$ and e for the determinant of the matrix e_μ^I. The minus sign comes from the determinant of the Minkowski metric.

Hence, the tetrad represents the square root of the metric and has, therefore, all the information about the geometry of the manifold. We can thus consider the tetrad as the fundamental description and the metric as a derived concept.

The spacetime indices are contracted with the metric $g_{\mu\nu}$, as usual, and the internal indices are contracted with the flat spacetime metric η_{IJ}, which, consisting of $0s$ and $\pm 1s$ is much easier to deal with than $g_{\mu\nu}$ — we will see that this is the whole point of the formalism.

Thinking of $e(x)$ as a square matrix, we can define its inverse e_I^μ such that

$$e_I^\mu e_\mu^J = \delta_I^J. \tag{5.4}$$

It can be sometimes confusing in the literature what does the term *tetrads* specifically refers to. Here, when we say tetrad, one can think of the local orthonormal basis vectors e_I, or its dual — the 1-form $e^I = e^I_\mu dx^\mu$ — or also the matrix e^I_μ containing the coefficients of the linear transformation.

5.3 Connections via tetrads

If we have a spacetime vector field v^μ and we take its derivative in a certain direction we get

$$(\nabla_\rho v)^\mu = \partial_\rho v^\mu + \Gamma^\mu_{\ \rho k} v^k, \tag{5.5}$$

where $\Gamma^\mu_{\ \rho k}$ is the Levi-Civita connection.

In a similar way, when we compute the derivative of vectors in the internal space, we expect something such as

$$(D_a v)^I = \partial_a v^I + \omega_{a\ J}^{\ I} v^J, \tag{5.6}$$

where the 1-form $\omega_{a\ J}^{\ I}$ is the *spin connection*. This is of course valid for a vector. For a general tensor we add a ω factor for each of its indices, just as in the covariant derivative with the Levi-Civita connection. So, for a rank (r, s) tensor we would have:

$$\begin{aligned} D_a T^{\mu_1 \dots \mu_r}_{\ \ \nu_1 \dots \nu_s} = \partial_a T^{\mu_1 \dots \mu_r}_{\ \ \nu_1 \dots \nu_s} + \\ + \omega^{\mu_1}_{\ \ ak} T^{k \dots \mu_r}_{\ \ \nu_1 \dots \nu_s} + \dots + \omega^{\mu_r}_{\ \ ak} T^{\mu 1 \dots k}_{\ \ \nu_1 \dots \nu_s} + \\ - \omega^k_{\ \ a\nu_1} T^{\mu_1 \dots \mu_r}_{\ \ k \dots \nu_s} + \dots - \omega^k_{\ \ a\nu_s} T^{\mu_1 \dots \mu_r}_{\ \ \nu_1 \dots k}. \end{aligned}$$

It can be easily seen that the spin connection is a 1-form since, on a curved manifold, when we move from a point x to a nearby point $x + dx$ it is expected that the local frame will rotate (in Euclidean space) or Lorentz transform (in Minkowski flat spacetime), thus, an infinitesimal translation has the effect of rotating the 1-form $e^I(x)$ infinitesimally. Hence, if we apply the exterior derivative d to this 1-form we should get

$$de^I = -\omega^I_{\ J} e^J, \tag{5.7}$$

for some antisymmetric ω_{IJ}, since the generators of rotations or Lorentz transformations are antisymmetric. The minus sign is just a convention. Since e^I is a 1-form, de^I is a 2-form and

$$\omega^I_{\ J} = \omega^I_{\ \mu J} dx^\mu, \tag{5.8}$$

is also a 1-form.

If we evaluate the covariant derivative of the Minkowski metric we get

$$\begin{aligned} D_a \eta_{IJ} &= \partial_a \eta_{IJ} + \omega_{a\ I}^{\ K} \eta_{KJ} + \omega_{a\ J}^{\ K} \eta_{IK} \\ &= \omega_{a\ I}^{\ K} \eta_{KJ} + \omega_{a\ J}^{\ K} \eta_{IK} \\ &= \omega_{a\ IJ} + \omega_{a\ JI}, \end{aligned} \tag{5.9}$$

and, if the covariant derivative is required to be compatible with η we get

$$\omega_{a\ IJ} = -\omega_{a\ JI}, \tag{5.10}$$

which attest the antisymmetry of the spin connection in its internal indices, as said before. This means that the coefficients of the connection take values in the Lie Algebra of the Lorentz group of that signature, as developed in appendix A.

We can built the relation between the spin connection ω on the internal space and the Levi Civita connection Γ on the manifold. Remember that we can always express a vector v at point P by a linear combination of the internal basis vectors e_I or by the spacetime basis vectors on the tangent space T_pM, the $\partial'_\mu s$: $v = v^I e_I = v^\mu \partial_\mu$.

Also, the connection ω on the internal space induces a connection on the tangent space T_pM for a given tetrad e. The covariant derivative $\tilde{D}$ on the internal space is defined via

$$\nabla v = e^{-1}[\tilde{D}e(v)]. \tag{5.11}$$

Developing these two derivatives leads us to

$$\begin{aligned}\nabla_\rho v = \nabla_\rho(v^I e_I) &= \nabla_\rho(v^\mu \partial_\mu)\\ e_I(\partial_\rho v^I) + v^K \omega_{\rho\ K}^{\ J} e_J &= (\partial_\rho v^\mu + v^k \Gamma^\mu_{\rho k})\partial_\mu,\end{aligned}$$

and, using $\partial_\mu = e^I_\mu e_I$ on the right side and $v^K = v^\mu e^K_\mu$ we get

$$\begin{aligned}e_I\partial_\rho(v^\mu e^I_\mu) + v^\mu e^K_\mu \omega_{\rho\ K}^{\ J} e_J &= (\partial_\rho v^\mu + v^k \Gamma^\mu_{\rho k})e^I_\mu e_I\\ e_I v^\mu(\partial_\rho e^I_\mu) + e_I e^I_\mu(\partial_\rho v^\mu) + e^K_\mu v^\mu \omega_{\rho\ K}^{\ J} e_J &= (\partial_\rho v^\mu)e^I_\mu e_I + v^k \Gamma^\mu_{\rho k} e^I_\mu e_I\\ v^\mu(\partial_\rho e^I_\mu) + e^K_\mu v^\mu \omega_{\rho\ K}^{\ I} &= v^k \Gamma^\mu_{\rho k} e^I_\mu\\ \partial_\rho e^I_\mu + e^K_\mu \omega_{\rho\ K}^{\ I} &= \Gamma^\nu_{\rho k} e^I_\nu,\end{aligned}$$

from where we can express the spin connection in terms of the Levi Civita connection:

$$\omega_{\nu\ J}^{\ I} = e^I_\rho \left(\partial_\nu e^\rho_J + e^\mu_J\, \Gamma^\rho_{\nu\mu}\right), \tag{5.12}$$

which will be useful later.

5.4 Curvature and Torsion via tetrads

If one defines the exterior covariant derivative as $D_\omega = d + \omega$ where d is the exterior derivative, it is possible to extract the curvature in the Cartan formalism. This is how one can take covariant derivatives of n-forms taking values in the internal space. We will use just D from now on to denote the exterior covariant derivative with respect to the connection ω.

Consider a 0-form ϕ^I, which has no spacetime index — only internal indices. Then [16]

$$D^I_{\ J}\phi^J = d\phi^I + \omega^I_{\ J}\phi^J. \tag{5.13}$$

If we calculate

$$D^K_I D^I_{\ J}\phi^J = d(d\phi^K + \omega^K_J\phi^J) + \omega^K_L(d\phi^L + \omega^L_{\ J}\phi^J), \tag{5.14}$$

the curvature will immediately emerge. The first term gives

$$dd\phi^I + (d\omega^K_J)\phi^J - \omega^K_J d\phi^J,$$

and the second term gives

$$\omega^K_L d\phi^L + \omega^K_L \omega^L_{\ J}\phi^J.$$

Then, since $d^2 = 0$, the sum gives, maintaining only the operators in the equation:

$$D^K_I D^I{}_J = d\omega^K{}_J + \omega^K{}_L \omega^L{}_J := F^K{}_J, \tag{5.15}$$

where F is the curvature 2-form:

$$F^{IJ} = F^{IJ}_{\mu\nu} dx^\mu dx^\nu, \tag{5.16}$$

i.e., this is the curvature of the connection 1-form ω^{IJ}_μ on the internal space. Since it is an antisymmetric tensor, its components are easily extracted from (5.15) and are given by

$$F^{IJ}_{\mu\nu} = \partial_\mu \omega^{IJ}_\nu - \partial_\nu \omega^{IJ}_\mu + [\omega_\mu, \omega_\nu]^{IJ}. \tag{5.17}$$

If we use equation (5.12) in (5.17) we can get the relation between the curvature 2-form F^{IJ} and the Riemann curvature tensor $R^\rho_{\mu\nu\sigma} = \partial_\mu \Gamma_{\nu\sigma} - \partial_\nu \Gamma_{\mu\sigma} + \Gamma^\rho_{\mu\alpha}\,\Gamma^\alpha_{\nu\sigma} - \Gamma^\rho_{\nu\alpha}\,\Gamma^\alpha_{\mu\sigma}$, which is

$$R^\rho_{\mu\nu\sigma} = e^\rho_I e^J_\sigma \, F_{\mu\nu}{}^I{}_J. \tag{5.18}$$

This relation shows that the Riemann curvature tensor $R^\rho_{\mu\nu\sigma}$ of the connection ∇ is just the spacetime image of the curvature $F_{\mu\nu}{}^I{}_J$ of the spin connection ω.

We can also convert all indices of $F^I_{\mu\nu J}$ to internal indices, which will be very useful to calculate the Ricci tensor and Ricci scalar. Thus, let us introduce the object

$$F_{MN}{}^I{}_J = F^I_{\mu\nu J}\, e^\mu_M e^\nu_N. \tag{5.19}$$

We can get the internal Ricci tensor by the usual contraction

$$F_{IJ} = F_{MI}{}^M{}_J, \tag{5.20}$$

and the Ricci tensor can be built from this via

$$R_{\mu\nu} = F_{IJ}\, e^I_\mu e^J_\nu. \tag{5.21}$$

We can also get the Ricci scalar in the usual way:

$$R = F_{IJ}\, \eta^{IJ}. \tag{5.22}$$

It is also easy to see that this scalar, in the internal structure, is the same as the Ricci scalar on the manifold:

$$\begin{aligned} R &= F_{IJ}\, \eta^{IJ} \\ &= R_{\mu\nu} e^\mu_I e^\nu_J\, \eta^{IJ} \\ &= R_{\mu\nu} g^{\mu\nu} \\ &= R, \end{aligned}$$

where, in the second line, we used equation (5.22) with inverses of e^I_μ applied.

We also may define the torsion in the local Minkowski space. First, the torsion T in the tangent bundle is given by

$$T(v,u) = \nabla_v u - \nabla_u v - [v,u], \tag{5.23}$$

which is, in coordinates:

$$T^\rho_{\mu\nu} = \nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu = \Gamma^\rho_{[\mu\nu]}.$$

However, from equation (5.12) we can write this as

$$\begin{aligned} T^{\rho}_{\mu\nu} &= e^{\rho}_{I}(\partial_\mu e^I_\nu - \partial_\nu e^I_\mu + \omega^I{}_{\mu J} e^J_\nu - \omega^I{}_{\nu J} e^J_\mu) \\ &= e^{\rho}_{I}(De^I)_{\mu\nu}, \end{aligned}$$

or, in spacetime index free notation

$$T^I = e^{-1}(De^I),$$

which states that the torsion in the tangent bundle is the pullback of De^I by the inverse of the tetrad. So, the torsion in the Minkowski bundle is just

$$T^I = De^I = de^I + \omega^I{}_J \wedge e^J. \tag{5.24}$$

5.5 Cartan's view of Riemannian geometry

Equations (5.7) and (5.15) are called first and second Cartan's structural equations, respectively. In Cartan's formalism, Riemannian geometry can be summarized by these two equations:

$$\begin{aligned} de^I + \omega^I_K \wedge e^K &= 0, \\ F^K_J &= d\omega^K_J + \omega^K_L \wedge \omega^L_J. \end{aligned} \tag{5.25}$$

The protocol goes as follows: given a metric, one chooses a basis of tetrads e^I satisfying equation (5.2). Then one can use the first of Cartan's equations to figure out the spin connection ω, and, finally, with the second of Cartan's equations, one has the curvature 2-form F. This is the easiest way of computing the Riemann curvature tensor, which can be done with the relations between F^{IJ} and $R^{\rho}_{\mu\nu\sigma}$ developed in the previous section.

5.5.1 The 2-sphere

It is instructive to develop an example to see how the formalism works, which we will do for the 2-sphere. The same can be done for the Schwarzschild metric or any other [9].

For the 2-sphere, writing the line element in spherical coordinates, we have

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = R^2 d\theta^2 + R^2 \operatorname{sen}^2\theta \, d\phi^2.$$

From $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$ we can immediately choose, for the tetrads:

$$\begin{aligned} e^1_\theta &= R \\ e^2_\phi &= R\operatorname{sen}\theta, \end{aligned}$$

with all the other components vanishing. For the 1-forms, we have $e^I = e^I_\mu dx^\mu$, so we get

$$\begin{aligned} e^1 &= R d\theta, \\ e^2 &= R\operatorname{sen}\theta d\phi. \end{aligned}$$

Now, from the first of Cartan's structural equations we get

$$\begin{aligned} de^1 + \omega^1_J \wedge e^J &= 0 \\ \omega^1_1 e^1 + \omega^1_2 e^2 &= 0 \\ \omega^1_1 R d\theta &= -\omega^1_2 R\operatorname{sen}\theta d\phi, \end{aligned} \tag{5.26}$$

and, since $\omega^1_1 = 0$, it follows that ω^1_2 is a 1-form proportional to $d\phi$.

Again, using the first of Cartan's structural equations:

$$\begin{aligned} de^2 + \omega^2_J \wedge e^J &= 0 \\ d(R\,\mathrm{sen}\theta d\phi) + \omega^2_1 e^1 + \omega^2_2 e^2 &= 0 \\ R\cos\theta(d\theta \wedge d\phi) + \omega^2_1 R d\theta + \omega^2_2 R\,\mathrm{sen}\theta d\phi &= 0, \end{aligned} \tag{5.27}$$

and again, since $\omega^k_k = 0$ (it is anti symmetric) we get

$$\begin{aligned} \omega^2_1 &= \cos\theta d\phi \\ \omega^1_2 &= -\cos\theta d\phi, \end{aligned}$$

with all the other components vanishing.

Now, from the second Cartan's structural equation:

$$F^I_{\ J} = d\omega^I_{\ J} + \omega^I_{\ K} \wedge \omega^K_{\ J},$$

we get directly the vanishing components $R^1_{\ 1} = R^2_{\ 2} = 0$ and the non-vanishing components:

$$\begin{aligned} F^1_{\ 2} &= d\omega^1_{\ 2} + \omega^1_{\ K} \wedge \omega^K_{\ 2} \\ &= \mathrm{sen}\theta(d\theta \wedge d\phi) + \omega^1_{\ 1} \wedge \omega^1_{\ 2} + \omega^1_{\ 2} \wedge \omega^2_{\ 2} \\ &= \mathrm{sen}\theta(d\theta \wedge d\phi), \end{aligned}$$

and, similarly, we get $F^2_{\ 1} = -\,\mathrm{sen}\theta(d\theta \wedge d\phi)$.

