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Two problems on Convex Geometry: isotropic measures and classification in valuation theory

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"A maior riqueza do homem é a sua incompletude".

Manoel de Barros

Resumo

Esta tese consiste em duas partes distintas, cada uma estudando um problema diferente na teoria dos corpos convexos. A primeira parte trata das medidas isotrópicas, mais especificamente, do problema de descrever explicitamente os pesos na decomposição da identidade para um corpo convexo na posição de John. Fazemos isso para a posição de John, ou seja, quando a bola Euclidiana unitária n-dimensional B^n, é o elipsóide com volume máximo dentro de K, e para a posição positiva de John em relação ao corpo convexo L, ou seja, quando $L \subseteq K$ e L tem volume máximo dentre todas as imagens TL em K, onde T é uma matriz definida-positiva. Também fazemos isso para elipsóides funcionais no sentido definido por Ivanov e Naszódi [30]. Consideramos funções log-côncavas próprias $h : \mathbb{R}^n \to \mathbb{R}$ (funções log-côncavas e semicontínuas superiores que possuem integral positiva finita). Por [30], para cada $s > 0$ existe (e é única no conjunto de funções log-côncavas próprias) uma função log-côncava com a maior integral sob a condição de que esta seja pontualmente menor ou igual a $h^{1/s}$. Essa função é chamada s-função de John de h. Novamente, por [30], existe uma caracterização dessa função semelhante àquela dada por John em seu teorema fundamental.

A segunda parte estuda o problema de caracterização de valuações semicontínuas superiores. Denote por $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ o espaço de funções convexas de valor finito em $\mathbb{R}$ que são afins por partes fora de uma conjunto compacto. Um funcional $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ é chamado uma valuação se

$$Z(u \vee v) + Z(u \wedge v) = Z(u) + Z(v)$$

para todo $u, v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ tal que $u \vee v, u \wedge v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$. Aqui, $u \vee v$ e $u \wedge v$ denotam as funções máximo e mínimo pontuais de $u, v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$, respectivamente. Uma classificação de valuações semicontínuas superiores, invariantes por translação e inalterada por adição de funções afins por partes no espaço $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ é estabelecida.

Palavras-chave: corpo convexo; posição de John; posição de Löwner; decomposição da identidade; medidas isotrópicas; funções log-côncava; elipsoides funcionais; valuações no espaço de funções convexas.

Abstract

This thesis consists in two separate parts, each studying a different problem in the theory of convex bodies. The first part deals with isotropic measures, more specifically, the problem of describing explicitly the weights in the decomposition of the identity for a convex body in John position. We do this for the John position, that is, when the n-dimensional unit Euclidean ball B^n, is the ellipsoid with maximum volume inside K, and for the positive John position with respect to the convex body L, that is, when $L \subseteq K$ and L has maximal volume among all images TL in K, where T is a positive-definite matrix. We also do this for functional ellipsoids in the sense defined by Ivanov and Naszódi [30]. We consider proper log-concave functions $h : \mathbb{R}^n \to \mathbb{R}$ (log-concave and upper semicontinuous functions that has finite positive integral). By [30], for every $s > 0$ there is (and is unique in the set of proper log-concave functions) one log-concave function with the largest integral under the condition that it is pointwise less than or equal to $h^{1/s}$. This function is called John s-function of h. Again, by [30] there exists a characterization of this function similar to the one given by John in his fundamental theorem.

The second part studies the problem of characterizing upper semicontinuous valuations. Denote by $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ the space of finite-valued, convex functions on $\mathbb{R}$ that are piecewise affine outside of a compact set. A functional $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ is called a valuation if

$$Z(u \vee v) + Z(u \wedge v) = Z(u) + Z(v)$$

for all $u, v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ such that $u \vee v, u \wedge v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$. Here, $u \vee v$ and $u \wedge v$ denote the pointwise maximum and minimum of $u, v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$, respectively. A classification of upper semicontinuous, translation invariant valuations and unchanged by the addition of piecewise affine functions on the space $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ is established.

Keywords: convex body; John position; Löwner position; decomposition of the identity; isotropic measures; log-concave functions; functional ellipsoids; valuations on the space of convex functions.

Contents

Introduction

In 1948 F. John showed that if K is a convex body then there exists a unique ellipsoid $\mathcal{E}_J \subseteq K$ of maximal volume inside K, know today as John's Ellipsoid. When the unit Euclidean ball B^n is the John ellipsoid we say that K is in John position. A construction that is dual to John ellipsoid is the Löwner ellipsoid $\mathcal{E}_L \supseteq K$ which is the unique ellipsoid of minimal volume containing K. The set K is in Löwner position if $\mathcal{E}_L = B^n$. F. John also showed a set of necessary conditions for $\mathcal{E}_J$ to be the unit Euclidean ball B^n. John's theorem can be stated as follows.

Theorem 0.1 ([32], Application 4, pag. 199 - 200)**.** *Assume K is in John position, then there exists a finite set of points $\{\xi_1, \dots, \xi_m\} \subset S^{n-1} \cap \partial K$, positive numbers $\{c_1, \dots, c_m\}$ and $\lambda \neq 0$, for which*

$$\sum_{i=1}^{m} c_i \xi_i \otimes \xi_i = \lambda \,\mathrm{Id} \quad \textit{and} \quad \sum_{i=1}^{m} c_i \xi_i = 0. \tag{1}$$

Here $v \otimes w$ is the rank-one matrix $(v \otimes w)_{i,j} = v_i w_j, 1 \leq i, j \leq n$.

The Theorem 0.1 guarantees that the atomic measure $\mu_K = \sum_{i=1}^{m} c_i \delta_{\xi_i}$ is centered and isotropic, but the existence of the measure μ_K is often show in a non-constructive way. Later Ball [11] proved that the existence of a non-negative centered isotropic measure μ_K in the set of contact points, guarantees that K is in John position if $B^n \subseteq K$, or in Löwner position if $K \subseteq B^n$. The literature around the John/Löwner position and its relation to isotropic measures, is vast. The relation between extremal position and isotropic measures was studied extensively in [23, 24, 25]. Extensions to related minimization problems were studied in [13, 14, 26, 34, 39]. Isotropic measures can also be used in combination with the Brascamp-Lieb inequality to find reverse isoperimetric inequalities, see [11, 9, 10, 40].

Artstein and Katzin showed that μ_K can be constructed as a weak approximation of uniform measures on subsets of S^{n-1}. Moreover, they introduced a new one-parameter family of positions: A convex body K is said to be in *maximal intersection position of radius r* if rB^n is the ellipsoid maximizing $\mathrm{vol}_n(rB^n \cap K)$ among all ellipsoids of same volume as rB^n. It is also shown that every centrally symmetric convex body K admits at least one of such position $T_r K$ with $T_r \in \mathrm{SL}_n(\mathbb{R})$, and in this case the uniform measure in $S^{n-1} \cap r^{-1} T_r K$ is isotropic. The theorem due to Artstein and Katzin is the following.

Theorem 0.2 ([6], Theorem 1.5)**.** *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body in John position such that $\mathrm{vol}_{n-1}(\partial K \cap \partial \mathcal{E}) = 0$ for all but finitely many ellipsoids $\mathcal{E}$. For every $r > 1$,* *denote by ν_r the uniform probability measure on $S^{n-1} \cap r^{-1}T_rK$, where T_rK is in maximal intersection position of radius r. Then there exists a sequence $r_j \searrow 1$ such that the sequence of measures ν_{r_j} weakly converges to an isotropic measure whose support is contained in $\partial K \cap S^{n-1}$.*

However it is worth mentioning that in practice there is a large body of computations to make. First we need to find the matrix T_r, for r close to 1, such that T_rK is in maximal intersection position of radius r. Later we need to take the limit of some subsequence of the measures ν_r, which is obtained from the matrices T_r.

Our first aim is to present a simple finite dimensional minimization problem whose solution (when it exists) can be used to construct a non-negative centered isotropic measure as above.

The next step is to consider two convex bodies both different from the unit Euclidean ball. Let K, L be convex bodies. We say that L is in *maximal volume position inside* K if L is its own maximal volume image inside K. A simple compactness argument shows that for every pair of convex bodies K and L there exists an affine image L_1 of L which is of maximal volume in K, but in this case the maximal volume position of L inside K is not necessarily unique. There is a generalization of classical John's Theorem 0.1 for the case where L is not the unit Euclidean ball. Giannopoulos, Perissinaki and Tsolomitis proved the following theorem.

Theorem 0.3 ([25], Theorem 2.5)**.** *Let K, L be smooth convex bodies in $\mathbb{R}^n$, such that L is of maximal volume in K. If $z \in \operatorname{int} L$, we can find contact points $v_1, \ldots, v_m$ of $K - z$ and $L - z$, contact points $u_1, \ldots, u_m$ of the polar bodies $(K - z)^\circ$ and $(L - z)^\circ$, and positive reals $c_1, \ldots, c_m$ such that*

$$\langle u_i, v_i \rangle = 1, \qquad \sum_{i=1}^{m} c_i u_i \otimes v_i = \mathrm{Id}, \qquad \sum_{i=1}^{m} c_i u_i = 0.$$

As mentioned before, Ball proved that for the classical John's theorem the existence of an isotropic measure supported on contact points is not only implied by, but also implies that K is in John position. For the setting in which both bodies are not the unit Euclidean ball, this characterization is not valid, since we do not have uniqueness of the maximal volume position. However, one does obtain an "if and only if" characterization of the position by the existence of a decomposition of the identity when considering a modification of the above position, namely the positive John position: Let K, L be convex bodies with non-empty interior. A positive image of L in K is a set of the form $PL + v$ contained in K, with $v \in \mathbb{R}^n$ and P a positive-definite matrix. We say that K *is in positive John position* with respect to L if $L \subseteq K$ and L has maximal volume among all positive images of L in K. The positive John position was defined by Artstein and Putterman in [7], see also [13]. The advantage of working with the positive John position is due to the following result.

Proposition 0.1 ([7], Proposition 3.1)**.** *Let K, L be convex bodies with the origin in the interior of K, and consider the set of positive images of L inside K,*

$$\mathcal{A}_{K,L} = \{PL + v : P \text{ is defined positive}, v \in \mathbb{R}^n \text{ and } PL + v \subseteq K\}.$$

Then there is a unique element in $\mathcal{A}_{K,L}$ of maximal volume.

From this result Artstein and Putterman presented a proof for the following theorem that has been proven by different methods in [13, 26].

Theorem 0.4 ([7], Theorem 1.2)**.** *Let K, L be convex bodies with the origin in the interior of K. Then K is in positive John position with respect to L if and only if $L \subseteq K$ and there exist contact points $x_1, \ldots, x_m$ of K and L, contact points $y_1, \ldots, y_m$ of the polar bodies K° and L° and $c_1, \ldots, c_m > 0$ such that*

$$\langle x_i, y_i \rangle = 1, \qquad \sum_{i=1}^{m} c_i (x_i \otimes y_i + y_i \otimes x_i) = \mathrm{Id}, \qquad \sum_{i=1}^{m} c_i y_i = 0.$$

We give an explicit representation for a centered and isotropic measure, supported on contact points between K and L, given that K is in positive John position in L.

In this way, we end our interest in constructing a centered and isotropic measure in the geometric version. Our next step is to look for definitions for functional ellipsoids in order to find out if there is a functional version of the decomposition of the identity like the one given in Theorem 0.1.

In 2018, Alonso-Gutiérrez, Gonzales Merino, Jiménez and Villa [4] extended the geometric notion of the John ellipsoid to the setting of log-concave functions. Their idea was as follows: Fixed an integrable log-concave function h on $\mathbb{R}^n$, first they take any constant $\beta \in (0, ||h||)$, where $||h||$ is the L_∞ norm of h, and consider the superlevel set $\{x \in \mathbb{R}^n : h(x) \geq \beta\}$ of h, which is a bounded convex set with non-empty interior. Then they show that there is a unique height $\beta_0 \in [0, ||h||]$ for which $\beta_0 \operatorname{vol}_n(\mathcal{E})$ is maximal, where $\mathcal{E}$ is maximal volume ellipsoid inside the level set. Then they define the John ellipsoid of h as the function $\mathcal{E}^{\beta_0}(x) = \beta_0 1_{\mathcal{E}}(x)$ obtained for this β_0.

Recently, in 2021, Ivanov and Naszódi [30] also extended the notion of the John ellipsoid to the setting of logarithmically concave functions. Unlike the first ones, they defined a class of functions on $\mathbb{R}^n$ indexed by $s > 0$. First they fix a proper log-concave function (log-concave and upper semicotinuous function that has finite positive integral) $h : \mathbb{R}^n \to [0, \infty)$ and $s > 0$. Later they prove that there is (and is unique in the set of proper log-concave functions) one function log-concave with the largest integral under the condition that it is pointwise less than or equal to $h^{1/s}$. They call it the *John s-function of h.* In [30, Theorem 6.1] it is shown that as $s \to 0$, the John s-functions converge to characteristic functions of ellipsoids, that is, there is a relationship between the first [4] and second approach [30]. Furthermore, they study the John s-functions as s tends to infinity. It is shown that the limit may only be a Gaussian density (is not necessarily unique).

Denote by $|x|_2$ the Euclidean norm of $x \in \mathbb{R}^n$. The height function of the unit ball B^n is given by $\hbar_{B^{n+1}}(x) = \sqrt{1 - |x|_2^2}$ if $x \in B^n$ and 0 otherwise. Moreover, this function is proper log-concave. An interesting fact about the second approach is that they give a characterization of the John s-function of h similar to the one given by F. John in his fundamental theorem. Namely,

Theorem 0.5 ([30], Theorem 5.2)**.** *Let h be a proper log-concave function on $\mathbb{R}^n$, $s > 0$. Assume $\hbar^s_{B^{n+1}} \leq h$, where $\hbar_{B^{n+1}}$ is the height function of the unit Euclidean ball B^{n+1}. Then the following are equivalent.*

(1) *The function* $\hbar^s_{B^{n+1}}$ *is the John* s*-function of* h*;*

(2) *There are points* $u_1, \dots, u_k \in B^n \subset \mathbb{R}^n$ *and positive weights* $c_1, \dots, c_k$ *such that*

(a) $h(u_i) = \hbar^s_{B^{n+1}}(u_i)$ *for all* $i = 1, \dots, k$*;*

(b) $\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{Id}$*;*

(c) $\sum_{i=1}^k c_i h(u_i)^{1/s} h(u_i)^{1/s} = s$*;*

(d) $\sum_{i=1}^k c_i u_i = 0$.

For this reason we will adopt this definition of ellipsoid, in order to obtain a "centered and isotropic measure" supported at the points of B^n where $h^{1/s}$ and its John s-function coincide.

In the second part of this thesis we study about valuations. A functional $Z : \mathcal{K}^n \to \mathcal{A}$ is called a *valuation* if

$$Z(K) + Z(L) = Z(K \cap L) + Z(K \cup L)$$

for all $K, L, K \cup L \in \mathcal{K}^n$, where $\mathcal{K}^n$ is the set of convex bodies in $\mathbb{R}^n$ and $\mathcal{A}$ is an abelian semigroup. Valuations play an important role in the geometry of convex bodies. For example, the intrinsic volumes $V_0(K), V_1(K), \dots, V_n(K)$ are valuations on $\mathcal{K}^n$ (see [28]). In particular, $V_0(K)$ is the Euler characteristic, $V_n(K)$ is the Lebesgue volume and $2V_{n-1}(K)$ is the surface area of K. Another important valuation on $\mathcal{K}^n$ is the support function $h_K(\cdot)$. On the space $\mathcal{K}^n_0$ of convex bodies with 0 as interior point, the function which associates each convex body $K \in \mathcal{K}^n_0$ with the polar body $K^\circ \in \mathcal{K}^n_0$ is a valuation on $\mathcal{K}^n_0$.

The interest in classifying valuations on $\mathcal{K}^n$ began with Hadwiger. Probably the most famous result on valuations is the Hadwiger's theorem. It classifies all continuous and rigid motion invariant valuations on the space $\mathcal{K}^n$ equipped with the Hausdorff distance (see [28]). Ludwig and Reitzner established an affine version of Hadwiger's theorem, proving a classification of upper semicontinuous valuations which are invariant under volume preserving maps (see [36]).

Currently, the notion of valuations has been extended to families of functions. We denote by $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ the space of finite-valued, convex functions on $\mathbb{R}^n$. We define valuations on the space $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ and its subspaces taking values in an abelian semigroup as

$$Z(u) + Z(v) = Z(u \wedge v) + Z(u \vee v)$$

for every $u, v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ such that also their pointwise maximum $u \vee v$ and pointwise minimum $u \wedge v$ belong to $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$.

In 2000, Ludwig characterized the rigid motion invariant and upper semicontinuous valuations defined on $\mathcal{K}^2$. Consider the set

$$\mathcal{W} = \left\{ \zeta : [0, +\infty) \to [0, \infty) : \ \zeta \text{ is concave, } \lim_{t \to 0} \zeta(t) = 0, \text{ and } \lim_{t \to +\infty} \zeta(t)/t = 0 \right\}.$$

Her theorem can be stated as follows.

Theorem 0.6 ([37])**.** *Let* $\mu : \mathcal{K}^2 \to \mathbb{R}$ *be an upper semicontinuous and rigid motion invariant* *valuation. Then there are constants* $c_0, c_1, c_2 \in \mathbb{R}$ *and a function* $\zeta \in \mathcal{W}$ *such that*

$$\mu(K) = c_0\chi(K) + c_1 L(K) + c_2 A(K) + \int_{S^1} \zeta(\rho(K,u)) d\mathcal{H}^1(u) \tag{2}$$

for every $K \in \mathcal{K}^2$.

Here $L(K)$ and $A(K)$ are the length and area of K, respectively, and $\rho(K,u)$ is the curvature radius of the boundary of K at the point with normal $u \in S^1$.

Our goal is to prove a functional version of Theorem 0.6 for $n = 1$.

This work is structured as follows. In Chapter 1, we give a very short introduction to basic convexity, we speak about the gauge and support functions. Later we present results already known in the literature that are necessary to prove our results.

In Chapter 2, we present in details the problem of explicit representations of centered and isotropic measures in John and Löwner positions as a simple finite dimensional minimization problem whose solution (when it exists) can be used to construct a non-negative centered isotropic measure. These results are published in the International Mathematics Research Notices https://doi.org/10.1093/imrn/rnac269. Later, we present the construction of centered and isotropic measures in positive John and positive Löwner positions. The results are not for publication due to similarity with the previous case.

In Chapter 3, we study the theory of the functional John ellipsoid due to Ivanov and Naszódi [30]. Next, we introduce some news concepts related to this theory in order to construct explicitly, as in the geometric case, a decomposition of the identity.

In Chapter 4, we give a introduction to valuations on convex bodies and on convex functions and we obtain a classification of upper semicontinuous and translation invariant valuations on the space of convex functions which is a piecewise linear function outside of a compact set of $\mathbb{R}$.

Chapter 1

Preliminaries

In this chapter, we briefly review basic definitions, clarify the main notations and collect some results that will be used in this work.

1.1 Basic convexity

We work in the space $\mathbb{R}^d$ equipped with the standard inner product

$$\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

defined as

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = x_1 y_1 + \cdots + x_n y_n,$$

and denote the canonical basis vectors by $e_1, \ldots, e_d$. Here, $|\cdot|_2 = \sqrt{\langle \cdot, \cdot \rangle}$ denotes the usual *Euclidean norm* in $\mathbb{R}^d$. The *unit Euclidean ball* in the normed space $(\mathbb{R}^d, |\cdot|_2)$ is the set $B^d = \{x \in \mathbb{R}^d : |x|_2 \leq 1\}$, and its boundary $S^{d-1} = \{x \in \mathbb{R}^d : |x|_2 = 1\}$ is the *unit sphere*.

A subset $A \subseteq \mathbb{R}^d$ is called *convex* if for any $x, y \in A$ and $\lambda \in [0,1]$ it holds $(1-\lambda)x + \lambda y \in A$. In others words, a subset A is convex when it contains any segment $[x, y]$, where $x, y \in A$.

A function $f : \mathbb{R}^d \to (-\infty, +\infty]$ is said to be *convex* if

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y) \tag{1.1}$$

for any $x, y \in \mathbb{R}^d$ and $\lambda \in [0,1]$. A function $f : \mathbb{R}^d \to [0, \infty)$ is *log-concave* if $f = e^{-\psi}$ for some convex function $\psi : \mathbb{R}^d \to (-\infty, \infty]$. We make the convention that $e^{-\infty} = 0$. The name justifies because a function $g : \mathbb{R}^d \to [-\infty, \infty)$ is said to be *concave* if -g is convex. A direct computation using (1.1) give us the following result.

Lemma 1.1. *A function $f : \mathbb{R}^d \to [0, \infty)$ is log-concave if and only if*

$$f(\lambda x + (1-\lambda) y) \geq f(x)^\lambda f(y)^{1-\lambda},$$

for any $x, y \in \mathbb{R}^d$ and every $\lambda \in (0,1)$.

For every convex function $f : \mathbb{R}^d \to \mathbb{R}$ and for any $t \in (-\infty, +\infty]$ we can consider the *sublevel sets*

$$\{f < t\} := \{x \in \mathbb{R}^d : f(x) < t\}, \quad \{f \leq t\} := \{x \in \mathbb{R}^d : f(x) \leq t\},$$

which are convex sets. Then, the *(effective) domain* of f is defined as the set

$$\operatorname{dom} f := \{f < +\infty\}.$$

The *epigraph* of f is the set

$$\operatorname{epi}(f) = \{(x,t) \in \mathbb{R}^{d+1} : f(x) \leq t\},$$

and $f : \mathbb{R}^d \to (-\infty, +\infty]$ is convex if and only if $\operatorname{epi}(f)$ is a convex subset of $\mathbb{R}^{d+1}$. We write $\operatorname{int} A$ for the *interior* of A. The *topological boundary* of A will be denoted by ∂A.

We say that a function $f : V \to \mathbb{R}$ defined in a vector subspace $V \subseteq \mathbb{R}^d$ is *coercive* if

$$\lim_{|x|_2 \to +\infty} f(x) = +\infty.$$

Theorem 1.1 ([46], Theorem 1.5.3)**.** *Every convex function $f : \mathbb{R}^d \to (-\infty, \infty]$ is continuous on* $\operatorname{int} \operatorname{dom} f$ *and Lipschitzian on any compact subset of* $\operatorname{int} \operatorname{dom} f$*.*

Proposition 1.1 ([46], Corollary 1.5.11)**.** *If $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is twice differentiable, then f is convex if and only if $f'' \geq 0$.*

In the general case, when $f : \mathbb{R}^d \to (-\infty, +\infty]$ is neither smooth nor strictly convex, the gradient of f, denoted by ∇f, exists almost everywhere in $\operatorname{int} \operatorname{dom} f$ by Rademacher's theorem (see, for example, [15]), and a theorem of Alexandrov [1] and Busemann and Feller [17] guarantees the existence of the Hessian, denoted by $\nabla^2 f$, almost everywhere in $\operatorname{int} \operatorname{dom} f$.

Theorem 1.2 ([44], Theorem 2.14)**.** *Let $f : U \to \mathbb{R}$ be twice differentiable on an open convex set $U \subset \mathbb{R}^d$. Then f is convex on U if and only if $\nabla^2 f(x)$ is positive-semidefinite for all $x \in U$.*

For $A \subseteq \mathbb{R}^d$, the set of all convex combinations

$$\operatorname{conv}(A) = \left\{ \sum_{j=1}^{m} \lambda_j x_j : x_j \in A \text{ and } \lambda_j \geq 0 \text{ for any } j = 1, \ldots, m, \text{ and } \sum_{j=1}^{m} \lambda_j = 1 \right\}$$

is called the *convex hull* of A.

The convex hull of a finite set of points is called a *polytope*.

Theorem 1.3 ([46], Theorem 1.1.2)**.** *If $A \subseteq \mathbb{R}^d$ is convex, then* $\operatorname{conv}(A) = A$*. For an arbitrary set $A \subseteq \mathbb{R}^d$,* $\operatorname{conv}(A)$ *is the intersection of all convex subsets of $\mathbb{R}^d$ containing A.*

We end this section with a definition of the central notion of this work.

Definition 1.1. A *convex body* is a set $K \subseteq \mathbb{R}^d$ which is convex, compact and has non-empty interior.

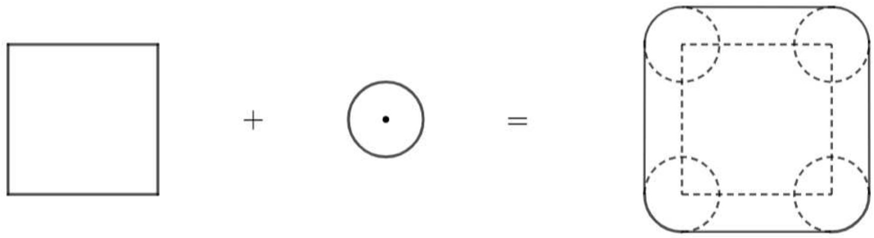
We define the *Minkowski sum* of sets $A, B \subseteq \mathbb{R}^d$ as

$$A + B = \{a + b : a \in A \text{ and } b \in B\}$$

which can be (geometrically) interpreted as the union of all translates of B by the points of A (and vice-versa). Moreover, the scalar multiplication is defined for a given set $A \subseteq \mathbb{R}^d$ and for some $\lambda \in \mathbb{R}$ as

$$\lambda A = \{\lambda a : a \in A\}.$$

Figure 1.1: The Minkowski sum of a square and a ball.



Source: Compiled by the author.

Throughout the text, the set of convex bodies in $\mathbb{R}^d$ will be denoted by $\mathcal{K}^d$, and the set of convex bodies which contain the origin as an interior point will be denoted by $\mathcal{K}^d_0$. Regarding the space $\mathcal{K}^d$ as a metric space is one of the most powerful techniques in convex geometry. For example, it often allows us to solve problems by approximating arbitrary bodies by "well-behaved" bodies, such as polytopes or smooth bodies. There is also a notion of distance between sets in $\mathbb{R}^d$ given by

$$d(A, B) = \inf\{a - b : a \in A \text{ and } b \in B\},$$

for any given $A, B \subset \mathbb{R}^d$. However, this is not completely satisfactory, since this does not define a metric. For example, $d(A, B) = 0$ whenever $A \cap B \neq \emptyset$.

We define the Hausdorff distance between two sets $K, L \in \mathcal{K}^d$ by

$$d_{\mathcal{H}}(K, L) := \inf\{\lambda \geq 0 : L \subseteq K + \lambda B^d \text{ and } K \subseteq L + \lambda B^d\}.$$

It follows immediately from compactness that the minimum exists and is finite. Hence the spaces $\mathcal{K}^d$ naturally become metric spaces with the Hausdorff metric $d_{\mathcal{H}}$. The space $\mathcal{K}^d$ of convex bodies of $\mathbb{R}^d$ is a subset of the space of non-empty compact sets, and we can define a metric in this more general family.

It is sometimes convenient to have a description of the convergence of convex bodies in terms of convergent sequences of points.

Theorem 1.4 ([46], Theorem 1.8.8)**.** *The convergence* $\lim_{j\to+\infty} K_j = K$ *in* $\mathcal{K}^d$ *is equivalent to the following conditions taken together:*

1. *Each point in* K *is the limit of a sequence* $(x_i)_i$ *with* $x_i \in K_i$ *for* $i \in \mathbb{N}$*;*

2. The limit of any convergent sequence $(x_{i_j})_j$ with $x_{i_j} \in K_{i_j}$ for each $i, j \in \mathbb{N}$ belongs to K.

1.2 Gauge and support functions

Let $A \subseteq \mathbb{R}^d$ a non-empty set. The *support function* h_A of A is defined by

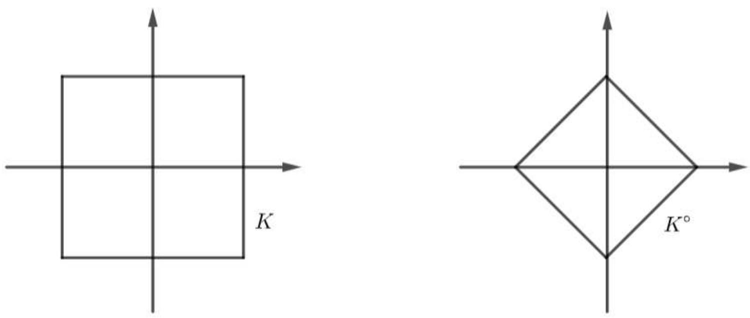
$$h_A(u) = \sup\{\langle x, u\rangle : x \in A\} \text{ for } u \in \mathbb{R}^d.$$

Observe that h_A enjoys the property $h_A(\lambda u) = \lambda h_A(u)$ for any $u \in \mathbb{R}^d$ and every $\lambda \geq 0$, that is, h_A is positively homogeneous. A function $f : \mathbb{R}^d \to (-\infty, +\infty]$ is *lower semicontinuous* if its epigraph is a closed set of $\mathbb{R}^d$. The support function $h_A : \mathbb{R}^d \to (-\infty, +\infty]$ of a non-empty set $A \subseteq \mathbb{R}^d$ is convex and lower semicontinuous. Moreover, for a non-empty set $A \subseteq \mathbb{R}^d$ it holds $h_A(u) < +\infty$ for every $u \in \mathbb{R}^d$ if and only if A is bounded.

Let $A \subseteq \mathbb{R}^d$ a non-empty set. The *polar set* of A is the set

$$A^\circ = \{x \in \mathbb{R}^d : \langle x, z\rangle \leq 1, \forall z \in A\} = \{x \in \mathbb{R}^d : h_A(x) \leq 1\}.$$

Figure 1.2: The convex body $K = [-1, 1]^2$ and its polar set $K^\circ = \text{conv}\{\pm(1, 0), \pm(0, 1)\}$.



Source: Compiled by the author.

Theorem 1.5 ([46], Theorem 1.6.1)**.** *If $K \in \mathcal{K}_0^d$, then $K^\circ \in \mathcal{K}_0^d$. The converse is also true.*

We define the *gauge function* of a set $K \in \mathcal{K}^d$ to be the function $||\cdot||_K : \mathbb{R}^d \to \mathbb{R}$ such that

$$||x||_K = \inf\{\lambda > 0 : x \in \lambda K\}. \tag{1.2}$$

The next results can be found in [46, Section 1.7].

Proposition 1.2. *Let K be a convex body containing the origin as an interior point. Then the gauge function $||\cdot||_K$ is a non-negative and finite-valued function satisfying:*

1. $||x||_K \geq 0$ *for all* $x \in \mathbb{R}^d$, *with equality if and only if* $x = 0$;
2. $||\lambda x||_K = \lambda||x||_K$ *for every* $\lambda \geq 0$;
3. $||x + y||_K \leq ||x||_K + ||y||_K$ *for any* $x, y \in \mathbb{R}^d$.

Remark 1.1. Notice that, in particular, the gauge function of a convex body $K \in \mathcal{K}^d$ is sublinear. Also, it only fails to be a norm because it is not homogeneous, but only positively homogeneous.

Corollary 1.1. *Let $K \in \mathcal{K}_0^d$. Then K is centered at the origin if and only if its gauge function is a norm. In this case, the unit ball is K.*

Proposition 1.3. *Let $K \in \mathcal{K}_0^d$. Then $||\cdot||_K$ is differentiable almost everywhere. We have that $||\cdot||_K$ is differentiable at $x \in \partial K$ if and only if it is differentiable at tx for any $t > 0$. In this case, we have the equality*

$$\nabla ||tx||_K = \nabla ||x||_K.$$

Moreover, if $||\cdot||_K$ is differentiable at $x \in \partial K$, then K has a unique unit outer normal vector $n^K(x)$ at x, and

$$\nabla ||x||_K = \frac{n^K(x)}{h_K(n^K(x))}. \tag{1.3}$$

Consequently, $\nabla ||x||_K$ is non-zero and

$$|\nabla ||x||_K| = \frac{1}{h_K(n^K(x))} \tag{1.4}$$

whenever $||\cdot||_K$ is differentiable at $x \in \partial K$.

We say that a convex body K is C^k or that it has a C^k-smooth boundary if $||x||_K$ is a C^k function in $\mathbb{R}^d \setminus \{0\}$. A boundary point x of K is said to be a smooth point if K has a unique unit outer normal vector at x. As a consequence of the previous proposition, for any $K \in \mathcal{K}_0^d$ a point $x \in \partial K$ is a smooth point of ∂K if and only if $||\cdot||_K$ is differentiable at x.

1.3 Measure

In what follows, we recall that

$$\mathrm{vol}_d(A) = \int_{\mathbb{R}^d} 1_A(x)dx,$$

for every Borel measurable set $A \subseteq \mathbb{R}^d$, where $\mathrm{vol}_d(\cdot)$ denotes the usual Lebesgue measure and dx stands for the integration with respect to that measure. We also will denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure for $0 < k \leq d$.

Let $K \subset \mathbb{R}^d$ be a convex body. Since K is compact and has non-empty interior, then

$$0 < \mathrm{vol}_d(K) < +\infty.$$

It is worth mentioning that $\mathrm{vol}_d(K)$ is the same as the d-dimensional Hausdorff measure $\mathcal{H}^d(K)$. The surface area of K is defined to be the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}(\partial K)$ of its boundary. It is very common to denote the $(d-1)$-Hausdorff measure on $\mathbb{R}^d$ by $\mathrm{vol}_{d-1}(K)$, and sometimes we will adopt this notation.

Theorem 1.6 ([41], Theorem 8.5)**.** *If $A \subseteq \mathbb{R}^d$ is a Borel set and $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear map,* *then $T(A)$ is a Borel set and*

$$\mathrm{vol}_d(T(A)) = |\det(T)|\,\mathrm{vol}_d(A).$$

The co-area formula gives the integral (with respect to the Lebesgue measure) of an integrable function over an open subset of $\mathbb{R}^d$ in terms of integrals of this function over the level sets of a given Lipschitz function.

Theorem 1.7 ([41], Theorem 18.1)**.** *(Co-area formula) Let $U \subset \mathbb{R}^d$ be an open set and $u : U \to \mathbb{R}$ be a Lipschitz function. For any integrable function $f : U \to \mathbb{R}$ we have*

$$\int_U f(x)|\nabla u(x)|dx = \int_{\mathbb{R}} \left(\int_{u^{-1}(\{t\})} f(y) d\mathcal{H}^{d-1}(y) \right) dt.$$

For us, the most important consequence of the co-area formula is the formula for integration by polar coordinates.

Theorem 1.8. *(Integration by polar coordinates) Let $f : \mathbb{R}^d \to \mathbb{R}$ be an integrable function. Then*

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty \int_{S^{d-1}} f(t\xi)t^{d-1} d\mathcal{H}^{d-1}(\xi)dt.$$

An other consequence of the co-area formula is that one can derive a similar formula where the unit sphere S^{d-1} is replaced by a convex body having the origin as interior point.

Proposition 1.4. *Let $K \subset \mathbb{R}^d$ be a convex body which has the origin as interior point. Then*

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty \int_{\partial K} f(tz)t^{d-1} h_K(n^k(z)) d\mathcal{H}^{d-1}(z)dt$$

for any integrable function $f : \mathbb{R}^d \to \mathbb{R}$.

We define the *support* of a function $f : \mathbb{R}^d \to (-\infty, +\infty]$ to be the set

$$\mathrm{supp}(f) = \{x \in \mathbb{R}^d : f(x) \neq 0\}.$$

Let $\Omega \subseteq \mathbb{R}^d$ be an open set on $\mathbb{R}^d$. We say that a Borel measure μ is a *Radon measure* if $\mu(C) < +\infty$ for every compact set $C \subset \Omega$. Consider μ_k a sequence of Radon measures in $\Omega \subseteq \mathbb{R}^d$. We say that μ_k *converges weakly* to a Radon measure μ in Ω if

$$\lim_{k \to +\infty} \int_\Omega \beta(x) d\mu_k(x) = \int_\Omega \beta(x) d\mu(x) \tag{1.5}$$

for every function β which is continuous with compact support on Ω.

