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On the Boundedness of Partial Sums of Multiplicative Functions

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Resumo

Na teoria analítica dos números, a discrepância de uma função $f : \mathbb{N} \to \mathbb{C}$ é definida como:

$$\sup_{n,d} \left| \sum_{j=1}^{n} f(jd) \right| .$$

O "Problema da Discrepância de Erdős" pergunta se a discrepância de uma função $f : \mathbb{N} \to \{-1, 1\}$ é infinita. Tao mostrou em [18] que esse é, de fato, o caso. Consequentemente, toda função totalmente multiplicativa que toma valores em $\{-1, 1\}$ possui somas parciais ilimitadas. Isso nos leva a uma pergunta natural: o que acontece se considerarmos funções multiplicativas ao invés de totalmente multiplicativas? Klurman [11] forneceu uma classificação completa de funções multiplicativas com somas parciais limitadas, um resultado conhecido como a conjectura de Erdős–Coons–Tao.

Outra questão relacionada é o estudo do que acontece se permitirmos o codomínio ser $\mathbb{C}$ ao invés de $\{-1, 1\}$. Nesse caso, não se conhece nenhuma classificação completa, porém alguns resultados foram estudados por Aymone [1].

O principal objetivo desta dissertação é entender os passos chave da demonstração da conjectura de Erdős-Coons-Tao e também investigar questões relacionadas no trabalho de Aymone [1], quando o codomínio é $\mathbb{C}$. O texto foi escrito com a intenção de ser o mais autocontido possível, portanto, todas as ferramentas necessárias são construídas do zero, tornando-o acessível a qualquer pessoa com conhecimento básico de matemática superior.

Palavras-Chave: Erdős-Coons-Tao, somas parciais, funções multiplicativas, conjunto limitado, teoria analítica dos números.

Abstract

In number theory the discrepancy of a function $f : \mathbb{N} \to \mathbb{C}$ is defined as:

$$\sup_{n,d} \left| \sum_{j=1}^{n} f(jd) \right|.$$

The *Erdős Discrepancy Problem* asks whether the discrepancy of a function $f : \mathbb{N} \to \{-1, 1\}$ is infinite. Tao showed in [18] that this is indeed the case. Consequently, every totally multiplicative function that takes values in $\{-1, 1\}$ has unbounded partial sums. This leads to a natural question: What happens when we consider multiplicative functions instead? Klurman [11] provided a complete classification of multiplicative functions with bounded partial sums, a statement known as the Erdős–Coons–Tao conjecture.

Another important related question is the study of what happens when we allow the codomain to be $\mathbb{C}$ instead of $\{-1, 1\}$. In this case, there is no complete classification, but some results in this direction were studied by Aymone [1].

The main goal of this dissertation is to understand the key steps in the proof of the Erdős-Coons-Tao conjecture and also investigate some related questions in Aymone's work [1] when the codomain is $\mathbb{C}$. The text is meant to be self-contained so we build all the necessary tools to understand the main results from the ground up, making this text accessible to anyone with basic undergraduate mathematics knowledge.

Keywords: Erdős-Coons-Tao; partial sums; multiplicative functions; boundedness; analytic number theory.

Contents

1 Introduction

In modern mathematics, analytic number theory is an area that studies what we call arithmetic functions, i.e., functions $f : \mathbb{N} \to \mathbb{C}$. The codomain being the complex numbers allows us to use many tools from complex analysis to approach our problems. Many problems in this field concern prime numbers, such as the famous *Prime Number Theorem*, which describes the asymptotic distribution of prime numbers. Another well-known problem is the *Riemann Hypothesis*, which remains an open problem and concerns the zeros of the zeta function. If proven true, it would imply many results regarding the distribution of prime numbers. The relationship between prime numbers and arithmetic functions arises from a special type of arithmetic function called multiplicative functions, which we are going to define shortly. Essentially, the idea is to utilize the *Fundamental Theorem of Arithmetic* (see Theorem 2.1) to construct the function based on its values for powers of primes.

There are two important classes of arithmetic functions that we are going to use in this work: multiplicative and totally multiplicative functions. We say that an arithmetic function f is multiplicative if $f(1) = 1$ and $f(ab) = f(a)f(b)$ for all a, b such that $(a, b) = 1$, where (a, b) denotes the greatest common divisor of a and b. We say that an arithmetic function f is totally multiplicative if $f(1) = 1$ and $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{N}$. An important property of multiplicative functions is that their values are completely determined by their values at prime powers. Now let us define some important multiplicative functions in number theory.

Definition 1.1. The Euler totient function $\phi(n)$ is defined as the number of positive integers not greater than n that are coprime with n, i.e:

$$\phi(n) = \#\{k \in \mathbb{N}, k \leq n \mid (n, k) = 1\}$$

We say that n is square-free if the only square number that divides n is 1.

Definition 1.2. The Möbius function $\mu(n)$ is defined as $\mu(n) = (-1)^{\omega(n)}$ if n is square-free and $\mu(n) = 0$ otherwise, where $\omega(n)$ denotes the number of prime divisors of n.

Remark 1.3. Since multiplicative functions are completely determined by their values at prime powers, we can also define $\phi(n)$ as the multiplicative function such that $\phi(p^k) = p^k - p^{k-1}$ and $\mu(n)$ as the multiplicative function such that $\mu(p) = -1$ and $\mu(p^k) = 0$ for $k > 1$.

Definition 1.4. The discrepancy of an arithmetic function f is defined as:

$$\sup_{n,d} \left| \sum_{j=1}^{n} f(jd) \right|$$

The *Erdős Discrepancy Problem* was proposed by the famous mathematician Paul Erdős around 1930. In his 1957 paper [7], Erdős listed many unsolved problems, including his discrepancy problem. The problem aims to show that an arithmetic function that takes values on the set $\{-1, 1\}$ must have infinite discrepancy. For decades, the problem remained open with little progress. Until, in 2010, the Polymath5 project[15] was initiated as an online collaboration of mathematicians who shared their insights on the problem. Although the project did not prove the discrepancy problem, it made significant progress towards its solution. Eventually, in 2015, Terence Tao [18] found a solution, which was a much more general version of the problem where the function could take values on the unit sphere of an arbitrary Hilbert space. Therefore, we have the following results:

Theorem 1.5. (General form of *Erdős Discrepancy Problem*) Let H be a real or complex Hilbert space, and let $f : \mathbb{N} \longrightarrow H$ be a function such that $||f(n)||_H = 1$ for all $n \in \mathbb{N}$. Then the discrepancy of f is infinite. Here we used the following more general notion of discrepancy:

$$\sup_{n,d} \left\| \sum_{j=1}^{n} f(jd) \right\|_H$$

Corollary 1.6. (Original formulation of *Erdős Discrepancy Problem*) Let $f : \mathbb{N} \longrightarrow \{-1, 1\}$ be an arithmetic function. Then f has infinite discrepancy.

Let $\mathbb{U}$ denote the set of complex numbers with absolute value 1. An important consequence of Corollary 1.5 is the following theorem:

Theorem 1.7. Let $f : \mathbb{N} \longrightarrow \mathbb{U}$ be a totally multiplicative function. Then f has unbounded partial sums.

Proof. We have

$$\sum_{j=1}^{n} f(jd) = f(d) \sum_{j=1}^{n} f(j).$$

Therefore

$$\sup_{n,d} \left| \sum_{j=1}^{n} f(jd) \right| = \sup_{n} \left| \sum_{j=1}^{n} f(j) \right|.$$

□

The next two examples show what might happen if we weaken the hypothesis of the previous theorem.

Example 1.8. A Dirichlet character χ modulo q is a totally multiplicative function, q-periodic such that $\chi(n) = 0$ if and only if $(q, n) > 1$. We say that χ is a principal Dirichlet character if $(q, n) = 1$ implies $\chi(n) = 1$. A non-principal Dirichlet character is a classical example of a totally multiplicative function with bounded partial sums. Using that χ has bounded partial sums we can go a step further and state that it must be bounded by q on every homogeneous arithmetic progression and therefore has bounded discrepancy:

$$\left|\sum_{j=1}^{n} \chi(jd)\right| = \left|\chi(d) \sum_{j=1}^{n} \chi(j)\right| \leq q$$

Of course this does not violate the previous theorem as χ vanishes when $(n, q) \neq 1$. For readers interested in delving deeper into properties of Dirichlet characters, I recommend referring to analytical number theory books such as Montgomery's [14].

Remark 1.9. The constant function, $f(x) = 0$, is also an example of a completely multiplicative function with bounded partial sums. What makes the Dirichlet character special is that it is a non-trivial one. In the sense that it is zero for only finitely many primes. Therefore, in some sense, we can say that it is a near counter-example to the Erdős Discrepancy Problem. In 2018, Klurman and Mangerel [13] proved Chudakov's conjecture, which states that if f is a completely multiplicative function that is zero for only finitely many primes and whose image is a finite set, then f is a Dirichlet character.

Example 1.10. (Coons)[6] We can define a multiplicative function $\chi_2 : \mathbb{N} \to \{-1, 1\}$ by $\chi_2(p^k) = (-1)^{p+1}$, where p is prime and $k \in \mathbb{N}$. So $\chi_2(n) = 1$ if n is odd an $\chi_2(n) = -1$ if n is even. We have that the partial sums of χ_2 are bounded by 1, so we conclude that being totally multiplicative instead of only multiplicative is also important on Theorem 1.7.

Example 1.11. (Borwein-Choi-Coons)[4] Let χ_3 be the non principal Dirichlet character modulo 3, i.e, $\chi(n) = 0$ if $n = 0$ mod 3, $\chi_3(n) = 1$ if $n = 1$ mod 3 and $\chi_3(n) = -1$ if $n = 2$ mod 3. We define an slightly different totally multiplicative function f by: $f(p) = 1$ if $p = 3$ and $f(p) = \chi_3(p)$ otherwise. Now let us investigate the partial sums of this function f for values of the form $n = 1 + 3 + 3^2 + \ldots + 3^k$. Separating the sum according to the largest power of 3 that divides the number we have:

$$\sum_{m=1}^{n} f(m) = \sum_{i=0}^{k} \sum_{1 \leq m \leq n/3^i} f(3^i m).$$

Using that f is completely multiplicative we have:

$$\begin{aligned} &= \sum_{i=0}^{k} \sum_{1\leq m\leq n/3^i} f(3^i)f(m) \\ &= \sum_{i=0}^{k} f(3^i) \sum_{1\leq m\leq n/3^i} \chi_3(m) \\ &= \sum_{i=0}^{k} \sum_{1\leq m\leq n/3^i} \chi_3(m). \end{aligned}$$

By definition $n/3^i \bmod 3 = 1$, so $\sum_{1\leq m\leq n/3^i} \chi_3(m) = 1$, as the partial sums of χ_3 vanishes at multiples of 3. Therefore the value of the partial sum is $k+1$, and we can conclude that $\log n \ll \sum_{k\leq n} f(k)$. So the discrepancy of this function grows logarithmically.

We know, by the *Erdős Discrepancy Problem* that the partial sums must diverge, but we do not know how fast. It is believed that the partial sums of a totally multiplicative function $f : \mathbb{N} \to \{-1, 1\}$ grows at least logarithmically like in the previous example.

Given Theorem 1.7 and Examples 1.8 and 1.10 a natural question arises: What are sufficient conditions to guarantee that f has bounded or unbounded partial sums when f is only multiplicative? Before proceeding in stating results in this direction we must introduce an important tool that will be used along this dissertation that is the notion of pretentiousness introduced by Granville and Soundararajan [8]:

Definition 1.12. Let f and g be two multiplicative functions bounded by 1. We define the distance between f and g as:

$$\mathbb{D}(f, g, x) = \left(\sum_{p\leq x} \frac{1 - \mathrm{Re}(f(p)\overline{g(p)})}{p} \right)^{1/2}$$

where p is prime. It is worth noting that $\mathbb{D}$ is not a distance in the classical sense, as we may have $\mathbb{D}(f, g) = 0$, even though $f \neq g$. What makes this definition important and motivates calling it a distance is that, for a fixed value of x, $\mathbb{D}$ satisfies the triangle inequality. If f and g satisfy $\mathbb{D}(f, g, \infty) < \infty$, we say that f pretends to be g or that f is g-pretentious.

In the same paper where he solved the discrepancy problem [18], Tao shows an important result towards the classification of multiplicative functions with bounded partial sums, namely:

Theorem 1.13. Let $f : \mathbb{N} \to \{-1, 1\}$ be a multiplicative function with bounded partial sums. Then $f(2^j) = -1$ for all $j \geq 1$ and f pretends to be 1, i.e. f pretends to be the constant function $f(x) = 1$.

Some time later, Klurman [11] provided a complete classification of multiplicative functions with bounded partial sums.

Theorem 1.14. (Erdős–Coons–Tao) Let $f : \mathbb{N} \to \{-1, 1\}$ be a multiplicative function. Then f has bounded partial sums if and only if there exists $m \geq 1$ such that $f(n) = f(n+m)$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{m} f(n) = 0$.

In the previous case, we were interested in multiplicative functions $f : \mathbb{N} \to \{-1, 1\}$. A natural generalization is to consider multiplicative functions that take values on larger sets, such as the complex numbers. However, a complete classification of when a multiplicative function has bounded partial sums in this more general setting is still unknown. Nonetheless, researchers have made progress in studying specific cases.

For instance, Aymone [1] gave sufficient conditions for a multiplicative function to have bounded partial sums. Results in the other direction have also been established, for example Aymone [2] and Klurman [12] have shown that totally multiplicative functions f supported on square-free integers, where $f(p) = \pm 1$ for all primes p, must have unbounded partial sums. Aymone [1] demonstrated that if f_1 and f_2 are periodic multiplicative functions with bounded partial sums. Then $f = f_1 * f_2$ has unbounded partial sums, where * stands for the Dirichlet product of those functions (See Definition 2.28).

