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A CONTROL CHART FOR THE JOINT MONITORING OF MEAN AND VARIANCE WITH KLEIN'S SUPPLEMENTARY RULE

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Orientador: Prof.Dr. Marcelo Azevedo Costa
Coorientador: Prof.Dr. Anderson Laécio Galindo Trindade

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FOLHA DE APROVAÇÃO

Uma carta de controle para o monitoramento conjunto da média e variância utilizando a regra suplementar de Klein

VÍCTOR BRITO QUININO

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Com sincera gratidão,

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Resumo

Regras suplementares de execução têm sido utilizadas desde a década de 1950 para melhorar o desempenho de cartas de controle. Em uma publicação anterior, Quinino et al.(QUININO et al., 2023), sugerimos uma melhoria para um procedimento conhecido como "Klein's 2-of-2", uma regra suplementar simples e eficiente que sinaliza que o processo está fora de controle apenas quando duas médias sequenciais de amostras são observadas acima do limite de controle superior (UCL) ou abaixo do limite de controle inferior (LCL). O uso dessa regra resulta em uma redução substancial no *average run length* (ARL) na detecção de pequenos deslocamentos no processo, embora essa melhoria tenha sido atestada apenas em comparação com a carta de controle tradicional $\bar{X}$, que visa apenas monitorar a média de um processo. Este trabalho propõe implementar o "Klein's 2-of-2" para um tipo mais complexo de carta de controle, a carta de controle conjunta $\bar{X} - S^2$, que visa monitorar simultaneamente a média e a variância de um processo, uma aplicação importante em processos de alta qualidade. O uso dessa regra para a carta $\bar{X} - S^2$ resulta em uma redução substancial no *average run length* (ARL) na detecção de pequenos deslocamentos no processo, em comparação com a carta de controle tradicional $\bar{X} - S^2$, mantendo também um alto nível de aplicabilidade devido à facilidade de uso da regra 2-of-2 em ambientes práticos. A implementação da regra de execução de Klein para a carta $\bar{X} - S^2$ foi realizada por meio de uma Cadeia de Markov de 15 estados, e seu desempenho foi comparado tanto com a carta de controle básica $\bar{X} - S^2$ quanto com duas aplicações da média móvel ponderada exponencialmente (EWMA), conhecida por ter excelentes capacidades na detecção de pequenos deslocamentos médios do processo, ao custo de um esquema de controle mais complexo, proporcionando um desempenho competitivo mesmo em comparação com a carta de controle EWMA mais complexa.

Palavras-chave: cartas de controle; regras suplementares; ARL; cadeia de Markov.

Abstract

Supplementary run-rules have been used since the 1950s to improve the performance of control charts. In a prior publication, Quinino et al.(QUININO et al., 2023), we have suggested an improvement for a procedure known as Klein's 2-of-2, a simple and efficient supplementary rule which signals that the process is out-of-control only when two sequential sample averages are observed above the upper control limit (UCL) or below the lower control limit (LCL). The use of this rule results in a substantial reduction of the average run length (ARL) in detecting small process shifts, although this improvement has only been attested when compared to the traditional $\bar{X}$ control chart, which aims to only monitor a process's mean. This work proposes to implement Klein's 2-of-2 for a more complex type of control chart, the $\bar{X}-S^2$ joint control chart, which aims to monitor both the mean and the variance of a process simultaneously, an important application in high quality processes. The use of this rule for the $\bar{X}-S^2$ control chart results in a substantial reduction of the average run length (ARL) in detecting small process shifts when compared to the traditional $\bar{X}-S^2$ control chart while also maintaining a high level of applicability due to the 2-of-2 rule's ease of use in practical environments. The implementation of Klein's run rule for the $\bar{X}-S^2$ control chart was done through a 15 state Markov Chain, and its performance was compared to both the basic $\bar{X}-S^2$ control chart as well as two applications of the exponentially weighted moving average (EWMA), known to have excellent capabilities in detecting small process average shifts at the cost of a more complex control scheme, providing a competitive performance even when compared to the more complex EWMA control chart.

Keywords: *control charts; run rules; ARL; Markov chain*.

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1 Introduction

Controlling the process mean and variance has been a challenge since the industrialization of our society, as they are a fundamental component of quality control in most processes, allowing for a consistent production and alerting an industry of significant changes. How to control these variables is a discussion with hundreds of years of history, but one of the most popular and efficient ways to do so are by using control charts. They are graphical displays created to allow a practitioner to determine whether a process is in-control or out-of-control by collecting samples at specified intervals and representing the values of some statistics on a graphic which includes decision lines, called control limits. The main performance indicator for control charts is the Average Run Length (ARL), defined by Montgomery (MONTGOMERY, 2020) as the average number of points that must be plotted before a point indicates an out-of-control condition. This performance indicator is normally divided in two metrics: 1- An in-control ARL, ARL_0, that is the average number of samples taken until the control chart indicates that process is out-of-control when it is not, a false alarm; 2- An out-of-control ARL, ARL_1, that is the average number of samples collected after the process is out-of-control, for the control chart to indicate that the process is indeed out-of-control. Normally the ARL_0 is fixed at certain values, usually $250, 370$ or 500, and the ARL_1 while maintaining the previously defined $\text{ARL}_0 s$ represents the performance of the control chart.

The first control chart aiming to control a process mean was proposed in the 1930's by Walter A. Shewhart in his seminal book (SHEWHART, 1932), where he described what ended up being the standard control chart used to control a process mean to this day, the $\bar{X}$ control chart. Using the same principles of the $\bar{X}$ control chart we can construct a control chart to monitor the process variance, named S^2 control chart. Notice that both the $\bar{X}$ and the S^2 control charts aim to monitor a single parameter, the mean or the variance, respectively. As it is often desirable to control both of these parameters at the same time, these two control charts are almost always used together, as noted in Gan (GAN, 1995). The problem with this approach, as mentioned by Gan (GAN, 1995), is that we are essentially looking at a bi-variate problem using two uni-variate procedures, since both parameters can shift at the same time and changes in the variance can affect the control limits of the mean chart. Thus, when using both control charts together, a common practice, it is necessary to adjust their control limits to account for the fact that we are considering a set of statistical inferences simultaneously, what is called the multiple comparisons problem.

Control schemes that aim to link these two control charts are called joint monitoring control schemes for the mean and variance. According to Chao and Chang (CHAO; CHENG, 1996), a control chart should be "simple to use, easy to understand, and quick to implement" while also clearly indicating which parameter is out-of-control, as added in McCracken and

Chakraborti (MCCRACKEN; CHAKRABORTI, 2013). Hence, one of the simplest joint monitoring tools used widely nowadays is the $\bar{X} - S^2$ control chart, where the two individual control charts, $\bar{X}$ and S^2, are used at the same time with adjusted limits. This control chart and its details will be discussed below.

The $\bar{X} - S^2$ control chart is well-known for its use in the joint monitoring of the mean and variance of processes in which the quality characteristic of interest (X) follows a normal distribution with mean μ and standard deviation σ. The process is considered in-control when $\mu = \mu_0$, the mean is equal to an in-control mean, and $\sigma = \sigma_0$, the standard deviation is equal to an in-control standard deviation. The process is considered out-of-control when the mean μ shifts from its target value, μ_0, to an out-of-control $\mu_1 = \mu_0 + \delta\sigma_0$, that is, it shifts from an in-control state to an out-of-control state by δ standard deviations, where $\delta \neq 0$. The process may also be considered out-of-control if the standard deviation shifts from its target value of σ_0 to an out-of-control $\sigma_1 = \gamma\sigma_0$, that is, it shifts from an in-control state to an out-of-control state by γ standard deviations, where $\gamma > 1$. Costa and Rahim(COSTA; RAHIM, 2004) discuss that it is usually assumed that the primary interest in variance quality control is in detecting increases in σ, since an increase corresponds to deterioration in quality, therefore $\gamma > 1$.

As previously stated, the $\bar{X} - S^2$ control chart consists of two control charts, the $\bar{X}$ control chart and the S^2 control chart. The $\bar{X}$ control chart is the most commonly used tool for controlling a process mean in the industry, while the S^2 control chart is considered the most popular method for controlling the process variance. They are so ubiquitous due to their simplicity and ability to detect large shifts quickly. However, both the $\bar{X}$ and the S^2 control charts lack sensitivity in detecting small sustained shifts in the mean or variance, respectively, that is, they present a high ARL_1 value for small shifts in the mean, δ, or small shifts in the variance, γ. To improve their sensitivity, several authors have proposed alternative control chart approaches, such as the exponentially weighted moving average (EWMA), proposed in Roberts (ROBERTS, 1959), cumulative sum (CUSUM) proposed in Page (PAGE, 1961), and double exponentially weighted moving average (DEWMA), presented in Shamma and Shamma (SHAMMA; SHAMMA, 1992). Such methods do present high sensitivity to detect small shifts in the process mean(variance), small $\delta(\gamma)$ values, but are often considered too complex for implementation in the industry, as discussed in Klein (KLEIN, 2000), Haq and Woodall (HAQ; WOODALL, 2023) and harking back to Gan's (GAN, 1995) statement about a control chart's simplicity, ease to implement and be understood. Aiming to improve the $\bar{X}$ and S^2 control chart's sensitivity while maintaining their already established and widely used system, supplementary run rules have been suggested throughout the history of quality control. Run rules are a set of new procedures which aim to increase the sensitivity of a control chart, according to Koutras et al. (KOUTRAS; BERSIMIS; MARAVELAKIS, 2007)

Different types of run rules, as described in Shmueli and Cohen (SHMUELI; COHEN, 2003), Champ (CHAMP, 1992), and Walker et al. (WALKER; PHILPOT; CLEMENT, 1991), have been proposed and compared in Palm (PALM, 1990). More recently, researchers such as Rocha et al. (ROCHA; MEDEIROS; HO, 2015), Kim and Cho (KIM; CHO, 2020), Adeoti and Malela-Majika (ADEOTI; MALELA-MAJIKA, 2020), Tran (TRAN, 2018), Ruiz-Tamayo et al. (RUIZ-TAMAYO et al., 2021), Malela-Majika et al. (MALELA-MAJIKA; MALANDALA; GRAHAM, 2018) and Karavigh and Amiri (KARAVIGH; AMIRI, 2022) have also suggested different complex run rules aiming to improve the performance of control charts. Jalilibal et al. (JALILIBAL et al., 2023) performs a recent literature review on run rules schemes for statistical process monitoring. Unfortunately, as discussed in Klein (KLEIN, 2000), the $\bar{X}$ and S^2 control charts without any run rule are still the most used tools for quality control of a process mean (variance) in practice, possibly because of a general perception that the implementation of modern methodologies with excellent ARL_1 values is too complex and difficult to interpret in an industrial environment.

To address this difficulty, simpler run rules have been proposed, that is, run rules which feature procedures that do not differ much from the standard $\bar{X}$ and S^2 control charts. One of the most prominent of these is the 2-of-2 rule proposed by Klein (KLEIN, 2000). In this rule, the process is only signaled as out-of-control when two successive points are above an upper control limit (UCL) or two successive points are below a lower control limit (LCL), thus improving the control chart's sensibility to small shifts. This paper proposes to apply the 2-of-2 rule to the $\bar{X}$ – S^2 control chart in order to improve its ability to detect small shifts in the mean and variance while maintaining its simplicity, as, according to Hurwitz and Mathur (HURWITZ; MATHUR, 1992), the 2-of-2 rule is simple and well-accepted in an industrial environment, avoiding operational difficulties.

The remainder of the article is organized as follows. Section 2 develops the newly proposed $\bar{X}$ – S^2 control chart with klein's 2-of-2 run rule. Section 3 discusses the performance of the newly proposed control chart when compared to the standard $\bar{X}$ – S^2 control chart, in section 3.1, and the exponentially weighted moving average (EWMA) control scheme in section 3.2. An illustrative numerical example is the objective of Section 4. Section 5 closes this article with a discussion and final remarks. Finally, all seven programs, developed in R and identified from A to G, utilized throughout this paper are included in Appendix A to G.

2 The improved $\bar{X} - S^2$ control chart

As discussed in section 1, the $\bar{X} - S^2$ control chart is widely used in the joint monitoring of the mean and variance of processes. It consists of two different control charts, the $\bar{X}$ and the S^2 control charts, with control limits such that the global ARL_0, that is, average number of samples collected until the $\bar{X} - S^2$ control chart indicates that the process is out-of-control when it is not, is fixed at a desired number, usually 370.4.

For the $\bar{X}$ chart, the means $\bar{x}$ of random samples of size n are computed and compared with the upper control limit, $\text{UCL}^*_{\bar{x}} = \mu_0 + Z_{1-\alpha/2}\dfrac{\sigma}{\sqrt{n}}$, and lower control limit, $\text{LCL}^*_{\bar{x}} = \mu_0 - Z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}}$, where $Z_{1-\alpha/2}$ is the $100(1-\alpha/2)\%$ percentile of the standard normal distribution. If $LCL_{\bar{X}} \leq \bar{X} \leq UCL_{\bar{X}}$, the process is said to be in control, otherwise the process is considered out-of-control.