Theorem 1.9 ([41], Proposition 4.26)**.** *If $\{\mu_k\}_{k\in\mathbb{N}}$ and μ are Radon measures on $\mathbb{R}^d$, then the following three statements are equivalent.*

(i) μ_k weakly converges to μ;

(ii) If K is compact and A is open, then

$$\mu(K) \geq \limsup_{k\to+\infty} \mu_k(K)$$
$$\mu(A) \geq \liminf_{k\to+\infty} \mu_k(A);$$

Theorem 1.10 ([12], Fatou's Lemma 4.8)**.** *Let (X, Σ_X, μ) be a measure space and $(f_n)_n$ be a sequence of non-negative measurable functions. Then*

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu,$$

where $\liminf_{n\to\infty} f_n$ denotes the function which associates each $x \in X$ to the inferior limit of the sequence $(f_n(x))_n$.

Theorem 1.11 ([12], Lebesgue dominated convergence theorem 5.6)**.** *Let (X, Σ_X, μ) be a measure space, and let $g : X \to [0, +\infty)$ be a non-negative integrable function. Let $\{f_m\}_m$ be a sequence of real functions satisfying:*

1. *$|f_m(x)| \leq g(x)$ for any $m \in \mathbb{N}$ and every $x \in X$, and*
2. *for μ-almost every $x \in X$ the sequence $(f_m(x))_m$ converges in $\mathbb{R}$.*

Then the function $f : X \to \mathbb{R}$ defined as

$$f(x) = \lim_{m\to\infty} f_m(x)$$

for each $x \in X$ is integrable and

$$\int_X f d\mu = \lim_{m\to\infty} \int_X f_m d\mu.$$

Proposition 1.5 ([12], Corollary 4.9)**.** *Let (X, Σ_X, μ) be a measure space and let $g : X \to [0, \infty]$ be an Σ_X-measurable function. Then,*

$$\mathcal{V} : \Sigma_X \to [0, \infty], \quad A \mapsto \int_A g d\mu$$

is an outer measure.

Theorem 1.12 ([45], Theorem 3.3)**.** *(Jensen's inequality) Consider (X, Σ_X, μ) a probability space. Let $D \subseteq \mathbb{R}$ be an open interval, and let $\varphi : D \to \mathbb{R}$ a convex function. If X is a topological space with a Borel probability measure and $f : X \to D$ is an integrable function, then*

$$\varphi\left(\int_X f d\mu\right) \leq \int_X \varphi \circ f d\mu. \tag{1.6}$$

Let λ, μ be positive measures on a σ-algebra $\mathcal{F}$. We say that λ is absolutely continuous with respect to μ, and write $\lambda \ll \mu$ if $\lambda(E) = 0$ for every $E \in \mathcal{F}$ for which $\mu(E) = 0$. If there exists a pair of disjoint sets A and B such that λ is concentrated on A and μ is concentrated on B, then we say that λ and μ are mutually singular, and write $\lambda \perp \mu$.

Theorem 1.13 ([45], Theorem of Lebesgue-Radon-Nikodym 6.10). *Let μ be a positive σ-finite measure on a σ-algebra Σ_X in a set X, and let λ be a positive measure on Σ_X. Then λ has a unique decomposition as $\lambda_1 + \lambda_2$, where λ_1 and λ_2 are positive measures on Σ_X such that $\lambda_1 \ll \mu, \lambda_2 \perp \mu$.*

Theorem 1.14 ([45], Theorem 21.4). *(Lusin's Theorem) Let (X, Σ_X, μ) be a Radon measure space and Y be a second-countable topological space equipped with a Borel algebra, and let $f : X \to Y$ be a measurable function. Given $\varepsilon > 0$, for every $A \in \Sigma_X$ of finite measure there is a closed set E with $\mu(A \setminus E) < \varepsilon$ such that f restricted to E is continuous.*

1.4 Linear Algebra

We denote by $\mathrm{M}_d(\mathbb{R})$ the collection of all matrices M of order $d \times d$ with entries in $\mathbb{R}$ and we consider this space equipped with the Frobenius inner product given by

$$\langle A, B \rangle_F = \mathrm{tr}(A^T B) = \sum_{i,j} A_{i,j} B_{i,j} \tag{1.7}$$

where $A = (A_{i,j}), B = (B_{i,j})$ and tr is the trace function defined in $\mathrm{M}_d(\mathbb{R})$. The $d \times d$ identity matrix will be denoted by Id. Note that the trace of a matrix M can be simply described as the Frobenius inner product of M with Id.

The subgroup of $\mathrm{M}_d(\mathbb{R})$ which consists the invertible matrices will be denoted by $\mathrm{GL}_d(\mathbb{R})$. The subgroup of $\mathrm{GL}_d(\mathbb{R})$ consisting of all matrices whose determinant equals 1 will be denoted by $\mathrm{SL}_d(\mathbb{R})$ and the subgroup consisting of the orthogonal matrices will be denoted by $\mathrm{O}_d(\mathbb{R})$. By Theorem 1.6 the matrices in $\mathrm{SL}_d(\mathbb{R})$ preserve Lebesgue volume, that is, if $A \subseteq \mathbb{R}^d$ is a Borel set and $T \in \mathrm{SL}_d(\mathbb{R})$, then $\mathrm{vol}_d(A) = \mathrm{vol}_d(TA)$.

The space $\mathrm{M}_d(\mathbb{R})$ with the operator norm

$$||T||_{op} = \max\{|Tx|_2 : x \in B^d\}$$

is a normed space with dimension d^2. The general linear group $\mathrm{GL}_d(\mathbb{R})$ is an open subset of $\mathrm{M}_d(\mathbb{R})$ since $\mathrm{GL}_d(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ and the determinant is a continuous function of $\mathrm{M}_d(\mathbb{R})$. Since $\mathrm{SL}_d(\mathbb{R}) = \det^{-1}(\{1\})$, we get that the special linear group is a closed group of $\mathrm{M}_d(\mathbb{R})$.

Observe that the topology defined on $\mathrm{M}_d(\mathbb{R})$ by the Frobenius inner product is the same topology induced by the operator norm, since all norms in a finite-dimensional vector space give rise to the same topology.

The next result is a classic result that can be found in most undergraduate linear algebra books (see, for example, [41, page 78]).

Lemma 1.2. *(Polar Decomposition) For every $T \in \mathrm{GL}_d(\mathbb{R})$ there exist $U, P \in \mathrm{GL}_d(\mathbb{R})$ where U is orthogonal and P is symmetric and positive-definite such that $T = PU$.*

We will use the notation $\mathrm{Sym}_d(\mathbb{R})$ for the subgroup of $\mathrm{M}_d(\mathbb{R})$ of symmetric matrices and

$\mathrm{Sym}_{d,+}(\mathbb{R})$ for the subgroup of symmetric and positive-definite matrices. Also define $\mathrm{Sym}_{d,a}(\mathbb{R}) = \{A \in \mathrm{Sym}_d(\mathbb{R}) : \mathrm{tr}(A) = a\} \subset \mathrm{M}_d(\mathbb{R})$ for $a \in \mathbb{R}$. Since we can "choose" $\frac{1}{2}d(d+1)$ entries for a symmetric matrix, it follows that

$$\dim(\mathrm{Sym}_d(\mathbb{R})) = \frac{1}{2}d(d+1).$$

Notice that if $T \in \mathrm{SL}_d(\mathbb{R})$ and $T = UP$ for $U \in \mathrm{O}_d(\mathbb{R})$ and $P \in \mathrm{Sym}_{d,+}(\mathbb{R})$, then we have that $P \in \mathrm{SL}_d(\mathbb{R})$.

We denote by $v \otimes w$ the rank-one matrix $(v \otimes w)_{i,j} = v_i w_j$. Under the canonical isomorphism between the linear map and its matrix representation in the canonical basis of $\mathbb{R}^d$, the matrix $v \otimes w = v \cdot w^T$ is identified with the map $(v \otimes w)(x) = \langle x, w\rangle v$.

Lemma 1.3. *Let $u, v \in \mathbb{R}^d$ and $T \in \mathrm{M}_d(\mathbb{R})$. Then*

$$\langle Tu, v\rangle = \langle T, v \otimes u\rangle_F.$$

Proof. Since both sides in the equality are linear in u and v (and also in T), it is sufficient to consider the case where $u = e_i$ and $v = e_k$ for some $i, k = 1, \ldots, d$. If $T = (t_{i,j})_{i,j=1}^d$, then

$$\langle Tu, v\rangle = \langle (t_{1i}, \ldots, t_{di}), e_k\rangle = t_{ki}.$$

On the other hand, a direct computation shows that the matrix $e_k \cdot e_i^T$ which represents $v \otimes u$ has 0 in all entries, except for the entry in the position ki, which is 1. Hence, by the definition of matrix inner product, we have

$$\langle T, v \otimes u\rangle_F = t_{ki} = \langle Tu, v\rangle,$$

as we wanted to prove. □

We consider the space $\mathrm{M}_d(\mathbb{R}) \times \mathbb{R}^l$, where $l \in \mathbb{N}$, equipped with the inner product

$$\langle (A, v), (B, w)\rangle = \langle A, B\rangle_F + \langle v, w\rangle_2,$$

and for simplicity we will denote only

$$\langle (A, v), (B, w)\rangle = \langle A, B\rangle + \langle v, w\rangle. \tag{1.8}$$

For $(A, v) \in \mathrm{M}_d(\mathbb{R}) \times \mathbb{R}^l$, we use $||(A, v)|| = \sqrt{||A||_F^2 + |v|_2^2}$ which is the norm induced by the inner product (1.8).

Lemma 1.4 ([5], Lemma 2.1.5)**.** *Let $A, B \in \mathrm{GL}_d(\mathbb{R})$ be symmetric and positive-definite linear matrices, and let $\lambda \in (0, 1)$. Then*

$$\det((1-\lambda)A + \lambda B) \geq \det(A)^{1-\lambda} \det(B)^{\lambda}, \tag{1.9}$$

and equality holds if and only if $A = B$.

The set $\mathrm{Sym}_{d,+}(\mathbb{R})$ is an open convex cone in $\mathrm{Sym}_d(\mathbb{R})$ with apex at the origin. The inequality (1.9) shows that the set

$$\mathcal{D} = \{T \in \mathrm{Sym}_{d,+}(\mathbb{R}) : \det(T) \geq 1\} \tag{1.10}$$

is a convex set. The following numerical inequality, which is a particular case of Theorem 1.12 (Jensen's inequality) is known as the arithmetic-geometric inequality.

Lemma 1.5 ([45], page 63). *Let $a, b > 0$ and let $\lambda \in (0, 1)$. Then*

$$\lambda a + (1 - \lambda)b \geq a^{\lambda} b^{1-\lambda}.$$

Equality holds if and only if $a = b$.

Theorem 1.15 ([29], Theorem 9). *(Spectral theorem) Let $T \in \mathrm{M}_d(\mathbb{R})$ be symmetric. There exists an orthonormal basis $\{v_1, \ldots, v_d\}$ of $\mathbb{R}^d$ and numbers $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$, such that $Tv_j = \lambda_j v_j$ for each $j = 1, \ldots, d$. In others words, $\mathbb{R}^d$ has an orthogonal basis of eigenvectors of T and*

$$T = U[\mathrm{diag}(\lambda_1, \ldots, \lambda_d)]U^{-1}$$

where U is the orthogonal matrix whose columns are $v_1, \ldots, v_d$ and, in particular $\det(T) = \lambda_1 \cdots \lambda_d$. Moreover, if T is also positive-definite, then $\lambda_1, \ldots, \lambda_d > 0$.

Here $\mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ is the matrix whose entry $x_{i,j}$ is zero for $i \neq j$ and it is λ_i for $i = j$, for each $i, j = 1, \ldots, d$.

1.5 Additional results

The technique of Lagrange multipliers allows us to maximize/minimize a function, subject to an implicit constraint. Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be a C^1 function, $C \in \mathbb{R}^n$ and $M = \{f = C\} \subseteq \mathbb{R}^d$. Now suppose we are given a function $h : \mathbb{R}^d \to \mathbb{R}$, and we want to find the local extremum of h on M. That is, we want to minimize or maximize h subject to the constraint $f = C$.

Theorem 1.16 ([44], Theorem 6.12). *(Lagrange multipliers) Let $h : \mathbb{R}^d \to \mathbb{R}, f : \mathbb{R}^d \to \mathbb{R}^n$ be C^1 functions, $C \in \mathbb{R}^n$ and $M = \{f = C\} \subseteq \mathbb{R}^d$. Assume that for all $x \in M$, rank$(f'(x)) = n$. If h attains a constrained local extremum at a, subject to the constraint $f = C$, then there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that*

$$\nabla h(a) = \sum_{i=1}^{n} \lambda_i \nabla f_i(a).$$

Theorem 1.17 ([44], Theorem 9.60). *(Rademacher) Let $U \subseteq \mathbb{R}^d$ be an open set and $f : U \subseteq \mathbb{R}^d \to \mathbb{R}$ a Lipschitz function. Then f is differentiable almost everywhere.*

Theorem 1.18 ([18], Theorem 5.21). *(Taylor expansion) A function $f : \mathbb{R}^d \to \mathbb{R}$ of class C^2 around x_0 admits at x_0 the following Taylor expansion of order two*

$$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0) \cdot Hf(x_0)(x - x_0) + o(|x - x_0|_2^2),$$

where $x \to x_0, \lim_{x\to x_0} \frac{o(|x-x_0|_2^2)}{|x-x_0|_2^2} = 0$ *and* $Hf(x_0)$ *is the Hessian matrix of* f *at* x_0*.*

Lemma 1.6 (22, Lemma A.1)**.** *Let* $A, B \in M_d(\mathbb{R})$*, and assume that* A *is invertible. Then,*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \det(A+\varepsilon B) = \det(A)\langle A^{-1}, B\rangle.$$

Theorem 1.19 ([33], Proposition 3.1)**.** *Let* $U \subseteq \mathbb{R}^d$ *be open and* $f : U \subseteq \mathbb{R}^d \to \mathbb{R}^{d-m}$ *of class* $C^k, k \geq 1$*. If* $c \in \mathbb{R}^{d-m}$ *is a regular value of* f*, then* $f^{-1}(c)$ *is either empty or an* m*-dimensional surface. For very* $p \in f^{-1}(c), T_p(f^{-1}(c))$ *is the Kernel of* $f'(p) : \mathbb{R}^d \to \mathbb{R}^{d-m}$*.*

Chapter 2

On Explicit representations of Isotropic Measures in some positions

This chapter is based in the work "On Explicit Representations of Isotropic Measures in John and Löwner Positions", published in the International Mathematics Research Notices, https://doi.org/10.1093/imrn/rnac269. We construct a non-negative centered isotropic measure from the convex body K, which is in John position, whose existence is guaranteed by John's Theorem 2.3. The method we propose requires the minimization of a convex function defined in a $\frac{n(n+3)}{2}$-dimensional vector space. Furthermore, we find a geometric interpretation of the minimizer of this convex function.

2.1 Existence of isotropic measures in John and Löwner positions

The main objective in this section is to present the so-called John's theorem, characterizing the John ellipsoid of a convex body.

An *invertible affine transformation* A is a linear function composed with a translation, that is, $A(\cdot) = T(\cdot) + v_0$ where $T \in \mathrm{GL}_n(\mathbb{R})$ and $v_0 \in \mathbb{R}^n$. The image of a set $U \subseteq \mathbb{R}^n$ under an affine transformation is called an *affine image* of U. An affine image of a convex body $K \subset \mathbb{R}^n$ is called a *position* of K. An affine image of the unit Euclidean ball B^n is called an *ellipsoid*.

By Polar Decomposition 1.2, each operator $A \in \mathrm{GL}_n(\mathbb{R})$ can be written as $A = PU$ where P is symmetric and positive-definite, and U is an orthogonal map. Then to obtain the ellipsoids of $\mathbb{R}^n$ it need not to consider all matrices in $\mathrm{M}_n(\mathbb{R})$, but only the symmetric and positive-definite ones. The advantage of working in the space $\mathrm{Sym}_{n,+}(\mathbb{R})$ is that by the Spectral Theorem 1.15 each matrix $P \in \mathrm{Sym}_{n,+}(\mathbb{R})$ is diagonalizable with an orthogonal basis of vectors.

The first result of this section states that any convex body $K \subset \mathbb{R}^n$ contains a unique ellipsoid of maximal volume.

Theorem 2.1 ([5], Proposition 2.1.6)**.** *If K is a convex body in $\mathbb{R}^n$, then there exists a unique* *ellipsoid* $\mathcal{E}_J^K \subseteq K$ *such that*

$$\mathrm{vol}_n(\mathcal{E}_J^K) = \sup\{\mathrm{vol}_n(\mathcal{E}) : \mathcal{E} \subseteq K \textit{ and } \mathcal{E} \textit{ is an ellipsoid}\}.$$

We call $\mathcal{E}_J^K$ *the John ellipsoid of* K.

A construction that is dual to John ellipsoid is the Löwner ellipsoid which is the unique ellipsoid of minimal volume containing the convex body K.

Theorem 2.2 ([5], Proposition 2.1.7)**.** *Let* K *be a convex body. There exists a unique ellipsoid* $\mathcal{E}_L^K \supseteq K$ *such that*

$$\mathrm{vol}_n(\mathcal{E}_L^K) = \inf\{\mathrm{vol}_n(\mathcal{E}) : K \subseteq \mathcal{E} \textit{ and } \mathcal{E} \textit{ is an ellipsoid}\}.$$

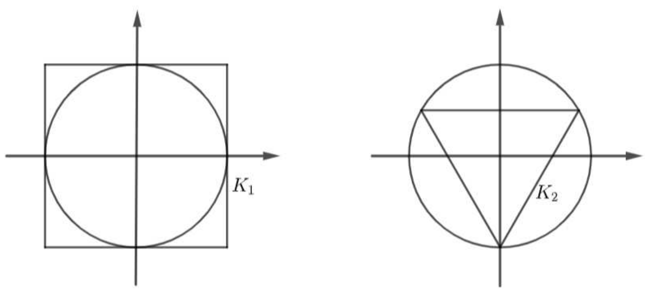
We call $\mathcal{E}_L^K$ *the Löwner ellipsoid of* K.

We say that a convex body is in *John position* if $\mathcal{E}_J^K = B^n$, and in *Löwner position* if $\mathcal{E}_L^K = B^n$. Notice that these definitions make sense, because an affine transformation preserves inclusion, transforms ellipsoids in ellipsoids and by Theorem 1.6 multiply every volume by the same constant. In other words, if $\mathcal{E}_J^K = T(B^n) + v_0$ is the John ellipsoid of K, then we have

$$B^n = T^{-1}(\mathcal{E}_J^K) - T^{-1}(v_0) \subseteq T^{-1}(K) - T^{-1}(v_0)$$

and B^n is the John ellipsoid of the position $T^{-1}(K) - T^{-1}(v_0)$ of K. Moreover, if K is in John position, then $T(K) + w_0$ is in John position if and only if $w_0 = 0$ and $T \in \mathrm{O}_n(\mathbb{R})$, that is, the John position of a convex body is unique up to orthogonal transformations. The same holds for the Löwner position.

Figure 2.1: K_1 is in John position and K_2 is in Löwner position.



Source: Compiled by the author.

Let $K, L \subseteq \mathbb{R}^n$ be convex bodies. A point $x \in \partial K \cap \partial L$ with the property that K and L are supported by a common hyperplane at x is called a *contact point* of K, L. When $L \subseteq K$, then any point $x \in \partial K \cap \partial L$ is a contact point between K and L, because any hyperplane which supports K at x also supports L at x.

Notice that if K is a convex body in John (Löwner) position, then the intersection $S^{n-1} \cap \partial K$

must be non-empty, otherwise we would have that $\alpha := \mathrm{dist}(S^{n-1}, \partial K) > 0$ from where

$$\left(1 + \frac{\alpha}{2}\right) B^n \subseteq K,$$

and this would contradict the fact that B^n is the ellipsoid of maximum volume contained in K.

In what follows we introduce a theorem due to Fritz John which characterizes how the contact points of a convex body in John position and its John ellipsoid are distributed.

Theorem 2.3 ([32], Application 4, pag. 199-200)**.** *Assume $K \in \mathcal{K}^n$ is in John (resp. Löwner) position, then there exists a finite set of points $\{\xi_1, \ldots, \xi_m\} \subset S^{n-1} \cap \partial K$, positive numbers $\{c_1, \ldots, c_m\} \subset \mathbb{R}$ and $\lambda \neq 0$, for which*

$$\sum_{i=1}^{m} c_i \xi_i \otimes \xi_i = \lambda \,\mathrm{Id} \quad \textit{and} \quad \sum_{i=1}^{m} c_i \xi_i = 0. \tag{2.1}$$

The equality in (2.1) is called a *decomposition of the identity.* Note that taking traces in the first equality of (2.1) we obtain $\sum_{i=1}^m c_i = n\lambda$, since $\mathrm{tr}(\xi_i \otimes \xi_i) = |\xi_i|_2 = 1$. This determines the value of λ.

Recalling that integration of $\mathbb{R}^n$ and $\mathrm{M}_n(\mathbb{R})$-valued functions is understood to be coordinatewise, we have the following definition.

Definition 2.1. A measure μ on the sphere S^{n-1} is said to be *isotropic* if for some $\lambda \neq 0$ holds

$$\int_{S^{n-1}} (\xi \otimes \xi) d\mu = \lambda \,\mathrm{Id} \tag{2.2}$$

and *centered* if

$$\int_{S^{n-1}} \xi d\mu = 0. \tag{2.3}$$

Then one can see that equation (2.1) can be expressed as the fact that the atomic measure $\mu_K = \sum_{i=1}^m c_i \delta_{\xi_i}$ is centered and isotropic.

Later Ball [11] proved the sufficiency part, that is, that the existence of a non-negative centered isotropic measure μ_K in the set of contact points, guarantees that K is in John position if $B^n \subseteq K$, or in Löwner position if $K \subseteq B^n$.

The existence of the measure μ_K in Theorem 2.3 is often shown in a non-constructive way. The first proof is due to Fritz John. In [32] he proves the necessity part of the theorem using the following result, which is an extension of the method of Lagrange multipliers to the case where the number of constraints may be infinite.

Theorem 2.4 ([32], Necessary conditions for a minimum, pag. 198-200)**.** *Let V be a real vector space of dimension n and U an open neighborhood in $V, F : U \to \mathbb{R}$ a C^1 function, S a compact metric space and $G : U \times S \to \mathbb{R}$ a continuous function such that $\nabla_u G(u, s)$ exists for every $u \in U, s \in S$ and $\nabla_u G$ is continuous on S. (In optimization terms, F is the objective function* *and G represents the set of constraints.) Let $A = \{u \in U : G(u,s) \leq 0,\ \forall s \in S\}$ (the feasible set) and $u_0 \in A$ such that $F(u_0) = \max_{u \in A} F(u)$. Then either $\nabla_u F(u_0) = 0$, or there exist $s_1, \dots, s_m \in S, m \leq n$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}^+$ such that $G(u_0, s_i) = 0$ for all $i = 1, \dots, m$ and*

$$\nabla_u F(u_0) = \sum_{i=1}^{m} \lambda_i \nabla_u G(u_0, s_i).$$

Other authors usually prove first that it is impossible to separate $\left(\frac{\mathrm{Id}}{n}, 0\right) \in \mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n$ from the set

$$\{(\xi \otimes \xi, \xi) \in \mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n : \xi \in S^{n-1} \cap \partial K\}$$

with linear functionals. The same approach can be found in [5, Section 2.1.3] and in [23, 27].

2.2 Maximal intersection position of radius r

Artstein and Katzin recently defined in [6] a one parameter family of positions of a convex body: the so-called maximum intersection position of radius r. They show that such positions induces an isotropic measure on the sphere when there are some good conditions in K. From that they give an interpretation of John's theorem as a limit case of the measures induced from the maximal intersection positions. The one-parametric family of extremal positions defined by them is the following.

Definition 2.2 ([6], Definition 1.2)**.** For a centrally symmetric convex body $K \subset \mathbb{R}^n$, the ellipsoid $\mathcal{E}_r$ of volume $r^n \operatorname{vol}_n(B^n)$ is a *maximum intersection ellipsoid of radius* r, if

$$\operatorname{vol}_n(K \cap \mathcal{E}_r) \geq \operatorname{vol}_n(K \cap \mathcal{E})$$

for all ellipsoids $\mathcal{E}$ of same volume $r^n \operatorname{vol}_n(B^n)$. We say that K is in *maximal intersection position of radius* r if rB^n is a maximum intersection ellipsoid of radius r.

Recalling that $\mathcal{E}_J^K$ and $\mathcal{E}_L^K$ denote the John ellipsoid and the Löwner ellipsoid of K, respectively, if r_J is a positive number satisfying $\operatorname{vol}_n(\mathcal{E}_J^K) = r_J^n \operatorname{vol}_n(B^n)$ and r_L is such that $\operatorname{vol}_n(\mathcal{E}_L^K) = r_L^n \operatorname{vol}_n(B^n)$, then K is in maximal intersection position of radius r_J if and only if $r_J^{-1}K$ is in John position, and the same holds for the Löwner position, that is, up to a scaling, the maximal intersection position of radius r_J is the John position, and the maximal intersection position of radius r_L is the Löwner position.

Their first result in [6] is the following.

Theorem 2.5 ([6], Theorem 1.3)**.** *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body such that $\operatorname{vol}_n(\partial K \cap \partial \mathcal{E}) = 0$ for all but finitely many ellipsoids $\mathcal{E}$, $\operatorname{vol}_{n-1}(\partial K \cap rS^{n-1}) = 0$, and $\operatorname{vol}_{n-1}(K \cap rS^{n-1}) > 0$. If K is in maximal intersection position of radius r, then the restriction of the surface area measure on the sphere to $S^{n-1} \cap r^{-1}K$ is an isotropic measure.*

When K is in John position we have $r = r_J = 1$ and $S^{n-1} \subset K$. Hence the theorem does not include this case, since it is already known to everyone that the surface area measure on the sphere is isotropic. Other result obtained in [6] is that if K is a convex body in John position and centrally symmetric, the measure given in Theorem 2.3 may be constructed as a limit of the isotropic measures from Theorem 2.5.

Theorem 2.6 ([6], Theorem 1.5). *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body in John position such that $\mathrm{vol}_{n-1}(\partial K \cap \partial \mathcal{E}) = 0$ for all but finitely many ellipsoids $\mathcal{E}$. For every $r > 1$, denote by μ_r the uniform probability measure on $S^{n-1} \setminus r^{-1}T_rK$, where T_rK is in maximal intersection position of radius r. Then there exists a sequence $r_j \searrow 1$ such that the sequence of measures μ_{r_j} weakly converges to an isotropic measure whose support is contained in $\partial K \cap S^{n-1}$.*

For Löwner position, we take $r < 1$ and μ_r as the uniform probability measure on the set $S^{n-1} \cap r^{-1}T_rK$. It is shown, similarly the John position, that under the hypothesis $\mathrm{vol}_{n-1}(\partial K \cap \partial \mathcal{E}) = 0$ for all but finitely many ellipsoids $\mathcal{E}$, there exists a sequence $r_j \nearrow 1$ such that the sequence of measures μ_{r_j} weakly converges to an isotropic measure whose support is contained in $\partial K \cap S^{n-1}$.

About the maximum intersection position of radius r it is also worth mentioning that Artstein and Katzin proved that such a position exists, but it is not yet known about the uniqueness of this position for $r_J < r < r_L$. What is already known, of course, is that if $0 < r < r_J$ or $r > r_L$ then the maximum intersection ellipsoid $\mathcal{E}_r$ of radius r is not unique and by John's theorem if $r = r_J$ or $r = r_L$ then $\mathcal{E}_r$ is unique. However, the case $r_J < r < r_L$ is a consequence of a well-known conjecture:

Conjecture 2.1. For a convex body $K \subset \mathbb{R}^n$ and a diagonal $n \times n$ matrix Λ, the function

$$\phi(t) = \mathrm{vol}_n(e^{t\Lambda}K \cap B^n)$$

is log-concave in t, i.e.,

$$\mathrm{vol}_n\left(e^{\frac{t}{2}\Lambda}K \cap B^n\right)^2 \geq \mathrm{vol}_n(e^{t\Lambda}K \cap B^n)\mathrm{vol}_n(K \cap B^n)$$

for all $t \in \mathbb{R}$ and all diagonal matrix Λ. Furthermore, equality is attained if and only if one of the following hold: $K \subset B^n, B^n \subset K$, or $\Lambda = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{R}$.

Proposition 2.1 ([6], Proposition 4.2). *Assuming Conjecture 2.1 is true, if K is a centrally symmetric convex body, the maximum intersection ellipsoid of radius r is unique for $r_J < r < r_L$.*

To conclude this section, it is important to make it clear that the isotropic measure is thus constructed (in the symmetric case), but one can argue that in practice there is a large body of computations to make. First of all, one has to solve a one-parameter family of minimization problems (find the matrix T_r for r close to 1), and then take the limit of (some subsequence of) all these measures μ_r.

2.3 Explicit representations of Isotropic Measures in John and Löwner positions

In this section we will construct a non-negative centered isotropic measure from the convex body K, which is in John position, whose existence is guaranteed by John's Theorem 2.3. A natural question about the existence of an isotropic measure would be about its usefulness. And the answer is that we can often determine if a body is in John position just observing the distribution of its contact points.

Throughout the text we will consider the following family of functions: for any $r \in (1/2, 1)$ and any function $\gamma : \mathbb{R} \to \mathbb{R}$ define

$$\gamma_r(s) = \gamma\left(\frac{s-1}{1-r}\right).$$

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two measurable functions. We define the functional $L_r : \mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ by

$$L_r(A, v) = \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||Ax+v||_K) g_r(|x|_2)dx,$$

and the functional $I_r : B_r \times \mathbb{R}^n \subseteq \mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ by

$$I_r(M, w) = \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||x||_K) g_r(|(\mathrm{Id} + (1-r)M)^{-1}(x - (1-r)w)|_2)dx,$$

where the domain B_r is the set of matrices M such that $\mathrm{Id} + (1-r)M$ is invertible, and in particular it contains the ball $B(0, (1-r)^{-1})$ in the operator norm.

Observe that if $(A, v) \in \mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^n \subset \mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n$, then

$$\begin{aligned} I_r\left(\frac{A - \mathrm{Id}}{1-r}, \frac{v}{1-r}\right) &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||x||_K) g_r(|A^{-1}(x-v)|_2)dx \\ &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||Ax+v||_K) g_r(|x|_2)dx \\ &= L_r(A, v). \end{aligned} \tag{2.4}$$

Let O be an orthogonal matrix. Since

$$\begin{aligned} L_r(AO, v) &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||AOx+v||_K) g_r(|x|_2)dx \\ &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||Ax+v||_K) g_r(|O^{-1}x|_2)dx \\ &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||Ax+v||_K) g_r(|x|_2)dx \\ &= L(A, v), \end{aligned}$$

then by Polar Decomposition 1.2, it suffices to know the behaviour of L_r restricted to $\mathrm{Sym}_{n,+}(\mathbb{R}) \times \mathbb{R}^n$. In particular, a global minimum of the restriction of L_r to $\mathrm{Sym}_{n,+}(\mathbb{R}) \times \mathbb{R}^n$ is also a global minimum in $\mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n$.

The reason why we define these functionals is because it is known that isotropic measures often show up if some suitable functional is maximized or minimized over all positions of a convex body. Therefore, our objective is to construct such a measure from the functionals L_r and I_r, which, as we have seen, are related to $(A, v) \in \mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^n$. The idea appear from the following lemma.

Lemma 2.1. *Assume that the functional L_r is smooth. Let (A_r, v_r) be a global minimum of the restriction of L_r to $(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$. Then there exists $\lambda_r \neq 0$ such that*

$$\frac{1}{1-r}\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)\nabla||x||_K \otimes x dx = (1-r)\lambda_r \,\mathrm{Id}$$
$$\frac{1}{1-r}\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)\nabla||x||_K dx = 0.$$

Note that if (A_r, v_r) is a global minimum of the restriction of L_r to $(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$, then by Lemma 1.2, it is a global minimum of the restriction of L_r to the smooth hypersurface $\mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^n \subseteq \mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n$. Moreover, if $x \in S^{n-1} \cap \partial K$ then $\nabla||x||_K = x$.

Hence

$$\begin{aligned}\frac{1}{1-r}\int_{S^{n-1}\cap\partial K}&(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)\nabla||x||_K \otimes x dx\\ &= \frac{1}{1-r}\int_{S^{n-1}\cap\partial K}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2) x \otimes x dx \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\frac{1}{1-r}\int_{S^{n-1}\cap\partial K}&(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)\nabla||x||_K dx\\ &= \frac{1}{1-r}\int_{S^{n-1}\cap\partial K}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2) x dx. \end{aligned} \tag{2.6}$$

What we are going to do is assume some properties about f, g that will give us good properties on the functionals L_r, I_r and allow us to show that the measure

$$\frac{1}{1-r}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)dx \tag{2.7}$$

concentrates near $S^{n-1} \cap \partial K$ as $r \to 1^-$ and converges for some sequence $r_k \to 1^-$ to a centered isotropic measure, as in Theorem 2.6.

Proof of Lemma 2.1. Let $\psi : \mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ be the function defined by $\psi(A, v) = \det(A)$. We know that $\mathrm{SL}_n(\mathbb{R}) = \det^{-1}(\{1\})$, where $c = 1$ is a regular value of the differentiable map ψ. Since $(A_r, v_r) \in \mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^n = \psi^{-1}(\{1\})$ is a singular point of L_r restricted to $\mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^n$, then by Theorem 1.16 there exists $\lambda_r \neq 0$ such that

$$\nabla L_r(A_r, v_r) = \lambda_r \nabla\psi(A_r, v_r), \tag{2.8}$$

where the gradients are taken with respect to the whole space $\mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n$. By Lemma 1.6, for $(V, w) \in T_{(A_r,v_r)}(\mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^n)$ we have

$$\psi'(A_r, v_r)[V, w] = \det(A_r)\langle A_r^{-T}, V\rangle = \langle A_r^{-T}, V\rangle.$$

Therefore,

$$\nabla\psi(A_r, v_r) = (A_r^{-T}, 0). \tag{2.9}$$

On the other hand, if $(V, w) \in T_{(A_r,v_r)}(\mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^n)$, we have

$$\begin{aligned} L_r'(M,v)[V,w] &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r'(||Mx+v||_K)\langle \nabla||Mx+v||_K, Vx+w\rangle g_r(|x|_2)dx \\ &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r'(||Mx+v||_K)(\langle \nabla||Mx+v||_K, Vx\rangle + \langle \nabla||Mx+v||_K, w\rangle)g_r(|x|_2)dx. \end{aligned}$$

By Lemma 1.3, we get

$$L_r'(M,v)[V,w] = \frac{1}{1-r}\int_{\mathbb{R}^n} f_r'(||Mx+v||_K)\langle(\nabla||Mx+v||_K \otimes x, \nabla||Mx+v||_K), (V,w)\rangle g_r(|x|_2)dx.$$

Since

$$f_r'(s) = \frac{d}{ds}f\left(\frac{s-1}{1-r}\right) = \frac{1}{1-r}f'\left(\frac{s-1}{1-r}\right) = \frac{1}{1-r}(f')_r(s),$$

then making $C_r = \dfrac{1}{(1-r)^2}$, we arrive at

$$\begin{aligned} &L_r'(M,v)[V,w] = \\ &\qquad \left\langle C_r\int_{\mathbb{R}^n} (f')_r(||Mx+v||_K)g_r(|x|_2)(\nabla||Mx+v||_K \otimes x, \nabla||Mx+v||_K)dx, (V,w)\right\rangle. \end{aligned}$$

Therefore,

$$\nabla L_r(A_r, v_r) = C_r\int_{\mathbb{R}^n} (f')_r(||A_rx+v_r||_K)g_r(|x|_2)(\nabla||A_rx+v_r||_K \otimes x, \nabla||A_rx+v_r||_K)dx. \tag{2.10}$$

Substituting equalities (2.9) and (2.10) in equality (2.8), we get

$$\begin{aligned} \lambda_r(A_r^{-T}, 0) &= C_r\int_{\mathbb{R}^n} (f')_r(||A_rx+v_r||_K)g_r(|x|_2)(\nabla||A_rx+v_r||_K \otimes x, \nabla||A_rx+v_r||_K)dx \\ &= C_r\int_{\mathbb{R}^n} (f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)(\nabla||x||_K \otimes A_r^{-1}(x-v_r), \nabla||x||_K)dx \\ &= C_r\int_{\mathbb{R}^n} (f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)((\nabla||x||_K \otimes x - \nabla||x||_K \otimes v_r)A_r^{-T}, \nabla||x||_K)dx. \end{aligned}$$

Note that in the last equality above we used the property

$$x \otimes Ay = (x \otimes y)A^T,$$

which is easily proved by noting that for any $z \in \mathbb{R}^n$ it holds

$$x \otimes Ay(z) = \langle z, Ay\rangle x = \langle A^Tz, y\rangle x = x \otimes y(A^Tz) = x \otimes yA^T(z).$$

By vector equality, we have

$$\frac{1}{(1-r)^2}\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)(\nabla||x||_K\otimes x-\nabla||x||_K\otimes v_r)dx=\lambda_r\,\mathrm{Id}$$
$$\frac{1}{(1-r)^2}\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)\nabla||x||_K dx=0.$$

Since

$$\begin{aligned}0&=\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)\nabla||x||_K dx\otimes v_r\\&=\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)\nabla||x||_K\otimes v_r dx,\end{aligned}$$

we conclude the proof. □

2.3.1 Basic Results

In order to construct the measure (2.7), we will assume the following properties for f and g:

f1 f is locally Lipschitz;

f2 f is convex;

f3 $f(x)=0$ for $x\leq -1$;

f4 f is strictly increasing in $[-1,\infty)$.

g1 g is locally Lipschitz;

g2 g is non-increasing;

g3 $g(x)=1$ for $x\leq -1$;

g4 $g(x)>0$ for $x\in(-1,1)$;

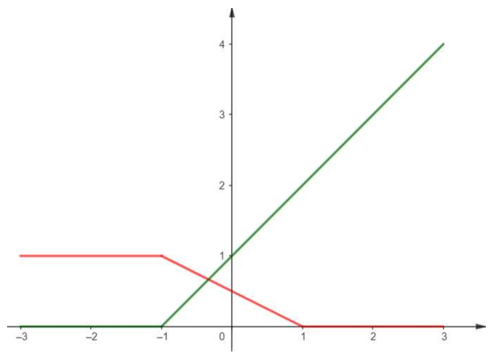
g5 $g(x)=0$ for $x\geq 1$.

