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HOLOMORPHIC FOLIATIONS OF DEGREE FOUR ON THE COMPLEX PROJECTIVE SPACE

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FOLHA DE APROVAÇÃO

Holomorphic foliations of degree four on the complex projective space

VÂNGELLIS OLIVEIRA SAGNORI MAIA

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Resumo

Neste trabalho, estudaremos folheações holomorfas de grau quatro no espaço projetivo complexo $\mathbb{P}^n$, com $n \geq 3$, com especial foco em obter um teorema estrutural para essas folheações. Mais ainda, para uma folheação $\mathcal{F}$ de grau $d \geq 4$ com $k°$-jato suficientente alto, provamos que $\mathcal{F}$ é transversalmente afim fora de uma hipersuperfície compacta, ou $\mathcal{F}$ é transversalmente projetiva fora de uma hipersuperfície compacta, ou $\mathcal{F}$ é o Pull-back de uma folheação em $\mathbb{P}^2$ por um mapa racional.

Palavras-chaves: folheação holomorfa; integral primeira racional; estrutura transversal afim; estrutura transversal projetiva pura; pull-back de folheações; seqüencias de godbillon-vey.

Abstract

In this work, we study holomorphic foliations of degree four on complex projective space $\mathbb{P}^n$, where $n \geq 3$, with a special focus on obtaining a structural theorem for these foliations. Furthermore, for a foliation $\mathcal{F}$ of degree $d \geq 4$ with a sufficiently high k^{th}-jet, we prove that either $\mathcal{F}$ is transversely affine outside a compact hypersurface, or $\mathcal{F}$ is transversely projective outside a compact hypersurface, or $\mathcal{F}$ is the pull-back of a foliation on $\mathbb{P}^2$ by a rational map.

Key words: holomorphic foliation; rational first integral; affine transverse structure; pure projective transverse structure; pull-back of foliations; godbillon-vey sequences.

Contents

1 Introduction

The study of Holomorphic Foliations has attracted the attention of many mathematicians and has been gaining more attention in recent decades. For instance, foliations are known to play an important role in the study of subvarieties of projective varieties. One beautiful example is a result given by Bogomolov [1] about the famous Green-Griffiths-Lang conjecture. On the other hand, techniques from algebraic geometry have been extremely useful in the study singular holomorphic foliations, for instance, J.-P. Jouanolou, in his celebrated Lectures Notes [11] proved that a generic vector field of degree than one on the complex projective plane does not admit any invariant algebraic curve.

Specifically, in Brazil, the theory of holomorphic foliations has had its beginnings with the works of Ivan Kupka, Airton Medeiros, César Camacho, Jacob Palis, Alcides Lins Neto, Paulo Sad, Marcio Soares, among others. They were important to consolidate a very active research area in Brazil.

Among the attention that researchers of holomorphic foliations dedicate to the theory, much effort has been given to the problem of classifying holomorphic foliations into complex manifolds, in particular, to foliations on complex projective spaces, mostly in codimension one foliations. More specifically, an example of such interest is the following conjecture which is attributed to different authors (Marco Brunella, Alcides Lins Neto,...):

Main Conjecture. Any codimension one holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^n$, with $n \geq 3$,

(*) either $\mathcal{F}$ admits a transverse projective structure with poles on some invariant hypersurface

(**) or $\mathcal{F}$ is a pull-back of a holomorphic foliation $\mathcal{G}$ on $\mathbb{P}^2$ by a rational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$.

The concepts used in the previous conjecture will be explained throughout this work. We emphasize that a codimension one singular holomorphic foliation $\mathcal{F}$ on the complex projective space $\mathbb{P}^n$ have a special characteristic, in an affine chart $\mathbb{C}^n \subset \mathbb{P}^n$, it can be defined by the vanishing of a 1-form, $\omega = 0$, which is integrable, that is, satisfying $\omega \wedge d\omega = 0$, whose coefficients are complex polynomials. If we establish the *degree* d of $\mathcal{F}$ as the number of tangencies (counted with multiplicity) of a generic linearly embedded $\mathbb{P}^1$ with $\mathcal{F}$ then we can consider the space of codimension one holomorphic foliations of a specific degree. The Zariski closure of this set can be identified with an algebraic set and therefore naturally has irreducible components.

With a focus on explaining these irreducible components to certain degrees, several works have been done and much progress has already been made. The first case, which could be considered is the space of codimension one foliations of degree-zero on $\mathbb{P}^n$, has already been proved that it has only one component and it is isomorphic to the Grassmannian of lines in $\mathbb{P}^n$, a proof of this fact can be found in [9]. In $\mathbb{P}^n$, $n \geq 3$, the space of holomorphic foliations of degree-one has two irreducible components, this fact was proved by Jouanolou in [11].

Much later, Dominique Cerveau and Alcides Lins Neto resumed studies on irreducible components and published a result that brought studies on irreducible components back into the spotlight. In [5], they proved that in $\mathbb{P}^n$, $n \geq 3$, the space of codimension one foliations of degree-two has six irreducible components, but more than that, they explicitly show the generic element of each one of these irreducible components. These components are called *Linear pull-back foliations, Rational components, Logarithmic components, and a Exceptional component.* Cerveau-Lins Neto's work was an invaluable contribution and served as a motivation for several researchers to focus their studies on the classification of irreducible components of the space of holomorphic foliations on complex projective spaces.

Despite the advances in these studies, the classification problem of irreducible components of the space of holomorphic foliations is not simple to be solved. For instance, after the complete description of the irreducible components of the degree-two foliations space on $\mathbb{P}^n$, with $n \geq 3$, it has not yet been possible to explain all the irreducible components of the space of codimension one holomorphic foliations on $\mathbb{P}^n$, $n \geq 3$, of degree $d \geq 3$. However, Cerveau and Lins Neto has proved a structural theorem for degree-three foliations in [6]:

Theorem 1. *Let $\mathcal{F}$ be a holomorphic codimension one foliation of degree-three on $\mathbb{P}^n$, with $n \geq 3$. Then:*

- *either $\mathcal{F}$ has a rational first integral,*
- *or $\mathcal{F}$ has an affine transverse structure with poles on an invariant hypersurface,*
- *or $\mathcal{F} = \Phi^*(\mathcal{G})$, where $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a rational map and $\mathcal{G}$ is a foliation on $\mathbb{P}^2$.*

According to Cerveau and Lins Neto [6], the main conjecture seems to be reasonable (at least for codimension one foliations of small degree) for the following reasons: first of all, if $\mathbb{K}$ is a field of positive characteristic, every codimension one holomorphic foliation on a projective manifold over $\mathbb{K}$, in particular on $\mathbb{P}^n_{\mathbb{K}}$, is defined by a closed global 1-form (see for instance [7, Section 6]). On the other hand, if $\mathcal{F}$ is a codimension one foliation on $\mathbb{P}^n$ and p is a prime number then it is possible to define $\mathcal{F}_p$, the reduction modulo p of $\mathcal{F}$. The idea to construct $\mathcal{F}_p$ is the following: both $\mathcal{F}$ and $\mathbb{P}^n$, can be defined over a finitely generated $\mathbb{Z}$-algebra R and then we can reduce modulo the prime ideal $\langle p \rangle \subset R$ to obtain a foliation on a variety over a field of characteristic p, (see for instance Shepherd-Barron [19]). Having now the foliation $\mathcal{F}_p$, there is a conjecture of *Grothendieck-Katz*-type which says that if for almost all p the foliation $\mathcal{F}_p$ has a non-constant rational first integral then $\mathcal{F}$ itself has a non-constant rational first integral. In [6], Cerveau and Lins Neto announcement a result due to F. Touzet:

Theorem 2 (F. Touzet)**.** *The Grothendieck-Katz conjecture implies that any foliation of degree $\leq n-1$ on $\mathbb{P}^n$, either admits a projective transverse structure, or is a pull-back of some foliation on $\mathbb{P}^k$, $k < n$, by some rational map.*

Recently in [15], F. Loray, J. V. Pereira and F. Touzet proved a more accurate structural theorem for codimension one foliations of degree-three on $\mathbb{P}^3$, and in [8], Raphael Constant da Costa, Ruben Lizarbe and J. V. Pereira extend this result to $\mathbb{P}^n$, $n \geq 3$, as follows:

Theorem 3. *If $\mathcal{F}$ is a codimension one singular holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$, of degree three. Then*

- *either $\mathcal{F}$ is defined by a closed rational 1-form without codimension one zeros;*
- *or there exists an algebraically integrable codimension two foliation of degree one tangent to $\mathcal{F}$;*
- *or $\mathcal{F}$ is a linear pull-back of a degree-three foliation on $\mathbb{P}^2$;*
- *or $\mathcal{F}$ admits a rational first integral.*

Furthermore, Da Costa, Lizarbe and Pereira [8, Theorem B] provide a complete list of the irreducible components of the space of foliations of degree-three on $\mathbb{P}^n$, $n \geq 3$, whose general elements do not admit a rational first integral:

Theorem 4. *The space of codimension one foliations of degree-three on $\mathbb{P}^n$, $n \geq 3$, has exactly 18 distinct irreducible components whose general elements correspond to foliations which do not admit a rational first integral.*

However, even in [8], a complete classification of the irreducible components of the space of foliations of degree-three on $\mathbb{P}^n$, $n \geq 3$ is not known, but they establish the following result.

Theorem 5. *The space of codimension one foliations of degree-three on $\mathbb{P}^n$, $n \geq 3$, has at least 24 distinct irreducible components.*

Motivated by the above results, this work is devoted to study of codimension one holomorphic foliations of degree four on $\mathbb{P}^n$, with $n \geq 3$. One of the goals of this thesis is to obtain a structural theorem for degree-four foliations on $\mathbb{P}^n$, $n \geq 3$, similar to Cerveau-Lins Neto's theorem [6, Theorem 1].

Our main theorem is the following:

Theorem A. *Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree four on $\mathbb{P}^n$, with $n \geq 3$. Then,*

(i) either $\mathcal{F}$ admits a rational first integral;

(ii) or $\mathcal{F}$ is transversely affine outside a compact hypersurface;

(iii) or $\mathcal{F}$ is a pure transversely projective outside a compact hypersurface;

(iv) or $\mathcal{F} = \Phi^*(\mathcal{G})$, where $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a rational map and $\mathcal{G}$ is a holomorphic foliation on $\mathbb{P}^2$.

(v) or there exists a birational map $\Psi : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ such that the foliation $\Psi^*(\mathcal{F})$ is defined by a 1-form described as follows:

$$\eta_t = \beta_0 + t\beta_1 + t^2\beta_2 + t^3\beta_3 + t^4\beta_4 - tdt,$$

where the 1-forms β_j do not depend on $t \in \mathbb{P}^1$, for all $0 \leq j \leq 4$.

In order to prove our main result, we will use the tools of the proof of [6] and techniques concerning foliations which admit a finite *Godbillon-Vey* sequence (cf. [3] and [7]).

Note that in order to confirm the Main Conjecture for degree four codimension one foliations on $\mathbb{P}^n$, $n \geq 3$, it is necessary to prove that item (v) is equivalent to some of the previous items. We hope to prove this fact in the near future. One of the difference between Theorem A and the structural theorem of degree three foliations given by Cerveau-Lins Neto [6] is that pure transversely projective foliations exist. That is, there are foliations of degree four with projective transverse structure that is not affine. See the example given in [7, Section 5.4].

Now, let us focus on foliations of degree $d \geq 4$. Let $\mathcal{F}$ be a degree d foliation on $\mathbb{P}^n$. Then, $\mathcal{F}$ can be represented in an affine coordinate system $\mathbb{C}^n \simeq E \subset \mathbb{P}^n$ by an integrable polynomial 1-form

$$\omega_E = \sum_{j=0}^{d+1} \omega_j$$

where the coefficients of the 1-forms ω_j are polynomials homogeneous of degree j, $0 \leq j \leq d+1$, and $i_R(\omega_{d+1}) = 0$. Given $p \in E$, let $j_p^k(\omega_E)$ be the k^{th}-jet of ω_E at p, and let

$$\mathcal{J}(\mathcal{F}, p) = min\{k \geq 0 : j_p^k(\omega_E) \neq 0\}.$$

Note that $\mathcal{J}(\mathcal{F}, p)$ depends only on p and $\mathcal{F}$, and not on E and ω_E. Moreover, the singular set of $\mathcal{F}$ is given by

$$(\mathcal{F}) = \{p \in \mathbb{P}^n : \mathcal{J}(\mathcal{F}, p) \geq 1\}.$$

It is well-known that $(\mathcal{F})$ is an algebraic set and always contains irreducible components of codimension two, (cf. [12]).

Motivated by the family of foliations of [7, Section 5.4], where the first k^{th}-jets ($k < 3$) of the 1-form defining this family are all zero, we propose the following structural theorem for foliations of degree $d \geq 4$ in $\mathbb{P}^n$, with $n \geq 3$.

Theorem B. *Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree $d \geq 4$ on $\mathbb{P}^n$, with $n \geq 3$. Suppose that one of the two conditions is satisfied:*

1. *for all $p \in (\mathcal{F})$, we have $\mathcal{J}(\mathcal{F}, p) = 1$;*
2. *there exists $p \in (\mathcal{F})$ such that $\mathcal{J}(\mathcal{F}, p) \geq d-1$.*

Then,

- *(i) either $\mathcal{F}$ admits a rational first integral;*
- *(ii) or $\mathcal{F}$ is transversely affine outside a compact hypersurface;*
- *(iii) or $\mathcal{F}$ is a pure transversely projective outside a compact hypersurface;*
- *(iv) or $\mathcal{F} = \Phi^*(\mathcal{G})$, where $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a rational map and $\mathcal{G}$ is a holomorphic foliation on $\mathbb{P}^2$.*

The Thesis is organized as follows: in Chapter 2, we define the concept of holomorphic foliations and state some important results about codimension one foliations on $\mathbb{P}^n$. We also define the concept of the affine and projective transverse structure of a foliation. Moreover, we define the notion of a Godbillon-Vey sequence and establish important results of foliations that admit a Godbillon-Vey sequence with finite length. Chapter 1 ends with some formulas of indices of foliations that we will use throughout this text. Chapter 3 is devoted to prove Theorem A. In order to prove our main result, we will divide it into several lemmas. Our proof is given step to step. Each step is according to length of a Godbillon-Vey sequence adapted to a degree-four foliation. Finally, in Chapter 4, we will establish some open problems related to Theorem A.

2 Holomorphic Foliations

In this chapter, we will state some definitions and results well known in Foliation theory on complex manifolds that will be useful to prove our results in the forward chapters.

2.1 Preliminaries about Foliations

We start this section introducing the basic concepts of holomorphic foliations on complex manifolds. The dimension of the complex manifolds in this work always be the complex dimension, unless otherwise noted.

Definition 2.1. Let M be a complex manifold of dimension $n \geq 2$. A holomorphic foliation of dimension $1 \leq k < n$, is a decomposition $\mathcal{F}$ of M in complex submanifolds (call the leaves of the foliation $\mathcal{F}$) of dimension k, biunivocally immersed, with the following proprieties:

(i) $\forall p \in M$ there is only one submanifold L_p of the decomposition passing through p;

(ii) $\forall p \in M$ there is a holomorphic chart of M, called distinguished chart of $\mathcal{F}$, (φ, U), $p \in U$, $\varphi : U \to \varphi(U) \subset \mathbb{C}^n$, such that $\varphi(U) = P \times Q$, where P and Q are open subsets in $\mathbb{C}^k$ and $\mathbb{C}^{n-k}$ respectively;

(iii) If L is a leaf of $\mathcal{F}$ such that $L \cap U \neq \emptyset$, then $L \cap U = \bigcup_{q \in D_{L,U}} \varphi^{-1}(P \times \{q\})$, where $D_{L,U}$ is a countable subset of Q.

The subsets U of the form $\varphi(P \times \{q\})$ are called distinguished chart plaques (φ, U).

Remark 2.1. A holomorphic foliation $\mathcal{F}$ of dimension k in M induces a distribution of planes of dimension k on M, denoted by $T\mathcal{F}$, which is defined by

$$T_p\mathcal{F} = T_p(L_p) = \text{tangent space of the leaf } L_p \text{ of } \mathcal{F} \text{ at } p.$$

It follows from (iii) that the distribution $T\mathcal{F}$ is holomorphic. It defines a holomorphic vector subbundle of the holomorphic tangent bundle TM of M.

