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The Multiscale Hybrid Mixed Method for Parabolic Problems

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Resumo

Essa Tese tem como objetivo generalizar o método Multiescala Híbrido Misto (MHM) para equações diferenciais parciais parabólicas. Esse método numérico se baseia na formulação variacional híbrida primal do problema, onde a continuidade das soluções na fronteira dos elementos da malha espaço-tempo é imposta pelo uso de multiplicadores de Lagrange tanto para o espaço, quanto para o tempo. Tal abordagem conduz na formulação de um sistema acoplado de equações locais e globais, cuja solução é a mesma do problema original. As soluções das equações locais formam uma base para o problema global, podendo ser calculadas numericamente em paralelo e naturalmente incorporando as informações das escalas mais finas. Como as soluções são obtidas através de um esquema de marcha no tempo, a flexibilidade do método se reflete na possibilidade de utilizar diferentes partições espaço-tempo para aproximar a solução numérica em cada intervalo de tempo. Além disso, as estimativas de erro obtidas pela análise de convergência do primeiro nível de discretização mostram que as taxas de convergência espacial e temporal estão ligadas aos parâmetros de discretização da malha espaço-tempo, bem como aos graus dos polinômios utilizados para aproximar os multiplicadores de Lagrange na fronteira da malha.

Palavras-chave: MHM. MHM parabólico. Métodos Multiescala.

Abstract

This thesis aims to generalize the Multiscale Hybrid Mixed method (MHM) for parabolic partial differential equations. This numerical method is based on a primal variational formulation of the problem, where the continuity of the solution on the boundary of the space-time mesh is enforced thru the use of Lagrange multipliers either for space and time. Such approach leads to the formulation of a coupled system of global-local equations, where the solution is the same as the solution of the original problem. The solutions of the local equations turn into a basis used to solve the global problem, and due to the independence of such solutions, they can be numerically approximated in parallel, while capturing the in formation from the fine scales. Since the solutions are obtained thru a time marching scheme, the flexibility of the method reflects on the possibility to use different space-time partitions to approximate numerically the solution on each time interval. Besides, the error estimates for the first level discretization obtained in this work show that the spatial and temporal convergence rates are related to the discretization parameters of the space-time mesh, as well as the degree of the polynomials used to approximate the Lagrange multipliers over the boundary of the mesh.

Keywords: MHM. Parabolic MHM. Multiscale methods.

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1 Introduction

The finite element method (FEM) is a numerical technique used to approximate solutions to partial differential equations (PDEs). It is widely employed in various fields of engineering and science to solve complex problems where analytical solutions are difficult or impossible to obtain.

PDEs are mathematical equations that describe a variety of phenomena, such as heat transfer, fluid flow, and structural mechanics. They involve partial derivatives, which represent rates of change with respect to multiple variables. Analytical solutions to PDEs are often challenging to find due to their complexity, nonlinearities, or irregular geometries. This is where numerical methods like the finite element method come into play.

The finite element method discretizes the domain of the problem into smaller geometrical elements. These elements are interconnected and share a node, an edge or a face in 3D. The PDEs are then approximated by a system of algebraic equations, typically represented by a matrix equation. The unknowns in this system are the values of the solution, in some cases, at the nodes of the mesh. This method is very flexible and can handle problems with complex geometries, material properties, and boundary conditions. It is widely used in diverse fields, including structural analysis, heat transfer, fluid dynamics, electromagnetics, and many others. The method has been extensively studied and developed, leading to a rich body of theory and efficient computational algorithms.

To handle phenomena that exhibit variations across multiple scales, multiscale finite element methods (MsFEM) are widely used to minimize computational costs while capturing the nuances on parts of the domain. These methods, first introduced by (BABUSKA; OSBORN, 1983), acknowledge that certain systems involve intricate details at local levels alongside broader behaviors at larger scales. Rather than using a uniformly fine mesh throughout the entire domain, multiscale approaches employ coarse representations where global features dominate and introduce finer details only where necessary. This adaptability to different scales is particularly crucial in fields such as materials science, geophysics, and fluid dynamics, where phenomena can occur at vastly different levels of detail. The essence of multiscale numerical methods is summarized as follows: the domain is decomposed into a series of coarse element problems. Appropriate boundary conditions are applied to these local problems and they are solved on the fine scale to obtain the (coarse scale) multiscale finite element basis. The global solution is then approximated through this basis by a reduced degree-of-freedom globally coupled system. The independence of the localized subproblems (local problems) implies in a computational efficiency of the method.

Multiscale methods has been widely studied for elliptic equations, with consistent results in the works (CHU; GRAHAM; HOU, 2009), (WHEELER; XUE; YOTOV, 2012), (ARBOGAST; XIAO, 2013), (HOU; WU; CAI, 1999). We also mention here the generalized multiscale methods (GMsFEM), that introduce the construction of coarse scale spaces for MsFEMs that results in accurate coarse-scale solutions. The basis functions of such coarse spaces are computed using eigenvectors of an eigenvalue problem and partition of unity functions. This approach, first proposed in (HOU; WU, 1997) and then explored in the works of (EFENDIEV; GALVIS; WU, 2011), (EFENDIEV; GALVIS; HOU, 2013), (CHUNG; EFENDIEV; LI, 2014), (CHUNG; EFENDIEV; LEUNG, 2015), (CHUNG; EFENDIEV; LEUNG, 2017), to cite a few, successfully deal with heterogeneous coefficient with high-contrast.

A class of these multiscale numerical methods that is the main object of this work is called Multiscale Hybrid Mixed method (MHM), proposed at first by (ARAYA et al., 2013) to solve elliptic equations. The method is based on the primal hybrid variational formulation first introduced by (RAVIART; THOMAS, 1977), where its approach to solving PDEs simultaneously consider both the primary variables (such as displacements, temperatures, or concentrations) and auxiliary variables (such as fluxes or traction). The solution of the elliptic equation is therefore posed in a weaker, broken space which relaxes continuity, allows reconstruction of the dual variable, and localizes computations. The primary variables are typically approximated using standard finite element basis functions, while the auxiliary variables are introduced to capture additional information related to the solution, often at the element edges or faces. The MHM method, as a multiscale method, addresses two different meshes: a fine-scale mesh to capture the fine-scale details by incorporating information through local basis functions computed in parallel, and a coarse-scale mesh to capture the overall behavior and impose interelement continuity on faces and edges. The MHM method shares similarities with some works, such as (ARBOGAST et al., 2007) and (COCKBURN; GOPALAKRISHNAN, 2004). These works also adopt a divide-and-conquer approach which couples local basis computed in fine scales of coarse elements into a global problem in order to ensure continuity. On the other hand, they rely on a dual-hybrid procedure, i.e., instead of perform a hybridization on the elliptic model problem, they hybridize its mixed version. As a result, the Lagrange multipliers allow for relaxing the continuity of the flux and driving local problems as they prescribe Dirichlet boundary conditions at a local level, not Neumann condition as in the MHM.

In this work we present a generalization of the MHM method for the following linear parabolic equation

$$\begin{cases} \partial_t u - \nabla \cdot (A \nabla u) = f \quad \text{in} \quad \Omega \times [0,T], \\ \\ u = 0 \quad \text{on} \quad \partial\Omega \times [0,T], \\ \\ u = u_0 \quad \text{at} \quad \Omega \times \{t = 0\}, \end{cases} \tag{1.1}$$

where $f \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and $A \in [L^\infty(\Omega)]^{d\times d}$, $d \in \{2,3\}$, a symmetric matrix satisfying the parabolicity condition: there exists constants $c_{min}, c_{max} > 0$ such that

$$c_{min}|\xi|^2 \leq A(x,t)\xi_i\xi_j \leq c_{max}|\xi|^2, \text{ a.e. in } \Omega, \tag{1.2}$$

for all $(x,t) \in \Omega \times [0,T]$, $\xi \in \mathbb{R}^d$. Our method is based on the primal hybrid variational formulation of problem (1.1), which is defined on spaces where discontinuities are allowed either at the edges or faces of the space discretization and at the end points of the intervals of the time discretization. Then, we obtain a coupled system of global-local problems, where the basis functions incorporate fine scale contributions in space and time, upscalling this local behaviour. We also perform an error analysis of the method, alongside with numerical simulations to validate the theoretical convergence rates obtained in terms of the space and time partition parameters. Although the numerical method does not require regularity of the matrix A, we assume in our error analysis that the matrix is time independent and its coefficients are smooth. See remark 5.9 for more details.

The bibliography on multiscale methods for parabolic equations is not as extensive as for elliptic problems, but it has been increasing over the years. Most of them are generalizations of existing methods for elliptic equations, which we cite a few in the sequence. We start dividing the parabolic multiscale methods into two groups: a group where the time is discretized using some time-stepping method and then a multiscale method is applied only for the spatial variable; and another group where the multiscale basis functions are of space-time type. In the work of (TAN; HOANG, 2019), a sparse tensor product FE method for monotone parabolic problems is developed. It is introduced the multiscale homogenized equation that depends on n separated microscopic scales, containing all the course and fine scale information. Then, a full tensor product FEs and a sparse tensor product FEs for the backward Euler method is applied to these equations and the level of accuracy of such approaches are showed to be essentially equivalent. The same idea is used for the Crank–Nicolson method. In (SRINIVASAN; LAZAROV; MINEV, 2016) a direction splitting method in time is applied to the parabolic equation in order to evolve the solution from one time step to the next by intermediate semidiscrete elliptic problems. Then, for the space discretization two methods are employed: a finite volume multiscale method and a coarse-grid approximation by projecting the fine scale components (operator/right hand side) onto a coarse space. The constraint energy minimizing generalized multiscale finite element method (CEM-GMsFEM) combines the construction of spatial multiscale basis functions to capture the fine-scales heterogeneities by solving local constraint-minimizing problems with time discretization schemes. In special, (POVEDA; GALVIS; CHUNG, 2023) analyzes an exponential time integration method for semilinear parabolic problems in high-contrast multiscale media addressing a convergence analysis for such approach. The primal hybrid formulation for parabolic boundary initial value problems are introduced for the semidiscrete case in (ACHARYA; PATEL, 2016). Optimal order estimates for the primal and hybrid variables are established. A fully discrete scheme using backward Euler method is derived, along with optimal order error estimates. This work is particularly interesting, since some of its ideas were employed in the semidiscrete primal hybrid formulation contained in this thesis.

A space-time domain decomposition mixed finite element method for parabolic problems is proposed in (JAYADHARAN et al., 2023). It allows non-matching spatial grids and local time stepping via space-time mortar finite elements. This setting provides high flexibility with individual discretizations of each space-time coarse element, and in particular for local time stepping. Space-time parallelization is obtained by reducing to a space-time interface problem requiring the solution of the local problems on each space-time macroelement to exchange boundary data through transmission conditions, in the essence of space-time domain decomposition methods. In (LJUNG; MAIER; MÅLQVIST, 2022), a space-time multiscale method for parabolic problems with a coefficient A that is highly oscillatory in space and time is developed. Based on the framework of the Variational Multiscale Method, a course-scale representation of the differential operator is enriched by space-time corrector functions, providing well approximated discrete solutions for multiple source terms. A proof of the first-order convergence independently of the oscillatory scales of the coefficient A is provided, and computations of the space-time correctors can be localized since it is shown that they decay exponentially in both space and time. Another interesting work with multiscale basis functions is proposed in (CHUNG et al., 2018). The approach is based on a GMsFEM using space-time coarse cells, where space-time snapshot and offline spaces are constructed. The solutions of the space-time snapshot spaces, obtained using randomized boundary conditions and oversampling, are combined with local spectral problems, also in space-time domains, in order to build the multiscale offline basis functions. It is also proposed a possible construction of the called online basis functions, which uses the residual information to build new multiscale basis functions adptively in order to descrease the error rapidly with few iterations.

The thesis is structured in the following sequence: we start recovering some classical results in Chapter 2, related to finite element methods applied in parabolic PDE's, containig the error analysis of the semidiscrete and fully discrete approaches. Chapter 3 is dedicated to build the primal hybrid variational formulation for parabolic equations, generalizing the strategy employed for elliptic equations. In chapter 4 the parabolic MHM is posed. The discrete spaces used to approximate the unknows of the coupled system of global-local equations are defined, and an algorithm of the method is presented as well. This chapter ends with the equivalence of two time discretization schemes of the method. Chapter 5 is dedicated to the numerical analysis of the method defined in the previous chapter. We start with the well-posedeness of the discrete method, and in the sequence we prove error estimates, for the space and time first level parameters, in order to obtain a fully discrete convergence estimate. The dependence of such estimates on the numerical method is studied in the end. Numerical validation of the estimates obtained previously are the main subject of chapter 6, illustrated with tables containing the error in the natural norms of the problem for each value of the space and time parameters, error curves and screenshots of the numerical solution as well. In chapter 7 we summarize the advantages of the parabolic MHM method and talk about the next steps to be studied in more detail.

Notation: throughout this thesis c or C are generic constants independent of the mesh parameters.

2 Classical Results

This chapter provides an overview of well known results of Finite Element Method (FEM) applied to parabolic equations that, in this present work, will be widely used to numerically approximate the multiscale basis functions of our two level method, developed in the following chapters, on the second level discretization. The theory presented in this chapter is based on the book of (THOMEE, 2013) and is displayed here for the reader's convenience.

Parabolic equations commonly describe phenomena that evolve over time, such as heat conduction, diffusion, or the flow of fluids. To solve such equations using the FEM, a time-stepping approach is typically employed. This involves discretizing both the spatial domain and the time interval into smaller elements and time steps, respectively.

It is worth noting that parabolic equations often exhibit stability and convergence requirements due to the presence of time derivatives. The time step size and the spatial mesh size should be chosen carefully to ensure numerical stability and accuracy. Additionally, appropriate treatment of boundary conditions is essential to obtain physically meaningful solutions.

We start with the Galerkin FEM to approximate the solution of the model initial boundary value problem (1.1), under the assumption that Ω is a polygonal convex domain in $\mathbb{R}^d$.

The solutions of (1.1) are usually defined in spaces called Sobolev Spaces, in special the ones denoted by $H^r(\Omega) = W^{r,2}(\Omega)$, which are Hilbert spaces that contains all the real valued functions $v \in L^2(\Omega)$, with $D^\alpha v \in L^2(\Omega)$, $|\alpha| \leq r$. Here, α is a multiindex given by a vector of positive integers $(\alpha_1, \cdots, \alpha_d)$ and $|\alpha|$ stands for $\sum_{j=1}^{d} \alpha_j$. The symbol D^α refers to the multiindex weak derivative of a function v, which is defined by

$$D^\alpha v = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_d)^{\alpha_d} v$$

and satisfies the identity

$$\int_\Omega v D^\alpha w dx = (-1)^{|\alpha|} \int_\Omega D^\alpha v w dx \tag{2.1}$$

for all $w \in C_0^\infty(\Omega)$. The norms adopted to these spaces are the following ones,

$$||v|| = ||v||_{L^2(\Omega)} = \left(\int_\Omega v^2 \ dx \right)^{\frac{1}{2}}, \tag{2.2}$$

and, for a positive integer r,

$$||v||_r = ||v||_{H^r(\Omega)} = \left(\sum_{|\alpha| \leq r} ||D^\alpha v||^2 \right)^{\frac{1}{2}}. \tag{2.3}$$

See Appendix C for more details.

The space $H_0^1(\Omega) \subset H^1(\Omega)$ contains all the functions $v \in L^2(\Omega)$ with $\nabla v \in L^2(\Omega)$, that vanishes on $\partial\Omega$. A relevant result is that if $v \in H_0^1(\Omega)$, then the norm $||\nabla v||$ is equivalent to $||v||_1$ (see Theorem 3 of section 5.6.1 in (EVANS, 2010)) and the following inequality holds

$$c||v||_1 \leq ||\nabla v|| \leq ||v||_1, \quad \forall\, v \in H_0^1(\Omega), \quad c > 0. \tag{2.4}$$

Let $\{\mathcal{T}_h\}$ be a family of partitions of Ω into small disjoint triangles K (we call it a triangulation of Ω) and let the index h denote the maximum length of the edges of the triangulation $\mathcal{T}_h$, meaning that when we refine $\mathcal{T}_h$, the parameter h decreases. We also assume that the triangulation satisfies the conditions

a) None vertex of any triangle K lies on the interior of an edge of another triangle;

b) The measure of the internal angles of each triangle of $\mathcal{T}_h$ is bounded below by a positive constant independent of h;

c) $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$.

In addition, we assume that $\mathcal{T}_h$ is a *regular* triangulation, which means that there exists a constant $c > 0$ such that every $K \in \mathcal{T}_h$ contains a circle with radius ρ_K satisfying the condition

$$\rho_K > \frac{h_K}{c},$$

where h_K is half of the diameter of element K.

We now consider a finite dimensional space S_h composed of continuous functions on $\overline{\Omega}$ which are linear on each triangle of $\mathcal{T}_h$ and vanish outside $\overline{\Omega}$. Let $\{P_j\}_{j=1}^{N_h}$ be the set of interior vertices of $\mathcal{T}_h$.

Remark 2.1. A function in S_h is uniquely determined by its values at the points P_j and therefore depends on the number N_h. For a proof of this see (BRAESS, 2007) chapter II remark 5.4.

Let ϕ_j be functions in S_h, called *shape functions*, that has a pyramidal shape and satisfy

$$\phi_j(P_i) = \delta_{ji}. \tag{2.5}$$

The set $\{\phi_j\}_{j=1}^{N_h}$ of these functions form a basis for S_h and we can write every function $\xi \in S_h$ as

$$\xi(x) = \sum_{j=1}^{N_h} \alpha_j \phi_j, \text{ where } \alpha_j = \xi(P_j). \tag{2.6}$$

With these tolls, any smooth function v on Ω that vanishes on $\partial\Omega$ can be approximated by the interpolant operator $I_h v \in S_h$ defined as

$$I_h v(x) = \sum_{j=1}^{N_h} v(P_j)\phi_j(x). \tag{2.7}$$

The interpolant just defined satisfy the following error estimates, which can be found in (BRENNER; SCOTT, 2007): for $v \in H^2(\Omega) \cap H_0^1(\Omega)$ we have

$$||v - I_h v||_{L^2(\Omega)} \leq C\, h^2 ||v||_{H^2(\Omega)}, \tag{2.8}$$

$$||\nabla(v - I_h v)||_{L^2(\Omega)} \leq C\, h ||v||_{H^2(\Omega)}. \tag{2.9}$$

The proof of these estimates on an element $K \in \mathcal{T}_h$ is achieved by using Bramble-Hilbert Lemma and the generalization for the whole domain Ω uses affine transformations from a reference element K to all the other ones in $\mathcal{T}_h$.

From now on we assume that the family $\{S_h\}$ of finite dimensional subspaces of $H_0^1(\Omega)$ that we deal here are such that, for some integer $r \geq 2$ and small h, the following estimate holds

$$\inf_{v_h \in S_h} \{||v - v_h||_{L^2(\Omega)} + h||\nabla(v - v_h)||_{L^2(\Omega)}\} \leq Ch^s ||v||_{H^s(\Omega)} \tag{2.10}$$

for $1 \leq s \leq r$ when $v \in H^s \cap H_0^1(\Omega)$. The optimal orders where the functions and its gradients achieve under (2.10) are, respectively, $O(h^r)$ and $O(h^{r-1})$.

Now we turn our attention to the problem (1.1). We begin writing it in the weak form: we multiply the heat equation by a test function $v \in H_0^1(\Omega)$, integrate over Ω and apply Green's identity on $(\nabla \cdot (A\nabla u), v)$ to obtain

$$(u_t, v) + (A\nabla u, \nabla v) = (f, v),\ \forall\, v \in H_0^1(\Omega). \tag{2.11}$$

We then say that a function $u(x,t)$ is a weak solution of (1.1) on $[0,T]$ if (2.11) holds with $u \in L^2(0,T; H_0^1(\Omega))$, $f \in L^2(0,T; H^{-1}(\Omega))$ and $u(x,0) = u_0$. Once the boundary is smooth, the solution u is smooth provided that the data f and u_0 are smooth and satisfy some compatibility conditions at $t = 0$. A *parabolic regularity* estimate for such solutions is given in (EVANS, 2010) (section 7.1.3), considering the entries of matrix A smooth enough on $\overline{\Omega}$, with $u^{(j)} = (\partial/\partial t)^j u$ and $C = C(m,T)$, by

$$\sum_{j=0}^{m+1} \int_0^T ||u^{(j)}||^2_{2(m-j)+2} dt \leq C \left(||u_0||^2_{2m+1} + \sum_{j=0}^{m} \int_0^T ||f^{(j)}||^2_{2(m-j)} dt \right) \tag{2.12}$$

for $m \geq 0$, along with the compatibility conditions

$$\begin{aligned} g_0 &= u_0 \in H_0^1(\Omega), \\ g_1 &= f(0) - \nabla \cdot (A\nabla g_0) \in H_0^1(\Omega), \\ &\vdots \\ g_m &= \frac{d^{m-1} f}{dt^{m-1}}(0) - \nabla \cdot (A\nabla g_{m-1}) \in H_0^1(\Omega). \end{aligned} \tag{2.13}$$

To come up with the analysis of the numerical solution, we proceed following two steps. The first one is to approximate the solution $u(x,t)$ only in the spatial variable x, for each fixed t. This implies that for each $t \in [0,T]$ $u_h(x,t)$ belongs to the finite linear dimensional space S_h. Due to this, the solution u_h is referred as a *spatially discrete*, or just *semidiscrete*, solution. The second step consists in discretizing this system of equations also in time by a *time stepping* method to produce a fully discrete approximation to $u(x,t)$. In our work this approximation in time will be achieved by a Finite Difference approximation of the time derivative.

Based on the weak formulation (2.11), we define our semidiscrete problem for $u_h(t) = u_h(.,t) \in S_h$ and t fixed as

$$(\partial_t u_h, v_h) + (A\nabla u_h, \nabla v_h) = (f, v_h),\ \forall\, v_h \in S_h,\ t \in [0,T], \tag{2.14}$$

with discrete initial condition $u_h(0) = u_{0,h}$, where $u_{0,h}$ is some approximation of u_0 in S_h. Considering the basis $\{\phi_j\}_{j=1}^{N_h}$ of S_h, we state the semidiscrete problem as: Find the coefficients $\alpha_j(t)$ in $u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\phi_j(x)$ such that

$$\sum_{j=1}^{N_h} \alpha_j'(t)(\phi_j, \phi_l) + \sum_{j=1}^{N_h} \alpha_j(t)(A\nabla\phi_j, \nabla\phi_l) = (f, \phi_l),\ \ k = 1, \cdots, N_h, \tag{2.15}$$

with initial condition $\alpha_j(0) = \gamma_j$, where γ_j, $j = 1, \cdots, N_h$, are the coefficients of the aprroximation $u_{0,h}$ of u_0 in S_h. Expressing in matrix notation we have the system

$$\mathcal{M}\alpha'(t) + \mathcal{K}\alpha(t) = \widetilde{f}(t), \quad t \in [0,T], \quad \alpha(0) = \gamma, \tag{2.16}$$

where $\mathcal{M}_{jl} = (\phi_j, \phi_l)$, $\mathcal{K}_{jl} = (A\nabla\phi_j, \nabla\phi_l)$ and $\widetilde{f}_l = (f, \phi_l)$. Matrix $\mathcal{K}$ is called *stiffness matrix* and matrix $\mathcal{M}$ is called *mass matrix*. Both of them are positive definite and invertible, and then the system can be rewritten as

$$\alpha'(t) + \mathcal{M}^{-1}\mathcal{K}\alpha(t) = \mathcal{M}^{-1}\widetilde{f}(t), \quad t \in [0,T], \quad \alpha(0) = \gamma, \tag{2.17}$$

which has an unique solution for $t \in [0,T]$. The following result gives an estimate for the error between the solution of the semidiscrete problem and the solution of the continuous one.

Theorem 2.2. *Let u and u_h be solutions to (1.1) and (2.11) respectively. Assume that $u_0 = 0$ on $\partial\Omega$ and that $u \in H^l(0,T; H^r \cap H_0^1(\Omega))$, with $l \geq 1$ and r implicitly defined by (2.10). Therefore*

$$||u(t) - u_h(t)|| \leq ||u_0 - u_{0,h}|| + Ch^r \left(||u_0||_r + \int_0^t ||u_t||_r ds \right),\ \ for\ \ t \in [0,T]. \tag{2.18}$$

To obtain the proof of such estimate we use the interpolator called *Ritz projection* R_h onto S_h, defined by the orthogonal projection of u with respect to the inner product $(A\nabla u, \nabla v)$ as

$$(A\nabla R_h u, \nabla v_h) = (A\nabla u, \nabla v_h),\ \ \forall\ v_h \in S_h,\ u \in H_0^1(\Omega). \tag{2.19}$$

The strategy used to obtain the result is to write the error in the parabolic problem as a sum of two terms

$$u(t) - u_h(t) = \theta(t) + \rho(t), \text{ where } \theta = u_h - R_h u \text{ and } \rho = R_h u - u, \tag{2.20}$$

and then bound each of them separately.

We observe that the Ritz projector is stable in $H_0^1(\Omega)$ because if we replace v_h in (2.19) by $R_h u$ we have

$$\begin{aligned} c_{min} \; ||\nabla R_h u||^2_{L^2(\Omega)} \leq (A\nabla R_h u, \nabla R_h u) &= (A\nabla u, \nabla R_h u) \\ &\leq ||\nabla R_h u||_{L^2(\Omega)} ||A\nabla u||_{L^2(\Omega)} \\ &\leq \sqrt{c_{max}} \; ||\nabla R_h u||_{L^2(\Omega)} ||\nabla u||_{L^2(\Omega)} \end{aligned}$$

which can be written as

$$||\nabla R_h u||_{L^2(\Omega)} \leq C_A \; ||\nabla u||_{L^2(\Omega)}, \; \forall \; v \in H_0^1(\Omega), \tag{2.21}$$

where $C_A = \frac{\sqrt{c_{max}}}{c_{min}}$. The following lemma is a consequence of (2.10):

Lemma 2.3. *If (2.10) holds, then we have the estimate involving R_h defined in (2.19) given by*

$$||v - R_h v||_{L^2(\Omega)} + h \; ||\nabla(v - R_h v)||_{L^2(\Omega)} \leq Ch^s ||v||_{H^s(\Omega)}, \; \textit{for } v \in H^s \cap H_0^1(\Omega), \; 1 \leq s \leq r. \tag{2.22}$$

Proof. From the minimality property of the projection R_h and the boundedness of matrix A we have

$$\begin{aligned} ||A\nabla(v - R_h v)||^2 &\leq C(A\nabla(v - R_h v), \nabla(v - R_h v)) \\ &\leq C(A\nabla(v - v_h), \nabla(v - v_h)) \\ &\leq C \; ||\nabla(v - v_h)||^2, \end{aligned} \tag{2.23}$$

where C depends on the matrix A. Then, we combine (2.23) with (2.10) to obtain

$$||A\nabla(v - R_h v)|| \leq C \inf_{v_h \in S_h} ||\nabla(v - v_h)|| \leq Ch^{s-1} ||v||_s \tag{2.24}$$

For the other term we use a duality argument. Let $\xi \in L^2(\Omega)$ and $\psi \in H^2 \cap H_0^1(\Omega)$ be the solution of $-\nabla \cdot (A\nabla\xi) = \psi$ in Ω with $\psi = 0$ on $\partial\Omega$. It is well known from the theory that the H^2 norm of ξ is bounded by the L^2 norm of ψ, i.e.,

$$||\xi||_2 \leq C||\psi||. \tag{2.25}$$

We now consider $v_h \in S_h$ and the ortogonality $(A\nabla(v - R_h v), \nabla v_h) = 0$ to obtain

$$\begin{aligned}(v - R_h v, \psi) &= -(v - R_h v, \nabla \cdot (A\nabla\xi)) \\ &= (A\nabla(v - R_h v), \nabla\xi) \\ &= (A\nabla(v - R_h v), \nabla(\xi - v_h)) \\ &\leq ||A\nabla(v - R_h v)|| \cdot ||\nabla(\xi - v_h)||\end{aligned} \tag{2.26}$$

Applying (2.24) to $||A\nabla(v - R_h v)||$ and (2.10) to $||\nabla(\xi - v_h)||$ with $s = 2$, combined with (2.25), we have

$$(v - R_h v, \psi) \leq Ch^{s-1}||v||_s \; h||\xi||_2 \leq Ch^s||v||_s \; ||\psi||. \tag{2.27}$$

The result is obtained after replacing ψ by $v - R_h v$ in (2.27). □

Using this lemma an writing $u - u_h = \theta + \rho$ we can prove Theorem 2.2.

Proof of Theorem 2.2.

Applying Lemma 2.3 on $\rho(t)$ we have

$$\begin{aligned}||\rho(t)|| &\leq Ch^r \; ||u(t)||_r \\ &\leq Ch^r(||u_0 + \int_0^t u_t \; ds||_r) \\ &\leq Ch^r \left(||u_0||_r + \int_0^t ||u_t||_r ds\right).\end{aligned} \tag{2.28}$$

To bound θ we first observe that, for $v_h \in S_h$,

$$\begin{aligned}(\theta_t, v_h) + (A\nabla\theta, \nabla v_h) &= (\partial_t u_h, v_h) + (A\nabla u_h, \nabla v_h) \\ &\quad - (R_h u_t, v_h) - (A\nabla R_h u, \nabla v_h) \\ &= (f, v_h) - (R_h u_t, v_h) - (A\nabla u, \nabla v_h) \\ &= (u_t - R_h u_t, v_h)\end{aligned} \tag{2.29}$$

where in the process we use $(\partial_t u_h, v_h) + (A\nabla u_h, \nabla v_h) = (f, v_h)$, $(A\nabla R_h u, \nabla v_h) = (A\nabla u, \nabla v_h)$ and the fact that R_h commutes with the time derivative. We observe also that since $\rho_t = R_h u_t - u_t$, we can rewrite (2.29) as

$$(\theta_t, v_h) + (A\nabla\theta, \nabla v_h) = -(\rho_t, v_h), \; \forall \; v_h \in S_h, \; t \in [0, T]. \tag{2.30}$$

In the sequence we replace v_h by θ in (2.30) to obtain

$$(\theta_t, \theta) + (A\nabla\theta, \nabla\theta) = -(\rho_t, \theta). \tag{2.31}$$

We then use the properties

$$(\theta_t, \theta) = \frac{1}{2}\partial_t||\theta||^2 = \frac{1}{2} \cdot 2\partial_t||\theta|| \cdot ||\theta|| = ||\theta||\partial_t||\theta||,$$

$$(A\nabla\theta, \nabla\theta) \geq c_{min}||\nabla\theta||^2,$$

along with $-(\rho_t, \theta) \leq ||\rho_t||\ ||\theta||$ to write

$$\partial_t||\theta|| \leq ||\rho_t||. \tag{2.32}$$

In the sequence we integrate (2.32) in time over $[0, t)$, $t < T$, and apply the fundamental theorem of calculus to get

$$||\theta(t)|| \leq ||\theta(0)|| + \int_0^t ||\rho_t||\ ds. \tag{2.33}$$

We now write $\theta(0) = u_h(0) - R_h u(0) = u_{0,h} - u_0 + u_0 - R_h u_0$ and observe that

$$||\theta(0)|| \leq ||u_{0,h} - u_0|| + ||u_0 - R_h u_0|| \leq ||u_0 - u_{0,h}|| + Ch^r||u_0||_r \tag{2.34}$$

after applying Lemma 2.3 to $||u_0 - R_h u_0||$. Applying once again Lemma 2.3 to write

$$||\rho_t|| = ||u_t - R_h u_t|| \leq Ch^r||u_t||_r, \tag{2.35}$$

we bound $\theta(t)$ from above as

$$||\theta(t)|| \leq ||u_{0,h} - u_0|| + Ch^r\left(||u_0||_r + \int_0^t ||u_t||_r\ ds\right). \tag{2.36}$$

Combining (2.28) and (2.36) we have the result. □

Remark 2.4. The term $||u_0 - u_{0,h}||$ is bounded by $Ch^r||u_0||_r$ if we consider $u_{0,h}$ as $R_h u_0$ or $I_h u_0$ and apply Lemma 2.3.

The strategy of writing $u - u_h = \theta + \rho$ can also be employed to prove the error estimate of the L^2 norm of the gradient $\nabla(u - u_h)$:

Theorem 2.5. *Under the assumptions of Theorem 2.2 we obtain*

$$\begin{aligned} ||\nabla u(t) - \nabla u_h(t)|| &\leq C||\nabla u_0 - \nabla u_{0,h}|| \\ &+ Ch^{r-1}\left(||u_0||_r + ||u(t)||_r + \left(\int_0^t ||u_t||^2_{r-1}\ ds\right)^{\frac{1}{2}}\right), \quad \textit{for } t \geq 0. \end{aligned} \tag{2.37}$$

Proof. We once again write $u - u_h = \theta + \rho$ and observe that from Lemma 2.3 we have

$$||\nabla\rho(t)|| = ||\nabla(u(t) - R_h u(t))|| \leq Ch^{r-1}||u(t)||_r. \tag{2.38}$$

To bound the term $\nabla\theta$ we use (2.30) with $v_h = \theta_t$ to get

$$||\theta_t||^2 + \frac{1}{2}\frac{d}{dt}(A\nabla\theta, \nabla\theta) = -(\rho_t, \theta_t) \leq \frac{1}{2}(||\rho_t||^2 + ||\theta_t||^2),$$

from which we conclude that

$$\frac{d}{dt}||\nabla\theta||^2 \leq ||\rho_t||^2. \tag{2.39}$$

Integrating the last inequality from 0 to t and applying the fundamental theorem of calculus we have

$$\begin{aligned} ||\nabla\theta(t)||^2 &\leq ||\nabla\theta(0)||^2 + \int_0^t ||\rho_t||^2 \, ds \\ &\leq (||\nabla(u_0 - u_{0,h})|| + ||\nabla(u_0 - R_h u_0)||)^2 + \int_0^t ||\rho_t||^2 \, ds. \end{aligned} \tag{2.40}$$

Applying Lemma 2.3 to $\nabla(u_0 - R_h u_0)$ and using estimate (2.35), we end up with

$$||\nabla\theta(t)||^2 \leq C\left(||\nabla(u_0 - u_{0,h})||^2 + h^{2(r-1)}\left(||u_0||_r^2 + \int_0^t ||u_t||_{r-1}^2 \, ds\right)\right). \tag{2.41}$$

Combining this last inequality with (2.38) we prove the result. □

As in Remark 2.4, the term $||\nabla(u_0 - u_{0,h})||$ can be bounded by $Ch^{r-1}||u_0||_r$ if $u_{0,h} = R_h u_0$ or $I_h u_0$.

We now introduce a time step Δt and time points $t_n = n\Delta t$, where n is a nonnegative integer, to discretize the time interval $[0, T]$. We define $U^n = U_h^n \in S_h$ the approximation of $u(t_n)$. Writing the weak formulation of the discrete problem from (2.14) we have

$$(\partial_t U^n, v_h) + (A\nabla U^n, \nabla v_h) = (f(t_n), v_h), \ \forall \ v_h \in S_h, \ n \geq 1, \quad U^0 = u_{0,h}. \tag{2.42}$$

The *implicit Euler scheme* used to discretize the time derivative is given by the quotient

$$\partial_t U^n = \frac{U^n - U^{n-1}}{\Delta t}, \tag{2.43}$$

that, when applied to (2.42), becomes

$$(U^n, v_h) + (A\nabla U^n, \nabla v_h) = (U^{n-1} + \Delta t f(t_n), v_h), \ \forall v_h \in S_h, \tag{2.44}$$

which can be seen as a finite element formulation of the elliptic equation $u - \Delta t \ \nabla\cdot(A\nabla u) = g$, with $g = U^{n-1} + f(t_n)$. In matrix notation, we rewrite (2.44) as

$$(\mathcal{B} + \Delta t\mathcal{A})\alpha^n = \mathcal{B}\alpha^{n-1} + \Delta t\widetilde{f}(t_n) \tag{2.45}$$

where $\mathcal{B} + \Delta t\mathcal{A}$ is positive definite and, therefore, invertible. We are now ready to prove the following result:

Theorem 2.6. *Let u be the solution of (1.1) and U^n the numerical solution of (2.42). Assume that $u \in H^l(0, T; H^r \cap H_0^1(\Omega))$, with $l \geq 2$ and r implicitly defined by (2.10). If $||u_0 - u_{0,h}|| + h||\nabla(u_0 - u_{0,h})|| \leq Ch^r||u_0||_r$ and $u_0 = 0$ on $\partial\Omega$, we have the following error estimates*

$$||u(t_n) - U^n|| \leq Ch^r\left(||u_0||_r + \int_0^{t_n} ||u_t||_r \, ds\right) + \Delta t\int_0^{t_n} ||u_{tt}|| \, ds, \tag{2.46}$$

$$||\nabla(u(t_n) - U^n)|| \leq Ch^{r-1}\left(||u_0||_r + \int_0^{t_n} ||u_t||_r \, ds\right) + \Delta t\int_0^{t_n} ||u_{tt}|| \, ds, \ \textit{for } n \geq 0. \tag{2.47}$$

Proof. We start writing

$$U^n - u(t_n) = (U^n - R_h u(t_n)) + (R_h u(t_n) - u(t_n)) = \theta^n + \rho^n, \tag{2.48}$$

where $\rho^n = \rho(t_n)$ has the bound

$$||\rho(t_n)|| \leq Ch^r \left(||u_0||_r + \int_0^{t_n} ||u_t||_r \, ds \right). \tag{2.49}$$

To bound θ we first write

$$\begin{aligned}(\partial_t \theta^n, v_h) + (A\nabla\theta^n, \nabla v_h) &= (\partial_t U^n - R_h \partial_t u(t_n), v_h) + (A\nabla(U^n - R_h u(t_n)), \nabla v_h) \\ &= (f(t_n) - R_h \partial_t u(t_n) - A\nabla u(t_n), v_h) \\ &= (u_t(t_n) - R_h \partial_t u(t_n), v_h)\end{aligned} \tag{2.50}$$

which can be rewritten as

$$(\partial_t \theta^n, v_h) + (A\nabla\theta^n, \nabla v_h) = -(w^n, v_h), \ \forall \ n \geq 1, \ v_h \in S_h, \tag{2.51}$$

where

$$w^n = w_1^n + w_2^n := (R_h - I)\partial_t u(t_n) + (\partial_t u(t_n) - u_t(t_n)). \tag{2.52}$$

Replacing v_h by θ^n in (2.51) we have

$$(\partial_t \theta^n, \theta^n) = \left(\frac{\theta^n - \theta^{n-1}}{\Delta t}, \theta^n \right) = \frac{1}{\Delta t} \left[||\theta^n||^2 - (\theta^{n-1}, \theta^n) \right]$$

and

$$-(w^n, \theta^n) \leq ||w^n|| \cdot ||\theta^n||$$

that, combined, turns into

$$||\theta^n|| \leq ||\theta^{n-1}|| + \Delta t ||w^n||. \tag{2.53}$$

Applying recursively this procedure we end up with

$$||\theta^n|| \leq ||\theta^0|| + \Delta t \sum_{j=1}^n ||w_1^j|| + \Delta t \sum_{j=1}^n ||w_2^j||. \tag{2.54}$$

The bound on θ^0 is given by (2.34). Now we observe that, since $\partial_t u(t_n) = (\Delta t)^{-1}(u(t_n) - u(t_{n-1}))$, we have

$$w_1^j = (R_h - I)(\Delta t)^{-1} \int_{t_{j-1}}^{t_j} u_t \, ds = (\Delta t)^{-1} \int_{t_{j-1}}^{t_j} (R_h - I) u_t \, ds. \tag{2.55}$$

Therefore, from Lemma 2.3 we get

$$\Delta t \sum_{j=1}^n ||w_1^j|| \leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} Ch^r ||u_t||_r \, ds = Ch^r \int_0^{t_n} ||u_t||_r \, ds. \tag{2.56}$$

For w_2^j we first write using integration by parts

$$\begin{aligned}\Delta t \cdot w_2^j &= \Delta t \left(\frac{u(t_j) - u(t_{j-1})}{\Delta t} - u_t(t_j) \right) \\ &= -\int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s)\ ds\end{aligned} \tag{2.57}$$

and, then, we obtain the bound

$$\Delta t \sum_{j=1}^{n} ||w_2^j|| \leq \sum_{j=1}^{n} || \int_{t_{j-1}}^{t_j} (s - t_{j-1}) u_{tt}(s)\ ds|| \leq \Delta t \int_0^{t_n} ||u_{tt}||\ ds. \tag{2.58}$$

Combining estimates (2.52), (2.56) and (2.58) with the estimate

$$||\theta(0)|| \leq ||u_0 - u_{0,h}|| + Ch^r ||u_0||_r \leq Ch^r ||u_0||_r, \tag{2.59}$$

the estimate follows.