Two functions satisfying **f1** to **g5** are

$$f(x)=\begin{cases}0, & \text{if } x\leq -1\\ x+1, & \text{if } x>-1\end{cases}\quad,\qquad g(x)=\begin{cases}1, & \text{if } x\leq -1\\ \frac{1-x}{2}, & \text{if } x\in(-1,1)\\ 0, & \text{if } x\geq 1\end{cases}\quad,$$

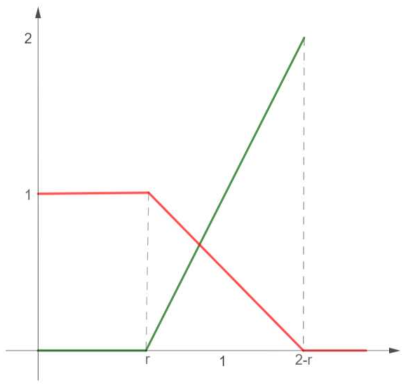
see Figures 2.2 and 2.3.

Figure 2.2: Functions f(green) and g(red).



Source: Compiled by the author.

Figure 2.3: Functions f_r(green) and g_r(red).



Source: Compiled by the author.

These conditions (specially **f2**) will guarantee that the functional L_r has a unique global minimum $(A_r, v_r) \in (\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$ for $r \in (1/2, 1)$.

Throughout this section we will fix a convex body $K \subseteq \mathbb{R}^n$ in John position. Since $B^n \subseteq K$, then 0 is an interior point of K.

We start by establishing basic properties of functionals L_r and I_r that will be useful in proving the main results. In order to justify the hypotheses requested about f and g, in each proposition we detail the properties of f, g that are necessary.

Proposition 2.2. *Assume* **f1**, **g1**, **g5** *are satisfied, then* L_r, I_r *are* C^1 *for* $r \in (1/2, 1)$.

Proof. By Rademacher's Theorem 1.17, since $f, g, ||\cdot||_K, |\cdot|_2$ are locally Lipschitz, then $f, g, ||\cdot||_K, |\cdot|_2$ are differentiable almost everywhere. In particular, $f(||Mx+v||_K)g(|x|_2)$ is differentiable almost everywhere, where $(A, v) \in \mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n$ and $f_r(||x||_K)g_r(|(\mathrm{Id}+(1-r)M)^{-1}(x-(1-r)w)|_2)$ is differentiable almost everywhere in $B_r \times \mathbb{R}^n$. $\square$

Definition 2.3. We say that a function $f : U \subseteq \mathbb{R}^d \to \mathbb{R}$ is coercive if $\lim_{|x|_2 \to \infty} f(x) = +\infty$. A one-parameter family of functions $f_r : U \subseteq \mathbb{R}^d \to \mathbb{R}$ is said to be coercive uniformly in r if $f_r(x) \geq f(x)$ for every $|x|_2 \geq C$ for some $C > 0$ and some coercive function f.

For a function $f : \mathbb{R} \to \mathbb{R}$ we say that f is coercive to the right if $\lim_{x \to \infty} f(x) = +\infty$.

Observe that a coercive and convex function defined in a convex set must have points of minimum, and that if the convexity of the function is strict the minimum is unique.

Lemma 2.2 ([8], Lemma 12)**.** *A convex function* $\psi : \mathbb{R}^d \to \mathbb{R}$ *with an isolated local minimum must be coercive.*

Recall that by (1.10), the set

$$\mathcal{D} = \{T \in \mathrm{Sym}_{n,+}(\mathbb{R}) : \det(T) \geq 1\}$$

is a convex set.

Proposition 2.3. *Assume* **f2**, **f3**, **f4**, **g3**, *then the family of functionals* L_r *restricted to* $\mathcal{D} \times \mathbb{R}^n$ *is coercive, uniformly for* $r \in (1/2, 1)$.

Proof. Let $(A, v) \in \mathcal{D} \times \mathbb{R}^n$ be arbitrarily given. Since A is symmetric and positive-definite, there exists w an eigenvector of A of eigenvalue $||A||_{op} = \max\{\lambda : \lambda \text{ is an eigenvalue of A}\}$, with Euclidean norm $\frac{1}{2}||A||_{op}$ and such that $\langle v, w\rangle \geq 0$. Consider the half-space

$$S = \{x \in \mathbb{R}^n : \langle x, v + w\rangle \geq \langle v + w, v + w\rangle\}$$

where clearly $v + w \in \partial S$. Let $\bar{S} = A^{-1}(S - v)$ be a half-space. Since

$$A^{-1}\left((v + w) - v\right) = A^{-1}(w) = \frac{1}{||A||_{op}} w = \frac{1}{2}\frac{w}{|w|_2},$$

then $\frac{1}{2}\frac{w}{|w|_2} \in \partial\bar{S}$. Applying the inverse of the affine transformation,

$$\mathrm{vol}_n\left(\left(\frac{1}{2}AB^n + v\right) \cap S\right) = \det(A)\,\mathrm{vol}_n\left(\frac{1}{2}B^n \cap \bar{S}\right).$$

Since the volume of the intersection of $\frac{1}{2}B^n$ with a half-space (not containing the origin) is a decreasing function of the distance of this half-space to the origin, and since

$$d(0, \bar{S}) \leq \left|\frac{1}{2}\frac{w}{|w|_2}\right|_2 = 1/2,$$

we have $\mathrm{vol}_n\left(\frac{1}{2}B^n \cap \bar{S}\right) \geq C_n$ where $C_n > 0$ is a dimensional constant. Also, $\langle v, w\rangle \geq 0$ implies

$$|x|_2 \geq |v + w|_2 \geq \sqrt{|v|_2^2 + |w|_2^2}$$

for every $x \in S$.

Since the convex body K is contained in the unit Euclidean ball, there exists a constant $C > 0$ such that

$$||Ax + v||_K \geq C|Ax + v|_2$$

for all $(A, v) \in \mathcal{D} \times \mathbb{R}^n$. Using **g3**, that f, g are non-negative, and that f is non-decreasing,

$$\begin{aligned} L_r(A, v) &= \frac{1}{1-r} \int_{\mathbb{R}^n} f_r(||Ax+v||_K) g_r(|x|_2) dx \\ &\geq \frac{1}{1-r} \int_{rB^n} f_r(||Ax+v||_K) dx \\ &\geq 2\det(A)^{-1} \int_{\left(\frac{1}{2}AB^n+v\right)\cap S} f_r(C|x|_2) dx \\ &\geq 2\det(A)^{-1} \operatorname{vol}_n\left(\left(\frac{1}{2}AB^n+v\right)\cap S\right) f_r\left(C\sqrt{|v|_2^2+|w|_2^2}\right) \\ &\geq 2\det(A)^{-1} \operatorname{vol}_n\left(\left(\frac{1}{2}AB^n+v\right)\cap S\right) f_r\left(C\sqrt{|v|_2^2+\frac{1}{4}||A||_{op}^2}\right) \\ &\geq 2C_n f\left(\frac{C\sqrt{|v|_2^2+\frac{1}{4}||A||_{op}^2}-1}{1-r}\right). \end{aligned}$$

For $C\sqrt{|v|_2^2+\frac{1}{4}||A||_{op}^2} \geq 1$, we obtain

$$L_r(A, v) \geq C_n f\left(\sqrt{|v|_2^2+\frac{1}{4}||A||_{op}^2}-1\right),$$

and since by **f2** and **f4** we have that f is coercive to the right, it follows that L_r is coercive to the right as well. $\square$

Proposition 2.4. *Let $r \in (1/2, 1)$ and assume* $\mathbf{g3}, \mathbf{g4}, \mathbf{f2}, \mathbf{f3}, \mathbf{f4}$. *The function L_r restricted to $\mathcal{D} \times \mathbb{R}^n$ is positive and convex.*

Proof. *Positive:* Take $(A, v) \in \mathcal{D} \times \mathbb{R}^n$ and assume $L_r(A, v) = 0$. Since the functions f_r, g_r are non-negative, we have that if $g_r(|x|_2) > 0$ then $f_r(||Ax+v||_K) = 0$. In other words,

$$x \in (2-r)B^n \Rightarrow Ax+v \in rK.$$

Hence, $(2-r)AB^n + v \subseteq rK$. Since K is in John position, $\det\left(\frac{2-r}{r}A\right) \leq 1$ from where

$$\det(A) \leq \left(\frac{r}{2-r}\right)^n < 1,$$

which contradicts the fact that $A \in \mathcal{D}$.

Convex: Let $(A, v), (B, w) \in \mathcal{D} \times \mathbb{R}^n$, and $\lambda \in [0, 1]$. By $\mathbf{f3}, \mathbf{f4}$, f is non-decreasing. Using this fact, the convexity of the gauge function of K and **f2**,

$$\begin{aligned}
L_r(\lambda(A,v)+(1-\lambda)(B,w)) &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||(\lambda A+(1-\lambda)B)x+(\lambda v+(1-\lambda)w)||_K)g_r(|x|_2)dx\\
&= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||\lambda(Ax+v)+(1-\lambda)(Bx+w)||_K)g_r(|x|_2)dx\\
&\leq \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(\lambda||Ax+v||_K+(1-\lambda)||Bx+w||_K)g_r(|x|_2)dx \qquad (2.11)\\
&\leq \frac{1}{1-r}\int_{\mathbb{R}^n} (\lambda f_r(||Ax+v||_K)+(1-\lambda)f_r(||Bx+w||_K))g_r(|x|_2)dx\\
&= \lambda L_r(A,v)+(1-\lambda)L_r(B,w),
\end{aligned}$$

and this is the desired inequality. □

Proposition 2.5. *Assume* **g5**, **f3**, *then for* $r\in(1/2,1)$ *we have* $L_r(\mathrm{Id},0)\leq C$ *where* C *is a constant depending only on* f *and* n.

Proof. By **g5** and polar coordinates, we get

$$\begin{aligned}
L_r(\mathrm{Id},0) &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||x||_K)g_r(|x|_2)dx\\
&\leq \frac{1}{1-r}\int_{(2-r)B^n} f_r(||x||_K)dx\\
&= \frac{1}{1-r}\int_{S^{n-1}}\int_0^{2-r} s^{n-1}f_r(s||\xi||_K)ds\,d\mathcal{H}^{n-1}(\xi).
\end{aligned}$$

Since K is in John position, then $S^{n-1}\subset K$. From where it follows $||\xi||_K\leq 1$ for all $\xi\in S^{n-1}$. Furthermore, since f_r is non-decreasing

$$L_r(\mathrm{Id},0)\leq \frac{1}{1-r}\int_{S^{n-1}}\int_0^{2-r} s^{n-1}f_r(s)ds\,d\mathcal{H}^{n-1}(\xi).$$

Making the substitution $s=1+(1-r)t$, recalling that $f_r(s)=f(t)$ and using **f3** we arrive at

$$\begin{aligned}
L_r(\mathrm{Id},0) &\leq \int_{S^{n-1}}\int_{-\frac{1}{1-r}}^{1} (1+(1-r)t)^{n-1}f(t)dt\,d\mathcal{H}^{n-1}(\xi)\\
&\leq \int_{S^{n-1}}\int_{-1}^{1} (1+(1-r)t)^{n-1}f(t)dt\,d\mathcal{H}^{n-1}(\xi)\\
&\leq 2^{n-1}\operatorname{vol}_{n-1}(S^{n-1})\int_{-1}^{1} f(t)dt\leq C.
\end{aligned}$$

□

Lemma 2.3. *Let* $n\geq 2, A,B\in \mathrm{GL}_n(\mathbb{R})$ *and* $v,w\in\mathbb{R}^n$ *be such that* $Ax+v$ *is a multiple of* $Bx+w$ *for every* x *in an open set* $U\subseteq\mathbb{R}^n$. *Then there exists* $a\neq 0$ *for which* $A=aB$ *and* $v=aw$.

Proof. First, we can assume that $Ax+v\neq 0$ for every $x\in U$, because if it does not we consider the open set $U\setminus\{-A^{-1}v\}$. The same we can assume for (B, w), that is, $Bx+w \neq 0$ for all $x \in U$. Now since $Ax + v$ is a multiple of $Bx + w$ for every x, then there is a function $a : U \subseteq \mathbb{R}^n \to \mathbb{R}$ such that

$$Ax + v = a(x)(Bx + w). \tag{2.12}$$

For any $x_0 \in U$, choose a coordinate i for which $(Bx_0 + w)_i \neq 0$. Thus,

$$a(x) = \frac{(Ax + v)_i}{(Bx + w)_i}$$

guarantees that $a(x)$ is a C^∞ function near x_0.

Taking the directional derivative of equation (2.12) with respect to x, in the direction of a vector x_1, we arrive at

$$Ax_1 = \langle \nabla a(x), x_1 \rangle (Bx + w) + a(x)Bx_1. \tag{2.13}$$

Now taking in (2.13), the directional derivative with respect to x, in the direction of x_2, at $x = x_0$

$$0 = (x_1^T Ha(x_0)x_2)(Bx_0 + w) + \langle \nabla a(x_0), x_1 \rangle Bx_2 + \langle \nabla a(x_0), x_2 \rangle Bx_1.$$

We claim that $\nabla a(x_0) = 0$. Indeed, if $\nabla a(x_0) \neq 0$ and $B\nabla a(x_0)$ is parallel to $Bx_0 + w$ it is enough for us to take $x_1 = \nabla a(x_0)$ and x_2 orthogonal to $\nabla a(x_0)$ to obtain

$$0 = (\nabla a(x_0)^T Ha(x_0)x_2)(Bx_0 + w) + \langle \nabla a(x_0), \nabla a(x_0) \rangle Bx_2,$$

which implies $B\nabla a(x_0) = 0$ because Bx_2 is not parallel to $B\nabla a(x_0)$. This contradicts the fact that B is invertible.

If $B\nabla a(x_0)$ is not parallel to $Bx_0 + w$ we take $x_1 = x_2 = \nabla a(x_0)$ to get

$$0 = (\nabla a(x_0)^T Ha(x_0)\nabla a(x_0))(Bx_0 + w) + 2\langle \nabla a(x_0), \nabla a(x_0) \rangle B\nabla a(x_0),$$

which implies $B\nabla a(x_0) = 0$ and we conclude that $\nabla a(x_0) = 0$.

For $x = x_0$, by (2.13)

$$\begin{aligned} Ax_1 &= \langle \nabla a(x_0), x_1 \rangle (Bx_0 + w) + a(x_0)Bx_1 \\ &= ((Bx_0 + w) \otimes \nabla a(x_0))x_1 + a(x_0)Bx_1 \end{aligned}$$

and the equality for every x_1 implies

$$A = (Bx_0 + w) \otimes \nabla a(x_0) + a(x_0)B. \tag{2.14}$$

By (2.14) becomes $A = a(x_0)B$, and by (2.12) for $x = x_0$ we get

$$a(x_0)Bx_0 + v = a(x_0)(Bx_0 + w)$$

which implies $v = a(x_0)w$.

Finally $A \in \mathrm{GL}_n(\mathbb{R})$ implies $a = a(x_0) \neq 0$. The proof is complete. □

2.3.2 Main Results in the geometric setting

Our first result is an immediate consequence of the Lagrange multipliers and explicitly gives us a centered and isotropic measure from a function F satisfying some conditions. Consider the set

$$\mathcal{F} = \{F : \mathbb{R} \to [0, \infty) : \; F \text{ is non-decreasing, convex, strictly convex in } [0, \infty), \text{ and } F'(0) > 0\}. \tag{2.15}$$

Theorem 2.7. *Let K be a convex body in John position. Choose any finite positive and non-zero measure ν in S^{n-1} with support inside $S^{n-1} \cap \partial K$, and any C^1 function $F \in \mathcal{F}$. Consider the convex functional $I_\nu : \mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$I_\nu(M, w) = \int_{S^{n-1}} F(\langle \xi, M\xi + w \rangle) d\nu(\xi).$$

If the restriction of I_ν to $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ is coercive then for any global minimum (M_0, w_0), the measure

$$F'(\langle \xi, M_0\xi + w_0 \rangle) d\nu(\xi)$$

is non-negative, non-zero, centered and isotropic.

By Lemma 2.2, if I_ν has an isolated local minimum, then it must be coercive so the coercivity can be established locally once a minimum is found.

Let us consider the situation where $S^{n-1} \cap \partial K$ is finite. In this case, a natural choice of ν is the counting measure c.

Corollary 2.1. *Let K be a convex body in John position and assume*

$$S^{n-1} \cap \partial K = \{\xi_1, \dots, \xi_m\}.$$

Choose any C^1 function $F \in \mathcal{F}$. Consider the convex functional $I_c : \mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$I_c(M, w) = \sum_{i=1}^{m} F(\langle \xi_i, M\xi_i + w \rangle).$$

If the restriction of I_c to $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ is coercive then for any global minimum (M_0, w_0), the numbers

$$c_i = F'(\langle \xi_i, M_0\xi_i + w_0 \rangle), i = 1, \dots, m,$$

together with the vectors $\xi_i, i = 1, \dots, m$, satisfy equation (2.1).

For a convex body L, the 0-th curvature measure $C_0(L, \cdot)$ is a measure in ∂L that generalizes the Gauss-Kronecker curvature $\kappa(x)$ of ∂L, for sets with non-smooth boundary (see, [46, Section

4 and formula (4.10)]). By [46, (4.25)], if L is C^2-smooth, and $A \subseteq \partial L$, we have

$$C_0(L, A) = \int_A \kappa(x) d\mathcal{H}^{n-1}(x). \tag{2.16}$$

We shall need the following property.

Proposition 2.6 ([46], Theorem 4.5.1)**.** *The support of the measure* $C_0(\operatorname{conv} A, \cdot)$ *is exactly* A*, where* $A \subseteq \mathbb{R}^n$ *is any set.*

For the convex body K there is a canonical choice of measure ν given by

$$\nu_K = C_0(\operatorname{conv}(S^{n-1} \cap \partial K), \cdot),$$

and will play a special role in Theorem 2.9 below. Depending on the set $S^{n-1} \cap \partial K$ and the measure ν, the function I_ν might or might not have a minimum. This can be a consequence of a "bad choice" of ν, or of the fact that $S^{n-1} \cap \partial K$ is degenerate in some sense. To make this precise we recall the following properties about John position. A proof can be found for the symmetric case in [5, proof of Theorem 2.1.10 and Lemma 2.1.13]. In the general case, the proof is analogue.

Theorem 2.8. *Let* L *be any convex body. The following statements are equivalent*

(i) L *is in John position;*

(ii) $B^n \subseteq L$ *and for every* $(M, w) \in (\operatorname{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n) \setminus \{(0, 0)\}$ *there exists* $\xi \in S^{n-1} \cap \partial L$ *for which* $\langle \xi, M\xi + w \rangle \geq 0$*;*

(iii) $B^n \subseteq L$ *and* $\left(\frac{1}{n} \operatorname{Id}, 0\right) \in \operatorname{conv}(\{(\xi \otimes \xi, \xi) : \xi \in S^{n-1} \cap \partial L\})$.

Theorem 2.9. *The following statements are equivalent*

(i) The restriction of I_ν *to* $\operatorname{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ *is coercive;*

(ii) For every $(M, w) \in (\operatorname{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n) \setminus \{(0, 0)\}$

$$\nu(\{\xi \in S^{n-1} \cap \partial K : \langle \xi, M\xi + w \rangle > 0\}) > 0;$$

If $\nu = \nu_K$ *or if* $S^{n-1} \cap \partial K$ *is finite and* $\nu = c$*, the statements above are also equivalent to the following:*

(iii) $\left(\frac{1}{n} \operatorname{Id}, 0\right)$ *lies in the interior of* $\operatorname{conv}(\{(\xi \otimes \xi, \xi) : \xi \in S^{n-1} \cap \partial K\}) \subseteq \operatorname{Sym}_{n,1}(\mathbb{R}) \times \mathbb{R}^n$*, where the interior is taken with respect to* $\operatorname{Sym}_{n,1}(\mathbb{R}) \times \mathbb{R}^n$.

Proof. We start by proving that $(i) \Rightarrow (ii)$. Assume by contradiction that for some $(M, w) \in (\operatorname{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n) \setminus \{(0, 0)\}$ it holds $\langle \xi, M\xi + w \rangle \leq 0$ for ν-almost every ξ. For $\lambda > 1$, since F is non-decreasing,

$$\begin{aligned}
I_\nu(\lambda(M,w)) &= \int_{S^{n-1}\cap\partial K} F(\lambda\langle\xi, M\xi + w\rangle)d\nu \\
&\le \int_{S^{n-1}\cap\partial K} F(\langle\xi, M\xi + w\rangle)d\nu \\
&= I_\nu(M,w),
\end{aligned}$$

which contradicts the coercivity of I_ν.

Now suppose that it holds (ii). Denote by $x_+ = \max\{x, 0\}$ the positive part of x and consider the function

$$E(M,w) = \int_{S^{n-1}} \langle\xi, M\xi + w\rangle_+ d\nu.$$

By hypothesis, E is positive in $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n \setminus \{(0,0)\}$. By Dominated Convergence Theorem 1.11, the function E is continuous, and there is $\varepsilon > 0$ such that $E(M,w) \ge \varepsilon$ for every $(M,w) \in \mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ with $||(M,w)|| = 1$. Using the comparison $F(x) \ge F'(0)x_+$ and writing $(\bar{M}, \bar{w}) = ||(M,w)||^{-1}(M,w)$, we deduce that

$$\begin{aligned}
I_\nu(M,w) &\ge \int_{S^{n-1}} F'(0)\langle\xi, M\xi + w\rangle_+ d\nu \\
&\ge ||(M,w)|| \int_{S^{n-1}} F'(0)\langle\xi, \bar{M}\xi + \bar{w}\rangle_+ d\nu \\
&\ge ||(M,w)||F'(0)\varepsilon,
\end{aligned}$$

which implies the coercivity of I_ν.

So far we have shown that $(i) \Leftrightarrow (ii)$. Now we assume (iii) for $\nu = \nu_K$ and show that it holds (ii). Let $(M,w) \in \mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n \setminus \{(0,0)\}$. First we show that there exists $\xi_0 \in S^{n-1} \cap \partial K$ such that $\langle\xi_0, M\xi_0 + w\rangle > 0$. Indeed, $\left(\frac{\mathrm{Id}}{n}, 0\right)$ belongs to the boundary of the half-space $F = \{(N,u) \in \mathrm{Sym}_{n,1}(\mathbb{R}) \times \mathbb{R}^n : \langle(N,u),(M,w)\rangle \le 0\}$, so by the hypothesis, we cannot have $\mathrm{conv}(\{\xi \otimes \xi : \xi \in S^{n-1} \cap \partial K\}) \subseteq F$.

Now we may find $\varepsilon > 0$ such that $|\xi - \xi_0|_2 < \varepsilon$ implies $\langle\xi, M\xi + w\rangle > 0$ as well. Since ξ_0 is in the support of ν_K, by Proposition 2.6 we have $\nu_K(B(\xi_0, \varepsilon)) > 0$. This implies that

$$\nu_K(\{\xi \in S^{n-1} \cap \partial K : \langle\xi, M\xi + w\rangle > 0\}) > 0.$$

The implication $(ii) \Rightarrow (iii)$ for $\nu = \nu_K$ follows because $\nu(\{\xi \in S^{n-1}\cap\partial K : \langle\xi, M\xi+w\rangle > 0\}) > 0$ implies $\{\xi \in S^{n-1} \cap \partial K : \langle\xi, M\xi + w\rangle > 0\}$ is non-empty.

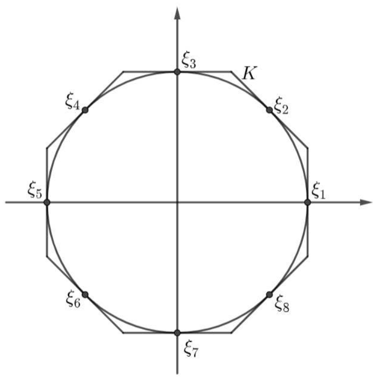
To finish, if $S^{n-1} \cap \partial K$ is finite and $\nu = c$ then $\mathrm{conv}(S^{n-1} \cap \partial K)$ is a polytope and ν_K is an atomic measure supported in $S^{n-1}\cap\partial K$, by Proposition 2.6 again. The equivalence then follows from the case $\nu = \nu_K$. □

Notice that these conditions do not depend on the choice of F but only on the measure ν. Theorem 2.9 shows that the condition of coercivity in Theorem 2.7 corresponds to a generic situation.

Example 2.1. Consider K the Octagon that is in John position. The set of contact points between K and B^2 is given by

$$S^1 \cap \partial K = \{\pm(1,0), \pm(0,1), \pm(1,1), \pm(1,-1)\}.$$

Figure 2.4: Contact points between K and B^2.



Source: Compiled by the author.

For each matrix $M = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \mathrm{Sym}_{2,0}(\mathbb{R})$ and vector $w = (c,d) \in \mathbb{R}^2$, we can show that for some $i = 1, \ldots, 8$, it holds

$$\langle \xi_i, M\xi_i + w \rangle > 0.$$

Here $a, b, c, d \in \mathbb{R}$ are such that at least one of them is non-zero. Therefore, by Theorem 2.9, the functional

$$I_c(M,w) = \sum_{i=1}^{8} e^{\langle \xi_i, M\xi_i + w \rangle},$$

given in Corollary 2.1, is coercive.

Proof of Theorem 2.7. By Theorem 1.19, the set $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ is the orthogonal complement of $(\mathrm{Id}, 0)$ in $\mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n$. Now we compute the derivative of I_ν in the direction of (V,w), at the point (M,v) and we use the Lemma 1.3

$$\begin{aligned}
\langle \nabla I_\nu(M,v), (V,w) \rangle &= \int_{S^{n-1}} F'(\langle \xi, M\xi + v \rangle) \langle \nabla(\langle \xi, M\xi + v \rangle), (V,w) \rangle d\nu \\
&= \int_{S^{n-1}} F'(\langle \xi, M\xi + v \rangle) \langle \xi, V\xi + w \rangle d\nu \\
&= \int_{S^{n-1}} F'(\langle \xi, M\xi + v \rangle) (\langle \xi \otimes \xi, V \rangle + \langle \xi, w \rangle) d\nu \\
&= \int_{S^{n-1}} F'(\langle \xi, M\xi + v \rangle) \langle (\xi \otimes \xi, \xi), (V,w) \rangle d\nu \\
&= \left\langle \int_{S^{n-1}} F'(\langle \xi, M\xi + v \rangle)(\xi \otimes \xi, \xi) d\nu, (V,w) \right\rangle.
\end{aligned}$$

We deduce that

$$\nabla I_\nu(M,v) = \int_{S^{n-1}} F'(\langle \xi, M\xi + v\rangle)(\xi \otimes \xi, \xi) d\nu$$

since this expression already in $\mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n$.

The gradient of the function $\psi(M,v) = \mathrm{tr}(M)$ is $\nabla\psi(M,v) = (\mathrm{Id}, 0)$. Since $\psi^{-1}(\{0\}) = \mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$, $(M_0, w_0) \in \mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n$ is a singular point of I_ν and 0 is a regular value of ψ, then by Theorem 1.16 there exists $\lambda > 0$ such that

$$\nabla I_\nu(M_0, w_0) = \lambda \nabla\psi(M_0, w_0),$$

that is,

$$\int_{S^{n-1}} F'(\langle \xi, M_0\xi + w_0\rangle)(\xi \otimes \xi, \xi) d\nu = \lambda(\mathrm{Id}, 0).$$

Equivalently,

$$\int_{S^{n-1}} F'(\langle \xi, M_0\xi + w_0\rangle)(\xi \otimes \xi) d\nu = \lambda\,\mathrm{Id} \tag{2.17}$$
$$\int_{S^{n-1}} F'(\langle \xi, M_0\xi + w_0\rangle)\xi d\nu = 0.$$

Since F is non-decreasing, $F'(\langle \xi, M_0\xi + w_0\rangle) \geq 0$. Taking traces in equation (2.17) we get

$$\lambda = \frac{1}{n}\int_{S^{n-1}} F'(\langle \xi, M_0\xi + w_0\rangle) d\nu.$$

By Theorem 2.9, we know that $\langle \xi, M_0\xi + w_0\rangle > 0$ for a set of positive ν-measure. Since $F'(x) \geq 0$ for every x and $F'(x) > 0$ for $x \geq 0$, we deduce that $\lambda > 0$ and the proof is complete. ☐

Theorem 2.10. *Let K be a convex body in John position and let f, g satisfy all the properties* **f1** *to* **g5***, then for every $r \in (1/2, 1)$ the restriction of L_r to $(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$ has a unique minimum (A_r, v_r) with $\lim_{r\to 1^-}(A_r, v_r) = (\mathrm{Id}, 0)$. Likewise, the restriction of the I_r to*

$$\mathcal{S}_r = \left(\frac{(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) - \mathrm{Id}}{1-r}\right) \times \mathbb{R}^n$$

has the unique minimum $(M_r, v_r) = \left(\frac{A - \mathrm{Id}}{1-r}, \frac{v_r}{1-r}\right)$ *with* $tr\left(\frac{M_r}{||M_r||_F}\right) \to 0$ *as* $r \to 1^-$.

Remark 2.1. The fact that $(A_r, v_r) \to (\mathrm{Id}, 0)$ is saying that the position of K that minimizes L_r, converges to the John position as $r \to 1^-$. The functional I_r is a "blowup" of L_r, by a change of coordinates that concentrates near $(\mathrm{Id}, 0)$ and stretches the distances by a factor of $\frac{1}{1-r}$. The restriction of L_r to the smooth surface $(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$ takes the same values as the restriction of I_r to $\mathcal{S}_r$. Now notice that in a neighborhood of the origin, as $r \to 1^-$, the surface $\mathcal{S}_r$ approaches the tangent space to $(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$ at $(\mathrm{Id}, 0)$, which is $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$. At the same time, -$(M_r, w_r)$ is the incremental quotient of (A_r, v_r) at $r = 1$. The statement that

$\operatorname{tr}\left(\frac{M_r}{||M_r||_F}\right) \to 0$ means that the line going through Id and M_r gets more and more parallel to $\mathrm{Sym}_{n,0}(\mathbb{R})$.

Proof of Theorem 2.10. First we show that for $r \in (1/2, 1)$, the restriction of L_r to $\mathcal{D} \times \mathbb{R}^n$ has exactly one global minimum (A_r, v_r).

Since by Proposition 2.3 L_r is coercive, by Proposition 2.4 it is convex and $\mathcal{D} \times \mathbb{R}^n$ is closed and convex, then L_r admits at least one minimum. Assume the minimum is attained in two different points (A, v) and (B, w). Since by Proposition 2.4 L_r is convex, then there is equality in equation (2.11) for every $\lambda \in [0,1]$ and since f_r is strictly increasing in $[r, +\infty)$, $Ax + v$ and $Bx + w$ are multiples for every $x \in (2-r)B^n$ such that $||Ax + v||_K > r$ or $||Bx + w||_K > r$. We claim that $((2-r)AB^n + v) \setminus rK$ has non-empty interior. Indeed, since $(2-r)AB^n + v$ and rK are convex bodies, then to say that $((2-r)AB^n + v) \setminus rK$ has empty interior is the same as to say that $((2-r)AB^n + v) \subseteq rK$, that is, $\left(\frac{2-r}{r}AB^n + v\right) \subseteq K$. And since K is in John position, then $\det\left(\frac{2-r}{r}A\right) \leq 1$, meaning

$$\det(A) \leq \left(\frac{r}{2-r}\right)^n < 1$$

which contradicts the fact that $A \in \mathcal{D}$. Thus, by Lemma 2.3 there exists $a > 0$ such that

$$A = aB, v = aw.$$

If $a = 1$ we are done. Assume without loss of generality that $a > 1$. Since $(AB^n + v) \setminus rK$ has non-empty interior (because K is in John position and $\det(A) = a\det(B) > 1$) then there exists $x \in B^n$ (where g_r is non-zero) such that $Ax + v \in (rK)^c$ (where f_r is strictly increasing). From where it follows that

$$L_r(B, w) < L_r(A, v),$$

which is absurd.

Now we show that $A_r \in \mathrm{SL}_n(\mathbb{R})$. If $\det(A_r) > 1$ then again $(A_rB^n + v_r) \setminus rK$ has non-empty interior and

$$\begin{aligned} L_r(\det(A_r)^{-1/n}(A_r, v_r)) &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||\det(A_r)^{-1/n}(A_r x + v_r)||_K)g_r(|x|_2)dx \\ &< \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||A_r x + v_r||_K)g_r(|x|_2)dx \\ &= L_r(A_r, v_r). \end{aligned}$$

This last inequality contradicts the minimality of (A_r, v_r). We conclude then that $A_r \in \partial\mathcal{D} = \mathrm{SL}_n(\mathbb{R}) \cap \mathrm{Sym}_{n,+}(\mathbb{R})$.