These findings highlight the complexities involved in determining the behavior of multiplicative functions with larger codomains, and further research is needed to obtain a complete understanding of when bounded or unbounded partial sums arise in such cases.

Another interesting area of research is to determine the rate at which the partial sums grow when a function has unbounded partial sums. In this regard, Aymone [1] has made some contributions showing big omega and big O bounds (Definitions 2.3, 2.5) for multiplicative functions of the form $f = f_1 * f_2$ where f_1 and f_2 are periodic multiplicative functions with bounded partial sums. In a later paper [3] Aymone, Maiti, Ramaré and Srivastav improved the omega bound.

2 Preliminaries

2.1 Conventions and Basic Notation

Throughout all this dissertation prime numbers are going to be extensively used, so it is convenient to reserve p and p_i represent prime numbers, unless explicitly mentioned the opposite. We use the notation $p \mid a$ when p divides a and $p \nmid a$ otherwise. Also we denote by $p^k \mid\mid a$ if $p^k \mid a$ and $p^{k+1} \nmid a$, i.e k is the largest number such that p^k divides a.

We use the letter s to denote a complex number and σ and t its real and imaginary parts, i.e $s = \sigma + it$. We use $\lfloor . \rfloor$, $\lceil . \rceil$, $\{ . \}$, $\|.\|$ to denote the floor, ceiling, fractional part and distance to nearest integers functions respectively.

2.2 Elementary Results

Our first result is so elementary that we make use all the time without explicitly mentioning. It is this result that guarantees that multiplicative functions are well defined.

Theorem 2.1. (Fundamental Theorem of Arithmetic) Every integer $n > 1$ has an unique factorization as a product of primes, apart from the order of factors.

Proof. We use induction on n. The result holds for $n = 2$ as it can only be written in one way. Suppose the result is valid for all natural number less than n. We are going to show that the result holds for n. If n is prime its factorization is just the own number. If n is composite, then n is divisible by a prime number $p < n$, but we know by the inductive hypothesis that n/p can be written as a product of primes and so can $n = p \cdot \frac{n}{p}$. It remains to show that such factorization is unique, let $n = p_1 p_2 \dots p_k = q_1 q_2 \dots q_l$ be two ways to write n as a product of primes, as $p_1 \mid n$ it must divide some of it's factors denote this factor by q_{n_0}, so $p_1 \mid q_{n_0}$ which implies that $p_1 = q_{n_0}$, for convenience let us relabel q_{n_0} as q_1. Note that $n/p_1 < n$ and that $n/p_1 = p_2 \dots p_k = q_2 \dots q_l$, so applying the induction hypothesis we conclude that both of these ways of writing n/p_1 have the same prime factors and so multiplying back by p_1 we conclude that both factorizations of n have the same prime factors. □

Theorem 2.2. (Division Algorithm) Let n and k be integers with $k \neq 0$. Then there exists an unique pair of integers q, r, with $0 \leq r < k$ such that $n = kq + r$.

Proof. Let $S = \{q \in \mathbb{Z} : n - kq \geq 0\}$, by the well ordering principle there exists an element $q_0 \in S$ such that $n - kq_0 \leq n - kq$ for all $q \in S$, define $r = n - kq_0$. We need to show that $0 \leq r < k$. The first inequality follows straight from the fact that $q_0 \in S$. For the second inequality, suppose that $r \geq k$. Then we have $n - kq_0 = r \geq k$ so $n - k(q_0 + 1) \geq 0$, this implies that $q_0 + 1 \in S$ but this contradicts the minimality of $n - kq$.

For the uniqueness suppose that $n = kq_1 + r_1 = kq_2 + r_2$, where $0 \leq r_1, r_2 < k$. Then $k(q_1 - q_2) = r_2 - r_1$ and therefore $k \mid (r_2 - r_1)$. But this only happens if $r_1 = r_2$ as $|r_2 - r_1| \in [0, k)$. Defining $q_1 = \frac{n-r_1}{k}$ and $q_2 = \frac{n-r_2}{k}$ we conclude also that $q_1 = q_2$. □

2.3 Asymptotic Notation

In order to study the behaviour of functions in the limit we make extensive use of asymptotic notation, which we now define:

Definition 2.3. Let f and g be functions, with g being a positive function. We say that "f is big O of g" and denote it by $f(x) = O(g(x))$ if there exist constants C and x_0 such that for all $x > x_0$, it holds that $|f(x)| < Cg(x)$.

Definition 2.4. Let f and g be functions, with g being a positive function. We say that "f is little o of g" and denote it by $f(x) = o(g(x))$ if for all $\epsilon > 0$ there exists an constant x_0 such that for all $x > x_0$, it holds that $|f(x)| < \epsilon g(x)$.

Definition 2.5. Let f and g be functions, with g being a positive function. We say that "f is big omega of g" and denote it by $f(x) = \Omega(g(x))$ as $x \to \infty$ if $f(x) \neq o(g(x))$.

Remark 2.6. The above notation for the big Oh is know as Landau's notation, we will also make use of Vinogradov's notation denoted by the symbol $\ll$: $f(x) = O(g(x)) \Leftrightarrow f(x) \ll g(x)$.

Remark 2.7. To indicate that the implicit constant used on the definition might depend on some parameter δ we use O_δ and $\ll_\delta$.

Remark 2.8. We might be interested in study the asymptotic as f approaches some number a instead of infinity, the change on the definitions is that we change the existence of the x_0 by the existence of some neighborhood of a. The notation changes by an subscript $x \to a$ or by explicitly saying $O(g(x))$ as $x \to a$.

Theorem 2.9. We have the following alternative definitions:

(i) $f(x) = O_{x\to a}(g(x)) \Leftrightarrow \limsup_{x\to a} \frac{|f(x)|}{g(x)} < \infty$,

(ii) $f(x) = o_{x\to a}(g(x)) \Leftrightarrow \lim_{x\to a} \frac{|f(x)|}{g(x)} = 0$,

(iii) $f(x) = \Omega_{x\to a}(g(x)) \Leftrightarrow \limsup_{x\to a} \frac{|f(x)|}{g(x)} > 0$.

Remark 2.10. $f(x) = O(1)$ means that f is bounded, and $f(x) = o(1)$ means that not only f is bounded but also $\lim_{x\to a} f(x) = 0$.

Theorem 2.11. The following properties are satisfied:

(i) $f(x) \ll g(x) \implies c \cdot f(x) \ll g(x)$, for all $c > 0$;

(ii) $f(x) \ll h(x)$ and $g(x) \ll h(x)$. Then $f(x) + g(x) \ll h(x)$;

(iii) $f(x) \ll g(x) \ll h(x)$. Then $f(x) \ll h(x)$.

Proof. By definition, if $f(x) \ll g(x)$, then there exists C such that $|f(x)| < Cg(x)$, therefore $cf(x) \ll (cC)g(x)$, i.e, $cf(x) \ll g(x)$.

If $f(x) \ll h(x)$ and $g(x) \ll h(x)$, then there exists constants C_1 and C_2 such that $f(x) < C_1h(x)$ and $g(x) < C_2h(x)$, therefore $f(x) + g(x) < (C_1 + C_2)h(x)$, i.e $f(x) + g(x) \ll h(x)$.

If $f(x) \ll g(x) \ll h(x)$, then there exists constants C_1 and C_2 such that $f(x) < C_1g(x)$ and $g(x) < C_2h(x)$, therefore $f(x) < (C_1C_2)h(x)$, i.e $f(x) \ll h(x)$. □

Remark 2.12. It is common to write equalities and inequalities using big O and little o like they were numbers. Beware that we need to be careful as these terms might absorb others, so the term represented by this notation might change between lines. So expressions like $x + O(x^2) = O(x^2)$, makes sense in the sense that x was absorbed by the big O term.

Example 2.13. The logarithm function is $O(x^\epsilon)$ for every $\epsilon > 0$ as:

$$\lim_{n\to\infty} \frac{\ln x}{x^\epsilon} = \lim_{n\to\infty} \frac{1}{\epsilon x^\epsilon} = 0. \tag{2.1}$$

This result shows that the logarithm grows slower than any polynomial function.

Example 2.14. As we will see in Equation 2.5, it is possible to show that $\ln x \ll \sum_{n=1}^{x} \frac{1}{n} \ll \ln x$, so we might say that $\sum_{n=1}^{x} \frac{1}{n}$ and $\ln x$ are asymptotic equivalent and denote it by $\sum_{n=1}^{x} \frac{1}{n} \asymp \ln x$.

2.4 The Riemann Stieltjes Integral

The Riemann Integral is a fundamental tool for every undergraduate student in exact sciences. With little effort, it is possible to extend the idea of Riemann integration to that of Riemann-Stieltjes integration, which allows us to apply integration techniques to summations. This might seem surprising at first, as integration is not typically applied to functions with a discrete domain when we first learn it.

Definition 2.15. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ and let $C = \{c_1, \cdots, c_n\}$, where each $c_i \in [x_{i-1}, x_i]$. For two bounded functions f and g define:

$$S(f, g, P, C) = \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})].$$

Define $||P|| = \max_{1\leq i\leq n} x_i - x_{i-1}$. If there exists a constant K such that for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$||P|| < \delta \implies |S(f, g, P, C) - K| < \epsilon,$$

then we say that f is Riemann-Stieltjes integrable on $[a, b]$ with respect to g and its integral is K. Denote this integral by:

$$\int_a^b f(x)dg(x).$$

Clearly this definition extends the concept of Riemann integration as we can always take $g(x) = x$.

Definition 2.16. The variation V of a function f on the interval $[a, b]$ is defined as:

$$V(f, a, b) = \int_a^b |f(x)|dx.$$

If $V(f, a, b)$ exists and is finite, then we say that f is of bounded variation.

We are going to enunciate some results, for the proof of these results and more properties on Riemann-Stieltjes integration we refer the interest reader to Rudin's book [16].

Theorem 2.17. Let f be continuous and g of bounded variation on $[a, b]$. Then f is Riemann-Stieltjes integrable on $[a, b]$ with respect to g.

Theorem 2.18. The following results holds for the Riemann-Stieltjes integral:

(i) Suppose that both $\int_a^b fdh$ and $\int_a^b gdh$ exists and let α, β be scalars. Then $\int_a^b(\alpha f+\beta g)dh$ exists and $\int_a^b(\alpha f + \beta g)dh = \alpha \int_a^b fdh + \beta \int_a^b gdh$;

(ii) Suppose that both $\int_a^b fdg$ and $\int_a^b fdh$ exists and let α, β be scalars. Then $\int_a^b fd(\alpha g+\beta h)$ exists and $\int_a^b fd(\alpha g + \beta h) = \alpha \int_a^b fdg + \beta \int_a^b fdh$;

(iii) If $\int_a^b fdg$ exists and $c \in [a, b]$, then both $\int_a^c fdg$ and $\int_c^b fdg$ exists and $\int_a^c fdg+\int_c^b fdg = \int_a^b fdg$;

(iv) If $f_1 \leq f_2$ on $[a, b]$, both f_1, f_2 are Riemann-Stieltjes integrable on $[a, b]$ and g is non-decreasing. Then $\int_a^b f_1dg \leq \int_a^b f_2dg$;

(v) If g is of bounded variation and differentiable on $[a, b]$ and $\int_a^b f(x)dg(x)$ exists. Then $\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx$.

Theorem 2.19. (Integration by Parts) Let f and g be two differentiable functions such that both $\int_a^b f(x)dg(x)$ and $\int_a^b g(x)df(x)$ exists. Then the following equality is valid:

$$f(b)g(b) - f(a)g(a) = \int_a^b f(x)dg(x) + \int_a^b g(x)df(x).$$

As we can see, Riemann-Stieltjes integration retains most properties of classical Riemann Integration. In the context of number theory, we use Riemann-Stieltjes integration to rewrite any summation as an integral. Therefore, we can apply the same tools available for integration to finite sums and series. A common technique that we will use several times is integration by parts to rewrite sums in more useful forms. Suppose we want to express $\sum_a^b a_n$ in integral form. First, we define $f(x) = a_{\lfloor x \rfloor}$. Then:

$$\sum_{n=a}^{b} a_n = \sum_{n=a}^{b} f(n) = \int_{a^-}^{b} f(x)d\lfloor x \rfloor.$$

It is important to notice that the a^- on the limit of integration is crucial. It allows us to get the value of f at a, as the floor function varies between a^- and a.

2.5 Topics in Complex Analysis

Here we remember some definitions and basic results in complex analysis. For the proofs not covered in this dissertation we refer the reader to Conway's book [5].

Definition 2.20. Let $U \subset \mathbb{C}$ be an open set. We say that $f : U \longrightarrow \mathbb{C}$ is analytic if for every $z \in U$ there exists some r such that f can be represented by a convergent power series on $B(z_0, r) \subset U$, i.e, there exists a sequence of numbers $c_i \in \mathbb{C}$, such that:

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots ,$$

for all $z \in B(z_0, r)$.

Theorem 2.21. For a power series centered around z_0, there exists a number R, which we call the radius of convergence of the series, with the following properties:

(i) If $|z - z_0| < R$, then the series converges absolutely.

(ii) If $|z - z_0| > R$, then the series diverges.

(iii) The series converges uniformly for every compact contained inside $B(x_0, R)$.

Theorem 2.22. If $f : U \subset \mathbb{C} \to \mathbb{C}$ is analytic, then f has continuous derivatives of all orders and its power series expansion around z_0 is given by:

$$f(z) = f(z_0) + f'(z_0)\frac{(z-z_0)}{1!} + f''(z_0)\frac{(z-z_0)^2}{2!} + f'''(z_0)\frac{(z-z_0)^3}{3!} + \cdots,$$

for z sufficiently close to z_0.