There are other two ways to define foliations, equivalent to the above definition, are as follows:

Proposition 2.1 (Lins Neto - Scárdua [13]). *A dimension k holomorphic foliation $\mathcal{F}$ on M can also be defined in the following equivalent ways:*

(1) Description given by distinguished charts: $\mathcal{F}$ is given by a holomorphic atlas of M, $\mathcal{A} = \{(\varphi_\alpha, U_\alpha) : \alpha \in I\}$ where

(1.1) $\varphi_\alpha(U_\alpha) = P_\alpha \times Q_\alpha$, where P_α, Q_α are open subsets of $\mathbb{C}^k$ and $\mathbb{C}^{n-k}$ respectively.

(1.2) If $U_\alpha \cap U_\beta \neq \emptyset$, then the change of charts $\varphi_\beta \circ \varphi_\alpha^{-1}$ is locally of the form

$$\varphi_\beta \circ \varphi_\alpha^{-1}(x_\alpha, y_\alpha) = (h_{\alpha\beta}(x_\alpha, y_\alpha), g_{\alpha\beta}(y_\alpha))$$

In this case the plaques of $\mathcal{F}$ in U_α are the subsets of the form $\varphi_\alpha^{-1}(P_\alpha \times \{q\})$.

(2) Description by local submersions: $\mathcal{F}$ is given by an open cover $M = \bigcup_{\alpha \in I} U_\alpha$ and by collections $\{y_\alpha\}_{\alpha \in I}$ and $\{g_{\alpha\beta}\}_{U_\alpha\beta \neq \emptyset}$ that satisfy:

(2.1) $\forall \alpha \in I$, $y_\alpha : U_\alpha \to \mathbb{C}^{n-k}$ is a holomorphic submersion.

(2.2) If $U_\alpha \cap U_\beta \neq \emptyset$, then $y_\alpha = g_{\alpha\beta}(y_\beta)$ where $g_{\alpha\beta} : y_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^k \to y_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^k$ is a local biholomorphism.
In this case the plaques of $\mathcal{F}$ in U_α are subsets of the form $y_\alpha^{-1}(q)$, $q \in V_\alpha = y_\alpha(U_\alpha)$.

Definition 2.2. Given two complex manifolds M and N, a holomorphic map $f : M \to N$ and a holomorphic foliation $\mathcal{F}$ in N of codimension k, we say that f is transversal to $\mathcal{F}$, if for every point $q \in N$, the vector subspaces $df_q(T_qM)$ and $T_p\mathcal{F}$ generate the tangent space T_pN, where $p = f(q)$.

Another important concept that we will use is pullback foliation.

Definition 2.3. Let M and N be complex manifolds and $f : M \to N$ be a holomorphic map transversal to a foliation $\mathcal{F}$ in N of codimension k. Then there is a holomorphic foliation $f^*(\mathcal{F})$ in M, of codimension k, whose leaves are the connected components of the inverse images of the leaves L of $\mathcal{F}$, $f^{-1}(L)$ in N. The foliation $f^*(\mathcal{F})$ is called pullback foliation of $\mathcal{F}$ under f.

Pullback foliations are a type of foliations that are very important in the study of foliation theory, they form an important family in the space of holomorphic foliations that we will discuss later.

Let M be a complex manifold of dimension n, and ω be a non-identically zero holomorphic 1-form in M. Let $Sing(\omega) = \{p \in M : \omega_p = 0\}$ be the *singular set* of ω. In this case, ω induces a holomorphic distribution of hyperplanes Ω in the open $N = M \backslash Sing(\omega)$, defined by:

$$\Omega_p = ker(\omega_p) = \{v \in T_pM : \omega_p(v) = 0\}.$$

Definition 2.4. We say that ω (or Ω) is *integrable*, if there is a holomorphic foliation $\mathcal{F}$ in N such that $T\mathcal{F} = \Omega$. In other words, the tangent space of the leaf of $\mathcal{F}$ at p coincides with Ω_p.

A characterization that replaces the above definition is the well-known Frobenius Theorem which tells us that ω is integrable if, and only if, $\omega \wedge d\omega = 0$. This characterization will be used a lot in this work. We commonly say that the foliation $\mathcal{F}$ is defined by the differential equation $\omega = 0$.

Remark 2.2. If η is a holomorphic 1-form such that $\eta = f\omega$, where f is a holomorphic map on N that does not vanish, then the hyperplane distribution induced by η coincides with ω, in particular, the foliation induced by $\omega = 0$ coincides with the foliation induced by $\eta = 0$.

The next result is what allows us to locally use holomorphic 1-forms to define codimension one holomorphic foliations by working with them rather than dealing with the foliation itself.

Proposition 2.2 (Lins Neto - Scárdua [13])**.** *Let M be a complex manifold of dimension $n \geq 2$ and $\mathcal{F}$ be a holomorphic foliation of codimension one on M. Then there are collections $\mathcal{W} = \{\omega_\alpha\}_{\alpha \in I}$, $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ and $\mathcal{G} = \{g_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \emptyset}$ such that:*

(i) $\mathcal{U}$ is an open cover of M.

(ii) ω_α is an integrable holomorphic 1-form into U_α that does not vanish at any point.

(iii) $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, set of holomorphic functions that does not vanish in $U_\alpha \cap U_\beta$.

(iv) In $U_\alpha \cap U_\beta \neq \emptyset$, we have $\omega_\alpha = g_{\alpha\beta} \cdot \omega_\beta$.

(v) If $p \in U_\alpha$, then $T_p\mathcal{F} = ker(\omega_\alpha(p))$.

Conversely, if there are collections $\mathcal{W}$, $\mathcal{U}$ and $\mathcal{G}$ satisfying (*i*), (*ii*), (*iii*) *and* (*iv*), *then there is a holomorphic foliation $\mathcal{F}$ on M that satisfies* (*v*)*.*

In view of the Proposition 2.2, we can define *singular codimension one holomorphic foliations* as follows:

Definition 2.5. Let M be a complex manifold of dimension $n \geq 2$. A singular codimension one holomorphic foliation on M is an object $\mathcal{F}$ given by collections $\{\omega_\alpha\}_{\alpha \in I}$, $\{U_\alpha\}_{\alpha \in I}$ and $\{g_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \emptyset}$ such that:

(i) $\{U_\alpha\}_{\alpha \in I}$ is a open cover of M.

(ii) ω_α is a holomorphic 1-form integrable in U_α that does not identically zero in U_α.

(iii) $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$.

(iv) If $U_\alpha \cap U_\beta \neq \emptyset$, then $\omega_\alpha = g_{\alpha\beta} \cdot \omega_\beta$ in $U_\alpha \cap U_\beta$.

For each 1-form ω_α, let us consider its singular set given by:

$$Sing(\omega_\alpha) = \{p \in U_\alpha : \omega_\alpha(p) = 0\} := S_\alpha$$

Note that S_α is a complex subvariety in U_α, $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ and $\omega_\alpha = g_{\alpha\beta} \cdot \omega_\beta$ in $U_\alpha \cap U_\beta \neq \emptyset$, then we have $S_\alpha \cap U_\alpha \cap U_\beta = S_\beta \cap U_\alpha \cap U_\beta$, hence

$$Sing(\mathcal{F}) := \bigcup_{\alpha \in I} S_\alpha$$

is a complex subvariety in M.

Codimention one holomorphic foliations with singularities is an important field of study in Foliation theory and in this work we will deal only with this type of foliations.

Definition 2.6. Let $\mathcal{F}$ be a codimension one holomorphic foliation in M. A *meromorphic (holomorphic) first integral* of $\mathcal{F}$ is a non-constant meromorphic (holomorphic) function in M, say f, such that f is constant along the leaves of $\mathcal{F}$.

Remark 2.3. If $\mathcal{F}$ is a codimension one holomorphic foliation, given by an integrable holomorphic 1-form ω in M, then a meromorphic (holomorphic) function f is a first integral of $\mathcal{F}$ if, and only if , $\omega \wedge df \equiv 0$.

2.2 Interior product and Lie Derivative

Let M be a connected compact complex manifold of dimension n. We shall denote by Θ_M the set of holomorphic vector fields over M.

Definition 2.7. Let $X \in \Theta_M$ be a holomorphic vector field on M. We will denote by $i_X(\omega)$ the contraction of $\omega \in \Omega^k_M$, $1 < k \leq n$, in the direction of the vector field X or the interior product of the vector field X and the form ω, it is defined as follows:

$$\begin{aligned} i_X : \Omega^k_M &\longrightarrow \Omega^{k-1}_M \\ \omega &\mapsto i_X(\omega) \end{aligned} \tag{2.1}$$

where $i_X(\omega)_p(v_1, \cdots, v_{k-1}) = \omega_p(X(p), v_1, \cdots, v_{k-1})$, for each $p \in M$ and $v_1, \cdots, v_{k-1} \in T_pM$.

The following result compiles two important properties of the interior product.

Proposition 2.3 (Tu [20]). *For a vector field $X \in \Theta_M$, $i_X : \Omega^*_M \longrightarrow \Omega^{*-1}_M$, we have the following properties:*

(i) $i_X \circ i_X = 0$*;*

(ii) For $\omega \in \Omega^k_M$ *and* $\eta \in \Omega^t_M$,

$$i_X(\omega \wedge \eta) = i_X(\omega) \wedge \eta + (-1)^k \omega \wedge i_X(\eta).$$

Now, we present the concepts of Lie derivative of a form in the direction of a vector field that will be of great importance for the proof of the main result in the following chapter.

Let X be a vector field on M, then there is a neighborhood U of a point $p \in M$, where X has a local flow, that is, there exists a small disk Δ centered at $0 \in \mathbb{C}$ and a holomorphic map

$$\varphi : \Delta \times U \to M$$

such that if we put $\varphi_t(q) = \varphi(t, q)$, then

$$\frac{\partial}{\partial t}\varphi_t(q) = X(\varphi_t(q)), \quad \varphi_0(q) = q \;\; for\ q \in U.$$

Definition 2.8. For X a holomorphic vector field and $\omega \in \Omega^k_M$ a k-form on a complex manifold M, the Lie derivative $\mathcal{L}_X(\omega)$ at $p \in M$ is

$$\mathcal{L}_X(\omega)_p = \lim_{t \to 0} \frac{\varphi_t^*(\omega_{\varphi_t(p)}) - \omega_p}{t} = \lim_{t \to 0} \frac{(\varphi_t^*\omega)_p - \omega_p}{t} = \frac{d}{dt}\bigg|_{t=0} (\varphi_t^*\omega)_p.$$

If $X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j}$ is a local holomorphic vector field on M, then for each holomorphic function f on M we have

$$X(f) = \sum_{j=1}^{n} a_j \frac{\partial f}{\partial z_j}.$$

Some calculations that will be made in the next chapter are facilitated by the following proposition:

Proposition 2.4 (Tu [20]). *If f is a holomorphic map and X a vector field on M, then*

$$\mathcal{L}_X(f) = X(f).$$

Now we state the main properties of the Lie derivative.

Theorem 6 (Tu [20]). *Assume that X is a holomorphic vector field on a complex manifold M.*

(i) The Lie derivative $\mathcal{L}_X : \Omega^*_M \to \Omega^*_M$ is a derivation: it's a $\mathbb{C}$-linear map and if $\omega \in \Omega^k_M$ and $\rho \in \Omega^l_M$, then

$$\mathcal{L}_X(\omega \wedge \rho) = (\mathcal{L}_X\omega) \wedge \rho + \omega \wedge (\mathcal{L}_X\rho).$$

(ii) The Lie derivative $\mathcal{L}_X$ commutes with the exterior derivative d.

(iii) (Cartan's magic formula) $\mathcal{L}_X = di_X + i_Xd$.

And to finish this section we have a global formula for Lie derivative.

Theorem 7 (Tu [20])**.** *For a holomorphic k-form ω and vector fields $X, Y_1, \cdots, Y_k$ on a complex manifold M, we have*

$$(\mathcal{L}_X\omega)(Y_1, \cdots, Y_k) = X(\omega(Y_1, \cdots, Y_k)) - \sum_{i=1}^{k} \omega(Y_1, \cdots, [X, Y_i], \cdots, Y_k).$$

2.3 Codimension one holomorphic foliations on the complex projective space

We will denote complex projective space by $\mathbb{P}^n$ as there will be no confusion with the real projective space as we will not mention it in this text. We will also use the notation $\mathcal{F}_1(n, d)$ instead of $\mathcal{F}_1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d + 2))$ for the space of codimension one holomorphic foliations of degree d in the complex projective space of dimension n.

One of the main results that we will use for codimension one holomorphic foliations on $\mathbb{P}^n$ is the following:

Theorem 8 (Lins Neto - Scárdua [13])**.** *Let $\mathcal{F}$ be a codimension one holomorphic foliation on $\mathbb{P}^n$ and $\mathcal{F}^* = \Pi^*(\mathcal{F})$, where $\Pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ is the canonical projection. Then there exists an integrable holomorphic 1-form in $\mathbb{C}^{n+1}$,*

$$\omega = \sum_{j=0}^{n} \omega_j dx_j$$

whose coefficients $\omega_0, \cdots, \omega_n$ are homogeneous complex polynomials of the same degree, such that $\omega = 0$ defines $\mathcal{F}^*$ in $\mathbb{C}^{n+1}\backslash\{0\}$. In particular, for every affine chart $E \subset \mathbb{P}^n$, $\mathcal{F}_{|_E}$ can be defined by an integrable polynomial 1-form.

We say that the 1-form ω represents $\mathcal{F}$ in homogeneous coordinates of $\mathbb{P}^n$.

Remark 2.4. Since $\Pi^{-1}([p])$ is a line that passes through the origin of $\mathbb{C}^{n+1}$ for every $[p] \in \mathbb{P}^n$, we have every line $\Pi^{-1}([p])$ is contained in the leaves of $\mathcal{F}^*$. In terms of the 1-form ω this can be expressed by the relationship:

$$i_R(\omega) = \sum_{j=0}^{n} x_j\omega_j \equiv 0$$

where $R = \sum_{j=0}^{n} x_j \frac{\partial}{\partial x_j}$ denotes the radial vector field in $\mathbb{C}^{n+1}$.

Let us fix a holomorphic foliation $\mathcal{F}$ of codimension one and a line $L \subset \mathbb{P}^n$, not invariant by $\mathcal{F}$, that is, such that L is not contained in a leaf of $\mathcal{F}$ nor in $sing(\mathcal{F})$. Let $p \in L$ and take an affine chart $\mathbb{C}^n \simeq E$ such that $p \in E$. Let ω be a complex polynomial 1-form representing $\mathcal{F}$ in E. We say that p is a *tangency point* of $\mathcal{F}$ with L, if the restriction $\omega_{|_L}$ vanishes at 0, (here p is identify with the origin 0 of $E \simeq \mathbb{C}^n$). The *tangency multiplicity* of $\mathcal{F}$ with L at p is, by definition, the order at p as zero of $\omega_{|_L}$. It is easy to prove that the above concepts are independent of the affine chart E and the 1-form ω which represents $\mathcal{F}$. With this, the following definition is natural:

Definition 2.9. The degree of a codimension one holomorphic foliation $\mathcal{F}$ in $\mathbb{P}^n$, is the number of tangencies, counted with multiplicity, of $\mathcal{F}$ with a generic non-invariant line by $\mathcal{F}$.

Remark 2.5. Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree d in $\mathbb{P}^n$ and ω be a 1-form representing $\mathcal{F}$ in homogeneous coordinates. Suppose that $cod(sing(\mathcal{F})) \geq 2$ (codimension of the singular set of $\mathcal{F}$). We have:

1. If ω_1 is another 1-form that represents $\mathcal{F}$ in homogeneous coordinates, then $\omega_1 = a\omega$, where $a \in \mathbb{C}^*$.
2. The degree of the coefficients of ω is $d+1$.

Note that Remark 2.5 implies that the space of codimension one holomorphic foliations of degree d on $\mathbb{P}^n$ is naturally identified with the projectivized of the following set of 1-forms in $\mathbb{C}^{n+1}$:

$$\mathcal{F}_1(n,d) = \mathbb{P}\left(\{\omega : \omega \wedge d\omega = 0,\ i_R(\omega) = 0,\ \omega = \sum_{j=0}^{n} \omega_j dx_j, \quad \text{and} \quad cod(sing(\omega)) \geq 2\} \right)$$

where ω_j are homogeneous polynomials of degree $d+1$. Note that $\mathcal{F}_1(n,d)$ can be seen as an algebraic subset of a space of complex polynomials.

Let us give some examples of foliations on $\mathbb{P}^n$.

Exemple 2.1 (Foliations with rational first integral)**.** Let $P, Q : \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomials such that $\deg(P) = \deg(Q) = k \geq 1$. Then $\omega = QdP - PdQ$ defines a codimension-one foliation $\mathcal{F}$ on $\mathbb{P}^n$ with a rational first integral P/Q.