Denote $M_r = \dfrac{A_r - \mathrm{Id}}{1-r}$, $w_r = \dfrac{v_r}{1-r}$. Since $A_r \in \mathrm{SL}_n(\mathbb{R})$, then by equation (2.4) we have $I_r(M_r, w_r) = L_r(A_r, v_r)$ and (M_r, w_r) is the unique global minimum of the restriction of I_r to

$$\left(\frac{(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) - \mathrm{Id}}{1-r}\right) \times \mathbb{R}^n.$$

Now let us prove that $(A_r, v_r) \to (\mathrm{Id}, 0)$. Assume that (A_r, v_r) does not converge to $(\mathrm{Id}, 0)$. By Propositions 2.3 and 2.5, the sequence $\{(A_r, v_r)\}_r$ is bounded. Then there is a sequence $r_k \to 1^-$such that $\{(A_{r_k}, v_{r_k})\}_k$ converges. Assume that $(A_{r_k}, v_{r_k}) \to (A^*, v^*) \in (\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$, with $(A^*, v^*) \neq (\mathrm{Id}, 0)$. Since the John position is unique up to orthogonal transformations and $A^* \in \mathrm{Sym}_{n,+}(\mathbb{R})$, the set $(A^* B^n + v^*) \setminus K$ has positive Lebesgue measure. Take $\mu < 1$ such that $\mu A^* B^n + v^* \setminus K$ has positive Lebesgue measure. For large k, we have $\mu A^* B^n + v^* \subseteq A_{r_k} B^n + v_{r_k}$. By Fatou's Lemma 1.10,

$$\begin{aligned}
\liminf_{k\to\infty} L_{r_k}(A_{r_k}, v_{r_k}) &\geq \int_{\mathbb{R}^n \setminus B^n} \liminf_{k\to\infty} \frac{1}{1-r_k} f_{r_k}(||x||_K) g_{r_k}(|A_{r_k}^{-1}(x - v_r)|_2) dx \\
&\geq \int_{\mu A^* B^n + v^* \setminus K} \liminf_{k\to\infty} \frac{1}{1-r_k} f_{r_k}(||x||_K) g(0) dx \\
&= +\infty,
\end{aligned}$$

which is absurd because by minimality of (A_{r_k}, v_{r_k}) and Proposition 2.5,

$$L_{r_k}(A_{r_k}, v_{r_k}) \leq L_{r_k}(\mathrm{Id}, 0) \leq C.$$

Note that we used for large k

$$g_{r_k}(|A_{r_k}^{-1}(x - v_{r_k})|_2) \geq g(0)$$

in $\mu A^* B^n + v^* \setminus K$, because if $x = \mu A^* \tilde{x} + v^*$, where $\tilde{x} \in B^n, ||x||_K > 1$, we have

$$x \in \mu A^* B^n + v^* \setminus K \Rightarrow A_{r_k}^{-1}(\mu A^* \tilde{x} + v^* - v_{r_k}) \subseteq A_{r_k}^{-1}(A_{r_k} B^n) = B^n \Rightarrow |A_{r_k}^{-1}(x - v_{r_k})|_2 \leq 1$$

and by **g2** it follows that

$$g_{r_k}(|A_{r_k}^{-1}(x - v_r)|_2) = g\left(\frac{|A_{r_k}^{-1}(x - v_r)|_2 - 1}{1-r}\right) \geq g(0).$$

It remains to show that $\mathrm{tr}\left(\frac{M_r}{||M_r||_F}\right) \to 0$. Recall that the trace is the differential of det at $\mathrm{Id} \in \mathrm{M}_n(\mathbb{R})$ and by Taylor 1.18,

$$\det(\mathrm{Id} + V) = 1 + \mathrm{tr}(V) + o(||V||_F)$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$.

Taking $V = (1-r)M_r$, we get

$$\begin{aligned}
1 &= \det(A_r) \\
&= \det(\mathrm{Id} + (1-r)M_r) \\
&= 1 + (1-r)\mathrm{tr}(M_r) + o((1-r)||M_r||_F).
\end{aligned}$$

Therefore,

$$\operatorname{tr}\left(\frac{M_r}{||M_r||_F}\right) = \frac{\operatorname{tr}(M_r)}{||M_r||_F} = -\frac{o((1-r)||M_r||_F)}{(1-r)||M_r||_F} \to 0$$

as $r \to 1^-$. $\square$

The functional I_r captures the asymptotic behaviour of the minimizers (A_r, v_r) when $r \to 1^-$, as following theorem explains.

Theorem 2.11. *Assume K has a C^1-smooth boundary and all the properties* **f1** *to* **g5** *are satisfied. The functional $I_r(M, w)$ is extends continuously to $r = 1$ as*

$$I_1(M, w) = \int_{S^{n-1}\cap\partial K} F(\langle \xi, M\xi + w\rangle) d\mathcal{H}^{n-1}\xi,$$

where F is the convolution $F(x) = f * \bar{g}(x), \bar{g}(x) = g(-x)$ and satisfies the conditions of Theorem 2.7. Moreover, $I_r \to I_1$ as $r \to 1^-$, uniformly in compact sets.

Thus we obtain $I_1 = I_{\mathcal{H}^{n-1}}$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure restricted to $S^{n-1} \cap \partial K$.

Proof. By Proposition 1.4, we have

$$\begin{aligned} I_r(M, w) &= \frac{1}{1-r}\int_{\mathbb{R}^n} f_r(||x||_K) g_r(|(\operatorname{Id} + (1-r)M)^{-1}(x - (1-r)w)|_2) dx \\ &= \frac{1}{1-r}\int_{\partial K}\int_0^\infty s^{n-1} f_r(s||z||_K) g_r(|(\operatorname{Id} + (1-r)M)^{-1}(sz - (1-r)w)|_2) \\ &\times h_K(n^K(z)) ds d\mathcal{H}^{n-1}(z). \end{aligned}$$

By Taylor expansion we obtain for any $x, w \in \mathbb{R}^n$,

$$|x + w|_2 = |x|_2 + \left\langle \frac{x}{|x|_2}, w \right\rangle + o(|w|_2). \tag{2.18}$$

We will denote by $o((1-r)^a)$(resp. $o(1)$) any function of the involved parameters M, w, r, s, t, z, satisfying

$$\lim_{r\to 1^-} \frac{o((1-r)^a)}{(1-r)^a} = 0 \left(\text{resp. } \lim_{r\to 1^-} o(1) = 0\right),$$

where the limits are uniform in compact sets with respect to the parameters. Likewise, $O(1)$ will denote any bounded function. For any $z, w \in \mathbb{R}^n, s \geq r > 0, M \in B_r \subseteq \mathrm{M}_n(\mathbb{R})$ (recall that B_r is the domain of functional I_r),

$$\begin{aligned} (\operatorname{Id} + (1-r)M)^{-1}(sz - (1-r)w) &= (sz - (1-r)w) - (1-r)M(sz - (1-r)w) + o(1-r) \\ &= sz - (1-r)(sMz + w) + o(1-r). \end{aligned} \tag{2.19}$$

Using (2.19) and (2.18),

$$\begin{aligned} |(\mathrm{Id}+(1-r)M)^{-1}(sz-(1-r)w)|_2 &= |sz-(1-r)(sMz+w)+o(1-r)|_2 \\ &= s\left|z-(1-r)\left(Mz+\frac{1}{s}w\right)+o(1-r)\right|_2 \\ &= s\left(|z|_2-(1-r)\left\langle\frac{z}{|z|_2}, Mz+\frac{1}{s}w\right\rangle\right)+o(1-r). \end{aligned}$$

Putting all together and making the substitution $s=1+(1-r)t$, we get

$$\begin{aligned} I_r(M,w) &= \frac{1}{1-r}\int_{\partial K}\int_0^\infty s^{n-1}f_r(s)g_r(|(\mathrm{Id}+(1-r)M)^{-1}(sz-(1-r)v)|_2) \\ &\quad \times h_K(n^K(z))dsd\mathcal{H}^{n-1}(z) \\ &= \int_{\partial K}\int_{-\frac{1}{1-r}}^\infty (1+(1-r)t)^{n-1}f(t)h_K(n^K(z)) \\ &\quad \times g_r\left(\left|(1+(1-r)t)\left(\left(|z|_2-(1-r)\left\langle\frac{z}{|z|_2}, Mz+\frac{1}{s}w\right\rangle\right)+o(1-r)\right)\right|_2\right)dtd\mathcal{H}^{n-1}(z) \\ &= \int_{\partial K}\int_{-\frac{1}{1-r}}^\infty (1+(1-r)t)^{n-1}f(t)h_K(n^K(z)) \\ &\quad \times g\left(\frac{|z|_2-1}{1-r}+t(|z|_2+o(1))-\left\langle\frac{z}{|z|_2}, Mz+v+o(1)\right\rangle+o(1)\right)dtd\mathcal{H}^{n-1}(z). \end{aligned}$$

Notice that $|z|_2=1$ for $z\in S^{n-1}\cap\partial K$, $|z|_2>1$ for $z\in\partial K\setminus S^{n-1}$ and that $n^K(z)=z, h_K(n^K(z))=h_K(z)=1$ for every $z\in S^{n-1}\cap\partial K$. Moreover,

$$\lim_{r\to1^-}\frac{|z|_2-1}{1-r}\to\infty$$

in $\partial K\setminus S^{n-1}$.

Also, by **f3** the integrand is 0 for $t<-1$, then

$$\begin{aligned} I_r(M,w) &= \int_{S^{n-1}\cap\partial K}\int_{-1}^\infty (1+(1-r)t)^{n-1}f(t)h_K(n^K(z)) \\ &\quad \times g\left(-\langle\xi, M\xi+w+o(1)\rangle+t(1+o(1))+o(1)\right)dtd\mathcal{H}^{n-1}(\xi) \\ &\quad + \int_{\partial K\setminus S^{n-1}}\int_{-\frac{1}{1-r}}^\infty (1+(1-r)t)^{n-1}f(t)h_K(n^K(z)) \\ &\quad \times g\left(\frac{|z|_2-1}{1-r}+t(|z|_2+o(1))+O(1)+o(1)\right)dtd\mathcal{H}^{n-1}(z). \end{aligned}$$

To prove the uniform convergence in compact sets, consider a convergent sequence $(M_k,w_k)\to(M,w)$ as $r_k\to1^-$. By property **g5**, the function in the second integral is zero for $t\geq C$ where C is independent of k. The functions f,g are thus uniformly bounded in the support of both integrals, and we may apply the Dominated Convergence Theorem 1.11 to obtain (thanks to property **g5**)

$$\lim_{k\to\infty} I_{r_k}(M_k, w_k) = \int_{S^{n-1}\cap\partial K}\int_{-\infty}^{\infty} f(t)g(-\langle\xi, M\xi + w\rangle + t)dtd\mathcal{H}^{n-1}(\xi).$$

Finally, we must show that F satisfies the properties of Theorem 2.7. First, F is non-negative because $f(t)g(t-x) \geq 0$, for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Second F is non-decreasing because since g is non-increasing then $\bar{g}$ is non-decreasing and

$$F'(x) = -\int_{-\infty}^{\infty} f(t)g'(t-x)dt = \int_{-\infty}^{\infty} f(x-t)g'(-t)dt = \int_{-\infty}^{\infty} f(x-t)\bar{g}'(t)dt \geq 0.$$

By $\mathbf{f1}, \mathbf{g1}$, f and g are locally Lipschitz and hence absolutely continuous and differentiable a.e., then F is twice differentiable a.e. and by $\mathbf{f4}, \mathbf{g3}, \mathbf{g4}, \mathbf{g5}$

$$F''(x) = \int_{-1}^{1} f'(x-t)\bar{g}'(t)dt \geq 0,$$

showing that F is convex. To see the strict convexity in $[0,\infty)$ take any $x > 0$. If $F''(x) = 0$, since $\bar{g}'(t) > 0$ in $(-1,1)$, the last inequality implies that $f' = 0$ in a set of positive measure inside $(x-1, x+1)$, and this contradicts $\mathbf{f4}$. ☐

As in the last remark made at the end of Section 2.2 with respect to the measure ν_r, to obtain the measure (2.7) one requires the computation of (A_r, v_r) for every r close to 1. The reason why Theorem 2.7 follows directly, is that the information of the curve (A_r, v_r) that is necessary to compute the isotropic measure, is contained in (M_0, w_0). That is the content of the last theorem.

Theorem 2.12. *Assume all the properties* **f1** *to* **g5** *are satisfied and the function I_1 restricted to* $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ *has a unique global minimum (M_0, w_0), then* $\left.\dfrac{\partial(A_r, v_r)}{\partial r}\right|_{r=1}$ *exists and is equal to* -(M_0, w_0).

In this case, if $(\bar{A}_r, \bar{v}_r)$ is any curve in $\mathrm{Sym}_{n,+}(\mathbb{R}) \times \mathbb{R}^n$ *of the form*

$$(\bar{A}_r, \bar{v}_r) = (\mathrm{Id}, 0) + (1-r)(M_0, w_0) + o(1-r),$$

the measure

$$\frac{1}{1-r}(f')_r(||x||_K)g_r(|\bar{A}_r^{-1}(x-\bar{v}_r)|_2)dx$$

converges weakly to centered and isotropic measure $F'(\langle\xi, M_0\xi + w_0\rangle)d\mathcal{H}^{n-1}(\xi)$.

In particular this is true for its linear part $(\bar{A}_r, \bar{v}_r) = (\mathrm{Id} + (1-r)M_0, (1-r)w_0)$ and for $(\bar{A}_r, \bar{v}_r) = (A_r, v_r)$ as well .

Proof. First we prove that if I_1 has a unique global minimum (M_0, w_0), then (M_r, w_r) converges to (M_0, w_0). By Lemma 2.2, I_1 is coercive. Then there exists $R > 0$ such that $(M, w) \in \mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n, ||(M,w)|| \geq R$ implies

$$I_1(M, w) \geq C + 2,$$

where C given in Proposition 2.5 is such that $L_r(\mathrm{Id}, 0) \leq C$.

Let $B_R = \{(M, w) \in \mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n : ||(M, w)|| \leq R\}$. Since I_1 is continuous in the compact set B_R, there is $\varepsilon > 0$ such that

$$I_1(M, w) \geq C + 1$$

for every $(M, w) \in B_R$ with $|\mathrm{tr}(M)| < \varepsilon$. By Theorem 2.11, there is $r_0 \in (1/2, 1)$ such that for every $r \in (r_0, 1)$ and $(M, w) \in B_R$,

$$|I_r(M, w) - I_1(M, w)| \leq 1/2.$$

Increasing r_0 if necessary, we may assume that for every $r \in (r_0, 1)$ and $\lambda \in [0, 1]$,

$$\det(\lambda A_r + (1 - \lambda)\,\mathrm{Id}) \leq \frac{C + 1/2}{C + 1/4} = 1 + \frac{1}{4C + 1}$$

and that $\left|\mathrm{tr}\left(\frac{M_r}{||M_r||_F}\right)\right| \leq \frac{\varepsilon}{R}$ (last part of Theorem 2.10). First we claim that $(M_r, w_r) \in B_R$ for $r \in (r_0, 1)$. Assume by contraction that $(M_r, w_r) \notin B_R$ for some $r \in (r_0, 1)$, and consider $\lambda < 1$ such that $||\lambda(M_r, w_r)|| = R$, then since $|\mathrm{tr}(\lambda M_r)| \leq \frac{||\lambda M_r||_F}{R}\varepsilon \leq \varepsilon$,

$$I_r(\lambda(M_r, w_r)) \geq I_1(\lambda(M_r, w_r)) - \frac{1}{2} \geq C + 1 - \frac{1}{2} = C + \frac{1}{2}.$$

By the convexity of $\mathcal{D}$,

$$(\mathrm{Id}, 0) + (1 - r)\lambda(M_r, w_r) = \lambda(A_r, v_r) + (1 - r)(\mathrm{Id}, 0) \in \mathcal{D}$$

from where it follows that

$$\begin{aligned}
L_r(\lambda(A_r, v_r) + (1 - \lambda)(\mathrm{Id}, 0)) &= \det(\lambda A_r + (1 - \lambda)\,\mathrm{Id})^{-1} I_r(\lambda(M_r, w_r)) \\
&\geq \left(\frac{C + 1/2}{C + 1/4}\right)^{-1} (C + 1/2) \\
&= C + 1/4 \\
&\geq L_r(\mathrm{Id}, 0) + 1/4 \\
&\geq L_r(A_r, v_r) + 1/4.
\end{aligned}$$

We obtain the inequalities

$$L_r(\lambda(A_r, v_r) + (1 - \lambda)(\mathrm{Id}, 0)) > L_r(\mathrm{Id}, 0) \geq L_r(A_r, v_r)$$

contradicting the convexity of L_r, that is, $(M_r, w_r) \in B_R$ for $r \in (r_0, 1)$ and the claim is proved.

For any matrix $M \in \mathrm{Sym}_{n,0}(\mathbb{R})$ consider

$$M^{(r)} = \frac{\det(\mathrm{Id} + (1 - r)M)^{-1/n}(\mathrm{Id} + (1 - r)M) - \mathrm{Id}}{1 - r}$$

and notice that it belongs to $\dfrac{(\mathrm{SL}_n(\mathbb{R}) \cap \mathrm{Sym}_{n,+}(\mathbb{R})) - \mathrm{Id}}{1-r}$ for r close to 1. We also have

$$\begin{aligned}\lim_{r\to 1^-} M^{(r)} &= \lim_{r\to 1^-} \frac{\det(\mathrm{Id}+(1-r)M)^{-1/n}-1}{1-r}\mathrm{Id} + \det(\mathrm{Id}+(1-r)M)^{-1/n}M \\ &= \frac{\partial}{\partial t}\bigg|_{t=0} (\det(\mathrm{Id}+tM)^{-1/n}\,\mathrm{Id}) + M \\ &= -\frac{1}{n}\mathrm{tr}(-M)\,\mathrm{Id}+M \\ &= M.\end{aligned}$$

Now that $\{(M_r, v_r)\}_r$ is bounded, for every convergent sequence $(M_{r_k}, w_{r_k}) \to (M_0, w_0)$ with $r_k \to 1^-$, and for every $(M, w) \in \mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$

$$I_1(M_0, w_0) \leftarrow I_{r_k}(M_{r_k}, w_{r_k}) \leq I_{r_k}(M^{(r_k)}, w) \to I_1(M, w)$$

so that (M_0, w_0) is the (unique) minimum of I_1, and we deduce $(M_r, w_r) \to (M_0, w_0)$ as desired.

Finally, we write

$$\begin{aligned}\frac{\partial(A_r, v_r)}{\partial r}\bigg|_{r=1} &= \lim_{r\to 1^-} \frac{(A_r, v_r) - (\mathrm{Id}, 0)}{r-1} \\ &= \lim_{r\to 1^-} (-M_r, -w_r) \\ &= -(M_0, w_0).\end{aligned}$$

For the second part of the theorem take δ any continuous function with compact support and write, as in the proof of Theorem 2.11,

$$\begin{aligned}&\frac{1}{1-r}\int_{\mathbb{R}^n} (f')_r(||x||_K) g_r(|\bar{A}_r^{-1}(x-\bar{v}_r)|_2)dx \\ &= \int_{S^{n-1}\cap\partial K}\int_{-1}^{\infty} \delta((1+(1-r)t)\xi) f'(t) g(-\langle \xi, M_0\xi + w_0 + o(1)\rangle + t(1+o(1)) + o(1)) \\ &\quad \times (1+(1-r)t)^{n-1} h_K(n^K(\xi)) dt d\mathcal{H}^{n-1}(\xi) \\ &\quad + \int_{\partial K\setminus S^{n-1}}\int_{-1}^{\infty} \delta((1+(1-r)t)z) f'(t) g\left(\frac{|z|_2 - 1}{1-r} + O(1) + t(|z|_2 + o(1))\right) \\ &\quad \times (1+(1-r)t)^{n-1} h_K(n^K(\xi)) dt d\mathcal{H}^{n-1}(\xi)\end{aligned}$$

hence by the Dominated Convergence Theorem 1.11,

$$\frac{1}{1-r}\int_{\mathbb{R}^n} (f')_r(||x||_K) g_r(|\bar{A}_r^{-1}(x-\bar{v}_r)|_2)dx \to \int_{S^{n-1}\cap\partial K} \delta(\xi) F'(\langle \xi, M_0\xi + w_0\rangle d\mathcal{H}^{n-1}(\xi) \quad (2.20)$$

as $r \to 1^-$.

Since (A_r, v_r) minimizes the functional L_r, then Lemma 2.1 guarantees the existence of $\lambda_r > 0$

such that

$$\frac{1}{1-r}\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)\nabla||x||_K\otimes xdx=(1-r)\lambda_r\operatorname{Id}$$
$$\frac{1}{1-r}\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)\nabla||x||_K dx=0.$$

Moreover, by equations (2.5) and (2.6) we obtain

$$\frac{1}{1-r}\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)1_{S^{n-1}\cap\partial K}(x)x\otimes xdx=(1-r)\lambda_r\operatorname{Id}$$
$$\frac{1}{1-r}\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)1_{S^{n-1}\cap\partial K}(x)xdx=0.$$

By Proposition 1.5, the function $\frac{1}{1-r}(f')_r(||x||_K)g_r(|A_r^{-1}(x-v_r)|_2)dx$ is a measure. By (2.20) we conclude that this measure weakly converges to $F'(\langle\xi,M_0\xi+w_0\rangle d\mathcal{H}^{n-1}(\xi)$ and by Theorem 2.7 this last measure is centered and isotropic. The proof is complete. □

Remark 2.2. This type of construction is also valid for the Löwner position. To do so, it is enough to consider the following functionals

$$\tilde{L}_r(A,v)=\frac{1}{1-r}\int_{\mathbb{R}^n}f_r(|Ax+v|_2)g_r(||x||_K)dx$$

and

$$\tilde{I}_r(M,w)\frac{1}{1-r}\int_{\mathbb{R}^n}f_r(|x|_2)g_r(||(\operatorname{Id}+(1-r)M)^{-1}(x-(1-r)w)||_K).$$

The results follow in a similar way, and in most cases it is enough to switch roles K for B^n. Here, we consider f and g as before. This construction is described in [8].

2.3.3 Geometric Interpretation of the minimizer

Consider the functions $f=1_{[-1,+\infty)}, g=1_{(-\infty,0]}$. We have

$$f_r(||Ax+v||_K)=\begin{cases}1, & \text{if } Ax+v\notin rK\\ 0, & \text{if } Ax+v\in rK\end{cases}, \qquad g_r(|x|_2)=\begin{cases}1, & \text{if } x\in B^n\\ 0, & \text{if } x\notin B^n\end{cases}.$$

Then for $(A,v)\in\operatorname{SL}_n(\mathbb{R})\times\mathbb{R}^n$

$$f_r(||Ax+v||_K)g_r(|x|_2)=1_{AB^n+v\setminus rK}(x)$$

and, hence

$$L_r(A,v)=\frac{1}{1-r}\operatorname{vol}_n(AB^n+v\setminus rK).$$

This meaning that a minimum (A_r,v_r) of the restriction of L_r to $(\operatorname{Sym}_{n,+}(\mathbb{R})\cap\operatorname{SL}_n(\mathbb{R}))\times\mathbb{R}^n$ induces a maximal intersection position of radius r.

Unfortunately, this choice of f and g do not give L_r the desirable properties to work with critical point theory. For instance, it is not known in general if the maximal intersection position of radius r is unique, because L_r is not convex.

As we know, assuming Conjecture 2.1 is true, we have that Proposition 2.1 holds. So a possible way to proceed with this work would be to assume uniqueness of the maximal intersection position of radius r and check if there is a possibility of weakening the hypothesis about f and g requested in the previous section. However, we chose to continue studying others things, as we will see.

Another geometric interpretation that we have seen (Theorem 2.12) for the minimizer (A_r, v_r) of the functional L_r is that the derivative of (A_r, v_r) for $r = 1$ equals to minus the minimizer of the functional I_1, which is the limit of the functionals I_r for $r \in (1/2, 1)$.

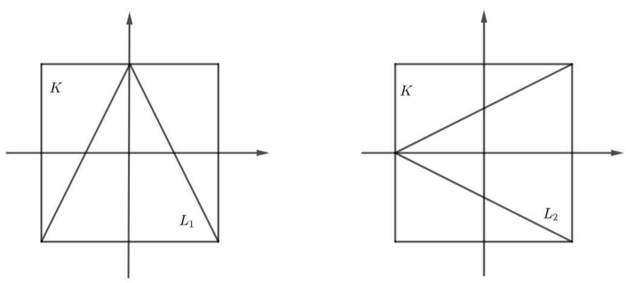
2.4 Explicit representations of Isotropic Measures in positive John and positive Löwner positions

One of the questions raised was if instead of considering a convex body and the unit Euclidean ball, we consider two convex bodies both different from the unit ball, and looking at the position of one of these bodies contained in the other, is it possible to construct a centered and isotropic measure supported at the points of contact of these bodies? We will see what it was possible to observe in this Section.

Let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin as an interior point. Replacing the Euclidean ball by L we can consider the affine image of L contained in K and ask us if there is a position with maximal volume among all such images. The answer is affirmative and this position was studied by many authors, including Giannopoulos, Perissinaki and Tsolomitis [25], Bastero and Romance [13], Gordon, Litvak, Meyer and Pajor [26] and Gruber and Schuster [27].

We say that $L \in \mathcal{K}^n$ is in *maximal volume position inside in* $K \in \mathcal{K}^n$ if L is its own maximal volume image inside K. Note that the maximal volume position of L inside K is not unique, as it can be seen by the example of a triangle inside the cube:

Figure 2.5: L_1 and L_2 are in maximum volume position inside in K.



Source: Compiled by the author.

In this case, triangles L_1 and L_2 have the same volume and one is an affine position to the other.

There is a generalization of the classical John's Theorem 2.3 for the case where L is not the unit Euclidean ball. Giannopoulos, Perissinaki and Tsolomitis [25] proved the following theorem.

Theorem 2.13 ([25], Theorem 2.5)**.** *Let K, L be smooth enough convex bodies in $\mathbb{R}^n$, such that L is of maximal volume in K. If $z \in \operatorname{int} L$, we can find contact points $v_1, \dots, v_m$ of $K - z$ and $L - z$, contact points $u_1, \dots, u_m$ of the polar bodies $(K - z)^\circ$ and $(L - z)^\circ$, and positive real numbers $c_1, \dots, c_m$ such that*

$$\langle u_i, v_i \rangle = 1, \qquad \sum_{i=1}^{m} c_i u_i \otimes v_i = \mathrm{Id}, \qquad \sum_{i=1}^{m} c_i u_i = 0.$$

When we finished the study presented in Section 2.3.2, our question was: Are we able to construct a centered and isotropic measure, supported at the points of contact between K and L, given that L is contained in K and has maximum volume among all its positions? Unfortunately the answer is not entirely straightforward. In fact, the method used by us does not fit into this setting (namely, Theorem 2.10), since the position of L in K of maximum volume is not unique and then the functionals L_r and I_r would not necessarily have a single minimizer. Furthermore, if we take a family that minimizes L_r, say $\{(A_r, v_r)\}_r$, we would not be able to guarantee the convergence $(A_r, v_r) \to (\mathrm{Id}, 0)$. And an important consequence of uniqueness that was heavily used in Theorem 2.10 is that the set $((2 - r)AL + v) \setminus rK$ has non-empty interior. For these reasons, we do not treat the general problem.

As mentioned before, Ball proved that for the classical John's theorem the existence of an isotropic measure supported on contact points is not only implied by, but also implies that K is in John position. For the setting in which both bodies are not the unit Euclidean ball, this characterization is not valid, since we do not have uniqueness of the maximal volume position. Note that this differs from the classic case, because the rotation of the Euclidean ball is the same ball, different from the general case. However, one does obtain an "if and only if" characterization of the position by the existence of a decomposition of the identity when considering a modification of the above position, namely the positive John position.

Definition 2.4. Let K, L be convex bodies with non-empty interior. We define a positive image of L in K to be a set of the form $PL + v$ contained in K, with $v \in \mathbb{R}^n$ and P a positive-definite matrix. We say that K *is in positive John position* with respect to L if $L \subseteq K$ and L has maximal volume among all positive images of L in K.

The positive John position was defined by Artstein-Avidan and Putterman in [7], see also [13]. The advantage of working with the positive John position is due to the following proposition.

Proposition 2.7 ([7], Proposition 3.1)**.** *Let K, L be convex bodies with the origin in the interior of K, and consider the set of positive images of L inside K,*

$$\mathcal{A}_{K,L} = \{PL + v : P \textit{ is defined positive}, v \in \mathbb{R}^n \textit{ and } PL + v \subseteq K\}.$$

Then there is a unique element in $\mathcal{A}_{K,L}$ of maximal volume.

From this result we are able to use our tools to construct an isotropic and centered measure, supported on contact points between K and L, given that K is in positive maximal volume in L. And we can exactly because in this case we are going to have the "good properties" of Theorem

2.10 that are missing in the general case mentioned earlier.

Following Artstein and Putterman, instead of using the term "position of maximal volume among positive images" we call the image of L with maximal volume in K guaranteed by the above proposition, the *positive John image* of L in K.

A subtlety observed by Artstein and Putterman in [7] is that the positive-definite matrices do not form a group, that is, for a given image $L_1 = PL + v$ of L, the family of positive images of L_1, namely $\{ML_1 + w : M \in \mathrm{Sym}_{n,+}(\mathbb{R}), w \in \mathbb{R}^n\}$, does not coincide with the family of positive images of L, while in the case of the usual position of maximal volume between the bodies K and L it holds that $L_1 = ML + z$, where $(M, z) \in \mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n$, is the affine image of L of maximal volume of L contained in K if and only if L_1 is itself in maximal volume position in K. And due to this observation, they obtained the following characterization.

Proposition 2.8 ([7], Proposition 3.2)**.** *For any two convex bodies K, L with non-empty interior, $v \in \mathbb{R}^n$ and a positive-definite matrix P, the body $L_1 = PL + v$ is the positive John image of L in K if and only if $P^{\frac{1}{2}}L + P^{-\frac{1}{2}}v$ is in positive John position inside $P^{-\frac{1}{2}}K$.*

Throughout the text the symmetric part of the matrix $x \otimes y$ will be denoted by $(x \otimes y)_{sym} := (x \otimes y - y \otimes x)$. Following [13], we say that (x, y) is a *contact pair* of K, L if $x \in \partial K \cap \partial L, y \in \partial K^\circ \cap \partial L^\circ$, and $\langle x, y\rangle = 1$. In other words, x is a common boundary point of K, L and y defines a supporting hyperplane to K and L at x.

The next theorem characterizes John positive position of K with respect to L in terms of contact points. It was first given in [13, Theorem 4], and reproven by different methods as [26, Corollary 4.4].

Theorem 2.14 ([7], Theorem 1.2)**.** *Let $K \in \mathcal{K}_0^n, L \in \mathcal{K}^n$. Then K is in positive John position with respect to L if and only if $L \subseteq K$ and there are contact pairs $(x_1, y_1), \ldots, (x_m, y_m)$ of K, L and $c_1, \ldots, c_m > 0$ such that*

$$\sum_{i=1}^{m} c_i (x_i \otimes y_i)_{sym} = \mathrm{Id}, \qquad \sum_{i=1}^{m} c_i y_i = 0. \tag{2.21}$$

For this position we can construct a centered and isotropic measure in the contact pairs. Our next step is to understand how to do this.

As before, consider $f, g : \mathbb{R} \to \mathbb{R}$ functions satisfying the conditions **f1** to **g5**. We define the functional $\tilde{L}_r : \mathrm{M}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{L}_r(A, v) = \frac{1}{1-r} \int_{\mathbb{R}^n} f_r(||Ax + v||_K) g_r(||x||_L) dx,$$

and the functional $\tilde{I}_r : B_r \times \mathbb{R}^n \subseteq \mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{I}_r(M, w) = \frac{1}{1-r} \int_{\mathbb{R}^n} f_r(||x||_K) g_r(||(\mathrm{Id} + (1-r)M)^{-1}(x - (1-r)w)||_L) dx,$$

where again B_r is the set of matrices M such that $\mathrm{Id} + (1-r)M$ is invertible. Note that the difference between these operators and the operators in Section 2.3 is that we replace the Euclidean norm with the gauge function of L.

By construction, if $(A,v) \in \mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^n$ we have

$$\tilde{L}_r(A,v) = \tilde{I}_r\left(\frac{A-\mathrm{Id}}{1-r}, \frac{v}{1-r}\right).$$

Unlike the classical case, $\tilde{L}_r$ is not invariant by orthogonal transformations. But that is not a problem, since we are considering the positive John position.

Again, consider (A_r, v_r) a global minimum of the restriction of $\tilde{L}_r$ to the space $(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$. Arguing similarly to that made in the classical case and following the same notation, we arrive at the following equations due to the Lagrange multipliers:

$$\begin{aligned}
&\lambda_r(A_r^{-T},0) = \frac{1}{1-r}\int_{\mathbb{R}^n} f_r'(||A_r x+v_r||_K) g_r(||x||_L)(\nabla||A_r x+v_r||_K \otimes x, \nabla||A_r x+v_r||_K)dx\\
&= \frac{1}{(1-r)^2}\int_{\mathbb{R}^n}(f')_r(||x||_K) g_r(||A_r^{-1}(x-v_r)||_L)(\nabla||x||_K\otimes A_r^{-1}(x-v_r), \nabla||x||_K)dx\\
&= \frac{1}{(1-r)^2}\int_{\mathbb{R}^n}(f')_r(||x||_K) g_r(||A_r^{-1}(x-v_r)||_L)((\nabla||x||_K\otimes x - \nabla||x||_K\otimes v_r)A_r^{-T}, \nabla||x||_K)dx
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{1}{1-r}\int_{\mathbb{R}^n}(f')_r(||x||_K) g_r(||A_r^{-1}(x-v_r)||_L)(\nabla||x||_K\otimes x)dx &= (1-r)\lambda_r\,\mathrm{Id}\\
\frac{1}{1-r}\int_{\mathbb{R}^n}(f')_r(||x||_K) g_r(||A_r^{-1}(x-v_r)||_L)\nabla||x||_K dx &= 0.
\end{aligned}$$

We will show that the measure $\frac{1}{1-r}(f')_r(||x||_K)g_r(||A_r^{-1}(x-v_r)||_L)dx$ concentrates near $\partial K \cap \partial L$ as $r \to 1^-$ and converges for some sequence $r_k \to 1^-$ to a centered isotropic measure, as in Theorem 2.14. This is because if $x \in \partial K \cap \partial L$, then $\nabla||x||_K \in \partial K^\circ \cap \partial L^\circ$ and $\langle \nabla||x||_K, x\rangle = 1$.

Since the gauge function has the property of being convex as the Euclidean norm (which in particular is a gauge function) and as we are going to assume the same properties for f and g, then the functionals $\tilde{L}_r, \tilde{I}_r$ have the same good properties as the functionals L_r, I_r, respectively, and therefore the proofs of the results in this setting of positive John position follow very similarly. In this way, in order to avoid repetition, we will only mention the main points.

The following result, for example, follows as in Proposition 2.2.

Proposition 2.9. *Assume* $\mathbf{f1}, \mathbf{g1}, \mathbf{g5}$ *are satisfied and* K *has a* C^1*-smooth boundary, then* $\tilde{L}_r, \tilde{I}_r$ *are* C^1 *for* $r \in (1/2, 1)$.

Throughout this section we assume that $K, L \subseteq \mathcal{K}_0^n$ are fixed convex bodies and K is in positive John position with respect to L.

Proposition 2.10. *Assume* $\mathbf{f2}, \mathbf{f3}, \mathbf{f4}, \mathbf{g3}$*, then the family of functionals* $\tilde{L}_r$ *restricted to* $\mathcal{D} \times \mathbb{R}^n$ *is coercive, uniformly for* $r \in (1/2, 1)$.

Proof. Take any $(A, v) \in \mathcal{D} \times \mathbb{R}^n$. Since L is a convex body containing the origin, there exists some centered ball B such that $B \subset \frac{1}{2}L$. The rest of the proof follows as in Proposition 2.3. □

Proposition 2.11. *Let $r \in (1/2, 1)$ and assume* **g3**, **g4**, **f2**, **f3**, **f4**. *The function $\tilde{L}_r$ restrict to $\mathcal{D} \times \mathbb{R}^n$ is positive and convex.*

Proof. We proceed similar to Proposition 2.4. □

Proposition 2.12. *Assume* **g5**, **f3**, *then for $r \in (1/2, 1)$ we have $\tilde{L}_r(\mathrm{Id}, 0) \leq C$, where C is a constant depending only on f, K and n.*

Proof. The proof follows as in Proposition 2.5. □

2.4.1 Main Results

Theorem 2.15. *Let K, L be convex bodies in $\mathbb{R}^n$, where K has C^1-smooth boundary and is in positive John position in L. Choose any finite positive and non-zero measure ν in ∂K with support inside $\partial K \cap \partial L$, and any C^1 function $F \in \mathcal{F}$ (see (2.15)). Consider the convex functional $I_\nu : \mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$I_\nu(M, w) = \int_{\partial K} \frac{1}{|\nabla||z||_K|_2} F(\langle \nabla||z||_K, Mz + w\rangle) d\nu(z).$$

If the restriction of I_ν to $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ is coercive, then for any global minimum (M_0, w_0), the measure

$$\frac{1}{|\nabla||z||_K|_2} F'(\langle \nabla||z||_K, M_0 z + w_0\rangle) d\nu(z)$$

is non-negative, non-zero, centered and isotropic.