Definition 2.23. Let $U \subset \mathbb{C}$ and f be an analytic function on $U - \{a\}$. We say that f has a pole of order m at a if m is the smallest integer for which there exists an analytic function g on U such that $f(z) = (z - z_0)^{-m} g(z)$.

Definition 2.24. Let $f : U \to \mathbb{C}$ be analytic and $f(z_0) = 0$. We say that z_0 is a zero of multiplicity m if f(z) $= (z - z_0)^m g(z)$, where g is analytic on U and $g(z_0) \neq 0$.

Theorem 2.25. Let $f : U \to \mathbb{C}$ be an analytic function. Then z_0 is a zero of multiplicity k if and only if

$$f(z_0) = f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0,$$

and

$$f^{(k)}(z_0) \neq 0.$$

Proof. The proof follows straight from the power series expansion around z_0. □

Theorem 2.26. Let $f : U \to \mathbb{C}$ be an analytic function and suppose $f(z) = O(|z - z_0|^k)$. Then z_0 is a zero of multiplicity at least k.

Theorem 2.27. Let x_n be a sequence of complex numbers that converges absolutely to s. Then

$$\prod_{n=1}^{\infty}(1 + x_n)$$

converges and its absolute value is less or equal e^s.

Proof. Let $s_n = \sum_{k=1}^{n} |x_n|$ and $s = \sum s_n$, remember that the product of n distinct numbers that sum to some fixed value is maximized when all terms are equal. Therefore:

$$\begin{aligned}\left|\prod_{k=1}^{n}(1+x_k)\right| &= \prod_{k=1}^{n} |1+x_k| \\ &\leq \prod_{k=1}^{n}(1+|x_k|) \\ &\leq \prod_{k=1}^{n}\left(1+\frac{s_n}{n}\right) \\ &= \left(1+\frac{s_n}{n}\right)^n \\ &\leq \left(1+\frac{s}{n}\right)^n.\end{aligned}$$

Taking the limit on both sides we have

$$\left|\prod_{k=1}^{\infty}(1+x_k)\right| \leq e^s.$$

□

2.6 Dirichlet Convolution

As multiplicative functions are defined according to their values at their divisors, it makes sense to define some kind of product that combines the values of two functions over their divisors. The Dirichlet character, which is defined in this section, does exactly that and, as we are going to see, has many useful properties.

Definition 2.28. Let f and g be two arithmetic functions. We define its Dirichlet convolution and denote it by $f * g$ the following function:

$$\begin{aligned}f * g(n) &= \sum_{d|n} f(d) \cdot g\left(\frac{n}{d}\right) \\ &= \sum_{d_1 d_2 = n} f(d_1)g(d_2).\end{aligned}$$

Theorem 2.29. Let f, g, h be multiplicative functions. Then:

(i) $f * g$ is multiplicative,

(ii) $f * g = g * f$,

(iii) $(f * g) * h = f * (g * h)$,

(iv) ε defined by $\varepsilon(1) = 1$ and $\varepsilon(n) = 0$ for $n > 1$ is the identity for Dirichlet convolution.

Proof. Suppose $(a, b) = 1$. Then:

$$\begin{aligned} f * g(ab) &= \sum_{\substack{a_1a_2=a \\ b_1b_2=b}} f(a_1b_1)g(a_2b_2) \\ &= \sum_{\substack{a_1a_2=a \\ b_1b_2=b}} f(a_1)f(b_1)g(a_2)g(b_2) \\ &= \sum_{a_1a_2=a} f(a_1)g(a_2) \sum_{b_1b_2=b} f(b_1)g(b_2) \\ &= (f * g(a))\,(f * g(b))\,. \end{aligned}$$

So (i) holds. (ii) follows straight from the definition changing the roles of d_1 and d_2. For (iii), we have:

$$\begin{aligned} (f * g) * h(n) &= \sum_{kc=n} (f * g)(k)h(c) \\ &= \sum_{kc=n} \sum_{ab=k} f(a)g(b)h(c) \\ &= \sum_{abc=n} f(a)g(b)h(c) \\ &= \sum_{ak=n} \sum_{bc=k} f(a)g(b)h(c) \\ &= \sum_{ak=n} f(a)(g * h)(k) \\ &= f * (g * h)(n). \end{aligned}$$

(iv) follows direct from the definition of ε:

$$f * \varepsilon(n) = \sum_{d_1d_2=n} \varepsilon(d_1)f(d_2) = f(n).$$

□

Example 2.30. The divisor function $\tau(n)$ counts the number of divisors of n. We can use the previous theorem to show that τ is multiplicative as:

$$\begin{aligned} \tau(n) &= \sum_{d|n} 1 \\ &= \sum_{d|n} 1(d)1(n/d) \\ &= 1 * 1(n). \end{aligned}$$

The next result is very important and emphasizes the significance of the Möbius μ function on the theory of multiplicative functions.

Theorem 2.31. (Möbius Inversion Formula) $f * 1 = g \Leftrightarrow f = g * \mu$, i.e the following equalities are equivalent:

$$g(n) = \sum_{d|n} f(d),$$
$$f(n) = \sum_{d|n} g(d)\mu(n/d).$$

Proof. First we will prove that $1 * \mu(n) = \sum_{d|n} \mu(d) = \varepsilon(n)$. If $n = 1$ both equations vanishes into $f(1) = g(1)$. If $n \neq 1$, let $p^k \mid\mid n$, then we have:

$$\begin{aligned}
\sum_{d|n} \mu(d) &= \sum_{\substack{d|n \\ (p,d)=1}} \mu(d) + \mu(pd) + \cdots + \mu(p^k d) \\
&= \sum_{\substack{d|n \\ (p,d)=1}} \mu(d) + \mu(pd) \\
&= \sum_{\substack{d|n \\ (p,d)=1}} \mu(d) - \mu(d) \\
&= 0.
\end{aligned}$$

Using the above equality we have:

$$\begin{aligned}
f * 1 = g &\Leftrightarrow (f * 1) * \mu = g * \mu \\
&\Leftrightarrow f * (1 * \mu) = g * \mu \\
&\Leftrightarrow f = g * \mu.
\end{aligned}$$

□

Example 2.32. We can rewrite Euler's Totient Function as:

$$n = \sum_{d|n} \phi(d).$$

To verify this let $S_d = \{m \in \mathbb{Z}, 1 \leq m \leq n : (m,n) = d\}$. Note that $(m,n) = d$ if and only if $(m/d, n/d) = 1$. We know that there are $\phi(n/d)$ integers m satisfying such condition, so

S_d has $\phi(n/d)$ elements. Using that $\{S_d\}$ is a partition of the set of integers between 1 and n, we have:

$$n = \sum_{d|n} |S_d| = \sum_{d|n} \phi(n/d) = \sum_{d|n} \phi(d).$$

By applying Theorem 2.31 we obtain a new way of calculating ϕ:

$$\phi(n) = \sum_{d|n} \mu(d) \cdot \frac{n}{d}.$$

2.7 Dirichlet Series

We introduce in this section one of the most important tools in analytic number theory which are the Dirichlet series. We note that some concepts related to these series have analogue in the theory of power series. For example, for Dirichlet series, we have the concept of semi-plane of convergence whereas in power series we have the concept of radius of convergence, and these concepts somehow behaves similarly.

Definition 2.33. Given a sequence of complex numbers $a = (a_n)_n$. We define the Dirichlet series associate to a by:

$$D(a; s) := \sum_{n \geq 1} \frac{a_n}{n^s},$$

where $s \in \mathbb{C}$. We can see every sequence of complex numbers as the associated arithmetic function and vice versa. So it makes sense to say the Dirichlet series associated to the arithmetic function f and denote it by $D(f; s)$.

Example 2.34. The Riemman zeta function denoted by $\zeta(s)$ is defined by $\zeta(s) = D(1, s)$, for all s such that $\sigma > 1$, i.e.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The next result shows the intrinsic relation between Dirichlet series and Dirichlet convolutions, and provides a useful way of writing the product of two Dirichlet series as a new Dirichlet series.

Theorem 2.35. Let $F(s)$, $G(s)$ and $H(s)$ be the Dirichlet series associated with the arithmetic functions $f, g, h = f * g$, respectively. If $F(s_0)$ and $G(s_0)$ are both absolutely convergent, then we have that $H(s_0)$ is also absolutely convergent and $F(s_0)G(s_0) = H(s_0)$.

Proof.

$$\sum_{n\le x}\left|\frac{h(n)}{n^{s_0}}\right| = \sum_{ab\le x}\left|\frac{f(a)g(b)}{(ab)^{s_0}}\right|$$
$$\le \sum_{a\le x}\left|\frac{f(a)}{a^{s_0}}\right|\sum_{b\le x}\left|\frac{g(b)}{b^{s_0}}\right|$$

The absolute convergence of $H(s_0)$ follows by taking $x \to \infty$. Now note that absolute convergence allows the change of the order of summation. Therefore:

$$\begin{aligned}F(s_0)G(s_0) &= \left(\sum_{n=1}^{\infty}\frac{f(n)}{n^{s_0}}\right)\left(\sum_{m=1}^{\infty}\frac{g(m)}{m^{s_0}}\right)\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{f(n)g(m)}{(nm)^{s_0}}\\ &= \sum_{n=1}^{\infty}\sum_{k\cdot l=n}\frac{f(k)g(l)}{n^{s_0}}\\ &= \sum_{n=1}^{\infty}\frac{h(n)}{n^{s_0}}\\ &= H(s_0).\end{aligned}$$

□

Theorem 2.36. Let $F(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ and suppose that $F(s_0)$ converges. Then for all $H > 0$ $F(s)$ is uniformly convergent on the region $S = \{s : \sigma \ge \sigma_0, |t - t_0| \le H|\sigma - \sigma_0|\}$.

Proof. In this proof assume that $s \in S$. Let $H > 0$ be a constant and $R(x) = \sum_{n\ge x} a_n n^{-s_0}$, using Riemann-Stieltjes integral and integration by parts:

$$\begin{aligned}\sum_{n=M+1}^{N}\frac{a_n}{n^s} &= \int_M^N n^{s_0-s}dR(n)\\ &= R(N)N^{s_0-s} - R(M)M^{s_0-s} - \int_M^N R(n)dn^{s_0-s}\\ &= R(N)N^{s_0-s} - R(M)M^{s_0-s} - (s_0 - s)\int_M^N R(n)n^{s_0-s-1}dn.\end{aligned}$$

As the series converges at s_0, for every $\epsilon > 0$ there exists N_0 such that $|R(M)| < \epsilon/(3+H)$ for all $M \geq N_0$. Taking $N \geq M > N_0$ and observing that $\sigma > \sigma_0$ implies $N^{\sigma_0-\sigma} \leq 1$ and $M^{\sigma_0-\sigma} \leq 1$. We therefore have:

$$\begin{aligned}
\left|\sum_{n=M+1}^{N} \frac{a_n}{n^s}\right| &\leq |R(N)N^{s_0-s}| + |R(M)M^{s_0-s}| + |s_0 - s| \int_M^\infty |R(n)n^{s_0-s-1}|dn \\
&\leq \frac{\epsilon}{3+H} N^{\sigma_0-\sigma} + \frac{\epsilon}{3+H} M^{\sigma_0-\sigma} + \frac{\epsilon}{3+H}|s_0 - s| \int_M^\infty n^{\sigma_0-\sigma-1} dn \\
&\leq \frac{\epsilon}{3+H}\left(2 + \frac{|s_0 - s|}{\sigma - \sigma_0} M^{\sigma_0-\sigma}\right) \\
&\leq \frac{\epsilon}{3+H}\left(2 + \frac{|s_0 - s|}{\sigma - \sigma_0}\right) \\
&\leq \frac{\epsilon}{3+H}\left(2 + \frac{|\sigma - \sigma_0| + |t - t_0|}{\sigma - \sigma_0}\right) \\
&\leq \frac{\epsilon}{3+H}(3+H) \\
&= \epsilon.
\end{aligned}$$

Therefore the convergence is uniform on S. □

Corollary 2.37. Let $F(s)$ be a Dirichlet series and suppose that $F(s_0)$ converges. Then if $s > s_0$ there exists a neighborhood of s where the series converges uniformly.

Proof. Just take H large enough in the previous theorem so that the region S contains s in its interior. □

Definition 2.38. We say that σ_c is an abscissa of convergence of the Dirichlet series $F(s)$ if for all s with $\sigma > \sigma_c$ the series converges and for every s with $\sigma < \sigma_c$ the series diverges.

Definition 2.39. We say that σ_a is an abscissa of absolute convergence of the Dirichlet series $F(s)$ if σ_a is an abscissa of convergence of the Dirichlet series with coefficients given by the absolute value of coefficients of F.

The existence of abscissa of convergence and absolute convergence is guaranteed by Corollary 2.37.

Example 2.40. For the Dirichlet series that represents the Riemann Zeta function we have $\sigma_c = \sigma_a$, as for $s = 1$ the series is exactly the harmonic series which diverges, and for $s > 1$ we have a p-series which we know to be absolute convergent for $p > 1$.