For the S^2 control chart, the variances s^2 of each sample of size n and the statistics $\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$ are computed and compared to an upper control limit $\text{UCL}^*_{\chi^2} = \chi^2_{1-\alpha^*;n-1}$, where $\chi^2_{1-\alpha^*;n-1}$ is the $100(1-\alpha^*)\%$ percentile of the chi-squared distribution, with $n-1$ degrees of freedom. If $\chi^2 \leq \chi^2_{1-\alpha^*;n-1}$, the process is said to be in-control, otherwise the process is considered out-of-control, notice that we do not establish a lower control limit (LCL) for this control chart, harking back to Costa and Rahim(COSTA; RAHIM, 2004)'s discussion about the primary interest in quality control being to detect increases in the variance, σ^2.

As previously stated, to use the joint $\bar{X} - S^2$ control chart it is necessary to calibrate the false-alarm rate of each individual control chart (the $\bar{X}$ and S^2 control charts) in such a way that the in-control joint average run length (ARL_0) is as planned. Considering the false-alarm rate of the $\bar{X}$ control chart to be given by α and the false-alarm rate of the S^2 control chart to be given by α^*, the false-alarm rate of the $\bar{X} - S^2$ control chart is given by $[1-(1-\alpha)(1-\alpha^*)]$, where usually $\alpha = \alpha^*$, since the joint control chart would be out-of-control if any combination of the $\bar{X}$ and S^2 is out-of-control. As an example, if we seek an $ARL_0 \approx 370$, we would need a false-alarm rate of 0.00135081 for both the $\bar{X}$ and S^2 control charts, so that $ARL_0 = \frac{1}{1-(1-0.00135081)^2} \approx 370$. In this perspective, we will name the upper control limit and lower control limit for the mean control chart as $UCL_{\bar{X}}$ and $LCL_{\bar{X}}$, respectively, and the control limit for the variance control chart as UCL_{S^2}.

In this paper we aim to improve the performance of said $\bar{X} - S^2$ control chart by using a supplementary run rule, proposed in Klein (KLEIN, 2000), named the 2-of-2 rule. It proposes that an action must be taken only if there is a sequence of two $\bar{X}$ lying on the same side beyond the control limits (above or below), that is, if $\bar{X}_{i-1} > UCL^{Klein}_{\bar{X}}$ and $\bar{X}_i > UCL^{Klein}_{\bar{X}}$ or $\bar{X}_{i-1} < LCL^{Klein}_{\bar{X}}$ and $\bar{X}_i < LCL^{Klein}_{\bar{X}}$ or $\frac{S^2_{i-1}(n-1)}{\sigma_0^2} > UCL^{Klein}_{S^2}$ and $\frac{S^2_{i}(n-1)}{\sigma_0^2} > UCL^{Klein}_{S^2}$. This simple and efficient run rule can be applied to the $\bar{X} - S^2$ control chart in order to improve its detection of small shifts in the mean or variance while maintaining a similar level of complexity to that of the standard joint control chart, therefore being a simpler and more feasible improvement to be applied in practice.

The proposed procedure to implement said supplementary run rule in the $\bar{X} - S^2$ control chart can be described by a Markov chain with 15 states, **E**=[$XC - SC; XS - SC; XSS - SC; XI - SC; XII - SC; XC - SF; XS - SF; XSS - SF; XI - SF; XII - SF; XC - SFF; XS - SFF; XSS - SFF; XI - SFF; XII - SFF$], defined in Table 1.

Let p_{xc} be the probability of a sample's mean value being between the control limits $UCL_{\bar{X}}^{Klein}$ and $LCL_{\bar{X}}^{Klein}$; p_{xu} be the probability of a sample's mean value being above the upper control limit $UCL_{\bar{X}}^{Klein}$; p_{xl} be the probability of a sample's mean value being below the lower control limit $LCL_{\bar{X}}^{Klein}$; p_{sc} the probability of a sample's χ^2 being below the control limit $UCL_{S^2}^{Klein}$ and p_{sf} the probability of a sample's χ^2 being above the control limit $UCL_{S^2}^{Klein}$.

The transition matrix can be expressed by the following matrix Q. Notice that the elements of the matrix are the conditioned probabilities, that is, the probability of reaching a state of **E** depends only on the previous state. This is known as a markovian property.

Q =

	$XC-SC$	$XS-SC$	$XSS-SC$	$XI-SC$	$XII-SC$	$XC-SF$	$XS-SF$	$XSS-SF$	$XI-SF$	$XII-SF$	$XC-SFF$	$XS-SFF$	$XSS-SFF$	$XI-SFF$	$XII-SFF$
$XC-SC$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0	0	0	0	0	0
$XS-SC$	$p_{xc}*p_{sc}$	0	$p_{xu}*p_{sc}$	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	0	$p_{xu}*p_{sf}$	$p_{xl}*p_{sf}$	0	0	0	0	0	0
$XSS-SC$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0	0	0	0	0	0
$XI-SC$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	0	$p_{xl}*p_{sc}$	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	0	$p_{xl}*p_{sf}$	0	0	0	0	0
$XII-SC$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0	0	0	0	0	0
$XC-SF$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	0	0	0	0	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0
$XS-SF$	$p_{xc}*p_{sc}$	0	$p_{xu}*p_{sc}$	$p_{xl}*p_{sc}$	0	0	0	0	0	0	$p_{xc}*p_{sf}$	0	$p_{xu}*p_{sf}$	$p_{xl}*p_{sf}$	0
$XSS-SF$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0	0	0	0	0	0
$XI-SF$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	0	$p_{xl}*p_{sc}$	0	0	0	0	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	0	$p_{xl}*p_{sf}$
$XII-SF$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0	0	0	0	0	0
$XC-SFF$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0	0	0	0	0	0
$XS-SFF$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0	0	0	0	0	0
$XSS-SFF$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0	0	0	0	0	0
$XI-SFF$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0	0	0	0	0	0
$XII-SFF$	$p_{xc}*p_{sc}$	$p_{xu}*p_{sc}$	0	$p_{xl}*p_{sc}$	0	$p_{xc}*p_{sf}$	$p_{xu}*p_{sf}$	0	$p_{xl}*p_{sf}$	0	0	0	0	0	0

In order to improve the reader's understanding, Table 1 explains each state in the transition matrix.

Observe that states $3-XSS-SC$, $5-XII-SC$, $8-XSS-SF$, $10-XII-SF$, $11-XC-SFF$, $12-XS-SFF$, $13-XSS-SFF$, $14-XI-SFF$ and $15-XII-SFF$, marked in **bold**, indicate situations in which the process is adjusted.

The values of the probabilities p_{xc}, p_{xu}, p_{xl}, p_{sc} and p_{sf} depends on the upper control limit, $UCL_{\bar{X}}^{Klein}$, and lower control limit, $LCL_{\bar{X}}^{Klein}$ for the $\bar{X}$ control chart and $UCL_{S^2}^{Klein}$ for the S^2 control chart. Observe that the transition probabilities in Q are simple multiplications, since: 1) The samples of size n used for the control process are collected independently and with identical distribution (iid); ii) The sample's mean $\bar{X}$ is independent of the sample's variance S^2, as observed in Chen et al.(CHEN; CHENG; XIE, 2001).

Table 1 – Explanation of each state

State	Explanation
$1-XC-SC$	$UCL_{\bar{X}}^{Klein} \leq \bar{X}_i \leq LCL_{\bar{X}}^{Klein}$; $\frac{S_i^2(n-1)}{\sigma_0^2} \leq UCL_{S^2}^{Klein}$
$2-XS-SC$	$[\bar{X}_i \geq UCL_{\bar{X}}^{Klein} \cap \bar{X}_{i-1} \leq UCL_{\bar{X}}^{Klein}]$; $\frac{S_i^2(n-1)}{\sigma_0^2} \leq UCL_{S^2}^{Klein}$
3- XSS-SC	$[\bar{X}_{i-1} \cap \bar{X}_i] \geq UCL_{\bar{X}}^{Klein}$; $\frac{S_i^2(n-1)}{\sigma_0^2} \leq UCL_{S^2}^{Klein}$
$4-XI-SC$	$[\bar{X}_i \leq LCL_{\bar{X}}^{Klein} \cap \bar{X}_{i-1} \geq LCL_{\bar{X}}^{Klein}]$; $\frac{S_i^2(n-1)}{\sigma_0^2} \leq UCL_{S^2}^{Klein}$
5- XII-SC	$[\bar{X}_{i-1} \cap \bar{X}_i] \leq LCL_{\bar{X}}^{Klein}$; $\frac{S_i^2(n-1)}{\sigma_0^2} \leq UCL_{S^2}^{Klein}$
$6-XC-SF$	$UCL_{\bar{X}}^{Klein} \leq \bar{X}_i \leq LCL_{\bar{X}}^{Klein}$; $\frac{S_i^2(n-1)}{\sigma_0^2} \geq UCL_{S^2}^{Klein}$
$7-XS-SF$	$[\bar{X}_i \geq UCL_{\bar{X}}^{Klein} \cap \bar{X}_{i-1} \leq UCL_{\bar{X}}^{Klein}]$; $\frac{S_i^2(n-1)}{\sigma_0^2} \geq UCL_{S^2}^{Klein}$
8- XSS-SF	$[\bar{X}_{i-1} \cap \bar{X}_i] \geq UCL_{\bar{X}}^{Klein}$; $\frac{S_i^2(n-1)}{\sigma_0^2} \geq UCL_{S^2}^{Klein}$
$9-XI-SF$	$[\bar{X}_i \leq LCL_{\bar{X}}^{Klein} \cap \bar{X}_{i-1} \geq LCL_{\bar{X}}^{Klein}]$; $\frac{S_i^2(n-1)}{\sigma_0^2} \geq UCL_{S^2}^{Klein}$
10- XII-SF	$[\bar{X}_{i-1} \cap \bar{X}_i] \leq LCL_{\bar{X}}^{Klein}$; $\frac{S_i^2(n-1)}{\sigma_0^2} \geq UCL_{S^2}^{Klein}$
11- XC-SFF	$UCL_{\bar{X}}^{Klein} \leq \bar{X}_i \leq LCL_{\bar{X}}^{Klein}$; $[\frac{S_{i-1}^2(n-1)}{\sigma_0^2} \cap \frac{S_i^2(n-1)}{\sigma_0^2}] \geq UCL_{S^2}^{Klein}$
12- XS-SFF	$[\bar{X}_i \geq UCL_{\bar{X}}^{Klein} \cap \bar{X}_{i-1} \leq UCL_{\bar{X}}^{Klein}]$; $[\frac{S_{i-1}^2(n-1)}{\sigma_0^2} \cap \frac{S_i^2(n-1)}{\sigma_0^2}] \geq UCL_{S^2}^{Klein}$
13- XSS-SFF	$[\bar{X}_{i-1} \cap \bar{X}_i] \geq UCL_{\bar{X}}^{Klein}$; $[\frac{S_{i-1}^2(n-1)}{\sigma_0^2} \cap \frac{S_i^2(n-1)}{\sigma_0^2}] \geq UCL_{S^2}^{Klein}$
14- XI-SFF	$[\bar{X}_i \leq LCL_{\bar{X}}^{Klein} \cap \bar{X}_{i-1} \geq LCL_{\bar{X}}^{Klein}]$; $[\frac{S_{i-1}^2(n-1)}{\sigma_0^2} \cap \frac{S_i^2(n-1)}{\sigma_0^2}] \geq UCL_{S^2}^{Klein}$
15- XII-SFF	$[\bar{X}_{i-1} \cap \bar{X}_i] \leq LCL_{\bar{X}}^{Klein}$; $[\frac{S_{i-1}^2(n-1)}{\sigma_0^2} \cap \frac{S_i^2(n-1)}{\sigma_0^2}] \geq UCL_{S^2}^{Klein}$

Without loss of generality, standardized values of the quality characteristic, in which the data follows a standard normal distribution $N(\mu_0 = 0, \sigma_0 = 1)$ when in-control and a distribution $N(\mu_1 = \mu_0 + \delta\sigma_0, \sigma_1 = \gamma\sigma_0)$ when out-of-control, are used to quantify the ARL_1 values in this paper in relation to shifts in the mean of δ and shifts in the standard deviation of of a multiplication factor of γ.

In order to calculate the ARL_1 values for the $\bar{X} - S^2$ joint control chart with the 2-of-2 run rule, it is necessary to evaluate the percentage of time, when the system is in equilibrium, in which the process stays in each of the 15 states described in table 1. This can be calculated through the stationary distribution given by $\pi = \{\pi_1, \pi_2, \pi_3, \dots \pi_{15}\}$ with $\pi = \pi\mathbf{Q}$. Solving the system of equations with the restriction $\sum_{i=1}^{15} \pi_i = 1$ yields π. We highlight that the matrix Q is irreducible and aperiodic. That is: i) It is possible to go from any state to any other state (positive probability) in a finite number of n-steps, Q^n, guaranteeing irreducibility; ii) There are no fixed cycles where the process always returns to the same set of states in fixed intervals, guaranteeing that the the chain is aperiodic, that is, the greatest common divisor of all return paths to a specific state is equal to 1. Note that period is a property of a class, and since the chain is irreducible (having only one class), it suffices to evaluate a single state. For example, state XC-SC in Q connects to itself with a positive probability, ensuring a period equal to 1, and consequently, all states have a period equal to 1. According to Grimmett and Stirzaker (GRIMMETT; STIRZAKER, 2020), a Markov Chain that is irreducible and aperiodic, with a finite amount of states, guarantees the existence of a single stationary state being known as an ergodic chain. Consequently, Q does posses a stationary state. For further details, including computational discussions using the R software, refer to Spedicato (SPEDICATO,).

