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Complete Intersection Curves in Biprojective Spaces

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Orientador: Prof. Dr. André Luís Contiero

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ATA DA CENTÉSIMA NONAGÉSIMA SEXTA DEFESA DE TESE DE DOUTORADO DO ALUNO MAXWELL DA PAIXÃO DE JESUS SANTOS, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA DO INSTITUTO DE CIÊNCIAS EXATAS DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA DIA 02 DE MARÇO DE 2023.

Aos dois dias do mês de março de 2023, às 10h00, em reunião pública virtual na Plataforma Microsoft Teams pelo link: https://teams.microsoft.com/l/meetup-join/19%3ameeting_OWE0NDQ5M2MtMTM3NS00NjZhLTlmMzEtMGNiNGUzOWI5OWRj%40thread.v2/0?context=%7b%22Tid%22%3a%2264126139-4352-4cd7-b1fb-2a971c6f69a6%22%2c%22Oid%22%3a%2234ff42fb-a77c-4e6d-a89f-7661dc6efadc%22%7d (conforme mensagem eletrônica da Pró-Reitoria de Pós-Graduação de 26/03/2020, com orientações para a atividade de defesa de tese durante a vigência da Portaria nº 1819), reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese do aluno **Maxwell da Paixão de Jesus Santos**, intitulada: *"Complete Intersection Curves in Biprojective Spaces"*, requisito final para obtenção do Grau de Doutor em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. André Luis Contiero, após dar conhecimento aos presentes do teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se reservadamente, sem a presença do aluno, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 02 de março de 2023.

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FOLHA DE APROVAÇÃO

Complete Intersection Curves in Biprojective Spaces

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Resumo

Nessa Tese são classificadas todas as curvas que são interseção completas em espaços biprojetivos cujo feixe canônico é uma seção hiperplana. Em seguida, estuda-se a geometria destas curvas no que diz respeito à sua gonalidade e existência de sistema linear. Será estabelicida uma cota inferior para a gonalidade de certas curvas que são interseção completas em espaços biprojetivos, seguindo os passos de Lazarsfeld para gonalidade de curvas de interseção completas em espaços projetivos. Também são fornecidos alguns resultados sobre a estratificação de Mukai do moduli de curvas de gênero pequeno.

Palavras chaves: Curvas, Intersecções completas, Sistemas lineares & Gonalidade

Abstract

In this Thesis are classified all complete intersection curves in biprojective spaces whose canonical sheaf is a hyperplane section. Then it is studied the geometry of these curves with respect to their gonality and existence of linear system. A lower bound for the gonality of suitable complete intersection curves in biprojective spaces is provided, given biproduct version of Lazarsfeld's lower bound for gonality of complete intersection curves in projective spaces. It also provides some results concerning on Mukai stratification of the moduli of curves of small genus.

Keywords: Complete intersection curves, Linear systems & Gonality

Contents

1 | Introduction

Let $\mathcal{C}$ be a smooth complex projective curve. The *gonality* of $\mathcal{C}$ is the minimum degree of a surjective morphism from $\mathcal{C}$ to the projective line,

$$\operatorname{gon}(\mathcal{C}) := \min\{k \geq 1 \,|\, C \xrightarrow{\text{k:1}} \mathbb{P}^1\}.$$

It is an invariant of a curve and measures how far the curve is from being rational. For example, $\operatorname{gon}(\mathcal{C}) = 1$ if, and only if, $\mathcal{C}$ is a rational curve. We usually say that a $k-$gonal curve is a curve whose gonality is k. A $2-$gonal curve is a hyperelliptic curve.

There is significant interest in computing the gonality of various classes of curves. The most general upper bound for the gonality of a smooth curve $\mathcal{C}$ of genus g is derived from Brill–Noether theory [ACGH85], that is

$$\operatorname{gon}(\mathcal{C}) \leq \left\lfloor \frac{g+3}{2} \right\rfloor.$$

It is well known that a projective plane curve of degree d has gonality exactly $d-1$, and the morphism that computes the gonality is obtained by a projection from a point on $\mathcal{C}$, as predicted by the classical Noether's theorem, see Example 2.2.1 of this Thesis. In this direction, Basili [Ba96] shows that the gonality of a smooth complete intersection space curve $\mathcal{C} \subset \mathbb{P}^3$ is also computed by projection from a line.

Concerning complete intersection smooth curves, Lazarsfeld, c.f. [Laz97, Exercise 4.12], provides his famous lower bound.

Theorem 1.0.1 (Lazarsfeld)**.** *If $\mathcal{C} \subset \mathbb{P}^n$ is a complete intersection smooth curve given by the intersection of $n-1$ hypersurfaces of degrees a_i, $i = 1, \ldots, n-1$ with $2 \leq a_1 \leq a_2 \leq a_{n-1}$, then*

$$\operatorname{gon}(\mathcal{C}) \geq (a_1 - 1)a_2 \cdots a_{n-1}.$$

A way to define the gonality of a possibly singular irreducible projective curve $\mathcal{C}$ is to declare it to be the gonality of its normalization $\tilde{\mathcal{C}}$, as in [HLU20]. Alternatively, and the definition that we assume in this Thesis, it is the smallest k for which there exists a $\mathfrak{g}^1_k$ on $\mathcal{C}$, i.e. a torsion-free sheaf $\mathcal{F}$ of rank 1 on $\mathcal{C}$ of degree k and with $\dim \mathrm{H}^0(\mathcal{C}, \mathcal{F}) \geq 2$. The notion of linear systems on singular curves is characterized by interchanging line bundles by torsion-free sheaves of rank 1. Note that *non-removable base points* are allowed.

Regarding singular curves, Hartshorne and Schlesinger [HS11] generalized Basili's results to ACM curves that satisfy some assumption of generality. More recently, Hotchkiss, Ching Lau, and Ullery [HLU20] study morphisms from complete intersection curves $\mathcal{C} \subset \mathbb{P}^n$, not necessarily smooth, to a projective space $\mathbb{P}^r$ that are given by a projection from a linear subspace, their main result is the following.

Theorem 1.0.2 (Hotchkiss, Ching Lau and Ullery)**.** *Let $\mathcal{C} \subset \mathbb{P}^n$ be a complete intersection curve of type $(a_1, \dots, a_{n-1})$ with*

$$4 \leq a_1 \leq \cdots \leq a_{n-1}.$$

For each $1 \leq r < n$, any morphism $f : \mathcal{C} \to \mathbb{P}^r$ such that

$$\deg f^* \mathcal{O}_{\mathbb{P}^r}(1) < \deg \mathcal{C}$$

is given by the projection of a $(n-r-1)$-plane. Thus $\operatorname{gon}(\mathcal{C}) = \deg(\mathcal{C}) - \gamma$, where γ is the maximum number of points on $\mathcal{C}$ contained in an $(n-2)$-plane.

Concerning Deligne–Munford stable curves, Coelho and Sercio [CoSe21] established the following result.

Theorem 1.0.3 (Coelho and Sercio)**.** *Let $\mathcal{C}$ be a stable curve.*

1. *If $\mathcal{C}$ is irreducible with δ nodes and its normalization $\tilde{\mathcal{C}}$ admits a morphism of degree $\tilde{k}$ to $\mathbb{P}^1$, then $\mathcal{C}$ admits a morphism to $\mathbb{P}^1$ of some degree k such that*

$$\tilde{k} < k < \tilde{k} + \delta.$$

2. *If $\mathcal{C}_i$, $i = 1 \dots, \ell$ are the irreducible components of the stable curve $\mathcal{C}$ and each $\mathcal{C}_i$ admits a morphism $C_i \to \mathbb{P}^1$ of degree k_i, then there is a morphism $\mathcal{C} \to \mathbb{P}^1$ of degree k, for some k satisfying*

$$k_1 + \cdots + k_\ell - \delta < k < k_1 + \cdots + k_\ell + \delta - 2(\ell - 1).$$

where δ is the number of external nodes of $\mathcal{C}$.

The importance of the gonality can also be noticed in understanding the ambient space where a curve can be embedded. On one hand, it is well known that a non-hyperelliptic smooth curve of genus $g > 2$ can be embedded by its canonical sheaf as a degree $2g - 2$ curve in $\mathbb{P}^{g-1}$. On the other hand, a classical result due to Bertini, c.f. [Sch86, Thm. 2.5], assures that any smooth k-gonal curve embeds in a $(k-1)$-fold normal scroll. More generally, an irreducible Gorenstein curve $\mathcal{C}$ can also be embedded as a canonical curve in $\mathbb{P}^{g-1}$ by its dualizing sheaf, and in a $(k-1)-$fold scroll as well, where k stands for the gonality of $\mathcal{C}$. In this way, in [LMS19] the authors show that the canonical model, in the sense of [KM09], of a singular irreducible non-Gorenstein curve embeds in a suitable $(k-1)$-fold scroll, where k is the gonality of $\mathcal{C}$. More recently, Contiero, Fontes, and Teles generalize a result due to Schlessinger [Sch86], by showing that any tetragonal Gorenstein curve $\mathcal{C}$ of genus $g \geq 6$ is a complete intersection in its associated rational normal $3-$fold scroll, see [CFT22, Thm. 3.1].

An application of the notion of gonality is observed in the theory of moduli space of curves as follows. Let us first consider a filtration of the moduli space $\mathcal{M}_g$ of smooth curves of genus $g \geq 3$, namely

$$\mathcal{H}_g := \mathcal{M}_g(2) \subseteq \mathcal{M}_g(3) \subseteq \ldots \subseteq \mathcal{M}_g\left(\lfloor (g+3)/2 \rfloor\right) = \mathcal{M}_g,$$

where $\mathcal{M}_g(k)$ is the space of curves admitting a $\mathfrak{g}^1_k$ and has dimension $2g + 2k - 5$. Note that a curve $\mathcal{C}$ has gonality k, if it belongs to $\mathcal{M}_g(k) \setminus \mathcal{M}_g(k-1)$. The space $\mathcal{H}_g$ is the space of hyperelliptic curves. The space of bielliptic curves $\mathcal{B}_g$ is the space of curves admitting a map of degree 2 onto an elliptic curve, i.e. onto a curve of genus 1.

In a sequence of celebrate papers, Mukai [Muk92, Muk95] made a deep study of canonical models of the non-hyperelliptic smooth curve of genus 7, providing a stratification of the space of smooth curves with fixed gonality, giving necessary conditions for a smooth curve to be in a suitable stratum. Next, we summarize the results presented in [Muk95] that are relevant to this Thesis.

Theorem 1.0.4 (Mukai). *Let C be a smooth curve of genus 7.*

2-gonal: *If C is hyperelliptic, then it is given by the zero locus of an isobaric form of degree 16 in the weighted projective plane $\mathbb{P}(1:1:8)$.*

3-gonal: *If C is trigonal*

3.1) and admits a unique $\mathfrak{g}^2_6$, then its canonical model is a hypersurface of degree 9 in the weighted projective space $\mathbb{P}(1:1:3)$.

3.2) and admits two $\mathfrak{g}^2_6$, then its canonical model is a complete intersection of two divisors of bidegrees $(1,1)$ and $(3,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$.

4-gonal: *If C is tetragonal curve*

4.1) and bielliptic, then its canonical model is a complete intersection of two divisors of degrees 3 and 4 in the weighted projective space $\mathbb{P}(1:1:1:2)$. Moreover, C has infinitely many $\mathfrak{g}^2_6$ and $\mathfrak{g}^1_4$.

4.2) and non-bielliptic with a unique $\mathfrak{g}^2_6$, then it is a complete intersection of two divisors of degrees 3 and 4 in the weighted projective space $\mathbb{P}(1:1:1:2)$.

4.3) and admits two $\mathfrak{g}^2_6$, then the canonical model of C is a complete intersection of three divisors of bidegrees $(1,1)$, $(1,1)$ and $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

4.4) having no $\mathfrak{g}^2_6$ with a unique $\mathfrak{g}^1_4$, then its canonical model is a complete intersection of three divisors of bidegrees $(1,1)$, $(1,2)$ and $(1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^3$.

5-gonal: *(Main Theorem, [Muk95]). A smooth curve of genus 7 is a transversal linear section of the orthonormal Grassmannian $X \subset \mathbb{P}^{15}$ of dimension 10 if, and only if, C has no $\mathfrak{g}^1_4$. Moreover, the transversal linear subspaces that cut out C are unique up to the action of* $\mathrm{SO}(10)$.

In a later paper [MukId03], S. Mukai & M. Ide studied the canonical models of non-hyperelliptic curves of genus 8, providing a stratification of $\mathcal{M}_8$ in terms of the gonality and the existence of suitable linear systems on C.

Theorem 1.0.5 (Mukai–Ide)**.** *Let C be a smooth curve of genus eight.*

2-gonal: *If C is hyperelliptic, then it is given by an isobaric form of degree 18 in the weighted projective plane $\mathbb{P}(1:1:9)$.*

3-gonal: *If C is trigonal, then it is a complete intersection in its respective 2-fold rational normal scroll, that is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$.*

4-goal: *If C is a tetragonal curve*

4.1) and bielliptic, then it is a complete intersection of the two divisors $\overline{S}$ and $\pi^{-1}(E_7)$ in $\mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-K_S))$. Here S is a del Pezzo surface in $\mathbb{P}^7$ containing an elliptic curve E_7, $\pi : \overline{S} \to S$ is a double covering with $E_7 \sim -K_S$ and $C \to E_7$ is also a double covering.

4.2) non-bielliptic and admitting a unique g^2_6, then C is a complete intersection given by the two divisors $-\frac{1}{2}K_{\mathcal{X}}$ and $-K_{\mathcal{X}}$ in the blowup $\mathcal{X} := B_q\mathbb{P}^2$ of $\mathbb{P}^2$ at *a point.*

4.3) not admitting a g^2_6, then C is a complete intersection of four divisors of type $(1,1)$, $(1,1)$, $(1,2)$ and $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^4$.

5-gonal: *If C is a pentagonal curve*

5.1) admitting a self adjoint g^2_7, then C is a complete intersection of a weighted projective Grassmannian $G(2,5)$ and a weighted projective space $\mathbb{P}(1^3 : 2^2)$.

5.2) with a non self adjoint g^2_7, then it is a complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$ of the two divisors $(1,1)$, $(1,2)$ and $(2,1)$.

5.3) does not admitting a g^2_7 if, and only if, there is a rank 2 bundle E over C, generated by its global sections, such that $\det(E) = \omega_C$ and C is a transversal section of the Grassmannian $G(2,V) \subset \mathbb{P}(\wedge^2 V)$.

It can be claimed that this Ph.D. Thesis was firstly inspired by the works of Mukai [Muk95] and Mukai–Ide [MukId03], followed by the lower bound given by Lazarsfeld [Laz97]. At the same time by the work due to Penev and Vakil [PeVa15], where the authors parameterize each of Mukai's stratum of $\mathcal{M}_6$ and then, through a theorem due to Vistoli [Vis87], show that the Chow ring with coefficients in $\mathbb{Q}$ of each stratum inside $\mathcal{M}_6$ is tautological. Hence the rational Chow ring of $\mathcal{M}_6$ is tautological as well. Therefore, the Mukai and Mukai–Ide stratification of $\mathcal{M}_7$ and $\mathcal{M}_8$ can be relevant to the intersection theory of moduli spaces of curves.

It is simple to conclude that a canonical curve of genus $g \geq 6$ is not a complete intersection in $\mathbb{P}^{g-1}$. On the other hand, an important feature of Mukai and Mukai–Ide results lies in the fact that canonical curves, with a given gonality, are realized as complete intersections in suitable smooth projective varieties, especially in the product of projective spaces $\mathbb{P}^r \times \mathbb{P}^s$. The way the authors show their results is to take a $\mathfrak{g}^r_d$ on a curve $\mathcal{C}$, given by a divisor D on $\mathcal{C}$ and consider its Serre dual $\mathrm{K}_{\mathcal{C}} - D$. Then they show that $\mathcal{C}$ can be embedded in $\mathbb{P}^r \times \mathbb{P}^s$, where $s = h^0(\mathcal{C}, \mathrm{K}_{\mathcal{C}} - D)$, and note that the canonical sheaf $\mathrm{K}_{\mathcal{C}}$ is always a hyperplane section on $\mathbb{P}^r \times \mathbb{P}^s$. Naturally, the following questions arise.

- Given a (big enough) positive integer g. What are the genus g complete intersections curves in biprojective spaces whose canonical sheaves are hyperplane sections?
- What is the behavior of the gonality on complete intersection curves in $\mathbb{P}^n \times \mathbb{P}^m$?

This thesis is developed focusing on the above two questions. In the first section of Chapter 2 we present some preliminary concepts on divisors, fixing some notation. The Lefschetz hyperplane theorem 2.1.6 plays a central role to compute the Picard group of some varieties. In Section 2.2 we review linear systems on curves with special attention to the gonality of curves. We recall the Brill–Noether theory and derive the well-known upper bound for gonality, c.f. Corollary 2.2.5. The Castelnuovo–Severi inequality 2.2.2 is also recalled because we used it a lot of times in this Thesis. At the end of this section, we present the Lazarsfeld lower bound for the gonality of a complete intersection curve in $\mathbb{P}^n$, c.f. Theorem 2.2.6. In Section 2.3 of Chapter 2 we recall some required results on the theory of bundles over varieties, in special over 3-folds. In particular, we recall the Nakai–Moishezon criterion for ampleness, Theorem 2.3.2, Bogomolov instability theorem 2.3.4, and a generalization due to Miyaoka of Bogomolov instability, c.f. Theorem 2.3.5.

Chapter 3 is the main chapter of this Thesis, where we give a partial answer to the above two questions. In Section 3.1 we address our study to complete intersection curves in biprojective spaces, i.e. a smooth curve $\mathcal{C} \subset \mathbb{P}^n \times \mathbb{P}^m$ given by $n+m-1$ divisors $D_i \in |\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^m}(a_i, b_i)|$, and such that its canonical sheaf is a hyperplane section. Due to the restriction on the canonical sheaf, we can provide that there is only a finite number of classes of smooth complete curves in $\mathbb{P}^n \times \mathbb{P}^m$, namely Theorem 3.1.1, that is

Theorem. *If C is a complete intersection smooth curve in $\mathbb{P}^n \times \mathbb{P}^m$ of genus g, with $n+m \geq 3$ and whose canonical sheaf is $\mathrm{K}_C = \mathcal{O}_C(1,1)$, then $m+n \leq 6$, $5 \leq g \leq 11$ and $\mathcal{C}$ can be one of twenty one cases presented in Table 1.1.*

The remaining Section 3.1 is devoted to studying the geometry of the curves in Theorem 3.1.1 with respect to gonality and the existence of suitable linear systems. Among the results we mention the following:

Lemma. *Let $\mathcal{C} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a smooth complete intersection of genus $g \geq 5$ with multi-degree $(d_1, \ldots, d_k)$. If $\mathcal{C}$ is trigonal, then there is some $i \in \{1, \ldots, k\}$ such that $d_i \equiv 0 \mod 3$ and $d_i \geq 3\, n_i$.*

Lemma. *Let $\mathcal{C} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a bielliptic smooth curve of multi-degree $(d_1, \ldots, d_k)$. If some $n_i \geq 2$, then d_i is even and $d_i \geq 2(n_i + 1)$.*

Theorem. *There is no embedding $u = (u_1, u_2) : \mathcal{C} \to \mathbb{P}^1 \times \mathbb{P}^4$ of a bielliptic curve of genus 8 such that $K_{\mathcal{C}} \cong u^*(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^4}(1,1))$, $u_1(\mathcal{C})$ spans a $\mathbb{P}^1$ and $u_2(\mathcal{C})$ spans $\mathbb{P}^4$.*

Several Remarks are also made on such curves in $\mathbb{P}^n \times \mathbb{P}^m$. The results of Section 3.1 are summarized in Table 1.2.