Corollary 2.2. *Let K be in positive John position in L and assume*

$$\partial K \cap \partial L = \{x_1, \ldots, x_m\}.$$

Choose any C^1 function $F \in \mathcal{F}$. Consider the convex functional $I_c : \mathrm{Sym}_n(\mathbb{R}) \to \mathbb{R}$ defined by

$$I_c(M, w) = \sum_{i=1}^{m} \frac{1}{|\nabla||x_i||_K|_2} F(\langle \nabla||x_i||_K, Mx_i + w\rangle).$$

If the restriction of I_c to $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ is coercive then for any global minimum (M_0, w_0) the numbers

$$c_i = \frac{1}{|\nabla||x_i||_K|_2} F'(\langle \nabla||x_i||_K, M_0 z + w_0\rangle), i = 1, \ldots, m,$$

together with the vectors $x_i, \nabla||x_i||_K, i = 1, \ldots, m$, are a decomposition of the identity as in (2.21).

Consider $\nu = \nu_{K_L} = C_0(\mathrm{conv}(\partial K \cap \partial L), \cdot)$, where C_0 is the 0-th curvature measure of $\mathrm{conv}(\partial K \cap \partial L)$ given by (2.16). Similar to Theorem 2.9, we have the following result.

Theorem 2.16. *Under the assumptions of Theorem 2.15 the following statements are equivalent.*

1. *The restriction of* I_ν *to* $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ *is coercive;*
2. *For every* $(M, w) \in (\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n) \setminus \{(0,0)\}$

$$\nu(\{z \in \partial K \cap \partial L : \langle \nabla ||z||_K, Mz + w\rangle > 0\}) > 0;$$

If $\nu = \nu_{K_L}$ *or if* $\partial K \cap \partial L$ *is finite and* $\nu = c$*, the statements above are also equivalent to the following:*

3. $\left(\frac{1}{n}\,\mathrm{Id}, 0\right)$ *lies in the interior of* $\mathrm{conv}(\{((x \otimes y)_{sym}, y) : (x, y)$ *is a contact pair of* K*,* $L\}) \subseteq \mathrm{Sym}_{n,1}(\mathbb{R}) \times \mathbb{R}^n$*, where the interior is taken with respect to* $\mathrm{Sym}_{n,1}(\mathbb{R}) \times \mathbb{R}^n$*.*

Proof. The proof is similar to the proof of Theorem 2.9, replacing B^n by L and applying the Proposition 2.6 with the set $A = \partial K \cap \partial L$. □

Proof of Theorem 2.15. We compute the derivative of I_ν in the direction of $(\bar{M}, \bar{v}) \in \mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n$, at the point (M, v)

$$\begin{aligned}
\langle \nabla I_\nu(M, v), (\bar{M}, \bar{v})\rangle &= \int_{\partial K} \frac{1}{|\nabla ||z||_K|_2} F'(\langle \nabla ||z||_K, Mz + w\rangle)\langle \nabla ||z||_K, \bar{M}z + \bar{v}\rangle d\nu \\
&= \int_{\partial K} \frac{1}{|\nabla ||z||_K|_2} F'(\langle \nabla ||z||_K, Mz + w\rangle)(\langle \nabla ||z||_K \otimes z, \bar{M}\rangle + \langle \nabla ||z||_K, \bar{v}\rangle) d\nu \\
&= \left\langle \int_{\partial K} \frac{1}{|\nabla ||z||_K|_2} F'(\langle \nabla ||z||_K, Mz + w\rangle)(\nabla ||z||_K \otimes z, \nabla ||z||_K) d\nu, (\bar{M}, \bar{v}) \right\rangle.
\end{aligned}$$

As we are working in the ambient space which is $\mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n$, the gradient of I_ν in the first variable is the symmetric part of this matrix, that is,

$$\nabla I_\nu(M, v) = \int_{\partial K} \frac{1}{|\nabla ||z||_K|_2} F'(\langle \nabla ||z||_K, Mz + w\rangle)((\nabla ||z||_L \otimes z)_{sym}, \nabla ||z||_K) d\nu.$$

We already known that $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ is the orthogonal complement of $(\mathrm{Id}, 0)$ in $\mathrm{Sym}_n(\mathbb{R}) \times \mathbb{R}^n$ and that the gradient of the function $T(M, v) = \mathrm{tr}(M)$ is $\nabla T(M, v) = (\mathrm{Id}, 0)$. Then by Lagrange multipliers 1.16, we have the equality at a minimum (M_0, w_0) given by

$$\int_{\partial K} \frac{1}{|\nabla ||z||_K|_2} F'(\langle \nabla ||z||_K, M_0 z + w_0\rangle)((\nabla ||z||_K \otimes z)_{sym}, \nabla ||z||_K) d\nu = \lambda(\mathrm{Id}, 0).$$

This clearly implies that

$$\begin{gathered}
\int_{\partial K} \frac{1}{|\nabla ||z||_K|_2} F'(\langle \nabla ||z||_K, M_0 z + w_0\rangle)(\nabla ||z||_K \otimes z)_{sym} d\nu = \lambda\,\mathrm{Id} \qquad (2.22) \\
\int_{\partial K} \frac{1}{|\nabla ||z||_K|_2} F'(\langle \nabla ||z||_L, M_0 z + w_0\rangle)\nabla ||z||_K d\nu = 0.
\end{gathered}$$

Since F is non-decreasing, $F'(\langle \nabla ||z||_K, M_0 z + w_0\rangle) \geq 0$. Taking traces in equation (2.22) we get

$$\lambda = \frac{1}{n} \int_{\partial K} \frac{1}{|\nabla ||z||_K|_2} F'(\langle \nabla ||z||_K, M_0 z + w_0\rangle) d\nu.$$

By Theorem 2.16, we know that $\langle \nabla ||z||_K, M_0 z + w_0 \rangle > 0$ for a set of positive ν-measure. To finish we use the same argument made in Theorem 2.7, that is, since $F'(x) \geq 0$ for every x and $F'(x) > 0$ for $x \geq 0$, then $\lambda > 0$ and the proof is complete. □

Theorem 2.17. *Let K be a convex body in positive John position in the convex body L and f, g satisfy all the properties* **f1** *to* **g5***, then for every $r \in (1/2, 1)$ the restriction of $\tilde{L}_r$ to $(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$ has a unique minimum (A_r, v_r) with $\lim_{r \to 1^-}(A_r, v_r) = (\mathrm{Id}, 0)$. Likewise, the restriction of $\tilde{I}_r$ to*

$$\left(\frac{(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) - \mathrm{Id}}{1-r} \right) \times \mathbb{R}^n$$

has the unique minimum $(M_r, v_r) = \left(\frac{A_r - \mathrm{Id}}{1-r}, \frac{v_r}{1-r} \right)$ *with* $tr\left(\frac{M_r}{||M_r||_F} \right) \to 0$ *as* $r \to 1^-$.

Proof. The proof that there is a unique minimum of $\tilde{L}_r$ in $(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$ follows as in Theorem 2.10, replacing B^n by L and use the results obtained previously. For the convergence of (A_r, v_r) to $(\mathrm{Id}, 0)$ as $r \to 1^-$ we need to notice that if (A_{r_k}, v_{r_k}) is the unique minimum of $\tilde{L}_{r_k}$ in $(\mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})) \times \mathbb{R}^n$ and it is such that $(A_{r_k}, v_{r_k}) \to (A^*, v^*)$ with $(A^*, v^*) \neq (\mathrm{Id}, 0)$, then $A^* \in \mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})$. Furthermore, since by Proposition 2.7 the positive John position is unique, then $A^*L + v^* \setminus K$ has positive Lebesgue measure. But $A^* \in \mathrm{Sym}_{n,+}(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{R})$ and $A^* \neq \mathrm{Id}$ which is a contradiction with the minimality given by Proposition 2.7.

To finish the proof we simply proceed as in Theorem 2.10, replacing B^n by L and use the results obtained in this section. □

Theorem 2.18. *Assume L has a C^1-smooth boundary and all the properties of f and g are satisfied. The functional $\tilde{I}_r$ is extended continuously to $r = 1$ as*

$$\tilde{I}_1(M, v) = \int_{\partial K \cap \partial L} \frac{1}{|\nabla ||z||_K|_2} F(\langle \nabla ||z||_K, Mz + w \rangle) d\mathcal{H}^{n-1}(z),$$

where F is the convolution $F(x) = f * \bar{g}(x), \bar{g}(x) = g(-x)$ and satisfies the conditions of Theorem 2.15. Moreover, $\tilde{I}_r \to \tilde{I}_1$ as $r \to 1^-$, uniformly in compact sets.

Proof. By Proposition 1.4, we have

$$\begin{aligned}
\tilde{I}_r(M, v) &= \frac{1}{1-r} \int_{\mathbb{R}^n} f_r(||x||_K) g_r(||(\mathrm{Id} + (1-r)M)^{-1}(x - (1-r)w)||_L) dx \\
&= \frac{1}{1-r} \int_{\partial K} \int_0^\infty s^{n-1} h_K(n^K(z)) f_r(s||z||_K) \\
&\quad \times g_r(||(\mathrm{Id} + (1-r)M)^{-1}(sz - (1-r)w)||_L) ds d\mathcal{H}^{n-1}(z).
\end{aligned}$$

Since L is smooth, by Taylor 1.18 and formula (1.3), we obtain for any $x, v \in \mathbb{R}^n$

$$\begin{aligned} ||x+v||_L &= ||x||_L + \langle \nabla ||x||_L, v\rangle + o(|v|_2) \\ &= ||x||_L + \left\langle \frac{||x||_L}{h_L(n^L(x))} n^L(x), v \right\rangle + o(|v|_2). \end{aligned} \tag{2.23}$$

By formulas (2.19) and (2.23), we arrived at

$$\begin{aligned} ||(\mathrm{Id} + (1-r)M)^{-1}(sz - (1-r)w)||_L = s\left(||z||_L - (1-r)\left\langle \frac{||z||_L}{\langle n^L(z), z\rangle} n^L(z), Mz + \frac{1}{s}w \right\rangle\right) \\ + o(1-r). \end{aligned}$$

Using this equality in the last integral and making the substitution $s = 1 + (1-r)t$, we get

$$\begin{aligned} \tilde{I}_r(M,w) &= \frac{1}{1-r}\int_{\partial K}\int_0^\infty s^{n-1} h_K(n^K(z)) f_r(s) \\ &\times g_r\left(s\left(||z||_L - (1-r)\left\langle \frac{||z||_L}{\langle n^L(z), z\rangle} n^L(z), Mz + \frac{1}{s}w\right\rangle\right) + o(1-r)\right) ds d\mathcal{H}^{n-1}(z) \\ &= \int_{-\frac{1}{1-r}}^\infty \int_{\partial K} (1 + (1-r)t)^{n-1} h_K(n^K(z)) f(t||z||_K) \\ &\times g\left(\frac{||z||_L - 1}{1-r} - \left\langle \frac{||z||_L}{\langle n^L(z), z\rangle} n^L(z), Mz + w + o(1)\right\rangle + t(||z||_L + o(1)) + o(1)\right) dt d\mathcal{H}^{n-1}(z). \end{aligned}$$

Note that $||z||_L = 1$ for $z \in \partial K \cap \partial L$, $||z||_L > 1$ for $z \in \partial K \setminus \partial L$ and that for all $z \in \partial K \cap \partial L$, $n^L(z) = \nabla ||z||_K h_K(n^K(z))$.

We have $\lim_{r\to 1^-} \frac{||z||_L - 1}{1-r} \to \infty$ in $\partial K \setminus \partial L$ and since 0 is in the interior of L, $\langle n^L(z), z\rangle$ and $||z||_L$ are bounded from below. Also, by **f3** the integrand is 0 for $t < -1$, then

$$\begin{aligned} I_r(M,w) &= \int_{\partial K \cap \partial L}\int_{-1}^\infty (1 + (1-r)t)^{n-1} f(t) h_K(n^K(z)) \\ &\times g\left(-\langle \nabla ||z||_K, Mz + w + o(1)\rangle + t(1 + o(1)) + o(1)\right) dt d\mathcal{H}^{n-1}(z) \\ &+ \int_{\partial K \setminus \partial L}\int_{-1}^\infty (1 + (1-r)t)^{n-1} f(t) h_K(n^K(z)) \\ &\times g\left(\frac{||z||_L - 1}{1-r} - \left\langle \frac{||z||_L}{\langle n^L(z), z\rangle} n^L(z), Mz + w + o(1)\right\rangle + t(||z||_L + o(1)) + o(1)\right) dt d\mathcal{H}^{n-1}(z). \end{aligned}$$

The proof of uniform convergence on compact sets similarly follows from Theorem 2.11 and that F satisfies the assumptions of Theorem 2.15 also follows from Theorem 2.11 already F in Theorem 2.15 and in Theorem 2.7 have the same properties. □

Theorem 2.19. *Assume all the properties* **f1** *to* **g5** *are satisfied and the function* $\tilde{I}_1$ *restricted to* $\mathrm{Sym}_{n,0}(\mathbb{R}) \times \mathbb{R}^n$ *has a unique global minimum* (M_0, w_0), *then* $\left.\frac{\partial (A_r, v_r)}{\partial r}\right|_{r=1}$ *exists and is equal to* -(M_0, w_0).

In this case, if $(\bar{A}_r, \bar{v}_r)$ *is any curve in* $\mathrm{Sym}_{n,+}(\mathbb{R}) \times \mathbb{R}^n$ *of the form*

$$(\bar{A}_r, \bar{v}_r) = (\mathrm{Id}, 0) + (1-r)(M_0, w_0) + o(1-r),$$

the measure

$$\frac{1}{1-r}(f')_r(||x||_K)g_r(||A_r^{-1}(x-v_r)||_L)dx$$

converges weakly to $\dfrac{1}{|\nabla||z||_K|_2}F'(\langle\nabla||z||_K, M_0z+w_0\rangle)d\mathcal{H}^{n-1}(z)$.

Proof. Note that all properties needed to prove the first part of Theorem 2.12, that is, $\left.\dfrac{\partial(A_r, v_r)}{\partial r}\right|_{r=1}$ exists and is equal to -(M_0, w_0) are also valid for this case. Therefore we will omit the proof of this part. For the second part of the theorem, take δ any continuous function with compact support and write, as in the proof of Theorem 2.18,

$$\begin{aligned}
&\frac{1}{1-r}\int_{\mathbb{R}^n}(f')_r(||x||_K)g_r(||\bar{A}_r^{-1}(x-\bar{v}_r)||_L)dx\\
&= \int_{\partial K\cap\partial L}\int_{-1}^{\infty}\delta((1+(1-r)t)z)f'(t)g(-\langle\nabla||z||_K, M_0z+w_0+o(1)\rangle+t(1+o(1))+o(1))\\
&\qquad\times(1+(1-r)t)^{n-1}h_K(n^K(z))dtd\mathcal{H}^{n-1}(z)\\
&\quad+\int_{\partial K\setminus\partial L}\int_{-1}^{\infty}\delta((1+(1-r)t)z)f'(t)g\left(\frac{||z||_L-1}{1-r}+O(1)+t(||z||_L+o(1))\right)\\
&\qquad\times(1+(1-r)t)^{n-1}h_K(n^K(z))dtd\mathcal{H}^{n-1}(z)
\end{aligned}$$

hence by the Dominated Convergence Theorem 1.11,

$$\begin{aligned}
&\frac{1}{1-r}\int_{\mathbb{R}^n}\delta(z)(f')_r(||x||_K)g_r(||\bar{A}_r^{-1}(x-\bar{v}_r)||_L)dx\\
&\qquad\longrightarrow\int_{\partial K\cap\partial L}\delta(z)\frac{1}{|\nabla||z||_K|_2}F'(\langle\nabla||z||_K, M_0z+w_0\rangle d\mathcal{H}^{n-1}(z).
\end{aligned}$$

□

Remark 2.3. We could have done all this construction for positive Löwner position instead of positive John position. It was enough to change the roles of K and L, that is, define the operators:

$$\begin{aligned}
\tilde{L}_r(A, v) &= \frac{1}{1-r}\int_{\mathbb{R}^n}f_r(||Ax+v||_L)g_r(||x||_K)dx\\
\tilde{I}_r(M, w) &= \frac{1}{1-r}\int_{\mathbb{R}^n}f_r(||x||_L)g_r(||(\mathrm{Id}+(1-r)M)^{-1}(x-(1-r)w)||_K)dx.
\end{aligned}$$

In this way, we finish the construction of centered and isotropic measures for sets. Our next step is to look for definitions of functional ellipsoids in order to find out if there is a functional version of the decomposition of the identity like the one given in Theorem 2.3.

Chapter 3

Functional John Ellipsoids

In this chapter we study a recent theory about functional John ellipsoids by G. Ivanov and M. Naszódi in [30]. They showed, non-constructively, a "decomposition of the identity" as given in Theorem 2.3. We will introduce some new concepts and explicitly construct, as in the geometric case, a decomposition of the identity.

3.1 Notation and preliminary results

In this section we will discuss the ideas introduced by G. Ivanov and M. Naszódi in [30]. Some of these concepts will not be necessary in practice for our main results, but we will still mention them in order to contextualize John's functional theory in the approach given by them.

We identify the hyperplane in $\mathbb{R}^{n+1}$ spanned by the first n standard basis vectors with $\mathbb{R}^n$. We say that a set $\bar{C} \subset \mathbb{R}^{n+1}$ is n-symmetric if $(x,t) \in \bar{C}$ implies $(x,-t) \in \bar{C}$. Throughout this chapter det denotes the determinant function defined in $\mathrm{M}_n(\mathbb{R})$ and the determinant function defined in $\mathrm{M}_{n+1}(\mathbb{R})$ will be denoted by $\det_{n+1}$. The trace function in either matrix space $\mathrm{M}_n(\mathbb{R})$ or $\mathrm{M}_{n+1}(\mathbb{R})$ will be denoted only by tr.

For $A \in \mathrm{M}_n(\mathbb{R})$ and a scalar $\alpha \in \mathbb{R}$, we denote by $A \oplus \alpha$ the $(n+1)\times(n+1)$ matrix

$$A \oplus \alpha = \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}.$$

Notice that $\det_{n+1}(A \oplus \alpha) = \alpha \det(A)$ and $\mathrm{tr}(A \oplus \alpha) = \alpha + \mathrm{tr}(A)$.

We will say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is below a function $g : \mathbb{R}^n \to \mathbb{R}$ if $f(x) \leq g(x)$ for all $x \in \mathbb{R}^n$.

To improve readability, we will denote sets with a bar over it for subsets of $\mathbb{R}^{n+1}$. The same holds for matrices of order $(n+1)\times(n+1)$.

Let $s > 0$. For every $x \in \mathbb{R}^n$, we denote the line in $\mathbb{R}^{n+1}$ perpendicular to $\mathbb{R}^n$ at x by l_x and the one-dimensional Lebesgue measure in l_x by l.

Definition 3.1 ([30], Section 2.2)**.** Let $\bar{C} \subset \mathbb{R}^{n+1}$ be a n-symmetric Borel set. The *s-volume* of

$\bar{C}$ is defined by

$$^{(s)}\mu(\bar{C}) = \int_{\mathbb{R}^n} \left[\frac{1}{2} l(\bar{C} \cap l_x)\right]^s dx. \tag{3.1}$$

Note that $^{(s)}\mu(\cdot)$ is not a measure on $\mathbb{R}^{n+1}$.

Definition 3.2 ([30], Section 2.2)**.** For any n-symmetric Borel set $\bar{C}$ in $\mathbb{R}^{n+1}$, the *s-marginal* of $\bar{C}$ on $\mathbb{R}^n$ is defined for any Borel set B in $\mathbb{R}^n$ by

$$^{(s)}\mathrm{marginal}(\bar{C})(B) = \int_{B} \left[\frac{1}{2} l(\bar{C} \cap l_x)\right]^s dx. \tag{3.2}$$

Namely, in this case, the s-marginal is a measure on $\mathbb{R}^n$.

Note that for any matrix $\bar{A} = A \oplus \alpha$, where $A \in \mathrm{M}_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$, any n-symmetric set $\bar{C}$ in $\mathbb{R}^{n+1}$ and any Borel set B in $\mathbb{R}^n$, we have

$$\begin{aligned}
^{(s)}\mathrm{marginal}(\bar{A}\ \bar{C})(AB) &= \int_{AB} \left[\frac{1}{2} l(\bar{A}\ \bar{C} \cap l_x)\right]^s dx \\
&= |\det_n(A)| \int_{B} \left[\frac{1}{2} l(\bar{A}\ \bar{C} \cap l_{Ax})\right]^s dx \\
&= |\det_n(A)| \int_{B} \left[\frac{1}{2} l((\mathrm{Id} \oplus \alpha)\bar{C} \cap l_x)\right]^s dx \\
&= |\det_n(A)||\alpha|^s \int_{B} \left[\frac{1}{2} l(\bar{C} \cap l_x)\right]^s dx \\
&= |\det_n(A)||\alpha|^{s\,(s)}\mathrm{marginal}(\bar{C})(B)
\end{aligned}$$

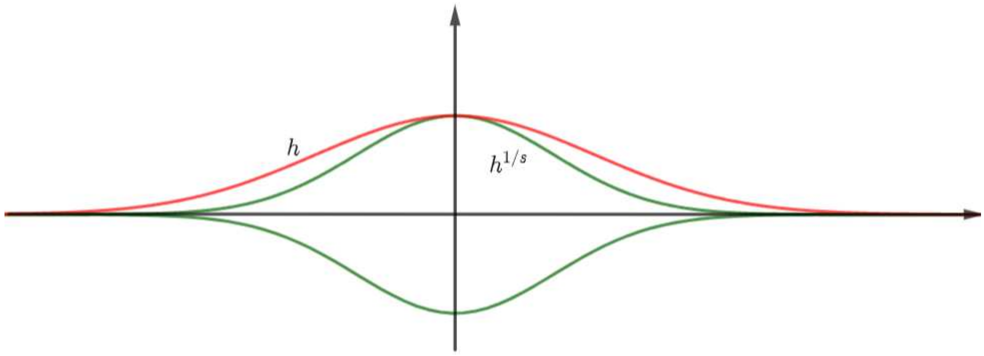
and therefore

$$^{(s)}\mu(\bar{A}\ \bar{C}) = |\det_n(A)||\alpha|^{s\,(s)}\mu(\bar{C}). \tag{3.3}$$

Definition 3.3 ([30], Section 2.3)**.** Let $h : \mathbb{R}^n \to [0, +\infty)$ be a function and $s > 0$. The *s-lifting* of h is a n-symmetric set in $\mathbb{R}^{n+1}$ defined by

$$^{(s)}\bar{h} = \{(x, \xi) \in \mathbb{R}^{n+1} : |\xi| \leq h(x)^{1/s}\}. \tag{3.4}$$

Figure 3.1: h and its s-lifting.



Source: Compiled by the author.

Note that for any Borel set B in $\mathbb{R}^n$,

$$^{(s)}\bar{h} \cap (B \times \mathbb{R}^n) = \{(x, \xi) \in \mathbb{R}^{n+1} : |\xi| \leq h(x)^{1/s} \text{ and } x \in B\}.$$

Since $^{(s)}\bar{h}$ is a n-symmetric set in $\mathbb{R}^{n+1}$, then $l(^{(s)}\bar{h} \cap l_x) = 2h(x)^{1/s}$ for all $x \in \mathbb{R}^n$. Hence,

$$\begin{aligned} ^{(s)}\mu\left(^{(s)}\bar{h} \cap (B \times \mathbb{R})\right) &= \int_{\mathbb{R}^n} \left[\frac{1}{2} l(^{(s)}\bar{h} \cap (B \times \mathbb{R}) \cap l_x)\right]^s dx \\ &= \int_B \left[\frac{1}{2} l(^{(s)}\bar{h} \cap l_x)\right]^s dx \\ &= \int_B \left[\frac{1}{2} 2h(x)^{1/s}\right]^s dx \\ &= \int_B h(x) dx, \end{aligned}$$

that is, $^{(s)}\text{marginal}\left(^{(s)}\bar{h}\right)$ is the measure on $\mathbb{R}^n$ with density h.

While Alonso-Gutiérrez, Gonzales Merino, Jiménez and Villa in [4] determine an ellipsoid defined by $A(B^n) + a$ where $A \in \mathrm{M}_n(\mathbb{R})$ is a positive-definite matrix and $a \in \mathbb{R}^n$, in [30] they consider n-symmetric ellipsoids in $\mathbb{R}^{n+1}$. To describe them, it is necessary to introduce the $\frac{(n+1)(n+2)}{2} + n$ dimensional vector space

$$\mathcal{M} = \{(\bar{A}, a) : \bar{A} \in \mathrm{Sym}_{n+1}(\mathbb{R}), a \in \mathbb{R}^n\}, \tag{3.5}$$

the subspace

$$\mathcal{E} = \{(A \oplus \alpha, a) \in \mathcal{M} : A \in \mathrm{Sym}_n(\mathbb{R}), \alpha > 0\}, \tag{3.6}$$

and the convex cone

$$\mathcal{E}_+ = \{(A \oplus \alpha, a) \in \mathcal{E} : A \text{ is positive-definite}, \alpha > 0\}. \tag{3.7}$$

Recall that the set of positive-definite matrices in $\mathrm{M}_n(\mathbb{R})$ is a convex cone.

We equip $\mathcal{M}$ with the inner product defined by (1.8). Thus, we may use the topology of $\mathcal{M}$ on the set $\mathcal{E}$ of ellipsoids in $\mathbb{R}^{n+1}$. Every n-symmetric ellipsoid in $\mathbb{R}^{n+1}$ is represented by

$$(A \oplus \alpha)B^{n+1} + a,$$

in a unique way, where $A \in \mathrm{GL}_n(\mathbb{R})$, since by Polar Decomposition every element of $\mathcal{E}_+$ uniquely determines each n-symmetric ellipsoid of $\mathbb{R}^{n+1}$. Here $\bar{v} + a$, where $\bar{v} \in \mathbb{R}^{n+1}$ and $a \in \mathbb{R}^n$, denotes $\bar{v} + (a, 0)$.

By (3.3), the s-volume of a n-symmetric ellipsoid can be expressed as

$$^{(s)}\mu((A \oplus \alpha)B^{n+1} + a) = {}^{(s)}\mu(B^{n+1})\alpha^s \det_n(A), \tag{3.8}$$

for any $(A \oplus \alpha, a) \in \mathcal{E}$. In [30] the authors show that

$$\lim_{s \to 0^+} {}^{(s)}\mu(B^{n+1}) = \mathrm{vol}_n(B^n).$$

In this chapter $h : \mathbb{R}^n \to [0, +\infty)$ is a log-concave and upper semicontinuous function and has finite positive integral. In this case we say that h is a *proper log-concave function.* Note that if $h = e^{-\psi}$, then ψ has the properties:

- $\lim_{|x|_2 \to \infty} \psi(x) = +\infty$ (otherwise, the integral of $e^{-\psi(x)}$ equals $+\infty$);
- $\mathrm{dom}\, \psi$ has positive measure (otherwise, the integral of $e^{-\psi(x)}$ equals zero).

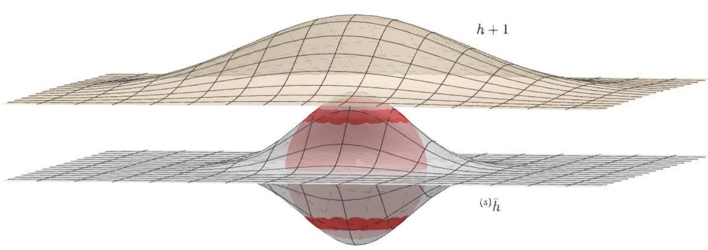
Fix $s > 0$ and let

$$z(h, s) = \sup\{{}^{(s)}\mu(\bar{E}) : \bar{E} \text{ is a } n\text{-symmetric ellipsoid in } \mathbb{R}^{n+1} \text{ with } \bar{E} \subseteq {}^{(s)}\bar{h}\}.$$

In [30] it is shown that this supremum is attained on a unique ellipsoid [30, Theorem 4.1]. This ellipsoid in $\mathbb{R}^{n+1}$ is called the *John s-ellipsoid* of h and is denoted by $\bar{E}(h, s)$. Moreover, they call the s-marginal of $\bar{E}(h, s)$ the *John s-function* of h, and denote its density by

$$^{(s)}J_h = \text{ the density of } {}^{(s)}\text{marginal}(\bar{E}(h, s)).$$

Figure 3.2: The John s-ellipsoid of h.



Source: Compiled by the author.

Let $(A \oplus \alpha, a) \in \mathcal{E}$. We say that α is the *height* of the ellipsoid $\bar{E} = (A \oplus \alpha)B^{n+1} + a$ and the *height function* of $\bar{E}$ is defined as

$$\hbar_{\bar{E}}(x) = \begin{cases} \alpha\sqrt{1 - \langle A^{-1}(x-a), A^{-1}(x-a)\rangle}, & \text{if } x \in AB^n + a \\ 0, & \text{otherwise} \end{cases}.$$

Lemma 3.1. *Let $\bar{E} = (A \oplus \alpha)B^{n+1} + a$ be a n-symmetric ellipsoid in $\mathbb{R}^n$. Then $\bar{E} \subset {}^{(s)}\bar{h}$ holds if and only if*

$$\hbar_{\bar{E}}(x+a) \leq h(x+a)^{1/s}, \tag{3.9}$$

for all $x \in AB^n$.

Proof. We have

$$\begin{aligned} (A \oplus \alpha)B^{n+1} + a = \bar{E} \subset {}^{(s)}\bar{h} &= \{(x,\xi) \in \mathbb{R}^{n+1} : |\xi| \leq h(x)^{1/s}\} \\ &\Leftrightarrow |\alpha| \leq h(x+a)^{1/s}, \forall x \in AB^n \\ &\Leftrightarrow \hbar_{\bar{E}}(x+a) = \alpha\sqrt{1 - \langle A^{-1}(x-a), A^{-1}(x-a)\rangle} \leq \alpha \leq h(x+a)^{1/s}, \end{aligned}$$

for all $x \in AB^n$. $\square$

Lemma 3.2. *The height function of a n-symmetric ellipsoid $\bar{E} = (A \oplus \alpha)B^{n+1} + a$ is a log-concave function.*

Proof. First consider the case $\bar{E} = B^{n+1}$. Note that $\hbar_{B^{n+1}}(x) = e^{-\psi(x)}$, where

$$\psi(x) = \begin{cases} -\ln\sqrt{1 - |x|_2^2} & , \text{ if } x \in \operatorname{int} B^n \\ +\infty & , \text{ otherwise} \end{cases}.$$

Since $\det \mathrm{D}^2\,\psi(x) = (1 - |x|_2^2)^{-\frac{n+2}{2}} > 0$ for all $x \in \operatorname{int} B^n$, by Theorem 1.2, ψ is a convex function on $\mathbb{R}^n$. Hence, by definition, $\hbar_{B^{n+1}}$ is a log-concave function. Now consider the case where $\bar{E} = (A \oplus \alpha)B^{n+1} + a$ is any n-symmetric ellipsoid. Since

$$\hbar_{\bar{E}}(x) = \alpha\hbar_{B^{n+1}}(A^{-1}(x-a))$$

for every $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$, then by Lemma 1.1 we obtain

$$\begin{aligned} \hbar_{\bar{E}}(\lambda x + (1-\lambda)y) &= \alpha\hbar_{B^{n+1}}(A^{-1}(\lambda x + (1-\lambda)y - a)) \\ &= \alpha\hbar_{B^{n+1}}(\lambda A^{-1}(x-a) + (1-\lambda)A^{-1}(y-a)) \\ &\geq \alpha\hbar_{B^{n+1}}(A^{-1}(x-a))^{\lambda}\hbar_{B^{n+1}}(A^{-1}(y-a))^{1-\lambda} \\ &= \left(\alpha\hbar_{B^{n+1}}(A^{-1}(x-a))\right)^{\lambda}\left(\alpha\hbar_{B^{n+1}}(A^{-1}(y-a))\right)^{1-\lambda} \\ &= \hbar_{\bar{E}}(x)^{\lambda}\hbar_{\bar{E}}(y)^{1-\lambda}, \end{aligned}$$

and, again by Lemma 1.1, we conclude that $\hbar_{\bar{E}}$ is a log-concave function. $\square$

For the geometric version, we consider the positions of the unit Euclidean ball B^n contained in a given convex body K. The John ellipsoid is the (unique) largest volume element of this family. In order to pose this problem less geometric, more analytical language, the classical John ellipsoid can be introduced as follows. The John s-function of a proper log-concave function h on $\mathbb{R}^n$ is the (unique) solution to the problem

$$\max_{h} \int_{\mathbb{R}^n} \psi^s,$$

where the maximum is taken over all positions $\psi(s) = \alpha \hbar_{B^{n+1}}(A^{-1}(x-a))$, where $a \in \mathbb{R}^n, A \in \mathrm{GL}_n(\mathbb{R}), \alpha > 0$, and

$$\hbar_{B^{n+1}}(x) = \begin{cases} \sqrt{1-|x|_2^2}, & \text{if } x \in B^n \\ 0, & \text{otherwise} \end{cases}. \tag{3.10}$$

In others words, ψ runs over all positions of the unit ball, under h.

3.2 Interpolation between ellipsoids

In the classical theory of the John ellipsoid, that is, where K is a convex body, the uniqueness of the largest volume ellipsoid contained in $K \subset \mathbb{R}^n$ follows from the convexity of K. In the setting given in [30], the set is not convex. Then, it is shown that if two ellipsoids in $\mathbb{R}^{n+1}$ of the same s-volume are contained in the s-lifting of a log-concave function h, there exists a third ellipsoid "between" the two ellipsoids which is of larger s-volume. This intermediate ellipsoid is obtained as a non-linear combination of the parameters determining the two ellipsoids.

The main tools used are the following two lemmas, that allow us to interpolate between two ellipsoids. Before stating them, we need the following definition.

Definition 3.4. We define the *Asplund sum* of two log-concave functions h_1 and h_2 on $\mathbb{R}^n$ by

$$(h_1 \star h_2)(x) = \sup_{x_1+x_2=x} h_1(x_1)h_2(x_2),$$

and the *epi-product* of a log-concave function h in $\mathbb{R}^n$ with a scalar $\lambda > 0$ by

$$(\lambda * h)(x) = h\left(\frac{x}{\lambda}\right)^{\lambda}.$$

Lemma 3.3 ([30], Lemma 4.1). *(Containment of the interpolated ellipsoid) Fix $s_1, s_2, \beta_1, \beta_2 > 0$ with $\beta_1 + \beta_2 = 1$. Let h_1 and h_2 be two proper log-concave functions on $\mathbb{R}^n$, and $\bar{E}_1$, $\bar{E}_2$ be two n-symmetric ellipsoids represented by $(A_1 \oplus \alpha_1, a_1), (A_2 \oplus \alpha_2, a_2) \in \mathcal{E}$, respectively, such that*

$$\bar{E}_1 \subset {}^{(s_1)}\bar{h}_1 \text{ and } \bar{E}_2 \subset {}^{(s_2)}\bar{h}_2.$$

Define

$$h = (\beta_1 * h_1) \star (\beta_2 * h_2) \text{ and } s = \beta_1 s_1 + \beta_2 s_2.$$

Set

$$(A \oplus \alpha, a) = \Big((\beta_1 A_1 + \beta_2 A_2) \oplus (\alpha_1^{\beta_1 s_1} \alpha_2^{\beta_2 s_2})^{1/s}, \beta_1 a_1 + \beta_2 a_2 \Big) \text{ and } \bar{E} = (A \oplus \alpha) B^{n+1} + a.$$

Then,

$$\bar{E} \subset {}^{(s)}\bar{h}.$$

Lemma 3.4 ([30], Lemma 4.2)**.** *(Volume of the interpolated ellipsoid) Under the conditions of Lemma 3.3 with $s = s_1 = s_2$, the following inequality holds*

$$^{(s)}\mu(\bar{E}) \geq ({}^{(s)}\mu(\bar{E}_1))^{\beta_1} ({}^{(s)}\mu(\bar{E}_2))^{\beta_2},$$

with equality if and only if $A_1 = A_2$.