Theorem 2.41. The Riemann Zeta function has the following analytic continuation for $\text{Re}(s) > 0$:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}} dx. \tag{2.2}$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^s} &= \int_{1^-}^{\infty} \frac{1}{x^s} d\lfloor x \rfloor \\
&= \int_{1^-}^{\infty} \frac{1}{x^s} dx - \int_{1^-}^{\infty} \frac{1}{x^s} d\{x\} \\
&= \frac{x^{1-s}}{1-s}\bigg|_{1^-}^{\infty} - \frac{\{x\}}{x^s}\bigg|_{1^-}^{\infty} + \int_1^{\infty} \{x\} dx^{-s} \\
&= \frac{1}{s-1} + 1 - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx \\
&= \frac{s}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx.
\end{aligned}$$

Now observe that:

$$0 \leq s\int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx \leq s\int_1^{\infty} \frac{1}{x^{s+1}} dx = 1,$$

therefore the integral converges for $\operatorname{Re} s > 0$ and we conclude that Equation 2.2 is indeed the desired analytic continuation. □

Corollary 2.42. As s approaches 1, we have the following equality:

$$\zeta(s) = \frac{1}{s-1} + O(1).$$

Theorem 2.43. Let $(a_n)_n$ be a sequence and F be its Dirichlet series. Then for $\sigma > \sigma_c$ we can write the derivative of the Dirichlet series by differentiating each term, i.e:

$$F'(s) = -\sum_{n=1}^{\infty} \frac{a_n \ln n}{n^s}.$$

Proof. Define $F_N = \sum_{n \leq N} \frac{a_n}{n^s}$, and let K be a compact contained in the semiplane $\sigma > \sigma_c$, by theorem 2.37 we know that this sequence converges uniformly to F, as each of these terms is holomorphic in K and its derivatives are given by:

$$F'_N = -\sum_{n \leq N} \frac{a_n \ln n}{n^s},$$

it follows from a classical result in complex analysis that F'_N also converges to F_N in K. □

Theorem 2.44. Let $F(s)$ be the Dirichlet Series of the arithmetic function f and suppose that f has bounded partial sums. Then $\sigma_c \leq 0$.

Proof. See Theorem 1.3 in [14]. □

Theorem 2.45. Let $F(s) = \sum_n \frac{f(n)}{n^s}$ be a Dirichlet series, then $\sigma_c \leq \sigma_a \leq \sigma_c + 1$.

Proof. The first equality is straightforward. For the second one note that for $\epsilon > 0$:

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_c+\epsilon}}$$

converges and therefore $f(n) \ll_\epsilon n^{\sigma+\epsilon}$. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma_c+1+2\epsilon}} &\ll \sum_{n=1}^{\infty} \frac{n^{1+\epsilon}}{n^{\sigma_c+1+2\epsilon}} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} \\ &< \infty.\end{aligned}$$

The result follows letting $\epsilon \to 0^+$. □

2.8 Euler Product

Let f be a multiplicative function. We would like to use the fact that f is multiplicative to rewrite the Dirichlet series in such a way that is more convenient in most cases. More precisely let $n = p_1^{k_1} p_2^{k_2} \dots p_l^{k_l}$. Then we have that

$$\frac{f(n)}{n^s} = \frac{f(p_1^{k_1}) f(p_2^{k_2}) \dots f(p_l^{k_l})}{p_1^{k_1 s} p_2^{k_2 s} \dots p_l^{k_l s}}.$$

Reordering the terms of the series we obtain what we call the Euler product of the series:

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}. \tag{2.3}$$

Let us remember that we must be careful reordering the terms of a series, so it is not always that Equation 2.3 is valid. The following result gives a condition that ensures the validity of this equality.

Theorem 2.46. Let f be multiplicative and $F(s)$ be its associated Dirichlet series. If $\sum_p \sum_{k=1}^{\infty} \frac{|f(p^k)|}{p^{k\sigma}}$ converges. Then $F(s)$ is absolutely convergent and Equation (2.3) is valid.

Proof. Let $P^+(n)$ denote the largest prime factor of n, and denote $\sum_p \sum_{k=1}^{\infty} \frac{|f(p^k)|}{p^{k\sigma}}$ by M. Then

$$\begin{aligned}\sum_{n\leq x} \frac{|f(n)|}{n^s} &\leq \sum_{P^+(n)\leq x} \frac{|f(n)|}{n^\sigma} \\ &= \prod_{p\leq x}\left(1+\sum_{k=1}^{\infty}\frac{|f(p^k)|}{p^{k\sigma}}\right) \\ &\ll 1,\end{aligned}$$

so $F(s)$ converges absolutely. For the other result observe that:

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\frac{f(n)}{n^s} - \prod_{p\leq x}\sum_{k\geq 0}\frac{f(p^k)}{p^{ks}}\right| &= \left|\sum_{P^+(n)>x}\frac{f(n)}{n^s}\right| \\ &\leq \sum_{n>x}\left|\frac{f(n)}{n^s}\right|.\end{aligned}$$

Letting $x \longrightarrow \infty$ we can make the right hand side as small as we want, so the theorem follows. □

If f is totally multiplicative we can simplify more Equation (2.3) by noticing that the terms of the series are in geometric progression:

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \prod_p \frac{1}{1-f(p)/p^s}. \tag{2.4}$$

Example 2.47. The Dirichlet series associated to the Möbius function μ has the following representation as an Euler product:

$$D(\mu,s) = \prod_p\left(\sum_{k=0}^{\infty}\frac{\mu(p^k)}{p^{ks}}\right) = \prod_p\left(1-\frac{1}{p^s}\right).$$

Example 2.48. let us write the Euler product of the Riemann ζ function. Observe that the function $f(n) = 1$ has the ζ function as its Dirichlet series. As f is totally multiplicative we have:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_p\left(1-\frac{1}{p^s}\right)^{-1}.$$

Remark 2.49. We can see that the Euler product of the Riemann Zeta function is the inverse of the Euler product of the Dirichlet series of the Möbius function, that is no coincidence and comes from the fact that $1 * \mu = \varepsilon$.

Remark 2.50. Let f and g be two arithmetic functions and F and G be their respective Dirichlet Series. Suppose F and G converges absolutely at s_0. Then we have by Theorem 2.27:

$$\begin{aligned}\prod_p \left(\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks_0}}\right)\left(\sum_{k=0}^{\infty} \frac{g(p^k)}{p^{ks_0}}\right) &= F(s_0)G(s_0) \\ &= H(s_0) \\ &= \prod_p \left(\sum_{k=0}^{\infty} \frac{f * g(p^k)}{p^{ks_0}}\right).\end{aligned}$$

2.9 The *Dirichlet Divisor Problem*

This section is a brief introduction to the *Dirichlet Divisor Problem*, which is a classical problem of analytic number theory. The importance of this function in this dissertation is the Δ function which we will define. This function shows up in the important Theorem 3.15. Let us first observe that

$$\begin{aligned}\sum_{n=1}^{x} \frac{1}{n} &= \int_{1^-}^{x} \frac{1}{n} d\lfloor n \rfloor \\ &= \int_{1^-}^{x} \frac{1}{n} dn - \int_{1^-}^{x} \frac{1}{n} d\{n\} \\ &= \ln x - \left.\frac{\{n\}}{n}\right|_{1^-}^{x} + \int_{1}^{x} \{n\} dn^{-1} \\ &= \ln x + 1 - \int_{1}^{x} \frac{\{n\}}{n^2} dn,\end{aligned}$$

which motivates the following definition:

Definition 2.51. The Euler–Mascheroni constant γ is defined as:

$$\gamma = 1 - \int_{1}^{\infty} \frac{\{x\}}{x^2} dx.$$

Using the above notation we can show the following expression:

$$\sum_{n \le x} \frac{1}{n} = \ln x + \gamma + O(1/x). \tag{2.5}$$

As x goes to infinity the $O(1/x)$ term becomes negligible, therefore the Euler-Mascheroni constant is the asymptotic difference between the logarithm function and the sequence of harmonic numbers.

The next lemma is an important tool for calculating the partial sums of the Dirichlet product of two arithmetic functions. The name of the result *Dirichlet Hyperbola Method* comes from a geometric way of looking at the partial sum of $f*g$ till t as summing $f(x)g(y)$ under the hyperbola $xy = t$.

Lemma 2.52. (Dirichlet Hyperbola Method) Let f and g be two arithmetic functions and define $F(x) = \sum_{n\le x} f(n)$ and $G(x) = \sum_{n\le x} g(n)$. Then for all $0 < y \le x$ the following equality holds:

$$\sum_{n\le x} f*g(n) = \sum_{n\le y} F(x/n)g(n) + \sum_{n\le x/y} f(n)G(x/n) - F(x/y)G(y).$$

Proof.

$$\begin{aligned}\sum_{n\le x} f*g(n) &= \sum_{ab\le x} f(a)g(b)\\ &= \sum_{mn\le x, n\le y} f(m)g(n) + \sum_{nm\le x, m>y} f(n)g(m)\\ &= \sum_{n\le y} F(x/n)g(n) + \sum_{n\le x/y} f(n)\left(G(x/n) - G(y)\right)\\ &= \sum_{n\le y} F(x/n)g(n) + \sum_{n\le x/y} f(n)G(x/n) - F(x/y)G(y).\end{aligned}$$

□

We are interested in an estimate for the value of the partial sums of τ. Using the previous result, we are able to establish the following result:

Theorem 2.53. We have the following expression for $\sum_{n\le x} \tau(n)$:

$$\sum_{n\le x} \tau(n) = 2x\ln x + (2\gamma - 1)x + O(\sqrt{x}).$$

Proof. Observe that $\tau = 1 * 1$, so taking $y = \sqrt{x}$ on Lemma 2.52 and using Equation (2.5) we have:

$$\begin{aligned}
\sum_{n \le x} \tau(n) &= 2 \sum_{n \le \sqrt{x}} \left\lfloor \frac{x}{n} \right\rfloor - \lfloor \sqrt{x} \rfloor^2 \\
&= 2 \sum_{n \le \sqrt{x}} \left(\frac{x}{n} + O(1) \right) - \lfloor \sqrt{x} \rfloor^2 \\
&= 2x \sum_{n \le \sqrt{x}} \frac{1}{n} - x + O(\sqrt{x}) \\
&= 2x \left(\ln x^{1/2} + \gamma + O(1/\sqrt{x}) \right) - x + O(\sqrt{x}) \\
&= x \ln x + (2\gamma - 1)x + O(\sqrt{x}).
\end{aligned}$$

□

Definition 2.54. We can denote the error term in the previous theorem by $\Delta(x)$ i.e:

$$\Delta(x) := \sum_{n \le x} \tau(x) - x \log x - (2\gamma - 1)x.$$

Definition 2.55. The *Dirichlet Divisor Problem* consists in finding estimates for the growth of Δ. More formally, we seek for α defined by:

$$\alpha := \inf\{a > 0 : \Delta(x) = O_a(x^a)\}.$$

The search for the value of α is an active area of research. From Theorem 2.53 we know that $\alpha \le 1/2$, in 1917 [9] Hardy showed that $\alpha \ge 1/4$. This bound was improved several times, and the current lower bound is 131/416 which was proved by Huxley in 2003 [10].

3 Multiplicative Functions With Codomain $\mathbb{C}$

We chose to start with the study of the more general case in this chapter and only in the next case we study the particular case where the codomain is $\{-1, 1\}$. This and the next chapter are almost independent, so the next chapter can be read already. The only result needed for the next chapter is Theorem 3.4, which can be easily assumed without any loss.

3.1 Necessary and Sufficient Conditions for Boundedness

The goal of this section is to give characterizations that help determining if a multiplicative function f has bounded or unbounded partial sums. Theorem 3.2 gives a necessary condition in the case f satisfies a property a little stronger than to be 1-pretentious. Theorem 3.4 gives an important characterization of boundedness of partial sums in the case f is periodic.

The following lemma formalizes the notion that if a multiplicative function has bounded partial sums, then it can't attain many large values.

Lemma 3.1. Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function with bounded partial sums. Then f is bounded and for all $\epsilon > 0$ there exists an M such that if $p \geq M$, then $|f(p^k)| < 1 + \epsilon$ for all $k \geq 1$.

Proof. Suppose the partial sums are bounded by C. Then:

$$\begin{aligned} |f(n)| &= \left| \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n-1} f(k) \right| \\ &\leq \left| \sum_{k=1}^{n} f(k) \right| + \left| \sum_{k=1}^{n-1} f(k) \right| \leq 2C. \end{aligned}$$

Suppose that there are infinitely many distinct primes p_n and a sequence k_n satisfying $\left| f(p_n^{k_n}) \right| \geq 1 + \epsilon$. Then the sequence $(f(\prod_{n \leq k} p_n^{k_n}))_k$ is unbounded as:

$$\begin{aligned} \left| f\left(\prod_{n \leq k} p_n^{k_n} \right) \right| &= \left| \prod_{n \leq k} f(p_n^{k_n}) \right| \\ &\geq (1+\epsilon)^k, \end{aligned}$$

which diverges as k goes to infinity, contradicting the hypothesis that the partial sums are bounded.

□

Theorem 3.2. Let f be a multiplicative function with bounded partial sums such that $\sum_p \frac{|1-f(p)|}{p} < \infty$. Then there exists a prime q such that $\sum_{k\geq 0} \frac{f(q^k)}{q^k} = 0$.