In Section 3.2 of Chapter 3 we provide a version of Lazarsfeld's Theorem 2.2.6 to complete intersection curves in the product of two projective spaces, whose proof is inspired by that

#	g	divisors	ambient	bidegree of $\mathcal{C}$
#1	5	$(2,3)$, $(1,1)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(3,5)$
#2	6	$(2,1)$, $(1,3)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(3,7)$
#3	6	$(2,2)$, $(1,2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(4,6)$
#4	6	two $(1,1)$, one $(1,3)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(3,7)$
#5	7	$(1,1)$, $(2,1)$, $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(3,9)$
#6	7	$(1,1)$, two $(1,2)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(4,8)$
#7	7	two $(1,1)$, $(2,1)$, $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(3,12)$
#8	7	three $(1,1)$, one $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(3,9)$
#9	7	$(2,2)$, $(1,1)$, $(1,1)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(6,6)$
#10	8	$(1,1)$, $(2,0)$, $(1,3)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(6,8)$
#11	8	$(1,1)$, $(1,1)$, $(1,2)$, $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(4,10)$
#12	8	$(2,1)$, $(1,2)$, $(1,1)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(7,7)$
#13	9	$(1,1)$, $(2,1)$ $(0,2)$, $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(4,12)$
#14	9	$(2,0)$, $(0,2)$, $(2,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(8,8)$
#15	9	$(2,0)$, $(1,2)$, $(1,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(8,8)$
#16	9	three $(1,1)$, two $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^5$	(4,12)
#17	9	$(1,2)$, $(1,1)$, $(1,1)$, $(1,1)$	$\mathbb{P}^2 \times \mathbb{P}^3$	$(7,9)$
#18	10	$(2,0)$, $(1,1)$, $(1,1)$, $(0,3)$	$\mathbb{P}^2 \times \mathbb{P}^3$	(6,12)
#19	11	four $(1,1)$, one $(0,2)$	$\mathbb{P}^2 \times \mathbb{P}^4$	(8,12)
#20	11	five $(1,1)$	$\mathbb{P}^3 \times \mathbb{P}^3$	(10,10)
#21	6	$(0,2)$, $(3,2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(4,6)$

Table 1.1: CI curves in $\mathbb{P}^n \times \mathbb{P}^m$ with $\mathrm{K}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(1,1)$

#	g	divisors	ambient	bidegree of $\mathcal{C}$	gonality	obs
#1	5	$(1,1)$, $(2,3)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(3,5)$	3	
#2	6	$(2,1)$, $(1,3)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(3,7)$	3	
#3	6	$(1,2)$, $(2,2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(4,6)$	?	$\mathrm{gon}(\mathcal{C}) \in \{3,4\}$
#4	6	two $(1,1)$, one $(1,3)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(3,7)$	3	
#5	7	$(1,1)$, $(2,1)$, $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(3,9)$	3	
#6	7	$(1,1)$, two $(1,2)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(4,8)$	4	
#7	7	two $(1,1)$, $(2,1)$, $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(3,12)$	3	
#8	7	three $(1,1)$, one $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(3,9)$	3	
#9	7	$(1,1)$, $(1,1)$, $(2,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(6,6)$	?	$\mathrm{gon}(\mathcal{C}) \in \{3,4\}$
#10	8	$(1,1)$, $(2,0)$, $(1,3)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(6,8)$	3	
#11	8	$(1,1)$, $(1,1)$, $(1,2)$, $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(4,10)$	4	non bielliptic
#12	8	$(1,1)$, $(2,1)$, $(1,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(7,7)$	?	$\exists\, \mathfrak{g}^2_7,\ \alpha^{\otimes 2} \neq \mathrm{K}_{\mathcal{C}}$
#13	9	$(1,1)$, $(2,1)$ $(0,2)$, $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(4,12)$	4	
#14	9	$(2,0)$, $(0,2)$, $(2,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(8,8)$	4	bielliptic, two $\mathfrak{g}^1_4$
#15	9	$(2,0)$, $(1,2)$, $(1,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(8,8)$	4	
#16	9	three $(1,1)$, two $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^5$	$(4,12)$	4	
#17	9	$(1,1)$, $(1,1)$, $(1,1)$, $(1,2)$	$\mathbb{P}^2 \times \mathbb{P}^3$	$(7,9)$	?	
#18	10	$(1,1)$, $(1,1)$, $(0,3)$, $(2,0)$	$\mathbb{P}^2 \times \mathbb{P}^3$	$(6,12)$	3	
#19	11	four $(1,1)$, one $(0,2)$	$\mathbb{P}^2 \times \mathbb{P}^4$	$(8,12)$	?	
#20	11	five $(1,1)$	$\mathbb{P}^3 \times \mathbb{P}^3$	$(10,10)$	5	one or two $\mathfrak{g}^1_5$
#21	6	$(0,2)$, $(3,2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(4,6)$	3	

Table 1.2: CI curves in $\mathbb{P}^n \times \mathbb{P}^m$ with $\mathrm{K}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(1,1)$

presented in [Laz97], using Miyaoka's Theorem on rank two vector bundles over 3-folds and Nakai–Moichezon's criterion for ampleness. The two main results of this section are the following

Theorem. *Let $\mathcal{C} \subset \mathbb{P}^n \times \mathbb{P}^m$ be a complete intersection smooth curve given by divisors of bidegrees (a_i, b_i), $i = 1, \ldots, n+m-1$. Let us assume that $(a_1, b_1) = (1,1)$, $a_2 b_2 > 0$ and that the 3-fold Y_0 given by the last $n+m-3$ divisors defining $\mathcal{C}$ is such $\mathrm{Pic}(Y_0) = \mathbb{Z} \times \mathbb{Z}$. Picking up $\kappa \in \mathbb{Q}_{\geq 0}$ such that $2H_1H_2 \leq H_1^2 + H_2^2 + 4\kappa$, we obtain*

$$gon(C) \geq H_1H_2 - \kappa,$$

provided that

- $a_2 = b_2$ *and* $H_1^2 + H_2^2 \leq 4H_1H_2$ *or*
- $a_2 \neq b_2$ *and* $(a_2 - 1)H_1^2 + (b_2 - 1)H_2^2 \leq (min\{3a_2 + b_2 - 4, 3b_2 + a_2 - 4\})H_1H_2$,

where H_1 and H_2 are hyperplane sections on a surface $S \subset Y_0$ given by a divisor in $|\mathcal{O}_{Y_0}(a_2, b_2)|$.

In addition, we also establish the following result:

Theorem. *Let $\mathcal{C} \subset \mathbb{P}^1 \times \mathbb{P}^m$ be a complete intersection smooth curve given by divisors of bidegrees (a_i, b_i), $i = 1, \ldots, n+m-1$. Assume that $(a_1, b_1) = (0, 2)$, $a_2 \geq 0$, $b_2 \geq 2$ and that the 3-fold Y_0 given by the last $n+m-3$ divisors defining $\mathcal{C}$ is such $\mathrm{Pic}(Y_0) = \mathbb{Z} \times \mathbb{Z}$. Let $X \subset Y_0$ cutting the divisor of bidegree (a_2, b_2) and fix the hyperplane sections H_1 and H_2 on X. If $\kappa \in \mathbb{Q}_{\geq 0}$ is such that $H_1H_2 \leq H_2^2 + \kappa$, then*

$$gon(C) \geq H_1H_2 - \kappa,$$

where H_1 and H_2 are hyperplane sections on a surface $S \subset Y_0$ given by a divisor in $|\mathcal{O}_{Y_0}(a_2, b_2)|$. In addition, if $b_2 \leq a_2$, then we can take κ equals to zero.

In section 3.3 of Chapter 3 we provide a few results on the locus of complete intersection smooth curves in the biprojective spaces with fixed genus and prescribed bidegrees. The main results of this section are the following.

Theorem. *A general trigonal curve of genus 10 is a complete intersection smooth curve given by the divisors $(2,0)$, $(1,1)$, $(1,1)$ and $(0,3)$ on $\mathbb{P}^2 \times \mathbb{P}^3$, as in case #18 of Table 1.1.*

Theorem. *A general trigonal curve of genus 8 is a complete intersection smooth curve*

#	g	divisors	ambient	bidegree of $\mathcal{C}$	gonality	$\lfloor \frac{g+3}{2} \rfloor$
#9	7	$(1,1)$, $(1,1)$, $(2,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(6,6)$	3 or 4	5
#12	8	$(1,1)$, $(2,1)$, $(1,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(7,7)$	4 or 5	5
#17	9	$(1,1)$, $(1,1)$, $(1,1)$, $(1,2)$	$\mathbb{P}^2 \times \mathbb{P}^3$	$(7,9)$	4 or 5	6
#19	11	four $(1,1)$, one $(0,2)$	$\mathbb{P}^2 \times \mathbb{P}^4$	$(8,12)$	5 or 6	7

Table 1.3: CI curves in $\mathbb{P}^n \times \mathbb{P}^m$ with $\mathrm{K}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(1,1)$

given by the divisors $(2,0)$, $(1,1)$ *and* $(1,3)$ *on* $\mathbb{P}^2 \times \mathbb{P}^2$, *case* #10 *in Table 1.1.*

Theorem. *A general element of* $\mathbb{M}_8^{(7,7)}$ *has gonality* 5 *and at least eight* $\mathfrak{g}^1_5$, *where* $\mathbb{M}_g^{(d_1,d_2)}$ *is the space of all smooth complete intersections curves in the biprojective space of genus* g *and bidegree* (d_1,d_2),

Due to the results of Sections 3.2 and 3.3, we can fill in some additional information on the four missing cases in Table 1.2, that are summarized in Table 1.3.

In Chapter 4 we provide some results concerning Mukai's stratification of genus 8 curves. The results presented are far from being a final study, it can be seen as the starting point to further works. We establish the following results.

Theorem. *Let* V *be the blowup of* $\mathbb{P}^3$ *at a point and* K_V *its the canonical sheaf. Assume that* $\mathcal{C}$ *is a complete intersection smooth curve given by* $-\frac{1}{2}\mathrm{K}_V$ *and* $-\mathrm{K}_V$. *Thus* $\mathcal{C}$ *has genus 8, does not admit any* $\mathfrak{g}^1_3$ *and every* $\mathfrak{g}^2_6$ *is linearly equivalent to* $\mathcal{O}_C(H-E)$. *In particular,* $\mathcal{C}$ *is tetragonal with a unique* $\mathfrak{g}^2_6$.

The last result of this Thesis is the following. Let V be a vector space of dimension 6. Consider the Grassmanian $G(2,V)$ embedded via Plücker in $\mathbb{P}(\wedge^2 V)$. The space of 7 of linear sections is parameterized by $G(7,\wedge^2 V)$. In the same way that $PGL(V)$ acts on $G(2,V)$, it determines an action on $G(7,\wedge^2 V)$.

Theorem. *The natural map*

$$\phi : (G(7,\wedge^2 V) \setminus \Delta)/PGL(V) \longrightarrow \mathcal{M}_8$$

is an open immersion (of Deligne-Mumford Stacks) whose image is $\mathcal{M}_8^{M_u}$. *Here* Δ *is a divisor of* $G(7,\wedge^2 V)$ *corresponding to the singular locus. Here* $\mathcal{M}_8^{Mu}$ *stands for the subspace of* $\mathcal{M}_8$ *consisting of pentagonal curves without a* $\mathfrak{g}^2_7$.

In Chapter 5 we give some possible directions and problems for future developments.

2 | Preliminaries

Throughout this Thesis, we work over the field of complex numbers $\mathbb{C}$. Varieties are assumed to be smooth and projective unless otherwise stated. In this chapter, we recall the objects and results that are required for a better understanding.

2.1. Divisor and Picard Groups

Let X be a smooth projective scheme. We recall that a *prime divisor* on X is a closed integral subscheme Y of codimension one. A *divisor* is an element of the free abelian group Div$X$ generated by the prime divisors. An element of DivX is usually denoted by

$$D = \sum n_i D_i,$$

where D_i are prime divisors and n_i are integers such that only finitely many are different from zero. If all $n_i \geq 0$, we say that D is *effective*.

Let Y be a prime divisor on X and η its generic point. The local ring $\mathcal{O}_{\eta,X}$ is a discrete valuation ring with a quotient ring K. We usually say that the discrete valuation ν_Y is the valuation of Y. Since X is separated, Y is uniquely determined by its valuation.

For a given $f \in K^*$ nonzero rational function on X, it is associated with the integer $\nu_Y(f)$ that is equal to zero for all except finitely many prime divisors Y. Hence we associated a divisor to f by setting

$$\mathrm{div}(f) := \sum_{Y \text{ is prime}} \nu_Y(f).$$

A divisor which is equal to the divisor of a rational function is called a *principal divisor*. The set of principal divisors forms a subgroup of Div$X$. The quotient of the group of divisors by the principal divisors is called the *divisor class group* of X. We also call it the Weill group of X.

Since not every scheme is regular in codimension one or projective, we need a more comprehensive definition of divisor. Let X be a scheme, a Cartier divisor on X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$, here $\mathcal{K}$ means the sheaf of total quotient rings of $\mathcal{O}$ and the operation on the group is the multiplication of global sections. A Cartier divisor is a pair $\{U_i, f_i\}$, where $\{U_i\}$ is an open cover of X and for each i an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$, such that for each i, j , $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$. A Cartier divisor is principal if it is in the image of the natural map $\Gamma(X, \mathcal{K}^*) \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$. Two Cartier divisors are linearly equivalent if their difference is principal.

Let X be a smooth projective variety and $\{U_i, f_i\}$ be a Cartier divisor of X. For each prime divisor Y, take the coefficient of Y to be $\nu_Y(f_i)$, where i is any index for which $U_i \cap Y \neq \emptyset$. If f_i/f_j is invertible on $U_i \cap U_j$, then $\nu_Y(f_i/f_j) = 0$, in other words $\nu_Y(f_i) = \nu_Y(f_j)$. So we get a Weill divisor $D = \sum \nu_Y(f_i)Y$.

Proposition 2.1.1. *Let X be a smooth projective variety. Then the group of Weill divisors on X is isomorphic to the group of Cartier divisors. Furthermore, the principal Weill divisors correspond to the principal Cartier divisors under this isomorphism.*

We also recall that the Picard group of X is the group of isomorphism classes of line bundles on X. Given a Cartier divisor on $D = \{U_i, f_i\}$ on X, we may associate an invertible sheaf $\mathcal{O}_X(D)$ on X that is defined by sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on U_i.

Theorem 2.1.2. *Let X be a smooth variety, the map*

$$D \longmapsto O_X(D)$$

defines an isomorphism from $\mathrm{Cl}(X)$ *to* $\mathrm{Pic}(X)$

Let $\mathcal{L}$ be a divisor, or an invertible sheaf, on X. We say that $\mathcal{L}$ is *very ample* if it induces an embedding of X into the projective space $\mathbb{P}(\mathrm{H}^0(X, \mathcal{L}))$ given by global sections of $\mathcal{L}$. In addition, we say that $\mathcal{L}$ is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there is an integer $n_0 > 0$ such that for every $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its *global sections*, i.e. there is a index set I and a surjective morphism of sheaves

$$\bigoplus_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes n}.$$

Remark 2.1.1. Let $\mathcal{D}$ and $\mathcal{L}$ be coherent sheaves on X. If $\mathcal{D}$ is ample, then there exists an integer $n_0 > 0$ such that $\mathcal{D} \otimes \mathcal{L}^{\otimes m}$ is ample for all $m \geq n_0$.

Example 2.1.1. The simplest example of a very ample sheaf is to take $X = \mathbb{P}^n$ and the *sheaf of hyperplane sections* $\mathcal{L} = \mathcal{O}_X(1)$. In addition, $\mathcal{O}_X(r)$ is very ample for every $r > 0$, inducing an embedding $X \longrightarrow \mathbb{P}^N$, where N is the dimension of the vector space of monomial forms of degree r in the variables $X_0, \dots, X_n$, namely $N = \binom{n+r}{n}$.

Example 2.1.2. Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and $D = \mathcal{O}_X(a,b)$ be a divisor on X with $a > 0$ and $b > 0$. It follows that D is very ample, embedding X in $\mathbb{P}^N$, where N is given by the dimension of the vector space of the monomial forms of bidegree (a,b), i.e $X_0^{r_0} \cdots X_n^{r_n} Y_0^{s_0} \cdots Y_m^{s_m}$ where $\sum r_i = a$ and $\sum s_i = b$.

The next theorem gives an equivalent definition of ample divisor, a proof is in Hartshorne‘s book [Hart77, p. 154].

Theorem 2.1.3. *Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes m}$ is very ample over* $\operatorname{Spec} A$ *for some $m > 0$.*

Theorem 2.1.4. *Let A be a noetherian ring and X be a proper scheme over* $\operatorname{Spec} A$. *For each invertible sheaf $\mathcal{L}$ on X, the following are equivalent:*

- *$\mathcal{L}$ is ample.*
- *For each coherent sheaf $\mathcal{F}$ on X, there is a integer n_0, depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \geq n_0$, $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$.*

Proof. See [Hart77, Proposition 5.4, pg. 229]. □

We include the proof of the following Proposition due to its simplicity, which also can be found in [Hart77].

Proposition 2.1.5. *Let $f : Y \longrightarrow X$ be a finite morphism of complete schemes and $\mathcal{L}$ an ample line bundle on X. Then the pullback $f^*\mathcal{L}$ is also an ample line bundle on Y.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on Y. Since f is finite, we obtain that the following right derived functor is null, $R^j f_*(\mathcal{F} \otimes f^*\mathcal{L}^m) = 0$, for $j > 0$ and $f_*(\mathcal{F} \otimes f^*\mathcal{L}^m) = f_*\mathcal{F} \otimes \mathcal{L}^m$. Hence $H^i(Y, \mathcal{F} \otimes f^*\mathcal{L}^m) = H^i(X, f_*\mathcal{F} \otimes \mathcal{L}^m)$ for all $i > 0$. Since $f_*\mathcal{F}$ is a coherent sheaf on X, we finish the proof by applying Theorem 2.1.4. □

Example 2.1.3. The inclusion $Y \hookrightarrow X$ morphism between two complete schemes Y and X satisfies the conditions of the previous Proposition. Hence, if H and E are two ample divisors on X, where $\dim X > 1$ and H is smooth, then $E|_H$ is ample over H.

The following version of Lefschetz hyperplane theorem is present in Lazarsfeld's book [Laz04, Thm. 3.1.17] and can also be useful to compute the Picard group of some varieties.

Theorem 2.1.6 (Lefschetz hyperplane theorem). *Let X be a nonsingular irreducible projective variety of dimension n, and D be any effective ample divisor on X. Then the restriction map*

$$\mathrm{H}^i(X,\mathbb{Z}) \longrightarrow \mathrm{H}^i(D,\mathbb{Z})$$

is an isomorphism for $i \le n-2$ and injective for $i = n-1$

Example 2.1.4. Setting $i = n-1$ and $X = \mathbb{P}^3$ in the Lefschtez hyperplane theorem, and taking the ample divisor $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ of the smooth quadratic surface in $\mathbb{P}^3$, we can deduce the map

$$\mathrm{H}^i(\mathbb{P}^3,\mathbb{Z}) \longrightarrow \mathrm{H}^i(Q,\mathbb{Z})$$

is not an isomorphism, because $\mathrm{Pic}(Q) = \mathbb{Z}\times\mathbb{Z}$ while $\mathrm{Pic}(\mathbb{P}^3) = \mathbb{Z}$.

Example 2.1.5. Let $X = \mathbb{P}^n \times \mathbb{P}^m$ and $D_i \in |\mathcal{O}_X(a_i,b_i)|$, $i = 1,\ldots,n+m-1$, divisors on X with $a_i, b_i > 0$ for all i. Assuming that $X_i = \bigcap_{j=1}^{i} D_j$ is smooth for all $j < i$, the Example 2.1.3 and Lefschetz hyperplane theorem imply that $\mathrm{Pic}(X_i) = \mathbb{Z}\times\mathbb{Z}$ for all $i \le n+m-4$.

2.2. Linear Systems and Gonality of Curves

We recall that a *complete linear system* on a nonsingular projective variety X is defined as the set of all divisors *linearly equivalent* to some given divisor D_0. Two divisors are said to be linearly equivalent if their difference is a principal divisor. The complete linear system associated with D_0 is denoted by $|D_0|$ and forms a projective space. A *linear system* is a subspace of a complete linear system.

It follows from the very definition that there is a $1-1$ correspondence between the complete linear system $|D_0|$ and the quotient of the global sections of D_0 by the action of the multiplicative group $\mathbb{C}^*$, $(\mathrm{H}^0(X,D_0) - 0)/\mathbb{C}^*$. Hence

$$\dim|D_0| = \dim(\mathrm{H}^0(X,D_0)) - 1.$$

If we assume that D_0 is a degree d divisor on a smooth curve X of genus g with $\dim|D_0| = r$, then we get a morphism of degree d from X to $\mathbb{P}^r$, $X \to \mathbb{P}^r$. In this case, the complete linear system is denoted by $\mathfrak{g}^r_d$.

A point $P \in X$ is a *base point* of a linear system $\mathfrak{D}$, if $P \in \mathrm{Supp}(D)$ for all $D \in \mathfrak{D}$. Here $\mathrm{Supp}(D)$ stands for the union of all prime divisors of D.

The next theorem can be found in [Laz04].

Theorem 2.2.1 (Asymptotic Riemann–Roch). *Let X be an irreducible projective variety of dimension n, and D and E two divisors on X. We have that*

$$p(m) = \chi(X, \mathcal{O}_X(mD+E)) = \frac{(D^n)m^n}{n!} + \frac{(D^{n-1})(K_X - 2E)m^{n-1}}{2(n-1)!} + \dots$$

is a polynomial of degree at most $n = \deg X$, where K_X stands for the canonical divisor on X.