The bound on $||\nabla(u(t_n) - U^n)||$ follows from bounding $\nabla\theta^n$ and $\nabla\rho^n$. The bound on $\nabla\rho_n$ is given by

$$||\nabla\rho(t_n)|| \leq Ch^{r-1} \left(||u_0||_r + \int_0^{t_n} ||u_t||_r\ ds \right). \tag{2.60}$$

To obtain an estimate for $\nabla\theta^n$ we replace $v_h = \partial_t \theta^n$ in (2.51) and show that

$$\partial_t ||\nabla\theta^n||^2 \leq ||w^n||^2$$

or, more appropriate,

$$||\nabla\theta^n||^2 \leq ||\nabla\theta^{n-1}||^2 + \Delta t\ ||w^n||^2. \tag{2.61}$$

We then apply this estimate recursively to end up with

$$\begin{aligned}||\nabla\theta^n||^2 &\leq ||\nabla\theta^0||^2 + \Delta t \sum_{j=1}^{n} ||w^j||^2 \\ &\leq ||\nabla\theta^0||^2 + Ch^{2s} \int_0^{t_n} ||u_t||_s^2\ dt + C(\Delta t)^2 \int_0^{t_n} ||u_{tt}||^2\ dt,\end{aligned} \tag{2.62}$$

for $1 \leq s \leq r$. Combining (2.62) with $s = r - 1$ with estimates on $||\nabla\theta(0)||$ used in (2.40) we have

$$||\nabla\theta^n||^2 \leq ||\nabla(u_0 - u_{0,h})|| + h^{2(r-1)} \left(||u_0||_r^2 + \int_0^t ||u_t||_{r-1}^2\ dt \right) + C(\Delta t)^2 \int_0^{t_n} ||u_{tt}||^2\ dt. \tag{2.63}$$

The estimate then comes from (2.60) and (2.63). □

3 Primal Hybrid Formulation

In the context of partial differential equations (PDEs), the primal hybrid formulation is a mixed finite element formulation that combines both the primal variables (such as the primary unknowns, such as displacement, temperature, etc.) and dual variables (such as fluxes or tractions) in a unified framework. It is an alternative approach to the classical primal formulation, where all variables are of the same type.

In the primal hybrid formulation, the PDE problem is reformulated by introducing additional variables that represent the gradients or fluxes associated with the primary unknowns. These additional variables are typically defined on element edges or faces, depending on the dimensionality of the problem.

The main advantage of the primal hybrid formulation is that it allows discontinuous functions for better approximation and stability properties, especially in problems with strong gradients or discontinuities. It can capture the local behavior of the solution more accurately, even with relatively coarse meshes. Additionally, the primal hybrid formulation often leads to the preservation of certain mathematical properties, such as the discrete maximum principle or energy conservation.

Section 3.1 is reserved to treat about the most relevant results involving the primal variational formulation for elliptic equations, which we take as a basis to build the primal formulations for parabolic equations. Section 3.2 contains generalizations of the results presented in 3.1, first for the semidiscrete case where the time is not discretized and later on for the space-time discretized problem.

3.1 Primal Hybrid Formulation for Elliptic Equations

The construction of finite element methods for second order elliptic equations based on a primal hybrid variational principle was first introduced by Pian, (PIAN et al., 1971) and (PIAN, 1972), and Tong (PIAN; TONG, 1969) and further generalized into a nonconforming finite element analysis setting by Raviart and Thomas (RAVIART; THOMAS, 1977). In order to capture the essence and main results of this approach, let us resume the steps presented in (RAVIART; THOMAS, 1977).

Consider the problem

$$\begin{cases} -\nabla \cdot (A\nabla u) = f \quad \text{in} \quad \Omega \\ \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

such that $\Omega \subset \mathbb{R}^d$ is an open bounded domain, $d \in \{2,3\}$; $f \in L^2(\Omega)$ and $A \in [L^\infty(\Omega)]^{d\times d}$ is a bounded positive definite matrix satisfying the usual properties of symmetry. The primal variational hybrid principle consists on guaranteeing the inter-element continuity by introducing a Lagrange multiplier. This approach allows us to look for solutions on broken spaces, bigger than the one where the problem is initially posed.

Let $\overline{\Omega} = \cup_{r=1}^R K_r$ be a decomposition of $\overline{\Omega}$ into subdomains K_r where

- K_r is an open subset of Ω with Lipschitz boundary ∂K_r, $1 \le r \le R$;
- $K_r \cap K_s = \emptyset$ if $r \neq s$.

A function $v \in L^2(\Omega)$ is in $H_0^1(\Omega)$ if, and only if,

a) The restriction v_r of v onto the subset K_r belongs to $H^1(K_r)$,

b) The traces of the functions v_r and v_s coincide in $\partial K_r \cap \partial K_s$,

c) The trace of the function v_r vanishes on $\partial K_r \cap \partial\Omega$ for all $1 \le r \le R$.

To relax (b) and (c) we introduce the space

$$X = \{v \in L^2(\Omega) |\; v_r \in H^1(K_r), 1 \le r \le R\} \tag{3.2}$$

with broken norm $||v||_X = \left(\sum_{r=1}^R ||v||_{1,K_r}^2\right)^{\frac{1}{2}}$ and

$$\begin{aligned} M = \Bigg\{\mu \in \prod_{r=1}^R H^{-\frac{1}{2}}(\partial K_r) \quad \Big| \quad & \text{there exists } \mathbf{q} \in H(div;\Omega) \\ \text{such that } \mathbf{q}\cdot\mathbf{n_r} \;\; = \;\; & \mu \text{ on } \partial K_r, 1 \le r \le R\Bigg\} \end{aligned} \tag{3.3}$$

with norm

$$||\mu||_M = \inf\{||\mathbf{q}||_{H(div;\Omega)}\Big|\; \mathbf{q} \in H(div;\Omega),\; \mathbf{q}\cdot\mathbf{n_r} = \mu \text{ on } \partial K_r,\; 1 \le r \le R\}. \tag{3.4}$$

Then, the hybrid weak formulation of the problem is: find $(u,\lambda) \in X \times M$ such that

$$\begin{cases} a(u,v) + b(v,\lambda) = (f,v) \;\text{ for all }\; v \in X, \\ b(u,\mu) = 0 \;\text{ for all }\; \mu \in M, \end{cases} \tag{3.5}$$

where $(.,.) = (.,.)_{L^2}$ and $a : X \times X \to \mathbb{R}$ and $b : X \times M \to \mathbb{R}$ are continuous bilinear forms given by

$$a(u,v) = \sum_{r=1}^R \int_{K_r} A\nabla u \nabla v dx \tag{3.6}$$

and

$$b(v,\mu) = -\sum_{r=1}^{R} \langle \mu, v \rangle_{\partial K_r}, \tag{3.7}$$

where notation $\langle .,. \rangle_{\partial K_r}$ refers to the dual product $H^{-\frac{1}{2}}, H^{\frac{1}{2}}$ on ∂K_r. The next theorem provides the existence and uniqueness of the solution for the hybrid weak formulation (3.5), as well as the characterization of the Lagrange multiplier λ:

Theorem 3.1. *The problem (3.5) has a unique solution $(u, \lambda) \in X \times M$. Furthermore, $u \in H_0^1(\Omega)$ is the solution of the problem (3.1) and*

$$\lambda = A\nabla u \cdot \mathbf{n_r} \quad on \quad \partial K_r, \; 1 \leq r \leq R. \tag{3.8}$$

Proof. Let $(u, \lambda) \in X \times M$ be the solution of (3.5). From the second equation of (3.5) we conclude that $u \in H_0^1(\Omega)$, a consequence from Lemma A.2. Replacing $v \in H_0^1(\Omega)$ into the first equation of (3.5) we get

$$\int_\Omega A\nabla u \nabla v dx = \int_\Omega f v dx.$$

Since this is true for all $v \in H_0^1(\Omega)$, we see that u is the weak solution of (3.1). Reciprocally, let $u \in H_0^1(\Omega)$ be the weak solution of (3.1) and consider the linear continuous functional

$$v \mapsto \int_\Omega f v dx - a(u, v).$$

This functional vanishes in $H_0^1(\Omega)$ by construction and, by Lemma A.2, there exists a unique $\lambda \in M$ such that

$$b(v, \lambda) = \int_\Omega f v dx - a(u, v),$$

for all $v \in X$. Therefore, the pair (u, λ) is the unique solution of (3.5).

Now, since $f = -\nabla \cdot (A\nabla u)$ in Ω, choosing $v \in X$ such that v is not zero in just one K_r, we get from Green's identity (Theorem A.1)

$$b(v, \lambda) = -\int_{K_r} div(A\nabla u) v \; dx - a(u, v) = -\int_{\partial K_r} A\nabla u \, v \cdot \mathbf{n_r} \; ds.$$

In other words, $\lambda = A\nabla u \cdot \mathbf{n_r}$ on ∂K_r, $1 \leq r \leq R$.

□

Another relevant result proved in (ARAYA et al., 2013) show an equivalent norm to the one defined in (3.4) on the space M:

Lemma 3.2. *Let $\mu \in M$. Then we have*

$$\frac{\sqrt{2}}{2} ||\mu||_M \leq \sup_{v \in X} \frac{b(\mu, v)}{||v||_X} \leq ||\mu||_M. \tag{3.9}$$

3.2 Primal Hybrid Formulation for Parabolic Equations

In this section we present the hybridization process for parabolic-type equations. We start first considering a discretization only on the spatial domain Ω, where the solution obtained is called *semidiscrete*. A Lagrange multiplier is therefore used to enforce interelement continuity over edges (or faces) almost everywhere in $(0, T)$. In the sequence, a space-time discretization is proposed and the primal hybrid formulation adds also a Lagrange multiplier to enforce continuity over time as well.

We start considering the problem (1.1) and the following two spaces where it is posed,

$$\mathcal{V} = \{u \in L^2(0,T;H^1(\Omega))|\ \partial_t u \in L^2(0,T;H^{-1}(\Omega))\}, \tag{3.10}$$

$$\mathcal{V}_0 = \{u \in L^2(0,T;H^1_0(\Omega))|\ \partial_t u \in L^2(0,T;H^{-1}(\Omega))\}. \tag{3.11}$$

We here call the attention to the fact that $(H^1)' \subset H^{-1}$.

Now let us motivate the weak formulation of (1.1). Considering Ω a Lipschitz domain we have that the following Green formula holds (Theorem A.1):

$$\int_\Omega A\nabla u(x)\nabla v(x)dx - \int_{\partial\Omega} \partial_n(A\nabla u(x))v(x)ds(x) = -\int_\Omega \nabla\cdot(A\nabla u(x))v(x)dx \tag{3.12}$$

for all $u, v \in C^2(\overline{\Omega})$ and $\partial_n(A\nabla u(x)) = A\nabla u(x)\cdot\mathbf{n}$. Now, considering $u, v \in C^1(0,T;C^2(\overline{\Omega}))$, integrating (3.12) over $(0,T)$, observing that $f := (\partial_t - \nabla\cdot(A\nabla))u \in C(\overline{\Omega\times(0,T]})$ we have

$$\int_0^T\int_\Omega A\nabla u(t,x)\nabla v(t,x)dxdt + \int_0^T\int_\Omega \partial_t u(t,x)v(t,x)dxdt$$
$$-\int_0^T\int_{\partial\Omega}\partial_n(A\nabla u(t,x))v(t,x)dtds(x) = \int_0^T\int_\Omega f(t,x)v(t,x)dxdt, \tag{3.13}$$

where the gradient is taken only on the spatial variable x. After an integration by parts in t of $\int_0^T\int_\Omega \partial_t u\, v\, dxdt$ in (3.13) and considering that the test space is $C_0^\infty(\mathbb{R}\times\Omega)$, we have the weak formulation of (1.1):

$$\int_0^T\int_\Omega (A\nabla u\nabla v - u\partial_t v)\ dxdt + \int_\Omega u(T)v(T)\ dx - \int_{\Omega_0} u_0 v(0)\ dx = \int_0^T\int_\Omega fv\ dxdt \tag{3.14}$$

for all $v \in C_0^\infty(\mathbb{R}\times\Omega)$.

If instead of using the test function space $C_0^\infty(\mathbb{R}\times\Omega)$ we use $C_0^\infty((-\infty,T)\times\Omega)$, we obtain from (3.14) the weak formulation

$$\int_0^T\int_\Omega (A\nabla u\nabla v - u\partial_t v)\ dxdt - \int_{\Omega_0} u_0 v\ dx = \int_0^T\int_\Omega fv\ dxdt, \tag{3.15}$$

which is equivalent to

$$\int_0^T\int_\Omega A\nabla u\nabla v\ dxdt + \int_0^T \langle\partial_t u, v\rangle\ dt = \int_0^T\int_\Omega fv\ dxdt, \tag{3.16}$$

if we choose not to integrate (3.13) by parts in t. Recall that, at each t, the notation $\langle .,. \rangle$ represents the pairing

$$\langle \partial_t u(t), v(t) \rangle = \langle \partial_t u(t), v(t) \rangle_{H^{-1}(\Omega), H^1(\Omega)}.$$

From Theorems 3 and 4 presented in chapter 7 of (EVANS, 2010), we have that (1.1) possesses an unique weak solution $u \in \mathcal{V}_0$.

Remark 3.3. For the case where A is time independent with smooth coefficients, $u_0 \in H_0^1(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$, Lemma A.8 shows that the time derivative $\partial_t u \in L^2(0,T;L^2(\Omega))$ and the pairing $\langle \partial_t u(t), v(t) \rangle$ is actually an L^2 product.

Remark 3.4. The strategy contained in (EVANS, 2010) to define a weak solution for the problem (1.1) requires that the weak solution $u : [0,T] \to H_0^1(\Omega)$ must satisfy the equality

$$\langle u_t, v \rangle + (A\nabla u, \nabla v) = (f, v) \tag{3.17}$$

for all $v \in H_0^1(\Omega)$ and a.e. in $(0,T)$ together with the initial condition

$$u(0) = u_0. \tag{3.18}$$

In this case, the pairing $\langle .,. \rangle$ represents the action of a function in $H^{-1} = (H_0^1)'$ over H_0^1. If instead of having a Dirichlet boundary condition we have a Neumann boundary condition given by $\dfrac{\partial u}{\partial \mathbf{n}} = \mu \in L^2(0,T;H^{-\frac{1}{2}}(\partial\Omega))$, formulation (3.17) would have an additional term at the right hand side

$$\langle u_t, v \rangle_{H^{-1}(\Omega),H^1(\Omega)} + (A\nabla u, \nabla v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)} - \langle \mu, v \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}, \tag{3.19}$$

where $\mu : [0,T] \to H^{-\frac{1}{2}}(\partial\Omega)$ a.e. in $(0,T)$ and $H^{-1} = (H^1)'$.

Remark 3.5. Formulation (3.17)-(3.18) is equivalent to (3.16) if we integrate over $(0,T)$.

3.2.1 Semidiscrete formulation

To obtain the primal hybrid variational formulation of (1.1), we proceed as in the elliptic case. Let $\{\mathcal{T}_H\}$ be a family of regular partitions of Ω into subdomains K whose diameter is $\leq H$ such that $\overline{\Omega} = \cup_{K\in\mathcal{T}_H} K$ satisfying:

i) K is an open subset of Ω with Lipschitz continuous boundary ∂K, $K \in \mathcal{T}_H$;

ii) $K_r \cap K_s = \emptyset$ if $r \neq s$.

Observe that $v \in L^2(0,T;L^2(\Omega))$ belongs to $L^2(0,T;H_0^1(\Omega))$ if, and only if,

a) The restriction v_K of v to the subset K belongs to $L^2(0,T;H^1(K))$,

b) The traces of the functions v_{K_1} and v_{K_2} coincide in $\partial K_1 \cap \partial K_2$ for almost every $t \in (0,T)$, whenever $\partial K_1 \cap \partial K_2 \neq \emptyset$,

c) The trace of the function v_K vanishes on $\partial K \cap \partial\Omega$ almost everywhere in $t \in (0,T)$, $K \in \mathcal{T}_H$.

Just like we have done in the elliptic case, we want to relax conditions (b) and (c) in order to seek solutions in a wider space than $\mathcal{V}_0$. Thus, we introduce the space

$$L^2(0,T;X),$$

with norm

$$||v||_{L^2(0,T;X)} = \left(\int_0^T ||v||_X^2 \; dt\right)^{\frac{1}{2}},$$

where X is the space defined in (3.2).

We also define the space of the Lagrange multipliers

$$L^2(0,T;M)$$

with the following norm

$$||\mu||_{L^2(0,T;M)} = \left(\int_0^T ||\mu(t)||_M^2 dt\right)^{\frac{1}{2}},$$

where M is the space defined in (3.3).

The next result is a generalization of Lemma A.2 to the space $L^2(0,T;H_0^1(\Omega))$:

Lemma 3.6. *A continuous linear functional L over the space $L^2(0,T;X)$ vanishes in $L^2(0,T;H_0^1(\Omega))$ if, and only if, there exists a unique $\mu \in L^2(0,T;M)$ such that*

$$L(v) = \int_0^T \sum_{K\in\mathcal{T}_H} \langle \mu, v\rangle_{\partial K} dt, \tag{3.20}$$

where $\langle .,.\rangle_{\partial K}$ is the duality product $H^{-\frac{1}{2}}, H^{\frac{1}{2}}$ on ∂K, for all $v \in L^2(0,T;X)$.

Proof. Assume that $L(v) = \int_0^T \sum_{K\in\mathcal{T}_H} \langle \mu, v\rangle_{\partial K} dt$, for $\mu \in L^2(0,T;M)$. We have from Lemma A.2 that

$$\sum_{K\in\mathcal{T}_H} \langle \mu(t), v\rangle_{\partial K} = 0 \;\; \forall \;\; v \in H_0^1(\Omega), \quad \text{for } t \text{ } a.e. \text{ in } [0,T].$$

Therefore, $L(v) = 0$ if $v \in L^2(0,T;H_0^1(\Omega))$.

In order to verify that L is continuous we define for each $\mu(t) \in M$, $t \in [0,T]$, $\epsilon > 0$ fixed, $\mathbf{q}(t) \in H(div;\Omega)$ such that $\mathbf{q(t)} \cdot \mathbf{n^K} = \mu$ on ∂K and

$$||\mu(t)||_M \leq ||\mathbf{q}(t)||_{H(div,\Omega)} \leq ||\mu(t)||_M + \epsilon. \tag{3.21}$$

First we use integration by parts and a triangular inequality to get

$$\begin{aligned} |L(v)| &= \left| \int_0^T \sum_{K\in\mathcal{T}_H} \langle \mu, v\rangle_{\partial K} dt \right| \\ &= \left| \int_0^T \sum_{k\in\mathcal{T}_H} \int_K \mathbf{q}\nabla v + v\nabla\cdot\mathbf{q} dxdt \right| \\ &\le \int_0^T \sum_{K\in\mathcal{T}_H} \int_K |\mathbf{q}\nabla v + v\nabla\cdot\mathbf{q}| dxdt \\ &\le \int_0^T \sum_{K\in\mathcal{T}_H} \int_K (|\mathbf{q}\nabla v| + |v\nabla\cdot\mathbf{q}|) dxdt. \end{aligned} \tag{3.22}$$

In the sequel we apply Hölder inequality on each element K on (3.22) to obtain the bound

$$\int_0^T \sum_{K\in\mathcal{T}_H} \int_K (|\mathbf{q}\nabla v| + |v\nabla\cdot\mathbf{q}|) dxdt \le \int_0^T \sum_{K\in\mathcal{T}_H} (||\mathbf{q}||_{L^2(K)}||\nabla v||_{L^2(K)} + ||v||_{L^2(K)}||\nabla\cdot\mathbf{q}||_{L^2(K)}) dt. \tag{3.23}$$

Using Hölder inequalities again we obtain

$$\begin{aligned} \int_0^T \sum_{K\in\mathcal{T}_H} ||\mathbf{q}||_{L^2(K)}||\nabla v||_{L^2(K)} dt &\le \int_0^T \left(\sum_{K\in\mathcal{T}_H} ||\mathbf{q}||^2_{L^2(K)}\right)^{\frac{1}{2}} \left(\sum_{K\in\mathcal{T}_H} ||\nabla v||^2_{L^2(K)}\right)^{\frac{1}{2}} dt \\ &\le \left(\int_0^T \sum_{K\in\mathcal{T}_H} ||\mathbf{q}||^2_{L^2(K)} dt\right)^{\frac{1}{2}} \left(\int_0^T \sum_{K\in\mathcal{T}_H} ||\nabla v||^2_{L^2(K)} dt\right)^{\frac{1}{2}} \\ &\le ||v||_{L^2(0,T;X)} \cdot \left(\int_0^T ||\mathbf{q}||^2_{H(div;\Omega)} dt\right)^{\frac{1}{2}}. \end{aligned} \tag{3.24}$$

Recalling (3.21), we observe, from $2\epsilon||\mu|| \le \epsilon^2 + ||\mu||^2$, that

$$\begin{aligned} \int_0^T ||\mathbf{q}||^2_{H(div;\Omega)} dt &\le \int_0^T (||\mu(t)||_M + \epsilon)^2 dt \\ &\le 2||\mu||^2_{L^2(0,T;M)} + 2\epsilon^2 T \end{aligned}$$

Therefore,

$$\int_0^T \sum_{K\in\mathcal{T}_H} ||\mathbf{q}||_{L^2(K)}||\nabla v||_{L^2(K)} dt \le ||v||_{L^2(0,T;X)}(\sqrt{2}||\mu||_{L^2(0,T;M)} + \epsilon\sqrt{2T}), \tag{3.25}$$

and, analogously,

$$\int_0^T \sum_{K\in\mathcal{T}_H} ||v||_{L^2(K)}||\nabla\cdot\mathbf{q}||_{L^2(K)} dt \le ||v||_{L^2(0,T;X)}(\sqrt{2}||\mu||_{L^2(0,T;M)} + \epsilon\sqrt{2T}), \tag{3.26}$$

since $\left(\int_0^T ||\nabla\cdot\mathbf{q}||^2_{L^2(\Omega)} dt\right)^{\frac{1}{2}} \le \left(\int_0^T ||\mathbf{q}||^2_{H(div;\Omega)} dt\right)^{\frac{1}{2}}$.

From (3.22)-(3.26) we conclude that

$$|L(v)| \le 2||v||_{L^2(0,T;X)}(\sqrt{2}||\mu||_{L^2(0,T;M)} + \epsilon\sqrt{2T}), \tag{3.27}$$

and taking the limit $\epsilon \to 0$, we obtain

$$|L(v)| \leq 2\sqrt{2}||v||_{L^2(0,T;X)}||\mu||_{L^2(0,T;M)}, \tag{3.28}$$

proving the continuity of L.

Conversely, let $L \in (L^2(0,T;X))'$. From the result found in Proposition 3.59 of (CIORANESCU; CIORANESCU; DONATO, 1999), we have that $(L^2(0,T;X))' = L^2(0,T;X')$ and L can be written as

$$L(v) = \int_0^T \langle \gamma(t), v(t) \rangle_{X' \times X} \; dt \tag{3.29}$$

for all $v \in L^2(0,T;X)$. If L vanishes in $L^2(0,T;H_0^1(\Omega))$, then

$$\langle \gamma(t), v \rangle_{X' \times X} = 0$$

for all $v \in H_0^1(\Omega)$ a.e. in $(0,T)$. Since $\gamma(t)$ defines a linear functional on X, we have from Lemma A.2 that there exists an unique $\mu(t) \in M$ such that

$$\langle \gamma(t), \tilde{v} \rangle_{X' \times X} = \sum_{K \in \mathcal{T}} \int_{\partial K} \mu(t) \tilde{v} ds$$

for all $\tilde{v} \in X$. Therefore, $L(v) = \int_0^T \sum_{K \in \mathcal{T}} \int_{\partial K} \mu v ds dt$. □

To introduce the primal hybrid formulation of the problem (1.1), instead of working with the space $L^2(0,T;X)$, we need to work with the space $\bar{X} = L^2(0,T;X) \cap H^1(0,T;X')$ in order to make sure that $\partial_t u$ is well defined.

We then set the bilinear forms $a : \bar{X} \times \bar{X} \to \mathbb{R}$ and $b : \bar{X} \times L^2(0,T;M) \to \mathbb{R}$ as

$$a(u,v) = \int_0^T \sum_{K \in \mathcal{T}} \int_K A \nabla u \cdot \nabla v dx dt,$$

$$b(v,\mu) = -\int_0^T \sum_{K \in \mathcal{T}} \langle \mu, v \rangle_{\partial K} dt.$$

As a consequence of Lemma 3.6, the space $L^2(0,T;H_0^1(\Omega))$ can be characterized as

$$L^2(0,T;H_0^1(\Omega)) = \{v \in L^2(0,T;X) | \; b(v,\mu) = 0 \; \forall \; \mu \in L^2(0,T;M)\}. \tag{3.30}$$

Therefore, our problem is to find a pair $(u, \lambda) \in \bar{X} \times L^2(0,T;M)$ such that

$$\begin{cases} \langle \partial_t u, v \rangle + a(u,v) + b(v,\lambda) = (f,v)_{L^2(0,T;L^2(\Omega))} \;\; \text{for all} \;\; v \in L^2(0,T;X), \\ b(u,\mu) = 0 \;\; \text{for all} \;\; \mu \in L^2(0,T;M), \\ (u(0),w)_{L^2(\Omega)} = (u_0,w)_{L^2(\Omega)} \;\; \text{for all} \;\; w \in X, \end{cases} \tag{3.31}$$

where $(.,.)_{L^2(0,T;L^2(\Omega))} = \int_0^T (.,.)_{L^2(\Omega)} dt$ and $\langle \partial_t u, v \rangle = \int_0^T \langle \partial_t u(t), v(t) \rangle \; dt$.

Remark 3.7. Observe that (3.31) still holds if instead of $L^2(0,T;X)$ we take $\bar{X}$ as a test function space.

The next Theorem will ensure that problem (3.31) is well defined and give a characterization of $\lambda \in L^2(0,T;M)$.

Theorem 3.8. *The problem (3.31) has a unique solution $(u,\lambda) \in \bar{X} \times L^2(0,T;M)$. Furthermore, $u \in L^2(0,T;H_0^1(\Omega))$ is a solution to (1.1) and we have*

$$\lambda = A\nabla u \cdot \mathbf{n} \quad on \quad \partial K, \;\; K \in \mathcal{T}_H, \;\; a.e. \; in \; (0,T). \tag{3.32}$$

Proof. If the pair $(u,\lambda) \in \bar{X} \times L^2(0,T;M)$ is the solution of (3.31), from its second equation we conclude that $u \in L^2(0,T;H_0^1(\Omega))$. Choosing $v \in L^2(0,T;H_0^1(\Omega))$ as a test function in the first equation of (3.31) we have

$$\int_0^T (A\nabla u, \nabla v)_{L^2(\Omega)} + (\partial_t u(t), v(t))_{L^2(\Omega)} \, dt = \int_0^T (f,v)_{L^2(\Omega)}.$$

Therefore, we conclude that u is the unique weak solution of (1.1).

Reciprocally, let $u \in \bar{X}$ be the unique weak solution of (1.1) and consider the linear continuous functional

$$v \mapsto \int_0^T (f - \partial_t u, v) \, dt - a(u,v).$$

It is important to recall that $\partial_t u \in L^2(0,T;L^2(\Omega))$ as observed in Remark (3.3). This functional vanishes in $L^2(0,T;H_0^1(\Omega))$ by construction and, from Lemma 3.6, there exists an unique $\lambda \in L^2(0,T;M)$ such that

$$b(v,\lambda) = \int_0^T (f - \partial_t u, v) \, dt - a(u,v), \tag{3.33}$$

for all $v \in L^2(0,T;X)$. Then, the pair (u,λ) is a solution of (3.31).

If $(\tilde{u},\tilde{\lambda})$ is another solution of (3.31), choosing again $v \in L^2(0,T;H_0^1(\Omega))$ as a test function in the first equation of (3.31) we would have

$$\int_0^T (A\nabla(u-\tilde{u}), \nabla v)_{L^2(\Omega)} + (\partial_t(u(t)-\tilde{u}(t)), v(t))_{L^2(\Omega)} \, dt = 0, \tag{3.34}$$

and from the third equation

$$(u(0) - \tilde{u}(0), w)_{L^2(\Omega)} = 0, \tag{3.35}$$

implying that $u = \tilde{u}$ since 0 is the unique solution of (1.1) with $f = 0$ and initial condition $u_0 = 0$. This fact also implies that $b(v, \lambda - \tilde{\lambda}) = 0$ for all $v \in L^2(0,T;X)$, concluding that $\lambda = \tilde{\lambda}$ and proving uniqueness.

Finally, since $f - \partial_t u = -\nabla \cdot (A\nabla u)$ in $\Omega \times (0,T)$, we have

$$b(v,\lambda) = -(\nabla \cdot (A\nabla u), v)_{L^2(Q)} - a(u,v) = -\int_0^T \sum_{K \in \mathcal{T}^H} \int_{\partial K} A\nabla u \cdot \mathbf{n}^K v \, ds$$

for all $v \in L^2(0,T;X)$. Therefore $\lambda = A\nabla u \cdot \mathbf{n}^K$ on ∂K a.e. in $(0,T)$. □

The following result is an estimate involving the norm $L^2(0,T;M)$ of the Lagrange multiplier λ:

Lemma 3.9. *The Lagrange multiplier λ can be bounded in the following way*

$$||\lambda||_{L^2(0,T;M)} \leq C(||u||_{L^2(0,T;H^1(\Omega))} + ||u_t||_{L^2(0,T;H^{-1}(\Omega))} + ||f||_{L^2(0,T;H^{-1}(\Omega))}). \tag{3.36}$$

Proof. Equality (3.33) holds a.e. in time, then we can obtain for all $v \in X$

$$\begin{aligned} b(v,\lambda(t)) &= \int_\Omega (f - u_t)v - A\nabla u \nabla v dx \\ &\leq C\ (||f||_{H^{-1}(\Omega)} + ||u_t||_{H^{-1}(\Omega)} + ||u||_{H^1(\Omega)})\ ||v||_{H^1(\mathcal{T}_H)} \end{aligned} \tag{3.37}$$

which implies that

$$||\lambda(t)||_M = \sup_{v\in X} \frac{b(v,\lambda)}{||v||_X} \leq C\ (||f||_{H^{-1}(\Omega)} + ||u_t||_{H^{-1}(\Omega)} + ||u||_{H^1(\Omega)}). \tag{3.38}$$

Squaring both sides of (3.38), integrating from 0 to T and then taking the square root of it we get the result. □

3.2.2 Fully discrete formulation

In the last subsection we have relaxed continuity over the element boundaries using Lagrange multipliers, which allows us to seek the solution u of (1.1) in a space larger than the one which the problem is posed. Now, we want also to relax continuity using the same approach on a discretization of the time interval $(0,T]$.

In addition to the family of regular partitions $\{\mathcal{T}_H\}$ of the domain Ω defined in the begining of subsection 3.2.1, we decompose the time domain $(0,T]$ in a (non necessarily) uniform partition $\mathcal{T}^{\Delta T}$, i.e., $0 = t_0 < t_1 < \ldots < t_M = T$. We define $I_n := (t_n, t_{n+1}] \in \mathcal{T}^{\Delta T}$, $n = 0,\ldots,N-1$, with $\partial I_n = \{t_n, t_{n+1}\}$ the boundary of I_n, $\Delta T_n = t_{n+1} - t_n$ and $\Delta T := \max_{n\in[0,N]} \Delta T_n$. We define the set $\partial\mathcal{T}^{\Delta T}$ as the union $\cup_{n=0}^{N-1} \partial I_n := \{t_0, t_1, \cdots, t_N\}$.

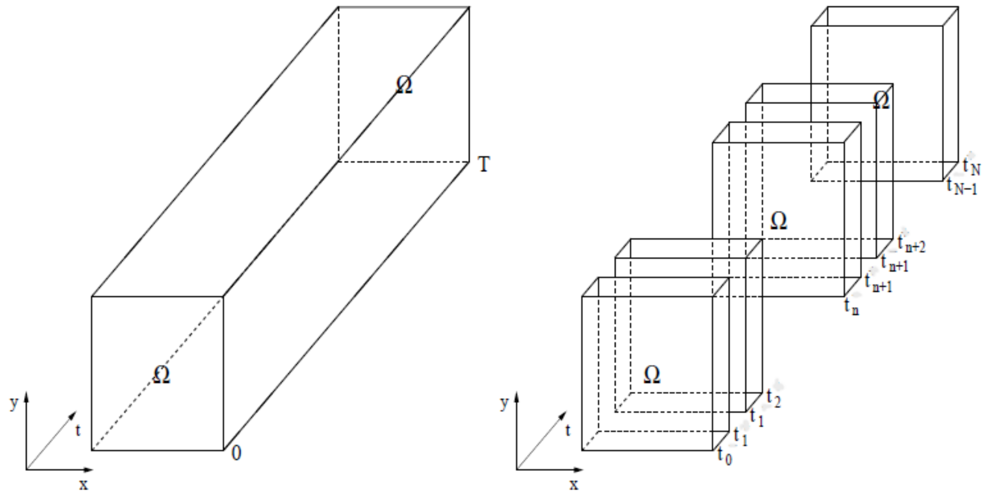


Figure 1 – Domain $\Omega \times (0, T)$ and subdomain $\Omega \times I_n$.

Without loss of generality, we shall use the terminology usually employed for three-dimensional domains to treat $\mathcal{T}_H$. As such, each element K has a boundary ∂K consisting of faces F, where we define $\mathcal{E}_H$ as the set of faces associated with $\mathcal{T}_H$. Let $\mathcal{E}_D$ be the set of faces in $\partial\Omega$ and $\mathcal{E}_0 = \mathcal{E}_H \setminus \mathcal{E}_D$ the set of internal faces. At each F, we associate a normal vector $\mathbf{n}$, taking care to ensure this is facing outward on $\partial\Omega$. For each $K \in \mathcal{T}_H$, we denote by $\mathbf{n}^K$ the outward normal vector on ∂K and let $\mathbf{n}_F^K := \mathbf{n}^k\big|_F$ for each $F \subset \partial K$. Then we observe that $\mathbf{n} \cdot \mathbf{n}_\mathbf{F}^\mathbf{K} = \pm 1$.

We denote the space-time partition $\mathcal{T}_H^{\Delta T}$ of $\Omega \times (0, T)$ as the set of elements $K_n = K \times I_n$, where $K \in \mathcal{T}_H$ and $I_n \in \mathcal{T}^{\Delta T}$. Then, we define the broken space-time domain $\mathcal{X}$ as

$$\mathcal{X} = \{v \in L^2(0, T; X) |\ v|_{I_n} \in L^2(I_n; X) \cap H^1(I_n; X'),\ \forall\ n = 0, \ldots, N-1\}.$$

The following figures display examples of a partition $\mathcal{T}_H^{\Delta T}$ and an element K_n.

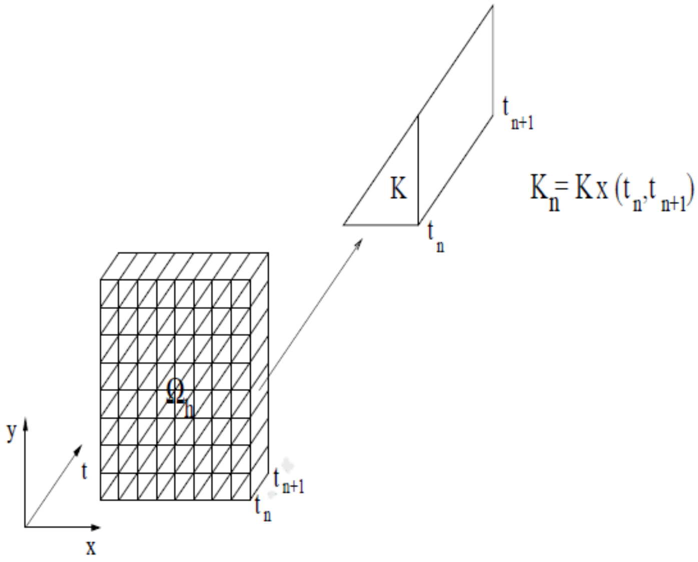


Figure 2 – Partition $\mathcal{T}_H^{\Delta T}$ and element K_n

We as well define the space-time Lagrange multipliers Λ as

$$\Lambda = L^2(0, T; M).$$

Here, X and M are the spaces defined in (3.2) and (3.3), repectively. Now we introduce a space of functions that lives on the skeleton of the time discretization, denoted by Σ, composed of functions in $L^2(\mathcal{T}_H)(= L^2(\Omega))$ at each point t_n of $\partial\mathcal{T}^{\Delta T}$, i.e.,

$$\Sigma = \{(\rho_0, \dots, \rho_N) \in [L^2(\Omega)]^N | \ \rho_n = \rho(t_n) \ \ \forall \ t_n \in \partial\mathcal{T}^{\Delta T}\}.$$

The subspace $\Sigma^0 \in \Sigma$, consisting of functions ρ such that $\rho_N = 0$, will be used in the sequence. Now we introduce the following notations

$$\begin{aligned}
(w, v)_{\mathcal{T}_H} &= \sum_{K \in \mathcal{T}_H} \int_K wv \ dx \ \ \forall \ w, v \in X, \\
(w, v)_{\partial\mathcal{T}_H} &= \sum_{K \in \mathcal{T}_H} \int_{\partial K} wv \ ds \ \ \forall \ w, v \in X, \\
(w, v)_{\partial\mathcal{T}_H} &= \sum_{K \in \mathcal{T}_H} \langle w, v \rangle_{H^{-1/2}(\partial K), H^{1/2}(\partial K)} \ \ \forall \ w \in M, v \in X, \\
(w, v)_{\mathcal{T}_H^{\Delta T}} &:= \sum_{n=0}^{N-1} \int_{I_n} (w, v)_{\mathcal{T}_H} dt \ \ \forall \ w, v \in L^2(I_n; X), n = 0, \dots, N, \\
(\partial_t w, v)_{\mathcal{T}_H^{\Delta T}} &:= \sum_{n=0}^{N-1} \int_{I_n} \sum_{K \in \mathcal{T}_H} \int_K \langle \partial_t w, v \rangle_{H^{-1}(K), H^1(K)} dt \ \ \forall \ w, v \in \mathcal{X}, \\
(w, v)_{\partial\mathcal{T}_H \times \mathcal{T}^{\Delta T}} &:= \sum_{n=0}^{N-1} \int_{I_n} (w, v)_{\partial\mathcal{T}_H} dt \ \ \forall \ w \in \Lambda, v \in \mathcal{X}, \\
(u, v)_{\mathcal{T}_H \times \partial\mathcal{T}^{\Delta T}} &:= \sum_{n=0}^{N-1} \left[(u, v)_{\mathcal{T}_H}(t_{n+1}^-) - (u, v)_{\mathcal{T}_H}(t_n^+) \right] \ \ \forall \ w, v \in \mathcal{X},
\end{aligned} \tag{3.39}$$

and define the norms of $\mathcal{X}$ and Λ, respectively, as

$$||v||^2_{\mathcal{X}} := ||v||^2_{L^2(0,T;X)} = \sum_{n=0}^{N-1} \int_{I_n} ||v||^2_{H^1(\mathcal{T}_H)} dt, \tag{3.40}$$

$$\begin{aligned} ||\mu||^2_{\Lambda} &:= \sum_{n=0}^{N-1} \int_{I_n} \left(\sup_{v \in H^1(\mathcal{T}_H)} \frac{(\mu, v)_{\partial \mathcal{T}_H}}{||v||_{H^1(\mathcal{T}_H)}} \right)^2 dt \\ &= \sum_{n=0}^{N-1} \int_{I_n} ||\mu||^2_M dt. \end{aligned} \tag{3.41}$$

After these preparations, we are ready to obtain the hybrid formulation of problem (1.1). In order to do that, we multiply the first equation of (1.1) by $v \in \mathcal{X}$ and integrate it to get

$$(\partial_t u, v)_{\mathcal{T}_H^{\Delta T}} - (\nabla \cdot (A \nabla u), v)_{\mathcal{T}_H^{\Delta T}} = (f, v)_{\mathcal{T}_H^{\Delta T}}. \tag{3.42}$$

Let us analyze each term of (3.42) individually, considering at first u, v continuous in time:

$$\begin{aligned} (\partial_t u, v)_{\mathcal{T}_H^{\Delta T}} &= -(u, \partial_t v)_{\mathcal{T}_H^{\Delta T}} + \sum_{n=0}^{N-1} \int_{I_n} \partial_t (u, v)_{\mathcal{T}_H} dt \\ &= -(u, \partial_t v)_{\mathcal{T}_H^{\Delta T}} + \sum_{n=0}^{N-1} \left[(u, v)_{\mathcal{T}_H}(t_{n+1}^-) - (u, v)_{\mathcal{T}_H}(t_n^+) \right] \end{aligned}$$

$$\begin{aligned} -(\nabla \cdot (A \nabla u), v)_{\mathcal{T}_H^{\Delta T}} &= -\sum_{n=0}^{N-1} \int_{I_n} (\nabla \cdot (A \nabla u), v)_{\mathcal{T}_H} dt \\ &= \sum_{n=0}^{N-1} \int_{I_n} \sum_{K \in \mathcal{T}_H} \int_K -\nabla \cdot (A \nabla u) v dx dt \\ &= \sum_{n=0}^{N-1} \int_{I_n} \sum_{K \in \mathcal{T}_H} \int_K A \nabla u \nabla v - \nabla \cdot (A \nabla u \ v) dx dt \\ &= \sum_{n=0}^{N-1} \int_{I_n} \sum_{K \in \mathcal{T}_H} \left[\int_K A \nabla u \nabla v dx - \int_{\partial K} A \nabla u \cdot \mathbf{n}^K v ds \right] dt \end{aligned}$$

Therefore, in order to relax continuity over the faces of the elements in $\mathcal{T}_H$ and at the points of the time mesh $\partial \mathcal{T}^{\Delta T}$, we impose the conditions

$$u(t_n^+) = u(t_n^-) = \tau_n, \tag{3.43}$$

for $\tau \in \Sigma$, and

$$A \nabla u \cdot \mathbf{n}^K = -\lambda \in \Lambda, \tag{3.44}$$

to enforce the solution u to belong to $L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^{-1}(\Omega))$. Then, applying (3.43) and (3.44), equation (3.42) writes

$$-(u,\partial_t v)_{\mathcal{T}_h^{\Delta t}}+(A\nabla u,\nabla v)_{\mathcal{T}_H^{\Delta T}}+(\lambda,v)_{\partial\mathcal{T}_H\times\mathcal{T}^{\Delta T}}+(\tau,v)_{\mathcal{T}_H\times\partial\mathcal{T}^{\Delta T}}=(f,v)_{\mathcal{T}_H^{\Delta T}} \tag{3.45}$$

Furthermore, from the equalities

$$(\mu,u)_{\partial\mathcal{T}_H\times\mathcal{T}^{\Delta T}}=0$$

for all $\mu\in\Lambda$ and

$$(w,u(0))_{L^2(\Omega)}=(w,u_0)_{L^2(\Omega)}$$

for all $w\in L^2(\Omega)$, we ensure that $u\in L^2(0,T;H^1(\Omega))$ and the initial condition $u(0)=u_0$ are satisfied.