The long term probability of an adjustment of the process is expressed as:

$$\Theta = \pi_3 + \pi_5 + \pi_8 + \pi_{10} + \pi_{11} + \pi_{12} + \pi_{13} + \pi_{14} + \pi_{15}. \tag{2.1}$$

Which is the sum of the probabilities to be in a state where the process is considered out-of-control. The states where the process is considered out-of-control are denoted in **bold** in Table 1.

When the process is in-control the value of Θ is calculated with the hypothesis that $\delta = 0$ and $\gamma = 1$, being defined as Θ_0. When the process is out-of-control the value of Θ will be calculated with the hypothesis that $\delta \neq 0$ and/or $\gamma > 1$, being defined as Θ_1.

With control limits $UCL_{\bar{X}}^{Klein}$, $LCL_{\bar{X}}^{Klein}$ and $UCL_{S^2}^{Klein}$ and considering a geometric distribution with probability of success given by the Eqs. (2.1), we are able to determine the values of ARL_0 and ARL_1 in relation to δ and γ, expressed respectively by:

$$\text{ARL}_0(\delta = 0; \gamma = 1) = \frac{1}{\Theta_0} \tag{2.2}$$

and

$$\text{ARL}_1(\delta \neq 0; \gamma > 1) = \frac{1}{\Theta_1}. \tag{2.3}$$

Note that the control limits $UCL_{\bar{X}}^{Klein}$, $LCL_{\bar{X}}^{Klein}$ and $UCL_{S^2}^{Klein}$ are determined in such a way to obtain a desired ARL_0 value, usually 250, 370.4 or 500. For example, if we set $\text{ARL}_0 = 370.4$ and use $n = 5$ with standardized data, then $UCL_{\bar{X}}^{Klein} = 0.87822$, $LCL_{\bar{X}}^{Klein} = -0.87822$ and $UCL_{S^2}^{Klein} = 10.051$. If we use these limits, we would have a value of $\text{ARL}_1 = 93.379$ for $\delta = 0.25$ and $\gamma = 1.05$. Using the traditional $\bar{X} - S^2$ control chart with the same definitions of ARL_0, n, δ and γ we would have $UCL_{\bar{X}} = 1.4309$, $LCL_{\bar{X}} = -1.4309$, $UCL_{S^2} = 17.842$ and $\text{ARL}_1 = 112.44$.

In order to motivate practitioners, ensure reproducibility of results and facilitate the expansion of research, two programs, developed in R, to find the control limits and the ARL of the standard $\bar{X} - S^2$ control chart and the newly proposed version are included in the supplementary material available in the Appendix, identified as appendix A and B, respectively.

3 Evaluating the performance of the improved $\bar{X} - S^2$ control chart

This Chapter consists of 2 sections. In Sec. 3.1, we compared the performance of the control chart proposed in this paper with the traditional $\bar{X} - S^2$ approach.

In Sec. 3.2, we compare the proposal of this work with a control chart using the exponentially weighted moving average (EWMA) control scheme, which is a complex but powerful alternative for detecting small shifts in both the mean and variance. The use of the EWMA control scheme coupled with Klein's 2-of-2 supplementary run rule is also discussed.

3.1 Comparison to the standard $\bar{X} - S^2$ control chart

Table 3 compares, in terms of ARL_1 and with an standard $ARL_0 = 370.4$, the performance of the standard $\bar{X} - S^2$ control chart to that of the proposed $\bar{X} - S^2$ control chart with Klein's 2-of-2 rule applied. In order to have a significant comparison, sample sizes of $n = [4, 5, 6, 7]$, shifts in the mean of $\delta = [0, 0.25, 0.5, 0.75, 1, 1.25, 1.5]$ and shifts in the variance of $\gamma = [1, 1.05, 1.1, 1.2, 1.3, 1.4, 1.5]$ were utilized. Additionally, Table 2 presents the control limits for each case discussed in Table 3.

Table 2 – Control limits for the $\bar{X} - S^2$ control chart for an ARL_0 of 370.4 and different sample sizes (n)

	$n = 4$	$n = 5$	$n = 6$	$n = 7$
$\text{UCL}_{\bar{X}}$	1.603	1.434	1.309	1.212
$\text{LCL}_{\bar{X}}$	-1.603	-1.434	-1.309	-1.212
UCL_{S^2}	15.621	17.791	19.811	21.729
$\text{UCL}_{\bar{X}}^{\text{Klein}}$	0.982	0.878	0.802	0.742
$\text{LCL}_{\bar{X}}^{\text{Klein}}$	-0.982	-0.878	-0.802	-0.742
$\text{UCL}_{S^2}^{\text{Klein}}$	8.335	10.051	11.671	13.227

The results presented in table 3 were obtained through the programs A and B available in the appendix. Program A calculates the ARL_1 values and control limits for the $\bar{X} - S^2$ control chart, while Program B calculates the ARL_1 values and control limits for the $\bar{X} - S^2$ control chart with Kleint's 2-of-2 rule, utilizing a Markov Chain approach.

Aiming to verify the coherence of the results presented in table 3 for the Markov Chain approach of the $\bar{X} - S^2$ control chart with Klein's 2-of-2 rule, we have also developed a program

C, also available in the Appendix, where results are obtained through Monte Carlo Simulations for the $\bar{X}$ – S^2 control chart with Klein's 2-of-2 rule. We have verified that the ARL_1 results obtained through a Monte Carlo simulation approach with 1000 different replicas, each with one million runs, fluctuate around the results obtained through the Markov Chain approach for the $\bar{X}$ – S^2 control chart with Klein's 2-of-2 rule. As an illustrative example: for n = 5, δ = 0.75 and γ = 1.5, the Markov Chain approach results in a value of ARL_1 = 5.287; Employing a Monte Carlo simulation featuring 1000 replicas with one million runs each, we obtain results that fluctuate around an ARL_1 of 5.287 following a normal distribution, as shown in Figure 1. All other values present in Table 3 were also verified through the same simulation approach and reach similar results.

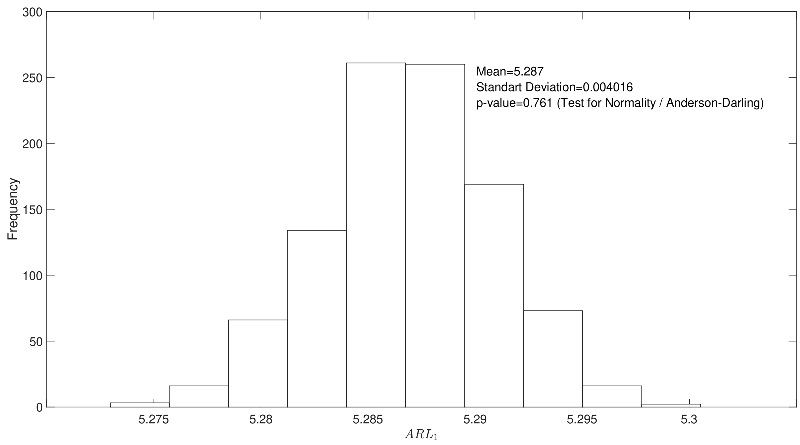


Figure 1 – Results obtained through a Monte Carlo simulation approach for n = 5; δ = 0.75; γ = 1.5

Table 3 – ARL_1 for the proposed improvement vs the standard $\bar{X} - S^2$ control chart

		$n = 4$		$n = 5$		$n = 6$		$n = 7$	
δ	γ	ARL_1	ARL_1 - Klein	ARL_1	ARL_1- Klein	ARL_1	ARL_1- Klein	ARL_1	ARL_1- Klein
0	1	370.400	370.400	370.400	370.400	370.400	370.400	370.400	370.400
0	1.05	**202.710**	208.780	**196.590**	199.160	191.290	**191.020**	186.560	**183.940**
0	1.1	**119.650**	126.240	**112.730**	115.280	106.860	**106.360**	101.710	**98.867**
0	1.2	**50.108**	55.295	**44.981**	47.182	**40.804**	41.067	37.300	**36.273**
0	1.3	**25.398**	29.388	**22.016**	24.051	**19.366**	20.260	**17.227**	17.437
0	1.4	**14.841**	18.055	**12.569**	14.458	**10.846**	12.008	**9.496**	10.245
0	1.5	**9.656**	12.357	**8.064**	9.812	**6.887**	8.130	**5.985**	6.947
0.25	1	204.080	**158.190**	180.380	**134.270**	160.620	**115.450**	143.950	**100.350**
0.25	1.05	126.600	**109.000**	112.440	**93.377**	100.850	**81.248**	91.157	**71.545**
0.25	1.1	82.260	**76.696**	72.894	**65.563**	65.376	**57.112**	59.175	**50.448**
0.25	1.2	**39.247**	40.893	34.231	**34.202**	30.322	**29.331**	27.173	**25.617**
0.25	1.3	**21.518**	24.251	**18.410**	19.801	**16.043**	16.680	**14.175**	14.375
0.25	1.4	**13.207**	15.879	**11.123**	12.764	**9.569**	10.647	**8.366**	9.126
0.25	1.5	**8.873**	11.300	**7.396**	9.023	**6.315**	7.515	**5.492**	6.453
0.5	1	65.619	**39.269**	49.996	**29.108**	39.303	**22.565**	31.676	**18.104**
0.5	1.05	48.292	**33.298**	37.801	**25.365**	30.440	**20.118**	25.065	**16.454**
0.5	1.1	36.395	**28.284**	29.090	**22.019**	23.881	**17.798**	20.013	**14.797**
0.5	1.2	21.928	**20.481**	17.976	**16.364**	15.118	**13.549**	12.961	**11.516**
0.5	1.3	**14.156**	15.043	**11.714**	12.121	**9.946**	10.133	**8.609**	8.700
0.5	1.4	**9.727**	11.363	**8.061**	9.166	**6.861**	7.690	**5.957**	6.637
0.5	1.5	**7.068**	8.891	**5.854**	7.175	**4.985**	6.041	**4.336**	5.242
0.75	1	21.798	**12.597**	15.341	**9.162**	11.397	**7.122**	8.825	**5.810**
0.75	1.05	18.097	**11.912**	13.139	**8.859**	10.014	**6.997**	7.921	**5.776**
0.75	1.1	15.236	**11.244**	11.365	**8.530**	8.859	**6.838**	7.142	**5.707**
0.75	1.2	11.136	**9.919**	8.661	**7.769**	6.997	**6.384**	5.818	**5.433**
0.75	1.3	**8.396**	8.634	**6.711**	6.908	**5.554**	5.777	**4.718**	4.988
0.75	1.4	**6.507**	7.467	**5.288**	6.051	**4.443**	5.119	**3.826**	4.467
0.75	1.5	**5.180**	6.473	**4.252**	5.287	**3.608**	4.511	**3.136**	3.969
1	1	8.596	**5.694**	5.926	**4.345**	4.393	**3.572**	3.439	**3.092**
1	1.05	7.756	**5.679**	5.504	**4.388**	4.171	**3.632**	3.321	**3.153**
1	1.1	7.050	**5.647**	5.133	**4.415**	3.969	**3.679**	3.209	**3.206**
1	1.2	5.911	**5.519**	4.492	**4.408**	**3.592**	3.723	**2.983**	3.269
1	1.3	**5.016**	5.304	**3.937**	4.311	**3.231**	3.684	**2.741**	3.261
1	1.4	**4.293**	5.021	**3.450**	4.135	**2.887**	3.569	**2.489**	3.183
1	1.5	**3.707**	4.705	**3.029**	3.913	**2.572**	3.405	**2.245**	3.057
1.25	1	4.113	**3.431**	2.909	**2.826**	**2.242**	2.496	**1.840**	2.304
1.25	1.05	3.928	**3.494**	**2.842**	2.886	**2.225**	2.547	**1.844**	2.345
1.25	1.1	3.761	**3.550**	**2.778**	2.941	**2.206**	2.595	**1.846**	2.385
1.25	1.2	**3.462**	3.629	**2.650**	3.030	**2.157**	2.677	**1.836**	2.456
1.25	1.3	**3.190**	3.660	**2.514**	3.081	**2.089**	2.731	**1.805**	2.505
1.25	1.4	**2.936**	3.641	**2.368**	3.088	**2.003**	2.748	**1.753**	2.526
1.25	1.5	**2.701**	3.579	**2.218**	3.056	**1.902**	2.731	**1.684**	2.517
1.5	1	**2.371**	2.559	**1.779**	2.276	**1.464**	2.139	**1.283**	2.069
1.5	1.05	**2.347**	2.613	**1.787**	2.315	**1.481**	2.166	**1.301**	2.087
1.5	1.1	**2.322**	2.664	**1.792**	2.354	**1.495**	2.193	**1.317**	2.106
1.5	1.2	**2.268**	2.757	**1.793**	2.428	**1.516**	2.249	**1.344**	2.146
1.5	1.3	**2.205**	2.829	**1.780**	2.490	**1.524**	2.298	**1.360**	2.183
1.5	1.4	**2.132**	2.877	**1.752**	2.533	**1.518**	2.334	**1.364**	2.212
1.5	1.5	**2.051**	2.897	**1.712**	2.556	**1.497**	2.355	**1.355**	2.230

The values in **bold** present in table 3 indicate the lower ARL_1 value between the $\bar{X} - S^2$ control chart and the $\bar{X} - S^2$ control chart with Klein's 2-of-2 supplementary run rule applied. It is possible to observe that, generally, small shifts in the mean and/or variance favour the use of the $\bar{X} - S^2$ with the 2-of-2 supplementary run rule, particularly when the sample size n increases.