Proof. The proof is done by induction on dim$X = n$. The case $n = 1$ is just the Riemann–Roch theorem for curves. Let H be a very ample divisor on X, by Remark 2.1.1 and Theorem 2.1.3 we can choose H such that $D + H = G$ is very ample as well. From the following two exact sequences

$$0 \longrightarrow \mathcal{O}_X(mD+E) \longrightarrow \mathcal{O}_X(mD+H+E) \longrightarrow \mathcal{O}_H(mD+H+E) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_X((m-1)D+E) \longrightarrow \mathcal{O}_X(mD+H+E) \longrightarrow \mathcal{O}_G(mD+H+E) \longrightarrow 0,$$

we obtain that

$$\chi(X, \mathcal{O}_X(mD+H+E)) = \chi(X, \mathcal{O}_X(mD+E)) + \chi(X, \mathcal{O}_H(mD+H+E))$$

and that

$$\chi(X, \mathcal{O}_X(mD+H+E)) = \chi(X, \mathcal{O}_X((m-1)D+E)) + \chi(X, \mathcal{O}_G(mD+H+E)).$$

Hence $p(m) - p(m-1) = \chi(X, \mathcal{O}_G(mD+H+E)) - \chi(X, \mathcal{O}_H(mD+H+E))$. By the induction hypothesis, $p(m) - p(m-1)$ is polynomial, then $\Delta(P(m)) = p(m) - p(m-1)$ is polynomial, which implies that $p(m)$ is polynomial. $\square$

As we stated in the Introduction of this Thesis, the *gonality* of a smooth curve $\mathcal{C}$ of genus g, is the smallest positive integer k for which there exists a surjective morphism from $\mathcal{C}$ to $\mathbb{P}^1$, i.e

$$\text{gon}(C) = \min\{k \mid \text{there is a } \mathfrak{g}^1_k \text{ on } \mathcal{C}\}.$$

In general, it's not simple to compute the gonality of a given curve. However, the gonality of a plane curve can be easily computed as follows.

Example 2.2.1 (Gonality of plane curves)**.** Let $\mathcal{C} \subset \mathbb{P}^2$ be a plane curve of degree $d > 1$. The gonality of $\mathcal{C}$ is equal to $d-1$.

Taking the projection from a point on $\mathcal{C}$, Bézout's Theorem implies $\mathrm{gon}(\mathcal{C}) \leq d-1$. Assume that $\mathrm{gon}(\mathcal{C}) = k \leq d-1$ and it is realized by the morphism $f : \mathcal{C} \to \mathbb{P}^1$. We also can assume that $\mathcal{C}$ is non-hyperelliptic, thus the canonical sheaf $K_{\mathcal{C}} \cong \mathcal{O}_C(d-3)$ embeds C in $\mathbb{P}^{g-1}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[rr, bend left, "\omega"] \arrow[dr, "f"'] & \mathbb{P}^2 \arrow[r, "\varphi"] & \mathbb{P}^{g-1} \\
& \mathbb{P}^1 &
\end{tikzcd}$$

The canonical morphism ω factors over $(d-3)$-fold Veronese φ, recall that the genus of a plane curve of degree d is $g = (d-1)(d-2)/2$. Let us take $p_1, \ldots, p_k$ a fiber over f and the divisor $D := p_1 + \cdots + p_k \in \mathrm{Pic}(\mathcal{C})$. By the Riemann–Roch theorem we obtain that $\mathrm{h}^0(\mathcal{C}, K_C - D) \geq (g-1) - k$, implying that $\varphi(p_1), \ldots, \varphi(p_k)$ are linearly independent in $\mathbb{P}^{g-1}$. We also can assume that $\varphi(p_i) = (x_i^{d-3} : x_i^{d-4} y_i : \cdots : y_i^{d-3})$, inducing a $k \times (d-2)$ Vandermonde matrix. If $k < d-1$ we obtain a contradiction, because the matrix has rank k. Hence $k = d-1$, and in this case, we obtain a $(d-1) \times (d-2)$ Vandermonde matrix of rank $d-2$, and we are done.

For the next example it is required the Castelnuovo–Severi inequality, whose proof can be found in [ACGH85, Ex. VIII C 1] or in [Kani84, Corollary at p. 26].

Theorem 2.2.2 (Castelnuovo–Severi inequality)**.** *Let $\mathcal{C}$ be a curve of genus g with two non-constant morphisms $f_i : \mathcal{C} \to \mathcal{C}_i$ where $\mathcal{C}_i$ is a curve of genus g_i. If f_1 and f_2 are disjoint, i.e. they don't factor over the same morphism $h : \mathcal{C} \to \mathcal{C}'$ with $\deg(h) > 1$, then*

$$g \leq d_1 g_1 + d_2 g_2 + (d_1 - 1)(d_2 - 1),$$

where $d_i = \deg(f_i)$.

Example 2.2.2. Let $\mathcal{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of genus g of bidegree (d_1, d_2), with $d_1 \leq d_2$. Assuming that d_1 is a prime number, we show that $\mathrm{gon}(\mathcal{C}) = d_1$. The first projection provides a morphism $\pi_1 : \mathcal{C} \longrightarrow \mathbb{P}^1$ of degree d_1. So $\mathrm{gon}(\mathcal{C}) \leq d_1$. If $\mathrm{gon}(\mathcal{C}) = k < d_1$, then there is a morphism f of degree k from $\mathcal{C}$ to $\mathbb{P}^1$. As π_1 does not factor by f since d_1 is prime, then by the Castelnuovo–Severi inequality, the morphism $\pi_1 \times f : \mathcal{C} \to \mathbb{P}^1 \times \mathbb{P}^1$ gives $g \leq (d_1 - 1)(k-1)$, which is a contradiction, because $g = (d_1 - 1)(d_2 - 1)$.

Let $\mathcal{C}$ be a smooth curve of genus g, and two integers $r \geq 0$ and $d \geq 1$, we set the well known Brill–Noether number

$$\rho(g,r,d) := \rho = g - (r+1)(g-d+r)$$

and the following two famous subvarieties of the Picard variety $\mathrm{Pic}(C)$, that parametrizes special linear series on $\mathcal{C}$, namely

$$W^r_d(\mathcal{C}) = \{|\mathcal{L}| \mid deg(\mathcal{L}) = \mathrm{d} \text{ and } h^0(\mathcal{L}) \geq r+1\},$$

whose support is the set of complete linear systems of degree d and dimension at least r and

$$G^r_d(\mathcal{C}) = \{\mathfrak{g}^r_d \text{ on } \mathcal{C}\}$$

parametrizes linear series of degree d and dimension exactly r on $\mathcal{C}$. The Brill–Noether number is the expected dimension of the $W^r_d(\mathcal{C})$.

The following two main results, for this section, can be found in [ACGH85] and in [GHa80], respectively.

Theorem 2.2.3 (Kempf and Fulton–Lazarsfeld)**.** *Let $\mathcal{C}$ be a smooth curve of genus g. If the Brill–Noether number $\rho \geq 0$ is non-negative, then $G^r_d(\mathcal{C})$, and hence $W^r_d(\mathcal{C})$, are non-empty. Furthermore, every component of $G^r_d(\mathcal{C})$ has dimension at least equal to ρ. The same is true for $W^r_d(\mathcal{C})$ provided $r \geq d-g$.*

Theorem 2.2.4 (Griffiths and Harris)**.** *If $\mathcal{C}$ is a generic curve, then $G^r_d(\mathcal{C})$ is reduced and all its components have the expected dimension ρ. In particular, $G^r_d(\mathcal{C})$ is empty when $\rho < 0$.*

With the above two results, we can derive also a well-known upper bound for the gonality of a smooth curve.

Corollary 2.2.5. *For a projective smooth curve $\mathcal{C}$ of genus $g > 1$, we have*

$$\mathrm{gon}(\mathcal{C}) \leq \left\lfloor \frac{g+3}{2} \right\rfloor.$$

Proof. The result follows from the two previous theorems by noting that $G^r_d(\mathcal{C})$ is non-empty of dimension at least ρ if, and only if, the Brill–Noether number is greater than zero. □

In the following, we transcribe Lazarsfeld's result on the gonality of a complete intersection curve exactly as it appears in the original paper. We believe that it's appropriate to include the steps presented in [Laz97, Exercise 4.12], because our two main results, Theorem 3.2.3 and Theorem 3.2.4 of Section 3.2 in Chapter 3, are inspired by Lazarsfeld's ideas.

Theorem 2.2.6 (Lazarsfeld's Theorem–Exercise 4.12)**.** *Let $\mathcal{C} \subset \mathbb{P}^r$ be a smooth complete intersection of hypersurfaces of degrees $2 \leq a_1 \leq a_2 \leq \cdots \leq a_{r-1}$. Let $\mathcal{L}$ be a base-point free line bundle on $\mathcal{C}$, of degree d, with $h^0(C, \mathcal{L}) \geq 2$. Then $d \geq (a_1 - 1)a_2 \dots a_{r-1}$.*

Proof Idea. The idea of the argument is this: let $S \subset \mathcal{C}$ be a general complete intersection surface of type $(a_2, \dots, a_{r-1})$. As in [Laz97, Exercise 3.18], one can associate to $\mathcal{L}$ a rank two vector bundle $\mathcal{F}$ on S. One finds that if $d < (a_1 - 1)a_2 \dots a_{r-1}$, then $\mathcal{F}$ is Bogomolov unstable, c.f. Theorem 2.3.4 below. It is easy to get a contradiction provided one knows that the destabilizing subsheaf is of the form $\mathcal{O}_S(k)$, but this doesn't seem to be guaranteed. To remedy this, instead of working on a surface we work on a complete intersection threefold $X \supset \mathcal{C}$, whose Picard group is controlled by the Lefschetz theorems. Related results, proved by more classical methods, appear in[CiLa90] and [Ba96], and the Theorem also connects with some of the conjectures in [EiGrHa93]. Paoletti [Paol] has extended the techniques of this exercise to deal with certain non-complete intersection curves. He proves the striking result that under suitable numerical hypotheses, the gonality of a space curve $\mathcal{C} \subset \mathbb{P}^3$ is governed by its Seshadri constant, which roughly speaking measures how positive the hyperplane bundle $\mathcal{O}_{\mathbb{P}^3}(1)$ is in a neighborhood of $\mathcal{C}$, see [Laz97, Section 5].

i) Put $\gamma = a_3 a_4 \dots a_{r-1}$, and let $X \supset \mathcal{C}$ be a general complete intersection threefold of type $(a_3, \dots, a_{r-1})$. If $r = 3$ take $X = \mathbb{P}^3$ and $\gamma = 1$. Let $f : Y \to X$ be the blowing-up of $\mathcal{C}$, and let $E \subset Y$ be the exceptional divisor, with $\pi : E \to \mathcal{C}$ the natural map. Consider on E the globally generated line bundle $\mathcal{G} = \pi^*\mathcal{L}$. Choosing a base-point free pencil in $\Gamma(E, \mathcal{G})$, we define in the usual way a rank two vector bundle $\mathcal{F}$ on Y via the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Y^2 \longrightarrow \mathcal{G} \longrightarrow 0.$$

Compute the Chern classes of $\mathcal{F}$.

ii) Denote by H the pull-back to Y of the hyperplane divisor on X, and for $0 \leq \varepsilon \in \mathbb{Q}$ consider the $\mathbb{Q}$-divisor $D_\varepsilon = (a_2 + \varepsilon)H - E$. Show that $D = D_0 = a_2 H - E$ is globally generated and that D_ε is ample if $\varepsilon > 0$. Now assume that $d < (a_1 - 1)a_2\gamma$. Prove then that for $0 < \varepsilon \ll 1$

$$c_1(\mathcal{F})^2 - 4c_2(\mathcal{F})D_\varepsilon = (a_1 - \varepsilon)a_1 a_2 \gamma - 4d > 0. \tag{2.1}$$

iii) Fixing ε for which equation (2.1) holds, an extension by Miyaoka of Bogomolov's instability theorem implies that there exists a rank one subsheaf $L \subset \mathcal{F}$ such that

$$(2c_1(L) - c_1(\mathcal{F}))D_\varepsilon D > 0.$$

Show that one can assume that L is locally free and that the vector bundle map $L \to \mathcal{F}$ drops rank (if at all) on a codimension two subset $Z \subset Y$. Prove that $L = \mathcal{O}_Y(-tH - E)$ for some integers $t \in \mathbb{Z}$.

iv) Now let $S \in |a_2H - E| = |D|$ be a general divisor, so that S is isomorphic to a complete intersection surface of type $(a_2, \dots, a_{r-1})$ through $\mathcal{C}$. Setting $F = \mathcal{F}|S$, show that $c_2(F) = d$, and that the restriction to S of the subsheaf $L \to F$ gives rise to an exact sequence

$$0 \longrightarrow \mathcal{O}_S(-sH) \longrightarrow F \longrightarrow \mathcal{O}_S(s - a_1) \otimes I_W \longrightarrow 0,$$

where $s = t + a_1$, and $W \subset S$ is some finite subscheme. Use instability to prove that $2s < a_1$. Then estimate $c_2(F)$ to deduce that $a_1 < s + 1$. But $s > 0$ since $h^0(S, F) = 0$, and this gives a contradiction.

v) Prove that the inequality in the Theorem is the best possible, in the sense that for any integers $2 \leq a_1 \leq \cdots \leq a_{r-1}$, there exists a complete intersection curve C that carries a base-point free pencil of degree $(a_1 - 1)a_2 \dots a_{r-1}$.

□

2.3. Some useful results on bundles

In this section, we summarize some known results on the theory of vector bundles and ample divisors that will be required later on to prove some main results of this Thesis. Here the word *bundle* stands for a locally free coherent sheaf of finite rank over an algebraic variety.

We start by recalling a very well-known criterion for ampleness due to Nakai, Moishezon, and Kleimann. Before that, we need the following lemma.

Lemma 2.3.1. *Let L be a globally generated bundle and $\phi : X \longrightarrow \mathbb{P}^r$ the induced morphism with $\phi = \phi_{|L|}$ and $\mathbb{P}^r = \mathbb{P}(H^0(X, L))$. Then L is ample if and only if ϕ is a finite morphism, or equivalently, if and only if $\int_C c_1(L) > 0$ for every irreducible curve $\mathcal{C} \subset X$.*

Proof. By assumption $L = \phi^*\mathcal{O}_{\mathbb{P}^r}(1)$, and the Proposition 2.1.5 assures that if ϕ is finite, then L is ample. Since the inclusion morphism is finite, $\int_C c_1(L) > 0$ for every irreducible curve $C \subset X$. Assuming that ϕ is not finite, there is a point $p \in \mathbb{P}^r$ such that $\phi^{-1}(p) = Z$ has positive dimension. Hence L restricts to a trivial line bundle on Z, because L is the pullback of a hyperplane section. In particular, $L|Z$ is not ample, and so by Proposition 2.1.5, L is not ample as well. And in this case, there is a curve $\mathcal{C} \in Z$ so $\int_C c_1(L) = 0$. □

Theorem 2.3.2 (Nakai–Moishezon–Kleiman)**.** *Let L be a line bundle on a projective scheme X. Then L is ample if only if $\int_V c_1(L)^{dim(V)} > 0$ for any subvariety $V \subseteq X$ with* $\dim V > 0$.

Proof. Suppose there is a m which $L^{\otimes m}$ is very ample and $m^{dim(V)}.\int_V c_1(L)^{dim(V)} = \int_V c_1(L^{\otimes m})^{dim(V)}$ is the degree of V in the projective embedding of X. That way we have the first part.

Now suppose $\int_V c_1(L)^{dim(V)} > 0$ for every variety $V \subseteq X$ with $dim(V) > 0$. We can assume without loss of generality X is reduced and irreducible. Let's prove by induction, if $dim(X) = 1$, then X is a curve and L has positive degree, which implies that L is ample. Assume that the theorem is true for every projective scheme Y with $dim(Y) \leq n - 1$.

Write $L = \mathcal{O}_X(D)$ for $D \in Pic(X)$. We assert first $H^0(X, \mathcal{O}_X(mD)) \neq 0$ for $m \gg 0$. Indeed, by the asymptotic Rieman-Roch theorem $\chi(X, \mathcal{O}_X(mD)) = m^n \frac{D^n}{n!} + O(m^{n-1})$ and $D^n > 0$ by assumption. By remark 2.1.1 we know that D is the difference between two ample divisors, so we can write $D = A - B$. Considering the product of $\mathcal{O}_X(mD - B)$ with A and with B we get the two exact sequences:

$$0 \longrightarrow \mathcal{O}_X(mD - B) \xrightarrow{.A} \mathcal{O}_X((m+1)D) \longrightarrow \mathcal{O}_A((m+1)D) \longrightarrow 0, \tag{2.2}$$

$$0 \longrightarrow \mathcal{O}_X(mD - B) \xrightarrow{.B} \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_B(mD) \longrightarrow 0. \tag{2.3}$$

By induction $\mathcal{O}_A(D)$ and $\mathcal{O}_B(D)$ are ample divisors, so choose $m \gg 0$ such that the higher cohomology of both vanish, then

$$H^i(X, \mathcal{O}_X(mD)) = H^i(X, \mathcal{O}_X(mD - B)) = H^i(X, \mathcal{O}_X((m+1)D))$$

for $i \geq 2$. So for $m \gg 0$ and $i \geq 2$, $h^i(X, \mathcal{O}_X(mD))$ are constant. Therefore $\chi(X, \mathcal{O}_X(mD)) = h^0(X, \mathcal{O}_X(mD)) - h^0(X, \mathcal{O}_X(mD)) + C$, where C is constant and $m \gg 0$. So by $\chi(X, \mathcal{O}_X(mD)) = m^n \frac{D^n}{n!} + O(m^{n-1})$, we get $H^0(X, \mathcal{O}_X(mD))$ not vanish for $m \gg 0$. Without loss of generality, we could replace D with mD, since D is ample if, only if, mD is ample. Therefore we can suppose that D is effective.

The next step is to show $\mathcal{O}_X(mD)$ is generated by its global sections if $m \gg 0$. We need to show that no point of D is base point of $|\mathcal{O}_X(mD)|$. By the exact sequence

$$0 \longrightarrow \mathcal{O}_X((m-1)D) \xrightarrow{\cdot D} \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD) \longrightarrow 0. \tag{2.4}$$

and by induction $\mathcal{O}_D(D)$ is ample.Therefore $\mathcal{O}_D(mD)$ is globally generated and, for sufficiently large m, $H^1(X, \mathcal{O}_D(mD)) = 0$. So the natural map

$$H^1(X, \mathcal{O}_D((m-1)D)) \longrightarrow H^1(X, \mathcal{O}_D(mD))$$

is surjective and since these spaces are finite-dimensional, for $m \gg 0$ these maps must be isomorphisms. Therefore, for $m \gg 0$, the map $H^0(X, \mathcal{O}_X(mD)) \longrightarrow H^0(X, \mathcal{O}_D(mD))$ is surjetive and since $\mathcal{O}_X(mD)$ is globally generate, not point of $Supp(D)$ is a base point of $|\mathcal{O}_X(mD)|$.

To finish the proof, we use Lemma 2.3.1, since $(mD.C) > 0$ for every irreducible curve. □

We also recall that a $\mathbb{Q}$-divisor on X is an element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. If D is a $\mathbb{Q}$-divisor on X, we write $D = \sum r_i D_i$, where D_i are prime divisors, $r_i \in \mathbb{Q}$ and only finitely many r_i are different from zero. We say that D is effective if all $r_i \geq 0$. If all r_i are integers, then we may say D is an integral divisor. Two $\mathbb{Q}$-divisors D_1 and D_2 are linearly equivalent if there is an integer r such that rD_1 and rD_2 are integral and linearly equivalent as divisors in $\mathrm{Div}(X)$.

Concerning the ampleness of $\mathbb{Q}$-divisors we have the following.

Proposition 2.3.3. *A $\mathbb{Q}$-divisor D is ample if any one of the following three equivalent conditions is satisfied:*

a) D is of the form $D = \sum r_i D_i$ where $r_i > 0$ is a rational number and D_i is an ample integral divisor

b) There is a positive integer $r > 0$ such that rD is integral and ample.

c) $D^{\dim V} \cdot V > 0$ for every irreducible subvariety $V \subset X$ of positive dimension.

Let X be a smooth projective surface and E a vector bundle over X of rank 2. The bundle E is *Bogomolov unstable* if there exists a finite sub-scheme $Z \subseteq X$ and line bundles $\mathcal{A}$

and $\mathcal{B}$ on X such that the sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow E \longrightarrow \mathcal{B} \otimes I_Z \longrightarrow 0$$

is exact, and

$$(\mathcal{A} - \mathcal{B})^2 > 0 \text{ and } (\mathcal{A} - \mathcal{B})H > 0$$

for all ample sheaves H on X.

Theorem 2.3.4 (Bogomolov's Instability Theorem)**.** *Let E be a rank two vector bundle on a smooth projective surface X. If $c_1(E)^2 - 4c_2(E) > 0$, then E is Bogomolov unstable. In particular, there is a saturated invertible subsheaf $F \hookrightarrow E$ with $L = \det E$ such*

$$(2F - L)^2 > 0 \text{ and } (2F - L)H > 0,$$

for some ample divisor H. Here $c_1(E)$ and $c_2(E)$ stands for the Chern classes of E.

Miyaoka [Miy87] generalizes the Bogomolov instability theorem by considering a vector bundle of rank two on a 3-fold variety.