With these components in hand we can use equation (5.16) to write

$$F^I_{\ J} = F^I_{\mu\nu\ J}\, dx^\mu dx^\nu,$$

so, from the non-vanishing components we get

$$\begin{aligned} F^1_{\ 2} = \mathrm{sen}(\theta) d\theta \wedge d\phi &= F^1_{\mu\nu\ 2} dx^\mu dx^\nu \\ &= F^1_{\theta\phi\ 2} d\theta \wedge d\phi, \end{aligned}$$

from where we get

$$F^1_{\theta\phi\ 2} = \mathrm{sen}\theta,$$

and, in a similar way,

$$F^2_{\theta\phi\ 1} = -\,\mathrm{sen}\theta.$$

Now, from (5.18), we can write the components of the Riemann curvature tensor,

$$\begin{aligned} R^\rho_{\mu\nu\sigma} &= e^\rho_I e^J_\sigma\ F_{\mu\nu\ J}^{\ I} \\ &= e^\rho_1 e^2_\sigma\ F_{\mu\nu\ 2}^{\ 1} + e^\rho_2 e^1_\sigma\ F_{\mu\nu\ 1}^{\ 2}. \end{aligned}$$

The only non-vanishing component, up to the symmetries of the Riemann tensor, is:

$$\begin{aligned} R^\theta_{\theta\phi\phi} &= e^\theta_1 e^2_\phi\ F_{\theta\phi\ 2}^{\ 1} + e^\theta_2 e^1_\phi\ F_{\theta\phi\ 1}^{\ 2} \\ &= e^\theta_1 e^2_\phi\ F_{\theta\phi\ 2}^{\ 1} \\ &= R^{-1} R\,\mathrm{sen}(\theta) F_{\theta\phi\ 2}^{\ 1} \\ &= \mathrm{sen}^2\theta. \end{aligned}$$

Also, from (5.19) and (5.20) we can write the component F_{11} of the Ricci tensor in internal indices:

$$\begin{aligned}F_{11} &= F_{\mu\theta\ 1}^{\ \ N} e_N^\mu e_1^\theta \\ &= F_{\mu\theta\ 1}^{\ \ 2} e_2^\mu R^{-1} \\ &= F_{\phi\theta\ 1}^{\ \ 2} e_2^\phi R^{-1} \\ &= \ \mathrm{sen}(\theta)(R\,\mathrm{sen}\theta)^{-1} R^{-1} \\ &= 1/R^2.\end{aligned}$$

The F_{22} component can be extracted in the same way:

$$\begin{aligned}F_{22} &= F_{\mu\phi\ 2}^{\ \ N} e_N^\mu e_2^\phi \\ &= F_{\mu\phi\ 2}^{\ \ 1} e_1^\mu (R\,\mathrm{sen}\theta)^{-1} \\ &= F_{\theta\phi\ 2}^{\ \ 1} e_1^\theta (R\,\mathrm{sen}\theta)^{-1} \\ &= (\ \mathrm{sen}\theta)(R)^{-1}(R\,\mathrm{sen}\theta)^{-1} \\ &= 1/R^2,\end{aligned}$$

with the other components being $F_{12} = F_{21} = 0$.

We can also use (5.21) to write the Ricci tensor in spacetime:

$$\begin{aligned}R_{\theta\theta} &= F_{IJ} e_\theta^I e_\theta^J \\ &= F_{11} e_\theta^1 e_\theta^1 + F_{22} e_\theta^2 e_\theta^2 \\ &= F_{11}(e_\theta^1)^2 \\ &= R^{-2} R^2 \\ &= 1,\end{aligned}$$

and, for the other non-vanishing component:

$$\begin{aligned}R_{\phi\phi} &= F_{IJ} e_\phi^I e_\phi^J \\ &= F_{11} e_\phi^1 e_\phi^1 + F_{22} e_\phi^2 e_\phi^2 \\ &= F_{22}(e_\phi^2)^2 \\ &= R^{-2}(R\,\mathrm{sen}\theta)^2 \\ &= \ \mathrm{sen}^2\theta.\end{aligned}$$

For the Ricci scalar, using (5.22), we have, finally

$$R = F_{IJ}\eta^{IJ} = F_{11}\eta^{11} + F_{22}\eta^{22} = F_{11} + F_{22} = \frac{2}{R^2}.$$

5.6 The Palatini action

The Palatini action for GR is just the Einstein-Hilbert action written as a function of the frame field e and the connection ω:

$$S[e,\omega] = \frac{1}{16\pi G}\int d^4x\sqrt{-\det g}\ R[\omega]. \tag{5.28}$$

From equation (5.3) we can write $\det g$ in terms of the tetrads. And, from equation (5.18) we can get the Ricci tensor by the usual contraction:

$$R_{\mu\sigma} = R^{\nu}_{\mu\nu\sigma} = F_{\mu\nu}{}^{I}{}_{J} e^{\nu}_{I} e^{J}_{\sigma}, \tag{5.29}$$

hence, the curvature scalar can be written as

$$\begin{aligned} R &= R_{\mu\sigma} g^{\mu\sigma} \\ &= F_{\mu\nu}{}^{I}{}_{J} e^{\nu}_{I} e^{J}_{\sigma} g^{\mu\sigma} \\ &= F_{\mu\nu}{}^{I}{}_{J} e^{\nu}_{I} e^{J}_{\sigma} \eta^{MN} e^{\mu}_{M} e^{\sigma}_{N} \\ &= F_{\mu\nu}{}^{I}{}_{J} \eta^{MJ} e^{\nu}_{I} e^{\mu}_{M} \\ &= F_{\mu\nu}^{IM} e^{\nu}_{I} e^{\mu}_{M} \\ &= F_{\mu\nu}^{IJ} e^{\nu}_{I} e^{\mu}_{J}, \end{aligned} \tag{5.30}$$

where, in the third line we used the contraction $e^{J}_{\sigma} e^{\sigma}_{N} = \delta^{J}_{N}$ and in the last line we just renamed a dummy index. Now we have $R[\omega]$. So, the Palatini action is

$$S[e,\omega] = \frac{1}{16\pi G} \int d^4x \; e \; e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu}, \tag{5.31}$$

where $e = \sqrt{-\det g}$.

We now just apply the variational principle to the action (5.31) to get the equations of motion. First, we vary the action with respect to the tetrad, i.e., we compute δS assuming $\delta\omega = 0$. This leads us to

$$\begin{aligned} \delta S &= \frac{1}{16\pi G} \int d^4x \left[e \, (\delta e^{\mu}_{I}) e^{\nu}_{J} F^{IJ}_{\mu\nu} + e \; e^{\mu}_{I} (\delta e^{\nu}_{J}) F^{IJ}_{\mu\nu} + (\delta e) e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu} \right] \\ &= \frac{1}{8\pi G} \int d^4x \; e \; \left[e^{\nu}_{J} F^{IJ}_{\mu\nu} (\delta e^{\mu}_{I}) - \frac{1}{2} e^{K}_{\sigma} e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu} (\delta e^{\sigma}_{K}) \right], \end{aligned} \tag{5.32}$$

where the last term is calculated via $\delta(\det A) = \det A \, \mathrm{Tr}(A^{-1}_{ji} \; \delta A_{ij})$:

$$\delta e = e \; e^{\sigma}_{K} \delta e^{K}_{\sigma} = -e \; e^{K}_{\sigma} \delta e^{\sigma}_{K}, \tag{5.33}$$

where in the last equality we used $\delta(e^{K}_{\sigma} e^{\sigma}_{K}) = 0$.

Hence, if we set $\delta S = 0$ we get

$$F^{KJ}_{\sigma\nu} e^{\nu}_{J} - \frac{1}{2} e^{K}_{\sigma} e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu} = 0.$$

If we act with $e_{\tau K} = \eta_{KJ} e^{J}_{\tau}$ on both sides we get:

$$F^{KJ}_{\sigma\nu} e^{\nu}_{J} e_{\tau K} - \frac{1}{2} (e^{K}_{\sigma} e^{J}_{\tau} \eta_{KJ}) e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu} = 0,$$

where, from (5.2), the term in parenthesis is just $g_{\tau\sigma}$, the last part of the second term, from equation (5.30), is just the Ricci scalar R, and the first term, from equation (5.18), is just the Ricci tensor $R_{\tau\sigma}$. We then derived Einstein's field equations in vacuum

$$R_{\tau\sigma} - \frac{1}{2} R g_{\tau\sigma} = 0.$$

We need also to vary (5.31) with respect to ω, assuming $\delta e = 0$:

$$\delta S = \frac{1}{16\pi G}\int d^4x \; e \; e^\mu_I e^\nu_J (\delta F^{IJ}_{\mu\nu}), \tag{5.34}$$

from (5.17) we get

$$\begin{aligned}\delta F^{IJ}_{\mu\nu} &= \partial_\mu(\delta\omega^{IJ}_\nu) - \partial_\nu(\delta\omega^{IJ}_\mu) + (\delta\omega^I_{\mu K})\omega^{KJ}_\nu + \omega^I_{\mu K}(\delta\omega^{KJ}_\nu) - (\delta\omega^I_{\nu K})\omega^{KJ}_\mu - (\delta\omega^{KJ}_\mu)\omega^I_K \\ &= 2\left(\partial_{[\mu}\delta\omega^{IJ}_{\nu]} + \delta\omega^I_{[\mu K}\omega^{KJ}_{\nu]} + \omega^I_{[\mu|K|}\delta\omega^{KJ}_{\nu]}\right) \\ &= 2\left(\partial_{[\mu}\delta\omega^{IJ}_{\nu]} + \omega^I_{[\mu|K|}\delta\omega^{KJ}_{\nu]} - \omega^J_{[\mu|K|}\delta\omega^{IK}_{\nu]}\right) = 2\left(\partial_{[\mu}\delta\omega^{IJ}_{\nu]} + 2\omega^{[I}_{[\mu|K|}\delta\omega^{|K|J]}_{\nu]}\right),\end{aligned}$$

where we used $A_{[\mu\nu]} = \frac{1}{2!}\left(A_{\mu\nu} - A_{\nu\mu}\right)$. Hence, in equation (5.34) we have

$$\delta S = \frac{1}{8\pi G}\int d^4x \; e \; e^\mu_I e^\nu_J (\partial_{[\mu}\delta\omega^{IJ}_{\nu]} + 2\omega^{[I}_{[\mu|K|}\delta\omega^{|K|J]}_{\nu]}). \tag{5.35}$$

The first term can be rewritten as

$$e \; e^\mu_I e^\nu_J \partial_{[\mu}\delta\omega^{IJ}_{\nu]} = -e \; e^\mu_{[I} e^\nu_{J]} \partial_\nu \delta\omega^\mu{}_{IJ},$$

then, integrating by parts we have (neglecting boundary terms)

$$\int d^4x \; e \; e^\mu_I e^\nu_J \partial_{[\mu}\delta\omega^{IJ}_{\nu]} = \int d^4x \; \partial_\nu(e \; e^\mu_{[I} e^\nu_{J]})\delta\omega^\mu{}_{IJ}. \tag{5.36}$$

The second term can be also rewritten as follow

$$2 \; e \; e^\mu_I e^\nu_J \omega^{[I}_{[\mu|K|}\delta\omega^{|K|J]}_{\nu]} = 2 \; e \; e^\mu_{[I} e^\nu_{J]} \omega^I_{[\mu K}\delta\omega^{|K|J}_{\nu]} = 2 \; e \; e^\mu_{[I} e^\nu_{J]} \omega^I_{\mu K}(\delta\omega^{KJ}_\nu),$$

where, in the last step, we just left out the antisymmetrization $[\mu, \nu]$ since the expression is already antisymmetric.

So, we have, for the second term of the action, renaming some dummy indices

$$-\int d^4x (2 \; e \; e^\nu_{[K} e^\mu_{J]} \omega^K_{\nu I})\delta\omega^{IJ}_\mu, \tag{5.37}$$

so, in (5.35) we have:

$$\delta S = \frac{1}{16\pi G}\int d^4x \left(\partial_\nu(e \; e^\mu_{[I} e^\nu_{J]}) - 2 \; e \; e^\nu_{[K} e^\mu_{J]} \omega^K_{\nu I}\right)\delta\omega^{IJ}_\mu, \tag{5.38}$$

then, if we set $\delta S/\delta\omega^{IJ}_\mu = 0$ we get

$$\partial_\nu(e \; e^\mu_{[I} e^\nu_{J]}) - 2 \; e \; e^\nu_{[K} e^\mu_{J]} \omega^K_{\nu I} = 0.$$

However, if we took the covariant derivative of the $e \; e^\mu_{[I} e^\nu_{J]}$ term we would get

$$D_\nu\left(e \; e^\mu_{[I} e^\nu_{J]}\right) = \partial_\nu\left(e \; e^\mu_{[I} e^\nu_{J]}\right) - \omega^K_{\nu I} e \; e^\mu_{[K} e^\nu_{J]} - \omega^K_{\nu J} e \; e^\mu_{[I} e^\nu_{K]},$$

which is exactly the expression above, whose value is zero:

$$D_\nu\left(e \; e^\mu_{[I} e^\nu_{J]}\right) = 0. \tag{5.39}$$

Since

$$e\, e^{\mu}_{[I} e^{\nu}_{J]} = \frac{1}{4}\epsilon_{IJKL}\epsilon^{\mu\nu\alpha\beta} e^{K}_{\alpha} e^{L}_{\beta},$$

we have, in (5.39):

$$\begin{aligned} D_\nu \left(e\, e^{\mu}_{[I} e^{\nu}_{J]} \right) &= \frac{1}{4}\epsilon_{IJKL}\epsilon^{\mu\nu\alpha\beta} \left[(D_\nu e^{K}_{\alpha}) e^{L}_{\beta} + (D_\nu e^{L}_{\beta}) e^{K}_{\alpha} \right] \\ &= \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\alpha\beta} (D_\nu e^{K}_{\alpha}) e^{L}_{\beta}, \end{aligned}$$

where, in the first line the two terms in the parenthesis are equivalent since the expression is antisymmetric in $[K, L]$ and $[\alpha, \beta]$, contributing with two factors of minus one to the last term, making it identical do the first one. We are left with

$$\epsilon^{\mu\nu\alpha\beta} (D_\nu e^{K}_{\alpha}) e^{L}_{\beta} = 0,$$

where the symbol ϵ_{IJKL} was removed since it's action on antisymmetric rank 2 tensors is invertible. If we act with e^{ρ}_{L} on both sides we are left with

$$D_{[\nu} e^{K}_{\alpha]} = 0. \tag{5.40}$$

This implies that the torsion is zero, and, since we have a metric compatibility — equation (5.10) — we know that we are talking about the Levi Civita connection.

This implies that the tetrad is constant with respect to the covariant derivative defined via the connection ω. So, just as it happened in the Palatini approach for the variables $g_{\mu\nu}$ and Γ, the variation of the action with respect to the connection told us that the metric was compatible with the covariant derivative defined by that connection. Here, the tetrad, playing the role of the metric, is compatible with the connection ω.

5.6.1 The covariant notation

We can write the same formalism using the notation of forms, in a coordinate independent way, which will be useful later.

We can write the Palatini action in this notation as

$$S[e, \omega] = \frac{1}{16\pi G} \int \frac{1}{2}\epsilon_{IJKL} e^{I} \wedge e^{J} \wedge F^{KL}. \tag{5.41}$$

If we open the integrated term in coordinates we will have

$$\begin{aligned} \frac{1}{2}\epsilon_{IJKL} e^{I} \wedge e^{J} \wedge F^{KL} &= \frac{1}{2}\epsilon_{IJKL} (e^{I}_{\mu} dx^{\mu}) \wedge (e^{J}_{\nu} dx^{\nu}) \wedge \left(\frac{1}{2} F_{\alpha\beta}{}^{KL} dx^{\alpha} \wedge dx^{\beta} \right) \\ &= \frac{1}{4}\epsilon_{IJKL} e^{I}_{\mu} e^{J}_{\nu} F_{\alpha\beta}{}^{KL} \left(dx^{\mu} \wedge dx^{\nu} \wedge dx^{\alpha} \wedge dx^{\beta} \right) \\ &= \frac{1}{4}\epsilon_{IJKL}\epsilon^{\mu\nu\alpha\beta} e^{I}_{\mu} e^{J}_{\nu} F_{\alpha\beta}{}^{KL} d^4x \\ &= e\, e^{[\alpha}_{K} e^{\beta]}_{L} F_{\alpha\beta}{}^{KL} d^4x \\ &= eR d^4x, \end{aligned}$$

where, in the fourth line we used one of the relations from appendix B, and in the last line we just removed the antisymmetrization in $[\alpha, \beta]$ since the expression is already antisymmetric.

To get the equations of motion we first vary the action with respect to the connection ω, which gives us (removing the constants):

$$\begin{aligned}
\delta S &= \int \epsilon_{IJKL} e^I \wedge e^J \wedge (\delta F^{KL}) \\
&= \int \epsilon_{IJKL} e^I \wedge e^J \wedge D(\delta\omega^{KL}) \\
&= \int D\left(\epsilon_{IJKL} e^I \wedge e^J \wedge \delta\omega^{KL}\right) - \int D\left(\epsilon_{IJKL} e^I \wedge e^J\right) \wedge \delta\omega^{KL} \\
&= -2\int \epsilon_{IJKL}\left(De^I\right) \wedge e^J \wedge \delta\omega^{KL},
\end{aligned}$$

where the boundary term was neglected from the third to the fourth line and we used the Palatini identity (see appendix C) in the second line.

So, if we set $\delta S/\delta\omega = 0$ we will get $De^I = 0$, which states that the connection ω is torsion free, as we already knew.

