UNIVERSIDADE FEDERAL DE MINAS GERAIS
Instituto de Ciências Exatas – ICEx
Departamento de Matemática

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Existence of robust non-uniformly hyperbolic endomorphism in homotopy classes

Belo Horizonte

2023

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Dissertação apresentada ao Programa de Pós Graduação em Matemática do Instituto de Ciências Exatas da Universidade Federal de Minas Gerais, como parte dos requisitos para a obtenção do título de Mestre em Matemática.

Orientador: Pablo Daniel Carrasco Correa

Belo Horizonte

2023

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Janeiro, Victor Gabriel Xavier.

J33e Existence of robust non-uniformly hyperbolic endomorphism in homotopy classes / Victor Gabriel Xavier Janeiro – 2023.
77 f. il.

Orientador: Pablo Daniel Carrasco Correa
Dissertação (mestrado) - Universidade Federal de Minas Gerais, Instituto de Ciências Exatas, Departamento de Matemática
Referências: f.76-77.

1. Matemática – Teses. 2. Teoria ergódica – Teses. 3. Sistemas dinâmicos – Teses. I. Correa, Pablo Daniel Carrasco II. Universidade Federal de Minas Gerais Instituto de Ciências Exatas, Departamento de Matemática. IIII. Título.

CDU 519.2(043)

Ficha catalográfica elaborada pela bibliotecária Belkiz Inez Rezende Costa
CRB 6/1510 Universidade Federal de Minas Gerais – ICEx

Universidade Federal de Minas Gerais
Departamento de Matemática
Programa de Pós-Graduação em Matemática

FOLHA DE APROVAÇÃO

Existence of robust non-uniformly hyperbolic endomorphism in homotopy classes

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Documento assinado digitalmente
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NILS MARTIN ANDERSSON
Data: 02/08/2023 14:31:45-0300
Verifique em https://validar.iti.gov.br

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Agradecimentos

Dedico este espaço para agradecer a todos aqueles que colaboraram em meu percurso para conseguir realizar este trabalho. É difícil mensurar o quanto cada um foi crucial em minha vida e me possibilitou chegar até aqui.

A meu orientador, Pablo Carrasco, por todos esses anos de variados ensimanetos. Você me apresentou á esta bela área dos sistemas dinâmicos e me ensinou tudo aquilo que eu fui capaz de aprender. Seu profissionalismo, seu entusiasmo pelo estudo da matemática e seu conhecimento são grandes inspirações que levarei para o resto de minha vida profissional. Muito obrigado por todas as oportunidades que você me proporcionou e todos os momentos de aprendizagem.

À minha família, meus pais Sandra e Marcos, meus irmãos Bruno e Gustavo e meu companheiro Mateus, por todo o apoio e confiança que sempre me deram. Além deles, a lista de famíliares e amigos que desejo agradecer se estende mais longa do que poderia citar aqui, muito obrigado a todos por todo o apoio incondicional.

Agradeço também aos pesquisadores Martin Andersson e Radu Saghin que, juntamente com meu orientador Pablo Carrasco, escreveram o artigo o qual esta dissertação se baseia e conjecturaram o principal resultado. Além disso, agradeço por suas observações e comentários que influenciaram significativamente o trabalho final.

Por fim, agradeço a todos os professores que participaram de forma determinante em minha formação. São tantos que prefiro agradecer a todo o Departamento de Matemática da UFMG por me proporcionar a rica formação que tive ao longo dos últimos anos. A qualidade do trabalho de todos como conjunto e de cada um individualmente mostram a excelência dessa Universidade a qual amo fazer parte. Agradeço em especial a professora Carmen Vergara, minha tutora no grupo PET ao longo da graduação, você sempre me inspirou com sua paixão pelo que faz.

Resumo

Nós estendemos os resultados expostos em [1] obtendo que qualquer endomorfismo linear em $\mathbb{T}^2$ induzido por uma homotetia é homotópico a um mapa conservativo e não uniformemente hiperbólico, desde que seu grau topológico seja ao menos 5^2. Nós também abordamos outros casos de grau topológico baixo que não foram considerados nesse artigo. Com isso, provamos a existência de um aberto da topologia C^1, formado por sistemas não uniformemente hiperbólicos, que intersecta essencialmente qualquer classe de homotopia de endomorfismos em $\mathbb{T}^2$, aberto no qual o expoente de Lyapunov varia continuamente.

Apresentamos detalhadamente todos os resultados de Andersson-Carrasco-Saghin. Tais resultados incluem a existência de endomorfismos estavelmente ergódicos (de fato são Bernoulli) em cada classe de homotopia na qual existência de robusta hiperbolicidade não uniforme é provada. Também incluímos aspectos gerais desta Teoria e algumas especificidades do toro bidimensional. Em particular, expomos aqui como a extensão natural de endomorfismos na mesma classe de homotopia pode ser canonicamente identificados com um Solenoide, desde que sejam recobrimentos normais. Esta é uma técnica de grande importância na teoria ergódica diferenciável.

Palavras Chave: hiperbolicidade não uniforme; expoentes de Lyapunov; ergodicidade estável; sistemas dinãmicos não invertíveis.

Abstract

We extend the results of [1] by showing that any linear endomorphism of $\mathbb{T}^2$ induced by a homothety is homotopic to a non-uniformly hyperbolic ergodic area preserving map, provided that its degree is at least 5^2. We also address other small topological degree cases not considered in the previous article. This proves the existence of a C^1 open set of non-uniformly hyperbolic systems, that intersects essentially every homotopy class in $\mathbb{T}^2$, where the Lyapunov exponents vary continuously.

We give here a detailed survey on Andersson-Carrasco-Saghin's results. Those includes the existence of stably ergodic (Bernoulli in fact) endomorphisms on each homotopy class where robust non-uniform hyperbolicity is achieved. We also includes generalized aspects of the theory and some specifications to the 2-torus case. In particular, we show how the natural extension of endomorphisms in the same homotopy class can be canonically identified with a Solenoidal manifold, provided that they are normal covers. This is a technique of great importance on the study of endomorphisms in the smooth ergodic theory.

Keywords: non-uniform hyperbolicity; Lyapunov exponents; stable ergodicity; non-invertible dynamical systems.

Contents

Introduction

An important concept in the study of dynamical systems is that of hyperbolicity. A differential dynamical system $f : M \to M$ on a compact Riemannian manifold is said to be uniformly hyperbolic if the tangent space splits into stable and unstable bundles, where the cocycle induced by the differential of the system has rate of contraction and expansion uniformly bounded away from zero.

Uniformly hyperbolic systems, also denominated Anosov systems, are known for their robust behavior. They present points approaching or distancing at an exponential rate, which is a local property that is preserved under sufficiently small (C^1) perturbations of the system. The study of this kind of stability is of great importance for predicting the behavior of systems from a wide range of different fields of study, as well for the advancement of mathematical theories.

A more general concept is the one of non-uniform hyperbolicity, where the weaker requirement is that the contractions and expansions may occur asymptotically, almost everywhere with respect to an invariant measure on M. That is the topic of Pesin's Theory, which provides fundamental tools for the development of the theory. Since for this systems we have a weaker requirement, they give rise to an even greater range of applications than its uniform counterpart.

However, it also makes it more difficult to obtain stability of these systems. Indeed, the Bochi-Mañe theorem [2] asserts a rigidity phenomena for these systems when M is a surface, that is, either a system is uniformly hyperbolic, or there exists, arbitrarily close, systems which are not non-uniformly hyperbolic.

We study conservative maps of the two-torus $\mathbb{T}^2$ from the point of view of smooth ergodic theory. We are interested in the Lyapunov exponents of these systems, in particular, in extending the results obtained in [1] to the homothety case and some cases with lower topological degree, which were not included in the previous results.

For a differentiable covering map $f : \mathbb{T}^2 \to \mathbb{T}^2$ and a pair $(x, v) \in T\mathbb{T}^2$, the number

$$\tilde{\lambda}(x, v) = \limsup_{n\to\infty} \frac{\log \|D_x f^n(v)\|}{n}$$

is the Lyapunov exponent of f at (x, v). Due to Oseledet's Theorem 1.1 [3], there is a full area set $\mathcal{R}$ on $\mathbb{T}^2$ where the previous limit exists for every v, and there exists a measurable bundle E^- defined on $\mathcal{R}$ such that for $x \in M_0$, $v \neq 0 \in E^-(x)$:

$$\lambda(x, v) := \lim_{n\to\infty} \frac{\log \|D_x f^n(v)\|}{n} = \lim_{n\to\infty} \frac{\log m(D_x f^n)}{n} := \lambda^-(x),$$

while for $v \in \mathbb{R}^2 \setminus E^-(x)$:

$$\lambda(x, v) = \lim_{n\to\infty} \frac{\log \|D_x f^n\|}{n} := \lambda^+(x),$$

Moreover, if μ denotes the Lebesgue (Haar) measure on $\mathbb{T}^2$, then:

$$\int (\lambda^+(x) + \lambda^-(x))d\mu(x) = \int \log|\det D_x f|\, d\mu(x) > 0, \tag{1}$$

so $\lambda^+(x) > 0$ almost everywhere. At last, we say that f is non-uniformly hyperbolic (NUH) (Definition 1.1) if $\lambda^-(x) < 0 < \lambda^+(x)$ almost everywhere.

Non uniformly hyperbolic systems provide a generalization of the classical Anosov surface maps [4]. Here, we will only be concerned with the non-invertible case in an attempt to aid the understanding of their statistical properties, which is still under development. For the general ergodic theory of endomorphisms, the reader is directed to [5].

Any map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is homotopic to a linear endomorphism $E : \mathbb{T}^2 \to \mathbb{T}^2$, induced by an integer matrix that we denote by the same letter. In [1], it is established the existence of a C^1 open set of non-uniformly hyperbolic systems that intersects every homotopy class that does not contain a homothety, provided that the degree is not too small. The authors then conjecture that the same is true for homotheties. In this dissertation, we prove this conjecture, provided that the degree is at least 5^2. There are other low topological degree cases not covered by Andersson, Carrasco and Saghin, which we also address here.

Let $\text{End}^r_\mu(\mathbb{T}^2)$ be the set of C^r local diffeomorphisms of $\mathbb{T}^2$ preserving the Lebesgue measure μ, that are not invertible. For $f \in \text{End}^r_\mu(\mathbb{T}^2)$, $(x, v) \in T^1\mathbb{T}^2$ define:

$$I(x, v; f^n) = \sum_{y \in f^{-n}(x)} \frac{\log \|(D_y f^n)^{-1} v\|}{\det(D_y f^n)}, \tag{2}$$

and

$$C_{\mathcal{X}}(f) = \sup_{n \in \mathbb{N}} \frac{1}{n} \inf_{(x,v) \in T^1\mathbb{T}^2} I(x, v; f^n). \tag{3}$$

Define the set

$$\mathcal{U} := \{f \in \text{End}^1_\mu(\mathbb{T}^2) \ : \ C_{\mathcal{X}}(f) > 0\},$$

which is open in the C^1-topology. As we shall prove in Chapter 2, every $f \in \mathcal{U}$ is non-uniformly hyperbolic. Thus, our following results gives us existence of C^1 open sets of NUH endomorphisms on the homotopy class of essentially any linear endomorphism E. Unlike the diffeomorphism case, there are no topological obstructions in $\mathbb{T}^2$ for the existence of robust NUH endomorphisms.

Theorem A. *For $E = k \cdot Id \in M_{2\times 2}(\mathbb{Z})$, with $|k| \geq 5$, the intersection $[E] \cap \mathcal{U}$ is non-empty and in fact contains maps that are real analytically homotopic to E.*

Theorem B. *For $E = (e_{ij}) \in M_{2\times 2}(\mathbb{Z})$ which is not a homothety, if $det(E)/\gcd(e_{ij}) > 4$ or $\gcd(e_{ij}) > 2$, the intersection $[E] \cap \mathcal{U}$ is non-empty and in fact contains maps that are real analytically homotopic to E.*

Our Theorem B is equivalent to the Theorem A of [1] but includes two cases of low topological degree which are not proved there, and Theorem A includes the cases of homotheties homotopy classes. The main difficulty for our results is that, in the case of a homothety, the induced projective action is trivial; non-triviality of this projective action is a central piece in the method of Andersson et al.

Then, we show that the families of maps constructed in Theorems A and B give rise to examples of stably ergodic endomorphisms, that is, ergodic endomorphisms for which every C^2 map C^1-close to them is also ergodic (Definition 3.1).

Theorem C. *For any linear endomorphism E as in Theorems A or B, if* ± 1 *is not an eigenvalue of E then* $[E] \cap \mathcal{U}$ *contains stably ergodic endomorphisms. That is, there exists a* C^1 *open set* $\mathcal{V}$*, such that every* $f \in \mathcal{V}$ *is ergodic.*

In fact, $[E] \cap \mathcal{U}$ *contains stably Bernoulli endomorphism (and in particular, maps that are mixing of all orders).*

For that, we first show a more general result, which has its own importance, concerning ergodicity of transitive area preserving maps on compact surfaces with large stable manifolds, that is, uniformly large diameter of the stable manifolds when measured inside the ambient space.

Theorem D. *Let* f *be a* C^2 *transitive, area preserving endomorphism, and non-uniformly hyperbolic on a compact surface M. If there exists* $\lambda > 0$ *such that for almost every* $x \in M$*, the diameter of the global stable manifold* $W^s(x)$ *is larger than* λ*. Then* f *is ergodic, even more* f *is Bernoulli.*

Finally, a natural question which arises in smooth ergodic theory is how the Lyapunov exponents depend on the map f. Classical results of Mañé-Bochi-Viana show that one cannot expect continuity on the C^1-topology for diffeomorphisms without dominated Oseledets splitting (see the Survey paper [6] for a more detailed discussion). However, the results we present here show that we can obtain better regularity of the Lyapunov exponents for cocycles over endomorphisms than its invertible counterpart.

Defining:

$$C_{\det}(f) := \sup_{n\in\mathbb{N}} \frac{1}{n} \inf_{x\in\mathbb{T}^2} \log(\det(D_x f^n)) > 0,$$

and:

$$\mathcal{U}_1 := \left\{ f \in \mathrm{End}^1_\mu(\mathbb{T}^2) \ : \ C_{\mathcal{X}}(f) > -\frac{1}{2} C_{\det}(f) \right\},$$

Clearly, $\mathcal{U}_1$ is a C^1 open set and contains the set $\mathcal{U}$ which is shown in Theorem 2.1 to contain only NUH endomorphisms. It holds:

Theorem E. *The maps* $\mathcal{U}_1 \ni f \mapsto \int_{\mathbb{T}^2} \lambda^+$ *and* $\mathcal{U}_1 \ni f \mapsto \int_{\mathbb{T}^2} \lambda^-$ *are continuous in the* C^1 *topology.*

This dissertation is organized as follows. Chapter 1 is a survey on the principal aspects we require here of the theory of smooth ergodic systems. We begin by defining the inverse limit space (also known as the natural extension), and showing how, for homotopic maps, these spaces can be canonically identified with a solenoidal space. This construction allows us to discuss the relation of the natural extension for different, but C^1 close, endomorphisms. We then present the measure theoretical properties of these spaces constructed. Finally, we present how to extend the classical Pesin theory for the endomorphism case. In particular we expose a construction of unstable manifolds for endomorphisms which is of great importance in the theory.

Chapter 2 is devoted for the proof of Theorems A and B. It starts by a proof that, indeed, every map $f \in \mathcal{U}$ is non-uniformly hyperbolic, and then follow for the proofs of the main theorems. Thus, resulting on the proof of existence of robust NUH endomorphism in essentially every homotopy class on $\mathbb{T}^2$. This Chapter gave rise to a paper submitted for publication.

Further, in Chapter 3, we prove Theorems C and D. For that, we rely on the classical Hopf argument, along with a new method introduced by Andersson-Carrasco-Saghin [1] to obtain intersections between stable and unstable manifolds.

At last, in Chapter 4 we prove Theorem E. We consider the projectivizations of the correspondent cocycles of a sequence $f_n \in \mathcal{U}_1$ converging to f and their lifts to the Solenoidal space constructed in the first chapter. The only way continuity may fail is if the sequence of stable lifts of the Haar measure on $\mathbb{T}^2$ for f_n (lifts supported on the stable Oseledets subspaces) does not converge to the stable lift of f. This would imply that this limit contains non-negligible parts of the unstable lift for f, in turn this would imply that there is a non-negligible part of the unstable lift for f which does not depends on the backward orbit. However, we show that the definition of $\mathcal{U}_1$ imposes that the unstable Oseledets subspaces cannot be independent of the past.

1 Smooth ergodic theory for endomorphisms

Let M be a smooth N-dimensional Riemannian manifold, μ a volume measure on M. We study conservative maps of M from the point of view of smooth ergodic theory. For a differentiable covering map $f : \mathbb{T}^2 \to \mathbb{T}^2$ and a pair $(x, v) \in TM$, the number

$$\tilde{\lambda}(x, v) = \limsup_{n\to\infty} \frac{\log \|D_x f^n(v)\|}{n}$$

is the Lyapunov exponent of f at (x, v), see [7] for more background in Smooth Ergodic Theory. The following Theorem is one of unique importance in this theory, it asserts the existence of the Lyapunov exponents and its properties.

Theorem 1.1. *(Oselelets Theorem [3]) There exists a full area set $\mathcal{R}$ on M, with $f(\mathcal{R}) = \mathcal{R}$, where the previous limit exists for every $v \neq 0$. Moreover, for every $x \in \mathcal{R}$ there exist a positive integer $s(x)$ and a measurable filtration*

$$T_x M = V^{(1)}(x) \supset \cdots \supset V^{(s(x))}(x) \supset V^{(s(x)+1)}(x) = \{0\},$$

with $dim(V^i(x) \setminus V^{i+1}(x)) := m^{(i)}(x)$, satisfying:

1. *For every $v \in V^i(x) \setminus V^{i+1}(x)$:*

$$\lim_{n\to\infty} \frac{\log \|D_x f^n(v)\|}{n} := \lambda^{(i)}(x).$$

2. *The filtration is invariant under Df, i.e. $D_x f V^{(i)}(x) = V^{(i)}(fx)$.*

3. *The maps $x \mapsto s(x)$, $x \mapsto \lambda^{(i)}(x)$ and $x \mapsto m^{(i)}(x)$, for $x \in \mathcal{R}$, are measurable and f-invariant, i.e. $s(fx) = s(x)$, $\lambda^{(i)}(fx) = \lambda^{(i)}(x)$ and $m^{(i)}(fx) = m^{(i)}(x)$.*

4. *If $\left\{ v_1^1, \cdots v^1_{m^{(1)}(x)}, v^2_{m^{(1)}(x)+1}, \cdots, v^2_{(m^{(1)}+m^{(2)})(x)}, \cdots, v^{s(x)}_{N-m^{s(x)}(x)+1}, \cdots, v_N^{s(x)} \right\}$ is any basis of $T_x M$ with:*

$$\lim_{n\to\infty} \frac{\|D_x f^n v_j^i\|}{n} = \lambda^{(i)}(x).$$

Then for every two non-empty disjoint sets $P, Q \subset \{1, \cdots, N\}$, we have:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \angle(Dx f^n E_P, D_x f^n E_Q) = 0, \tag{1.1}$$

where E_P and E_Q denote the subspaces spanned by the vectors $\{v_j^i\}_{j\in P}$ and $\{v_j^i\}_{j\in Q}$ respectively. And $\angle$ is the angle between two subspaces:

$$\angle(V, W) = \inf_{\substack{0\neq v\in V,\\ 0\neq w\in W}} \angle(v, w), \tag{1.2}$$

where $\angle(v, w)$ is the angle of two vectors obtained by the Riemannian structure of M.

Definition 1.1. *The numbers* $+\infty > \lambda^{(1)}(x) > \cdots > \lambda^{(s(x))}(x) > -\infty$ *are the Lyapunov exponents of* f.

We say that f *is non-uniformly hyperbolic (NUH) (for* μ*), or that* μ *is a hyperbolic measure, if for every* $i \in \{1, \cdots, s(x)\}$, $\lambda^{(i)}(x) \neq 0$ *for* μ*-almost every* $x \in M$.

Non uniformly hyperbolic systems provide a generalization of the classical Anosov maps [4]. Here, we will only be concerned with the non-invertible case in an attempt to aid the understanding of their statistical properties, which is still under development. For the general ergodic theory of endomorphisms, the reader is directed to [5].

1.1 Inverse Limit

Let $f : M \to M$ be a local diffeomorphism with d-sheets, in order to better understand the dynamics of f we consider its natural extension the set

$$\boldsymbol{L_f} = \{(x_0, x_1, x_2, \cdots) \in M^{\mathbb{N}} \ : \ f(x_{i+1}) = x_i, \text{ for every } i \geq 0\},$$

endowed with the product topology inherited from $M^{\mathbb{N}}$, and denote by $\pi_{ext} : \boldsymbol{L_f} \to M$ the projection onto the first coordinate. This way, the map:

$$\begin{aligned} \hat{f} : \boldsymbol{L_f} &\to \boldsymbol{L_f} \\ (x_0, x_1, x_2, \cdots) &\mapsto (f(x_0), x_0, x_1, \cdots) \end{aligned}$$

is a homeomorphism on $\boldsymbol{L_f}$ satisfying $\pi_{ext} \circ \hat{f} = f \circ \pi_{ext}$. We want to show that, under some hypothesis on f (normal cover - Def. 1.2), $\boldsymbol{L_f}$ is a solenoidal N-manifold, that is, it is locally homeomorphic to the product of an N-dimensional disk with a Cantor set. Even more $\boldsymbol{L_f}$ is a bundle over M whose fibres $\pi_{ext}^{-1}(x)$ are Cantor sets.

For that, we will define a group G acting freely and properly discontinuously on $\tilde{M} \times \Sigma$, where $\tilde{M}$ is the universal cover of M and $\Sigma = \{0, 1, \cdots, d-1\}^{\mathbb{N}}$ endowed with the product topology induced by the discrete topology on $\{0, 1, \cdots, d-1\}$. We may then form the orbit space

$$Sol = (\tilde{M} \times \Sigma)/G$$

called the solenoid of f. We shall verify that in fact it depends only on the induced homomorphism $f_* : \pi_1(M, x) \to \pi_1(M, f(x))$ and thus it can be utilized to study the inverse limit for any map in the same homotopy class. This solves the problem that for different maps $f, g : M \to M$ we have that $\boldsymbol{L_f}$ and $\boldsymbol{L_g}$ are different spaces, which makes it more difficult to discuss convergence of measures.

Definition 1.2. *A cover map* $f : M \to M$ *is a normal cover if* $f_*\pi_1(M, x)$ *is a normal subgroup of* $\pi_1(M, f(x))$*, for some* $x \in M$.

Theorem 1.2. *(Characterization of Normal Coverings [8]) The following are equivalent:*

1. $f : M \to M$ *is a normal cover;*

2. *For every* $y \in M$*, the subgroups* $f_*\pi_1(M, x)$ *are the same for every* $x \in f^{-1}(y)$*;*

3. *The subgroup* $f_*\pi_1(M, x)$ *is a normal subgroup of* $\pi_1(M, f(x))$ *for every* $x \in M$.

From now on, f is a normal cover with d sheets. Let $\tilde{f} : \tilde{M} \to \tilde{M}$ be a lift of f to the universal cover $\tilde{M}$ with the projection denoted by π_{cov}, we want to show that there is a canonical way to identify in the group $Aut_{\pi_{cov}}(\tilde{M}) = \{\phi : \tilde{M} \to \tilde{M} \mid \pi_{cov} \circ \phi = \pi_{cov}\}$ of deck transformations, the elements which preserve fibers of the map $\pi_{cov} \circ \tilde{f}^{-1}$. For that, we define the homomorphism:

$$\begin{aligned} \Phi : Aut_{\pi_{cov}}(\tilde{M}) &\to Aut_{\pi_{cov}}(\tilde{M}) \\ \phi &\mapsto \tilde{f} \circ \phi \circ \tilde{f}^{-1} \end{aligned} \tag{1.3}$$

Notice that if an automorphism $\psi \in Im(\Phi)$, then there is a $\phi \in Aut_{\pi_{cov}}(\tilde{M})$ such that $\psi = \tilde{f} \circ \phi \circ \tilde{f}^{-1}$, thus for every $e \in \tilde{M}$, $\tilde{f}^{-1}(\psi(e)) = \phi \circ \tilde{f}^{-1}(e)$. Hence $\tilde{f}^{-1}(\psi(e))$ and $\tilde{f}^{-1}(e)$ are in the same π_{cov}-fiber, for every $e \in \tilde{M}$.

Proposition 1.1. *The subgroup $\Phi(Aut_{\pi_{cov}}(\tilde{M}))$ is a normal subgroup of $Aut_{\pi_{cov}}(\tilde{M})$.*

Proof. For fixed points $x \in M$ and $e \in \pi_{cov}^{-1}(x)$, we have a canonical isomorphism

$$\Delta_x^e : \pi_1(M, x) \to Aut_{\pi_{cov}}(\tilde{M}),$$

that sends a loop $\alpha \in \pi_1(M, x)$ to the unique $\phi_\alpha \in Aut_{\pi_{cov}}(\tilde{M})$ such that if $\tilde{\alpha}$ is the lift of α to $\tilde{M}$ starting in e, then $\phi_\alpha(e) = \tilde{\alpha}(1)$.

Let $y = f(x)$, then $\tilde{f}(e) \in \pi_{cov}^{-1}(y)$. For this proof, we are particularly interested in the isomorphism $\Delta_y^{\tilde{f}(e)} : \pi_1(M, y) \to Aut_{\pi_{cov}}(\tilde{M})$. For $\phi \in \Phi(Aut_{\pi_{cov}}(\tilde{M}))$, let $\alpha \in \pi_1(M, y)$ be such that $\Delta_y^{\tilde{f}(e)}(\phi) = \alpha$. We claim that $\alpha \in f_* \pi_1(M, x)$.

Indeed if $\tilde{\alpha}$ is the lift of α to $\tilde{M}$ starting at $\tilde{f}(e)$, then by the observations made before, we have that $\tilde{f}^{-1} \circ \tilde{\alpha}(0) = e$ and $\tilde{f}^{-1} \circ \tilde{\alpha}(1) = \tilde{f}^{-1} \circ \phi(\tilde{f}(e))$ are in $\pi_{cov}^{-1}(x)$. Hence the path $\pi_{cov} \circ \tilde{f}^{-1} \circ \tilde{\alpha}$ in M is a loop in x. As

$$f \circ \pi_{cov} \circ \tilde{f}^{-1} \circ \tilde{\alpha} = \pi_{cov} \circ \tilde{f} \circ \tilde{f}^{-1} \circ \tilde{\alpha} = \pi_{cov} \circ \tilde{\alpha} = \alpha,$$

we conclude that $\alpha \in f_* \pi_1(M, x)$. Thus $\Delta_y^{\tilde{f}(e)} \circ \Phi(Aut_{\pi_{cov}}(\tilde{M})) \subseteq f_* \pi_1(M, x)$.

Conversely, if $\alpha \in f_* \pi_1(M, x)$, let $\phi_\alpha = \Delta_y^{\tilde{f}(e)}(\alpha)$. If α^* is the lift of α to M such that $f \circ \alpha^* = \alpha$ and $\alpha^*(0) = x$, we have that α^* is a loop in x since $f_*(\alpha^*) = \alpha \in f_* \pi_1(M, x)$. As before, we have $\alpha^* = \pi_{cov} \circ \tilde{f}^{-1} \circ \tilde{\alpha}$. Hence, $\tilde{f}^{-1} \circ \tilde{\alpha}(0) = e$ and $\tilde{f}^{-1} \circ \tilde{\alpha}(1) = \tilde{f}^{-1} \circ \phi_\alpha(\tilde{f}(e))$ are in $\pi_{cov}^{-1}(x)$.

Thus, there exists an unique $\phi \in Aut_{\pi_{cov}}(\tilde{M})$ such that $\phi(e) = \tilde{f}^{-1} \circ \phi_\alpha(\tilde{f}(e))$. We have $\phi_\alpha(\tilde{f}(e)) = \tilde{f} \circ \phi \circ \tilde{f}^{-1}(\tilde{f}(e))$, then $\phi_\alpha = \Phi(\phi)$ (automorphisms that agree at a point are identical [8]). We conclude $(\Delta_y^{\tilde{f}(e)})^{-1} \circ f_*(\pi_1(M, x)) \subseteq \Phi(Aut_{\pi_{cov}}(\tilde{M}))$.

Then, as $\Delta_y^{\tilde{f}(e)} \circ \Phi(Aut_{\pi_{cov}}(\tilde{M})) = f_* \pi_1(M, x)$ and f is a normal cover, we have that $\Phi(Aut_{\pi_{cov}}(\tilde{M}))$ is a normal subgroup.

□

Remark 1.1. *The group $\Phi(Aut_{\pi_{cov}}(\tilde{M}))$ does not depend on the choice of the lift $\tilde{f}$. Indeed, given $\tilde{f}_1$ another lift of f, there exists $\psi \in Aut_{\pi_{cov}}(\tilde{M})$ such that $\tilde{f}_1 = \psi \circ \tilde{f}$. Thus, if Φ_1 is the correspondent homomorphism induced by $\tilde{f}_1$, we obtain $Im(\Phi_1) = \psi Im(\Phi) \psi^{-1} = Im(\Phi)$, since $Im(\Phi)$ is a normal subgroup.*

Proposition 1.2. *The group $Aut_{\pi_{cov}}(\tilde{M})/\Phi(Aut_{\pi_{cov}}(\tilde{M}))$ is finite and has d elements.*

Proof. We fix points $y \in M$, $e \in \pi_{cov}^{-1}(y)$, in the proof of Prop. 1.1 we have seen that the canonical isomorphism Δ_y^e sends $\Phi(Aut_{\pi_{cov}}(\tilde{M}))$ to $f_*\pi_1(M,x)$, for some $x \in f^{-1}(y)$. As mentioned in Theorem 1.2, the normal subgroup $f_*\pi_1(M,x)$ is the same for every $x \in f^{-1}(y)$, let us denote this subgroup by N for simplicity.

Thus, we have a natural isomorphism, induced by Δ_y^e, from $Aut_{\pi_{cov}}(\tilde{M})/\Phi(Aut_{\pi_{cov}}(\tilde{M}))$ to $\pi_1(M,y)/N$. Then it is enough to prove that $\pi_1(M,y)/N$ has d elements.

For $\alpha \in \pi_1(M,y)$ we define the map:

$$\begin{aligned} L_\alpha \,:\, f^{-1}(y) &\to f^{-1}(y) \\ \alpha_x^*(1) &\mapsto x, \end{aligned}$$

where α_x^* is the unique lift of α ($f \circ \alpha^* = \alpha$) starting in x. The map $L(\alpha) = L_\alpha$ is a homomorphism from $\pi_1(M,y)$ to the group of permutations of $\{0,1,\cdots,d-1\}$, denoted S_d. Its kernel is N, because for every $x \in f^{-1}(y)$, $L_\alpha(x) = x$ if and only if $\alpha \in f_*\pi_1(M,x) = N$. We have from the first isomorphism theorem that $\pi_1(M,y)/N \cong L(\pi_1(M,y))$, which already gives us that this group is finite.

To see that it actually has d elements, we invoke Burnside's Counting Theorem [9], to this particular case where the action of $\pi_1(M,y)/N$ on $f^{-1}(y)$ is transitive [8]. In this case we get that the order of $\pi_1(M,y)/N$ equals the sum of the number of fixed points of its elements, which equals d since the only element with a fixed point is the identity. □

With this setting, we can choose maps $\psi_0 = Id$, $\psi_1, \cdots, \psi_{d-1} \in Aut_{\pi_{cov}}(\tilde{M})$ such that

$$Aut_{\pi_{cov}}(\tilde{M})/\Phi(Aut_{\pi_{cov}}(\tilde{M})) = \{Aut_{\pi_{cov}}(\tilde{M})\psi_i \;:\; i = 0,1,\cdots,d-1\}.$$

Putting $\Sigma = \{0,\cdots,d-1\}^{\mathbb{N}}$, we may define a group G of transformations of $\tilde{M} \times \Sigma$ given by:

$$G = \{(e,\boldsymbol{\omega}) \mapsto (\phi(e), \Omega_\phi(\boldsymbol{\omega})) \mid \phi \in Aut_{\pi_{cov}}(\tilde{M})\}, \tag{1.4}$$

where each $\Omega_\phi \,:\, \Sigma \to \Sigma$ is a homeomorphism given by:

Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots)$, and set $\phi_0 = \phi$. For $n \geq 1$:

- Let τ_n be the unique number in $\{0,\cdots,d-1\}$ such that:

$$\psi_{\tau_n}\phi_{n-1}\psi_{\omega_n}^{-1} \in Aut_{\pi_{cov}}(\tilde{M}), \text{ and}$$

- set ϕ_n as the unique element of $Aut_{\pi_{cov}}(\tilde{M})$ such that

$$\psi_{\tau_n}\phi_{n-1}\psi_{\omega_n}^{-1} = \Phi(\psi_n),$$

then $\Omega_\phi(\boldsymbol{\omega}) = \boldsymbol{\tau}$.

G is a group action of $Aut_{\pi_{cov}}(\tilde{M})$ on $\tilde{M} \times \Sigma$ and, as such, it acts freely and properly discontinuously, we may thus form a space

$$Sol = (\tilde{M} \times \Sigma)/G,$$

called the solenoid of f. It is a solenoidal N-manifold, N=dim(M), meaning that it is a fibre bundle over M in which the local trivializations are products of an N-dimensional disk with a Cantor set.

Proposition 1.3. *Define $\mathcal{F}_i = \tilde{f}^{-1} \circ \psi_i$. Given $\phi \in Aut_{\pi_{cov}}(\tilde{M})$, and $\boldsymbol{\omega}, \boldsymbol{\tau} \in \Sigma$, the following are equivalent:*

1. $\Omega_\phi(\boldsymbol{\omega}) = \boldsymbol{\tau}$,
2. $\mathcal{F}_{\tau_n} \circ \cdots \circ \mathcal{F}_{\tau_1} \circ \phi(e) = \phi_n \circ \mathcal{F}_{\omega_n} \circ \cdots \circ \mathcal{F}_{\omega_1}(e)$, *for every* $e \in \tilde{M}, n \geq 1$,
3. $\pi_{cov} \circ \mathcal{F}_{\tau_n} \circ \cdots \circ \mathcal{F}_{\tau_1} \circ \phi(e) = \pi_{cov} \circ \mathcal{F}_{\omega_n} \circ \cdots \circ \mathcal{F}_{\omega_1}(e)$, *for every* $e \in \tilde{M}, n \geq 1$.