Theorem 2.3.5 (Miyaoka's generalization of Bogomolov's theorem)**.** *Let $\mathcal{E}$ be a rank two vector bundles on a smooth 3-fold variety. Given two divisors D and D' with D globally generated and D' ample, if*

$$(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}))D' > 0,$$

then there is a rank one subsheaf $\mathcal{L} \subset \mathcal{E}$ such that

$$(2c_1(\mathcal{L}) - c_1(\mathcal{E}))D' \cdot D > 0.$$

3 | On the Gonality of CI Curves in Biprojectic Spaces

This is the main chapter of the Thesis and where we study the geometry of curves that are complete intersections in the biproduct of projective spaces in terms of their gonalities and the existence of suitable linear systems. The main inspiration is the two main results due to Mukai, c.f. Theorem 1.0.4, and Mukai & Ide, c.f. Theorem 1.0.5, where the authors stratify the moduli space of smooth curves of genus 7 and 8 using gonality and linear systems of higher rank. The curves that compound each stratum are curves whose canonical models are realized as complete intersections in certain algebraic varieties, in special the biproduct of projective spaces. So, we are studying the gonality of canonical curves embedded in the biproduct of projective spaces. In this setting, another inspiration is the lower bound for the gonality of complete intersection curves in projective spaces due to Lazarsfeld, Theorem 2.2.6.

Just to make clear what a complete intersection curve means here, we state the following definition.

Definition 3.0.1. Let $Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$ be a multi-projective space, with $n = n_1 + \cdots + n_k$ and $k \geq 2$. A smooth curve $\mathcal{C}$ is a complete intersection in X, when it is integral, given by the intersection of exactly $n-1$ divisors $D_j \in |\mathcal{O}_Y(a_{j1}, a_{j2}, \ldots, a_{jk})|$ and $\mathcal{C}$ is not degenerated, i.e. it is not contained in any hyperplane section of Y.

Remark 3.0.1. Let $\mathcal{C}$ be a complete intersection curve in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, with $k \geq 1$, and take the projections $\pi_i : \mathcal{C} \to \mathbb{P}^{n_i}$ for $i = 1, \ldots, k$. Let us assume that $\mathcal{C}$ is not hyperelliptic. Due to Definition 3.0.1, the image of $\mathcal{C}$ under each projection spans the whole projective space $\mathbb{P}^{n_i}$. Thus $\mathcal{C}$ is equipped with k distinct linear series $\mathfrak{g}^{n_i}_{d_i}$, with

$$d_i := \deg(\pi_{i|\mathcal{C}}) \deg(\pi_i(\mathcal{C})), \text{ for } i = 1, \ldots, k.$$

We also assume the convention that $\deg(\pi_i(\mathcal{C})) = 1$ whenever $n_i = 1$. The sequence of integers $(d_1, \ldots, d_k)$ is the *multi-degree* of $\mathcal{C}$.

#	g	divisors	ambient	bidegree of $\mathcal{C}$
#1	5	$(2,3)$, $(1,1)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(3,5)$
#2	6	$(2,1)$, $(1,3)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(3,7)$
#3	6	$(2,2)$, $(1,2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(4,6)$
#4	6	two $(1,1)$, one $(1,3)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(3,7)$
#5	7	$(1,1)$, $(2,1)$, $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(3,9)$
#6	7	$(1,1)$, two $(1,2)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(4,8)$
#7	7	two $(1,1)$, $(2,1)$, $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(3,12)$
#8	7	three $(1,1)$, one $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(3,9)$
#9	7	$(2,2)$, $(1,1)$, $(1,1)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(6,6)$
#10	8	$(1,1)$, $(2,0)$, $(1,3)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(6,8)$
#11	8	$(1,1)$, $(1,1)$, $(1,2)$, $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(4,10)$
#12	8	$(2,1)$, $(1,2)$, $(1,1)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(7,7)$
#13	9	$(1,1)$, $(2,1)$ $(0,2)$, $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(4,12)$
#14	9	$(2,0)$, $(0,2)$, $(2,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(8,8)$
#15	9	$(2,0)$, $(1,2)$, $(1,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(8,8)$
#16	9	three $(1,1)$, two $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^5$	(4,12)
#17	9	$(1,2)$, $(1,1)$, $(1,1)$, $(1,1)$	$\mathbb{P}^2 \times \mathbb{P}^3$	$(7,9)$
#18	10	$(2,0)$, $(1,1)$, $(1,1)$, $(0,3)$	$\mathbb{P}^2 \times \mathbb{P}^3$	(6,12)
#19	11	four $(1,1)$, one $(0,2)$	$\mathbb{P}^2 \times \mathbb{P}^4$	(8,12)
#20	11	five $(1,1)$	$\mathbb{P}^3 \times \mathbb{P}^3$	(10,10)
#21	6	$(0,2)$, $(3,2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(4,6)$

Table 3.1: CI curves in $\mathbb{P}^n \times \mathbb{P}^m$ with $\mathrm{K}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(1,1)$

3.1. Curves whose canonical bundle is a hyperplane section

As can be read from Theorems 1.0.4 and 1.0.5, the canonical curves of genus 7 or 8 that are complete intersections, c.i. for short, in the biproduct of projective spaces are such that their canonical divisors are given by hyperplane sections. So we start by studying c.i. curves $\mathcal{C} \subset \mathbb{P}^n \times \mathbb{P}^m$ whose canonical sheaf is $\mathcal{O}_{\mathcal{C}}(1,1)$. This condition on the canonical sheaf imposes stronger conditions on $\mathcal{C}$ and on the ambient space $\mathbb{P}^n \times \mathbb{P}^m$ in a way that only a finite number of cases are allowed.

Theorem 3.1.1. *If C is a complete intersection smooth curve in $\mathbb{P}^n \times \mathbb{P}^m$ of genus g, with $n+m \geq 3$ and whose canonical sheaf is $\mathrm{K}_C = \mathcal{O}_C(1,1)$, then $m+n \leq 6$, $5 \leq g \leq 11$ and $\mathcal{C}$ can be one of twenty one presented in Table 3.1.*

Proof. Let us assume that $\mathcal{C}$ is given by $m+n-1$ divisors $D_i \in |\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}(a_i, b_i)|$ and that

$a_i \geq a_{i+1}$. Since $\mathcal{C}$ is a complete intersection, its canonical sheaf is

$$\mathrm{K}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}\left(\sum a_i - n - 1, \sum b_i - m - 1\right).$$

By assumption $\mathrm{K}_{\mathcal{C}} = \mathcal{O}_{C}(1,1)$, so we obtain

$$\sum_{i=1}^{n+m-1} a_i = n+2 \quad \text{and} \quad \sum_{i=1}^{n+m-1} b_i = m+2. \tag{3.1}$$

We also can assume that if some $a_i = 0$ or $b_j = 0$, then the corresponding $b_i \geq 2$ or $a_j \geq 2$, otherwise $\mathcal{C}$ is degenerated. It follows from Equation (3.1) and the assumption $a_i \geq a_{i+1}$, that $a_j = 0$ for $j \geq n+3$.

Let a_{n+2-k} be the last term where $a_i \neq 0$, then we claim that $a_i = 1$ for $i \in \{k+1, \cdots, n+2-k\}$ and furthermore $b_1 + \cdots + b_{n+2} \geq n+2$. Indeed, we can assume that $k+1 \leq n+2-k$, the proof is the same if we have the opposite inequality. If $a_{k+1} \geq 2$, then

$$n+2 = a_1 + \cdots + a_{k+1} + a_{k+2} + \cdots + a_{n+2-k} \geq 2(k+1) + (n+2-k-k-1) = n+3,$$

which is an absurd, then $a_{k+1} = 1$ and by the assumption that $a_i \geq a_{i+1}$ we get $a_i = 1$ for $i \in \{k+1, \cdots, n+2-k\}$. Therefore $b_i \geq 1$ for $i \in \{k+1, \cdots, n+2-k\}$ and since $b_i \geq 2$ for $i \geq k+1$ we can see that

$$b_1 + \cdots + b_{n+2-k} + b_{n+3-k} + \cdots + b_{n+2} \geq (n+2-2k) + 2(n+2-n-2+k) = n+2.$$

Let us assume that $m \geq 3$. In this case, we obtain

$$b_j \geq 2 \text{ for } j = n+3, \ldots, m+n-1 \text{ and } n+2+2(m-3) \leq \textstyle\sum_{i=1}^{n+m-1} b_i = m+2.$$

Hence $n+m \leq 6$, and then there are only a finite number of cases to analyse, namely $\mathbb{P}^1 \times \mathbb{P}^m$ with $2 \leq m \leq 5$, $\mathbb{P}^2 \times \mathbb{P}^m$ with $m = 2,3,4$ and $\mathbb{P}^3 \times \mathbb{P}^3$.

Let us start by analyzing $\mathbb{P}^1 \times \mathbb{P}^m$. In this case we can assume that each $b_i \geq 1$, otherwise the projection of $\mathcal{C}$ to $\mathbb{P}^1$ is just a point and then we can assume that $\mathcal{C} \subset \mathbb{P}^m$. If $m = 3$ and $a_i = 0$, then $b_i > 2$, otherwise $\mathcal{C}$ would be contained in a divisor of bidegree $(0,1)$ or $(0,2)$ i.e $\mathcal{C} \subset \mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ which contradicts $\mathcal{C}$ being complete intersection. Hence by equation 3.1, we can determine all curves intersection complete in $\mathbb{P}^1 \times \mathbb{P}^3$.

Now let us assume that $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}^m$, with $2 \leq m \leq 4$. We claim that each $a_i \leq 2$. Let us assume that $a_1 = 4$, so $b_i > 1$ for all $i > 1$, which implies $m+2 = b_1 + \cdots + b_{m+1} \geq 2m$

and $a_i = 0$, $i = 2, \ldots, m+1$ i.e $m \leq 2$ the only possible case is that $\mathcal{C}$ is the complete intersection of divisor one divisor of bidegree $(4,0)$ and two divisor of bidegree $(0,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, which is an absurd, because in this case $\dim(\mathcal{C}) \geq 2$. If $a_1 = 3$, then $a_2 = 1$ and $a_i = 0$ for $i > 2$, and we get $m \leq 3$ so there are only two possibilities: $\mathcal{C}$ is a complete intersection of two divisor of bidegree $(0,2)$, one of bidegree $(3,0)$ and one of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ or $\mathcal{C}$ is a complete intersection of one divisor of bidegree $(3, b_1)$, one of bidegree $(1, b_2)$ and one of bidegree $(0, b_3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, where $b_2 \geq 1$ and $b_3 \geq 2$. In both cases $g(\mathcal{C}) > 3(m+1)$ which is a absurd.

Finally, lets us assume $C \subset \mathbb{P}^3 \times \mathbb{P}^3$. By equation (3.1) the only possible solution is $a_i = b_i = 1$.

Let's finish our proof with an idea of how to produce the table 3.1. Most of the calculation requires basic arithmetic, so let's exemplify by fixing an ambient. Take $X = \mathbb{P}^1 \times \mathbb{P}^2$ and $\mathcal{C} = \cap_{i=1}^{2}(a_i h + b_i H)$ a complete intersection curve of genus g in X. Since $K_{\mathcal{C}} = h + H$, then $g \leq (n+1)(m+1)$ and $\deg(\mathrm{K}_{\mathcal{C}}) = (h+H)(\mathcal{C}) = 2g-2$, so in our example $g \leq 6$ and $\deg(\mathrm{K}_{\mathcal{C}}) = (h+H)((a_1b_2 + a_2b_1)hH + b_1b_2H^2) = a_1b_2 + a_2b_1 + b_1b_2$, in this case $\mathcal{C}$ has bidegree $(b_1b_2, a_1b_2 + a_2b_1)$. By equation 3.1 we get

$$a_1 + a_2 = 3$$

$$b_1 + b_2 = 4,$$

If $a_1 = 0$, then $a_2 = 3$ and $b_1 > 1$ (since $\mathcal{C}$ is complete intersection), in case $b_1 > 2$ then $\deg(K_{\mathcal{C}}) = 12$ (so g=7), which means that the only solution is $(a_1, b_1) = (0, 2)$ and $(b_1, b_2) = (3, 2)$. Here $\mathcal{C}$ has bidegree $(4, 6)$ and genus 6. If $a_1 = 1$, then $a_2 = 2$, b_1 can't be zero because $\mathcal{C}$ is a complete intersection and b_2 can't be zero otherwise the first projection would be a point. That way we get the cases #1, #2 and #3 of table 3.1. $\square$

The next paragraphs and results are devoted to the study of the geometry, concerning the existence of suitable linear systems and the gonality, of complete intersections curves in $\mathbb{P}^n \times \mathbb{P}^m$ as in Theorem 3.1.1.

Lemma 3.1.2. *Let $\mathcal{C} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a smooth complete intersection of genus $g \geq 5$ with multi-degree $(d_1, \ldots, d_k)$. If $\mathcal{C}$ is trigonal, then there is some $i \in \{1, \ldots, k\}$ such that $d_i \equiv 0 \mod 3$ and $d_i \geq 3\, n_i$.*

Proof. Let us assume that $\mathcal{C}$ is trigonal. The hypothesis is that $g \geq 5$ implies a unique g_3^1 by the Castelnuovo–Severi inequality. Thus the canonical model $C \subset \mathbb{P}^{g-1}$ is contained in a ruled surface S of degree $g-2$ and its ruling cuts the $\mathfrak{g}_3^1$.

Set $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Since the Segre embedding $\nu : Y \to \mathbb{P}^r$, $r = -1 + \prod_{i=1}^{k}(n_i + 1)$, of Y is cut out by quadrics, we have $S \subset Y$. Since the linear spaces contained in $\nu(Y)$ are only the fibers of the k rulings of Y, there is $i \in \{1, \dots, k\}$ such that the $g^{n_i}_{d_i}$ factors through the g^1_3 and in particular $d_i \equiv 0 \pmod 3$ and $d_i \geq 3n_i$ with equality if, and only if, either $n_i = 1$ and the i-th projection π_i induces the g^1_3 or $n_i \geq 2$, $\pi_i(\mathcal{C})$ is a rational normal curve and $\pi_{i|\mathcal{C}} : \mathcal{C} \to \pi_i(\mathcal{C})$ is the $\mathfrak{g}^1_3$. □

As an immediate consequence of Lemma 3.1.2 we obtain:

Corollary 3.1.3. *Every curve occurring in #3 of genus 6, #6 of genus 7, #11 and #12 of genus 8, #14, #15 and #16 of genus 9 and #20 of genus 11 of Table 3.1 are non-trigonal. In addition, the cases #6, #11 and #16 are equipped with a $\mathfrak{g}^1_4$, Remark 3.0.1, or by Castelnuovo–Severi inequality, they are tetragonal.*

Remark 3.1.1. Follows from Remark 3.0.1 that a curve $\mathcal{C}$ as in cases #1, #2, #4, #5, #7, and #8 are equipped with a $\mathfrak{g}^1_3$. So they are trigonal.

Remark 3.1.2. By the Corollary 3.1.3 a curve $\mathcal{C}$ as in #14 and #15 are non-trigonal. If we assume $\mathcal{C}$ as in #14, then $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4,4)|$. Hence $\mathcal{C}$ is tetragonal with exactly two $\mathfrak{g}^1_4$'s. Now assume $\mathcal{C}$ as in case #15. Since the divisor of type $(0,2)$ corresponds to a smooth conic, then the first ruling of $\mathbb{P}^2 \times \mathbb{P}^2$ induces a $4:1$ morphism of $\mathcal{C}$ into a smooth conic. Hence $\mathcal{C}$ is tetragonal.

Remark 3.1.3. Let $\mathcal{C}$ be a curve as in #10 or #18. Taking the divisor $(2,0)$, the first projection $\pi_1(\mathcal{C})$ is a smooth conic and $\pi_{i|C} : \mathcal{C} \to \pi_1(\mathcal{C})$ is a g^1_3. Hence $\mathcal{C}$ is trigonal. By the same argument and taking divisor $(0,2)$, the curve in case #21 is trigonal.

Remark 3.1.4. Let $\mathcal{C}$ be a curve as in case #9 where $g = 7$ and $(d_1, d_2) = (6,6)$. Since a smooth plane sextic has genus 10, either $\mathcal{C}$ is trigonal or tetragonal.

Remark 3.1.5. Let $\mathcal{C}$ be a complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$ of the divisors $(1,1)$, $(1,2)$ and $(2,1)$, case #12 of genus 8 in Table 3.1. By Remark, 3.0.1 $\mathcal{C}$ is equipped with two $\mathfrak{g}^2_7$ and none of them is self-adjoint. Otherwise, if α is a $\mathfrak{g}^2_7$ with $\alpha^{\otimes 2} \cong \mathrm{K}_{\mathcal{C}}$, then $\mathrm{K}_{\mathcal{C}} \cong \mathcal{O}_{\mathcal{C}}(2,0)$, contrary to the assumption that the canonical is a hyperplane section.

The proof of the next result can be found in [CKM92, Corollary 2.2.2].

Theorem 3.1.4 (Coppens, Keem and Martens)**.** *Let $\mathcal{C} \to \mathcal{C}'$ be a double covering of a smooth curve C' of genus g'. A base point free linear system $\mathfrak{g}^r_d$ on $\mathcal{C}$ with $d \leq g - 1$, where g is the genus of $\mathcal{C}$, is induced by $\mathcal{C}'$ if $d \leq 2 - 2g'$ or if $r \geq 2g'$.*

Recall that a bielliptic curve is a smooth curve $\mathcal{C}$ admitting a degree two surjective morphism $\pi : \mathcal{C} \to E$ to an elliptic curve E. If $g \geq 6$, it follows by the Castelnuovo–Severi inequality that the morphism π is unique. The automorphism $i_{\mathcal{C}}$ of $\mathcal{C}$ interchanging the two points of each fiber of E over π is called the *bielliptic involution*. In this way its not difficult to see that E is isomorphic to the quotient $\mathcal{C}/(i_{\mathcal{C}})$. Taking a morphism $\rho : E \to \mathbb{P}^1$ of degree two, we produce a $4:1$ morphism $\phi : \mathcal{C} \to \mathbb{P}^1$ that is the composition of π and ρ.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\pi", "2:1"'] \arrow[rr, bend left, "4:1"] & E \arrow[r, "\rho", "2:1"'] & \mathbb{P}^1 \end{tikzcd}$$

Remark 3.1.6. It follows from the Castelnuovo–Severy inequality that a bielliptic curve does not admit a $\mathfrak{g}^1_3$ neither is hyperelliptic. We conclude that every bielliptic curve is tetragonal.

The next result provides a necessary condition for a curve described in Theorem 3.1.1 to be bielliptic.

Lemma 3.1.5. *Let $\mathcal{C} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a bielliptic smooth curve of multi-degree $(d_1, \dots, d_k)$. If some $n_i \geq 2$, then d_i is even and $d_i \geq 2(n_i + 1)$.*

Proof. Let $\mathcal{C} \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be complete intersection smooth curve and a double covering $\pi : \mathcal{C} \to C'$ with C' a smooth curve of genus $q > 0$. Fix a base point free g^r_d on $\mathcal{C}$. If either $d \leq g - 2q$ or $d \leq g - 1$ and $r \geq 2q$, then the g^r_d factors through π, c.f. Theorem 3.1.4. Now assume $q = 1$, i.e. assume that $\mathcal{C}$ is bielliptic. If either $d_i = d_{3-i}$ and $n_i \geq 2$ or $d_i < d_{3-i}$, then $\pi_{i|C}$ factors through π. Thus if $n_i \geq 2$, then $\deg(\pi_{i|C})$ is even, d_i is even and $d_i \geq 2(n_i + 1)$. □

Corollary 3.1.6. *It follows from Corollary 3.1.5 and Remark 3.1.6 that the possible bielliptic curves in Theorem 3.1.1 of Table 3.1 are among the cases #3, #6, #9, #11, #13, #14, #15, #16, #19 and #20.*

To prove of next Lemma, we need the classical extensions of Castelnuovo's theorem for curves in $\mathbb{P}^r$ due to Eisenbud and Harris [Ha82, Theorems 3.7 and 3.11].

Theorem 3.1.7 (Castelnuovo's Theorem - Parte I). *Let $\mathcal{C}$ be an irreducible non-degenerated projective curve of degree d in $\mathbb{P}^r$. Set $m = \lfloor \frac{d-1}{r-1} \rfloor$ and ε such that $d = m(r-1) + \varepsilon + 1$. The genus g of $\mathcal{C}$ satisfies*

$$g \leq \pi(r, d) := \binom{m}{2}(r-1) + m\varepsilon$$

and $g = \pi(r,d)$ if and only if $\mathcal{C}$ is arithmetically Cohen–Macaulay.

Theorem 3.1.8 (Castelnuovo's Theorem - Parte II). *Let $\mathcal{C}$ be an irreducible non-degenerated projective curve of degree d in $\mathbb{P}^r$ with $r \neq 5$. If $d \geq 2r+1$ and the genus of $\mathcal{C}$ is $\pi(r,d)$, then $\mathcal{C}$ lies in a normal rational surface Scroll S. In addition, if S is smooth then the class of $\mathcal{C}$ in S is given by*

$$[\mathcal{C}] = (m+1)H - (r-2-\varepsilon)R$$

or when $\varepsilon = 0$

$$[\mathcal{C}] = mH + R,$$

where H is the class of a hyperplane section and R is the class of a ruling on S.