This lemma is an immediate consequence of formula (3.8) and Lemma 1.4.

Theorem 3.1 ([30], Theorem 4.1)**.** *(Existence and uniqueness of the John s-ellipsoid) Let $s > 0$ and h be a proper log-concave function on $\mathbb{R}^n$. Then, there exists a unique John s-ellipsoid of h.*

To prove Theorem 3.1 it would be necessary to state additional results. For this reason we chose not to do so, because this is not the objective of this work. Lemmas 3.3 and 3.4 are stated here with the objective to convince the reader that if there are two ellipsoids of maximal s-volume in ${}^{(s)}\bar{h}$, it is possible to obtain a third ellipsoid still contained in ${}^{(s)}\bar{h}$ with larger s-volume than the others.

In [30], they proof the following theorem which is a extension of John's theorem for closed n-symmetric set, since in this case $\bar{K}$ does not need to be convex.

Theorem 3.2 ([30], Theorem 5.1)**.** *Let $\bar{K}$ be a closed n-symmetric set in $\mathbb{R}^{n+1}$, and let $s > 0$. Assume that $B^{n+1} \subseteq \bar{K}$. Then the following hold.*

1. *Assume that B^{n+1} is a locally maximal s-volume ellipsoid contained in $\bar{K}$, that is, in some neighborhood of B^{n+1}, no ellipsoid contained in $\bar{K}$ is of larger s-volume. Then there are contact points $\bar{u}_1, \ldots, \bar{u}_k \in \partial(B^{n+1}) \cap \partial(\bar{K})$ and positive weights $c_1, \ldots, c_k$ such that*

$$\sum_{i=1}^{k} c_i \bar{u}_i \otimes \bar{u}_i = \mathrm{Id} \oplus s \quad \text{and} \quad \sum_{i=1}^{k} c_i u_i = 0, \tag{3.11}$$

where u_i is the orthogonal projection of $\bar{u}_i$ onto $\mathbb{R}^n$. Moreover, the number of points k satisfies $n + 1 \leq k \leq \frac{(n+1)(n+2)}{2} + n + 1$;

2. *Assume that $\bar{K} = {}^{(s)}\bar{h}$ for a proper log-concave function h, and that there are contact points and positive weights satisfying (3.11). Then B^{n+1} is the unique ellipsoid of (globally) maximal s-volume among all n-symmetric ellipsoids contained in $\bar{K}$.*

To end this introduction to the theory of functional John ellipsoids, Theorem 3.2 can be rephrased as follows.

Theorem 3.3 ([30], Theorem 5.2)**.** *Let h be a proper log-concave function on $\mathbb{R}^n$, $s > 0$. Assume $\hbar^s_{B^{n+1}} \leq h$. Then the following are equivalent.*

(1) *The function* $\hbar^s_{B^{n+1}}$ *is the John* s*-function of* h*;*

(2) *There are points* $u_1, \dots, u_k \in B^n \subset \mathbb{R}^n$ *and positive weights* $c_1, \dots, c_k$ *such that*

   (*a*) $h(u_i) = \hbar^s_{B^{n+1}}(u_i)$ *for all* $i = 1, \dots, k$*;*

   (*b*) $\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{Id}$*;*

   (*c*) $\sum_{i=1}^k c_i h(u_i)^{1/s} h(u_i)^{1/s} = s$*;*

   (*d*) $\sum_{i=1}^k c_i u_i = 0$.

Definition 3.5. A measure μ on the unit Euclidean ball B^n is said to be *s-isotropic* if for some $\lambda > 0$ it holds

$$\int_{B^n} (u \otimes u \oplus (1 - |u|_2^2)) d\mu = \lambda(\mathrm{Id} \oplus s)$$

and it is called *centered* if

$$\int_{B^n} u d\mu = 0.$$

Our goal in this chapter is to construct a measure that satisfies the items of condition *(2)* of Theorem 3.3, that is, we will fix a proper log-concave function h on $\mathbb{R}^n$ such that $\hbar^s_{B^{n+1}}$ is its John s-function and we will prove that there exists a centered and s-isotropic measure supported in the set $\{h = \hbar^s_{B^{n+1}}\}$. In order to do this, we need to introduce new concepts.

Consider the $(n+1) \times (n+1)$ matrix $M \oplus \beta$, where $M \in \mathrm{M}_n(\mathbb{R})$ and $\beta \in (0, +\infty)$. We define the *s-determinant* of $M \oplus \beta$ by

$$^{(s)}\mathrm{det}_{n+1}(M \oplus \beta) = \beta^s \det(M), \tag{3.12}$$

and the *s-trace* of $M \oplus \beta$ by

$$^{(s)}\mathrm{tr}(M \oplus \beta) = s\beta + \mathrm{tr}(M). \tag{3.13}$$

Based on these definitions, it makes sense to define the sets

$$^{(s)}\,\mathrm{SL}_{n+1}(\mathbb{R}) = \left\{ M \oplus \beta \in \mathrm{M}_{n+1}(\mathbb{R}) : {}^{(s)}\mathrm{det}_{n+1}(M \oplus \beta) = 1 \right\},$$

$$^{(s)}\,\mathrm{Sym}_{n+1,0}(\mathbb{R}) = \{ M \oplus \beta \in \mathrm{Sym}_{n+1}(\mathbb{R}) : {}^{(s)}\mathrm{tr}(M \oplus \beta) = 0 \},$$

and

$$^{(s)}\mathcal{E}_+ = \{ (A \oplus \alpha, a) \in \mathcal{M} : A \in \mathrm{Sym}_{n,+}(\mathbb{R}), \alpha > 0, a \in \mathbb{R}^n \text{ and } {}^{(s)}\mathrm{det}_{n+1}(A \oplus \alpha) \geq 1 \}.$$

Note that ${}^{(s)}\mathcal{E}_+ \subset \mathcal{M}$ is a convex set. Indeed, take $A \oplus \alpha, B \oplus \beta \in {}^{(s)}\mathcal{E}_+$ and $\lambda \in [0, 1]$. By

Lemmas 1.4 and 1.5, we have

$$\begin{aligned}
{}^{(s)}\det_{n+1}(\lambda(A\oplus\alpha)+(1-\lambda)(B\oplus\beta)) &= {}^{(s)}\det_{n+1}((\lambda A+(1-\lambda)B)\oplus(\lambda\alpha+(1-\lambda)\beta)) \\
&= (\lambda\alpha+(1-\lambda)\beta)^s\det(\lambda A+(1-\lambda)B) \\
&\geq (\alpha^\lambda\beta^{1-\lambda})^s\det(A)^\lambda\det(B)^{1-\lambda} \qquad (3.14) \\
&= (\alpha^s\det(A))^\lambda(\beta^s\det(B))^{1-\lambda} \\
&\geq 1. \qquad (3.15)
\end{aligned}$$

The set ${}^{(s)}\operatorname{SL}_{n+1}(\mathbb{R})$ is important due (3.8), because we want to consider the ellipsoids contained in the s-lifting of h with same s-volume that the unit Euclidean ball B^{n+1}, since we are assuming that $\hbar_{B^{n+1}}$ is the John s-function of the log-concave function h and the set ${}^{(s)}\operatorname{Sym}_{n+1,0}(\mathbb{R})$ is important because it is the orthogonal complement of $(\operatorname{Id}\oplus s, 0)$ in $\mathcal{E}$ and in the proof of Theorem 3.4 we will use this fact.

Recall that $r\in(1/2,1), \gamma_r(s)=\gamma\left(\frac{s-1}{1-r}\right)$ and f,g are functions that satisfy the conditions **f1** to **g5**.

We define the functional $\bar{L}_r:\operatorname{M}_n(\mathbb{R})\oplus(0,+\infty)\times\mathbb{R}^n\to\mathbb{R}$ by

$$\bar{L}_r(A\oplus\alpha,v)=\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{\alpha y}{h(Ax+v)^{1/s}}\right)g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right)dydx, \qquad (3.16)$$

where $\operatorname{M}_n(\mathbb{R})\oplus(0,+\infty)$ denotes the set of matrices $A\oplus\alpha\in\operatorname{M}_{n+1}(\mathbb{R})$.

And for $\tilde{A}_r=\operatorname{Id}+(1-r)M, \tilde{\alpha}_r=1+(1-r)\beta$ we define the functional $\bar{I}_r:\bar{B}_r\times\mathbb{R}^n\subseteq\mathcal{E}\to\mathbb{R}$ by

$$\bar{I}_r(M\oplus\beta,w)=\frac{\tilde{\alpha}_r^{s-1}}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{y}{h(x)^{1/s}}\right)g_r\left(\frac{|\tilde{A}_r^{-1}(x-(1-r)w)|_2^2+(\tilde{\alpha}_r^{-1}y)^2-1}{2h(\tilde{A}_r^{-1}(x-(1-r)w))^{2/s}}+1\right)dydx,$$

where $\bar{B}_r=\{M\oplus\beta\in\mathcal{E}: M\in\operatorname{Sym}_n(\mathbb{R})$ is such that $(\operatorname{Id}+(1-r)M)$ is invertible$\}$.

Observe that if $(A\oplus\alpha,w)\in{}^{(s)}\operatorname{SL}_{n+1}(\mathbb{R})\times\mathbb{R}^n$, then

$$\begin{aligned}
\bar{I}_r\left(\frac{A\oplus\alpha-\bar{\operatorname{Id}}}{1-r},\frac{w}{1-r}\right) &= \frac{\alpha^{s-1}}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{y}{h(x)^{1/s}}\right)g_r\left(\frac{|A^{-1}(x-w)|_2^2+(\alpha^{-1}y)^2}{2h(A^{-1}(x-w))^{2/s}}+1\right)dydx \\
&= \frac{\alpha^s\det(A)}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{\alpha y}{h(Ax+w)^{1/s}}\right)g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right)dydx \\
&= \bar{L}_r(A\oplus\alpha,w). \qquad (3.17)
\end{aligned}$$

The idea is to minimize the functional $\bar{L}_r$ over all positions of the unit Euclidean ball B^{n+1} and thus obtain a sequence of measures that weakly converges to a centered and s-isotropic measure. Consider the following lemma.

Lemma 3.5. *Let $(A_r\oplus\alpha_r,v_r)$ be a global minimum of the restriction of $\bar{L}_r$ to $\mathcal{E}_+\cap({}^{(s)}\operatorname{SL}_{n+1}(\mathbb{R})\times$*

$\mathbb{R}^n$). *Then there exists* $\lambda_r \neq 0$ *such that*

$$(1-r)\lambda_r(\mathrm{Id}\oplus s) = \frac{\alpha_r^{s-1}}{1-r}\int_{\mathbb{R}^n}\int_0^\infty (f')_r\left(\frac{y}{h(x)^{1/s}}\right) g_r\left(\frac{|A_r^{-1}(x-v_r)|_2^2+(\alpha_r^{-1}y)^2-1}{2h(A_r^{-1}(x-v_r))^{2/s}}+1\right) \times \frac{y}{h(x)^{3/s}}\left(-\nabla h(x)^{1/s}h(x)^{1/s}\otimes x \oplus h(x)^{1/s}h(x)^{1/s}\right) dydx \tag{3.18}$$

and

$$0 = \frac{\alpha_r^{s-1}}{1-r}\int_{\mathbb{R}^n}\int_0^\infty (f')_r\left(\frac{y}{h(x)^{1/s}}\right) g_r\left(\frac{|A_r^{-1}(x-v_r)|_2^2+(\alpha_r^{-1}y)^2-1}{2h(A_r^{-1}(x-v_r))^{2/s}}+1\right)\frac{y}{h(x)^{3/s}} \times \left(-\nabla h(x)^{1/s}h(x)^{1/s}\right) dydx. \tag{3.19}$$

As we know, the height function of the unit Euclidean ball B^{n+1} is given by $\hbar_{B^{n+1}}(x) = \sqrt{1-|x|_2^2}$, for all $x\in B^n$. Thus, if $x \in \operatorname{int} B^n$,

$$\nabla \hbar_{B^{n+1}}(x) = -\frac{x}{\sqrt{1-|x|_2^2}} = -\frac{x}{\hbar_{B^{n+1}}(x)}.$$

In particular, if $x\in \operatorname{int} B^n$ is such that $h(x) = \hbar^s_{B^{n+1}}(x)$, we have that

$$-\nabla h(x)^{1/s}h(x)^{1/s} = x.$$

Now consider the set $\Lambda = \{x \in B^n : h(x) = \hbar^s_{B^{n+1}}(x)\}$ and the measure $\mu_r : \mathbb{R}^n \to [0,+\infty]$ given by

$$\mu_r(B) = \int_B \frac{\alpha_r^{s-1}}{1-r}\int_0^\infty f_r'\left(\frac{y}{h(x)^{1/s}}\right) g_r\left(\frac{|A_r^{-1}(x-v_r)|_2^2+(\alpha_r^{-1}y)^2-1}{2h(A_r^{-1}(x-v_r))^{2/s}}+1\right)\frac{y}{h(x)^{3/s}}dydx. \tag{3.20}$$

Then it holds that

$$\int_\Lambda \left(-\nabla h(x)^{1/s}h(x)^{1/s}\otimes x\oplus h(x)^{1/s}h(x)^{1/s}\right)d\mu_r(x) = \int_\Lambda \left(x\otimes x\oplus h(x)^{1/s}h(x)^{1/s}\right)d\mu_r(x)$$

and

$$\int_\Lambda \left(-\nabla h(x)^{1/s}h(x)^{1/s}\right)d\mu(x) = \int_\Lambda x d\mu_r(x).$$

We will show that the measure $\mu_r(B)$ concentrates near Λ as $r \to 1^-$ and converges weakly to a centered and s-isotropic measure.

Proof of Lemma 3.5. Let $\psi : \mathrm{M}_n(\mathbb{R}) \oplus (0,+\infty) \to \mathbb{R}$ be the function defined by

$$\psi(M\oplus\beta, w) = {}^{(s)}\mathrm{det}_{n+1}\left(M\oplus\beta\right).$$

We know that ${}^{(s)}\,\mathrm{SL}_{n+1}(\mathbb{R})\times\mathbb{R}^n = \psi^{-1}(\{1\})$, where $c=1$ is a regular value of the differentiable map ψ, then by Theorem 1.16 there exists $\lambda_r \neq 0$ such that

$$\nabla \bar{L}_r(A_r \oplus \alpha_r, v_r) = \lambda_r \nabla \psi(A_r \oplus \alpha_r, v_r), \tag{3.21}$$

where the gradients are taken with respect to the whole space $\mathrm{M}_n(\mathbb{R}) \oplus (0, +\infty)$.

Let $(V \oplus \alpha, w) \in T_{(A_r \oplus \alpha_r, v_r)}(\mathcal{E}_+ \cap ({}^{(s)}\operatorname{SL}_{n+1}(\mathbb{R}) \times \mathbb{R}^n))$. We have

$$\begin{aligned}\psi'(M \oplus \beta, v)\,[V \oplus \alpha, w] &= \beta^s \nabla \det(M) \cdot V + s\beta^{s-1}\alpha \det(M)\\ &= (\beta^s \nabla \det(M) \oplus s\beta^{s-1}\det(M), 0)\,[V \oplus \alpha, w]\,.\end{aligned} \tag{3.22}$$

Thus, since $\nabla \det(M) = \det(M)M^{-T}$, at the point $(A_r \oplus \alpha_r, v_r)$, we arrived at

$$\begin{aligned}\nabla \psi(A_r \oplus \alpha_r, v_r) &= \left(\alpha_r^s \det(A_r) A_r^{-T} \oplus s\alpha_r^{s-1}\det(A_r), 0\right)\\ &= \alpha_r^s \det(A_r) \left(A_r^{-T} \oplus \frac{s}{\alpha_r}, 0\right)\\ &= \left((\mathrm{Id} \oplus s)\left(A_r^{-T} \oplus \frac{1}{\alpha_r}\right), 0\right)\\ &= \left((\mathrm{Id} \oplus s)\,(A_r \oplus \alpha_r)^{-T}, 0\right).\end{aligned} \tag{3.23}$$

Denote the function $\dfrac{\alpha y}{h(Mx+v)^{1/s}}$ by $\varphi(M, \alpha, v)$. Deriving the function $\bar{L}_r$ at the point $(M \oplus \beta, v)$ in the direction of the vector $(V \oplus \alpha, w)$ and using Lemma 1.3, we have

$$\begin{aligned}\bar{L}_r'(M \oplus \beta, v)\,[V \oplus \alpha, w] &= \frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r'\left(\varphi(M, \beta, v)\right) g_r\left(\frac{|x|_2^2 + y^2 - 1}{2h(x)^{2/s}} + 1\right)\\ &\quad\times \left\langle \nabla\left(\frac{\beta y}{h(Mx+v)^{1/s}}\right), (V \oplus \alpha)(x,1) + w\right\rangle dydx\\ &= \frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r'\left(\varphi(M, \beta, v)\right) g_r\left(\frac{|x|_2^2 + y^2 - 1}{2h(x)^{2/s}} + 1\right)\left\langle \nabla\varphi\left(M, \beta, v\right), (V \oplus \alpha)(x,1) + (w,0)\right\rangle dydx\\ &= \frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r'\left(\varphi(M, \beta, v)\right) g_r\left(\frac{|x|_2^2 + y^2 - 1}{2h(x)^{2/s}} + 1\right)\left(\left\langle \nabla\varphi\left(M, \beta, v\right) \otimes (x,1), (V \oplus \alpha)\right\rangle\right.\\ &\quad\left. + \left\langle \nabla\varphi\left(M, \beta, v\right), (w,0)\right\rangle\right) dydx.\end{aligned}$$

Once

$$\nabla\varphi(M, \beta, v) = \nabla\left(\frac{\beta y}{h(Mx+v)^{1/s}}\right) = \left(\frac{-\beta y \nabla h(Mx+v)^{1/s}}{h(Mx+v)^{2/s}}, \frac{y}{h(Mx+v)^{1/s}}\right),$$

then

$$\bar{L}_r'(M\oplus\beta,v)\left[V\oplus\alpha,w\right]=\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r'\left(\varphi(M,\beta,v)\right)g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right)$$
$$\times\left\langle\left(\nabla\varphi(M,\beta,v)\otimes(x,1),\frac{-\beta y\nabla h(Mx+v)^{1/s}}{h(Mx+v)^{2/s}}\right),(V\oplus\alpha,w)\right\rangle dydx$$
$$=\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r'\left(\varphi(M,\beta,v)\right)g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right)$$
$$\times\left\langle\left(\frac{-\beta y\nabla h(Mx+v)^{1/s}}{h(Mx+v)^{2/s}}\otimes x\oplus\frac{y}{h(Mx+v)^{1/s}},\frac{-\beta y\nabla h(Mx+v)^{1/s}}{h(Mx+v)^{2/s}}\right),(V\oplus\alpha,w)\right\rangle dydx.$$

Thus,

$$\nabla\bar{L}_r(A_r\oplus\alpha_r,v_r)=\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r'\left(\frac{\alpha_r y}{h(A_rx+v_r)^{1/s}}\right)g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right)$$
$$\times\left(\left(\frac{-\alpha_r y\nabla h(A_rx+v_r)^{1/s}}{h(A_rx+v_r)^{2/s}}\otimes x\right)\oplus\frac{y}{h(A_rx+v_r)^{1/s}},\frac{-\alpha_r y\nabla h(A_rx+v_r)^{1/s}}{h(A_rx+v_r)^{2/s}}\right)dydx. \quad (3.24)$$

Substituting equalities (3.23) and (3.24) in equality (3.21) and using that $x\otimes Ay=(x\otimes y)A^T$, we get

$$\lambda_r\left((\mathrm{Id}\oplus s)\left(A_r\oplus\alpha_r\right)^{-T},0\right)=\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r'\left(\frac{\alpha_r y}{h(A_rx+v_r)^{1/s}}\right)g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right)$$
$$\times\left(\left(\frac{-\alpha_r y\nabla h(A_rx+v_r)^{1/s}}{h(A_rx+v_r)^{2/s}}\otimes x\right)\oplus\frac{y}{h(A_rx+v_r)^{1/s}},\frac{-\alpha_r y\nabla h(A_rx+v_r)^{1/s}}{h(A_rx+v_r)^{2/s}}\right)dydx$$
$$=\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r'\left(\frac{y}{h(x)^{1/s}}\right)g_r\left(\frac{|A_r^{-1}(x-v_r)|_2^2+(\alpha_r^{-1}y)^2-1}{2h(A_r^{-1}(x-v_r))^{2/s}}+1\right)$$
$$\times\left(\left(\frac{-y\nabla h(x)^{1/s}}{h(x)^{2/s}}\otimes A_r^{-1}(x-v_r)\right)\oplus\frac{y}{\alpha_r h(x)^{1/s}},\frac{-y\nabla h(x)^{1/s}}{h(x)^{2/s}}\right)\frac{1}{\alpha_r\det(A_r)}dydx$$
$$=\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r'\left(\frac{y}{h(x)^{1/s}}\right)g_r\left(\frac{|A_r^{-1}(x-v_r)|_2^2+(\alpha_r^{-1}y)^2-1}{2h(A_r^{-1}(x-v_r))^{2/s}}+1\right)$$
$$\times\left(\left(\frac{-y\nabla h(x)^{1/s}}{h(x)^{2/s}}\otimes(x-v_r)\right)A_r^{-T}\oplus\frac{y}{h(x)^{1/s}}\alpha_r^{-1},\frac{-y\nabla h(x)^{1/s}}{h(x)^{2/s}}\right)\alpha_r^{s-1}dydx$$
$$=\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r'\left(\frac{y}{h(x)^{1/s}}\right)g_r\left(\frac{|A_r^{-1}(x-v_r)|_2^2+(\alpha_r^{-1}y)^2-1}{2h(A_r^{-1}(x-v_r))^{2/s}}+1\right)$$
$$\times\left(\left(\frac{-y\nabla h(x)^{1/s}}{h(x)^{2/s}}\otimes(x-v_r)\oplus\frac{y}{h(x)^{1/s}}\right)(A_r\oplus\alpha_r)^{-T},\frac{-y\nabla h(x)^{1/s}}{h(x)^{2/s}}\right)\alpha_r^{s-1}dydx.$$

By vector equality, and using that $f_r'(s)=\dfrac{1}{1-r}(f')_r(s)$, we obtain

$$(1-r)\lambda_r(\mathrm{Id}\oplus s)=\frac{\alpha_r^{s-1}}{1-r}\int_{\mathbb{R}^n}\int_0^\infty(f')_r\left(\frac{y}{h(x)^{1/s}}\right)g_r\left(\frac{|A_r^{-1}(x-v_r)|_2^2+(\alpha_r^{-1}y)^2-1}{2h(A_r^{-1}(x-v_r))^{2/s}}+1\right)$$
$$\times\frac{y}{h(x)^{3/s}}\left(-\nabla h(x)^{1/s}h(x)^{1/s}\otimes x\oplus h(x)^{1/s}h(x)^{1/s}\right)dydx$$

and

$$0 = \frac{\alpha_r^{s-1}}{1-r}\int_{\mathbb{R}^n}\int_0^\infty (f')_r\left(\frac{y}{h(x)^{1/s}}\right) g_r\left(\frac{|A_r^{-1}(x-v_r)|_2^2+(\alpha_r^{-1}y)^2-1}{2h(A_r^{-1}(x-v_r))^{2/s}}+1\right)\frac{y}{h(x)^{3/s}}$$
$$\times\left(-\nabla h(x)^{1/s}h(x)^{1/s}\right)dydx.$$

□

3.3 Basic Results

Throughout this section we fix a proper log-concave function $h:\mathbb{R}^n\to[0,+\infty)$ such that $\hbar_{B^{n+1}}$ is its John s-function. Due to the good properties of the functions f and g we keep having good properties for the functionals $\bar{L}_r$ and $\bar{I}_r$. The only difference between the properties of L_r and $\bar{L}_r$ will be that while L_r is convex, $\bar{L}_r$ will have another property that we will call convex*. It is worth mentioning that the convexity of the functional L_r was necessary to conclude that it admitted a unique minimum (see Theorem 2.10).

Proposition 3.1. *Assume* **f1**, **g1**, **g5** *are satisfied, then* $\bar{L}_r, \bar{I}_r$ *are* C^1 *for* $r\in(1/2,1)$.

The proof of this proposition is similar to that given in Proposition 2.2, and so it will be omitted.

Proposition 3.2. *Assume* **f2**, **f3**, **f4**, **g3**, *then the family of functionals* $\bar{L}_r$ *restricted to* ${}^{(s)}\mathcal{E}_+\times\mathbb{R}^n$ *is coercive, uniformly for* $r\in(1/2,1)$.

Proof. Let $(x,y)\in B^{n+1}, y\geq 0$. Then

$$\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\leq 1$$

and by **g2** it holds

$$g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right)\geq g_r(1)=g(0).$$

Using **g4**, that f,g are non-negative and $r>1/2$, we obtain

$$\bar{L}_r(A\oplus\alpha,v)\geq\frac{1}{1-r}\int_{B^n}\int_0^{\sqrt{1-|x|_2^2}} f_r\left(\frac{\alpha y}{h(Ax+v)^{1/s}}\right)g(0)dydx$$
$$\geq 2\int_{B^n}\int_0^{\sqrt{1-|x|_2^2}} f_r\left(\frac{\alpha y}{h(Ax+v)^{1/s}}\right)g(0)dydx.$$

Since h is a log-concave function, there exists a convex function ψ such that

$$h(Ax+v)^{1/s}=e^{-\psi(Ax+v)/s}.$$

Then

$$f_r\left(\frac{\alpha y}{h(Ax+v)^{1/s}}\right) = f_r\left(\alpha y e^{\psi(Ax+v)/s}\right) = f\left(\frac{\alpha y e^{\psi(Ax+v)/s}-1}{1-r}\right)$$

and for $\alpha y e^{\psi(Ax+v)/s} \geq 1$

$$\begin{aligned}\bar{L}_r(A\oplus\alpha, v) &\geq 2\int_{B^n}\int_0^{\sqrt{1-|x|_2^2}} f\left(\frac{\alpha y e^{\psi(Ax+v)/s}-1}{1-r}\right) g(0)dydx \\ &\geq 2\int_{B^n}\int_0^{\sqrt{1-|x|_2^2}} f\left(\alpha y e^{\psi(Ax+v)/s}-1\right) g(0)dydx.\end{aligned}$$

By **f2** and **f4**, the function f is coercive to the right and by assumption ψ is a coercive function, hence

$$\begin{aligned}\lim_{||(A\oplus\alpha,v)||\to+\infty} \bar{L}_r(A\oplus\alpha, v) &\geq \lim_{||(A\oplus\alpha,v)||\to+\infty} 2\int_{B^n}\int_0^{\sqrt{1-|x|_2^2}} f\left(\alpha y e^{\psi(Ax+v)/s}-1\right) g(0)dydx \\ &= +\infty.\end{aligned}$$

□

Proposition 3.3. *Let $r \in (1/2, 1)$ and assume* $\mathbf{g3}, \mathbf{g4}, \mathbf{f2}, \mathbf{f3}, \mathbf{f4}$. *The functional $\bar{L}_r$ restricted to ${}^{(s)}\mathcal{E}_+$ is positive.*

Proof. First notice that since $g_r(s) = 0$ always that $s > 2-r$, where $r \in (1/2, 1)$, then $2-r > 1$ and

$$g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right) = 0 \quad \Leftrightarrow \quad \frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1 \geq 2-r.$$

Since for $(x, y) \in B^{n+1}, y \geq 0$, it holds that

$$\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1 \leq 1,$$

then $g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right) > 0$ for all $(x, y) \in B^{n+1}, y \geq 0$.

Now take $(A\oplus\alpha, v) \in {}^{(s)}\mathcal{E}_+$ and assume $\bar{L}_r(A\oplus\alpha, v) = 0$. Since $g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right) > 0$ for all $(x, y) \in B^{n+1}, y \geq 0$, then we must have $f_r\left(\frac{\alpha y}{h(Ax+v)^{1/s}}\right) = 0$ for all $(x, y) \in B^{n+1}\cap(\mathbb{R}^n\times[0,\infty))$, which is equivalent to

$$\frac{\alpha y}{h(Ax+v)^{1/s}} \leq r \quad \Leftrightarrow \quad \frac{\alpha y}{r} \leq h(Ax+v)^{1/s}.$$

Hence,

$$\left(Ax+v, \frac{\alpha y}{r}\right) \in {}^{(s)}\bar{h}$$

for all $(x,y)\in B^{n+1}, y\geq 0$, that is,

$$\left(A\oplus\frac{\alpha}{r}\right)B^{n+1}+v\subset {}^{(s)}\bar{h}.$$

By Lemma 3.1 and using that $\hbar_{B^{n+1}}$ is the John s-function of h, we have that

$${}^{(s)}\mu\left(\left(A\oplus\frac{\alpha}{r}\right)B^{n+1}+v\right)=\left(\frac{\alpha}{r}\right)^s\det(A)\ {}^{(s)}\mu(B^{n+1})\leq {}^{(s)}\mu(B^{n+1}).$$

Therefore,

$$\left(\frac{\alpha}{r}\right)^s\det(A)\leq 1,$$

that is,

$$\alpha^s\det(A)\leq r^s<1,$$

which is a contradiction since $A\oplus\alpha\in {}^{(s)}\mathcal{E}_+$. □

Proposition 3.4. *Let $r\in(1/2,1)$ and assume* $\mathbf{g3},\mathbf{g4},\mathbf{f2},\mathbf{f3},\mathbf{f4}$. *Take $(A\oplus\alpha,v),(B\oplus\beta,w)\in {}^{(s)}\mathcal{E}_+$. The functional $\bar{L}_r$ satisfies the property*

$$\bar{L}_r((\lambda A+(1-\lambda)B)\oplus\alpha^\lambda\beta^{1-\lambda},\lambda v+(1-\lambda)w)\leq\lambda\bar{L}_r(A\oplus\alpha,v)+(1-\lambda)\bar{L}_r(B\oplus\beta,w)$$

for all $\lambda\in[0,1]$.

We will call this property convex.*

Proof. First since h is log-concave, for all $\lambda\in[0,1]$,

$$h(\lambda(Ax+v)+(1-\lambda)(Bx+w))\geq h(Ax+v)^\lambda h(Bx+w)^{1-\lambda}.$$

Since $s>0$, we have

$$h(\lambda(Ax+v)+(1-\lambda)(Bx+w))^{1/s}\geq h(Ax+v)^{\lambda/s}h(Bx+w)^{(1-\lambda)/s},$$

from where it follows that

$$\frac{1}{h(\lambda(Ax+v)+(1-\lambda)(Bx+w))^{1/s}}\leq\frac{1}{h(Ax+v)^{\lambda/s}h(Bx+w)^{(1-\lambda)/s}}.$$

Now since f is non-decreasing, by Lemma 1.5 and using that f is convex, we arrive at

$$\begin{aligned}f_r\left(\frac{\alpha^\lambda\beta^{1-\lambda}y}{h(Ax+v)^{\lambda/s}h(Bx+w)^{(1-\lambda)/s}}\right)&=f_r\left(\left(\frac{\alpha y}{h(Ax+v)^{1/s}}\right)^\lambda\left(\frac{\beta y}{h(Bx+w)^{1/s}}\right)^{1-\lambda}\right)\\&\leq f_r\left(\lambda\frac{\alpha y}{h(Ax+v)^{1/s}}+(1-\lambda)\frac{\beta y}{h(Bx+w)^{1/s}}\right)\\&\leq\lambda f_r\left(\frac{\alpha y}{h(Ax+v)^{1/s}}\right)+(1-\lambda)f_r\left(\frac{\beta y}{h(Bx+w)^{1/s}}\right).\end{aligned}$$

From these inequalities and recalling that by inequalities (3.14) and (3.15) it holds that

$((\lambda A+(1-\lambda)B)\oplus\alpha^\lambda\beta^{1-\lambda},\lambda v+(1-\lambda)w)\in{}^{(s)}\mathcal{E}_+$, we obtain

$$\begin{aligned}
&\bar{L}_r((\lambda A+(1-\lambda)B)\oplus\alpha^\lambda\beta^{1-\lambda},\lambda v+(1-\lambda)w)\\
&=\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{\alpha^\lambda\beta^{1-\lambda}y}{h(\lambda(Ax+v)+(1-\lambda)(Bx+w))^{1/s}}\right)g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right)dydx\\
&\le\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty\left(\lambda f_r\left(\frac{\alpha y}{h(Ax+v)^{1/s}}\right)+(1-\lambda)f_r\left(\frac{\beta y}{h(Bx+w)^{1/s}}\right)\right)\\
&\quad\times g_r\left(\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\right)dydx\\
&=\lambda\bar{L}_r(A\oplus\alpha,v)+(1-\lambda)\bar{L}_r(B\oplus\beta,w),
\end{aligned}$$

as we wanted to prove. ☐

Proposition 3.5. *Assume* **g5, f3**, *then for* $r\in(1/2,1)$ *we have* $\bar{L}_r(\bar{\mathrm{Id}},0)\le C$ *where* C *is a constant depending on* f,h,n *and* s.

Proof. We know that

$$0<g_r(s)\le 1\quad\Leftrightarrow\quad s<2-r,$$

for all $r\in(1/2,1)$ and $1-r\le\frac{1}{2}$. Since

$$\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}+1\le 2-r\quad\Leftrightarrow\quad\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}\le 1-r$$

and

$$\frac{|x|_2^2+y^2}{2h(x)^{2/s}}\le\frac{1}{2}\quad\Rightarrow\quad\frac{|x|_2^2+y^2-1}{2h(x)^{2/s}}\le\frac{1}{2},$$

then if $\bar{C}=\left\{(x,y)\in\mathbb{R}^n\times[0,\infty):\frac{|x|_2^2+y^2}{h(x)^{2/s}}\le 1\right\}$, we have

$$\bar{L}_r(\bar{\mathrm{Id}},0)\le\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{y}{h(x)^{1/s}}\right)1_{\bar{C}}(x,y)dydx.$$

Now notice that $(x,y)\in\bar{C}$ implies

$$0\le\frac{y}{h(x)^{1/s}}\le 1.$$

Making the substitution $\frac{y}{h(x)^{1/s}}=1+(1-r)t$, we get

$$\begin{aligned}
\bar{L}_r(\bar{\mathrm{Id}},0)&\le\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{y}{h(x)^{1/s}}\right)1_{\bar{C}}(x,y)dydx\\
&\le\int_{\mathbb{R}^n}\int_{-\frac{1}{1-r}}^0 f_r(1+(1-r)t)1_{\bar{C}}(x,(1+(1-r)t)h(x)^{1/s})h(x)^{1/s}(1+(1-r)t)dtdx\\
&=\int_{\mathbb{R}^n}\int_{-\frac{1}{1-r}}^0 f(t)1_{\bar{C}}(x,(1+(1-r)t)h(x)^{1/s})h(x)^{1/s}(1+(1-r)t)dtdx.
\end{aligned}$$

Observe that

$$1_{\bar{C}}(x,(1+(1-r)t)h(x)^{1/s})=1 \Leftrightarrow \frac{|x|_2^2+(1+(1-r)t)^2h(x)^{2/s}}{h(x)^{2/s}} \leq 1 \Leftrightarrow \frac{|x|_2^2}{h(x)^{2/s}} \leq 1-(1+(1-r)t)^2.$$

Set

$$\bar{C}_1=\left\{(x,t)\in\mathbb{R}^n\times[-1,0] : \frac{|x|_2^2}{h(x)^{2/s}} \leq 1-(1+(1-r)t)^2\right\}$$

and

$$\bar{C}_2=\left\{(x,t)\in\mathbb{R}^{n+1} : \frac{|x|_2^2}{h(x)^{2/s}} \leq 1\right\}=\left\{(x,t)\in\mathbb{R}^{n+1} : \frac{|x|_2}{h(x)^{1/s}} \leq 1\right\}.$$

Since $\bar{C}_1 \subseteq \bar{C}_2, r\in(1/2,1)$ and $f(t)=0$ if $t<-1$, then

$$\bar{L}_r(\bar{\mathrm{Id}},0) \leq 2\int_{\mathbb{R}^n}\int_{-1}^{0} f(t)1_{\bar{C}_2}(x,(1+(1-r)t)h(x)^{1/s})h(x)^{1/s}dtdx.$$

Since h is a proper log-concave function, there exists a constant $\tilde{C}$ such that $h(x)^{1/s}\leq\tilde{C}$ for all $x\in\mathbb{R}^n$. Then,

$$(x,(1+(1-r)t)h(x)^{1/s})\in\bar{C}_2 \quad\Rightarrow\quad |x|_2\leq h(x)^{1/s}\leq\tilde{C}.$$

Therefore,

$$\begin{aligned}\bar{L}_r(\bar{\mathrm{Id}},0) &\leq 2\int_{\mathbb{R}^n}\int_{-1}^{0} f(t)1_{\bar{C}_2}(x,(1+(1-r)t)h(x)^{1/s})h(x)^{1/s}dtdx\\ &\leq 2\int_{\tilde{C}B^n}\int_{-1}^{0}\tilde{C}f(t)dtdx\\ &= 2\tilde{C}^{n+1}\,\mathrm{vol}_n(B^n)\int_{-1}^{0}f(t)dt\\ &\leq C.\end{aligned}$$

□

3.4 Main results in the functional setting

In this section we will present the results obtained for the functional setting in order to construct a centered and s-isotropic measure. Our goal is to make the results as similar as possible to the geometric version. Consider again the set

$$\mathcal{F}=\{F:\mathbb{R}\to[0,\infty) : F \text{ is non-decreasing, convex, strictly convex in } [0,\infty), \text{ and } F'(0)>0\}.$$

Theorem 3.4. *Let $h:\mathbb{R}^n\to\mathbb{R}$ be a proper log-concave function and $\hbar_{B^{n+1}}$ its John s-function. Choose any finite positive and non-zero measure ν in B^n with support inside the* *subset* $\Lambda = \{x \in B^n : h(x)^{1/s} = \hbar_{B^{n+1}}(x)\}$, *and any* C^1 *function* $F \in \mathcal{F}$. *Consider the convex functional* $\bar{I}_\nu : \mathcal{E} \to \mathbb{R}$ *defined by*

$$\bar{I}_\nu(M \oplus \beta, w) = \int_{B^n} h(x)^{1/s} F\left(\frac{\langle x, Mx + w\rangle}{h(x)^{2/s}} + \beta\right) d\nu(x).$$

If the restriction of $\bar{I}_\nu$ *to* ${}^{(s)}\operatorname{Sym}_{n+1,0}(\mathbb{R}) \times \mathbb{R}^n$ *is coercive then for any global minimum* $(M_0 \oplus \beta_0, w_0)$, *the measure*

$$\frac{1}{h(x)^{1/s}} F'\left(\frac{\langle x, M_0 x + w_0\rangle}{h(x)^{2/s}} + \beta_0\right) d\nu(x)$$

is non-negative, non-zero, centered and s-isotropic.