Proof. $1 \Rightarrow 2$: Assuming $\Omega_\phi(\boldsymbol{\omega}) = \boldsymbol{\tau}$, we have that τ_1 is the unique number in $\{0, \cdots, d-1\}$ such that $\psi_{\tau_1}\phi\psi_{\omega_1}^{-1} \in \Phi(Aut_{\pi_{cov}}(\tilde{M}))$ and $\phi_1 \in Aut_{\pi_{cov}}(\tilde{M})$ satisfies $\Phi(\phi_1) = \tilde{f} \circ \phi_1 \circ \tilde{f}^{-1} = \phi_{\tau_1}\phi\phi_{\omega_1}^{-1}$. Hence, for every $e \in \tilde{M}$:

$$\begin{aligned}\mathcal{F}_{\tau_1} \circ \phi(e) = \tilde{f}^{-1} \circ \psi_{\tau_1} \circ \phi(e) &= \tilde{f}^{-1} \circ \tilde{f} \circ \phi_1 \circ \tilde{f}^{-1} \circ \psi_{\omega_1}(e) \\ &= \phi_1 \circ \tilde{f}^{-1} \circ \psi_{\omega_1}(e).\end{aligned}$$

Inductively, assume that $\mathcal{F}_{\tau_{n-1}} \circ \cdots \circ \mathcal{F}_{\tau_1} \circ \phi(e) = \phi_{n-1} \circ \mathcal{F}_{\omega_{n-1}} \circ \cdots \circ \mathcal{F}_{\omega_1}(e)$. We have τ_n as the unique number such that $\psi_{\tau_n}\phi_{n-1}\phi_{\omega_n}^{-1} \in Aut_{\pi_{cov}}(\tilde{M})$ and $\phi_n \in Aut_{\pi_{cov}}(\tilde{M})$ satisfying $\Phi(\phi_n) = \tilde{f} \circ \phi_n \circ \tilde{f}^{-1} = \psi_{\tau_n}\phi_{n-1}\psi_{\omega_n}^{-1}$, therefore for every $e \in \tilde{M}$:

$$\begin{aligned}\mathcal{F}_{\tau_n} \circ \cdots \circ \mathcal{F}_{\tau_1} \circ \phi(e) &= \tilde{f}^{-1} \circ \psi_{\tau_n} \circ \mathcal{F}_{\tau_{n-1}} \circ \cdots \circ \mathcal{F}_{\tau_1} \circ \phi(e) \\ &= \tilde{f}^{-1} \circ \psi_{\tau_n} \circ \phi_{n-1} \circ \mathcal{F}_{\omega_{n-1}} \circ \cdots \circ \mathcal{F}_{\omega_1}(e) \\ &= \phi_n \circ \mathcal{F}_{\omega_n} \circ \cdots \circ \mathcal{F}_{\omega_1}(e)\end{aligned}$$

by the same argument used before.

$2 \Rightarrow 3$: That is a direct consequence of the fact that π_{cov} is invariant under Deck transformations.

$3 \Rightarrow 1$: We make an inductive process. Initially we have

$$\pi_{cov} \circ \tilde{f}^{-1} \circ \psi_{\tau_1} \circ \phi = \pi_{cov} \circ \tilde{f}^{-1} \circ \psi_{\omega_1},$$

then given $e \in \tilde{M}$, $\tilde{f}^{-1} \circ \psi_{\tau_1} \circ \phi(e)$ and $\tilde{f}^{-1} \circ \psi_{\omega_1}(e)$ are in the same fiber. Consequently, there exist a $\phi_1 \in Aut_{\pi_{cov}}(\tilde{M})$ that maps one point to the other, that is, such that:

$$\phi_1 \circ \tilde{f}^{-1} \circ \psi_{\omega_1}(e) = \tilde{f}^{-1} \circ \psi_{\tau_1} \circ \phi(e).$$

Thus, $\tilde{f}^{-1} \circ \phi_1 \circ \tilde{f}^{-1} \circ \psi_{\omega_1}(e) = \psi_{\tau_1} \circ \phi(e)$, and, since Deck transformations that coincides in a point are the same, we get $\Phi(\phi_1) = \psi_{\tau_1}\phi\psi_{\omega_1}^{-1}$, as desired. Even more, we have the relation $\mathcal{F}_{\tau_1} \circ \phi = \phi_1 \circ \mathcal{F}_{\omega_1}$.

Inductively, we assume $\psi_{\tau_{n-1}}\phi_{n-2}\psi_{\omega_{n-1}}^{-1} \in \Phi(Aut_{\pi_{cov}}(\tilde{M}))$, and that $\phi_{n-1} \in Aut_{\pi_{cov}}(\tilde{M})$ such that $\Phi(\phi_{n-1}) = \psi_{\tau_{n-1}}\phi_{n-2}\psi_{\omega_{n-1}}^{-1}$ satisfies the relation

$$\mathcal{F}_{\tau_{n-1}} \circ \cdots \circ \mathcal{F}_{\tau_1} \circ \phi = \phi_{n-1} \circ \mathcal{F}_{\omega_{n-1}} \circ \cdots \circ \mathcal{F}_{\omega_1}.$$

As we have:

$$\pi_{cov} \circ \mathcal{F}_{\tau_n} \circ \cdots \circ \mathcal{F}_{\tau_1} \circ \phi = \pi_{cov} \circ \mathcal{F}_{\omega_n} \circ \cdots \circ \mathcal{F}_{\omega_1},$$

given $e \in \tilde{M}$, there exists $\phi_n \in Aut_{\pi_{cov}}(\tilde{M})$ such that:

$$\phi_n \circ \mathcal{F}_{\omega_n} \circ \cdots \circ \mathcal{F}_{\omega_1}(e) = \mathcal{F}_{\tau_n} \circ \cdots \circ \mathcal{F}_{\tau_1} \circ \phi(e),$$

hence:

$$\tilde{f}^{-1} \circ \phi_n \circ \tilde{f}^{-1} \circ \psi_{\omega_n} \circ \mathcal{F}_{\omega_{n-1}} \circ \cdots \circ \mathcal{F}_{\omega_1}(e) = \psi_{\tau_n} \circ \mathcal{F}_{\tau_{n-1}} \circ \cdots \circ \mathcal{F}_{\tau_1} \circ \phi(e)$$
$$= \psi_{\tau_n} \circ \phi_{n-1} \circ \mathcal{F}_{\omega_{n-1}} \circ \cdots \circ \mathcal{F}_{\omega_1}(e).$$

We conclude, as before, that $\tilde{f}^{-1} \circ \phi_n \circ \tilde{f}^{-1} \circ \psi_{\omega_n} = \psi_{\tau_n} \circ \phi_{n-1}$. Thus, $\Phi(\phi_n) = \psi_{\tau_n}\phi_{n-1}\psi_{\omega_n}^{-1}$ as desired, and it also satisfies the relation:

$$\mathcal{F}_{\tau_n} \circ \cdots \circ \mathcal{F}_{\tau_1} \circ \phi = \phi_n \circ \mathcal{F}_{\omega_n} \circ \cdots \circ \mathcal{F}_{\omega_1}.$$

□

Define the map $\tilde{\Psi} : \tilde{M} \times \Sigma \to \boldsymbol{L_f}$ by setting:

$$\tilde{\Psi}(e, \omega_1, \omega_2, \cdots) = (x_0, x_1, x_2, \cdots),$$

where

$$x_i = \pi_{cov} \circ \mathcal{F}_{\omega_i} \circ \cdots \circ \mathcal{F}_{\omega_1}(e).$$

It follows from Prop. 1.3 that $\tilde{\Psi}$ is G-invariant, i.e.

$$\tilde{\Psi}(e, \boldsymbol{\omega}) = \tilde{\Psi}(\phi(e), \Omega_\phi(\boldsymbol{\omega})), \quad \text{for every } \phi \in Aut_{\pi_{cov}}(\tilde{M}).$$

Hence $\tilde{\Psi}$ induces a homeomorphism $\Psi : Sol \to \boldsymbol{L_f}$ such that $\tilde{\Psi} = \Psi \circ \pi_G$, where $\pi_G : \tilde{M} \times \Sigma \to Sol$ is the natural projection. This way, the natural extension $\hat{f} : \boldsymbol{L_f} \to \boldsymbol{L_f}$ induces a homeomorphism $Sf : Sol \to Sol$ by setting $Sf = \Psi^{-1} \circ \hat{f} \circ \Psi$. The expression of Sf is given by a quite nice way to visualize: we let $F, F^{\#} : \tilde{M} \times \Sigma \to \tilde{M} \times \Sigma$ be the maps:

$$(e, \omega_1, \omega_2, \cdots) \stackrel{F}{\mapsto} (\tilde{f}(e), 0, \omega_1, \omega_2, \cdots),$$
$$(e, \omega_1, \omega_2, \cdots) \stackrel{F^{\#}}{\mapsto} (\mathcal{F}_{\omega_1}(e), \omega_2, \omega_3, \cdots),$$

with $\mathcal{F}_i$ as in Prop. 1.3 ($\mathcal{F}_0(e) = \tilde{f}^{-1}(e)$). The map F is not surjective and the map $F^{\#}$ is not injective, but $F^{\#} \circ F$ is the identity on $\tilde{M} \times \Sigma$, we get:

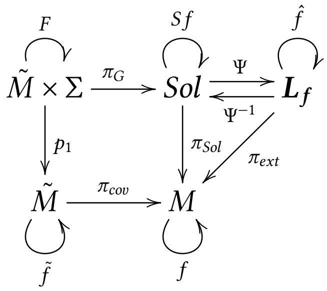
Proposition 1.4. *The map Sf acts as follows:*

$$Sf([(e, \boldsymbol{\omega})]_G) = [F(e, \boldsymbol{\omega})]_G,$$

and

$$Sf^{-1}([(e, \boldsymbol{\omega})]_G) = [F^{\#}(e, \boldsymbol{\omega})]_G$$

We exhibit the following commuting diagram in order to clarify the notations of the maps constructed and how they relate:



(1.5)

Finally, the problem of working in the natural extension $\boldsymbol{L_f}$ is that it depends on the map f. For some results, namely Theorem E in this dissertation, this can create a problem as the one remarked in the beginning of Section 4.2. The next result shows us that the Solenoidal space, indeed, only depends on the homotopy class of a map.

Theorem 1.3. *If $g : M \to M$ is a local diffeomorphism homotopic to f, and Φ_g is the homomorphism on $Aut_{\pi_{cov}}(\tilde{M})$ induced by a lift $\tilde{g} : \tilde{M} \to \tilde{M}$. Then g is a normal cover, and the images $\Phi_g(Aut_{\pi_{cov}}(\tilde{M})) = \Phi(Aut_{\pi_{cov}}(\tilde{M}))$ are the same.*

As a direct consequence, the space Sol constructed as before is the same for every g homotopic to f.

Proof. We first show that g is a normal cover. Let $H : [0,1] \times M \to M$ be such that $H(0,\cdot) = f$ and $H(1,\cdot) = g$. For a fixed $x \in M$, denote by $h_1(t) = H(t,x)$ the path from $f(x)$ to $g(x)$, and by $\mathcal{B}_{h_1} : \pi_1(M, f(x)) \to \pi_1(M, g(x))$ the change of base point induced by the path h_1 (which is an isomorphism). Our claim is that $\mathcal{B}_{h_1} \circ f_*(\pi_1(M,x)) = g_*(\pi_1(M,x))$, which, since f is a normal cover, proves that $g_*(\pi_1(M,x))$ is a normal subgroup of $\pi_1(M, g(x))$ for every x.

Indeed if $\alpha_1 \in \mathcal{B}_{h_1} \circ f_*(\pi_1(M,x))$ let $\alpha \in f_*(\pi_1(M,x))$ be such that $\mathcal{B}_{h_1}(\alpha) = \alpha_1$, and $\alpha_f \in \pi_1(M,x)$ be such that $f_*(\alpha_f) = \alpha$, we show that $g_*(\alpha_f) = \alpha_1$. For that, define:

$$G : [0,1] \times [0,1] \to M$$

$$(t,s) \mapsto \begin{cases} H(1-3st, x), \ t \in [0,1/3]; \\ H(1-s, \alpha_f(3t-1)), \ t \in [1/3,2/3]; \\ H(1-3s(1-t), x), \ t \in [2/3,1]. \end{cases}$$

Then, $G(0,s) = G(1,s) = H(1,x) = g(x)$ for every s, $G(t,0) \simeq g \circ \alpha_f$ and $G(t,1) \simeq h_1 \cdot \alpha \cdot -h_1 = \mathcal{B}_{h_1}(\alpha) = \alpha_1$, where $\simeq$ means that they are homotopic loops. This shows that $g \circ \alpha_f$ is homotopic to α_1, hence $g_*(\alpha_f) = \alpha_1$, that is, $\alpha_1 \in g_*(\pi_1(M,x))$. The other inclusion follows by an analogous argument.

Now, if we define $h : [0,1] \to M$ to be the unique lift of h_1 by f starting at the point x ($f \circ h = h_1$), we must have $h(1) = y \in f^{-1}(g(x))$. If we denote by $\mathcal{B}_h : \pi_1(M,x) \to \pi_1(M,y)$ the change of base point induced by h, it is a simple exercise to show that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(M,x) & \xrightarrow{\mathcal{B}_h} & \pi_1(M,y) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \pi_1(M,f(x)) & \xrightarrow{\mathcal{B}_{h_1}} & \pi_1(M,g(x)) \end{array}$$

Then the image $f_*(\pi_1(M,y)) = \mathcal{B}_{h_1}(f_*(\pi_1(M,x)) = g_*(\pi_1(M,x))$. This, along with item 2 of Theorem 1.2, gives us the following:

Lemma 1.1. *Let $g : M \to M$ be homotopic to the normal cover $f : M \to M$. Then, for every $y \in M$, the subgroups $g_*(\pi_1(M,x))$, and $f_*(\pi_1(M,z))$ are all the same for every $x \in g^{-1}(y)$ and $z \in f^{-1}(y)$.*

Finally, the argument utilized in the proof of Prop. 1.1 shows us that the image $\Phi(Aut_{\pi_{cov}}(\tilde{M}))$ only depends on the image of the induced homomorphism f_*. The same argument goes for Φ_G since we have already proved that it is a normal cover. This concludes that $Im(\Phi_G) = Im(\Phi)$, as we wanted.

□

1.2 Invariant measures

Until now, we have studied the lifts of f in a topological point of view, from now on we present the measure theoretical aspects of it. Let us assume that μ is an f-invariant Borel probability in M, i.e., $f_*\mu(A) = \mu(f^{-1}A) = \mu(A)$ for every $A \in \mathcal{B}(M)$. It induces an unique $\tilde{f}$-invariant Borel measure in $\tilde{M}$ by setting, for $A \in \mathcal{B}(\tilde{M})$, $\tilde{\mu}(A) = \mu(\pi_{cov}(A))$ whenever $\pi_{cov}|_A$ is injective, and extending it accordingly. In this case, we say that $\tilde{\mu}$ descends to μ.

Proposition 1.5. *There is a unique $\hat{f}$-invariant Borel measure on $\boldsymbol{L}_f$, which we denote by $\hat{\mu}$, such that $\pi_{ext*}\hat{\mu} = \mu$.*

Proof. For $A \in \mathcal{B}(M)$ and $n \geq 0$, define $A_n = \{\hat{x} \in \boldsymbol{L}_f \,:\, x_n \in A\}$. Then, $A_n \in \mathcal{B}(\boldsymbol{L}_f)$ and in fact the set $\{A_n \,:\, A \in \mathcal{B}(M),\ n \geq 0\}$ generates the Borel σ-algebra on $\boldsymbol{L}_f$. We set the measure $\hat{\mu}$ on $\boldsymbol{L}_f$ by

$$\hat{\mu}(A_n) = \mu(A),$$

it is direct to verify that it satisfy the properties required. It is unique since if $\hat{\nu}$ is another measure on $\boldsymbol{L}_f$ with these properties, by an iterate process with $\hat{f}$ we get $\hat{\nu}(A_n) = \hat{\mu}(A_n)$ for every $A \in \mathcal{B}(M)$, $n \geq 0$.

□

Now, since Sf is conjugated to $\hat{f}$ by $\Psi : Sol \to \boldsymbol{L}_f$, and $\pi_{ext} \circ \Psi = \pi_{Sol}$, it follows that there is a unique Sf-invariant measure $\boldsymbol{\mu}$ on Sol that projects to μ through π_{Sol}. This measure has a particularly intuitive description that follows.

Let $\tilde{\boldsymbol{\mu}}$ be the lift of $\boldsymbol{\mu}$ to $\tilde{M} \times \Sigma$ (which is a cover of Sol). For convenience purposes we present a schematic diagram of how $\mu, \tilde{\mu}, \hat{\mu}, \boldsymbol{\mu}, \tilde{\boldsymbol{\mu}}$ are related. A filled arrow ($\rightarrow$) indicates that measures are related by push-forward and a dashed arrow ($\dashrightarrow$) indicates that one descends to the other.

$$\begin{array}{ccccc} & & Sf \circlearrowright & & \hat{f} \circlearrowright \\ (\tilde{M}\times\Sigma, \tilde{\boldsymbol{\mu}}) & \overset{\pi_G}{\dashrightarrow} & (Sol,\boldsymbol{\mu}) & \underset{\Psi^{-1}}{\overset{\Psi}{\rightleftarrows}} & (\boldsymbol{L}_f,\hat{\mu}) \\ \downarrow p_1 & & \downarrow \pi_{Sol} & \swarrow \pi_{ext} & \\ (\tilde{M},\tilde{\mu}) & \overset{\pi_{cov}}{\dashrightarrow} & (M,\mu) & & \\ & & \circlearrowright f & & \end{array} \tag{1.6}$$

Since $\pi_{Sol*}\boldsymbol{\mu} = \mu$, we must have that $p_{1*}\tilde{\boldsymbol{\mu}} = \tilde{\mu}$, i.e., for every measurable $A \in \tilde{M}$

$$\tilde{\boldsymbol{\mu}}(A \times \Sigma) = \tilde{\mu}(A).$$

Thus, invariance of $\boldsymbol{\mu}$ under Sf together with Prop. 1.4 gives us that

$$\tilde{\boldsymbol{\mu}}(A \times C(\omega_1, \cdots, \omega_n)) = \tilde{\mu}(\mathcal{F}_{\omega_n} \circ \cdots \circ \mathcal{F}_{\omega_1}(A)), \tag{1.7}$$

where $C(\omega_1, \cdots, \omega_n) = \{\boldsymbol{\tau} \in \Sigma \ :\ \tau_i = \omega_i, \text{ for } i = 1, \cdots, n.\}$ is called a cylinder set. The cylinders generates the Borel σ-algebra on $\tilde{M} \times \Sigma$, hence this relation determines $\tilde{\boldsymbol{\mu}}$. Thus, for sets $A \subset \tilde{M}$ contained in a fundamental domain ($\phi(A) \cap A = \emptyset$, for every $\phi \in Aut_{\pi_{cov}}(\tilde{M})$), it also determines $\boldsymbol{\mu}$.

Remark 1.2. *The maps* $\tilde{f} : \tilde{M} \to \tilde{M}$ *and* $F : \tilde{M}\times\Sigma \to \tilde{M}\times\Sigma$ *have been left out from (1.6) as they are not necessarily measure preserving. In fact,* $\tilde{f}$ *preserves* $\tilde{\mu}$ *if, and only if,* f *has constant Jacobian with respect to* μ*, i.e., there is a constant* C *such that* $\mu(f(U)) = C\mu(U)$ *for every* $U \subset M$ *where* $f|_U$ *is 1-1. Also, the push-foward* $F_*\tilde{\boldsymbol{\mu}}$ *is the restriction of* $\tilde{\boldsymbol{\mu}}$ *to* $\tilde{M} \times C(0)$ *(the image of* F*).*

By a change of variables, the right hand side of (1.7) becomes:

$$\begin{aligned}\tilde{\boldsymbol{\mu}}(A \times C(\omega_1, \cdots, \omega_n)) &= \int_A |\det D_{\tilde{x}}(\mathcal{F}_{\omega_n} \circ \cdots \circ \mathcal{F}_{\omega_1})| \, d\tilde{\mu}(\tilde{x}) \\ &= \int_A \tilde{\boldsymbol{\mu}}_{\tilde{x}}(C(\omega_1, \cdots, \omega_n)) \, d\tilde{\mu}(\tilde{x}),\end{aligned} \tag{1.8}$$

where $\tilde{\boldsymbol{\mu}}_{\tilde{x}}$ is the unique measure on Σ such that for every cylinder set $C(\omega_1, \cdots, \omega_n) \subset \Sigma$:

$$\tilde{\boldsymbol{\mu}}_{\tilde{x}}(C(\omega_1, \cdots, \omega_n)) = |\det D_{\tilde{x}}(\mathcal{F}_{\omega_n} \circ \cdots \circ \mathcal{F}_{\omega_1})| \tag{1.9}$$

From item 2 of Prop. 1.3, we have that the right hand side of (1.9) is a G-invariant function on $\tilde{M} \times \Sigma$, hence:

$$\phi_*\tilde{\boldsymbol{\mu}}_{\tilde{x}} = \tilde{\boldsymbol{\mu}}_{\phi(\tilde{x})}, \quad \text{for every } \phi \in Aut_{\pi_{cov}}(\tilde{M}),$$

and $\tilde{\boldsymbol{\mu}}_{\tilde{x}}$ descends to a measure $\boldsymbol{\mu}_x$ on *Sol*, where $x = \pi_{cov}(\tilde{x})$, given by $\boldsymbol{\mu}_x = (\pi_G)_*\tilde{\boldsymbol{\mu}}_{\tilde{x}}$ for any $\tilde{x} \in \pi_{cov}^{-1}(x)$.

The push-forward $\hat{\mu}_x = \Psi_*\boldsymbol{\mu}_x$ is the same as the push-forward of $\tilde{\boldsymbol{\mu}}_{\tilde{x}}$ under $\tilde{\Psi}$. Indeed, for every $(\tilde{x}, \omega_1, \omega_2, \cdots) \in \tilde{\Psi}^{-1}(x_0, x_1, \cdots)$ and every $n \geq 1$, we have:

$$|\det D_{\tilde{x}}(\mathcal{F}_{\omega_n} \circ \cdots \circ \mathcal{F}_{\omega_1})| = |\det D_{x_n} f^n|^{-1},$$

thus, by setting $E(x_n) = \{(\xi_1, \xi_2, \cdots) \in \boldsymbol{L}_f : \xi_n = x_n\} \subset \boldsymbol{L}_f$:

$$\hat{\mu}_x(E(x_n)) = \boldsymbol{\mu}_x(\Psi^{-1}(E_n)) = \tilde{\boldsymbol{\mu}}_{\tilde{x}}(\pi_G^{-1} \circ \Psi^{-1}(E_n)) = \tilde{\boldsymbol{\mu}}_{\tilde{x}}(\tilde{\Psi}^{-1}(E_n)) = |\det D_{x_n} f^n|^{-1}. \tag{1.10}$$

Extending (1.7) to the full σ-algebra on $\tilde{M} \times \Sigma$ and descending to Sol accordingly, we get:

$$\boldsymbol{\mu}(A) = \int_M \boldsymbol{\mu}_x(A) d\mu(x), \quad \text{for every measurable } A \subset Sol. \tag{1.11}$$

Similarly,

$$\hat{\mu}(A) = \int_M \hat{\mu}_x(A) d\mu(x), \quad \text{for every measurable } A \subset \boldsymbol{L}_f. \tag{1.12}$$

We are now in condition to discuss the continuity of those measures. Of course the measures $\hat{\mu}, \hat{\mu}_x, \boldsymbol{\mu}_x, \tilde{\boldsymbol{\mu}}_{\tilde{x}}, \boldsymbol{\mu}$ and $\tilde{\boldsymbol{\mu}}$ all depend on the map f, which we evidence by denoting these measures by $\hat{\mu}^f, \hat{\mu}_x^f, \boldsymbol{\mu}_x^f, \tilde{\boldsymbol{\mu}}_{\tilde{x}}^f, \boldsymbol{\mu}^f$ and $\tilde{\boldsymbol{\mu}}^f$.

Proposition 1.6. *The measures* $\hat{\mu}_x^f, \boldsymbol{\mu}_x^f, \tilde{\boldsymbol{\mu}}_{\tilde{x}}^f, \boldsymbol{\mu}^f$ *and* $\tilde{\boldsymbol{\mu}}^f$ *all depend continuously on* f*, when seem as maps from* $End_\mu^1(M)$ *to their respect spaces endowed with the weak* topology.*

Furthermore, the measures $\hat{\mu}_x^f$*,* $\boldsymbol{\mu}_x^f$ *and* $\tilde{\boldsymbol{\mu}}_{\tilde{x}}^f$ *all depend continuously on* $x \in M$ *(or* $\tilde{x} \in \tilde{M}$*), in the weak* topology of their respective spaces.*

Finally, if we consider the measure $\hat{\mu}^f$ *as a measure on* $M^{\mathbb{Z}_+}$*, supported on* $\boldsymbol{L}_f$*, then it also depends continuously on* $f \in End_\mu^1(M)$*.*

Proof. The proof of the first affirmation goes as follows. The first equality of (1.8) gives us continuity for $\tilde{\boldsymbol{\mu}}^f$, which also implies continuity for $\boldsymbol{\mu}^f$. (1.9) gives us continuity for $\tilde{\boldsymbol{\mu}}^f_{\tilde{x}}$, which from the relations $\boldsymbol{\mu}^f_x = (\pi_G)_*\tilde{\boldsymbol{\mu}}^f_{\tilde{x}}$ and $\hat{\mu}^f_x = \Psi_*\boldsymbol{\mu}^f_x$, it also implies continuity for these measures.

The second affirmation is a direct consequence of (1.10). The third one comes as a consequence of the continuity of $\hat{\mu}^f_x$ in relation to f, together with (1.12).

□

At last, we show how to construct and understand the projectivized cocycles generated by the ones studied here. Each of the spaces $\boldsymbol{L}_f$, Sol and $\tilde{M} \times \Sigma$ comes with a fibre bundle:

$$\begin{aligned} &\mathbb{P}\mathbb{R}^N \to \mathbb{P}\boldsymbol{L}_f \xrightarrow{p} \boldsymbol{L}_f, && p^{-1}(\hat{x}) = \mathbb{P}T_{\pi_{ext}(\hat{x})}M, \\ &\mathbb{P}\mathbb{R}^N \to \mathbb{P}Sol \xrightarrow{p} Sol, && p^{-1}([(\tilde{x},\boldsymbol{\omega})]_G) = \mathbb{P}T_{\pi_{Sol}([(\tilde{x},\boldsymbol{\omega})]_G)}M, \\ &\mathbb{P}\mathbb{R}^N \to \mathbb{P}(\tilde{M}\times\Sigma) \xrightarrow{p} \tilde{M}\times\Sigma, && p^{-1}(\tilde{x},\boldsymbol{\omega}) = \mathbb{P}T_{\pi_{cov}(\tilde{x})}M, \end{aligned}$$

where we denote the three projections by p for simplicity, and $N = dimM$.

On each of these bundles, the derivative of f induces bundle maps $\mathbb{P}\hat{f}$, $\mathbb{P}Sf$ and $\mathbb{P}F$, given by:

$$\begin{aligned} \mathbb{P}\hat{f} &: (\hat{x},[v]) \mapsto (\hat{f}(\hat{x}), [D_{\pi_{ext}(\hat{x})}f \cdot v]), \\ \mathbb{P}Sf &: ([(\tilde{x},\boldsymbol{\omega})]_G,[v]) \mapsto (Sf([(\tilde{x},\boldsymbol{\omega})]_G), [D_{\pi_{Sol}([(\tilde{x},\boldsymbol{\omega})]_G)}f \cdot v]), \\ \mathbb{P}F &: ((\tilde{x},\boldsymbol{\omega}),[v]) \mapsto (F(\tilde{x},\boldsymbol{\omega}), [D_{\pi_{cov}(\tilde{x})}f \cdot v]). \end{aligned}$$

Remark 1.3. *When $TM \cong M \times \mathbb{R}^N$ is trivial, the projective bundles are also trivial.*

If we denote by i the identity map on the fibres, the following diagram commutes;

$$\begin{array}{ccccc} \overset{\mathbb{P}F}{\circlearrowright} & & \overset{\mathbb{P}Sf}{\circlearrowright} & & \overset{\mathbb{P}\hat{f}}{\circlearrowright} \\ \mathbb{P}(\tilde{M}\times\Sigma) & \xrightarrow{(\pi_G,i)} & \mathbb{P}Sol & \underset{(\Psi^{-1},i)}{\overset{(\Psi,i)}{\rightleftarrows}} & \mathbb{P}\boldsymbol{L}_f \\ \downarrow p & & \downarrow p & & \downarrow p \\ \tilde{M}\times\Sigma & \xrightarrow{\pi_G} & Sol & \underset{\Psi^{-1}}{\overset{\Psi}{\rightleftarrows}} & \boldsymbol{L}_f \\ \underset{F}{\circlearrowleft} & & \underset{Sf}{\circlearrowleft} & & \underset{\hat{f}}{\circlearrowleft} \end{array} \tag{1.13}$$

The study of these cocycles will be of particular interest when studying continuity of Lyapunov exponents (Chapter 4), thus we are also interest on their invariant measures. In particular, a $\mathbb{P}Sf$-invariant measure $\boldsymbol{\mu}^{\mathbb{P}}$ corresponds to a $\mathbb{P}\hat{f}$-invariant measure $\hat{\mu}^{\mathbb{P}} = (\Psi,i)_*\boldsymbol{\mu}^{\mathbb{P}}$ and vice-versa. Moreover, $\boldsymbol{\mu}^{\mathbb{P}}$ lifts to a unique measure $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}$ on $\mathbb{P}(\tilde{M}\times\Sigma)$ through (π_G,i), which is the same as the lift of $\hat{\mu}^{\mathbb{P}}$ through $(\tilde{\Psi},i)$, however this measure may not be invariant under $\mathbb{P}F$ as remarked in 1.2. Even though, we have:

Proposition 1.7. *Given a measure $\boldsymbol{\mu}^{\mathbb{P}}$ on $\mathbb{P}Sol$ and a sequence of measures $\boldsymbol{\mu}^{\mathbb{P}}_n$ on $\mathbb{P}Sol$, with corresponding lifts $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}$ and $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}_n$ on $\mathbb{P}(\tilde{M}\times\Sigma)$. Then, $\boldsymbol{\mu}^{\mathbb{P}}_n$ converges weakly* to $\boldsymbol{\mu}^{\mathbb{P}}$ if, and only if, $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}_n$ converges weakly* to $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}$.*

Remark 1.4. *Since the measures $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}_n$ and $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}$ are (ϕ,Ω_ϕ,i)-invariant for any $(\phi,\Omega_\phi) \in G$ as in* (1.4)*, in order to obtain that $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}_n$ converges weakly* to $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}$, it suffices to show that $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}_n|_Q$ converges to $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}|_Q$ where $Q \subset \mathbb{P}(\tilde{M}\times\Sigma)$ is any set containing a fundamental domain of the action of $\{(\phi,\Omega_\phi,i) : (\phi,\Omega_\phi) \in G\}$, and such that $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}(\partial Q) = 0$.*

1.3 Pesin theory for endomorphisms

The unstable and stable manifolds are powerful tools in the study of the dynamical properties of systems, first appearing in the study of hyperbolic ones. Pesin expanded the study of these manifolds for NUH diffeomorphisms by developing a solid theory for stable and unstable manifolds which correspond to its non-vanishing Lyapunov exponents [10]. Thus, translating the linear properties given by Oseledets Theorem into a non-linear theory of stable and unstable manifolds. Finally, Liu and Qian [11] and Shu Zhu [12] expanded those results for non invertible maps by studying the inverse limit space.

Throughout this section, M is a compact N-dimensional Riemannian manifold, with the distance $d(x,y)$ between $x, y \in M$ induced by the Riemannian metric. We assume that $f : M \to M$ is a C^2 endomorphism, non-uniformly hyperbolic with respect to an invariant smooth (normalized) volume μ, and we maintain the notations $\boldsymbol{L}_f, \hat{f}, \hat{\mu}, \cdots$ as in Section 1.1.

Theorem 1.4. *(Oseledets Theorem for $\hat{f}$)*

There exists a Borel set $\hat{\mathcal{R}} \subset \boldsymbol{L}_f$ such that $\hat{f}(\hat{\mathcal{R}}) = \hat{\mathcal{R}}$ and $\hat{\mu}(\hat{\mathcal{R}}) = 1$. Furthermore, for every $\hat{x} = (x_n)_{n\geq 0} \in \hat{\mathcal{R}}$, there is an integer $s(\hat{x})$ and a splitting of the tangent space

$$T_{x_n}M = E_n^{(1)}(\hat{x}) \oplus \cdots \oplus E_n^{(s(\hat{x}))}(\hat{x}),$$

numbers $+\infty > \lambda^{(1)}(\hat{x}) > \cdots > \lambda^{(s(\hat{x}))}(\hat{x})$ and $m^{(i)}(\hat{x})$, for $i = 1, \cdots, s(\hat{x}) > -\infty$, such that:

1. *$s(\hat{x})$, $\lambda^{(i)}(\hat{x})$ and $m^{(i)}(\hat{x})$ are $\hat{f}$-invariant;*
2. *$dim E_n^{(i)}(\hat{x}) = m^{(i)}(\hat{x})$ for all $n \geq 0$, $i = 1, \cdots, s(\hat{x})$.*
3. *The splitting is invariant under Df, i.e. $D_{x_n} f E_n^{(i)}(\hat{x}) = E_n^{(i)}(\hat{f}\hat{x})$. In particular, for $n \geq 1$, $D_{x_n} f E_n^{(i)}(\hat{x}) = E_{n-1}^{(i)}(\hat{x})$;*
4. *For $n \geq 0$, $m \in \mathbb{Z}$, if we set*

$$D_n^m(\hat{x}) = \begin{cases} D_{x_n} f^m : T_{x_n}M \to T_{f^m(x_n)}M, & \text{if } m > 0, \\ Id : T_{x_n}M \to T_{x_n}M, & \text{if } m = 0, \\ (D_{n-m}^{-m}(\hat{x}))^{-1} = (D_{x_{n-m}} f^{-m})^{-1} : T_{x_n}M \to T_{x_{n-m}}M, & \text{if } m < 0. \end{cases}$$

Then, $\lim\limits_{m\to\pm\infty} \frac{1}{m} \log \left| D_n^m(\hat{x}) v \right| = \pm\lambda^{(i)}(\hat{x})$, for every $0 \neq v \in E_n^{(i)}(\hat{x})$ and $i = 1, \cdots, s(\hat{x})$.

5. *For every $i \neq j \in \{1, \cdots, s(\hat{x})\}$ we have*

$$\lim_{m\to\pm\infty} \frac{1}{m} \log \angle(D_0^m E_0^{(i)}, D_0^m E_0^{(j)}) = 0,$$

where $\angle$ denotes the angle between two associated subspaces and $E_0^{(i)} = E_0^{(i)}(\hat{x})$ and $D_0^m = D_0^m(\hat{x})$, (we shall omit the dependence of the cocycle and of the associated subspace on $\hat{x}$ when it's implied).

6. *$x_0 = \pi_{ext}(\hat{x}) \in \mathcal{R}$ and the measurable functions s, $m^{(i)}$, $\lambda^{(i)}$ presented here equal their equivalents in Theorem 1.1 for x_0 (thus for every x_n).*

It is important to remark that the inverse limit space of f, $\mathbf{L}_f$ allows us to study the homeomorphism $\hat{f}$ instead of the non-invertible map f. As the previous Theorem shows, for invertible maps the filtration obtained in Osedelets Theorem becomes a decomposition of T_xM into invariant subspaces for which the vectors have the same average growth given by the Lyapunov exponents.

Definition 1.3. *For a non-uniformly hyperbolic f, we define the stable and unstable subspaces of $T_{x_0}M$ as:*

$$E^s(\hat{x}) = \bigoplus_{i\,:\lambda^{(i)}(\hat{x})<0} E_0^{(i)}(\hat{x}), \quad E^u(\hat{x}) = \bigoplus_{i\,:\lambda^{(i)}(\hat{x})>0} E_0^{(i)}(\hat{x}).$$

From the previous Theorem, the bundles E^s and E^u over $\hat{\mathcal{R}}$ are measurable.

The stable bundle E^s projects to a measurable bundle $(\pi_{ext})_* E^s$ as it depends only on the forward orbit of a point, that is, $E^s(\hat{x}^1) = E^s(\hat{x}^2)$ for every pair $\hat{x}^1, \hat{x}^2 \in \pi_{ext}^{-1}(x)$, and for every $x \in \mathcal{R}$. However, the unstable E^u may differ at different points of the same fiber, hence it does not project to M under π_{ext}. Even so, these bundles allows us to study dynamical properties of f.