Figure 2 shows the results of table 3 for $n = 5$ graphically, in order to illustrate and summarize the results. The X-axis represents shifts in the standart deviation, γ, while shifts in the mean, δ, were represented on the top of the graph's 6 sections, each for a specific shift in the mean. The Y-axis represents a ratio of the ARL_1 values of the $\bar{X} - S^2$ control chart with Klein's run rule, $\text{ARL}^{Klein}_{\bar{X}-S^2}$, and the standard $\bar{X} - S^2$ control chart, $\text{ARL}_{\bar{X}-S^2}$, that is, values lower than 1 on the Y-axis represent a situation in which the proposed $\bar{X} - S^2$ control chart with Klein's run rule had a better performance than the $\bar{X} - S^2$ standard joint control chart, in terms of ARL_1.

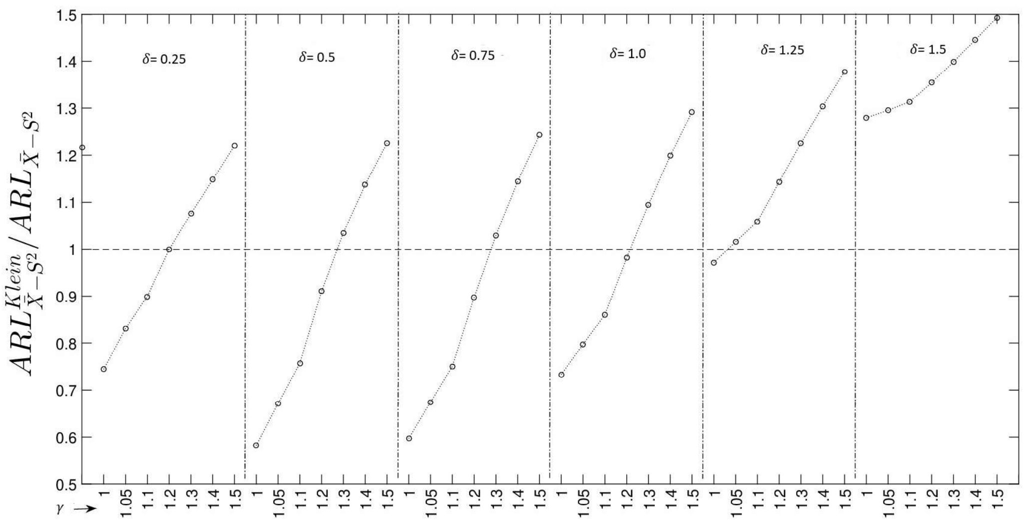


Figure 2 – Comparison of performance between the proposed $\bar{X} - S^2$ control chart and the standard $\bar{X} - S^2$ control chart

Figure 2 shows that the $\bar{X} - S^2$ control chart with Klein's run rule presents a lower average ARL_1 in relation to the standard $\bar{X} - S^2$ control chart for shifts in the mean of $\delta = 0.25, 0.5, 0.75, 1$ and shifts in variance of $\gamma = 1, 1.05, 1.1, 1.2$. It is also noticeable that the performance difference, in terms of ARL_1, of the proposed control chart and the standard $\bar{X} - S^2$ control chart is increased significantly for small shifts both in the mean and variance, such as the cases where the variance didn't change, $\gamma = 1$ and the shift in the mean was lower than 0.75, $\delta = 0.25, 0.5, 0.75$. In these situations the proportion between the performance of the newly proposed chart and the standard control chart was as low as 0.6.

For larger shifts in the mean, such as $\delta = 1.25, 1.50$, we can see that the performance of the standard joint control chart is superior thorough all shifts in variance, although it is also important to highlight that most control charts, including the $\bar{X} - S^2$, are already sensible to large shifts in the parameters being controlled, presenting low ARL_1 values. The difference in performance between both methods, although seeming to be large, is of 0.64 samples on average, for a $\delta = 1.50$. The same difference in performance is of 9.69 samples on average for a $\delta = 0.25$. The notion that a better performance for large shifts is generally not as important was discussed in Khoo (KHOO, 2003) where it is stated that the standard $\bar{X}$ control chart can detect large shifts, at the earliest, one observation earlier than the $\bar{X}$ control chart using the 2-of-2 rule, while "for small process average shifts, the difference in the time of detecting an out-of-control signal between the standard Shewhart and the other schemes are quite significant".

These results contribute to the initial hypothesis that the addition of Klein's 2-of-2 rule to the $\bar{X} - S^2$ control chart would improve itt's performance for small shifts both in the mean and variance, making it more suitable for processes requiring control of small changes in these two parameters while maintaining the simplicity which makes the standard $\bar{X} - S^2$ control chart so ubiquitous in the industry.

3.2 Comparison to the EWMA control chart

In this section we aim to compare the performance of the improved $\bar{X} - S^2$ control chart to the performance of a more complex control chart, well known for presenting good performance results, the exponentially weighted moving average (EWMA) control chart, initially proposed by Roberts (ROBERTS, 1959) to improve the sensitivity to small changes in the process mean and adapted for the use in the joint monitoring of mean and variance in Chen et al.(CHEN; CHENG; XIE, 2001).

According to Klein(KLEIN, 2000), exponentially weighted moving average (EWMA) schemes have excellent capabilities in detecting small process average shifts, as described in Montgomery(MONTGOMERY, 2009). However, until now, they do not appear to have gained widespread adoption beyond chemical process industries. This could be attributed to the perception that the necessary calculations are too intricate for regular shop floor operations and/or the common organizational inertia linked with procedural changes. Hence, in this paper, we explore variations of simpler traditional methods that might find more acceptance among practitioners. Nevertheless, we consider it important to compare performance results with the EWMA control scheme, since it is considered as the benchmark for detecting minor shifts in the mean and/or variance.

In order to make such a comparison, it is important to define how does the EWMA control scheme works for the $\bar{X} - S^2$ joint control chart. For this purpose, we cite the description given in Chen et al.(CHEN; CHENG; XIE, 2001), where the EWMA $\bar{X}$ chart and the EWMA $\log(S^2)$ control charts are described:

For the EWMA $\bar{X}$ chart, the control limits would be

$$LCL_{Ewma}^{\bar{X}} = \mu - K_1\sqrt{\frac{\lambda_1}{(2-\lambda_1)}} \times \frac{\sigma_0}{\sqrt{n}},$$
$$UCL_{Ewma}^{\bar{X}} = \mu + K_1\sqrt{\frac{\lambda_1}{(2-\lambda_1)}} \times \frac{\sigma_0}{\sqrt{n}}, \tag{3.1}$$

Where λ_1 and K_1 are the parameters that control the performance of the EWMA $\bar{X}$ chart. The plotted statistics are:

$$T_i = (1-\lambda_1)T_{i-1} + \lambda_1\bar{X}_i, 0 < \lambda_1 \leq 1, i = 1, 2, \ldots \tag{3.2}$$

where $T_0 = \mu_0$ is the starting value. For the modified EWMA $\log(S^2)$ chart, the control limits are:

$$LCL_{Ewma}^{S^2} = c - K_2\sqrt{\frac{d\lambda_2}{(2-\lambda_2)}},$$
$$UCL_{Ewma}^{S^2} = c + K_2\sqrt{\frac{d\lambda_2}{(2-\lambda_2)}}, \tag{3.3}$$

Where λ_2 and K_2 are the parameters that control the performance of the EWMA $\log(S^2)$ chart, c is the (approximate) mean of $\log(S_i^2)$, given by:

$$c = \ln(\sigma_0^2) - \frac{1}{n-1} - \frac{1}{3(n-1)^2} + \frac{1}{15(n-1)^4} \tag{3.4}$$

and d is the (approximate) variance of $\log(S_i^2)$, given by:

$$d = \frac{2}{n-1} + \frac{2}{(n-1)^2} + \frac{4}{3(n-1)^3} - \frac{16}{15(n-1)^5} \tag{3.5}$$

The plotting statistics are:

$$W_i = (1-\lambda_2)W_{i-1} + \lambda_2\log(S_i^2), 0 < \lambda_2 \leq 1, i = 1, 2, \ldots \tag{3.6}$$

where W_0 is the starting value which assumes the value c.

Table 5 compares the proposed $\bar{X} - S^2$ control chart with klein's 2-of-2 rule performance, in terms of ARL_1 to the performance of the $\bar{X} - S^2$ control chart using the EWMA proposition, found in Chen et al.(CHEN; CHENG; XIE, 2001). The ARL_1 values obtained by Chen were derived through a Monte Carlo simulation with 10 thousand runs. In order to compare both results, we have adopted the parameters used in Chen et al.(CHEN; CHENG; XIE, 2001) for the EWMA control chart, that is, an ARL_0 of 250, a sample size of $n = 5$, shifts in the mean of $\delta = (0, 0.25, 0.5, 1, 2)$ and shifts in the variance of $\gamma = (0.25, 0.5, 1, 1.5, 2)$ and $\lambda_1 = \lambda_2 = \lambda$. The parameters used for the EWMA control chart were $\lambda = 0.80$ in EWMA-A and $\lambda = 0.1$ in EWMA-B, which were selected to encompass distinct values of λ in the comparison, similarly to Costa and Rahim (COSTA; RAHIM, 2004). All bilateral control limits for the cases discussed in Table 5 are presented in Table 4.

Table 4 – Control limits for the different bilateral $\bar{X} - S^2$ control charts present in table 5, for an ARL_0 of 250

$\bar{X} - S^2$ Klein	EWMA ($\lambda = 0.8$)	EWMA ($\lambda = 0.1$)	$\bar{X} - S^2$
$\text{UCL}_{\bar{X}}^{\text{Klein}} = 0.917$	$\text{UCL}_{EWMA}^{\bar{X}} = 1.128$	$\text{UCL}_{EWMA}^{\bar{X}} = 0.288$	$\text{UCL}_{\bar{X}} = 1.380$
$\text{LCL}_{\bar{X}}^{\text{Klein}} = -0.917$	$\text{LCL}_{EWMA}^{\bar{X}} = -1.128$	$\text{LCL}_{EWMA}^{\bar{X}} = -0.288$	$\text{LCL}_{\bar{X}} = -1.380$
$\text{UCL}_{S^2}^{\text{Klein}} = 9.974$	$\text{UCL}_{EWMA}^{S^2} = 2.274$	$\text{UCL}_{EWMA}^{S^2} = 0.257$	$\text{UCL}_{S^2} = 18.509$
$\text{LCL}_{S^2}^{\text{Klein}} = 0.635$	$\text{LCL}_{EWMA}^{S^2} = -2.814$	$\text{LCL}_{EWMA}^{S^2} = -0.797$	$\text{LCL}_{S^2} = 0.090$

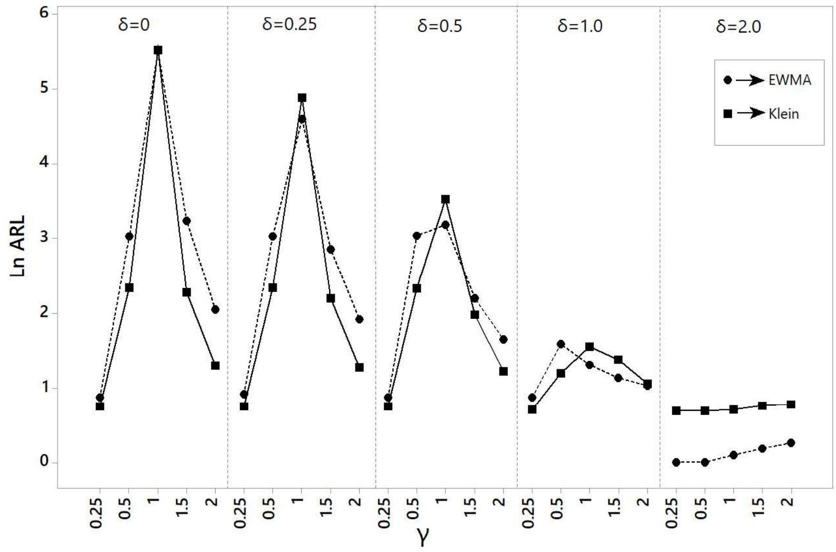
Notice that shifts in the variance where $\sigma_1 < \sigma_0$ were used, that is $\gamma < 1$, signifying that there is an interest in detecting decreases in the variance as well as increases. Thus, the S^2 component of the $\bar{X} - S^2$ control chart, which was previously unilateral, only concerned with detecting increases in the variance, was altered such that the $\bar{X} - S^2$ control chart is now bilateral and capable of detecting decreases in the variance, allowing for a proper comparison with the EWMA control chart described in Chen et al.(CHEN; CHENG; XIE, 2001). The bilateral $\bar{X} - S^2$ control chart with the 2-of-2 rule was implemented through a Markov Chain with 25 states. The bilateral ARL_1 values presented in Table 5 for the standard $\overline{X} - S^2$ control chart, $\overline{X} - S^2$ control chart using Klein's 2-of-2 run rule were calculated by using programs D and E respectively. The bilateral ARL_1 values presented in Table 5 for the $\overline{X} - S^2$ control chart utilizing the EWMA control scheme were taken from Chen et al.(CHEN; CHENG; XIE, 2001), however they can be reproduced through program F. Programs D, E and F are present in the supplementary material available in the Appendix.