Now, let $v\in L^2(0,T;X)$ and $v_n^+=v(t_n^+,x)$ and $v_n^-=v(t_n^-,x)$ be the upper and lower limits, respectively, when $t\to t_n$. We define the jump of v at t_n, for $n=0,\dots,N$, as

$$[[v]]_n:=v_n^+-v_n^-, \tag{3.46}$$

where $[[v]]_0:=v_0^+$ and $[[v]]_N:=-v_N^-$. Given $v,w\in\mathcal{X}$, it follows that

$$\begin{aligned}(w,v)_{\mathcal{T}_H\times\partial\mathcal{T}^{\Delta T}}&=\sum_{n=0}^{N-1}(w_{n+1}^-,v_{n+1}^-)_{\mathcal{T}_H}-(w_n^+,v_n^+)_{\mathcal{T}_H}\\&+\sum_{n=0}^{N-1}(w_{n+1}^+,v_{n+1}^-)_{\mathcal{T}_H}-(w_{n+1}^+,v_{n+1}^-)_{\mathcal{T}_H}\\&=\sum_{n=1}^{N-1}\Big[-([[w]]_n,v_n^-)_{\mathcal{T}_H}-(w_n^+,[[v]]_n)_{\mathcal{T}_H}\Big]\\&+(w_N^-,v_N^-)_{\mathcal{T}_H}-(w_0^+,v_0^+)_{\mathcal{T}_H}.\end{aligned} \tag{3.47}$$

For a given $\rho\in\Sigma$, it is convenient to denote

$$(\rho,v)_{\mathcal{T}_H\times\partial\mathcal{T}^{\Delta T}}=-\sum_{n=0}^{N}(\rho_n,[[v]]_n)_{\mathcal{T}_H}. \tag{3.48}$$

Moreover, if u is the solution of (1.1), then $[[u]]_n=0$ for $n=1,\dots,N-1$ and $[[u]]_0=u(0)=g$. Therefore,

$$(\rho,u)_{\mathcal{T}_H\times\partial\mathcal{T}^{\Delta T}}=-\sum_{n=0}^{N}(\rho_n,[[u]]_n)_{\mathcal{T}_H}=-(\rho_0,g)_\Omega$$

for all $\rho\in\Sigma^0$.

After these considerations, the hybrid formulation of problem (1.1) becomes: Find $(u,\lambda,\tau)\in\mathcal{X}\times\Lambda\times\Sigma$ such that

$$\begin{cases} -(u,\partial_t v)_{\mathcal{T}_H^{\Delta T}} + (A\nabla u,\nabla v)_{\mathcal{T}_H^{\Delta T}} + (\lambda, v)_{\partial\mathcal{T}_H\times\mathcal{T}^{\Delta T}} + (\tau, v)_{\mathcal{T}_H\times\partial\mathcal{T}^{\Delta T}} = (f,v)_{\mathcal{T}_H^{\Delta T}}, \\ (\mu, u)_{\partial\mathcal{T}_H\times\mathcal{T}^{\Delta T}} = 0, \\ (\rho, u)_{\mathcal{T}_H\times\partial\mathcal{T}^{\Delta T}} = -(\rho_0, g)_{0,\Omega}, \end{cases} \tag{3.49}$$

for all $(v,\mu,\rho) \in \mathcal{X}\times\Lambda\times\Sigma^0$.

The third equality of (3.49) is necessary to guarantee the continuity in time of the solution. Indeed, if we choose $\rho_i \neq 0$ and $\rho_0,\ldots,\rho_{i-1},\rho_{i+1},\ldots,\rho_{N-1} = 0$, we have

$$(\rho_i, [[u]]_i)_{\mathcal{T}_H} = 0 \tag{3.50}$$

and, this way, we conclude that $[[u]]_n = 0$ for $n = 1,\ldots,N-1$. From this fact, we can see that

$$\begin{aligned} (\partial_t u, v)_{\mathcal{T}_H^{\Delta T}} + (u,\partial_t v)_{\mathcal{T}_H^{\Delta T}} &= \sum_{n=0}^{N-1} (u_{n+1}^-, v_{n+1}^-)_{\mathcal{T}_H} - (u_n^+, v_n^+)_{\mathcal{T}_H} \\ &= -\sum_{n=1}^{N-1} (u_n^+, [[v]]_n)_{\mathcal{T}_H} + (u_N^-, v_N^-)_{\mathcal{T}_H} - (u_0^+, v_0^+)_{\mathcal{T}_H} \\ &= -\sum_{n=0}^{N} (u_n^+, [[v]]_n)_{\mathcal{T}_H}. \end{aligned} \tag{3.51}$$

Using the equivalences (3.48) and (3.51), we rewrite problem (3.49) in the following way: Find $(u,\lambda,\tau) \in \mathcal{X}\times\Lambda\times\Sigma$ such that

$$\begin{cases} (\partial_t u, v)_{\mathcal{T}_H^{\Delta T}} + (A\nabla u,\nabla v)_{\mathcal{T}_H^{\Delta T}} + (\lambda, v)_{\partial\mathcal{T}_H\times\mathcal{T}^{\Delta T}} + \sum_{n=0}^{M} (u_n^+ - \tau_n, [[v]]_n)_{\mathcal{T}_H} = (f,v)_{\mathcal{T}_H^{\Delta t}}, \\ (\mu, u)_{\partial\mathcal{T}_H\times\mathcal{T}^{\Delta T}} = 0, \\ \sum_{n=0}^{N} (\rho_n, [[u]]_n)_{\mathcal{T}_H} = (\rho_0, g)_{0,\Omega}, \end{cases} \tag{3.52}$$

for all $(v,\mu,\rho) \in \mathcal{X}\times\Lambda\times\Sigma^0$.

Formulation (3.52) is more suitable to generate the local basis that embed the fine scale contributions in the MHM method. The next result shows that problems (1.1) and (3.52) are equivalent.

Theorem 3.10. *Problem (3.52) has a unique solution $(u,\lambda,\tau) \in \mathcal{X}\times\Lambda\times\Sigma$. Furthermore, $u \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ is also a weak solution of problem (1.1) and, for all $K \in \mathcal{T}_H$, $I_n \in \mathcal{T}^{\Delta T}$, $n = 1,\ldots,N$, we have*

$$\lambda = -A\nabla u\cdot\mathbf{n}^K \quad \textit{on} \quad \partial K_n$$

and

$$\tau_n = u_n^+ = u_n^-.$$

Proof. Suppose that $(u, \lambda, \tau) \in \mathcal{X} \times \Lambda \times \Sigma$ is a solution of (3.52). From the last equation of (3.52) we have that u is continuous in time (see (3.50)) and if we consider test functions $v \in \bar{X}$ and $w \in X$ we can rewrite (3.52) as

$$\begin{cases} \langle \partial_t u, v\rangle + \int_0^T (A\nabla u, \nabla v)_{\mathcal{T}_H} dt + \int_0^T (\lambda, v)_{\partial \mathcal{T}_H} dt = (f, v)_{L^2(\Omega_T)}, \\ \int_0^T (\mu, u)_{\partial \mathcal{T}_H} dt = 0, \\ (u(0), w) = (g, w). \end{cases} \tag{3.53}$$

Theorem 3.8 guarantees that $(u, -\lambda)$ is the unique solution of (3.53) and, furthermore, u is the weak solution of (1.1). To prove uniqueness of τ we consider $(u, -\lambda, \tau_1)$ and $(u, -\lambda, \tau_2)$ solutions of (3.52), and conclude from its first equation that

$$\sum_{n=0}^{N} (\tau_1^n - u_n^+ - (\tau_2^n - u_n^+), [[v]]_n)_{\mathcal{T}_H} = \sum_{n=0}^{N} (\tau_1^n - \tau_2^n, [[v]]_n)_{\mathcal{T}_H} = 0$$

for all $v \in \mathcal{X}$ and, consequently, $\tau_1 = \tau_2$ since $[[v]]_n$ is arbitrary.

Now suppose that u is the weak solution of (1.1). Set $-\lambda = A\nabla u \cdot \mathbf{n}^K$ on ∂K_n. Theorem 3.8 ensures that the pair $(u, \lambda) \in \bar{X} \times \Lambda$ is the unique solution of (3.31) with $u(0) = g$. We define $\tau \in \Sigma$ such that $\tau_n = u_n^+$ for $n = 1, \ldots, N$, which implies that

$$\sum_{n=0}^{N} (\tau_n - u_n^+, [[v]]_n)_{\mathcal{T}_H} = 0$$

for all $v \in \mathcal{X}$. Then the triple $(u, -\lambda, \tau)$ satisfies (3.52), since $[[u]]_n = 0$, $n = 1, \ldots, N$, because u is continuous in time.

As a consequence of Theorem 3.8 we can characterize $-\lambda = A\nabla u \cdot \mathbf{n}^K$ on ∂K_n and from construction we have $\tau_n = u_n^+$. □

4 The Parabolic MHM Method

In this chapter we introduce the MHM for parabolic problems.The method involves solving a global problem obtained after applying the hybrid formulation, in a coarse-scale mesh, to capture global behavior, and local problems, in a fine-scale mesh within each coarse element, to capture local variations. On the fine scale mesh within each coarse element a local basis is build from a numerical method of choice (usually FEM is employed), and a combination of local basis at each coarse element generates a global basis used to solve, via FEM, the global problem for primary unknowns. The two scales are coupled through appropriate interface conditions.The solutions are post-processed to extract desired quantities of interest, providing an efficient way to accurately handle problems with disparate length scales.

We start this chapter with section 4.1, where we build the system of global-local equations that characterizes the MHM method for parabolic equations. Then, we discretize the spaces of Lagrange multipliers in section 4.2 and rewrite the system obtained previously for the discrete variables. In 4.3 we describe the steps considering a space-time basis functions computed exactly. In section 4.4 we show that two time discretization strategies for the method yield the same numerical approximation.

4.1 Global-Local Formulation

In this section we introduce the essence of the MHM method, which consists in rewrite (3.52) as a system of locally and globally-defined problems. Such an approach guides the definition of stable finite subspaces composed of functions which incorporate multiple scales into the basis functions.

First, we observe that problem (3.52) can be rewritten as: Find $(u, \lambda, \tau) \in \mathcal{X} \times \Lambda \times \Sigma$ such that

$$\begin{cases} (\mu, u)_{\partial\mathcal{T}_H \times \mathcal{T}^{\Delta T}} = 0, \\ \sum_{n=0}^{N-1} (\rho_n, [[u]]_n)_{\mathcal{T}_H} = (\rho_0, g)_{\mathcal{T}_H}, \end{cases} \tag{4.1}$$

for all $\mu \in \Lambda$ and all $\rho \in \Sigma^0$, and

$$(\partial_t u, v)_{K_n} + (A\nabla u, \nabla v)_{K_n} + (u_n^+, v_n^+)_K = (f, v)_{K_n} - (\lambda, v)_{\partial K_n} + (\tau_n, v_n^+)_K, \tag{4.2}$$

for all $v \in L^2(I_n; X)$, $K_n = K \times I_n$ and $\partial K_n = \partial K \times I_n$. Note that, once we know f, λ and τ, we can compute u in each element K_n through (4.2). The local problem corresponding to (4.2) reads

$$\begin{cases} u_t - \nabla \cdot (A\nabla u) = f, \quad \text{in} \quad K_n, \\ A\nabla u \cdot \mathbf{n}^K = -\lambda, \quad \text{on} \quad \partial K_n, \\ u_n = \tau_n \quad \text{in} \quad K \times \{t_n\}. \end{cases} \tag{4.3}$$

From (4.3), we define the operators $S : \Lambda \to \mathcal{X}$, $\bar{S} : \Sigma \to \mathcal{X}$ and $\hat{S} : L^2(Q) \to \mathcal{X}$ such that its restrictions on each $K_n \in \mathcal{T}_H^n$ are weak solutions, respectively, of the following problems

$$\begin{cases} \partial_t S\mu - \nabla \cdot (A\nabla S\mu) = 0, \quad \text{in} \quad K_n, \\ A\nabla S\mu \cdot \mathbf{n}^K = -\mu, \quad \text{on} \quad \partial K_n, \\ S\mu = 0 \quad \text{in} \quad K \times \{t_n\}, \end{cases} \tag{4.4}$$

$$\begin{cases} \partial_t \bar{S}\rho_n - \nabla \cdot (A\nabla \bar{S}\rho_n) = 0, \quad \text{in} \quad K_n, \\ A\nabla \bar{S}\rho_n \cdot \mathbf{n}^K = 0, \quad \text{on} \quad \partial K_n, \\ \bar{S}\rho_n = \rho_n \quad \text{in} \quad K \times \{t_n\}, \end{cases} \tag{4.5}$$

$$\begin{cases} \partial_t \hat{S}q - \nabla \cdot (A\nabla \hat{S}q) = q, \quad \text{in} \quad K_n, \\ A\nabla \hat{S}q \cdot \mathbf{n}^K = 0, \quad \text{on} \quad \partial K_n, \\ \hat{S}q = 0 \quad \text{in} \quad K \times \{t_n\}. \end{cases} \tag{4.6}$$

The weak formulations associated to problems (4.4)-(4.6) are

$$\begin{aligned} (\partial_t S\mu, v)_{K_n} + (A\nabla S\mu, \nabla v)_{K_n} &= -(\mu, v)_{\partial K_n} \quad \forall \quad v \in L^2(I_n; X), \\ S\mu(t_n) &= 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} (\partial_t \bar{S}\rho_n, v)_{K_n} + (A\nabla \bar{S}\rho_n, \nabla v)_{K_n} &= 0 \quad \forall \quad v \in L^2(I_n; X), \\ \bar{S}\rho_n(t_n) &= \rho_n, \end{aligned} \tag{4.8}$$

$$\begin{aligned} (\partial_t \hat{S}q, v)_{K_n} + (A\nabla \hat{S}q, \nabla v)_{K_n} &= (q, v)_{K_n} \quad \forall \quad v \in L^2(I_n; X), \\ \hat{S}q(t_n) &= 0, \end{aligned} \tag{4.9}$$

where $\mu \in \Lambda$, $\rho \in \Sigma$ and $q \in L^2(I_n; \Omega)$. From the linearity and uniqueness of the weak solutions of problems (4.2), (4.7)-(4.9), we have $u = S\lambda + \bar{S}\tau + \hat{S}f$ on each K_n.

Replacing $u = S\lambda + \bar{S}\tau + \hat{S}f$ in (4.1) and using that $u_n^+ = u_n^- = \tau_n$, since u is continuous in time, we have the following problem: Find $(\lambda, \tau) \in \Lambda \times \Sigma$ such that

$$\begin{cases} (\mu, S\lambda + \bar{S}\tau)_{\partial\mathcal{T}_H \times \mathcal{T}^{\Delta T}} = -(\mu, \hat{S}f)_{\partial\mathcal{T}_H \times \mathcal{T}^{\Delta T}}, \quad \forall \quad \mu \in \Lambda, \\ \sum_{n=1}^{N} (\rho, \tau_n)_{\mathcal{T}_H} = \sum_{n=1}^{N} (\rho, u_n)_{\mathcal{T}_H}, \quad \forall \quad \rho \in \Sigma. \end{cases} \tag{4.10}$$

Therefore, we have the following result

Lemma 4.1. *The weak formulations (3.49) or (3.52) are equivalent to the coupled system (4.4)-(4.10) with $\mu = \lambda$, $\rho = \tau$ and $q = f$. The unknowns λ and τ in (4.10) are then used to reconstruct $u \in C^0(0,T;H^1(\Omega))$ and the dual variable $\sigma \in C^0(0,T;H(div;\Omega))$ as follows:*

$$u = S\lambda + \bar{S}\tau + \hat{S}f \quad \text{and} \quad \sigma = A\nabla(S\lambda + \bar{S}\tau + \hat{S}f). \tag{4.11}$$

It is worthy pointing out that formulation (4.10) can be solved by a marching in time algorithm more appropriate to be discretized. Indeed, it is natural to observe that for $\Lambda^n := \Lambda|_{I_n}$ we can write $\Lambda = span(\Lambda^n)$ and, therefore, we rewrite (4.10) as a sequence of problems: Find $(\lambda_n, \tau_n) \in \Lambda^n \times L^2(\Omega)$ such that

$$\begin{cases} (\mu, S\lambda_n + \bar{S}\tau_n)_{\partial\mathcal{T}_H^n} = -(\mu, \hat{S}f)_{\partial\mathcal{T}_H^n}, \quad \forall \ \ \mu \in \Lambda^n, \\ \\ (w, \tau_n)_{\mathcal{T}_H} = (w, u_n)_{\mathcal{T}_H}, \quad \forall \ \ w \in L^2(\Omega), \end{cases} \tag{4.12}$$

for $n = 0, \ldots, N-1$, where $\partial\mathcal{T}_H^n = \partial\mathcal{T}_H \times I_n$.

4.2 Discretization

In this section we introduce the discrete spaces used to approximate the variables $\lambda_n \in \Lambda^n$ and $\tau_n \in W$, since they uniquely determine $S\lambda$ and $\bar{S}\tau_n$. To choose such spaces we observe that the flexibility of the method is expressed in the range of parameters used in the formulation of problem (4.12). The parameter H at first refers to the diameter of the partition $\mathcal{T}_H$ of Ω related to the hibridization proposed in subsection 3.2.1, while $\bar{h}$ is the parameter of $\mathcal{T}_{\bar{h}}$ of Ω, used to project the initial data $u_H^{n-1}(t_n)$. In this work we consider a submesh $\mathcal{T}_h \subset \mathcal{T}_H$, with $h \leq H$, that generates the space

$$\mathcal{E}_h := \{F' \in \partial\mathcal{T}_h | \ F' \subset F, \ F \in \mathcal{E}_H\}, \tag{4.13}$$

a refinement of $\mathcal{E}_H$, satifying the following assumption:

Assumption A1. *The trace on ∂K coincides on $\mathcal{E}_h$ for the shape regular triangulation $\mathcal{T}_h|_K$.*

We then build the discrete space $M_h \subset M$ given by

$$M_h = \{\mu \in M | \ \mu|_F \in \mathbb{P}_m(F) \text{ for all } F \in \mathcal{E}_h, m \geq 0\}. \tag{4.14}$$

Remark 4.2. In the case where $h = H$, we call it a **mesh-based** refinement of the space Λ. On the other hand, when we fix H and consider $h \to 0$, we say Λ_h^n is a **space-based** refinement. This nomenclature is similar to the one used in the elliptic case in (BARRENECHEA et al., 2020).

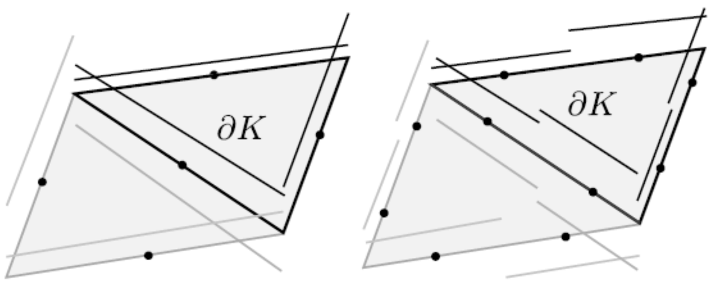


Figure 3 – A mesh-based discretization on the left and a space-based on the righ.

Therefore, we consider the following discrete spaces

$$\begin{aligned} \Lambda_h^n &= L^2(I_n; M_h), \\ W_{\bar{h}} &= \mathbb{P}_s(\mathcal{T}_{\bar{h}}), s \geq 1, \end{aligned} \tag{4.15}$$

used to obtain the numerical approximations $\lambda_{n,h}$ of λ_n and $\tau_{n,h}$ of τ_n and introduce the notation

$$(.,.)_{\partial\mathcal{T}_H^n} := \int_{I_n} (.,.)_{\partial\mathcal{T}_H} dt. \tag{4.16}$$

The MHM method for the parabolic problem reads: For $n = 0, \dots, N-1$, find $(\lambda_{n,h}, \tau_{n,h}) \in \Lambda_h^n \times W_{\bar{h}}$ such that

$$\begin{cases} (\mu_h, S\lambda_{n,h} + \bar{S}\tau_{n,h})_{\partial\mathcal{T}_H^n} = -(\mu_h, \hat{S}f)_{\partial\mathcal{T}_H^n}, \quad \forall \quad \mu_h \in \Lambda_h^n, \\ \\ (\rho_{\bar{h}}, \tau_{n,h})_{\mathcal{T}_{\bar{h}}} = \begin{cases} (\rho_{\bar{h}}, u_0)_{\mathcal{T}_{\bar{h}}}, \text{ if } n = 0 \\ (\rho_{\bar{h}}, u_h^{n-1}(t_n))_{\mathcal{T}_{\bar{h}}}, \text{ otherwise} \end{cases}, \quad \forall \quad \rho_{\bar{h}} \in W_{\bar{h}}, \end{cases} \tag{4.17}$$

where the operators S, $\bar{S}$ and $\hat{S}$ are defined locally by (4.7), (4.8) and (4.9), respectively, and u_H^n is an approximation for u restricted to the time interval I_n written as

$$u_h^n = S\lambda_{n,h} + \bar{S}\tau_{n,h} + \hat{S}f \notin H_0^1(\Omega) \text{ for all } t \in (0,T). \tag{4.18}$$

We also build approximations for σ in $\mathcal{T}_H^n$, $n = 0, \dots, N-1$, namely

$$\sigma_h^n = A\nabla(S\lambda_{n,h} + \bar{S}\tau_{n,h} + \hat{S}f) \in H(div, \Omega) \text{ for all } t \in (0,T). \tag{4.19}$$

It is important to highlight that h is a first level discretization parameter.

Remark 4.3. The parabolic MHM method consists in a two level discretization method, in space and time, where we call **first level discretization** the one represented by problem (4.17), while operators S, $\bar{S}$ and $\hat{S}$ are locally approximated through (4.7)-(4.9) in the **second level discretization**. In the coming sections we perform the error analysis for the first level, and for that the operators S, $\bar{S}$ and $\hat{S}$ are assumed to be exactly calculated.

Local problems (4.7)-(4.9) naturally embed heterogeneous or high contrasts in time-space features into the construction of the global weak formulation, which are not handled by the resolution of the space-time mesh $\mathcal{T}_H^n$. In practice we need a numerical approximation of these local problems. These approximations should be performed in a sufficiently fine mesh in order to capture the oscillation of the local problem solutions. The advantage of this strategy is that problems (4.7)-(4.9) can be computed in parallel. It is worth mentioning that the optimal convergence for u_h^n and σ_h^n on the natural norms relies only on the ability of (λ_n, τ_n) to be optimally interpolated by $(\lambda_{n,h}, \tau_{n,h})$ on time-space faces. Besides, the method addresses the flexibility to work with different space-time discretizations $\mathcal{T}_H^n$ on each slab $\Omega \times I_n$.

Remark 4.4. Formulation (4.17) can be decoupled and then solved more easily as follows:

- First, obtain $\tau_{n,h}$ by solving the second equation of (4.17) as a L^2 projection of $u_h^{n-1}(t_n)$ onto the space $W_{\bar{h}}$.
- Next, replace $\tau_{n,h}$ into the first equation of (4.17) and solve the following: Find $\lambda_{n,h} \in \Lambda_h^n$ such that

$$(\mu_h, S\lambda_{n,h})_{\partial\mathcal{T}_H^n} = -(\mu_h, \bar{S}\tau_{n,h})_{\partial\mathcal{T}_H^n} - (\mu_h, \hat{S}f)_{\partial\mathcal{T}_H^n}, \quad \forall \ \ \mu_h \in \Lambda_h^n. \tag{4.20}$$

  We reinforce that the problem (4.20) is defined in the time interval I_n and $\hat{S}f$ is locally defined in each I_n with initial data equals 0.

4.3 First level MHM

Before exploring the general steps used to compute the numerical solution of the parabolic MHM method, it is important to have a more detailed view of the operators $S\lambda_{n,h}$, $\bar{S}\tau_{n,h}$ and $\hat{S}f$. Suppose that $\{\psi_i\}_{i=1}^{dim\Lambda_h^n}$ is a basis for Λ_h^n and $\{\phi_i\}_{l=1}^{dimW_{\bar{h}}}$ is a basis for $W_{\bar{h}}$. We define the sets $\{\eta_i\}_{i=1}^{dim\Lambda_h^n} \subset \mathcal{X}$ and $\{\theta_l\}_{l=1}^{dimW_{\bar{h}}} \subset \mathcal{X}$ such that $S\psi_i = \eta_i$ and $\bar{S}\phi_l = \theta_l$ are solutions on each K_n of

$$\begin{aligned} (\partial_t\eta_i, v)_{K_n} + (A\nabla\eta_i, \nabla v)_{K_n} &= -(\psi_i, v)_{\partial K_n} \quad \forall \ \ v \in L^2(I_n; X), \\ \eta_i(t_n) &= 0, \end{aligned} \tag{4.21}$$

$$\begin{aligned} (\partial_t\theta_l, v)_{K_n} + (A\nabla\theta_l, \nabla v)_{K_n} &= 0 \quad \forall \ \ v \in L^2(I_n; X), \\ \theta_l(t_n) &= \phi_l. \end{aligned} \tag{4.22}$$

The function $\hat{S}f$ is the solution of

$$\begin{aligned} (\partial_t\hat{S}f, v)_{K_n} + (A\nabla\hat{S}f, \nabla v)_{K_N} &= (f, v)_{K_n} \quad \forall \ \ v \in L^2(I_n; X), \\ \hat{S}f(t_n) &= 0. \end{aligned} \tag{4.23}$$

Writing $\lambda_{n,h} = \sum_{i=1}^{dim\Lambda_h^n} \beta_i\psi_i$ and $\tau_{n,h} = \sum_{i=1}^{dimW_{\bar{h}}} \gamma_i\phi_i$, and then multiplying the last two equalities by β_i and γ_i, we have from linearity that

$$S\lambda_{n,h} = \sum_{i=1}^{dim\Lambda_h^n} \beta_i\eta_i \quad \text{and} \quad \bar{S}\tau_{n,h} = \sum_{i=1}^{dimW_{\bar{h}}} \gamma_i\theta_i.$$

It follows then

$$u_h^n = \sum_{i=1}^{dim\Lambda_h^n} \beta_i\eta_i + \sum_{l=1}^{dimW_{\bar{h}}} \gamma_l\theta_l + \hat{S}f.$$

In this sense, the method can be seen as a nonconforming method to find an approximation of the solution u of problem (1.1), restricted to a time interval I_n, in the space $X \not\subset H^1(\Omega)$. Also, the flow σ^n is approximated by σ_h^n as follows

$$\sigma_h^n = \sum_{i=1}^{dim\Lambda_h^n} \beta_i A\nabla\eta_i + \sum_{l=1}^{dimW_{\bar{h}}} \gamma_l A\nabla\theta_l + A\nabla\hat{S}f \in H(div;\Omega).$$

We next present a scheme that does not take into account the second level discretization at each element to obtain the multiscale basis functions. Consider M the number of partitions of the time interval $(0,T]$ represented by the union $\cup_{n=0}^{N-1} I_n$, with $I_n = (t_n, t_{n+1}]$. We then define the finite dimensional spaces

$$\Lambda_h^n := \mathbb{P}_r(I_n) \otimes M_h, \tag{4.24}$$
$$W_{\bar{h}} := \mathbb{P}_s(\mathcal{T}_{\bar{h}}), \tag{4.25}$$

where M_h is the space defined in (4.14) and the symbol $\otimes$ refers to the tensor product between vector spaces. Then, Λ_h^n stands for the set of the product of polynomials with degree at most r in time and m on $\mathcal{E}_h$ with zero jump on $F \in \mathcal{E}_h$. The set $\mathbb{P}_l(\mathcal{T}_{\bar{h}})$ is the set of piecewise continuous polynomials of degree at most $l \geq 1$ on each $K \in \mathcal{T}_{\bar{h}}$.

In the sequel, we describe in details the essence of the MHM algorithm:

(1) Let $u_{0,h} = u_0$ be the initial condition;

(2) Given the basis functions ψ_i and ϕ_i, for each $K_n \in \mathcal{T}_H^n$, we obtain the local basis functions η_i^n, η_τ^n and η_f^n exactly from local problems (4.21)-(4.23);

(3) Do $n = 0$ to $N-1$;

(4) Get the degrees of freedom γ_l^n of $\tau_{n,h}$ from

$$\sum_{l=1}^{dim\ W_{\bar{h}}} \gamma_l^n(\phi_l,\phi_j)_{\mathcal{T}_{\bar{h}}} = (\phi_j, u_h^{n-1}(t_n))_{\mathcal{T}_{\bar{h}}}$$

for $j = 1,\ldots, dim\ W_{\bar{h}}$, which leads to the linear system

$$\mathcal{A}\gamma^n = \mathcal{B}^n$$

where $\mathcal{A}_{l,j} = (\phi_l,\phi_j)_{\mathcal{T}_{\bar{h}}}$ and $\mathcal{B}_j^n = (\phi_j, u_h^{n-1}(t_n))_{\mathcal{T}_{\bar{h}}}$;

(5) Build $\bar{S}\tau_{n,h}$, in each K_n, from

$$\bar{S}\tau_{n,h} = \sum_{l=1}^{dim\ W_{\bar{h}}} \gamma_l^n \theta_l;$$

(6) Get the degrees of freedom β_i^n of $\lambda_{n,h}$ after solving

$$\sum_{i=1}^{dim\ \Lambda_h^n} \beta_i^n (\psi_j^n, \eta_i^n)_{\partial \mathcal{T}_H^n} = -(\psi_j^n, \eta_f^n + \bar{S}\tau_{n,h})_{\partial \mathcal{T}_H^n} \tag{4.26}$$

for $j = 1, \ldots, dim\ \Lambda_h^n$, leading to the linear system

$$\mathcal{C}^n \beta^n = \mathcal{D}^n$$

where $\mathcal{C}_{i,j}^n = (\psi_j^n, \eta_i^n)_{\partial \mathcal{T}_H^n}$ and $\mathcal{D}_j^n = -(\psi_j^n, \eta_f^n + \bar{S}\tau_{n,h})_{\partial \mathcal{T}_H^n}$;

(7) Compute u_h^n and σ_h^n from

$$u_h^n = \sum_{i=1}^{dim \Lambda_h^n} \beta_i^n \eta_i^n + \sum_{l=1}^{dim W_{\bar{h}}} \gamma_l^n \theta_l^n + \eta_f^n$$

and

$$\sigma_h^n = \sum_{i=1}^{dim \Lambda_h^n} \beta_i^n A \nabla \eta_i^n + \sum_{l=1}^{dim W_{\bar{h}}} \gamma_l^n A \nabla \theta_l^n + A \nabla \eta_f^n;$$

(8) End do.

This model requires that the basis functions η_i, θ_l and η_f are known exactly, which is not available in general. Therefore, a two level method is actually mandatory in order to approximate those basis functions in step (2) on each $K_n \in \mathcal{T}_H^n$. It is worthy pointing out that problems (4.21)-(4.23) may be naturally implemented in parallel and the systems associated to the local and global problems are symmetric and positive definite. We also need to choose a method to discretize the time derivative on each local problem (4.21)-(4.23).

In the appendix B, we show a fully discrete method where the local problems are discretized via FEM combined with an Euler Implicit method to approximate the time derivative on each local problem. This strategy is described in Chapter 2, and the convergence rates of such approach is displayed in Theorem 2.6.

4.4 Marching in Time Schemes

In this section we analyze how the discretization $W_{\bar{h}}$ of the space W affects the numerical solution obtained by the method. We show that when the space W is not discretized, i.e. $W = W_{\bar{h}}$, two different numerical time discretization of the space Λ yield the same numerical approximation. More precisely, given a time partition $\mathcal{T}^{\Delta T} = \cup_{n=1}^N I_n$ of $(0, T]$, where $I_n = (t_{n-1}, t_n]$, we consider the two following strategies:

(i) Local problems (4.7)-(4.9) are defined in time interval I_n and the basis functions of the discrete space $\Lambda_h^n = span_i\{\psi_i^n\}$ are defined in each time interval I_n.

(ii) Local problems (4.7)-(4.9) are defined in time interval $(0,T]$ and Λ_h, the discrete approximation of Λ, is understood as functions defined in the time interval [0,T] with compact support with respect to the time partition $\mathcal{T}^{\Delta T}$ and $\Lambda_h|_{I_n} = \Lambda_h^n$;

Let us see in more detail the peculiarities of these two approaches, and show that they are in fact equivalent.

Scheme 1. Let $\{\psi_i^n\}_{i=1}^R$ be a basis for Λ_h^n, with $R = dim(\Lambda_h^n)$, and let η_i^n, η_τ^n and η_f^n be the weak solutions of

$$\begin{cases} \partial_t \eta_i^n - \nabla\cdot(A\nabla\eta_i^n) = 0, \quad \text{in} \;\; K_n, \\ A\nabla\eta_i^n \cdot \mathbf{n}^K = -\psi_i^n, \quad \text{on} \;\; \partial K_n, \\ \eta_i^n = 0 \;\; \text{in} \;\; K\times\{t_n\}, \end{cases} \tag{4.27}$$

$$\begin{cases} \partial_t \eta_\tau^n - \nabla\cdot(A\nabla\eta_\tau^n) = 0, \quad \text{in} \;\; K_n, \\ A\nabla\eta_\tau^n \cdot \mathbf{n}^K = 0, \quad \text{on} \;\; \partial K_n, \\ \eta_\tau^n = \tau_{n,\bar{h}} \;\; \text{in} \;\; K\times\{t_n\}, \end{cases} \tag{4.28}$$

$$\begin{cases} \partial_t \eta_f^n - \nabla\cdot(A\nabla\eta_f^n) = f, \quad \text{in} \;\; K_n, \\ A\nabla\eta_f^n \cdot \mathbf{n}^K = 0, \quad \text{on} \;\; \partial K_n, \\ \eta_f^n = 0 \;\; \text{in} \;\; K\times\{t_n\}, \end{cases} \tag{4.29}$$

respectively. Then, we build the global system

$$\mathcal{A}^n\beta^n = b^n \tag{4.30}$$

where $\mathcal{A}$ is matrix of order R with $(\mathcal{A}^n)_{j,i} = (\psi_j^n, \eta_i^n)_{\partial\mathcal{T}_H\times I_n}$, b^n is a vector in $\mathbb{R}^R$ with $(b^n)_j = -(\psi_j^n, \eta_\tau^n + \eta_f^n)_{\partial\mathcal{T}_H\times I_n}$ and β^n is a vector in $\mathbb{R}^R$. The solution is then given by

$$u_h^n = \sum_{i=1}^R \beta_i^n\eta_i^n + \eta_\tau^n + \eta_f^n.$$

On I_1 we have the initial conditions $\eta_f^1(0) = 0$, $\eta_i^1(0) = 0$ and $\eta_\tau^1(0) = u_{0,h}$. After computing the basis we solve the global system $\mathcal{A}^1\beta^1 = b^1$ to obtain the values of β^1 to compute the solution u_h^1. In the next time slab I_2, we have the initial conditions $\eta_f^2(t_1) = 0$, $\eta_i^2(t_1) = 0$ and $\eta_\tau^2(t_1) = u_h^1(t_1)$ and we proceed as before to obtain β^2 and compute u_h^2. We keep on this process until the final slab I_M.

Scheme 2. Here we consider a discretization Λ_h in the way that $\Lambda_h|_{I_n} = \Lambda_h^n$. A natural basis for this space is given by

$$\widehat{\psi}_i^n = \begin{cases} \psi_i^n, \;\text{if}\; t\in I_n, \\ 0, \;\text{otherwise.} \end{cases}$$

Then we compute the MHM basis $\widehat{\eta_i}^n$, $\widehat{\eta_\tau}$ and $\widehat{\eta_f}$, weak solutions of

$$\begin{cases} \partial_t \widehat{\eta_i}^n - \nabla \cdot (A \nabla \widehat{\eta_i}^n) = 0 \ \text{ in } \ K_T, \\ \\ -A \nabla \widehat{\eta_i}^n \cdot \mathbf{n} = \widehat{\psi}_i^n \ \text{ on } \ \partial K_T, \\ \\ \widehat{\eta_i}^n = 0 \ \text{ at } \ K \times \{t = 0\}, \end{cases} \tag{4.31}$$

$$\begin{cases} \partial_t \widehat{\eta_\tau} - \nabla \cdot (A \nabla \widehat{\eta_\tau}) = 0 \ \text{ in } \ K_T, \\ \\ -A \nabla \widehat{\eta_\tau} \cdot \mathbf{n} = 0 \ \text{ on } \ \partial K_T, \\ \\ \widehat{\eta_\tau} = u_{0,h} \ \text{ at } \ K \times \{t = 0\}, \end{cases} \tag{4.32}$$

$$\begin{cases} \partial_t \widehat{\eta_f} - \nabla \cdot (A \nabla \widehat{\eta_f}) = f \ \text{ in } \ K_T, \\ \\ -A \nabla \widehat{\eta_f} \cdot \mathbf{n} = 0 \ \text{ on } \ \partial K_T, \\ \\ \widehat{\eta_f} = 0 \ \text{ at } \ K \times \{t = 0\}, \end{cases} \tag{4.33}$$

respectively. Now we analyze the terms $(\widehat{\psi}_j^n, \widehat{\eta_i}^m)_{\partial \mathcal{T}_H \times (0,T)}$.