If we now vary the action with respect to the tetrads we will have

$$\delta S = \int \epsilon_{IJKL}(\delta e^J) \wedge e^I \wedge F^{KL},$$

and, setting $\delta S/\delta e^J = 0$ will lead us to

$$\epsilon_{IJKL} e^I \wedge F^{KL} = 0.$$

and, when we open this equation in coordinates we get

$$\epsilon_{IJKL}(e^I_\sigma F^{KL}_{\mu\nu}) dx^\sigma \wedge dx^\mu \wedge dx^\nu = 0. \tag{5.42}$$

One should note that the only free index in the above equation is L, which then gives us four equations — one for each value of $L = 0, 1, 2, 3$ — stating that a certain 3-form vanishes. And, since in a n-dimensional space a p-form has $\binom{n}{p}$ independent components, our 3-forms will have $\binom{4}{3} = 4$ independent components. Equation (5.42) is then grouping 16 different equations, which may lead one to infer that this is probably Einstein's field equations — which indeed is, as we will now show.

Acting with dx^ρ in (5.42) gives us

$$\epsilon_{IJKL}(e^I_\sigma F^{KL}_{\mu\nu}) dx^\sigma \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho = 0.$$

Since $\epsilon^{\sigma\mu\nu\rho} d^4x = dx^\sigma \wedge dx^\mu \wedge dx^\nu \wedge dx^\rho$, and contracting $\epsilon_{IJKL} e^I_\sigma = e\, e^\alpha_J e^\beta_K e^\gamma_L \epsilon_{\sigma\alpha\beta\gamma}$, this leads us to

$$e^\alpha_J e^\beta_K e^\gamma_L F^{KL}_{\mu\nu} \epsilon^{\sigma\mu\nu\rho} \epsilon_{\sigma\alpha\beta\gamma} = 0.$$

Since J is a free index, one can act with e^J_θ to get $e^\beta_K e^\gamma_L F^{KL}_{\mu\nu} \epsilon^{\sigma\mu\nu\rho} \epsilon_{\sigma\alpha\beta\gamma} = 0$. However, $e^\beta_K e^\gamma_L F^{KL}_{\mu\nu} = R^{\beta\gamma}_{\mu\nu}$ and

$$\epsilon^{\sigma\mu\nu\rho}\epsilon_{\sigma\alpha\beta\gamma} = -2\left(\delta^{[\mu}_\alpha \delta^{\nu]}_\beta \delta^\rho_\gamma + \delta^{[\rho}_\alpha \delta^{\mu]}_\beta \delta^\nu_\gamma + \delta^{[\nu}_\alpha \delta^{\mu]}_\beta \delta^\rho_\gamma\right),$$

which gives us

$$R^{\beta\gamma}_{\mu\nu}\left(\delta^{[\mu}_\alpha \delta^{\nu]}_\beta \delta^\rho_\gamma + \delta^{[\rho}_\alpha \delta^{\mu]}_\beta \delta^\nu_\gamma + \delta^{[\nu}_\alpha \delta^{\mu]}_\beta \delta^\rho_\gamma\right) = 0.$$

Therefore,

$$R^\rho_\alpha - \frac{1}{2} R \delta^\rho_\alpha = 0,$$

where $R^\rho_\alpha := R^{\mu\rho}_{\mu\alpha}$ and $R := R^\mu_\mu$. Acting with $g_{\rho\mu}$ we then obtain Einstein's field equations in vacuum, as expected:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0.$$

5.7 The Holst action

One could think about the dual of the terms in the Palatini action, such as the dual of the curvature 2-form:

$$(\star F)^{IJ}_{ab} = \frac{1}{2}\epsilon^{IJ}{}_{KL}F^{KL}_{ab}.$$

Varying the action with this term in the same way we did before leads us to the same compatibility condition, i.e., we are still dealing with the connection that preserves the tetrads.

This allows us to generalize the Palatini action to

$$S[e,\omega] = \frac{1}{16\pi G}\int d^4x\; e\; e^\mu_I e^\nu_J P^{IJ}{}_{KL}F^{KL}_{\mu\nu}. \tag{5.43}$$

where the new term is defined as

$$P^{IJ}{}_{KL} := \delta^{[I}_K\delta^{J]}_L - \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL}. \tag{5.44}$$

Again, if we vary this action with respect to the connection, we are led to

$$\epsilon^{abcd}\epsilon_{IJKL}P^{KL}{}_{MN}D_a(e^M_c e^N_d) = 0,$$

which still remains being the compatibility condition.

Varying the action with respect to the tetrads, as we did in (5.32), will lead to the Einstein's field equation with an extra term $\epsilon^{Iabc}R_{abIL}$, which vanishes by the symmetries of the Riemann tensor.

Therefore, for any value of γ, we get the same equations of motion from this new action. The action in (5.43) is called the Holst action, and γ is the Barbero-Immirzi parameter.

This action is a step towards constructing the Ashtekar's formulation of general relativity.

5.7.1 Forms notation

It is also possible to write this action in the notation of forms, which would lead us to

$$S = \frac{1}{32\pi G}\int \left[\left(\star + \frac{1}{\gamma}\right) e_I \wedge e_J\right] \wedge F^{IJ}. \tag{5.45}$$

Varying this action with respect to the connection gives us

$$\begin{aligned}
\delta S &= \frac{1}{32\pi G}\int \left[\left(\star + \frac{1}{\gamma}\right) e_I \wedge e_J\right] \wedge \delta F^{IJ} \\
&= \frac{1}{32\pi G}\int \left[\left(\star + \frac{1}{\gamma}\right) e_I \wedge e_J\right] \wedge D(\delta\omega^{IJ}) \\
&= -\frac{1}{16\pi G}\int \left[\left(\star + \frac{1}{\gamma}\right) (De_I) \wedge e_J\right] \wedge \delta\omega^{IJ},
\end{aligned}$$

where we used $\delta F^{IJ} = D(\delta\omega^{IJ})$ in the second line and we integrated by parts and neglected the boundary term in the next line.

From equation (5.24) it can be seen that forcing this variation to vanish leads us to the torsion free condition:

$$T^I = De^I = 0.$$

This can also be taken as the definition of ω: our connection is the one that can be entirely determined by the tetrads, given this condition. So, the only independent variable in our theory is e^I. Defining the connection in this way would obviously make it satisfy the equation of motion — i.e. there is no variation with respect to ω since it is not an independent variable. This formulation is called first order, while the formulation where both e and ω is independent is called first order.

Finally, varying the action with respect to the tetrad gives us

$$\delta S = \frac{1}{16\pi G}\int (\delta e_I) \wedge e_J \wedge \left[\left(\star + \frac{1}{\gamma}\right) F^{IJ}\right].$$

Hence, setting $\delta S/\delta e = 0$ leads us to

$$e_J \wedge \left(\star + \frac{1}{\gamma}\right) F^{IJ} = 0. \tag{5.46}$$

The second term vanishes since, from the first Bianchi identity (see appendix C) we have that $e_J \wedge F^{IJ} = D^2 e_J = 0$, since the connection is torsion free. So, the γ term vanishes on-shell, i.e., when the torsion is zero.

Hence, we are left only with the first term:

$$e_J \wedge \left(\star F^{IJ}\right) = 0, \tag{5.47}$$

which is just Einstein's field equations in forms notation, which can be shown in a similar way as we did in the Palatini section, using coordinate notation.

Chapter 6

Ashtekar Formulation of GR

> Mathematics is a language plus reasoning; it is like a language plus logic. Mathematics is a tool for reasoning.
>
> *Richard Feynman*

6.1 Introduction

This formalism consists in developing a Hamiltonian formalism for the Holst action. Hence, one needs to talk about some field that is evolving in time, where we will enter with the (3+1) split, in a similar way as we did in the ADM formalism, however, now, with different variables.

While following the path to get a Hamiltonian field in this spacetime foliation with the local 3D frame field ε^I being our dynamical variable, we will also define new variables — the Ashtekar variables — the *densitized triad* and the *Ashtekar-Barbero connection*, which are going to play an important role when one follow the steps to quantize the theory.

6.2 Triads and Time Gauge

In the ADM formalism one splits the metric $g_{\mu\nu}$ — the main dynamical variable — into its spatial part, the induced metric h_{ab}, and define a time vector field t^a to study the spacetime evolution as the history of the spatial slices Σ_t evolving through the time t^a.

Here, similar to the spatial canonical metric h_{ab}, we define the spatial tensor field

$$\varepsilon^a_I = e^a_I + n^a n_I, \tag{6.1}$$

where n^a is the unitary normal vector to the spatial slice and $n_I := e^a_I n_a$.

This breaks the basis of tetrads e^a_I in its spatial part, the basis of triads ε^a_I, which lives in the spatial slice Σ_t, as is easily seen:

$$\begin{aligned}\varepsilon^a_I n_a &= e^a_I n_a + n^a n_I n_a \\ &= n_I - n_I \\ &= 0,\end{aligned} \tag{6.2}$$

which can also be seen in figure (6.1).

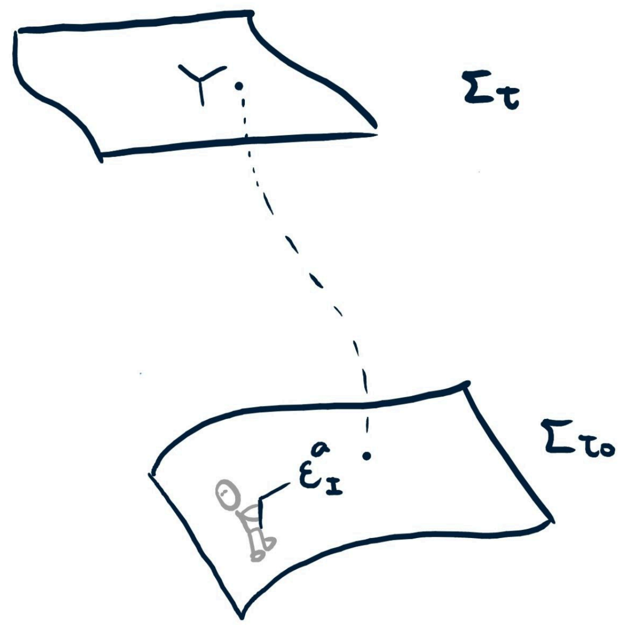

Figure 6.1: The triad

However, in the triad formalism, there is an additional condition to the usual spacetime split done in the ADM formalism, which is the split in the internal directions of the tetrad in Minkowski time and space components.

There are two ways of doing this gauge fixing. The first one is to require that $e^a_0 = n^I e^a_I = n^a$ be the unit normal to the foliation, this is known as the time gauge. Here we assumed $n^I = \delta^I_0$ to be a timelike internal vector field. Internal Lorentz transformations that preserve n^I are reduced to spatial rotations around the fixed direction n^I.

Another way of doing it, which will show to be more practical in the calculations we are going to develop, is to open the tetrads in its spatial and time components and set the time gauge directly from it.

First, let us consider the 1-forms e^I. For the Minkowski time component:

$$e^0 = e^0_\mu dx^\mu = e^0_0 dx^0 + e^0_a dx^a.$$

Here we set the spatial part to be zero:

$$e^0_a = 0, \tag{6.3}$$

leaving just the time component for this tetrad:

$$\begin{aligned} e^0 &= e^0_\mu dx^\mu \\ &= e^0_0 dx^0 + e^0_a dx^a \\ &= e^0_0 dx^0 \\ &= N dt, \end{aligned} \tag{6.4}$$

where we defined the lapse function $e^0_0 := N$. This fixes the Minkowski time in the internal space: the direction orthogonal to the spatial slice.

Now, for the spatial Minkowski components one can write

$$\begin{aligned} e^i &= e^i_\mu dx^\mu \\ &= e^i_0 dx^0 + e^i_a dx^a \\ &= N^i dx^0 + e^i_a dx^a, \end{aligned} \tag{6.5}$$

where we defined the shift vector $N^i := e^i_0$.

With these, the (3+1) split of the tetrad is done: one has the lapse function N, the shift vector N^i and the spatial triads e^i_a:

$$e^I_\mu = \begin{pmatrix} N & N^i \\ 0 & e^i_a \end{pmatrix}. \tag{6.6}$$

In the formalism that we are going to develop, the 3-metric of the ADM formalism is substituted by the new dynamical variable, the triads, and those contain the same geometrical information: $h_{ab} = \delta_{ij}\varepsilon^i_a\varepsilon^j_b$. The ADM formalism variables will then be replaced by

$$(h_{ab}, \pi^{ab}) \mapsto (\varepsilon^i_a, A^i_b),$$

where A^i_b is an $SU(2)$ connection, the canonical conjugate variable to the triad, which will appear shortly in our development.

6.3 The space-time split

6.3.1 The Holst Action

We will now expand the Holst action in equation (5.45) in order to do the 3+1 split in the triads variable.

First, let us note that

$$\begin{aligned} e^I \wedge e^J \wedge F^{KL} &= (e^I_\mu dx^\mu) \wedge (e^J_\nu dx^\nu) \wedge (F^{KL}_{\rho\sigma} dx^\rho dx^\sigma) \\ &= e^I_\mu e^J_\nu F^{KL}_{\rho\sigma} (dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma) \\ &= -\tilde{\epsilon}^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu F^{KL}_{\rho\sigma} \left(dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\right) \\ &= -\tilde{\epsilon}^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu F^{KL}_{\rho\sigma} dt \wedge d^3x, \end{aligned} \tag{6.7}$$

where, in the third line we used the fact the the term $dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$ is completely anti-symmetric in $\mu\nu\rho\sigma$, then, it can be written in terms of the Levi Civita symbol. The minus sign comes from the fact that we will use $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ to represent the Levi-Civita symbol and $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is defined as being the Levi-Civita symbol multiplied by $\text{sign}(g) = -1$. We can then plug this into the action in (5.45):

$$\begin{aligned} S &= \frac{1}{4}\int \left(\star + \frac{1}{\gamma}\right) e_I \wedge e_J \wedge F^{IJ} \\ &= \frac{1}{4}\int \left\{\frac{1}{2}\epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} + \frac{1}{\gamma}\eta_{IK}\eta_{JL} e^I \wedge e^J \wedge F^{KL}\right\} \\ &= -\frac{1}{4}\int dt \int d^3x\ \tilde{\epsilon}^{\mu\nu\rho\sigma} \left\{\frac{1}{2}\epsilon_{IJKL} e^I_\mu e^J_\nu F^{KL}_{\rho\sigma} + \frac{1}{\gamma}\eta_{IK}\eta_{JL} e^I_\mu e^J_\nu F^{KL}_{\rho\sigma}\right\}. \end{aligned} \tag{6.8}$$

6.3.2 The 3+1 split

Now, given the relations in (6.6), we can decompose the terms in the action (6.8) doing the (3+1) split in the spacetime indices. First, note that

$$\begin{aligned}
\epsilon_{IJKL}\tilde{\epsilon}^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu F^{KL}_{\ \ \rho\sigma} &= \epsilon_{IJKL}\left(\tilde{\epsilon}^{0abc}e^I_0 e^J_a F^{KL}_{\ \ bc} + \tilde{\epsilon}^{a0bc}e^I_a e^J_0 F^{KL}_{\ \ bc} + \tilde{\epsilon}^{ab0c}e^I_a e^J_b F^{KL}_{\ \ 0c} + \tilde{\epsilon}^{abc0}e^I_a e^J_b F^{KL}_{\ \ c0}\right)\\
&= \epsilon_{IJKL}\left(\tilde{\epsilon}^{0abc}e^I_0 e^J_a F^{KL}_{\ \ bc} + \tilde{\epsilon}^{a0bc}e^I_a e^J_0 F^{KL}_{\ \ bc} + \tilde{\epsilon}^{ab0c}e^I_a e^J_b F^{KL}_{\ \ 0c} + (-\tilde{\epsilon}^{ab0c})e^I_a e^J_b(-F^{KL}_{\ \ 0c})\right)\\
&= \epsilon_{IJKL}\left(2\tilde{\epsilon}^{0abc}e^I_0 e^J_a F^{KL}_{\ \ bc} + 2\tilde{\epsilon}^{ab0c}e^I_a e^J_b F^{KL}_{\ \ 0c}\right)\\
&= \epsilon_{IJKL}\tilde{\epsilon}^{abc}\left(2e^I_0 e^J_a F^{KL}_{\ \ bc} + 2e^I_a e^J_b F^{KL}_{\ \ 0c}\right),
\end{aligned} \tag{6.9}$$

where we defined in the last line $\tilde{\epsilon}^{abc} := \tilde{\epsilon}^{0abc}$, and the factor of 2 in the first term came from:

$$\begin{aligned}
\tilde{\epsilon}^{a0bc}\epsilon_{IJKL}e^I_a e^J_0 F^{KL}_{\ \ bc} &= -\tilde{\epsilon}^{0abc}\epsilon_{IJKL}e^I_a e^J_0 F^{KL}_{\ \ bc}\\
&= \tilde{\epsilon}^{0abc}\epsilon_{JIKL}e^I_a e^J_0 F^{KL}_{\ \ bc}\\
&= -\tilde{\epsilon}^{0abc}\epsilon_{JIKL}e^J_a e^I_0 F^{KL}_{\ \ bc}\\
&= \tilde{\epsilon}^{0abc}\epsilon_{IJKL}e^J_a e^I_0 F^{KL}_{\ \ bc},
\end{aligned}$$

hence, the second term in the second line of (6.9) is equal to the first. This was for the first term in (6.8), but the calculation is analogous for the second term with the η parameters.