Figure 3 shows the results of table 5 graphically, in order to illustrate and summarize the results. The X-axis represents shifts in the variance, γ, while shifts in the mean, δ, were represented on the top of the graph's 5 sections, each for a specific shift in the mean. The Y-axis represents the $ln(ARL_1)$ values for both control charts, the proposed control chart with Klein's 2-of-2 rule being represented by a square, ■, and the control scheme using the EWMA control chart being represented by a circle, ●. Subfigure 3(a) refers to a comparison between the proposed Klein method and the EWMA control scheme with $\lambda = 0.80$, while Subfigure 3(b) compares the proposed method with the EWMA control scheme using a $\lambda = 0.10$ instead, in

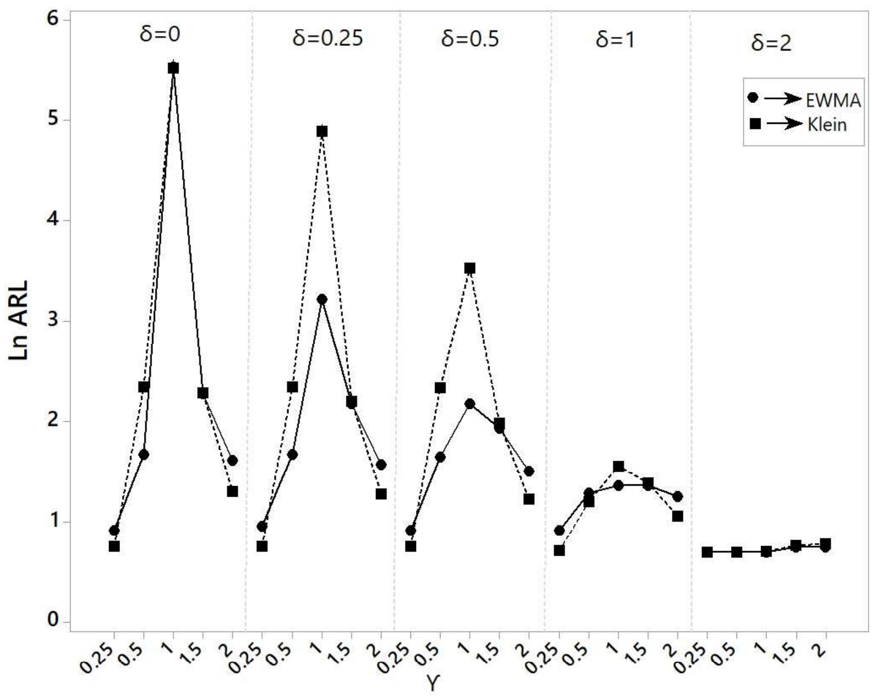
Table 5 – Comparison, in terms of ARL_1, between a joint monitoring scheme using the exponentially weighted moving average (EWMA) control chart using different λ values and the proposed improvement using Klein's 2-of-2 rule.

Parameters		Control charts			
δ	γ	Klein	EWMA-A ($\lambda = 0.8$)	EWMA-B ($\lambda = 0.1$)	$\overline{X} - S^2$
0	0.25	2.120	2.400	2.500	6.151
0	0.5	10.387	20.600	5.300	69.643
0	1	250.000	249.400	252.100	250.000
0	1.5	9.790	25.600	9.700	8.314
0	2	3.661	7.800	5.000	2.436
0.25	0.25	2.120	2.500	2.600	6.151
0.25	0.5	10.387	20.600	5.300	69.642
0.25	1	132.890	99.500	25.100	127.080
0.25	1.5	9.053	17.400	8.800	7.516
0.25	2	3.593	6.800	4.800	2.379
0.5	0.25	2.120	2.400	2.500	6.151
0.5	0.5	10.305	20.800	5.200	69.443
0.5	1	34.153	24.200	8.800	37.630
0.5	1.5	7.270	9.000	6.900	5.774
0.5	2	3.405	5.200	4.500	2.226
1	0.25	2.035	2.400	2.500	6.140
1	0.5	3.315	4.900	3.600	17.048
1	1	4.738	3.700	3.900	5.005
1	1.5	3.980	3.100	3.900	2.890
1	2	2.874	2.800	3.500	1.798
2	0.25	2.000	1.000	2.000	1.000
2	0.5	2.000	1.000	2.000	1.003
2	1	2.023	1.100	2.000	1.090
2	1.5	2.136	1.200	2.100	1.194
2	2	2.170	1.300	2.100	1.196

order to represent the EWMA method with different values of λ and ensure a fair comparison.



(a) Klein x EWMA-A (λ = 0.80)



(b) Klein x EWMA-B (λ = 0.1)

Figure 3 – Comparison of performance between the proposed $\bar{X} - S^2$ control chart and the $\bar{X} - S^2$ control chart using the EWMA proposition

The figure indicates that the proposed $\bar{X} - S^2$ control chart with the 2-of-2 rule presents a very competitive performance even in relation to complex methodologies known to have strong results, such as the EWMA, presenting a lower ARL value on several cases throughout the values tested. Notably, the performance of the EWMA control chart is superior for large shifts in the mean, but, as previously discussed in Section 3.1, large shifts may not be as relevant in terms of performance, as the differences in samples collected are generally small.

These results show the proposed $\bar{X} - S^2$ control chart with Klein's rule to be a very interesting option, as it has a competitive performance in relation to a significantly more complex control scheme, the EWMA, while maintaining a simplicity that made the standard $\bar{X} - S^2$ control chart so widely used, proposing only a small adjustment in the collection of samples and a small adjustment on the calculation of the control limits.

Observing the results in Table 3, a noteworthy point that deserves evaluation is to check if the performance of the discussed joint EWMA procedure in this section could be improved by incorporating Klein's procedure (2-of-2 rule). Thus, the EWMA procedure presented in Chen (CHEN; CHENG; XIE, 2001) was adapted to incorporate Klein's rule, i.e., the process will be judged out of control if we observe two consecutive points of T_i above (below) $UCL_{Ewma}^{\bar{X}-Klein}$ ($LCL_{Ewma}^{\bar{X}-Klein}$) and/or have two consecutive points of W_i above (below) $UCL_{Ewma}^{S^2-Klein}$ ($LCL_{Ewma}^{S^2-Klein}$).

The values of $UCL_{Ewma}^{\bar{X}-Klein}$, $LCL_{Ewma}^{\bar{X}-Klein}$, $UCL_{Ewma}^{S^2-Klein}$ and $LCL_{Ewma}^{S^2-Klein}$ were obtained in such a way as to ensure an ARL_0 close to 250, as adopted in the cases of Table 5. As in Chen (CHEN; CHENG; XIE, 2001), we used Monte Carlo simulations to estimate the values of ARL_1 (value based on 500000 runs) with $n = 5$ and $\lambda = 0.1$. The R program where we implemented the joint EWMA procedure adding Klein's 2-of-2 rule and obtained the values of ARL_1 through Monte Carlo simulation is available as program G in the Appendix. Using programs F and G, we calculated the ARL_1 values for the proposed joint EWMA and the proposed joint EWMA incorporating Klein's 2-of-2 rule. The δ and γ values used for the simulations were those of Table 5 as well as other small shifts in the mean and variance.

We can see in Figure 4 that the performance results for the two proposals are similar, with the joint EWMA control scheme's performance being inferior to the joint EWMA incorporating the Klein's 2-2 rule only in a few cases where the shifts are small: $(\delta; \gamma) = (0.05; 0.95), (0.05; 1), (0.05; 1.05), (0.25; 0.95), (0.25; 1), (0.1; 0.95), (0.1; 1), (0.1; 1.05)$. In this sense, we understand that the use of the joint EWMA proposal with the supplementary 2-of-2 rule would only be recommended in comparison to the use of the joint EWMA scheme in case it is necessary to detect very small changes in the mean and/or variance.

The control limits for the EWMA control scheme being used for the joint $\bar{X} - S^2$ control chart are presented in Table 4. For the EWMA control scheme with Klein's 2-of-2 run rule, the control limits were $UCL_{Ewma}^{\bar{X}-Klein} = 0.25957$, $LCL_{Ewma}^{\bar{X}-Klein} = -0.25957$, $UCL_{Ewma}^{S^2-Klein} = 0.20497$ and $LCL_{Ewma}^{S^2-Klein} = -0.7456$.

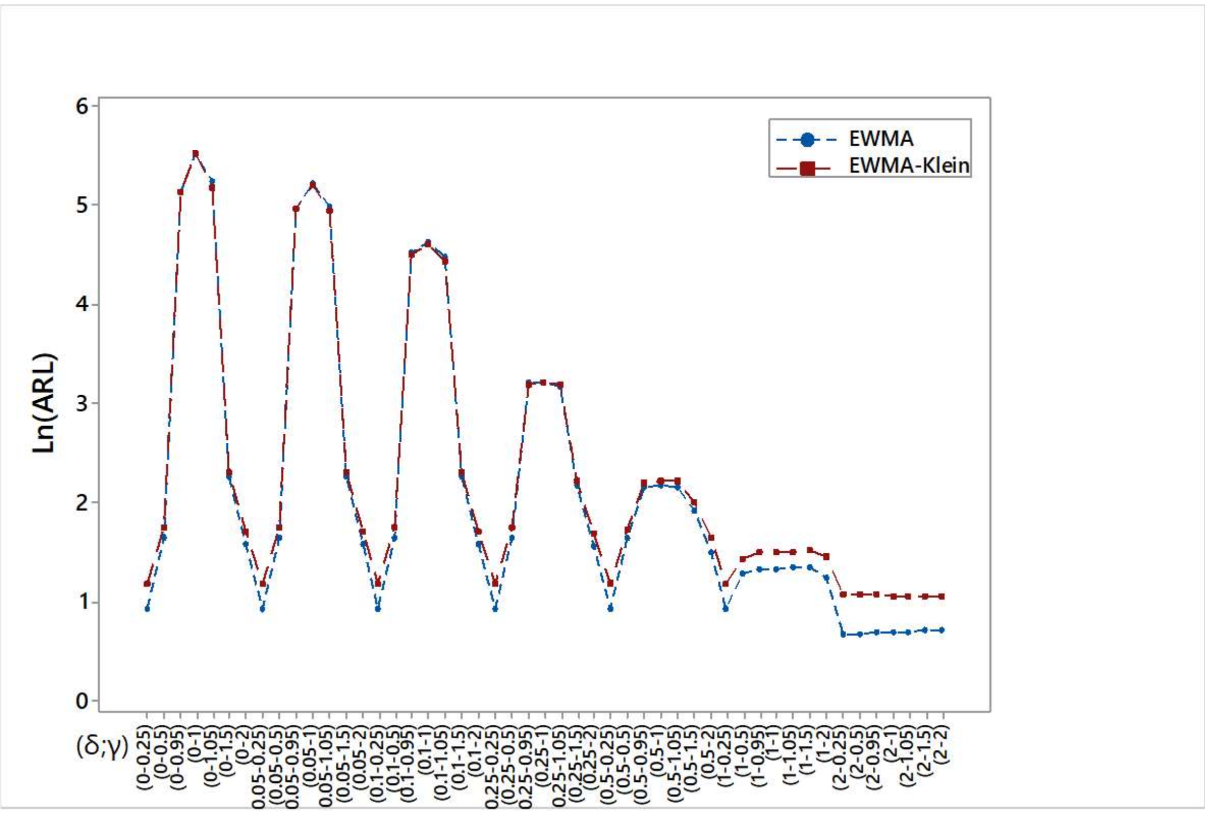


Figure 4 – Comparison of the joint EWMA control chart versus the joint EWMA control chart using the 2-of-2 run rule

4 Numerical Example

In this section, we provide a numerical example based on Montgomery (MONTGOMERY, 2020), assuming that the described process has been in operation for a long time. The example considers the production of piston rings for automotive engines through a forging process, where controlling the inner ring diameters (D) is crucial for ensuring product quality. Small changes in the average diameter of the pistons can lead to an increase in nonconforming units, hence the need to quickly detect such variations.