Exemple 2.2 (Foliations associated to closed meromorphic 1-forms)**.** If ω is a closed meromorphic 1-form on $\mathbb{P}^n$, $n \geq 2$, then it define a codimension-one holomorphic foliation on $\mathbb{P}^n$. By [14, Proposition. 1.2.5], we have ω has a decomposition

$$\omega = \sum_i \lambda_i \frac{df_i}{f_i} + dh,$$

where the λ_i's are complex numbers and the f_i's and h are rational functions. The leaves are (outside the singular set of the foliation) the connected components of the level sets of the multivalued function $\sum_i \lambda_i \log f_i + h$.

2.4 Affine and projective transverse structures

Let $\mathcal{F}$ be a codimension one holomorphic foliation on a complex manifold M of dimension n, with singular set $sing(\mathcal{F})$. Such foliation can be given outside its singular set by an atlas of holomorphic submersions $y_i : U_i \to \mathbb{C}$ such that if $U_i \cap U_j \neq \emptyset$, then $y_i = g_{ij}(y_j)$, for some biholomorphism g_{ij} between open subsets of $\mathbb{C}$.

Definition 2.10. We say that $\mathcal{F}$ is *transversely affine* or $\mathcal{F}$ admits an *affine transverse structure*, if it is possible to choose an atlas of submersions as above $\{y_i : U_i \to \mathbb{C}\}_{i\in I}$, defining $\mathcal{F}$ in $M\backslash sing(\mathcal{F})$, whose changes of coordinates are *affine*, that is, $y_i = a_{ij}y_j + b_{ij}$ over $U_i \cap U_j \neq \emptyset$, where a_{ij}, b_{ij} are constants.

The problem of deciding the existence of transverse structures for a given foliation is equivalent, in certain cases, to a problem in 1-forms, as shown in the following result:

Theorem 9 (Lins Neto - Scárdua [13])**.** *Let $\mathcal{F}$ be a codimension one holomorphic foliation on a complex manifold M. Suppose that $\mathcal{F}$ can be defined by a meromorphic 1-form, that is, that there exists an integrable meromorphic 1-form ω, which defines $\mathcal{F}$ outside its pole divisor, $(\omega)_\infty$. The foliation $\mathcal{F}$ is transversely affine in the open $U\backslash sing(\mathcal{F})$ if and only if there is a meromorphic 1-form η in M satisfying the following properties:*

(a) η *is closed;*

(b) $d\omega = \eta \wedge \omega$;

(c) $(\eta)_\infty = (\omega)_\infty$;

(d) The order of the pole of η *along any irreducible component of* $(\eta)_\infty$ *is one;*

(e) For every irreducible component L *of* $(\omega)_\infty$*, we have* $Res(\eta, L) = -(\text{order of } (\omega)_\infty|_L)$.

Furthermore, two pairs (Ω, η) and (Ω', η') define the same affine structure for $\mathcal{F}$ in U if and only if there is a meromorphic function $g : M \to \overline{\mathbb{C}}$ satisfying $\Omega' = g\Omega$ and $\eta' = \eta + \dfrac{dg}{g}$.

Definition 2.11. We shall call a pair (ω, η) that satisfies the properties of Theorem 9 for a foliation $\mathcal{F}$ defined by ω of *affine pair*.

Definition 2.12. Let $\mathcal{F}$ be a codimension one holomorphic foliation in M. We say that $\mathcal{F}$ is *transversely projective* on M or $\mathcal{F}$ admits a *projective transverse structure* if it is possible to choose an atlas of holomorphic submersions $y_i : U_i \to \mathbb{C}$ adapted to $\mathcal{F}$ such that $M \backslash sing(\mathcal{F}) = \bigcup_i U_i$ and having projective relations,

$$y_i = \frac{a_{ij} y_j + b_{ij}}{c_{ij} y_j + d_{ij}}$$

over $U_i \cap U_j \neq \emptyset$, where $a_{ij}, b_{ij}, c_{ij}, d_{ij} : U_i \cap U_j \to \mathbb{C}$ are locally constants with $a_{ij}d_{ij} - b_{ij}c_{ij} = 1$ in $U_i \cap U_j$.

As in the affine case, there is a formulation of the existence of projective transverse structures for a foliation in terms of 1-forms:

Theorem 10 (Lins Neto - Scárdua [13])**.** *Let $\mathcal{F}$ be a codimension one holomorphic foliation on M given by an integrable holomorphic 1-form ω, suppose that there is a holomorphic 1-form η in M such that $d\omega = \eta \wedge \omega$. The foliation $\mathcal{F}$ is transversely projective in M if and only if there is a holomorphic 1-form ξ in M satisfying:*

(i) $d\eta = \omega \wedge \xi$;

(ii) $d\xi = \xi \wedge \eta$.

Furthermore, (ω, η, ξ) and (ω', η', ξ') are two triple that define the same projective structure for $\mathcal{F}$ if, and only if:

$$\begin{aligned} \omega' &= f\omega \\ \eta' &= \eta + \frac{df}{f} + 2g\omega \\ \xi' &= \frac{1}{f}(\xi - 2dg - 2g\eta - 2g^2\omega) \end{aligned}$$

for some holomorphic functions $f : M \to \mathbb{C}^*$ e $g : M \to \mathbb{C}$.

Definition 2.13. A tripe (ω, η, ξ) of 1-meromorphic forms in M is called a *projective triple* if it satisfies the projective relations:

$$\begin{aligned} d\omega &= \eta \wedge \omega \\ d\eta &= \omega \wedge \xi \\ d\xi &= \xi \wedge \eta \end{aligned}$$

Let $\mathcal{F}$ be a codimension one holomorphic foliation in M. We say that (ω, η, ξ) is a projective triple for $\mathcal{F}$, if $\mathcal{F}$ is defined by ω outside the pole divisor $(\omega)_\infty$.

Note that given a codimension one holomorphic foliation $\mathcal{F}$ that admits a projective transverse structure, it can also have an affine transversely structure, but in general this does not happen. On the other hand, if the foliation $\mathcal{F}$ admits an affine transversely structure we can say that it admits a projective transversely structure because, given the affine pair (ω, η) that defines the affine transverse structure, we can define a natural projective triple (ω', η', ξ), where $\omega = \omega'$, $\eta = \eta'$ and $\xi \equiv 0$. Thus, we have the following definition:

Definition 2.14. Given a codimension one holomorphic foliation $\mathcal{F}$, we say that it admits a *pure projective transverse structure* when it admits a projective transversely structure but does not admit an affine transversely structure.

We also have a local normal form for a foliation which admits a projective transverse structure, see for instance [10, III 3.16]:

Theorem 11. *Given a codimension one foliation $\mathcal{F}$ defined on M, where M is a complex manifold and U an open subset in M, suppose there is a projective triple (ω, η, ξ) for $\mathcal{F}$, we can then find in the neighborhood of each point of U holomorphic functions (f, g, h), such that:*

$$\begin{cases} \omega = & -gdf \\ \eta = & \frac{dg}{g} - h\omega \\ \xi = & \frac{1}{2}h^2\omega + h\eta + dh \end{cases}$$

If (f', g', h') is another triple satisfying the same relations, then $f' = \varphi \circ f$ for some automorphism φ.

To continue, we give an example of a foliation with a pure projective transverse structure which not admit an affine transverse structure.

Exemple 2.3. A Riccati foliation $\mathcal{F}$ in $\mathbb{P}^2$ in an affine chat $(x, y) \in \mathbb{C}^2 \hookrightarrow \mathbb{P}^2$ is given by the polynomial 1-form $\Omega = p(x)dy - (y^2c(x) - yb(x) - a(x))dx$ where p, a, b and c are polynomials. Let us now calculate a projective triplet for Ω.

Suppose $\eta = Adx + Bdy$, where A and B are meromorphic functions in $\mathbb{C}^2$ such that $d\Omega = \eta \wedge \Omega$. On the one hand,

$$d\Omega = (p'(x) + 2yc(x) - b(x))dx \wedge dy.$$

On the other hand, we have

$$\begin{aligned} \eta \wedge \omega &= [Adx + Bdy] \wedge [p(x)dy - (y^2c(x) - yb(x) - a(x))dx] \\ &= Ap(x)dx \wedge dy + B(y^2 - yb(x) - a(x))dx \wedge dy \\ &= [Ap(x) + B(y^2 - yb(x) - a(x))]dx \wedge dy \end{aligned} \tag{2.2}$$

Choosing $B = 0$ we get

$$\begin{aligned}
&Ap(x)dx \wedge dy = (p'(x) + 2yc(x) - b(x))dx \wedge dy \\
&\Rightarrow Ap(x) = p'(x) + 2yc(x) - b(x) \\
&\Rightarrow A = \frac{p'(x) + 2yc(x) - b(x)}{p(x)} \qquad (2.3)
\end{aligned}$$

Therefore, $\eta = \dfrac{p'(x) + 2yc(x) - b(x)}{p(x)}dx$. Now we have $d\eta = \frac{2c(x)}{p(x)}dx \wedge dy$ and in a similar way to what we did above let us define $\xi = Cdx + Ddy$ where D and C are meromorphic functions such that $d\eta = \Omega \wedge \xi$ and computing the second part of the last equality

$$\begin{aligned}
\Omega \wedge \xi &= [p(x)dy - (y^2c(x) - yb(x) - a(x))dx] \wedge [Cdx + Ddy] \\
&= -p(x)Cdx \wedge dy - D(y^2c(x) - yb(x) - a(x))dx \wedge dy \qquad (2.4)
\end{aligned}$$

taking $D = 0$, we get

$$\begin{aligned}
&- p(x)Cdx \wedge dy = \frac{2c(x)}{p(x)}dx \wedge dy \\
&\Rightarrow -p(x)C = \frac{2c(x)}{p(x)} \\
&\Rightarrow C = -\frac{2c(x)}{p^2(x)} \qquad (2.5)
\end{aligned}$$

Thus, $\xi = -\frac{2c(x)}{p^2(x)}dx$ and (Ω, η, ξ) it's a projective triple, the last condition being trivially satisfied $d\xi = 0 = \xi \wedge \eta$.

2.5 Godbillon-Vey Sequences

In this section we are going to deal with an important tool that will be widely used in the demonstration of the main result, which is the Godbillon-Vey sequence. The reader may consult references [10] and [3] for a detailed study of Godbillon-Vey sequences.

Definition 2.15. Let $\mathcal{F}$ be a codimension one holomorphic foliation on a complex manifold M. A *Godbillon-Vey Sequence* (abbreviated G-V-S) associated to $\mathcal{F}$ is a sequence of meromorphic 1-forms in M, say $(\Omega_j)_{j\geq 0}$, such that:

1) $\mathcal{F}$ is defined by Ω_0, outside its pole set $(\Omega_0)_\infty$. In particular, Ω_0 is integrable, i.e.,

$$\Omega_0 \wedge d\Omega_0 = 0.$$

2) The 1-form defined by the formal power series

$$\Omega = dz + \Omega_0 + z\Omega_1 + \frac{z^2}{2}\Omega_2 + \sum_{j\geq 3} \frac{z^j}{j!}\Omega_j \qquad (2.6)$$

is integrable.

When there exists $N \in \mathbb{N}$ such that $\Omega_N \neq 0$ but $\Omega_j = 0$ for all $j > N$ then we say that $\mathcal{F}$ admits a finite G-V-S of length N. In this case, the 1-form in (2.6) is meromorphic and can be extended meromorphically to $M \times \mathbb{P}^1$. Since it is integrable, it defines a codimension one foliation $\mathcal{H}$ in $M \times \mathbb{P}^1$ such that $\mathcal{H}|_{M\times\{0\}} = \mathcal{F}$.

Remark 2.6. Let $\mathcal{F}$ and $\mathcal{G}$ be foliations on complex manifolds M and Y, respectively. Suppose that $\mathcal{G}$ admits a finite G-V-S of length N and that $\mathcal{F} = \Phi^*(\mathcal{G})$, where $\Phi : M \dashrightarrow Y$ is a rational map. Then $\mathcal{F}$ also admits a G-V-S of length N. A proof of this fact can be found in [7].

Remark 2.7. When $\mathcal{F}$ admits a G-V-S of length $N \leq 2$ then $\mathcal{F}$ has a projective transverse structure with poles in a hypersurface. When $N = 1$ then the structure is in fact affine, see for instance [18].

For foliations that admit G-V-S of length ≥ 3, we have the following result:

Theorem 12 (Cerveau - Lins-Neto - Loray - Pereira - Touzet [7])**.** *Let $\mathcal{F}$ be a codimension one foliation on a complex manifold M that admits a G-V-S of length $N \geq 3$. Then*

- *or $\mathcal{F}$ is transversely affine;*
- *or there is a compact Riemann surface S, 1-meromorphic forms $\alpha_0, \cdots, \alpha_N$ in S and a rational map $\phi : M \to S \times \mathbb{P}^1$ such that $\mathcal{F}$ is defined by the 1-form $\Omega = \phi^*(\eta)$, where*

$$\eta = dz + \alpha_0 + z\alpha_1 + \cdots + z^N\alpha_N. \tag{2.7}$$

When $M = \mathbb{P}^n$, $n \geq 3$, necessarily $S = \mathbb{P}^1$ and the 1-form in 2.7 can be written as

$$\eta = dz - P(t,z)dt,$$

where $P \in \mathbb{C}(t)[z]$ and $\mathcal{F} = \phi^*(\mathcal{G})$, where $\mathcal{G}$ is defined in $\mathbb{P}^1 \times \mathbb{P}^1$ by the differential equation $\frac{dz}{dt} = P(t,z)$.

There is a very interesting construction of a G-V-S associated to a codimension one foliation when there exists a vector field transverse to it. More precisely, let ω be a 1-form defining $\mathcal{F}$ and assume that there exists a vector field X satisfying $\omega(X) = 1$. Then, the integrability condition of ω is equivalent to

$$\omega \wedge d\omega = 0 \Leftrightarrow d\omega = \omega \wedge \mathcal{L}_X(\omega). \tag{2.8}$$

In fact, from $\mathcal{L}_X\omega = d(\omega(X)) + d\omega(X,\cdot) = d\omega(X,\cdot)$, we derive

$$0 = \omega \wedge d\omega(X,\cdot,\cdot) = \omega(X)\cdot d\omega - \omega \wedge (d\omega(X,\cdot)) = d\omega - \omega \wedge \mathcal{L}_X\omega.$$

The converse is trivial. Applying this identity to the formal 1-form

$$\Omega = dz + \omega_0 + z\omega_1 + \frac{z^2}{2}\omega_2 + \ldots + \frac{z^k}{k!}\omega_k + \ldots \tag{2.9}$$

together with the vector field $X = \partial_z$, we derive

$$\Omega \wedge d\Omega = 0 \Leftrightarrow \sum_{k=0}^{\infty} \frac{z^k}{k!} d\omega_k = \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\omega_k \right) \wedge \left(\sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!}\omega_k \right). \tag{2.10}$$

Therefore, we get from the integrability condition for Ω:

$$\begin{aligned}
d\omega_0 &= \omega_0 \wedge \omega_1 \\
d\omega_1 &= \omega_0 \wedge \omega_2 \\
d\omega_2 &= \omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_2 \\
d\omega_3 &= \omega_0 \wedge \omega_4 + 2\omega_1 \wedge \omega_3 \\
&\cdot \\
&\cdot \\
&\cdot \\
d\omega_k &= \omega_0 \wedge \omega_{k+1} + \sum_{j=1}^{k} \binom{j}{k} \omega_j \wedge \omega_{k+1-j} \\
&\cdot \\
&\cdot
\end{aligned}$$

Hence, if we start with ω and X satisfying $\omega(X) = 1$, then the iterated Lie derivates $\omega_k := \mathcal{L}_X^k \omega$ define a Godbillon-Vey sequence for $\mathcal{F}$. For a recent account of the Godbillon-Vey sequences we refer the reader [7].

2.6 Baum-Bott Theory

Let $\mathcal{F}$ be a one-dimensional holomorphic foliation with isolated singularities in $U \subset \mathbb{C}^2$. Let us fix the holomorphic vector field $X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ that defines $\mathcal{F}$ in U and $\Omega = P(x,y)dy - Q(x,y)dx$ the dual 1-form to X.

Lemma 2.1 (Lins Neto - Scárdua [13])**.** *There exists a (1,0)-form η which is $\mathcal{C}^\infty$ in $V = U \backslash sing(\mathcal{F})$ satisfying the following properties:*

(a) $d\Omega = \eta \wedge \Omega$*;*

(b) $\eta \wedge d\eta$ *is closed;*

(c) the cohomology class of $\eta \wedge d\eta$ in $H^3_{DR}(V)$ depends only of $\mathcal{F}$.