For $n = 1$ we have

$$(\widehat{\psi}_j^1, \widehat{\eta_i}^1)_{\partial \mathcal{T}_H \times (0,T)} = (\widehat{\psi}_j^1, \widehat{\eta_i}^1)_{\partial \mathcal{T}_H \times I_1}, \tag{4.34}$$

$$(\widehat{\psi}_j^1, \widehat{\eta_i}^2)_{\partial \mathcal{T}_H \times (0,T)} = (\widehat{\psi}_j^1, \widehat{\eta_i}^2)_{\partial \mathcal{T}_H \times I_1} = 0. \tag{4.35}$$

Here, to obtain the last equality we use the fact that $\widehat{\eta_i}^2$ is zero on I_1. Indeed, $\widehat{\eta_i}^2$ is the weak solution of (4.31) with data $\widehat{\psi}_i^2 = 0$. In I_1, we have that $\widehat{\psi}_i^2 = 0$ and the right-hand side of each equation of (4.31) is zero on I_1 and, therefore, from the theory of parabolic equations we have $\widehat{\eta_i}^2 = 0$ in I_1. Following a similar idea we conclude that

$$(\widehat{\psi}_j^n, \widehat{\eta_i}^m)_{\partial \mathcal{T}_H \times (0,T)} = 0, \text{ for all } n < m. \tag{4.36}$$

For $n = 2$ we have

$$(\widehat{\psi}_j^2, \widehat{\eta_i}^1)_{\partial \mathcal{T}_H \times (0,T)} = (\widehat{\psi}_j^2, \widehat{\eta_i}^1)_{\partial \mathcal{T}_H \times I_2}, \tag{4.37}$$

$$(\widehat{\psi}_j^2, \widehat{\eta_i}^2)_{\partial \mathcal{T}_H \times (0,T)} = (\widehat{\psi}_j^2, \widehat{\eta_i}^2)_{\partial \mathcal{T}_H \times I_2}, \tag{4.38}$$

$$(\widehat{\psi}_j^2, \widehat{\eta_i}^M)_{\partial \mathcal{T}_H \times (0,T)} = 0, \ m > 2. \tag{4.39}$$

In this case, it is important to point out that $\widehat{\eta_i}^m$ is not zero in I_n, when $m \leq n$, due to the fact that $\widehat{\psi}_i^m \neq 0$ in $I_m \leq I_n$. Therefore we have

$$(\widehat{\psi}_j^n, \widehat{\eta_i}^m)_{\partial \mathcal{T}_H \times (0,T)} \neq 0, \text{ for all } m \leq n. \tag{4.40}$$

From (4.36) and (4.40) we build the stiffness block matrix of order N given by

$$\mathcal{A} = \begin{bmatrix} (\widehat{\psi}^1, \widehat{\eta_\lambda}^1)_{I_1} & 0 & 0 & \dots & 0 \\ (\widehat{\psi}^2, \widehat{\eta_\lambda}^1)_{I_2} & (\widehat{\psi}^2, \widehat{\eta_\lambda}^2)_{I_2} & 0 & \dots & 0 \\ (\widehat{\psi}^3, \widehat{\eta_\lambda}^1)_{I_3} & (\widehat{\psi}^3, \widehat{\eta_\lambda}^2)_{I_3} & (\widehat{\psi}^3, \widehat{\eta_\lambda}^3)_{I_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\widehat{\psi}^N, \widehat{\eta_\lambda}^1)_{I_N} & (\widehat{\psi}^N, \widehat{\eta_\lambda}^2)_{I_N} & (\widehat{\psi}^N, \widehat{\eta_\lambda}^3)_{I_N} & \dots & (\widehat{\psi}^N, \widehat{\eta_\lambda}^N)_{I_N} \end{bmatrix}, \tag{4.41}$$

where $(\widehat{\psi}^n, \widehat{\eta}_i{}^m)_{I_n}$ corresponds to a matrix of order R whose element ji is obtained by $((\widehat{\psi}^n, \widehat{\eta_\lambda}^m)_{I_n})_{j,i} = (\widehat{\psi}^n_j, \widehat{\eta}_i{}^m)_{\partial\mathcal{T}_H \times I_n}$.

We then build the global matrix system of **Scheme 2** with the terms $b^n = -(\widehat{\psi}^n, \widehat{\eta_\tau} + \widehat{\eta_f})_{I_n}$ as

$$\begin{bmatrix} (\widehat{\psi}^1, \widehat{\eta_\lambda}^1)_{I_1} & 0 & 0 & \dots & 0 \\ (\widehat{\psi}^2, \widehat{\eta_\lambda}^1)_{I_2} & (\widehat{\psi}^2, \widehat{\eta_\lambda}^2)_{I_2} & 0 & \dots & 0 \\ (\widehat{\psi}^3, \widehat{\eta_\lambda}^1)_{I_3} & (\widehat{\psi}^3, \widehat{\eta_\lambda}^2)_{I_3} & (\widehat{\psi}^3, \widehat{\eta_\lambda}^3)_{I_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\widehat{\psi}^N, \widehat{\eta_\lambda}^1)_{I_N} & (\widehat{\psi}^N, \widehat{\eta_\lambda}^2)_{I_N} & (\widehat{\psi}^N, \widehat{\eta_\lambda}^3)_{I_N} & \dots & (\widehat{\psi}^N, \widehat{\eta_\lambda}^N)_{I_N} \end{bmatrix} \begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \alpha_3 \\ \vdots \\ \alpha^N \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \\ b^3 \\ \vdots \\ b^N \end{bmatrix}, \tag{4.42}$$

where α^n, $n = 1, \dots, N$, are constant vectors. We then have the system of equations

$$\begin{aligned} (\widehat{\psi}^1, \alpha^1\widehat{\eta_\lambda}^1)_{I_1} &= -(\widehat{\psi}^1, \widehat{\eta_\tau} + \widehat{\eta_f})_{I_1} \\ (\widehat{\psi}^2, \alpha^1\widehat{\eta_\lambda}^1 + \alpha^2\widehat{\eta_\lambda}^2)_{I_2} &= -(\widehat{\psi}^2, \widehat{\eta_\tau} + \widehat{\eta_f})_{I_2} \\ &\vdots \\ (\widehat{\psi}^N, \alpha^1\widehat{\eta_\lambda}^1 + \dots + \alpha^N\widehat{\eta_\lambda}^N)_{I_N} &= -(\widehat{\psi}^N, \widehat{\eta_\tau} + \widehat{\eta_f})_{I_N}. \end{aligned} \tag{4.43}$$

We now show that both schemes produce the same numerical solution.

In the first time slab I_1 we observe that $\widehat{\eta}_i{}^1|_{I_1} = \eta_i^1$, $\widehat{\eta_\tau}|_{I_1} = \eta_\tau^1$ and $\widehat{\eta_f}|_{I_1} = \eta_f^1$ since $\widehat{\psi}_i^1 = \psi_i^1$ in I_1 and the initial data in $t_0 = 0$ of the MHM basis are all the same. Therefore, the first equation of (4.43) is equivalent to (4.30) with $n = 1$ from **scheme 1** and we conclude that $\alpha^1 = \beta^1$.

Once we have obtained the vector α_1, we rewrite the second equation of (4.43) as

$$(\widehat{\psi}^2, \alpha^2\widehat{\eta_\lambda}^2)_{I_2} = -(\widehat{\psi}^2, \alpha^1\widehat{\eta_\lambda}^1 + \widehat{\eta_\tau} + \widehat{\eta_f})_{I_2} \tag{4.44}$$

to compute the next solution α^2. In the sequel we observe that in I_2 the function $\widehat{\eta}_i{}^1$ is the solution of

$$\begin{cases} \partial_t\widehat{\eta}_i{}^1 - \nabla\cdot(A\nabla\widehat{\eta}_i{}^1) = 0 \ \text{ in } \ K \times I_2, \\ -A\nabla\widehat{\eta}_i{}^1 \cdot \mathbf{n} = 0 \ \text{ on } \ \partial K \times I_2, \\ \widehat{\eta}_i{}^1 = \widehat{\eta}_i(t_1) \ \text{ at } \ K \times \{t = t_1\}, \end{cases} \tag{4.45}$$

and the functions $\widehat{\eta_\tau}$ and $\widehat{\eta_f}$ are solutions, respectively, of

$$\begin{cases} \partial_t \widehat{\eta_\tau} - \nabla \cdot (A \nabla \widehat{\eta_\tau}) = 0 \ \text{ in } \ K \times I_2, \\ \\ -A \nabla \widehat{\eta_\tau} \cdot \mathbf{n} = 0 \ \text{ on } \ \partial K \times I_2, \\ \\ \widehat{\eta_\tau} = \widehat{\eta_\tau}(t_1) \ \text{ at } \ K \times \{t = t_1\}, \end{cases} \tag{4.46}$$

and

$$\begin{cases} \partial_t \widehat{\eta_f} - \nabla \cdot (A \nabla \widehat{\eta_f}) = f \ \text{ in } \ K \times I_2, \\ \\ -A \nabla \widehat{\eta_f} \cdot \mathbf{n} = 0 \ \text{ on } \ \partial K \times I_2, \\ \\ \widehat{\eta_f} = \widehat{\eta_f}(t_1) \ \text{ at } \ K \times \{t = t_1\}. \end{cases} \tag{4.47}$$

We then rewrite $\alpha^1 \widehat{\eta_\lambda}^1 + \widehat{\eta_\tau} + \widehat{\eta_f}$ as

$$\alpha^1 \widehat{\eta_\lambda}^1 + \widehat{\eta_\tau} + \widehat{\eta_f} = \tilde{\eta_\tau} + \tilde{\eta_f}, \tag{4.48}$$

where these functions satisfy the following PDEs

$$\begin{cases} \partial_t \tilde{\eta_\tau} - \nabla \cdot (A \nabla \tilde{\eta_\tau}) = 0 \ \text{ in } \ K \times I_2, \\ \\ -A \nabla \tilde{\eta_\tau} \cdot \mathbf{n} = 0 \ \text{ on } \ \partial K \times I_2, \\ \\ \tilde{\eta_\tau} = (\alpha^1 \widehat{\eta_\lambda}^1 + \widehat{\eta_\tau} + \widehat{\eta_f})(t_1) \ \text{ at } \ K \times \{t = t_1\}, \end{cases} \tag{4.49}$$

and

$$\begin{cases} \partial_t \tilde{\eta_f} - \nabla \cdot (A \nabla \tilde{\eta_f}) = f \ \text{ in } \ K \times I_2, \\ \\ -A \nabla \tilde{\eta_f} \cdot \mathbf{n} = 0 \ \text{ on } \ \partial K \times I_2, \\ \\ \tilde{\eta_f} = 0 \ \text{ at } \ K \times \{t = t_1\}. \end{cases} \tag{4.50}$$

Since $\alpha^1 = \beta^1$, we see that $u_h^1(t_1) = (\alpha^1 \widehat{\eta_\lambda}^1 + \widehat{\eta_\tau} + \widehat{\eta_f})(t_1)$ and the restrictions $\widehat{\eta_\lambda}^2|_{I_2} = \eta_i^2$, $\tilde{\eta_\tau}|_{I_2} = \eta_\tau^2$ and $\tilde{\eta_f}|_{I_2} = \eta_f^2$ hold, where η_i^2, η_τ^2 and η_f^2 are the MHM basis on I_2 of **Scheme 1**. Therefore, the system

$$(\widehat{\psi}^2, \alpha^2 \widehat{\eta_\lambda}^2)_{I_2} = -(\widehat{\psi}^2, \tilde{\eta_\tau} + \tilde{\eta_f})_{I_2}, \tag{4.51}$$

is equivalent to (4.30) for $n = 2$ and we again conclude that $\alpha^2 = \beta^2$.

Following the same strategy used earlier we conclude that $\alpha^n = \beta^n$, $n = 1, \ldots, N$, and, thus, the numerical solution obtained from **Scheme 1** and from **Scheme 2** are the same.

5 Error Analysis

This chapter is dedicated to the numerical analysis of the parabolic MHM method.

We start by proving in section 5.1 the solvability of the discrete system obtained in section 4.2. Error estimates in the natural norms of the problem are treated in section 5.2, where the estimates are still not ideal since it relies on the norms of a function dependent of the numerical method. This issue is addressed in section 5.3, where the equivalence of the march in time schemes showed in section 4.4 is used to avoid such dependence.

5.1 Existence and Uniqueness

In this section we are going to establish the existence and uniqueness of the problem (4.17). To do that we first prove that problem (4.20) possesses an unique solution.

Theorem 5.1. *Let $f \in L^2(0,T;L^2(\Omega))$ and $\tau_{n,h}$ be a function on $W_{\bar{h}}$. Also, let $\bar{S}\tau_{n,h}$ and $\hat{S}f$ be solutions of problems (4.8)-(4.9), respectively, with initial conditions $\bar{S}\tau_{n,h}(t_n) = \tau_{n,h}$ and $\hat{S}f(t_n) = 0$. Therefore, problem (4.20) possesses an unique solution $\lambda_{n,h} \in \Lambda_h^n$.*

Proof. Define the discrete bilinear form $a_h : \Lambda_h^n \times \Lambda_h^n \to \mathbb{R}$ as

$$a_h(\mu_h, \lambda_{n,h}) = -(\mu_h, S\lambda_{n,h})_{\partial\mathcal{T}_H^n},$$

and rewrite (4.20) in the following way: find $\lambda_{n,h} \in \Lambda_h^n$ such that

$$a_h(\mu_h, \lambda_{n,h}) = (\mu_h, \bar{S}\tau_{n,\bar{h}} + \hat{S}f)_{\partial\mathcal{T}_H^n}, \quad \forall \quad \mu_h \in \Lambda_h^n. \tag{5.1}$$

Using (4.7) and the fact that $S\mu_h = 0$ at t_n, we can see that

$$\begin{aligned}
a_h(\mu_h,\mu_h) &= (\partial_t S\mu_h, S\mu_h)_{\mathcal{T}_H^n} + (A\nabla S\mu_h, \nabla S\mu_h)_{\mathcal{T}_H^n} \\
&\geq c_{min}||\nabla S\mu_h||^2_{L^2(\mathcal{T}_H^n)} + \frac{1}{2}\int_{t_n}^{t_{n+1}} \partial_t(||S\mu_n||^2_{L^2(\mathcal{T}_H)})dt \\
&= c_{min}||\nabla S\mu_h||^2_{L^2(\mathcal{T}_H^n)} + \frac{1}{2}||S\mu_h(t_{n+1})||^2_{L^2(\mathcal{T}_H)} \\
&\geq c_{min}||\nabla S\mu_h||^2_{L^2(\mathcal{T}_H^n)}.
\end{aligned}$$

Now suppose that $\lambda_{n,h}^{(1)}$ and $\lambda_{n,h}^{(2)}$ are both solutions of (5.1). Then,

$$\begin{aligned}
a_h(\mu_h, \lambda_{n,h}^{(1)} - \lambda_{n,h}^{(2)}) &= a_h(\mu_h, \lambda_{n,h}^{(1)}) - a_h(\mu_h, \lambda_{n,h}^{(2)}) \\
&= (\mu_h, \bar{S}\tau_{n,\bar{h}} + \hat{S}f)_{\partial\mathcal{T}_H^n} - (\mu_h, \bar{S}\tau_{n,\bar{h}} + \hat{S}f)_{\partial\mathcal{T}_H^n} \\
&= 0.
\end{aligned}$$

Taking $\mu_h = \lambda_{n,h}^{(1)} - \lambda_{n,h}^{(2)}$ we have

$$0 = a_h(\lambda_{n,h}^{(1)} - \lambda_{n,h}^{(2)}, \lambda_{n,h}^{(1)} - \lambda_{n,h}^{(2)}) \geq c_{min}||\nabla S(\lambda_{n,h}^{(1)} - \lambda_{n,h}^{(2)})||^2_{L^2(\mathcal{T}_H^n)}.$$

From the last inequality we have that $||\nabla S(\lambda_{n,h}^{(1)} - \lambda_{n,h}^{(2)})||_{L^2(\mathcal{T}_H^n)} = 0$, implying that $S(\lambda_{n,h}^{(1)} - \lambda_{n,h}^{(2)}) = c$. However, since $S(\lambda_{n,h}^{(1)} - \lambda_{n,h}^{(2)}) = 0$ at t_n, we have that $c = 0$ and, then, $\lambda_{n,h}^{(1)} = \lambda_{n,h}^{(2)}$ due to the injectivity of operator S.

Let $dim\ \Lambda_h^n = R$ and $\{\psi_i^n\}_{i=1}^R$ be a basis for Λ_h^n. Observe that the operator $A : \Lambda_h^n \to \mathbb{R}^R$ defined by $(A\mu_h)_i = a_h(\psi_i^n, \mu_h)$ is injective, from the previous argument. Therefore, it is also surjective due to the fact that injective linear maps between vector spaces with the same dimension are surjective. This implies that the matrix A where $A_{ij} = a_h(\psi_i^n, \psi_j^n)$ is invertible and, therefore, the system $A\alpha^n = b$, where $\alpha^n = (\alpha_1^n, \ldots, \alpha_N^n)$ are the coefficients of $\lambda_{n,h}$ and $b_i = (\psi_i^n, \bar{S}\tau_{n,\bar{h}} + \hat{S}f)_{\partial\mathcal{T}_H^n}$ is uniquely solvable. This proves existence and uniqueness of problem (5.1). □

Corollary 5.2. *Problem (4.17) is well-posed.*

Proof. The well posedeness of problem (4.17) follows from the fact that $\bar{S}\tau_{n,h}$ is uniquely defined through the initial condition given by the L^2 projection of $u_h^{n-1}(t_n)$ (or u_h^0 if $n = 0$) onto the space $W_{\bar{h}}$.

□

5.2 Intermediate Error Estimates

In this section we derive some approximation error estimates for the MHM in the natural norms defined for the spaces $\mathcal{X}$ and Λ in (3.40). We consider the matrix A of problem (1.1) time independent. To obtain such estimates we work with a modified version of the original problem (1.1), and we apply the MHM method to obtain basis functions that can be written as a product of the original basis for (1.1) by an exponential function to be defined. We then prove error estimates for such basis and recover the results for the MHM basis of the original problem (1.1). In the sequence, we split the error analysis first for the semidiscrete in space case in subsection 5.2.1, then for the semidiscrete in time case in subsection 5.2.2 and, combining the error estimates for the two cases, we obtain an error estimate for the MHM method presented in Theorem 5.14.

Let u be the solution of problem (1.1) and define $\tilde{u} = e^{-\beta t}u$, $\beta > 0$. Observe that

$$\partial_t \tilde{u} = e^{-\beta t}(\partial_t u - \beta u) \tag{5.2}$$

and that $\tilde{u}$ satisfies the following problem

$$\begin{cases} \partial_t \tilde{u} - \nabla \cdot (A\nabla \tilde{u}) + \beta \tilde{u} = \tilde{f} \ \text{ in } \ \Omega \times (0,T), \\ \\ \tilde{u} = 0 \ \text{ on } \ \partial\Omega \times (0,T), \\ \\ \tilde{u} = u_0 \ \text{ at } \ \Omega \times \{t=0\}, \end{cases} \tag{5.3}$$

where $\tilde{f} = e^{-\beta t} f$ and, when restricted to a time interval I_n, the initial condition writes $\tilde{\tau}_n = \tilde{u}(t_n) = e^{-\beta t_n}\tau_n$. The hybrid formulation applied to the original problem (1.1) is naturally extended to problem (5.3), and considering new local operators S', $\bar{S}'$ and $\hat{S}'$ that are locally defined as weak solutions of

$$\begin{aligned} (\partial_t S'\tilde{\mu}, v)_{K_n} + (A\nabla S'\tilde{\mu}, \nabla v)_{K_n} + \beta(S'\tilde{\mu}, v)_{K_n} &= -(\tilde{\mu}, v)_{\partial K_n} \ \ \forall \ \ v \in L^2(I_n; X), \\ S'\tilde{\mu}(t_n) &= 0, \end{aligned} \tag{5.4}$$

$$\begin{aligned} (\partial_t \bar{S}'\tilde{\rho}_n, v)_{K_n} + (A\nabla \bar{S}'\tilde{\rho}_n, \nabla v)_{K_n} + \beta(\bar{S}'\tilde{\rho}_n, v)_{K_n} &= 0 \ \ \forall \ \ v \in L^2(I_n; X), \\ \bar{S}'\tilde{\rho}_n(t_n) &= \tilde{\rho}_n, \end{aligned} \tag{5.5}$$

$$\begin{aligned} (\partial_t \hat{S}'\tilde{q}, v)_{K_n} + (A\nabla \hat{S}'\tilde{q}, \nabla v)_{K_n} + \beta(\hat{S}'\tilde{q}, v)_{K_n} &= (\tilde{q}, v)_{K_n} \ \ \forall \ \ v \in L^2(I_n; X), \\ \hat{S}'\tilde{q}(t_n) &= 0, \end{aligned} \tag{5.6}$$

where $\tilde{\mu} \in \Lambda$, $\tilde{\rho} \in \Sigma$ and $\tilde{q} \in L^2(I_n; L^2(\Omega))$. The trace of $\tilde{u}$ on the boundary ∂K_n is $-A\nabla\tilde{u}\cdot\mathbf{n} = \tilde{\lambda}_n$ and

$$-A\nabla\tilde{u}\cdot\mathbf{n} = -A\nabla(e^{-\beta t}u)\cdot\mathbf{n} = e^{-\beta t}(-A\nabla u\cdot\mathbf{n}) = e^{-\beta t}\lambda_n,$$

from what we can conclude that $\tilde{\lambda}_n = e^{-\beta t}\lambda_n$. The global-local formulation of (5.3) is, therefore, given by

$$\begin{cases} (\mu, S'\tilde{\lambda}_n + \bar{S}'\tilde{\tau}_n)_{\partial\mathcal{T}_H^n} = -(\mu, \hat{S}'\tilde{f})_{\partial\mathcal{T}_H^n}, \ \ \forall \ \ \mu \in \Lambda, \\ \\ (z, \tilde{\tau}_n)_{\mathcal{T}_H} = (z, \tilde{u}_n)_{\mathcal{T}_H}, \ \ \forall \ \ z \in L^2(\Omega), \end{cases} \tag{5.7}$$

along with local problems (5.4)-(5.6). Therefore, we write $\tilde{u} = S'\tilde{\lambda}_n + \bar{S}'\tilde{\tau}_n + \hat{S}'\tilde{f}$ on each K_n after replacing μ, ρ_n and q by $\tilde{\lambda}_n, \tilde{\tau}_n$ and $\tilde{f}$, respectively.

We now derive some properties relating the operators $S, \bar{S}, \tilde{S}$ to the operators $S', \bar{S}', \tilde{S}'$, respectively. Set $w = e^{-\beta t}S\lambda_n$ (see (4.7)) and observe that w is the weak solution of

$$\begin{cases} \partial_t w - \nabla\cdot(A\nabla w) + \beta w = 0, \ \text{ in } \ K_n, \\ -A\nabla w\cdot\mathbf{n}^K = e^{-\beta t}\lambda_n = \tilde{\lambda}_n, \ \text{ on } \ \partial K_n, \\ w = 0 \ \text{ in } \ K \times \{t_n\}. \end{cases} \tag{5.8}$$

On the other hand, from (5.4) we have $S'\tilde{\lambda}_n$ weak solution of

$$\begin{cases} \partial_t S'\tilde{\lambda}_n - \nabla \cdot (A\nabla S'\tilde{\lambda}_n) + \beta S'\tilde{\lambda}_n = 0, \quad \text{in} \quad K_n, \\ -A\nabla S'\tilde{\lambda}_n \cdot \mathbf{n}^K = \tilde{\lambda}_n, \quad \text{on} \quad \partial K_n, \\ S'\tilde{\lambda}_n = 0 \quad \text{in} \quad K \times \{t_n\}. \end{cases} \tag{5.9}$$

From uniqueness of problems (5.8) and (5.9), we obtain $S'\tilde{\lambda}_n = w = e^{-\beta t} S\lambda_n$. Following a similar strategy for the others, we obtain the following identities

$$\begin{aligned} S'\tilde{\lambda}_n &= e^{-\beta t} S\lambda_n, \\ \bar{S}'\tilde{\tau}_n &= e^{-\beta t}\bar{S}\tau_n, \\ \hat{S}'\tilde{f} &= e^{-\beta t}\hat{S}f. \end{aligned} \tag{5.10}$$

In sequence, let $\{\psi_i\}_{i=1}^L$ be a basis for $\Lambda_h^{\Delta T}$ and $\{\phi_l\}_{l=1}^R$ a basis for $W_{\bar{h}}$. Define $\tilde{\Lambda}_h^{\Delta T} := span\{\tilde{\psi}_i\}_{i=1}^M$, where $\tilde{\psi}_i = e^{-\beta t}\psi_i$. It is important to keep in mind that the discrete spaces $\Lambda_h^{\Delta T}$ and $\tilde{\Lambda}_h^{\Delta T}$ are not the same. From identities (5.10), we relate the multiscale basis functions $\eta_i = S\psi_i$ with $\tilde{\eta}_i = S'\tilde{\psi}_i$, and $\theta_l = \bar{S}\phi_l$ with $\tilde{\theta}_l = \bar{S}'(e^{-\beta t_n}\phi_l)$ as

$$\begin{aligned} \tilde{\eta}_i &= e^{-\beta t}\eta_i, \\ \tilde{\theta}_l &= e^{-\beta t}\theta_l. \end{aligned} \tag{5.11}$$

Then, the discrete operators $S'\tilde{\lambda}_{n,h} = \sum_{i=1}^M \alpha_i^n \tilde{\eta}_i$ and $\bar{S}\tilde{\tau}_{n,h} = \sum_{l=1}^R \beta_l^n \tilde{\theta}_l$ satisfy the discrete identities given by

$$\begin{aligned} S'\tilde{\lambda}_{n,h} &= e^{-\beta t} S\lambda_{n,h}, \\ \bar{S}'\tilde{\tau}_{n,h} &= e^{-\beta t}\bar{S}\tau_{n,h}. \end{aligned} \tag{5.12}$$

Before stating any results about the convergence of the method, we show that the error analysis of the original problem can be transferred to the error analysis of (5.3) by the following Lemma:

Lemma 5.3. *Let u^n be the solution of problem (1.1) restricted to the time interval I_n and u_h^n be the solution of the system (4.7)-(4.9) and (4.12). Also let $\tilde{u}^n$ be the solution of problem (5.3) and $\tilde{u}_h^n$ be the solution of the coupled system (5.4)-(5.6) and (5.7). Then, the following estimates hold*

$$\begin{aligned} ||u^n - u_h^n||_{L^2(I_n;X)} &\le e^{\beta t_{n+1}}\, ||\tilde{u}^n - \tilde{u}_h^n||_{L^2(I_n;X)}, \\ ||u^n - u_h^n||_{L^2(0,T;X)} &\le e^{\beta T}\, ||\tilde{u}^n - \tilde{u}_h^n||_{L^2(0,T;X)}, \\ ||\tau_n - \tau_{n,h}||_{L^2(\Omega)} &= e^{\beta t_n}||\tilde{\tau}_n - \tilde{\tau}_{n,h}||_{L^2(\Omega)}. \end{aligned} \tag{5.13}$$

Proof. From (5.10) and (5.12) we see that $u_h^n = e^{\beta t} \cdot e^{-\beta t}(S\lambda_{n,h} + \bar{S}\tau_{n,\bar{h}} + \hat{S}f) = e^{\beta t}\tilde{u}_h^n$ and the first estimate of (5.13) comes from

$$\begin{aligned} ||u^n - u_h^n||^2_{L^2(I_n;X)} &= \int_{I_n} ||e^{\beta t}(\tilde{u}^n - \tilde{u}_h^n)||_X^2 dt \\ &= \int_{I_n} e^{2\beta t}||\tilde{u}^n - \tilde{u}_h^n||_X^2 dt \\ &\leq e^{2\beta t_{n+1}}||\tilde{u}^n - \tilde{u}_h^n||^2_{L^2(I_n;X)}. \end{aligned} \tag{5.14}$$

The second estimate of (5.13) is a consequence of summing over all the time intervals I_n the last inequality, bounding $e^{2\beta t_{n+1}}$ by $e^{2\beta T}$, to get

$$\begin{aligned} ||u^n - u_h^n||^2_{L^2(0,T;X)} &= \sum_{n=0}^{N-1} ||u^n - u_h^n||^2_{L^2(I_n;X)} \\ &\leq e^{2\beta T} \sum_{n=0}^{N-1} ||\tilde{u}^n - \tilde{u}_h^n||^2_{L^2(I_n;X)} \\ &\leq e^{2\beta T}||\tilde{u}^n - \tilde{u}_h^n||^2_{L^2(0,T;X)}. \end{aligned} \tag{5.15}$$

The last estimate of (5.13) is obtained from

$$||\tau_n - \tau_{n,\bar{h}}||^2_{L^2(\Omega)} = \int_\Omega |e^{\beta t_n}(\tilde{\tau}_n - \tilde{\tau}_{n,\bar{h}})|^2 dx = e^{2\beta t_n}||\tilde{\tau}_n - \tilde{\tau}_{n,\bar{h}}||^2_{L^2(\Omega)}. \tag{5.16}$$

□

In the sequence we define the bilinear form $\tilde{a} : \tilde{\Lambda}^n \times \tilde{\Lambda}^n \to \mathbb{R}$ such that

$$\tilde{a}(\mu, \tilde{\lambda}_n) = -(\mu, S'\tilde{\lambda}_n)_{\partial \mathcal{T}_H^n}$$

and the respective problems

$$\tilde{a}(\mu, \tilde{\lambda}_n) = (\mu, \bar{S}'\tilde{\tau}_n + \hat{S}'\tilde{f})_{\partial \mathcal{T}_H^n} \quad \forall\ \mu \in \tilde{\Lambda}^n, \tag{5.17}$$

and

$$\tilde{a}(\tilde{\mu}_h^{\Delta T}, \tilde{\lambda}_{n,h}) = (\tilde{\mu}_h^{\Delta T}, \bar{S}'\tilde{\tau}_{n,h} + \hat{S}'\tilde{f})_{\partial \mathcal{T}_H^n} \quad \forall\ \tilde{\mu}_h^{\Delta T} \in \tilde{\Lambda}_h^{\Delta T}. \tag{5.18}$$

The next result displays a bound for $\tilde{u}^n - \tilde{u}_h^n$ in the norms $L^2(I_n; X)$ and $L^2(0, T; X)$ for the discrete spaces $\tilde{\Lambda}_h^{\Delta T}$ and $W_{\bar{h}}$.

Lemma 5.4 (Best Approximation Result)**.** *Let $\tilde{u}^n$ be the solution of (5.3) and $\tilde{u}_h^n$ be its numerical approximation of (5.4)-(5.7). Recall that $\tilde{\tau}_n = \tilde{u}(t_n)$ and $\tilde{\tau}_{n,h} = \mathcal{P}_{W_{\bar{h}}}(\tilde{u}_h^{n-1}(t_n))$, where $\mathcal{P}_{W_{\bar{h}}}$ is the L^2 projection onto $W_{\bar{h}}$. Then, we have the following error estimate in the time interval I_n given by*

$$\begin{aligned} ||\tilde{u}^n - \tilde{u}_h^n||^2_{L^2(I_n;X)} &+ ||\tilde{\tau}_{n+1,\bar{h}} - \tilde{\tau}_{n+1,h}||^2_{L^2(\Omega)} \\ &\leq ||\tilde{\tau}_n - \tilde{\tau}_{n,\bar{h}}||^2_{L^2(\Omega)} + ||\tilde{\tau}_{n,\bar{h}} - \tilde{\tau}_{n,h}||^2_{L^2(\Omega)} \\ &\quad + \gamma^{-2}||\tilde{\lambda}^* - \tilde{\mu}_h^{\Delta T}||^2_{L^2(I_n,M)}, \end{aligned} \tag{5.19}$$

where $\gamma = \min\{c_{min}, \beta, \frac{1}{2}\}$, $\tilde{\tau}_{n,\bar{h}} = \mathcal{P}_{W_{\bar{h}}}\tilde{u}(t_n)$ *and* $\tilde{\lambda}^* = -A\nabla\tilde{u}^*\cdot\mathbf{n}$ *is the Lagrange multiplier associated with the problem*

$$\begin{cases} \partial_t\tilde{u}^* - \nabla\cdot(A\nabla\tilde{u}^*) + \beta\tilde{u}^* = \tilde{f} \quad in \quad \Omega_n = \Omega\times I_n, \\ \tilde{u}^* = 0 \quad on \quad \partial\Omega_N = \partial\Omega\times I_n, \\ \tilde{u}^* = e^{-\beta t_n}\tau_{n,h} \quad at \quad \Omega\times\{t = t_n\}. \end{cases} \tag{5.20}$$

Furthermore, the following estimate holds

$$\begin{aligned} ||\tilde{u} - \tilde{u}_h||_{L^2(0,T;X)} + ||\tilde{\tau}_{N,\bar{h}} - \tilde{\tau}_{N,h}||_{L^2(\Omega)} \\ \leq \sum_{n=0}^{N-1} ||\tilde{\tau}_n - \tilde{\tau}_{n,\bar{h}}||_{L^2(\Omega)} + \gamma^{-1}||\tilde{\lambda}^* - \tilde{\mu}_h^{\Delta T}||_\Lambda. \end{aligned} \tag{5.21}$$

Proof. Let us begin with some estimates used in this proof involving $S'\mu$, $\mu\in\Lambda$. From (5.4), replacing v by $S'\mu$, we have

$$\begin{aligned} -(\mu, S'\mu)_{\partial\mathcal{T}_H^n} &= (\partial_t S'\mu, S'\mu)_{\mathcal{T}_H^n} + (A\nabla S'\mu, \nabla S'\mu)_{\mathcal{T}_H^n} + \beta(S'\mu, S'\mu)_{\mathcal{T}_H^n} \\ &\geq c_{min}||\nabla S'\mu||^2_{L^2(I_n;L^2(\mathcal{T}_H))} + \beta||S'\mu||^2_{L^2(I_n;L^2(\mathcal{T}_H))} \\ &\quad + \frac{1}{2}\left(||S'\mu(t_{n+1}))||^2_{L^2(\Omega)} - ||S'\mu(t_n))||^2_{L^2(\Omega)}\right) \\ &\geq \gamma\left(||S'\mu||^2_{L^2(I_n;X)} + ||S'\mu(t_{n+1}))||^2_{L^2(\Omega)}\right) \end{aligned} \tag{5.22}$$

where $\gamma = \min\{c_{min}, \beta, \frac{1}{2}\}$ and we used that $S'\mu(t_n) = 0$. Therefore we have the estimate

$$\gamma\left(||S'\mu||^2_{L^2(I_n;X)} + ||S'\mu(t_{n+1}))||^2_{L^2(\Omega)}\right) \leq -(\mu, S'\mu)_{\partial\mathcal{T}_H^n}. \tag{5.23}$$

We also have the following estimate

$$\begin{aligned} -(\mu, S'\lambda)_{\partial\mathcal{T}_H^n} &\leq \int_{I_n}\sup_{v\in X}\frac{-(\mu, v)_{\partial\mathcal{T}_H}}{||v||_X}||S'\lambda||_X dt \\ &\leq ||\mu||_{L^2(I_n;M)}||S'\lambda||_{L^2(I_n;X)}. \end{aligned} \tag{5.24}$$

Now we consider the auxiliary global problem (5.20) and write $\tilde{u}^* = S'\tilde{\lambda}^* + \bar{S}'\tilde{\tau}_{n,\bar{h}} + \hat{S}'\tilde{f}$ on each $K_n\in\mathcal{T}_H^n$, where the operators S', $\bar{S}'$ and $\hat{S}'$ are solutions of the local problems (5.4)-(5.6).