Proof. Define $F(s) = \sum_{n\geq 1} \frac{f(n)}{n^s}$. As f has bounded partial sums we know by Theorem 2.44 that F is analytic for $\mathrm{Re}(s) > 0$, so $\sigma_c \leq 0$, and by Theorem 2.45, we have $\sigma_a \leq 1$, so if $\mathrm{Re}(s) > 1$. Then F can be represented by its Euler product:

$$F(s) = \prod_p \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}.$$

By Lemma 3.1, choose M such that $|f(p^k)| \leq 1+\epsilon$ for all $p \geq M$ and $k \geq 1$ and consider the following tail product:

$$\begin{aligned}\frac{1}{\zeta(s)} \prod_{p\geq M} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} &= \prod_{p<M}\left(1-\frac{1}{p^s}\right) \prod_{p\geq M}\left(1-\frac{1}{p^s}\right) \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \\ &= \prod_{p<M}\left(1-\frac{1}{p^s}\right) \prod_{p\geq M}\left(1+\frac{f(p)-1}{p^s}+\frac{O(1)}{p^{2s}}\right).\end{aligned}$$

By the pretentiousness hypothesis we know by theorem 2.27, that the previous product converges at $s = 1$. Using this convergence and Corollary 2.42, we note that:

$$\begin{aligned}\prod_{p\geq M} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} &= \zeta(s)\left(\frac{1}{\zeta(s)} \prod_{p\geq M} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}\right) \\ &= \left(\frac{1}{s-1} + O_{s\to 1}(1)\right)\left(\frac{1}{\zeta(s)} \prod_{p\geq M} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}\right) \\ &= \frac{c + o_{s\to 1}(1)}{s-1},\end{aligned}$$

for some constant $c \neq 0$. But we know that F is analytic at 1 and in order for not being a pole we must have:

$$\prod_{p<M} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} = O(s-1)$$

hence the following equality holds:

$$\prod_{p<M} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^k} = 0$$

and we conclude that one of the terms must vanish. □

Theorem 3.3. (Klurman) Let f be a multiplicative function such that $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. Then f is periodic and sums to 0 at its period if and only if $f(2^k) = -1$ for all $k \geq 1$ and there exists M such that if $p^k > M$, then $f(p^k) = f(p^{k-1})$.

Theorem 3.3 is an important tool established by Klurman [11] to investigate the partial sums of multiplicative functions $f : \mathbb{N} \to \{-1, 1\}$. To prove that f is periodic with bounded partial sums, we just need to check some rigidity condition on large powers of primes, which can be much simpler. Later Aymone [1] noted that the same proof used by Klurman could be extended to prove a stronger result which is given by the following theorem:

Theorem 3.4. (Aymone) Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function with period m and bounded partial sums. If $f(m) \neq 0$, then we have the following:

(i) For some prime $q \mid m$, $\sum_{k=0}^{\infty} \frac{f(q^k)}{q^k} = 0$;

(ii) If $p^a \mid\mid m$, $f(p^k) = f(p^a)$ for all $k \geq a$;

(iii) If $(p, m) = 1$, then $f(p^k) = 1$ for all $k \geq 1$.

Conversely, if f is multiplicative and conditions (i)-(iii) are satisfied. Then f has bounded partial sums and f has period m.

Proof. Suppose there exists an integer $m > 0$ such that $f(n + m) = f(n)$ for all $n \in \mathbb{N}$. Additionally, assume that $\sum_{k=1}^{m} f(k) = 0$ and $f(m) \neq 0$. Let $\prod_p p^{k_p} = m$ be the prime factorization of m. By exploiting the properties of f being m-periodic and multiplicative, we can derive the following relationship for any prime q and non-negative integer α:

$$\begin{aligned}\prod_p f(p^{k_p}) &= f(m) \\ &= f(q^\alpha m) \\ &= f(q^{\alpha+k_q}) \prod_{p\neq q} f(p^{k_p})\end{aligned}$$

Since $f(m) \neq 0$, we can conclude that $f(p^{k_p}) \neq 0$. Therefore, by dividing both sides by $\prod_{p\neq q} f(p^{k_p})$, we arrive at $f(q^{\alpha+k_q}) = f(q^{k_q})$, so items (ii) and (iii) are proved. Now let $p^a \mid\mid n$ and $p^b \mid\mid m$. If $a \geq b$, then by item (ii) we have $f(p^a) = f(p^b)$, therefore

$f(p^{\min(a,b)}) = f(p^a)$. By applying the multiplicative property of f we can conclude that if $(n,m) = d$, then $f(n) = f(d)$. So

$$\begin{aligned}\sum_{n\le m} f(n) &= \sum_{d|m} \sum_{\substack{n\le m\\(n,m)=d}} f(n)\\ &= \sum_{d|m} \sum_{\substack{n\le m\\(n,m)=d}} f(d)\\ &= \sum_{d|m} \phi(m/d) f(d)\\ &= f * \phi(m). \end{aligned} \tag{3.1}$$

The function $f * \phi$ is the product of two multiplicative functions so it is also multiplicative. Let us investigate its values at powers of primes. First, remember that $\phi(p^k) = p^k(1 - 1/p)$. Then:

$$\begin{aligned} f * \phi(p^a) &= f(p^a) + f(p^{a-1})\phi(p) + \cdots + f(1)\phi(p^a)\\ &= f(p^a) + f(p^{a-1})p\left(1 - \frac{1}{p}\right) + f(p^{a-2})p^2\left(1 - \frac{1}{p}\right) + \cdots + p^a\left(1 - \frac{1}{p}\right)\\ &= p^a\left(1 - \frac{1}{p}\right)\left(\frac{f(p^0)}{p^0} + \frac{f(p)}{p^1} + \cdots + \frac{f(p^{a-1})}{p^{a-1}} + \frac{f(p^a)}{p^a(1-1/p)}\right)\\ &= p^a\left(1 - \frac{1}{p}\right)\left(\sum_{k=0}^{a-1} \frac{f(p^k)}{p^k} + \frac{f(p^a)}{p^a(1-1/p)}\right)\\ &= p^a\left(1 - \frac{1}{p}\right)\left(\sum_{k=0}^{a-1} \frac{f(p^k)}{p^k} + \sum_{k=0}^{\infty} \frac{f(p^a)}{p^a}\frac{1}{p^k}\right)\\ &= p^a\left(1 - \frac{1}{p}\right)\left(\sum_{k=0}^{a-1} \frac{f(p^k)}{p^k} + \sum_{k=a}^{\infty} \frac{f(p^k)}{p^k}\right)\\ &= p^a\left(1 - \frac{1}{p}\right)\sum_{k=0}^{\infty} \frac{f(p^k)}{p^k}\\ &= \phi(p^a)\sum_{k=0}^{\infty} \frac{f(p^k)}{p^k}. \end{aligned} \tag{3.2}$$

Using (3.1) in (3.2) we obtain:

$$\begin{aligned}\sum_{n\le m} f(n) &= \prod_{p^l||m} f * \phi(p^l)\\ &= \prod_{p^l||m} \phi(p^l)\sum_{k=0}^{\infty} \frac{f(p^k)}{p^k}\\ &= \phi(m)\prod_{p|m}\sum_{k=0}^{\infty} \frac{f(p^k)}{p^k}. \end{aligned} \tag{3.3}$$

By hypothesis the above sum is zero so we conclude item (i). Now assume f is multiplicative and satisfies items (i), (ii) and (iii). Then by items (ii) and (iii) we have that if $(a, m) = d$, then $f(a) = f(d)$, so if $a = km$ is a positive multiple of m we conclude that $f(m) = f(km)$, so f is m-periodic. Observe that we can repeat the same calculations done in Equations (3.1), (3.2) and (3.3), therefore using item (i) we conclude that f has bounded partial sums. □

Note that condition (iii) is weaker than Klurman's requirement of $f(2^k) = -1$ in the sense that if $f(2^k) = -1$, then condition (iii) is satisfied for $q = 2$.

We end this section by showing some consequences of the previous theorem. The next two results unfolds rigidity conditions that a multiplicative function must satisfy in order to have bounded partial sums. In particular, Corollary 3.6 gives a partial demonstration of Tao's Theorem 1.13.

Corollary 3.5. Let q be a prime number. Then there exists only one q-periodic function with bounded partial sums such that $f(q) \neq 0$.

Proof. Let q be a prime number and f be a q-periodic multiplicative function with bounded partial sums and $f(q) \neq 0$. By Theorem 3.4 item (iii) we have that $f(p) = 1$ for all $p \neq q$, using items (i) and (ii) we have:

$$\begin{aligned} 0 &= \sum_{k\geq 0} \frac{f(q^k)}{q^k} \\ &= 1 + \sum_{k\geq 1} \frac{f(q)}{q^k} \\ &= 1 + \frac{f(q)}{q-1} \end{aligned}$$

Therefore $f(q^k) = 1 - q$ for all $k \geq 1$. □

Corollary 3.6. Let $f : \mathbb{N} \to \{-1, 1\}$ be a m-periodic multiplicative function with bounded partial sums. Then $f(2^j) = -1$ for all $j \geq 1$.

Proof. By Theorem 3.4, we have for some prime $q \mid m$ that $\sum_{j\geq 0} f(q^j)/q^j = 0$, so:

$$\begin{aligned} \sum_{k\geq 0} \frac{f(q^k)}{q^k} &= 1 + \sum_{k\geq 1} \frac{f(q)}{q^k} \\ &\geq 1 - \sum_{k\geq 1} \frac{1}{q^k} \\ &= 1 - \frac{1}{q-1}. \end{aligned}$$

We conclude that the above sum can only be 0 if $q = 2$. Besides that, observe that if $f(2^k) = 1$ for some $k > 0$, then the above sum would be at least $\frac{1}{2^{k-1}}$, so $f(2^k) = -1$ for all $k \geq 1$. □

Remark 3.7. As pointed out by Aymone, Theorem 3.4 is a powerful tool to build multiplicative functions with bounded partial sums. For example suppose we define a multiplicative function by $f(p^k) = 1$ if $(p, 5) = 1$, $f(5) = \pi$ and we want f to have bounded partial sums. Then in order to satisfy the theorem's hypothesis we can define $f(25) = 25 + 5\pi$ and $f(5^k) = 0$, for all $k > 2$.

3.2 An Important Lemma

For the remaining of this chapter we are interest in the study of functions of the form $f = f_1 * f_2$, where f_1 and f_2 satisfies conditions (i)-(iii) of Theorem 3.4. In this section we introduce the function $g = f * \mu * \mu$ and prove some important properties of then which will be important on the remaining of this chapter when we investigate the asymptotic growth of $f = f_1 * f_2$.

First let us remember a classical result of analysis that will be important in the proof of the main result of this section.

Lemma 3.8. (Kroeneker's Lemma) Let $(x_n)_n$ be a sequence of real number such that the series $\sum_{n=1}^{\infty} x_n$ converges, and let $(b_n)_n$ be a non-decreasing sequence of positive numbers such that $b_n \longrightarrow \infty$. Then:

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k x_k = 0$$

Proof. See Shiryaev's book [17], lemma 2 on section 3 of chapter iv. □

Lemma 3.9. Let $f = f_1 * f_2$, where f_1 and f_2 are both functions satisfying hypothesis (i)-(iii) of Theorem 3.4. Let m_1 and m_2 be their respective periods and define $g = f * \mu * \mu = f * \tau^{-1}$. Then g satisfies the following properties:

(i) $\sum_{n \leq x} |g(n)| = O_\epsilon(x^\epsilon)$, for all $\epsilon > 0$;

(ii) $g(n) = \varepsilon(n)$ whenever $(n, m_1 m_2) = 1$;

(iii) $\sum_{n=1}^{\infty} \frac{g(n)}{n} = \sum_{n=1}^{\infty} \frac{g(n)\log n}{n} = 0$.

Proof. Using the fact that p is prime we have for each integer a, either $p|a$ or $(p,a)=1$, holds. Using this observation, and the Euler product formula for the Dirichlet series of the convolution we have for all $\sigma > 1$:

$$\begin{aligned}
F(s) &= \sum_{n=1}^{\infty} \frac{f_1 * f_2(n)}{n^s} \\
&= \prod_p \left(\sum_{k=0}^{\infty} \frac{f_1(p^k)}{p^{ks}} \right) \left(\sum_{k=0}^{\infty} \frac{f_2(p^k)}{p^{ks}} \right) \\
&= \prod_{p|m_1m_2} \left(\sum_{k=0}^{\infty} \frac{f_1(p^k)}{p^{ks}} \right) \left(\sum_{k=0}^{\infty} \frac{f_2(p^k)}{p^{ks}} \right) \prod_{(p,m_1m_2)=1} \left(\sum_{k=0}^{\infty} \frac{f_1(p^k)}{p^{ks}} \right) \left(\sum_{k=0}^{\infty} \frac{f_2(p^k)}{p^{ks}} \right) \\
&= \prod_{p|m_1m_2} \left(\sum_{k=0}^{\infty} \frac{f_1(p^k)}{p^{ks}} \right) \left(\sum_{k=0}^{\infty} \frac{f_2(p^k)}{p^{ks}} \right) \prod_{(p,m_1m_2)=1} \left(1 - \frac{1}{p^s} \right)^{-2},
\end{aligned}$$

where the last equality comes from the fact that $f_i(p^k) = 1$ if $(p, m_i) = 1$. Define $G(s) = F(s)/\zeta(s)^2$. Using that $\zeta(s) = \prod_p (1 - 1/p^s)^{-1}$ we have:

$$G(s) = F(s)/\zeta(s)^2 = \prod_{p|m_1m_2} \left(\sum_{k=0}^{\infty} \frac{f_1(p^k)}{p^{ks}} \right) \left(\sum_{k=0}^{\infty} \frac{f_2(p^k)}{p^{ks}} \right) \left(1 - \frac{1}{p^s} \right)^2. \tag{3.4}$$

Now that all terms are defined for the same primes p, we can see that $G(s)$ is exactly the Dirichlet series of $g = f * \mu * \mu$. For all p coprime with m_1m_2 we have that the corresponding Euler product term must be 1. It follows that $g(p^k) = 0$ for all $k \geq 1$, which proves item (ii).