Therefore, plugging (6.9) in (6.8) we are led to

$$\begin{aligned}
S &= -\frac{1}{4}\int dt\int d^3x\ \tilde{\epsilon}^{\mu\nu\rho\sigma}\left\{\frac{1}{2}\epsilon_{IJKL}e^I_\mu e^J_\nu F^{KL}_{\ \ \rho\sigma} + \frac{1}{\gamma}\eta_{IK}\eta_{JL}e^I_\mu e^J_\nu F^{KL}_{\ \ \rho\sigma}\right\}\\
&= -\frac{1}{4}\int dt\int d^3x\ \tilde{\epsilon}^{abc}\left\{2\frac{1}{2}\epsilon_{IJKL}e^I_0 e^J_a F^{KL}_{\ \ bc} + 2\frac{1}{2}\epsilon_{IJKL}e^I_a e^J_b F^{KL}_{\ \ 0c} + 2\frac{1}{\gamma}\eta_{IK}\eta_{JL}e^I_0 e^J_a F^{KL}_{\ \ bc} + 2\frac{1}{\gamma}\eta_{IK}\eta_{JL}e^I_a e^J_b F^{KL}_{\ \ 0c}\right\}\\
&= -\frac{1}{2}\int dt\int d^3x\ \tilde{\epsilon}^{abc}\left\{\frac{1}{2}\epsilon_{IJKL}e^I_0 e^J_a F^{KL}_{\ \ bc} + \frac{1}{2}\epsilon_{IJKL}e^I_a e^J_b F^{KL}_{\ \ 0c} + \frac{1}{\gamma}\eta_{IK}\eta_{JL}e^I_0 e^J_a F^{KL}_{\ \ bc} + \frac{1}{\gamma}\eta_{IK}\eta_{JL}e^I_a e^J_b F^{KL}_{\ \ 0c}\right\}.
\end{aligned} \tag{6.10}$$

We now do the same (3+1) split in the internal indices. For the first term inside the brackets in (6.10) we get

$$\begin{aligned}
\frac{1}{2}\epsilon_{IJKL}e^I_0 e^J_a F^{KL}_{\ \ bc} &= \frac{1}{2}\epsilon_{0ijk}e^0_0 e^i_a F^{jk}_{\ \ bc} + \frac{1}{2}\epsilon_{i0jk}e^i_0 e^0_a F^{jk}_{\ \ bc} + \frac{1}{2}\epsilon_{ij0k}e^i_0 e^j_a F^{0k}_{\ \ bc} + \frac{1}{2}\epsilon_{ijk0}e^i_0 e^j_a F^{k0}_{\ \ bc}\\
&= \frac{1}{2}\epsilon_{0ijk}e^0_0 e^i_a F^{jk}_{\ \ bc} + \frac{1}{2}\epsilon_{0ijk}e^i_0 e^j_a F^{0k}_{\ \ bc} - \frac{1}{2}\epsilon_{0ijk}e^i_0 e^j_a F^{k0}_{\ \ bc}\\
&= \frac{1}{2}\epsilon_{0ijk}e^0_0 e^i_a F^{jk}_{\ \ bc} + \frac{1}{2}\epsilon_{0ijk}e^i_0 e^j_a F^{0k}_{\ \ bc} + \frac{1}{2}\epsilon_{0ijk}e^i_0 e^j_a F^{0k}_{\ \ bc}\\
&= \frac{1}{2}\epsilon_{0ijk}e^0_0 e^i_a F^{jk}_{\ \ bc} + \epsilon_{0ijk}e^i_0 e^j_a F^{0k}_{\ \ bc}\\
&= \frac{1}{2}\epsilon_{ijk}N e^i_a F^{jk}_{\ \ bc} + \epsilon_{ijk}N^d e^i_d e^j_a F^{0k}_{\ \ bc},
\end{aligned} \tag{6.11}$$

where the second term in the first line vanishes because of the time gauge ($e^0_a = 0$) and in the last term we recovered the definitions of the lapse function $N = e^0_0$ and the shift vector $e^i_0 = N^i = N^d e^i_d$, and we also defined $\epsilon_{0ijk} := \epsilon_{ijk}$.

Now, for the second term in parenthesis in equation (6.10) we have

$$\begin{aligned}
\frac{1}{2}\epsilon_{IJKL}e^I_a e^J_b F^{KL}_{\ \ 0c} &= \frac{1}{2}\epsilon_{0ijk}e^0_a e^i_b F^{jk}_{\ \ 0c} + \frac{1}{2}\epsilon_{i0jk}e^i_a e^0_b F^{jk}_{\ \ 0c} + \frac{1}{2}\epsilon_{ij0k}e^i_a e^j_b F^{0k}_{\ \ 0c} + \frac{1}{2}\epsilon_{ijk0}e^i_a e^j_b F^{k0}_{\ \ 0c} \\
&= \frac{1}{2}\epsilon_{ij0k}e^i_a e^j_b F^{0k}_{\ \ 0c} + \frac{1}{2}\epsilon_{ijk0}e^i_a e^j_b F^{k0}_{\ \ 0c} \\
&= \frac{1}{2}\epsilon_{0ijk}e^i_a e^j_b F^{0k}_{\ \ 0c} - \frac{1}{2}\epsilon_{0ijk}e^i_a e^j_b \left(-F^{0k}_{\ \ 0c}\right) \\
&= \epsilon_{ijk}e^i_a e^j_b F^{0k}_{\ \ 0c},
\end{aligned} \tag{6.12}$$

where the first two terms in the first line vanishes due to the time gauge ($e^0_a = 0$) and in the last term we also used $\epsilon_{0ijk} \coloneqq \epsilon_{ijk}$.

For the last two terms in (6.10) we will open the indices in space and time using the metric relations above:

$$\eta_{IK}\eta_{JL} = \begin{cases} \eta_{00}\eta_{00} = 1 \\ \eta_{00}\eta_{ij} = -\delta_{ij} \\ \eta_{ij}\eta_{00} = -\delta_{ij} \\ \eta_{ik}\eta_{jl} = \delta_{ik}\delta_{jl} \end{cases} \tag{6.13}$$

Then, for the third term in (6.10) we get

$$\begin{aligned}
\frac{1}{\gamma}\eta_{IK}\eta_{JL}e^I_0 e^J_a F^{KL}_{\ \ bc} &= \frac{1}{\gamma}\left[\eta_{00}\eta_{00}e^0_0 e^0_a F^{00}_{\ \ bc} + \eta_{00}\eta_{jl}e^0_0 e^j_a F^{0l}_{\ \ bc} + \eta_{00}\eta_{ik}e^i_0 e^0_a F^{k0}_{\ \ bc} + \eta_{ik}\eta_{jl}e^i_0 e^j_a F^{kl}_{\ \ bc}\right] \\
&= \frac{1}{\gamma}\left[e^0_0 e^0_a F^{00}_{\ \ bc} - \delta_{jl}e^0_0 e^j_a F^{0l}_{\ \ bc} - \delta_{ik}e^i_0 e^0_a F^{k0}_{\ \ bc} + \delta_{ik}\delta_{jl}e^i_0 e^j_a F^{kl}_{\ \ bc}\right] \\
&= \frac{1}{\gamma}\left[-\delta_{jl}e^0_0 e^j_a F^{0l}_{\ \ bc} - \delta_{ik}e^i_0 e^0_a F^{k0}_{\ \ bc} + \delta_{ik}\delta_{jl}e^i_0 e^j_a F^{kl}_{\ \ bc}\right] \\
&= \frac{1}{\gamma}\left[\delta_{ik}\delta_{jl}N^d e^i_d e^j_a F^{kl}_{\ \ bc} - \delta_{jl}N e^j_a F^{0l}_{\ \ bc}\right],
\end{aligned} \tag{6.14}$$

where the first term in the second line vanishes since the curvature 2-form is anti-symmetric ($F^{\mu\mu} = 0$), in the third line the second term also vanishes due to the time gauge, and in the last line we plugged in the definitions of N and $N^i = e^i_0 = N^d e^i_d$.

Finally, for the fourth term in parenthesis in equation (6.10) we get

$$\begin{aligned}
\frac{1}{\gamma}\eta_{IK}\eta_{JL}e^I_a e^J_b F^{KL}_{\ \ 0c} &= \frac{1}{\gamma}\left[\eta_{00}\eta_{00}e^0_a e^0_b F^{00}_{\ \ 0c} + \eta_{00}\eta_{jl}e^0_a e^j_b F^{0l}_{\ \ 0c} + \eta_{ik}\eta_{00}e^i_a e^0_b F^{k0}_{\ \ 0c} + \eta_{ik}\eta_{jl}e^i_a e^j_b F^{kl}_{\ \ 0c}\right] \\
&= \frac{1}{\gamma}\left[e^0_a e^0_b F^{00}_{\ \ 0c} - \delta_{jl}e^0_a e^j_b F^{0l}_{\ \ 0c} - \delta_{ik}e^i_a e^0_b F^{k0}_{\ \ 0c} + \delta_{ik}\delta_{jl}e^i_a e^j_b F^{kl}_{\ \ 0c}\right] \\
&= \frac{1}{\gamma}\left[-\delta_{jl}e^0_a e^j_b F^{0l}_{\ \ 0c} - \delta_{ik}e^i_a e^0_b F^{k0}_{\ \ 0c} + \delta_{ik}\delta_{jl}e^i_a e^j_b F^{kl}_{\ \ 0c}\right] \\
&= \frac{1}{\gamma}\delta_{ik}\delta_{jl}e^i_a e^j_b F^{kl}_{\ \ 0c},
\end{aligned} \tag{6.15}$$

where the first two terms in the third line vanishes due to the time gauge.

Now, plugging (6.11), (6.12), (6.14) and (6.15) in the action (6.10) leads us to

$$\begin{aligned}
S &= -\frac{1}{2}\int dt \int d^3x\ \tilde{\epsilon}^{abc}\left\{\frac{1}{2}\epsilon_{IJKL}e^I_0 e^J_a F^{KL}_{\ \ bc} + \frac{1}{2}\epsilon_{IJKL}e^I_a e^J_b F^{KL}_{\ \ 0c} + \frac{1}{\gamma}\eta_{IK}\eta_{JL}e^I_0 e^J_a F^{KL}_{\ \ bc} + \frac{1}{\gamma}\eta_{IK}\eta_{JL}e^I_a e^J_b F^{KL}_{\ \ 0c}\right\} \\
&= -\frac{1}{2\gamma}\int dt \int d^3x\ \tilde{\epsilon}^{abc}\left\{\right\},
\end{aligned} \tag{6.16}$$

where the term in parenthesis is

$$\begin{aligned} \{\} = &\left(\frac{1}{2}\gamma\epsilon_{ijk}Ne^i_aF^{jk}_{\ bc} + \gamma\epsilon_{ijk}N^de^i_de^j_aF^{0k}_{\ bc}\right) + \left(\gamma\epsilon_{ijk}e^i_ae^j_bF^{0k}_{\ 0c}\right) \\ &+ \left(\delta_{ik}\delta_{jl}N^de^i_de^j_aF^{kl}_{\ bc} - \delta_{jl}Ne^j_aF^{0l}_{\ bc}\right) + \left(\delta_{ik}\delta_{jl}e^i_ae^j_bF^{kl}_{\ 0c}\right), \end{aligned} \tag{6.17}$$

which, by grouping terms proportional to N and N^d, can be written as

$$\begin{aligned} \{\} = &\left(\delta_{ik}\delta_{jl}e^i_ae^j_bF^{kl}_{\ 0c} + \gamma\epsilon_{ijk}e^i_ae^j_bF^{0k}_{\ 0c}\right) + \\ &+ N^d\left(\delta_{ik}\delta_{jl}e^i_de^j_aF^{kl}_{\ bc} + \gamma\epsilon_{ijk}e^i_de^j_aF^{0k}_{\ bc}\right) + \\ &- N\left(\delta_{jl}e^j_aF^{0l}_{\ bc} - \frac{1}{2}\gamma\epsilon_{ijk}e^i_aF^{jk}_{\ bc}\right). \end{aligned} \tag{6.18}$$

Now, with (6.18) in (6.16) we can break the integral in three terms:

$$S = \frac{1}{\gamma}\int dt \int d^3x(L_1 + L_2 + L_3), \tag{6.19}$$

where we have

$$L_1 = -\frac{1}{2}\tilde{\epsilon}^{abc}e^i_ae^j_b\left(\delta_{ik}\delta_{jl}F^{kl}_{\ 0c} + \gamma\epsilon_{ijk}F^{0k}_{\ 0c}\right), \tag{6.20}$$

$$L_2 = -\frac{1}{2}N^d\tilde{\epsilon}^{abc}e^i_de^j_a\left(\delta_{ik}\delta_{jl}F^{kl}_{\ bc} + \gamma\epsilon_{ijk}F^{0k}_{\ bc}\right), \tag{6.21}$$

$$L_3 = \frac{1}{2}Ne^i_a\tilde{\epsilon}^{abc}\left(\delta_{ik}F^{0k}_{\ bc} - \frac{1}{2}\gamma\epsilon_{ijk}F^{jk}_{\ bc}\right). \tag{6.22}$$

We will treat those terms separately since each of them will be responsible for a different constraint in our Hamiltonian formalism, as we will be soon developing.

6.4 The Ashtekar-Barbero Variables

Having defined the triads let us now do some minor modifications in those to define the Ashtekar-Barbero variables, which are the variables in terms of which we will write the action and the Hamiltonian.

6.4.1 Densitized Triad

The densitized triad $\tilde{E}^a_i$ is defined as

$$\tilde{E}^a_i := \det(e)\, e^a_i, \tag{6.23}$$

where $\det(e)$ stands for the determinant of e^i_a.

There are some useful identities relating the densitized triad that will be useful along the development. We will prove some of them here.

First, let us manipulate the determinant identity for a 3-dimensional matrix e:

$$\epsilon_{ijk}e^i_ae^j_be^k_c = \det(e)\,\tilde{\epsilon}_{abc}.$$

One can multiply it by e^a_l and use $e^i_ae^a_l = \delta^i_l$ to get to

$$\epsilon_{ljk}e^j_be^k_c = \det(e)e^a_l\,\tilde{\epsilon}_{abc} = \tilde{E}^a_l\tilde{\epsilon}_{abc}.$$

Contracting now with $\tilde{\epsilon}^{bcd}$ and using $\tilde{\epsilon}_{abc}\tilde{\epsilon}^{bcd} = 2\delta_a^d$ we are left with

$$\tilde{\epsilon}^{bcd}\epsilon_{ljk}e_b^j e_c^k = 2\tilde{E}_l^d,$$

and, renaming some dummy indices, we get

$$\tilde{E}_i^a = \frac{1}{2}\tilde{\epsilon}^{abc}\epsilon_{ijk}e_b^j e_c^k. \tag{6.24}$$

One can also show that

$$e_a^i = \frac{\epsilon^{ijk}\tilde{\epsilon}_{abc}\tilde{E}_j^b\tilde{E}_k^c}{2\det(e)}. \tag{6.25}$$

For that, let us compute $e_a^i e_l^a$:

$$\begin{aligned}
e_a^i e_l^a &= \frac{\epsilon^{ijk}\tilde{\epsilon}_{abc}\tilde{E}_j^b\tilde{E}_k^c}{2\det(e)}e_l^a \\
&= \frac{\det(e)}{2}\epsilon^{ijk}\tilde{\epsilon}_{abc}e_j^b e_k^c e_l^a \\
&= \frac{\det(e)}{2}\epsilon^{ijk}\epsilon_{ljk}\det(e^{-1}) \\
&= \frac{\det(e)}{2}2\delta_l^i\det(e)^{-1} \\
&= \delta_l^i,
\end{aligned}$$

where in the second line we just used (6.23). In the third line we used the equation for the 3-dimensional determinant (see Appendix B) and in the fourth line we used $\det(A^{-1}) = (\det A)^{-1}$.