Definition 1.4. *(Unstable and stable manifolds for endomorphisms) Let $\hat{x} \in \hat{\mathcal{R}}$, $x = \pi_{ext}(\hat{x})$.*

1. *A $C^{1,1}$ embedded submanifold $W^u_{loc}(\hat{x}) \subset M$ is a local unstable manifold at $\hat{x}$ if there exist constants $\lambda > 0$, $0 < \epsilon < \frac{\lambda}{200}$, $0 < C_1 \leq 1 < C_2$ so that $y_0 \in W^u_{loc}(\hat{x})$ if, and only if, there exists a unique $\hat{y} \in \pi_{ext}^{-1}(y_0)$ satisfying, for every $n \geq 0$:*

$$d(x_n, y_n) \leq C_1 e^{-n\epsilon}, \quad d(x_n, y_n) \leq C_2 e^{-n\lambda}.$$

   The lift of $W^u_{loc}(\hat{x})$ to $\mathbf{L}_f$ is denoted by $\hat{W}^u_{loc}(\hat{x})$.

2. *The unstable manifold of f at $\hat{x}$ is*

$$W^u(\hat{x}) = \left\{ y_0 = \pi_{ext}(\hat{y}) \ : \ \limsup_{n\to\infty} \frac{1}{n} \log d(x_n, y_n) < 0 \right\}.$$

   The lift of $W^u(\hat{x})$ to $\mathbf{L}_f$ by π_{ext} is denoted by $\hat{W}^u(\hat{x})$.

3. *$C^{1,1}$ embedded submanifold $W^s_{loc}(x) \subset M$ is a local stable manifold at x if there exist constants $\lambda > 0$, $0 < \epsilon < \frac{\lambda}{200}$, $0 < C_1 \leq 1 < C_2$ so that $y \in W^s_{loc}(x)$ if and only if for every $n \geq 0$*

$$d(f^n(x), f^n(y)) \leq C_1 e^{-n\epsilon}, \quad d(f^n(x), f^n(y)) \leq C_2 e^{-n\lambda}.$$

   The lift of $W^s_{loc}(x)$ to $\mathbf{L}_f$ by π_{ext} is denoted by $\hat{W}^s_{loc}(\hat{x})$.

4. *The stable manifold of f at x is*

$$W^s(x) = \pi_{ext}\left(\bigcup_{n=0}^{\infty} \hat{f}^{-n} \hat{W}^s_{loc}(\hat{f}^n \hat{x}) \right)$$

   And its lift for $\mathbf{L}_f$ is denoted by $\hat{W}^s(\hat{x})$.

The following Theorem guarantees the existence of stable and unstable manifolds for non-uniformly hyperbolic maps and gives continuity of these manifolds on positive measured sets whose union has full measure.

Theorem 1.5. *[12] There exists a countable family $\{\hat{\Lambda}_k\}_{k\geq 0}$ of subsets of $\hat{\mathcal{R}}$ in $\boldsymbol{L_f}$ satisfying that $\hat{\mu}(\bigcup_k \hat{\Lambda}_k) = 1$, and such that:*

1. *For fixed k, there exists a continuous family $\{W^u_{loc}(\hat{x}) \subset M \ : \ \hat{x} \in \hat{\Lambda}_k\}$ of local unstable manifolds so that for every $\hat{x} \in \hat{\Lambda}_k$ it holds:*
   - *(a) $T_{x_0}W^u_{loc}(\hat{x}) = E^u(\hat{x})$, in particular $E^u(\hat{x})$ depends continuously on $\hat{x} \in \hat{\Lambda}_k$.*
   - *(b) There exists a sequence of $C^{1,1}$ submanifolds $\{W^u(\hat{x}, n)\}_{n\geq 0}$ in M with:*
     - $W^u(\hat{x}, 0) = W^u_{loc}(\hat{x})$,
     - $fW^u(\hat{x}, n) \supset W^u(\hat{x}, n-1)$, *for every* $n \geq 1$,
     - $W^u(\hat{x}) = \bigcup\limits_{n\geq 0} f^n W^u(\hat{x}, n)$.
2. *If $\Lambda_k = \pi_{ext}(\hat{\Lambda}_k)$, then for every k, there exists a continuous family of local stable manifolds $\{W^s_{loc}(x) \ : \ x \in \Lambda_k\}$ so that for every $x \in \Lambda_k$ it holds:*
   - $T_x W^s_{loc}(x) = E^s(x)$, *in particular* $E^s(x)$ *depends continuously on* $x \in \Lambda_k$,
   - $fW^s_{loc}(x) \subset W^s_{loc}(fx)$.

Definition 1.5. *The sets Λ_k, $\hat{\Lambda}_k$ above are called the Pesin blocks of $f, \hat{f}$, respectively.*

Before proving this theorem, we give some extra results which will be of importance for the proof of Theorems C and D.

Corollary 1.1. *For a fixed k the unstable local manifolds $W^u_{loc}(\hat{x})$ have diameter uniformly bounded from below for points $\hat{x} \in \hat{\Lambda}_k$. Also for a fixed k, the stable local manifolds have diameter uniformly bounded from below for points $x \in \Lambda_k$.*

As $fW^s_{loc}(x) \subset W^s_{loc}(fx)$, we have that $W^s(x)$ is an immersed submanifold of M, and these manifolds W^s and $\hat{W}^s$ form an invariant lamination of M and $\boldsymbol{L_f}$ respectively. Note that the global unstable manifolds W^u do not form a lamination of M as they may intersect, however in $\boldsymbol{L_f}$ we have no such intersections and $\hat{W}^u$ do form an invariant lamination. The following results give us absolute continuity of these laminations, in the following sense.

In view of Corollary 1.1 we may enumerate the Pesin blocks $\hat{\Lambda}_k$, $\Lambda_k = \pi_{ext}\hat{\Lambda}_k$, in a way that for every $\hat{x} \in \hat{\Lambda}_k$, $W^u_{loc}(\hat{x})$ and $W^s_{loc}(x_0)$ have diameters greater then k^{-1}. For $\hat{x} \in \hat{\Lambda}_k$, and a fixed number $0 < q << k^{-1}$, we set:

$$\mathcal{Q}^u(\hat{x}, q) := \bigcup_{\hat{y}\in\hat{\Lambda}_k\cap\pi_{ext}^{-1}B(x_0,q)} W^u_{loc}(\hat{y}), \quad \mathcal{Q}^s(x_0, q) := \bigcup_{y_0\in\Lambda_k\cap B(x_0,q)} W^s_{loc}(y_0),$$

and the families:

$$\mathcal{W}^u(\hat{x}, q) := \{W^u_{loc}(\hat{y}) \ : \ \hat{y} \in \hat{\Lambda}_k \cap \pi_{ext}^{-1}B(x_0, q)\},$$

$$\mathcal{W}^s(x_0, q) := \{W^s_{loc}(y_0) \ : \ y_0 \in \Lambda_k \cap B(x_0, q)\}.$$

Given W^1, W^2 two local open submanifolds of M which are uniformly transversal to $\mathcal{W}^s(x_0, q)$, we define the holonomy map between W^1, W^2 as:

$$\begin{aligned} \mathcal{H} \ : \ \mathcal{Q}^s(x_0, q) \cap W^1 &\to \mathcal{Q}^s(x_0, q) \cap W^2 \\ W^1 \cap W^s_{loc}(y_0) &\mapsto W^2 \cap W^s_{loc}(y_0), \end{aligned} \tag{1.14}$$

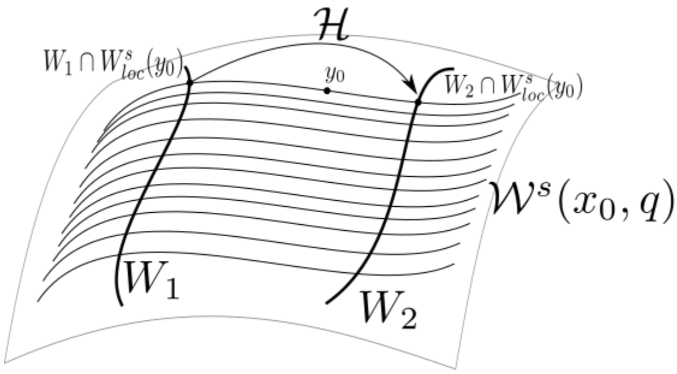


Figure 1.1: Stable Holonomy map

Remark 1.5. *Due to Theorem 1.5, $\mathcal{W}^s(x_0, q)$ is a continuous family of submanifolds, the holonomy map between any two transversal as defined above is a homeomorphism onto its image.*

Definition 1.6. *A measurable partition ξ of M is said to be subordinate to the stable lamination if for μ-a.e. $x \in M$ one has $\xi(x) \subset W^s(x)$ and $\xi(x)$ contains an open neighbourhood of x inside the arc connected component of $W^s(x)$ which contains x (this being taken with respect to the submanifold topology).*

Analogously, a measurable partition η of $\mathbf{L}_f$ is said to be subordinate to to the unstable lamination $\hat{W}^u$ if for $\hat{\mu}$-a.e. $\hat{x} \in \mathbf{L}_f$ the element $\eta(\hat{x})$ of η that contains $\hat{x}$ has the following properties:

1. *$\pi_{ext}|_{\eta(\hat{x})} \ : \ \eta(\hat{x}) \to \pi_{ext}\eta(\hat{x})$ is bijective.*
2. *If $k(\hat{x}) = dim(E^u_0)$, there exists a $k(\hat{x})$-dimensional $C^{1,1}$ embedded submanifold $W_{\hat{x}}$ on M, with $W_{\hat{x}} \subset W^u(\hat{x})$, satisfying:*
    - *$\pi_{ext}(\eta(\hat{x})) \subset W_{\hat{x}}$,*
    - *$\pi_{ext}(\eta(\hat{x}))$ contains an open neighborhood of x_0 in $W_{\hat{x}}$ (with respect to the submanifold topology of $W_{\hat{x}}$.*

We have the following results on the absolute continuity of these laminations.

Theorem 1.6. *[13] The stable lamination W^s is absolutely continuous in the following sense. Given any Pesin block Λ_k taken as above, $x \in \Lambda_k$, and any $q << k^{-1}$ the holonomy map $\mathcal{H}^s$ between any two submanifolds W^1, W^2 transversal to the family $\mathcal{W}^s(x_0, q)$ is absolutely continuous with respect to the Lebesgue measure of the two transversals.*

As a consequence, given a partition of M subordinate to the stable lamination, the disintegration of the Lebesgue measure on M along the elements of the partition are absolutely continuous with respect to the Lebesgue measure on stable manifolds.

Theorem 1.7. *[14] The unstable lamination $\hat{W}^u$ is absolutely continuous in the following sense. Given any partition of $\boldsymbol{L}_f$ subordinate to the Pesin unstable lamination $\hat{W}^u$, the disintegrations of $\hat{\mu}$ along the elements of the partition are absolutely continuous with respect to the Lebesgue measure on the unstable manifolds.*

We now devote the rest of this section to prove Theorem 1.5. For $r > 0$ let

$$\hat{\Lambda}_r := \{\hat{x} \in \hat{\mathcal{R}} \ : \ \lambda^{(i)}(\hat{x}) \notin [0, r] \ \forall\, 1 \le i \le s(\hat{x})\},$$

clearly $\hat{f}(\hat{\Lambda}_r) = \hat{\Lambda}_r$. For $\hat{x} \in \hat{\Lambda}_r$, $n \geq 0$ and $m \in \mathbb{Z}$, we use the following notations:

$$E^u_n(\hat{x}) = \bigoplus_{\lambda^{(i)}(\hat{x}) > r} E^{(i)}_n(\hat{x}), \qquad E^s_n(\hat{x}) = \bigoplus_{\lambda^{(i)}(\hat{x}) < 0} E^{(i)}(\hat{x}).$$

E^u_n is called the unstable bundle, and E^s_n the stable bundle at $\hat{x} \in \boldsymbol{L}_f$.

Remark 1.6. *We are assuming that f is non-uniformly hyperbolic, thus $E^u_n(\hat{x}) \oplus E^s_n(\hat{x}) = T_{x_n}M$ for every $\hat{x} \in \hat{\Lambda}_r$ and every $n \geq 0$. We make this assumption for simplicity purposes only, the general case is completely similar and can be found in [12] or [11].*

From now on, we fix $0 < \epsilon \leq \min\left\{1, \frac{r}{200}\right\}$ and $K \in \{1, \cdots, N\}$, ($N = dim(M)$), and assume that the set

$$\hat{\Lambda}_{r,K} := \{\hat{x} \in \hat{\Lambda}_r \ : \ dim(E^u_0(\hat{x})) = K\} \neq \emptyset.$$

Remember the definition of $D^m_n(\hat{x})$ in Theorem 1.4.

Lemma 1.2. *There exists a measurable function $\ell : \hat{\Lambda}_{r,K} \times \mathbb{N} \to [1, +\infty)$ such that for any $\hat{x} \in \hat{\Lambda}_{r,K}$ and $n, m \geq 0$ we have*

1. $|D^{-m}_n(\hat{x})\xi| \leq \ell(\hat{x}, n)e^{(-r+\epsilon)m}|\xi|$, $\xi \in E^u_n(\hat{x})$;
2. $|D^{-m}_n(\hat{x})\eta| \geq \ell(\hat{x}, n)^{-1}e^{-\epsilon m}|\eta|$, $\eta \in E^s_n(\hat{x})$;
3. $\angle(E^s_{n+m}(\hat{x}), E^u_{n+m}(\hat{x})) \geq \ell(\hat{x}, n)^{-1}e^{-\epsilon m}$;
4. $\ell(\hat{x}, n + m) \leq \ell(\hat{x}, n)e^{\epsilon m}$.

Proof. We introduce the notation $\rho^{(1)}(\hat{x}) \geq \cdots \geq \rho^{(N)}(\hat{x})$, $N = dim(M)$ to denote

$$\lambda^{(1)}(\hat{x}), \cdots, \lambda^{(1)}(\hat{x}), \cdots, \lambda^{(i)}(\hat{x}), \cdots, \lambda^{(i)}(\hat{x}), \cdots, \lambda^{s(\hat{x})}, \cdots, \lambda^{(s(\hat{x}))},$$

with $\lambda^{(i)}(\hat{x})$ being repeated $m^{(i)}(\hat{x})$ times. Due to Theorem 1.4 we may choose $\{\xi_1, \cdots, \xi_N\}$ a basis of $T_{x_0}M$ such that $\{\xi_j\}^K_{j=1} \subset E^u_0(\hat{x})$, $\{\xi_j\}^N_{j=K+1} \subset E^s_0(\hat{x})$ and for each ξ_i:

$$\lim_{m \to \infty} |D^{-m}_0(\hat{x})\xi_i| = -\rho^{(i)}(\hat{x}). \tag{1.15}$$

Also, for any two non-empty disjoint subsets $P, Q \subset \{1, \cdots, N\}$, we must have

$$\lim_{m\to\infty} \frac{1}{m} \log \angle(D_0^{-m}(\hat{x})E_P, D_0^{-m}E_Q) = 0, \tag{1.16}$$

where E_P, E_Q are the subspaces of $T_{x_0}M$ spanned by $\{\xi_i\}_{i\in P}$ and $\{\xi_i\}_{i\in Q}$ respectively. From this, it follows:

$$A(\tilde{x}, n) := \inf_{P,Q} \inf_{m\geq 0} \angle(D_0^{-(n+m)}(\hat{x})E_P, D_0^{-(n+m)}(\hat{x})E_Q)e^{\frac{\epsilon}{2N}m} > 0.$$

We may define $\ell_1(\hat{x}, n) = \inf_{m\geq 0} \angle(E^u_{n+m}(\hat{x}), E^s_{n+m}(\hat{x}))e^{\frac{\epsilon}{2N}m}$, which is an everywhere positive measurable function on $\hat{\Lambda}_{r,K} \times \mathbb{N}$.

From a simple geometrical argument we can check that if E is a vector space with an inner product $\langle , \rangle$, and $\|\cdot\|$ is the norm induced by $\langle , \rangle$. If $\eta, \xi \in E$ satisfy $\angle(\eta, \xi) \geq q$ for some $q > 0$, then $\|\eta\| + \|\xi\| \leq 4q^{-1}\|\eta + \xi\|$. Hence, for $m \geq 0$ and any $\xi = \sum_i \alpha_i D_0^{-m}(\hat{x})\xi_i \in T_{x_m}M$, we have:

$$|\xi| \leq \sum_i |\alpha_i||D_0^{-m}(\hat{x})\xi_i| \leq B(\hat{x}, m)|\xi|, \tag{1.17}$$

where $B(\hat{x}, m) = (4A(\hat{x}, m)^{-1})^N$ satisfying

$$B(\hat{x}, n + m) \leq B(\hat{x}, n)e^{\frac{\epsilon}{2}m}, \ \forall m \geq 0. \tag{1.18}$$

From (1.15), there exists $C(\hat{x}, n) > 0$ such that for every ξ_i and $m \geq 0$:

$$C(\hat{x}, n)^{-1}e^{\left(-\rho^{(i)}(\hat{x})-\frac{\epsilon}{4}\right)m} \leq |D_0^{-(n+m)}(\hat{x})\xi_i| \leq C(\hat{x}, n)e^{\left(-\rho^{(i)}(\hat{x})+\frac{\epsilon}{4}\right)m}, \tag{1.19}$$

therefore, for every $m, l \geq 0$, we have the following upper bounds:

$$|D_0^{-(n+l+m)}(\hat{x})\xi_i| \leq C(\hat{x}, n)^2|D_0^{-(n+l)}(\hat{x})\xi_i|e^{\left(-r+\frac{\epsilon}{4}\right)m+\frac{\epsilon}{2}l}, \tag{1.20}$$

for every $\xi_i \in E^u_0(\hat{x})$, and

$$|D_0^{-(n+l)}(\hat{x})\xi_i| \leq C(\hat{x}, n)^2|D_0^{-(n+l+m)}(\hat{x})\xi_i|e^{\frac{\epsilon}{4}m+\frac{\epsilon}{2}l}, \tag{1.21}$$

for every $\xi_i \in E^s_0(\hat{x})$. Putting together (1.17), (1.18) and (1.20) for $\xi = \sum_{i=0}^{K} \alpha_i D_0^{-n}(\hat{x})\xi_i \in E^u_n(\hat{x})$ and $m, l \geq 0$ we get:

$$\begin{aligned}
|D_n^{-(l+m)}(\hat{x})\xi| &= \sum_{i=0}^{K} |\alpha_i||D_o^{-(n+l+m)}(\hat{x})\xi_i| \\
&\leq \left(\sum_{i=0}^{K} |\alpha_i||D_0^{-(n+l)}(\hat{x})\xi_i|\right) C(\hat{x}, n)^2 e^{\left(-r+\frac{\epsilon}{4}\right)m+\frac{\epsilon}{2}l} \\
&\leq |D_n^{-l}(\hat{x})\xi|B(\hat{x}, n)C(\hat{x}, n)^2 e^{\left(-r+\frac{\epsilon}{4}\right)m+\epsilon l}.
\end{aligned}$$

Hence, the function

$$\ell_2(\hat{x}, n) = \sup\left\{ \frac{|D_n^{-(l+m)}(\hat{x})\xi|}{|D_n^{-l}(\hat{x})\xi|} e^{\left(r-\frac{\epsilon}{4}\right)m-\epsilon l} \ : \ l, m \geq 0, \ 0 \neq \xi \in E^u_n(\hat{x}) \right\},$$

is finite at each point of $\hat{\Lambda}_{r,K}$. Similarly but using (1.21), we get that the function

$$\ell_3(\hat{x}, n) = \sup \left\{ \frac{|D_n^{-l}(\hat{x})\xi|}{|D_n^{-(l+m)}(\hat{x})\xi|} e^{-\frac{3\epsilon}{4}m-\epsilon l} \ : \ l, m \geq 0, \ 0 \neq \xi \in E_n^s(\hat{x}) \right\}$$

is also finite at each point of $\hat{\Lambda}_{r,K}$. Finally we define:

$$\ell(\hat{x}, n) = \max \{\ell_i(\hat{x}, n) \ : \ i = 1, 2, 3.\},$$

from the way the function is defined, it is direct to verify that ℓ satisfies the properties required.

□

For $L \geq 1$, due to the previous Lemma, we may define the measurable sets

$$\hat{\Lambda}_{r,K}^{\epsilon,L} := \{\hat{x} \in \hat{\Lambda}_{r,K} \ : \ \ell(\hat{x}, 0) \leq L\}.$$

(it depends on ϵ as for different values of ϵ we get different definitions of the function ℓ).

Proposition 1.8. *The vector bundles E_0^u and E_0^s depend continuously on $\hat{x} \in \hat{\Lambda}_{r,K}^{\epsilon,L}$.*

Proof. Let $\{\hat{x^n}\}$ be a sequence in $\hat{\Lambda}_{r,K}^{\epsilon,L}$ such that $\hat{x^n}$ converges to $\hat{x} \in \hat{\Lambda}_{r,K}^{\epsilon,L}$. We may assume that $E_0^u(\hat{x^n})$ converges to a subspace $E \subset T_{x_0}M$ by taking a subsequence if necessary. For $\xi \in E$, taking $\xi^n \in E_0^u(\hat{x^n})$ converging to ξ, we have for $m \geq 0$:

$$\begin{aligned} |D_0^{-m}\xi| = \lim_n |D_0^{-m}(\hat{x^n})\xi_n| &\leq \limsup_n \ell(\hat{x^n}, m)|\xi_n| e^{(-r+\epsilon)m} \\ &\leq L e^{\epsilon m} |\xi| e^{(-b+\epsilon)m}, \end{aligned}$$

thus $\xi \in E_0^u(\hat{x})$, implying $E = E_0^u(\hat{x})$. The argument for E_0^s is the same.

□

Lemma 1.2 allows us to define, for every $\hat{x} \in \hat{\Lambda}_{r,K}^{\epsilon,L}$ and $n \geq 0$ an inner product $\langle , \rangle_{\hat{x},n}$ on $T_{x_n}M$ for which we see contraction (resp. expansion) on the stable (resp.unstable) bundle in the first iterate, as follows:

$$\langle \zeta, \zeta' \rangle_{\hat{x},n} := \sum_{l=0}^{+\infty} e^{2(r-\epsilon)l} \langle D_n^{-l}(\hat{x})\zeta, D_n^{-l}(\hat{x})\zeta' \rangle, \ \ \zeta, \zeta' \in E_n^u(\hat{x}), \tag{1.22}$$

$$\langle \eta, \eta' \rangle_{\hat{x},n} := \sum_{l=0}^{n} e^{-4\epsilon l} \langle D_n^{l}(\hat{x})\eta, D_n^{l}(\hat{x})\eta' \rangle, \ \ \eta, \eta' \in E_n^s(\hat{x}), \tag{1.23}$$

with $E_n^u(\hat{x})$ and $E_n^s(\hat{x})$ as orthogonal subspaces with respect to $\langle , \rangle_{\hat{x},n}$. Furthermore, defining the norm $\| \cdot \|_{\hat{x},n}$ on $T_{x_n}M$ as:

$$\|\xi\|_{\hat{x},n} = (\langle \xi, \xi \rangle_{\hat{x},n})^{\frac{1}{2}}, \qquad \text{for } \xi \in E_n^s(\hat{x}) \text{ or } \xi \in E_n^u(\hat{x}), \tag{1.24}$$

$$\|\xi\|_{\hat{x},n} = \max\{\|\zeta\|_{\hat{n},n}, \ \|\eta\|_{\hat{x},n}\}, \qquad \text{for } \xi = \zeta + \eta \in E_n^u(\hat{x}) \oplus E_n^s(\hat{x}), \tag{1.25}$$

we get, from Prop. 1.8, that for a fixed $n \geq 0$ the norm $\| \cdot \|_{\hat{x},n}$ depends continuously on $\hat{x} \in \hat{\Lambda}_{r,K}^{\epsilon,L}$.

Definition 1.7. *The sequence of norms* $\{\|\cdot\|_{\hat{x},n}\}_{n=0}^{+\infty}$ *is called the (backwards) Lyapunov metric at* $\hat{x}$.

The following Lemma gives us the required property of the Lyapunov metric.

Lemma 1.3. *For* $\hat{x} \in \hat{\Lambda}_{r,K}^{\epsilon,L}$, *the Lyapunov metric satisfies for each* $n \geq 0$*:*

1. *For* $\xi \in E_n^u(\hat{x})$, $\|D_n^{-1}(\hat{x})\xi\|_{\hat{x},n+1} \leq e^{-r+2\epsilon}\|\xi\|_{\hat{x},n}$;
2. *For* $\xi \in E^s(\hat{x})$, $\|D_n^{-1}(\hat{x})\xi\|_{\hat{x},n+1} \geq e^{-2\epsilon}\|\xi\|_{\hat{x},n}$;
3. *For every* $\xi \in T_{x_n}M$, $\frac{1}{2}|\xi| \leq \|\xi\|_{\hat{x},n} \leq Ae^{2\epsilon n}|\xi|$, *where* $A = 4L^2(1-e^{-2\epsilon})^{-\frac{1}{2}}$.

Proof. For item 1 it is direct from the definition that we have, for $\xi \in E_n^u(\hat{x})$:

$$\|D_n^{-1}(\hat{x})\xi\|_{\hat{x},n+1}^2 = \sum_{l=0}^{+\infty} e^{2(r-\epsilon)l}|D_n^{-(l+1)}(\hat{x})\xi|^2 \leq e^{-2(r-2\epsilon)}\|\xi\|_{\hat{x},n},$$

and for item 2 the argument is the same. Now for item 3, we start noticing that for $\xi \in E_n^s(\hat{x})$ or $\xi \in E_n^u(\hat{x})$, it is direct from the definition that $|\xi| \leq \|\xi\|_{\hat{x},n}$. Then, for $\xi = \xi_u + \xi_s \in E_n^u(\hat{x}) \oplus E_n^s(\hat{x})$, we get:

$$|\xi| \leq |\xi_u| + |\xi_s| \leq \|\xi_u\|_{\hat{x},n} + \|\xi_s\|_{\hat{x},n} \leq 2\|\xi\|_{\hat{x},n}.$$

For the right part of the inequality, by items 1,2 and 4 of Lemma 1.2 we have

$$\|\xi_u\|_{\hat{x},n} \leq L(1-e^{-2\epsilon})^{-\frac{1}{2}}e^{\epsilon n}|\xi_u|, \text{ and}$$

$$\|\xi_s\|_{\hat{x},n} \leq L(1-e^{-2\epsilon})^{-\frac{1}{2}}e^{\epsilon n}|\xi_s|.$$

Moreover, from item 3 of the same Lemma, we have $\angle(E_n(\hat{x}), E_u(\hat{x})) \geq (Le^{\epsilon n})^{-1}$ which implies $|\xi_u| + |\xi_s| \leq 4Le^{\epsilon n}|\xi|$. From these, we have:

$$\|\xi\|_{\hat{x},n} \leq \|\xi_u\|_{\hat{x},n} + \|\xi_s\|_{\hat{x},n} \leq L(1-e^{-2\epsilon})^{-\frac{1}{2}}e^{\epsilon n}(|\xi_u| + |\xi_s|) \leq Ae^{2\epsilon n}|\xi|.$$

□

Now, to build the unstable and stable manifolds of f on M, we need to relate the results obtained for $D_n^m(\hat{x})$ to the correspondent f-orbit on M. For that we state the next Lemma, the proof can be found in [12].

Lemma 1.4. *There exists a universal number* $\rho > 0$ *such that the map*

$$\begin{aligned} H_x \ : \ T_xM(\rho) &\to T_{fx}M \\ \xi &\mapsto exp_{fx}^{-1} \circ f \circ exp_x(\xi) \end{aligned} \tag{1.26}$$

is well defined, where $T_xM(\rho)$ *is the ball of radius* ρ *centered at* $0 \in T_xM$.

Furthermore, there exists a Borel set $\hat{\mathcal{R}}_0 \subset \hat{\mathcal{R}}$ *with* $\hat{\mu}(\hat{\mathcal{R}}_0) = 1$ *and* $\hat{f}(\hat{\mathcal{R}}_0) = \hat{\mathcal{R}}_0$, *and a measurable function* $R : \hat{\mathcal{R}}_0 \to (0, +\infty)$ *with the following properties:*

1. *For any* $\hat{x} = (x_n)_n \in \hat{\mathcal{R}}_0$ *the map* $G_{\hat{x},0} := H_{x_1}^{-1} : T_{x_0}M(R(\hat{x})^{-1}) \to T_{x_1}M$ *is well defined.*

2. *The map* $DG_{\hat{x},0} : T_{x_0}M \ni \xi \mapsto D_\xi G_{\hat{x},0}$ *is Lipschitz with Lipschitz constant satisfying* $Lip(DG_{\hat{x},0}) \le R(\hat{x})$.

3. *The map* $g_{\hat{x},0} := exp_{x_1} \circ G_{\hat{x},0} \circ exp_{x_0}^{-1}$ *is well defined and* $f \circ g_{\hat{x},0} = Id|_{B(x_0,\ R(\hat{x})^{-1})}$.

4. *For every* $n \ge 0$, $\hat{x} \in \hat{\mathcal{R}}_0$, $R(\hat{f}^{-n}\hat{x}) \le R(\hat{x})e^{\epsilon n}$.

Consequently, for $\hat{x} \in \hat{\mathcal{R}}_0$ *and* $n \ge 0$*, the maps:*

$$G_{\hat{x},n} := H_{x_{n+1}}^{-1} \ : T_{x_n}M(R(\hat{x})^{-1}e^{-\epsilon n}) \to T_{x_{n+1}}M, \ \ and$$

$$g_{\hat{x},n} := exp_{x_{n+1}} \circ G_{\hat{x},n} \circ exp_{x_n}^{-1} \ : B(x_n, R(\hat{x})^{-1}e^{-\epsilon n}) \to M$$

are well defined and $f \circ g_{\hat{x},n} = Id|_{B(x_n,\ R(\hat{x})^{-1}e^{-\epsilon n})}$.

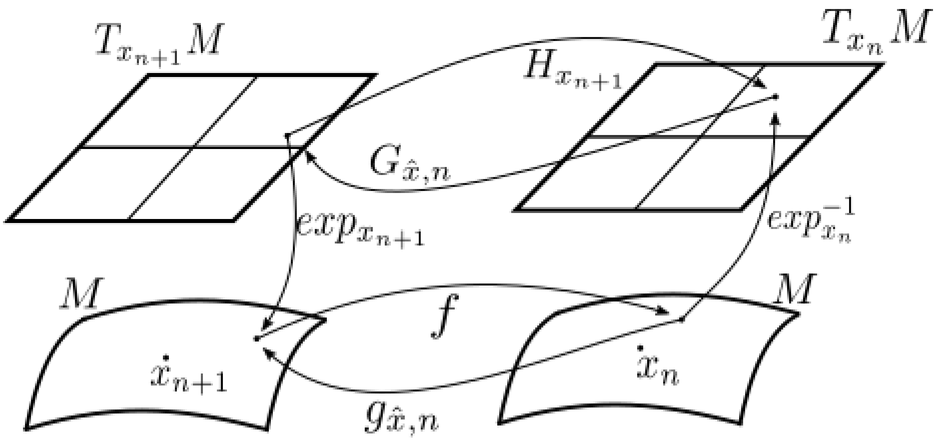


Figure 1.2: $G_{\hat{x},n} : T_{x_n}M \to T_{x_{n+1}}M$

For $R > 2\rho^{-1}$, define $\hat{\Lambda}_{r,K}^{\epsilon,L,R} := \{\hat{x} \in \hat{\Lambda}_{r,K}^{\epsilon,L} \cap \hat{\mathcal{R}}_0 \ : \ R(\hat{x}) < R\}$

Remark 1.7. *Taking increasing sequences of positive numbers* $R_k \underset{k}{\to} +\infty$, $L_k \underset{k}{\to} +\infty$*, we have* $\hat{\Lambda}_{r,K}^{\epsilon,L_k,R_k} \subseteq \hat{\Lambda}_{r,K}^{\epsilon,L_{k+1},R_{k+1}}$ *for every* $k \ge 0$ *and*

$$\hat{\Lambda}_{r,K} \cap \hat{\mathcal{R}}_0 = \bigcup_{k=1}^{\infty} \Lambda_{r,K}^{\epsilon,L_k,R_k}.$$

Then, by taking a decreasing sequence of positive numbers $1 > r_n \underset{n}{\to} 0$ *and* $\epsilon_n = \frac{r_k}{100}$*, we have* $\bigcup_{k=1}^{+\infty} \hat{\Lambda}_{r_n,K}^{\epsilon_n,L_k,R_k} \subseteq \bigcup_{k=1}^{+\infty} \hat{\Lambda}_{r_{n+1},K}^{\epsilon_{n+1},L_k,R_k}$ *for every* $n \ge 0$ *and:*

$$\hat{\mathcal{R}} \cap \hat{\mathcal{R}}_0 = \bigcup_{K=0}^{N} \bigcup_{n=1}^{+\infty} \bigcup_{k=1}^{+\infty} \hat{\Lambda}_{r_n,K}^{\epsilon_n,L_k,R_k}.$$

Clearly we have $\hat{\mu}(\hat{\mathcal{R}} \cap \hat{\mathcal{R}}_0) = 1$*. The sets* $\hat{\Lambda}_{r_n,K}^{\epsilon_n,L_k,R_k}$ *are the ones that we will prove to satisfy Theorem 1.5, thus choosing these sequences and enumerating then accordingly we have the required Pesin blocks.*

Now, we move on to the construction of the local unstable manifold at a point $\hat{x} \in \boldsymbol{L}_f$. For that, we use a graph transform method analogous to the hyperbolic case. From now on, we fix a set $\hat{\Lambda}^{\epsilon,L,R}_{r,K}$ and the numbers involved in its definition, we denote it by $\hat{\Lambda}' := \hat{\Lambda}^{\epsilon,L,R}_{r,K}$ for simplicity. By the previous Remark, it is enough to prove Theorem 1.5 in $\hat{\Lambda}'$. We begin by the following Lemma on the properties of the maps $G_{\hat{x},n}$.

Lemma 1.5. *For $\epsilon_0 = e^{-r+4\epsilon} - e^{-r+2\epsilon}$, $c_0 = 4ARe^{2\epsilon}$ (A as in Lemma 1.3) and $C = c_0^{-1}\epsilon_0$, we have for every $\hat{x} \in \hat{\Lambda}'$ and $l \geq 0$:*

1. *$G_{\hat{x},l} : \{\xi \in T_{x_l}M \ : \ \|\xi\|_{\hat{x},l} \leq Ce^{-3\epsilon l}\} \to T_{x_{l+1}}M$ is well defined;*
2. *If $Lip_{\|\cdot\|}(\cdot)$ is defined with respect to the norms $\|\cdot\|_{\hat{x},l}$ and $\|\cdot\|_{\hat{x},l+1}$, then:*

$$Lip_{\|\cdot\|}(DG_{\hat{x},l}) \leq c_0 e^{3\epsilon l}.$$

Hence, $Lip_{\|\cdot\|}(G_{\hat{x},l} - D_0 G_{\hat{x},l}) \leq \epsilon_0$.