By Theorem 3.4 we have that f_i has bounded partial sums and therefore $f_i = O(1)$, so:

$$\begin{aligned}
|g(p^k)| &= |f_1 * f_2 * \mu * \mu(p^k)| \\
&= |\sum_{k_1+k_2+k_3+k_4=k} f_1(p^{k_1}) f_2(p^{k_2}) \mu(p^{k_3}) \mu(p^{k_4})| \\
&= |\sum_{\substack{k_1+k_2+k_3+k_4=k \\ k_3,k_4 \leq 1}} f_1(p^{k_1}) f_2(p^{k_2}) \mu(p^{k_3}) \mu(p^{k_4})| \\
&= |\sum_{\substack{k_1+k_2+k_3+k_4=k \\ k_3,k_4 \leq 1}} O(1)| \\
&\ll k,
\end{aligned}$$

as we are summing $O(k)$ terms. We conclude that the following sum is bounded:

$$\sum_{p|m_1m_2} \sum_{k=1}^{\infty} \frac{|g(p^k)|}{p^{k\sigma}}.$$

But by Theorem 2.46, this implies that $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $\sigma > 0$ and its value is given by the Euler product (3.4). Now observe that in particular $G(\epsilon)$ converges absolutely, i.e, $\sum_{n=1}^{\infty} \frac{|g(n)|}{n^{\epsilon}} < \infty$. Let $x_n = \frac{|g(n)|}{n^{\epsilon}}$ and $b_n = n^{\epsilon}$. By Kroenecker's lemma we conclude that:

$$0 = \lim_{x\to\infty} \frac{1}{b_x} \sum_{n\le x} b_n x_n = \lim_{x\to\infty} \frac{1}{x^{\epsilon}} \sum_{n\le x} |g(n)|.$$

But this is equivalent to $\sum_{n\le x} |g(n)| = o(x^{\epsilon})$, as this holds for every $\epsilon > 0$ we conclude item (i).

By item (i) of hypothesis, there exist primes q_i such that $q_i \mid m_i$ and the following holds:

$$\sum_{k=0}^{\infty} \frac{f_i(q_i^k)}{q_i^k} = 0.$$

Define $F_{q_i}(s) = \sum_{k=0}^{\infty} \frac{f_i(q_i^k)}{q_i^{ks}}$ and observe that as f_i has bounded partial sums, we have by Theorem 2.44, that this function is analytic at 1, i.e, $F_{q_i}(s)$ admits an expansion in power series around 1 and therefore:

$$\begin{aligned} F_{q_i}(s) &= F_{q_i}(1) + F'_{q_i}(1)\frac{(s-1)}{1!} + F''_{q_i}(1)\frac{(s-1)^2}{2!} + F'''_{q_i}(1)\frac{(s-1)^3}{3!} + \dots \\ &= F'_{q_i}(1)\frac{(s-1)}{1!} + F''_{q_i}(1)\frac{(s-1)^2}{2!} + F'''_{q_i}(1)\frac{(s-1)^3}{3!} + \dots \\ &= O(|s-1|). \end{aligned}$$

Observe that $\sum_{k=0}^{\infty} \frac{f_i(p^k)}{p^{ks}} = O(1)$ and $(1 - 1/p^s)^2 = O(1)$, now using these observations and Equation (3.4) we conclude that $G(s) = O(|s-1|^2)$ for s sufficiently close to 1, but this is by Theorem 2.26 the same as saying that 1 is a zero of order 2 and therefore $G(1) = G'(1) = 0$, which completes the proof of item (iii). □

3.3 An Omega Bound on the Partial Sums of $f = f_1 * f_2$

The goal of this section is to establish an omega bound for $f = f_1 * f_2$. An important consequence of this bound is that the convolution of multiplicative functions with bounded partial sums can be unbounded. Before going to the main result we need some results about maximal order that we enunciate next.

Definition 3.10. We say that g is a maximal order of f if:

$$\limsup_{n\to\infty} \frac{|f(n)|}{|g(n)|} = 1.$$

Saying that f has maximal order g is stronger than saying that $f = O(g)$, as the imposed condition requires f to be arbitrarily close to g for infinitely many x, and being much greater than g for only finitely many x. More formally, we can state the following:

Remark 3.11. If g is a maximal order for f, then we have:

(i) $f(n) \leq (1 + o(1))g(n)$ for all n;

(ii) There exists a sequence $(x_n)_n$ such that $f(x_n) = (1 + o(1))g(x_n)$.

In this dissertation, the only function for which we are concerned about the maximal order is the prime omega function ω, which we will use to obtain estimates for the partial sums of $f_1 * f_2$. The following lemma determines its maximal order:

Lemma 3.12. A maximal order for the prime omega function ω is:

$$\frac{\log n}{\log \log n}.$$

Proof. See Theorem 5.5 in [19]. □

Theorem 3.13. (Aymone) Let f_1 and f_2 be two multiplicative functions with periods m_1 and m_2 respectively, such that f_1 and f_2 satisfy conditions (i)-(iii) of Theorem 3.4. Let $f = f_1 * f_2$. Then there exists a constant $d > 0$ such that:

$$\sum_{n \leq x} f(n) = \Omega\left(\exp\left(d \frac{\log x}{\log \log x} \right) \right).$$

Proof. By triangle inequality we have:

$$|f(x)| \leq |\sum_{n \leq x-1} f(n)| + |\sum_{n \leq x} f(n)|.$$

We have that one of the above terms is at least $|f(x)|/2$. Suppose without loss of generality that $|\sum_{n \leq x} f(n)| \geq |f(x)|/2$ and take x such that $(x, m_1 m_2) = 1$. By Lemma 3.9 we can rewrite the function f as $g * \tau$, and if $(n, m_1 m_2) = 1$, then $g(n) = \varepsilon(n)$. Therefore:

$$\begin{aligned}
2|\sum_{n \leq x} f(n)| &\geq |f(x)| \\
&= |g * \tau(x)| \\
&= |\sum_{ab=x} g(a) * \tau(b)| \\
&= |\sum_{ab=x} \varepsilon(a) * \tau(b)| \\
&= |\tau(x)| \\
&\geq 2^{\omega(x)}.
\end{aligned}$$

By Lemma 3.12 there exists a sequence x_n such that $\omega(x_n) = \frac{(1+o(1))\log x_n}{\log\log x_n}$, which implies our theorem. □

It's important to note that Theorem 3.13 does not give the best known omega bound for this multiplicative function. In his paper [1] Aymone noted the existence of a big gap between the omega and big O bounds established by him and conjectured that the omega bound could be improved to $\Omega(x^{1/4})$. In his recent work [3] Aymone together with Maiti, Ramaré and Srivastav proved this result that we present in the form of the following theorem.

Theorem 3.14. Let f_1 and f_2 be two multiplicative functions satisfying the hypothesis of theorem 3.13. Let $f = f_1 * f_2$. Then:

$$\sum_{n \le x} f(n) = \Omega(x^{1/4}).$$

3.4 An Upper Bound on the Partial Sums of $f = f_1 * f_2$

Now that we know an omega bound for the partial sums of f its natural to look for a big O bound, in this section we do exactly this. The first result gives a powerful expression to evaluate the partial sums of f, more precisely it says that for x sufficiently big these partial sums are given as a linear combination of terms involving the Δ function of *Dirichlet's Divisor Problem*. The big O bound follows straight from this result by using big O estimates for Δ.

Theorem 3.15. Let f_1 and f_2 be two multiplicative functions with periods m_1 and m_2 respectively, such that f_1 and f_2 satisfy conditions (i)-(iii) of Theorem 3.4. Let $f = f_1 * f_2$. Then, for all $x > m_1 m_2$, we have:

$$\sum_{n \le x} f(n) = \sum_{n | m_1 m_2} f * \mu * \mu(n) \Delta\left(\frac{x}{n}\right),$$

where Δ is the function of Dirichlet Divisor problem.

Proof. Using Lemma 3.9 to rewrite f as $g * \tau$ and Theorem 2.53 to estimate the partial sums of τ, we have:

$$
\begin{aligned}
\sum_{n\le x} f(n) &= \sum_{n\le x} g * \tau(n) \\
&= \sum_{n\le x} g(n) \sum_{m\le x/n} \tau(m) \\
&= \sum_{n\le x} g(n)\left(\frac{x}{n}\ln\frac{x}{n} + (2\gamma-1)\frac{x}{n} + \Delta(x/n)\right) \\
&= x\ln x \sum_{n\le x}\frac{g(n)}{n} - x\sum_{n\le x}\frac{g(n)\ln n}{n} + (2\gamma-1)x\sum_{n\le x}\frac{g(n)}{n} + \sum_{n\le x} g(n)\Delta(x/n).
\end{aligned}
\tag{3.5}
$$

By condition (ii) we have that if $p^{k_i} \mid\mid m_i$, for $i = 1, 2$, then $f_i(p^t) = f_i(p^{k_i})$ for all $t > k_i$. Therefore we conclude that $f_i * \mu(p^t) = f_i(p^t) - f_i(p^{t-1}) = 0$ for all $t \ge k_i + 1$. Taking $t \ge k_1 + k_2 + 1$ and using that $g = (f_1 * \mu) * (f_2 * \mu)$ we conclude that $g(p^t) = 0$ as each term of the convolution vanishes.

Now let us show that $g(n) = 0$, whenever $n > m_1m_2$. Suppose $g(n) \ne 0$, let $p^k \mid\mid n$. If $(p, m_1m_2) \ne 0$, then we know by the previous observation that $k < k_1 + k_2 + 1$, i.e, $p^k \mid\mid m_1m_2$. If $(p, m_1m_2) = 0$, then by item (ii) of Lemma 3.9, we have that $g(n) = 0$. Therefore the only prime factors of n are of the form p^k, with $p^k \mid\mid m_1m_2$, and we conclude that $n \le m_1m_2$.

By Lemma 3.9 item (iii) and the observation that $g(n)$ vanishes whenever $n > m_1m_2$, we know that:

$$
0 = \sum_{n\le m_1m_2}\frac{g(n)}{n} = \sum_{n\le m_1m_2}\frac{g(n)\ln n}{n}. \tag{3.6}
$$

Combining Equations (3.5), (3.6) and using Lemma 3.9 (ii) we have:

$$
\begin{aligned}
\sum_{n\le x} f(n) &= \sum_{n\le m_1m_2} g(n)\Delta(x/n) \\
&= \sum_{n\mid m_1m_2} g(n)\Delta(x/n)
\end{aligned}
$$

□

The result given by the previous theorem is an exact formula for the partial sums of f. By using estimates for Δ, we can establish the asymptotic behaviour of this formula.

Corollary 3.16. Let f be defined according to Theorem 3.15 and α be the constant of Dirichlet Divisor Problem. Then for all $\epsilon > 0$ we have $\sum_{n\le x} f(n) = O(x^{\alpha+\epsilon})$.

Proof. By Theorem 3.15, we have for $x > m_1 m_2$:

$$\begin{aligned} |\sum_{n\leq x} f(n)| &= |\sum_{n|m_1m_2} g(n)\Delta(x/n)| \\ &\leq \sum_{n|m_1m_2} |g(n)||\Delta(x/n)| \\ &\ll \sum_{n|m_1m_2} |\Delta(x/n)| \\ &\ll \Delta(x) \\ &\ll x^{\alpha+\epsilon}. \end{aligned}$$

□

Such result makes reasonable to believe that the improved big omega bound is in fact tight, as we believe that the constant of the *Dirichlet Divisor Problem* satisfies $\alpha = \frac{1}{4}$.

4 The Erdős Coons Tao Conjecture

The proof of Erdős-Coons-Tao conjecture is long, and we won't present it completely, just the core steps. In this chapter we begin by defining in Section 4.1 a function that relates to our problem and showing some of its nice properties. In Section 4.2 we establish an important lower bound for this function. In Section 4.3 we show that under the boundedness hypothesis of f some series which we will define must converge. Finally in Section 4.4 we use the tools of the previous sections to give a proof of the conjecture.

4.1 The G Function

Definition 4.1. Let f be multiplicative and $g = f * \mu$. We define G as:

$$G(a) = \prod_{p^k || a} \left(|g(p^k)|^2 + 2 \sum_{j>k} \frac{g(p^k)g(p^j)}{p^{j-k}} \right).$$

In the above sum we use the convention that k can be zero. From now on we are going to denote the term inside the product by E_p:

$$E_p(a) = |g(p^k)|^2 + 2 \sum_{j>k} \frac{g(p^k)g(p^j)}{p^{j-k}}.$$

The main reason to define G the way we did is the following important result which provides a decomposition of the mean value of a very specific case of correlation of multiplicative functions. We refer the interested reader to Klurman's paper [11], where it is established similar results for more complex cases.

Lemma 4.2. Let $f : \mathbb{N} \to \mathbb{U}$ be multiplicative and $\mathbb{D}(1, f, \infty) < \infty, m \in \mathbb{N}$. Then:

$$\frac{1}{x} \sum_{n \leq x} f(n)\overline{f(n+m)} = \sum_{r|m} \frac{G(r)}{r} + o_{x \to \infty}(1),$$

where the term $o(1)$ does depend of the particular choice of m.

Proof. See Corollary 3.3 in [11]. □

The remaining of this section is dedicated to establish basic results of the functions we defined. These results are essential in the remaining of this chapter.

Lemma 4.3. The function $g = f * \mu$ has the following properties:

(i) $g(p^k) = f(p^k) - f(p^{k-1})$ for all $k \geq 1$, and therefore $f(p^k) \in \{-2, 0, 2\}$, for all $k \geq 1$;

(ii) Let k be the smallest positive integer such that $g(p^k) \neq 0$. Then $g(p^k) = -2$;

(iii) If $g(p^k) \neq 0, g(p^l) \neq 0$ and $g(p^i) = 0$ for all $k < i < l$. Then $g(p^k)g(p^l) < 0$.

Proof. For $k \geq 1$ we have:

$$\begin{aligned} g(p^k) &= f * \mu(p^k) \\ &= \sum_{n=0}^{k} f(p^n) \cdot \mu(p^{k-n}) \\ &= f(p^k) - f(p^{k-1}). \end{aligned}$$

So as f assumes only the values -1 and 1, we have that result (i) follows.