Some other useful identities that follow from those definitions are

$$\tilde{\epsilon}^{abc}e_a^j = e_p^b e_q^c \epsilon^{jpq}\det e, \tag{6.26}$$

and

$$\tilde{\epsilon}^{abc}e_a^i = \frac{\epsilon^{ijk}\tilde{E}_j^b\tilde{E}_k^c}{\sqrt{\det\tilde{E}}}, \tag{6.27}$$

which follows from (6.25). Those identities will be useful along further development.

6.4.2 Ashtekar-Barbero Connection

One can use the spatial components of the (3+1) split on the spin connection $\omega_\mu^{\ ij}$ to define a new connection on the spatial slice.

We start by defining the extrinsic curvature K_a^i via

$$K_a^i := \omega_a^{\ i0} = -\omega_a^{\ 0i}.$$

Secondly, let us remember that the spin connection is anti-symmetric in its internal indices: $\omega_\mu^{\ ij} = -\omega_\mu^{\ ji}$. Hence, it is a 2-form on the internal space. Taking its Hodge dual we obtain the *dual spin connection* Γ_a^i:

$$\Gamma_a^i := -\frac{1}{2}\epsilon^i_{\ jk}\omega_a^{\ jk},$$

or, inverting the equation:

$$\omega_a^{\ jk} = -\epsilon_i^{\ jk}\Gamma_a^i.$$

With those elements in hand we can define the Ashtekar-Barbero connection A_a^i:

$$A_a^i := \Gamma_a^i + \gamma K_a^i, \tag{6.28}$$

where γ is the Barbero-Irimizi parameter. This object is a 1-form in space, not spacetime, since we have done the (3+1) split in defining those quantities.

6.5 The Curvature Terms

We will now do the (3+1) split in the curvature terms that appear in equations (6.20), (6.21) and (6.22). Let us remind the definition of the curvature 2-form

$$F^{IJ} = \frac{1}{2} F^{IJ}_{\mu\nu} dx^\mu dx^\nu,$$

whose components are given by

$$\frac{1}{2} F^{IJ}_{\mu\nu} = \partial_{[\mu} \omega_{\nu]}^{\ IJ} + \eta_{KL} \omega_{[\mu}^{\ IK} \omega_{\nu]}^{\ LJ}.$$

Doing the (3+1) split in the spacetime indices:

$$\frac{1}{2} F^{IJ}_{0c} = \partial_{[0} \omega_{c]}^{\ IJ} + \eta_{KL} \omega_{[0}^{\ IK} \omega_{c]}^{\ LJ}, \tag{6.29}$$

$$\frac{1}{2} F^{IJ}_{bc} = \partial_{[b} \omega_{c]}^{\ IJ} + \eta_{KL} \omega_{[b}^{\ IK} \omega_{c]}^{\ LJ}. \tag{6.30}$$

With those, we can decompose it in the internal space, using $\omega^{00} = 0$, $\eta_{00} = -1$ and $\eta_{ij} = \delta_{ij}$. Looking into equations (6.20), (6.21) and (6.22) we see that we have four different terms involving the curvature. Let us write down in the (3+1) internal split each of those terms. For the first term we get

$$\begin{aligned}
\frac{1}{2} F^{0k}_{0c} &= \partial_{[0} \omega_{c]}^{\ 0k} + \eta_{ml} \omega_{[0}^{\ 0m} \omega_{c]}^{\ kl} \\
&= -\partial_{[0} K_{c]}^{\ k} - \delta_{ml} K_{[0}^{\ m} \omega_{c]}^{\ kl} \\
&= -\partial_{[0} K_{c]}^{\ k} + \delta_{ml} \epsilon^{lk}{}_{q} K_{[0}^{\ m} \Gamma_{c]}^{\ q} \\
&= -\partial_{[0} K_{c]}^{\ k} - \epsilon^{k}{}_{pq} K_{[0}^{\ P} \Gamma_{c]}^{\ Q},
\end{aligned} \tag{6.31}$$

where we used $\omega_a^{i0} = K_a^i$ in the second line and in the third we used $\omega_a^{ij} = -\epsilon^{ij}{}_{k} \Gamma^k_a$.

The second curvature term which appears in the integral is

$$\begin{aligned}
\frac{1}{2} F^{kl}_{0c} &= \partial_{[0} \omega_{c]}^{\ kl} + \eta_{00} \omega_{[0}^{\ k0} \omega_{c]}^{\ 0l} + \eta_{mn} \omega_{[0}^{\ km} \omega_{c]}^{\ nl} \\
&= \partial_{[0} \omega_{c]}^{\ kl} + K_{[0}^{\ k} K_{c]}^{\ l} + \delta_{mn} \omega_{[0}^{\ km} \omega_{c]}^{\ nl} \\
&= -\epsilon^{kl}{}_{p} \partial_{[0} \Gamma_{c]}^{\ p} + K_{[0}^{\ k} K_{c]}^{\ l} + \delta_{mn} \epsilon^{km}{}_{s} \epsilon^{nl}{}_{r} \Gamma_{[0}^{\ s} \Gamma_{c]}^{\ r} \\
&= -\epsilon^{kl}{}_{p} \partial_{[0} \Gamma_{c]}^{\ p} + K_{[0}^{\ k} K_{c]}^{\ l} - \Gamma_{[0}^{\ k} \Gamma_{c]}^{\ l}.
\end{aligned} \tag{6.32}$$

For the third curvature term we have

$$\begin{aligned}
\frac{1}{2} F^{k}_{bc} &= \partial_{[b} \omega_{c]}^{\ 0k} + \eta_{ml} \omega_{[b}^{\ 0m} \omega_{c]}^{\ lk} \\
&= -\partial_{[b} K_{c]}^{\ k} - \delta_{ml} K_{[b}^{\ m} \omega_{c]}^{\ lk} \\
&= -\partial_{[b} K_{c]}^{\ k} + \delta_{ml} \epsilon^{lk}{}_{q} K_{[b}^{\ m} \Gamma_{c]}^{\ q} \\
&= -\partial_{[b} K_{c]}^{\ k} - \epsilon^{k}{}_{pq} K_{[b}^{\ p} \Gamma_{c]}^{\ q},
\end{aligned} \tag{6.33}$$

and, finally, the last curvature term appearing in the integral is

$$\begin{aligned}
\frac{1}{2} F^{kl}_{bc} &= \partial_{[b} \omega_{c]}^{\ kl} + \eta_{00} \omega_{[b}^{\ k0} \omega_{c]}^{\ 0l} + \eta_{mn} \omega_{[b}^{\ km} \omega_{c]}^{\ nl} \\
&= -\epsilon^{kl}{}_{p} \partial_{[b} \Gamma_{c]}^{\ p} + K_{[b}^{\ k} K_{c]}^{\ l} + \delta_{mn} \omega_{[b}^{\ km} \omega_{c]}^{\ ln} \\
&= -\epsilon^{lk}{}_{p} \partial_{[b} \Gamma_{c]}^{\ p} + K_{[b}^{\ k} K_{c]}^{\ l} + \delta_{mn} \epsilon^{km}{}_{p} \epsilon^{ln}{}_{q} \Gamma_{[b}^{\ p} \Gamma_{c]}^{\ q} \\
&= -\epsilon^{kl}{}_{p} \partial_{[b} \Gamma_{c]}^{\ p} + K_{[b}^{\ k} K_{c]}^{\ l} - \Gamma_{[b}^{\ k} \Gamma_{c]}^{\ l}.
\end{aligned} \tag{6.34}$$

6.6 The Ashtekar action and the equations of motion

Now, inserting equations (6.31), (6.32), (6.33) and (6.34) in (6.20), (6.21) and (6.22) we get, finally, the partial terms of the action completely decomposed in the (3+1) split — in internal and spacetime indices. As said before, each of these three terms, when developed, will lead us to constraints in the system, as we will now show.

6.6.1 L_1: The first term and the Gauss constraint

From equation (6.20) we have

$$\begin{aligned} L_1 = & -\frac{1}{2}\tilde{\epsilon}^{abc}e^i_a e^j_b\left(\delta_{ik}\delta_{jl}F^{kl}_{\ \ 0c}+\gamma\epsilon_{ijk}F^{0k}_{\ \ 0c}\right)= \\ &= \tilde{\epsilon}^{abc}e^i_a e^j_b\delta_{ik}\delta_{jl}\left(\epsilon^{kl}{}_p\partial_{[0}\Gamma^{\ p}_{c]}-K^{\ k}_{[0}K^{\ l}_{c]}+\Gamma^{\ k}_{[0}\Gamma^{\ l}_{c]}\right)+ \\ &+\tilde{\epsilon}^{abc}e^i_a e^j_b\gamma\epsilon_{ijk}\left(\partial_{[0}K^{\ k}_{c]}+\epsilon^k{}_{pq}K^{\ p}_{[0}\Gamma^{\ q}_{c]}\right). \end{aligned}$$

Opening the anti-symmetrizers and using $\tilde{E}^i_a=\frac{1}{2}\tilde{\epsilon}^{abc}\epsilon_{ijk}e^i_a e^j_b$ we get

$$\begin{aligned} L_1 = &\frac{1}{2}\epsilon^{ijm}\tilde{E}^c_m\delta_{ik}\delta_{jl}\left[\epsilon^{kl}{}_p\left(\partial_0\Gamma^{\ p}_c-\partial_c\Gamma^{\ p}_0\right)-K^{\ k}_0K^{\ l}_c+K^{\ k}_cK^{\ l}_0+\Gamma^{\ k}_0\Gamma^{\ l}_c-\Gamma^{\ k}_c\Gamma^{\ l}_0\right]+ \\ &+\tilde{E}^c_k\gamma\left[\partial_0K^{\ k}_c-\partial_cK^{\ k}_0+\epsilon^k{}_{pq}\left(K^{\ p}_0\Gamma^{\ q}_c-K^{\ p}_c\Gamma^{\ q}_0\right)\right], \end{aligned}$$

and using $\delta_{ik}\delta_{jl}\epsilon^{ijm}\epsilon^{kl}{}_p=2\delta^m_p$ we are left with

$$\begin{aligned} L_1 = &\tilde{E}^c_p\partial_0\Gamma^p_c-\tilde{E}^c_p\partial_c\Gamma^p_0+\frac{1}{2}\tilde{E}^c_m\epsilon^m{}_{kl}\left(\Gamma^{\ k}_0\Gamma^{\ l}_c-\Gamma^{\ k}_c\Gamma^{\ l}_0-K^{\ k}_0K^{\ l}_c+K^{\ k}_cK^{\ l}_0\right)+ \\ &+\gamma\tilde{E}^c_k\left[\partial_0K^{\ k}_c-\partial_cK^{\ k}_0+\epsilon^k{}_{pq}\left(K^{\ p}_0\Gamma^{\ q}_c-K^{\ p}_c\Gamma^{\ q}_0\right)\right]. \end{aligned}$$

Now remember from equation (6.19) that the L_1 term is being integrated in space and time. Hence, integrating by parts we get, neglecting boundary terms:

$$\begin{aligned} -\int\tilde{E}^c_p\partial_c\Gamma^p_0 &= \int\Gamma^p_0\partial_c\tilde{E}^c_p \\ -\int\gamma\tilde{E}^c_k\partial_cK^{\ k}_0 &= \int\gamma K^{\ k}_0\partial_c\tilde{E}^c_k. \end{aligned}$$

Then, we have

$$\begin{aligned} L_1 = &\tilde{E}^c_p\partial_0\Gamma^p_c+\Gamma^p_0\partial_c\tilde{E}^c_p+\tilde{E}^c_m\epsilon^m{}_{kl}\left(\Gamma^{\ k}_0\Gamma^{\ l}_c-K^{\ k}_0K^{\ l}_c\right)+ \\ &+\gamma\tilde{E}^c_k\partial_0K^{\ k}_c+\gamma K^{\ k}_0\partial_c\tilde{E}^c_k+\gamma\tilde{E}^c_k\epsilon^k{}_{pq}\left(K^{\ p}_0\Gamma^{\ q}_c-K^{\ p}_c\Gamma^{\ q}_0\right), \end{aligned}$$

where in the parentheses of the first line we used the fact that $\epsilon^m{}_{kl}$ is anti-symmetric in $[k,l]$

and that $A_{kl} = \frac{1}{2}(A_{kl} - A_{lk})$. Grouping now some terms and relabeling some indices we get

$$
\begin{aligned}
L_1 &= \left(\tilde{E}^c_k \partial_0 \Gamma^k_c + \gamma \tilde{E}^c_k \partial_0 K_c{}^k\right) + \left(\Gamma^k_0 \partial_c \tilde{E}^c_k + \gamma K_0{}^k \partial_c \tilde{E}^c_k\right) + \tilde{E}^c_k \epsilon^k{}_{ij} \Gamma^i_0 \left(\Gamma^j_c + \gamma K^j_c\right) - \tilde{E}^c_k \epsilon^k{}_{ij} K^i_0 \left(K^j_c - \gamma \Gamma^j_c\right) = \\
&= \tilde{E}^c_k \partial_0 \left(\Gamma^k_c + \gamma K_c{}^k\right) + \partial_c \tilde{E}^c_k \left(\Gamma^k_0 + \gamma K^k_0\right) + \tilde{E}^c_k \epsilon^k{}_{ij} \Gamma^i_0 \left(\Gamma^j_c + \gamma K^j_c\right) - \tilde{E}^c_k \epsilon^k{}_{ij} K^i_0 \left(K^j_c - \gamma \Gamma^j_c\right) = \\
&= \tilde{E}^c_k \partial_0 \left(\Gamma^k_c + \gamma K_c{}^k\right) + \Gamma^i_0 \left\{\partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \left(\Gamma^j_c + \gamma K^j_c\right)\right\} + \gamma K^i_0 \left\{\partial_c \tilde{E}^c_i - \tilde{E}^c_k \epsilon^k{}_{ij} \left(\frac{1}{\gamma} K^j_c - \Gamma^j_c\right)\right\} = \\
&= \tilde{E}^c_k \partial_0 \left(\Gamma^k_c + \gamma K_c{}^k\right) + \Gamma^i_0 \left\{\partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \left(\Gamma^j_c + \gamma K^j_c\right)\right\} - \frac{1}{\gamma} K^i_0 \left\{-\gamma^2 \partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \left(\gamma K^j_c - \gamma^2 \Gamma^j_c\right)\right\} = \\
&= \tilde{E}^c_k \partial_0 \left(\Gamma^k_c + \gamma K_c{}^k\right) + \Gamma^i_0 \left\{\partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \left(\Gamma^j_c + \gamma K^j_c\right)\right\} - \frac{1}{\gamma} K^i_0 \left\{-\gamma^2 \partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \gamma K^j_c - \tilde{E}^c_k \epsilon^k{}_{ij} \gamma^2 \Gamma^j_c\right\} = \\
&= \tilde{E}^c_k \partial_0 \left(\Gamma^k_c + \gamma K_c{}^k\right) + \Gamma^i_0 \left\{\partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \left(\Gamma^j_c + \gamma K^j_c\right)\right\} - \frac{1}{\gamma} K^i_0 \left\{\partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \left(\Gamma^j_c + \gamma K^j_c\right)\right\} + \\
&\quad - \frac{1}{\gamma} K^i_0 \left\{-\gamma^2 \partial_c \tilde{E}^c_i - \tilde{E}^c_k \epsilon^k{}_{ij} \Gamma^j_c - \tilde{E}^c_k \epsilon^k{}_{ij} \gamma^2 \Gamma^j_c - \partial_c \tilde{E}^c_i\right\} = \\
&= \tilde{E}^c_k \partial_0 \left(\Gamma^k_c + \gamma K_c{}^k\right) + \left(\Gamma^i_0 - \frac{1}{\gamma} K^i_0\right) \left\{\partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \left(\Gamma^j_c + \gamma K^j_c\right)\right\} + \\
&\quad + K^i_0 \left\{\partial_c \tilde{E}^c_i \left(\gamma + \frac{1}{\gamma}\right) + \tilde{E}^c_k \epsilon^k{}_{ij} \Gamma^j_c \left(\gamma + \frac{1}{\gamma}\right)\right\} = \\
&= \tilde{E}^c_k \partial_0 \left(\Gamma^k_c + \gamma K_c{}^k\right) + \left(\Gamma^i_0 - \frac{1}{\gamma} K^i_0\right) \left\{\partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \left(\Gamma^j_c + \gamma K^j_c\right)\right\} + K^i_0 \left(\gamma + \frac{1}{\gamma}\right) \left\{\partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \Gamma^j_c\right\},
\end{aligned}
$$

where, from the fifth to the sixth line we added and subtracted the term $\partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \Gamma^j_c$ inside the last parentheses of the equation.