Lemma 3.1.9. *A curve $\mathcal{C}$ as in #20 of Table 3.1 is pentagonal with exactly two $\mathfrak{g}^1_5$, if both projections $\pi_{i|\mathcal{C}}$ belong to $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(5,5)|$, or just one $\mathfrak{g}^1_5$, if both $\pi_{i|\mathcal{C}}$ are in the Hirzebruch surface F_2.*

Proof. From Lemma 3.1.2 it follows that $\mathcal{C}$ is non-trigonal. Now we will analyze three cases. First, let us assume that both projections $\pi_{1|C}$ and $\pi_{2|\mathcal{C}}$ are birational onto their images. Thus $\pi_i(\mathcal{C}) \subset \mathbb{P}^3$ is an integral non-degenerate curve of degree 10 and arithmetic genus at least 11. By Castelnuovo's Theorems 3.1.7 and 3.1.8, $\pi_{i|\mathcal{C}}$ is an embedding and $\pi_i(\mathcal{C})$ is the complete intersection of an integral quadric surface $T \subset \mathbb{P}^3$ and a degree 5 surface S. First, assume that T is smooth. In this case $\pi_{i|\mathcal{C}} \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(5,5)|$, $\mathcal{C}$ is 5-gonal and it has exactly two g^1_5 by the Brill-Noether theory of curves on $\mathbb{P}^1 \times \mathbb{P}^1$. Now assume that T is a quadric cone. Since the complete intersection $\pi_i(\mathcal{C}) = S \cap T$ is smooth, S does not contain the vertex of T. The minimal desingularization of T is the Hirzebruch surface F_2. By the Brill-Noether theory of curves on F_2 the curve $\pi_i(\mathcal{C})$ is 5-gonal and it has a unique g^1_5. Thus either $\pi_1(\mathcal{C})$ and $\pi_2(\mathcal{C})$ are contained in a smooth quadric or both are contained in a quadric cone. Both cases may occur with the following construction. We fix an integral quadric surface T and take a smooth curve $\mathcal{C}$ which is the complete intersection of T and a quintic surface. Then we embed $\mathcal{C} \subset \mathbb{P}^3$ into $\mathbb{P}^3 \times \mathbb{P}^3$ using the diagonal embedding of $\mathbb{P}^3$ into $\mathbb{P}^3 \times \mathbb{P}^3$.

Now assume that exactly one among $\pi_{1|\mathcal{C}}$ and $\pi_{2|\mathcal{C}}$ is not birational onto its image, say $\deg(\pi_{1|\mathcal{C}}) > 1$. Since $\mathcal{C}$ has bidegree $(10,10)$, we obtain $\deg(\pi_{1|\mathcal{C}}) = 2$ and $\deg(\pi_1(\mathcal{C})) = 5$. Thus $\pi_1(\mathcal{C})$ has geometric genus at most 2. The first case above applied for π_2 gives that $\mathcal{C}$ is 5-gonal, contradicting Castelnuovo-Severi inequality applied to the g^1_5 and the degree 2 map $\mathcal{C} \to D$, where D is the normalization of $\pi_1(\mathcal{C})$. Hence this case did not occur.

Finally we assume that neither $\pi_{1|\mathcal{C}}$ nor $\pi_{2|\mathcal{C}}$ are birational onto their images. Both have degree 2 and their images have at most geometric genus 2. By the Castelnuovo-Severi inequality $\pi_{1|\mathcal{C}}$ and $\pi_{2|\mathcal{C}}$ are the same, contradicting the assumption that $\mathcal{C}$ is embedded in $\mathbb{P}^3 \times \mathbb{P}^3$. □

Remark 3.1.7. The above Lemma 3.1.9 and Remark 3.1.6 assure that a curve as in case #20 is non-bielliptic.

For the next result, it is required another result due to Coppens and Martens, namely [CM00, part (v) of Proposition 2.2], that assures the following:

Proposition 3.1.10. *Let $s(r) = s(r, \mathcal{C})$ be the minimal degree of complete, base point free, and simple linear series of dimension $r \geq 2$ on a smooth curve $\mathcal{C}$. If $\mathcal{C}$ is a bielliptic curve of genus $g \geq 4$, then*

$$s(r) = g + r - 1 \text{ for } 2 \leq r \leq g.$$

Theorem 3.1.11. *There is no embedding $u = (u_1, u_2) : \mathcal{C} \to \mathbb{P}^1 \times \mathbb{P}^4$ of a bielliptic curve of genus 8 such that $K_{\mathcal{C}} \cong u^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^4}(1,1))$, $u_1(\mathcal{C})$ spans $\mathbb{P}^1$ and $u_2(\mathcal{C})$ spans $\mathbb{P}^4$.*

Proof. Let's do by contradiction, start by taking the bundles $L_1 := u_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $L_2 := u_2^*(\mathcal{O}_{\mathbb{P}^4}(1))$. By assumption, we have $h^0(\mathcal{C}, L_1) \geq 2$, $h^0(\mathcal{C}, L_2) \geq 5$ and $L_2 \cong K_C - L_1$. Set $a := \deg(L_1)$. Thus $\deg(L_2) = 14 - a$. Since $\mathcal{C}$ is bielliptic and $g = 8$, we obtain $a \geq 4$, and the double covering $\pi : \mathcal{C} \to E$ is unique by the Castelnuovo–Severi inequality. In particular, $\mathcal{C}$ is a double covering of a unique elliptic curve. Now we analyze the cases when $u_2|\mathcal{C}$ is birational onto its image and the case when it is not.

Assuming that $u_{2|\mathcal{C}}$ is birational onto its image, it follows from Proposition 3.1.10, that $14 - a \geq 8 + 4 - 1$, providing a contradiction.

Now let us assume that $u_{2|\mathcal{C}}$ is not birational onto its image. Note that $14 - a = \deg(u_2)\deg(u_2(\mathcal{C}))$. Since $u_2(\mathcal{C})$ spans $\mathbb{P}^4$, $\deg(u_2(\mathcal{C})) \geq 4$. Hence $a = 4$. Since $\mathcal{C}$ is not hyperelliptic, $u_2(\mathcal{C})$ has positive geometric genus. Thus $u_2(\mathcal{C})$ is the curve E embedded in $\mathbb{P}^4$ by a degree 5 complete linear system, and u_2 is the composition of π with this embedding. Write $L_2 = \pi^*(R_2)$ with $R_2 \in \mathrm{Pic}^5(E)$. Since $a = 4$, $L_1 = \pi^*(R_1)$ for some $R_1 \in \mathrm{Pic}^2(E)$. Thus $K_{\mathcal{C}} \cong \pi^*(R)$ for some $R \in \mathrm{Pic}^7(E)$. In characteristic not 2 this is impossible for the following reason. Since E has genus 1, $h^0(E, R) = 7$. Since $\mathcal{C}$ has genus 8, $h^0(\mathcal{C}, K_{\mathcal{C}}) = 8$. The Riemann-Hurwitz formula says that π is ramified at 14 distinct points, say $p_1, \ldots, p_{14}$, and $\pi_*(\mathcal{O}_{\mathcal{C}}) \cong \mathcal{O}_E \oplus \mathcal{O}_E(-p_1 - \cdots - p_{14})$. Thus $8 =$

#	g	divisors	ambient	bidegree of $\mathcal{C}$	gonality	obs
#1	5	$(1,1)$, $(2,3)$	$\mathbb{P}^1\times\mathbb{P}^2$	$(3,5)$	3	
#2	6	$(2,1)$, $(1,3)$	$\mathbb{P}^1\times\mathbb{P}^2$	$(3,7)$	3	
#3	6	$(1,2)$, $(2,2)$	$\mathbb{P}^1\times\mathbb{P}^2$	$(4,6)$	?	$\mathrm{gon}(\mathcal{C})\in\{3,4\}$
#4	6	two $(1,1)$, one $(1,3)$	$\mathbb{P}^1\times\mathbb{P}^3$	$(3,7)$	3	
#5	7	$(1,1)$, $(2,1)$, $(0,3)$	$\mathbb{P}^1\times\mathbb{P}^3$	$(3,9)$	3	
#6	7	$(1,1)$, two $(1,2)$	$\mathbb{P}^1\times\mathbb{P}^3$	$(4,8)$	4	
#7	7	two $(1,1)$, $(2,1)$, $(0,3)$	$\mathbb{P}^1\times\mathbb{P}^4$	$(3,12)$	3	
#8	7	three $(1,1)$, one $(0,3)$	$\mathbb{P}^1\times\mathbb{P}^4$	$(3,9)$	3	
#9	7	$(1,1)$, $(1,1)$, $(2,2)$	$\mathbb{P}^2\times\mathbb{P}^2$	$(6,6)$	?	$\mathrm{gon}(\mathcal{C})\in\{3,4\}$
#10	8	$(1,1)$, $(2,0)$, $(1,3)$	$\mathbb{P}^2\times\mathbb{P}^2$	$(6,8)$	3	
#11	8	$(1,1)$, $(1,1)$, $(1,2)$, $(0,2)$	$\mathbb{P}^1\times\mathbb{P}^4$	$(4,10)$	4	non bielliptic
#12	8	$(1,1)$, $(2,1)$, $(1,2)$	$\mathbb{P}^2\times\mathbb{P}^2$	$(7,7)$	?	$\exists\,\mathfrak{g}^2_7,\ \alpha^{\otimes 2}\neq \mathrm{K}_{\mathcal{C}}$
#13	9	$(1,1)$, $(2,1)$ $(0,2)$, $(0,2)$	$\mathbb{P}^1\times\mathbb{P}^4$	$(4,12)$	4	
#14	9	$(2,0)$, $(0,2)$, $(2,2)$	$\mathbb{P}^2\times\mathbb{P}^2$	$(8,8)$	4	bielliptic, two $\mathfrak{g}^1_4$
#15	9	$(2,0)$, $(1,2)$, $(1,2)$	$\mathbb{P}^2\times\mathbb{P}^2$	$(8,8)$	4	
#16	9	three $(1,1)$, two $(0,2)$	$\mathbb{P}^1\times\mathbb{P}^5$	$(4,12)$	4	
#17	9	$(1,1)$, $(1,1)$, $(1,1)$, $(1,2)$	$\mathbb{P}^2\times\mathbb{P}^3$	$(7,9)$	?	
#18	10	$(1,1)$, $(1,1)$, $(0,3)$, $(2,0)$	$\mathbb{P}^2\times\mathbb{P}^3$	(6,12)	3	
#19	11	four $(1,1)$, one $(0,2)$	$\mathbb{P}^2\times\mathbb{P}^4$	$(8,12)$	?	
#20	11	five $(1,1)$	$\mathbb{P}^3\times\mathbb{P}^3$	$(10,10)$	5	one or two $\mathfrak{g}^1_5$
#21	6	$(0,2)$, $(3,2)$	$\mathbb{P}^1\times\mathbb{P}^2$	$(4,6)$	3	

Table 3.2: CI curves in $\mathbb{P}^n\times\mathbb{P}^m$ with $\mathrm{K}_{\mathcal{C}}=\mathcal{O}_{\mathcal{C}}(1,1)$

$h^0(E,\pi_*(\pi^*(R))=h^0(E,R)+h^0(E,R(-P_1-\cdots-p_{14}))$. Since $\deg(R(-P_1-\cdots-p_{14})=-7$, $h^0(E,R(-P_1-\cdots-p_{14}))=0$, that provides a contradiction. $\square$

Remark 3.1.8. It follows from Theorem 3.1.11 and Remark 3.1.6 that a curve as in case #11 of Table 3.1 is non-bielliptic.

Lemma 3.1.12. *A smooth curve of genus 8 in $\mathbb{P}^2\times\mathbb{P}^2$ given by divisors of bidegrees $(2,0)$, $(0,2)$ and $(2,2)$, case #14 of Table 3.1, is bielliptic.*

Proof. To see this we just have to consider the morphism $\pi: C\to E$, where E is the elliptic curve in the quadric surface $\mathbb{P}^1\times\mathbb{P}^1$ given by a divisor in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)|$. $\square$

We finish this section by collecting all the previous results and remarks on smooth curves in the biproduct of projective spaces.

Theorem 3.1.13. *The complete intersection smooth curves in $\mathbb{P}^n\times\mathbb{P}^m$ of genus g, with $n+m\geq 3$ and whose canonical sheaf is $\mathrm{K}_C=\mathcal{O}_C(1,1)$ are exactly those presented in Table 3.2.*

Note that there are four missing cases in Table 3.2 that we are not able to manage. In the next two sections, we try to handle these cases using another approach.

3.2. On the Gonality of C.I. Curves in Biprojectic Spaces

Next, we provide an extension of Lazarsfeld's Theorem 2.2.6 to complete intersection curves in the product of two projective spaces, whose proof is inspired by that presented in [Laz97], using Miyaoka's Theorem on rank two vector bundles over 3-folds and Nakai–Moichezon's criterion for ampleness. We also must refer to the lecture notes by B. Ullery [Ull17] on linear systems and positivity of vector bundles, these notes helped us to understand the main Lazarsfeld's techniques to show Theorem 2.2.6.

Since our curve $\mathcal{C}$ it is assumed to be in $\mathbb{P}^n \times \mathbb{P}^m$, we have to impose some restriction on the divisors providing $\mathcal{C}$.

Definition 3.2.1. Let $\mathcal{C}$ be a complete intersection curve in a smooth variety $\mathcal{Y}$, with $\dim \mathcal{Y} = n$, given by $n-1$ divisors $D_i \in |\mathcal{O}_{\mathcal{Y}}(a_i, b_i)|$. We say that the pair $(\mathcal{C}, \mathcal{Y})$ is *k-Lefschetz* if the k-fold Y_0 given by the divisors D_i, for $i = k, \ldots, n-1$ is such that its Picard group is isomorphic to the Picard group of $\mathcal{Y}$,

$$\mathrm{Pic}(Y_0) \cong \mathrm{Pic}(\mathcal{Y}).$$

Example 3.2.1. The Lefschetz hyperplane theorem, c.f. Theorem 2.1.6, assures that a complete intersection curve $\mathcal{C}$ in $\mathbb{P}^n \times \mathbb{P}^m$ given by $n+m-1$ divisors $D_i \in |\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}(a_i, b_i)|$ with $a_i, b_i > 0$ for $i = 1, \ldots, n+m-1$ and the picard of three-fold generated for $2 \leq k \leq n+m-2$ is $\mathbb{Z} \times \mathbb{Z}$, provided that the varieties given by any $\ell \leq n+m-2$ divisors D_i is smooth. In this case $(\mathcal{C}, \mathbb{P}^n \times \mathbb{P}^m)$ k-Lefschetz for any $2 \leq k \leq n+m-2$.

Remark 3.2.1. Let us fix a complete intersection curve $\mathcal{C}$ in $\mathbb{P}^n \times \mathbb{P}^m$ given by divisors D_i of bidegrees (a_i, b_i), with $i = 1, \ldots, n+m-1$, and assume that

$$a_1 \leq a_2 \quad \text{and} \quad b_1 \leq b_2.$$

Also assume that the 3-fold $Y_0 \supset C$ given by D_i, $i = 3, \ldots, n+m-1$, is smooth, or more generally that $(\mathcal{C}, \mathbb{P}^n \times \mathbb{P}^m)$ is 3-Lefschetz. Hence

$$\mathrm{Pic}(Y_0) = \mathbb{Z} \times \mathbb{Z}.$$

We also fix the classes $H_1 = \mathcal{O}_{Y_0}(1,0)$ and $H_2 = \mathcal{O}_{Y_0}(0,1)$ in Y_0.

Now let us take $\pi : Y \longrightarrow Y_0$ the blow-up along $\mathcal{C}$ with exceptional divisor E. By abuse of notation, we also denote $\pi : E \longrightarrow C$ the restriction of π to E. Let $\mathcal{B}$ be a divisor on $\mathcal{C}$ giving a non constant degree d morphism $\psi : C \longrightarrow \mathbb{P}^1$ and take its pullback to Y, $\mathcal{L} = \pi^*(\mathcal{B})$. Finally denote $\mathcal{L} = \mathcal{O}_Y(B)$, where B is an effective divisor.

With the above notation and construction, we obtain an exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{O}_Y^2 \longrightarrow \mathcal{L} \longrightarrow 0, \tag{3.2}$$

where F is rank 2 vector bundle over Y whose first Chern class is $c_1(F) = -E$, because $\det(F) = \mathcal{O}_Y(-E)$. The second Chern class is $c_2(F) = B$. To show this we start by taking the ideal exact sequence of E,

$$0 \to \mathcal{O}_Y(-E) \to \mathcal{O}_Y \to \mathcal{O}_E \to 0$$

and so $-c_2(\mathcal{O}_E) = c_2(\mathcal{O}_{-E}) + c_1(\mathcal{O}_Y(-E))c_1(\mathcal{O}_E) = 0 - E^2$. Now, taking the pushforward of the above ideal sequence we obtain

$$0 \to \mathcal{O}_E \to \mathcal{L} \to \mathcal{O}_B(\mathcal{L}) \to 0.$$

By Whitney formula, the total Chern class of $\mathcal{L}$ is $c_t(\mathcal{L}) = (1+Et+Et^2+\dots)(1-Bt^2+\dots)$ and so $c_2(\mathcal{L}) = E^2 - B$. Now $c_2(F) = -c_1(F)c_1(\mathcal{L}) - c_2(\mathcal{L}) = E^2 - E^2 + B = B$.

Let H_1 and H_2 be the pullback to Y of the two hyperplane sections on Y_0. For each rational number $\epsilon \geq 0$, let us consider the following two $\mathbb{Q}$-divisors on Y:

$$D_\epsilon := (a_2 + \epsilon)H_1 + (b_2 + \epsilon)H_2 - E$$

and

$$D := a_2 H_1 + b_2 H_2 - E.$$

We have to notice that $|D| = |V|$ with $V \subset \mathrm{H}^0(Y, \mathcal{O}_Y(a_2, b_2))$ the locus containing $\mathcal{C}$ and $|D|$ is globally generated. To avoid a heavy notation, we use the same symbols H_1, H_2, and E to denote their classes in the Chow ring of Y.

The next result can be found in Eisenbud–Harris 3264 book [EH16, Section 13.6.3] and it will be useful to prove Lemma 3.2.2 below.

Lemma 3.2.1. *Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be a smooth 3-fold whose Picard group is $\mathbb{Z} \times \mathbb{Z}$. If Y*

is the blow-up of X along a smooth curve $\mathcal{C}$ with exceptional divisor E, then in the Chow ring of Y the following hold:

1. $\deg(H_1^2 \cdot E) = \deg(H_2^2 \cdot E) = \deg(H_1 \cdot H_2 \cdot E) = 0$,

   where H_1 and H_2 are the pullback to Y of hyperplane sections h_1 and h_2 generating $\mathrm{Pic}(X)$, *and E is the class of the exceptional divisor;*

2. $\deg(H_1 \cdot E^2) = -d_1$ *and* $\deg(H_2 \cdot E^2) = -d_2$, *where d_i is* $\deg(C \cdot h_i)$*;*

3. $\deg(E^3) = -deg(\mathcal{N}_{\mathcal{C}/X})$, *where $\mathcal{N}_{\mathcal{C}/X}$ stands for the normal sheaf of $\mathcal{C} \subset X$.*

Lemma 3.2.2. *D_ϵ is ample for every $\epsilon > 0$.*

Proof. The proof we will use is the Nakai–Moichezon's criterion for ampleness, c.f. Theorem 2.3.2. Then we have to show that

$$D_\epsilon^{\dim X} \cdot X > 0$$

for every integral subscheme X.

Let us first assume that X is an integral curve on Y. If X is not contained in the exceptional divisor E, then there is a hyperplane section on Y_0 containing the image of X and not containing $\mathcal{C}$. Thus, pulling back this section we obtain $\epsilon(H_1 + H_2)X > 0$. Then we are able to choose a surface $S \in |D|$, away from E, such that $D_\epsilon = S + \epsilon(H_1 + H_2)$, and $X \not\subset S$. Hence $D_\epsilon \cdot X > 0$. Now let us assume that $X \subset E$. Then take a surface S not containing X. In this case $S \cdot X \geq 0$. If $S \cdot X > 0$ we proceed as before. Now, if $S \cdot X = 0$, then X is a horizontal divisor on E, i.e. intersects every fiber of $\pi : E \to C$. In this case, we can choose a hyperplane section on Y_0 intersecting $\mathcal{C}$ transversely. Taking the pullback of this section we obtain again that $\epsilon(H_1 + H_2)X > 0$, and we are done.