Let p be a singular point of $\mathcal{F}$. Since p is an isolated singularity, let us fix a ball $B = B(P, \rho) \subset U$ such that the only singularity of $\mathcal{F}$ in B is p.

Definition 2.16. The *Baum-Bott index* of $\mathcal{F}$ at p is the complex number

$$BB(\mathcal{F}, p) = Res(\eta \wedge d\eta, p)$$

where $Res(\eta \wedge d\eta, p)$ is the residue of $\eta \wedge d\eta$ at p: let $0 < r < \rho$ and $S_r = S^3(P, r) = \partial B(p, \rho)$ (3-sphere in $\mathbb{C}^2$ centered at p). Then

$$Res(\eta \wedge d\eta, p) = \frac{1}{8V} \int_{S_r} \eta \wedge d\eta,$$

where V is the volume of the ball of radius 1 in $\mathbb{C}^2$ with the Euclidean metric.

Remark 2.8. The integral above is invariant over r, it does not depend on r since $\eta \wedge d\eta$ is closed. In fact for any compact K with regular boundary $M = \partial K$, such that $p \in int(K)$ is the unique singularity of $\mathcal{F}$ in K holds

$$Res(\eta \wedge d\eta, p) = \frac{1}{8V} \int_{S_r} \eta \wedge d\eta = \frac{1}{8V} \int_{M} \eta \wedge d\eta$$

It follows that $BB(\mathcal{F}, p)$ is invariant by change of coordinates, that is, if $\varphi : V \to U$ is a biholomorphism then

$$BB(\varphi^*(\mathcal{F}), \varphi^{-1}(p)) = BB(\mathcal{F}, p).$$

Proposition 2.5 (Lins Neto - Scárdua [13])**.** *Let $\mathcal{F}$ be a holomorphic foliation with isolated singularities in $U \subset \mathbb{C}^2$ and A an open with compact closure $\overline{A} \subset U$, whose boundary ∂A is regular by parts and $\partial A \cap sing(\mathcal{F}) = \emptyset$. Let η be as in the previous Lemma 2.1. Then*

$$\sum_{p \in (sing(\mathcal{F}) \cap A)} BB(\mathcal{F}, p) = \frac{1}{8V} \int_{\partial A} \eta \wedge d\eta.$$

Theorem 13 (Baum-Bott's theorem for foliations on $\mathbb{P}^2$)**.** *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^2$ of degree d, with isolated singularities. Then*

$$\sum_{p \in sing(\mathcal{F})} BB(\mathcal{F}, p) = (d+2)^2.$$

Let $\mathcal{F}$ be a codimension one holomorphic foliations on $\mathbb{P}^n$, $n \geq 3$ and $sing(\mathcal{F})$ its singular set, we supose that $sing(\mathcal{F})$ is an analytic subset of $\mathbb{P}^n$ of codimension at least 2. In this sense, we can consider $sing_2(\mathcal{F})$ the union of the irreducible components of $sing(\mathcal{F})$ whose codimension is precisely 2. This subset $sing_2(\mathcal{F})$ represents the most important part of $sing(\mathcal{F})$ (cf. [2]) and moreover we can state a fundamental lemma.

Lemma 2.2 (Lins Neto [12])**.** *Let $\mathcal{F}$ be a codimension-one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$. Then the singular set $sing(\mathcal{F})$ has non trivial components of complex dimension $n-2$. That is $sing_2(\mathcal{F}) \neq \emptyset$.*

Let $\Gamma \in sing_2(\mathcal{F})$. Given a smooth point $p \in \Gamma$ and a germ of embedding $i : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{P}^n, p)$ transverse to Γ, we define $BB(\mathcal{F}, \Gamma, i, p) := BB(i^*(\mathcal{F}), 0)$. We then have the following result:

Theorem 14 (Cerveau - Lins Neto [6]). *There is a proper analytic subset $\Gamma_1 \subset \Gamma$ such that:*

(a) if $p \in \Gamma \backslash \Gamma_1$ then $BB(\mathcal{F}, \Gamma, i, p)$ does not depend on the embedding $i : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{P}^n, p)$ transverse to Γ. We will then denote $BB(\mathcal{F}, \Gamma, p) := BB(\mathcal{F}, \Gamma, i, p)$;

(b) the map $p \in \Gamma \backslash \Gamma_1 \longmapsto BB(\mathcal{F}, \Gamma, p) \in \mathbb{C}$ is constant.

We denote by $BB(\mathcal{F}, \Gamma) := BB(\mathcal{F}, \Gamma, p)$, where $p \in \Gamma \backslash \Gamma_1$.

3 Foliations of degree four on $\mathbb{P}^n$, $n \geq 3$

According to Cerveau [4], all known foliations $\mathcal{F}$ (of codimension-one) on $\mathbb{P}^n$, $n \geq 3$, satisfy the following alternative:

(*) either $\mathcal{F}$ admits a projective transverse structure

(**) or $\mathcal{F}$ is a rational pull-back of a foliation $\mathcal{F}_0$ on $\mathbb{P}^2$.

It is not known if the previous alternative is always satisfied or if there exist other types of foliations on $\mathbb{P}^n$. In this chapter we will present some results about a classification of codimension one foliations of degree four in $\mathbb{P}^n$. In our context, we will show that codimension one holomorphic foliations of degree four on $\mathbb{P}^n$, $n \geq 3$, satisfy the previous alternative, up to rational first integral.

The main result of this chapter is as follows:

Theorem A. *Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree four on $\mathbb{P}^n$, with $n \geq 3$. Then,*

(i) either $\mathcal{F}$ admits a rational first integral;

(ii) or $\mathcal{F}$ is transversely affine outside a compact hypersurface;

(iii) or $\mathcal{F}$ is a pure transversely projective outside a compact hypersurface;

(iv) or $\mathcal{F} = \Phi^*(\mathcal{G})$, where $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a rational map and $\mathcal{G}$ is a holomorphic foliation on $\mathbb{P}^2$.

(v) or there exists a birational map $\Psi : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ such that the foliation $\Psi^*(\mathcal{F})$ is defined by a 1-form described as follows:

$$\eta_t = \beta_0 + t\beta_1 + t^2\beta_2 + t^3\beta_3 + t^4\beta_4 - tdt,$$

where the 1-forms β_j do not depend on $t \in \mathbb{P}^1$, for all $0 \leq j \leq 4$.

We emphasize that Theorem A has been motivated by the main result of Cerveau - Lins Neto [6], (see Theorem 15 to continuation). They provided a classification of codimension one holomorphic foliations of degree three on the complex projective space.

Theorem 15 (Cerveau - Lins Neto [6])**.** *Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree three on $\mathbb{P}^n$, $n \geq 3$. Then:*

(i) either $\mathcal{F}$ has a rational first integral;

(ii) or $\mathcal{F}$ has an affine transversely structure with poles on some an invariant hypersurface;

(iii) or $\mathcal{F} = g^*(\mathcal{G})$, where $g : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a rational map and $\mathcal{G}$ a holomorphic foliation on $\mathbb{P}^2$.

By Definition 2.9 and Theorem 8, a holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^n$ of degree d can be represented by an affine coordinate system $\mathbb{C}^n \simeq E \subset \mathbb{P}^n$ by an integrable polynomial 1-form $\omega_E = \sum_{j=0}^{d+1} \omega_j$ where the coefficients of the 1-forms ω_j are polynomials homogenous of degree j, $0 \leq j \leq d+1$, and $i_R(\omega_{d+1}) = 0$, where R is the radial vector field. The 1-form ω_E can be considered as a meromorphic 1-form in $\mathbb{P}^n$ with poles of order $d+2$ in the hyperplane at infinity of E. Given $p \in E$, we define the set

$$\mathcal{J}(\mathcal{F}, p) = min\{k \geq 0 \mid j^k_p(\omega_E) \neq 0\}.$$

It is proved that $\mathcal{J}(\mathcal{F}, p)$ depends only on p and $\mathcal{F}$ and does not depends on E or ω_E.

Another important theorem due to Cerveau and Lins Neto in [6] that we will use as an important tool is the following result:

Theorem 16 (Cerveau - Lins Neto [6])**.** *Let $\mathcal{F}$ be a codimension one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$. Assume that $sing(\mathcal{F})$ has an irreducible component of codimension two Γ such that*

(i) $BB(\mathcal{F}, \Gamma) \neq 0$;

(ii) The algebraic set $\{p \in \Gamma | \mathcal{J}(\mathcal{F}, p) > 1\}$ has codimension four in $\mathbb{P}^n$.

Then $\mathcal{F}$ has a rational first integral.

Remark 3.1. As a consequence of Lemma 2.2 that $sing(\mathcal{F})$ has at least one component of codimension two, say Γ, such that $BB(\mathcal{F}, \Gamma) \neq 0$, where $\mathcal{F}$ is a codimension one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$.

As a consequence we have the following corollary:

Corollary 3.1 (Cerveau - Lins Neto [6])**.** *Let $\mathcal{F}$ be a codimension one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$. If $\mathcal{J}(\mathcal{F}, p) \leq 1$, $\forall\ p \in \mathbb{P}^n$, then $\mathcal{F}$ has a rational first integral.*

Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree four in $\mathbb{P}^n$, $n \geq 3$. In order to prove Theorem A, we consider two possibilities: either there exists a point $p \in \mathbb{P}^n$ such that $\mathcal{J}(\mathcal{F}, p) \geq 2$, or $\mathcal{J}(\mathcal{F}, p) = 1$ for all $p \in \mathbb{P}^n$. In the second case, we have $\mathcal{F}$ admits a rational first integral by Corollary 3.1. Hence we do not noting to prove in Theorem A.

Therefore, we shall assume that there exists a point $p \in \mathbb{P}^n$ such that $\mathcal{J}(\mathcal{F}, p) \geq 2$. Taking affine coordinates $(z_1, \cdots, z_n) \in \mathbb{C}^n \subset \mathbb{P}^n$, where $p = 0 \in \mathbb{C}^n$, we can assume that $\mathcal{F}|_{\mathbb{C}^n}$ is given by the polynomial 1-form

$$\omega = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \tag{3.1}$$

where α_j are homogeneous polynomial 1-forms of degree j, with $j = 2, 3, 4, 5$ and

$$i_R(\alpha_5) = 0, R = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}. \tag{3.2}$$

We now look at the pull-back of ω by the blow-up of $\mathbb{P}^n$ at zero $0 \in \mathbb{C}^n \subset \mathbb{P}^n$. Let $\pi : \tilde{\mathbb{P}}^n \to \mathbb{P}^n$ be the punctual blow-up at $0 \in \mathbb{C}^n \subset \mathbb{P}^n$. We calculate $\pi^*(\mathcal{F})$ in the chart

$$(\tau_1, \cdots, \tau_{n-1}, x) = (\tau, x) \in \mathbb{C}^{n-1} \times \mathbb{C} \mapsto (x\tau, x) = (z_1, \cdots, z_n) \in \mathbb{C}^n \subset \mathbb{P}^n.$$

Putting $\alpha_j(z) = \sum_{i=1}^{n} P_{ji}(z) dz_i$ and writing

$$F_j(\tau, 1) = \sum_{i=1}^{n-1} P_{j-1i}(\tau, 1)\tau_i + P_{j-1i}(\tau, 1),$$

where P_{j-1i} are homogeneous polynomials of degree $j-1$, we obtain

$$\begin{aligned} \pi^*(\omega) &= \pi^*(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \\ &= x^2[x\theta_2 + x^2\theta_3 + x^3\theta_4 + x^4\theta_5 + (F_3(\tau, 1) + xF_4(\tau, 1) + x^2F_5(\tau, 1))dx] \end{aligned} \tag{3.3}$$

where $\theta_j = \sum_{i=1}^{n-1} P_{ji}(\tau, 1) d\tau_i$ and it is important to note that $F_6(\tau, 1) = i_R(\alpha_5) \equiv 0$ by equation (3.2).

Now, we will use the following notation $F_i(\tau, 1) := F_i$. Under the conditions described above we have the following possibilities for ω and $F_i's$:

1) $i_R(\omega) = 0$ or equivalently $F_3 = F_4 = F_5 = 0$;

2) $F_4 = F_5 = 0$ and $F_3 \neq 0$;

3) $F_3 = F_4 = 0$ and $F_5 \neq 0$;

4) Possibilities solved in an analogous way:
   a) $F_3 = F_5 = 0$ and $F_4 \neq 0$;
   b) $F_3 = 0$ and $F_4 \neq 0 \neq F_5$;
5) $F_5 = 0$ and $F_3 \neq 0 \neq F_4$;
6) $F_4 = 0$ and $F_3 \neq 0 \neq F_5$;
7) $F_3 \neq 0$ and $F_4 \neq 0$ and $F_5 \neq 0$.

We will explore these possibilities through the following lemmas considering ω and $\mathcal{F}$ under the conditions described above unless otherwise noted.

Note that

$$\begin{aligned} i_R(\omega) =& i_R(\omega) \\ =& i_R(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \\ =& i_R(\alpha_2) + i_R(\alpha_3) + i_R(\alpha_4) + i_R(\alpha_5) \\ =& F_3 + F_4 + F_5 + 0 = 0 \end{aligned}$$

How $F_i's$ are polinimial maps of diferent degrees, we have $F_i = 0$, for all $i = 3, 4, 5$.

Lemma 3.1 (Case 1)**.** *Suppose that $\mathcal{F}|_{\mathbb{C}^n}$ is defined by ω as in (3.1) and $i_R(\omega) = 0$ or, equivalently, $F_3 = F_4 = F_5 = 0$. Then $\mathcal{F}$ is a pullback of a foliation on $\mathbb{P}^{n-1}$ by a linear map.*

Proof. First of all, it follows from (3.1) that $\alpha_5 \neq 0$, otherwise $\mathcal{F}$ would have a degree less than or equal to 3, an absurd. Now, note that the integrability condition of ω implies that

$$\begin{aligned} \omega \wedge d\omega = 0 &\Rightarrow i_R(\omega \wedge d\omega) = 0 \\ &\Rightarrow i_R(\omega) \wedge d\omega - \omega \wedge i_R(d\omega) = 0. \end{aligned} \tag{3.4}$$

Hence, since $i_R(\omega) = 0$, we have

$$\omega \wedge i_R(d\omega) = 0. \tag{3.5}$$

On the other hand, by Cartan's magic formula

$$\begin{aligned} \mathcal{L}_R(\omega) &= d i_R(\omega) + i_R(d\omega) \\ \mathcal{L}_R(\omega) &= i_R(d\omega), \end{aligned} \tag{3.6}$$

thus, the Lie derivative of ω with respect to radial vector field R is

$$\begin{aligned} \mathcal{L}_R(\omega) &= \mathcal{L}_R(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \\ &= \mathcal{L}_R(\alpha_2) + \mathcal{L}_R(\alpha_3) + \mathcal{L}_R(\alpha_4) + \mathcal{L}_R(\alpha_5) \\ &= 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5. \end{aligned} \tag{3.7}$$

Applying (3.5) and (3.6), we get

$$\begin{aligned} (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \wedge (3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5) &= 4\alpha_2 \wedge \alpha_3 + 5\alpha_2 \wedge \alpha_4 + 6\alpha_2 \wedge \alpha_5 \\ &+ 3\alpha_3 \wedge \alpha_2 + 5\alpha_3 \wedge \alpha_4 + 6\alpha_3 \wedge \alpha_5 \\ &+ 3\alpha_4 \wedge \alpha_2 + 4\alpha_4 \wedge \alpha_3 + 6\alpha_4 \wedge \alpha_5 \\ &+ 3\alpha_5 \wedge \alpha_2 + 4\alpha_5 \wedge \alpha_3 + 5\alpha_5 \wedge \alpha_4 \\ &= \alpha_2 \wedge \alpha_3 + 2\alpha_2 \wedge \alpha_4 + 3\alpha_2 \wedge \alpha_5 \\ &+ \alpha_3 \wedge \alpha_4 + 2\alpha_3 \wedge \alpha_5 + \alpha_4 \wedge \alpha_5 = 0 \end{aligned} \tag{3.8}$$

From the last equality follows

$$\alpha_2 \wedge \alpha_3 = \alpha_2 \wedge \alpha_4 = \alpha_2 \wedge \alpha_5 = \alpha_3 \wedge \alpha_4 = \alpha_3 \wedge \alpha_5 = \alpha_4 \wedge \alpha_5 = 0.$$

Since the coefficients of α_j's are homogeneous polynomials of degree j, each of these outer products generates a homogeneous 2-form polynomial of a degree different from each other, so none of them can be a combination of the others. Since $\alpha_5 \neq 0$ and

$$\alpha_2 \wedge \alpha_5 = \alpha_3 \wedge \alpha_5 = \alpha_4 \wedge \alpha_5 = 0$$

then there exists meromorphic functions f_j, $j = 2, 3, 4$, such that $\alpha_j = f_j \alpha_5$. We assert that $f_j = 0$ for all $j = 2, 3, 4$. Indeed, suppose by contradiction that some $f_j \neq 0$. Since

$$\omega = f_2\alpha_5 + f_3\alpha_5 + f_4\alpha_5 + \alpha_5 = (f_2 + f_3 + f_4 + 1)\alpha_5,$$

we have that the coefficients of $(f_2 + f_3 + f_4 + 1)\alpha_5$ could be of a degree different than 5, it is an absurd, because $\mathcal{F}$ has degree 4 by hypothesis. The assertion is proved.