Table 6 presents the data of the inner ring diameters for the 25 samples, each of size $n = 5$, taken from the production process, presenting each individual value as well as each sample's mean $\bar{X}$ and variance S^2. Samples in **bold** indicate sequences in which the process is considered out-of-control in the $\bar{X} - S^2$ control chart with Kleins supplementary rule. Observe that, for each sample, a decision is made on the process being in statistical control or out of statistical control.

To evaluate the future capacity of the process, it is essential to establish statistical control. Under statistical control, the inner ring diameters follow a normal distribution with mean $\mu_0 = 74.0508$ mm and variance $\sigma_0^2 = 0.4748^2$ mm. If samples of size $n = 5$ are collected each hour, the control limits aiming for an $\text{ARL}_0 = 370.4$ can be obtained through program B, presented in the Appendix, and Table 6, where, for the $\bar{X}$ control chart, $UCL_{\bar{X}}^{Klein}$ = 74.468 and $LCL_{\bar{X}}^{Klein}$ = 73.634 and, for the S^2 control chart, $UCL_{S^2}^{Klein} = 10.051$.

In order to more directly compare the standard deviation (S) of the samples instead of using $\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$ for the control limits in the S^2 control chart, $UCL_{S^2}^{Klein}$ was transformed in $UCL_S^{Klein^*}$, where $UCL_S^{Klein^*} = \sqrt{\frac{UCL_S^{Klein} \times \sigma_0^2}{n-1}}$, which can be directly compared to the standard deviation, resulting in a value of $UCL_S^{Klein^*} = 0.752$.

In order to facilitate the reader's understanding, Figure 5 presents the results of Table 6 in two graphs, for the mean and the standard deviation of the samples. The first graph represents the $\bar{X}$ control chart, used to control the sample mean, and the second graph represents the S^2 control chart, used to control the sample variance/standard deviation. The calculated upper and lower control limits, in the case of the $\bar{X}$ control chart, or the upper control limit, in the case of the S^2 control chart, are also represented in each graph. Samples with an in-control average value are represented by a circle, ●, and samples with an out-of-control average value are represented by a square, ■. Lastly, situations in which the process would be considered out-of-control, two consecutive samples are beyond the same side of the control limits (above or below), are circled. Notice that if any of the two control charts indicate that the process is out-of-control using this run rule, the joint control chart will indicate that the process is out-of-control.

Table 6 – Example - inner diameter measurements (mm) of automobile engine piston rings

Sample	X_1	X_2	X_3	X_4	X_5	$\bar{X}$	$\bar{X}$ beyond control limits?	S	S beyond control limits?	Decision
1	74.4357	74.5999	73.9342	73.4610	74.7280	74.2318	No	0.5258	No	in-control
2	73.9415	74.2705	73.8179	74.0556	74.3251	74.0821	No	0.2149	No	in-control
3	74.1459	74.9491	73.9810	74.5521	74.8735	74.5003	Yes (above UCL)	0.4298	No	in-control
4	74.0997	74.5921	74.4758	74.2567	73.9703	74.2789	No	0.2573	No	in-control
5	**75.2493**	**74.7503**	**73.5058**	**73.4198**	**74.2975**	**74.2445**	**No**	**0.7896**	**Yes (above UCL)**	**in-control**
6	**75.1547**	**73.3587**	**74.4780**	**75.1246**	**73.6130**	**74.3458**	**No**	**0.8351**	**Yes (above UCL)**	**out-of-control**
7	74.3572	73.9932	73.8009	73.6307	74.5064	74.0577	No	0.3686	No	in-control
8	73.5098	74.0014	74.1039	73.3648	74.6444	73.9249	No	0.5102	No	in-control
9	73.5265	73.4815	73.1418	74.0177	74.2900	73.6915	No	0.4577	No	in-control
10	75.1802	74.5007	75.1190	75.1474	74.8506	74.9596	Yes (above UCL)	0.2880	No	in-control
11	73.7025	73.6221	74.2995	73.5349	73.6022	73.7522	No	0.3118	No	in-control
12	73.4093	73.9984	74.2685	73.5359	73.5414	73.7507	No	0.3661	No	in-control
13	73.3414	74.0958	74.6182	75.5084	74.0438	74.3215	No	0.8040	Yes (above UCL)	in-control
14	74.2642	74.5148	74.0881	73.4742	74.0550	74.0793	No	0.3843	No	in-control
15	73.4588	72.5344	72.1376	72.7350	72.7788	72.7289	Yes (Below LCL)	0.5106	No	in-control
16	75.0715	74.1691	74.3394	73.0938	74.0899	74.1527	No	0.7079	No	in-control
17	74.4718	74.4386	73.7068	74.1232	73.8066	74.1094	No	0.3513	No	in-control
18	73.7300	74.2468	74.4114	74.1108	73.8559	74.0710	No	0.2787	No	in-control
19	74.2566	73.7013	74.3251	73.8000	74.3087	74.0783	No	0.3021	No	in-control
20	**74.4414**	**74.8761**	**74.4717**	**74.4444**	**74.4130**	**74.5293**	**Yes (above UCL)**	**0.1949**	**No**	**in-control**
21	**75.8819**	**74.4186**	**74.6863**	**74.8287**	**73.8999**	**74.7431**	**Yes (above UCL)**	**0.7286**	**No**	**out-of-control**
22	73.4192	74.4942	74.5731	73.4163	74.4422	74.0690	No	0.5963	No	in-control
23	73.5922	74.3617	74.2836	73.2267	74.8433	74.0615	No	0.6457	No	in-control
24	**75.1241**	**73.3756**	**74.5663**	**74.9275**	**73.7670**	**74.3521**	**No**	**0.7531**	**Yes (above UCL)**	**in-control**
25	**75.0859**	**73.3279**	**74.5162**	**75.0222**	**73.6491**	**74.3203**	**No**	**0.7988**	**Yes (above UCL)**	**out-of-control**

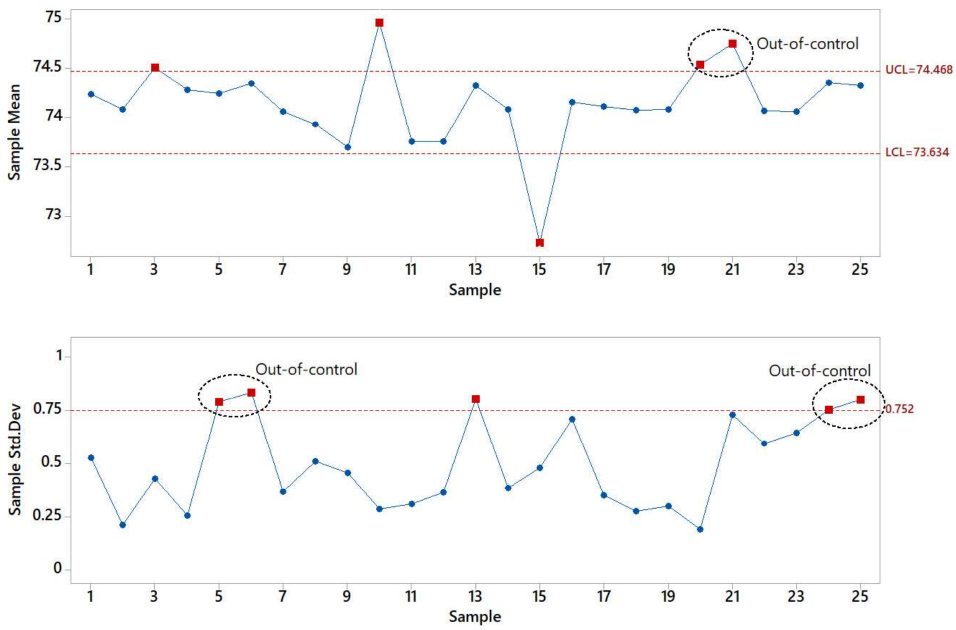


Figure 5 – joint control chart - inner diameter measurements (mm) of the automobile engine piston rings.

5 Final Remarks

In this paper we propose an improvement to the $\bar{X} - S^2$ control chart, aimed to control the mean and variance of a process, while attempting to maintain the simplicity which made it so widely used throughout the industry. Several propositions to improve the joint control chart's performance can be found throughout the literature, but few have been well-accepted in the industry, demonstrating a resistance to implement complex solutions such as the introduction of new control limits beyond the usual ones, new decision rules, etc, even if they present good performance.

In order to improve the $\bar{X} - S^2$ performance while maintaining a simple approach, we have applied a run rule proposed in Klein (KLEIN, 2000) named the 2-of-2 rule. The idea of klein's 2-of-2 rule is that the process is only signaled as out-of-control when two successive points are above an upper control limit (UCL) or two successive points are below a lower control limit (LCL), thus improving the control chart's sensibility to small shifts. According to Hurwitz and Mathur (HURWITZ; MATHUR, 1992), the 2-of-2 rule is simple and well-accepted in an industrial environment, since it avoids operational difficulties and presents similar complexity to that of the $\bar{X} - S^2$ control chart.

In order to apply the 2-of-2 rule to the joint $\bar{X} - S^2$ control chart, we have used a Markov Chain approach. The performance results, in terms of ARL_1, of the newly proposed control chart were compared to the standard $\bar{X} - S^2$ control chart and to the more complex $\bar{X} - s^2$ control chart with the exponentially weighted moving average (EWMA) control scheme. The proposed control chart presented superior results for small shifts in the mean and variance when compared to the standard joint control chart and very competitive performance results when compared to the more complex EWMA joint control chart, all while being simple and easy to implement.

Additionally, we have also evaluated combining the EWMA control scheme with Klein's 2-of-2 run rule for the joint $\bar{X} - S^2$ control chart. According to the results found in section 3.2, we concluded that the joint EWMA control chart and joint EWMA control chart with the 2-of-2 run rule present very similar ARL_1 values, leading to the conclusion that Klein's supplementary run rule did not present a significant improvement when combined with the EWMA control scheme.

A numeric example was presented and all seven programs utilized throughout this paper, developed in R, are available in Appendix A to G.

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Appendix

APPENDIX A – Control limits and ARL of the $\bar{X}$- S^2 unilateral control chart for a given sample size "n" and target ARL_0 of "ARL_0"

```
#A control chart for the joint monitoring of mean and variance
rm(list = ls())
##Markov chain

u0=0 #In-control average
u1=0.25 #out-of-control average (defined in the previous vector)
s0=1 #in-control standard deviation
s1=1.05 # out-of-control standard deviation (defined in the previous vector)
n=5 #Sample size
ARL0=370.4 #Target ARL0


OtiUCL <- function(alfa){
  alfag=1-(1-alfa[1])*(1-alfa[2])
  ARL=1/alfag
  ARLphi=(ARL-ARL0)^2
  return(ARLphi)
}
#upper limit to be used in the optimize function. Change as needed.
LSa=1
LSb=1
#initial value. Change as needed.
ivxbar=0.04
ivs2=0.04
par_optim <- nlminb(c(ivxbar,ivs2),OtiUCL,lower=c(0,0),upper =c(LSa,LSb))
alfaa=par_optim[[1]][1] #Prob Xbar
alfab=par_optim[[1]][2] #Prob S2

#ARL1
LSCxb=qnorm((1-alfaa/2),u0,s0/(n^0.5))
LICxb=qnorm(alfaa/2,u0,s0/(n^0.5))
LSCqui=qchisq((1-alfab),(n-1))

pc=pnorm(LICxb,u1,s1/(n^0.5))
pa=1-pnorm(LSCxb,u1,s1/(n^0.5))
pb=1-pa-pc

pas=1-pchisq(LSCqui*(s0^2/s1^2),(n-1))
pbs=1-pas


ARL1<-1/(1-pb*pbs)

cat('LSCxb=',LSCxb,"\n")
cat('LICxb=',LICxb,"\n")
cat('LSCqui=',LSCqui,"\n")
cat('Probxbar=',alfaa,"\n")
cat('Probs2=',alfab,"\n")
cat('ARL1=',ARL1,"\n")
```