Now let us assume that $X = aH_1 + bH_2 - cE$ is a surface that is given by the pullback of a surface in Y_0. We have to compute $D_\epsilon^2 \cdot X$. To organize the computations, let us first note that since $\mathcal{C}$ is a complete intersection of divisors of bidegrees (a_1, b_1) and (a_2, b_2) on Y_0, we have $\mathcal{N}_{C/Y_0} = \mathcal{O}_{\mathcal{C}}(a_1, b_1) \oplus \mathcal{O}_{\mathcal{C}}(a_2, b_2)$. By Lemma 3.2.1 item (3) we obtain

$$-\deg(E^3) = \deg(\mathcal{N}_{C/Y_0}) = \deg(\det(\mathcal{N}_{C/Y_0})) = d_1(a_1 + a_2) + d_2(b_1 + b_2). \tag{3.3}$$

The second item of Lemma 3.2.1 assures that

$$\deg(H_i \cdot E^2) = -d_i = -\deg(C \cdot h_i),$$

where $h_1 = \mathcal{O}_{Y_0}(1,0)$ and $h_2 = \mathcal{O}_{Y_0}(0,1)$ are hyperplane sections on Y_0. Hence

$$d_1 = a_1a_2H_1^3 + (a_1b_2 + a_2b_1)H_1^2H_2 + b_1b_2H_1H_2^2 \tag{3.4}$$

and

$$d_2 = a_1a_2H_1^2H_2 + (a_1b_2 + a_2b_1)H_1H_2^2 + b_1b_2H_2^3. \tag{3.5}$$

Going back to $D_\epsilon^2 \cdot X$, by item (1) of Lemma 3.2.1, we may write

$$\begin{aligned} D_\epsilon^2 \cdot X = & \; (\,(a_2+\epsilon)H_1 + (b_2+\epsilon)H_2 - E\,)^2 \cdot (aH_1 + bH_2 - cE) \\ = & \; (aH_1+bH_2)((a_2+\epsilon)H_1 + (b_2+\epsilon)H_2)^2 \\ & -2E(aH_1+bH_2)((a_2+\epsilon)H_1 + (b_2+\epsilon)H_2) \\ & +(aH_1+bH_2)E^2 - cE(\,(a_2+\epsilon)H_1 + (b_2+\epsilon)H_2\,)^2 \\ & +2cE^2(\,(a_2+\epsilon)H_1 + (b_2+\epsilon)H_2\,) - cE^3 \\ = & \; (aH_1+bH_2)((a_2+\epsilon)H_1 + (b_2+\epsilon)H_2)^2 + (aH_1+bH_2)E^2 \\ & +2cE^2(\,(a_2+\epsilon)H_1 + (b_2+\epsilon)H_2\,) - cE^3 \end{aligned}$$

Now, by items (2) and (3) of Lemma 3.2.1 and equation 3.3 we obtain

$$\begin{aligned} D_\epsilon^2 \cdot X = & \; (aH_1+bH_2)((a_2+\epsilon)H_1 + (b_2+\epsilon)H_2)^2 - ad_1 - bd_2 \\ & -2c(\,d_1(a_2+\epsilon) + d_2(b_2+\epsilon)\,) + c(\,d_1(a_1+a_2) + d_2(b_1+b_2)\,). \end{aligned}$$

Substituting the values of d_1 and d_2, equations 3.4 and 3.5, in the above expression of $D_\epsilon^2 \cdot X$, we get an expression with 48 terms. The way we choose to order this expression is to consider $D_\epsilon^2 \cdot X$ as a polynomial in the indeterminate ϵ, say

$$D_\epsilon^2 \cdot X = \mathfrak{p}_2\epsilon^2 + \mathfrak{p}_1\epsilon + \mathfrak{p}_0,$$

where in $\mathfrak{p}_0$ is the constant term, and $\mathfrak{p}_1$ and $\mathfrak{p}_2$ collect the terms attached to ϵ and ϵ^2. The polynomial $\mathfrak{p}_0$ has 26 terms, while $\mathfrak{p}_1$ and $\mathfrak{p}_2$ 16 and 6 terms, respectively. We implemented these computations in Maple software.

Thus we have

$$\mathfrak{p}_2 = aH_1^3 + 2aH_1^2H_2 + aH_1H_2^2 + bH_1^2H_2 + 2bH_1H_2^2 + bH_2^3$$

and we easily conclude that $\mathfrak{p}_2 > 0$, because it is assumed $\mathbb{P}^n \times \mathbb{P}^m$ with $n+m > 2$ and a, b are two nonnegatives integers with $a \cdot b > 0$.

Now we consider the polynomial $\mathfrak{p}_1$ as a polynomial in the variable H_1. The constant term and the coefficients attached to H_1, H_1^2 and H_1^3 of $\mathfrak{p}_1$ are, respectively

$$\begin{aligned}
&2b_2(b-cb_1)H_2^3,\\
&(2b_2(a-ca_1)+2(a_2+b_2)(b-cb_1))H_2^2,\\
&(2a_2(b-cb_1)+2(a_2+b_2)(a-ca_1))H_2 \ \text{ and}\\
&2a_2(a-ca_1).
\end{aligned}$$

Recall it is assumed $a_2 \geq a_1 \geq 0$ and $b_2 \geq b_1 \geq 0$. Since $\mathcal{C} \subset Y_0$ is given by the intersection of two divisors of bidegrees (a_1, b_1) and (a_2, b_2) and it is not a plane curve, it follows that in the blowup Y_0 the numbers $(a - ca_1)$ and $(b - cb_1)$ are both non negatives. Hence $\mathfrak{p}_1 \geq 0$.

Let us also take $\mathfrak{p}_0$ as a polynomial in H_1, its constant term and its coefficients attached to H_1, H_1^2 and H_1^3 of $\mathfrak{p}_0$ are, respectively

$$\begin{aligned}
&b_2(b_2-b_1)(b-cb_1)H_2^3,\\
&((a_2(b_2-b_1)+b_2(a_2-a_1))\,(b-cb_1)+b_2(b_2-b_1)(a-ca_1))H_2^2,\\
&((b_2(a_2-a_1)+a_2(b_2-b_1))\,(a-ca_1)+a_2(a_2-a_1)(b-cb_1))H_2 \ \text{ and}\\
&a_2(a_2-a_1)(a-ca_1)
\end{aligned}$$

By our assumptions, we also conclude that $\mathfrak{p}_0 \geq 0$. Hence we have shown that

$$D_\epsilon^2 \cdot X = \mathfrak{p}_2\epsilon^2 + \mathfrak{p}_1\epsilon + \mathfrak{p}_0 > 0$$

whenever X is the pullback of a surface in Y_0. The remaining case in dimension two is when X is the exceptional divisor. In this case, it is very simple to conclude that

$$D_\epsilon^2 \cdot E = d_1(a_2 - a_1) + d_2(b_2 - b_1) + 2\epsilon(d_1 + d_2) > 0.$$

The computations to show that $D_\epsilon^3 > 0$ are completely analogous. By considering it as a polynomial in ϵ, the term of degree 3 is

$$H_1^3 + 3H_1^2H_2 + 3H_1H_2^2 + H_2^3,$$

so it is positive. To prove that the coefficients attached to ϵ^i with $i \leq 2$ are non-negative, we proceed exactly as before, assuming that they are polynomials in H_1. We do not display this computation, because they are analogous to the previous case. □

Note that in the proof of Lemma 3.2.2 the terms of the polynomials $\mathfrak{p}_1$ and $\mathfrak{p}_0$ are symmetric, and are realized by the permutation cycles $(a\,b)$, $(a_1\,b_1)$ and $(a_2\,b_2)$.

For future use, we have to make the following technical Remark.

Remark 3.2.2. Recall from Remark 3.2.1 that $\mathcal{L} = \mathcal{O}_Y(B)$, with B an effective divisor, is the pullback to Y of a line bundle on $\mathcal{C}$ providing a $\mathfrak{g}^1_d$. We will apply Miyaoka instability theorem 2.3.5 by considering the 3-fold Y, the globally generated divisor $D := a_2H_1 + b_2H_2 - E$ and the ample divisor $D_\epsilon := (a_2+\epsilon)H_1 + (b_2+\epsilon)H_2 - E$.

We have to compute $(c_1(F)^2 - 4c_2(F))D_\epsilon = (E^2 - 4B)D_\epsilon$, where F is the rank two bundle introduced in exact sequence 3.2. We can choose the hyperplane sections H_1 and H_2 avoiding the points of the g^1_d, i.e $H_1 \cdot B = H_2 \cdot B = 0$. Taking a divisor $D_0 = a_2H_1 + b_2H_2$ such that $B \cdot D = B \cdot (D_0 - E) = d$, we obtain

$$
\begin{aligned}
(c_1(F)^2 - 4c_2(F))D_\epsilon &= (E^2 - 4B)D_\epsilon \\
&= E^2(a_2+\epsilon)H_1 + E^2(b_2+\epsilon)H_2 - E^3 - 4d \\
&= (a_1-\epsilon)d_1 + (b_1-\epsilon)d_2 - 4d.
\end{aligned}
$$

Now consider the surface $Z \subset Y_0$ given by the divisor of bidegree (a_2, b_2) on Y_0. We can write each d_i in terms of the surface Z, that is $\mathcal{C} = a_1h_1 + b_1h_2$ in the Chow ring of Z, where the $h_i = H_i|_Z$, and then $d_i = h_i \cdot \mathcal{C}$. Hence we can write

$$
\begin{aligned}
(c_1(F)^2 - 4c_2(F))D_\epsilon = \;& a_1(a_1-\epsilon)h_1^2 + b_1(b_1-\epsilon)h_2^2 \\
&+ [a_1(b_1-\epsilon) + b_1(a_1-\epsilon)]h_1h_2 - 4d
\end{aligned} \tag{3.6}
$$

Now, if we assume that $(c_1(F)^2 - 4c_2(F))D_\epsilon > 0$, then Miyaoka's theorem assures that there is a subsheaf L of F such that

$$(2c_1(L) - c_1(F)) \cdot D_\epsilon \cdot D > 0.$$

Next, we study the sheaf L. Arguing in terms of biduals, we can assume that $L = \mathcal{O}_Y(-t_1H_1 - t_2H_2 - \mu E)$ is a line bundle in Y. We can assume that L drops rank on a codimension 2 subset of Y, otherwise, if $L \longrightarrow F$ does not drop rank along an effective divisor, say A, we can change L for $L(A)$ which drops rank.

Setting $S = a_2H_1 + b_2H_2 - E$ and $F' = F|_S$, we obtain

$$L|_S = \mathcal{O}_S(-t_1H_1 - t_2H_2 - \mu E) = \mathcal{O}_S(-t_1H_1 - t_2H_2 - \mu C) = \mathcal{O}_S(-\alpha H_1 - \beta H_2),$$

where $\alpha = t_1 + a_1\mu$ and $\beta = t_2 + b_1\mu$. Then we get the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-\alpha H_1 - \beta H_2) \longrightarrow F' \longrightarrow M \oplus I_W \longrightarrow 0$$

where M is a line bundle and W is finite. We know that

$$c_1(F') = c_1(F)|_S = -a_1 H_1 - b_1 H_2$$

and using the exact sequence we obtain $c_1(M) = (\alpha - a_1)H_1 + (\beta - b_1)H_2$. Note that $c_2(F') = c_2(F)|_S = d$ and then

$$c_2(F') = -(\alpha H_1 + \beta H_2)((\alpha - a_1)H_1 + (\beta - b_1)H_2) + \text{length}(W).$$

By construction $\mathrm{H}^0(Y, F) = 0$. Since $F \longrightarrow \mathcal{O}_Y^2$ drops rank along E, we have

$$0 \longrightarrow F' \longrightarrow \mathcal{O}_S^2 \longrightarrow \mathcal{L} \longrightarrow 0$$

and $\mathrm{H}^0(S, F') = 0$. By the inclusion $L|_S \longrightarrow F'$ and the fact that $\mathrm{H}^0(S, F') = 0$ we have $\mathrm{H}^0(S, L|_S) = 0$.

Hence we conclude that $\alpha \geq 0$ and $\beta \geq 0$ and both can not be zero simultaneously. And thus we finish the technical Remark, where the most important part is:

If $(c_1(F)^2 - 4c_2(F))D_\epsilon > 0$, then there exists L subsheaf of F, such that its restriction to the surface S is $L|_S = \mathcal{O}_S(-\alpha H_1 - \beta H_2)$ with $\alpha \geq 0$ and $\beta \geq 0$ and both cannot be zero simultaneously.

Now we are ready to establish the two main results of this section. For both proofs we assume all the notation fixed in this section, in particular the Remarks 3.2.1 and 3.2.2.

Theorem 3.2.3. *Let $\mathcal{C} = \cap_{i=1}^{n+m-1} D_i$ be a complete intersection curve in $\mathbb{P}^n \times \mathbb{P}^m$ where D_i are divisors of bidegrees (a_i, b_i). Assume that $\mathcal{C}$ is a curve described in Remark 3.2.1 and with $(a_1, b_1) = (1, 1)$. Taking $\kappa \in \mathbb{Q}_{\geq 0}$ such that $2H_1H_2 \leq H_1^2 + H_2^2 + 4\kappa$, where H_i are restrictions of hyperplane to the surface $\cap_{i=2}^{n-m-1} D_i$, we obtain*

$$\text{gon}(C) \geq H_1H_2 - \kappa$$

whenever:

- *$a_2 = b_2$ and $H_1^2 + H_2^2 \leq 4H_1H_2$ or*
- *$a_2 \neq b_2$ and $(a_2 - 1)h_1^2 + (b_2 - 1)H_2^2 \leq (min\{3a_2 + b_2 - 4, 3b_2 + a_2 - 4\})H_1H_2$.*

Proof. Let us assume that d is such that $d < H_1H_2 - \kappa$. Let Y be the blow-up of the 3-fold Y_0 at $\mathcal{C}$ and F the respective rank two vector bundle. Denote $D = a_2h_1 + b_2h_2 - E$, we know $D_\epsilon = D + \epsilon(h_1 + h_2)$ is ample. It follows from the fact that $2H_1H_2 \leq H_1^2 + H_2^2 + 4\kappa$ and Equation 3.6 that we can choose $\epsilon > 0$ such that $(c_1(F)^2 - 4c_2(F))D_\epsilon > 0$. Hence Miyaoka Theorem for 3-folds applied to D_ϵ and D assures that there is a subsheaf L of F such that

$$(2c_1(L) - c_1(F))D_\epsilon D > 0,$$

since $L|_D = \mathcal{O}_D(-\alpha, -\beta)$, we obtain

$$\begin{aligned}(2c_1(L) - c_1(F))D_\epsilon D =&(1-2\alpha)[(a_2 - 1 + \epsilon)H_1^2 + (b_2 - 1 + \epsilon)H_1H_2]+ \\ &(1-2\beta)[(a_2 - 1 + \epsilon)H_1H_2 + (b_2 - 1 + \epsilon)H_2^2] > 0\end{aligned} \tag{3.7}$$

We can make computations in the original surface inside Y_0, because they only depend on H_i. Since the terms inside the brackets are always positive, we get $2\alpha < 1$ or $2\beta < 1$. From the fact that, $H^0(F) = 0$, we obtain $\alpha = 0$ or $\beta = 0$. Let us assume that $\alpha = 0$. By the two distinct computations of $c_2(F')$, c.f. Remark 3.6, we obtain

$$(\alpha - \alpha^2)H_1^2 + (\alpha + \beta - 2\alpha\beta)H_1H_2 + (\beta - \beta^2)H_2^2 \leq d$$

and

$$(1 - \beta)H_1H_2 + (-\beta + \beta^2)H_2^2 > \kappa \geq 0,$$

Hence we get $\beta > 1$ and $\beta H_2^2 > H_1H_2$. If $a_2 = b_2$ in equation 3.7, we obtain

$$H_1^2 + H_1H_2 > (2\beta - 1)(H_2^2 + H_1H_2).$$

Since $\beta > 1$ and $\beta H_2^2 > H_1H_2$, we obtain $H_1^2 + H_2^2 > 4H_1H_2$, contradicting the first item of the statement of the Theorem.

If $a_2 \neq b_2$, then taking ϵ is sufficiently small in Equation 3.7, we obtain

$$(a_2 - 1)H_1^2 + (b_2 - 1)H_1H_2 \geq (2\beta - 1)[(a_2 - 1)H_1H_2 + (b_2 - 1)H_2^2].$$

Again, since $\beta > 1$ and $\beta H_2^2 > H_1H_2$, then $(a_2 - 1)H_1^2 + (b_2 - 1)H_2^2 > (3a_2 + b_2 - 4)H_1H_2$. Now, if $\beta = 0$, then the right side is $(3b_2 + a_2 - 4)H_1H_2$, contradicting the second item of the statement of the Theorem. □

Note if our ambient is $\mathbb{P}^1 \times \mathbb{P}^n$, then $H_1^2 = 0$, simplifying our computations. This is what we do in the next theorem.

Theorem 3.2.4. *Let $\mathcal{C} = \cap_{i=1}^{n} D_i$ be a complete intersection curve in $\mathbb{P}^1 \times \mathbb{P}^n$ where D_i are divisors of bidegrees (a_i, b_i). Assume $\mathcal{C}$ satisfies the conditions of 3.2.1, and additionally $(a_1, b_1) = (0, 2)$. Taking X to be the surface $\cap_{i=2}^{n+m-1} D_i$, with hyperplane sections $H_1 = \mathcal{O}_X(1, 0)$ and $H_2 = \mathcal{O}_X(0, 1)$, if $\kappa \in \mathbb{Q}_{\geq 0}$ is such that $H_1 H_2 \leq H_2^2 + \kappa$, then*

$$gon(C) \geq H_1 H_2 - \kappa.$$

Moreover, if $b_2 \leq a_2$, then we can choose κ equal to zero.

Proof. Following the above Theorem, let us assume that $d < H_1 H_2 - \kappa$. Since $H_1 H_2 \leq H_2^2 + \kappa$ we can take ϵ such that $(c_1(F)^2 - 4c_2(F))D_\epsilon > 0$. Hence Miyaoka's theorem applied to D_ϵ and D assures that there is a subsheaf $L = \mathcal{O}_Y(-\alpha, -\beta)$ of F such that

$$(2c_1(L) - c_1(F))D_\epsilon D > 0,$$

Thus

$$(-2\alpha)[(b_2 - 1 + \epsilon)H_1 H_2] + (2 - 2\beta)[(a_2 - 1 + \epsilon)H_1 H_2 + (b_2 - 1 + \epsilon)H_2^2] > 0, \tag{3.8}$$

implying that $\beta = 0$. Since

$$2\alpha H_1 H_2 \leq d < H_1 H_2 - \kappa,$$

we obtain $(1 - 2\alpha)H_1 H_2 > \kappa \geq 0$, which is a absurd because $\alpha > 0$.

If we assume $b_2 \leq a_2$, since $H_1 H_2 = b_2 b_3 \dots b_n$ and $H_2^2 = \sum_{i=2}^{n} a_i b_2 \dots b_{i-1} b_{i+1} \dots b_n$, we get $H_1 H_2 \leq H_2^2$, and we are done. □

Corollary 3.2.5. *Let $\mathcal{C} = \cap_{i=1}^{n+m-1} D_i$ be a complete intersection curve in $\mathbb{P}^n \times \mathbb{P}^m$ where D_i are divisors of bidegrees (a_i, b_i). If the canonical sheaf of $\mathcal{C}$ is a hyperplane section, $\mathrm{K}_{\mathcal{C}} = \mathcal{O}_C(1, 1)$, then there is a surface $Z := \cap_{i=2}^{n+m-1} D_i$ such that*

$$H_1 H_2 - 1 \leq gon(C) \leq H_1 H_2,$$

where H_1 and H_2 are hyperplane sections of Z.

Proof. Taking the divisors exactly in the order that they appear in Table 3.2, the surface $Z = \cap_{i=2}^{n+m-1} D_i$ works for almost all curves, except for the cases #9, #12, #17 and #19.

For the case #17. Let Y be a three-fold generated by two divisors of bidegrees $(1, 1)$ and

$(1,2)$, arguing in the same way as in example 2.1.5, we obtain $\text{Pic}(Y) = \mathbb{Z} \times \mathbb{Z}$, and then pick the surface $Z = D \cap Y$, where D has bidegree $(1,1)$. Thus $H_1^2 = 2$, $H_2^2 = 4$ and $H_1H_2 = 5$. Now in Theorem 3.2.3 take $\kappa = 1$, then we get $gon(C) \geq H_1H_2 - 1$. Taking the first projection of $\mathbb{P}^2 \times \mathbb{P}^2$, we get a $\mathfrak{g}_7^2$ on $\mathcal{C}$. Since $\mathcal{C}$ is not a plane curve, then the divisor $D = \mathcal{O}_C(1,0)$ its not very ample, i.e there $p, q \in \mathcal{C}$ so $D - p - q$ is a g_5^1, that imply $gon(C) \leq H_1H_2 = 5$.

The case #19. We know that $\text{Pic}(D) = \mathbb{Z} \times \mathbb{Z}$, where D is given by the divisor of bidegree $(0,2)$, and by Lefschetz Theorems $\text{Pic}(Y) = \mathbb{Z} \times \mathbb{Z}$, where Y is given by three divisors of bidegrees $(1,1)$, $(1,1)$ and $(0,2)$. Then we can argue as in the case #17.