Proof. By item 4 of Lemma 1.4 and the definition of $\hat{\Lambda}'$ we have that $G_{\hat{x},l}$ is defined on $T_{x_l}M(R^{-1}e^{-\epsilon l})$. By 3 of Lemma 1.3 we have:

$$\{\xi \in T_{x_l}M \ : \ \|\xi\|_{\hat{x},l} \leq Ce^{-3\epsilon l}\} \subseteq T_{x_l}M(R^{-1}e^{-\epsilon l}).$$

Again, due to item 3 of Lemma 1.3 and items 2, 4 of Lemma 1.4, we have for ξ, η in the domain of $G_{\hat{x},l}$ and $\zeta \in T_{x_l}M$:

$$\begin{aligned}\frac{\|(D_\xi G_{\hat{x},l} - D_\eta G_{\hat{x},l})\zeta\|_{\hat{x},l+1}}{\|\zeta\|_{\hat{x},l}} &\leq 2Ae^{2\epsilon l}\frac{|(D_\xi G_{\hat{x},l} - D_\eta G_{\hat{x},l})\zeta|}{|\zeta|} \\ &\leq 2Ae^{2\epsilon l}R(\hat{f}^{-l}(\hat{x})) \leq 2ARe^{3\epsilon l} \leq c_0 e^{3\epsilon l}.\end{aligned}$$

□

We may now define our graph transform, we fix a point $\hat{x} \in \Lambda'$, a number $n \geq 0$ and let $t = e^{-r+6\epsilon}$. We define:

$$\begin{aligned}\Gamma_u &= \left\{\sigma = (\sigma_l)_{l=1}^{+\infty} \ : \ \sigma_l \in E^u_{n+l}(\hat{x}) \ \forall\, l \geq 1, \text{ and } \|\sigma\| := \sup_{l\geq 1}\|t^{-l}\sigma_l\|_{\hat{x},n+l} < +\infty\right\}, \\ \Gamma_s &= \left\{\tau = (\tau_l)_{l=0}^{+\infty} \ : \ \tau_l \in E^s_{n+l}(\hat{x}) \ \forall\, l \geq 0, \text{ and } \|\tau\| := \sup_{l\geq 0}\|t^{-l}\tau_l\|_{\hat{x},n+l} < +\infty\right\}.\end{aligned} \tag{1.27}$$

We can easily check that $(\Gamma_u, \|\cdot\|)$ and $(\Gamma_s, \|\cdot\|)$ are Banach spaces. We define:

$$\begin{aligned}X &= \{\sigma \in \Gamma_u \ : \ \|\sigma\| \leq Ce^{-3\epsilon n}\}, \\ Y &= \{\tau \in \Gamma_s \ : \ \|\tau\| \leq Ce^{-3\epsilon n}\}, \\ Z &= \{\xi \in E^u_n(\hat{x}) \ : \ \|\xi\|_{\hat{x},n} \leq Ce^{-3\epsilon n}\},\end{aligned}$$

which are closed subsets of Γ_u, Γ_s and $E^u_n(\hat{x})$, respectively. We equip the products $Z\times X\times Y$ and $X \times Y$ with the maximum norm, and define the transformation:

$$\begin{aligned}\Theta : Z \times X \times Y &\to X \times Y \\ (\xi, \sigma, \tau) &\mapsto (\sigma', \tau'),\end{aligned} \tag{1.28}$$

with:

$$\sigma_1' = \pi_u G_{\hat{x},n}(\xi, \tau_0), \qquad \tau_0' = D^1_{n+1}(\tau_1 + D_n^{-1}\tau_0 - \pi_s G_{\hat{x},n}(\xi, \tau_0)),$$
$$\sigma_l' = \pi_u G_{\hat{x},n+l-1}(\sigma_{l-1}, \tau_{l-1}),\ l \geq 2, \quad \tau_l' = D^1_{n+l+1}(\tau_{l+1} + D_{n+l}^{-1}\tau_l - \pi_s G_{\hat{x},n+l}(\sigma_l, \tau_l)),\ l \geq 1,$$

where $\pi_u : T_{x_k}M \to E^u_k(\hat{x})$ and $\pi_s : T_{x_k}M \to E^s_k(\hat{x})$ are the projections to each coordinate of $(\xi_u, \xi_s) \in T_{x_k}M = E^u_k(\hat{x}) \oplus E^s_k(\hat{x})$, and $D_i^j = D_i^j(\hat{x})$ for the initially fixed point $\hat{x}$. In Figure 1.3, we illustrate the construction of σ_1' and τ_0', the rest goes analogously:

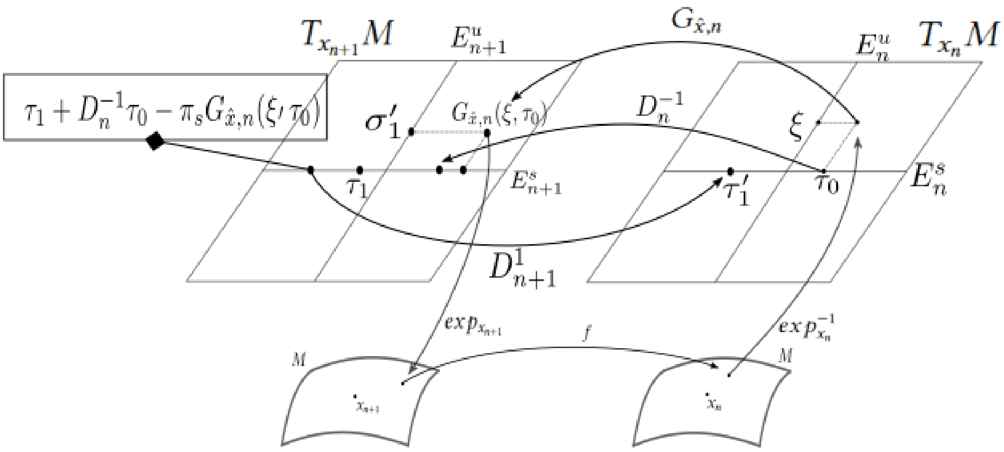


Figure 1.3: Construction of $\sigma_1' \in E^u_{n+1}$ and $\tau_0' \in E^s_n$.

Remark 1.8. *The transform Θ is quite similar as the more known graph transform from the proof of the stable manifold theorem for hyperbolic diffeomorphisms. In that scenario, the analogous to this map would be to define $\tau_l' = \pi_s G_{\hat{x},n+l-1}(\sigma_{l-1}, \tau_{l-1})$, it would work just fine as well here but to get the graph of the unstable manifold on x_0, it would be necessary that we made the entire construction in a full orbit of x_0 (not only in a backwards orbit). To avoid this extra work, we define it in the presented way.*

Lemma 1.6. *The map Θ defined above is a well defined Lipschitz map with $Lip(\Theta) \leq e^{-2\epsilon}$ with respect to the maximum norm on $Z \times X \times Y$ and $X \times Y$. Moreover, Θ is C^1 and $D\Theta$ is Lispchitz.*

Proof. We denote $\|\cdot\|_{\hat{x},n} = \|\cdot\|_n$ and $G_{\hat{x},n} = G_n$ for simplicity as $\hat{x}$ is fixed. We start by noticing that for each $l \geq 0$, we have for $(\xi^1, \sigma^1, \tau^1), (\xi^2, \sigma^2, \tau^2) \in Z \times X \times Y$, $t = e^{-r+6\epsilon}$ as in (1.27) and $\epsilon_0 = e^{-r+4\epsilon} - e^{-r+2\epsilon}$ as in Lemma 1.5:

$$\begin{aligned}
\|t^{-1}((\sigma_1^1)' - (\sigma_1^2)')\|_{n+1} &= t^{-1}\|\pi_u G_n(\xi^1, \tau_0^1) - \pi_u G_n(\xi^2, \tau_0^2)\|_{n+1} \\
&\leq t^{-1}\|(\pi_u G_n - \pi_u D_0 G_n)(\xi^1, \tau_0^1) - (\pi_u G_n - \pi_u D_0 G_n)(\xi^2, \tau_0^2)\|_{n+1} \\
&\quad + t^{-1}\|\pi_u D_0 G_n((\xi^1, \tau_0^1) - (\xi^2, \tau_0^2))\|_{n+1} \\
&\leq t^{-1} Lip_{\|\cdot\|}(G_n - D_0 G_n)\|(\xi^1, \tau_0^1) - (\xi^2, \tau_0^2)\|_n \\
&\quad + t^{-1} e^{-r+2\epsilon}\|(\xi^1, \tau_0^1) - (\xi^2, \tau_0^2)\|_n \\
&\leq t^{-1}(\epsilon_0 + e^{-r+2\epsilon})\|(\xi^1, \tau_0^1) - (\xi^2, \tau_0^2)\|_n \leq e^{-2\epsilon}\|(\xi^1, \tau_0^1) - (\xi^2, \tau_0^2)\|_n ,
\end{aligned}$$

where the second inequality is due to item 2 of Lemma 1.5 and item 1 of Lemma 1.3, noticing that $\pi_u D_0 G_n = D_n^{-1}|_{E^u_n}$. Analogously, for every $l \geq 2$, we have:

$$\|t^{-l+1}((\sigma_l^1)' - (\sigma_l^2)')\|_{n+l+1} \leq e^{-2\epsilon}\|t^{-l}((\sigma_{l-1}^1, \tau_{l-1}^2) - (\sigma_{l-1}^2, \tau_{l-1}^2))\|_{n+l},$$

hence $\|(\sigma^1)' - (\sigma^2)'\| \leq e^{-2\epsilon}\|(\xi^1, \sigma^1, \tau^1) - (\xi^2, \sigma^2, \tau^2)\|$.

Now, for the stable part we initially have from item 2 of Lemma 1.3 :

$$\begin{aligned} t^{-l}\|((\tau^1)' - (\tau^2)')\|_{n+l} &= t^{-l}\|D_{n+l+1}(\tau^1_{l+1} - \tau^2_{l+1} + D^{-1}_{n+l}(\tau^1_l - \tau^2_l) \\ &\quad - (\pi_s G_{n+l}(\sigma^1_l, \tau^1_l) - \pi_s G_{n+l}(\sigma^2_l, \tau^2_l)))\|_{n+l} \\ &\leq e^{-2\epsilon} t^{-l}\|\tau^1_{l+1} - \tau^2_{l+1} \\ &\quad + D^{-1}_{n+l}(\tau^1_l - \tau^2_l) - (\pi_s G_{n+l}(\sigma^1_l, \tau^1_l) - \pi_s G_{n+l}(\sigma^2_l, \tau^2_l))\|_{n+l+1} \\ &\leq e^{-2\epsilon} t\|\tau^1 - \tau^2\| \\ &\quad + e^{-2\epsilon}\|D^{-1}_{n+l}(\tau^1_l - \tau^2_l) - (\pi_s G_{n+l}(\sigma^1_l, \tau^1_l) - \pi_s G_{n+l}(\sigma^2_l, \tau^2_l))\|_{n+l+1}. \end{aligned}$$

Since the elements τ^1_l and τ^2_l are in the stable subspace E^s_{n+l}, we have $D^{-1}_{n+l}(\tau^1_l - \tau^2_l) = \pi_s D^{-1}_{n+l}(\tau^1_l - \tau^2_l) = \pi_s D_0 G_{n+l}((\sigma^1_l, \tau^1_l) - (\sigma^2_l, \tau^2_l))$. Thus, the last term of the previous inequality becomes:

$$\|\pi_s(D_0 G_{n+l} - G_{n+l})(\sigma^1_l, \tau^1_l) - \pi_s(D_0 G_{n+l} - G_{n+l})(\sigma^2_l, \tau^2_l)\|_{n+l+1} \leq \epsilon_0\|(\sigma^1_l, \tau^1_l) - (\sigma^2_l, \tau^2_l)\|_{n+l},$$

where the upper bound comes from item 2 of Lemma 1.5. We conclude:

$$\|((\tau^1)' - (\tau^2)')\| \leq e^{-2\epsilon}(t + \epsilon_0)\|(\xi^1, \sigma^1, \tau^1) - (\xi^2, \sigma^2, \tau^2)\| \leq e^{-2\epsilon}\|(\xi^1, \sigma^1, \tau^1) - (\xi^2, \sigma^2, \tau^2)\|,$$

which proves that Θ is a Lipschitz map with $Lip(\Theta) \leq e^{-2\epsilon}$.

Now, to see that Θ is C^1 with $D\Theta$ Lipschitz, we note that taking

$$\begin{aligned} D(\xi^0, \sigma^0, \tau^0) \, : \, E^u_n \times \Gamma_u \times \Gamma_s &\to \Gamma_u \times \Gamma_s \\ (\xi, \sigma, \tau) &\mapsto (\sigma'', \tau''), \end{aligned}$$

where:

$$\begin{aligned} \sigma''_1 &= \pi_u D_{(\xi^0, \tau^0_0)} G_n(\xi, \tau_0), \\ \sigma''_l &= \pi_u D_{(\sigma^0_{l-1}, \tau^0_{l-1})} G_{n+l-1}(\sigma_{l-1}, \tau_{l-1}), \ l \geq 2, \\ \tau''_0 &= D^1_{n+1}(\tau_1 + D^{-1}_n \tau_0 - \pi_s D_{(\xi^0, \tau^0_0)} G_n(\xi, \tau_0)), \\ \tau''_l &= D^1_{n+l+1}(\tau_{l+1} + D^{-1}_{n+l}\tau_l - \pi_s D_{(\sigma^0_l, \tau^0_l)} G_{n+l}(\sigma_l, \tau_l)), \ l \geq 1, \end{aligned}$$

it follows from Lemma 1.5 that $D(\xi^0, \sigma^0, \tau^0)$ is a well defined bounded linear operator and is the derivative of Θ at the point $(\xi^0, \sigma^0, \tau^0) \in int(Z \times X \times Y)$, it also follows that $(\xi, \sigma, \tau) \mapsto D(\xi, \sigma, \tau)$ is a Lipschitz map. □

From the previous Lemma, we make use of Lemmas 2.1 and 2.2 of the Chapter 3 in [11] to obtain a Lipschitz map $\Delta \, : \, Z \to X \times Y$ with $Lip(\Delta) \leq e^{-2\epsilon}$ such that for any $\xi \in Z$, $\Delta(\xi)$ is the unique fixed points of the map $\Theta_\xi \, : \, X \times Y \to X \times Y$ with $\Theta_\xi(\sigma, \tau) = \Theta(\xi, \sigma, \tau)$. Moreover, we have that Δ is C^1 on $int(Z)$ and $D\Delta$ is Lipschitz with $Lip(D\Delta) \leq De^{3\epsilon n}$ where $D = (1 - e^{-2\epsilon})^{-3}(1 + e^{-2\epsilon})^2 c_0 e^r$. Denote $\Delta(\xi)$ by $(\sigma(\xi), \tau(\xi)) \in X \times Y$, and define:

$$\begin{aligned} h_{\hat{x},n} \, : \, int(Z) \subset E^u_n(\hat{x}) &\to E^s_n(\hat{x}) \\ \xi &\mapsto \tau(\xi)_0. \end{aligned}$$

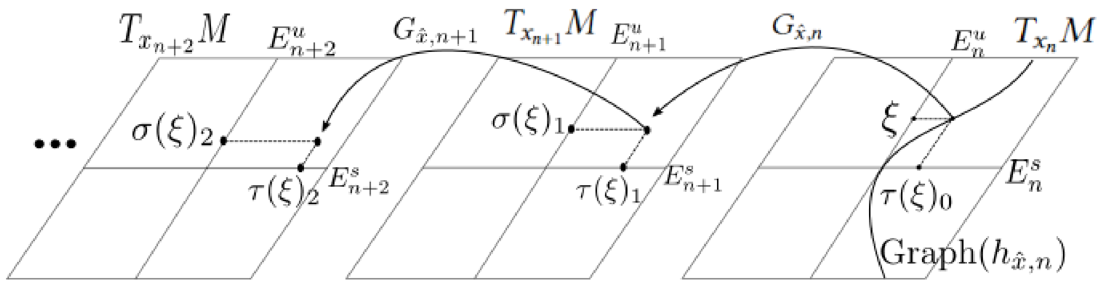


Figure 1.4: For every $\xi \in Z$, $\Delta(\xi) = (\sigma(\xi), \tau(\xi))$ is the unique point in $X \times Y$ such that $G_{\hat{x},n+l-1}(\sigma(\xi)_{l-1}, \tau(\xi)_{l-1}) = (\sigma(\xi)_l, \tau(\xi)_l)$ for every $l \geq 1$.

It follows directly from the properties of Δ that $h_{\hat{x},n}$ is a $C^{1,1}$ map satisfying that $h_{\hat{x},n}(0) = 0$, and from the proof of Lemma 1.6 that $D_0 h_{\hat{x},n} = 0$, and:

$$Lip_{\|\cdot\|}(h_{\hat{x},n}) \leq e^{-2\epsilon}, \quad Lip_{\|\cdot\|}(Dh_{\hat{x},n}) \leq De^{-3\epsilon n}. \tag{1.29}$$

For $l \geq 1$, if we take

$$\begin{cases} G_n^0 = Id, \\ G_n^l = G_{\hat{x},n+l-1} \circ \cdots \circ G_{\hat{x},n}, \end{cases}$$

wherever it makes sense for those maps to be defined, we have from (1.28) and the definition of Δ that $G_n^l(\xi, \tau(\xi)_0) = (\sigma(\xi)_l, \tau(\xi)_l)$, for every $l \geq 1$. Hence, from item 4 of Lemma 1.4, item 1 of Lemma 1.5 and the domain of definition of Θ, we get that $h_{\hat{x}_n}(\xi) = \tau(\xi)_0$ is the unique point in $E_n^s(\hat{x})$ such that:

$$\|G_n^l(\hat{x})(\epsilon, \tau(\xi)_0)\|_{\hat{x},n+l} < Ce^{-3\epsilon n} e^{(-r+6\epsilon)l}, \quad \text{for every } l \geq 0,$$

which implies that the graph of $h_{\hat{x},n}$ has the form:

$$Graph(h_{\hat{x},n}) = \{(\xi, \eta) \in E_n^u(\hat{x}) \oplus E_n^s(\hat{x}) \ : \ \|G_n^l(\xi, \eta)\|_{\hat{x},n+l} < Ce^{-3\epsilon n} e^{(-r+6\epsilon)l}, \ \forall\, l \geq 0\}. \tag{1.30}$$

We remark that we can make the entire process to arbitrary $n \geq 0$ to obtain (1.30). As $G_{n+1}^l \circ G_n^1 = G_n^{l+1}$, it follows:

$$G_n^1 Graph(h_{\hat{x},n}) \subset Graph(h_{\hat{x},n+1}), \ \forall n \geq 0, \tag{1.31}$$

which allows us to construct the required local unstable manifolds on M.

Proposition 1.9. *For $\hat{x} \in \hat{\Lambda}'$ and $n \geq 0$, defining:*

$$W^u(\hat{x}, n) := exp_{x_n} Graph(h_{\hat{x},n}),$$

then $W^u(\hat{x}, , n)_{n \geq 0}$ is a sequence of $C^{1,1}$ submanifolds in M satisfying:

1. *$T_{x_n} W^u(\hat{x}, n) = E_n^u(\hat{x})$ for every $n \geq 0$;*

2. *$fW^u(\hat{x}, n) \supset W^u(\hat{x}, n-1)$ for every $n \geq 1$.*

3. *Defining $W_{loc}^u(\hat{x}) = W^u(\hat{x}, 0)$, we have that for every $y_0 \in W_{loc}^u(\hat{x})$, there exists a unique point $\hat{y} \in \pi_{ext}^{-1}(y_0)$ such that for every $n \geq 0$:*

$$d(x_n, y_n) \leq 2Ce^{(-r+6\epsilon)n}.$$

4. *$W^u_{loc}(\hat{x})$ depends continuously on $\hat{x} \in \hat{\Lambda}'$.*

5. *The unstable manifold $W^u(\hat{x})$ of f at $\hat{x}$ defined in Definition 1.4 satisfy:*

$$W^u(\hat{x}) = \bigcup_{n\geq 0} f^n W^u(\hat{x}, n).$$

Proof. Since $D_0 h_{\hat{x},n} = 0$, we have item 1. From (1.31), we have $g_{\hat{x},n} W^u(\hat{x}, n) \subset W^u(\hat{x}, n+1)$, thus $f \circ g_{\hat{x},n} = Id$ gives us item 2.

For $y_0 \in W^u_{loc}(\hat{x})$, we define inductively $y_n = g_{\hat{x},n-1}(y_{n-1})$ for $n \geq 1$. By definition, $exp^{-1}_{x_0} y_0 \in Graph(h_{\hat{x},0})$, from (1.31) we have that y_n is well defined, and since $f \circ g_{\hat{x},n} = Id$, it satisfies $\hat{y} = (y_n)_{n\geq 0} \in \pi^{-1}_{ext}(y_0)$. Moreover, we have $exp^{-1}_{x_n}(y_n) = G^n_0 \exp^{-1}_{x_0}(y_0)$ for every $n \geq 0$. Thus, from (1.30):

$$\|exp^{-1}_{x_n}(y_n)\|_{\hat{x},n} < Ce^{(-r+6\epsilon)n}, \quad \forall n \geq 0. \tag{1.32}$$

By item 3 of Lemma 1.3, we get:

$$|exp^{-1}_{x_n}(y_n)| < 2r_o e^{(-r+6\epsilon)n}, \quad \forall n \geq 0, \tag{1.33}$$

which gives us item 3, the uniqueness of $\hat{y}$ is given from the uniqueness of Δ.

For item 4, we already know from Prop.1.8 that $T_{x_0} W^u_{loc}(\hat{x}) = E^u_0(\hat{x})$ depends continuously on $\hat{x} \in \hat{\Lambda}'$. By the compactness of $\boldsymbol{L_f}$, we can find a finite partition $\{\hat{\Lambda}'_l\}_{l=1}^{m_0}$ with $\bigcup_l \hat{\Lambda}'_l = \hat{\Lambda}'$ such that for each $\hat{\Lambda}'_l$ we can find a basis of $T_{\pi_{ext}(\hat{x})} M$ that depends continuously on $\hat{x} \in \hat{\Lambda}'_l$.

Furthermore, since the inner product $\langle , \rangle_{\hat{x},n}$ varies continuously on $\hat{x} \in \hat{\Lambda}'$, we can fix an orthonormal frame $\{\xi_i\}_{i=0}^N$, with respect to $\langle , \rangle_{\hat{x},0}$, where $N = dimM$, that depends continuously on $\hat{x} \in \hat{\Lambda}'_l$ and such that $\{\xi_i(\hat{x})\}_{i=0}^k$ is a basis of $E^u_0(\hat{x})$ and $\{\xi_j\}_{j=k+1}^N$ is a basis of $E^s_0(\hat{x})$. Thus, defining the linear isometry $T(\hat{x}) : \mathbb{R}^k \oplus \mathbb{R}^{N-k} \to E^u_0(\hat{x}) \oplus E^s_0(\hat{x})$ by $T(e_i) = \xi_i(\hat{x})$, we get that T varies continuously on $\hat{x} \in \hat{\Lambda}'_l$.

Defining $\theta_l : \hat{\Lambda}'_l \to Emb^1(D_k, M)$ by $\theta_l(\hat{x}) = exp_{x_0} \circ (Id, h_{\hat{x},0}) \circ T(\hat{x})|_{D_k}$, where $D_k = \{v \in \mathbb{R}^k : |v| < C\}$, we get that for every $\hat{x} \in \hat{\Lambda}'_l$, $\theta_l(\hat{x})$ is a $C^{1,1}$ embedding with $\theta_l(\hat{x}) = W^u_{loc}(\hat{x})$. Finally, we note that (1.29) allows us to invoke Arzela-Ascoli theorem and get that if $\{\hat{x^m}\}_{m\in\mathbb{N}} \subset \hat{\Lambda}'_l$ converges to $\hat{x} \in \hat{\Lambda}'_l$ then $h_{\hat{x^m},0}$ and $Dh_{\hat{x^m},0}$ converges uniformly to $h_{\hat{x},0}$ and $Dh_{\hat{x},0}$ respectively. This, along with continuity of T, gives us that θ_l is continuous, completing the proof of item 4.

Finally, for item 5, we want to prove:

$$W^u(\hat{x}) := \left\{ y_0 = \pi_{ext}(\hat{y}) \ : \ \limsup_{n\to\infty} \frac{1}{n} \log d(x_n, y_n) < 0 \right\} = \bigcup_{n\geq 0} f^n W^u(\hat{x}, n). \tag{1.34}$$

We begin by the left inclusion. Let $y_0 \in \bigcup_{n\geq 0} f^n W^u(\hat{x}, n)$, $n_0 \geq 0$ be such that $y_0 \in f^{n_0} W^u(\hat{x}, n_0)$, and $y_{n_0} \in W^u(\hat{x}, n_0)$ such that $f^{n_0}(y_{n_0}) = y_0$. Since $W^u(\hat{x}, n_0) = W^u(\hat{f}\hat{x}, 0) = W^u_{loc}(\hat{f}\hat{x})$, item 3 gives us a unique point $\hat{y}' = (y_{n_0}, y_{n_0+1}, \cdots) \in \pi^{-1}_{ext}(y_{n_0})$ satisfying

$$d(x_n, y_n) \leq 2Ce^{(-r+6\epsilon)n},$$

for every $n \geq 0$. Hence, $\hat{y} = \hat{f}^{n_0} \hat{y}'$ is in $\pi^{-1}_{ext}(y_0)$ and satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \log d(x_n, y_n) \leq -r + 6\epsilon < 0 \tag{1.35}$$

Now, given $\pi_{ext}(\hat{y}) = y_0 \in W^u(\hat{x})$, we first note that since $\hat{x} \in \hat{\Lambda}' = \hat{\Lambda}^{\epsilon,L,R}_{r,K}$, we must have

$$\left\{ y_0 = \pi_{ext}(\hat{y}) \ : \ \limsup_{n\to\infty} \frac{1}{n} \log d(x_n, y_n) \leq -r + 6\epsilon \right\} \subseteq \bigcup f^n W^u(\hat{x}, n).$$

Indeed, if $y_0 = \pi_{ext}(y)$ satisfies $\limsup\limits_{n\to\infty} \frac{1}{n} \log d(x_n, y_n) \leq -r + 6\epsilon$, there is an n_0 satisfying $e^{(-r+6\epsilon)n_0} \leq 2C$ such that, for every $l \geq 0$:

$$d(x_{n_0+l}, y_{n_0+l}) \leq e^{(-r+6\epsilon)(n_0+l)} \leq 2Ce^{(-r+6\epsilon)l},$$

which by (1.30), using the same argument used in item 3, implies $y_{n_0} \in W^u(\hat{x}, n_0)$, as we wanted. Taking $r_j = \frac{r}{j}$, and $\epsilon_j = \frac{\epsilon}{j}$, by Remark 1.7 we have that $\hat{x} \in \Lambda^{\epsilon_j,L,R}_{r_j,K}$, and the exact same construction made in this section gives us, for every $j \geq 0$, and for the same $\hat{x}$:

$$W^u_j(\hat{x}) := \left\{ y_0 = \pi_{ext}(\hat{y}) \ : \ \limsup_{n\to\infty} \frac{1}{n} \log d(x_n, y_n) \leq -r_j + 6\epsilon_j \right\} \subseteq \bigcup f^n W^u(\hat{x}, n).$$

Since $W^u(\hat{x}) = \bigcup_j W^u_j(\hat{x})$, this concludes the proof of item 5.

□

The previous Proposition concludes the proof of Theorem 1.5 to the unstable case (item 1). We leave out the proof of the stable case as it is completely analogous and can be found in [11].

2 Non Uniformly Hyperbolic Endomorphisms

In order to obtain that Theorems A and B imply existence of robust non uniformly hyperbolic endomorphisms on the homotopy classes, we need need first to prove that, indeed, every map in $\mathcal{U}$ is non-uniformly hyperbolic. So, let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a endomorphism of the 2-torus preserving the Haar measure μ, and non-uniformly hyperbolic. We denote by $\lambda^+(x) > 0 > \lambda^-(x)$ the Lyapunov exponents on regular points $x \in \mathcal{R}$ and $E^+(\hat{x}), E^-(\hat{x})$ the correspondent Lyapunov subspaces of regular $\hat{x} \in \pi_{ext}^{-1}(x) \cap \hat{\mathcal{R}}$. Of course, from Theorem 1.4, it holds for every $\hat{x} \in \hat{\mathcal{R}}$, $x = \pi_{ext}(x)$:

$$\lambda^+_{\hat{f}}(\hat{x}) = -\lambda^-_{\hat{f}^{-1}}(\hat{x}) = \lambda^+_f(x), \quad \lambda^-_{\hat{f}}(\hat{x}) = -\lambda^+_{\hat{f}^{-1}}(\hat{x}) = \lambda^-_f(x). \tag{2.1}$$

Recall the definition of $I(x, v, f^n)$ from (2), by the relation (1.10), we obtain that for every $(x, v) \in T^1\mathbb{T}^2$:

$$I(x, v, f^n) = \sum_{y \in f^{-n}(x)} \frac{\log \|(D_y f)^{-1} \cdot v\|}{det(D_y f^n)} = \int_{\pi_{ext}^{-1}(x)} \log \|D_{\hat{x}} \hat{f}^{-n} \cdot v\| \, d\hat{\mu}_x(\hat{x}). \tag{2.2}$$

Proposition 2.1. *For any $n \in \mathbb{N}$, it holds:*

$$I(x, v; f^n) = \sum_{i=0}^{n-1} \sum_{y \in f^{-i}(x)} \frac{I(y, F_y^{-i} v; f)}{\det(D_y f^i)}, \tag{2.3}$$

where $F_y^{-i} v = \frac{(D_y f^i)^{-1} v}{\|(D_y f^i)^{-i} v\|}$. Hence, $\frac{1}{n} I(x, v; f^n)$ is a convex combination of others $I(y, w; f)$.

Proof. We compute:

$$\begin{aligned}
I(x, v; f^{n+1}) &= \sum_{y \in f^{-(n+1)}(x)} \frac{\log \|(D_y f^{n+1})^{-1} v\|}{\det(D_y f^{n+1})} = \sum_{y \in f^{-n}(x)} \sum_{z \in f^{-1}(y)} \frac{\log \|(D_z f)^{-1} (D_y f^n)^{-1} v\|}{\det(D_z f) \det(D_y f^n)} \\
&= \sum_{y \in f^{-n}(x)} \sum_{z \in f^{-1}(y)} \frac{\log \|(D_z f)^{-1} F_y^{-n} v\| + \log \|(D_y f^n)^{-1} v\|}{\det(D_z f) \det(D_y f^n)} \\
&= \sum_{y \in f^{-n}(x)} \frac{1}{\det(D_y f^n)} \sum_{z \in f^{-1}(y)} \frac{\log \|(D_z f)^{-1} F_y^{-n} v\|}{\det(D_z f)} \\
&\quad + \sum_{y \in f^{-n}(x)} \frac{\log \|(D_y f^n)^{-1} v\|}{\det(D_y f^n)} \sum_{z \in f^{-1}(y)} \frac{1}{\det(D_z f)}
\end{aligned}$$

From the definition of $I(x, v; f^n)$, in (2), we obtain:

$$I(x, v; f^{n+1}) = \sum_{y \in f^{-n}(x)} \frac{1}{\det(D_y f^n)} I(y, F_y^{-n} v; f) + \sum_{y \in f^{-n}(x)} \frac{\log \|(D_y f^n)^{-1} v\|}{\det(D_y f^n)}$$
$$= \sum_{y \in f^{-n}(x)} \frac{1}{\det(D_y f^n)} I(y, F_y^{-n} v; f) \ + I(x, v; f^n)$$

Thus, by an inductive process, if (2.3) holds for n, we obtain:

$$I(x, v; f^{n+1}) = \sum_{y \in f^{-n}(x)} \frac{1}{\det(D_y f^n)} I(y, F_y^{-n} v; f) + \sum_{i=0}^{n-1} \sum_{y \in f^{-i}(x)} \frac{I(y, F_y^{-i} v; f)}{\det(D_y f^i)}$$
$$= \sum_{i=0}^{n} \sum_{y \in f^{-i}(x)} \frac{I(y, F_y^{-i} v; f)}{\det(D_y f^i)},$$

which concludes the proof. Of course $\sum_{y \in f^{-i}(x)} \frac{1}{\det(D_y f^i)}$ equals 1 for every i, thus the sum of the coefficients of $I(x, v; f^n)$ is equal to n. Hence, indeed, $\frac{1}{n} I(x, v; f^n)$ is a convex combination of others $I(y, w; f)$.

□

Corollary 2.1. *For every $(x, v) \in T^1\mathbb{T}^2$ and any integers $n, m \geq 0$, $\frac{1}{n} I(x, v; f^{nm})$ is a convex combination of other $I(y, w, f^m)$*

Remember that $C_{\mathcal{X}}(f) = \sup_{m \in \mathbb{N}} \frac{1}{m} \inf_{(x,v) \in T^1\mathbb{T}^2} I(x, v, f^m)$. A immediate consequence of the definition and Corollary 2.1 is that the definition of $C_{\mathcal{X}}(f)$ is independent of the election of norm on $T\mathbb{T}^2$, and it can be rewritten as:

$$C_{\mathcal{X}}(f) = \lim_{m \to \infty} \frac{1}{m} \inf_{(x,v) \in T^1\mathbb{T}^2} I(x, v, f^m), \tag{2.4}$$

moreover, for the same reasons, we have:

$$C_{det}(f) = \sup_{n \in \mathbb{N}} \frac{1}{n} \inf_{x \in \mathbb{T}^2} \log(det(D_x f^n)) = \lim_{n \to \infty} \frac{1}{n} \inf_{x \in \mathbb{T}^2} \log(det(D_x f^n)). \tag{2.5}$$

We give a slightly more complete result which will be also useful in the proof of Theorem E.

Theorem 2.1. *For μ almost every $x \in \mathbb{T}^2$ we have $\lambda_f^-(x) \leq -C_{\mathcal{X}}(f)$ and $\lambda_f^+(x) \geq C_{\mathcal{X}}(f) + C_{det}(f)$. In particular:*

1. *If $f \in \mathcal{U}$ ($C_{\mathcal{X}}(f) > 0$), then $\lambda_f^-(x) < 0 < \lambda_f^+(x)$ for almost every $x \in \mathbb{T}^2$, i.e. f is non-uniformly hyperbolic.*

2. *If $f \in \mathcal{U}_1$ ($C_{\mathcal{X}}(f) > -\frac{1}{2} C_{det}(f)$), then $\lambda_f^-(x) < \lambda_f^+(x)$ for almost every $x \in \mathbb{T}^2$.*

Proof. For μ-a.e. $x \in \mathbb{T}^2$, it holds:

$$\lambda_f^+(x) + \lambda_f^-(x) = \lim_{n \to \infty} \frac{1}{n} \log(\det(D_x f^n)) \geq C_{\det}(f).$$

Thus, if $\lambda_f^-(x) \le -C_{\mathcal{X}}(f)$ for μ-a.e. $x \in \mathbb{T}^2$, then $\lambda_f^+(x) \ge C_{\mathcal{X}}(f) + C_{\det}(f)$ also holds μ almost everywhere.

By defining $C_m = \frac{1}{m} \inf_{(x,v)\in T^1\mathbb{T}^2} I(x,v;f^m)$, from the definition $\sup_m C_m = C_{\mathcal{X}}(f)$. From Corollary 2.1, we have that for every $n, m \ge 1$ and $(x,v) \in T^1\mathbb{T}^2$:

$$\frac{I(x,v;f^{nm})}{nm} = \frac{1}{nm} \int_{\pi_{ext}^{-1}(x)} \log \|D_{\hat{x}} \hat{f}^{-nm} \cdot v\| \, d\hat{\mu}_x(\hat{x}) \ge C_m.$$

Hence, by (2.1) and Dominated Convergence Theorem, we have for μ-a.e. $x \in \mathbb{T}^2$ and any unit vector $v \in T_x\mathbb{T}^2$:

$$\int_{\pi_{ext}^{-1}(x)} \tilde{\lambda}_{\hat{f}^{-1}}(\hat{x}, v) \, d\hat{\mu}_x(\hat{x}) \ge C_m,$$

where $\tilde{\lambda}_{\hat{f}^{-1}}(\hat{x}, v) = \lim_{n\to\infty} \frac{\log \|D_{\hat{x}} \hat{f}^{-n} \cdot v\|}{n}$ is the Lyapunov exponent of $\hat{f}$ at $(\hat{x}, v)$ for $\hat{x} \in \hat{\mathcal{R}}$ where the limit exists.