Therefore, all $f_j = 0$ and hence $\alpha_j = 0$, for all $j = 2, 3, 4$. In particular, we have $\omega = \alpha_5$, since α_5 is integrable (by Theorem 8) it defines a foliation of degree 4 on $\mathbb{P}^{n-1}$ which we shall denote by $\mathcal{F}_{n-1}$. Whereas $\mathbb{P}^{n-1}$ is the set of lines that pass through $0 \in \mathbb{C}^n \subset \mathbb{P}^n$, there exists a natural projection $\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$ such that

$$\mathcal{F} = \pi^*(\mathcal{F}_{n-1}).$$

Finally, $\mathcal{F}$ is a linear pull-back from a codimension foliation on $\mathbb{P}^{n-1}$ of four degree. □

Lemma 3.2 (Case 2)**.** *Suppose that $F_4 = F_5 = 0$ and $F_3 \neq 0$. Then, either $\mathcal{F}$ has an affine transverse structure, or $\mathcal{F}$ is a pullback by a rational map of a foliation on $\mathbb{P}^2$.*

Proof. Putting $\beta_j = \dfrac{\theta_{j+1}}{F_3}$, it follows from (4.4) that

$$x^2\pi^*(\omega) = x\theta_2 + x^2\theta_3 + x^3\theta_4 + x^4\theta_5 + F_3 dx$$

$$\eta = \frac{x^2\pi^*(\omega)}{F_3} = x\beta_1 + x^2\beta_2 + x^3\beta_3 + x^4\beta_4 + dx. \tag{3.9}$$

Note that β_j, for all $1 \leq j \leq 4$, does not depend on x. We can obtain a finite G-V-S

$$(\eta_0 = \eta, \eta_1, \eta_2, \eta_3, \eta_4),$$

where $\eta(\frac{\partial}{\partial x}) = 1$, $\eta_j = \mathcal{L}^j_{\partial_x}(\eta)$, here $\mathcal{L}^j_{\partial_x}(\eta)$ denotes the j-th Lie derivative along ∂_x of the form η.
We have two subcases:
Subcase 1. ($\beta_4 = 0$ and $\beta_3 \neq 0$) or ($\beta_4 \neq 0$). In this case, $\mathcal{F}$ admits a finite G-S-V of length ≥ 3. Therefore, Theorem 12 implies that either $\mathcal{F}$ has an affine transverse structure or it is a pull-back by a rational map of a foliation on $\mathbb{P}^2(\mathbb{P}^1 \times \mathbb{P}^1$ birrational to $\mathbb{P}^2)$.
Subcase 2. $\beta_4 = \beta_3 = 0$. In this case, $\mathcal{F}$ admits a finite G-V-S of length ≤ 2, where

$$\eta = dx + x\beta_1 + x^2\beta_2.$$

Taking the change of coordinates $x = \frac{1}{w}$, we get $\Omega = -\frac{\eta}{w^2} = dw - \beta_2 - w\beta_1$ and

$$d\Omega = -\, d\beta_2 - dw \wedge \beta_1 - w d\beta_1 \tag{3.10}$$

We assert that β_1 is closed, that is, $d\beta_1 = 0$. In fact, we have

$$-\beta_1 \wedge \Omega = -\beta_1 \wedge (dw - \beta_2 - w\beta_1) = -\beta_1 \wedge dw + \beta_1 \wedge \beta_2. \tag{3.11}$$

From the integrability condition of Ω, we get

$$\begin{aligned} 0 = \Omega \wedge d\Omega &= (dw - \beta_2 - w\beta_1) \wedge (-d\beta_2 - dw \wedge \beta_1 - w d\beta_1) \\ &= -dw \wedge d\beta_2 - w dw \wedge d\beta_1 + \beta_2 \wedge d\beta_2 - \beta_2 \wedge \beta_1 \wedge dw \\ &\quad + w\beta_2 \wedge d\beta_1 + w\beta_1 \wedge d\beta_2 + w^2\beta_1 \wedge d\beta_1 \end{aligned} \tag{3.12}$$

Using Lie derivative $\mathcal{L}^2_{\partial_w}(\Omega \wedge d\Omega) = 2\beta_1 \wedge d\beta_1 = 0 \Rightarrow w^2\beta_1 \wedge d\beta_1 = 0$. Again using Lie derivative

$$\mathcal{L}_{\partial_w}(\Omega \wedge d\Omega) = -dw \wedge d\beta_1 + \beta_2 \wedge d\beta_1 + \beta_1 \wedge d\beta_2 = 0 \tag{3.13}$$

From the equality (3.13), we obtain that $dw \wedge d\beta_1 = 0$, because $\beta_2 \wedge d\beta_1 + \beta_1 \wedge d\beta_2$ does not depend on dw. Since $dw \neq 0$, and $d\beta_1$ does not depend on w, we get $d\beta_1 = 0$ and the assertion is proved.

Therefore (3.10) it really comes down to

$$d\Omega = -d\beta_2 - dw \wedge \beta_1. \tag{3.14}$$

Let us show that equations (3.14) and (3.11) are equals, which implies that Ω has an affine transversely structure, and therefore $\mathcal{F}$ has too. To show the equality is sufficient to prove that

$$d\beta_2 = \beta_2 \wedge \beta_1.$$

It follows from (3.13) that $\beta_2 \wedge d\beta_1 + \beta_1 \wedge d\beta_2 = 0 \Rightarrow w\beta_2 \wedge d\beta_1 + w\beta_1 \wedge d\beta_2 = 0$. Then we summarize $\Omega \wedge d\Omega$ in

$$\Omega \wedge d\Omega = -dw \wedge d\beta_2 + \beta_2 \wedge d\beta_2 - \beta_2 \wedge \beta_1 \wedge dw = 0$$

of the above equal $\beta_2 \wedge d\beta_2 = 0$, as the other parcels depend on dw. Finally

$$(d\beta_2 - \beta_2 \wedge \beta_1) \wedge dw = 0 \Rightarrow d\beta_2 - \beta_2 \wedge \beta_1 = 0 \Rightarrow d\beta_2 = \beta_2 \wedge \beta_1. \tag{3.15}$$

As consequence from (3.10) and (3.11), we get

$$d\Omega = -\beta_1 \wedge \Omega.$$

This implies that η has an affine transverse structure and thus $\mathcal{F}$ has too. $\square$

Lemma 3.3 (Case 3)**.** *Suppose $F_4 = F_3 = 0$ and $F_5 \neq 0$. Then either $\mathcal{F}$ has an affine transverse structure, or $\mathcal{F}$ is a pullback by a rational map of a foliation on $\mathbb{P}^2$, or $\mathcal{F}$ has a pure projective transverse structure.*

Proof. From hypothesis, we obtain

$$\begin{aligned} x^{-3}\pi^*(\omega) &= \theta_2 + x\theta_3 + x^2\theta_4 + x^3\theta_5 + xF_5dx \\ \Rightarrow \eta = \frac{\pi^*(\omega)}{F_5x^3} &= \frac{\theta_2}{F_5} + \frac{x\theta_3}{F_5} + \frac{x^2\theta_4}{F_5} + \frac{x^3\theta_5}{F_5} + xdx \end{aligned} \tag{3.16}$$

Applying a transformation on η by the map $\psi(\tau, z) = (\tau, \frac{1}{z}) = (\tau, x)$ we get

$$\begin{aligned} \psi^*(\eta) &= \left[\frac{\theta_2}{F_5} + \frac{\theta_3}{zF_5} + \frac{\theta_4}{z^2F_5} + \frac{\theta_5}{z^3F_5} - \frac{1}{z^3}dz\right] \\ \Rightarrow \psi^*(\eta) &= -\frac{1}{z^3}\left[-\frac{z^3\theta_2}{F_5} - \frac{z^2\theta_3}{F_5} - \frac{z\theta_4}{F_5} - \frac{\theta_5}{F_5} + dz\right] \\ \Rightarrow -z^3\psi^*(\eta) &= \left[z^3\frac{-\theta_2}{F_5} + z^2\frac{-\theta_3}{F_5} + z\frac{-\theta_4}{F_5} + \frac{-\theta_5}{F_5} + dz\right] \\ \Rightarrow -z^3\psi^*(\eta) &= \beta_0 + z\beta_1 + z^2\beta_2 + z^3\beta_3 + dz, \end{aligned}$$

where $\beta_0 = -\dfrac{\theta_5}{F_5}$, $\beta_1 = -\dfrac{\theta_4}{F_5}$, $\beta_2 = -\dfrac{\theta_3}{F_5}$ and $\beta_3 = -\dfrac{\theta_2}{F_5}$.

Note that, $-z^3\psi^*(\eta)$ and therefore η admits a finite G-V-S of length ≥ 3 if $\beta_3 \neq 0$. Hence, in this case, by applying Theorem 12, $\mathcal{F}$ has an affine transverse structure or it is pullback by a rational map of a foliation on $\mathbb{P}^2$.

On the other hand, suppose that $\beta_3 = 0$, that is, $\theta_2 = 0$. It follows from (3.16) that

$$\begin{aligned} x^{-3}\pi^*(\omega) &= \theta_2 + x\theta_3 + x^2\theta_4 + x^3\theta_5 + xF_5 dx \\ \Rightarrow \eta = \frac{x^{-4}\pi^*(\omega)}{F_5} &= \beta_0 + x\beta_1 + x^2\beta_2 + dx \end{aligned} \tag{3.17}$$

where $\beta_j = \dfrac{\theta_{j+3}}{F_5}$ for $j = 0, 1, 2$. Calculating now the iterated Lie derivatives with respect to the vector field ∂_x we have:

$$\mathcal{L}_{\partial_x}(\eta) = \beta_1 + 2x\beta_2 \tag{3.18}$$

$$\mathcal{L}^2_{\partial_x}(\eta) = 2\beta_2 \tag{3.19}$$

From the integrability condition of η, we obtain

$$\begin{aligned} 0 = \eta \wedge d\eta =& (\beta_0 + x\beta_1 + x^2\beta_2 + dx) \wedge (d\beta_0 + dx \wedge \beta_1 + xd\beta_1 + 2xdx \wedge \beta_2 + x^2 d\beta_2) \\ =& \beta_0 \wedge d\beta_0 + \beta_0 \wedge dx \wedge \beta_1 + x\beta_0 \wedge d\beta_1 + 2x\beta_0 \wedge dx \wedge \beta_2 + x^2\beta_0 \wedge d\beta_2 \\ &+ x\beta_1 \wedge d\beta_0 + x^2\beta_1 \wedge d\beta_1 + 2x^2\beta_1 \wedge dx \wedge \beta_2 + x^3\beta_1 \wedge d\beta_2 \\ &+ x^2\beta_2 \wedge d\beta_0 + x^2\beta_2 \wedge dx \wedge \beta_1 + x^3\beta_2 \wedge d\beta_1 + x^4\beta_2 \wedge d\beta_2 \\ &+ dx \wedge d\beta_0 + xdx \wedge d\beta_1 + x^2 dx \wedge d\beta_2 \\ =& \beta_0 \wedge d\beta_0 + \beta_0 \wedge dx \wedge \beta_1 + dx \wedge d\beta_0 \\ &+ x\beta_0 \wedge d\beta_1 + 2x\beta_0 \wedge dx \wedge \beta_2 + x\beta_1 \wedge d\beta_0 + xdx \wedge d\beta_1 \\ &+ x^2\beta_0 \wedge d\beta_2 + x^2\beta_1 \wedge d\beta_1 + 2x^2\beta_1 \wedge dx \wedge \beta_2 + x^2\beta_2 \wedge d\beta_0 + x^2\beta_2 \wedge dx \wedge \beta_1 + x^2 dx \wedge d\beta_2 \\ &+ x^3\beta_1 \wedge d\beta_2 + x^3\beta_2 \wedge d\beta_1 \\ &+ x^4\beta_2 \wedge d\beta_2. \end{aligned} \tag{3.20}$$

Now let us calculate some iterates of the Lie derivatives of $\eta \wedge d\eta$ with respect to the vector field ∂_x. First, by calculating $\mathcal{L}^4_{\partial_x}(\eta \wedge d\eta)$, we have

$$\beta_2 \wedge d\beta_2 = 0. \tag{3.21}$$

Calculating now $\mathcal{L}^3_{\partial_x}(\eta \wedge d\eta)$ and using (3.21) we have

$$\beta_1 \wedge d\beta_2 + \beta_2 \wedge d\beta_1 = 0 \Rightarrow d(\beta_1 \wedge \beta_2) = 0. \tag{3.22}$$

In $\mathcal{L}^2_{\partial_x}(\eta \wedge d\eta)$ and using (3.22) and (3.21), one has

$$\begin{aligned} &\beta_0 \wedge d\beta_2 + \beta_1 \wedge d\beta_1 + 2\beta_1 \wedge dx \wedge \beta_2 + \beta_2 \wedge d\beta_0 + \beta_2 \wedge dx \wedge \beta_1 + dx \wedge d\beta_2 = 0 \\ \Rightarrow &\beta_0 \wedge d\beta_2 + \beta_1 \wedge d\beta_1 + \beta_2 \wedge d\beta_0 - \beta_1 \wedge \beta_2 \wedge dx + d\beta_2 \wedge dx = 0 \\ \Rightarrow &(\beta_0 \wedge d\beta_2 + \beta_1 \wedge d\beta_1 + \beta_2 \wedge d\beta_0) + (-\beta_1 \wedge \beta_2 + d\beta_2) \wedge dx = 0. \end{aligned} \tag{3.23}$$

Remark 3.2. Note that the expression $(\beta_0 \wedge d\beta_2 + \beta_1 \wedge d\beta_1 + \beta_2 \wedge d\beta_0)$ does not depend on dx while $(-\beta_1 \wedge \beta_2 + d\beta_2) \wedge dx$ depends, but $(-\beta_1 \wedge \beta_2 + d\beta_2)$ does not depend on dx, then

$$\begin{aligned} (-\beta_1 \wedge \beta_2 + d\beta_2) \wedge dx &= 0 \\ \Rightarrow \beta_1 \wedge \beta_2 &= d\beta_2 \end{aligned} \tag{3.24}$$

Now in $\mathcal{L}^1_{\partial_x}(\eta \wedge d\eta)$, using the equations (3.21), (3.22), (3.23), and an argument similar to Remark 3.2 we have

$$\beta_0 \wedge d\beta_1 + 2\beta_0 \wedge dx \wedge \beta_2 + \beta_1 \wedge d\beta_0 + dx \wedge d\beta_1 = 0 \tag{3.25}$$

$$\Rightarrow 2\beta_0 \wedge dx \wedge \beta_2 + dx \wedge d\beta_1 = 0$$

$$\Rightarrow dx \wedge (-2\beta_0 \wedge \beta_2 + d\beta_1) = 0$$

$$\Rightarrow 2\beta_0 \wedge \beta_2 = d\beta_1 \tag{3.26}$$

Hence $\mathcal{L}^1_{\partial_x}(\eta \wedge d\eta)$ with the equations (3.21), (3.22), (3.23), and (3.25) gives us

$$\beta_0 \wedge d\beta_0 + \beta_0 \wedge dx \wedge \beta_1 + dx \wedge d\beta_0 = 0$$

which gives us two equations

$$\beta_0 \wedge d\beta_0 = 0 \quad \text{and}$$
$$\beta_0 \wedge dx \wedge \beta_1 + dx \wedge d\beta_0 = 0,$$

again by Remark 3.2. Of the second equality we have

$$\beta_0 \wedge \beta_1 = d\beta_0. \tag{3.27}$$

Finally, it follows from (3.24), (3.26) and (3.27):

$$\begin{cases} d\beta_0 = \beta_0 \wedge \beta_1 \\ d\beta_1 = 2\beta_0 \wedge \beta_2 \\ d\beta_2 = \beta_1 \wedge \beta_2 \end{cases}$$

We shall now show the triple (η, Ω, ψ) which defines a projective transverse structure for η satisfying

$$\begin{cases} d\eta = \Omega \wedge \eta \\ d\Omega = \eta \wedge \xi \\ d\xi = \xi \wedge \Omega \end{cases}$$

By taking $\Omega = -\mathcal{L}_{\partial_x}(\eta) = -\beta_1 - 2x\beta_2$, we have

$$\begin{aligned} \Omega \wedge \eta &= (-\beta_1 - 2x\beta_2) \wedge (\beta_0 + x\beta_1 + x^2\beta_2 + dx) \\ &= -\beta_1 \wedge \beta_0 - x^2\beta_1 \wedge \beta_2 - \beta_1 \wedge dx - 2x\beta_2 \wedge \beta_0 - 2x^2\beta_2 \wedge \beta_1 - 2x\beta_2 \wedge dx \\ &= d\beta_0 - x^2 d\beta_2 - \beta_1 \wedge dx + xd\beta_1 + 2x^2 d\beta_2 - 2x\beta_2 \wedge dx \\ &= d\beta_0 + dx \wedge \beta_1 + xd\beta_1 + 2xdx \wedge \beta_2 + x^2 d\beta_2 \\ &= d\eta. \end{aligned} \tag{3.28}$$

On the other hand, by calculating $d\Omega$, we have

$$\begin{aligned} d\Omega &= -d\beta_1 - 2dx \wedge \beta_2 - 2xd\beta_2 \\ &= -2\beta_0 \wedge \beta_2 - 2dx \wedge \beta_2 - 2x\beta_1 \wedge \beta_2 \\ &= \beta_0 \wedge (-2\beta_2) + dx \wedge (-2\beta_2) + x\beta_1 \wedge (-2\beta_2) \\ &= (\beta_0 + dx + x\beta_1) \wedge (-2\beta_2) \\ &= (\beta_0 + dx + x\beta_1 + x^2\beta_2) \wedge (-2\beta_2) \\ &= \eta \wedge (-2\beta_2). \end{aligned} \tag{3.29}$$

The second equality is obtained from the relations found in (3.24), (3.26), and (3.27).