In order to prove our result we start with the following inequality

$$||\tilde{u}^n - \tilde{u}_h^n||_{L^2(I_n;X)} \leq ||\tilde{u}^n - \tilde{u}^*||_{L^2(I_n;X)} + ||\tilde{u}^* - \tilde{u}_h^n||_{L^2(I_n;X)}. \tag{5.25}$$

First, we estimate the second term on the right-hand side of (5.25). Observe that

$$\tilde{u}^* - \tilde{u}_h^n = S'(\tilde{\lambda}^* - \tilde{\lambda}_{n,h}) + \bar{S}'(\tilde{\tau}_{n,h} - \tilde{\tau}_{n,h}) + \hat{S}'(\tilde{f} - \tilde{f}) = S'(\tilde{\lambda}^* - \tilde{\lambda}_{n,h}) \tag{5.26}$$

on each K_n and, then, from (5.23), we have

$$\gamma\Big(||S'(\tilde\lambda^* - \tilde\lambda_{n,h})||^2_{L^2(I_n;X)} + ||S'(\tilde\lambda^* - \tilde\lambda_{n,h})(t_{n+1}))||^2_{L^2(\Omega)}\Big) \leq -(\tilde\lambda^* - \tilde\lambda_{n,h}, S'\tilde\lambda^* - S'\tilde\lambda_{n,h})_{\partial\mathcal{T}^n_H}$$
$$= -(\tilde\lambda^* - \tilde\mu_h^{\Delta T}, S'\tilde\lambda^* - S'\tilde\lambda_{n,h})_{\partial\mathcal{T}^n_H} - (\tilde\mu_h^{\Delta T} - \tilde\lambda_{n,h}, S'\tilde\lambda^* - S'\tilde\lambda_{n,h})_{\partial\mathcal{T}^n_H}, \quad (5.27)$$

where we summed and subtracted an arbitrary element $\tilde\mu_h^{\Delta T} \in \tilde\Lambda_h^{\Delta T}$ to obtain the last identity.
From (5.17)-(5.18) we have, for all $\tilde\mu_h^{\Delta T} \in \tilde\Lambda_h^{\Delta T}$,

$$-(\tilde\mu_h^{\Delta T}, S'\tilde\lambda^*)_{\partial\mathcal{T}^n_H} = (\tilde\mu_h^{\Delta T}, \bar S'\tilde\tau_{n,\bar h} + \hat S'f)_{\partial\mathcal{T}^n_H} = -(\tilde\mu_h^{\Delta T}, S'\tilde\lambda_{n,h})_{\partial\mathcal{T}^n_H},$$

and, therefore, we see that $-(\tilde\mu_h^{\Delta T} - \tilde\lambda_{n,h}, S'\tilde\lambda^* - S'\tilde\lambda_{n,h})_{\partial\mathcal{T}^n_H} = 0$ since $\tilde\mu_h^{\Delta T} - \tilde\lambda_{n,h} \in \tilde\Lambda_h^{\Delta T}$.
Estimate (5.27) therefore becomes

$$\gamma\left(||S'(\tilde\lambda^* - \tilde\lambda_{n,h})||^2_{L^2(I_n;X)} + ||S'(\tilde\lambda^* - \tilde\lambda_{n,h})(t_{n+1}))||^2_{L^2(\Omega)}\right) \leq -(\tilde\lambda^* - \tilde\mu_h^{\Delta T}, S'\tilde\lambda^* - S'\tilde\lambda_{n,h})_{\partial\mathcal{T}^n_H}, \quad (5.28)$$

for all $\tilde\mu_h^{\Delta T} \in \tilde\Lambda_h^{\Delta T}$. We then apply estimate (5.24) to $-(\tilde\lambda^* - \tilde\mu_h^{\Delta T}, S'\tilde\lambda^* - S'\tilde\lambda_{n,h})_{\partial\mathcal{T}^n_H}$ to obtain

$$-(\tilde\lambda^* - \tilde\mu_h^{\Delta T}, S'\tilde\lambda^* - S'\tilde\lambda_{n,h})_{\partial\mathcal{T}^n_H} \leq ||\tilde\lambda^* - \tilde\mu_h^{\Delta T}||_{L^2(I_n;M)}||S'(\tilde\lambda^* - \tilde\lambda_{n,h})||_{L^2(I_n;X)}. \quad (5.29)$$

Now we combine (5.28) with (5.29) and manipulate it to get

$$\begin{aligned}\gamma \quad \Big(\quad & ||S'(\tilde\lambda^* - \tilde\lambda_{n,h})||^2_{L^2(I_n;X)} + ||S'(\tilde\lambda^* - \tilde\lambda_{n,h})(t_{n+1}))||^2_{L^2(\Omega)}\Big) \leq \\ \leq \quad & ||\tilde\lambda^* - \tilde\mu_h^{\Delta T}||_{L^2(I_n;M)}||S'(\tilde\lambda^* - \tilde\lambda_{n,h})||_{L^2(I_n;X)} \\ \leq \quad & ||\tilde\lambda^* - \tilde\mu_h^{\Delta T}||_{L^2(I_n;M)}\left(||S'(\tilde\lambda^* - \tilde\lambda_{n,h})||^2_{L^2(I_n;X)} + ||S'(\tilde\lambda^* - \tilde\lambda_{n,h})(t_{n+1}))||^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\end{aligned}$$

which becomes

$$||S'(\tilde\lambda^* - \tilde\lambda_{n,h})||^2_{L^2(I_n;X)} + ||S'(\tilde\lambda^* - \tilde\lambda_{n,h})(t_{n+1}))||^2_{L^2(\Omega)} \leq \gamma^{-2}||\tilde\lambda^* - \tilde\mu_h^{\Delta T}||^2_{L^2(I_n;M)}. \quad (5.30)$$

Next, we estimate the first term on the right-hand side of (5.25). For $n = 0, \ldots, N-1$, the term $\tilde u^n - \tilde u^*$ is the solution of the problem

$$\begin{cases} \partial_t(\tilde u^n - \tilde u^*) - \nabla\cdot(A\nabla(\tilde u^n - \tilde u^*)) + \beta(\tilde u^n - \tilde u^*) = 0 \ \text{ in } \ \Omega_n = \Omega\times I_n, \\ \tilde u^n - \tilde u^* = 0 \ \text{ on } \ \partial\Omega_n = \partial\Omega\times I_n, \\ \tilde u^n - \tilde u^* = \tilde\tau_n - \tilde\tau_{n,h} \ \text{ at } \ \Omega\times\{t = t_n\}. \end{cases} \quad (5.31)$$

The weak formulation of (5.31) writes

$$\begin{aligned}\langle\partial_t(\tilde u^n - \tilde u^*), v\rangle + (A\nabla(\tilde u^n - \tilde u^*), \nabla v) \\ +\beta((\tilde u^n - \tilde u^*), v) = 0\end{aligned} \quad (5.32)$$

for all $v \in L^2(I_n;X)$ where $\langle .,. \rangle = \int_{I_n} \langle .,. \rangle_{X',X} dt$ and $(.,.) = \int_{I_n} (.,.)_{L^2}$. Replacing $v = (\tilde{u}^n - \tilde{u}^*)$ in (5.32) and bounding it from below

$$\begin{aligned} 0 &= \langle \partial_t(\tilde{u}^n - \tilde{u}^*), (\tilde{u}^n - \tilde{u}^*) \rangle + (A\nabla(\tilde{u}^n - \tilde{u}^*), \nabla(\tilde{u}^n - \tilde{u}^*)) \\ &+ \beta((\tilde{u}^n - \tilde{u}^*), (\tilde{u}^n - \tilde{u}^*)) \\ &\geq \frac{1}{2}\left(||(\tilde{u}^n - \tilde{u}^*)(t_{n+1})||^2_{L^2(\Omega)} - ||(\tilde{u}^n - \tilde{u}^*)(t_n)||^2_{L^2(\Omega)} \right) \\ &+ c_{min}||\nabla(\tilde{u}^n - \tilde{u}^*)||^2_{L^2(I_n,X)} + \beta||(\tilde{u}^n - \tilde{u}^*)||^2_{L^2(I_n;X)} \\ &\geq \gamma \left(||(\tilde{u}^n - \tilde{u}^*)(t_{n+1})||^2_{L^2(\Omega)} - ||(\tilde{u}^n - \tilde{u}^*)(t_n)||^2_{L^2(\Omega)} + ||\tilde{u}^n - \tilde{u}^*)||^2_{L^2(I_n;X)} \right) \end{aligned}$$

we obtain

$$||\tilde{u}^n - \tilde{u}^*||^2_{L^2(I_n;X)} + ||(\tilde{u}^n - \tilde{u}^*)(t_{n+1})||^2_{L^2(\Omega)} \leq ||(\tilde{u}^n - \tilde{u}^*)(t_n)||^2_{L^2(\Omega)}. \tag{5.33}$$

Now, we observe that

$$\begin{aligned} &||\tilde{u}^n - \tilde{u}^n_h||^2_{L^2(I_n;X)} + ||(\tilde{u}^n - \tilde{u}^n_h)(t_{n+1})||^2_{L^2(\Omega)} \\ &\quad \leq ||\tilde{u}^n - \tilde{u}^*||^2_{L^2(I_n;X)} + ||(\tilde{u}^n - \tilde{u}^*)(t_{n+1})||^2_{L^2(\Omega)} \\ &\qquad + ||\tilde{u}^* - \tilde{u}^n_h||^2_{L^2(I_n;X)} + ||(\tilde{u}^* - \tilde{u}^n_h)(t_{n+1})||^2_{L^2(I_n;X)} \qquad \text{adding and subtracting } \tilde{u}^* \\ &\quad \leq ||\tilde{u}^n - \tilde{u}^*||^2_{L^2(I_n;X)} + ||(\tilde{u}^n - \tilde{u}^*)(t_{n+1})||^2_{L^2(\Omega)} \\ &\qquad + \ ||S'\tilde{\lambda}^* - S'\tilde{\lambda}_{n,h}||^2_{L^2(I_n;X)} + ||(S'\tilde{\lambda}^* - S'\tilde{\lambda}_{n,h})(t_{n+1})||^2_{L^2(\Omega)} \qquad \text{using (5.26)} \\ &\quad \leq ||(\tilde{u}^n - \tilde{u}^*)(t_n)||^2_{L^2(\Omega)} + \gamma^{-2}||\tilde{\lambda}^* - \tilde{\mu}^{\Delta T}_h||^2_{L^2(I_n,M)}, \qquad \text{using (5.30) and (5.33)} \end{aligned}$$

and, therefore, we have

$$||\tilde{u}^n - \tilde{u}^n_h||^2_{L^2(I_n;X)} + ||(\tilde{u}^n - \tilde{u}^n_h)(t_{n+1})||^2_{L^2(\Omega)} \leq ||(\tilde{u}^n - \tilde{u}^*)(t_n)||^2_{L^2(\Omega)} + \gamma^{-2}||\tilde{\lambda}^* - \tilde{\mu}^{\Delta T}_h||^2_{L^2(I_n,M)}. \tag{5.34}$$

We now use the L^2 projection operator $\mathcal{P} : L^2(\Omega) \to W_{\bar{h}}$ to obtain, in one hand,

$$\begin{aligned} ||(\tilde{u}^n - \tilde{u}^n_h)(t_{n+1})||^2_{L^2(\Omega)} &= ||(\tilde{u}^n - \tilde{u}^n_h)(t_{n+1}) - \mathcal{P}((\tilde{u}^n - \tilde{u}^n_h)(t_{n+1})) \\ &\quad + \mathcal{P}((\tilde{u}^n - \tilde{u}^n_h)(t_{n+1}))||^2_{L^2(\Omega)} \\ &= ||(\tilde{u}^n - \tilde{u}^n_h)(t_{n+1}) - \mathcal{P}((\tilde{u}^n - \tilde{u}^n_h)(t_{n+1}))||^2_{L^2(\Omega)} \\ &\quad + ||\mathcal{P}((\tilde{u}^n - \tilde{u}^n_h)(t_{n+1}))||^2_{L^2(\Omega)} \\ &= ||\tilde{\xi}_{n+1}||^2_{L^2(\Omega)} + ||\tilde{\tau}_{n+1,\bar{h}} - \tilde{\tau}_{n+1,h}||^2_{L^2(\Omega)}, \end{aligned} \tag{5.35}$$

where $\tilde{\xi}_{n+1} = (\tilde{u}^n - \tilde{u}^n_h)(t_{n+1}) - \mathcal{P}((\tilde{u}^n - \tilde{u}^n_h)(t_{n+1}))$, $\tilde{\tau}_{n+1,\bar{h}} = \mathcal{P}(\tilde{u}^n(t_{n+1})) = \mathcal{P}(\tilde{u}(t_{n+1}))$ and $\tilde{\tau}_{n+1,h} = \mathcal{P}(\tilde{u}^n_h(t_{n+1}))$. On the other hand, from the initial condition of problem (5.31) and using again the projection $\mathcal{P}$, we have

$$\begin{aligned} ||(\tilde{u}^n - \tilde{u}^*)(t_n)||^2_{L^2(\Omega)} &= ||\tilde{\tau}_n - \tilde{\tau}_{n,h}||^2_{L^2(\Omega)} \\ &= ||\tilde{\tau}_n - \tilde{\tau}_{n,h} - \mathcal{P}(\tilde{\tau}_n - \tilde{\tau}_{n,h})||^2_{L^2(\Omega)} + ||\mathcal{P}(\tilde{\tau}_n - \tilde{\tau}_{n,h})||^2_{L^2(\Omega)} \\ &= ||\tilde{\tau}_n - \tilde{\tau}_{n,\bar{h}}||^2_{L^2(\Omega)} + ||\tilde{\tau}_{n,\bar{h}} - \tilde{\tau}_{n,h}||^2_{L^2(\Omega)}, \end{aligned} \tag{5.36}$$

where we use the fact that $\mathcal{P}\tilde{\tau}_{n,h} = \tilde{\tau}_{n,h}$ in the second equality of (5.36).

From (5.35) we observe that $||\tilde{\tau}_{n+1,\bar{h}} - \tilde{\tau}_{n+1,h}||^2_{L^2(\Omega)} \leq ||(\tilde{u}^n - \tilde{u}^n_h)(t_{n+1})||^2_{L^2(\Omega)}$ and, combining this fact with (5.36) and (5.34), we have prove estimate (5.19).

Now, summing (5.19) over all the time intervals we have

$$\begin{aligned} ||\tilde{u} - \tilde{u}_h||^2_{L^2(0,T;X)} &+ \sum_{n=0}^{N-1} ||\tilde{\tau}_{n+1,\bar{h}} - \tilde{\tau}_{n+1,h}||^2_{L^2(\Omega)} \\ &\leq \sum_{n=0}^{N-1} ||\tilde{\tau}_n - \tilde{\tau}_{n,\bar{h}}||^2_{L^2(\Omega)} + \sum_{n=0}^{N-1} ||\tilde{\tau}_{n,\bar{h}} - \tilde{\tau}_{n,h}||^2_{L^2(\Omega)} \\ &\quad + \gamma^{-2} \sum_{n=0}^{N-1} ||\tilde{\lambda}^* - \tilde{\mu}_h^{\Delta T}||^2_{L^2(I_n;M)}. \end{aligned} \tag{5.37}$$

After observing that

$$\begin{aligned} \sum_{n=0}^{N-1} (||\tilde{\tau}_{n+1,\bar{h}} - \tilde{\tau}_{n+1,h}||^2_{L^2(\Omega)} - ||\tilde{\tau}_{n,\bar{h}} - \tilde{\tau}_{n,h}||^2_{L^2(\Omega)}) &= ||\tilde{\tau}_{N,\bar{h}} - \tilde{\tau}_{N,h}||^2_{L^2(\Omega)} - ||\tilde{\tau}_{0,\bar{h}} - \tilde{\tau}_{0,h}||^2_{L^2(\Omega)} \\ &= ||\tilde{\tau}_{N,\bar{h}} - \tilde{\tau}_{N,h}||^2_{L^2(\Omega)} \end{aligned} \tag{5.38}$$

since $\tilde{\tau}_{0,\bar{h}} = \tilde{\tau}_{0,h} = \mathcal{P}(u_0)$, we obtain the estimate

$$\begin{aligned} &||\tilde{u} - \tilde{u}_h||^2_{L^2(0,T;X)} + ||\tilde{\tau}_{N,\bar{h}} - \tilde{\tau}_{N,h}||^2_{L^2(\Omega)} \\ &\qquad \leq \sum_{n=0}^{N-1} ||\tilde{\tau}_n - \tilde{\tau}_{n,\bar{h}}||^2_{L^2(\Omega)} + \gamma^{-2} \sum_{n=0}^{N-1} ||\tilde{\lambda}^* - \tilde{\mu}_h^{\Delta T}||^2_{L^2(I_n;M)}, \end{aligned} \tag{5.39}$$

yielding (5.21). □

At this point, the best approximation estimate of Lemma 5.4 is obtained using the space $\tilde{\Lambda}_h^{\Delta T}$. However, we need to recover a similar result for the original problem (1.1) using the space $\Lambda_h^{\Delta T}$. To do that, we define the problem

$$\begin{cases} \partial_t u^* - \nabla \cdot (A \nabla u^*) = f \ \text{ in } \ \Omega_n = \Omega \times I_n, \\ u^* = 0 \ \text{ on } \ \partial\Omega_n = \partial\Omega \times I_n, \\ u^* = \tau_{n,h} \ \text{ at } \ \Omega \times \{t = t_n\}, \end{cases} \tag{5.40}$$

and observe that applying MHM on it we can write $u^* = S\lambda^* + \bar{S}\tau_{n,\bar{h}} + Sf$ on each $K_n \in \mathcal{T}_H^n$. Once again, writing $w = e^{-\beta t} u^*$ we can observe that w is the weak solution of (5.20), $\tilde{\lambda}^* = e^{-\beta t}\lambda^*$, and, therefore, from identities (5.10) we can see that $S'\tilde{\lambda}^* = e^{-\beta t} S\lambda^*$. We then record that a function $\tilde{\mu}_h^{\Delta T} \in \tilde{\Lambda}_h^{\Delta T}$ can be written as

$$\tilde{\mu}_h^{\Delta T} = \sum_{i=1}^{N} \alpha_i \tilde{\psi}_i = e^{-\beta t} \sum_{i=1}^{N} \alpha_i \psi_i = e^{-\beta t} \mu_h^{\Delta T}$$

and, thus, we can estimate

$$\begin{aligned}
||\tilde{\lambda}^* - \tilde{\mu}_h^{\Delta T}||^2_{L^2(I_n;M)} &= ||e^{-\beta t}(\lambda^* - \mu_h^{\Delta T})||^2_{L^2(I_n;M)} \\
&= \int_{I_n} ||e^{-\beta t}(\lambda^* - \mu_h^{\Delta T})||^2_M dt \\
&= \int_{I_n} e^{-2\beta t}||\lambda^* - \mu_h^{\Delta T}||^2_M dt \\
&\leq e^{-2\beta t_n}||\lambda^* - \mu_h^{\Delta T}||^2_{L^2(I_n;M)}.
\end{aligned} \tag{5.41}$$

In the end, we can transfer the error analysis from problem (5.3) using space $\tilde{\Lambda}_h^{\Delta T}$ to problem (1.1) using space $\Lambda_h^{\Delta T}$ with the following result,

Theorem 5.5 (Best Approximation Result)**.** *Let u^n be the solution of (1.1) and u_h^n be its numerical approximation given by (4.7)-(4.10). Recall that $\tau_n = u(t_n)$, $\tau_{n,\bar{h}} = \mathcal{P}_{W_{\bar{h}}} u(t_n)$ and $\tau_{n,h} = \mathcal{P}_{W_{\bar{h}}} u_h^{n-1}(t_n)$. Therefore we have the estimate*

$$\begin{aligned}
&||u - u_h||_{L^2(0,T;X)} + ||\tau_{N,\bar{h}} - \tau_{N,h}||_{L^2(\Omega)} \\
&\qquad \leq C\ e^{\beta T}\left(\sum_{n=0}^{N-1} ||\tau_n - \tau_{n,\bar{h}}||_{L^2(\Omega)} + \gamma^{-1} \inf_{\mu_h^{\Delta T}\in\Lambda_h^{\Delta T}} ||\lambda^* - \mu_h^{\Delta T}||_\Lambda\right),
\end{aligned} \tag{5.42}$$

where $\gamma = \min\{c_{min}, \beta, \frac{1}{2}\}$ comes from Lemma 5.4 and $\beta > 0$ is an arbitrary constant.

Proof. We employ Lemma 5.3 and estimate (5.21), recalling that $t_N = T$, to get

$$\begin{aligned}
||u - u_h||^2_{L^2(0,T;X)} + ||\tau_{N,\bar{h}} - \tau_{N,h}||^2_{L^2(\Omega)} &\leq e^{2\beta T}||\tilde{\tau}_{N,\bar{h}} - \tilde{\tau}_{N,h}||^2_{L^2(\Omega)} + e^{2\beta T}||\tilde{u} - \tilde{u}_h||^2_{L^2(0,T;X)} \\
&\leq e^{2\beta T}\left(\sum_{n=0}^{N-1} ||\tilde{\tau}_n - \tilde{\tau}_{n,\bar{h}}||^2_{L^2(\Omega)} + \gamma^{-2}||\tilde{\lambda}^* - \tilde{\mu}_h^{\Delta T}||^2_\Lambda\right) \\
&\leq e^{2\beta T}\left(\sum_{n=0}^{N-1} ||\tau_n - \tau_{n,\bar{h}}||^2_{L^2(\Omega)} + \gamma^{-2}||\lambda^* - \mu_h^{\Delta T}||^2_\Lambda\right)
\end{aligned} \tag{5.43}$$

and we have the result. □

The following corollary rewrites the estimates of Theorem 5.5 for the case where the space W is discretized and Λ is not. In this perspective, u^* is still the solution of problem (5.40), with initial data given by $\tau_{n,h}$.

Corollary 5.6. *Under hypothesis of Theorem 5.5, if W is discretized into $W_{\bar{h}}$ and $\Lambda_h^{\Delta T} = \Lambda$ we have the estimate*

$$||u - u_h||_{L^2(0,T;X)} + ||\tau_{N,\bar{h}} - \tau_{N,h}||_{L^2(\Omega)} \leq e^{\beta T} \sum_{n=0}^{N-1} ||\tau_n - \tau_{n,\bar{h}}||_{L^2(\Omega)}. \tag{5.44}$$

Proof. In this configuration we observe that the MHM solutions of (1.1) and (5.3) writes $u_h^n = S\lambda_n + \bar{S}\tau_{n,h} + \hat{S}f$ and $\tilde{u}_h^n = S'\tilde{\lambda}_n + \bar{S}'\tilde{\tau}_{n,h} + \hat{S}'\tilde{f}$, respectively, which eliminate the term $\lambda^* - \tilde{\mu}_h^{\Delta T}$ in the estimates of Lemma 5.4. □

Now, in this corollary, we obtain the estimates of Theorem 5.5 for the case where the space Λ is discretized and W is not. In this situation, u^* becomes the solution of problem (5.40) in $(0,T]$ instead, with initial data $\mathcal{P}_{W_{\bar{h}}} u_0$.

Corollary 5.7. *Once again, under hypothesis of Theorem 5.5, if* $W_{\bar{h}} = W$ *and* Λ *is discretized into* $\Lambda_h^{\Delta T}$*, we have the estimate*

$$||u - u_h||_{L^2(0,T;X)} \leq C\ e^{\beta \Delta T} \gamma^{-1} \inf_{\mu_h^{\Delta T} \in \Lambda_h^{\Delta T}} ||\lambda^* - \mu_h^{\Delta T}||_\Lambda. \tag{5.45}$$

In addition, if we project only the initial data u_0 *onto* $W_{\bar{h}}$ *we have the estimate*

$$||u - u_h||_{L^2(0,T;X)} \leq C\ e^{\beta \Delta T} (||u_0 - \mathcal{P}_{W_{\bar{h}}}(u_0)||_{L^2(\Omega)} + \gamma^{-1} \inf_{\mu_h^{\Delta T} \in \Lambda_h^{\Delta T}} ||\lambda^* - \mu_h^{\Delta T}||_\Lambda). \tag{5.46}$$

Proof. In this case the MHM solutions of (1.1) and (5.3) writes $u_h^n = S\lambda_{n,h} + \bar{S}\tau_n + \hat{S}f$ and $\tilde{u}_h^n = S'\tilde{\lambda}_{n,h} + \bar{S}'\tilde{\tau}_n + \hat{S}'\tilde{f}$, respectively, eliminating the terms involving τ in the estimates of Lemma 5.4. □

In the coming subsections, we bound the term $||\lambda^* - \mu_h^{\Delta T}||_\Lambda$ in order to obtain an error estimate based on the discretization parameters h and ΔT and the norms of u^* and its time derivatives.

5.2.1 Error Estimates: The Semidiscrete in Space Case

In this subsection we consider a discretization only in the spatial domain of Λ, in order to obtain a bound for the term $||\lambda^* - \mu_h^{\Delta T}||_\Lambda$ presented in Lemma 5.5 based on the discretization parameter h, displayed in Lemma 5.8.

Recalling from (4.14) the definition of M_h, we define the semidiscrete space

$$\Lambda_h := L^2(0,T;M_h), \tag{5.47}$$

and then define locally on $F_n \subset \mathcal{E}_H^n = \mathcal{E}_H \times I_n$, where $\mathcal{E}_H$ is the set of all faces of the hybridization partition $\mathcal{T}_H$, the projection over the space $L^2(I_n;M_h)$ given by

$$\begin{aligned} \mathcal{P}_{h,F}^n \quad &: \quad L^2(I_n;L^2(F)) \to L^2(I_n;M_h) \\ \int_{I_n \times F} \mathcal{P}_{h,F}^n v \cdot w\ ds\ dt \quad &= \quad \int_{I_n \times F} v \cdot w\ ds\ dt, \quad \forall\ w \in L^2(I_n;M_h), \end{aligned} \tag{5.48}$$

and globally by

$$\begin{aligned} &\mathcal{P}_h : L^2(0,T;L^2(\mathcal{E}_H)) \to \Lambda_h \\ &\mathcal{P}_h|_{I_n \times F} = \mathcal{P}_{h,F}^n. \end{aligned} \tag{5.49}$$

It is important to observe that, for a fixed $t \in (0,T)$, we have that $\mathcal{P}^n_{h,F}$ coincides with the L^2 projection operator

$$\begin{aligned} \mathcal{P}_{h,F} &: L^2(F) \to M_h \\ \int_F \mathcal{P}_{h,F} v \cdot w \, ds &= \int_F v \cdot w \, ds, \quad \forall\, w \in M_h. \end{aligned} \tag{5.50}$$

Then, recalling that the matrix A is time independent, we have the following estimate:

Lemma 5.8. *Let u^* be the solution of (5.40) and assume that $u^* \in L^2(0,T;H^{l+1}(\mathcal{T}_h) \cap H^1_0(\Omega))$, $A\nabla u^* \in L^2(0,T;H^l(\mathcal{T}_h))$ and $A\nabla u^* \in L^2(0,T;H(div,\Omega))$. Then, for $\mu_h = \mathcal{P}_h \lambda^*$, where $\lambda^* = A\nabla u^* \cdot \mathbf{n}$ on $\partial K \in \partial \mathcal{T}_H$, there exists a constant C independent of h, H and A, such that the following estimate holds*

$$||\lambda^* - \mu_h||_\Lambda \leq C\ h^l\ ||A\nabla u^*||_{L^2(0,T;H^l(\mathcal{T}_H))}, \tag{5.51}$$

for all $1 \leq l \leq m+1$, m being the degree of the polinomials in Λ_h. Here, $||v||_{H^l(\mathcal{T}_H)} = \left(\sum_{K \in \mathcal{T}_H} ||v||^2_{H^l(K)}\right)^{\frac{1}{2}}$.

Remark 5.9. The regularity conditions on u^* are satisfied if we assume the coefficients of A to be smooth and time independent. In particular, if the coefficients of A are $C^{2l+1}(\overline{\Omega})$, $\partial\Omega \in C^{2l}$, $u_0 \in H^{2l+1}(\Omega)$ and $\frac{d^k f}{dt^k} \in L^2(0,T;H^{2l-2k}(\Omega))$ with $k = 0,\ldots,l$, satisfying the compatibility conditions

$$\begin{cases} g_0 := u_0 \in H^1_0(\Omega), g_1 := f(0) - Lg_0 \in H^1_0(\Omega), \\ \ldots, g_l := \frac{d^{l-1} f}{dt^{l-1}}(0) - Lg_{l-1} \in H^1_0(\Omega), \end{cases} \tag{5.52}$$

where $Lw := \partial_t w - \nabla \cdot (A\nabla w)$, we have from Theorem 6 of section 7.1.3 of (EVANS, 2010) that $u^* \in L^2(0,T;H^{2l+2-2k}(\Omega))$, $k = 0,\ldots,l+1$. Furthermore, we have the estimate

$$||A\nabla u^*||_{L^2(0,T;H^{2l+1-2k}(\Omega))} \leq C\ ||u^*||_{L^2(0,T;H^{2l+2-2k}(\Omega))} \tag{5.53}$$

where C depends on the coefficients of A.

Proof. In the proof of Lemma 3 of (BARRENECHEA et al., 2020) we have the inequality

$$(\lambda^* - \mu_h, v)_{\partial K} \leq C\ h^l |A\nabla u^*|_{H^l(K)} ||v||_{H^1(K)},$$

where $\mu_h = \mathcal{P}_{h,F}(A\nabla u^* \cdot \mathbf{n}^K) = \mathcal{P}_{h,F}\lambda^*$, with $\mathcal{P}_{h,F}$ being defined by (5.50). From this inequality we obtain

$$||\lambda^* - \mu_h||_M \leq C\ h^l ||A\nabla u^*||_{H^l(\mathcal{T}_H)}. \tag{5.54}$$

Integrating over I_n we have

$$\begin{aligned} ||\lambda^* - \mu_h||^2_{L^2(I_n;M)} &= \int_{I_n} ||\lambda^* - \mu_h||^2_M dt \\ &\leq \int_{I_n} (C\ h^l\ ||A\nabla u^*||_{H^l(\mathcal{T}_H)})^2 dt \\ &= (C\ h^l)^2 ||A\nabla u^*||^2_{L^2(I_n;H^l(\mathcal{T}_H))}. \end{aligned}$$

The result then follows after summing the previous inequality over all the time intervals I_n. □

5.2.2 Error Estimates: The Semidiscrete in Time Case

This subsection is dedicated to establish a bound for $||\lambda^* - \mu_h^{\Delta T}||_\Lambda$ presented in Lemma 5.5 for a discretization only in the time domain of Λ, proved in Lemma 5.12, based on the parameter ΔT. We reinforce here that the matrix A is time independent and its coefficients are smooth in $\overline{\Omega}$.

We first define the discrete in time space

$$\Lambda_0^{\Delta T} = \{\xi(x,t) \in L^2(0,T;M)|\ \xi(x,t)|_{I_n} = w_n(t) \cdot v_n(x) \text{ for each } I_n \in \mathcal{T}^{\Delta T} \text{ where } w_n \in \mathbb{P}_0(I_n), \text{ and } v_n \in M\}, \tag{5.55}$$

where $\mathcal{T}^{\Delta T}$ is the interval partition of $(0,T)$, with $\Delta T = (t_{n+1} - t_n)$. The space $\Lambda_0^{\Delta t}$ is the one where we prove the estimates presented in the sequence. First, we need to make the following assumption on the regularity of u^* and f:

Assumption A2. *Let u^* be defined by (5.40) and assume that the time derivatives u_{tt}^* and f_t belong to the space $L^2(0,T;H^{-1}(\Omega))$, u_t^* is in $L^2(0,T;H^1(\Omega))$ and $A\nabla u_t^* \in L^2(0,T;H(div,\Omega))$.*

Then, we have u_t^* being the weak solution of

$$\begin{cases} u_{tt}^* - \nabla \cdot (A\nabla u_t^*) = f_t \quad \text{in} \quad \Omega_n, \\ \\ u_t^* = f(t_n) - A\nabla \tau_{n,h} \quad \text{at} \quad \Omega \times \{t = t_n\}. \end{cases} \tag{5.56}$$

From Theorems 3.8 and 3.10 we observe that

$$-\lambda^* = A\nabla u^* \cdot \mathbf{n} \text{ a.e. in } I_n, \tag{5.57}$$

and, therefore, differentiating it with respect to time we obtain

$$-\lambda_t^* = A\nabla u_t^* \cdot \mathbf{n} \text{ a.e. in } I_n. \tag{5.58}$$

The next Lemma provides an estimate for the Λ norm of λ_t^* in terms of the norms of u_t^*, u_{tt}^* and f_t,

Lemma 5.10. *Let u^* be a solution of problem (5.40) and define λ_t^* as in (5.58). If assumption A2 holds, λ_t^* is then bounded by*

$$||\lambda_t^*||_{L^2(0,T;M)} \leq C(||f_t||_{L^2(0,T;H^{-1}(\Omega))} + ||u_{tt}^*||_{L^2(0,T;H^{-1}(\Omega))} + ||u_t^*||_{L^2(0,T;H^1(\Omega))}). \tag{5.59}$$

Proof. Multiplying the first equation of (5.56) by a test function $v \in L^2(0,T;X)$ and integrating by parts we get

$$-\langle \lambda_t^*, v\rangle_{\partial K} = \int_K (f_t - u_{tt}^*)v - A\nabla u_t^* \nabla v dx$$

which becomes, after summing over all $K \in \mathcal{T}_H$, recalling the definition of $(.,.)_{\partial \mathcal{T}_H}$ in (3.39),

$$-(\lambda_t^*, v)_{\partial \mathcal{T}_H} = \int_{\mathcal{T}_H} (f_t - u_{tt}^*)v - A\nabla u_t^* \nabla v \; dx. \tag{5.60}$$

The term $\int_\Omega (f_t - u_{tt}^*)v \; dx$ is then estimated by

$$\begin{aligned} \int_\Omega (f_t - u_{tt}^*)v \; dx &\leq ||f_t - u_{tt}^*||_{H^{-1}(\mathcal{T}_h)} \; ||v||_{H^1(\mathcal{T}_H)} \\ &\leq (||f_t||_{H^{-1}(\Omega)} + ||u_{tt}^*||_{H^{-1}(\Omega)}) \; ||v||_{H^1(\mathcal{T}_H)} \end{aligned} \tag{5.61}$$

since $H^{-1}(\mathcal{T}_H) \subset H^{-1}(\Omega)$ due to the fact that $H^1(\Omega) \subset H^1(\mathcal{T}_H)$. The other term $\int_{\mathcal{T}_H} A\nabla u_t^* \nabla v \; dx$ is estimated by

$$\int_{\mathcal{T}_H} A\nabla u_t^* \nabla v \; dx \leq C||u_t^*||_{H^1(\Omega)} \; ||v||_{H^1(\mathcal{T}_H)} \tag{5.62}$$

where C is related to the boundedness of matrix A. If we combine (5.60), (5.61) and (5.62) we end up with the following estimate for $t \in [0,T]$ almost everywhere

$$-(\lambda_t^*(t), v)_{\partial \mathcal{T}_H} \leq C(||f_t(t)||_{H^{-1}(\Omega)} + ||u_{tt}^*(t)||_{H^{-1}(\Omega)} + ||u_t^*(t)||_{H^1(\Omega)})||v||_{H^1(\mathcal{T}_H)}. \tag{5.63}$$

From the previous inequality we conclude that

$$||\lambda_t^*(t)||_M = \sup_{v \in H^1(\mathcal{T}_H)} \frac{-(\lambda_t^*(t), v)_{\partial \mathcal{T}_H}}{||v||_{H^1(\mathcal{T}_H)}} \leq C(||f_t(t)||_{H^{-1}(\Omega)} + ||u_{tt}^*(t)||_{H^{-1}(\Omega)} + ||u_t^*(t)||_{H^1(\Omega)}). \tag{5.64}$$

Squaring both sides of (5.64) and integrating over (I_n) we have

$$||\lambda_t^*||^2_{L^2(I_n;M)} \leq C \; (||f_t||^2_{L^2(I_n;H^{-1}(\Omega))} + ||u_{tt}^*||^2_{L^2(I_n;H^{-1}(\Omega))} + ||u_t^*||^2_{L^2(I_n;H^1(\Omega))}), \tag{5.65}$$

from which the result follows after summing over all time intervals I_n. □

The Lemma just proved is combined with the next result in order to bound the term $||\lambda^* - \mu_h^{\Delta T}||_{L^2(I_n;M)}$ in terms of ΔT and the norms of f, u^* and its time derivates.

The following Lemma provides an error estimate for a piece-wise constant in time interpolation operator $\Gamma : [0,T] \to \mathcal{M}$, where $\mathcal{M}$ is a Banach space.

Lemma 5.11. *Let $\mathcal{M}$ be a Banach space and let φ be a function in $H^1(0,T;\mathcal{M})$. Define $\Gamma\varphi : [0,T] \to \mathcal{M}$ such that, restricted to a time interval $I_n \in \mathcal{T}^{\Delta T}$, we have*

$$\Gamma\varphi(t) := \varphi(t_n) \quad \text{if} \;\; t \in I_n = [t_n, t_{n+1}]. \tag{5.66}$$

Therefore, the following local estimate holds

$$||\varphi - \Gamma\varphi||_{L^2(I_n;\mathcal{M})} \leq \Delta T \; ||\varphi||_{H^1(I_n;\mathcal{M})}. \tag{5.67}$$

Also, we have the global one

$$||\varphi - \Gamma\varphi||_{L^2(0,T;\mathcal{M})} \leq \Delta T \; ||\varphi||_{H^1(0,T;\mathcal{M})}. \tag{5.68}$$

Proof. First we observe that

$$\begin{aligned} ||\varphi - \Gamma\varphi||^2_{L^2(0,T;\mathcal{M})} &= \int_0^T ||\varphi - \Gamma\varphi||^2_{\mathcal{M}} dt \\ &\leq \sum_{n=0}^{N-1} \int_{I_n} ||\varphi - \varphi(t_n)||^2_{\mathcal{M}} dt. \end{aligned} \tag{5.69}$$

From Theorem 2 of section 5.9.2 of (EVANS, 2010) we can write $\varphi(t)$ on each I_n as

$$\varphi(t) = \varphi(t_n) + \int_{t_n}^t \varphi'(s)ds, \quad t_n \leq t \leq t_{n+1}. \tag{5.70}$$

Writing $\varphi(t) - \varphi(t_n) = \int_{t_n}^t \varphi'(s)ds$ and applying Jensen's inequality together with Holder's inequality in time we have

$$\begin{aligned} \int_{I_n} ||\varphi - \varphi(t_n)||^2_{\mathcal{M}} dt &= \int_{I_n} || \int_{t_n}^t \varphi'(s)ds||^2_{\mathcal{M}} dt \\ &\leq \int_{I_n} \left(\int_{I_n} ||\varphi'(s)||_{\mathcal{M}} ds \right)^2 dt \\ &\leq \int_{I_n} \left(\Delta T \; \int_{I_n} ||\varphi'(s)||^2_{\mathcal{M}} ds \right) dt \\ &\leq \Delta T \int_{I_n} ||\varphi||^2_{H^1(I_n;\mathcal{M})} dt \\ &\leq (\Delta T)^2 \; ||\varphi||^2_{H^1(I_n;\mathcal{M})}. \end{aligned} \tag{5.71}$$

If we take the square root on both sides of (5.71) we obtain estimate (5.67). Now, summing (5.71) from 0 to $N-1$ we get

$$\begin{aligned} \sum_{n=0}^{N-1} \int_{I_n} ||\varphi - \varphi(t_n)||^2_{\mathcal{M}} dt &\leq \sum_{n=0}^{N-1} (\Delta T)^2 \; ||\varphi||^2_{H^1(I_n;\mathcal{M})} \\ &= (\Delta T)^2 \sum_{n=0}^{N-1} ||\varphi||^2_{H^1(I_n;\mathcal{M})} \\ &= (\Delta T)^2 \; ||\varphi||^2_{H^1(0,T;\mathcal{M})}. \end{aligned} \tag{5.72}$$

Combining (5.69) and (5.72) and taking the square root of both sides we obtain estimate (5.68). □

From Lemmas 5.10 and 5.11 we obtain the following result

Lemma 5.12. *Let u^* be a solution of problem (5.40) and define λ^* as in (5.57). Let the space $\Lambda_0^{\Delta T}$ be defined by (5.55) and let $\Gamma : [0,T] \to M$ be defined as in (5.66). If assumption A2 holds, we have the following local and global estimates*

$$||\lambda^* - \mu^{\Delta T}||_{L^2(I_n;M)} \leq C\ \Delta T(||u^*||_{H^1(I_n;H^1(\Omega))} + ||u^*_{tt}||_{L^2(I_n;H^{-1}(\Omega))} + ||f||_{H^1(I_n;H^{-1}(\Omega))}), \tag{5.73}$$

and

$$||\lambda^* - \mu^{\Delta T}||_{\Lambda} \leq C\ \Delta T(||u^*||_{H^1(0,T;H^1(\Omega))} + ||u^*_{tt}||_{L^2(0,T;H^{-1}(\Omega))} + ||f||_{H^1(0,T;H^{-1}(\Omega))}). \tag{5.74}$$

Proof. First we define $\mu^{\Delta t} = \Gamma\lambda^*$ and then apply Lemmas 3.9, 5.10 and 5.11 to obtain

$$\begin{aligned} ||\lambda^* - \mu^{\Delta T}||_{L^2(I_n;M)} &\leq C\ \Delta T\ ||\lambda^*||_{H^1(I_n;M)} \\ &\leq C\ \Delta T\ (||\lambda^*||_{L^2(I_n;M)} + ||\lambda^*_t||_{L^2(I_n;M)}) \\ &\leq C\ \Delta T\ \Big(||u^*||_{L^2(I_n;H^1(\Omega))} + ||f||_{L^2(I_n;H^{-1}(\Omega))} \\ &\quad + ||u^*_t||_{L^2(I_n;H^1(\Omega))} + ||u^*_{tt}||_{L^2(I_n;H^{-1}(\Omega))} + ||f_t||_{L^2(I_n;H^{-1}(\Omega))}\Big), \end{aligned} \tag{5.75}$$

and the result follows. □

5.2.3 Error Estimates: The Fully Discrete Case

From the results obtained in subsections 5.2.1 and 5.2.2, we prove the next Theorem which provides an error estimate for a fully discrete approximation of λ^*. To do that, using the notation of (HOUSTON; SCHWAB; SÜLI, 2002) we introduce the space

$$\Lambda_{h,0}^{\Delta T} := \mathbb{P}_0(\mathcal{T}^{\Delta T}) \otimes M_h \tag{5.76}$$

where the symbol $\otimes$ refers to the tensor product of the polynomial spaces M_h, defined in (4.14), and $\mathbb{P}_0(\mathcal{T}^{\Delta T}) := \{\eta \in L^2(0,T)|\ \eta|_{I_n} \in \mathbb{P}_0(I_n),\ \text{for all}\ I_n \in \mathcal{T}^{\Delta T}\}$.

Theorem 5.13. *Assume that the hypothesis of Lemma 5.8 hold and define λ^* as in (5.57). If assumption A2 holds, there exists a constant C independent of h, H, A and ΔT such that*

$$||\lambda^* - \mu_h^{\Delta T}||_{L^2(I;M)} \leq C\Big(h^l||A\nabla u^*||_{H^1(I;H^l(\mathcal{T}_H))} + \Delta T\Big(||u^*||_{H^1(I;H^1(\Omega))} + ||u^*_{tt}||_{L^2(I;H^{-1}(\Omega))} + ||f||_{H^1(I;H^{-1}(\Omega))}\Big)\Big), \tag{5.77}$$

where I can be either the time interval I_n or the whole $(0,T)$, for all $1 \leq l \leq m+1$, m being the degree of the polynomials in M_h.