We conclude:

$$-\lambda_f^-(x) = \int_{\pi_{ext}^{-1}(x)} \lambda_{\hat{f}^{-1}}^+(\hat{x}) \, d\hat{\mu}_x(\hat{x}) \ge \int_{\pi_{ext}^{-1}(x)} \tilde{\lambda}_{\hat{f}^{-1}}(\hat{x}, v) \, d\hat{\mu}_x(\hat{x}) \ge \sup_m C_m = C_{\mathcal{X}}(f), \tag{2.6}$$

which, by the initial observation, proves item 1, that is, proves that every $f \in \mathcal{U}$ is non-uniformly hyperbolic.

For item 2, if $C_{\mathcal{X}}(f) > -\frac{1}{2} C_{\det}(f)$, also by the initial observation and (2.6), we have for μ-a.e. $x \in \mathbb{T}^2$:

$$\lambda_f^+(x) - \lambda_f^-(x) \ge 2C_{\mathcal{X}}(f) + C_{\det}(f) > 0.$$

□

2.1 Shears

For fixed points $z_1, z_2, z_3, z_4 \in \mathbb{T}^1$, in this order, take the closed intervals $I_1 = [z_1, z_2]$, $I_3 = [z_3, z_4]$, and the open intervals $I_2 = (z_2, z_3)$ and $I_4 = (z_4, z_1)$.

Definition 2.1. *We define the horizontal and vertical critical regions in* $\mathbb{T}^2$ *as* $C_h = (I_1 \cup I_3) \times \mathbb{T}^1$, $C_v = \mathbb{T}^1 \times (I_1 \cup I_3)$ *and its complements* $\mathcal{G}_h = \mathbb{T}^2 \setminus C_h$, $\mathcal{G}_v = \mathbb{T}^2 \setminus C_v$ *are respectively the horizontal and vertical good region.*

We then divide the good regions into $\mathcal{G}_h^+ = I_2 \times \mathbb{T}^1$, $\mathcal{G}_h^- = I_4 \times \mathbb{T}^1$, $\mathcal{G}_v^+ = \mathbb{T}^1 \times I_2$ *and* $\mathcal{G}_v^- = \mathbb{T}^1 \times I_4$.

For fixed numbers $0 < a < b$, we take $s : \mathbb{T}^1 \to \mathbb{R}$ as an analytic map satisfying the following conditions:

1. If $z \in I_2$, then $a < s'(z) < b$;

2. If $z \in I_4$, then $-b < s'(z) < -a$;

3. If $z \in I_1 \cup I_3$, then $|s'(z)| < b$.

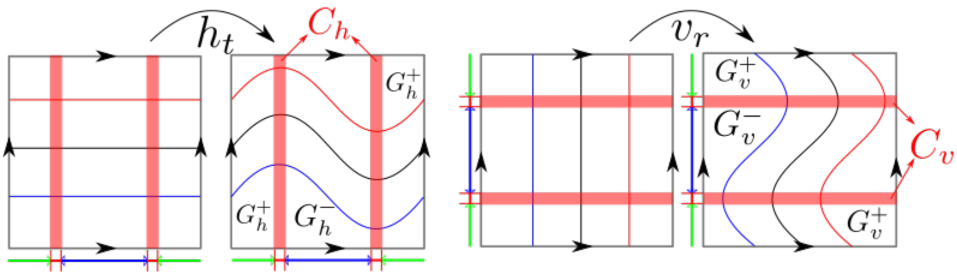


Figure 2.1: Actions of the shears h_t and v_r, for $v, r > 0$.

Consider the two families of conservative diffeomorphisms of the torus given by:

$$h_t(x_1, x_2) = (x_1, x_2 + ts(x_1)), \quad v_r(x_1, x_2) = (x_1 + rs(x_2), x_2), \quad t, r \in \mathbb{R}. \tag{2.7}$$

Note that:

$$D_{(x_1,x_2)}h_t = \begin{pmatrix} 1 & 0 \\ ts'(x_1) & 1 \end{pmatrix}, \quad D_{(x_1,x_2)}v_r = \begin{pmatrix} 1 & rs'(x_2) \\ 0 & 1 \end{pmatrix}.$$

In order to simplify the computations we will consider the maximum norm on $T\mathbb{T}^2$ as $\|(u_1, u_2)\| = \max\{|u_1|, |u_2|\}$, and all the computations from now on are performed using this norm. This way, we get, for every $x \in \mathbb{T}^2$:

$$\|D_x h_t\| < bt + 1, \quad \text{and} \quad \|D_x v_r\| < bt + 1.$$

Definition 2.2. *Given $\alpha > 0$, the corresponding horizontal cone is $\Delta_\alpha^h = \{(u_1, u_2) \in \mathbb{R}^2 : |u_2| \leq \alpha |u_1|\}$, while the corresponding vertical cone is its complement $\Delta_\alpha^v = \mathbb{R}^2 \setminus \Delta_\alpha^h$,*

Lemma 2.1. *For $\alpha > 1$, let Δ_α^h and Δ_α^v be the corresponding horizontal and vertical cones. Then, for every $t, r > \frac{2\alpha}{a}$, and, for every unit vector $u \in T_x\mathbb{T}^2$, the following holds:*

1. *If $u \in \Delta_\alpha^v$, and:*
   (a) *$x \in \mathcal{G}_v$, then*
      - $(D_x v_r)^{-1} u \in \Delta_\alpha^h$ $\;((D_x v_r)^{-1} \Delta_\alpha^v \subset \Delta_\alpha^h)$;
      - $\|(D_x v_r)^{-1} u\| > \frac{ar - \alpha}{\alpha} = r\frac{a - \frac{\alpha}{r}}{\alpha}$;

   (b) *$x \in C_v$, then $\|(D_x v_r)^{-1} u\| > \frac{1}{\alpha}$.*
2. *If $u = \pm(1, u_2) \in \Delta_\alpha^h$, then:*
   (a) *either for every $x \in \mathcal{G}_v^+$ (if $u_2 \leq 0$) or for every $x \in \mathcal{G}_v^-$ (if $u_2 \geq 0$) it holds:*
      - $(D_x v_r)^{-1} u \in \Delta_\alpha^h$;
      - $\|(D_x v_r)^{-1} u\| > 1$;

   (b) *for every other x, we have $\|(D_x v_r)^{-1} u\| > \frac{1}{br+1}$.*
3. *If $u \in \Delta_\alpha^h$, and:*
   (a) *$x \in \mathcal{G}_h$, then*

- $(D_x h_t)^{-1}u \in \Delta^v_\alpha$ $((D_x h_t)^{-1}\Delta^h_\alpha \subset \Delta^v_\alpha)$;
- $\|(D_x h_t)^{-1}u\| > \frac{at-\alpha}{\alpha} = t\frac{a-\frac{\alpha}{t}}{\alpha}$;

(b) $x \in C_h$, *then* $\|(D_x h_t)^{-1}u\| > \frac{1}{\alpha}$.

4. *If* $u = \pm(u_1, 1) \in \Delta^v_\alpha$, *then:*

   (a) either for every $x \in \mathcal{G}_h^+$ *(if* $u_1 \leq 0$*) or for every* $x \in \mathcal{G}_h^-$ *(if* $u_1 \geq 0$*) it holds:*
   - $(D_x h_t)^{-1}u \in \Delta^v_\alpha$;
   - $\|(D_x h_t)^{-1}u\| > 1$;

   (b) for every other x, *we have* $\|(D_x h_t)^{-1}u\| > \frac{1}{bt+1}$.

Proof. We prove items 1 and 2, the case for h_t is analogous. Let $x = (x_1, x_2) \in \mathcal{G}_v$, and $u^\pm = (1, \pm\alpha)$ then:

$$(D_x v_r)^{-1}u^\pm = \begin{pmatrix} 1 & -rs'(x_2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \pm\alpha \end{pmatrix} = \begin{pmatrix} 1 \mp rs'(x_2)\alpha \\ \pm\alpha \end{pmatrix},$$

also since $x \in \mathcal{G}_v$, $a < |s'(x_2)| < b$, we also have $\alpha > 1$ and $r > \frac{2\alpha}{a}$, hence:

$$|1 \mp rs'(x_2)\alpha| \geq r\alpha a - 1 > 2\alpha^2 - 1 > \alpha > 1,$$

which shows that $(D_x v_r)^{-1}\Delta^v_\alpha \subset \Delta^h_\alpha$. Also, $\|(D_x v_r)^{-1}u\| = |1 \mp rs'(x_2)\alpha| > ra\alpha - 1$. Now, noticing that the minimal expansion of vectors in Δ^v_α occurs on either of $(1, \pm\alpha)$, we have for every unit vector $u \in \Delta^v_\alpha$:

$$\|(D_x v_r)^{-1}u\| \geq \frac{\|(D_x v_r)^{-1}(1, \pm\alpha)\|}{\|(1, \pm\alpha)\|} > \frac{r\alpha - 1}{\alpha}.$$

For part 2 (a), we have for x $\in \mathcal{G}_v^+$ $s'(x_2) > a > 0$, and for $x \in \mathcal{G}_v^-$, $s'(x_2) < -a < 0$, thus, by simple calculations analogous to the last one, we get the results. Finally, for (b) we just use $m((D_x v_r)^{-1}) = \frac{1}{\|D_x v_r\|} > \frac{1}{br+1}$ for every $x \in \mathbb{T}^2$. □

In Figure 2.2, we show the action of the derivative $(D_x h_t)^{-1}$ for $x \in \mathcal{G}_h$. The action of $(D_x v_r)^{-1}$ for $x \in \mathcal{G}_v$ is analogous but in the opposite direction.

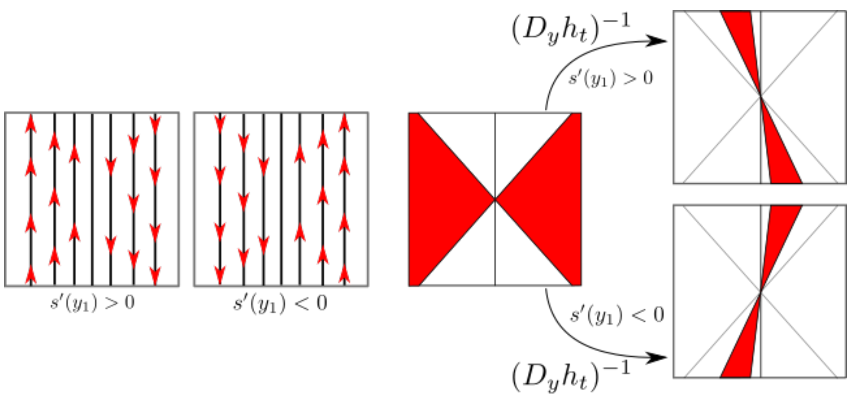


Figure 2.2: Action of $(D_x h_t)^{-1}$ for $x \in \mathcal{G}_h$.

2.2 Construction of robust NUH homotopic to a homothety

In this section, we present the proof of Theorem A. Fix $E = k \cdot Id$, for some $k \in \mathbb{N}$ (we shall make the entire argument on $k \in \mathbb{N}$ for the sake of simplicity of notation, we emphasize that the entire argument works for $k \in \mathbb{Z}$ by replacing k for $|k|$ when necessary). Fix $L < \frac{1}{4k}$ and define the critical and good regions as in Def. 2.1 for points $z_1, z_2, z_3, z_4 \in \mathbb{T}^1$ such that:

- $I_1 = [z_1, z_2]$ and $I_3 = [z_3, z_4]$ have size L;
- The translation of I_1 by a multiple of $\frac{1}{k}$ does not intersect I_3.
- $I_2 = (z_2, z_3)$ and $I_4 = (z_4, z_1)$ have size strictly larger than $\frac{1}{k}\left[\frac{k-1}{2}\right]$, where $[p]$ denotes the floor of p.

It is obtained directly from the definitions that:

Proposition 2.2. *For every $x = (x_1, x_2) \in \mathbb{T}^2$, $E^{-1}(x)$ has k^2 points given by:*

$$E^{-1}(x_1, x_2) = \left\{ \left(\frac{x_1 + i}{k}, \frac{x_2 + j}{k} \right) \; : \; i, j = 0, \cdots, k-1 \right\}.$$

At least $k\left[\frac{k-1}{2}\right]$ are inside each of $\mathcal{G}_v^+$, $\mathcal{G}_v^-$, $\mathcal{G}_h^+$ and $\mathcal{G}_h^-$, and at most k of them are inside each of C_v, C_h.

From now on, in this section, we fix any $\alpha > 1$ and the corresponding cones as in Def. 2.2. We consider the analytic maps:

$$f_{(t,r)} = E \circ v_r \circ h_t, \tag{2.8}$$

which we shall denote only by $f = f_{(t,r)}$. Clearly f is an area preserving endomorphism isotopic to E. We observe that, given $x \in \mathbb{T}^2$ and $y \in f^{-1}(x)$, we have:

$$(D_y f)^{-1} = (D_y h_t)^{-1} (D_{h_t(y)} v_r)^{-1} E^{-1}.$$

The goal is for $(D_{h_t(y)} v_r)^{-1}$ to take vectors in the vertical cone and expand them in the horizontal direction and then $(D_y h_t)^{-1}$ takes its images and expands them in the vertical direction, resulting in $(D_y f)^{-1}$ expanding in the vertical direction for most points in $f^{-1}(x)$. Thus, in order to keep track of this derivative, we must localize the points $y \in f^{-1}(x)$ in regard to which of $\mathcal{G}_h$ or C_h they belong, and $\{h_t(y) \; : \; y \in f^{-1}(x)\} = (E \circ v_r)^{-1}(x)$ regarding which of $\mathcal{G}_v$ or C_v they belong.

Lemma 2.2. *For every $x \in \mathbb{T}^2$, we have:*

1. *$(v_r \circ E)^{-1}(x)$ has k^2 points of which at least $k\left[\frac{k-1}{2}\right]$ of them are in each one of $\mathcal{G}_v^+$ and $\mathcal{G}_v^-$ and at most k of them are in C_v;*
2. *$f^{-1}(x)$ has k^2 points of which at least $k\left[\frac{k-1}{2}\right]$ of them are in each one of $\mathcal{G}_h^+$ and $\mathcal{G}_h^-$ and at most k of them are in C_h.*

3. *There exists a point $y \in f^{-1}(x)$ such that $d(y, C_h) > \frac{1}{10}$ and $d(h_t(y), C_v) > \frac{1}{10}$.*

Proof. 1. It is a direct consequence of Prop. 2.2 along with the fact that the regions $\mathcal{G}_v^+$, $\mathcal{G}_v^-$ and C_v are invariant under v_r.

2. Notice that in each row of pre-images by E of a point $x = (x_1, x_2)$ given by

$$\left\{ \left(\frac{x_1 + i}{k}, \frac{x_2 + j_0}{k} \right) : i = 0, \cdots, k-1 \right\},$$

for a fixed $j_0 \in \{0, \cdots, k-1\}$, v_r^{-1} is a rotation by $-rs\left(\frac{x_2+j_0}{k}\right)$ in the circle $\mathbb{T}^1 \times \left\{\frac{x_2+j_0}{k}\right\}$. Hence, at least $\left[\frac{k-1}{2}\right]$ of the k points of this row are inside each one of $\mathcal{G}_h^+$ and $\mathcal{G}_h^-$, and at most 1 is in C_h.

As this is also true for all the k rows of pre-images by E, we get at least $k\left[\frac{k-1}{2}\right]$ pre-images by $E \circ v_r$ are inside each one of $\mathcal{G}_h^+$ and $\mathcal{G}_h^-$, and at most k pre-images by $E \circ v_r$ are inside C_h. Finally, since these sets are invariant under h_t, we get the desired result.

3. It results directly from the argument used in item 2 along with the fact that $k > 5$.

□

Remark 2.1. *Even knowing which regions is a point $y \in (E \circ v_r)^{-1}(x)$, we cannot determine the region which $h_t^{-1}(y)$ is inside, as t is varying. That is, there may be points $y \in f^{-1}(x)$ that are in $\mathcal{G}_h$ such that $h_t(y) \in C_v$ and vice-versa.*

Definition 2.3. *In order to keep track of the vectors, define:*

- *For $u = (u_1, u_2) \in \mathbb{R}^2$ with $u_2 \neq 0$:*

$$* (u) = \begin{cases} -sgn\left(\frac{u_1}{u_2}\right), & \text{if } u_1 \neq 0, \\ -sgn(u_2), & \text{if } u_1 = 0. \end{cases}$$

Notice that $*(u) = *(E^{-1}u)$, for every $u \in \mathbb{R}^2$.

- *For $x \in \mathbb{T}^2$, $y \in f^{-1}(x)$ and $u \in \mathbb{R}^2$, let $(w_1, w_2) = (D_{h_t(y)}v_r)^{-1}E^{-1}u$:*

$$*_y (u) = \begin{cases} -sgn\left(\frac{w_1}{w_2}\right), & \text{if } w_1, w_2 \neq 0, \\ -sgn(w_2), & \text{if } w_2 \neq 0,\ w_1 = 0, \\ -sgn(w_1), & \text{if } w_1 \neq 0,\ w_2 = 0. \end{cases}$$

In view of item 4 of Lemma 2.1, even though $(D_{h_t(y)}v_r)^{-1}$ may not send a vector $u \in \Delta_\alpha^v$ to the horizontal cone if $h_t(y) \in C_v$, we can still end up having expansion in the vertical direction, depending on whether $y \in \mathcal{G}_h^{*_y(u)}$ or not. In this regard, from Lemma 2.2, there are k points $y \in f^{-1}(x)$ such that $h_t(y)$ are in C_v, and these points $(h_t(y))$ are all in the same circle $\mathbb{T}^1 \times \left\{\frac{x_2+j_0}{k}\right\}$, hence the derivative $(D_{h_t(y)}v_r)^{-1}$ is the same for those points. We get:

Proposition 2.3. *For every $x \in \mathbb{T}^2$ it holds: Given $u \in \mathbb{R}^2$, the sign $*_y(u) = sg\left(\frac{w_1}{w_2}\right)$ is the same for every point $y \in f^{-1}(x)$ that satisfies $h_t(y) \in C_v$, where $*_y(u)$ is as in Definition 2.3.*

Proof. From Lemma 2.2, there are k points $y \in f^{-1}(x)$ such that $h_t(y)$ are in $\mathcal{C}_v$, and these points $(h_t(y))$ are all in the same circle $\mathbb{T}^1 \times \left\{\frac{x_2+j_0}{k}\right\}$, hence the derivative $(D_{h_t(y)}v_r)^{-1}$ is the same for those points. □

Definition 2.4. *For a fixed $x \in \mathbb{T}^2$ and:*

- $u \in \Delta_\alpha^v$, *define:* $\begin{cases} A = \{y \in f^{-1}(x) \ : \ y \in \mathcal{G}_h, h_t(y) \in \mathcal{G}_v\}. \\ B = \{y \in f^{-1}(x) \ : \ y \in \mathcal{G}_h^{*_y(u)}, h_t(y) \in \mathcal{C}_v\}, \\ \mathcal{V}_v = A \cup B, \\ \mathcal{V}_h = f^{-1}(x) \setminus \mathcal{V}_v. \end{cases}$
- $u \in \Delta_\alpha^h$, *define:* $\begin{cases} C = \{y \in f^{-1}(x) \ : \ y \in \mathcal{G}_h, h_t(y) \in \mathcal{G}_v^{*(u)}\}. \\ D = \{y \in f^{-1}(x) \ : \ y \in \mathcal{G}_h^{*_y(u)}, h_t(y) \in \mathcal{C}_v \cup \mathcal{G}_v^{-*(u)}\}, \\ \mathcal{H}_v = C \cup D, \\ \mathcal{H}_h = f^{-1}(x) \setminus \mathcal{H}_v. \end{cases}$

A consequence of Lemma 2.2 and Prop. 2.3, having Remark. 2.1 in mind, is the following:

Lemma 2.3. *For a fixed $(x, u) \in T\mathbb{T}^2$, $f^{-1}(x)$ has k^2 points, of which:*

1. *For $u \in \Delta_\alpha^v$, at most $2k - 1 - \left[\frac{k-1}{2}\right]$ of them are in $\mathcal{V}_h$ and at least $(k-1)^2 + \left[\frac{k-1}{2}\right]$ are inside $\mathcal{V}_v$. Moreover, if there is a point $y \in f^{-1}(x)$ with $h_t(y) \in \mathcal{C}_v$, then:*
   - *At least $(k-1)^2$ are in A and,*
   - *at least $\left[\frac{k-1}{2}\right]$ are in B.*
2. *For $u \in \Delta_\alpha^h$, at most $k^2 - \left[\frac{k-1}{2}\right]\left(k + \left[\frac{k-1}{2}\right]\right)$ are in $\mathcal{H}_h$ and at least $\left[\frac{k-1}{2}\right]\left(k + \left[\frac{k-1}{2}\right]\right)$ are in $\mathcal{H}_v$, where:*
   - *At least $(k-1)\left[\frac{k-1}{2}\right]$ are in C and,*
   - *at least $\left[\frac{k-1}{2}\right]\left(1 + \left[\frac{k-1}{2}\right]\right)$ are in D.*

Proof. We prove item 1, the item 2 goes analogously. Given $x \in \mathbb{T}^2$, we need to count the pre-images $y \in A$, that is, $y \in f^{-1}(x)$ that satisfies $y \in \mathcal{G}_h$ and $h_t(y) \in \mathcal{G}_v$. We show an example for $k = 5$ in Figure 2.3.

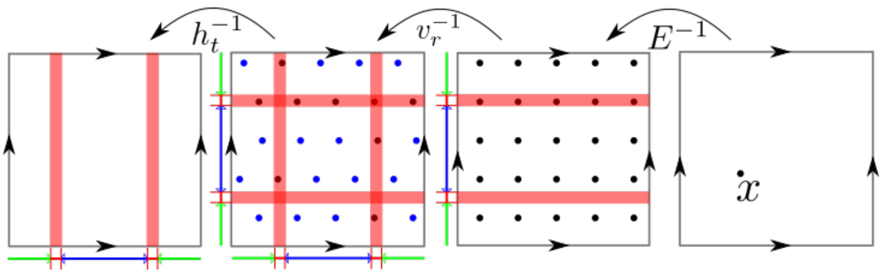


Figure 2.3: In blue is the set $h_t(A)$.

Since we want a lower bound that holds for every $x \in \mathbb{T}^2$, we must analyse the worst possible scenario, that is, when there exists a point $y \in f^{-1}(x)$ with its image by h_t in the critical region $\mathcal{C}_v$. Remember that for a point $y \in f^{-1}(x)$, $h_t(y) \in v_r \circ E^{-1}(x)$.

As in Lemma 2.2, we have $v_r^{-1} \circ E^{-1}(x)$ composed by k rows, thus the existence of a point of $v_r^{-1} \circ E^{-1}(x)$ in C_v implies that the entire row (k points) are in C_v. That is, we have at least $k-1$ rows of points in $v^{-1} \circ E^{-1}(x)$ which are in $\mathcal{G}_v$, each row composed by k points. Now, since h_t^{-1} preservers C_h, it is enough to point out that each row of $v_r^{-1} \circ E^{-1}(x)$ can have at most one point in C_h since the points are separated by a distance of $1/k$ in the first coordinate. Hence, we have at least $k(k-1)-(k-1)=(k-1)^2$ of pre-images of x by f which are in A.

To count points in B, we want points $y \in f^{-1}(x)$ with $y \in \mathcal{G}_h^{*_y(u)}$ and $h_t(y) \in C_v$. We already know that, in this scenario, there is exactly one row of points of $v_r^{-1} \circ E^{-1}(x)$ in C_v. If we denote by $R_v = C_v \cap v_r^{-1} \circ E^{-1}(x)$, from the same argument used in Lemma 2.2, we have that $h_t^{-1}(R_v)$ has at least $\left[\frac{k-1}{2}\right]$ points in each $\mathcal{G}_v^+, \mathcal{G}_v^-$. Finally, from Prop. 2.3, $*_y\,(u)$ is constant in $\mathcal{R}_v$, hence there are at least $\left[\frac{k-1}{2}\right]$ points in B.

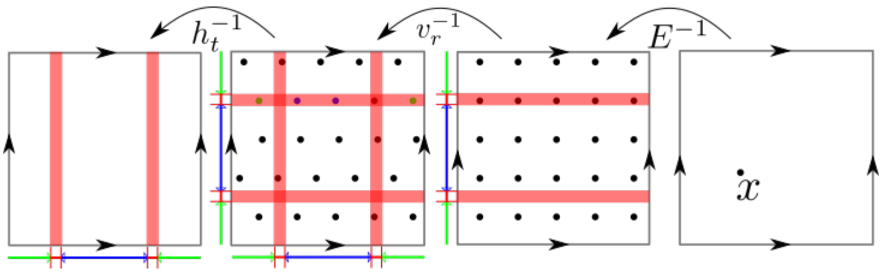


Figure 2.4: In blue, the set $h_t(B)$ for $*_y\,(u) = +$, and in green the set $h_t(B)$ for $*_y\,(u) = -$.

$\square$

Knowing that for every unit vector $u \in \mathbb{R}^2$ we have $\|E^{-1}u\| = \frac{1}{k}$ (maximum norm), from Lemma 2.1 we get:

Lemma 2.4. *For $t, r > \frac{2\alpha}{a}$ and for fixed $x \in \mathbb{T}^2$, it holds:*

1. *If $u \in \Delta_\alpha^v$, then for all $y \in \mathcal{V}_v$ we have $(D_y f)^{-1}u \in \Delta_\alpha^v$;*
2. *If $u \in \Delta_\alpha^v$ is a unit vector, then:*

$$\|(D_y f)^{-1}u\| > \begin{cases} \left(\frac{a-\frac{\alpha}{t}}{\alpha}\right)\left(\frac{a-\frac{\alpha}{r}}{\alpha}\right)\frac{tr}{k}, & y \in A, \\ \frac{1}{\alpha k}, & y \in B, \\ \frac{1}{(bt+1)\alpha k}, & y \in \mathcal{V}_h; \end{cases}$$

3. *If $u \in \Delta_\alpha^h$, then for all $y \in \mathcal{H}_v$ we have $(D_y f)^{-1}u \in \Delta_\alpha^v$;*
4. *If $u \in \Delta_\alpha^h$ is a unit vector, then:*

$$\|(D_y f)^{-1}u\| > \begin{cases} \left(\frac{a-\frac{\alpha}{t}}{\alpha}\right)\frac{t}{k}, & y \in C, \\ \frac{1}{(br+1)k}, & y \in D, \\ \frac{1}{(bt+1)(br+1)k}, & y \in \mathcal{H}_h. \end{cases}$$

Proof. Denote by $w = E^{-1}u$, since $E = k \cdot Id$ is a homothety, if $u \in \Delta_\alpha^\sigma$, then $w \in \Delta_\alpha^\sigma$, $\sigma = v, h$, and $\|w\| = \frac{\|u\|}{k}$. For $y \in f^{-1}(x)$, we have:

$$(D_y f)^{-1}u = (D_y h_t)^{-1}(D_{h_t(y)}v_r)^{-1}w.$$

For $u \in \Delta_\alpha^v$, $y \in \mathcal{V}_v$, either $y \in A$ or $y \in B$:

- If $y \in A$, then $y \in \mathcal{G}_h$ and $h_t(y) \in \mathcal{G}_v$. From item $1(a)$ of Lemma 2.1, it follows that $w' := (D_{h_t(y)}v_r)^{-1}w \in \Delta_\alpha^h$, and $\|(D_{h_t(y)}v_r)^{-1}w\| > r\frac{a-\frac{\alpha}{r}}{\alpha}\|w\|$.

  From item $3(a)$ of the same Lemma, we get $(D_y f)^{-1}u = (D_y h_t)^{-1}w' \in \Delta_\alpha^v$ and $\|(D_y h_t)^{-1}w'\| > t\frac{a-\frac{\alpha}{t}}{\alpha}\|w'\|$. We conclude that $(D_y f)^{-1}u \in \Delta_\alpha^v$ and

$$\|(D_y f)^{-1}u\| > \left(\frac{a-\frac{\alpha}{t}}{\alpha}\right)\left(\frac{a-\frac{\alpha}{r}}{\alpha}\right)\frac{tr}{k}.$$

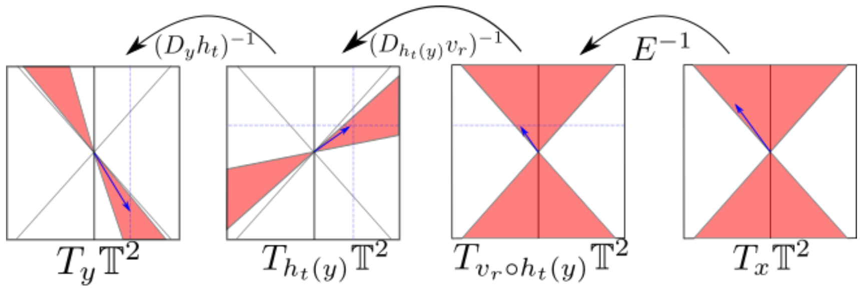


Figure 2.5: The action of $(D_y f)^{-1}$ for $y \in A$.

- If $y \in B$, then $y \in \mathcal{G}_h^{*_y(u)}$ and $h_t(y) \in \mathcal{C}_v$. Since $h_t(y)$ is in the critical zone, we are not able to control if $w' = (D_{h_t(y)}v_r)^{-1}w$ is a vertical or horizontal cone, we only have that $\|(D_{h_t(y)}v_r)^{-1}w\| > \frac{1}{\alpha}\|w\|$ from item $1(b)$ of Lemma 2.1.

  Now, it is not important if w' is vertical or horizontal. If $w' \in \Delta_\alpha^v$ we apply item $1(a)$ of 2.1 to conclude $(D_y h_t)^{-1}w' \in \Delta_\alpha^v$. If $w' \in \Delta_\alpha^h$, remembering the definition of $*_y(u)$, we apply item $4(a)$ to also conclude $(D_y h_t)^{-1}w' \in \Delta_\alpha^v$. To estimate $\|(D_y h_t)^{-1}w'\|$ we compare the both cases and use the worst one, which is the case $w \in \Delta_\alpha^h$. Then, from item $4(a)$ again, we have $\|(D_y h_t)^{-1}w'\| > \|w'\|$. In conclusion, $(D_y f)^{-1}u \in \Delta_\alpha^v$ and

$$\|(D_y f)^{-1}u\| > \frac{1}{\alpha k}.$$

  In Figure 2.6, we show the action of $(D_y f)^{-1}$, for $y \in B$. In the example, we take $(D_{h_y(y)v_r})^{-1} = Id$, and we show how, in such cases, we lost the track of cones, but we can keep track of vectors.

Finally, for $y \in \mathcal{V}_h$, $u \in \Delta_\alpha^v$, we cannot control if $(D_y f)^{-1}u$ is in the vertical or horizontal cone. To estimate $\|(D_y f)^{-1}u\|$ we study the worst possible scenario, since w is a vertical cone, the worst estimate is given by item $1(b)$ of Lemma 2.1 to get $\|(D_{h_t(y)}v_r)^{-1}w\| > \frac{1}{\alpha}\|w\|$. As we cannot tell if $w' = (D_{h_t(y)}v_r)^{-1}w$ is vertical or horizontal, we work with the

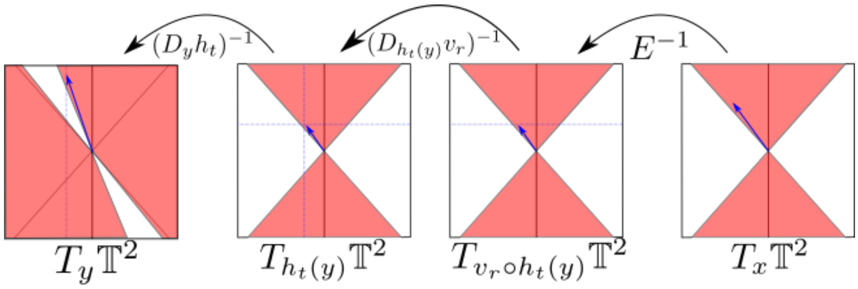


Figure 2.6: The action of $(D_y f)^{-1}$, for $y \in B$.

worst estimate which is the one given by item 4(b) of the same Lemma, it gives us $\|(D_y h_t)^{-1} w'\| > \frac{1}{bt+1}\|w'\|$. We conclude:

$$\|(D_y f)^{-1} u\| > \frac{1}{(bt+1)\alpha k}.$$

This finishes the proof of items 1 and 2. The proofs of items 3 and 4 follow analogously by applying the corresponding cases given by Lemma 2.1.

□

Non-uniform hyperbolicity

For $(x, u) \in T\mathbb{T}^2$ with $u \neq 0$ and for $n \in \mathbb{N}$ denote by

$$Df^{-n}(x, u) = \{(y, w) \in T\mathbb{T}^2 \ : \ f^n(y) = x, D_y f^n w = u\}.$$

For any non-zero tangent vector (x, u) and $n \geq 0$, define:

$$\begin{aligned}
\mathcal{G}_n &= \{(z, w) \in Df^{-n}(x, u) \ : \ w \in \Delta^v_\alpha\},\\
\mathcal{B}_n &= Df^{-n}(x, u) \setminus \mathcal{G}_n,\\
g_n &= \#\mathcal{G}_n,\\
b_n &= \#\mathcal{B}_n = k^{2n} - g_n.
\end{aligned}$$

From Lemmas 2.3, 2.4 one deduces:

Lemma 2.5. *Let $(x, u) \in T\mathbb{T}^2$.*

1. *If $u \in \Delta^v_\alpha$, then at least $(k-1)^2 + \left[\frac{k-1}{2}\right]$ of its pre-images under Df are also in Δ^v_α;*
2. *If $u \in \Delta^h_\alpha$, then at least $\left[\frac{k-1}{2}\right]\left(k + \left[\frac{k-1}{2}\right]\right)$ of its pre-images under Df are in Δ^v_α.*

Proof. 1. If u is in the vertical cone, from item 1 of Lemma 2.4 we get that a pre-image $y \in f^{-1}(x)$ such that $y \in \mathcal{V}_v$ satisfies $(D_y f)^{-1} u \in \Delta^v_\alpha$. From item 1 of Lemma 2.3, we have at least $(k-1)^2 + \left[\frac{k-1}{2}\right]$ of pre-images $y \in \mathcal{V}_v$.

2. If u is in the horizontal cone, item 3 of Lemma 2.4 gives us $(D_y f)^{-1} u \in \Delta^v_\alpha$ for every $y \in \mathcal{H}_v$. Item 2 of Lemma 2.3 gives us that there are at least $\left[\frac{k-1}{2}\right]\left(k + \left[\frac{k-1}{2}\right]\right)$ pre-images $y \in \mathcal{H}_v$.

□

By the lemma above, we get:

$$\begin{aligned} g_{n+1} &\geq \left((k-1)^2 + \left[\frac{k-1}{2}\right]\right) g_n + \left[\frac{k-1}{2}\right]\left(k + \left[\frac{k-1}{2}\right]\right) b_n \\ &= \left((k-1)^2 - \left[\frac{k-1}{2}\right]\left(k-1+\left[\frac{k-1}{2}\right]\right)\right) g_n + \left[\frac{k-1}{2}\right]\left(k+\left[\frac{k-1}{2}\right]\right) k^{2n}, \end{aligned}$$

hence:

$$\begin{aligned} \frac{g_{n+1}}{k^{2(n+1)}} \geq \frac{1}{k^2}\left((k-1)^2 - \left[\frac{k-1}{2}\right]\left(k-1+\left[\frac{k-1}{2}\right]\right)\right)\frac{g_n}{k^{2n}} \\ + \frac{1}{k^2}\left[\frac{k-1}{2}\right]\left(k+\left[\frac{k-1}{2}\right]\right). \end{aligned}$$

Denoting by $a_n = \frac{g_n}{k^{2n}}$ and

$$\begin{aligned} c &= \frac{1}{k^2}\left((k-1)^2 - \left[\frac{k-1}{2}\right]\left(k-1+\left[\frac{k-1}{2}\right]\right)\right), \\ e &= \frac{1}{k^2}\left[\frac{k-1}{2}\right]\left(k+\left[\frac{k-1}{2}\right]\right), \end{aligned}$$

the inequality above becomes:

$$a_{n+1} \geq c \cdot a_n + e.$$

Lemma 2.6. *For every $(x,u) \in T\mathbb{T}^2$, $u \neq 0$, and $n \geq 0$ it holds:*

$$\begin{aligned} a_n &\geq \frac{e}{1-c}(1-c^n) \\ &= \frac{\left[\frac{k-1}{2}\right]\left(k+\left[\frac{k-1}{2}\right]\right)}{2k-1+\left[\frac{k-1}{2}\right]\left(k-1+\left[\frac{k-1}{2}\right]\right)}(1-c^n) \end{aligned}$$

In particular,

$$\liminf a_n \geq \frac{\left[\frac{k-1}{2}\right]\left(k+\left[\frac{k-1}{2}\right]\right)}{2k-1+\left[\frac{k-1}{2}\right]\left(k-1+\left[\frac{k-1}{2}\right]\right)} := L(k),$$

uniformly in $(x,u) \in \mathbb{T}^2$.