Now, we take $\xi = -\mathcal{L}^2_{\partial_x}(\eta) = -2\beta_2$, which satisfies the following equality

$$\begin{aligned} d\xi =& -2d\beta_2 = -2\beta_1 \wedge \beta_2 \\ =&(-2\beta_2) \wedge (-\beta_1) \\ =&(-2\beta_2) \wedge (-\beta_1 - 2x\beta_2) \\ =&\xi \wedge \Omega. \end{aligned} \tag{3.30}$$

Note that Ω is closed if and only if $\beta_2 = 0$ which, consequently, we will have $\xi = 0$ and the structure will be affine. On the other hand, if $\beta_2 \neq 0$, then the foliation has a pure projective transverse structure. □

Lemma 3.4 (Case 4)**.** *Suppose $F_3 = 0$ and $F_4 \neq 0$. Then, either $\mathcal{F}$ has an affine transverse structure, or $\mathcal{F}$ is a pull-back by a rational map of a foliation on $\mathbb{P}^2$.*

Proof. We will divide the proof in two subcases and in each of them we shall get a 1-form similar to

$$\eta = \beta_0 + x\beta_1 + x^2\beta_2 + dx \tag{3.31}$$

satisfying $\mathcal{L}_{\partial_x}(\beta_j) = 0$ and $i_{\partial_x}(\beta_j) = 0$ for all $j = 0, 1, 2$. With this in hand, we shall thus obtain some of the expected results.

Subcase 1: Suppose that $F_5 = 0$.

$$\begin{aligned} x^{-2}\pi^*(\omega) &= x\theta_2 + x^2\theta_3 + x^3\theta_4 + x^4\theta_5 + xF_4dx \\ \Rightarrow \eta &= \frac{x^{-3}\pi^*(\omega)}{F_4} = \beta_0 + x\beta_1 + x^2\beta_2 + x^3\beta_3 + dx \end{aligned} \tag{3.32}$$

where $\beta_j = \frac{\theta_{j+2}}{F_4}$, $j = 0, 1, 2, 3$. Suppose then that $\beta_3 = 0$, otherwise we would get a finite G-V-S of length ≥ 3 and we would have nothing else to do, so we get an expression like (3.31).

Subcase 2: Suppose now $F_5 \neq 0$.

$$x^{-3}\pi^*(\omega) = \theta_2 + x\theta_3 + x^2\theta_4 + x^3\theta_5 + (F_4 + xF_5)dx = \tilde{\eta}$$

Consider the map $\Psi(\tau, z) = (\tau, \frac{F_4}{F_5}\frac{z}{1-z}) = (\tau, x)$ and we calculate the pull-back of $\tilde{\eta}$ for this map:

$$\begin{aligned}
\Psi^*(\tilde{\eta}) &= \theta_2 + \frac{F_4}{F_5}\frac{z}{1-z}\theta_3 + \frac{F_4^2}{F_5^2}\frac{z^2}{(1-z)^2}\theta_4 + \frac{F_4^3}{F_5^3}\frac{z^3}{(1-z)^3}\theta_5 + \left(F_4 + \frac{F_4}{F_5}\frac{z}{1-z}F_5\right) d\left(+\frac{F_4}{F_5}\frac{z}{1-z}\right) \\
&= \theta_2 + \frac{F_4 z}{F_5(1-z)}\theta_3 + \frac{F_4^2 z^2}{F_5^2(1-z)^2}\theta_4 + \frac{F_4^3 z^3}{F_5^3(1-z)^3}\theta_5 \\
&\quad + \left(F_4 + \frac{F_4 z}{1-z}\right)\left[\frac{z dF_4}{(1-z)F_5} + \frac{F_4 dz}{(1-z)F_5} - \frac{zF_4 dF_5}{(1-z)F_5^2} + \frac{F_4 z dz}{(1-z)^2 F_5}\right] \\
&= \theta_2 + \frac{F_4 z}{F_5(1-z)}\theta_3 + \frac{F_4^2 z^2}{F_5^2(1-z)^2}\theta_4 + \frac{F_4^3 z^3}{F_5^3(1-z)^3}\theta_5 \\
&\quad + \frac{F_4 z dF_4}{(1-z)F_5} + \frac{F_4^2 dz}{(1-z)F_5} - \frac{zF_4^2 dF_5}{(1-z)F_5^2} + \frac{F_4^2 z dz}{(1-z)^2 F_5} \\
&\quad + \frac{F_4 z^2 dF_4}{(1-z)^2 F_5} + \frac{zF_4^2 dz}{(1-z)^2 F_5} - \frac{z^2 F_4^2 dF_5}{(1-z)^2 F_5^2} + \frac{F_4^2 z^2 dz}{(1-z)^3 F_5} \\
&= \frac{F_4^2}{F_5(1-z)^3}\Big[(1 - 3z + 3z^2 - z^3)\frac{F_5\theta_2}{F_4^2} + (z - 2z^2 + z^3)\frac{\theta_3}{F_4} + (z^2 - z^3)\frac{\theta_4}{F_5} + z^3\frac{F_4\theta_5}{F_5^2} \\
&\quad + (z - 2z^2 + z^3)\frac{dF_4}{F_4} - (z - 2z^2 + z^3)\frac{dF_5}{F_5} + (z^2 - z^3)\frac{dF_4}{F_4} - (z^2 - z^3)\frac{dF_5}{F_5} + dz\Big]
\end{aligned}$$

$$\begin{array}{lllll}
= \frac{F_4^2}{F_5(1-z)^3}\Big[\frac{F_5\theta_2}{F_4^2} & +z\left(-3\frac{F_5\theta_2}{F_4^2}\right) & +z^2\left(3\frac{F_5\theta_2}{F_4^2}\right) & +z^3\left(-\frac{F_5\theta_2}{F_4^2}\right) \\
 & +z\left(\frac{\theta_3}{F_4}\right) & +z^2\left(-2\frac{\theta_3}{F_4}\right) & +z^3\left(\frac{\theta_3}{F_4}\right) \\
 & & +z^2\left(\frac{\theta_4}{F_5}\right) & +z^3\left(-\frac{\theta_4}{F_5}\right) \\
 & & & +z^3\left(\frac{F_4\theta_5}{F_5^2}\right) \\
 & +z\left(\frac{dF_4}{F_4}\right) & +z^2\left(-2\frac{dF_4}{F_4}\right) & +z^3\left(\frac{dF_4}{F_4}\right) \\
 & +z\left(-\frac{dF_5}{F_5}\right) & +z^2\left(2\frac{dF_4}{F_4}\right) & +z^3\left(-\frac{dF_5}{F_5}\right) \\
 & & +z^2\left(\frac{dF_4}{F_4}\right) & +z^3\left(-\frac{dF_4}{F_4}\right) \\
 & & +z^2\left(-\frac{dF_5}{F_5}\right) & +z^3\left(\frac{dF_5}{F_5}\right) + dz\Big]
\end{array}$$

Omitting $\dfrac{F_4^2}{F_5(1-z)^3}$ and grouping the terms that have the same power as z we obtain:

$$\begin{aligned}
&= \tfrac{F_5\theta_2}{F_4^2} + z\left(-3\tfrac{F_5\theta_2}{F_4^2} + \tfrac{\theta_3}{F_4} + \tfrac{dF_4}{F_4} - \tfrac{dF_5}{F_5}\right) + z^2\left(3\tfrac{F_5\theta_2}{F_4^2} - 2\tfrac{\theta_3}{F_4} + \tfrac{\theta_4}{F_5} - \tfrac{dF_4}{F_4} + \tfrac{dF_5}{F_5}\right) \\
&+ z^3\left(-\tfrac{F_5\theta_2}{F_4^2} + \tfrac{\theta_3}{F_4} - \tfrac{\theta_4}{F_5} + \tfrac{F_4\theta_5}{F_5^2}\right) + dz \\
&= \beta_0 + z\beta_1 + z^2\beta_2 + z^3\beta_3 + dz = \eta
\end{aligned}$$

As in the previous case we assume $\beta_3 = 0$ for the same reason, we get an expression equal to (3.31).

We will now work both cases at the same time as they both fell into the form (3.31). Remembering that in the initial blow-up we have $\pi^*(\alpha_2) = x^2[x\theta_2 + F_3(\tau, 1)dx] = x^3\theta_2$. When $\alpha_2 = 0$ we have $\beta_0 = 0$ and $\eta = x\beta_1 + x^2\beta_2 + dx$ where we fall into the Lemma 3.2, so let us assume $\alpha_2 \neq 0$. The integrability condition of ω gives us

$$\begin{aligned}(\alpha_2+\alpha_3+\alpha_4+\alpha_5)\wedge(d\alpha_2+d\alpha_3+d\alpha_4+d\alpha_5) =& \alpha_2\wedge d\alpha_2+\alpha_2\wedge d\alpha_3+\alpha_2\wedge d\alpha_4+\alpha_2\wedge d\alpha_5\\ &+\alpha_3\wedge d\alpha_2+\alpha_3\wedge d\alpha_3+\alpha_3\wedge d\alpha_4+\alpha_3\wedge d\alpha_5\\ &+\alpha_4\wedge d\alpha_2+\alpha_4\wedge d\alpha_3+\alpha_4\wedge d\alpha_4+\alpha_4\wedge d\alpha_5\\ &+\alpha_5\wedge d\alpha_2+\alpha_5\wedge d\alpha_3+\alpha_5\wedge d\alpha_4+\alpha_5\wedge d\alpha_5=0 \end{aligned} \tag{3.33}$$

The coefficients of α_j are homogeneous polynomials of degree j, then the coefficients of $\alpha_2 \wedge d\alpha_2$ are homogeneous polynomials of degree 3 and none of the other 2-forms in the other parcels have coefficients of degree 3, then $\alpha_2 \wedge d\alpha_2 = 0$. In particular, either $Cod(Sing(\alpha_2)) \geq 2$ and α_2 define a degree 1 foliation in $\mathbb{P}^{n-1}$; or $\alpha_2 = h\alpha_1$, where α_1 defines a zero degree foliation in $\mathbb{P}^{n-1}$ ($\alpha_2 = h\alpha_1$, in this case $Cod(Sing(\alpha_2)) < 2$, and the foliation needs to be saturated and therefore what is left is a 1-form polynomial of degree 1). In both cases α_2 has an integrating factor, that is, there exists a function f such that $d(f^{-1}\alpha_2) = 0$.

In **Subcase 1**, we have:

$$\pi^*\left(\frac{\alpha_2}{f}\right) = \frac{\theta_2}{f(\tau,1)} \Rightarrow d\left(\frac{\theta_2}{f(\tau,1)}\right) = 0 \Rightarrow d\left(\frac{F_4(\tau,1)}{f(\tau,1)}\beta_0\right) = 0 \tag{3.34}$$

We will put $F_1(\tau) := \dfrac{f(\tau,1)}{F_4(\tau,1)}$ for this case.

In **Subcase 2**, we have:

$$\pi^*\left(\frac{\alpha_2}{f}\right) = \frac{\theta_2}{f(\tau,1)} \Rightarrow d\left(\frac{\theta_2}{f(\tau,1)}\right) = 0 \Rightarrow d\left(\frac{F_4^2(\tau,1)}{F_5(\tau,1)f(\tau,1)}\beta_0\right) = 0 \tag{3.35}$$

We will put $F_2(\tau) := \dfrac{F_5(\tau,1)f(\tau,1)}{F_4^2(\tau,1)}$ for this case.

Now consider the map $\Phi_i(\tau,z) = (\tau, F_i(\tau)z) = (\tau, x)$, $i = 1,2$. If η is the one that appears in **Subcase 1**, just use Φ_1, if it is η in **Subcase 2**, just use Φ_2, so we will omit the indices of Φ_i and F_i because the calculation we will make with these applications in one case is identical to the other. If η is like in (3.31), a direct calculation gives us:

$$\begin{aligned}\Phi^*(\eta) =& F\cdot\left(dz + F^{-1}\beta_0 + \left(\beta_1 + \frac{dF}{F}\right)z + F\beta_2 z^2\right)\\ =& F\cdot\left(dz + \tilde{\beta}_0 + \tilde{\beta}_1 z + \tilde{\beta}_2 z^2\right) = F\tilde{\eta}.\end{aligned}$$

Let us consider the birational map $\phi(\tau,w) = (\tau, \frac{1}{w}) = (\tau, z)$. Then

$$\phi^*(\tilde{\eta}) = -w^2\hat{\eta}$$

where $\hat{\eta} = dw - \tilde{\beta}_2 - w\tilde{\beta}_1 - w^2\tilde{\beta}_0$. Making calculations in $\hat{\eta}$ analogous to the calculations made in Lemma 3.3, we find:

$$\begin{cases} d\tilde{\beta}_0 = \tilde{\beta}_0 \wedge \tilde{\beta}_1 \\ d\tilde{\beta}_1 = 2\tilde{\beta}_0 \wedge \tilde{\beta}_2 \\ d\tilde{\beta}_2 = \tilde{\beta}_1 \wedge \tilde{\beta}_2 \end{cases} \tag{3.36}$$

From (3.34) and (3.35) we have $d\tilde{\beta}_0 = 0$ and by the first equation in (3.36) we obtain $\tilde{\beta}_0 \wedge \tilde{\beta}_1 = 0$. Denoting by $\mathcal{M}_k$ the set of meromorphic functions in $\mathbb{P}^k$. It follows from $\tilde{\beta}_0 \wedge \tilde{\beta}_1 = 0$ that there exists $g \in \mathcal{M}_{n-1}$ such that $\tilde{\beta}_1 = g\tilde{\beta}_0$.

The second relation in (3.36) gives us

$$\begin{aligned} d\tilde{\beta}_1 = dg \wedge \tilde{\beta}_0 = 2\tilde{\beta}_0 \wedge \tilde{\beta}_2 &\Rightarrow dg \wedge \tilde{\beta}_0 + 2\tilde{\beta}_2 \wedge \tilde{\beta}_0 = 0 \\ &\Rightarrow (dg + 2\tilde{\beta}_2) \wedge \tilde{\beta}_0 = 0. \end{aligned}$$

Therefore, there exists $h \in \mathcal{M}_{n-1}$ such that

$$\frac{dg}{2} + \tilde{\beta}_2 = h\tilde{\beta}_0 \Rightarrow \tilde{\beta}_2 = h\tilde{\beta}_0 - \frac{dg}{2}.$$

The third relation in (3.36) implies

$$\begin{aligned} d\tilde{\beta}_2 = dh \wedge \tilde{\beta}_0 = &\tilde{\beta}_1 \wedge \tilde{\beta}_2 = g\tilde{\beta}_0 \wedge \left(h\tilde{\beta}_0 - \frac{dg}{2} \right) = g\frac{dg}{2} \wedge \tilde{\beta}_0 \\ \Rightarrow &dh \wedge \tilde{\beta}_0 = g\frac{dg}{2} \wedge \tilde{\beta}_0 \Rightarrow \left(dh - g\frac{dg}{2} \right) \wedge \tilde{\beta}_0 = 0 \\ \Rightarrow &d\left(h - \frac{g^2}{4} \right) \wedge \tilde{\beta}_0 = 0. \end{aligned} \tag{3.37}$$

Let us denote by $\mathcal{G}$ the foliation generated by $\tilde{\beta}_0$ in $\mathbb{P}^{n-1}$. Note that $\tilde{\beta}_0$ generates a foliation in $\mathbb{P}^{n-1}$ because it is defined in $\mathbb{C}^{n-1}$ and moreover it is closed.