Assume that $\{x \in B^n : h(x)^{1/s} = \hbar_{B^{n+1}}(x)\}$ is finite and that ν is the counting measure c. As a consequence of the previous theorem we get the following result.

Corollary 3.1. *Let* $h : \mathbb{R}^n \to \mathbb{R}$ *be a proper log-concave function and* $\hbar_{B^{n+1}}$ *its John s-function. Assume*

$$\{x \in B^n : h(x)^{1/s} = \hbar_{B^{n+1}}(x)\} = \{x_1, \dots, x_m\}.$$

Choose any C^1 *function* $F \in \mathcal{F}$. *Consider the convex functional* $\bar{I}_c : \mathcal{E} \to \mathbb{R}$ *defined by*

$$\bar{I}_c(M \oplus \beta, w) = \sum_{i=1}^{m} h(x_i)^{1/s} F\left(\frac{\langle x_i, Mx_i + w\rangle}{h(x_i)^{2/s}} + \beta\right).$$

If the restriction of $\bar{I}_c$ *to* ${}^{(s)}\operatorname{Sym}_{n+1,0}(\mathbb{R}) \times \mathbb{R}^n$ *is coercive then for any global minimum* $(M_0 \oplus \beta_0, w_0)$, *the numbers*

$$c_i = \frac{1}{h(x_i)^{1/s}} F'\left(\frac{\langle x_i, M_0 x_i + w_0\rangle}{h(x_i)^{2/s}} + \beta_0\right), i = 1, \dots, m,$$

together with the vectors $x_i, i = 1, \dots, m$, *satisfy the conditions of item* (2) *of Theorem 3.3.*

Theorem 3.5. *Let* $h : \mathbb{R}^n \to \mathbb{R}$ *be a proper log-concave function and* $\hbar_{B^{n+1}}$ *its John s-function. Consider* F *as in Theorem 3.4. The following statements are equivalent.*

(a) *The restriction of* $\bar{I}_\nu$ *to* ${}^{(s)}\operatorname{Sym}_{n+1,0}(\mathbb{R}) \times \mathbb{R}^n$ *is coercive;*

(b) *For every* $(M \oplus \beta, w) \in ({}^{(s)}\operatorname{Sym}_{n+1,0}(\mathbb{R}) \times \mathbb{R}^n) \setminus \{(0,0)\}$

$$\nu\left(\left\{x \in B^n : h(x)^{1/s} = \hbar_{B^{n+1}}(x) \text{ and } \left(\frac{\langle x, Mx + w\rangle}{h(x)^{2/s}} + \beta\right) > 0\right\}\right) > 0.$$

Proof. The proof follows as in Theorem 2.9. □

Proof of Theorem 3.4. First we will calculate the derivative of $\bar{I}_\nu$ at the point $(M \oplus \beta, w) \in \mathcal{E}$, in the direction of $(V \oplus \alpha, v) \in T_{(M \oplus \beta, w)}\mathcal{E}$. By Lemma 1.3 and the inner product given by (1.8), we obtain

$$\begin{aligned}
\bar{I}'_\nu(M\oplus\beta,w)[V\oplus\alpha,v] &= \int_{B^n} h(x)^{1/s}F'\left(\frac{\langle x,Mx+w\rangle}{h(x)^{2/s}}+\beta\right)\left(\frac{\langle x,Vx+v\rangle}{h(x)^{2/s}}+\alpha\right)d\nu(x)\\
&= \int_{B^n} h(x)^{1/s}F'\left(\frac{\langle x,Mx+w\rangle}{h(x)^{2/s}}+\beta\right)\left(\left\langle\left(\frac{x\otimes x}{h(x)^{2/s}},\frac{x}{h(x)^{2/s}}\right),(V,v)\right\rangle+\frac{h(x)^{1/s}h(x)^{1/s}\alpha}{h(x)^{2/s}}\right)d\nu(x)\\
&= \int_{B^n} h(x)^{1/s}F'\left(\frac{\langle x,Mx+w\rangle}{h(x)^{2/s}}+\beta\right)\left\langle\left(\frac{x\otimes x\oplus h(x)^{1/s}h(x)^{1/s}}{h(x)^{2/s}},\frac{x}{h(x)^{2/s}}\right),(V\oplus\alpha,v)\right\rangle d\nu(x)\\
&= \int_{B^n} \frac{1}{h(x)^{1/s}}F'\left(\frac{\langle x,Mx+w\rangle}{h(x)^{2/s}}+\beta\right)\left\langle\left(x\otimes x\oplus h(x)^{1/s}h(x)^{1/s},x\right),(V\oplus\alpha,v)\right\rangle d\nu(x).
\end{aligned}$$

Since $(x\otimes x\oplus h(x)^{1/s}h(x)^{1/s},x)\in\mathcal{E}$, then we conclude that

$$\nabla\bar{I}_\nu(M\oplus\beta,w)=\int_{B^n}\frac{1}{h(x)^{1/s}}F'\left(\frac{\langle x,Mx+w\rangle}{h(x)^{2/s}}+\beta\right)(x\otimes x\oplus h(x)^{1/s}h(x)^{1/s},x)d\nu(x).$$

The gradient of the function $\psi\,(V\oplus\alpha,v) = {}^{(s)}\mathrm{tr}\,(V\oplus\alpha)$ is $\nabla\psi\,(V\oplus\alpha,v)=(\mathrm{Id}\oplus s,0)$. By Theorem 1.19 we have that ${}^{(s)}\,\mathrm{Sym}_{n+1,0}(\mathbb{R})\times\mathbb{R}^n$ is the orthogonal complement of $(\mathrm{Id}\oplus s,0)$ in $\mathcal{E}$. And since $(M_0\oplus\beta_0,w_0)\in{}^{(s)}\,\mathrm{Sym}_{n+1,0}(\mathbb{R})\times\mathbb{R}^n$ is a singular point of $\bar{I}_\nu$ and 0 is a regular value of ψ, then by Theorem 1.16 there exists $\lambda\in\mathbb{R}$ such that

$$\nabla\bar{I}_\nu(M_0\oplus\beta_0,w_0)=\lambda\nabla\psi(M_0\oplus\beta_0,w_0),$$

that is,

$$\int_{B^n}\frac{1}{h(x)^{1/s}}F'\left(\frac{\langle x,M_0x+w_0\rangle}{h(x)^{2/s}}+\beta_0\right)(x\otimes x\oplus h(x)^{1/s}h(x)^{1/s},x)d\nu(x)=\lambda(\mathrm{Id}\oplus s,0).$$

Equivalently,

$$\int_{B^n}\frac{1}{h(x)^{1/s}}F'\left(\frac{\langle x,M_0x+w_0\rangle}{h(x)^{2/s}}+\beta_0\right)(x\otimes x\oplus h(x)^{1/s}h(x)^{1/s})d\nu(x)=\lambda(\mathrm{Id}\oplus s)\tag{3.25}$$

$$\int_{B^n}\frac{1}{h(x)^{1/s}}F'\left(\frac{\langle x,M_0x+w_0\rangle}{h(x)^{2/s}}+\beta_0\right)xd\nu(x)=0.$$

Finally, we only need to show that λ is positive and to do it we following as in the proof of Theorem 2.7. Since F is non-decreasing, then $F'\left(\frac{\langle x,M_0x+w_0\rangle}{h(x)^{2/s}}+\beta_0\right)\geq 0$. Taking the trace function in equation (3.25) and recalling that the support of measure ν is a subset of points of B^n where $h(x)^{1/s}=\hbar_{B^{n+1}}(x)=\sqrt{1-|x|_2^2}$, we arrive at

$$\begin{aligned}
\lambda &= \frac{1}{n+s}\int_{B^n}\frac{1}{h(x)^{1/s}}F'\left(\frac{\langle x,M_0x+w_0\rangle}{h(x)^{2/s}}+\beta_0\right)(|x|_2^2+1-|x|_2^2)d\nu(x)\\
&= \frac{1}{n+s}\int_{B^n}\frac{1}{h(x)^{1/s}}F'\left(\frac{\langle x,M_0x+w_0\rangle}{h(x)^{2/s}}+\beta_0\right)d\nu(x).
\end{aligned}$$

By Theorem 3.5, we know that $\frac{\langle x,M_0x+w_0\rangle}{h(x)^{2/s}}+\beta_0>0$ for a set of positive ν-measure. Since

$F'(x) \geq 0$ for every x and $F'(x) > 0$ for $x \geq 0$, we deduce that $\lambda > 0$ and the proof is complete. $\square$

Lemma 3.6. *If $(A_r \oplus \alpha_r, v_r) \in {}^{(s)}\mathcal{E}_+$ minimizes $\bar{L}_r$, then ${}^{(s)}\det_{n+1}(A_r \oplus \alpha_r) = 1$.*

Proof. We already know that
$$^{(s)}\det_{n+1}(A_r \oplus \alpha_r) \geq 1.$$
Assume that ${}^{(s)}\det_{n+1}(A_r \oplus \alpha_r) > 1$, that is, $\alpha_r^s \det(A_r) > 1$. Take $\bar{A}_r = A_r \oplus \dfrac{1}{\det(A_r)^{1/s}}$. Then, $(\bar{A}_r, v_r) \in {}^{(s)}\mathcal{E}_+ \cap ({}^{(s)}\mathrm{SL}_{n+1}(\mathbb{R}) \times \mathbb{R}^n)$. Notice that since B^{n+1} is the John s-ellipsoid of h and ${}^{(s)}\det_{n+1}(\bar{A}) \geq 1$, then
$$\bar{A}_r B^{n+1} + v_r \setminus {}^{(s)}r^{\bar{s}}h$$
must have non-empty interior. In fact, to say that $\bar{A}_r B^{n+1} + v_r \setminus {}^{(s)}r^{\bar{s}}h$ has empty interior is the same as to say that $(\mathrm{Id} \oplus r)^{(s)}\bar{h} \supseteq \bar{A}_r B^{n+1} + v_r$. But
$$^{(s)}\det_{n+1}\left((\mathrm{Id} \oplus r)^{-1}\bar{A}_r\right) = {}^{(s)}\det_{n+1}\left(A_r \oplus \frac{1}{r\det(A_r)^{1/s}}\right) = \frac{1}{r^s} > 1$$
and this is a contradiction with the fact that B^{n+1} is the John s-ellipsoid of h. Since $\bar{A}_r B^{n+1} + v_r \setminus {}^{(s)}r^{\bar{s}}h$ has non-empty interior, then there exists a subset $\bar{C}$ of B^{n+1} such that $\mathrm{vol}_{n+1}(\bar{C}) > 0$ and $(x, y) \in \bar{C}$ implies the following inequality
$$rh\left(A_r x + v_r\right)^{1/s} < \frac{y}{\det(A_r)^{1/s}}$$
and hence implies that $f_r\left(\dfrac{y}{\det(A_r)^{1/s}h(A_r x + v_r)^{1/s}}\right)$ is positive. Moreover, in this set $\bar{C}$ it holds $g_r\left(\dfrac{|x|_2^2 + y^2 - 1}{2h(x)^{2/s}} + 1\right)$ is positive since $\dfrac{|x|_2^2 + y^2 - 1}{2h(x)^{2/s}} \leq 0$. Thus,
$$\begin{aligned}
\bar{L}_r(\bar{A}_r, v_r) &= \frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{y}{\det(A_r)^{1/s}h(A_r x + v_r)^{1/s}}\right) g_r\left(\frac{|x|_2^2 + y^2 - 1}{2h(x)^{2/s}} + 1\right) dydx \\
&< \frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{\alpha_r y}{h(A_r x + v_r)^{1/s}}\right) g_r\left(\frac{|x|_2^2 + y^2 - 1}{2h(x)^{2/s}} + 1\right) dydx \\
&= \bar{L}_r(A_r \oplus \alpha_r, v_r),
\end{aligned}$$
which contradicts the minimality of $(A_r \oplus \alpha_r, v_r)$. Therefore ${}^{(s)}\det_{n+1}(A_r \oplus \alpha_r) = 1$. $\square$

Theorem 3.6. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a proper log-concave function and $\hbar_{B^{n+1}}$ its John s-function. Consider f, g functions that satisfy all the properties* **f1** *to* **g5**. *Then for every $r \in (1/2, 1)$ the restriction of $\bar{L}_r$ to ${}^{(s)}\mathcal{E}_+ \cap ({}^{(s)}\mathrm{SL}_{n+1}(\mathbb{R}) \times \mathbb{R}^n)$ has a unique minimum $(A_r \oplus \alpha_r, v_r)$, up to horizontal translation, with $\lim_{r\to 1^-}(A_r \oplus \alpha_r, v_r) = (\bar{\mathrm{Id}}, 0)$. Likewise, the restriction of $\bar{I}_r$ to*
$$\frac{{}^{(s)}\mathcal{E}_+ \cap ({}^{(s)}\mathrm{SL}_{n+1}(\mathbb{R}) \times \mathbb{R}^n) - \bar{\mathrm{Id}} \times \mathbb{R}^n}{1-r}$$

has the unique minimum $(M_r \oplus \beta_r, w_r) = \left(\frac{A_r \oplus \alpha_r - \bar{\mathrm{Id}}}{1-r}, \frac{v_r}{1-r}\right)$*, up to horizontal translation, with* ${}^{(s)}tr\left(\frac{M_r \oplus \beta_r}{||M_r \oplus \beta_r||_F}\right) \to 0$ *as* $r \to 1^-$.

Proof. The existence of one minimum of the functional $\bar{L}_r$ follows of the fact that it is coercive in ${}^{(s)}\mathcal{E}_+$ and that this set is a closed convex set. Now we assume that there are two distinct minimum of $\bar{L}_r$ in ${}^{(s)}\mathcal{E}_+$, say $(A \oplus \alpha, v)$ and $(B \oplus \beta, w)$. Then, by Proposition 3.4, it holds that

$$\bar{L}_r((\lambda A + (1-\lambda)B) \oplus \alpha^\lambda \beta^{1-\lambda}, \lambda v + (1-\lambda)w) = \lambda \bar{L}_r(A \oplus \alpha, v) + (1-\lambda)\bar{L}_r(B \oplus \beta, w),$$

that is, $((\lambda A + (1-\lambda)B) \oplus \alpha^\lambda \beta^{1-\lambda}, \lambda v + (1-\lambda)w) \in {}^{(s)}\mathcal{E}_+$ also minimizes the functional $\bar{L}_r$ and, by Lemma 3.6,

$${}^{(s)}\mathrm{det}_{n+1}((\lambda A + (1-\lambda)B) \oplus \alpha^\lambda \beta^{1-\lambda}) = 1.$$

By (1.9), we have

$$\begin{aligned} 1 = {}^{(s)}\mathrm{det}_{n+1}((\lambda A + (1-\lambda)B) \oplus \alpha^\lambda \beta^{1-\lambda}) &= (\alpha^s)^\lambda (\beta^s)^{1-\lambda} \det(\lambda A + (1-\lambda)B) \\ &\geq (\alpha^s)^\lambda (\beta^s)^{1-\lambda} \det(A)^\lambda \det(B)^{1-\lambda} \\ &= (\alpha^s \det(A))^\lambda (\beta^s \det(B))^{1-\lambda} \\ &= 1. \end{aligned}$$

This last equality implies that $\det(\lambda A + (1-\lambda)B) = \det(A)^\lambda \det(B)^{1-\lambda}$ and hence we have $A = B$. Since

$$\beta^s \det(B) = 1 = \alpha^s \det(A),$$

it follows that $\alpha = \beta$. Then, up to horizontal translation, the minimizers $(A \oplus \alpha, v)$ and $(B \oplus \beta, w)$ coincide.

Denote $M_r \oplus \beta_r = \frac{A_r \oplus \alpha_r - \bar{\mathrm{Id}}}{1-r}, w_r = \frac{v_r}{1-r}$. Since $(A_r \oplus \alpha_r, v_r) \in {}^{(s)}\mathcal{E}_+ \cap ({}^{(s)}\,\mathrm{SL}_{n+1}(\mathbb{R}) \times \mathbb{R}^n)$, we have

$$\bar{L}_r(A_r \oplus \alpha_r, v_r) = \alpha_r^{s-1} \bar{I}_r(M_r \oplus \beta_r, w_r),$$

and $(M_r \oplus \beta_r, w_r)$ is, up to horizontal translation, the unique global minimum of the restriction of $\bar{I}_r$ to

$$\frac{{}^{(s)}\mathcal{E}_+ \cap ({}^{(s)}\,\mathrm{SL}_{n+1}(\mathbb{R}) \times \mathbb{R}^n) - \bar{\mathrm{Id}} \times \mathbb{R}^n}{1-r}.$$

Our next step is to prove that $(A_r \oplus \alpha_r, v_r) \to (\bar{\mathrm{Id}}, 0)$. Assume that $(A_r \oplus \alpha_r, v_r)$ does not converge to $(\bar{\mathrm{Id}}, 0)$. Since by Propositions 3.2 and 3.5 the sequence $\{(A_r \oplus \alpha_r, v_r)\}_r$ is bounded, then there is a sequence $r_k \to 1^-$such that $\{(A_{r_k} \oplus \alpha_{r_k}, v_{r_k})\}_k$ converges. Assume that $(A_{r_k} \oplus \alpha_{r_k}, v_{r_k}) \to (A^* \oplus \alpha^*, v^*) \in {}^{(s)}\mathcal{E}_+ \cap ({}^{(s)}\,\mathrm{SL}_{n+1}(\mathbb{R}) \times \mathbb{R}^n)$ with $(A^* \oplus \alpha^*, v^*) \neq (\bar{\mathrm{Id}}, 0)$.

Again, because B^{n+1} is the John s-ellipsoid of h and ${}^{(s)}\mathrm{det}_{n+1}(A^* \oplus \alpha^*) = 1$, then the set $(A^* \oplus \alpha^*)B^{n+1} + v^* \setminus {}^{(s)}\bar{h}$ has positive Lebesgue measure. Take $\rho < 1$ such that the set $\rho(A^* \oplus \alpha^*)B^{n+1} + v^* \setminus {}^{(s)}\bar{h}$ has positive Lebesgue measure. For large k, we have

$\rho(A^* \oplus \alpha^*)B^{n+1} + v^* \subseteq (A_{r_k} \oplus \alpha_{r_k})B^{n+1} + v_{r_k}$. By Fatou's lemma,

$$
\begin{aligned}
&\liminf_{k\to+\infty} \bar{L}_{r_k}(A_{r_k} \oplus \alpha_{r_k}, v_{r_k}) \\
&= \liminf_{k\to+\infty} \frac{1}{1-r_k} \int_{\mathbb{R}^n} \int_0^\infty f_{r_k}\left(\frac{\alpha_{r_k} y}{h(A_{r_k}x + v_{r_k})^{1/s}}\right) g_{r_k}\left(\frac{|x|_2^2 + y^2 - 1}{2h(x)^{2/s}} + 1\right) dydx \\
&\geq \liminf_{k\to+\infty} \frac{\alpha_{r_k}^{s-1}}{1-r_k} \int_{\mathbb{R}^n \setminus {}^{(s)}\bar{h}} \int_0^\infty f_{r_k}\left(\frac{y}{h(x)^{1/s}}\right) g_{r_k}\left(\frac{|A_{r_k}^{-1}(x - v_{r_k})|_2^2 + (\alpha_{r_k}^{-1}y)^2 - 1}{2h(A_{r_k}^{-1}(x - v_{r_k}))^{2/s}} + 1\right) dydx.
\end{aligned}
$$

Notice that if $(\tilde{x}, \tilde{y}) \in B^{n+1}$ and $(x, y) = \rho(A^* \oplus \alpha^*)(\tilde{x}, \tilde{y}) + v^*$, then

$$
(A_{r_k}^{-1} \oplus \alpha_{r_k}^{-1})(\rho(A^* \oplus \alpha^*)(\tilde{x}, \tilde{y}) + v^* - v_{r_k}) \in (A_{r_k} \oplus \alpha_{r_k})^{-1}(A_{r_k} \oplus \alpha_{r_k})B^{n+1} = B^{n+1},
$$

from where $|A_{r_k}^{-1}(x - v_{r_k})|_2^2 + (\alpha_{r_k}^{-1}y)^2 \leq 1$. And by **g2** it follows that

$$
g_{r_k}\left(\frac{|A_{r_k}^{-1}(x - v_{r_k})|_2^2 + (\alpha_{r_k}^{-1}y)^2 - 1}{2h(A_{r_k}^{-1}(x - v_{r_k}))} + 1\right) \geq g_{r_k}(1) = g(0).
$$

Thus,

$$
\begin{aligned}
\liminf_{k\to+\infty} \bar{L}_{r_k}(A_{r_k} \oplus \alpha_{r_k}, v_{r_k}) \geq& \liminf_{k\to+\infty} \frac{\alpha_{r_k}^{s-1}}{1-r_k} \int_{(\rho(A^* \oplus \alpha^*)B^{n+1} + v^*) \setminus {}^{(s)}\bar{h}} \int_0^\infty f_{r_k}\left(\frac{y}{h(x)^{1/s}}\right) g(0)dydx \\
=& +\infty,
\end{aligned}
$$

which contradicts the bounded of the minimizer $(A_r \oplus \alpha_r, v_r)$, since by Proposition 3.5

$$
\bar{L}_{r_k}(A_{r_k} \oplus \alpha_{r_k}, v_{r_k}) \leq \bar{L}_r(\bar{\mathrm{Id}}, 0) \leq C.
$$

To finish we need to prove that ${}^{(s)}\mathrm{tr}\left(\frac{M_r \oplus \beta_r}{||M_r \oplus ||_F}\right) \to 0$. A simple calculation shows that ${}^{(s)}\mathrm{tr}$ is the differential of ${}^{(s)}\mathrm{det}_{n+1}$ at $\bar{\mathrm{Id}} \in \mathrm{M}_{n+1}(\mathbb{R})$.

By Taylor,

$$
{}^{(s)}\mathrm{det}_{n+1}(\bar{\mathrm{Id}} + \bar{V}) = 1 + \langle \mathrm{Id} \oplus s, \bar{V} \rangle + o(||\bar{V}||_F) = 1 + {}^{(s)}\mathrm{tr}(\bar{V}) + o(||\bar{V}||_F)
$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. Taking $\bar{V} = (1-r)(M_r \oplus \beta_r)$ we get

$$
\begin{aligned}
1 &= {}^{(s)}\mathrm{det}_{n+1}(A_r \oplus \alpha_r) \\
&= {}^{(s)}\mathrm{det}_{n+1}(\bar{\mathrm{Id}} + (1-r)(M_r \oplus \beta_r)) \\
&= 1 + (1-r)^{(s)}\mathrm{tr}(M_r \oplus \beta_r) + o((1-r)||M_r \oplus \beta_r||_F).
\end{aligned}
$$

Therefore,

$$
{}^{(s)}\mathrm{tr}\left(\frac{M_r \oplus \beta_r}{||M_r \oplus \beta_r||_F}\right) = \frac{{}^{(s)}\mathrm{tr}(M_r \oplus \beta_r)}{||M_r \oplus \beta_r||_F} = -\frac{o((1-r)||M_r \oplus \beta_r||_F)}{(1-r)||M_r \oplus \beta_r||_F} \to 0
$$

as $r \to 1^-$. □

Theorem 3.7. *Assume all the properties* **f1** *to* **g5** *are satisfied and that* h *is continuous. The functional* $\bar{I}_r(M \oplus \beta, w)$ *is extended continuously to* $r = 1$ *as*

$$\bar{I}_1(M \oplus \beta, w) = \int_\Lambda h(x)^{1/s} \bar{F}\left(\frac{\langle x, Mx + w\rangle}{h(x)^{2/s}} + \beta\right) dx,$$

where $\Lambda = \{x \in B^n : h(x)^{1/s} = \hbar_{B^{n+1}}(x)\}$ *and* F *is the convolution* $F(x) = f * \bar{g}(x), \bar{g}(x) = g(-x)$ *and satisfies the conditions of Theorem 3.4. Moreover,* $\bar{I}_r \to \bar{I}_1$ *as* $r \to 1^-$*, uniformly in compact sets.*

Proof. As before, we denote by $o((1-r)^a)$ (resp.$o(1)$) any function of the involved parameters M, β, w, r, s, t, x, satisfying

$$\lim_{r\to 1^-} \frac{o((1-r)^a)}{(1-r)^a} = 0 \left(\text{resp. } \lim_{r\to 1^-} o(1) = 0\right),$$

where the limits are uniform in compact sets with respect to the parameters. Likewise, $O(1)$ denote any bounded function.

By Taylor expansion it holds that for all $x, w \in \mathbb{R}^n$ and $M \oplus \beta \in \bar{B}_r$ ($\bar{B}_r$ is the domain of the functional $\bar{I}_r$)

$$\begin{aligned}(\mathrm{Id} + (1-r)M)^{-1}(x - (1-r)w) &= (x - (1-r)w) - (1-r)M(x - (1-r)w) + o(1-r)\\ &= x - (1-r)(Mx + w) + o(1-r)\end{aligned}$$

and

$$\begin{aligned}|(\mathrm{Id} + (1-r)M)^{-1}(x - (1-r)w)|_2 &= |x - (1-r)(Mx + w) + o(1-r)|_2\\ &= |x|_2 - (1-r)\left\langle \frac{x}{|x|_2}, Mx + w\right\rangle + o(1-r).\end{aligned}$$

By simplicity, denote

$$\begin{aligned}&\psi(M;\beta;w;(x,t))\\ &= \frac{|(\mathrm{Id} + (1-r)M)^{-1}(x - (1-r)w)|_2^2 + ((1 + (1-r)\beta)^{-1}(1 + (1-r)t)h(x)^{1/s})^2 - 1}{2h((\mathrm{Id} + (1-r)M)^{-1}(x - (1-r)w))^{2/s}} + 1.\end{aligned}$$

Since

$$\begin{aligned}\bar{I}_r(M \oplus \beta, w) = \frac{1}{1-r}&\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{y}{h(x)^{1/s}}\right)\\ &\times g_r\left(\frac{|(\mathrm{Id} + (1-r)M)^{-1}(x - (1-r)w)|_2^2 + ((1 + (1-r)\beta)^{-1}y)^2 - 1}{2h((\mathrm{Id} + (1-r)M)^{-1}(x - (1-r)w))^{2/s}} + 1\right) dydx,\end{aligned}$$

making the substitution $\frac{y}{h(x)^{1/s}} = 1 + (1-r)t$, we get

$$\bar{I}_r(M \oplus \beta, w) = \int_{\mathbb{R}^n}\int_{-\frac{1}{1-r}}^\infty h(x)^{1/s}(1 + (1-r)t) f_r(1 + (1-r)t)\, g_r(\psi(M;\beta;w;(x,t)))\, dtdx.$$

To calculate $\psi(M;\beta;w;(x,t))$, note that

- $$|(\mathrm{Id}+(1-r)M)^{-1}(x-(1-r)w)|_2^2 = |x|_2^2 + (1-r)^2\left\langle \frac{x}{|x|_2}, Mx+w\right\rangle^2 - 2|x|_2(1-r)\left\langle \frac{x}{|x|_2}, Mx+w\right\rangle + o(1-r)^2 + 2o(1-r)\left(|x|_2 - (1-r)\left\langle \frac{x}{|x|_2}, Mx+w\right\rangle\right);$$
- $$((1+(1-r)\beta)^{-1}h(x)^{1/s}(1+(1-r)t))^2 = \frac{h(x)^{2/s}(1+2(1-r)t+(1-r)^2t^2)}{(1+(1-r)\beta)^2};$$
- $$2h((\mathrm{Id}+(1-r)M)^{-1}(x-(1-r)w))^{2/s} = 2h(x-(1-r)(Mx+w)+o(1-r))^{2/s}.$$

Then,

$$\begin{aligned}\psi(M;\beta;w;(x,t)) &= \frac{(|x|_2^2+h(x)^{2/s}-1)(2\beta(|x|_2^2-1)(1-r)}{2(1+(1-r)\beta)^2h(x-(1-r)(Mx+w)+o(1-r))^{2/s}(1-r)}\\ &+ \frac{(1-r)\left\langle \frac{x}{|x|_2}, Mx+w\right\rangle - 2\langle x, Mx+w\rangle}{2h(x-(1-r)(Mx+w)+o(1-r))^{2/s}}\\ &+ \frac{2h(x)^{2/s}t+(1-r)h(x)^{2/s}t^2+(1-r)\beta^2(|x|_2^2-1)}{2(1+(1-r)\beta)^2h(x-(1-r)(Mx+w)+o(1-r))^{2/s}}\\ &+ \left[\frac{o(1-r)^2}{1-r} + \frac{2o(1-r)}{1-r}\left(|x|_2-(1-r)\left\langle\frac{x}{|x|_2}, Mx+w\right\rangle\right)\right]\\ &\quad\times \frac{1}{2h(x-(1-r)(Mx+w)+o(1-r))^{2/s}}.\end{aligned}$$

Consider the following sets:

$$\Lambda = \{x\in\mathbb{R}^n : |x|_2^2 + h(x)^{2/s} - 1 \le 0\}$$
$$\Lambda^c = \{x\in\mathbb{R}^n : |x|_2^2 + h(x)^{2/s} - 1 > 0\}.$$

Notice that

(*i*) If $x\in\Lambda^c$, since h is bounded and $(1+(1-r)\beta)\le(1+\beta)$, we have

$$\frac{|x|_2^2+h(x)^{2/s}-1}{2(1+(1-r)\beta)^2h(x-(1-r)(Mx+w)+o(1-r))^{2/s}(1-r)} \longrightarrow +\infty$$

as $r\to 1^-$. Then by **g5** it holds $g\mid_{\Lambda^c} \overset{r\to1^-}{\longrightarrow} 0$;

(*ii*) If $x\in\Lambda$, then $|x|_2^2+h(x)^{2/s}=1$. Indeed,

$$|x|_2^2+h(x)^{2/s}<1 \Leftrightarrow \sqrt{|x|_2^2+h(x)^{2/s}}<1 \Leftrightarrow (x,h(x)^{1/s})\in\mathrm{int}(B^{n+1})\subset\mathrm{int}({}^{(s)}\bar{h}).$$

But, as we know $(x,h(x)^{1/s})\in\partial^{(s)}\bar{h}$, for all $x\in\mathbb{R}^n$. Hence,

$$\Lambda=\{x\in\mathbb{R}^n : |x|_2^2+h(x)^{2/s}-1\le 0\} = \{x\in\mathbb{R}^n : |x|_2^2+h(x)^{2/s}-1=0\};$$

(iii) $h(x-(1-r)(Mx+w)+o(1-r))^{2/s} \overset{r\to 1^-}{\longrightarrow} h(x)^{2/s}$ since h is continuous.