From now on we shall denote by $L(k) = \frac{\left[\frac{k-1}{2}\right]\left(k+\left[\frac{k-1}{2}\right]\right)}{2k-1+\left[\frac{k-1}{2}\right]\left(k-1+\left[\frac{k-1}{2}\right]\right)}$. As another consequence of Lemmas 2.3 and 2.4 we have the following:

Lemma 2.7. *If $r, t > \frac{2\alpha}{a}$, then for all $(x,u) \in T\mathbb{T}^2$ we have:*

1. *If $u \in \Delta^v_\alpha$, then:*

$$\begin{aligned} I(x,u;f) \geq & \frac{(k-1)^2}{k^2}\log r + \left(\frac{k^2-4k+2+\left[\frac{k-1}{2}\right]}{k^2}\right)\log t \\ &+ \log\left(\frac{1}{\alpha k}\left(\left(a-\frac{\alpha}{t}\right)\left(a-\frac{\alpha}{r}\right)\right)^{\frac{(k-1)^2}{k^2}}\left(b+\frac{1}{t}\right)^{-\frac{1}{k^2}\left(2k-1-\left[\frac{k-1}{2}\right]\right)}\right). \end{aligned}$$

2. If $u \in \Delta_\alpha^h$, then:

$$I(x,u;f) \geq -\left(\frac{k^2-(k-1)\left[\frac{k-1}{2}\right]}{k^2}\right)\log r - \left(\frac{k^2-\left[\frac{k-1}{2}\right]\left(2k-1+\left[\frac{k-1}{2}\right]\right)}{k^2}\right)\log t$$
$$+\log\left(\left(\frac{1}{k}\left(\frac{1}{\alpha}\left(a-\frac{\alpha}{t}\right)\right)\right)^{\frac{k-1}{k^2}\left[\frac{k-1}{2}\right]-1}\left(b+\frac{1}{t}\right)^{\frac{1}{k^2}\left[\frac{k-1}{2}\right]\left(k+\left[\frac{k-1}{2}\right]\right)-1}\right).$$

Proof. For $u \in \Delta_\alpha^v$ a unit vector, we have:

$$I(x,u;f) = \sum_{y\in f^{-1}(x)} \frac{\log\|(D_yf)^{-1}u\|}{k^2} = \sum_{y\in\mathcal{V}_v} \frac{\log\|(D_yf)^{-1}u\|}{k^2} + \sum_{y\in\mathcal{V}_h} \frac{\log\|(D_yf)^{-1}u\|}{k^2}$$
$$= \sum_{y\in A} \frac{\log\|(D_yf)^{-1}u\|}{k^2} + \sum_{y\in B} \frac{\log\|(D_yf)^{-1}u\|}{k^2} + \sum_{y\in\mathcal{H}_v} \frac{\log\|(D_yf)^{-1}u\|}{k^2}$$

Then, combining item 1 of Lemma 2.3 and item 2 of Lemma 2.4:

$$I(x,u;f) \geq \frac{(k-1)^2}{k^2}\log\left(\left(\frac{a-\frac{\alpha}{t}}{\alpha}\right)\left(\frac{a-\frac{\alpha}{r}}{\alpha}\right)\frac{tr}{k}\right) + \frac{1}{k^2}\left[\frac{k-1}{2}\right]\log\left(\frac{1}{\alpha k}\right)$$
$$+\frac{1}{k^2}\left(2k-1-\left[\frac{k-1}{2}\right]\right)\log\left(\frac{1}{(bt+1)\alpha k}\right).$$

Then we just simplify the expression to obtain item 1. The proof of item 2 follows analogously by dividing the pre-images of x into $y \in C$, $y \in D$ and $y \in \mathcal{H}_h$ and using the corresponding results of Lemmas 2.3 and 2.4.

□

Now, to calculate $C_{\mathcal{X}}(f)$, we use Prop. 2.1 to compute:

$$I(x,u;f^n) = \sum_{i=0}^{n-1}\sum_{y\in f^{-i}(x)} \frac{I(y,(D_yf^i)^{-1}u;f)}{k^{2i}} := \sum_{i=0}^{n-1} J_i,$$

and, if $t, r > \frac{2\alpha}{a}$, for each i we obtain:

$$J_i = \frac{1}{k^{2i}}\sum_{y\in f^{-1}(x)} I(y,(D_yf^i)^{-1}u;f) = \frac{1}{k^{2i}}\sum_{(y,w)\in\mathcal{G}_i} I(y,w;f) + \frac{1}{k^{2i}}\sum_{(y,w)\in\mathcal{B}_i} I(y,w;f)$$
$$\geq a_iV(t,r,k) + (1-a_i)H(t,r,k),$$

where V and H are the right side of the inequalities obtained in Lemma 2.7 for $u \in \Delta_\alpha^v$ and $u \in \Delta_\alpha^h$ respectively. It follows from Lemma 2.6, with $L(k)$ as above and $c_k = \left[\frac{k-1}{2}\right]$, to simplify the notation, that:

$$\lim_{i\to\infty} J_i \geq L(k)V(t,r,k) + (1-L(k))H(t,r,k)$$
$$= C(t,r,k) + \frac{1}{k^2}\left(L(k)\left((k-1)(2k-c_k)+1\right) - \left(k^2-(k-1)c_k\right)\right)\log r +$$
$$\frac{1}{k^2}\left(L(k)\left(2(k-1)^2 - c_k\left(2(k-1)+c_k\right)\right) - \left(k^2 - c_k\left(2k-1+c_k\right)\right)\right)\log t \quad ,$$

where

$$C(t,r,k) = L(k)C_1(t,r,k) + (1 - L(k))C_2(t,r,k),$$

with

$$C_1(t,r,k) = \log\left(\frac{1}{\alpha k}\left(\left(a - \frac{\alpha}{t}\right)\left(a - \frac{\alpha}{r}\right)\right)^{\frac{(k-1)^2}{k^2}}\left(b + \frac{1}{t}\right)^{-\frac{1}{k^2}\left(2k-1-\left[\frac{k-1}{2}\right]\right)}\right)$$

$$C_2(t,r,k) = \log\left(\frac{1}{k}\left(\frac{1}{\alpha}\left(a - \frac{\alpha}{t}\right)\right)^{\frac{k-1}{k^2}\left[\frac{k-1}{2}\right]-1}\left(b + \frac{1}{t}\right)^{\frac{1}{k^2}\left[\frac{k-1}{2}\right]\left(k+\left[\frac{k-1}{2}\right]\right)-1}\right),$$

as in Lemma 2.7. From this, we get that for any k, $C(t,r,k)$ is growing as t and r grow, then for $t, r > \frac{2\alpha}{a}$, $C(t,r,k) > C$ is uniformly bounded from below by some constant C.

Now, in order to get $\lim_{i\to\infty} J_i > 0$, we can either make t or r large, depending on whether the constant (which depends on k) multiplying $\log t$ or $\log r$ is positive or negative. However, for both of them, we only get positivity of the constant if $k \geq 5$.

Thus, for $k \geq 5$, since all the bounds above are uniform for all non-zero tangent vectors (x,u), we obtain that for t (or r) sufficiently large, for all i greater than some i_0, and for all nonzero tangent vectors (x,u), $J_i(x,u) > N > 0$ for some constant N. Hence, there exists some n_0 such that

$$\frac{1}{n_0} I(x,u;f^{n_0}) = \frac{1}{n_0}\sum_{i=0}^{n_0-1} J_i(x,u) > \frac{N}{2} > 0,$$

for all nonzero tangent vectors (x,u). Therefore, $C_{\mathcal{X}}(f) > 0$ which by Theorem 2.1 concludes the proof of Theorem A.

We finish this section by including some examples for a better visualization that for a fixed $k \in \mathbb{N}$, the bounds obtained in this section are quite simple. For that, we fix $k = 5$, we get $L(5) = \frac{2}{3}$, the limitations of our last calculations become:

$$\lim_{i\to\infty} J_i \geq C(t,r,5) + 5\log r + 5\log t,$$

with

$$C(t,r,5) = \log\left(\frac{1}{5}\frac{\alpha^{\frac{17}{25}}}{a^{2/3}}\left(a - \frac{\alpha}{t}\right)^{\frac{1}{5}}\left(a - \frac{\alpha}{r}\right)^{\frac{32}{75}}\left(b + \frac{1}{t}\right)^{-\frac{18}{25}}\right)$$

Thus, taking the map $s : \mathbb{T}^1 \to \mathbb{R}$ as $s(u) = \sin(2\pi u)$, $L = \frac{1}{20}$, $a = 2\pi\sin(\frac{\pi}{10})$, $b = 2\pi$, and $\alpha = 1.1$, we get that for every $t, r \gtrapprox \frac{2a}{\alpha} \approx 1.77$ the number $C(t,r,5) + 5\log r + 5\log t$ is positive. Thus, the maps $f_{(t,r)} = E \circ v_r \circ h_t$ satisfy the results of Theorem A.

2.3 Construction of NUH endomorphism in the general case

In this section, we prove Theorem B. For $k \cdot Id \neq E \in M_{2\times 2}(\mathbb{Z})$, let $\tau_1(E)$ be the greatest common divisor of the entries of E, $\tau_2(E) = \det(E)/\tau_1(E)$, so that $d = \tau_1 \cdot \tau_2$ coincides with the topological degree of the induced endomorphism $E : \mathbb{T}^2 \to \mathbb{T}^2$. We will focus on the case $\tau_1 = \gcd(e_{ij}) > 2$, since the other one is proved in [1].

We want to make a slight change in the argument used in [1] so that for every $x \in \mathbb{T}^2$, $f^{-1}(x)$ has at most one point in the critical zone. This includes the cases where the pair (τ_1, τ_2) is $(3,3)$ or $(4,4)$. Thus, only five cases remain out of the conclusions of Theorem B: $(1,2), (1,3), (1,4), (2,2)$ and $(2,4)$, where even with this improvement in the argument, the proportion we obtain for vectors in the good region is still insufficient to obtain expansion in the vertical direction, given the small amount of pre-images.

The numbers $\tau_1,\ \tau_2$ are the elementary divisors of E and, as in Section 2.4 of [1], there exists $P \in GL_2(\mathbb{Z})$ such that the matrix $G = P^{-1} \cdot E \cdot P$ satisfies:

$$G^{-1}(\mathbb{Z}) = \left\{ \begin{pmatrix} \frac{i}{\tau_2} \\ \frac{j}{\tau_1} \end{pmatrix} : i, j \in \mathbb{Z} \right\}$$

Moreover, as E is not a homothety, by another change of coordinates if necessary we may assume that E does not have $(0,1)$ as an eigenvector.

With this in mind, we assume that $\mathbb{P}E$ does not fix $[(0,1)]$ and that $E^{-1}\mathbb{Z}^2 = \frac{1}{\tau_2}\mathbb{Z} \times \frac{1}{\tau_1}\mathbb{Z}$. So there exists an $\alpha > \tau_2 > 1$ such that if Δ^h_α and Δ^v_α are the corresponding horizontal and vertical cones as in Def. 2.2, then $\overline{E^{-1}\Delta^v_\alpha} \subset Int(\Delta^h_\alpha)$. From now on, we fix such $\alpha > \tau_2$.

Let $L < \min\left\{ \frac{1}{4\tau_2}, \frac{\tau_2^{-1} - \alpha^{-1}}{2}, \frac{1}{d} \right\}$, choose points $z_1, z_2, z_2, z_4 \in \mathbb{T}^1$, in this order, such that:

- $I_1 = [z_1, z_2]$ and $I_3 = [z_3, z_4]$ have size L;
- the translation of I_1 by a multiple of $1/\tau_2$ does not intersect I_3;
- $I_2 = (z_2, z_3)$ and $I_4 = (z_4, z_1)$ have size strictly larger than $\frac{1}{\tau_2}\left[\frac{\tau_2 - 1}{2}\right]$,

and define the critical and good regions $\mathcal{C}_h$, $\mathcal{G}_h$ and $\mathcal{G}^\pm_h$ as in Def. 2.1. As an immediate consequence of the definition we get:

Proposition 2.4. *For every $x \in \mathbb{T}^2$, $E^{-1}(x)$ has d points of which at least $\frac{1}{\tau_2}\left[\frac{\tau_2 - 1}{2}\right]$ are inside each of $\mathcal{G}^+_h$ and $\mathcal{G}^-_h$, and at most τ_1 of them are inside of $\mathcal{C}_h$.*

In order to have at most one pre-image of each point in the critical zone of the shear $h_t(x_1, x_2) = (x_1, x_2 + ts(x_1)$ defined as before, we define the conservative diffeomorphism of the torus $v(x_1, x_2) = (x_1 + \tilde{s}(x_2), x_2)$, with $\tilde{s} : \mathbb{T}^1 \to \mathbb{R}$ an analytic map which we shall impose restrictions later. We then study the family:

$$f_t = E \circ v \circ h_t, \tag{2.9}$$

of area preserving endomorphism of the torus isotopic to E. We shall denote $f = f_t$ to simplify the notation.

Given $x \in \mathbb{T}^2$, the set $f^{-1}(x) = h_t^{-1} \circ v^{-1} \circ E^{-1}(x)$ is composed by d points, and given $y \in f^{-1}(x)$, we have $(D_y f)^{-1} = (D_y h_t)^{-1} \circ (D_{h_t(y)} v)^{-1} \circ E^{-1}$.

In order to define v in a way that only one pre-image of x by f remains in the critical zone, we notice that $E^{-1}(x)$ is composed by d points which, by the change of coordinates made initially, are aligned in a lattice of height τ_1 and length τ_2. We also notice that the map h_t^{-1} keeps the vertical lines invariant. Therefore, the map v^{-1} needs to act in a way that it moves points on a vertical line enough so that only one remains in the critical zone, and, also, it cannot move them so much that we have new points entering the critical zone.

In this way, we take the analytic map $\tilde{s} : \mathbb{T}^1 \to \mathbb{R}$ satisfying:

1. If L is the size of the intervals I_1, I_3 then $|\tilde{s}(u)| < \frac{1}{2}\left(\frac{1}{\tau_2} - L\right)$, for all $u \in \mathbb{T}^1$.

2. For all $u \in \mathbb{T}^1$, we have that $\left|\tilde{s}\left(u + \frac{j}{\tau_1}\right)\right| > L$ for all $j \in \{0, 1, \cdots, \tau_1 - 1\}$ except at most two indices.

3. For every $u \in \mathbb{T}^1$ and $i \in \{0, 1, \cdots, \tau_1 - 1\}$, there is at most one $j \neq i$ that satisfies $\left|\tilde{s}\left(u + \frac{i}{\tau_2}\right) - \tilde{s}\left(u + \frac{j}{\tau_2}\right)\right| \leq L$.

4. $|\tilde{s}'(u)| < (2\alpha)^{-1}$, for all $u \in \mathbb{T}^1$, where α is the size of the cones fixed in the previous subsection.

Notice that conditions 2, 3 and 4 are not mutually exclusives due to the conditions for α and L imposed in Section 2.1 and initially in this section. Now, conditions 1 and 2 give us:

Lemma 2.8. *For every $x \in \mathbb{T}^2$, $f^{-1}(x)$ is composed by d points of which at most two are inside C_h. At least $d - 2$ of the pre-images are inside $\mathcal{G}$ of which at least $\tau_1 \left[\frac{\tau_2 - 1}{2}\right]$ are inside each of $\mathcal{G}_h^+$ and $\mathcal{G}_h^-$. Finally, there exists one point $y \in f^{-1}(x)$ such that $d(y, C_h) > \frac{1}{10}$.*

Proof. We have $f^{-1}(x) = h_t^{-1} \circ v^{-1} \circ E^{-1}(x)$. Since h_t preserves the critical and good regions, we must analyse the set $v^{-1} \circ E^{-1}(x)$. Remember that $E^{-1}(x)$ form a lattice of d points vertically spaced by a distance of $1/\tau_1$ and horizontally spaced by $1/\tau_2$.

In the case where $E^{-1}(x)$ has no points in the critical zone, due to condition 1, there is at most one column that can be sent into C_h by v^{-1}. The condition 3 implies that, in this column, at most 2 points can be sent into C_h. This, together with the fact that h_t preserves vertical lines, gives that the map $h_t^{-1} \circ v^{-1}$ can send at most two points to the critical zone.

In the case where $E^{-1}(x)$ has a point in the critical zone, it implies that we have exactly τ_1 points there. Due to condition 2, at most two of those points are able to remain there.

For the minimum amount of points in each of $\mathcal{G}_h^+$ and $\mathcal{G}_h^-$, we notice that, by Prop. 2.4, $E^{-1}(x)$ already has at least $\tau_1 \left[\frac{\tau_2 - 1}{2}\right]$ points inside each one, and, due to condition 1, those points must remain there.

Finally, by noticing that $E^{-1}(x)$ is a lattice of points of height τ_1 and length τ_2 separated horizontally by a distance of $\frac{1}{\tau_2}$. Since v^{-1} and h_t^{-1} are only rotations in the horizontal and vertical directions respectively, and as $\tau_2 > 4$, we get that there exists a point $y \in f^{-1}(x)$ with $d(y, C_h) > \frac{1}{10}$. □

At last, condition 4 gives us the next lemma, required for the whole construction to work:

Lemma 2.9. *There exists $\beta > \alpha$ such that for all $y \in \mathbb{T}^2$, $\overline{(D_y v)^{-1} \circ E^{-1} \Delta_\beta^v} \subset \Delta_\beta^h$, where Δ_β^v and Δ_β^h are the corresponding vertical and horizontal cones of size β as in Def. 2.2.*

Proof. For $y = (y_1, y_2)$, $D_y v = \begin{pmatrix} 1 & \tilde{s}'(y_2) \\ 0 & 1 \end{pmatrix}$. Then, due to condition 4, for all $\lambda \in \mathbb{R}$, $D_y v \cdot \lambda e_2 = \lambda(\tilde{s}'(y_2), 1) \in \Delta_{2\alpha}^v$. Since, by the definition of α, we have $E^{-1} \cdot \lambda e_2 \in int(\Delta_\alpha^h)$, we conclude that for every $y \in \mathbb{T}^2$, $\mathbb{P}((D_y v)^{-1} \circ E^{-1}) \cdot [e_2]$ is uniformly away from $[e_2]$, hence there exists such β as we wanted. □

Remark 2.2. *Items 3 and 4 of Lemma 2.1 also works in this cases for Δ^v_β and Δ^h_β.*

We give the correspondent to Lemma 2.4 for this case, as a consequence of items 3 and 4 of Lemma 2.1, Remark 2.2 and Lemma 2.9 . From now on, we fix $\beta > \alpha$ as in Lemma 2.9 and let:

$$\begin{aligned} e_v &= \inf\left\{ \|(D_x v)^{-1} \circ E^{-1} u\| \;:\; (x,u) \in T^1\mathbb{T}^2,\; u \in \Delta^v_\beta \right\}, \\ e_h &= \inf\left\{ \|(D_x v)^{-1} \circ E^{-1} u\| \;:\; (x,u) \in T^1\mathbb{T}^2,\; u \in \Delta^h_\beta \right\}. \end{aligned} \tag{2.10}$$

Lemma 2.10. *For $t > \frac{2\beta}{a}$ it holds:*

1. *if $y \in \mathcal{G}_h$ then $\overline{(D_y f)^{-1}\Delta^v_\beta} \subset \Delta^v_\beta$, it is strictly invariant.*
2. *if $u \in \Delta^v_\beta$ is a unit vector, then*

$$\|(D_y f)^{-1} u\| > \begin{cases} \frac{e_v(a-\beta/t))}{\beta} t, & y \in \mathcal{G}_h, \\ \frac{e_v}{\beta}, & y \in \mathcal{C}_h. \end{cases}$$

3. *if $u \in \Delta^h_\beta$, and $(D_{h_t(y)} v)^{-1} \circ E^{-1} \cdot u = (w_1, w_2)$ let $*_y\ (u)$ be as in Def. 2.3. Then if $y \in \mathcal{G}_h^{*_y(u)}$ we have $(D_y f)^{-1}(u) \in \Delta^v_\beta$.*
4. *if $u \in \Delta^h_\beta$ is a unit vector, then*

$$\|(D_y f)^{-1} u\| > \begin{cases} e_h, & y \in \mathcal{G}_h^{*_y(u)}, \\ \frac{e_h}{b+\frac{1}{t}} t^{-1}, & y \notin \mathcal{G}_h^{*_y(u)}. \end{cases}$$

We notice that, analogously to the homothety case, we have the problem that $*_y\ (u)$ depends on $y \in f^{-1}(x)$, therefore even though we have at least $\tau_1 \left[\frac{\tau_2-1}{2}\right]$ points in each of $\mathcal{G}_h^\pm$, there could be a vector $u \in \mathbb{R}^2$ such that for all $y \in \mathcal{G}_h^+$, $*_y\ (u) = -$ and vice-versa. However, we can see that this is not the case:

Proposition 2.5. *For every $x \in \mathbb{T}^2$, $u \in \mathbb{R}^2$, there are at least $\tau_1 \left[\frac{\tau_2-1}{2}\right]$ points $y \in f^{-1}(x)$ such that $y \in \mathcal{G}_h^{*_y(u)}$, where $*_y\ (u)$ is as in Def. 2.3 changing v_r for v.*

Proof. By the same argument used in Prop. 2.3, we can see that $*_y\ (u)$ is constant for points $y \in f^{-1}(x)$ such that $h_t(y)$ lies in the same horizontal line. There are exactly τ_2 pre-images y' such that $h_t(y)$ and $h_t(y')$ are in the same horizontal line, hence at least $\left[\frac{\tau_2-1}{2}\right]$ of these lies in $\mathcal{G}_h^{*_y(u)}$. As $v^{-1} \circ E^{-1}(x)$ has τ_1 different vertical lines, we get the result. □

Non-uniform hyperbolicity

We end up having calculations completely mirrored in those made in Subsection 2.2, and for that reason we will skip the details. For $(x,u) \in T\mathbb{T}^2$ with $u \neq 0$ and for $n \in \mathbb{N}$, we define the sets $Df^{-n}(x,u)$, $\mathcal{G}_n$, $\mathcal{B}_n$, and the numbers g_n, $b_n = d^n - g_n$ as before. From Lemmas 2.8, 2.10 and Prop. 2.5 we deduce:

Lemma 2.11. *Let $(x,u) \in T\mathbb{T}^2$.*

1. *If $u \in \Delta^v_\beta$, then at least $d-2$ of its pre-images under Df are also in Δ^v_β.*

2. *If $u \in \Delta^h_\beta$, then at least $\tau_1 \left[\frac{\tau_2-1}{2}\right]$ of its pre-images under Df are in Δ^v_β.*

For that, we get for all $n \in \mathbb{N}$:

$$g_{n+1} \geq \left(d - 2 - \tau_1 \left[\frac{\tau_2 - 1}{2}\right]\right) g_n + \tau_1 \left[\frac{\tau_2 - 1}{2}\right] d^n,$$

hence, putting $a_n = \frac{g_n}{d^n}$:

$$a_{n+1} \geq \left(\frac{d-2}{d} - \frac{1}{\tau_2}\left[\frac{\tau_2 - 1}{2}\right]\right) a_n + \frac{1}{\tau_2}\left[\frac{\tau_2 - 1}{2}\right].$$

Thus, we get:

Lemma 2.12. *For every $(x, u) \in T\mathbb{T}^2$, $u \neq 0$, and $n \geq 0$, it holds:*

$$\liminf a_n \geq \frac{1}{\tau_2}\left[\frac{\tau_2 - 1}{2}\right] \frac{d}{2 + \tau_1 \left[\frac{\tau_2 - 1}{2}\right]} := L(\tau_1, \tau_2).$$

Remark 2.3. *This is where we are able to verify that this argument will include the cases (τ_1, τ_2) as $(3, 3)$ and $(4, 4)$, as we have $L(\tau_1, \tau_2)$ as $3/5$ and $2/3$, respectively. And it won't work for the other cases $(1, 2), (1, 3), (1, 4), (2, 2)$ and $(2, 4)$. As we will see, for the rest of the argument to work, we need this lower bound strictly greater than $1/2$.*

As another consequence of Lemmas 2.8, 2.10 and Prop. 2.5, we get:

Lemma 2.13. *If $t > \frac{2\beta}{a}$, then for all $(x, u) \in T\mathbb{T}^2$, it holds:*

1. *If $u \in \Delta^v_\beta$, then:*

$$I(x, u; f) \geq \frac{d-2}{d} \log t + \log\left(\left(\frac{e_v}{\beta}\right)^2 \left(a - \frac{\beta}{t}\right)^{\frac{d-2}{d}}\right).$$

2. *If $u \in \Delta^h_\beta$, then:*

$$I(x, u; f) \geq -\left(1 - \frac{1}{\tau_2}\left[\frac{\tau_2 - 1}{2}\right]\right) \log t + \log\left(e_h \left(b + \frac{1}{t}\right)^{-\left(1 - \frac{1}{\tau_2}\left[\frac{\tau_2 - 1}{2}\right]\right)}\right).$$

Again, by Prop. 2.1, we have:

$$I(x, u; f^n) = \sum_{i=0}^{n-1} \sum_{y \in f^{-i}(x)} \frac{I(y, (D_y f^i)^{-1} u; f)}{k^{2i}} := \sum_{i=0}^{n-1} J_i,$$

we compute, for $t > \frac{2\beta}{a}$, for all $i \geq 0$:

$$\begin{aligned} J_i &= \frac{1}{d} \sum_{(y,w) \in \mathcal{G}_i} I(y, w; f) + \frac{1}{d} \sum_{(y,w) \in \mathcal{B}_i} I(y, w; f) \\ &\geq a_i V(t, \tau_1, \tau_2) + (1 - a_i) H(t, \tau_1, \tau_2), \end{aligned}$$

where a_i is as in Lemma 2.12, V and H are the right side of the inequalities obtained in Lemma 2.13 for $u \in \Delta_\beta^v$ and $u \in \Delta_\beta^h$ respectively. It follows:

$$\begin{aligned}\lim_{i\to\infty} J_i &\geq L(\tau_1,\tau_2)V(t,\tau_1,\tau_2) + (1 - L(\tau_1,\tau_2))H(t,\tau_1,\tau_2)\\ &= \frac{\tau_1\left[\frac{\tau_2-1}{2}\right]-2}{\tau_1\left[\frac{\tau_2-1}{2}\right]+2}\log t + C(t,\tau_1,\tau_2),\end{aligned}$$

where:

$$\begin{aligned}C(t,\tau_1,\tau_2) =& L(\tau_1,\tau_2)\log\left(\left(\frac{e_v}{\beta}\right)^2\left(a-\frac{\beta}{t}\right)^{\frac{d-2}{d}}\right)\\ &+ (1-L(\tau_1,\tau_2))\log\left|e_h\left(b+\frac{1}{t}\right)^{-\left(1-\frac{1}{\tau_2}\left[\frac{\tau_2-1}{2}\right]\right)}\right| > C,\end{aligned}$$

for all $t > \frac{2\beta}{a}$, that is, $C(t,\tau_1,\tau_2)$ is uniformly bounded from below by some constant C.

Since $\tau_1 = \gcd(e_{ij}) > 2$, the constant multiplying $\log t$ is positive. Therefore, since all the bounds above are uniform for all non-zero tangent vectors (x,u), as in the homothety case we obtain that for t sufficiently large, for all n greater than some n_0, and for all nonzero tangent vectors (x,u):

$$\frac{1}{n}I(x,u;f^n) = \frac{1}{n}\sum_{i=0}^{n-1} J_i(x,u) > 0,$$

hence, $C_{\mathcal{X}}(f) > 0$ which by Theorem 2.1 concludes the proof of Theorem B.

3 Stable Ergodicity

The purpose of this Chapter is to prove Theorem C, and to do so we first prove Theorem D. Before digging into the proofs, let us clarify the notion of stable ergodicity.

Definition 3.1. *We say that an endomorphism f is stably ergodic (for μ) if there exists a C^1 neighborhood $\mathcal{U}'$ of f in $End^1_\mu(M)$ such that for every $g \in \mathcal{U}'$ of class C^2, μ is ergodic.*

Of course, the more natural way one would expect for this definition is for every map C^1-close to a stably ergodic one to be also ergodic, indeed that is a stronger condition than the one we present. However, by asking only C^1 regularity, the conclusions of Section 1.3 may not hold, these results are crucial for the construction of the arguments in the proofs of Theorems D and C.

Let us state now what we mean by diameter of the stable manifolds in Theorem D.

Definition 3.2. *Let M be a compact Riemannian manifold, with $d(x, y)$ the distance induce by its Riemannian metric. For a embedded submanifold W of M, the diameter of W is the number:*

$$\sup \{d(x, y) \ : \ x, y \in W\}.$$

Given $f : M \to M$ satisfying the conditions of Theorem 1.5, we say that f has large stable manifolds if there exists $\lambda > 0$ such that for almost every $x \in M$, $W^s(x)$ has diameter grater than λ.

It is important to remark that what we call diameter is not the intrinsic diameter of the submanifold, that is, the one measured with the induced metric. Instead, we consider the diameter of a submanifold measured inside the ambient space M.

3.1 Ergodicity of NUH maps on surfaces

We devote this section to the proof of Theorem D, thus throughout the entire section M is a compact Riemannian surface with volume μ, and $f : M \to M$ is a C^2 transitive, area preserving, and non-uniformly hyperbolic endomorphism, with the property that for almost every $x \in M$, the diameter of the global stable manifold is larger than some $\lambda > 0$. As in Section 1.3, $\boldsymbol{L_f}$ is the natural extension space, with $\hat{f}$ the lift of f to $\boldsymbol{L_f}$ and $\hat{\mu}$ the lift of the (normalized) volume measure μ. We start by noticing that reducing the results of Theorems 1.5 for a dimension two manifold, and a NUH endomorphism, we obtain that the set of Lebesgue regular points satisfies $\hat{\mathcal{R}} = \bigcup_k \hat{\Lambda}_k$ and a splitting of the tangent spaces:

$$T_{x_0}M = E^s \oplus E^u, \quad x_0 = \pi_{ext}(\hat{x}),\ \hat{x} \in \mathcal{R}$$

which varies continuously for $\hat{x} \in \Lambda_k$. From the NUH hypothesis it follows that both E^u and E^s are one dimensional subspaces of $T_{x_0}M$. At last, the unstable and stable manifolds $W^s(x)$, $W^u(\hat{x})$ given by Theorem 1.5 are both one dimensional submanifolds of M.

We define $\hat{\mathcal{R}}_0 \subset \hat{\mathcal{R}}$ to be such that for $\hat{x} \in \hat{\mathcal{R}}_0$, the Birkhoff averages of the Dirac measures $\left(\frac{1}{n}\sum_{i=0}^{n-1} \hat{f}_*^i \delta_{\hat{x}} \text{ and} \frac{1}{n}\sum_{i=0}^{n-1} \hat{f}_*^{-i}\delta_{\hat{x}}\right)$ converges to the same limit for both $\hat{f}$ and $\hat{f}^{-1}$. Then, from the Birkhoff's ergodic theorem, $\hat{\mathcal{R}}_0$ has full $\hat{\mu}$ measure.

We also define $\hat{\mathcal{R}}_1 \subset \hat{\mathcal{R}}$ to be such that for $\hat{x} \in \hat{\mathcal{R}}_1$, Lebesgue almost every point in $\hat{W}^u(\hat{x})$ is in $\hat{\mathcal{R}}_0$, i.e. $\hat{\mathcal{R}}_1 = \{\hat{x} \in \hat{\mathcal{R}} \ : \ \hat{\mu}^u_{\hat{x}}(\hat{W}^u(\hat{x}) \setminus \hat{\mathcal{R}}_0) = 0\}$, where we denote by $\hat{\mu}^u_{\hat{x}}$ the Lebesgue measure on $\hat{W}^u(\hat{x})$. The absolute continuity of the unstable foliation (Theorem 1.7) gives us that $\hat{\mathcal{R}}_1$ has full $\hat{\mu}$-measure. It is clear that both $\hat{\mathcal{R}}_0$ and $\hat{\mathcal{R}}_1$ are $\hat{f}$-invariant, and that $\mathcal{R}_0 = \pi_{ext}(\hat{\mathcal{R}}_0)$, $\mathcal{R}_1 = \pi_{ext}(\hat{\mathcal{R}}_1)$ are both invariant under f, and have full μ-measure on M.

From the Ergodic Decomposition Theorem, it is known that any invariant measure of a measurable transformation can be disintegrated into ergodic components. In our case, as f is a non-uniformly hyperbolic C^2 endomorphism, Pesin [15] proved that there exists a countable partition of $\mathcal{R}$ given by f-invariant Borel sets $\{A_j\}_{j\geq 0}$, and a family of f-invariant and ergodic measures μ_j on A_j such that each μ_j is the normalization of μ on A_j. Moreover the basin of μ_j, given by

$$B(\mu_j) = \left\{ x \in M \ : \ \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f_*^i \delta_x = \mu_j \right\},$$

is saturated by stable leaves of A_j, i.e. $B(\mu_j) = \bigcup_{x\in A_j} W^s(x)$ $(mod\ 0)$. Another proof is given by Pugh and Shub in [16].

Remark 3.1. *Note that, for any $\hat{x} \in \hat{\mathcal{R}}$, given two points $\hat{y}_1, \hat{y}_2 \in \hat{W}^u(\hat{x}) \cap \hat{\mathcal{R}}_0$, as they are in the same unstable manifold we must have $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \hat{f}_*^{-i}\delta_{\hat{y}_1} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \hat{f}_*^{-i}\delta_{\hat{y}_2}$. Thus, since they belong to $\hat{\mathcal{R}}_0$, we also have $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \hat{f}_*^{i}\delta_{\hat{y}_1} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \hat{f}_*^{i}\delta_{\hat{y}_2}$. Hence, for every $y_1, y_2 \in W^u(\hat{x}) \cap \mathcal{R}_0$, we also have $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f_*^{i}\delta_{y_1} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f_*^{i}\delta_{y_2}$.*

From that, we conclude that if $\hat{x} \in \hat{\mathcal{R}}_1$, then $\mu^u_{\hat{x}}$-almost every point in $W^u(\hat{x})$ is in the basin of the same ergodic component μ_j for some j, where $\mu^u_{\hat{x}} = (\pi_{ext})_*\hat{\mu}^u_{\hat{x}}$ is the Lebesgue measure on $W^u(\hat{x})$.

Definition 3.3. *The ergodic Pesin blocks are the sets $\Lambda_{k,\mu_j} := B(\mu_j) \cap \Lambda_k$, where Λ_k are the Pesin blocks given by Theorem 1.5.*

It is clear that $\mu(A_j) > 0$ for some j, we assume that $\mu(A_0) > 0$, our goal is to show that in fact $A_0 = \mathcal{R}$ and $\mu_0 = \mu$, which implies that μ is ergodic.