APPENDIX B – Control limits and ARL of the improved $\bar{X}$- S^2 unilateral control chart for a given sample size "n" and target ARL_0 of "ARL_0"

```
#A control chart for the joint monitoring of mean and variance with Klein's
    supplementary rule
require(pracma)
rm(list = ls())
##############################################################################
    u0=0 #In-control average
    u1=0.25 #out-of-control average (defined in the previous vector)
    s0=1 #in-control standard deviation
    s1=1.05 # out-of-control standard deviation (defined in the previous vector)
    n=5 #Sample size
    ARL0=370.4 #Target ARL0

#############################################################################
    OtiUCL <- function(U){#Optimization function used to find the UCL and LCL
     UCLxb=qnorm((1-U[1]/2),u0,s0/(n^0.5))
     LCLxb=qnorm(U[1]/2,u0,s0/(n^0.5))
     LCs2=qchisq((1-U[2]),(n-1))
     #X-bar control chart
     pxi=pnorm(LCLxb,u0,s0/(n^0.5))
     pxs=1-pnorm(UCLxb,u0,s0/(n^0.5))
     pxc=1-pxs-pxi

     #S2 control chart
     psf=1-pchisq(LCs2,(n-1))
     psc=1-psf
     #Markov chain
     size<- 15 #Size of the Markov chain
     MarkovChain<- matrix(0,nrow=size,ncol=size,byrow=TRUE)
     MarkovChain[1,]<-
        c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
     MarkovChain[2,]<-
        c(pxc*psc,0,pxs*psc,pxi*psc,0,pxc*psf,0,pxs*psf,pxi*psf,0,0,0,0,0,0)
     MarkovChain[3,]<-
        c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
     MarkovChain[4,]<-
        c(pxc*psc,pxs*psc,0,0,pxi*psc,pxc*psf,pxs*psf,0,0,pxi*psf,0,0,0,0,0)
     MarkovChain[5,]<-
        c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
     MarkovChain[6,]<-
        c(pxc*psc,pxs*psc,0,pxi*psc,0,0,0,0,0,0,pxc*psf,pxs*psf,0,pxi*psf,0)
     MarkovChain[7,]<-
        c(pxc*psc,0,pxs*psc,pxi*psc,0,0,0,0,0,0,pxc*psf,0,pxs*psf,pxi*psf,0)
     MarkovChain[8,]<-
        c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
     MarkovChain[9,]<-
        c(pxc*psc,pxs*psc,0,0,pxi*psc,0,0,0,0,0,pxc*psf,pxs*psf,0,0,pxi*psf)
     MarkovChain[10,]<-
        c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
     MarkovChain[11,]<-
        c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
     MarkovChain[12,]<-
        c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
     MarkovChain[13,]<-
        c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
```

```
  MarkovChain[14,]<-
     c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
  MarkovChain[15,]<-
     c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)

  #The stationary distribution
  A = t(MarkovChain) - eye(15)
  A[15,] = ones(1,15)
  B = zeros(15, 1)
  B[15,1] = 1
  Solved_markov_chain = solve(A)%*%B

  ARL<- 1/sum(Solved_markov_chain[3,],Solved_markov_chain[5,],
           Solved_markov_chain[8,],Solved_markov_chain[10,],
           Solved_markov_chain[11,],Solved_markov_chain[12,],
           Solved_markov_chain[13,],Solved_markov_chain[14,],
           Solved_markov_chain[15,])

  ARLphi=(ARL-ARL0)^2

  return(ARLphi)
}

################################################################################
#Limit used in the optimize function for the x-bar control chart
LSa=1
LSb=1
#initial value. Change as needed.
ivxbar=0.04
ivs2=0.04
par_optim <- nlminb(c(ivxbar,ivs2),OtiUCL,lower=c(0,0),upper =c(LSa,LSb))#
Ua=par_optim[[1]][1] #Prob Xbar
Ub=par_optim[[1]][2] #Prob S2
UCLxb=qnorm((1-Ua/2),u0,s0/(n^0.5)) #upper control limit for the x-bar
LCLxb=qnorm(Ua/2,u0,s0/(n^0.5)) #lower control limit for the x-bar
LCs2=qchisq((1-Ub),(n-1)) #control limit for the s2

#X-bar control chart
pxi=pnorm(LCLxb,u1,s1/(n^0.5))
pxs=1-pnorm(UCLxb,u1,s1/(n^0.5))
pxc=1-pxs-pxi

#S2 control chart
psf=1-pchisq(LCs2*(s0^2/s1^2),(n-1))
psc=1-psf

###############################################################################
#Now that we have the probabilities for this specific case, we solve it
   through a Markov Chain again
size<- 15 #Size of the Markov chain
MarkovChain<- matrix(0,nrow=size,ncol=size,byrow=TRUE)

MarkovChain[1,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
MarkovChain[2,]<-
    c(pxc*psc,0,pxs*psc,pxi*psc,0,pxc*psf,0,pxs*psf,pxi*psf,0,0,0,0,0,0)
MarkovChain[3,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
MarkovChain[4,]<-
    c(pxc*psc,pxs*psc,0,0,pxi*psc,pxc*psf,pxs*psf,0,0,pxi*psf,0,0,0,0,0)
MarkovChain[5,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
MarkovChain[6,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,0,0,0,0,0,pxc*psf,pxs*psf,0,pxi*psf,0)
MarkovChain[7,]<-
    c(pxc*psc,0,pxs*psc,pxi*psc,0,0,0,0,0,0,pxc*psf,0,pxs*psf,pxi*psf,0)
MarkovChain[8,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
MarkovChain[9,]<-
    c(pxc*psc,pxs*psc,0,0,pxi*psc,0,0,0,0,0,pxc*psf,pxs*psf,0,0,pxi*psf)
```

```
MarkovChain[10,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
MarkovChain[11,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
MarkovChain[12,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
MarkovChain[13,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
MarkovChain[14,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)
MarkovChain[15,]<-
    c(pxc*psc,pxs*psc,0,pxi*psc,0,pxc*psf,pxs*psf,0,pxi*psf,0,0,0,0,0,0)

#The stationary distribution
A = t(MarkovChain) - eye(15)
A[15,] = ones(1,15)
B = zeros(15, 1)
B[15,1] = 1
Solved_markov_chain = solve(A)%*%B

ARL1<- 1/sum(Solved_markov_chain[3,],Solved_markov_chain[5,],
          Solved_markov_chain[8,],Solved_markov_chain[10,],
          Solved_markov_chain[11,],Solved_markov_chain[12,],
          Solved_markov_chain[13,],Solved_markov_chain[14,],
          Solved_markov_chain[15,])


options(digits=5)
cat('UCLxb=',UCLxb,"\n")
cat('LCLxb=',LCLxb,"\n")
cat('LCs2=',LCs2,"\n")
cat('ARL1=',ARL1,"\n")
```

APPENDIX C – ARL values for the improved $\bar{X}$- S^2 unilateral control chart for a given sample size "n" and target $ARL_0 = 250$ using Monte Carlo simulations

```
library(pracma)
clear()
tic()

# Definition of parameters
# Target ARL0=370.4
# In-control
u0 = 0
sd0 = 1

# Out-of-control - Table 2
u1 = 0.5
sd1 = 1.5
n = 5

# Limits obtained by experimentation to reach target ARL0= 250
# LCL1=-0.98188; UCL1=0.98188; UCL2=8.3347 #n=4
LCL1 = -0.87822; UCL1 = 0.87822; UCL2 = 10.051 #n=5
# LCL1=-0.8017; UCL1=0.8017; UCL2=11.671 #n=6
# LCL1=-0.7422; UCL1=0.7422; UCL2=13.227 #n=7

runs = 1000000 # number of simulations for Monte Carlo
Result <- matrix(0, runs, 1)

for (i in 1:runs) {
  s1 = 0
  s2 = 0
  s1a = 0
  D <- c()

  while (s1 < 2 & s2 < 2 & s1a < 2) { # klein rule
    RR <- rnorm(n, u1, sd1)
    R = mean(RR)
    V = var(RR)
    T = R
    W = V * (n - 1) / sd0

    if (T > UCL1 & W > UCL2) {
      D <- rbind(D, 1)
      s1 = s1 + 1
      s2 = 0
      s1a = s1a + 1
    }

    if (T < LCL1 & W > UCL2) {
      D <- rbind(D, 1)
      s1 = 0
      s2 = s2 + 1
      s1a = s1a + 1
    }
```

```
    if (T >= LCL1 & T <= UCL1 & W > UCL2) {
      D <- rbind(D, 1)
      s1 = 0
      s2 = 0
      s1a = s1a + 1
    }

    if (T > UCL1 & W < UCL2) {
      D <- rbind(D, 1)
      s1 = s1 + 1
      s2 = 0
      s1a = 0
    }

    if (T < LCL1 & W < UCL2) {
      D <- rbind(D, 1)
      s1 = 0
      s2 = s2 + 1
      s1a = 0
    }

    if (T >= LCL1 & T <= UCL1 & W < UCL2) {
      D <- rbind(D, 1)
      s1 = 0
      s2 = 0
      s1a = 0
    }

    ua = T
    sa = W
  }
  Result[i, 1] = sum(D)
}

ARL <- mean(Result[, 1])
cat("ARL1=", ARL, "\n")
toc()
```

APPENDIX D – Control limits and ARL for the $\bar{X}$- S^2 bilateral control chart for a given sample size "n" and target ARL_0 of "ARL_0"

```
#A control chart for the joint monitoring of mean and variance
rm(list = ls())
##Markov chain

u0=0 #In-control average
u1=2 #out-of-control average (defined in the previous vector)
s0=1 #in-control standard deviation
s1=0.25 # out-of-control standard deviation (defined in the previous vector)
n=5 #Sample size
ARL0=250 #Target ARL0


OtiUCL <- function(alfa){
  alfag=1-(1-alfa[1])*(1-alfa[2])
  ARL=1/alfag
  ARLphi=(ARL-ARL0)^2
  return(ARLphi)
}
#upper limit to be used in the optimize function. Change as needed.
LSa=1
LSb=1
#initial value. Change as needed.
ivxbar=0.04
ivs2=0.04
par_optim <- nlminb(c(ivxbar,ivs2),OtiUCL,lower=c(0,0),upper =c(LSa,LSb))
alfaa=par_optim[[1]][1] #Prob Xbar
alfab=par_optim[[1]][2] #Prob S2

#ARL1
LSCxb=qnorm((1-alfaa/2),u0,s0/(n^0.5))
LICxb=qnorm(alfaa/2,u0,s0/(n^0.5))
LSCqui=qchisq((1-alfab/2),(n-1))
LICqui=qchisq(alfab/2,(n-1))

pc=pnorm(LICxb,u1,s1/(n^0.5))
pa=1-pnorm(LSCxb,u1,s1/(n^0.5))
pb=1-pa-pc

pas=1-pchisq(LSCqui*(s0^2/s1^2),(n-1))
pcs=pchisq(LICqui*(s0^2/s1^2),(n-1))
pbs=1-pas-pcs


ARL1<-1/(1-pb*pbs)

cat('LSCxb=',LSCxb,"\n")
cat('LICxb=',LICxb,"\n")
cat('LSCqui=',LSCqui,"\n")
cat('Probxbar=',alfaa,"\n")
cat('Probs2=',alfab,"\n")
cat('ARL1=',ARL1,"\n")
```