Proof. The proof is a direct application of Lemmas 5.8 and 5.12, after defining $\mu_h^{\Delta T} = \Pi\lambda^*$ where

$$\begin{aligned} \Pi = \Gamma \circ \mathcal{P}_h : H^1(0,T;M) \to \Lambda_{h,0}^{\Delta T} \\ \Pi\ \mu|_{F_n} = \Gamma(\mathcal{P}_{h,F}\mu) = (\mathcal{P}_{h,F}\mu)(t_n) \end{aligned} \tag{5.78}$$

and observing that

$$\begin{aligned} ||\lambda^* - \Pi\lambda^*||^2_{L^2(I_n;M)} &\leq ||\lambda^* - \Gamma\lambda^*||^2_{L^2(I_n;M)} + ||\Gamma(\lambda^* - \mathcal{P}_h^n\lambda^*)||^2_{L^2(I_n;M)} \\ &\leq ||\lambda^* - \Gamma\lambda^*||^2_{L^2(I_n;M)} + ||(\lambda^* - \mathcal{P}_h^n\lambda^*)(t_n)||^2_{L^2(I_n;M)} \\ &\leq ||\lambda^* - \Gamma\lambda^*||^2_{L^2(I_n;M)} + \Delta T\ ||(\lambda^* - \mathcal{P}_h\lambda^*)(t_n)||^2_M \\ &\leq ||\lambda^* - \Gamma\lambda^*||^2_{L^2(I_n;M)} + \Delta T\ h^l||A\nabla u^*(t_n)||^2_{H^l(\mathcal{T}_H)} \\ &\leq ||\lambda^* - \Gamma\lambda^*||^2_{L^2(I_n;M)} + h^l||\Gamma(A\nabla u^*)||^2_{L^2(I_n;H^l(\mathcal{T}_H))} \\ &\leq ||\lambda^* - \Gamma\lambda^*||^2_{L^2(I_n;M)} + C\ h^l||A\nabla u^*||^2_{H^1(I_n;H^l(\mathcal{T}_H))} \end{aligned} \tag{5.79}$$

□

As a consequence of Theorem 5.13, we obtain the following a priori estimate,

Theorem 5.14. *Let u^n be the solution of problem (1.1) restricted to the time interval I_n and let u_h^n be the numerical approximation of the system (4.7)-(4.9),(4.17). Also, let u^* be defined by (5.40), $\tau_n = u(t_n)$, $\tau_{n,\bar{h}} = \mathcal{P}_{W_{\bar{h}}}u(t_n)$ and $\tau_{n,h} = \mathcal{P}_{W_{\bar{h}}}(u_h^{n-1}(t_n))$ (or the projection of u_0 in I_0). Under the hypothesis of Lemma 5.8 and assumption A2, and for all $1 \leq l \leq m+1$, m being the degree of the polynomials in M_h, there exists a constant C independent of h, H, A and ΔT such that*

$$\begin{aligned} ||u - u_h||_{L^2(0,T;X)} + ||\tau_{N,\bar{h}} - \tau_{N,h}||_{L^2(\Omega)} \leq C\Bigg(\sum_{n=0}^{N-1} ||\tau_n - \tau_{n,\bar{h}}||_{L^2(\Omega)} + h^l||A\nabla u^*||_{H^1(0,T;H^l(\mathcal{T}_H))} \\ + \Delta T\Big(||u^*||_{H^1(0,T;H^1(\Omega))} + ||u^*_{tt}||_{L^2(0,T;H^{-1}(\Omega))} + ||f||_{H^1(0,T;H^{-1}(\Omega))}\Big)\Bigg). \end{aligned} \tag{5.80}$$

If, in addition, we consider an uniform time partition of $(0,T)$ with $\Delta T = t_{n+1} - t_n$ for all $n = 0,\dots,N-1$, we replace $\sum_{n=0}^{N-1} ||\tau_n - \tau_{n,\bar{h}}||_{L^2(\Omega)}$ by

$$\sum_{n=0}^{N-1} ||\tau_n - \tau_{n,\bar{h}}||_{L^2(\Omega)} \leq \frac{\bar{h}^{l+1}}{\Delta T}||u||_{L^\infty(0,T;H^{l+1}(\Omega))}. \tag{5.81}$$

Proof. Since $\tau_n = u(t_n)$ and $\tau_{n,\bar{h}} = \mathcal{P}_{W_{\bar{h}}}u(t_n)$, we can estimate the term $||\tau_n - \tau_{n,\bar{h}}||_{L^2(\Omega)}$ based on the properties of the L^2 projection. From the fact that $u(t_n) \in H^{l+1}(\Omega)$, $1 \leq l \leq m+1$, and $\mathcal{P}_{W_{\bar{h}}}$ is the L^2 projection of $L^2(\Omega)$ onto $W_{\bar{h}} = \mathbb{P}_s(\mathcal{T}_H)$, whenever $1 \leq l \leq s$, we have from the estimate proved in (ERN; GUERMOND, 2004) that

$$||\tau_n - \tau_{n,\bar{h}}||_{L^2(\Omega)} \leq C\ \bar{h}^{l+1}||u(t_n)||_{H^{l+1}(\Omega)}. \tag{5.82}$$

Then, the sum becomes

$$\sum_{n=0}^{N-1} ||\tau_n - \tau_{n,\bar{h}}||_{L^2(\Omega)} \leq C\ \bar{h}^{l+1} \sum_{n=0}^{N-1} ||u(t_n)||_{H^{l+1}(\Omega)} \leq C\ \bar{h}^{l+1}\ N||u||_{L^\infty(0,T;H^{l+1}(\Omega))}, \quad (5.83)$$

where the the estimate follows after writing $N = \dfrac{T}{\Delta T}$. □

Remark 5.15. The estimate just proved involves the rate of $\frac{\bar{h}^{l+1}}{\Delta T}$ related to the term $\tau_n - \tau_{n,\bar{h}}$ at each time interval I_n. This means that the projection of the solution u at time t_n can pollute the convergence if we choose $\Delta T < \bar{h}^{l+1}$. In the next section we show that we need to worry only with the projection of the initial data u_0, and not with $u(t_n)$ at each I_n.

5.3 Further Error Estimates

The estimates proved so far are related to norms of u^*, which is not ideal since u^* depends on the L^2 projection of the MHM solution $u_h^{n-1}(t_n)$ onto the space $W_{\bar{h}}$. In order to overcome this issue, we use the equivalence of the two time schemes discretization studied in section (4.4). Recall first that u^* is the weak solution of (5.40) with initial data $\tau_{n,h}$ on every time interval I_n, and u is the solution of problem (1.1).The idea here is to observe that the MHM is usually seen through **Scheme 1**, visited in section (4.4), but we can also understand it through **Scheme 2**.

When we analyze the MHM under the perspective of **Scheme 2**, the function u^* becomes the solution of problem

$$\begin{cases} \partial_t u^* - \nabla \cdot (A\nabla u^*) = f \quad \text{in} \quad \Omega \times (0,T), \\ \\ u^* = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ \\ u^* = u_{0,h} \quad \text{at} \quad \Omega \times \{t = 0\}, \end{cases} \quad (5.84)$$

and we have the following result:

Theorem 5.16. *Let u be the solution of problem (1.1) and let u_h be the numerical approximation of the coupled system (4.31)-(4.33),(4.42). Also, let u^* be the solution of (5.84) and $u_{0,h} = \mathcal{P}_{W_{\bar{h}}} u_0$. Under the hypothesis of Lemma 5.8 and assumption A2, there exists a constant C independent of h, H, A and ΔT such that*

$$||u - u_h||_{L^2(0,T;X)} + ||u(T) - u_h(T)||_{L^2(\Omega)} \leq C\Big(||u_0 - u_{0,h}||_{L^2(\Omega)} + h^l ||A\nabla u^*||_{L^2(0,T;H^l(\mathcal{T}_H))} \\ + \Delta T\Big(||u^*||_{H^1(0,T;H^1(\Omega))} + ||u^*_{tt}||_{L^2(0,T;H^{-1}(\Omega))} + ||f||_{H^1(0,T;H^{-1}(\Omega))}\Big)\Big). \quad (5.85)$$

Remark 5.17. Observe that now the function u^* depends only on the projection of the initial data u_0 onto $W_{\bar{h}}$, and not on the numerical method anymore.

Remark 5.18. The MHM is a method where the basis functions need to be computed on a second level discretization inside each macroelement $K_n \in \mathcal{T}_H^{\Delta T}$. If $W_{\bar{h}}$ is the same space used to discretize the second level, the error analysis of the fully discrete method related to the first level variables coincide with the case treated in the Theorem 5.16.

Remark 5.19. Finally, to avoid the dependence on the initial data $u_{0,h}$, we consider $u_0 \in H^2(\Omega)$, $f_t \in L^2(0,T;L^2(\Omega))$ and $u_{0,h}$ as the L^2 projection of u_0 onto

$$W_{\bar{h}} \subset H^2(\Omega).$$

From the improved regularity estimate presented in Theorem 5 of section 7.1 of (EVANS, 2010), we have

$$\begin{aligned} \text{ess sup}_{0\leq t\leq T}(||u^*(t)||_{H^2(\Omega)} + ||u_t^*(t)||_{L^2(\Omega)}) + ||u_t^*||_{L^2(0,T;H^1(\Omega))} \\ + ||u_{tt}^*||_{L^2(0,T;H^{-1}(\Omega))} \leq C(||f||_{H^1(0,T;L^2(\Omega))} + ||u_{0,h}||_{H^2(\Omega)}). \end{aligned} \tag{5.86}$$

We name $\Phi = (||f||_{H^1(0,T;L^2(\Omega))} + ||u_{0,h}||_{H^2(\Omega)})$ and then observe in (5.85) that if $l = 1$ we can obtain the following bound for $||A\nabla u^*||_{L^2(0,T;H^1(\mathcal{T}_H))}$, under remark 5.9:

$$||A\nabla u^*||_{L^2(0,T;H^1(\mathcal{T}_H))} \leq ||A\nabla u^*||_{L^2(0,T;H^1(\Omega))} \leq C||u^*||_{H^2(\Omega)} \leq C\Phi. \tag{5.87}$$

Combining this last inequality with the stability estimates for the L^2 projection

$$\begin{aligned} ||u_{0,h}||_{H^2(\Omega)} \leq ||u_0||_{H^2(\Omega)}, \\ ||u_0 - u_{0,h}||_{L^2(\Omega)} \leq \bar{h}^2||u_0||_{H^2(\Omega)}, \end{aligned} \tag{5.88}$$

we end up with the following estimate

$$\begin{aligned} ||u - u_h||_{L^2(0,T;X)} + ||u(T) - u_h(T)||_{L^2(\Omega)} \leq C\Bigg(\bar{h}^2||u_0||_{H^2(\Omega)} + \\ (h^l + \Delta T)(||u_0||_{H^2(\Omega)} + ||f||_{H^1(0,T;L^2(\Omega))})\Bigg). \end{aligned} \tag{5.89}$$

6 Numerical Validation

To obtain the numerical solution we make use of the MHM Library Solver (or just MSL), a library created to implement the MHM method originaly in $C++$ language, and adapted for python as well. The python version is the one used to run the experiments we present in the sequence, and it can be accessed in <https://github.com/lumath93/MSL_Python_MHM.git>.

Another important library used in python to adapt the MSL code is the FEniCS library. FEniCS is an open-source computational platform written in Python for solving partial differential equations. It provides a high-level interface for expressing and solving PDEs using finite element methods. FEniCS allows users to define complex mathematical models and discretize them automatically, enabling efficient and accurate numerical computations.

For the examples presented in the sequel, we do not use parallelism to compute the MHM basis functions. However, this possibility is not difficult to be implemented due to the independence of the local problems.

In order to validate the estimate obtained in (5.89), we consider two different experiments for the following problem:

$$\begin{cases} \partial_t u - \Delta u = f \quad \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u = u_0, \quad \text{at} \quad \Omega \times \{t=0\}. \end{cases} \tag{6.1}$$

The parameters are describe in the sequence and the experiment names are related to the initial data adopted,

Pol: Set $u_0 = 0$ and $f(x,t) = 16xy(1-x)(1-y) + 32t(x(1-x) + y(1-y))$ to obtain as exact solution $u_e(x,t) = 16txy(1-x)(1-y)$;

Trig: Set $u_0 = \sin(\pi x)\sin(\pi y)$ and $f(x,t) = (2\pi - 1)e^{-t}\sin(\pi x)\sin(\pi y)$ to obtain as exact solution $u_e(x,t) = e^{-t}\sin(\pi x)\sin(\pi y)$.

In all the tests we adopt $m = 0$ in (5.89) and $\mathbb{P}_1(\mathcal{T}_h)$ in the spatial discretization of each $K_n \in \mathcal{T}_H^n$. We here employ the notation H and ΔT for first level discretization parameters, while h and Δt refers to the second one. We here consider the space $W_{\bar{h}} = \mathbb{P}_1(\mathcal{T}_h)$, the same one used in the second level. The local basis are then computed at each second level time step through a combination of an Euler Implicit scheme for the time variable with standard finite element method for the spatial one. In the coming sections we analyze the convergence rates of each parameter, H and ΔT, in the norms $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$. Error curves and tables containing the error information are displayed to confirm the expected behavior. At last, we run a test with oscillatory coefficient A in order to observe how the multiscale basis functions capture the fine-scales heterogeineties.

6.1 Spatial Convergence

In this section we check the error curves to observe the convergence rate of the parameter H, which by estimate (5.89) should be of order H in the norm $L^2(0,T;H^1(\Omega))$ and H^2 in the norm $L^2(0,T;L^2(\Omega))$.

The list of parameters defined to run the tests are displayed in the blue boxes in the sequence. In both experiments, the parameters used to compute the numerical solution are the same.

Parameters: ΔT, H, Δt, and h

$T = 0.5,\ \ \Omega = [0,1]^2$
$\Delta T = \frac{T}{100}$
$H = \sqrt{0.5}, \frac{\sqrt{0.5}}{2}, \frac{\sqrt{0.5}}{4}, \frac{\sqrt{0.5}}{8}$
$\Delta t = \{\frac{\Delta T}{10}, \frac{\Delta T}{20}, \frac{\Delta T}{40}, \frac{\Delta T}{80}\}$
$h = \{\frac{H^2}{2}, \frac{H^2}{4}, \frac{H^2}{8}, \frac{H^2}{16}\}$

6.1.1 Experiment Pol

The tables contain the information of the error in the convergence norms and also the convergence rates, while the error curves are displayed right after the tables. We also show some screenshots of the numerical solution at time $T = 0.5$ for each Δt. This test is related to the exact solution $16txy(1-x)(1-y)$.

Table 1 – Spatial Error in the norm $L^2(0,T;L^2/H^1)$

i	H_i	h	Δt	$\|\|u-u_h\|\|_{L^2(L^2)}$	$\|\|u-u_h\|\|_{L^2(H^1)}$
1	$7,071\times10^{-1}$	0.2286	$\frac{\Delta T}{40}$	1.834×10^{-2}	1.881×10^{-1}
2	$3,536\times10^{-1}$	0.08081	$\frac{\Delta T}{15}$	4.327×10^{-3}	9.354×10^{-2}
3	$1,768\times10^{-1}$	0.01542	$\frac{\Delta T}{40}$	1.210×10^{-3}	4.357×10^{-2}
4	$8,839\times10^{-2}$	0.003855	$\frac{\Delta T}{80}$	4.592×10^{-4}	2.572×10^{-2}

The rate is then computed by the difference

$$\frac{\ln e_{H_i} - \ln e_{H_{i+1}}}{\ln H_i - \ln H_{i+1}}. \tag{6.2}$$

Table 2 – Spatial Rates in the norm $L^2(0,T;L^2/H^1)$

i	Rate $L^2(L^2)$	Rate $L^2(H^1)$
1	—	—
2	2,084	1,008
3	1,838	1,102
4	1,397	0,760

We observe that $H_{i+1} = H_i/2$, and the error should decay approximately at the rate of 1, in the norm $L^2(0,T;H^1(\Omega))$, and 2, in the norm $L^2(0,T;L^2(\Omega))$, as expected theoretically in the estimate (5.85). From Table 2 we observe that the expected rates for the norms are not very accurate for small H. This is probably related to a low refinement of the second level discretization used to compute the local space-time basis at each space-time element. The discrepancy is higher when we take a look at the $L^2(L^2)$ norm. This suggests that in order to achieve more accurate rates we need to refine the second level more and more as the first parameter becomes smaller. However, when we compare the error curves with the theoretical lines of slopes H and H^2, to confront the numerical results with the expected behavior, we observe that the tendency for small H is to adjust the lines, as we can observe in the sequence

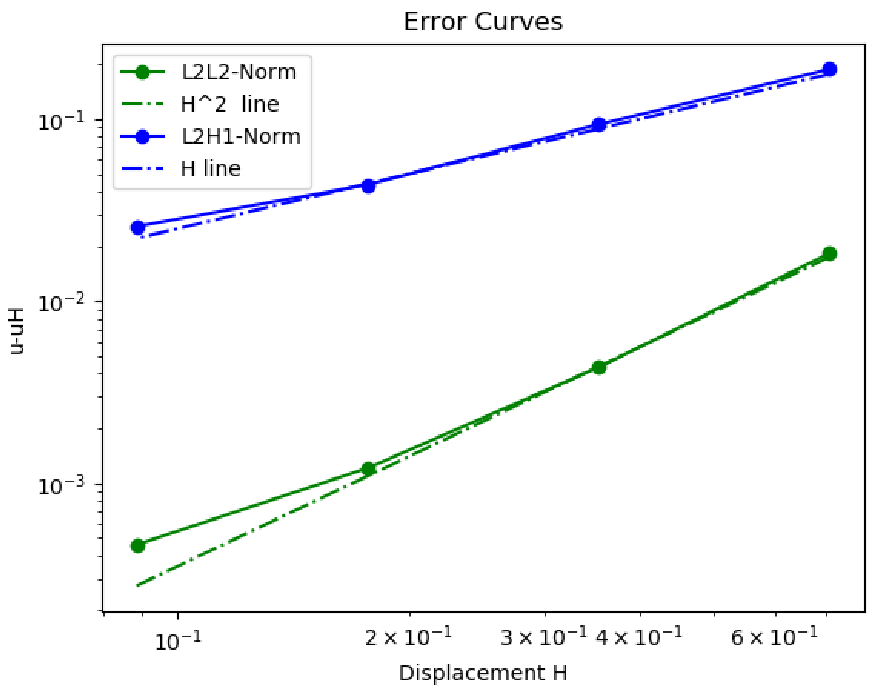
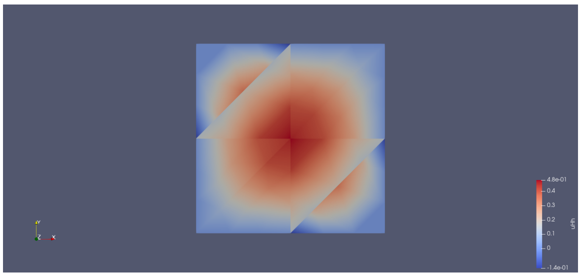
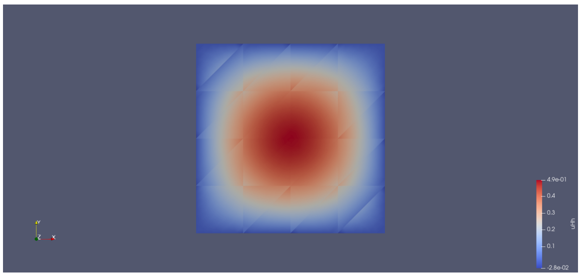


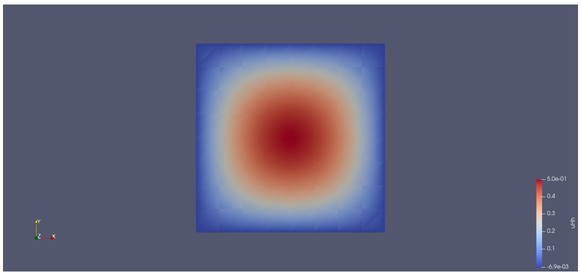
Figure 4 – Error curves in the norms $L^2(H^1/L^2)$.



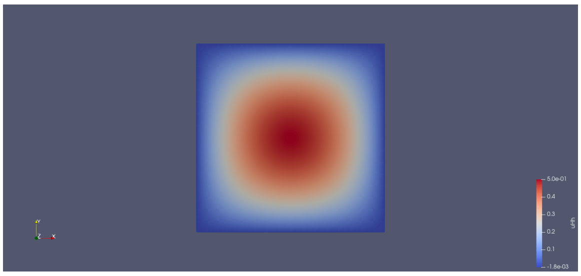
(a) $H = 7,071 \times 10^{-1}$



(b) $H = 3,536 \times 10^{-1}$



(c) $H = 1,768 \times 10^{-1}$



(d) $H = 8,839 \times 10^{-2}$

Figure 5 – Numerical solution u_h at time $T = 0.5$.

6.1.2 Experiment Trig

This is the test for the solution given by $e^{-t}\sin(\pi x)\sin(\pi y)$. The error tables and convergence curves are presented in the sequence.

Table 3 – Spatial Error in the norm $L^2(0,T;L^2/H^1)$

i	H_i	h	Δt	$\|\|u-u_h\|\|_{L^2(L^2)}$	$\|\|u-u_h\|\|_{L^2(H^1)}$
1	$7,071 \times 10^{-1}$	0.2286	$\frac{\Delta T}{10}$	4.543×10^{-2}	4.643×10^{-1}
2	$3,536 \times 10^{-1}$	0.08081	$\frac{\Delta T}{15}$	1.021×10^{-2}	2.350×10^{-1}
3	$1,768 \times 10^{-1}$	0.01542	$\frac{\Delta T}{40}$	2.830×10^{-3}	1.105×10^{-1}
4	$8,839 \times 10^{-2}$	0.002577	$\frac{\Delta T}{100}$	9.455×10^{-4}	6.257×10^{-2}

Table 4 – Spatial Rates in the norm $L^2(0,T;L^2/H^1)$

i	Rate $L^2(L^2)$	Rate $L^2(H^1)$
1	—	—
2	2,154	0,983
3	1,851	1,089
4	1,581	0,820

We also observe here that the decay of the error in the norms $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;H^1(\Omega))$ displayed in Table 4 are around 1 in the norm $L^2(H^1)$ and 2 in the norm $L^2(L^2)$. Once again, the divergence observed for small H may come from the refinement that was not enough to observe more precisely the convergence rates expected theoretically by estimate (5.85). The comparison between the error curves with the lines H and H^2, displayed in the sequence, confirm again the tendency of the error to follow those lines.

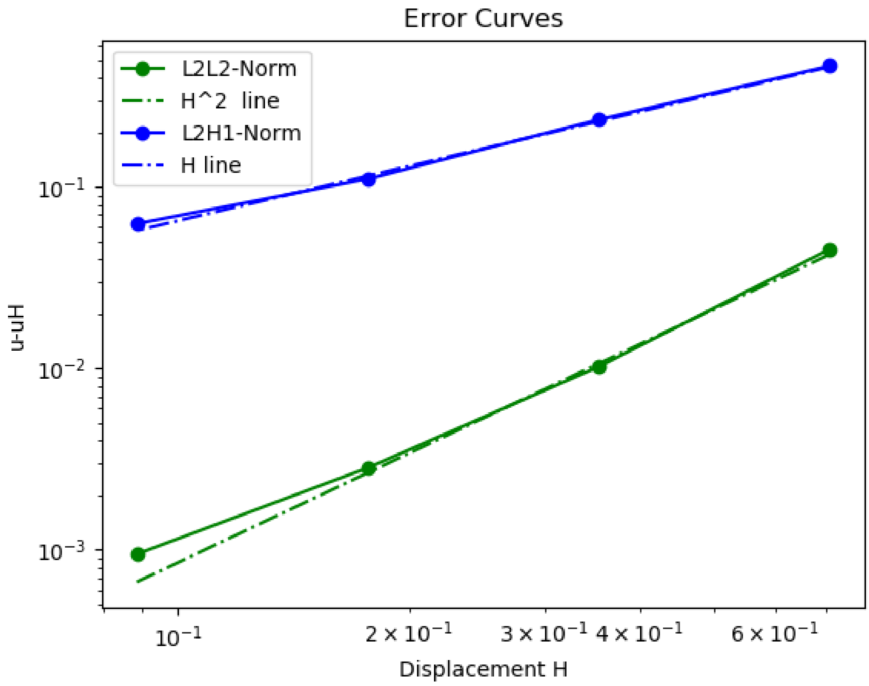
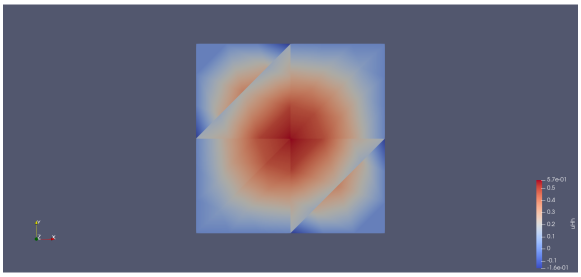
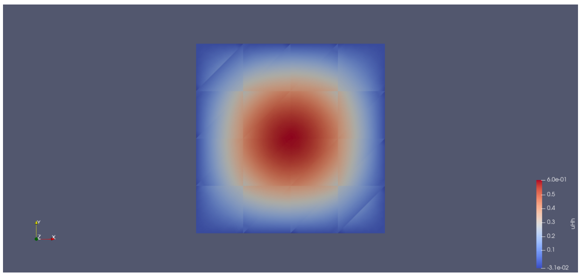


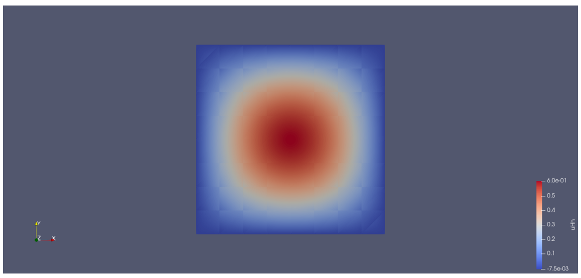
Figure 6 – Error curves in the norms $L^2(H^1/L^2)$.



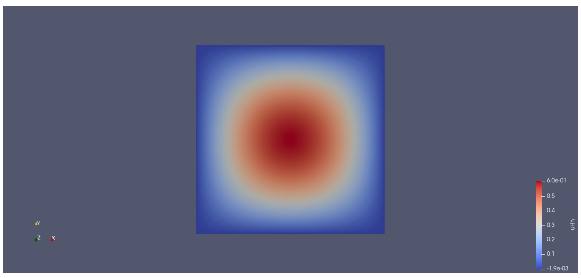
(a) $H = 7,071 \times 10^{-1}$



(b) $H = 3,536 \times 10^{-1}$



(c) $H = 1,768 \times 10^{-1}$



(d) $H = 8,839 \times 10^{-2}$

Figure 7 – Numerical solution u_h at time $T = 0.5$.

6.2 Time Convergence

In this section we check the error curves to observe the convergence rate of the parameter ΔT, which by estimate (5.89) should be of order ΔT in the norm $L^2(0,T;H^1(\Omega))$ and ΔT^2 in the norm $L^2(0,T;L^2(\Omega))$.

Parameters: ΔT, H, Δt, and h

$T = 2,\ \Omega = [0,1]^2$
$\Delta T = \frac{T}{2}, \frac{T}{4}, \frac{T}{8}, \frac{T}{16}$
$H = \frac{1}{16}$
$\Delta t = \frac{\Delta T}{10}$
$h = \frac{H}{8}$

6.2.1 Experiment Trig

Following the sequence adopted for the spatial convergence, the tables display the information of the error in the convergence norms and the comparison between the error curves and the lines of slopes ΔT and ΔT^2 are illustrated in the sequence. We also show some screenshots of the numerical solution at time $T = 2$ right after the error curves. This is the test related to the exact solution $e^{-t}\sin\pi x \sin\pi y$.

Table 5 – Global error for $H = \frac{1}{16}$

i	H	ΔT_i	$\|\|u - u_h\|\|_{L^2(L^2)}$	$\|\|u - u_h\|\|_{L^2(H^1)}$
1	1/16	1.0	$5,562 \times 10^{-1}$	$7,850 \times 10^{-1}$
2	1/16	0.5	$1,265 \times 10^{-1}$	$3,087 \times 10^{-1}$
3	1/16	0.25	$3,187 \times 10^{-2}$	$1,580 \times 10^{-1}$
4	1/16	0.125	$1,830 \times 10^{-2}$	$1,141 \times 10^{-1}$

The error curves in the next figure are compared to the lines with slopes ΔT and ΔT^2 in the sequel.

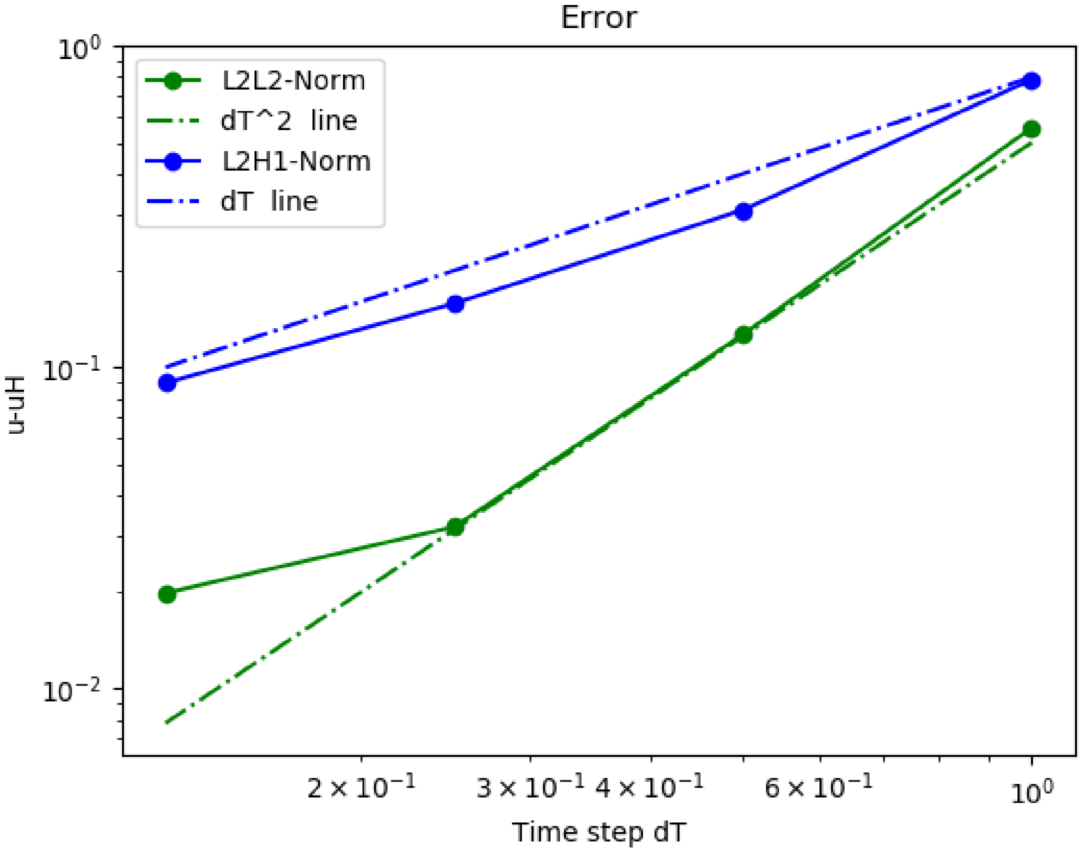


Figure 8 – Error curves in the norms $L^2(H^1/L^2)$ for $H = \frac{1}{16}$.

The error in the norm $L^2(0,T;L^2(\Omega))$ related to $\Delta T_4 = 0,125$ is visually not following the curve ΔT^2. We, therefore, refine the space partition from 1/16 to 1/32 and the second level time parameter Δt from 1/10 to 1/50 in order to achieve the theoretical convergence rate expected. The error obtained was

Table 6 – Global error for $H = \frac{1}{32}$

ΔT_4	$\|\|u - u_h\|\|_{L^2(L^2)}$	$\|\|u - u_h\|\|_{L^2(H^1)}$
0.125	$6,130 \times 10^{-3}$	$7,225 \times 10^{-2}$

Table 7 – Spatial Rates in the norm $L^2(0,T;L^2/H^1)$

i	Rate $L^2(L^2)$	Rate $L^2(H^1)$
1	—	—
2	2,136	1,347
3	1,989	0,967
4	2,378	1,129

The rates displayed in Table 7 follows the convergence rates 1 in the norm $L^2(H^1)$ and 2 in the norm $L^2(L^2)$, expected theoretically by (5.89). The error curves in the sequence is compared with the lines ΔT and ΔT^2 to confirm the behavior expected.

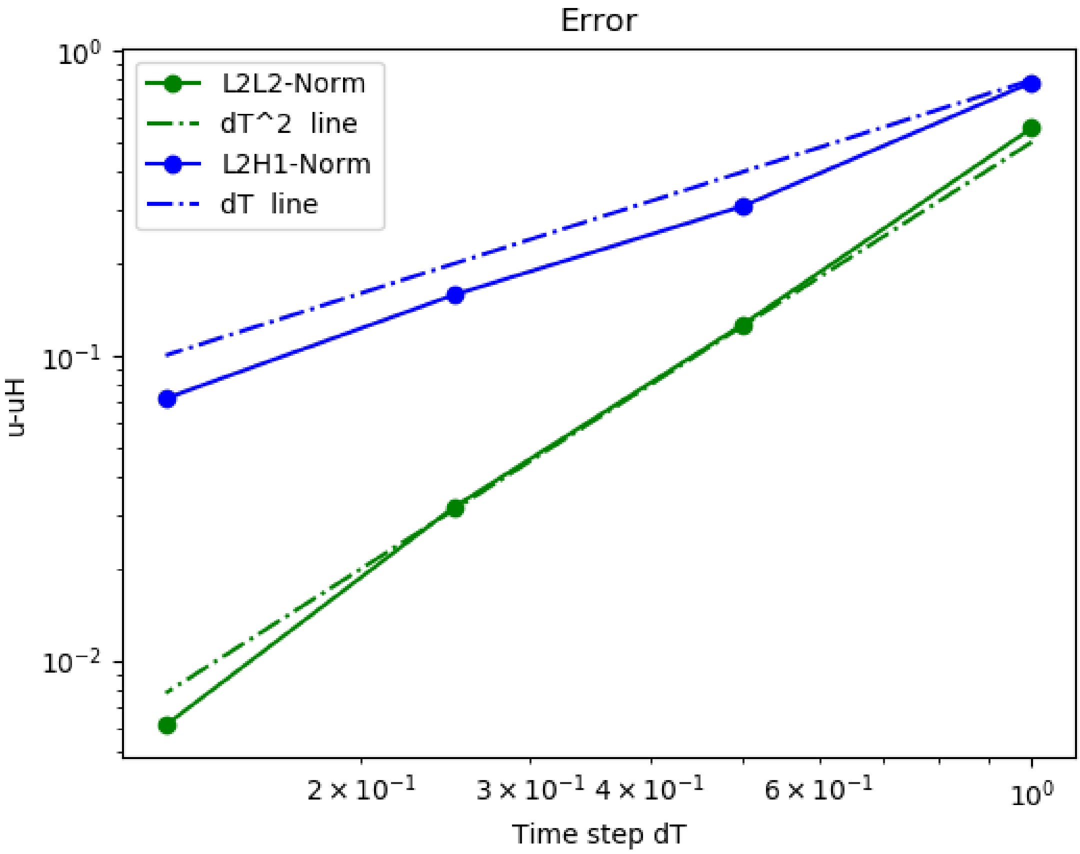
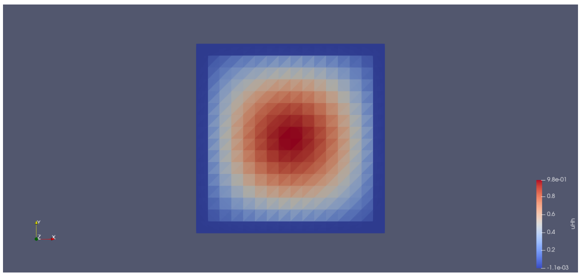
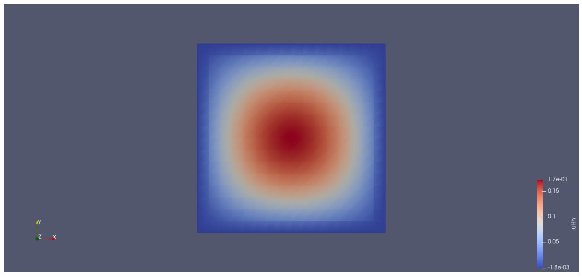


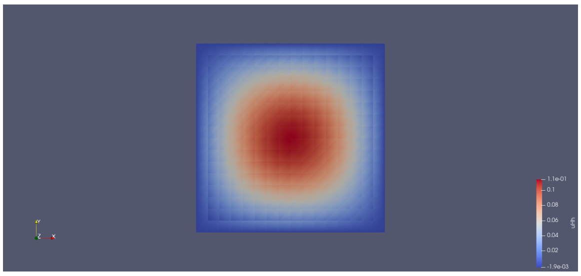
Figure 9 – Error curves in the norms $L^2(H^1/L^2)$ after refining the last point.



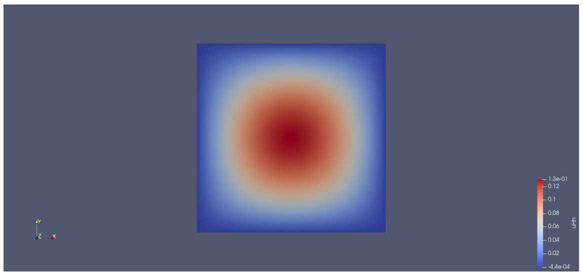
(a) $\Delta T = 1,0$ and $H = 1/16$



(b) $\Delta T = 0,5$ and $H = 1/16$



(c) $\Delta T = 0,25$ and $H = 1/16$



(d) $\Delta T = 0,125$ and $H = 1/32$

Figure 10 – Numerical solution u_h at time $T = 2$.

6.2.2 Experiment Pol

The numerical error in the convergence norms are displayed in the following table, along with the error curves and some screenshots of the numerical solution at time $T = 2$. This is the test for the exact solution $txy(1-x)(1-y)$. To avoid the refinement issue addressed in the previous experiment, we already computed the last point using a space partition of size $H = 1/32$. We display the results in the following table.

Table 8 – Error in the norm $L^2(0,T;L^2/H^1)$

i	H	ΔT_i	Δt_i	$\|\|u-u_h\|\|_{L^2(L^2)}$	$\|\|u-u_h\|\|_{L^2(H^1)}$
1	1/16	1.0	$\Delta T_i/10$	$8,687 \times 10^{-1}$	$1,386 \times 10^{0}$
2	1/16	0.5	$\Delta T_i/10$	$1,899 \times 10^{-1}$	$5,537 \times 10^{-1}$
3	1/16	0.25	$\Delta T_i/10$	$5,095 \times 10^{-2}$	$3,463 \times 10^{-1}$
4	1/32	0.125	$\Delta T_i/10$	$1,194 \times 10^{-2}$	$1,698 \times 10^{-1}$

Table 9 – Spatial Rates in the norm $L^2(0,T;L^2/H^1)$

i	Rate $L^2(L^2)$	Rate $L^2(H^1)$
1	—	—
2	2,194	1,324
3	1,898	0,677
4	2,093	1,028

The rate in the norm $L^2(0,T;H^1(\Omega))$ is not close to 1 between ΔT_2 and ΔT_3. Once again this suggest a refinement in the second level parameters and the parameter H in order to observe the convergence expected. The error curves confronted with the lines ΔT and ΔT^2 are displayed in the sequence.