Lemma 3.1. *Almost every point in M belongs to an ergodic Pesin block of positive μ-measure.*

Proof. For each k, let Λ'_k be the set consisting of Lebesgue density points of Λ_k. By Poincaré's Recurrence Theorem, $\mu(\Lambda_k \setminus \Lambda'_k) = 0$. As $\mu(\bigcup_k \Lambda_k) = 1$, we can fix k with

$\mu(\Lambda_k) > 0$ and let $x \in \Lambda_k \cap \Lambda'_k \cap \mathcal{R}_1$. As $x \in \mathcal{R}_1$, there exists a point $\hat{x} \in \hat{\mathcal{R}}_1 \cap \pi_{ext}^{-1}(x)$. Since $W^s_{loc}(x)$ is transverse to $W^u_{loc}(\hat{x})$, from proposition 1.8 we have that there exists a sufficiently small $r > 0$ such that for every $y \in B(x,r) \cap \Lambda_k$, $W^u_{loc}(\hat{x})$ intersects transversely $W^s_{loc}(y)$.

Even more, since $x \in \Lambda'_k$, we have $\mu(B(x,k) \cap \Lambda_k) > 0$, and as the local stable lamination is absolutely continuous (Theorem 1.6), we have that

$$\mu^u_{\hat{x}}\left(\bigcup_{y \in B(x,r) \cap \Lambda_k} (W^s_{loc}(y) \cap W^u_{loc}(\hat{x})) \right) > 0, \tag{3.1}$$

where $\mu^u_{\hat{x}}$ is the Lebesgue measure along $W^u_{loc}(\hat{x})$.

From Remark 3.1, $\mu^u_{\hat{x}}$-almost every point of the type $W^s_{loc}(y) \cap W^u_{loc}(\hat{x})$ is in the basin of the same ergodic component μ_j for some j. Finally, as $B(\mu_j)$ is saturated by stable leaves, we conclude that it must contain μ-almost every point of $B(x,r) \cap \Lambda_k$, i.e. $\mu(\Lambda_{k,\mu_j}) > 0$ and x is a density point of Λ_{k,μ_j}.

Finally, as we can repeat the process for every $x \in \Lambda_k \cap \Lambda'_k \cap \mathcal{R}_1$, and since

$$\bigcup_{k\,:\,\mu(\Lambda_k)>0} (\Lambda_k \cap \Lambda'_k \cap \mathcal{R}_1)$$

has total μ-measure, it follows that μ-a.e. point of M is a density point of Λ_{k,μ_j} for some j satisfying $\mu(\Lambda_{k,\mu_j}) > 0$, hence μ-a.e. point of M is in Λ_{k,μ_j} for some j with $\mu(\Lambda_{k,\mu_j}) > 0$. □

Now, we want to show that, in fact, almost every point in M belongs to the ergodic Pesin block of the same (thus unique) ergodic component of f. For that, we rely on the classical Hopf argument.

Definition 3.4. *A su-rectangle is a piecewise C^1 simple closed curve in M consisting of two pieces of local stable manifolds and two pieces of local unstable manifolds.*

Given μ_j as before, a su-rectangle is μ_j-regular if the two pieces of unstable manifolds are contained in $\hat{\mathcal{R}}_1$ (which implyes that Lebesgue almost every point of the unstable pieces is in $\mathcal{R}_0$), and Lebesgue almost every point in the unstable pieces is in $B(\mu_j)$.

Lemma 3.2. *Almost every point in M is in the interior of an arbitrarily small regular su-rectangle.*

Proof. We consider the set of regular points with the property that almost every point in the local stable manifold is in $\mathcal{R}_1$, given by:

$$\mathcal{R}_2 := \{x \in \mathcal{R}_1 \,:\, \mu^s_x(W^s_{loc}(x) \setminus \mathcal{R}_1) = 0\},$$

where μ^s_x is the Lebesgue measure on $W^s_{loc}(x)$. Since $\mathcal{R}_1$ has full μ-measure, and the stable manifolds form an absolutely continuous lamination (Theorem 1.6), $\mathcal{R}_2$ also has full μ-measure.

Now, as in the previous Lemma, let Λ'_k be the set of density points of the Pesin block Λ_k. We fix k such that $\mu(\Lambda_k) > 0$ and we show that every $x \in \Lambda'_k \cap \Lambda_k \cap \mathcal{R}_2$ is in the interior of an arbitrarily small regular su-rectangle. So, let $x \in \Lambda'_k \cap \Lambda_k \cap \mathcal{R}_2$ be fixed.

As illustrated in Figure 3.1, since $x \in \mathcal{R}_2$ and $W^s_{loc}(x)$ is one dimensional, there exist, arbitrarily close to x, points $y, z \in W^s_{loc}(x) \cap \mathcal{R}_1$ satisfying that x lies between y and z. Even more, as y, z are in $\mathcal{R}_1$, there exist $\hat{y} \in \pi^{-1}_{ext}(y) \cap \hat{\mathcal{R}}_1$ and $\hat{z} \in \pi^{-1}_{ext}(z) \cap \hat{\mathcal{R}}_1$. Since $x \in \mathcal{R}_1$ and x is a density point of Λ_k, applying Remark 3.1, we can choose $\hat{x} \in \pi^{-1}_{ext}(x) \cap \hat{\mathcal{R}}_1$ and $p_1, p_2 \in W^u_{loc}(\hat{x}) \cap \Lambda_k$ arbitrarily close to x satisfying that x lies between p_1, p_2 and p_1, p_2 are in the basin of the same ergodic component μ_j for some j.

At last, the continuity of the stable and unstable bundles and manifolds in Λ_k implies that if $r > 0$ is sufficiently small, then choosing $y, z, p_1, p_2 \in B(x, r)$, we have that $W^s_{loc}(p_1)$ and $W^s_{loc}(p_2)$ must intersect transversely both $W^u_{loc}(\hat{y})$ and $W^u_{loc}(\hat{z})$. Thus, it defines a μ_j-regular su-rectangle, with diameter smaller than $r > 0$, and such that x in its interior.

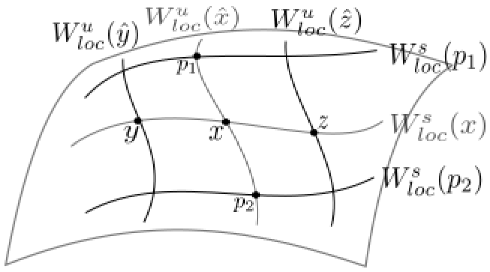


Figure 3.1: An arbitrarily small μ_j-regular su-rectangle with x in its interior.

Finally, to conclude the proof of the Lemma, we note that $\bigcup_{k:\mu(\Lambda_k)>0} (\Lambda'_k \cap \Lambda_k \cap \mathcal{R}_2)$ has total μ-measure on M.

□

We remark that Lemmas 3.1 and 3.2 did not require the hypothesis of large stable manifolds neither transitivity given in Theorem D. Hence, they still hold for any f area preserving non-uniformly hyperbolic C^2 endomorphism of a compact surface. The next Lemma is the first one for which the property of large stable manifolds will be required.

Lemma 3.3. *Given a μ_j-regular su-rectangle, with diameter smaller then λ (as in Theorem D), μ-almost every point in the interior of the rectangle is also inside the basin of μ_j.*

Proof. Let $R = W^s_1 \cup W^u_1 \cup W^s_2 \cup W^u_2$ be a μ_0-regular su-rectangle of diameter smaller than λ, and denote by V the interior of R. From Lemma 3.1, we can cover almost all V with ergodic Pesin blocks of positive measure, what we need to prove is that every one of those corresponds to the same ergodic component μ_0 of μ.

Suppose it is not true, that is, that there exists a different ergodic component, say μ_1, of μ such that $B = \Lambda_{k,\mu_1} \cap V$ has positive μ-measure. Let B' be the set of Lebesgue density points of B and $Rec(B \cap B')$ the recurrent points of $B \cap B'$, that is, the points in $B \cap B'$ that return to $B \cap B'$ infinitely many times. By Poincaré Recurrence Theorem, $Rec(B \cap B')$ has full measure on $B \cap B'$, thus has full measure on B.

Since $Rec(B \cap B')$ has positive μ-measure, we can find a point $x \in Rec(B \cap B')$ such that its global stable manifold $W^s(x)$ has diameter larger than λ. As R has diameter smaller than λ, this implies that $W^s(x)$ must intersect at least one of the unstable pieces of R, suppose then that it intersects W^u_1. Let $y \in W^s(x) \cap W^u_1$ be a point where this intersection is topologically transverse (it may be a tangency but $W^s(x)$ must cross W_1^u and leave the interior of the rectangle).

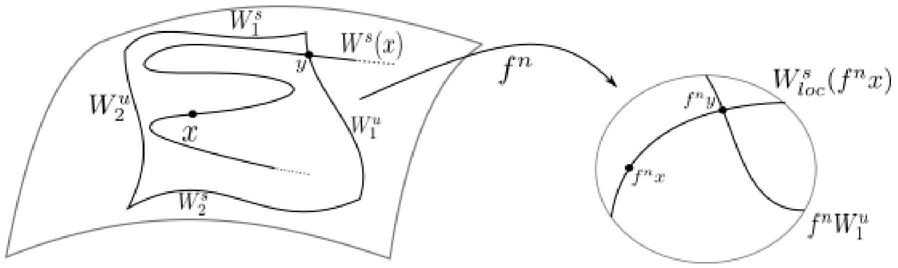


Figure 3.2: Intersection of a large stable manifold with a su rectangle

The point x is recurrent to $B \cap B'$, and the piece of $W^s(x)$ between x and y is contracting under forward iterates of f. Hence, there exists $n \geq 0$ such that $f^n(x) \in B \cap B'$ and $f^n(y) \in W_{loc}^s(f^n x)$. As f is C^1, the intersection between $f^n(W_1^u)$ and $W_{loc}^s(f^n x)$ at the point $f^n(y)$ is also topologically transverse as shown in Figure 3.2. We are now able to apply the same argument of Proposition 5.1 in [17], which goes as follows.

The lamination of stable manifolds of points in B can be extended locally to a C^1 foliation box which we also denote by $\mathcal{W}^s$ for simplicity. Taking T, with $f^n(x) \in T$, a smooth submanifold transverse to the foliation, we consider the holonomy map $\mathcal{H} : T \to f^n(W_1^u)$ as in (1.14), shown in Figure 3.3. This map is C^1 since the foliation is C^1, and Sard's Theorem implies that the set of critical values of $\mathcal{H}$ have zero Lebesgue measure (measured on W_1^u). Critical values of $\mathcal{H}$ correspond exactly to tangencies between $f^n(W_1^u)$ and the stable foliation, thus there is a set of stable leaves with positive μ-measure which are transverse to $f^n(W_1^u)$ in a small neighborhood of x.

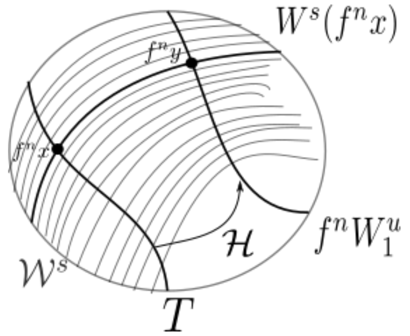


Figure 3.3: Intersections between stable manifolds of points in B with $f^n(W_1^u)$,

Since x in a density point of B and the stable foliation is absolutely continuous (Theorem 1.6), we obtain a positive μ-measured set of points $p \in B$ such that $W_{loc}^s(p)$ intersects transversely $f^n(W_1^u)$ in a set of positive Lebesgue measure (measured on $f^n(W_1^u)$). Now, since the basin of μ_1 is saturated by stable leaves, this implies that a positive measured set of $f^n(W_1^u)$ is in $B(\mu_1)$, hence a positive measured set of W_1^u is in $B(\mu_1)$, contradicting the fact that R is a μ_0-regular su-rectangle. We conclude that almost every point in B is in the basin of μ_0.

□

Now, we are in conditions to conclude the proof of Theorem D, we remark that now will be the first part of the proof to require transitivity of μ. The previous Lemma 3.3 tell us that given a regular su-rectangle with diameter smaller than λ, almost every point in its interior is in the basin of the same ergodic component of μ.

Given two of those rectangles, transitivity of f implies that the forward iterate of the interior of one must intersect the interior of the other in an open set (in particular a set of positive μ-measure). Thus, every regular su-rectangle with diameter smaller than λ has the interior (up to zero measured sets) inside the basin of the same ergodic component of f. Finally, Lemma 3.2 tell us that the interior of such rectangles cover all M up to zero μ-measured sets. In conclusion, μ-almost every point in M is in the basin of the same ergodic component, hence (f, μ) is ergodic, which finishes the proof of Theorem D.

3.2 Large stable manifolds of endomorphisms on $\mathbb{T}^2$

The objective of this section is to prove the first part of Theorem C, that is, for any linear endomorphism E on $\mathbb{T}^2$ as in Theorems A or B, if ± 1 is not an eigenvalue of E then $[E] \cap \mathcal{U}$ contains stably ergodic endomorphisms. To do so, we rely on the more general Theorem D, hence it is enough to prove that there exists a C^1 open set $\mathcal{V} \subset \mathcal{U}$ with the property that every $g \in \mathcal{V}$ has large stable manifolds.

Transitivity of those maps is a result of Andersson [18]:

Theorem 3.1. *(Andersson [18]) Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ is an area preserving endomorphism of degree at least two. If f is not homotopic to a linear map which has a real eigenvalue of modulus one, then f is transitive.*

Our approach will rely on the constructions made in Chapter 2. In this sense, from now on we fix a linear endomorphism E on $\mathbb{T}^2$ satisfying the conditions either of Theorem A or B, which we shall denote case A and case B, respectively. We keep the notations from that Chapter, that is:

- $f_{(t,r)} = E \circ v_r \circ h_t$, as in (2.8) (case A), $f_t = E \circ v \circ h_t$ as in (2.9) (case B), with the correspondent limitations $0 < a < b$;
- $\alpha > 1$ for case A and $\alpha > \tau_2 > 1$ for case B, denoting the angle of the vertical and horizontal cones Δ^v_α and Δ^h_α as in Definition 2.2;
- $L < 1/4k$, for case A, and $L < \max\left\{\frac{1}{4\tau_2}, \frac{\tau_2^{-1} - \alpha^{-1}}{2}\right\}$ for case B, denoting the size of the intervals I_1, I_3 which defines the critical zones C_v and C_h.
- e_h, e_v as in (2.10).

Since in Chapter 2 we only worked with the maximum norm, we will also consider lengths of curves in this norm. That is, given $\gamma : I \to \mathbb{T}^2$, $\gamma = (\gamma_1, \gamma_2)$ a C^1 curve with its euclidean length given by $\ell(\gamma)$, we will consider the length:

$$\ell_m(\gamma) := \int_I \max\{|\gamma_1'(t)|, |\gamma_2'(t)|\} dt,$$

which clearly satisfies $\ell_m(\gamma) \leq \ell(\gamma) \leq \sqrt{2}\ell_m(\gamma)$. From now on, we fix the number $\lambda = \frac{\alpha k}{5}$ for case A, and $\lambda = \frac{\alpha}{5e_v}$ for case B.

Definition 3.5. *A v-segment is a C^1 curve $\gamma : I \to \mathbb{T}^2$ which is tangent to Δ^v_α, i.e., such that $\gamma'(t) \in \Delta^v_\alpha$ for every $t \in I$, and whose length $\ell_m(\gamma)$ is equal to $\lambda = \frac{\alpha k}{5}$ for case A, and $\lambda = \frac{\alpha}{5e_v}$ for case B.*

Remark 3.2. *Note that since $\alpha > 1$, if $\gamma = (\gamma_1, \gamma_2)$ is a v-segment, then in fact its length is given by the length of its projection on the vertical axis γ_2, thus $\ell_m(\gamma_2) = \lambda$. Hence, the diameter (Definition 3.2) of a v-segment is greater or equal than the minimum between λ and the diameter of $\mathbb{T}^2$.*

In order to prove that there is a C^1 open set $\mathcal{V} \subset \mathcal{U}$ such that every $f \in \mathcal{V}$ satisfies the conditions of Theorem D, we will prove the following:

Proposition 3.1. *Assume $t, r > \max\left\{\frac{3\alpha}{a}, \frac{2\alpha^2 k+\alpha}{a}\right\}$ in case A, or $t > \max\left\{\frac{3\alpha}{a}, \frac{2\alpha^2+\alpha e_v}{a e_v}\right\}$ for case B.*

There exists a C^1 open set $\mathcal{V} \subset \mathcal{U}$ of area preserving, non-uniformly hyperbolic endomorphism containing $f_{(t,r)}$ for case A, or containing f_t for case B, such that any $f \in \mathcal{V}$ of class C^2 satisfies the following property: for μ-almost every point $x \in \mathbb{T}^2$, $W^s(x)$ contains a v-segment.

Remark 3.2 implies that every $f \in \mathcal{V}$ of class C^2 has large stable manifolds (Definition 3.2).

Before proving the Proposition, we show how it implies the first part of Theorem C:

Proof of the first part Theorem C. Since ± 1 are not eigenvalues of E, Theorem 3.1 gives us transitivity of any area preserving endomorphism homotopic to E. This, along with Proposition 3.1, gives us that any map in $\mathcal{V} \cap [E]$ satisfies the conditions of Theorem D, hence they are stably ergodic. □

Remark 3.3. *We notice that if f satisfies the hypothesis of Theorem D, then the same holds for f^n for every $n \geq 1$. It follows that f^n is ergodic (with respect to the ares μ) for every $n \geq 1$*

We devote the rest of this section for the proof of Proposition 3.1.

Lemma 3.4. *Assume $t, r > \max\left\{\frac{3\alpha}{a}, \frac{2\alpha^2 k+\alpha}{a}\right\}$ in case A, $t > \max\left\{\frac{3\alpha}{a}, \frac{2\alpha^2+\alpha e_v}{a e_v}\right\}$ in case B.*

There exists a C^1 open set $\tilde{\mathcal{U}}$ containing $f_{(t,r)}$ for case A, or f_t for case B, such that any map $f \in \tilde{\mathcal{U}}$ satisfies the following property: if γ is a v-segment in $\mathbb{T}^2$, then every lift of γ by f contains a v-segment. That is, every C^1 curve $\tilde{\gamma}$ with $f(\tilde{\gamma}) = \gamma$ contains a v-segment.

Proof. We begin by case A, assuming $E = k \cdot Id$, $|k| > 5$, our required open set is the following:

$$\tilde{\mathcal{U}} := \{E \circ v \circ h : v, h : \mathbb{T}^2 \to \mathbb{T}^2 \text{ are } C^1 \text{ diffeomorphisms satisfying property } (H_1)\},$$

where property (H_1) is given by:

(H_1) If $\mathcal{G}_v$ and $\mathcal{G}_h$ are as in Definition 2.1, then

1. For every unit vector $u \in \Delta^v_\alpha$ and every $x \in \mathcal{G}_v$, we have $D_x v^{-1} u \in \Delta^h_\alpha$ and $\|D_x v^{-1} u\| > 2$;

2. For every unit vector $u \in \Delta^h_\alpha$ and every $x \in \mathcal{G}_h$, we have $D_x h^{-1}u \in \Delta^v_\alpha$ and $\|D_x h^{-1}u\| > 2\alpha k$.

Clearly, $\tilde{\mathcal{U}}$ is a C^1 open set, since (H_1) is an open property. As $t, r > \max\left\{\frac{3\alpha}{a}, \frac{2\alpha^2 k+\alpha}{a}\right\}$, from Lemma 2.1 along with the fact that $\mathcal{G}_h$ is h_t-invariant and $\mathcal{G}_v$ is v_r-invariant, it follows that $f_{(t,r)} \in \tilde{\mathcal{U}}$. Hence, we need to prove that for $f = E \circ v \circ h \in \tilde{\mathcal{U}}$, if γ is a v-segment, then every lift of γ by f contains a v-segment. We study the sequence of curves illustrated in Figure 3.4.

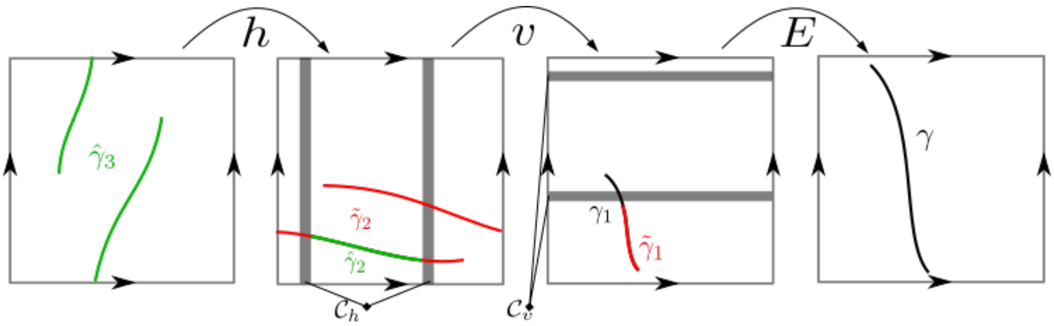


Figure 3.4: Pre-image of a v-segment.

Let γ_1 be any lift of γ by E, with $\gamma(s) = E \circ \gamma_1(s)$. As γ is a v-segment and E^{-1} only multiply vectors by $1/k$, $\gamma_1'(s) \in \Delta^v_\alpha$ for every s and $\|\gamma_1'(s)\| = \frac{\|\gamma'(s)\|}{k}$ for every s, thus $\ell_m(\gamma_1) = \frac{\ell_m(\gamma)}{k} = \frac{\alpha}{5} > \frac{1}{5}$. Therefore, since $I_1 \cup I_3$ have length $L < \frac{1}{4k} < \frac{1}{10}$, it follows that there exists a restriction $\tilde{\gamma}_1 \subset \gamma_1$ such that $\tilde{\gamma}_1(s) \in \mathcal{G}_v$ for every s, and with $\ell_m(\tilde{\gamma}_1) > \frac{1}{10}$.

Taking $\gamma_2 = v^{-1} \circ \gamma_1$ and $\tilde{\gamma}_2 = v^{-1} \circ \tilde{\gamma}_1$, from (H_1) item 1, $\tilde{\gamma}_2{}'(s) = D_{\tilde{\gamma}_1(s)} v^{-1}\tilde{\gamma}_1'(s) \in \Delta^h_\alpha$ for every s. Moreover $\|\tilde{\gamma}_2{}'(s)\| > 2\|\tilde{\gamma}_1'(s)\|$ for every s, hence $\ell_m(\tilde{\gamma}_2) > 2\ell_m(\tilde{\gamma}_1) > \frac{1}{5}$. Thus, by the same argument used before, there exists a restriction $\hat{\gamma}_2 \subset \tilde{\gamma}_2$ such that $\hat{\gamma}_2(s) \in \mathcal{G}_h$ for every s, and with $\ell_m(\hat{\gamma}_2) > \frac{1}{10}$.

Finally, taking $\gamma_3 = h^{-1} \circ \gamma_2$ and $\hat{\gamma}_3 = h^{-1} \circ \hat{\gamma}_2$, from (H_1) item 2, it follows that $\hat{\gamma}_3{}'(s) = D_{\hat{\gamma}_2(s)} h^{-1}\hat{\gamma}_2'(s) \in \Delta^v_\alpha$ is a vertical cone, and $\|\hat{\gamma}_3{}'(s)\| > 2\alpha k\|\hat{\gamma}_2{}'(s)\|$ for every s. Hence, $\ell_m(\hat{\gamma}_3) > 2\alpha k \ell_m(\hat{\gamma}_2) = \frac{\alpha k}{5} = \lambda$, which implies that it contains a v-segment as we wanted.

To prove the Lemma for case B, the argument goes completely analogous as before, thus we only present the open set:

$$\tilde{\mathcal{U}} := \{E \circ v \circ h \,:\, v, h \,:\, \mathbb{T}^2 \to \mathbb{T}^2 \text{ are } C^1 \text{ diffeomorphisms satisfying property } (H_2)\},$$

where the property (H_2) is given by:

(H_2)

1. $(E \circ v)^{-1}\Delta^v_\alpha \subset \Delta^h_\alpha$;
2. For every unit vector $u \in \Delta^h_\alpha$ and every $x \in \mathcal{G}_h$, we have $D_x h^{-1}u \in \Delta^v_\alpha$ and $\|D_x h^{-1}u\| > \frac{2\alpha}{e_v}$.

Again, $\tilde{\mathcal{U}}$ is a C^1 open set, since (H_2) is an open property. As $t > \max\left\{\frac{3\alpha}{a}, \frac{2\alpha^2+\alpha e_v}{a e_v}\right\}$, from Lemma 2.9 and Remark 2.2 (only by changing the notation β with α) along with the fact that $\mathcal{G}_h$ is h_t-invariant, it follows that $f_t \in \tilde{\mathcal{U}}$. The rest of the argument is completely analogous to case A.

□

Lemma 3.5. *Assume $t, r > \max\left\{\frac{3\alpha}{a}, \frac{2\alpha^2 k+\alpha}{a}\right\}$ in case A, $t > \max\left\{\frac{3\alpha}{a}, \frac{2\alpha^2+\alpha e_v}{a e_v}\right\}$ in case B.*

There exists a C^1 open set $\tilde{\mathcal{V}} \subset \tilde{\mathcal{U}}$ containing $f_{(t,r)}$ for case A, or containing f_t for case B, such that every $f \in \tilde{\mathcal{V}}$ satisfies the following property: for any $x \in \mathbb{T}^2$ and any C^1 curve γ passing through x, there exists $n \in \mathbb{N}$, $y \in f^{-n}(x)$, and a v-segment $\tilde{\gamma}$ passing through y such that $f^n(\tilde{\gamma}) = \gamma$.

In other words, for every C^1 curve γ, some pre-image of γ contains a v-segment.

Proof. We begin by case A, thus $f \in \tilde{\mathcal{U}}$ satisfies property (H_1). We define:

$$\tilde{\mathcal{V}} := \{f = E \circ v \circ h \in \tilde{\mathcal{U}} \ : \ f \text{ satisfies the property } (H_3)\},$$

where property (H_3) is given by:

(H_3)

1. For any $(x, v) \in T\mathbb{T}^2$, there exists a pre-image (y, w) by Df such that $(y, w) \in \mathcal{G}_h \times \Delta^v_\alpha$.
2. For any $x \in \mathbb{T}^2$, there exists a pre-image $y \in f^{-1}(x)$ such that $y \in \mathcal{G}_h$, $h_t(y) \in \mathcal{G}_v$ with $d(y, C_h) > \frac{1}{10}$ and $d(h_t(y), C_v) > \frac{1}{10}$

These are clearly C^1 open properties, hence $\tilde{\mathcal{V}}$ is a C^1 open set. From Lemma 2.3, $f_{(t,r)}$ satisfies item 1 (we take y in the set A if $v \in \Delta^v_\alpha$ and y in the set C if $v \in \Delta^h_\alpha$), and by 2.2, $f_{(t,r)}$ satisfies item 2, hence $f_{(t,r)} \in \tilde{\mathcal{V}}$. Thus, it is enough to show that any $f \in \mathcal{V}$ satisfies the conclusion of the Lemma.

Let $x \in \mathbb{T}^2$, γ a C^1 curve passing through x, and $v = \gamma'(x)$. From item 1 of (H_3), there exists a pre-image x_1 of x such that $v_1 = (D_{x_1} f)^{-1} v \in \Delta^v_\alpha$ and $x_1 \in \mathcal{G}_h$. Furthermore, we can construct a sequence of pre-images of x, $\{x_n\}_{n \in \mathbb{N}}$, such that:

- $f(x_{n+1}) = x_n$;
- $x_n \in \mathcal{G}_h$ with $d(x_n, C_h) > \frac{1}{10}$, and $h(x_n) \in \mathcal{G}_v$ with $d(h(x_n), C_v) > \frac{1}{10}$.

We then take $\tilde{\gamma_1}$ to be the pre-image of γ passing through x_1. Since it is a C^1 curve and $v_1 = \tilde{\gamma_1}'(x_1) \in \Delta^v_\alpha$, we can find a restriction γ_1 of $\tilde{\gamma}_1$ which contains x_1, is tangent to the vertical cone Δ^v_α and satisfies $\gamma_1 \subset \mathcal{G}_h$ and $h(\gamma_1) \subset \mathcal{G}_v$.

Thus, we can take the corresponding pre-images of γ_1 passing through x_n obtaining a sequence of curves γ_n, having two possibilities:

1. If every γ_n satisfies $\gamma_n \subset \mathcal{G}_h$ and $h(\gamma_n) \subset \mathcal{G}_v$, then by the condition (H_1), γ_n is tangent to the vertical cone Δ^v_α for every n and their length grow exponentially on n. Hence, we eventually obtain a v-segment as we wanted.

2. If not, then there exists $n \in \mathbb{N}$ such that $\gamma_n \subset \mathcal{G}_h$, $h(\gamma_n) \subset \mathcal{G}_v$, but this is not satisfied for γ_{n+1}. Here, it does not matter which one of those properties is not satisfied for γ_{n+1}, because for either case we must have that the length of the horizontal projection of γ_{n+1} inside $\mathcal{G}_h$ is greater than $\frac{1}{10}$ and the length of the vertical projection of $h(\gamma_{n+1})$ inside $\mathcal{G}_v$ is also greater than $\frac{1}{10}$.

   These properties, along with condition (H_1), by the same argument used in the proof of Lemma 3.4, gives us that γ_{n+1} must contain a v-segment, as we wanted.

For case B, any $f \in \tilde{\mathcal{U}}$ satisfies condition (H_2), we define:

$$\tilde{\mathcal{V}} := \{f = E \circ v \circ h \in \tilde{\mathcal{U}} \ : \ f \text{ satisfies the property } (H_4)\},$$

where property (H_4) is given by:

(H_4)
1. For any $(x, v) \in T\mathbb{T}^2$, there exists a pre-image (y, w) by Df such that $(y, w) \in \mathcal{G}_h \times \Delta^v_\alpha$.
2. For any $x \in \mathbb{T}^2$, there exists a pre-image $y \in f^{-1}(x)$ such that $y \in \mathcal{G}_h$ with $d(y, \mathcal{C}_h) > \frac{1}{10}$.

Again, these conditions are clearly C^1-open. From Lemma 2.10 f_t satisfies item 1, and from Lemma 2.8 f_t satisfies item 2. Thus, $\tilde{\mathcal{V}}$ is a C^1-open set containing f_t. The rest of the argument goes completely analogous as the homothety case.

□

We define now $\mathcal{V} := \tilde{\mathcal{V}} \cap \mathcal{U}$, any $f \in \mathcal{V}$ preserves the area μ and is non-uniformly hyperbolic. We show that $\mathcal{V}$ satisfies the conclusions of Proposition 3.1. Thus, let $f \in \mathcal{V}$ be a C^2 endomorphisms, every result obtained in Section 1.3 still holds for f. Then, we can define $V := \{x \in \mathbb{T}^2 \ : \ W^s(x) \text{ contains a v-segment}\}$. The two previous Lemmas give us that:

1. $f^{-1}(V) \subset V$;
2. $\bigcup_{n=0}^{\infty} f^n(V)$ has full Lebesgue measure.

The following Lemma gives us that V has full Lebesgue measure.

Lemma 3.6. *Assume that $f : M \to M$ is a measurable map preserving a Borel probability μ. If a measurable set E satisfies $f^{-1}(E) \subset E$ and $\mu\left(\bigcup_{n=0}^{\infty} f^n(E)\right) = 1$, then E has full μ-measure.*

Proof. The second condition and the fact that μ is f-invariant imply that $\mu(E) > 0$.

We take E' to be the set of recurrent points of E

$$E' := Rec(E) = \{x \in E \ : \ \forall n, \exists m \text{ with } m > n \text{ such that } f^m(x) \in E\},$$

then, by Poincaré's Recurrence Theorem, $\mu(E') = \mu(E)$, hence $\mu\left(\bigcup_{n=0}^{\infty} f^n(E')\right) = 1$.

We show that $\bigcup_{n=0}^{\infty} f^n(E') \subset E$, therefore $\mu(E) = 1$. Indeed, if $x \in \bigcup_{n=0}^{\infty} f^n(E')$, then there exists n such that $f^n(x) \in E$. Since $f^{-1}(E) \subset E$, we have $f^{-n}(E) \subset E$, thus $x \in E$.

□

In conclusion, $V = \{x \in \mathbb{T}^2 \ : \ W^s(x) \text{ contains a v-segment}\}$ has full Lebesgue measure in $\mathbb{T}^2$, which completes the proof of Proposition 3.1.

3.3 Bernoulli property of endomorphisms on $\mathbb{T}^2$

Here we show how to deduce the Bernoulli property of endomorphisms in the C^1 open set $\mathcal{V}$ constructed in the previous section for Theorem C. We recall:

Definition 3.6. *Consider an endomorphism $f : M \to M$ preserving a measure μ, and the corresponding lifts $\hat{f} : \boldsymbol{L_f} \to \boldsymbol{L_f}$ and $\hat{\mu}$. We say that the system (f, μ) is Bernoulli if $(\hat{f}, \hat{\mu})$ is metrically isomorphic to a Bernoulli process.*

A classical result to deduce the Bernoulli property of systems is the following:

Theorem 3.2. *(Ledrappier [19]) Let L be a compact Riemannian manifold, $F : L \to L$ a C^2 diffeomorphisms preserving a hyperbolic probability ν on L such that ν satisfies the conclusion of Theorem 1.7 (such measures are called SRB measures). If (F^n, ν) is ergodic for every $n \geq 1$, then (F, ν) is Bernoulli.*

In order to make use of the previous Theorem, we show a more general classical construction as explained for example by Viana-Yang in [20].

Lemma 3.7. *The natural extension $\hat{g} : \boldsymbol{L_g} \to \boldsymbol{L_g}$ of any C^k local diffeomorphism $g : M \to M$ on a compact manifold admits a $\boldsymbol{C^k}$ realization. That is, there exists a smooth manifold $L = M \times D^m$, where D^m is the m-dimensional open unit ball for some $m > 0$, and $G : M \times D^m \to M \times D^m$ a C^k skew-product over g, with an attractor $\Lambda = \bigcap_{n\geq 0} G^n(L)$, with $G|_\Lambda$ a diffeomorphism topologically conjugated to $\hat{g}$. The map $G|_\Lambda$ is called the smooth model of L_g.*

Proof. Since M is compact and g is a local diffeomorphism, there exist families of open sets $\{U_1, \cdots, U_m\}$ and $\{V_1, \cdots, V_m\}$ such that $\{U_i\}_{i=1}^m$ covers M, $\overline{U_i} \subset V_i$ and $g|_{V_i}$ is injective for every $i = 1, \cdots, m$. We take smooth functions $h_i : M \to [0, 1]$ with $h_i|_{U_i} \equiv 1$ and $h_i|_{V_i^c} \equiv 0$, and define $h(x) = (h_1(x), \cdots, h_m(x))$ for $x \in M$.

This way, $h : M \to [0, 1]^m$ satisfies $h(x) \neq h(y)$ for every pair $(x, y) \in A$ with $A = \{(x, y) \in M^2 : x \neq y, \ g(x) = g(y)\}$. As g is a local diffeomorphism, the set A is a compact subset of M^2, hence there exists $0 < \delta < 1/2$ such that $\|h(x) - h(y)\| \geq \delta$ for any $(x, y) \in A$. We fix $\lambda < \delta/4k$ and define:

$$G : M \times D^m \to M \times D^m, \ \ G(x, v) = \left(g(x), \frac{h(x)}{2k} + \lambda v \right).$$

It is clear that G is a well defined C^k local diffeomorphism, and that the image $G(M \times D^m)$ is relatively compact in $M \times D^m$. Moreover, if $G(x, v) = G(y, w)$, then

$$g(x) = g(y), \ \ h(x) - h(y) = 2k\lambda(w - v).$$

Hence, $\|h(x) - h(y)\| \leq 4nk < \delta$, which, by the definition of δ, implies $x = y$, thus $h(x) = h(y)$ and $w = v$. This proves that G is injective, hence an embedding.