If k is the smallest positive integer such that $g(p^k) \neq 0$. Then $f(p^i) = 1$ for all $i < k$ and $f(p^k) = -1$, so result (ii) follows by (i).

In order to prove item (iii), we separate in two cases. If $k = 0$, then the result follows directly from item (ii). If $k \neq 0$, we observe that $g(p^i) = 0$ implies that $f(p^i) = f(p^{i-1})$ and therefore $f(p^k) = \cdots = f(p^{l-1})$, as $g(p^l) \neq 0$. By item (i) we conclude that $f(p^k) = -f(p^{k-1})$ and $f(p^l) = -f(p^{l-1})$, therefore $g(p^l) = f(p^l) - f(p^{l-1}) = f(p^l) - f(p^k)$ implies $f(p^l) = -f(p^{l-1}) = -f(p^k) = f(p^{k-1})$, and so $g(p^l) = -g(p^k)$. □

The next result establishes bounds for the value of E_p and is, therefore, useful for bounding G.

Lemma 4.4. If $g(p^k) \neq 0$, then:

(i) $4 - \frac{8}{p} \leq E_p(p^k) \leq 4 + \frac{8}{p}$, for all $k \geq 1$;

(ii) $1 - \frac{4}{p} \leq E_p(1) \leq 1$.

Proof. All the proofs are based on the observation that Lemma 4.3 implies that the series on the definition of E_p alternates signs and the absolute values of its terms decreases, so the lowest/greatest value is obtained by summing at most one term.

$$\begin{aligned} E_p(p^k) &= |g(p^k)|^2 + 2\sum_{j>k} \frac{g(p^k)g(p^j)}{p^{j-k}} \\ &= 4 + 2\sum_{j>k} \frac{g(p^k)g(p^j)}{p^{j-k}} \\ &\leq 4 + \frac{8}{p}. \end{aligned}$$

$$E_p(p^k) = |g(p^k)|^2 + 2\sum_{j>k} \frac{g(p^k)g(p^j)}{p^{j-k}}$$
$$= 4 + 2\sum_{j>k} \frac{g(p^k)g(p^j)}{p^{j-k}}$$
$$\geq 4 - \frac{8}{p}.$$

$$E_p(1) = |g(1)|^2 + 2\sum_{j>0} \frac{g(1)g(p^j)}{p^{j-k}}$$
$$= 1 + 2\sum_{j>0} \frac{g(p^j)}{p^j}$$
$$\leq 1.$$

$$E_p(1) = |g(1)|^2 + 2\sum_{j>0} \frac{g(1)g(p^j)}{p^{j-k}}$$
$$= 1 + 2\sum_{j>0} \frac{g(p^j)}{p^j}$$
$$\geq 1 - \frac{4}{p}.$$

□

Corollary 4.5. The function G is well defined, i.e, the product on the definition converges.

Lemma 4.6. The function $G(a)$ has the following properties:

(i) $G(4a) = 0, a \in \mathbb{N}$;

(ii) $G(2a) = -4G(a), a$ odd;

(iii) $\sum_{a\geq 1} \frac{G(a)}{a^2} = 0$;

(iv) If $f(3) = 1$. Then $G(a) \leq 0$ for all a odd;

(v) $\sum_{a\geq 1} \frac{G(a)}{a} = 1$.

Proof. First observe that we know the value of g at powers of 2: $g(2) = f(2)\mu(1) + f(1)\mu(2) = f(2) - 1 = -2$ and $g(2^k) = f(2^k)\mu(1) + f(2^{k-1})\mu(2) = 0$, when $k > 1$.

Using the above remark and the definition of E_2 we have:

$$E_2(4a) = |g(2^k)|^2 + 2\sum_{j>k} \frac{g(2^k)g(2^j)}{2^{j-k}} = 0,$$

as $k > 1$. Therefore $G(4a) = 0$. In the same way, if a is odd, then:

$$\begin{aligned}
E_2(2a) &= |g(2)|^2 + 2\sum_{j>1} \frac{g(2)g(2^j)}{2^{j-1}} \\
&= 4 \\
&= -4\left(|g(1)|^2 + 2\sum_{j>0} \frac{g(1)g(2^j)}{2^j}\right) \\
&= -4E_2(a),
\end{aligned}$$

so $G(2a) = -4G(a)$. Using properties (i) and (ii) we prove the third:

$$\begin{aligned}
\sum_{a\geq 1} \frac{G(a)}{a^2} &= \sum_{\substack{a\geq 1 \\ a \text{ even}}} \frac{G(a)}{a^2} + \sum_{\substack{a\geq 1 \\ a \text{ odd}}} \frac{G(a)}{a^2} \\
&= \sum_{\substack{a\geq 1 \\ a \text{ odd}}} \frac{G(a)}{a^2} + \frac{G(2a)}{(2a)^2} + \frac{G(4a)}{(4a)^2} + \ldots \\
&= \sum_{\substack{a\geq 1 \\ a \text{ odd}}} \frac{G(a)}{a^2} + \frac{G(2a)}{(2a)^2} \\
&= \sum_{\substack{a\geq 1 \\ a \text{ odd}}} \frac{G(a)}{a^2} - \frac{4G(a)}{4a^2} \\
&= 0.
\end{aligned}$$

In order to prove (iv) note that

$$\begin{aligned}
E_2(a) &= |g(1)|^2 + 2\sum_{j>0} \frac{g(1)g(2^j)}{2^j} \\
&= 1 + 2\frac{-2}{2} \\
&= -1.
\end{aligned}$$

As $f(3) = 1$ we have $g(3) = f(3)\mu(1)+f(1)\mu(3) = f(3)-1 = 0$, consequently if $(a,3) = 1$, then:

$$\begin{aligned} E_3(a) &= 1 + 2\sum_{j>0} \frac{g(3^j)}{3^j} \\ &= 1 + 2\sum_{j>1} \frac{g(3^j)}{3^j} \\ &\geq 1 - \frac{4}{9} \\ &> 0. \end{aligned}$$

By Lemma 4.4 we know that $E_p(a) > 0$ if $(p,a) > 1$ and that $E_p(a) > 0$ if $p > 3$, therefore result (iv) follows from $G(a) = \prod E_p(a)$ and from the fact that 2 is the only p such that $E_p(a) < 0$.

In order to prove result (v) we use Lemma 4.2 with $m = 0$:

$$\begin{aligned} 1 &= \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} 1 \\ &= \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} f(n)f(n+0) \\ &= \sum_{a|0} \frac{G(a)}{a} \\ &= \sum_{a\geq 1} \frac{G(a)}{a}. \end{aligned}$$

□

4.2 A Lower Bound for G

An important step in the proof of Erdős-Coons-Tao conjecture is to show that a series that depends on G is divergent. In this section we establish an important non trivial lower bound for $|G(a)|$ in terms of $|G(1)|$. So in order to prove that $|G(a)| \gg 1$ we will just have to show that $G(a) \neq 0$ and $|G(1)| \gg 1$.

Lemma 4.7. Suppose that $G(a) \neq 0$. Then:

$$|G(a)| \gg \left(\frac{4}{3}\right)^{\omega(a)} |G(1)|. \tag{4.1}$$

Proof. By Lemma 4.4 we have that $E_p(1) \le 1$ and $E_p(p^k) \ge \frac{4}{3}$, so we conclude that $\frac{4}{3}E_p(1) \le E_p(a)$. Using this result we have:

$$\begin{aligned}|G(a)| &= \left|\prod_{p^k||a,k\ge1} E_p(a)\right| \cdot \left|\prod_{p^k||a,k=0} E_p(1)\right| \\ &\ge \left(\frac{4}{3}\right)^{\omega(a)} \left|\prod_{p^k||a,k\ge1} E_p(1)\right| \cdot \left|\prod_{p^k||a,k=0} E_p(1)\right| \\ &= \left(\frac{4}{3}\right)^{\omega(a)} |G(1)|.\end{aligned}$$

□

Remark 4.8. The bound established by Klurman [11] was actually the following one:

$$|G(a)| \gg \left(\frac{5}{4}\right)^{\omega(a)-1} \frac{2}{5}|G(1)|.$$

The choice of Equation (4.1) instead of Klurman's bound was made due to its clearness. Klurman's way of estimating E_p was not the most tight, so he had to deal separately the case $p = 3$ and $p > 3$, as the $\frac{2}{5}$ term would create the inconvenience of $\left(\frac{2}{5}\right)^{\omega(a)}$ going to zero as ω increases.

4.3 Estimating the Second Moment

In this section we show an equality involving the second moment and some series. A key step in the proof of Theorem 1.14 is that this series must converge as the second moment of a multiplicative function $f : \mathbb{N} \to \{-1, 1\}$ with bounded partial sums is finite, therefore we would obtain a contradiction if we show this series actually diverges. We start by proving an auxiliary equality used in our proof.

Lemma 4.9. Let $H = ra + s$ with $0 \le s < a$. Then:

$$\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right) - 4\left(\left\{\frac{H}{2a}\right\} - \left\{\frac{H}{2a}\right\}^2\right) = -2\left\|\frac{H}{2a}\right\|.$$

Proof. We separate in two cases. First suppose r is even. Then:

$$\begin{aligned}\left(\left\{\frac{H}{a}\right\}-\left\{\frac{H}{a}\right\}^2\right)-4\left(\left\{\frac{H}{2a}\right\}-\left\{\frac{H}{2a}\right\}^2\right)&=\left(\left\{\frac{s}{a}\right\}-\left\{\frac{s}{a}\right\}^2\right)-4\left(\left\{\frac{s}{2a}\right\}-\left\{\frac{s}{2a}\right\}^2\right)\\&=\frac{s}{a}-\frac{s^2}{a^2}-4\left(\frac{s}{2a}-\frac{s^2}{4a^2}\right)\\&=-\frac{s}{a}\\&=-2\left\|\frac{s}{2a}\right\|\\&=-2\left\|\frac{H}{2a}\right\|.\end{aligned}$$

Now suppose r is odd. Then:

$$\begin{aligned}\left(\left\{\frac{H}{a}\right\}-\left\{\frac{H}{a}\right\}^2\right)-4\left(\left\{\frac{H}{2a}\right\}-\left\{\frac{H}{2a}\right\}^2\right)&=\left(\left\{\frac{s}{a}\right\}-\left\{\frac{s}{a}\right\}^2\right)-4\left(\left\{0.5+\frac{s}{2a}\right\}-\left\{0.5+\frac{s}{2a}\right\}^2\right)\\&=\left(\frac{s}{a}-\frac{s^2}{a^2}\right)-4\left(\left(0.5+\frac{s}{2a}\right)-\left(0.5+\frac{s}{2a}\right)^2\right)\\&=\frac{s}{a}-\frac{s^2}{a^2}-4\left(0.5+\frac{s}{2a}-0.5^2-\frac{s}{2a}-\frac{s^2}{4a^2}\right)\\&=\frac{s}{a}-1\\&=-2\left\|0.5+\frac{s}{2a}\right\|\\&=-2\left\|\frac{H}{2a}\right\|.\end{aligned}$$

□

Lemma 4.10. Let $H \in \mathbb{N}$. Then:

$$\frac{1}{x}\sum_{n\le x}\left(\sum_{k=n}^{n+H} f(k)\right)^2=-2\sum_{a\ge1, a \text{ odd}} G(a)\left\|\frac{H}{2a}\right\|+o_{x\to\infty}(1)$$

Proof. First, let us open the products in $\sum_{n\le x}\left(\sum_{k=n}^{n+H} f(k)\right)^2$ and collect similar terms. Whenever n satisfies $H \le n \le n+h \le x-H$, the term $f(n)f(n+h)$ appears $2(H-h)$ times. Whenever n satisfies $H \le n \le x-H$, the term $f(n)f(n+h)$ appears H times. We have $O(H^2)$ remaining terms to take into account, as each one of these terms are bounded by 1, and H does not depend on x, we see that the contribution of these terms is $O(1)$. Taking these observations into account we have:

$$\begin{aligned}
\frac{1}{x}\sum_{n\le x}\left(\sum_{k=n}^{n+H} f(k)\right)^2 &= \frac{1}{x}\left(\sum_{n\le x} Hf(n)f(n) + 2\sum_{n\le x}\sum_{1\le h\le H}(H-h)f(n)f(n+h)\right) + o(1)\\
&= \frac{1}{x}\left(\sum_{n\le x} Hf(n)f(n) + 2\sum_{n\le x}\sum_{1\le h\le H}(H-h)f(n)f(n+h)\right) + o(1)\\
&= \sum_{a\ge 1}\frac{G(a)}{a}\left(H + 2\sum_{1\le h\le H, a|h}(H-h)\right) + o(1) \qquad (4.2)
\end{aligned}$$

The final equality is a direct application of Lemma 4.2. Using Theorem 2.2, let us rewrite H as $H = ra + s$, where $0 \le s < a$. Then:

$$\begin{aligned}
H + 2\sum_{1\le h\le H, a|h}(H-h) &= ra + s + 2\sum_{1\le m\le r}(ra + s - ma)\\
&= ra + s + 2rs + ar(r-1)\\
&= \frac{(ra+s)^2}{a} + a\left(\frac{s}{a} - \left(\frac{s}{a}\right)^2\right)\\
&= \frac{(ra+s)^2}{a} + a\left(\left\{\frac{s}{a}\right\} - \left\{\frac{s}{a}\right\}^2\right)\\
&= \frac{H^2}{a} + a\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right). \qquad (4.3)
\end{aligned}$$

Substituting (4.3) in (4.2), using results (i)-(iii) of Lemma 4.6 and Lemma 4.9, we obtain:

$$
\begin{aligned}
&\frac{1}{x}\sum_{n\le x}\left(\sum_{k=n}^{n+H} f(k)\right)^2 \\
&= H^2\sum_{a\ge 1}\frac{G(a)}{a^2} + \sum_{a\ge 1} G(a)\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right) + o(1) \\
&= \sum_{a\ge 1} G(a)\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right) + o(1) \\
&= \sum_{\substack{a\ge 1\\ a \text{ odd}}} G(a)\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right) + \sum_{\substack{a\ge 1\\ a \text{ even}}} G(a)\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right) + o(1) \\
&= \sum_{\substack{a\ge 1\\ a \text{ odd}}} G(a)\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right) + \sum_{\substack{a\ge 1\\ a \text{ odd}}} G(2a)\left(\left\{\frac{H}{2a}\right\} - \left\{\frac{H}{2a}\right\}^2\right) + o(1) \\
&= \sum_{\substack{a\ge 1\\ a \text{ odd}}} G(a)\left[\left(\left\{\frac{H}{a}\right\} - \left\{\frac{H}{a}\right\}^2\right) - 4\left(\left\{\frac{H}{2a}\right\} - \left\{\frac{H}{2a}\right\}^2\right)\right] + o(1) \\
&= -2\sum_{\substack{a\ge 1\\ a \text{ odd}}} G(a)\left\|\frac{H}{2a}\right\| + o(1).
\end{aligned}
$$

□

The importance of the previous result in the proof of the Erdős-Coons-Tao conjecture comes in the form of the following corollary.