APPENDIX E – Control limits and ARL for the improved $\bar{X}$- S^2 bilateral control chart for a given sample size "n" and target ARL_0 of "ARL_0"

```
#A control chart for the joint monitoring of mean and variance with Klein's
    supplementary rule

require(pracma)
rm(list = ls())

cat('N','\t', 'u1','\t','s1','\t','ARL1','\n')

m1f<- c(0.00,0.25,0.50,1,2) #Vector containing different out-of-control averages
m1fs<- size(m1f)
s1f<- c(0.25,0.50,1,1.50,2)#Vector containing different out-of-control standard
    deviations
s1fs<- size(s1f)
nf<- c(5) #Vector containing different sample sizes
nfs<- size(nf)

###########################################################################################
for(k in 1:nfs[2]){
  for(i in 1:m1fs[2]){
    for(j in 1:s1fs[2]){
      u0=0 #In-control average
      u1=m1f[i] #out-of-control average (defined in the previous vector)
      s0=1 #in-control standard deviation
      s1=s1f[j]# out-of-control standard deviation (defined in the previous vector)
      n=nf[k] #Sample size
      ARL0=250 #Target ARL0

      ####################################################################################
      OtiUCL <- function(U){#Optimization function used to find the UCL and LCL
        UCLxb=qnorm((1-U[1]/2),u0,s0/(n^0.5))
        LCLxb=qnorm(U[1]/2,u0,s0/(n^0.5))
        LCs2a=qchisq((1-U[2]),(n-1))
        LCs2b=qchisq(U[3],(n-1))

        #X-bar control chart
        pxl=pnorm(LCLxb,u0,s0/(n^0.5))
        pxu=1-pnorm(UCLxb,u0,s0/(n^0.5))
        pxc=1-pxu-pxl

        #S2 control chart
        psu=1-pchisq(LCs2a,(n-1))
        psl=pchisq(LCs2b,(n-1))
        psc=1-psu-psl

        #Markov chain
        size<- 25 #Size of the markov chain
        MarkovChain<- matrix(0,nrow=size,ncol=size,byrow=TRUE)

        MarkovChain[1,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,
        pxu*psl,0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
        MarkovChain[2,]<- c(pxc*psc,0,pxc*psl,pxc*psu,0,pxu*psc,0,pxu*psl,
        pxu*psu,0,pxl*psc,0,pxl*psl,pxl*psu,0,0,0,0,0,0,0,0,0,0,0)
        MarkovChain[3,]<- c(pxc*psc,pxc*psu,0,0,pxc*psl,pxu*psc,pxu*psu,
        0,0,pxu*psl,pxl*psc,pxl*psu,0,0,pxl*psl,0,0,0,0,0,0,0,0,0,0)
```

```
    MarkovChain[4,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,
    pxu*psl,0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[5,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,
    pxu*psl,0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[6,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,0,0,0,0,0,pxl*psc,
    pxl*psu,pxl*psl,0,0,pxu*psc,pxu*psu,pxu*psl,0,0,0,0,0,0,0)
    MarkovChain[7,]<- c(pxc*psc,0,pxc*psl,pxc*psu,0,0,0,0,0,0,pxl*psc,
    0,pxl*psl,pxl*psu,0,pxu*psc,0,pxu*psl,pxu*psu,0,0,0,0,0,0)
    MarkovChain[8,]<- c(pxc*psc,pxc*psu,0,0,pxc*psl,0,0,0,0,0,pxl*psc,
    pxl*psu,0,0,pxl*psl,pxu*psc,pxu*psu,0,0,pxu*psl,0,0,0,0,0)
    MarkovChain[9,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
    0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[10,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,
    pxu*psl,0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[11,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,
    pxu*psl,0,0,0,0,0,0,0,0,0,0,0,0,pxl*psc,pxl*psu,pxl*psl,0,0)
    MarkovChain[12,]<- c(pxc*psc,0,pxc*psl,pxc*psu,0,pxu*psc,0,pxu*psl,
    pxu*psu,0,0,0,0,0,0,0,0,0,0,0,pxl*psc,0,pxl*psl,pxl*psu,0)
    MarkovChain[13,]<- c(pxc*psc,pxc*psu,0,0,pxc*psl,pxu*psc,pxu*psu,0,0,
    pxu*psl,0,0,0,0,0,0,0,0,0,0,pxl*psc,pxl*psu,0,0,pxl*psl)
    MarkovChain[14,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
    0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[15,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
    0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[16,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
    0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[17,]<- c(pxc*psc,0,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,0,0,
    pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[18,]<- c(pxc*psc,pxc*psu,0,0,0,pxu*psc,pxu*psu,pxu*psl,0,0,
    pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[19,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
    0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[20,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
    0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[21,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
    0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[22,]<- c(pxc*psc,0,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,0,0,
    pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[23,]<- c(pxc*psc,pxc*psu,0,0,0,pxu*psc,pxu*psu,pxu*psl,0,0,
    pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[24,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,0,0,
    pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
    MarkovChain[25,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,0,0,
    pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)



    A = t(MarkovChain) - eye(25)
    A[25,] = ones(1,25)
    B = zeros(25, 1)
    B[25,1] = 1
    Solved_markov_chain = solve(A)%*%B

    ARL<- 1/sum(Solved_markov_chain[4,],Solved_markov_chain[5,],
             Solved_markov_chain[9,],Solved_markov_chain[10,],
             Solved_markov_chain[14,],Solved_markov_chain[15,],
             Solved_markov_chain[16,],Solved_markov_chain[17,],
             Solved_markov_chain[18,],Solved_markov_chain[19,],
             Solved_markov_chain[20,],Solved_markov_chain[21,],
             Solved_markov_chain[22,],Solved_markov_chain[23,],
             Solved_markov_chain[24,],Solved_markov_chain[25,])

    ARLphi=(ARL-ARL0)^2
    #ARLphi=ARL
    return(ARLphi)
  }

  ##################################################################################################
  LSa=0.06462 #Limit used in the optimize function for the x-bar control chart
  LSb=0.11456 #Limit used in the optimize function for the s2 control chart
```

```
par_optim <- nlminb(c(0.04,0.04,0.04),OtiUCL,lower=c(0,0,0),upper
    =c(LSa,LSb,LSb))#
Ua=par_optim[[1]][1] #Prob Xbar
Ub=par_optim[[1]][2] #Prob S2
Uc=par_optim[[1]][3]

UCLxb=qnorm((1-Ua/2),u0,s0/(n^0.5)) #upper control limit for the x-bar
LCLxb=qnorm(Ua/2,u0,s0/(n^0.5)) #lower control limit for the x-bar
LCs2a=qchisq((1-Ub),(n-1)) #control limit for the s2
LCs2b=qchisq(Uc,(n-1)) #control limit for the s2

#X-bar control chart
pxl<-pnorm(LCLxb,u1,s1/(n^0.5))
pxu=1-pnorm(UCLxb,u1,s1/(n^0.5))
pxc=1-pxu-pxl

#S2 control chart
psu=1-pchisq(LCs2a*(s0^2/s1^2),(n-1))
psl=pchisq(LCs2b*(s0^2/s1^2),(n-1))
psc=1-psu-psl
##########################################################################################
#Now that we have the probabilities for this specific case, we solve it
    through a Markov Chain again
size<- 25 #Size of the markov chain
MarkovChain<- matrix(0,nrow=size,ncol=size,byrow=TRUE)

MarkovChain[1,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[2,]<- c(pxc*psc,0,pxc*psl,pxc*psu,0,pxu*psc,0,pxu*psl,
pxu*psu,0,pxl*psc,0,pxl*psl,pxl*psu,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[3,]<- c(pxc*psc,pxc*psu,0,0,pxc*psl,pxu*psc,pxu*psu,0,0,
pxu*psl,pxl*psc,pxl*psu,0,0,pxl*psl,0,0,0,0,0,0,0,0,0,0)
MarkovChain[4,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[5,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[6,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,0,0,0,0,0,pxl*psc,pxl*psu,
pxl*psl,0,0,pxu*psc,pxu*psu,pxu*psl,0,0,0,0,0,0,0)
MarkovChain[7,]<- c(pxc*psc,0,pxc*psl,pxc*psu,0,0,0,0,0,0,pxl*psc,0,
pxl*psl,pxl*psu,0,pxu*psc,0,pxu*psl,pxu*psu,0,0,0,0,0,0)
MarkovChain[8,]<- c(pxc*psc,pxc*psu,0,0,pxc*psl,0,0,0,0,0,pxl*psc,pxl*psu,
0,0,pxl*psl,pxu*psc,pxu*psu,0,0,pxu*psl,0,0,0,0,0)
MarkovChain[9,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[10,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[11,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,0,0,0,0,0,0,0,0,0,0,pxl*psc,pxl*psu,pxl*psl,0,0)
MarkovChain[12,]<- c(pxc*psc,0,pxc*psl,pxc*psu,0,pxu*psc,0,pxu*psl,pxu*psu,
0,0,0,0,0,0,0,0,0,0,0,pxl*psc,0,pxl*psl,pxl*psu,0)
MarkovChain[13,]<- c(pxc*psc,pxc*psu,0,0,pxc*psl,pxu*psc,pxu*psu,0,0,pxu*psl,
0,0,0,0,0,0,0,0,0,0,pxl*psc,pxl*psu,0,0,pxl*psl)
MarkovChain[14,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,
pxu*psl,0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[15,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[16,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[17,]<- c(pxc*psc,0,pxc*psl,0,0,pxu*psc,
pxu*psu,pxu*psl,0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[18,]<- c(pxc*psc,pxc*psu,0,0,0,pxu*psc,
pxu*psu,pxu*psl,0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[19,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[20,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[21,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,
0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
MarkovChain[22,]<- c(pxc*psc,0,pxc*psl,0,0,pxu*psc,pxu*psu,
pxu*psl,0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
```

```
  MarkovChain[23,]<- c(pxc*psc,pxc*psu,0,0,0,pxu*psc,pxu*psu,
  pxu*psl,0,0,pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
  MarkovChain[24,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,0,0,
  pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)
  MarkovChain[25,]<- c(pxc*psc,pxc*psu,pxc*psl,0,0,pxu*psc,pxu*psu,pxu*psl,0,0,
  pxl*psc,pxl*psu,pxl*psl,0,0,0,0,0,0,0,0,0,0,0,0)

  #Solved_markov_chain<-MarkovChain%^%100000

  a<- replicate(25,0)
  for(l in 1:size){

   a[l]<- sum(MarkovChain[i,])
  }


  A = t(MarkovChain) - eye(25)
  A[25,] = ones(1,25)
  B = zeros(25, 1)
  B[25,1] = 1
  Solved_markov_chain = solve(A)%*%B

  ARL1<- 1/sum(Solved_markov_chain[4,],Solved_markov_chain[5,],
            Solved_markov_chain[9,],Solved_markov_chain[10,],
            Solved_markov_chain[14,],Solved_markov_chain[15,],
            Solved_markov_chain[16,],Solved_markov_chain[17,],
            Solved_markov_chain[18,],Solved_markov_chain[19,],
            Solved_markov_chain[20,],Solved_markov_chain[21,],
            Solved_markov_chain[22,],Solved_markov_chain[23,],
            Solved_markov_chain[24,],Solved_markov_chain[25,])


  options(digits=5)
  cat(n,'\t',u1,'\t',s1,'\t',ARL1,'\n')

}}}
```

APPENDIX F – ARL values for the $\bar{X}$-S^2 bilateral control chart with the EWMA control scheme using Monte Carlo simulations

```
library(pracma)
clear()
tic()
# Definition of parameters
#Target ARL0=250.
#In-control
u0=0
sd0=1
#out-of-control - Table 3.
u1=0.25
sd1=0.5
lambda1=0.1
lambda2=0.1
#Chen et. al. (2001)
k1=2.81
k2=2.86
n=5
LCL1=u0-k1*((lambda1/(2-lambda1))^0.5)*(sd0/(n^0.5))
UCL1=u0+k1*((lambda1/(2-lambda1))^0.5)*(sd0/(n^0.5))
c=log(sd0^2)-1/(n-1)-1/(3*(n-1)^2)+2/(15*(n-1)^4)
d=2/(n-1)+2/((n-1)^2)+4/(3*(n-1)^3)-16/(15*(n-1)^5)
LCL2=c-k2*(d*lambda2/(2-lambda2))^0.5
UCL2=c+k2*(d*lambda2/(2-lambda2))^0.5

runs=100000
Result<-matrix(0, runs, 1)

for(i in 1:runs){
  ua=0 #starting value
  sa=c #starting value
  s=0
  D<-c()
  s<-0
  while(s<1){
    RR<-rnorm(n,u1,sd1)
    R=mean(RR)
    V=var(RR)
    T=lambda1*R+(1-lambda1)*ua
    W=lambda2*log(V)+(1-lambda2)*sa

    if (T>UCL1 | T<LCL1 |W>UCL2 |W<LCL2){
      D<-rbind(D,1)
      s=s+1
    }else{
      D<-rbind(D,1)
      s=0
    }
    ua=T
    sa=W
  }
  Result[i,1]=sum(D)


}
```

```
ARL<-mean(Result[,1])
cat("ARL1=",ARL,"\n")

toc()
```

APPENDIX G – ARL values for the $\bar{X}$-S^2 bilateral control chart with the EWMA control scheme and Klein's 2-of-2 run rule using Monte Carlo simulations

```
library(pracma)
clear()
tic()
# Definition of parameters
#Target ARL0=250.
#In-control
u0=0
sd0=1
#out-of-control
u1=0
sd1=1
lambda1=0.1
lambda2=0.1
n=5
#Limits obtained by experimentation to reach target ARL0
LCL1=-0.25957
UCL1=0.25957
LCL2=-0.7456
UCL2=0.20497
c=log(sd0^2)-1/(n-1)-1/(3*(n-1)^2)+2/(15*(n-1)^4) #starting value
runs=10000
Result<-matrix(0, runs, 1)

for(i in 1:runs){
  print(i)
  ua=0 #starting value
  sa=c #starting value
  s1=0
  s2=0
  s1a=0
  s2a=0
  D<-c()
  while(s1<2 & s2<2 & s1a<2 & s2a<2){ #klein rule
    RR<-rnorm(n,u1,sd1)
    R=mean(RR)
    V=var(RR)
    T=lambda1*R+(1-lambda1)*ua
    W=lambda2*log(V)+(1-lambda2)*sa

    if (T>UCL1 & W>UCL2){
      D<-rbind(D,1)
      s1=s1+1
      s2=0
      s1a=s1a+1
      s2a=0
    }

    if (T>UCL1 & W<LCL2){
      D<-rbind(D,1)
      s1=s1+1
      s2=0
      s1a=0
      s2a=s2a+1
    }
```

```
    if (T<LCL1 & W>UCL2){
     D<-rbind(D,1)
     s1=0
     s2=s2+1
     s1a=s1a+1
     s2a=0
    }

    if (T<LCL1 & W<LCL2){
     D<-rbind(D,1)
     s1=0
     s2=s2+1
     s1a=0
     s2a=s2a+1
    }

    if (T>=LCL1 & T<= UCL1 & W>=LCL2 & W<= UCL2 ){
     D<-rbind(D,1)
     s1=0
     s2=0
     s1a=0
     s2a=0
    }


    if (T>=LCL1 & T<= UCL1 & W<=LCL2){
     D<-rbind(D,1)
     s1=0
     s2=0
     s1a=0
     s2a=s2a+1
    }

    if (T>=LCL1 & T<= UCL1 & W>=UCL2){
     D<-rbind(D,1)
     s1=0
     s2=0
     s1a=s1a+1
     s2a=0
    }

    if (W>=LCL2 & W<= UCL2 & T>=UCL1){
     D<-rbind(D,1)
     s1=s1+1
     s2=0
     s1a=0
     s2a=0
    }

    if (W>=LCL2 & W<= UCL2 & T<=LCL1){
     D<-rbind(D,1)
     s1=0
     s2=s2+1
     s1a=0
     s2a=0
    }

    ua=T
    sa=W
  }
  Result[i,1]=sum(D)


}

ARL<-mean(Result[,1])
cat("ARL1=",ARL,"\n")
toc()
```