Now, using

$$A^k_c = \Gamma^k_c + \gamma K^k_c \,, \tag{6.35}$$

and introducing the quantities

$$\alpha^i := \left(\frac{1}{\gamma} + \gamma\right) K^i_0, \tag{6.36}$$

$$\lambda^i := \Gamma^i_0 - \frac{1}{\gamma} K^i_0, \tag{6.37}$$

$$G_i := \partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} A^j_c, \tag{6.38}$$

we are left with

$$L_1 = \tilde{E}^c_k \partial_0 A^k_c + \lambda^i G_i + \alpha^i (\partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k{}_{ij} \Gamma^j_c). \tag{6.39}$$

The last term in parentheses is just $d_\Gamma E$, which vanishes. This happens because the connection Γ is torsionless, i.e. $\mathbf{T} = d_\Gamma \mathbf{e} = 0$, where $\mathbf{e}$ is the frame field in form notation. One could also write the densitized triad in forms notation via

$$E^i{}_{ab} = \epsilon^i{}_{jk} e^j_a e^k_b = \frac{1}{2} [\mathbf{e}, \mathbf{e}]^i{}_{ab},$$

hence, it is easy to see that

$$d_\Gamma \mathbf{E} = \frac{1}{2} d_\Gamma [\mathbf{e}, \mathbf{e}] = [d_\Gamma \mathbf{e}, \mathbf{e}] = [\mathbf{T}, \mathbf{e}] = 0$$

indeed vanishes.

Hence, the first term is just

$$L_1 = \tilde{E}^c_k \partial_0 A^k_c + \lambda^i G_i, \tag{6.40}$$

where G_i is called the Gauss Constraint, which generates $SU(2)$ gauge transformations as we will discuss later.

6.6.2 L_2: The second term and the Diffeomorphism constraint

From equations (6.33) and (6.34) in (6.21) we get

$$\begin{aligned} L_2 = &- \frac{1}{2} N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \left(\delta_{ik}\delta_{jl} F^{kl}_{\ \ bc} + \gamma \epsilon_{ijk} F^{0k}_{\ \ bc} \right) = \\ &= N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \delta_{ik}\delta_{jl} \left(\epsilon^{kl}_{\ \ p} \partial_{[b} \Gamma^{\ p}_{c]} - K^{\ k}_{[b} K^{\ l}_{c]} + \Gamma^{\ k}_{[b} \Gamma^{\ l}_{c]} \right) + \\ &+ N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \gamma \epsilon_{ijk} \left(\partial_{[b} K^{\ k}_{c]} + \epsilon^k_{\ pq} K^{\ p}_{[b} \Gamma^{\ q}_{c]} \right). \end{aligned}$$

Note that $\delta_{ik}\delta_{jl}\epsilon^{kl}_{\ \ p} = \epsilon_{ijp}$, and, relabelling some dummy indices we can write

$$\begin{aligned} L_2 = &N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \epsilon_{ijk} \partial_{[b} \Gamma^{\ k}_{c]} + N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \delta_{ik}\delta_{jl} \left(\Gamma^{\ k}_{[b} \Gamma^{\ l}_{c]} - K^{\ k}_{[b} K^{\ l}_{c]} \right) + \\ &+ N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \left(\gamma \epsilon_{ijk} \partial_{[b} K^{\ k}_{c]} + \gamma \epsilon_{ijk} \epsilon^k_{\ pq} K^{\ p}_{[b} \Gamma^{\ q}_{c]} \right), \end{aligned}$$

and, grouping some similar terms we are led to

$$\begin{aligned} L_2 = &N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \epsilon_{ijk} \left(\partial_{[b} \Gamma^{\ k}_{c]} + \gamma \partial_{[b} K^{\ k}_{c]} \right) + N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \delta_{ik}\delta_{jl} \left(\Gamma^{\ k}_{[b} \Gamma^{\ l}_{c]} - K^{\ k}_{[b} K^{\ l}_{c]} \right) + \\ &+ N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \gamma \epsilon_{ijk} \epsilon^k_{\ pq} K^{\ p}_{[b} \Gamma^{\ q}_{c]}. \end{aligned}$$

From the definition of the Ashtekar-Barbero connection one can easily write the first term in parentheses as $\partial_{[b} \Gamma^{\ k}_{c]} + \gamma \partial_{[b} K^{\ k}_{c]} = \partial_{[b} A^k_{c]}$. Also, in the last line we can write:

$$\epsilon_{ijk}\epsilon^k_{\ pq} = \eta^{mk}\epsilon_{ijk}\epsilon_{pqm} = \eta^{mk}(\delta_{ip}\delta_{jq}\delta_{km} - \delta_{iq}\delta_{jp}\delta_{mk}) = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}.$$

Then we are left with

$$\begin{aligned} L_2 = &N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \epsilon_{ijk} \partial_{[b} A^k_{c]} + N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \delta_{ik}\delta_{jl} \left(\Gamma^{\ k}_{[b} \Gamma^{\ l}_{c]} - K^{\ k}_{[b} K^{\ l}_{c]} \right) + \\ &+ N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \gamma (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) K^{\ p}_{[b} \Gamma^{\ q}_{c]} = \\ &= N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \epsilon_{ijk} \partial_{[b} A^k_{c]} + N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \delta_{ik}\delta_{jl} \left(\Gamma^{\ k}_{[b} \Gamma^{\ l}_{c]} - K^{\ k}_{[b} K^{\ l}_{c]} \right) + \\ &+ N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \gamma (K^{\ i}_{[b} \Gamma^{\ j}_{c]} - K^{\ j}_{[b} \Gamma^{\ i}_{c]}), \end{aligned}$$

and, since the entire expression is being multiplied by $\tilde{\epsilon}^{abc}$, which is already anti-symmetric in $[bc]$, we can drop out the anti-symmetrizers:

$$\begin{aligned} L_2 &= N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \left\{ \epsilon_{ijk} \partial_b A^k_c + \delta_{ik}\delta_{jl} \left(\Gamma_b^{\ k} \Gamma_c^{\ l} - K_b^{\ k} K_c^{\ l} \right) + \gamma (K_b^{\ i} \Gamma_c^{\ j} - K_b^{\ j} \Gamma_c^{\ i}) \right\} = \\ &= N^d \tilde{\epsilon}^{abc} e^i_d e^j_a \left\{ \epsilon_{ijk} \partial_b A^k_c + \left(\Gamma_b^{\ i} \Gamma_c^{\ j} - K_b^{\ j} K_c^{\ i} \right) + \gamma (K_b^{\ i} \Gamma_c^{\ j} - K_b^{\ j} \Gamma_c^{\ i}) \right\}. \end{aligned} \tag{6.41}$$

However, the curvature 2-form of the Ashtekar-Barbero connection A^k_c is defined, in index notation, as

$$\frac{1}{2} F^k_{bc} \coloneqq \partial_{[b} A^k_{c]} + \frac{1}{2} \epsilon^k_{\ lm} A^l_b A^m_c. \tag{6.42}$$

Hence, expanding $A^k_c = \Gamma^k_c + \gamma K^k_c$ and contracting both sides of (6.42) with $\tilde{\epsilon}^{abc}\epsilon_{ijk}$ we get

$$\begin{aligned}
\frac{1}{2}\tilde{\epsilon^{abc}}\epsilon_{ijk}F^k_{bc} &= \tilde{\epsilon}^{abc}\left[\epsilon_{ijk}\partial_{[b}A^k_{c]} + \frac{1}{2}\epsilon_{ijk}\epsilon^k{}_{lm}(\Gamma^l_b + \gamma K^l_b)(\Gamma^m_c + \gamma K^m_c)\right] \\
&= \tilde{\epsilon}^{abc}\left[\epsilon_{ijk}\partial_{[b}A^k_{c]} + \frac{1}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(\Gamma^l_b + \gamma K^l_b)(\Gamma^m_c + \gamma K^m_c)\right] \\
&= \tilde{\epsilon}^{abc}\left[\epsilon_{ijk}\partial_{[b}A^k_{c]} + \frac{1}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(\Gamma^l_b\Gamma^m_c + \Gamma^l_b\gamma K^m_c + \gamma K^l_b\Gamma^m_c + \gamma^2 K^l_b K^m_c)\right] \\
&= \tilde{\epsilon}^{abc}\left[\epsilon_{ijk}\partial_{[b}A^k_{c]} + \delta_{il}\delta_{jm}\left\{\Gamma^l_b\Gamma^m_c + \gamma\left(\Gamma^l_b K^m_c + K^l_b\Gamma^m_c\right) + \gamma^2 K^l_b K^m_c\right\}\right],
\end{aligned}$$

where in the last line we used again the notation for anti-symmetric objects $A_{\mu\nu} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$.

Therefore, subtracting $\tilde{\epsilon}^{abc}\delta_{il}\delta_{jm}(1+\gamma^2)K^l_b K^m_c$ from both sides we can write

$$\tilde{\epsilon}^{abc}\left\{\frac{1}{2}\epsilon_{ijk}F^k_{bc} - \delta_{il}\delta_{jm}(1+\gamma^2)K^l_b K^m_c\right\} = \tilde{\epsilon}^{abc}\left[\epsilon_{ijk}\partial_{[b}A^k_{c]} + \delta_{il}\delta_{jm}\left\{\left(\Gamma^l_b\Gamma^m_c - K^l_b K^m_c\right) + \gamma\left(\Gamma^l_b K^m_c + K^l_b\Gamma^m_c\right)\right\}\right].$$

Hence, in equation (6.41) we have

$$L_2 = N^d\tilde{\epsilon}^{abc}e^i_d e^j_a\left\{\frac{1}{2}\epsilon_{ijk}F^k_{bc} - \delta_{il}\delta_{jm}(1+\gamma^2)K^l_b K^m_c\right\},$$

and since $\tilde{\epsilon}^{abc}e^j_a = e^b_p e^c_q \epsilon^{jpq}\det e$ (equation (6.26)) we have

$$\begin{aligned}
L_2 &= N^d e^i_d e^b_p e^c_q |e|\epsilon^{jpq}\left\{\frac{1}{2}\epsilon_{ijk}F^k_{bc} - \delta_{il}\delta_{jm}(1+\gamma^2)K^l_b K^m_c\right\} \\
&= N^d e^i_d e^b_p e^c_q |e|\left\{\frac{1}{2}(\delta^p_k\delta^q_i - \delta^p_i\delta^q_k)F^k_{bc} - \delta_{il}\epsilon^{pq}{}_m(1+\gamma^2)K^l_b K^m_c\right\} \\
&= N^d|e|\left\{\frac{1}{2}e^i_d(e^b_p e^c_i F^p_{bc} - e^b_i e^c_q F^q_{bc}) - \delta_{il}e^i_d e^b_p e^c_q\epsilon^{pq}{}_m(1+\gamma^2)K^l_b K^m_c\right\} \\
&= N^d|e|\left\{\frac{1}{2}\left[e^b_p(e^i_d e^c_i)F^p_{bc} - (e^i_d e^b_i)e^c_q F^q_{bc})\right] - \delta_{il}e^i_d e^b_p e^c_q\epsilon^{pq}{}_m(1+\gamma^2)K^l_b K^m_c\right\} \\
&= N^d|e|\left\{\frac{1}{2}\left[e^b_p F^p_{bd} - e^c_q F^q_{dc})\right] - \delta_{il}e^i_d e^b_p e^c_q\epsilon^{pq}{}_m(1+\gamma^2)K^l_b K^m_c\right\} \\
&= N^d|e|\left\{e^b_p F^p_{bd} - \delta_{il}e^i_d e^b_p e^c_q\epsilon^{pq}{}_m(1+\gamma^2)K^l_b K^m_c\right\},
\end{aligned} \tag{6.43}$$

where we used $e^i_d e^c_i = \delta^c_d$ in the fifth line and $A_{\mu\nu} = (A_{\mu\nu} - A_{\nu\mu})/2$ in the last one.

We can also express this in terms of the densitized triad $\tilde{E}^a_i = |e|e^a_i$:

$$\begin{aligned}
L_2 &= N^a|e|\left\{e^b_p F^p_{ba} - \delta_{il}e^i_a e^b_p e^c_q\epsilon^{pq}{}_m(1+\gamma^2)K^l_b K^m_c\right\} \\
&= -N^a|e|\left\{e^b_p F^p_{ab} + \delta_{il}e^i_a e^b_p e^c_q\epsilon^{pq}{}_m(1+\gamma^2)K^l_b K^m_c\right\} \\
&= -N^a\left\{\tilde{E}^b_p F^p_{ab} + \delta_{il}e^i_a e^b_p(1+\gamma^2)K^l_b\epsilon^{pq}{}_m K^m_c\tilde{E}^c_q\right\}.
\end{aligned} \tag{6.44}$$

Note that, from (6.38) we can write:

$$\begin{aligned}
G_i &= \partial_c\tilde{E}^c_i + \tilde{E}^c_k\epsilon^k{}_{ij}(\Gamma^j_c + \gamma K^j_c) \\
&= \partial_c\tilde{E}^c_i + \tilde{E}^c_k\epsilon^k{}_{ij}\Gamma^j_c + \tilde{E}^c_k\epsilon^k{}_{ij}\gamma K^j_c \\
&= \tilde{E}^c_k\epsilon^k{}_{ij}\gamma K^j_c,
\end{aligned}$$

since $\partial_c\tilde{E}^c_i + \tilde{E}^c_k\epsilon^k{}_{ij}\Gamma^j_c = d_\Gamma E = 0$.

Therefore we can write that $\epsilon^{pq}_{\ \ m} K_c^m \tilde{E}_q^c = -\frac{1}{\gamma} G^p$. Hence, in (6.44) we get

$$L_2 = -N^a \left\{ \tilde{E}_p^b F_{ab}^p - \delta_{il} e_a^i e_p^b \left(\frac{1}{\gamma} + \gamma \right) K_b^l G^p \right\}.$$

The part with the Gauss constraint G^p is redundant, its content is already covered by L_1. We are then left with

$$L_2 = -N^a \tilde{E}_p^b F_{ab}^p, \tag{6.45}$$

or, defining the momentum constraint as

$$V_a := \tilde{E}_p^b F_{ab}^p, \tag{6.46}$$

we can write it as

$$L_2 = -N^a V_a. \tag{6.47}$$

This is called the *vector constraint*, which is related to spatial diffeomorphisms, as we will show and discuss later.