We have two possibilities:

Possibility I: $\mathcal{G}$ has no non-constant first integral. We claim that ω has an integral factor. In fact, by (3.37)

$$d\left(h - \frac{g^2}{4}\right) \wedge \tilde{\beta}_0 = 0 \Rightarrow h = \frac{g^2}{4} + c,$$

where $c \in \mathbb{C}$, otherwise $\mathcal{G}$ would have non-constant first integral. We can then write:

$$\begin{aligned} \hat{\eta} &= dw - \tilde{\beta}_2 - w\tilde{\beta}_1 - w^2\tilde{\beta}_0 = dw - \left(\frac{g^2}{4} + c\right)\tilde{\beta}_0 - \frac{dg}{2} - g\tilde{\beta}_0 w - \tilde{\beta}_0 w^2 \\ &= dw - \frac{dg}{2} - \left(\frac{g}{4} + c + gw + w^2\right)\tilde{\beta}_0 \\ &= d\left(w - \frac{g}{2}\right) - \left(\left[w + \frac{g}{2}\right]^2 + c\right)\tilde{\beta}_0, \end{aligned} \tag{3.38}$$

in particular, if we set $\mathcal{P} := \left(\left[w + \frac{g}{2}\right]^2 + c\right)^{-1} \hat{\eta}$, then

$$\mathcal{P} := \frac{d(w - \frac{g}{2})}{\left[w + \frac{g}{2}\right]^2 + c} - \tilde{\beta}_0 \Rightarrow d\tilde{\beta}_0 = 0.$$

Therefore $\hat{\eta}$ has an integral factor and therefore ω as well. Then there exists h such that

$$\begin{aligned} d(h\omega) = 0 &\Rightarrow dh \wedge \omega + hd\omega = 0 \\ &\Rightarrow -\frac{dh}{h} \wedge \omega = dw. \end{aligned} \tag{3.39}$$

Hence ω has an affine transverse structure.

Possibility II: $\mathcal{G}$ has non-constant first integral. We claim that $\mathcal{F}$ is the pull-back of a Riccati equation on $\mathbb{P}^1 \times \mathbb{P}^1$ by a birational map. In fact, by Stein's Factorization Theorem $\mathcal{G}$ has a meromorphic first integral, say f, with connected fibers: if $\phi \in \mathcal{M}_{n-1}$ and $d\phi \wedge df = 0$ then there exists $\psi \in \mathcal{M}_1$ such that $\phi = \psi(f)$ where $\psi(f) = \psi \circ f$.
On the other hand, the relation (3.37) implies that there exists $\phi_2 \in \mathcal{M}_1$ such that

$$d\left(h - \frac{g^2}{4}\right) \wedge \tilde{\beta}_0 = 0 \Rightarrow d\left(h - \frac{g^2}{4}\right) \wedge \phi_1 df = 0 \Rightarrow d\left(h - \frac{g^2}{4}\right) \wedge df = 0$$

and by Stein's Factorization Theorem

$$h - \frac{g^2}{4} = \psi_2(f) \Rightarrow h = \psi_2(f) + \frac{g^2}{4},$$

replacing in $\hat{\eta}$ we have

$$\begin{aligned} \hat{\eta} =& dw - \left(h\tilde{\beta}_0 - \frac{dg}{2}\right) - w(g\tilde{\beta}_0) - \tilde{\beta}_0 w^2 \\ =& d\left(w - \frac{g}{2}\right) + (-h - wg - w^2)\tilde{\beta}_0 \\ =& d\left(w - \frac{g}{2}\right) - \left(\frac{g^2}{4} + \psi_2(f) + wg + w^2\right)\psi_1(f)df \\ =& d\left(w - \frac{g}{2}\right) - \left(\left[w + \frac{g}{2}\right]^2 + \psi_2(f)\right)\psi_1(f)df \end{aligned} \tag{3.40}$$

Consider the rational map $\Phi_1 : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\Phi_1(\tau, w) = \left(f(\tau), w - \frac{g(\tau)}{2}\right) := (x, y)$$

Then $\hat{\eta} = \Phi^*(\theta)$, where

$$\theta = dy - (y^2 + \psi_2(x))\psi_1(x)dx$$

Note that θ is an integrable 1-form that defines a Riccati equation in $\mathbb{P}^1 \times \mathbb{P}^1$ and furthermore, it has an affine transverse structure as we wanted, then ω has an affine transverse structure. $\square$

Lemma 3.5 (Case 5)**.** *Suppose that $F_5 \equiv 0$, and $F_3 \not\equiv 0 \not\equiv F_4$. Then, there exists a birational map $\Psi : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ such that the foliation $\Psi^*(\mathcal{F})$ is defined by a 1-form described as follows:*

$$\eta_t = \beta_0 + t\beta_1 + t^2\beta_2 + t^3\beta_3 + t^4\beta_4 - tdt,$$

where the 1-forms β_j do not depend on $t \in \mathbb{P}^1$, for all $0 \leq j \leq 4$.

Proof. We consider $(\tau, x) \in \mathbb{C}^{n-1} \times \mathbb{CP}^{n-1} \times \mathbb{P}^1$ as outlined in (4.3), so that the blowing-up σ extends to a birational map $\psi_1 : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^1$. If $F_5 \equiv 0$, it can be inferred from (4.4) that

$$\eta = x\theta_2 + x^2\theta_3 + x^3\theta_4 + x^4\theta_5 + (F_3(\tau, 1) + xF_4(\tau, 1))dx, \tag{3.41}$$

and the pull-back foliation $\psi_1^*(\mathcal{F})$ is induced by η. We will perform a series of pull-backs by birational maps on η until the desired result is achieved. To begin, we initiate the pull-back by applying the birational map $\psi_z : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^1$ defined as $\psi_z(\tau, z) = (\tau, \frac{1}{z}) = (\tau, x)$. Consequently, $\psi_z^*(\eta) = \frac{\eta_z}{z^4}$, where

$$\eta_z = \theta_5 + z\theta_4 + z^2\theta_3 + z^3\theta_2 - (zF_4 + z^2F_3)dz.$$

Continuing, we consider the birational map $\psi_t : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^1$ defined as $\psi_t(\tau, t) = \left(\tau, \dfrac{\frac{F_4(\tau,1)}{F_3(\tau,1)}t}{(1-t)}\right) = (\tau, z)$. Calculatting the pull-back we have

$$\begin{aligned} \psi_z^*(\eta) &= \frac{1}{z}\theta_2 + \frac{1}{z}\theta_2 + \frac{1}{z}\theta_2 + \frac{1}{z}\theta_2 - \left(F_3 + \frac{1}{z}F_4\right) d\left(\frac{1}{z}\right) \\ &= \frac{1}{z^4}\left[z^3\theta_2 + z^2\theta_3 + z\theta_4 + \theta_5 - (z^2F_3 + zF_4)dz\right] \end{aligned}$$

Omitting $\dfrac{1}{z^4}$ we have

$$\eta_z = \theta_5 + z\theta_4 + z^2\theta_3 + z^3\theta_2 - (zF_4 + z^2F_3)dz.$$

We will now pullback η_z by the application $\psi_t(\tau, t) = \left(\tau, \frac{F_4}{F_3}\frac{t}{(1-t)}\right)$:

$$\begin{aligned}
\psi_t^*(\eta_z) &= \theta_5 + \left(\frac{F_4}{F_3}\frac{t}{(1-t)}\right)\theta_4 + \left(\frac{F_4^2}{F_3^2}\frac{t^2}{(1-t)^2}\right)\theta_3 + \left(\frac{F_4^3}{F_3^3}\frac{t^3}{(1-t)^3}\right)\theta_2 \\
&\quad - \left(\left(\frac{F_4}{F_3}\frac{t}{(1-t)}\right)F_4 + \left(\frac{F_4^2}{F_3^2}\frac{t^2}{(1-t)^2}\right)F_3\right)d\left(\frac{F_4}{F_3}\frac{t}{(1-t)}\right) \\
&= \theta_5 + \left(\frac{F_4}{F_3}\frac{t}{(1-t)}\right)\theta_4 + \left(\frac{F_4^2}{F_3^2}\frac{t^2}{(1-t)^2}\right)\theta_3 + \left(\frac{F_4^3}{F_3^3}\frac{t^3}{(1-t)^3}\right)\theta_2 \\
&\quad - \left(\left(\frac{F_4^2}{F_3}\frac{t}{(1-t)}\right) + \left(\frac{F_4^2}{F_3}\frac{t^2}{(1-t)^2}\right)\right)\left(\frac{F_4}{(1-t)F_3}dt + \frac{t}{(1-t)F_3}dF_4\right. \\
&\quad \left. + \frac{tF_4}{(1-t)^2F_3}dt + \frac{tF_4}{(1-t)F_3^2}dF_3\right) \\
&= \theta_5 + \left(\frac{F_4}{F_3}\frac{t}{(1-t)}\right)\theta_4 + \left(\frac{F_4^2}{F_3^2}\frac{t^2}{(1-t)^2}\right)\theta_3 + \left(\frac{F_4^3}{F_3^3}\frac{t^3}{(1-t)^3}\right)\theta_2 \\
&\quad - \left(\frac{tF_4^3}{(1-t)^2F_3^2}dt + \frac{t^2F_4^2}{(1-t)^2F_3^2}dF_4 + \frac{t^2F_4^3}{(1-t)^3F_3^2}dt - \frac{t^2F_4^3}{(1-t)^2F_3^3}dF_3\right. \\
&\quad \left. + \frac{t^2F_4^3}{(1-t)^3F_3^2}dt + \frac{t^3F_4^2}{(1-t)^3F_3^2}dF_4 + \frac{t^3F_4^3}{(1-t)^4F_3^2}dt - \frac{t^3F_4^3}{(1-t)^3F_3^3}dF_3\right) \\
&= \frac{F_4^3}{(1-t)^4F_3^2}\left[(1-t)^4\frac{F_3^2}{F_4^3}\theta_5 + t(1-t)^3\frac{F_3}{F_4^2}\theta_4 + t^2(1-t)^2\frac{\theta_3}{F_4} + t^3(1-t)\frac{\theta_2}{F_3}\right. \\
&\quad - t^2(1-t)^2\frac{dF_4}{F_4} + t^2(1-t)^2\frac{dF_3}{F_3} - t^3(1-t)\frac{dF_4}{F_4} + t^3(1-t)\frac{dF_3}{F_3} \\
&\quad \left. - \left(t(1-t)^2 + t^2(1-t) + t^2(1-t) + t^3\right)dt\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{F_4^3}{(1-t)^4F_3^2}\left[(1-4t+6t^2-4t^3+t^4)\frac{F_3^2}{F_4^3}\theta_5 + (t-3t^2+3t^3-t^4)\frac{F_3}{F_4^2}\theta_4 + (t^2-2t^3+t^4)\frac{\theta_3}{F_4}\right. \\
&\quad \left. + (t^3-t^4)\frac{\theta_2}{F_3} - (t^2-2t^3+t^4)\frac{dF_4}{F_4} + (t^2-2t^3+t^4)\frac{dF_3}{F_3} - (t^3-t^4)\frac{dF_4}{F_4} + (t^3-t^4)\frac{dF_3}{F_3} - tdt\right] \\
&= \frac{F_4^3}{(1-t)^4F_3^2}\left[\left(\frac{F_3^2}{F_4^3}\theta_5\right) + t\left(-4\frac{F_3^2}{F_4^3}\theta_5 + \frac{F_3}{F_4^2}\theta_4\right) + t^2\left(6\frac{F_3^2}{F_4^3}\theta_5 - 3\frac{F_3}{F_4^2}\theta_4 + \frac{\theta_3}{F_4} - \frac{dF_4}{F_4} + \frac{dF_3}{F_3}\right)\right. \\
&\quad + t^3\left(-4\frac{F_3^2}{F_4^3}\theta_5 + 3\frac{F_3}{F_4^2}\theta_4 - 2\frac{\theta_3}{F_4} + 2\frac{dF_4}{F_4} + \frac{dF_3}{F_3} + \frac{\theta_2}{F_3}\right) + t^4\left(\frac{F_3^2}{F_4^3}\theta_5 - \frac{F_3}{F_4^2}\theta_4 + \frac{\theta_3}{F_4} - \frac{\theta_2}{F_3}\right. \\
&\quad \left.\left. - \frac{dF_4}{F_4} + \frac{dF_3}{F_3} + \frac{dF_4}{F_4} - \frac{dF_3}{F_3}\right) - tdt\right] = \frac{F_4^3}{(1-t)^4F_3^2}\eta_t
\end{aligned}$$

$\psi_t^*(\eta_z) = \frac{F_4^3}{(1-t)^4F_3^2}\eta_t$, where

$$\eta_t = \beta_0 + t\beta_1 + t^2\beta_2 + t^3\beta_3 + t^4\beta_4 - tdt$$

with

$$\begin{aligned}
\beta_0 &= \frac{F_3^2}{F_4^3}\theta_5 \\
\beta_1 &= -4\frac{F_3^2}{F_4^3}\theta_5 + \frac{F_3}{F_4^2}\theta_4 \\
\beta_2 &= 6\frac{F_3^2}{F_4^3}\theta_5 - 3\frac{F_3}{F_4^2}\theta_4 + \frac{\theta_3}{F_4} - \frac{dF_4}{F_4} + \frac{dF_3}{F_3} \\
\beta_3 &= -4\frac{F_3^2}{F_4^3}\theta_5 + 3\frac{F_3}{F_4^2}\theta_4 - 2\frac{\theta_3}{F_4} + 2\frac{dF_4}{F_4} + \frac{dF_3}{F_3} + \frac{\theta_2}{F_3} \\
\beta_4 &= \frac{F_3^2}{F_4^3}\theta_5 - \frac{F_3}{F_4^2}\theta_4 + \frac{\theta_3}{F_4} - \frac{\theta_2}{F_3}
\end{aligned}$$

The proof concludes by observing that the 1-forms β_j are dependent solely on τ, for all $0 \leq j \leq 4$, and taking $\Psi := \psi_1 \circ \psi_z \circ \psi_t$. □

Lemma 3.6 (Cases 6 and 7)**.** *Suppose that $F_4 \equiv 0$, and $F_3 \not\equiv 0$, $F_5 \not\equiv 0$; or $F_3 \not\equiv 0$, $F_4 \not\equiv 0$, and $F_5 \not\equiv 0$. Then, the foliation $\mathcal{F}$ is either transversely affine or is the pull-back by a rational map of a foliation on $\mathbb{P}^2$, or there exists a birational map $\Psi : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ such that the foliation $\Psi^*(\mathcal{F})$ is defined by a 1-form described as follows:*

$$\eta_t = \beta_0 + t\beta_1 + t^2\beta_2 + t^3\beta_3 + t^4\beta_4 - tdt,$$

where the 1-forms β_j do not depend on $t \in \mathbb{P}^1$, for all $0 \leq j \leq 4$.

Proof. As in Lemma 3.5. the map σ extends to a birational map $\psi_1 : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^1$. Then, following from equation (4.4), the foliation $\psi^*(\mathcal{F})$ is defined by the 1-form:

$$\eta = x\theta_2 + x^2\theta_3 + x^3\theta_4 + x^4\theta_5 + (F_3(\tau,1) + xF_4(\tau,1) + x^2F_5(\tau,1))dx. \tag{3.42}$$

Since $F_5 \not\equiv 0$, we can divide the expression of η by $F_5(\tau,1)$ and obtain:

$$\eta_1 = x\alpha_2 + x^2\alpha_3 + x^3\alpha_4 + x^4\alpha_5 + \left(\frac{F_3(\tau,1)}{F_5(\tau,1)} + x\frac{F_4(\tau,1)}{F_5(\tau,1)} + x^2\right)dx, \tag{3.43}$$

where $\alpha_j = \theta_j/F_5(\tau,1)$, for all $2 \leq j \leq 5$. Now, we factorize the polynomial

$$x^2 + x\frac{F_4(\tau,1)}{F_5(\tau,1)} + \frac{F_3(\tau,1)}{F_5(\tau,1)} = (x - c_1(\tau))(x - c_2(\tau)).$$

Note that $c_1(\tau)$ and $c_2(\tau)$ are not identically zero, as assumed from the hypotheses $F_3 \not\equiv 0$, $F_5 \not\equiv 0$, and $c_1(\tau) \cdot c_2(\tau) = \frac{F_3(\tau,1)}{F_5(\tau,1)}$.