Finally the cases #9 and #12. Like the previous cases, we use Theorem 3.2.3 for $\kappa = 1$ and $\kappa = \frac{3}{2}$ respectively, so $gon(C) \geq H_1H_2 - 1$ and taking projections, and the fact that none of these curves is plane, we obtain $gon(C) \leq H_1H_2$. $\square$

Remark 3.2.3. Using Theorem 3.2.3 we can compute the gonality of almost all curves in Table 3.2. The only exception is case #2. But we can adapt the argument to achieve this case too. The steps are simple, first, we need a curve like in remark 3.2.1. For example, if $a_i > 0$, $b_i > 0$ and $(a_1, b_1) = (1,1)$, by Lefschetz Theorem $\text{Pic}(Y) = \mathbb{Z} \times \mathbb{Z}$, where $Y = \cap_{i=2}^{n+m-1}(a_i, b_i)$. Then compute H_1^2, H_2^2 and H_1H_2 in Y, and finally pick the smallest κ possible.

Using the results of this section we can update the information on curves in $\mathbb{P}^n \times \mathbb{P}^m$ whose canonical sheaf is a hyperplane section in Table 3.3.

3.3. On the locus of few CI curves in the biproduct

In this section, we provide a few results on the locus of complete intersection smooth curves in the biproduct with fixed genus and prescribed bidegrees. It is far from being a final study, however, we believe that some results are interesting and they can be the starting point for future developments. Let us start by fixing some notation.

Let $\mathbb{M}_g^{(d_1,d_2)}$ be the space of all smooth complete intersections curves in the biprojective space $\mathbb{P}^n \times \mathbb{P}^m$ of genus g and bidegree (d_1, d_2),

$$\mathbb{M}_g^{(d_1,d_2)} := \{\mathcal{C} \subset \mathbb{P}^n \times \mathbb{P}^m \,|\, \mathcal{C} \text{ is a ci , with } g(\mathcal{C}) = g \text{ and bidegree } (d_1, d_2)\}.$$

We also denote $\mathfrak{M}_g^{(d_1,d_2)}$ the image of $\mathbb{M}_g^{(d_1,d_2)}$ in $\mathcal{M}_g$, the moduli of smooth curves of genus g. If there is no confusion when the invariant g, and (d_1, d_2) are totally clear, for short we use $\mathbb{M} := \mathbb{M}_g^{(d_1,d_2)}$ and $\mathfrak{M} = \mathfrak{M}_g^{(d_1,d_2)}$.

#	g	divisors	ambient	bidegree of $\mathcal{C}$	gonality	$\lfloor \frac{g+3}{2} \rfloor$
#1	5	$(1,1)$, $(2,3)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(3,5)$	3	4
#2	6	$(2,1)$, $(1,3)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(3,7)$	3	4
#3	6	$(1,2)$, $(2,2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(4,6)$	3 or 4	4
#4	6	two $(1,1)$, one $(1,3)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(3,7)$	3	4
#5	7	$(1,1)$, $(2,1)$, $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(3,9)$	5	5
#6	7	$(1,1)$, two $(1,2)$	$\mathbb{P}^1 \times \mathbb{P}^3$	$(4,8)$	4	5
#7	7	two $(1,1)$, $(2,1)$, $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(3,12)$	3	5
#8	7	three $(1,1)$, one $(0,3)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(3,9)$	3	5
#9	7	$(1,1)$, $(1,1)$, $(2,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(6,6)$	3 or 4	5
#10	8	$(1,1)$, $(2,0)$, $(1,3)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(6,8)$	3	5
#11	8	$(1,1)$, $(1,1)$, $(1,2)$, $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(4,10)$	4	5
#12	8	$(1,1)$, $(2,1)$, $(1,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(7,7)$	4 or 5	5
#13	9	$(1,1)$, $(2,1)$ $(0,2)$, $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^4$	$(4,12)$	4	6
#14	9	$(2,0)$, $(0,2)$, $(2,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(8,8)$	4	6
#15	9	$(2,0)$, $(1,2)$, $(1,2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$(8,8)$	4	6
#16	9	three $(1,1)$, two $(0,2)$	$\mathbb{P}^1 \times \mathbb{P}^5$	$(4,12)$	4	6
#17	9	$(1,1)$, $(1,1)$, $(1,1)$, $(1,2)$	$\mathbb{P}^2 \times \mathbb{P}^3$	$(7,9)$	4 or 5	6
#18	10	$(1,1)$, $(1,1)$, $(0,3)$, $(2,0)$	$\mathbb{P}^2 \times \mathbb{P}^3$	$(6,12)$	3	6
#19	11	four $(1,1)$, one $(0,2)$	$\mathbb{P}^2 \times \mathbb{P}^4$	$(8,12)$	5 or 6	7
#20	11	five $(1,1)$	$\mathbb{P}^3 \times \mathbb{P}^3$	$(10,10)$	5	7
#21	6	$(0,2)$, $(3,2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$(4,6)$	3	4

Table 3.3: CI curves in $\mathbb{P}^n \times \mathbb{P}^m$ with $\mathrm{K}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(1,1)$

Since $\mathbb{M}$ is an integral quasi-projective variety, we may apply the semicontinuity theorem to study the gonality of an element of $\mathbb{M}$. A non-empty open subset of $\mathcal{V}$ has the maximal gonality among all $X \in \mathbb{M}$.

For all integers $d > 0$ and $0 \leq \delta \leq (d-1)(d-2)/2$, let $V(d,\delta)$ denote the set of all degree d integral plane curves with exactly δ ordinary nodes as only singularities,

$$V(d,\delta) := \{C \subset \mathbb{P}^2 \mid \deg(C) = d,\ C \text{ nodal with } \delta \text{ ordinary nodes}\}.$$

Thus the normalization of any $D \in V(d,\delta)$ has geometric genus $(d-2)(d-1)/2$. The algebraic set $V(d,\delta)$ is an irreducible variety of dimension $\binom{d+2}{2} - 1 - \delta$, see [Ha86]. Its closure in the Hilbert scheme of $\mathbb{P}^2$ contains the set $W(d,\delta)$ of all integral plane curves of degree d and geometric genus $(d-2)(d-1)/2 - \delta$.

For the next we recall that for positive integers g, r, and d the number $\rho(g,r,d) := (r+1)d - rg - r(r+1)$ stands for the Brill–Noether number.

Theorem 3.3.1. *A general trigonal curve of genus* 10 *is a complete intersection smooth curve given by four divisors of bidegree* $(2,0)$, $(1,1)$, $(1,1)$ *and* $(0,3)$ *on* $\mathbb{P}^2 \times \mathbb{P}^3$, *case* #18 *in Table 3.2.*

Proof. We starting by noting that a trigonal smooth curve (X, R) of genus 10, where $|R|$ a $\mathfrak{g}^1_3$, is in $\mathbb{V}_{10}$, if and only if $\mathrm{K}_X - 2R$ is base point free. It also is well known that the trigonal locus of smooth curves has dimension 21. Let $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}^3$ be a trigonal complete intersection smooth curve given by the divisors of bidegree $(2,0)$, $(1,1)$, $(1,1)$ and $(0,3)$. We known that $\mathcal{C}$ is trigonal and the Castelnouvo–Severi inequality assures that there is a unique $\mathfrak{g}^1_3$, say $|L|$. If we assume that the embedding of $\mathcal{C}$ is the full canonical embedding we also need to assume $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2,0)) = 3$, i.e. $\dim|2L| = 2$. Even if we do not assume the canonical embedding, the fact that $\pi_{1|\mathcal{C}}$ is a $3:1$ gives that we are using only $|L| + |L|$ to map C inside $\mathbb{P}^2 \times \mathbb{P}^3$. The pair of divisors $(|2L|, |\mathrm{K}_{\mathcal{C}} - 2L|)$ provides an embedding, if and only if, $\mathrm{K}_{\mathcal{C}} - 2L$ is base point free and

$$\dim|\mathrm{K}_{\mathcal{C}} - 2L - p_1 - p_2| = \dim|\mathrm{K}_{\mathcal{C}} - 2L| - 2, \tag{3.9}$$

for all $p_1, p_2 \in \mathcal{C}$ such that there is $p_3 \in \mathcal{C}$ with $p_1 + p_2 + p_3 \in |L|$.

By duality, the condition in Equation (3.9) is equivalent to $\dim|2L + p_1 + p_2| = \dim|2L|$ for all $p_1, p_2 \in \mathcal{C}$ such that there is $p_3 \in \mathcal{C}$ with $p_1 + p_2 + p_3 \in |L|$. Since $|3L|$ is base point free, this is the case if and only if $\dim|LR| = \dim|2L| + 1$. Hence, $\dim|2L| = 2$. In summary, a trigonal smooth curve $\mathcal{C}$ of genus 10 can be canonically embedded in $\mathbb{P}^2 \times \mathbb{P}^3$

if, and only if, $\dim|3R| = 3$. In particular, a general trigonal curve is as described in case #18. □

Following the same steps of Theorem 3.3.1, we are able to also establish the following.

Theorem 3.3.2. *A general trigonal curve of genus 8 is a complete intersection smooth curve given by divisors of bidegree $(2,0)$, $(1,1)$ and $(1,3)$ on $\mathbb{P}^2 \times \mathbb{P}^2$, #10 in Table 3.2.*

Let $\mathcal{N}_{\mathcal{C}}$ be the normal bundle of $\mathcal{C}$ in the multi-projective space $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $\mathcal{N}_{\mathcal{C}/\mathbb{P}^N}$ the normal bundle of a curve in a projective space $\mathbb{P}^N$. It is known that for a general curve we have $h^1(\mathcal{C}, \mathcal{N}_{\mathcal{C}/\mathbb{P}^N}) > 0$. So the set of all complete intersections curves in $\mathbb{P}^N$ with prescribed degrees of the hypersurfaces is smooth of dimension $h^0(\mathcal{C}, \mathcal{N}_{\mathcal{C}/\mathbb{P}^N})$. There is a possibility that it remains true if we consider complete intersection curves in Y instead of $\mathbb{P}^N$. Let $(d_1, \ldots, d_k)$ be the multidegree of $\mathcal{C}$. Assume that $\mathcal{C}$ is the complete intersection of $n-1$ divisors D_j, $1 \leq i \leq n-1$, with $D_j \in |\mathcal{O}_Y(a_{j1}, \ldots, a_{jk})|$. The assumption that $\mathcal{C}$ is not contained in a smaller multiprojective space gives $\sum_{i=1}^k a_{ji} \geq 2$ for all j. The normal bundle $\mathcal{N}_{\mathcal{C}}$ is a direct sum of $n-1$ line bundles. Those normal bundles with $a_{ji} > 0$ for all j, i and $a_{ji} \geq 2$ for at least one (i,j) have $h^1 = 0$. In all cases which I saw so far $(0,2)$ or $(0,3)$ has degree $> 2g-2$ and hence $h^1 = 0$. The factor $\mathcal{O}_X(1, \ldots, 1))$ give 1 to h^1, but it is easy to count the dimension they give; if there are $e \geq 1$ of them and $N+1 = \prod_{i=1}^k (n_i + 1)$, they give the dimension $e(N+1-e)$ of the Grassmannian $G(e, N+1)$ of e-dimensional linear subspaces of $\mathbb{K}^{N+1}$. We should do a general proposition, if true. In single cases, it is easy.

To the variety $V(d,\delta)$, introduced above, is attached the morphism

$$p_\delta : V(d,\delta) \longrightarrow \mathrm{Sym}^\delta(\mathbb{P}^2)$$

mapping a plane curve to the set of its nodes. The following result is due to Treger and can be found in [Tre89, Theorem 3.9, item (i)].

Theorem 3.3.3 (Treger)**.** *Let $\delta \leq (d-1)(d-2)/2$ and $(d,\delta) \neq (6,9)$. The morphism*

$$p_\delta : V(d,\delta) \longrightarrow \mathrm{Sym}^\delta(\mathbb{P}^2)$$

maps $V(d,\delta)$ onto its image, and for a general $C \in V(d,\delta)$ the pre-image $p_\delta^{-1}(p_\delta(C))$ consists of a point.

Remark 3.3.1. A element of $\mathbb{M}_8^{(7,7)}$ is a curve like case #12 of table 3.2.1. Indeed, let $\mathcal{C}$ be a complete intersection curve of $X = \mathbb{P}^n \times \mathbb{P}^m$ given by divisors D_i of bidegree (a_i, b_i).

A curve of bidegree $(7,7)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ has genus 36, so $m > 1$. Define $h = \mathcal{O}_X(1,0)$ and $H = \mathcal{O}_X(0,1)$, so $K_\mathcal{C} = (ah + bH)|_\mathcal{C}$ where $a = -n - 1 + \sum a_i$ and $b = -m - 1 + \sum b_i$. If $\mathcal{C}$ has bidegree $(7,7)$ and genus 8, then $\deg(K_\mathcal{C}) = (ah + bH)\mathcal{C} = 14$, so the bidegree of canonical sheaf of $\mathcal{C}$ is $(2-b, b)$ where $b \in \mathbb{Z}$. If $b = 1$, the proof is done. Without loss of generality suppose $b > 1$, then $\sum a_i = n + 3 - b$ and $\sum b_i = m + 1 + b$, if $a_i = 0$, then $\mathcal{C}H = (D_1 D_2 \cdots b_i H \cdots D_{n+m-1})H = 7$, and since $b_i > 1$ we get $b_i = 7$ so

$$m + 1 + b = b_1 + \cdots + b_{n+3-b} + b_{n+4-b} + \cdots + b_{n+m-1} \geq 7(n + m - 1 - (n + 3 - b)),$$

thus $6m \leq 29 - 6b$. As $m > 1$, it follows that $b = 2$, so $K_\mathcal{C} = \mathcal{O}_\mathcal{C}(0,2)$, which implies $m = 2$. If $\mathcal{C}$ is a curve of bidegree $(7,7)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, then $\mathcal{C}h = b_1 b_2 = 7$ that means $b_1 + b_2 = 8$, since $K_\mathcal{C} = \mathcal{O}_\mathcal{C}(0,2)$ we get $b_1 + b_2 = m + 1 + b = 5$. Lastly in $\mathbb{P}^2 \times \mathbb{P}^2$ there two case for curves with $K_\mathcal{C} = \mathcal{O}_C(0,2)$: $\mathcal{C}$ is given by two divisors of bidegree $(1,2)$ and one of bidegree $(1,1)$ or $\mathcal{C}$ is given by two divisors of bidegree $(1,1)$ and one of bidegree $(1,3)$ in both cases $\mathcal{C}$ it does not have bidegree $(7,7)$.

Theorem 3.3.4. *A general element of* $\mathbb{M}_8^{(7,7)}$ *has gonality* 5 *and at least eight* $\mathfrak{g}^1_5$.

Proof. Let $\mathcal{C}$ a curve in $\mathbb{P}^2 \times \mathbb{P}^2$ of genus 8 of of bidegree $(7,7)$. Since 7 is a prime, the two projections, $\pi_{1|\mathcal{C}}$ and $\pi_{2|\mathcal{C}}$, are birational onto their images and hence $\mathcal{C}$ is equipped with at least 2 base point free g^2_7, c.f. Remark 3.0.1. Thus $\pi_i(\mathcal{C}) \in W(7,8)$, $i = 1,2$. These two g^2_7 are complete, because 7 is a prime and 6 is the maximal genus of a degree 7 non-degenerate space curve.

Let us assume that $\mathcal{C}$ is canonically embedded, say $u = (u_1, u_2) : \mathcal{C} \to \mathbb{P}^2 \times \mathbb{P}^2$, and take $L_i := u_i^* \mathcal{O}_{\mathbb{P}^2}(1)$, with $i = 1,2$, the two lines bundles associated. Call $D_1, D_2 \subset \mathbb{P}^2$ the images associated to these $\mathfrak{g}^2_7$'s. Since a smooth degree 7 plane curve has genus 15, each D_i is singular.

Since $\mathcal{C}$ it is assumed canonically embedded, we obtain $L_2 \cong \mathrm{K}_\mathcal{C} - L_1$. Conversely each degree 7 plane curve $D \subset \mathbb{P}^2$ with geometric genus 8 is associated to a base point free line bundle L with $h^0(\mathcal{C}, L) = 3$, where $\mathcal{C}$ is its normalization, and Riemann–Roch theorem implies $h^0(\mathcal{C}, \mathrm{K}_\mathcal{C} - L) = 3$. If L is a theta-characteristic, i.e. if $L^{\otimes 2} \cong K_\mathcal{C}$, then the pair $(L, K_\mathcal{C} - L)$ does not provides an embedding in $\mathbb{P}^2 \times \mathbb{P}^2$, because $|L|$ does not give an embedding. If L is not a theta-characteristic, then L and $K_\mathcal{C} - L$ give a morphism $f : \mathcal{C} \to \mathbb{P}^2 \times \mathbb{P}^2$ which is birational onto its image, but a priori it may be not injective or ramified. The complete intersections give smooth examples by the Bertini theorem.

Since $\rho(8,1,5) = 10 - 8 - 2 = 0$ and $\rho(8,1,4) = -2$, Brill–Noether theory provides that a general element of $\mathcal{M}_8$ has gonality 5 and that the set of all curves in $\mathcal{M}_8$ with gonality smaller than 4 has codimension 2 in $\mathcal{M}_8$. Since $\rho(8,2,7) = 21 - 16 - 6 = -1$, the set of all curves $\mathcal{M}_8$ with a g^2_7 forms a hypersurface of $\mathcal{M}_8$. Note that $\mathbb{M} := \mathbb{M}_8^{(7,7)}$ is irreducible and hence we may compute its dimension at some $\mathcal{C} \in \mathbb{M}$ contained in a smooth element Σ of $|\mathcal{O}_Y(1,1)|$, where Y stands for $\mathbb{P}^2 \times \mathbb{P}^2$. Since $\dim|\mathcal{O}_Y(1,1)| = 8$ and $\mathcal{N}_{\mathcal{C},\Sigma} \cong \mathcal{O}_{\mathcal{C}}(2,1) \oplus \mathcal{O}_{\mathcal{C}}(1,2)$, we obtain $\dim\mathbb{M} = h^0(\mathcal{C}, \mathcal{N}_{\mathcal{C}})$, because $\mathcal{N}_{\mathcal{C}} \cong \mathcal{O}_{\mathcal{C}}(2,1) \oplus \mathcal{O}_{\mathcal{C}}(1,2) \oplus \mathcal{O}_{\mathcal{C}}(1,1)$ and hence $h^1(\mathcal{C}, \mathcal{N}_{\mathcal{C}}) = 1$ and $h^0(\mathcal{C}, \mathcal{N}_{\mathcal{C}}) = 14 + 14 + 8 = 36$.

From the fact that $\dim \mathrm{Aut}(\mathbb{P}^2 \times \mathbb{P}^2) = 16$ and that a curve has a unique canonical line bundle, it follows that the image of $\mathbb{M}$ in $\mathcal{M}_8$ has dimension 20, while $\dim\mathcal{M}_8 = 21$. Hence a general element of $\mathbb{M}$ is 5-gonal.

The normalization map induces a morphism $\phi : V(7,8) \to \mathcal{M}_8$. It is clear that projectively equivalent plane curves have isomorphic normalizations. Thus all fibers of ϕ have dimension at least 8. Since $\dim V(7,8) - 8 \leq 21 = \dim\mathcal{M}_8$, $\dim\phi(V(7,8)) = \dim V(7,8) - 8 = 20$, c.f. Theorem 3.3.3. Thus the normalization of a general $D \in V(7,8)$ gives the general element of $\mathbb{M}$. They have at least eight $\mathfrak{g}^1_5$. $\square$

Remark 3.3.2. Due to Theorem 3.3.4, we expect that the gonality of any curve as in case #12 of Table 3.3 is five.

4 | On Mukai stratum of genus 8

This thesis started with the project to study the rational Chow rings of $\mathcal{M}_7$ and $\mathcal{M}_8$ using Mukai's stratification. The main idea was to parameterize each Mukai stratum and then show that its image in $\mathcal{M}_g$ is tautological. This is the way that Penev–Vakil [PeVa15] choose to show that the rational Chow ring of $\mathcal{M}_6$ is tautological. In a very beautiful paper, Canning & Larson [CanLar21] show that $\mathcal{M}_7$, $\mathcal{M}_8$, and $\mathcal{M}_9$ are tautological using a different approach. Hence we moved our project to study projective curves in biprojective spaces. But the problem on parameterize the Mukai strata remains.

The loci of hyperelliptic and trigonal curves are well understood for any genus, we refer to [PeVa15]. The locus of bielliptic curves is a fascinating object of study, its image in $\mathcal{M}_g$ and its Chow ring is far from being well understood for larger genus. Hence we address the following loci in $\mathcal{M}_8$: tetragonal non-bielliptic locus with a $\mathfrak{g}^2_6$, the Pentagonal locus in $\mathcal{M}_8$ with a non self-adjoint g^2_7 and the pentagonal locus without a $\mathfrak{g}^2_7$. We also note, a *a priori*, that these classes of curves in such Mukai's strata are not realized as a complete intersection in biprojective spaces.