6.6.3 L_3: The third term and the Hamiltonian constraint

Finally we get, for the third term, from equations (6.33) and (6.34) in (6.22):

$$\begin{aligned} L_3 &= \frac{1}{2} N e_a^i \tilde{\epsilon}^{abc} \left(\delta_{ik} F^{0k}_{\ \ bc} - \frac{1}{2} \gamma \epsilon_{ijk} F^{jk}_{\ \ bc} \right) \\ &= N e_a^i \tilde{\epsilon}^{abc} \delta_{ik} \left(-\partial_{[b} K_{c]}^{\ k} - \epsilon^k_{\ pq} K_{[b}^{\ p} \Gamma_{c]}^{\ q} \right) + \\ &\quad - \frac{1}{2} N e_a^i \tilde{\epsilon}^{abc} \gamma \epsilon_{ijk} \left(-\epsilon^{jk}_{\ \ p} \partial_{[b} \Gamma_{c]}^{\ p} + K_{[b}^{\ j} K_{c]}^{\ k} - \Gamma_{[b}^{\ j} \Gamma_{c]}^{\ k} \right). \end{aligned}$$

Using equation (6.27) we are left with

$$\begin{aligned} L_3 = &- \frac{N \epsilon^{imn} \tilde{E}_m^b \tilde{E}_n^c}{\sqrt{\det \tilde{E}}} \delta_{ij} \left(\partial_b K_c^{\ j} + \epsilon^j_{\ pq} K_b^{\ p} \Gamma_c^{\ q} \right) + \\ &+ \frac{1}{2} \frac{N \epsilon^{imn} \tilde{E}_m^b \tilde{E}_n^c}{\sqrt{\det \tilde{E}}} \gamma \left[\epsilon_{ijk} \epsilon^{jk}_{\ \ p} \partial_b \Gamma_c^{\ p} + \epsilon_{ijk} \left(\Gamma_b^{\ j} \Gamma_c^{\ k} - K_b^{\ j} K_c^{\ k} \right) \right], \end{aligned}$$

where we dropped the anti-symmetrizers since the expression is already anti-symmetric in $[b, c]$. Also, since $\epsilon_{ijk} \epsilon^{jk}_p = 2\delta_{ip}$, and renaming some dummy indices, we get

$$\begin{aligned} L_3 = &- \frac{N \epsilon^{imn} \tilde{E}_m^b \tilde{E}_n^c}{\sqrt{\det \tilde{E}}} \left(\delta_{ij} \partial_b K_c^{\ j} + \epsilon_{ijk} K_b^{\ j} \Gamma_c^{\ k} \right) + \\ &+ \frac{N \epsilon^{imn} \tilde{E}_m^b \tilde{E}_n^c}{\sqrt{\det \tilde{E}}} \gamma \left[\delta_{ij} \partial_b \Gamma_c^{\ j} + \frac{1}{2} \epsilon_{ijk} \left(\Gamma_b^{\ j} \Gamma_c^{\ k} - K_b^{\ j} K_c^{\ k} \right) \right], \end{aligned}$$

and, regrouping some terms:

$$\begin{aligned} L_3 = &\frac{N \epsilon^{imn} \tilde{E}_m^b \tilde{E}_n^c}{\sqrt{\det \tilde{E}}} \left[\delta_{ij} \gamma \partial_b \left(\Gamma_c^{\ j} - \frac{1}{\gamma} K_c^{\ j} \right) \right] + \\ &- \frac{N \epsilon^{imn} \tilde{E}_m^b \tilde{E}_n^c}{\sqrt{\det \tilde{E}}} \epsilon_{ijk} \left[K_b^{\ j} \Gamma_c^{\ k} + \frac{1}{2} \gamma \left(K_b^{\ j} K_c^{\ k} - \Gamma_b^{\ j} \Gamma_c^{\ k} \right) \right]. \end{aligned}$$

Now, plugging $K^j_c = \frac{1}{\gamma}(A^j_c - \Gamma^j_c)$ into the equation we are led to

$$L_3 = \frac{N\epsilon^{imn}\tilde{E}^b_m\tilde{E}^c_n}{\sqrt{\det\tilde{E}}}\left[\delta_{ij}\gamma\partial_b\left(\Gamma_c^{\ j} - \frac{1}{\gamma^2}(A^j_c - \Gamma^j_c)\right)\right] + \\ - \frac{N\epsilon^{imn}\tilde{E}^b_m\tilde{E}^c_n}{\sqrt{\det\tilde{E}}}\epsilon_{ijk}\left[\frac{1}{\gamma}(A^j_b - \Gamma^j_b)\Gamma_c^{\ k} + \frac{1}{2}\gamma\left\{\frac{1}{\gamma^2}(A^j_b - \Gamma^j_b)(A^k_c - \Gamma^k_c) - \Gamma_b^{\ j}\Gamma_c^{\ k}\right\}\right],$$

and, developing the equation:

$$L_3 = \frac{N\epsilon^{imn}\tilde{E}^b_m\tilde{E}^c_n}{\sqrt{\det\tilde{E}}}\left[\partial_b\left(\gamma\Gamma_c^{\ i} - \frac{1}{\gamma}(A^i_c - \Gamma^i_c)\right) - \epsilon_{ijk}\frac{1}{\gamma}(A^j_b - \Gamma^j_b)\Gamma_c^{\ k}\right] + \\ - \frac{N\epsilon^{imn}\tilde{E}^b_m\tilde{E}^c_n}{\sqrt{\det\tilde{E}}}\epsilon_{ijk}\left[\frac{1}{2}\left\{\frac{1}{\gamma}(A^j_bA^k_c - A^j_b\Gamma^k_c + \Gamma^j_b\Gamma^k_c - \Gamma^j_bA^k_c) - \gamma\Gamma_b^{\ j}\Gamma_c^{\ k}\right\}\right] = \\ = \frac{N\epsilon^{imn}\tilde{E}^b_m\tilde{E}^c_n}{\gamma\sqrt{\det\tilde{E}}}\left[\gamma^2\partial_b\Gamma_c^{\ i} - \partial_bA^i_c + \partial_b\Gamma^i_c - \epsilon_{ijk}(A^j_b - \Gamma^j_b)\Gamma_c^{\ k}\right] + \\ - \frac{N\epsilon^{imn}\tilde{E}^b_m\tilde{E}^c_n}{\gamma\sqrt{\det\tilde{E}}}\epsilon_{ijk}\frac{1}{2}\left[(A^j_bA^k_c - A^j_b\Gamma^k_c + \Gamma^j_b\Gamma^k_c - \Gamma^j_bA^k_c) - \gamma^2\Gamma_b^{\ j}\Gamma_c^{\ k}\right].$$

Note that the term proportional to ϵ_{ijk} is

$$-A^j_b\Gamma_c^{\ k} + \Gamma_c^{\ k}\Gamma^j_b - \frac{1}{2}A^j_bA^k_c + \frac{1}{2}A^j_b\Gamma^k_c - \frac{1}{2}\Gamma^j_b\Gamma^k_c + \frac{1}{2}\Gamma^j_bA^k_c + \frac{1}{2}\gamma^2\Gamma_b^{\ j}\Gamma_c^{\ k},$$

which can be simplified as

$$-\frac{1}{2}A^j_b\Gamma_c^{\ k} + \frac{1}{2}\Gamma_c^{\ k}\Gamma^j_b - \frac{1}{2}A^j_bA^k_c + \frac{1}{2}\Gamma^j_bA^k_c + \frac{1}{2}\gamma^2\Gamma_b^{\ j}\Gamma_c^{\ k},$$

or, finally

$$\frac{1}{2}\Gamma_c^{\ k}\Gamma^j_b - \frac{1}{2}A^j_bA^k_c + \frac{1}{2}\gamma^2\Gamma_b^{\ j}\Gamma_c^{\ k} = \frac{1}{2}(1+\gamma^2)\Gamma_c^{\ k}\Gamma^j_b - \frac{1}{2}A^j_bA^k_c.$$

Hence, we can write the Lagrangian as

$$L_3 = -\frac{N\epsilon^{imn}\tilde{E}^b_m\tilde{E}^c_n}{\gamma\sqrt{\det\tilde{E}}}\left[\partial_bA^i_c - (1+\gamma^2)(\partial_b\Gamma^i_c)\right] + \\ - \frac{N\epsilon^{imn}\tilde{E}^b_m\tilde{E}^c_n}{\gamma\sqrt{\det\tilde{E}}}\epsilon_{ijk}\left[\frac{1}{2}A^j_bA^k_c - \frac{1}{2}(1+\gamma^2)\Gamma_c^{\ k}\Gamma^j_b\right],$$

then,

$$L_3 = -\frac{N\epsilon^{imn}\tilde{E}^b_m\tilde{E}^c_n}{\gamma\sqrt{\det\tilde{E}}}\left[\left(\partial_bA^i_c + \frac{1}{2}\epsilon_{ijk}A^j_bA^k_c\right) - (1+\gamma^2)\left(\partial_b\Gamma^i_c + \frac{1}{2}\epsilon_{ijk}\Gamma_c^{\ k}\Gamma^j_b\right)\right]$$

or, using $\epsilon^{imn} = \delta^{ip}\epsilon_p^{\ mn}$:

$$L_3 = -\frac{N\epsilon_i^{\ mn}\tilde{E}^b_m\tilde{E}^c_n}{\gamma\sqrt{\det\tilde{E}}}\left[\left(\partial_bA^i_c + \frac{1}{2}\epsilon^i_{\ jk}A^j_bA^k_c\right) - (1+\gamma^2)\left(\partial_b\Gamma^i_c + \frac{1}{2}\epsilon^i_{\ jk}\Gamma_c^{\ k}\Gamma^j_b\right)\right],$$

However, the expressions in parentheses are the equations in components for the curvature 2-forms of the connections A^i_j and Γ^i_j, respectively:

$$\frac{1}{2}F^i_{\ bc} := \partial_b A^i_c + \frac{1}{2}\epsilon^i_{\ jk} A^j_b A^k_c, \tag{6.48}$$
$$\frac{1}{2}R^i_{\ bc} := \partial_b \Gamma^i_{\ c} + \frac{1}{2}\epsilon^i_{\ jk} \Gamma_c^{\ k} \Gamma^j_b.$$

Hence, the expression can be written as

$$L_3 = -\frac{N\epsilon_i^{\ mn}\tilde{E}^b_m \tilde{E}^c_n}{2\gamma\sqrt{\det \tilde{E}}}\left[F^i_{\ bc} - (1+\gamma^2)R^i_{\ bc}\right]. \tag{6.49}$$

If one defines the scalar constraint C as

$$C := -\frac{\epsilon_i^{\ mn}\tilde{E}^b_m \tilde{E}^c_n}{2\gamma\sqrt{\det \tilde{E}}}\left[F^i_{\ bc} - (1+\gamma^2)R^i_{\ bc}\right], \tag{6.50}$$

then

$$L_3 = NC, \tag{6.51}$$

where C is the scalar constraint — or Hamiltonian constraint — and N is just a Lagrangian multiplier.

6.7 The Hamiltonian as a Linear Combination of Constraints

Back to equation (6.19) we can now write the action as

$$S = \frac{1}{\gamma}\int dt \int d^3x \left(\tilde{E}^a_i \partial_t A^i_a + \lambda^i G_i + N^a V_a + NC\right), \tag{6.52}$$

which is the Ashtekar action for classical gravity. From the first term we can see that the Ashtekar-Barbero connection A^i_b and the densitized triad $\tilde{E}^b_i$ are conjugate variables. Here, λ^i, N^a and N are Lagrange multipliers. The following terms — already previously defined — deserve to be highlighted again for the sake of clarity:

- Gauss constraint: $G_i := \partial_c \tilde{E}^c_i + \tilde{E}^c_k \epsilon^k_{\ ij} A^j_c$
- Vector constraint: $V_a := \tilde{E}^b_i F^i_{ab}$
- Hamiltonian constraint: $C := -\dfrac{\epsilon_i^{\ mn}\tilde{E}^b_m \tilde{E}^c_n}{2\gamma\sqrt{\det \tilde{E}}}\left[F^i_{\ bc} - (1+\gamma^2)R^i_{\ bc}\right]$

We can then get the Hamiltonian

$$H[A^i_a; \tilde{E}^a_i] = \int d^3x \left(\lambda^i G_i + N^a V_a + NC\right), \tag{6.53}$$

with the first class constraints, which generates the expected gauge freedom: the triad rotations and spacetime diffeomorphisms, which is discussed in the next section.

If one writes the Hamilton equations that result from this Hamiltonian, one will indeed reproduce Einstein field equations, as expected.

6.8 Geometrical interpretation of the Constraints

We will now conclude our discussion by developing the geometrical interpretation of the constraints in the Hamiltonian on Ashtekar's formulation of General Relativity.

6.8.1 Electromagnetism

Let us first develop the geometrical interpretation of the constraints for a more familiar theory, classical electromagnetism. The Lagrangian for electromagnetism is

$$L = \frac{1}{4} \int d^3x F_{\mu\nu} F^{\mu\nu}, \tag{6.54}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor and A_μ is the vector potential. It is easy to see that the zeroth component of the vector potential will not appear with a time derivative, since $F_{\mu\nu}$ vanishes for $\mu = \nu$. Hence, only spatial derivatives of A_0 appear in the Lagrangian, which means that it does not play a dynamical role. It is, actually, a Lagrange multiplier, as we will show later.

Hence, being A_b our dynamical variable, its conjugate momentum is given by

$$\pi^b = \frac{\delta L}{\delta \dot{A}_b}, \tag{6.55}$$

which, when one takes the functional derivative of (6.54), gives $\tilde{E}^b$ — the electric field. The canonical variables are then $A_b(x)$ and $\tilde{E}^b(x)$, where a stands for spatial coordinates, as usual: $a = 1, 2$ or 3 (remember that A_0 is not a dynamical variable).

When taking the functional derivative we get a density, as expected, since the Lagrangian is a volume integral, hence, its integrand must be a scalar density. When one takes a functional derivative, the integral disappears and the result must be a density. The Poisson bracket of $\tilde{E}^b(x)$ and $A_b(x)$ (which is not a density) is again a density, as expected

$$\left\{ A_b(x), \tilde{E}^b(x) \right\} = \delta^a_b \delta^3(x - y).$$

By the usual Legendre transformation one can build the Hamiltonian

$$H := \int d^3x \left(\tilde{E}^b(x) \dot{A}_b(x) - \tilde{\mathcal{L}} \right)$$

which, when written in terms of the canonical pairs, gives

$$H = \int d^3x \left(\frac{1}{2} \left[E^a(x) E^b(x) + B^a(x) B^b(x) \right] \delta_{ab} - A_0 \partial_a E^a \right) \tag{6.56}$$

where $B^a = \dfrac{1}{2} \epsilon^{abc} F_{bc}$ is the magnetic field, which is a function of A_a.

Working out the equation of motion for π^0 we get [8]

$$\begin{aligned} \dot{\pi}^0 &= \left\{ \pi^0, H \right\} \\ &= \partial_a E^a, \end{aligned} \tag{6.57}$$

which should vanish, since $\pi^\alpha = \dfrac{\delta L}{\delta \dot{A}_\alpha} = F^{\alpha 0}$ is zero for $\alpha = 0$. Hence, its time evolution should also vanish. Therefore we get $\partial_a E^a = 0$, which is Gauss's law without the presence of charges.

This is a constraint, since it implies that we cannot have any E^a for a electric field, but only configurations for which the divergence is zero. We can now see how constraints are generators of symmetries. Here, it is good to introduce the idea of a smeared constraint

$$G(\lambda) := \int d^3x \lambda \partial_a E^a,$$

where the parameter λ is an arbitrary smooth and differentiable function of x. Requiring that the smeared constraint $G(\lambda)$ vanishes for all λ is equivalent to requiring that the constraint itself vanishes at all points of the manifold. This is important to do since we are dealing with densities and distributions and not with functions itself. Since distributions behave better under an integral, it will be in most cases easier to deal with the smeared constraint than with the constraint itself.

Taking the Poisson bracket of the smeared constraint and the Hamiltonian, one finds out that it vanishes

$$\Big\{G(\lambda), H\Big\} = 0,$$

as expected, which means that the Hamiltonian does not change under the transformation generated by the constraint; hence, the theory and the physics is unmodified, and this is indeed a symmetry, as expected.

Taking the Poisson bracket of the smeared constraint with the conjugate variables we get

$$\Big\{G(\lambda), E^a\Big\} = 0,$$

which means that the electric field is unchanged under the transformation generated by the constraint. And, finally, one can compute that

$$\Big\{G(\lambda), A_a\Big\} = \partial_a \lambda,$$

which means that the vector potential may change by the gradient of a function $\lambda(x)$. We already knew that the vector potential is defined up to the gradient of a function, which is a gauge freedom of the theory. Therefore, we see that the constraints give rise to symmetries, which are revealed in the gauge freedom of the system. In this context, Gauss law is called the generator of gauge transformations, since it comes up as a constraint and it gives rise to the gauge freedom that we have in choosing the potential vector A_μ. Moreover, A^0 is the Lagrange multiplier of the constraint $\partial_a E^a$, as we mentioned previously.

By computing the time evolution of the canonical variables A_a and E^a one can recover the rest of Maxwell equations

$$\begin{aligned} \dot{A}_a &= \Big\{A_a, H\Big\} = E_a + \partial_a A_0 \\ \dot{E}^a &= \Big\{E^a, H\Big\} = \epsilon^{abc} \partial_b B_c \end{aligned}$$

where it is easy to note that the evolution depends on the choice of the Lagrange multiplier A_0, which makes the vector potential defined only up to the gradient of a function $\lambda(x)$, which is the gauge symmetry of Maxwell's theory.

6.8.2 Gravity

We may now apply the same reasoning to the Hamiltonian in (6.53) in terms of the conjugate canonical pair $\tilde{E}^a_i$ and A^i_a.

We can introduce a combination of the vector and Gauss constraints, which we call the diffeomorphism constraint:

$$C_a = V_a - A_a^i G_i. \tag{6.58}$$

Writing the smeared diffeomorphism constraint V as

$$C(\vec{N}) = \int d^3x N^a C_a$$

and computing the Poisson bracket of this smeared constraint with a function of the canonical coordinates $f(\tilde{E}, A)$, one gets

$$\left\{C(\vec{N}), f(\tilde{E}, A)\right\} \sim \mathcal{L}_{\vec{N}} f, \tag{6.59}$$

which states that the orbit generated by the constraint in phase space is just the Lie derivative along N^a, up to a constant factor that depends on the choice of renormalization. Therefore, this is called the diffeomorphism constraint, since it generates infinitesimal spatial diffeomorphism transformations.