First, we consider the birational map $\psi_z : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^1$ defined by $\psi_z(\tau,z) = \left(\tau, \frac{c_1(\tau)z}{z-1}\right) = (\tau,x)$. A direct calculation yields $\psi_z^*(\eta_1) = \frac{c_1^2}{(z-1)^4}\eta_z$, where

$$\eta_z = z\gamma_1 + z^2\gamma_2 + z^3\gamma_3 + z^4\gamma_4 - [(c_1(\tau) - c_2(\tau))z + c_2(\tau)]dz \tag{3.44}$$

with

$$\begin{aligned}
\gamma_1 &= \frac{\alpha_2}{c_1(\tau)} - c_2(\tau)\frac{dc_1(\tau)}{c_1(\tau)} \\
\gamma_2 &= \frac{3\alpha_2}{c_1(\tau)} + \alpha_3 + (2c_2(\tau) - c_1(\tau))\frac{dc_1(\tau)}{c_1(\tau)} \\
\gamma_3 &= \frac{-3\alpha_2}{c_1(\tau)} - 2\alpha_3 - c_1(\tau)\alpha_4 + (c_1(\tau) - c_2(\tau))\frac{dc_1(\tau)}{c_1(\tau)} \\
\gamma_4 &= \frac{\alpha_2}{c_1(\tau)} + \alpha_3 + c_1(\tau)\alpha_4 + c_1^2(\tau)\alpha_5
\end{aligned}$$

Now we will analyze the expression of η_z in (3.44), observing that we have the following subcases:

Subcase I. $c_1(\tau) = c_2(\tau)$. In this situation, we have that

$$\eta_z = z\gamma_1 + z^2\gamma_2 + z^3\gamma_3 + z^4\gamma_4 + c_2(\tau)dz \tag{3.45}$$

Since $c_2(\tau)$ is not identically zero, we can divide η_z by $c_2(\tau)$ and obtain a 1-form $\tilde{\eta}$ that is equivalent to the 1-form derived in equation (3.9) of Lemma 3.2. Hence, we can conclude that the foliation $\mathcal{F}$ is either transversely affine or is the pull-back by a rational map of a foliation on $\mathbb{P}^2$.

Subcase II. $c_1(\tau) \neq c_2(\tau)$. In this subcase, the 1-form η_z is equivalent to the 1-form derived in equation (3.41) of Lemma 3.5. Therefore, we can conclude that there exists a birational map $\Psi : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^1$ such that the foliation $\Psi^*(\mathcal{F})$ is defined by a 1-form described as follows:

$$\eta_t = \beta_0 + t\beta_1 + t^2\beta_2 + t^3\beta_3 + t^4\beta_4 - tdt,$$

where the 1-forms β_j do not depend on $t \in \mathbb{P}^1$, for all $0 \leq j \leq 4$. □

We can condense the results obtained in the Lemmas 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6 in the following proposition: In the above situation we have the five possibilities:

(i) either $\mathcal{F}$ is transversely affine outside a compact hypersurface;

(ii) or $\mathcal{F}$ is pure transversely projective outside a compact hypersurface;

(iii) or $\mathcal{F}$ is a pull-back by a rational map of a foliation on $\mathbb{P}^2$;

(iv) or $\mathcal{F}$ is a pull-back by a linear map $\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$ of a foliation of degree four on $\mathbb{P}^{n-1}$.

(v) or there exists a birational map $\Psi : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ such that the foliation $\Psi^*(\mathcal{F})$ is defined by a 1-form described as follows:

$$\eta_t = \beta_0 + t\beta_1 + t^2\beta_2 + t^3\beta_3 + t^4\beta_4 - tdt,$$

where the 1-forms β_j do not depend on $t \in \mathbb{P}^1$, for all $0 \leq j \leq 4$.

In particular, if $n = 3$ then $\mathcal{F}$ satisfies (i), (ii), (v), or (iii). Finally, we give a proof of Theorem A.

3.1 Proof of Theorem A

We give the proof by induction on the dimension $n \geq 3$. If $n = 3$, then Theorem A follows from Corollary 3.1 and Proposition 3. Let us assume that Theorem A is true for $n - 1 \geq 3$ and prove that it holds for n.

Let $\mathcal{F}$ be a codimension one foliation of degree four on $\mathbb{P}^n$, $n \geq 4$. It follows from Corollary 3.1 and Proposition 3 that, either $\mathcal{F}$ satisfies one of the conclusions of Theorem A, or $\mathcal{F}$ is the pull-back by a linear map $\pi : \mathbb{P}^n \to \mathbb{P}^{n-1}$ of a foliation $\mathcal{F}_{n-1}$ of degree four on $\mathbb{P}^{n-1}$. In this last case, as Theorem A holds true for $n - 1$, it follows that one the five possibilities outlined below must also be true:

(i) $\mathcal{F}_{n-1}$ has a rational first integral, say $F : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^1$. In this case, $F \circ \pi$ is a rational first integral of $\mathcal{F}$ and we are done.

(ii) $\mathcal{F}_{n-1}$ is transversely affine. In this case, $\mathcal{F}_{n-1}$ admits a G-V-S of length one. Hence, $\mathcal{F}$ also admits a G-V-S of length one by Remark **??**.

(iii) $\mathcal{F}_{n-1}$ is transversely projective. In this case, $\mathcal{F}_{n-1}$ admits a G-V-S of length two. Hence, $\mathcal{F}$ also admits a G-V-S of length two by Remark **??**.

(iv) $\mathcal{F}_{n-1} = \Phi^*(\mathcal{G})$, where $\mathcal{G}$ is a foliation on $\mathbb{P}^2$ and $\Phi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^2$ a rational map. In this case, we get $\mathcal{F} = (\Phi \circ \pi)^*(\mathcal{G})$ and we are done.

(v) There exists a birational map $\Psi_1 : \mathbb{P}^{n-2} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^{n-1}$ such that the foliation $\Psi_1^*(\mathcal{F}_{n-1})$ is defined by a 1-form described as follows:

$$\eta_t = \beta_0 + t\beta_1 + t^2\beta_2 + t^3\beta_3 + t^4\beta_4 - tdt,$$

where the 1-forms β_j do not depend on $t \in \mathbb{P}^1$, for all $0 \leq j \leq 4$. Now consider any rational map $\phi : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^{n-2} \times \mathbb{P}^1$ such that it fixes the variable at $\mathbb{P}^1$, and choose any birational map Ψ such that $\pi \circ \Psi = \Psi_1 \circ \phi$. With this, we have that

$$\Psi^*(\mathcal{F}) = \Psi^*(\pi^*(\mathcal{F}_{n-1})) = (\pi \circ \Psi)^*(\mathcal{F}_{n-1}) = (\Psi_1 \circ \phi)^*(\mathcal{F}_{n-1}) = \phi^*(\Psi_1^*(\mathcal{F}_{n-1}))$$

Using the fact that ϕ fixes the variable over $\mathbb{P}^1$, we deduce that $\Psi^*(\mathcal{F})$ is defined by a 1-form similar to item (v). This concludes the proof of Theorem A.

4 Foliations of hight degree

We decided to include this chapter with a new result that emerged as a result of the main theorem. The result that will be demonstrated is the Theorem B:

Theorem B. *Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree $d \geq 4$ on $\mathbb{P}^n$, with $n \geq 3$. Suppose that one of the two conditions is satisfied:*

1. *for all $p \in (\mathcal{F})$, we have $\mathcal{J}(\mathcal{F}, p) = 1$;*
2. *there exists $p \in (\mathcal{F})$ such that $\mathcal{J}(\mathcal{F}, p) \geq d - 1$.*

Then,

(i) either $\mathcal{F}$ admits a rational first integral;

(ii) or $\mathcal{F}$ is transversely affine outside a compact hypersurface;

(iii) or $\mathcal{F}$ is a pure transversely projective outside a compact hypersurface;

(iv) or $\mathcal{F} = \Phi^*(\mathcal{G})$, where $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a rational map and $\mathcal{G}$ is a holomorphic foliation on $\mathbb{P}^2$.

Proof. Let $\mathcal{F}$ be a codimension one holomorphic foliation of degree $d \geq 4$ in $\mathbb{P}^n$, $n \geq 3$. Suppose that one of the two conditions is satisfied:

1. for all $p \in (\mathcal{F})$, we have $\mathcal{J}(\mathcal{F}, p) = 1$;
2. there exists $p \in (\mathcal{F})$ such that $\mathcal{J}(\mathcal{F}, p) \geq d - 1$.

In the first case, $\mathcal{F}$ admits a rational first integral by invoking Corollary 3.1. This consequently establishes the validity of assertion (i) within Theorem B.

Therefore, we shall assume that there exists a point $p \in \mathbb{P}^n$ such that $\mathcal{J}(\mathcal{F}, p) \geq d - 1$. By employing affine coordinates $(z_1, \ldots, z_n) \in \mathbb{C}^n \subset \mathbb{P}^n$, where $p = 0 \in \mathbb{C}^n$, we can conveniently consider $\mathcal{F}|_{\mathbb{C}^n} : \omega = 0$, where ω is a polynomial 1-form in $\mathbb{C}^n$ expressed as follows:

$$\omega = \alpha_{d-1} + \alpha_d + \alpha_{d+1}, \tag{4.1}$$

here, α_j corresponds to homogeneous polynomial 1-forms of degree j, $d - 1 \leq j \leq d + 1$, and

$$i_R(\alpha_{d+1}) = 0, \qquad \text{with} \quad R = \sum_{i=1}^{n} z_i \partial z_i. \tag{4.2}$$

Once again, we will express α_j as: $\alpha_j(z) := \sum_{i=1}^{n} P_{ji}(z)dz_i$, with $d-1 \leq j \leq d+1$. Additionally, we introduce

$$F_j(z) := i_R(\alpha_{j-1}) = \sum_{i=1}^{n} z_i \cdot P_{j-1i}(z), \quad d-1 \leq j \leq d+1,$$

where P_{j-1i} are homogeneous polynomials of degree $j-1$. Note that $F_{d+2} \equiv 0$ by (4.2).

We proceed to examine the pull-back of ω through the process of blowing-up of $\mathbb{P}^n$ at $0 \in \mathbb{C}^n \subset \mathbb{P}^n$. Let $\sigma : \tilde{\mathbb{P}}^n \to \mathbb{P}^n$ denote the blow-up at $0 \in \mathbb{C}^n \subset \mathbb{P}^n$, and let $\tilde{\mathcal{F}}$ represent the strict transform of $\mathcal{F}$ by σ. Our objective is to calculate $\sigma^*(\omega)$ within the chart

$$(\tau_1, \ldots, \tau_{n-1}, x) = (\tau, x) \in \mathbb{C}^{n-1} \times \mathbb{C} \mapsto (x\tau, x) = (z_1, \ldots, z_n) \in \mathbb{C}^n \subset \mathbb{P}^n. \tag{4.3}$$

We have

$$\sigma^*(\omega) \;=\; x^{d-1}(x\theta_{d-1} + x^2\theta_d + x^3\theta_{d+1} + (F_d(\tau,1) + xF_{d+1}(\tau,1) + x^2F_{d+2}(\tau,1)))$$

where

$$\theta_j = \sum_{i=1}^{n-1} P_{ji}(\tau,1)d\tau_i, \qquad d-1 \leq j \leq d+1$$

depends only on τ. Utilizing the condition $F_{d+2}(\tau,1) \equiv 0$, we derive the 1-form η as follows:

$$\eta = x\theta_{d-1} + x^2\theta_d + x^3\theta_{d+1} + (F_d(\tau,1) + xF_{d+1}(\tau,1))dx. \tag{4.4}$$

This 1-form serves to define the foliation $\tilde{\mathcal{F}}$ in the chart (τ, x).

Given the aforementioned conditions, we are presented with the subsequent possibilities for F_i:

(1) $F_d \equiv 0$, $F_{d+1} \not\equiv 0$;

(2) $F_d \not\equiv 0$, $F_{d+1} \equiv 0$;

(3) $F_d \not\equiv 0$, $F_{d+1} \not\equiv 0$.

In the case (1), after dividing η by $x \cdot F_{d+1}$, we have

$$\eta_1 = \gamma_0 + x\gamma_1 + x^2\gamma_2 + dx,$$

where $\gamma_0 = \theta_{d-1}/F_{d+1}$, $\gamma_1 = \theta_d/F_{d+1}$, and $\gamma_2 = \theta_{d+1}/F_{d+1}$ depends only on τ. The 1-form η_1 is equivalent to the 1-form from (3.17) of Lemma 3.3. Therefore, we can conclude either $\mathcal{F}$ is transversely affine, or $\mathcal{F}$ is the pull-back by a rational map of a foliation on $\mathbb{P}^2$, or $\mathcal{F}$ is pure transversely projective.

In the case (2), after dividing η by F_d, we have

$$\eta_1 = x\gamma_1 + x^2\gamma_2 + x^3\gamma_3 + dx,$$

where $\gamma_1 = \theta_{d-1}/F_d$, $\gamma_2 = \theta_d/F_d$, and $\gamma_3 = \theta_{d+1}/F_d$ depends only on τ. The 1-form η_1 is equivalent to the 1-form from (3.9) of Lemma 3.2, Subcase II. Consequently, we can deduce that either $\mathcal{F}$ is transversely affine, or $\mathcal{F}$ is the pull-back by a rational map of a foliation on $\mathbb{P}^2$.

In the case (3), we consider the birational map $\psi_z : \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^1$ defined as $\psi_z(\tau, z) = \left(\tau, \frac{F_d(\tau,1)z}{1-F_{d+1}(\tau,1)z}\right)$. A direct calculation yields $\psi_z^*(\eta_1) = \frac{F_d^2}{(1-F_{d+1}(\tau,1)z)}\eta_z$, where

$$\eta_z = z\beta_1 + z^2\beta_2 + z^3\beta_3 + dz$$

with

$$\begin{aligned} \beta_1 &= \frac{\theta_{d-1}}{F_d} + \frac{dF_d}{F_d} \\ \beta_2 &= -\frac{F_{d+1}}{F_d}\theta_{d-1} + \theta_d - F_{d-1}\frac{dF_d}{F_d} + dF_{d+1} \\ \beta_3 &= -F_{d+1}\theta_d + F_d\theta_{d+1} \end{aligned}$$

Once again, the 1-form η_z is equivalent to the 1-form from (3.9) of Lemma 3.2, Subcase II. Thus, we can deduce that either $\mathcal{F}$ is transversely affine, or $\mathcal{F}$ is the pull-back by a rational map of a foliation on $\mathbb{P}^2$. This finishes the proof of Theorem B.

□

5 Open Problems

In this chapter we will talk about some open problems about the results proved in this work and others that are related to them.

Of course, the interesting questions arises and we can pose the following problem:

Problem 1. The item (v) of the Theorem A is equivalent to any of the other items (i), (ii), (iii) or (iv)?

In particular, every codimension one foliation on $\mathbb{P}^5$ of degree-four also has a trivial canonical bundle. Thus, we can pose another problem:

Problem 2. It is possible to provide a classification of codimension one holomorphic foliations on $\mathbb{P}^5$ of degree-four?

It seems reasonable to hope that Theorem A will give a clue to a classification of the irreducible components of the space of degree-four foliations on $\mathbb{P}^n$, $n \geq 3$, which are not the pull-back by rational maps of foliations on $\mathbb{P}^2$. However, the analysis of the irreducible components of rational pull-back or logarithmic type seems to be more delicate, since we have no control on the degrees of the objects that appear in ours proofs.

Problem 3. Classify the irreducible components of the space of codimension one foliations of degree-four on $\mathbb{P}^n$, with $n \geq 3$.

According to [8], it is known at least 24 distinct irreducible components of the space of foliations of degree-three on $\mathbb{P}^n$, with $n \geq 3$. Moreover, they assert that there are missed irreducible components that correspond to foliations with rational first integrals. One problem on the subject that seems interesting is the following:

Problem 4. Does the number of irreducible components of the space of codimension one foliations of degree-four on $\mathbb{P}^n$, with $n \geq 3$ vary with n?

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