Corollary 4.11. If f has bounded partial sums. Then the following equality holds:

$$
-2\sum_{\substack{a\ge 1\\ a \text{ odd}}} G(a)\left\|\frac{H}{2a}\right\| = O_{H\to\infty}(1).
$$

Proof. Let $C = \sup_x |\sum_{n\le x} f(n)|$. Then:

$$
\begin{aligned}
\frac{1}{x}\sum_{n\le x}\left(\sum_{k=n}^{n+H} f(k)\right)^2 &\le \frac{1}{x}\sum_{n\le x}(2C)^2 \\
&= 4C^2.
\end{aligned}
$$

And the result follows by applying Lemma 4.10. □

4.4 Proof of Erdős-Coons-Tao Conjecture

In this section we prove Theorem 1.14, which is the main theorem of this dissertation.

Proof of Theorem 1.14. Suppose there exists m such that $f(n) = f(n+m)$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{m} f(n) = 0$. Then for all $x \in \mathbb{N}$:

$$\left|\sum_{n\leq x} f(n)\right| \leq m,$$

therefore f has bounded partial sums. Now suppose f has bounded partial sums. Then by Theorem 1.13 we have $f(2^k) = -1$ for all $k \geq 1$ and f is 1-pretentious. By Corollary 4.11 we have:

$$-2 \sum_{\substack{a\geq 1 \\ a \text{ odd}}} G(a) \left\|\frac{H}{2a}\right\| = O_{H\to\infty}(1). \tag{4.4}$$

Suppose that $g(n) = 0$ for all odd n except for some finite set. Let $a_1, \ldots, a_n$ be all odd indexes such that $g(a_i) \neq 0$ and suppose $a_k < a_{k+1}$ for all $k < n$. Then letting $M = 2a_n$ we have that $0 = g(p^k) = f(p^k) - f(p^{k-1})$ for all $p^k > M$. Applying Theorem 3.4 we conclude the theorem for this case.

Let $(a_n)_n$ be an increasing sequence containing all odd numbers such that $g(a_n) \neq 0$.

Now suppose that $f(3) = 1$. By Lemma 4.7 item (iv), we have that $g(3) = 0$, therefore, using the same technique of Lemma 4.4 we have:

$$\begin{aligned} E_3(1) &= 1 + 2\sum_{j\geq 1} \frac{g(3^j)}{3^j} \\ &= 1 + 2\sum_{j\geq 2} \frac{g(3^j)}{3^j} \\ &\geq 1 - \frac{4}{9} \\ &= \frac{5}{9}, \end{aligned}$$

so by Lemma 4.4, we conclude that $|G(1)| \gg 1$. By Lemma 4.7, we have that $|G(a_n)| \gg 1$. Let $H_M = lcm[a_1, a_2, \ldots, a_M]$. Observe that $\frac{H_M}{a_n}$ is odd, and therefore, $\left\|\frac{H_M}{2a_n}\right\| = \frac{1}{2}$. By

Lemma 4.6 item (iv) we have that each $G(a_n)$ is negative and, therefore as we are summing positive terms of the form $-G(a_n)$, we can drop some values to obtain a smaller sum:

$$\begin{aligned} -2 \sum_{a\geq 1, a \text{ odd}} G(a) \left\| \frac{H_M}{2a} \right\| &= -2 \sum_{n=1}^{\infty} G(a_n) \left\| \frac{H_M}{2a_n} \right\| \\ &\geq -2 \sum_{n=1}^{M} G(a_n) \left\| \frac{H_M}{2a_n} \right\| \\ &= 2 \sum_{n=1}^{M} |G(a_n)| \left\| \frac{H_M}{2a_n} \right\| \\ &= \sum_{n=1}^{M} |G(a_n)| \\ &\gg M. \end{aligned} \tag{4.5}$$

But we know by Equation (4.4) that the left hand side of equation 4.5 is bounded, so we have a contradiction for large M.

We are left with the case where $f(3) = -1$. First suppose that $E_3(1) = 0$. In this case $g(3^k) = (-1)^k$ and therefore $|G(3^k)| \gg 1$ for all $k > 0$. Now suppose that $E_3(1) > 0$. Then by Lemma 4.4, $|G(a_n)| \gg 1$ holds. In both cases we have infinitely many a_k such that $|G(a_k)| \gg 1$ and we can just follow the same steps made on the case $f(3) = 1$ to arrive on the same contradiction.

Now assume $E_3(1) < 0$. Let us define two auxiliary functions:

$$G^*(a) = \prod_{p^k || a, p>3} \left(|g(p^k)|^2 + 2 \sum_{i \geq k+1} \frac{g(p^k) g(p^i)}{p^{i-k}} \right)$$

and

$$S(H) = 2 \sum_{\substack{a \geq 1 \\ (a,6)=1}} G^*(a) \left\| \frac{H}{2a} \right\| .$$

Using that $E_p(a)$ depends only on the value of $p^k \mid\mid a$, we have that if $(b,p)=1$, then $E_p(a)=E_p(ab)$. Therefore:

$$\begin{aligned}
\sum_{i\geq 0} E_3(3^i) S\left(\frac{H}{3^i}\right) &= 2\sum_{i\geq 0} E_3(3^i) \sum_{\substack{a\geq 1\\(a,6)=1}} G^*(a) \left\|\frac{H}{2a3^i}\right\| \\
&= 2\sum_{i\geq 0}\sum_{\substack{a\geq 1\\(a,6)=1}} E_3(3^i) G^*(a) \left\|\frac{H}{2a3^i}\right\| \\
&= 2\sum_{i\geq 0}\sum_{\substack{a\geq 1\\(a,6)=1}} E_3(a3^i) G^*(a3^i) \left\|\frac{H}{2a3^i}\right\| \\
&= -2\sum_{\substack{a\geq 1\\ a \text{ odd}}} G(a) \left\|\frac{H}{2a}\right\| \qquad (4.6)
\end{aligned}$$

Combining Equations (4.4), (4.6) and using the observation that that S is non negative, $E_3(3^i)\geq 0$ for $i\geq 1$ and $E_3(1)<0$ we can arrive at:

$$\begin{aligned}
S(H) &= \frac{E_3(3)S(H/3)}{-E_3(1)} - \frac{1}{E_3(1)}\sum_{i=2}^{\infty} E_3(3^i) S\left(\frac{H}{3^i}\right) + O_{H\to\infty}(1) \\
&\geq \frac{E_3(3)S(H/3)}{-E_3(1)} + O_{H\to\infty}(1). \qquad (4.7)
\end{aligned}$$

By Lemma 4.4 we have:

$$E_3(3)\geq \frac{4}{3}$$

and

$$E_3(1)\geq -\frac{1}{3}.$$

Using these bounds on Equation (4.7) we obtain:

$$S(H)\geq 4S\left(\frac{H}{3}\right) + O_{H\to\infty}(1).$$

We can change the $O(1)$ term by a sufficiently large M obtaining:

$$S(H)\geq 4S\left(\frac{H}{3}\right) - M.$$

Suppose there are infinitely many numbers b such that $(b,6)=0$ and let $(b_n)_n$ be an increasing sequence containing all numbers b_n such that $(b_n,6)=1$ and $g(b_n)\neq 0$. Now define $H_0 = lcm[b_1, \ldots, b_K]$. Note that by Lemma 4.4, $|G^*(b_n)| \gg 1$. Therefore, for sufficiently large K, $S(H_0) \geq 2M$ holds. So:

$$\begin{aligned} S(3H_0) &\geq 4S(H_0) - M \\ &\geq 4S(H_0) - \frac{S(H_0)}{2} \\ &= \frac{7}{2}S(H_0). \end{aligned}$$

Now we use induction on n to conclude that for all $n \geq 1$ we have

$$S(3^n H_0) \geq \left(\frac{7}{2}\right)^n S(H_0).$$

Defining $H_n = 3^n H_0$ and taking $c > 0$ such that $3^{1+c} \leq \frac{7}{2}$. We have:

$$\begin{aligned} H_n^{1+c} &= (3^n H_0)^{1+c} \\ &\ll 3^{(1+c)n} \\ &\ll \left(\frac{7}{2}\right)^n \\ &\ll \left(\frac{7}{2}\right)^n S(H_0) \\ &\ll S(3^n H_0) \\ &= S(H_n). \end{aligned}$$

Now using the fact that $\sum_{n\geq 1} \frac{G(a)}{a}$ is absolutely convergent, we have:

$$\sum_{\substack{a \geq H \\ (a,6)=1}} \frac{G^*(a)}{a} = o_{H\to\infty}(1). \tag{4.8}$$

So:

$$
\begin{aligned}
S(H) &= 2 \sum_{\substack{a\geq 1\\(a,6)=1}} G^*(a) \left\| \frac{H}{2a} \right\| \\
&= 2 \sum_{\substack{a<H\\(a,6)=1}} G^*(a) \left\| \frac{H}{2a} \right\| + H \sum_{\substack{a\geq H\\(a,6)=1}} \frac{G^*(a)}{a} \\
&\ll \sum_{\substack{a<H\\(a,6)=1}} G^*(a) + H \sum_{\substack{a\geq H\\(a,6)=1}} \frac{G^*(a)}{a} \\
&\ll \sqrt{H} \sum_{\substack{a<\sqrt{H}\\(a,6)=1}} \frac{G^*(a)}{a} + H \sum_{\substack{\sqrt{H}\leq a<H\\(a,6)=1}} \frac{G^*(a)}{a} + H \sum_{\substack{a\geq H\\(a,6)=1}} \frac{G^*(a)}{a} \\
&\ll \sqrt{H} \log \sqrt{H} + H \sum_{\substack{a\geq \sqrt{H}\\(a,6)=1}} \frac{G^*(a)}{a} \\
&= o(H).
\end{aligned}
$$

From $S(H_n) \gg H_n^{1+c}$ there exist constants k and n_0 such that $n > n_0$ implies $H_n^{1+c} \leq kS(H_n)$, and from $S(H) = o(H)$ we have that, for H large enough, $S(H) < H/k$ holds. So we have:

$$
H_n^{1+c} \leq kS(H_n) < H_n.
$$

But this is clearly a contradiction. Therefore we are left with the case where $b_1, b_2, \ldots b_l$ are all indices b_i such that $(b_i, 6) = 1$ and $g(b_i) \neq 0$. We may assume without loss of generality that $b_i < b_{i+1}$ for all $i < l$. Note that there exists at least one such element b_i as g being multiplicative implies $g(1) = 1$.

Let U be the set of integers k such that $g(3^k) \neq 0$. Then, $V = \{3^u \cdot b_i : u \in U, 1 \leq i \leq l\}$ is exactly the set of odd integers m for which $g(m) \neq 0$. In other words, the sequence $(a_n)_n$ and the set V contains the same elements. Note that $|V| = l|U|$, however by the existence of (a_n), we know that $|V|$ is infinite, we conclude that $|U|$ is infinite, i.e, $g(3^k) \neq 0$, for infinitely many integers k.

Define $H_0 = lcm[b_1, b_2, \ldots, b_l]$. Observe that $\frac{H_0}{b_i}$ is odd, and therefore, $\left\|\frac{H_0}{2b_i}\right\| = \frac{1}{2}$. Therefore:

$$\begin{aligned} S(3^i H_0) &= 2 \sum_{(a,6)=1} G^*(a) \left\| \frac{3^i H_0}{2a} \right\| \\ &= 2 \sum_{j=1}^{l} G^*(b_j) \left\| \frac{3^i H_0}{2b_j} \right\| \\ &= \sum_{j=1}^{l} G^*(b_j). \end{aligned}$$

Therefore, defining $M = \sum_{j=1}^{l} G^*(b_j)$. We have:

$$\begin{aligned} -2 \sum_{\substack{a \geq 1 \\ a \text{ odd}}} G(a) \left\| \frac{3^K H_0}{2a} \right\| &= \sum_{i \geq 0} E_3(3^i) S\left(\frac{3^K H_0}{3^i} \right) \\ &\geq \sum_{i \leq K} E_3(3^i) S\left(3^{K-i} H_0 \right) \\ &= M \sum_{i \leq K} E_3(3^i). \end{aligned}$$

Now observe that if $g(3^k) \neq 0$ for $k \geq 1$, then $E_3(3^k) \geq 1$, as $g(3^k) \neq 0$ for infinitely many values of k. Taking K large enough makes the last sum unbounded, but this contradicts Equation (4.4). □

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