The results presented here are far from being a final study of Mukai's strata in genus 8, but we believe that this Chapter can be the starting point to further works.

4.1. Tetragonal non bielliptic with a g^2_6

Let $\mathcal{C}$ be a smooth curve, we define the index of Clifford of $\mathcal{C}$ by

$$\mathrm{Cliff}(\mathcal{C}) = min\{\deg(D) - 2(h^0(\mathcal{C}, D) - 1) \mid D \in \mathrm{Pic}(\mathcal{C}),\, h^0(\mathcal{C}, D) > 0 \text{ and } h^1(\mathcal{C}, D) > 0\}.$$

Let $\mathcal{C}$ be a tetragonal non-bielliptic curve of genus 8, equipped with a $\mathfrak{g}^2_6$, say α and fix its Serre dual $\beta := K_{\mathcal{C}} \otimes \alpha^{-1}$, that is a $\mathfrak{g}^3_8$. Note that both α and β are base point free because $\mathrm{Cliff}(\mathcal{C}) = 2$. Let $\overline{\mathcal{C}}$ be the image of $\phi_{|\alpha|} : \mathcal{C} \longrightarrow \mathbb{P}^2$. Thus $\overline{\mathcal{C}}$ is a plane curve of degree 6 without triple points and with two double points. Let us take π the composition of two blow-ups at each double point of $\overline{\mathcal{C}}$,

$$B \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} \mathbb{P}^2.$$

Let E_i be the exceptional divisors in each blow-up and h the pullback of a line. Hence $\mathcal{C}$ belongs to the linear system $|6h - 2\sum_{i=1}^{2} E_i|$ and $K_B = -3h + \sum_{i=1}^{2} E_i$. By the adjunction formula $K_{\mathcal{C}} = (K_B + \mathcal{C})|_{\mathcal{C}} = (3h - \sum_{i=1}^{2} E_i)|_C = h|_{\mathcal{C}} + (2h - \sum_{i=1}^{2} E_i)|_{\mathcal{C}}$. Then $\alpha = h|_{\mathcal{C}}$ and $\beta = (2h - \sum_{i=1}^{2} E_i)|_{\mathcal{C}}$.

Now consider the morphism induced by β,

$$\phi_{|\beta|} : \mathcal{C} \longrightarrow \mathcal{C}_8 \subset \mathbb{P}^3,$$

where $\mathcal{C}_8$ is a degree 8 curve in $\mathbb{P}^3$ with a double point $q \in \mathcal{C}_8$. Let us take $V := Bl_q\mathbb{P}^3$ the blow-up of $\mathbb{P}^3$ at q, H the pullback of hyperplane in $\mathbb{P}^3$ and E the exceptional divisor. In [MukId03], the authors show that $\mathcal{C}$ is the complete intersection of $2H - E \sim -\frac{1}{2}K_V$ and $4H - 2E \sim -K_V$.

Let us consider the opposite:

Assumption 1. Let V be the blowup of $\mathbb{P}^3$ at a point and K_V its canonical sheaf. Let $\mathcal{C}$ be a complete intersection smooth curve given by $-\frac{1}{2}\mathrm{K}_V = 2H - E$ and $-\mathrm{K}_V = 4H - 2E$.

We know that, for $n \geq 2$, the chow ring of blow up of $\mathbb{P}^n$ at a point is isomorph to $\mathbb{Z}[H, E]/I$, where $I = (EH, H^n - (-1)^n E^n)$. In the Chow ring $A^*(V)$ we can write $[\mathcal{C}] = 8H^2 + 2E^2$, implying

$$K_{\mathcal{C}} = (2H - E)|_{\mathcal{C}} + (4H - 2E)|_{\mathcal{C}} - (4H - 2E)|_{\mathcal{C}} = (2H - E)|_{\mathcal{C}},$$

and so $\deg(K_{\mathcal{C}}) = (2H - E)(8H^2 + 2E^2) = 16H^3 - 2E^3 = 14$ and $g = 8$. Since $2H - E$ is very ample, it embeds V in $\mathbb{P}^8 = \mathbb{P}(H^0(V, \mathcal{O}_V(2H - E)))$. The kernel of the the restriction map

$$\mathrm{H}^0(V, \mathcal{O}_V(2H - E)) \longrightarrow \mathrm{H}^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2H - E)),$$

embeds $\mathcal{C}$ in $\mathbb{P}^7$ as a canonical curve.

Considering the exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathcal{C}}(H - E) \longrightarrow \mathcal{O}_V(H - E) \longrightarrow \mathcal{O}_{\mathcal{C}}(H - E) \longrightarrow 0, \tag{4.1}$$

by the vanishing $H^0(V, \mathcal{I}_{\mathcal{C}}(H - E)) = 0$, we conclude that $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(H - E)) \geq 3$. Since

$\mathcal{C} \cdot (H - E) = (8H^2 + 2E^2) \cdot (H - E) = 6$ in $A^*(V)$, we obtain that $\mathcal{O}_{\mathcal{C}}(H - E)$ is a g_6^2 on $\mathcal{C}$.

Remark 4.1.1. Let $\pi : V \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ at p_0. We gonna call $Y \subset V$ a surface type $2H - E$ or $(2H - E)$-surface if $[Y] \sim (2H - E)$ in $\text{Pic}(V)$. Let $\Gamma = \{p_1, \cdots, p_d\}$ be distinct points in V, we say Γ imposes independent conditions on surfaces of type $2H - E$ whenever $h^0(V, \mathcal{I}_\Gamma(2H - E)) = h^0(V, \mathcal{O}_V(2H - E)) - d$. We could see a element of $|2H - E|$ as a quadric passing through p_0, let's say $F = \sum a_{i,j} X_i^{n_i} X_j^{n_j}$ where $n_i + n_j = 2$, so "imposes independent conditions" means that the system $\{F(p_k) = 0\}_{1 \leq k \leq d}$ is linearly independent. Thus, if every quadric passing through p_0 that vanishes at $d - 1$ points of Γ also vanishes at the other point, then Γ fails to impose independent conditions on $(2H - E)$-surfaces.

The next Lemma is an adaptation of an exercise from [ACGH85, pg. 199], where it asks to examine $W_d^r(\mathcal{C})$, where $\mathcal{C}$ is a smooth curve in $\mathbb{P}^3$ given by the intersection of a smooth complete intersection of a smooth quadric a smooth quartic.

Lemma 4.1.1. *Let $\pi : V \to \mathbb{P}^3$ the blowup of $\mathbb{P}^3$ at p_0, $\Gamma = \{p_1, \cdots, p_d\}$ be distinct points in V and $\Gamma' = \{p_0, p_1', \cdots, p_d'\}$,with $\pi(p_i) = p_i'$. Assume that $2 \leq \#\Gamma' \leq 6$. If $\#\Gamma'$ is 2 or 3, then Γ imposes independent conditions on surfaces of type $2H - E$. If $\#\Gamma' \geq 4$ and Γ does not impose independent conditions on $(2H - E)$-surfaces, then either:*

- *Four points of Γ' are collinear; or*
- *Six points of Γ' are coplanar.*

Proof. Let $\mathcal{I}_\Gamma$ be the ideal sheaf at Γ, by the exact sequence

$$0 \longrightarrow \mathcal{I}_\Gamma(2H - E) \longrightarrow \mathcal{O}_V(2H - E) \longrightarrow \mathcal{O}_\Gamma \longrightarrow 0, \tag{4.2}$$

we have $h^0(V, \mathcal{I}_\Gamma(2H - E)) \geq h^0(V, \mathcal{O}_V(2H - E)) - d$, to show that Γ imposes independent conditions on $(2H - E)$-surfaces we have to show the opposite inequality. To do this we construct quadrics through p_0 and all other points except one.

If $\#\Gamma' = 2$ take quadric containing p_0 and not containing p_1'. If $\#\Gamma' = 3$ take a hyperplane H_1 containing p_0, but not p_2' and a H_2 containing p_1, but not p_2', now just take the quadric given by these two hyperplanes. The second part of the lemma follows by contrapositive.

- $\#\Gamma' = 4$. Let L be the line containing p_0 and p_1'. We can assume that p_3' is not in this line. Take hyperplane H_1 containing L, but not p_3' and H_2 a hyperplane containing p_2, but not p_3'.

- $\#\Gamma' = 5$. We have two cases, three points are collinear or not. If so, let L be the line containing these points and we can assume p'_4 away from this line. Take a hyperplane H_1 containing L, but not p'_4 and H_2 a hyperplane containing the other point away from L, but not p'_4. Otherwise, take a hyperplane H_1 containing p_0 and p'_1, but not p'_4 and H_2 a hyperplane containing p'_2 and p'_3, but not p'_4.
- $\#\Gamma' = 6$. Note that if three points are collinear, then the remaining are not. Then take a hyperplane H_1 containing the three points but not p'_5 and hyperplane H_2 containing the other two but not p'_5. If any three points are not collinear, take H_1 containing p_0, p'_1 and p'_2, and we can assume p'_5 away from this hyperplane. And so take H_2 containing p'_3 and p'_4, but not p'_5.

In all of the cases above, the result follows by taking the quadric given by H_1 and H_2. □

Theorem 4.1.2. *If $\mathcal{C}$ is a curve as in Assumption 1, then $\mathcal{C}$ has genus 8, does not admit any $\mathfrak{g}^1_3$ and every $\mathfrak{g}^2_6$ is linearly equivalent to $\mathcal{O}_C(H-E)$. In particular, $\mathcal{C}$ is tetragonal with a unique $\mathfrak{g}^2_6$.*

Proof. Take $\Gamma = \{p_1, \cdots, p_6\} \subset \mathcal{C}$. Taking the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{H}^0(V, \mathcal{I}_S(2H-E)) & \longrightarrow & \mathrm{H}^0(V, \mathcal{O}_V(2H-E)) & \longrightarrow & H^0(V, \mathcal{O}_S) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{H}^0(C, \mathcal{I}_S(2H-E) & \longrightarrow & \mathrm{H}^0(C, \mathcal{O}_{\mathcal{C}}(2H-E)) & \longrightarrow & \mathrm{H}^0(\mathcal{C}, \mathcal{O}_S) \end{array}$$

we have that the morphism of the right column is an isomorphism and of the middle column is surjective, implying that the morphism of the left column is surjective. Thus given a $\mathfrak{g}^r_d$ on $\mathcal{C}$, written as the sum of distinct points, it follows that the set of its points does not impose independent conditions on surfaces of the type $2H-E$. Let $D = p_1 + \cdots + p_6$ be a $\mathfrak{g}^2_6$ on $\mathcal{C}$. We can assume that all p_i are distinct. Applying Lemma 4.1.1 for all $D - p_i$, we obtain that the image of those points in $\mathbb{P}^3$ are in a plane containing p_0. Since $\mathcal{C} \cdot (H-E) = 6$, we have $D \sim \mathcal{O}_{\mathcal{C}}(H-E)$.

Assume that $\mathcal{C}$ has a $\mathfrak{g}^1_3$, again, write $D = p_1 + p_2 + p_3$. If any of these points are in E, then the first part of Lemma 4.1.1 implies that the set of points imposes independent conditions on $2H-E$. Then assume that these points are not in E. By Lemma 4.1.1, these points are on the same line by p_0, so they are on a plane passing through p_0. Since $(H-E)\cdot\mathcal{C} = 6$, follows that there is $q \in \mathcal{C}$ such that $D + E|_{\mathcal{C}} + q = (H-E)|_{\mathcal{C}}$. Since $2D$ is a $\mathfrak{g}^2_6$, we obtain that $2D \sim (H-E)_{\mathcal{C}} \sim D + E|_{\mathcal{C}} + q$, then $D \sim E|_{\mathcal{C}} + q$, finishing the proof. □

Next, we provide a way to parameterize tetragonal curves of genus 8 with a unique $\mathfrak{g}^2_6$.

Let $\pi : Y \longrightarrow \mathbb{P}^8 = \mathbb{P}(V, H^0(V, \mathcal{O}_V(2H-E)))$ be a bundle such that

$$\pi^{-1}(D) = \frac{H^0(V, \mathcal{O}_V(4H-2E))}{< 2D >},$$

for every $D \in \mathbb{P}^8$. Denote by Δ the divisor in Y where the curves are nonsmooth and G the subgroup of $GL(4)$ fixing a point let's say q, take $Z = (Y \backslash \Delta)/G$. Denote G^2_6 the space of tetragonal non-bielliptic curves with a unique $\mathfrak{g}^2_6$ in $\mathcal{M}_8$, so we get a natural morphism $G^2_6 \longrightarrow Z$.

4.2. Pentagonal with a non self-adjoint g^2_7

Let $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}^2 = X$ be a curve given by the intersection of three divisors of bidegrees $(1,2)$, $(2,1)$ e $(1,1)$. We know from Chapter 3, c.f. Table 3.3, that $\mathcal{C}$ has gonality 4 or 5, and that $\mathcal{C}$ admits a g^2_7 non self-adjoint.

Let $\pi : Y \longrightarrow \mathbb{P}^8 = \mathbb{P}(H^0(X, \mathcal{O}_X(1,1)))$ be the rank 30 bundle where

$$\pi^{-1}(H) = \frac{H^0(X, \mathcal{O}_X(1,2))}{< H.y_i >} \times \frac{H^0(X, \mathcal{O}_X(2,1))}{< H.x_i >},$$

for every $H \in \mathbb{P}^8$, Since we can see $H^0(X, \mathcal{O}_X(1,2))$ is a vector space generated forms of bidegree $(1,2)$ in x_i and y_i, then $< H.y_i >$ is a subspace of $H^0(X, \mathcal{O}_X(1,2))$. Denote Δ the divisor in Y where the curves are non smooth and take $V = (Y \backslash \Delta)/(GL(3) \times GL(3))$ and $\dim(V) = 38 - 18 = 20$.

Let G^2_7 be the locus of pentagonal curves with a non self-adjoint g^2_7 in $\mathcal{M}_8$. We know by Theorem 1.0.5 that every $\mathcal{C} \in G^2_7$ is a complete intersection of divisors of bidegrees $(1,2)$, $(2,1)$ e $(1,1)$ in $X = \mathbb{P}^2 \times \mathbb{P}^2$. The place of curves with a g^2_7 has dimension 20 in $\mathcal{M}_8$ and since every curve of G^2_7 is general, then $\dim(G^2_7) = 20$.

4.3. Mukai Locus

The Mukai Locus $\mathcal{M}^{Mu}_8$ in genus 8, is the subspace of $\mathcal{M}_8$ consisting of pentagonal curves without a $\mathfrak{g}^2_7$. In this section we adapt the arguments used by Peneve–Vakil in [PeVa15] to provide a parameterization of $\mathcal{M}^{Mu}_8$.

Let us first recall the Mukai–Ide results [Muk92] and [MukId03].

Theorem 4.3.1 (Mukai–Ide). *A smooth curve $\mathcal{C}$ belongs to $\mathcal{M}^{Mu}_8$ if and only if there* *is a vector bundle E of rank 2 on $\mathcal{C}$, generated by its global sections with* $\det(E) = \mathrm{K}_{\mathcal{C}}$, *such that $\mathcal{C}$ is a transversal section of the Grassmanian $G(2,V) \subset \mathbb{P}(\wedge^2 V)$. Here $V :=$ $\mathrm{H}^0(\mathcal{C}, E)$ and a transversal section means that there are hyperplanes sections H_i such that $\mathcal{C} = G(2,V) \cap H_1 \cap \cdots \cap H_7$.*

Remark 4.3.1. Since $\mathrm{K}_{G(2,V)} = \mathcal{O}_{\mathbb{P}(\wedge^2 V)}(-6)|_{G(2,V)}$, it follows that $\mathrm{K}_{\mathcal{C}} = \mathcal{O}_{\mathbb{P}(\wedge^2 V)}(1)|_{\mathcal{C}}$. Denoting by S the linear space generated by the hyperplanes H_i, we obtain an exact sequence:

$$0 \longrightarrow S \longrightarrow \wedge^2 V \longrightarrow Q \longrightarrow 0. \tag{4.3}$$

Here Q is the quotient whose dimension is 8. By the above sequence S cut $\mathbb{P}Q \subset \mathbb{P} \wedge^2 V$, and so $\mathcal{C}$ is non degenerate in $\mathbb{P}Q$. We then identify $\mathbb{P}Q$ with $\mathbb{P}\mathrm{H}^0(\mathcal{C}, K_{\mathcal{C}})$, which implies that Q differs from $\mathrm{H}^0(\mathcal{C}, K_{\mathcal{C}})$ only by a scalar.

Let V be a vector space of dimension 6. Consider the Grassmanian $G(2,V)$ embedded via Plücker in $\mathbb{P}(\wedge^2 V)$. The space of dimension 7 of linear sections of $G(2,V)$ is parameterized by $G(7, \wedge^2 V)$. In the same way that $PGL(V)$ acts on $G(2,V)$, it determines an action on $G(7, \wedge^2 V)$.

Theorem 4.3.2. *The natural map*

$$\phi : (G(7, \wedge^2 V) \setminus \Delta)/PGL(V) \longrightarrow \mathcal{M}_8$$

is an open immersion (of Deligne-Mumford Stacks) whose image is $\mathcal{M}_8^{Mu}$. Here Δ is a divisor of $G(7, \wedge^2 V)$ corresponding to the singular place.

Proof. First note that the image of ϕ is $\mathcal{M}_8^{Mu}$. This fact follows from Mukai's work [Muk92]. If $\mathcal{C}$ is a Mukai curve, just apply the Theorem 4.3.1, for $V = \mathrm{H}^0(\mathcal{C}, E)$.

Now we have to show that ϕ is representable. Note that for each curve $\mathcal{C}$, we have a bundle E and an isomorphism class. In the work due to Mukai [Muk92], he proved that the bundle E is unique up to isomorphism, so we just have to show that this isomorphism is given by a scalar product.

Let $\mathcal{L}$ be a bundle giving a $\mathfrak{g}_5^1$ and $\mathcal{M} = \mathcal{K}_{\mathcal{C}} \otimes \mathcal{L}^\vee$ its Serre dual, which is a $\mathfrak{g}_9^3$. Let $\alpha : E \longrightarrow E$ be an automorphism. So we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r, "\beta"] & E \arrow[r] \arrow[d, "\alpha"] & \mathcal{M} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r] & E \arrow[r] & \mathcal{M} \arrow[r] & 0
\end{tikzcd}$$

By [Muk92, Lemma 3.10], we obtain $\dim \operatorname{Hom}(\mathcal{L}, E) \leq 1$, thus the inclusion $\mathcal{L} \xrightarrow{\beta} E$ is unique up to scaling. Thus, by multiplying α by a suitable scalar we have

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r, "\beta"] \arrow[d, "="] & E \arrow[r] \arrow[d, "\alpha"] & \mathcal{M} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{L} \arrow[r, "\beta"] & E \arrow[r] & \mathcal{M} \arrow[r] & 0 \end{tikzcd}$$

Finally, we can see that there is a unique (up to scalar) nontrivial extension of $\mathcal{L}$ by $\mathcal{M}$. To see this, just apply the functor $\operatorname{Hom}(-, \mathcal{M})$, in the above exact sequence and use [Muk92, Lemma 3.6]. Using a suitable scalar change in $\mathcal{M}$ we finally obtain

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r, "\beta"] \arrow[d, "="] & E \arrow[r] \arrow[d, "\alpha"] & \mathcal{M} \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \mathcal{L} \arrow[r, "\beta"] & E \arrow[r] & \mathcal{M} \arrow[r] & 0 \end{tikzcd}$$

So E is unique up to scalar. Since ϕ is a (representable) birational bijection onto its image, and $\mathcal{M}_8^M$ is normal, it follows that ϕ is an isomorphism. $\square$

5 | Future Developments

In this chapter, we briefly provide some directions and problems that could possibly be resolved in a couple of years.

Problem 1. *We believe that it is simple to adapt our computations on Section 3.1 of Chapter 3 to provide analogous results for multiprojective spaces* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ *with* $k > 2$.

Problem 2. *Is it possible to obtain a simpler proof of Lemma 3.2.2, avoiding heavy computations? Maybe a more geometrical one.*

Problem 3. *Is there a way to obtain a more general statement for Theorem 3.2.3, avoiding the two technical assumptions?*

Problem 4. *Due to Lemma 3.1.12. What is the image of the locus of curves in* $\mathbb{P}^2 \times \mathbb{P}^2$ *given by three divisors of bidegrees* $(2, 0)$, $(0, 2)$ *and* $(2, 2)$ *inside the space* $\mathcal{M}_8^{bi}$ *of bielliptic curves of genus* 8?

Problem 5. *Is there a way to provide a lower bound for the gonality for c.i. curves in* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ *with* $k > 2$.

Problem 6. *Is there a way to define a Mukai locus in* $\mathcal{M}_g$, *for large* g *or any* $g \geq 6$, *in order to have an analogous of Theorem 4.3.2? For sure that the gonality of a curve in this locus is equal to* $\lfloor (g + 3)/3 \rfloor$. *The main questions lies in what kind must be avoided.*

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