One can also smear the Gauss constraint G_i and get

$$G(\lambda) = \int d^3x \lambda^i G_i,$$

which generates the infinitesimal gauge transformation

$$\left\{G(\lambda), \mathbf{A}\right\} \sim d_{\mathbf{A}} \lambda \tag{6.60}$$

and also

$$\left\{G(\lambda), \mathbf{E}\right\} \sim [\lambda, \mathbf{E}] \tag{6.61}$$

which are the $SU(2)$ gauge transformations.

Chapter 7

Conclusion

> Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. "Immortality" may be a silly word, but probably a mathematician has the best chance of whatever it may mean.
>
> *G.H. Hardy*

We have built a full development of GR in many different ways. Each approach allows us to see different sides of some symmetries and some results of the theory.

We first presented the foundations of GR from a somehow historical point of view, discussing the equivalence principle stated by Einstein and working out the Newtonian limit as a limiting case. Exploring Bianchi's identity, we showed that one can build an equation that is generally covariant, relating the metric $g_{\mu\nu}$ to the curvature $R_{\mu\nu}$ and the energy momentum tensor $T_{\mu\nu}$. We have concluded that the only way to do this in an equation that would contain the equivalence principle, reduce to Newtonian gravity in the classical limit, be manifestly covariant and also, beyond satisfying the conservation laws, would also include the physical idea that the presence of mass — $T_{\mu\nu}$ — is responsible for the curvature — $R_{\mu\nu}$ — of spacetime was in Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu}.$$

We have also shown how one can get to the same equation using the Lagrangian formalism. We first did this for the classic Einstein-Hilbert action — an action where the dynamical variable is the metric itself. We then developed the same formalism for the so called Palatini action, where not just the metric $g_{\mu\nu}$ but also the connection Γ plays a dynamical role. We then got two equations of motion. The first one, obtained when varying the action with respect to the metric, led us to Einstein field equations. The second one, obtained when varying the action with respect to the connection, stated that the connection in play was the one compatible with the metric — the Levi-Civita connection.

Next, we developed the theory of constrained Hamiltonian systems. We introduced a $(3+1)$-split by switching the metric g_{ab} as a dynamical variable to the 3-metric h_{ab}, living in spatial slices Σ_t, for t constant. In this way, we were able to tell the history of spacetime as the evolution of the spatial slices Σ_t through time t. This needs to be done since there is no absolute time in

GR, so the decomposition of the metric g_{ab} in its spatial part h_{ab} will be necessary for one to define a time parameter in order to talk about the evolution of the system, which is necessary if one wants to build a Hamiltonian representation of the system.

This allowed us to develop a Hamiltonian formalism for GR — the ADM formalism — where we could write a Hamiltonian as a sum of constraints multiplied by Lagrange multipliers. This provided a Hamiltonian representation of the dynamics of the spacetime geometry. The canonical variables here are the induced metric h_{ab} and its conjugate momenta π_{ab}. With this Hamiltonian it becomes possible to study the spacetime dynamics in a canonical way, using every tool of the Hamiltonian formalism.

We then developed the tetrads formalism. Here, we replaced the metric $g_{\mu\nu}$ as the main dynamical variable for the tetrads e^I_μ and e^J_ν. Since Riemannian manifolds are locally flat, one can always choose an orthonormal basis of vectors $\{e_0, e_1, e_2, e_3\}$ for each point P on the manifold M, and that is the starting point of this new formalism. The tetrads contain the same geometrical information from the manifold as the metric $g_{\mu\nu}$, since they are related via $g_{\mu\nu} = e^I_\nu e^J_\nu \eta_{IJ}$. Hence, by taking the determinant, we get $g = -e^2$ and the tetrad is just the square root of the metric and it has, therefore, all the information about the geometry of the manifold. We can thus consider the tetrad as the fundamental description and the metric as a derived concept.

We then built the two Cartan's structural equations. The first one was $de^I + \omega^I{}_K e^K = 0$, which allowed us to find the spin connection ω if we had the tetrads e. The second one was $F^K{}_J = d\omega^K{}_J + \omega^K{}_L \omega^L{}_J$, which allowed us to find the curvature 2-form F^K_J. So, given a metric, one chooses a basis of tetrads e^I, founds the spin connection and then the curvature 2-form with both of Cartan's structural equations.

Since we had the relation between F^{IJ} and the Riemann curvature tensor $R^\rho_{\mu\nu\sigma}$, we were then able to translate the Palatini action in terms of the tetrads and the curvature 2-form $\mathbf{F}$. This action led us to the same conclusions that we arrived before, only now in a different language — the Einstein field equations and the metric compatibility equation in the notation of differential forms. Finally, we did a slight modification in this action, by introducing a parameter γ, the Barbero-Immirzi parameter. This did not change the equations of motion, but we were led to a more general action — the Holst action.

Finally, we developed the Hamiltonian formalism using the Holst action, which led us to the formulation of GR in terms of some new variables — the Ashtekar-Barbero connection A^i_a and the densitized triad $\tilde{E}^b_j$. In a way, we mixed what was developed in the last two chapters, since the main idea here consisted in developing a Hamiltonian formalism after we did the (3+1)-split of the geometry. However we have used the Holst action instead of the Einstein-Hilbert one, hence, our variables were in terms of the triads — the spatial part of the tetrads — and not the 3-metric h_{ab}. Although the development was extensive, the steps followed in this part led us to the construction of a constrained Hamiltonian for GR, which allowed us to extract some symmetries of the system, the constraints giving rise to gauge transformations.

This puts us one step behind the quantization of gravity. In the Hamiltonian formalism, one can promote the Poisson brackets to commutators and the conjugate variables to operators, and, at least in theory, quantize gravity. The path followed for the canonical quantization of gravity in this approach with the Ashtekar formulation of gravity is known as loop quantum gravity, which can be studied in a future work.

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Appendices

Appendix A

Spin Connection and Lie Algebra

A transformation Λ is said to be a Lorentz transformation if

$$\Lambda^T \eta \Lambda = \eta, \tag{A.1}$$

where η is the Minkowski metric.

The group of all Lorentz transformations is called de Lorentz group, and it is the $O(3,1)$ group.

If one wants to characterize the Lorentz group in terms of properties of its associated Lie Algebra, one needs to consider an infinitesimal transformation

$$\Lambda = e^{\omega} \approx 1 + \omega,$$

from where we get, in (A.1):

$$\begin{aligned} \eta &= \Lambda^T \eta \Lambda \\ &= (1+\omega)^T \eta (1+\omega) \\ &= (\eta + \omega^T \eta)(1+\omega) \\ &= \eta + \eta\omega + \omega^T \eta, \end{aligned}$$

where, in the last line, we have neglected second order terms in ω.

So, we are left with

$$\eta\omega + \omega^T \eta = 0 \tag{A.2}$$

or, in terms of coordinates:

$$\eta_{\alpha\beta}\omega^{\beta}{}_{\rho} + \omega^{\mu}{}_{\nu}\eta_{\mu\sigma} = 0,$$

or

$$\eta_{\alpha\beta}\omega^{\beta}{}_{\rho} = -\omega^{\mu}{}_{\nu}\eta_{\mu\sigma}$$

which leads as to

$$\omega_{\alpha\rho} = -\omega_{\nu\sigma}. \tag{A.3}$$

So, as we see from (5.10), the spin connection coefficients assume values in the Lie algebra of the Lorentz group, as told before.

Appendix B

Useful formulas for tetrads relating the Levi Civita symbol

For any square matrix $A_{\mu\nu}$ in 4D we have

$$\det A = \epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}A_{\mu\alpha}A_{\nu\beta}A_{\rho\gamma}A_{\sigma\delta}. \tag{B.1}$$

From that, one can prove the following identities:

$$\begin{aligned}
e\, e_I^{[\mu}e_J^{\nu}e_K^{\rho}e_L^{\sigma]} &= \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL} \\
e\, e_I^{[\mu}e_J^{\nu}e_K^{\rho]} &= \frac{1}{3!}\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e_\sigma^L \\
e\, e_I^{[\mu}e_J^{\nu]} &= \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e_\sigma^L e_\rho^K \\
e\, e_I^{\mu} &= \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e_\sigma^L e_\rho^K e_\nu^J \\
e &= \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e_\sigma^L e_\rho^K e_\nu^J e_\mu^I
\end{aligned} \tag{B.2}$$

For the 3-dimensional case the determinant can be calculated via

$$\det(e) = \epsilon^{ijk}\tilde{\epsilon}_{abc}e_i^a e_j^b e_k^c \tag{B.3}$$

or, for the inverse of e_i^a:

$$\det(e^{-1}) = \epsilon_{ijk}\tilde{\epsilon}^{abc}e_a^i e_b^j e_c^k \tag{B.4}$$

Appendix C

Bianchi and Palatini identities

The Bianchi identities in forms notation can be easily deduced by computing the exterior covariant derivative of the two structure equations.

For the first one, let us take two covariant derivatives of the an arbitrary vector V^I:

$$\begin{aligned}
D^2V^I &= D(dV^I + \omega^I{}_K \wedge V^K) + \omega^I{}_J \wedge (dV^I + \omega^I{}_K \wedge V^K) \\
&= (d\omega^I{}_K \wedge V^K - \omega^I{}_K \wedge dV^K) + (\omega^I{}_J \wedge dV^I + \omega^I{}_J \wedge \omega^I{}_K \wedge V^K) \\
&= d\omega^I{}_K \wedge V^K + \omega^I{}_J \wedge \omega^I{}_K \wedge V^K \\
&= (d\omega^I{}_K + \omega^I{}_J \wedge \omega^I{}_K) \wedge V^K \\
&= F^I_K V^K,
\end{aligned}$$

which is the first Bianchi identity. This could be done just by taking the derivative of the torsion $T^I = e^I + \omega^I_j e^J$.

Now, the second Bianchi identity can be built by taking the derivative of the second of Cartan's structural equations $F^K_J = d\omega^K_J + \omega^K_L \wedge \omega^L_J$:

$$\begin{aligned}
DF^K_J &= dF^K_J + \omega^M_J \wedge F^K_M + \omega^K_N \wedge F^N_J \\
&= (d\omega^K_M \wedge \omega^M_J - d\omega^N_J \wedge \omega^K_N) + \omega^M_J \wedge (d\omega^K_M + \omega^K_L \wedge \omega^L_M) + \omega^K_N \wedge (d\omega^K_J + \omega^N_L \wedge \omega^L_J) \\
&= 0
\end{aligned}$$

Note that the first terms on the last two parenthesis cancel out with the terms in the first parenthesis. Also, the expression $\omega^M_J \wedge (\omega^K_L \wedge \omega^L_M) + \omega^K_N \wedge (\omega^N_L \wedge \omega^L_J) = -\omega^L_M \wedge \omega^M_J \wedge \omega^K_L + \omega^K_N \wedge \omega^N_L \wedge \omega^L_J = 0$.

The equation $DF = 0$ is the second Bianchi identity.

The Bianchi identities hold, in general, in a geometric structure satisfying the metric compatibility condition $\nabla g = 0$, even in the case of non-vanishing torsion.

It is also useful to have the Palatini identity written in tensor coordinate notation

$$\delta R_{\sigma\nu} = \nabla_\rho(\delta\Gamma^\rho{}_{\nu\sigma} - \nabla_\nu(\delta\Gamma^\rho{}_{\rho\sigma}) \tag{C.1}$$

or in forms notation

$$\delta F^{KL} = D(\delta\omega^{KL}). \tag{C.2}$$

To see that this is indeed true, let us compute $\delta F^{IJ}_{\mu\nu}$. First, since

$$F^{IJ}_{\mu\nu} = \partial_\mu \omega^{IJ}_\nu - \partial_\nu \omega^{IJ}_\mu + \omega_{\mu K}\omega^{KJ}_\nu - \omega^I_{\nu K}\omega^{KJ}_\mu$$

we have that

$$\begin{aligned}\delta F^{IJ}_{\mu\nu} &= \partial_{[\mu}\delta\omega^{IJ}_{\nu]} + \omega^{I}_{[\mu K}\delta\omega^{KJ}_{\nu]} - \omega^{J}_{[\mu K}\delta\omega^{IK}_{\nu]} \\ &= D(\delta\omega^{IJ})_{\mu\nu}\end{aligned}$$

or, suppressing the spacetime indices

$$\delta F^{IJ} = D(\delta\omega^{IJ})$$

Appendix D

Tensor vs Tensor Densities

The object $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is not a tensor, but a tensor density. Note for the 2 dimensional case that

$$dx^1 \wedge dx^2 = \frac{1}{2!}(dx^1 \otimes dx^2 - dx^2 \otimes dx^1)$$

If we change for the coordinate system $d\tilde{x}^\mu$, the we forms will transform as

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu$$

therefore, the object $dx^1 \wedge dx^2$ will be written, in those new coordinates, as

$$\begin{aligned} dx^1 \wedge dx^2 &= \frac{1}{2!}(dx^1 \otimes dx^2 - dx^2 \otimes dx^1) \\ &= \frac{1}{2!}\left[\left(\frac{\partial x^1}{\partial \tilde{x}^1} d\tilde{x}^1 + \frac{\partial x^1}{\partial \tilde{x}^2} d\tilde{x}^2\right) \otimes \left(\frac{\partial x^2}{\partial \tilde{x}^1} d\tilde{x}^1 + \frac{\partial x^2}{\partial \tilde{x}^2} d\tilde{x}^2\right) - \left(\frac{\partial x^2}{\partial \tilde{x}^1} d\tilde{x}^1 + \frac{\partial x^2}{\partial \tilde{x}^2} d\tilde{x}^2\right) \otimes \left(\frac{\partial x^1}{\partial \tilde{x}^1} d\tilde{x}^1 + \frac{\partial x^1}{\partial \tilde{x}^2} d\tilde{x}\right.\right. \\ &= \frac{1}{2!}\left[\left(\frac{\partial x^1}{\partial \tilde{x}^1}\frac{\partial x^2}{\partial \tilde{x}^2} - \frac{\partial x^2}{\partial \tilde{x}^2}\frac{\partial x^1}{\partial \tilde{x}^2}\right) d\tilde{x}^1 \otimes d\tilde{x}^2 - \left(\frac{\partial x^1}{\partial \tilde{x}^2}\frac{\partial x^2}{\partial \tilde{x}^1} - \frac{\partial x^2}{\partial \tilde{x}^2}\frac{\partial x^1}{\partial \tilde{x}^1}\right) d\tilde{x}^2 \otimes d\tilde{x}^1\right] \\ &= \frac{1}{2!}(d\tilde{x}^1 \otimes d\tilde{x}^2 - d\tilde{x}^2 \otimes d\tilde{x}^1)|J| \end{aligned}$$

and we see that the determinant of the Jacobian $|J|$ appears in the transformation, making the object a tensor density.

We call it a tensor density of weight 1, since the determinant of the Jacobian appears raised to the first power. A tensor density of weight ω transforms as a tensor except that it is additionally multiplied (or weighted) by a power ω of the Jacobian determinant of the coordinate change function.

Therefore, the object $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is also a tensor density of weight 1, and it can be written in terms of the Levi Civita symbol as

$$dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = -\tilde{\epsilon}_{\mu\nu\sigma\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\rho$$

Here, the tilde on the Levi Civita symbol stands for the fact that is a tensor density, and not a tensor itself. The minus sign comes from that fact that $sign(g) = -1$. The symbol is defined as

$$\tilde{\epsilon}_{\mu\nu\sigma\rho} \coloneqq \begin{cases} +1 \text{ if } (\mu\nu\sigma\rho) \text{ is an even permutation of } (0,1,2,3) \\ \text{-1 if } (\mu\nu\sigma\rho) \text{ is an odd permutation of } (0,1,2,3) \\ 0 \text{ if there is any repetition of two indices} \end{cases}$$

This quantity is the same in any coordinate system, which is why it is not a tensor. However we can relate the tensor density $\tilde{A}$ to the proper tensor A via

$$\tilde{A} = \left(|g|^{1/2}\right)^{\omega} A$$

where ω is the weight of the tensor density and g stands for the determinant of the metric.

One can show that

$$\tilde{\epsilon}_{\mu\nu\sigma\rho} = |g|^{1/2}\epsilon_{\mu\nu\sigma\rho}$$

and thus the Levi Civita symbol is a tensor of weight +1.

For the indices $(IJKL)$, we have not used the tilde since those indices are of the internal flat space, and, hence, ϵ^{IJKL} is actually a tensor and not a density.

It is also possible to define the 3-dimensional symbol as $\tilde{\epsilon}^{abc} := \tilde{\epsilon}^{0abc}$, which could also be done for 2D or 1D. The same could be done for the $(IJKL)$ indices in the internal space.