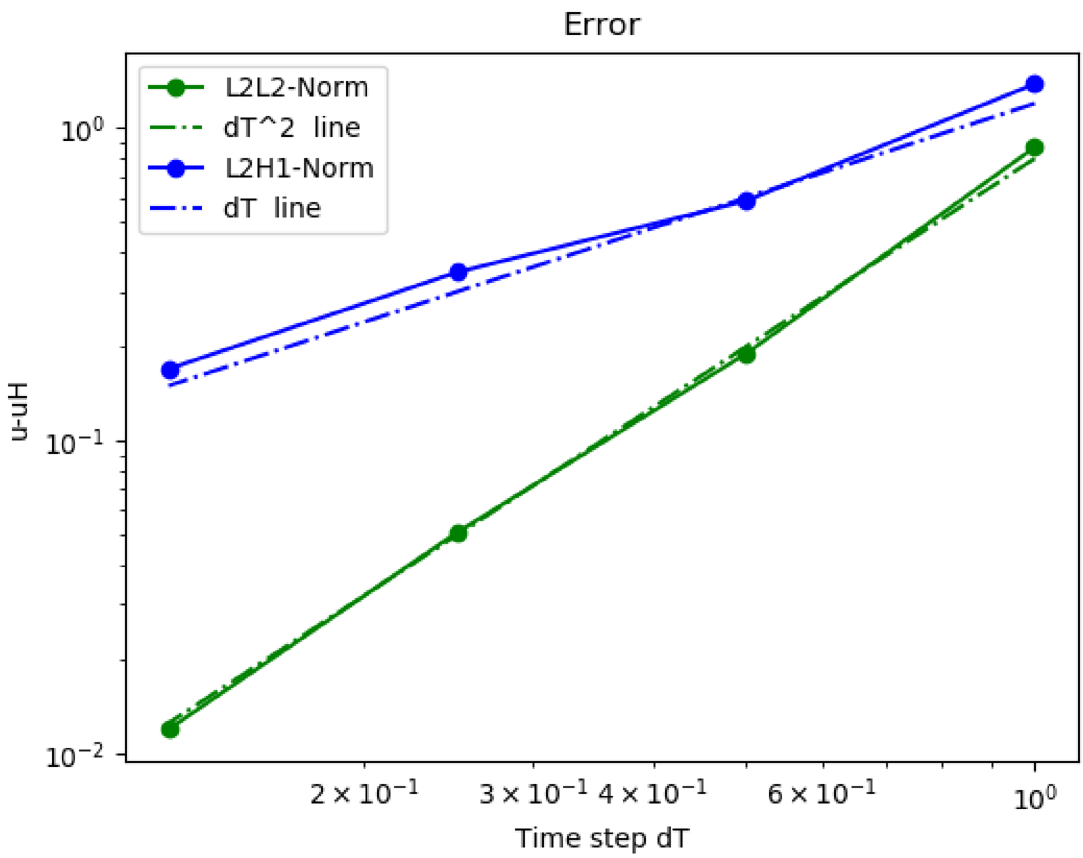
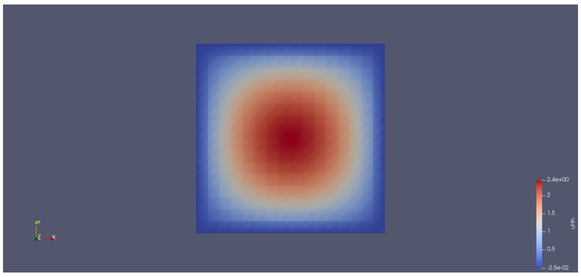
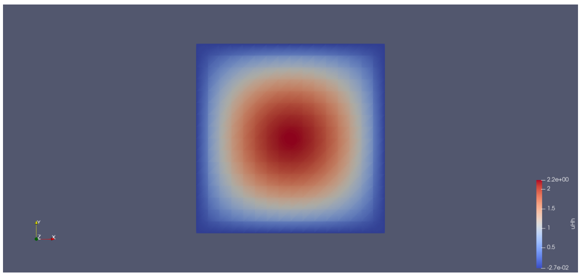


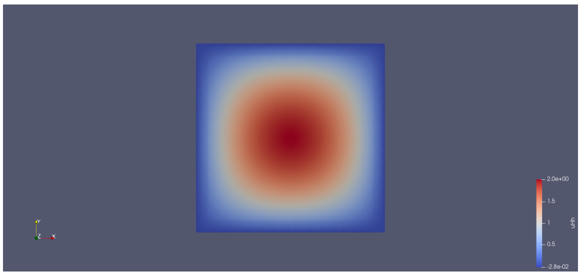
Figure 11 – Error curves in the norms $L^2(H^1/L^2)$ after refining the last point.



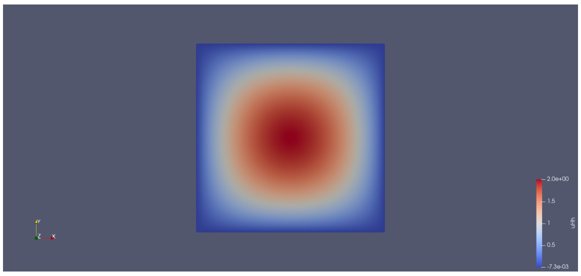
(a) $\Delta T = 1,0$



(b) $\Delta T = 0,5$



(c) $\Delta T = 0,25$



(d) $\Delta T = 0,125$ and $H = 1/32$

Figure 12 – Numerical solution u_h at time $T = 2$.

6.3 An Oscillatory Experiment

Multiscale methods are designed to deal with oscillations in the fine-scales of the local problems, in order to generate finite element basis capable of incorporate such details and upscale them into the global formulation. The following experiment shows how the parabolic MHM basis incorporates the fine-scales contribution for an oscillatory coefficient $A(x, y)$. The parameters and terms used to run this test are displayed in the sequence.

Parameters

$T = 0.5,\ \Omega = [0, 1]^2$

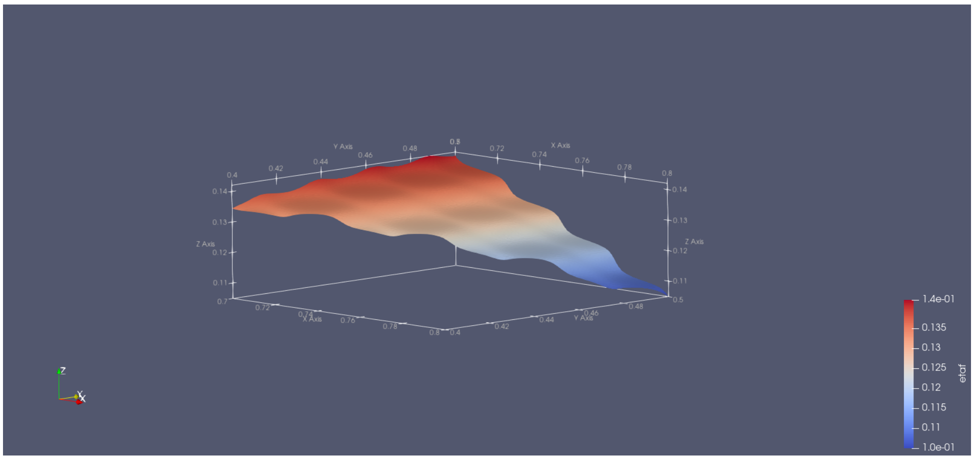
$A(x, y) = 1 + 10\sin^2(\frac{\pi x}{\epsilon})\cos^2(\frac{\pi y}{\epsilon}), \quad \epsilon = \frac{1}{40}$

$\Delta T = \frac{T}{50}\ , \quad H = \frac{1}{10}$

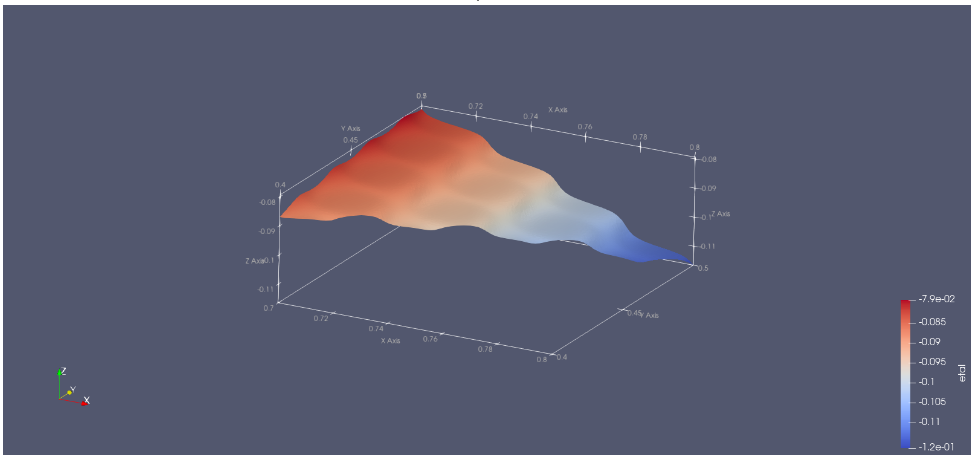
$\Delta t = \frac{\Delta T}{100}\ , \quad h = \frac{H}{64}$

$f(t, x, y) = \sin(x)\sin(y)\ , \quad u_0(x, y) = y(1 - y)\sin(\pi x)$

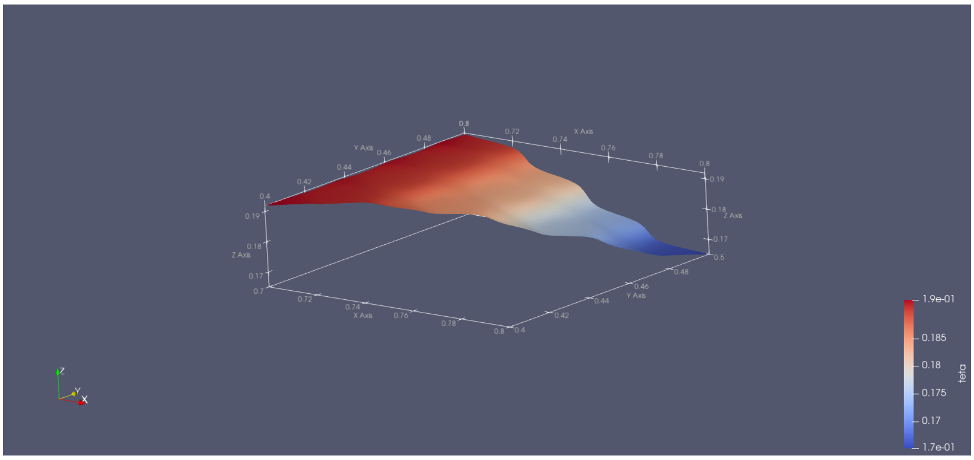
The following figures show some screenshots of the multiscale basis functions on an element K at a time t_i of the discretization of time interval I_0.



(a) Multiscale basis η_f at a time point of I_0.

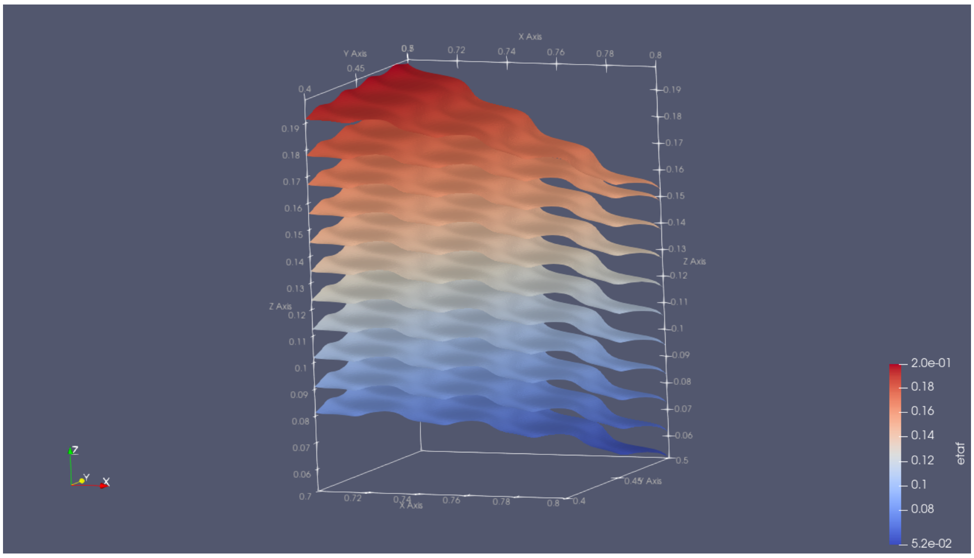


(b) Multiscale basis η_λ at a time point of I_0.

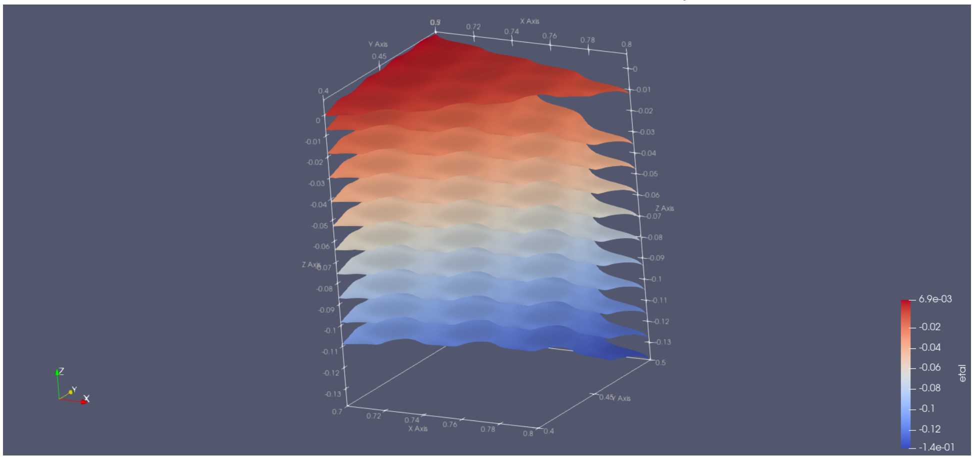


(c) Multiscale basis θ at a time point of I_0.

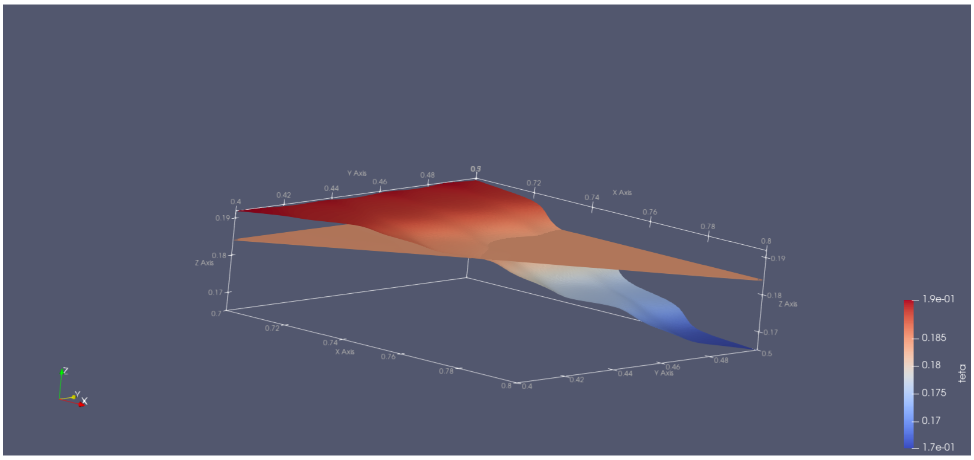
It is interesting to observe how the multiscale functions capture the oscillations of the coefficient A. This is what we expect when working with multiscale basis functions. In the coming plots, we observe how these functions evolve over time in slab I_0. The first local basis is the one on the top of all, and the other ones, from the top to the bottom, are the next ones.



(a) Evolution of the multiscale basis η_f at I_0.

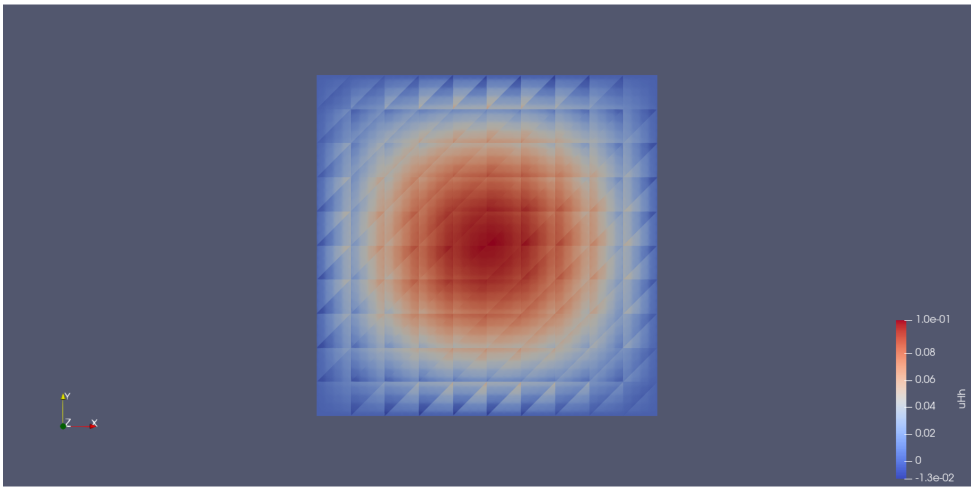


(b) Evolution of the multiscale basis η_λ at I_0.

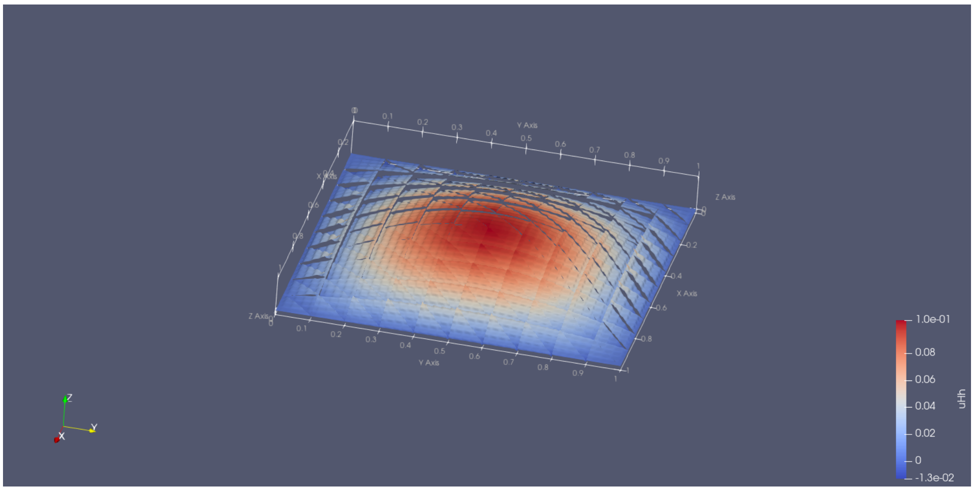


(c) Evolution of the multiscale basis θ at I_0.

We finish this section with the screenshots of the solution u_h at the end of the time interval I_0, to observe the global solution combining all the basis functions.



(a) Numerical solution u_h at the endpoint of I_0.



(b) Surface plot of the solution.

7 Conclusions and Future Perspectives

The Multiscale Hybrid Mixed Method (MHM) is a byproduct of a primal hybrid formulation applied to the PDE, that starts at the continuous level posed on a coarse partition. It consists in a decomposition of the exact solution into local and global contributions. When discretized, such a characterization decouples in local and global problems: the global formulation turns out to be responsible for the degrees of freedom over the skeleton of the coarse partition and ensure the continuity of the solution over it, and the local problems provide the multiscale basis functions, that are solutions of local boundary value problems with Neumann boundary condition on the faces of the macroelements. These multiscale basis functions naturally embed the fine-scale heterogeneties of the domain and can be computed in parallel, which is computationally interesting, due to the independence of the local problems.

The multiscale mortar mixed finite element method, developed in (JAYADHARAN et al., 2023), shares some similarities with our method. The local basis functions are of space-time type and they are computed on each macroelement as the solution of a mixed formulation of problem (1.1) with Dirichlet boundary condition. Like our method, their approach allows asynchronous time steps on each space-time element as well as different local space partitions. The continuity of the flux over the faces of the space mesh is imposed via coarse-scale space-time mortar variable.

This present thesis achieves the goal of generalizing the MHM method for parabolic linear problems with Dirichlet boundary condition. To the best of our knowledge, such result has not been established in the literature so far. The error analysis of the method displays the convergence rates on the spatial and time first level parameters, where the numerical validation confirm such orders. Our error analysis employs some classical analysis techniques and we assume that the coefficients of the matrix A are time independent and smooth in order to prove the error estimates. See remark 5.9 for more details.

The flexibility and accuracy of the method relies in some important points:

1. The framework allows the parallelization of the local problems, which is very useful computationally speaking;

2. The space-time basis functions incorporate information from the fine scales in space and time within each macroelement, upscaling them to the global problem;

3. The space-time mesh $\mathcal{T}_H^n$ does not need to be the same at each slab $\Omega \times I_n$;

4. The accuracy of the method depends basically on the time parameters h and ΔT and on the order of the polynomial approximations of spaces $L^2(I_n; M_h)$ and $\mathbb{P}_l(\mathcal{T}_{\bar{h}})$.

A range of possibilities when dealing with this approach can be explored in the future. We list the next steps to be developed, in order to expand the parabolic MHM and to fully understand its capabilities:

1. Error analysis for the case where $W_{\bar{h}}$ is coarser;
2. A second level error analysis of the method;
3. Implementation of a space-based approach in space;
4. The influence of high oscillatory and high-contrast coefficients in the method;
5. High order in time convergence rates;
6. A posteriori error estimate;
7. The impact of different time schemes to obtain the basis functions.

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Appendix

APPENDIX A – Auxiliary Results

Here we highlight some relevant results we use to prove the results presented in this thesis. The demonstrations of some of them are referenced, while others we exhibit their proofs.

First we start recalling the Green identities for smooth functions,

Theorem A.1. *Let $u, v \in C^2(\overline{\Omega})$. Then*

(i) $\int_\Omega \Delta u \; dx = \int_{\partial\Omega} \partial_n u \; ds$;

(ii) $\int_\Omega \nabla u \nabla v \; dx = -\int_\Omega u\Delta v \; dx + \int_{\partial\Omega} \partial_n uv \; ds$;

(iii) $\int_\Omega u\Delta v - v\Delta u \; dx = \int_{\partial\Omega} u\partial_n v - v\partial_n u \; ds$.

Proof. The proof is in (EVANS, 2010). □

The next result characterizes a continuous linear functional of X' whose kernel contains the space H_0^1. The spaces X and M of the next theorem are defined in chapter 3.

Lemma A.2. *A continuous linear functional L on the space X vanishes in $H_0^1(\Omega)$ if, and only if, there exists a unique $\mu \in M$ such that*

$$L(v) = \sum_{r=1}^{R} \int_{\partial\Omega_r} \mu v \; ds,$$

for all $v \in X$.

Proof. For the proof see (RAVIART; THOMAS, 1977). □

The next Lemma is a generalization of Lax-Milgram theorem:

Lemma A.3 (Lions projection lemma)**.** *Let H be a Hilbert space and $\Phi \subset H$ a dense subset. Let $a : H \times \Phi \to \mathbb{R}$ be a bilinear form with the following properties:*

i) for every $\phi \in \Phi$, the linear functional $u \mapsto a(u, \phi)$ is continuous in H;

ii) there exists $\alpha > 0$ such that

$$a(\phi, \phi) \geq \alpha ||\phi||_H^2 \quad \textit{for all} \quad \phi \in \Phi.$$

Then, for each $f \in H'$, there exists a unique $u \in H$ such that

$$a(u,\phi) = \langle f,\phi\rangle \quad \text{for all} \quad \phi \in \Phi$$

and

$$||u||_H \leq \frac{1}{\alpha}||f||_{H'}.$$

If Φ is not dense in H, we also have uniqueness.

Proof. The proof of the lemma can be found in (LIONS, 2013). □

As a consequence of Lemma C.5 for the Neumann boundary value problem, we have the following result present in (COSTABEL, 1990),

Theorem A.4. *For each $f \in L^2(Q)$, with $Q := (0,T)\times\Omega$, and $h \in L^2((0,T);H^{-\frac{1}{2}}(\Gamma))$ there exists a unique $u \in \tilde{\mathcal{V}}(Q)$ such that*

$$\begin{cases} \partial_t u - \nabla\cdot(A\nabla u) &= f \quad \text{in} \quad Q \\ \\ \gamma_1 u &= h \quad \text{on} \quad \Gamma, \end{cases} \tag{A.1}$$

where $\tilde{\mathcal{V}}(Q) = \{u \in L^2((-\infty,T);H^1(\Omega)) \mid \partial_t u \in L^2((-\infty,T);H^{-1}(\Omega))\}$. This Neumann problem is given by the following weak form

$$\int_Q A\nabla u\nabla v dxdt - \int_Q u\partial_t v dxdt = \int_Q f v dxdt + \int_\Gamma h v dsdt \tag{A.2}$$

for all $v \in C_0^\infty((-\infty,T)\times\mathbb{R}^n)$.

In the sequence, the theorems A.5, A.6 and A.8 deal with existence and regularity of solutions of parabolic equations.

Theorem A.5. *Let $u \in L^2(t_i,t_j;H^1(\Omega))\cap H^1(t_i,t_j;H^{-1}(\Omega))$ be the solution of the boundary value problem*

$$\begin{cases} u_t - \nabla\cdot(A\nabla u) = f & \text{in} \quad \Omega_T = (t_i,t_j)\times\Omega \\ A\nabla u\cdot\boldsymbol{n} = \mu & \text{on} \quad \partial\Omega_T = (t_i,t_j)\times\partial\Omega, \\ u = g & \text{at} \quad t = t_i, \end{cases} \tag{A.3}$$

where $\Omega \subset \mathbb{R}^d$ is an open bounded set, $f \in L^2(t_i,t_j;L^2(\Omega))$, $g \in L^2(\Omega)$, $\mu \in L^2(t_i,t_j;H^{-\frac{1}{2}}(\partial\Omega))$ and $A \in [L^\infty(\Omega)]^{d\times d}$ is such that there exists $\beta,\gamma > 0$ satisfying the following

$$\beta|\xi|^2 \leq A(x)\xi_i\xi_j \leq \gamma|\xi|^2,$$

for all $\xi \in \mathbb{R}^d$. Then, we have the following energy estimate

$$\begin{aligned} \max_{t_i\leq t\leq t_j} ||u(t,.)||_{L^2(\Omega)} + ||u||_{L^2(0,T;H^1(\Omega))} + ||u_t||_{L^2(0,T;H^{-1}(\Omega))} \\ \leq C(||g||_{L^2(\Omega)} + ||f||_{L^2(t_i,t_j;L^2(\Omega))} + ||\mu||_{L^2(t_i,t_j;\Lambda)}) \end{aligned} \tag{A.4}$$

where $C = C(\beta, \gamma, t_j - t_i)$ *and* $||.||_{L^2(t_i,t_j;\Lambda)}$ *stands for*

$$||\mu||_{L^2(t_i,t_j;\Lambda)} = \left(\int_{t_i}^{t_j} ||\mu(t)||_\Lambda^2 dt\right)^{\frac{1}{2}} \tag{A.5}$$

with $||\mu(t)||_\Lambda = \sup_{v \in H^1(\Omega)} \dfrac{(\mu, v)_{\partial\Omega}}{||v||_{H^1(\Omega)}}$ *and* $(.,.)_{\partial\Omega}$ *refers to the dual pair* $H^{-\frac{1}{2}}(\partial\Omega) - H^{\frac{1}{2}}(\partial\Omega)$*. The result remains true if* $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$*, where* $u|_{\partial\Omega_D} = 0$ *and* $A\nabla u \cdot \mathbf{n}|_{\partial\Omega_N} = \mu$*.*

Proof. The following proof is an adaptation of the proofs presented in Theorems 1, 2 and 3 of subsection 7.1.2 of (EVANS, 2010).

We start with the Galerkin's method. Let $\{w_k(x)\}$, $k = 1, 2, \dots,$ be a set of functions such that

i) $\{w_k(x)\}_{k=1}^\infty$ is an orthogonal set in $H^1(\Omega)$;

ii) $\{w_k(x)\}_{k=1}^\infty$ is an orthonormal set in $L^2(\Omega)$.

Fixing an integer m, we define a function $u_m : [t_i, t_j] \to H^1(\Omega)$ of the form

$$u_m(t) := \sum_{k=1}^m d_m^k(t) w_k, \tag{A.6}$$

$$d_m^k(t_i) = (g, w_k), \quad k = 1, \dots, m \tag{A.7}$$

satisfying the weak form of (A.3) given by

$$(u_m', u_m)_\Omega + (A\nabla u_m, \nabla u_m)_\Omega = (f, u_m)_\Omega + (\mu, u_m)_{\partial\Omega} \tag{A.8}$$

where $(.,.)_\Omega$ and $(.,.)_{\partial\Omega}$ are the L^2 products over Ω and its boundary, respectively, and $' = \dfrac{d}{dt}$. Let us also consider $f : [t_i, t_j] \to L^2(\Omega)$ and $\mu : [t_i, t_j] \to H^{-\frac{1}{2}}(\partial\Omega)$. First, we observe that

$$\begin{aligned}
(u_m', u_m)_\Omega &= \frac{1}{2}\frac{d}{dt}(||u_m||_{L^2(\Omega)}^2) \\
(A\nabla u_m, \nabla u_m)_\Omega &\geq \beta ||\nabla u_m||_{L^2(\Omega)}^2 \\
(f, u_m)_\Omega &\leq ||f||_{L^2(\Omega)} ||u_m||_{H^1(\Omega)} \\
(\mu, u_m)_{\partial\Omega} &= \frac{(\mu, u_m)_{\partial\Omega}}{||u_m||_{H^1(\Omega)}} \cdot ||u_m||_{H^1(\Omega)} \\
&\leq \sup_{v \in H^1(\Omega)} \frac{(\mu, v)_{\partial\Omega}}{||v||_{H^1(\Omega)}} \cdot ||u_m||_{H^1(\Omega)}
\end{aligned}$$

and, therefore, defining $||\mu||_\Lambda = \sup_{v \in H^1(\Omega)} \dfrac{(\mu, v)_{\partial\Omega}}{||v||_{H^1(\Omega)}}$, we have the following inequality

$$\frac{1}{2}\frac{d}{dt}(||u_m||_{L^2(\Omega)}^2) + \beta ||\nabla u_m||_{L^2(\Omega)}^2 \leq \Big(||f||_{L^2(\Omega)} + ||\mu||_\Lambda\Big) \cdot ||u_m||_{H^1(\Omega)}. \tag{A.9}$$

Applying Cauchy's inequality with ε on the right side of (A.9) we have

$$(||f||_{L^2(\Omega)} + ||\mu||_\Lambda) \cdot ||u_m||_{H^1(\Omega)} \leq \varepsilon ||u_m||^2_{H^1(\Omega)} + \frac{1}{4\varepsilon}(||f||_{L^2(\Omega)} + ||\mu||_\Lambda)^2.$$

We can use the fact that $(a+b)^2 \leq 2(a^2+b^2)$, $a, b \geq 0$, to get that

$$(||f||_{L^2(\Omega)} + ||\mu||_\Lambda) \cdot ||u_m||_{H^1(\Omega)} \leq \varepsilon ||u_m||^2_{H^1(\Omega)} + \frac{1}{2\varepsilon}||f||^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}||\mu||^2_\Lambda.$$

Choosing $\varepsilon < \beta$ at the previous inequality we obtain from (A.9), after multiplying by 2, the following

$$\frac{d}{dt}(||u_m||^2_{L^2(\Omega)}) + 2(\beta - \varepsilon)||\nabla u_m||^2_{L^2(\Omega)} \leq C_1||u_m||^2_{L^2(\Omega)} + C_2||f||^2_{L^2(\Omega)} + C_3||\mu||^2_\Lambda \quad \text{(A.10)}$$

where $C_1 = 2\varepsilon$, $C_2 = C_3 = \dfrac{1}{\varepsilon}$.

Now, we set the functions $\eta(t) = ||u_m||^2_{L^2(\Omega)}$ and $\xi(t) = ||f||^2_{L^2(\Omega)} + ||\mu||^2_\Lambda$, in addition to setting $C_4 = (C_2 + C_3)$ to get from (A.10) that

$$\eta'(t) \leq C_1\eta(t) + C_4\xi(t)$$

almost everywhere in t. Consequently, from Gronwall's inequality we have

$$\eta(t) \leq e^{C_1 t}\left(\eta(t_i) + C_4\int_{t_i}^{t}\xi(s)ds\right). \quad \text{(A.11)}$$

Since $\eta(t_i) = ||u_m(t_i)||^2_{L^2(\Omega)} \leq ||g||^2_{L^2(\Omega)}$ using (A.7), we can conclude that

$$\begin{aligned}\max_{t_i \leq t \leq t_j} ||u_m(t)||^2_{L^2(\Omega)} &\leq e^{C_1 T}\left(||g||^2_{L^2(\Omega)} + C_4\int_{t_i}^{t_j}(||f||^2_{L^2(\Omega)} + ||\mu||^2_\Lambda)dt\right) \\ &\leq C_5(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(t_i,t_j;L^2(\Omega))} + ||\mu||^2_{L^2(t_i,t_j;\Lambda)}) \quad \text{(A.12)}\end{aligned}$$

where $C_5 = e^{C_1 T}(1 + C_4)$.

Let's estimate the norm $L^2(t_i, t_j; H^1(\Omega))$ of u_m. Observe that

$$\begin{aligned}||u_m||^2_{L^2(t_i,t_j;H^1(\Omega))} &= \int_{t_i}^{t_j}(||u_m||^2_{L^2(\Omega)} + ||\nabla u_m||^2_{L^2(\Omega)})dt \\ &\leq \int_{t_i}^{t_j}\Bigg[||u_m||^2_{L^2(\Omega)} + \frac{C_1}{2(\beta-\varepsilon)}||u_m||^2_{L^2(\Omega)} \\ &+ \frac{C_2}{2(\beta-\varepsilon)}||f||^2_{L^2(\Omega)} + \frac{C_3}{2(\beta-\varepsilon)}||\mu||^2_\Lambda\Bigg]dt \quad \text{(A.13)}\end{aligned}$$

isolating $||\nabla u_m||^2_{L^2(\Omega)}$ in (A.10) on the left side of the inequality. We can also see, from (A.12), that

$$\begin{aligned}\int_{t_i}^{t_j}\left(||u_m||^2_{L^2(\Omega)} + \frac{C_1}{2(\beta-\varepsilon)}||u_m||^2_{L^2(\Omega)}\right)dt &= \left(1 + \frac{C_1}{2(\beta-\varepsilon)}\right)\int_{t_i}^{t_j}||u_m||^2_{L^2(\Omega)}dt \\ &\leq \left(1 + \frac{C_1}{2(\beta-\varepsilon)}\right)\max_{t_i \leq t \leq t_j}||u_m||^2_{L^2(\Omega)}\int_{t_i}^{t_j}dt \\ &\leq C_6(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(t_i,t_j;L^2(\Omega))} \\ &\quad + ||\mu||^2_{L^2(t_i,t_j;\Lambda)}). \quad \text{(A.14)}\end{aligned}$$

with $C_6 = \left(1 + \dfrac{C_1}{2(\beta - \varepsilon)}\right)(t_j - t_i)C_5$. Combining the two last inequalities we have

$$\begin{aligned} ||u_m||^2_{L^2(t_i,t_j;H^1(\Omega))} &\leq C_6(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(t_i,t_j;L^2(\Omega))} + ||\mu||^2_{L^2(t_i,t_j;\Lambda)}) \\ &+ \frac{C_2}{2(\beta-\varepsilon)}||f||^2_{L^2(t_i,t_j;L^2(\Omega))} + \frac{C_3}{2(\beta-\varepsilon)}||\mu||^2_{L^2(t_i,t_j;\Lambda)} \\ &\leq C_7(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(t_i,t_j;L^2(\Omega))} + ||\mu||^2_{L^2(t_i,t_j;\Lambda)}) \end{aligned} \tag{A.15}$$

where $C_7 = C_6 + \dfrac{C_2}{2(\beta-\varepsilon)} + \dfrac{C_3}{2(\beta-\varepsilon)}$.

Now fix $v \in H^1(\Omega)$ such that $||v||_{H^1(\Omega)} \leq 1$ and write $v = v^{(1)} + v^{(2)}$ where $v^{(1)} \in span\{w_k\}_{k=1}^{\infty}$ and $(v^{(2)}, w_k)_\Omega = 0$ for all $k = 1, \ldots, m$. Since $\{w_k\}_{k=1}^{\infty}$ is orthogonal in $H^1(\Omega)$ we have $||v^{(1)}||_{H^1(\Omega)} \leq ||v||_{H^1(\Omega)} \leq 1$. For almost all t in $[t_i, t_j]$, plugging v into (A.8) we get

$$\langle u_m', v\rangle = (u_m', v)_\Omega = (u_m', v^{(1)})_\Omega = (f, v^{(1)})_\Omega - (\mu, v^{(1)})_\Omega - (A\nabla u_m, \nabla v^{(1)})_\Omega.$$

Consequently

$$\begin{aligned} \langle u_m', v\rangle &\leq ||f||_{L^2(\Omega)}||v^{(1)}||_{H^1(\Omega)} + ||\mu||_\Lambda ||v^{(1)}||_{H^1(\Omega)} + \gamma||u_m||_{H^1(\Omega)}||v^{(1)}||_{H^1(\Omega)} \\ &\leq (1+\gamma)(||f||_{L^2(\Omega)} + ||\mu||_\Lambda + ||u_m||_{H^1(\Omega)}) \end{aligned}$$

because $||v^{(1)}||_{H^1(\Omega)} \leq 1$. Therefore,

$$||u_m'||_{H^{-1}(\Omega)} \leq (1+\gamma)(||f||_{L^2(\Omega)} + ||\mu||_\Lambda + ||u_m||_{H^1(\Omega)}).$$

From the fact that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, $a, b, c \geq 0$, we have

$$||u_m'||^2_{H^{-1}(\Omega)} \leq C_8(||f||^2_{L^2(\Omega)} + ||\mu||^2_\Lambda + ||u_m||^2_{H^1(\Omega)})$$

with $C_8 = 3(1+\gamma)^2$. Then, using (A.15), we get

$$\begin{aligned} ||u_m'||^2_{L^2(t_i,t_j;H^{-1}(\Omega))} &= \int_{t_i}^{t_j} ||u_m'||^2_{H^{-1}(\Omega)} dt \\ &\leq C_8 \int_{t_i}^{t_j} (||f||^2_{L^2(\Omega)} + ||\mu||^2_\Lambda + ||u_m||^2_{H^1(\Omega)}) dt \\ &\leq C_8\Big[||f||^2_{L^2(t_i,t_j;L^2(\Omega))} + ||\mu||^2_{L^2(t_i,t_j;\Lambda)} + C_7(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(t_i,t_j;L^2(\Omega))} \\ &\quad + ||\mu||^2_{L^2(t_i,t_j;\Lambda)})\Big] \\ &\leq C_9(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(t_i,t_j;L^2(\Omega))} + ||\mu||^2_{L^2(t_i,t_j;\Lambda)}) \end{aligned} \tag{A.16}$$

where $C_9 = (1 + C_7)C_8$.

Finally, from the boundedness of $||u_m||_{L^2(t_i,t_j;H^1(\Omega))}$ and $||u_m'||_{L^2(t_i,t_j;H^{-1}(\Omega))}$ we can use weak limits to obtain a solution $u \in L^2(t_i, t_j; H^1(\Omega)) \cap H^1(t_i, t_j; H^{-1}(\Omega))$ of (A.3) and from the estimates (A.12), (A.15) and (A.16), this solution satisfies estimate (A.4).