Finally, for each $\hat{x} = (x_n)_{n\geq 0} \in \boldsymbol{L_g}$ and $n \geq 1$ the set $G^n(\{x_n\} \times D^m) := D_n(\hat{x})$ is a disk of radius λ^n contained in $\{x_0\} \times D^m$. These disks satisfy $D_{n+1}(\hat{x}) \subset D_n(\hat{x})$ and each $D_{n+1}(\hat{x})$ is relatively compact in $D_n(\hat{x})$. Hence, the intersection $\bigcap_{n\geq 0} D_n(\hat{x})$ consists of exactly one point. We define:

$$\iota : L_g \to M \times D^n, \ \ \iota(\hat{x}) = \bigcap_{n\geq 0} D_n(\hat{x}).$$

It is clear that $\iota(\boldsymbol{L_g}) = \bigcap_{n\geq 0} G^n(M \times D^n) = \Lambda$, and that $G \circ \iota = \iota \circ \hat{g}$.

□

Getting back to our case, let $f \in \mathcal{V}$ as in Theorem C and $F : \mathbb{T}^2 \times D^m \to \mathbb{T}^2 \times D^m$, $\Lambda = \bigcap_{n\geq 0} F^n(\mathbb{T}^2 \times D^m)$ be as in the previous Lemma, and let $\iota : \boldsymbol{L_f} \to M \times D^m$ be such that $F \circ \iota = \iota \circ \hat{f}$. Since F is an injective local diffeomorphism, it is a diffeomorphism onto its image. Let $\nu = \iota_* \hat{\mu}$, it can be easily checked that ν is an F-invariant probability with $supp(\nu) = \Lambda$, and that it projects to μ on $\mathbb{T}^2$.

It remains now to show that ν satisfy the SRB property (absolutely continuous disintegration along unstable manifolds, as in Theorem 1.7). For that we make use of another classical result:

Theorem 3.3. *(Ledrappier-Young [21]) Let $f : M \to M$ be a C^2 diffeomorphism of a compact Riemannian manifold M preserving a Borel measure μ. Then, μ satisfies the SRB property if, and only if, it satisfies the Pesin entropy formula:*

$$h_\mu(f) = \int \sum_i (\lambda^{(i)})^+(x) m^{(i)}(x) d\mu(x), \tag{3.2}$$

where $(\lambda^{(i)})^+(x) = \max\{\lambda^{(i)}(x), 0\}$ and $m^{(i)}(x)$ is the dimension of the corresponding Lyapunov subspace.

It is proved by Thieullen [22] and Liu [23] that for any C^2 endomorphism $g : M \to M$ preserving a Borel probability μ absolutely continuous with respect to the Lebesgue measure, (g, μ) satisfies the Pesin entropy formula. Moreover, Qian and Zhu [14] gives us the analogous as Theorem 3.3 for the endomorphism case.

It is easy to verify that $h_{\hat{\mu}}(\hat{f}) = h_\mu(f)$ and since $(\hat{f}, \hat{\mu})$ and (F, ν) are metrically isomorphic, $h_\nu(F) = h_{\hat{\mu}}(\hat{f}) = h_{\hat{\mu}}(f)$. Because F is a contraction on the fibers $\{x\} \times D^m$, we obtain that the Lyapunov exponents of F for ν are exactly $\lambda^+(f)$, $\lambda^-(f)$ and m others negative exponents. Hence the Pesin entropy formula holds for (F, ν), resulting that ν satisfies the SRB property.

Finally, since (f^n, μ) is ergodic for every n (Remark 3.3), we have that both $(\hat{f}^n, \hat{\mu})$ and (F^n, ν) are ergodic for every n. We are then in the hypothesis of Ledrappier's Theorem 3.2, which concludes the proof of Theorem C.

4 Continuity of the characteristic exponents

The purpose of this Chapter is to prove Theorem E. Initially, we present some context for the elements utilized throughout the proof. We begin by a well known lemma in ergodic theory which characterizes the lifts of measures for the correspondent projectivized cocycle.

Lemma 4.1. *(Projectivized cocycles [24]) Let $f : M \to M$ be a C^1 map on a 2 dimensional manifold, with an invariant measure μ and different Lyapunov exponents $\lambda^-(x) < \lambda^+(x)$ at μ-almost every $x \in M$, and let $E^-(x)$ and $E^+(x)$ be the corresponding Lyapunov subspaces. Consider $\mathbb{P}f : \mathbb{P}M \to \mathbb{P}M$ the projectivization of f, which is itself a bundle map over f. Then:*

1. *If μ is ergodic, then there are exactly two ergodic lifts of μ to $\mathbb{P}M$, $\mu^{\mathbb{P}^+}$ and $\mu^{\mathbb{P}^-}$. The disintegrations of $\mu^{\mathbb{P}^+}$ (resp. $\mu^{\mathbb{P}^-}$) along the fibers of $\mathbb{P}M$ are exactly the Dirac measures at the Lyapunov spaces E^+ (resp. E^-).*

2. *If $\mu^{\mathbb{P}}$ is a lift of μ to $\mathbb{P}M$ (μ is not necessarily ergodic), then there is a measurable f-invariant function $\rho : M \to [0,1]$ such that the disintegrations of $\mu^{\mathbb{P}}$ along the fibers of $\mathbb{P}M$ are $\rho(x)\delta_{E^+(x)} + (1-\rho(x))\delta_{E^-(x)}$*

4.1 Product Measures

For the proof of continuity, we will be interested in the product structure of the disintegrations of the lifts of the Haar measure. Let us remark that what we are calling product measures here is a concept stronger than the more known notion of measure with product structure.

Definition 4.1. *Let X, Y be measurable spaces and μ a measure on the product $X \times Y$. We say that μ is a product measure if $\mu = (\pi_X)_*\mu \times (\pi_Y)_*\mu$, where π_X, π_Y are the respective projections.*

In particular, with this notion, we ask the density to be constant, so that the measure on the product space is exactly the product of two measures on the factor spaces. The following result evidence this property:

Lemma 4.2. *(Characterization of product measures) Let X, Y be compact metric spaces and μ a Borel probability on $M \times N$. Then μ is a product measure if, and only if, for any $f \in C(X, \mathbb{R})$ and $g \in C(Y, \mathbb{R})$, we have:*

$$\int_{X\times Y} (f \circ \pi_X) \cdot (g \circ \pi_Y)\, d\mu = \int_X f\, d(\pi_X)_*\mu \cdot \int_Y g\, d(\pi_Y)_*\mu \tag{4.1}$$

Corollary 4.1. *(Limits of Product measures) A weak* limit of Borel product probability measures is a Borel product probability measure.*

As stated before, our approach will require us to lift the Haar measure on $\mathbb{T}^2$ to the natural extension $\boldsymbol{L_f}$, to the Solenoidal representation of it *Sol* presented in Section 1.1, and then to lift again to the proctivized spaces $\mathbb{P}\boldsymbol{L_f}$ and $\mathbb{P}Sol$ presented in Section 1.2. We then show that these measures disintegrate along the fibers as product measures, therefore we present here some properties of these disintegrations. This idea can be generalized for any continuous bundle, however, as remarked in 1.3, we may only consider here the simplification on product spaces.

Lemma 4.3. *(Measures with product disintegrations) Let X, Y, Z be compact metric spaces and μ be a Borel probability on the product $X \times Y \times Z$. Denote by μ_x the conditional of μ along $\{x\} \times Y \times Z$, for $x \in X$. Then μ_x is a product measure for $(\pi_X)_*\mu$-a.e. $x \in X$ if, and only if, for every $f \in C(X, \mathbb{R})$, $g \in C(Y, \mathbb{R})$ and $h \in C(Z, \mathbb{R})$ we have:*

$$\int_W f(x) \cdot g(y) \cdot h(z)\, d\mu(x, y, z) = \int_W f(x) \cdot h(z) \cdot \left(\int_Y g\, d(\pi_Y)_*\mu_x \right) d\mu(x, y, z) \tag{4.2}$$

Proof. By the previous Lemma, μ_x is a product measure if, and only if, for every $g \in C(Y, \mathbb{R})$, $h \in C(Z, \mathbb{R})$, it holds:

$$\int_{\{x\}\times Y\times Z} g(y)h(z)\, d\mu_x(y, z) = \int_Y g\, d(\pi_Y)_*\mu_x \cdot \int_Z h\, d(\pi_Z)_*\mu_x.$$

This equality holds for $(\pi_X)_*\mu$-a.e. $x \in X$ if, and only if, for every $f \in C(X, \mathbb{R})$ it holds:

$$\begin{aligned} &\int_X f(x) \cdot \left(\int_{\{x\}\times Y\times Z} g(y)h(z)\, d\mu_x(y, z) \right) d(\pi_X)_*\mu \\ &\qquad = \int_X f(x) \cdot \left(\int_Y g\, d(\pi_Y)_*\mu_x \cdot \int_Z h\, d(\pi_Z)_*\mu_x \right) d(\pi_X)_*\mu(x). \end{aligned}$$

Furthermore, since $\int_Z h(z)\, d(\pi_Z)_*\mu_x = \int_{Y\times Z} h(z)\, d\mu_x(y, z)$, the right hand side of the previous equality may be rewritten as:

$$\begin{aligned} &\int_X f(x) \cdot \left(\int_Y g\, d(\pi_Y)_*\mu_x \cdot \int_{Y\times Z} h(z)\, d\mu_x(y, z) \right) d(\pi_X)_*\mu(x) \\ &\qquad = \int_X f(x) \cdot \left(\int_{Y\times Z} h(z) \cdot \left(\int_Y g(y)\, d(\pi_Y)_*\mu_x \right) d\mu_x(y, z) \right) d(\pi_X)_*\mu \\ &\qquad = \int_W f(x) \cdot h(z) \cdot \left(\int_Y g d(\pi_Y)_*\mu_x \right) d\mu(x, y, z). \end{aligned}$$

Finally, by the definition of conditional measure, the initial expression given by $\int_X f(x) \cdot \left(\int_{\{x\}\times Y\times Z} g(y)h(z)\, d\mu_x(y, z) \right) d(\pi_X)_*\mu$ equals the integral of the product on W, that is, equals $\int_W f(x)g(y)h(z) d\mu(x, y, z)$, which concludes the proof.

□

At last, since our final goal is to prove continuity of integrated Lyapunov exponents, we would like to pass the property of having product disintegrations to the weak* limit of a measure. It turns out that this simply is not always achieved and one can easily construct examples of sequence with product disintegrations converging to a measure without this property. For this to happen, we require some extra conditions on the convergence, we have the following result.

Lemma 4.4. *(Limits of measures with product disintegrations) Let X, Y, Z be compact metric spaces. Let μ^k be a sequence of Borel probability measures on the product $X \times Y \times Z$ such that μ^k converges to μ in the weak* topology, and such that for $(\pi_X)_*\mu^k$-a.e. $x \in X$, the disintegration μ^k_x is a product measure on $Y \times Z$. Suppose, additionally, that one of the following two conditions is verified:*

1. *Given any $g \in C(Y, \mathbb{R})$, the functions:*

$$\alpha^k(x) = \int_Y g \, d(\pi_Y)_*\mu^k_x, \quad \alpha(x) = \int_Y g \, d(\pi_Y)_*\mu_x$$

   can be extended continuously to all X, and α^k converges uniformly to α.

2. *The measures $(\pi_X)_*\mu^k$ are equivalent to $(\pi_X)_*\mu$ with the Jacobian $J^k = \frac{d(\pi_X)_*\mu^k}{d(\pi_X)_*\mu}$ uniformly bounded from above, and $(\pi_Y)_*\mu^k_x$ converges in the weak* topology to $(\pi_Y)_*\mu_x$ for $(\pi_X)_*\mu$-a.e. $x \in X$.*

Then the disintegrations μ_x of μ are product measures for $(\pi_X)_*\mu$-a.e. $x \in X$.

Proof. From Lemma 4.3, to prove this Lemma we must prove that relation (4.2) is satisfied for the limit measure μ. So, we fix $f \in C(X, \mathbb{R})$, $g \in C(Y, \mathbb{R})$ and $h \in C(Z, \mathbb{R})$. Since each μ^k has product disintegration, it holds for every k:

$$\int_W f(x)g(y)h(z) \, d\mu^k(x, y, z) = \int_W f(x) \cdot h(z) \cdot \left(\int_Y g \, d(\pi_Y)_*\mu^k_x \right) \, d\mu^k(x, y, z).$$

The first expression converges to $\int_W f(x)g(y)h(z) \, d\mu(x, y, z)$ from weak* convergence of the measures, thus it suffices to show that the latter expression converges to $\int_W f(x) \cdot h(z) \cdot \left(\int_Y g \, d(\pi_Y)_*\mu_x \right) \, d\mu(x, y, z)$.

Consider the functions $\beta^k, \beta : W \to \mathbb{R}$ given by:

$$\beta^k(x, y, z) = \alpha^k(x)f(x)h(z), \quad \beta(x, y, z) = \alpha(x)f(x)h(z),$$

we have:

$$\begin{aligned} \left| \int_W \beta^k \, d\mu^k - \int_W \beta \, d\mu \right| &\leq \left| \int_W \beta^k \, d\mu^k - \int_W \beta \, d\mu^k \right| + \left| \int_W \beta \, d\mu^k - \int_W \beta \, d\mu \right| \\ &\leq \|f\|_\infty \cdot \|h\|_\infty \cdot \int_X |\alpha^k - \alpha| \, d(\pi_X)_*\mu^k + \left| \int_W \beta \, d\mu^k - \int_W \beta \, d\mu \right| \end{aligned} \tag{4.3}$$

If hypothesis 1 is satisfied, the functions α^k are continuous and converge uniformly to α, which gives us convergence to zero of the first term of the last expression. Also, the function β is continuous which gives us convergence to zero of the latter term, by convergence of μ^k. This concludes the proof in this case.

On the other hand, if hypothesis 2 is satisfied, then there exists a set $X' \subset X$ with $(\pi_X)_*\mu(X') = 1$ such that for every $x \in X'$, $\alpha^k(x)$ and $\alpha(x)$ are well defined and satisfy $\lim_{k\to\infty} \alpha^k(x) = \alpha(x)$. This, along with the fact that α^k, α, and J^k are uniformly bounded from above, we conclude:

$$\lim_{k\to\infty} \int_X |\alpha^k - \alpha| \, d(\pi_X)_*\mu^k = \lim_{k\to\infty} \int_X |\alpha^k - \alpha| \cdot J^k \, d(\pi_X)_*\mu = 0,$$

which from (4.3) concludes the proof in this case. □

Remark 4.1. *The conditions of Lemma 4.4 mean, in fact, that one of the two factors of the product disintegrations (corresponding to Y) converges weakly to the factor of the disintegrations of the limit measure (uniformly or pointwise). However, the other factor (corresponding to Z) of the disintegrations may not converge.*

4.2 Proof of Continuity

Before starting the proof of Theorem E, let us make a technical comment on its proof.

Let (f_n) be a sequence in $\mathcal{U}_1$ converging to $f \in \mathcal{U}_1$. For each n, let $\hat{\mu}_n$ be the unique $\hat{f}_n$ invariant measure on $\boldsymbol{L}_{f_n}$ projecting to μ. By Theorem 2.1, we know that all these cocycles have different Lyapunov exponents $\lambda^-(f_n) < \lambda^+(f_n)$ at almost every point. Let $E_n^-(\hat{x})$ and $E_n^+(\hat{x})$ be the corresponding Lyapunov subspaces at the point $x = \pi_{ext}(\hat{x})$ determined by f_n (do not mistake this notation of $E_n^{\pm}(\hat{x})$, which refer to the Lyapunov subspaces at $x = \pi_{ext}(\hat{x})$ corresponding to the map f_n, with the same notation utilized in Section 1.3 where they referred to subspaces at the point x_n, for $\hat{x} = (x_1, x_2, \cdots)$, for one fixed map).

Take, for each n, the measure $\hat{\mu}_n^{\mathbb{P}^-}$ on $\mathbb{P}\boldsymbol{L}_{f_n}$, called the stable measure of $\mathbb{P}\hat{f}_n$ defined as follows, for every E Borel set on $\mathbb{P}\boldsymbol{L}_{f_n}$:

$$\hat{\mu}_n^{\mathbb{P}^-}(E) = \int_{\boldsymbol{L}_{f_n}} \delta_{E_n^-(\hat{x})}(p_n^{-1}(\hat{x}) \cap E) \, d\hat{\mu}_n(\hat{x}),$$

where $p_n : \mathbb{P}\boldsymbol{L}_{f_n} \to \boldsymbol{L}_{f_n}$ are the projections, and $\delta_{E_n^-(\hat{x})}$ are the Dirac measures on the Lyapunov subspaces. And denote the corresponding objects for f by $\boldsymbol{L}_f$, $\hat{f}$, $\hat{\mu}$, $\mathbb{P}\hat{f}$ and $\hat{\mu}^{\mathbb{P}^-}$, and $p : \mathbb{P}\boldsymbol{L}_f \to \boldsymbol{L}_f$ the projection. Clearly, $p_{n*}\hat{\mu}_n^{\mathbb{P}^-} = \hat{\mu}_n$ and $p_*\hat{\mu}^{\mathbb{P}^-} = \hat{\mu}$.

Thus, denoting by $v_n^s(x)$ the unit vector in the Lyapunov space $E_n^-(\hat{x}) = E_n^-(x)$, $x = \pi_{ext}(\hat{x})$, we have:

$$\begin{aligned}\int_{\mathbb{T}^2} \lambda^-(f_n) \, d\mu = \int_{\boldsymbol{L}_{f_n}} \lambda^-(\hat{f}_n) \, d\hat{\mu}_n &= \int_{\boldsymbol{L}_{f_n}} \log \|D_{\hat{x}} f_n \cdot v_n^s\| \, d\hat{\mu}_n(\hat{x}) \\ &= \int_{\mathbb{P}\boldsymbol{L}_{f_n}} \log \|D_{\hat{x}} f_n \cdot v\| \, d\hat{\mu}_n^{\mathbb{P}^-}(\hat{x}, v),\end{aligned} \tag{4.4}$$

and similarly for f.

To prove continuity of the negative Lyapunov exponent (which implies continuity of the positive one as well), we would like to pass the right hand side of (4.4) to the limit, saying that $\hat{\mu}_n^{\mathbb{P}^-}$ converges to $\hat{\mu}^{\mathbb{P}^-}$. However, that is not possible in this setting, since each measure $\hat{\mu}_n$, and of course each $\hat{\mu}_n^{\mathbb{P}^-}$, is in the space $\boldsymbol{L}_{f_n}$ and $\mathbb{P}\boldsymbol{L}_{f_n}$ respectively, which depends on the function f_n. Then, those are measures on different spaces and it makes no sense to speak of their limits.

For this reason, it is required that we work on the more abstract solenoidal manifold *Sol* whose construction is made in Section 1.1. Theorem 1.3 gives us that this space only depends on the homotopy class we are working on. Since we are working with a converging sequence of maps (f_n), we may assume that they all lie in the same homotopy class. Thus, we have for each n, the following:

$$\int_{\mathbb{T}^2} \lambda^-(f_n)\, d\mu = \int_{Sol} \lambda^-(Sf_n) d\boldsymbol{\mu}_n = \int_{\mathbb{P}Sol} \log \|D_x f_n \cdot v\|\, d\boldsymbol{\mu}_n^{\mathbb{P}^-}(\boldsymbol{x}, v), \tag{4.5}$$

where $\boldsymbol{\mu}_n$ is the unique Sf_n-invariant measure on *Sol* projecting on μ, and $\boldsymbol{\mu}_n^{\mathbb{P}^-}$ is the corresponding stable measure of $\mathbb{P}Sf_n$ on $\mathbb{P}Sol$, we have the same for f. This way, it makes sense to argue about the weak* limit of the measures $\boldsymbol{\mu}_n^{\mathbb{P}^-}$.

We turn now, and devote the rest of this Section, to the proof of Theorem E.

Let (f_n) be a sequence in $\mathcal{U}_1$ converging to $f \in \mathcal{U}_1$ in the C^1 topology. As remarked before we may, and do, assume that f_n are all in the same homotopy class. Let $\boldsymbol{\mu}_n$, $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_n^{\mathbb{P}^-}$, $\boldsymbol{\mu}^{\mathbb{P}^-}$ be the lifts to *Sol* and $\mathbb{P}Sol$ as before. Due to (4.5), to prove Theorem E, it suffices to show that $\boldsymbol{\mu}_n^{\mathbb{P}^-}$ converges weakly* to $\boldsymbol{\mu}^{\mathbb{P}^-}$. Finally, due to Prop. 1.7 this is equivalent to show that $\tilde{\boldsymbol{\mu}}_n^{\mathbb{P}^-}$ converges weakly* to $\tilde{\boldsymbol{\mu}}^{\mathbb{P}^-}$, where $\tilde{\boldsymbol{\mu}}_n^{\mathbb{P}^-}$ and $\tilde{\boldsymbol{\mu}}^{\mathbb{P}^-}$ are the lifts of $\boldsymbol{\mu}_n^{\mathbb{P}^-}$ and $\boldsymbol{\mu}^{\mathbb{P}^-}$ to $\mathbb{P}(\mathbb{R}^2 \times \Sigma)$.

We suppose that the Theorem does not hold, that is, that $\boldsymbol{\mu}_n^{\mathbb{P}^-}$ does not converge to $\boldsymbol{\mu}^{\mathbb{P}^-}$. Since the space of probabilities on a compact space is compact in the weak* topology, we assume, by passing to a subsequence if necessary, that $\boldsymbol{\mu}_n^{\mathbb{P}^-}$ converges to some $\boldsymbol{\mu}^{\mathbb{P}}$ different from $\boldsymbol{\mu}^{\mathbb{P}^-}$. Note that since f_n converges to f in the C^1 topology, $\mathbb{P}Sf_n$ converges to $\mathbb{P}Sf$ as well, thus as $\boldsymbol{\mu}_n^{\mathbb{P}^-}$ is $\mathbb{P}Sf_n$-invariant for each n this implies that $\boldsymbol{\mu}^{\mathbb{P}}$ must be $\mathbb{P}Sf$-invariant.

We start by claiming that the limit $\boldsymbol{\mu}^{\mathbb{P}}$ must have product disintegrations, i.e.

$$\boldsymbol{\mu}^{\mathbb{P}} = \int_{\mathbb{T}^2} \boldsymbol{\mu}_x \times \nu_x \, d\mu(x), \tag{4.6}$$

for some family of measures ν_x on $\mathbb{PR}^2$. Of course, to prove this claim we must study the convergence of the sequence $\boldsymbol{\mu}_n^{\mathbb{P}^-}$ to $\boldsymbol{\mu}^{\mathbb{P}}$. Even better, we define:

$$\begin{aligned} &I^2 = [0,1]^2, &&\tilde{\boldsymbol{\eta}}_n = \tilde{\boldsymbol{\mu}}_n|_{I^2\times\Sigma}, &&\tilde{\boldsymbol{\eta}}_n^{\mathbb{P}^-} = \tilde{\boldsymbol{\mu}}_n^{\mathbb{P}^-}|_{I^2\times\Sigma\times\mathbb{PR}^2},\\ &\tilde{\eta} = \tilde{\mu}|_{I^2}, &&\tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\mu}}|_{I^2\times\Sigma}, &&\tilde{\boldsymbol{\eta}}^{\mathbb{P}} = \tilde{\boldsymbol{\mu}}^{\mathbb{P}}|_{I^2\times\Sigma\times\mathbb{PR}^2}. \end{aligned} \tag{4.7}$$

Remember that, as stated in Remark 1.3, the projective bundle $\mathbb{P}(\mathbb{R}^2\times\Sigma)$ is trivial since $T\mathbb{T}^2 \equiv \mathbb{T}^2 \times \mathbb{R}^2$. Note that the boundary of $Q = I^2 \times \Sigma \times \mathbb{PR}^2$ has zero $\tilde{\boldsymbol{\mu}}^{\mathbb{P}}$-measure. Thus, from Prop. 1.7 and Remark 1.4, instead of studying the convergence of the sequence $\boldsymbol{\mu}_n^{\mathbb{P}^-}$ to $\boldsymbol{\mu}^{\mathbb{P}}$, we may study the convergence of $\tilde{\boldsymbol{\eta}}_n^{\mathbb{P}^-}$ to $\tilde{\boldsymbol{\eta}}^{\mathbb{P}}$. Indeed, if we take G the group as in (1.4) and i to be the identity in $\mathbb{PR}^2$ as in (1.13), clearly $Q = I^2 \times \Sigma \times \mathbb{PR}^2$ contains a fundamental domain of the $(\{G\} \times i)$-action. Thus, we obtain that $(\pi_G \times i)_* \tilde{\boldsymbol{\eta}}_n^{\mathbb{P}^-} = \boldsymbol{\mu}^{\mathbb{P}^-}$ and $(\pi_G \times i)_* \tilde{\boldsymbol{\eta}}^{\mathbb{P}} = \boldsymbol{\mu}^{\mathbb{P}}$.

The convenience of working with these measure instead is that the underlying space $I^2 \times \Sigma \times \mathbb{PR}^2$ is a product of compact spaces, which allows us to apply Lemma 4.4. Let us now evidence the details of the situation we have:

1. The measures $\tilde{\boldsymbol{\eta}}_n^{\mathbb{P}^-}$ on $I^2\times\Sigma\times\mathbb{PR}^2$ project to $\tilde{\boldsymbol{\eta}}_n$ on $I^2\times\Sigma$. As defined, the disintegrations of $\tilde{\boldsymbol{\eta}}_n^{\mathbb{P}^-}$ along each fiber $\{(\tilde{x},\boldsymbol{\omega})\}\times\mathbb{PR}^2$ are the Dirac measures on the Lyapunov subspaces $\delta_{E_n^-(\pi_{cov}(\tilde{x}))}$.

2. The projection of each $\tilde{\boldsymbol{\eta}}_n$ to I^2 is $\tilde{\eta}$, and the disintegration of $\tilde{\boldsymbol{\eta}}_n$ along $\{\tilde{x}\}\times\Sigma$ is $\tilde{\boldsymbol{\mu}}_{n,\tilde{x}}$ defined in (1.9) (the dependence on n comes, of course, from the dependence of (1.9) on the function f_n).

3. From the previous items, the projection of each $\tilde{\boldsymbol{\eta}}_n^{\mathbb{P}^-}$ to I^2 is also $\tilde{\eta}$, and the disintegration of $\tilde{\boldsymbol{\eta}}_n^{\mathbb{P}^-}$ along the fiber $\{\tilde{x}\}\times\Sigma\times\mathbb{PR}^2$ is $\tilde{\boldsymbol{\mu}}_{n,\tilde{x}}\times\delta_{E_n^-(\pi_{cov}(\tilde{x}))}$

4. The disintegrations $\tilde{\boldsymbol{\mu}}_{n,\tilde{x}}$ vary continuously with respect to $\tilde{x}\in I^2$, and $\tilde{\boldsymbol{\mu}}_{n,\tilde{x}}$ converges weakly* to $\tilde{\boldsymbol{\mu}}_{\tilde{x}}$ (Prop. 1.6).

These considerations show us that we are within the hypothesis of Lemma 4.4, with I^2 in place of X, Σ in place of Y, $\mathbb{PR}^2$ in place of Z, $\tilde{\boldsymbol{\eta}}_n^{\mathbb{P}^-}$ in place of μ^n, $\tilde{\boldsymbol{\eta}}^{\mathbb{P}^-}$ in place of μ, and $\tilde{\boldsymbol{\mu}}_{n,\tilde{x}}\times\delta_{E_n^-(\pi_{cov}(\tilde{x}))}$ in place of μ_x^n. Indeed, due to items 2 and 4 of the previous observations we can easily check that hypothesis 2 of Lemma 4.4 is satisfied in our case. In fact, due to the previous observations, along with the relation (1.9) and the fact that f_n converges to f on the C^1-topology, we may check that hypothesis 1 is also satisfied here. Hence, we obtain that the disintegrations $\tilde{\boldsymbol{\eta}}_{\tilde{x}}^{\mathbb{P}}$ of the limit measure $\tilde{\boldsymbol{\eta}}^{\mathbb{P}}$ along $\{\tilde{x}\}\times\Sigma\times\mathbb{PR}^2$ are product measures, given by:

$$\tilde{\boldsymbol{\eta}}^{\mathbb{P}} = \int_{I^2} \tilde{\boldsymbol{\mu}}_{\tilde{x}} \times \nu_{\tilde{x}}\, d\tilde{\eta}(\tilde{x}). \tag{4.8}$$

Consequently:

$$\begin{aligned}\boldsymbol{\mu}^{\mathbb{P}} = (\pi_G\times i)_*\tilde{\boldsymbol{\eta}}^{\mathbb{P}} &= \int_{I^2}(\pi_G\times i)_*(\tilde{\boldsymbol{\mu}}_{\tilde{x}}\times\nu_{\tilde{x}})\, d\tilde{\eta}(\tilde{x})\\ &= \int_{\mathbb{T}^2}\boldsymbol{\mu}_x\times\nu_x\, d\mu(x),\end{aligned} \tag{4.9}$$

where $x=\pi_{cov}(\tilde{x})$, and $\boldsymbol{\mu}_x=(\pi_G)_*\tilde{\boldsymbol{\mu}}_{\tilde{x}}$ does not depend on the choice of $\tilde{x}\in\pi_{cov}^{-1}(x)$ since $\tilde{\boldsymbol{\mu}}_{\tilde{x}}$ does not, which gives us (4.6).

With (4.6) proved, we now move from *Sol* to $\boldsymbol{L_f}$ in order to simplify the calculations. Thus, we take $\hat{\mu}^{\mathbb{P}}=(\Psi\times i)_*\boldsymbol{\mu}^{\mathbb{P}}$ on $\boldsymbol{L_f}\times\mathbb{PR}^2$, where $\Psi : Sol\to\boldsymbol{L_f}$ is the homeomorphism defined in Section 1.1. By the same argument as (4.9), we obtain:

$$\hat{\mu}^{\mathbb{P}} = \int_{\mathbb{T}^2}\hat{\mu}_x\times\nu_x\, d\mu(x).$$

Since $\boldsymbol{\mu}^{\mathbb{P}}$ is $\mathbb{P}Sf$-invariant and from the relation (1.13), the measure $\hat{\mu}^{\mathbb{P}}$ is $\mathbb{P}\hat{f}$-invariant. Lemma 4.1 gives us an $\hat{f}$-invariant measurable function $\hat{\rho} : \boldsymbol{L_f}\to[0,1]$ such that:

$$\hat{\mu}^{\mathbb{P}} = \int_{\boldsymbol{L_f}}\hat{\rho}(\hat{x})\delta_{E^+(\hat{x})} + (1-\hat{\rho}(\hat{x}))\delta_{E^-(\hat{x})}\, d\hat{\mu}(\hat{x}). \tag{4.10}$$

Since the ergodic components of $\hat{f}$ are exactly the pre-images by $(\pi_{ext})_*$ of the ergodic components of f, we have that $\hat{\rho}$ is constant on the fibers $\pi_{ext}^{-1}(x)$, i.e. there exists an f-invariant measurable function $\rho : \mathbb{T}^2\to[0,1]$ with $\hat{\rho}=\rho\circ\pi_{ext}$. Thus, since $E^-(\hat{x})$ is also constant on the fibers, we have:

$$\begin{aligned}\int_{L_f} (\rho \circ \pi_{ext})\delta_{E^+}\, d\hat{\mu} &= \hat{\mu}^{\mathbb{P}} - \int_{L_f} (1-\rho\circ\pi_{ext})\delta_{E^-}\, d\hat{\mu} \\ &= \int_{\mathbb{T}^2} \hat{\mu}_x \times \nu_x \, d\mu - \int_{\mathbb{T}^2} \hat{\mu}_x \times ((1-\rho)\delta_{E^-})\, d\mu \\ &= \int_{\mathbb{T}^2} \hat{\mu}_x \times (\nu_x - (1-\rho)\delta_{E^-})\, d\mu \\ &= \int_{L_f} \nu_{\pi_{ext}(\hat{x})} - (1-\rho\circ\pi_{ext})\delta_{E^-}\, d\hat{\mu}.\end{aligned}$$

Hence, for $\hat{\mu}$ almost every $\hat{x} \in \boldsymbol{L_f}$, it holds:

$$(\rho \circ \pi_{ext}(\hat{x}))\delta_{E^+(\hat{x})} = \nu_{\pi_{ext}(\hat{x})} - (1-\rho\circ\pi_{ext}(\hat{x}))\delta_{E^-(\hat{x})}. \tag{4.11}$$

Since $\boldsymbol{\mu}^{\mathbb{P}}$ is different from $\boldsymbol{\mu}^{\mathbb{P}^-}$ we must have $\hat{\mu}^{\mathbb{P}}$ different from $\hat{\mu}^{\mathbb{P}^-}$, which from (4.10) and the definition of $\hat{\mu}^{\mathbb{P}^-}$ implies that $\rho(x) > 0$ in a set A of positive μ-measure on $\mathbb{T}^2$. Since the right hand side of (4.11) is constant on every fiber by π_{ext}, we have that for $x \in A$, $E^+(\hat{x})$ is constant on $\pi_{ext}^{-1}(x)$, that is $E^+(\hat{x}) = [v^+(x)] \in \mathbb{P}\mathbb{R}^2$ for a unit vector $v^+(x) \in \mathbb{R}^2$ which only depends on $x \in A$.

Finally, we remember that the set of regular points $\hat{x} \in \hat{\mathcal{R}}$ where $\lambda^+(\hat{x}) = \lambda^+(\pi_{ext}(x))$ has full $\hat{\mu}$ measure. Thus, for μ-a.e. $x \in A$, we must have that $\hat{\mu}_x$-a.e. $\hat{x} \in \pi_{ext}^{-1}(x)$ is regular, we assume that this actually holds for every $x \in A$ (by removing a zero measured subset if necessary). Thus, for every $x \in A$, we have:

$$\begin{aligned}\lim_{k\to\infty} \frac{I(x, v^+(x), f^k)}{k} &= \lim_{k\to\infty} \frac{1}{k} \int_{\pi_{ext}^{-1}(x)} \log \|(D_{\hat{x}}\hat{f})^{-k} \cdot v^+(\pi_{ext}(\hat{x}))\|\, d\hat{\mu}_x \\ &= \int_{\pi_{ext}^{-1}(x)} \lim_{k\to\infty} \frac{1}{k} \log \|(D_{\hat{x}}\hat{f})^{-k} \cdot v^+(\pi_{ext}(\hat{x}))\|\, d\hat{\mu}_x \\ &= \int_{\pi_{ext}^{-1}(x)} -\lambda^+(\hat{x})\, d\hat{\mu}_x = -\lambda^+(x).\end{aligned}$$

From this, with the fact that $C_{\mathcal{X}}(f)$ may be taken as a limit (2.4), we obtain that for every $x \in A$:

$$\lambda^+(x) = -\lim_{k\to\infty} \frac{I(x, v^+(x), f^k)}{k} \leq -C_{\mathcal{X}}(f).$$

Even more, we know from Theorem 2.1 that for μ-a.e. $x \in \mathbb{T}^2$:

$$\lambda^-(x) \leq -C_{\mathcal{X}}(f).$$

Hence, for μ-a.e. $x \in A$, which implies that in a positive μ-measured set, we have:

$$-2C_{\mathcal{X}}(f) \geq \lambda^+(x) + \lambda^-(x) = \lim_{n\to infty} \log(\det(D_x f^n)) \geq C_{\det}(f),$$

which contradicts the fact that $f \in \mathcal{U}_1 = \{g \in End^1_\mu(\mathbb{T}^2) \; : \; C_{\mathcal{X}}(g) > -\frac{1}{2}C_{\det}(g)\}$. This completes the proof of Theorem E.

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