If we suppose that $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, where $\partial\Omega_D$ is the part of the boundary with Dirichlet's condition and $\partial\Omega_N$ is the one with Neumann's, we can follow the same steps of the previous demonstration but instead of working with a fundamental set in $H^1(\Omega)$ we work with a fundamental set in the subspace $V = \{v \in H^1(\Omega) \mid v|_{\partial\Omega_D} = 0\}$. Thus, equation (A.8) becomes

$$(u'_m, u_m)_\Omega + (A\nabla u_m, \nabla u_m)_\Omega = (f, u_m)_\Omega + (\mu, u_m)_{\partial\Omega_N} \tag{A.17}$$

and the norm of $\mu \in \Lambda$ becomes

$$||\mu||_\Lambda = \sup_{v\in V} \frac{(\mu, v)_{\partial\Omega_N}}{||v||_{H^1(\Omega)}}.$$

□

Theorem A.6. *Let $u \in L^2(t_i, t_j; H_0^1(\Omega)) \cap H^1(t_i, t_j; H^{-1}(\Omega))$ be the solution of the boundary value problem*

$$\begin{cases} u_t - \nabla\cdot(A\nabla u) = f & in \ \ \Omega_T = (t_i, t_j)\times\Omega \\ u = 0 & on \ \ \partial\Omega_T = (t_i, t_j)\times\partial\Omega, \\ u = g & at \ \ t = t_i, \end{cases} \tag{A.18}$$

where $\Omega \subset \mathbb{R}^d$ is an open bounded set, $f \in L^2(t_i, t_j; L^2(\Omega))$, $g \in L^2(\Omega)$ and $A \in [L^\infty(\Omega)]^{d\times d}$ is such that there exists $\beta, \gamma > 0$ satisfying the following

$$\beta|\xi|^2 \le A(x)\xi_i\xi_j \le \gamma|\xi|^2,$$

for all $\xi \in \mathbb{R}^d$. Then, we have the following energy estimate

$$\begin{aligned}\max_{t_i\le t\le t_j} ||u(t,.)||_{L^2(\Omega)} + ||u||_{L^2(t_i,t_j;H_0^1(\Omega))} + ||u_t||_{L^2(t_i,t_j;H^{-1}(\Omega))} \\ \le C(||g||_{L^2(\Omega)} + ||f||_{L^2(t_i,t_j;L^2(\Omega))})\end{aligned} \tag{A.19}$$

where $C = C(\beta, \gamma, t_j - t_i)$.

Proof. The full proof can be found in (EVANS, 2010). Here we are interested in showing how $t_j - t_i$ appears on C.

This comes from Gronwall's inequality (A.11) with $\eta(t) = ||u_m(t)||^2_{L^2(\Omega)}$ and $\xi(t) = ||f(t)||^2_{L^2(\Omega)}$ that implies

$$\max_{t_i\le t\le t_j} ||u_m(t,.)||^2_{L^2(\Omega)} \le C(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(t_i,t_j;L^2(\Omega))}) \tag{A.20}$$

and using inequality

$$\frac{d}{dt}(||u_m||^2_{L^2(\Omega)}) + 2\beta||u_m||^2_{H_0^1(\Omega)} \le C_1||u_m||^2_{L^2(\Omega)} + C_2||f||^2_{L^2(\Omega)} \tag{A.21}$$

a.e. in $[t_i, t_j]$ to obtain

$$\begin{aligned} ||u_m||^2_{L^2(t_i,t_j;H^1_0(\Omega))} &= \int_{t_i}^{t_j} ||u_m||^2_{H^1_0(\Omega)} dt \\ &\leq \int_{t_i}^{t_j} (C_1||u_m||^2_{L^2(\Omega)} + C_2||f||^2_{L^2(\Omega)}) dt \\ &\leq C_1 \max_{t_i\leq t\leq t_j} ||u(t,.)||^2_{L^2(\Omega)} \int_{t_i}^{t_j} dt + C_2||f||^2_{L^2(t_i,t_j;L^2(\Omega))} \\ &\leq C_3(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(t_i,t_j;L^2(\Omega))}) \end{aligned} \tag{A.22}$$

where $C_3 = CC_1(t_j - t_i) + C_2$. Now for u_t we use inequality

$$||u'_m||^2_{H^{-1}(\Omega)} \leq C'(||f||^2_{L^2(\Omega)} + ||u_m||^2_{H^1_0(\Omega)}) \tag{A.23}$$

combined with estimate (A.22) to show that

$$\begin{aligned} ||u'_m||^2_{L^2(t_i,t_j;H^{-1}(\Omega))} &\leq \int_{t_i}^{t_j} C'(||f||^2_{L^2(\Omega)} + ||u_m||^2_{H^1_0(\Omega)}) dt \\ &\leq C_4(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(t_i,t_j;L^2(\Omega))}) \end{aligned} \tag{A.24}$$

with $C_4 = C'C_3 + C'$. □

Remark A.7. If we consider $f = 0$ in Theorem A.6, we would have the bounds

$$\begin{aligned} ||u_m||^2_{L^2(t_i,t_j;H^1_0(\Omega))} &\leq C(t_j - t_i)||g||^2_{L^2(\Omega)}, \\ ||u'_m||^2_{L^2(t_i,t_j;H^{-1}(\Omega))} &\leq C(t_j - t_i)||g||^2_{L^2(\Omega)}. \end{aligned} \tag{A.25}$$

□

Theorem A.8. *Let $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ be the weak solution of the boundary value problem*

$$\begin{cases} u_t - \nabla\cdot(A\nabla u) = f & in \;\; \Omega_T = (0,T)\times\Omega \\ u = 0 & on \;\; \partial\Omega_T = (0,T)\times\partial\Omega, \\ u = g & at \;\; t = 0, \end{cases} \tag{A.26}$$

where $\Omega \subset \mathbb{R}^d$ is an open bounded set with $\partial\Omega \in C^2$, $f \in L^2(0,T;L^2(\Omega))$ and $g \in H^1_0(\Omega)$. Assume that $A \in [L^\infty(\Omega)]^{d\times d}$ is such that its coefficients $a_{ij}(x)$ are smooth on $\overline{\Omega}$ and satisfies the condition: there exists $\beta,\gamma > 0$ such that

$$\beta|\xi|^2 \leq A(x)\xi_i\xi_j \leq \gamma|\xi|^2,$$

for all $\xi \in \mathbb{R}^d$. Then, in fact,

$$u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)) \;\; with \;\; u' \in L^2(0,T;L^2(\Omega)), \tag{A.27}$$

and we have the following energy estimate

$$\begin{aligned} ess\ sup_{0\leq t\leq T}||u(t,.)||_{H^1_0(\Omega)} + ||u||_{L^2(0,T;H^2(\Omega))} + ||u_t||_{L^2(0,T;L^2(\Omega))} \\ \leq C(||g||_{H^1_0(\Omega)} + ||f||_{L^2(0,T;L^2(\Omega))}) \end{aligned} \tag{A.28}$$

where $C = C(\beta,\gamma,\Omega,T)$.

Proof. In order to use Galerkin's method we consider the complete collection $\{w_k\}_{k=1}^{\infty}$ of eigenfunctions for $-\nabla\cdot(A\nabla)$ on $H_0^1(\Omega)$.

Fixing $m\geq 1$, writing $u_m=\sum_{k=1}^m d_m^k(t)w_k(x)$, we multiply equality

$$(u_m', w_k)_\Omega + (A\nabla u_m, \nabla w_k)_\Omega = (f, w_k)_\Omega, \quad a.e.\ 0\leq t\leq T$$

by $(d_m^k(t))'$, where $(.,.)_\Omega=(.,.)_{L^2(\Omega)}$, and sum from $k=1,\ldots,m$ to get

$$(u_m', u_m')_\Omega + (A\nabla u_m, \nabla u_m')_\Omega = (f, u_m')_\Omega, \quad for\ a.e.\ 0\leq t\leq T. \tag{A.29}$$

Now we observe that

$$\int_\Omega A\nabla u_m \nabla u_m' dx \geq \beta\int_\Omega u_m u_m' dx \geq \frac{\beta}{2}\partial_t(||u_m||^2_{H_0^1(\Omega)}). \tag{A.30}$$

From (A.29) we use Cauchy's inequality with $\varepsilon=\frac{1}{4}$ to obtain

$$\begin{aligned} ||u_m'||^2_{L^2(\Omega)} &\leq ||f||_{L^2(\Omega)}||u_m'||_{L^2(\Omega)} \\ &\leq ||f||^2_{L^2(\Omega)} + \frac{1}{4}||u_m'||^2_{L^2(\Omega)} \end{aligned}$$

and, therefore,

$$||u_m'||^2_{L^2(\Omega)} \leq \frac{4}{3}\ ||f||^2_{L^2(\Omega)}. \tag{A.31}$$

Integrating (A.31) with respect to t we get

$$||u_m'||^2_{L^2(0,T;L^2(\Omega))} \leq \frac{4}{3}\ ||f||^2_{L^2(0,T;L^2(\Omega))}. \tag{A.32}$$

We have from (A.29), (A.30) and (A.31) that

$$\begin{aligned} \frac{\beta}{2}\partial_t(||u_m||^2_{H_0^1(\Omega)}) &\leq (f, u_m')_\Omega \\ &\leq \frac{2}{\sqrt{3}}\ ||f||^2_{L^2(\Omega)} \end{aligned}$$

or equivalently

$$\partial_t(||u_m||^2_{H_0^1(\Omega)}) \leq \frac{4}{\beta\sqrt{3}}\ ||f||^2_{L^2(\Omega)}.$$

Therefore, from Gronwall's Lemma we have that

$$\begin{aligned} ||u_m||^2_{H_0^1(\Omega)} &\leq \left(||u_m(0)||^2_{H_0^1(\Omega)} + \frac{4}{\beta\sqrt{3}}\int_0^t ||f||^2_{L^2(\Omega)}dt\right) \\ &\leq C_1\left(||g||^2_{H_0^1(\Omega)} + ||f||^2_{L^2(0,T;L^2(\Omega))}\right), \end{aligned}$$

where $C_1=1+\frac{4}{\beta\sqrt{3}}$ a.e. in t, which implies that

$$\text{ess sup}_{0\leq t\leq T}||u_m||^2_{H_0^1(\Omega)} \leq C_1\left(||g||^2_{H_0^1(\Omega)} + ||f||^2_{L^2(0,T;L^2(\Omega))}\right). \tag{A.33}$$

The fact that $||u_m(0)||_{H_0^1(\Omega)} \leq ||g||_{H_0^1(\Omega)}$ can be seen in the proof of Theorem A.5.

Now, passing to weak limits we conclude from (A.32) and (A.33) that $u \in L^\infty(0,T;H_0^1(\Omega))$ and $u' \in L^2(0,T;L^2(\Omega))$.

In particular, for a.e. in t we have

$$(u',v)_\Omega + (A\nabla u, \nabla v)_\Omega = (f,v)_\Omega$$

for all $v \in H_0^1(\Omega)$. We can rewrite this equality in the form

$$(A\nabla u, \nabla v)_\Omega = (h,v)_\Omega \tag{A.34}$$

for $h = f - u' \in L^2(\Omega)$ a.e. in $[0,T]$. From the elliptic regularity Theorem 4 of section 6.3.2 in (EVANS, 2010) we see that $u(t) \in H^2(\Omega)$ a.e. in $[0,T]$, with the estimate

$$\begin{aligned}
||u||^2_{H^2(\Omega)} &\leq C\left(||h||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}\right)\\
&\leq C\left(||h||^2_{L^2(\Omega)} + ||u'||^2_{L^2(\Omega)} + ||u||^2_{L^2(\Omega)}\right).
\end{aligned}$$

Integrating in t and using (A.32) and (A.33) we have the estimate (A.28).

□

Lemma A.9. *Given $\beta > 0$ and $u \in L^2(0,T;H^1(\Omega))$, there exists $w \in L^2(0,T;H^1(\Omega))$ such that $u = e^{\beta t} w$.*

Proof. Define $w = e^{-\beta t} u$ and observe that

$$\begin{aligned}
||w||^2_{L^2(0,T;H^1(\Omega))} &= \int_0^T ||w||^2_{H^1(\Omega)} dt\\
&= \int_0^T \left(||e^{-\beta t} u||^2_{L^2(\Omega)} + ||\nabla e^{-\beta t} u||^2_{L^2(\Omega)}\right) dt\\
&= \int_0^T e^{-2\beta t} ||u||^2_{H^1(\Omega)} dt\\
&\leq \int_0^T ||u||^2_{H^1(\Omega)} dt < \infty
\end{aligned}$$

and, therefore, $w \in L^2(0,T;H^1(\Omega))$. □

Lemma A.10. *Let a_i, $i = 0,\ldots,k$, positive numbers. We therefore have*

$$\left(\sum_{i=0}^{k-1} a_i\right)^2 \leq k \sum_{i=0}^{k-1} a_i^2.$$

Proof. We prove the assertion using induction on k.

First, for $k = 1$ we have $(a_0)^2 = 1 \cdot a_0^2$. Now, suppose that for k it holds

$$\left(\sum_{i=0}^{k-1} a_i\right)^2 \leq k \sum_{i=0}^{k-1} a_i^2.$$

For $k+1$ we have

$$\begin{aligned}\left(\sum_{i=0}^{k} a_i\right)^2 &= \left(\sum_{i=0}^{k-1} a_i + a_k\right)^2 \\ &= \left(\sum_{i=0}^{k-1} a_i\right)^2 + 2a_k \sum_{i=0}^{k-1} a_i + a_k^2 \\ &\leq k\sum_{i=0}^{k-1} a_i^2 + a_0^2 + a_k^2 + \cdots + a_{k-1}^2 + a_k^2 + a_k^2 \\ &\leq k\sum_{i=0}^{k-1} a_i^2 + \sum_{i=0}^{k} a_i^2 + k a_k^2 \\ &= (k+1)\sum_{i=0}^{k} a_i^2,\end{aligned}$$

where to get the inequality we use the induction hypothesis and the fact that $2a_ia_k \leq a_i^2+a_k^2$. Thus, the result is proved. □

The following Lemma is contained in reference (CROUZEIX; RAVIART, 1973), but we will repeat the demonstration in the sequel in order to be able to use some arguments adopted along the proof in the demonstration of the best approximation result in chapter 3.

Let $K \in \mathbb{R}^N$ be a N-simplex and K' be a $(N-1)$-dimensional face of K. Let $\mathcal{P}'_l$ be the space of restrictions to K' of all polynomials of degree l and let $\Pi^l_{K'}$ be the orthogonal projection from $L^2(K')$ to $\mathcal{P}'_l$ satisfying

$$\int_{K'} w \cdot \Pi^l_{K'} v \; ds = \int_{K'} w \cdot v \; ds \quad \forall\, w \in \mathcal{P}'_l. \tag{A.35}$$

Let us also recall the notations

$$\begin{aligned} h(K) &= \text{diameter of K}, \\ \rho(K) &= \text{diameter of the biggest sphere inscribed in K}, \\ \sigma(K) &= \frac{h(K)}{\rho(K)}. \end{aligned}$$

Lemma A.11. *For any integer m with $0 \leq m \leq l$, there exists a constant C independent of K such that*

$$\left| \int_{K'} w \cdot (v - \Pi^l_{K'} v) \; ds \right| \leq C\sigma(K)(h(K))^{m+1} |w|_{H^1(K)} |v - \Pi^l_{K'} v|_{H^{m+1}(K)} \tag{A.36}$$

for all $w \in H^1(K)$ and $v \in H^{m+1}(K)$.

Proof. Let $K \in \mathbb{R}^N$ be a N-simplex and K' be a $(N-1)$-dimensional face of K. To simplify notation we assume that K' and $\hat{K}'$ are in the same supporting hyperplane $x_N = 0$. Denote

$$F(x) = Bx + b,\ B \in \mathcal{L}(\mathbb{R}^N),\ b \in \mathbb{R}^N,$$

the affine invertible map such that $F(\hat{K}) = K$ and $F(\hat{K}') = K'$. Let B' be the $(N-1) \times (N-1)$ matrix obtained after crossing out the N^{th} column and row of matrix B. For any function f on K(or K') we write

$$\hat{f} = f \circ F.$$

Then we observe that

$$\widehat{\Pi^l_{K'} v} = \Pi^l_{\hat{K}'} \hat{v}.$$

Performing a change of variables we can see that

$$\int_{K'} w \cdot (v - \Pi^l_{K'} v)\ ds = |det(B')| \int_{\hat{K}'} \hat{w} \cdot (\hat{v} - \Pi^l_{\hat{K}'} \hat{v})\ ds. \tag{A.37}$$

For a fixed $v \in H^{m+1}(K)$, $0 \le m \le l$, we observe that the functional on $H^1(K)$ given by

$$\hat{w} \mapsto \int_{\hat{K}'} \hat{w} \cdot (\hat{v} - \Pi^l_{\hat{K}'} \hat{v})\ ds$$

is continuous, has norm $||\hat{v} - \Pi^l_{\hat{K}'} \hat{v}||_{L^2(\hat{K}')}$ and vanishes for all $\hat{w} \in \mathcal{P}_0$ from (A.35). By Lemma 6 in (CIARLET; RAVIART, 1972) we get

$$\left| \int_{\hat{K}'} \hat{w} \cdot (\hat{v} - \Pi^l_{\hat{K}'} \hat{v})\ ds \right| \le c_1\ |\hat{w}|_{H^1(\hat{K})} ||\hat{v} - \Pi^l_{\hat{K}'} \hat{v}||_{L^2(\hat{K}')} \tag{A.38}$$

for some constant $c_1 = c_1(\hat{K})$. Since $\Pi^l_{\hat{K}'} \hat{v} = \hat{v}$ for all $\hat{v} \in \mathcal{P}_m$ we get from (CIARLET; RAVIART, 1972), Lemma 7, that

$$||\hat{v} - \Pi^l_{\hat{K}'} \hat{v}||_{L^2(\hat{K}')} \le c_2\ |\hat{v}|_{H^{m+1}(\hat{K})} \tag{A.39}$$

where $c_2 = c_2(\hat{K})$. Combining (A.37)-(A.39) we obtain

$$\left| \int_{K'} w \cdot (v - \Pi^l_{K'} v)\ ds \right| \le c_1 c_2\ |det(B')|\ |\hat{w}|_{H^1(\hat{K})}\ |\hat{v}|_{H^{m+1}(\hat{K})}. \tag{A.40}$$

From inequality (4.15) in (CIARLET; RAVIART, 1972), given by

$$|\hat{v}|_{H^l(\hat{K})} \le |det(B)|^{-\frac{1}{2}}\ ||B||^l\ |v|_{H^l(K)}, \tag{A.41}$$

we have

$$\left| \int_{K'} w \cdot (v - \Pi^l_{K'} v)\ ds \right| \le c_1 c_2\ |det(B')|\ |det(B)|^{-1}\ ||B||^{m+2}\ |w|_{H^1(K)}\ |v|_{H^{m+1}(K)}, \tag{A.42}$$

where $||B||$ is the euclidean norm of a matrix.

Let e_N be the basis canonical vector of $\mathbb{R}^N$. The N^{th} component of the vector $B^{-1} e_N$ is

$$(B^{-1} e_N)_N = det(B')(det(B))^{-1}$$

so that

$$|det(B')| \leq |det(B)|\ ||B^{-1}||. \tag{A.43}$$

By (A.42) and (A.43) we get

$$\left| \int_{K'} w \cdot (v - \Pi^l_{K'} v)\ ds \right| \leq c_1 c_2\ ||B^{-1}||\ ||B||^{m+2}\ |w|_{H^1(K)}\ |v|_{H^{m+1}(K)}. \tag{A.44}$$

From Lemma 2 of (CIARLET; RAVIART, 1972) we employ the bounds

$$||B|| \leq \frac{h(K)}{\rho(\widehat{K})},\ ||B^{-1}|| \leq \frac{h(\widehat{K})}{\rho(K)}, \tag{A.45}$$

in inequality (A.44) with $C = c_1 c_2 \dfrac{h(\widehat{K})}{\rho(\widehat{K})^{m+2}}$, which depends only on $\widehat{K}$, to get the desired inequality.

□

For functions in H^1 defined on a polygonal element K with diameter h_K, we have the following inequality:

Theorem A.12. *Given a function $v \in H^1(K)$, there exists a constant C such that*

$$||v||_{L^2(\partial K)} \leq C \left(\frac{1}{h_K} ||v||^2_{L^2(K)} + h_K ||\nabla v||^2_{L^2(K)} \right)^{\frac{1}{2}}. \tag{A.46}$$

Proof. This result is a consequence of a more general one given in Theorem 3.10 of (AGMON, 2010). □

APPENDIX B – Fully Discrete Parabolic MHM

In this section we present the fully discrete scheme for the space-time MHM method. As observed in Chapter 4, a second level is necessary to approximate the multiscale basis functions η_i, θ_l and η_f. Since the local problems involve time derivatives, we employ a finite difference Euler Implicit scheme to discretize time derivative, combine with FEM to approximate spatially the basis.

Then, we consider a space partition $\mathcal{T}_h \subset \mathcal{T}_H$ and for each interval of the time partition $\mathcal{T}^{\Delta T} = \cup_{N=0}^{M-1} I_n$, we consider a finer one on each I_n, designated by $\mathcal{T}^{\Delta t}$, where Δt is the local time step. This means the we are partitioning I_n as the set of points

$$t_n = t_0, t_1, \ldots, t_m = t_{n+1},$$

with $\Delta t = t_{l+1} - t_l$ for all $l = 0, \ldots, m-1$. We list below the dimensions of the global spaces and the geometrical parameters to enlighten the following definitions.

Table 10 – List of Parameters

Symbol	Definition
n_F	number of faces in $\mathcal{E}_H$
n_F^K	number of faces in $\mathcal{E}_H \cap \partial K$
n_K	number of elements in $\mathcal{T}_H$
$n_{\Delta T}$	number of intervals in $\mathcal{T}^{\Delta T}$

Table 11 – List of Dimensions

Symbol	Dimension of	Value
$m_\Lambda^{F_n}$	$\Lambda_H^n(F)$	$d\frac{(l+d-1)!}{(d-1)!l!}$
$m_\Lambda^{K_n}$	$\Lambda_H^n(K)$	$n_F^K \cdot m_\Lambda^{F_n}$
m_Λ^n	Λ_H^n	$n_F \cdot m_\Lambda^{F_n}$
m_Λ	$\Lambda_H^0 \times \cdots \times \Lambda_H^{M-1}$	$n_{\Delta T} \cdot n_F \cdot m_\Lambda^{F_n}$

We then consider the local basis $\{\psi_i^{K_n}\}_{i=1}^{m_\Lambda^{K_n}}$ of $\Lambda_H^n(K)$ and write $\lambda_{n,H}^K = \sum_{i=1}^{m_\Lambda^{K_n}} \beta_i^N \psi_i^{K_n}$. We also consider the finite dimensional space $X_h(K) \subset H^1(K)$ and take its basis to be the same one used to project the initial data of the local problems. This means that we do not need to worry for now with writing η_τ^N as a linear combination of basis θ_l since we are considering $W_{\bar{h}} = X_h$.

To perform the space-time integration necessary to assemble the left and right hand side of global problem (4.26), we seek to obtain the value of the basis functions at every point $t_n \in \mathcal{T}^{\delta_N}$. We then employ the Euler implicit method to discretize the time derivative on each local problem (4.21)-(4.23), obtaining at each time step point t_n the corresponding spatial vector.

The Euler implicit method applied to $v_t = f$, where $f = f(t, v(t))$ is a function, writes

$$v^{n+1} - v^n = v(t_{n+1}) - v(t_n) \approx \delta_N \cdot f(t_{n+1}, v(t_{n+1})). \tag{B.1}$$

Writing from (B.1) the discrete derivative

$$\partial_t u_{Hh}^{n+1} \approx \frac{u_{Hh}^{n+1} - u_{Hh}^n}{\Delta t}, \tag{B.2}$$

we seek to satisfy equation (4.2) in the form

$$\begin{aligned}\left(\frac{u_{Hh}^{n+1} - u_{Hh}^n}{\Delta t}, v_h\right)_K + (A\nabla u_{Hh}^{n+1}, \nabla v_h)_K = \\ (f(t_{n+1}), v_h)_K - (\lambda_H^{n+1}, v_h)_{\partial K},\end{aligned} \tag{B.3}$$

for all $v_h \in X_h(K)$ with initial data $u_{Hh}^0 = u_{Hh}^{N-1}(t_N)$ (or $= u_h^0$ if $N = 0$). The local discrete operators $S_h^n : \Lambda_H^N \to X_h$, $\bar{S}_h^n : W_H \to X_h$ and $\hat{S}_h^n : L^2(Q) \to X_h$ satisfy the local problems on each K given by

$$(S_h^{n+1}\lambda_{N,H}, v_h)_K + \delta_N(A\nabla S_h^{n+1}\lambda_{N,H}, \nabla v_h)_K = (S_h^n\lambda_{N,H}, v_h)_K - \delta_N(\lambda_{N,H}(t_{n+1}), v_h)_{\partial K}, \tag{B.4}$$

$$(\bar{S}_h^{n+1}\tau_{N,H}, v_h)_K + \delta_N(A\nabla \bar{S}_h^{n+1}\tau_{N,H}, \nabla v_h)_K = (\bar{S}_h^n\tau_{N,H}, v_h)_K, \tag{B.5}$$

$$(\hat{S}_h^{n+1}f^N, v_h)_K + \delta_N(A\nabla \hat{S}_h^{n+1}f^N, \nabla v_h)_K = (\hat{S}_h^n f^N, v_h)_K + \delta_N(f(t_{n+1}), v_h)_K. \tag{B.6}$$

Denoting $S_h^n\psi_i = \eta_i^n$, $\bar{S}_h^n\tau_{N,H} = \eta_\tau^n$ and $\hat{S}_h^n f = \eta_f^n$, solutions of

$$(\eta_i^{n+1}, v_h)_K + \Delta t(A\nabla\eta_i^{n+1}, \nabla v_h)_K = (\eta_i^n, v_h)_K - \Delta t(\psi_i^{n+1}, v_h)_{\partial K}, \tag{B.7}$$

$$(\eta_\tau^{n+1}, v_h)_K + \Delta t(A\nabla\eta_\tau^{n+1}, \nabla v_h)_K = (\eta_\tau^n, v_h)_K, \tag{B.8}$$

$$(\eta_f^{n+1}, v_h)_K + \Delta t(A\nabla\eta_f^{n+1}, \nabla v_h)_K = (\eta_f^n, v_h)_K + \Delta t(f(t_{n+1}), v_h)_K, \tag{B.9}$$

with initial conditions $\eta_i^0 = 0$, $\eta_\tau^0 = u_{Hh}^{N-1}(t_N)$ and $\eta_f^0 = 0$, we have, from linearity of (B.7), that

$$S_h^n\lambda_{N,H}^K = \sum_{i=1}^{m_\Lambda^{K_N}} \beta_i^N S_h^n\psi_i = \sum_{i=1}^{m_\Lambda^{K_N}} \beta_i^N \eta_i^n, \tag{B.10}$$

$$\bar{S}_h^n\tau_{N,H} = \eta_\tau^N, \tag{B.11}$$

$$\hat{S}_h f^N = \eta_f^N. \tag{B.12}$$

Remark B.1. The initial condition $u_{Hh}^{N-1}(t_N)$ stands for the L^2 projection of the initial data onto X_h, used to start the evolution of the local basis η_τ^n once we know from the second level time discretization that $t_N = t_0$. In the time slab $I_0 = (t_0, t_1]$, we have that $u_{H,h}^0$ is the projection of the initial data u_0 from (1.1).

After computing the basis functions on each t_n, $n = 1, \ldots, m$, we obtain the space-time basis given in the matrix form by

$$\eta_i^N := \begin{bmatrix} \eta_i^1 \\ \eta_i^2 \\ \vdots \\ \eta_i^m \end{bmatrix}, i = 1, \ldots, R, \tag{B.13}$$

$$\eta_\tau^N := \begin{bmatrix} \eta_\tau^1 \\ \eta_\tau^2 \\ \vdots \\ \eta_\tau^m \end{bmatrix}, \tag{B.14}$$

$$\eta_f^N := \begin{bmatrix} \eta_f^1 \\ \eta_f^2 \\ \vdots \\ \eta_f^m \end{bmatrix}, \tag{B.15}$$

where each vector η_i^n, η_τ^n and η_f^n represent the coefficients related to the linear combination of the local basis of $X_h(K)$. In order to obtain the degrees of freedom β_i^N of $\lambda_{N,H}$ we apply the trapezoidal rule in time to compute the following integrals

$$\begin{aligned} (\psi_j, \eta_i^N)_{\partial K_N} &= (\psi_j, \eta_i^N)_{\partial K \times I_N} \\ &= \sum_{n=1}^{m} \frac{(\psi_j^n, \eta_i^n)_{\partial K} + (\psi_j^{n-1}, \eta_i^{n-1})_{\partial K}}{2} \cdot \Delta t \\ &= \Delta t \sum_{n=1}^{m-1} (\psi_j^n, \eta_i^n)_{\partial K} + \frac{\Delta t}{2} (\psi_j^m, \eta_i^m)_{\partial K}, \end{aligned} \tag{B.16}$$

$$\begin{aligned} (\psi_j, \eta_\tau^N)_{\partial K_N} &= (\psi_j, \eta_\tau^N)_{\partial K \times I_N} \\ &= \sum_{n=1}^{m} \frac{(\psi_j^n, \eta_\tau^n)_{\partial K} + (\psi_j^{n-1}, \eta_\tau^{n-1})_{\partial K}}{2} \cdot \Delta t \\ &= \frac{\Delta t}{2} (\psi_j^0, u_{Hh}^{N-1}(t_N))_{\partial K} + \Delta t \sum_{n=1}^{m-1} (\psi_j^n, \eta_\tau^n)_{\partial K} \\ &+ \frac{\Delta t}{2} (\psi_j^m, \eta_\tau^m)_{\partial K}, \end{aligned} \tag{B.17}$$

$$
\begin{aligned}
(\psi_j, \eta_f^N)_{\partial K_N} &= (\psi_j, \eta_f^N)_{\partial K \times I_N} \\
&= \sum_{n=1}^{m} \frac{(\psi_j^n, \eta_f^n)_{\partial K} + (\psi_j^{n-1}, \eta_f^{n-1})_{\partial K}}{2} \cdot \Delta t \\
&= \Delta t \sum_{n=1}^{m-1} (\psi_j^n, \eta_f^n)_{\partial K} + \frac{\Delta t}{2} (\psi_j^m, \eta_f^m)_{\partial K},
\end{aligned} \tag{B.18}
$$

since $\eta_i(x, t_0) = 0$, $\eta_f(x, t_0) = 0$ and $\eta_\tau(x, t_0) = u_{Hh}^{N-1}(t_N)$.

Then, we solve the problem $A^N \beta^N = B^N$ where A^N and B^N are assembled using the local terms

$$
A_{ji}^{K_N} = (\psi_j^{K_N}, \eta_i^N)_{\partial K_N}, \tag{B.19}
$$

$$
B_j^{K_N} = -(\psi_j^{K_N}, \eta_\tau^N)_{\partial K_N} - (\psi_j^{K_N}, \eta_f^N)_{\partial K_N}. \tag{B.20}
$$

After obtaining the degrees of freedom β_i^N, we then compute the numerical solution u_{Hh}^N at the time slab I_N performing the linear combination

$$
u_{Hh}^N = \sum_{i=1}^{R} \beta_i^N \eta_i^N + \eta_\tau^N + \eta_f^N, \tag{B.21}
$$

which turns out to be represented by the matrix

$$
\begin{bmatrix}
\sum_{i=1}^R \beta_i^N \eta_i^1 + \eta_\tau^1 + \eta_f^1 \\
\sum_{i=1}^R \beta_i^N \eta_i^2 + \eta_\tau^2 + \eta_f^2 \\
\vdots \\
\sum_{i=1}^R \beta_i^N \eta_i^m + \eta_\tau^m + \eta_f^m
\end{bmatrix} \tag{B.22}
$$

with m rows.

To measure the error, we write the numerical solution as a vector $u_{Hh}^M = [u_{Hh}^N(t_l)]_{l=0}^{Mm}$, where $t_l = l \cdot \delta_N$ are all the time step points t_n of each I_N. Then, knowing the exact solution which we designate here by $u_E(x, t)$, we make use of the following discrete norm on each time slab I_N

$$
||u_E - u_{Hh}^M||_{L^2(0,T;H^r(\mathcal{T}_h))} = \left(\sum_{l=1}^{Mm} ||(u_E - u_{Hh}^N)(t_n)||^2_{H^r(\mathcal{T}_h)} \right)^{\frac{1}{2}}, \quad r = 0, 1. \tag{B.23}
$$

Remark B.2. In order to deal with the boundary conditions of the global problem we can choose between two approaches:

1. The first one is to incorporate the boundary conditions into the local problems (B.7)-(B.9) and not worry with it when assembling the global one;

2. The other one is to obtain the local basis with only the conditions imposing by the local problems and then apply the global boundary conditions to the global problem $A^N \beta^N = B^N$.

Algorithm 1 MHM algorithm

for all $K_N \in \mathcal{T}_h^N$ **do**
  for all $n = 0, \cdots, m$ **do**
    Compute the local solutions η_f^n, η_i^n and η_τ^n, for each $i = 1, \cdots, m_\Lambda^{K_N}$
  end for
  Compute the local terms A^{K_N} and B^{K_N}
  Assemble the local terms into A and B, respectively
end for
Solve the global linear system

$$A^N \beta^N = B^N$$

for all $K_N \in \mathcal{T}_h^N$ **do**
  Extract the local solution vector $\lambda_{N,H}^K \in \mathbb{R}^{m_\Lambda^{K_N}}$ from $\lambda_{N,H}$
  for all $n = 0, \cdots, m$ **do**
    Compute the global solution

$$u_{Hh}^N(t_n) = \sum_{i=1}^{m_\Lambda^K} \beta_i^N \, \eta_i^n + \eta_f^n + \eta_\tau^n.$$

  end for
end for

APPENDIX C – Sobolev Spaces

This appendix reinforce some definitions and important results of Sobolev spaces.

Definition C.1. *A function u belongs to the space $H^m(\Omega)$ if for each multiindex $|\alpha| \leq m$, the weak derivative $D^\alpha u$ exists and belongs to $L^2(\Omega)$. Moreover, we can define the norms in these spaces as*

- $||u||_{H^m(\Omega)} := \left(\sup_{|\alpha|\leq m} \int_\Omega |D^\alpha u|^2 dx\right)^{\frac{1}{2}}$ *if* $1 \leq m < \infty$,
- $||u||_{H^\infty(\Omega)} := ess\ sup_\Omega |D^\alpha u|$ *if* $m = \infty$.

We know from the theory ((CIORANESCU; CIORANESCU; DONATO, 1999), (EVANS, 2010)) that the spaces $H^m(\Omega)$ are Hilbert spaces. Other important Sobolev spaces used to define solution of parabolic type PDE's are the time dependent ones, defined in sequence,

Definition C.2. *The space $L^p(0,T;X)$, where X is a Banach space, consists of all strongly measurable functions from $[0,T] \to X$ such that*

i) $||u||_{L^p(0,T;X)} := \left(\int_0^T (||u(t)||)^p\right)^{\frac{1}{p}} < \infty$ *if* $1 \leq p < \infty$ *and*

ii) $||u||_{L^\infty(0,T;X)} := ess\ sup_{0\leq t\leq T} ||u(t)|| < \infty$.

When functions of such spaces have more regularity in time, we can define the following spaces,

Definition C.3. *The Sobolev space $H^r(0,T;X)$, X a Banach space, consists of all functions $u \in L^2(0,T;X)$ such that $\dfrac{d^l u}{dt^l}$ exists for $0 \leq l \leq r$ and belong to $L^2(0,T;X)$ in the weak sense. Furthermore, the norm on those spaces are given by*

$$||u||_{H^r(0,T;X)} := \left(\int_0^T \sum_{l=0}^r \left\|\frac{d^l u(t)}{dt^l}\right\|^2 dt\right)^{\frac{1}{2}}.$$

Some other important spaces that will show up in the next sections are:

Definition C.4. *i) The dual space of $H_0^1(\Omega)$ or $H^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ (the dual spaces of each one of them are different but the notation will be the same);*

ii) The space $H(div,\Omega)$ consists of all functions $u \in [L^2(\Omega)]^n$ such that $\nabla \cdot u \in L^2(\Omega)$ and the norm is defined as

$$||u||_{H(div,\Omega)} := (||u||^2_{L^2(\Omega)} + ||\nabla \cdot u||^2_{L^2(\Omega)})^{\frac{1}{2}};$$

iii) The dual space of $L^2(0,T;H)$, H a Hilbert space, is the space $L^2(0,T;H')$ where H' is the dual of H.

Item *iii)* is a result from Proposition 3.59 of (CIORANESCU; CIORANESCU; DONATO, 1999).

The following Lemma can be seen as a generalization of Lax-Milgram Theorem for time dependent functions, called Lions' Projection Lemma, proved by Jacques-Louis Lions in (LIONS, 2013).

Lemma C.5. *Let H be a Hilbert space and $\Phi \subset H$ a dense subset. Let $a : H \times \Phi \to \mathbb{R}$ be a bilinear form with the following properties:*

i) for every $\phi \in \Phi$, the linear functional $u \mapsto a(u,\phi)$ is continuous in H;

ii) there exists $\alpha > 0$ such that

$$a(\phi,\phi) \geq \alpha ||\phi||_H^2 \quad \text{for all} \quad \phi \in \Phi.$$

Then, for each $f \in H'$, there exists a unique $u \in H$ such that

$$a(u,\phi) = \langle f,\phi \rangle \quad \text{para todo} \quad \phi \in \Phi$$

and

$$||u||_H \leq \frac{1}{\alpha}||f||_{H'}.$$

For functions in H^1 that do not vanish on the boundary of the domain, a characterization is needed since we have to perform integration of those functions over the boundary. To establish the nature of such functions over the boundary's domain, we have the following Theorem, whose proof is in (CIORANESCU; CIORANESCU; DONATO, 1999):

Theorem C.6. *Let Ω be an open bounded set in $\mathbb{R}^n$ such that $\partial\Omega$ is Lipschitz continuous. Then, there exists an unique continuous linear map*

$$\eta : H^1(\Omega) \to L^2(\partial\Omega),$$

such that for any $u \in H^1(\Omega) \cap C(\overline{\Omega})$ one has $\eta(u) = u|_{\partial\Omega}$. The function $\eta(u)$ is the trace of u on $\partial\Omega$.

From this result, whenever we need to deal with functions u in H^1 on the boundary of the domain, we take its representative $\eta(u)$, called *trace of u*, which lives in the space:

Definition C.7. *Let $\partial\Omega$ be Lipschitz continuous. We define the set $H^{\frac{1}{2}}(\partial\Omega)$ as the range of the map η from Theorem C.6, i.e.,*

$$H^{\frac{1}{2}}(\partial\Omega) = \eta(H^1(\Omega)).$$

When we obtain the variational formulation of PDE's with natural boundary conditions, the following Green's identity come in handy and will be very useful in the next sections. The proof of such identity can be seen in Theorem 3.33 of (CIORANESCU; CIORANESCU; DONATO, 1999).

Theorem C.8. *Let $\Omega \in \mathbb{R}^N$ an open bounded set and $\partial\Omega$ a Lipschitz continuous boundary. For any $u, v \in H^1(\Omega)$ and $1 \leq i \leq N$ we have*

$$\int_\Omega u \frac{\partial v}{\partial x_i}\, dx = -\int_\Omega \frac{\partial u}{\partial x_i} v\, dx + \int_{\partial\Omega} \eta(u)\eta(v)\mathbf{n}_i\, ds \tag{C.1}$$

or, in vector form,

$$\int_\Omega u \cdot \nabla v\, dx = -\int_\Omega \nabla u \cdot v\, dx + \int_{\partial\Omega} \eta(u)\eta(v) \cdot \mathbf{n}\, ds \tag{C.2}$$

where $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_N)$ denotes the unit outward normal vector of Ω.

Remark C.9. Instead of writing $\eta(u)\eta(v)$ on $\partial\Omega$, it is usual to write the integral over $\partial\Omega$ without indicating the trace map, i.e.,

$$\int_{\partial\Omega} u\, v \cdot \mathbf{n}_i\, ds.$$