By **f3**, the integrand is 0 for $t<-1$ and by (ii), we obtain

$$\begin{aligned}
\bar{I}_r(M\oplus\beta,w) &= \int_{\mathbb{R}^n}\int_{-1}^{\infty} h(x)^{1/s}(1+(1-r)t)f(t)g\left(\psi(M;\beta;w;(x,t))\right)dtdx \\
&= \int_{\Lambda^c}\int_{-1}^{\infty} h(x)^{1/s}(1+(1-r)t)f(t)g\left(\psi(M;\beta;w;(x,t))\right)dtdx \\
&\quad + \int_{\Lambda}\int_{-1}^{\infty} h(x)^{1/s}(1+(1-r)t)f(t)g\left(\psi(M;\beta;w;(x,t))\right)dtdx \\
&= \int_{\Lambda^c}\int_{-1}^{\infty} h(x)^{1/s}(1+(1-r)t)f(t) \\
&\quad \times g\left(\frac{|x|_2^2+h(x)^{2/s}-1+(1-r)O(1)+(1-r)t(2h(x)^{2/s}+o(1))+o(1)}{2(1+(1-r)\beta)^2h(x-(1-r)(Mx+w)+o(1-r))^{2/s}(1-r)}\right)dtdx \\
&\quad + \int_{\Lambda}\int_{-1}^{\infty} h(x)^{1/s}(1+(1-r)t)f(t) \\
&\quad \times g\left(\frac{-2\beta(1-|x|_2^2+o(1))-2\langle x,Mx+w+o(1)\rangle+t(2h(x)^{2/s}+o(1))+o(1)}{2(1+(1-r)\beta)^2h(x-(1-r)(Mx+w)+o(1-r))^{2/s}}\right)dtdx.
\end{aligned} \tag{3.26}$$

To prove that $\bar{I}_r$ converges to $\bar{I}_1$, when $r\to 1^-$, in compact sets, consider a convergent sequence $(M_k\oplus\beta_k,w_k)\to(M\oplus\beta,w)$ and $r_k\to 1^-$. By (i) and **g5**, the function g in the first integral is zero for $t>C$ where C is independent of k. Since the functions f,g are thus uniformly bounded in the support of both integrals and it holds (iii), we may apply the Dominated Convergence Theorem 1.11 in (3.26), to obtain

$$\begin{aligned}
\lim_{k\to\infty}\bar{I}_{r_k}(M_k\oplus\beta_k,w_k) &= \int_{\Lambda}\int_{-1}^{\infty} h(x)^{1/s}f(t)g\left(t-\frac{\beta(1-|x|_2^2)}{h(x)^{2/s}}-\frac{\langle x,Mx+w\rangle}{h(x)^{2/s}}\right)dtdx \\
&= \int_{\Lambda}\int_{-1}^{\infty} h(x)^{1/s}f(t)g\left(t-\frac{\langle x,Mx+w\rangle}{h(x)^{2/s}}-\beta\right)dtdx \\
&= \int_{\Lambda} h(x)^{1/s}\int_{-1}^{\infty} f(t)g\left(t-\frac{\langle x,Mx+w\rangle}{h(x)^{2/s}}-\beta\right)dtdx.
\end{aligned}$$

Let $\bar{g}(x)=g(-x)$ and let $*$ be the convolution function. Define the function $F(x)=f*\bar{g}(x)$ and

$$\bar{I}_1(M\oplus\beta,w)=\int_{\Lambda} h(x)^{1/s}F\left(\frac{\langle x,Mx+w\rangle}{h(x)^{2/s}}+\beta\right)dx.$$

Finally, we must show that F satisfies the properties of Theorem 3.4, but it follows from Theorem 2.11. □

Theorem 3.8. *Assume all the properties* **f1** *to* **g5** *are satisfied and the function $\bar{I}_1$ restricted to $\mathcal{E}\cap{}^{(s)}\,\mathrm{Sym}_{n+1,0}(\mathbb{R})\times\mathbb{R}^n$ has a unique global minimum $(M_0\oplus\beta_0,w_0)$, then $\left.\frac{\partial(A_r\oplus\alpha_r,v_r)}{\partial r}\right|_{r=1}$ exists and is equal to -$(M_0\oplus\beta_0,w_0)$.*

In this case, if $(\tilde{A}_r \oplus \tilde{\alpha}_r, \tilde{v}_r)$ *is any curve in* $\mathcal{E}_+$ *of the form*

$$(\tilde{A}_r \oplus \tilde{\alpha}_r, \tilde{v}_r) = (\bar{\mathrm{Id}}, 0) + (1-r)(M_0 \oplus \beta_0, w_0) + o(1-r),$$

the measure

$$\frac{\tilde{\alpha}_r^{s-1}}{1-r}\int_0^\infty (f')_r\left(\frac{y}{h(x)^{1/s}}\right) g_r\left(\frac{|\tilde{A}_r^{-1}(x-v_r)|_2^2 + (\tilde{\alpha}_r^{-1}y)^2 - 1}{h(\tilde{A}_r^{-1}(x-\tilde{v}_r))^{2/s}} + 1\right)\frac{y}{h(x)^{3/s}}dydx$$

converges weakly to the centered and s-isotropic measure

$$\frac{1}{h(x)^{1/s}}F'\left(\frac{\langle x, M_0x + w_0\rangle}{h(x)^{1/s}} + \beta_0\right)dx.$$

In particular, this is true for $(\tilde{A}_r \oplus \tilde{\alpha}_r, \tilde{v}_r) = (A_r \oplus \alpha_r, v_r)$, *and for its linear part* $(\tilde{A}_r \oplus \tilde{\alpha}_r, \tilde{v}_r) = (\bar{\mathrm{Id}} + (1-r)(M_0 \oplus \beta_0), (1-r)w_0)$.

In order to prove Theorem 3.8, we before need to prove that the family of minimizers of the functionals $\bar{I}_r$ admits a convergent subsequence.

Lemma 3.7. *For every* $r \in (1/2, 1)$, *let* $(M_r \oplus \beta_r, w_r)$ *be a minimizer of the functional* $\bar{I}_r$ *given by Theorem 3.6. The sequence* $\{(M_r \oplus \beta_r, w_r)\}_r$ *is bounded.*

Proof. By Lemma 2.2, the functional $\bar{I}_1$ is coercive. Thus there exists $R > 0$ such that if $(M \oplus \beta, w) \in {}^{(s)}\,\mathrm{Sym}_{n+1,0}(\mathbb{R}) \times \mathbb{R}^n$ and $||(M \oplus \beta, w)|| \geq R$, then

$$\bar{I}_1(M \oplus \beta, w) \geq C + 2$$

where $C \geq \bar{L}_r(\bar{\mathrm{Id}}, 0)$ is given by Proposition 3.5.

Let $\bar{B}_{2R} = \{(M \oplus \beta, w) \in \mathrm{Sym}_{n+1}(\mathbb{R}) \times \mathbb{R}^n : ||(M \oplus \beta, w)|| \leq 2R\}$. By Theorem 3.7, there is $r_0 \in (1/2, 1)$, such that for every $r \in (r_0, 1)$ and $(M \oplus \beta, w) \in \bar{B}_{2R}$,

$$|\bar{I}_r(M \oplus \beta, w) - \bar{I}_1(M \oplus \beta, w)| \leq 1/2.$$

We will show that $(M_r \oplus \beta_r, w_r) \in \bar{B}_{2R}$ for $r \in (r_0, 1)$. Assume by contradiction that $(M_r \oplus \beta_r, w_r) \notin \bar{B}_{2R}$ for some $r \in (r_0, 1)$, and consider $\lambda < 1$ such that $||\lambda(M_r \oplus \beta_r, w_r)|| = 2R$. Since $\left.\frac{\partial}{\partial t}(1 + t\beta_r)^\lambda\right|_{t=0} = \lambda\beta_r$ and by (1.5) holds $(1+t\beta_r)^\lambda \leq 1 + t\lambda\beta_r$ for $t \geq 0$, then for $r \to 1^-$, we have

$$R \leq \rho = \left|\left|\left(\lambda M_r \oplus \left(\frac{(1 + (1-r)\beta_r)^\lambda - 1}{1-r}\right), \lambda w_r\right)\right|\right| \leq ||\lambda(M_r \oplus \beta_r, w_r)|| = 2R.$$

Since $\bar{I}_1$ is continuous in the compact set $\bar{B}_{2R}$, there is $\varepsilon > 0$ such that

$$\bar{I}_1(M \oplus \beta, w) \geq C + 1$$

for every $(M \oplus \beta, w) \in \partial \bar{B}_\rho = \{(M \oplus \beta, w) \in \mathrm{Sym}_{n+1}(\mathbb{R}) \times \mathbb{R}^n : ||(M \oplus \beta, w)|| = \rho, R \leq \rho \leq 2R\}$ with ${}^{(s)}\mathrm{tr}(M \oplus \beta) < \varepsilon$.

By Theorem 3.6, it holds $A_r \oplus \alpha_r \to \bar{\mathrm{Id}}$ as $r \to 1^-$, then increasing r_0 if necessary, we may assume for every $r \in (r_0, 1)$ and $\lambda \in [0,1]$

$$\det_{n+1}(\lambda(A_r \oplus \alpha_r) + (1-\lambda)\bar{\mathrm{Id}}) \leq \frac{C + 1/2}{C + 1/4} = 1 + \frac{1}{4C+1}$$

and again by Theorem 3.6 we have that $\left|{}^{(s)}\mathrm{tr}\left(\frac{M_r \oplus \beta_r}{||M_r \oplus \beta_r||_F}\right)\right| \leq \frac{\varepsilon}{2R}$.

Moreover,

$$\left|{}^{(s)}\mathrm{tr}\left(\lambda M_r \oplus \left(\frac{(1+(1-r)\beta_r)^\lambda - 1}{1-r}\right)\right)\right| \leq |{}^{(s)}\mathrm{tr}(\lambda(M_r \oplus \beta_r))| \leq \frac{||\lambda(M_r \oplus \beta_r)||_F}{2R}\varepsilon \leq \varepsilon,$$

then we obtain

$$\begin{aligned} \bar{I}_r\left(\lambda M_r \oplus \left(\frac{(1+(1-r)\beta_r)^\lambda - 1}{1-r}\right), \lambda w_r\right) &\geq \bar{I}_1\left(\lambda M_r \oplus \left(\frac{(1+(1-r)\beta_r)^\lambda - 1}{1-r}\right), \lambda w_r\right) - 1/2 \\ &\geq C + 1 - 1/2 \\ &= C + 1/2. \end{aligned}$$

Using that $(M_r \oplus \beta_r, w_r) = \left(\frac{A_r \oplus \alpha_r - \bar{\mathrm{Id}}}{1-r}, \frac{v_r}{1-r}\right)$, it holds

$$(((1-r)\lambda M_r + \mathrm{Id}) \oplus (1+(1-r)\beta)^\lambda, (1-r)\lambda w_r) = ((\lambda A_r + (1-\lambda)\,\mathrm{Id}) \oplus \alpha_r^\lambda, \lambda v_r)$$

and since $\lambda A_r + (1-\lambda)\,\mathrm{Id} \in \mathrm{Sym}_{n,+}(\mathbb{R})$ for $r \to 1^-$ we obtain

$$\begin{aligned} {}^{(s)}\det_{n+1}((\lambda A_r + (1-\lambda)\,\mathrm{Id}) \oplus \alpha_r^\lambda) &= (\alpha_r^s)^\lambda \det(\lambda A_r + (1-\lambda)\,\mathrm{Id}) \\ &\geq (\alpha_r^s)^\lambda \det(A_r)^\lambda \det(\mathrm{Id})^{1-\lambda} \\ &= {}^{(s)}\det_{n+1}(A_r \oplus \alpha_r)^{\lambda\,(s)}\det_{n+1}(\bar{\mathrm{Id}}) \\ &\geq 1. \end{aligned}$$

Hence $((\lambda A_r + (1-\lambda)\,\mathrm{Id}) \oplus \alpha_r^\lambda, \lambda v_r) \in {}^{(s)}\mathcal{E}_+$. We have

$$
\begin{aligned}
&\bar{L}_r((\lambda A_r + (1-\lambda)\,\mathrm{Id}) \oplus \alpha_r^\lambda, \lambda v_r) \\
&= \frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty f_r\left(\frac{\alpha_r^\lambda y}{h((\lambda A_r + (1-\lambda)\,\mathrm{Id})x + \lambda v_r)^{1/s}}\right) g_r\left(\frac{|x|_2^2 + y^2 - 1}{2h(x)^{2/s}} + 1\right) dydx \\
&= \frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty \frac{1}{\alpha_r^\lambda \det(\lambda A_r + (1-\lambda)\,\mathrm{Id})} f_r\left(\frac{y}{h(x)^{1/s}}\right) \\
&\quad\times g_r\left(\frac{|(\lambda A_r + (1-\lambda)\,\mathrm{Id})^{-1}(x-\lambda v_r)|_2^2 + (\alpha_r^{-\lambda} y)^2 - 1}{2h((\lambda A_r + (1-\lambda)\,\mathrm{Id})^{-1}(x-\lambda v_r))^{2/s}} + 1\right) dydx \\
&= \frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty \frac{1}{\alpha_r^\lambda \det(\lambda A_r + (1-\lambda)\,\mathrm{Id})} f_r\left(\frac{y}{h(x)^{1/s}}\right) \\
&\quad\times g_r\left(\frac{|(\mathrm{Id} + (1-r)\lambda M_r)^{-1}(x-(1-r)\lambda w_r)|_2^2 + ((1+(1-r)\beta_r)^\lambda)^{-1}y)^2 - 1}{2h((\mathrm{Id} + (1-r)\lambda M_r)^{-1}(x-(1-r)\lambda w_r))^{2/s}} + 1\right) dydx.
\end{aligned}
$$

A simple calculation using Lemma 1.5 shows the inequality

$$\alpha_r^\lambda \det(\lambda A_r + (1-\lambda)\,\mathrm{Id}) \le \det_{n+1}(\lambda(A_r \oplus \alpha_r) + (1-\lambda)\bar{\mathrm{Id}}).$$

From where it follows that

$$
\begin{aligned}
&\bar{L}_r((\lambda A_r + (1-\lambda)\,\mathrm{Id}) \oplus \alpha_r^\lambda, \lambda v_r) \ge \frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty \frac{1}{\det_{n+1}(\lambda(A_r \oplus \alpha_r) + (1-\lambda)\bar{\mathrm{Id}})} f_r\left(\frac{y}{h(x)^{1/s}}\right) \\
&\qquad\times g_r\left(\frac{|(\mathrm{Id} + (1-r)\lambda M_r)^{-1}(x-(1-r)\lambda w_r)|_2^2 + ((1+(1-r)\beta_r)^\lambda)^{-1}y)^2 - 1}{2h((\mathrm{Id} + (1-r)\lambda M_r)^{-1}(x-(1-r)\lambda w_r))^{2/s}} + 1\right) dydx \\
&= \frac{\bar{I}_r\left(\lambda M_r \oplus \left(\frac{(1+(1-r)\beta_r)^\lambda - 1}{1-r}\right), \lambda w_r\right)}{\det_{n+1}(\lambda(A_r \oplus \alpha_r) + (1-\lambda)\bar{\mathrm{Id}})} \\
&\ge \left(\frac{C+1/2}{C+1/4}\right)^{-1}(C+1/2) \\
&\ge \bar{L}_r(\bar{\mathrm{Id}}, 0) + 1/4.
\end{aligned}
$$

Since $\bar{L}_r(\bar{\mathrm{Id}}, 0) \ge \bar{L}_r(A_r \oplus \alpha_r, v_r)$, we obtain the inequalities

$$\bar{L}_r((\lambda A_r + (1-\lambda)\,\mathrm{Id}) \oplus \alpha_r^\lambda, \lambda v_r) > \bar{L}_r(A_r \oplus \alpha_r, v_r)$$

and

$$\bar{L}_r((\lambda A_r + (1-\lambda)\,\mathrm{Id}) \oplus \alpha_r^\lambda, \lambda v_r) > \bar{L}_r(\bar{\mathrm{Id}}, 0),$$

which contradicts the fact that $\bar{L}_r$ is *convex** (see Proposition 3.4). Therefore, $(M_r \oplus \beta_r, w_r) \in \bar{B}_{2R}$ for all $r \in (r_0, 1)$ and we conclude the proof. $\square$

Lemma 3.8. *If $(M_0 \oplus \beta_0, w_0)$ is the unique global minimum of $\bar{I}_1$, then $(M_r \oplus \beta_r, w_r)$ converges to $(M_0 \oplus \beta_0, w_0)$.*

Proof. Take $M \oplus \beta \in {}^{(s)}\operatorname{Sym}_{n+1,0}(\mathbb{R})$ and define

$$^{(s)}(M\oplus\beta)^{(r)} = \frac{^{(s)}\mathrm{det}_{n+1}(\bar{\mathrm{Id}}+(1-r)(M\oplus\beta))^{-1/(n+s)}(\bar{\mathrm{Id}}+(1-r)(M\oplus\beta))-\bar{\mathrm{Id}}}{1-r}.$$

Notice that $({}^{(s)}(M\oplus\beta)^{(r)}, w)$ belongs to

$$\frac{^{(s)}\mathcal{E}_+ \cap ({}^{(s)}\operatorname{SL}_{n+1}(\mathbb{R})\times\mathbb{R}^n) - \bar{\mathrm{Id}}\times\mathbb{R}^n}{1-r}$$

for r close to 1. We also have

$$\begin{aligned}
\lim_{r\to 1^-} {}^{(s)}(M\oplus\beta)^{(r)} &= \lim_{r\to 1^-}\Bigg[\frac{^{(s)}\mathrm{det}_{n+1}(\bar{\mathrm{Id}}+(1-r)(M\oplus\beta))^{-1/(n+s)}-1}{1-r}\bar{\mathrm{Id}} \\
&\quad +{}^{(s)}\mathrm{det}_{n+1}(\bar{\mathrm{Id}}+(1-r)(M\oplus\beta))^{-1/(n+s)}M\oplus\beta\Bigg] \\
&= \frac{\partial}{\partial t}\bigg|_{t=0}({}^{(s)}\mathrm{det}_{n+1}(\bar{\mathrm{Id}}+t(1-r)(M\oplus\beta))^{-1/(n+s)}\bar{\mathrm{Id}})+M\oplus\beta \\
&= \frac{-1}{n+s}{}^{(s)}\mathrm{tr}(-M\oplus\beta)\bar{\mathrm{Id}}+M\oplus\beta \\
&= M\oplus\beta.
\end{aligned}$$

By Lemma 3.7, the sequence $(M_r\oplus\beta_r, w_r)$ is bounded, then for every convergent sequence $(M_{r_k}\oplus\beta_{r_k}, w_{r_k}) \to (M_0\oplus\beta_0, w_0)$ with $r_k\to 1^-$, and for every $(M\oplus\beta, w)\in {}^{(s)}\operatorname{Sym}_{n+1,0}(\mathbb{R})\times\mathbb{R}^n$, we have

$$\bar{I}_{r_k}(M_{r_k}\oplus\beta_{r_k}, w_{r_k}) \to \bar{I}_1(M_0\oplus\beta_0, w_0)$$

and

$$\bar{I}_{r_k}(M_{r_k}\oplus\beta_{r_k}, w_{r_k}) \le \bar{I}_{r_k}({}^{(s)}(M\oplus\beta)^{(r_k)}, w) \to \bar{I}_1(M\oplus\beta, w),$$

so that $(M_0\oplus\beta_0, w_0)$ is the (unique) minimum of $\bar{I}_1$, and we deduce $(M_r\oplus\beta_r, w_r)\to(M_0\oplus\beta_0, w_0)$ as desired. □

Proof of Theorem 3.8. By Lemma 3.8, we have

$$\frac{\partial(A_r\oplus\alpha_r, v_r)}{\partial r}\bigg|_{r=1} = \lim_{r\to 1^-}\frac{(A_r\oplus\alpha_r, v_r)-(\bar{\mathrm{Id}},0)}{r-1} = \lim_{r\to 1^-}(-M_r\oplus\beta_r, -w_r) = -(M_0\oplus\beta_0, w_0).$$

Now take δ any continuous function with compact support and, as in the proof of Theorem 3.7, consider the sets $\Lambda^c = \{x\in\mathbb{R}^n : |x|_2^2 + h(x)^{2/s} > 1\}, \Lambda = \{x\in\mathbb{R}^n : |x|_2^2 + h(x)^{1/s} = 1\}$. We have

$$\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty \delta(x)(f')_r\left(\frac{y}{h(x)^{1/s}}\right)g_r\left(\frac{|\tilde{A}_r^{-1}(x-\tilde{v}_r)|_2^2+(\tilde{\alpha}_r^{-1}y)^2-1}{h(\tilde{A}_r^{-1}(x-\tilde{v}_r))^{2/s}}+1\right)\frac{y}{h(x)^{3/s}}\tilde{\alpha}_r^{s-1}dydx$$
$$=\int_{\Lambda^c}\int_{-1}^\infty \delta(x)(f')_r(1+(1-r)t)g_r\left(\frac{|\tilde{A}_r^{-1}(x-\tilde{v}_r)|_2^2+(\tilde{\alpha}_r^{-1}(1+(1-r)t)h(x)^{1/s})^2-1}{h(\tilde{A}_r^{-1}(x-\tilde{v}_r))^{2/s}}+1\right)$$
$$\times h(x)^{1/s}\frac{(1+(1-r)t)h(x)^{1/s}}{h(x)^{3/s}}\tilde{\alpha}_r^{s-1}dydx$$
$$+\int_{\Lambda}\int_{-1}^\infty \delta(x)(f')_r(1+(1-r)t)g_r\left(\frac{|\tilde{A}_r^{-1}(x-\tilde{v}_r)|_2^2+(\tilde{\alpha}_r^{-1}(1+(1-r)t)h(x)^{1/s})^2-1}{h(\tilde{A}_r^{-1}(x-\tilde{v}_r))^{2/s}}+1\right)$$
$$\times h(x)^{1/s}\frac{(1+(1-r)t)h(x)^{1/s}}{h(x)^{3/s}}\tilde{\alpha}_r^{s-1}dydx$$
$$=\int_{\Lambda^c}\int_{-1}^\infty \delta(x)f'(t)g\left(\frac{|x|_2^2+h(x)^{2/s}-1+(1-r)O(1)+(1-r)t(2h(x)^{2/s}+o(1))+o(1)}{2(1+(1-r)\beta)^2h(x-(1-r)(Mx+w)+o(1-r))^{2/s}(1-r)}\right)$$
$$\times\frac{(1+(1-r)t)}{h(x)^{1/s}}\tilde{\alpha}_r^{s-1}dydx$$
$$+\int_{\Lambda}\int_{-1}^\infty \delta(x)f'(t)g\left(\frac{-2\beta(1-|x|_2^2+o(1))-2\langle x,Mx+w+o(1)\rangle+t(2h(x)^{2/s}+o(1))+o(1)}{2(1+(1-r)\beta)^2h(x-(1-r)(Mx+w)+o(1-r))^{2/s}}\right)$$
$$\times\frac{(1+(1-r)t)}{h(x)^{1/s}}\tilde{\alpha}_r^{s-1}dydx$$

hence by the Dominated Convergence Theorem 1.11,

$$\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty \delta(x)(f')_r\left(\frac{y}{h(x)^{1/s}}\right)g_r\left(\frac{|\tilde{A}_r^{-1}(x-v_r)|_2^2+(\tilde{\alpha}_r^{-1})y)^2-1}{h(\tilde{A}_r^{-1}(x-\tilde{v}_r))^{2/s}}+1\right)\frac{y}{h(x)^{3/s}}\tilde{\alpha}_r^{s-1}dydx$$
$$\longrightarrow\int_\Lambda \delta(x)\frac{1}{h(x)^{1/s}}F'\left(\frac{\langle x,M_0x+w_0\rangle}{h(x)^{2/s}}+\beta_0\right)dx \tag{3.27}$$

as $r\to 1^-$.

For us to finish, since $(A_r\oplus\alpha_r,v_r)$ minimizes the functional $\bar{L}_r$, then by Lemma 3.5 there exists $\lambda_r>0$ such that

$$\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty (f')_r\left(\frac{y}{h(x)^{1/s}}\right)g_r\left(\frac{|\tilde{A}_r^{-1}(x-v_r)|_2^2+(\tilde{\alpha}_r^{-1})y)^2-1}{h(\tilde{A}_r^{-1}(x-\tilde{v}_r))^{2/s}}+1\right)\frac{y}{h(x)^{3/s}}\tilde{\alpha}_r^{s-1}$$
$$\times\left(-\nabla h(x)^{1/s}h(x)^{1/s}\otimes x\oplus h(x)^{1/s}h(x)^{1/s}\right)dydx=\lambda_r(\mathrm{Id}\oplus s,0)$$

and

$$\frac{1}{1-r}\int_{\mathbb{R}^n}\int_0^\infty (f')_r\left(\frac{y}{h(x)^{1/s}}\right)g_r\left(\frac{|\tilde{A}_r^{-1}(x-v_r)|_2^2+(\tilde{\alpha}_r^{-1})y)^2-1}{h(\tilde{A}_r^{-1}(x-\tilde{v}_r))^{2/s}}+1\right)\frac{y}{h(x)^{3/s}}\tilde{\alpha}_r^{s-1}$$
$$\times\left(-\nabla h(x)^{1/s}h(x)^{1/s}\right)dydx=0.$$

And by (3.27) we conclude the wished. □

3.5 Functional Löwner Ellipsoids

In 2019, Li, Schütt and Werner [35] extended a notion of Löwner ellipsoids for functional setting. We say that a function is non-degenerate if int supp$(h) \neq \emptyset$. They showed that if $h : \mathbb{R}^n \to \mathbb{R}$ is a non-degenerate integrable log-concave function then there exists a unique pair (A_0, t_0), where A_0 is an invertible affine transformation and $t_0 \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} e^{-|A_0x|_2+t_0} dx = \min \left\{ \int_{\mathbb{R}^n} e^{-|Ax|_2+t} dx : e^{-|Ax|_2+t} \geq h(x) \right\}.$$

The uniqueness of A_0 is up to left multiplication by orthogonal transformations. They call $e^{-|A_0x|_2+t_0}$ the *Löwner function of* h and denote it by

$$L(h)(x) = e^{-|A_0x|_2+t_0}.$$

For a convex body $K \subset \mathbb{R}^n$ denote its characteristic function by $1_K(x)$. In [35] it is shown that the super-level set $\{L(1_K(x)) \geq 1\}$ is exactly the Löwner ellipsoid of K.

As in the setting of convex sets, the connection between these two optimization problems is via polar duality. Let $h = e^{-\psi} : \mathbb{R}^n \to [0, +\infty]$, then its *log-conjugate* (or *polar*) function is defined by

$$h^\circ(y) = \inf_{x \in \mathbb{R}^n} \frac{e^{-\langle x,y \rangle}}{h(x)}.$$

Note that the log-conjugate function of any function is log-concave. It is easy to show that the log-conjugate function of a proper log-concave function containing the origin in the interior of the support is a proper log-concave function. Also, if f and g are log-concave functions, then

$$f \leq g \quad \Leftrightarrow \quad g^\circ \leq f^\circ.$$

The polar function of the characteristic function of the unit Euclidean ball is $e^{-|x|_2}$. Therefore, the class of functions considered in [4] and [35] consists of translates of functions that are polar to each other.

Combining ideas of [35] and [30], Ivanov and Tsiutsiurupa consider the "dual" problem and define the Löwner s-function below [31]. For a function $\psi : [0, +\infty) \to (-\infty, +\infty]$, they consider the following class of functions

$$\mathcal{E}^n[\psi] = \left\{ \alpha e^{-\psi(|A(x-a)|_2)} : A \in \mathrm{GL}_n(\mathbb{R}), \alpha > 0, a \in \mathbb{R}^n \right\}.$$

One may consider the functional class $\mathcal{E}^n[\psi]$ as the class of "affine" positions of the function $x \mapsto e^{-\psi(|x|_2)}, x \in \mathbb{R}^n$. Now we can say that the classes of "affine" positions of the characteristic function of the unit ball and the function $x \mapsto e^{-|x|_2}, x \in \mathbb{R}^n$, were considered in [4] and [35], respectively.

They require for functions of $\mathcal{E}^n[\psi]$ to be reasonably good log-concave functions. We say that $\psi : [0, +\infty) \to (-\infty, +\infty]$ is an *admissible function* if the function $t \mapsto e^{-\psi(|t|)}, t \in \mathbb{R}$, is an upper semicontinuous log-concave function of finite positive integral.

For an admissible function $\psi : [0,+\infty) \to (-\infty,+\infty]$ and an upper semicontinuous log-concave function $h : \mathbb{R}^n \to [0,+\infty)$ of finite positive integral, they study the following optimization problem:

$$\min_{l \in \mathcal{E}^n[\psi]} \int_{\mathbb{R}^n} l \ \text{ subject to } \ h \le l. \tag{3.28}$$

They study the dual problem to that considered in [30]. For any $s \in [0,+\infty]$, they define $\psi_s : [0,+\infty) \to [0,+\infty)$ by

$$\psi_s(t) = \begin{cases} t, & s = 0 \\ \frac{s}{2}\left[\sqrt{1+4\left(\frac{t}{s}\right)^2} - \ln\left(\frac{1+\sqrt{1+4\left(\frac{t}{s}\right)^2}}{2}\right) - 1\right], & s \in (0,+\infty) \\ t^2, & s = +\infty \end{cases}.$$

This function is an admissible function for any fixed $s \in [0,+\infty]$. One sees that $\mathcal{E}^n[\psi_s]$ with $s = 0$ coincides with the class of functions considered in [35], and that the Löwner function in the sense of [35] is a solution to problem (3.28) with $\psi = \psi_0$. The class $\mathcal{E}^n[\psi_{+\infty}]$ consists of Gaussian densities. The cumbersome definition of ψ_s in the case $s \in (0,+\infty)$ is caused by polar duality, since the problem (3.28) with $\psi = \psi_s$ and $s \in (0,+\infty)$ is dual to the problem of finding the John s-function considered in [30].

They proved the following theorem.

Theorem 3.9 ([31], Theorem 1.2)**.** *Fix $s \in [0,+\infty)$ and let $h : \mathbb{R}^n \to [0,+\infty)$ be an upper semicontinuous log-concave function of finite positive integral. Then, there exists a unique solution to problem*

$$\min_{l \in \mathcal{E}^n[\psi_s]} \int_{\mathbb{R}^n} l \ \textit{ subject to } \ h \le l. \tag{3.29}$$

The solution to problem (3.29) for a fixed $s \in [0,+\infty)$ is called *Löwner s-function* of f, and denoted by ${}^{(s)}L_f$. Note that the Löwner function in the sense of [35] is precisely ${}^{(0)}L_f$. And as in the case of John s-functions, they showed that ${}^{(s)}L_f \to {}^{(0)}L_f$ as $s \to 0$, uniformly on $\mathbb{R}^n$ (see [31, Theorem 1.3]).

Probably, the most striking difference between the John s-function and the Löwner s-function appears in the case $s = +\infty$. This is because in [31] it is shown that as $s \to +\infty$ the limit may only be a Gaussian density and it is necessarily unique, while in [30] it was shown that the Gaussian density of maximal integral below a given upper semicontinuous log-concave function of positive integral is not necessarily unique.

We say that the functions of $\mathcal{E}^n[\psi]$ are *ψ-ellipsoidal* functions. If ψ is an admissible function, then all the functions of $\mathcal{E}^n[\psi]$ are proper log-concave functions. We use $l_{\psi,\bar{E}}$ to denote the ψ-ellipsoidal function represented by $\bar{E} = (A \oplus \alpha, a) \in \mathcal{E}_+$. The reason they define the classes of

s-ellipsoidal functions is the following result.

Lemma 3.9 ([31], Lemma 6.1). *Let* $\bar{E} = (A \oplus \alpha, 0) \in \mathcal{E}_+$ *and* $s \in [0, +\infty]$. *Then*

$$\left({}^{(s)}h_{\bar{E}}\right)^{\circ} = {}^{(s)}l_{\bar{E}} \quad \textit{and} \quad \left({}^{(s)}l_{\bar{E}}\right)^{\circ} = {}^{(s)}h_{\bar{E}},$$

where ${}^{(s)}h_{\bar{E}} = \alpha^{-(s+1)}\hbar^{s}_{\bar{E}}$.

Chapter 4

Upper semicontinuous valuations on the space of functions

In this chapter, a classification of upper semicontinuous and translation invariant valuations which is unchanged by the addition of piecewise affine functions on the space of convex functions which is a piecewise affine function outside of a compact set of $\mathbb{R}$, is established in Theorem 4.6. This is a joint work with Monika Ludwig.

Following the standard notations in valuation theory, in this chapter we will adopt the notation $\mathcal{K}^n$ for the set of all non-empty, compact, convex subsets of $\mathbb{R}^n$ and V_n for n-dimensional volume on $\mathbb{R}^n$.

4.1 Preliminaries

A functional $Z : \mathcal{K}^n \to \mathbb{R}$ is called a *valuation* if

$$Z(K \cup L) + Z(K \cap L) = Z(K) + Z(L)$$

whenever $K, L, K \cup L \in \mathcal{K}^n$.

We say that Z is translation invariant if $Z(K + x) = Z(K)$ for every vector x, and Z is rotation invariant if $Z(\phi K) = Z(K)$ for every rotation $\phi : \mathbb{R}^n \to \mathbb{R}^n$. We consider continuous and upper semicontinuous valuations, where $\mathcal{K}^n$ and its subspaces are equipped with the topology induced by the Hausdorff metric. We say that a valuation $Z : \mathcal{K}^n \to \mathbb{R}$ is upper semicontinuous if for every $K_m \in \mathcal{K}^n$ converging to $K \in \mathcal{K}^n$,

$$\limsup_{m \to +\infty} Z(K_m) \leq Z(K).$$

The n-dimensional volume $V_n : \mathcal{K}^n \to [0, +\infty)$ is an example of continuous valuation, since if $K, L, K \cup L$ are convex bodies then

$$V_n(K \cup L) + V_n(K \cap L) = V_n(K) + V_n(L).$$

Other examples of valuations on $\mathcal{K}^n$ are the intrinsic volumes $V_0, \dots, V_{n-1}$ (see [46]). They are fundamental in convex geometry since they carry important geometric information. For example, V_0 is the Euler characteristic, that is, $V_0(K) = 1$ for $K \in \mathcal{K}^n$, and $2V_{n-1}(K)$ is the surface area of K. Furthermore, $V_j(K)$ is the j-dimensional volume of K if K is contained in a j-dimensional plane, and V_j is j-homogeneous, that is, $V_j(tK) = t^j V_j(K)$ for $t > 0$ and $K \in \mathcal{K}^n$.

There are many results that characterize the valuations defined on $\mathcal{K}^n$. For example, Mc Mullen showed a fundamental result in the theory of translation invariant valuations on convex bodies.

Theorem 4.1 ([42]). *If $Z : \mathcal{K}^n \to \mathbb{R}$ is continuous, translation invariant valuation, then there exist $Z_j : \mathcal{K}^n \to \mathbb{R}$ that is a continuous, translation invariant and j-homogeneous valuation, for $j = 1, \dots, n$, such that*

$$Z = Z_0 + \cdots + Z_n.$$

Hadwiger showed this result under the additional assumption that Z is simple, that is, $Z(P) = 0$ for all polytopes that are not full-dimensional. Probably the most famous result on valuations, and one of the most important results in this field is Hadwiger's characterization theorem.

Theorem 4.2 ([28], Section 6.1.10). *A functional $Z : \mathcal{K}^n \to \mathbb{R}$ is a continuous, translation and rotation invariant valuation if and only if there are constants $c_0, \dots, c_n \in \mathbb{R}$ such that*

$$Z(K) = c_0 V_0(K) + \cdots + c_n V_n(K)$$

for every $K \in \mathcal{K}^n$.

Note that the planar case of Theorem 4.2 states that every continuous and rigid motion invariant valuation $Z : \mathcal{K}^2 \to \mathbb{R}$ can be written as a linear combination of the Euler characteristic $V_0 = \chi$, the length $V_1 = L$ and the area $V_2 = A$ of the convex body, i.e., there are constants $c_0, c_1, c_2 \in \mathbb{R}$ such that

$$Z(K) = c_0 \chi(K) + c_1 L(K) + c_2 A(K) \tag{4.1}$$

for every $K \in \mathcal{K}^2$.

The next result is due to M. Ludwig [37] and characterizes the rigid motion invariant and upper semicontinuous valuations defined on $\mathcal{K}^2$. Consider the set

$$\mathcal{W} = \left\{ \zeta : [0, +\infty) \to [0, \infty) : \ \zeta \text{ is concave}, \ \lim_{t \to 0} \zeta(t) = 0, \ \text{and} \ \lim_{t \to +\infty} \zeta(t)/t = 0 \right\}. \tag{4.2}$$

Theorem 4.3 ([37]). *Let $\mu : \mathcal{K}^2 \to \mathbb{R}$ be an upper semicontinuous and rigid motion invariant valuation. Then there are constants $c_0, c_1, c_2 \in \mathbb{R}$ and a function $\zeta \in \mathcal{W}$ such that*

$$\mu(K) = c_0 \chi(K) + c_1 L(K) + c_2 A(K) + \int_{S^1} \zeta(\rho(K, u)) \, d\mathcal{H}^1(u) \tag{4.3}$$

for every $K \in \mathcal{K}^2$.

Here $\rho(K, u)$ is the curvature radius of the boundary of K at the point with normal $u \in S^1$.

Currently, the notion of valuations has been extended to families of functions due to their intimate relation to valuations on convex bodies. We denote by

$$\mathrm{Conv}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to (-\infty, +\infty] : u \text{ is l.s.c and convex, } u \not\equiv +\infty\}$$

the space of lower semicontinuous, convex, proper functions defined on $\mathbb{R}^n$, by $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ the space of finite-valued, convex functions on $\mathbb{R}^n$ and by $\mathrm{Conv}_{\mathrm{lip}}(\mathbb{R}^n;\mathbb{R})$ the space of Lipschitz functions. Note that $\mathrm{Conv}_{\mathrm{lip}}(\mathbb{R}^n;\mathbb{R}) \subset \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \subset \mathrm{Conv}(\mathbb{R}^n)$.

We define valuations on the space $\mathrm{Conv}(\mathbb{R}^n)$ and its subspaces as follows. We say that $Z : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ is a *valuation* if

$$Z(u) + Z(v) = Z(u \wedge v) + Z(u \vee v)$$

for every $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ such that also $u \wedge v, u \vee v \in \mathrm{Conv}(\mathbb{R}^n)$. Here, $u \vee v$ and $u \wedge v$ denote the pointwise maximum and minimum of $u, v \in \mathrm{Conv}(\mathbb{R}^n)$, respectively.

We say that Z is *translation invariant* if

$$Z(u \circ \tau^{-1}) = Z(u)$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and translation τ on $\mathbb{R}^n$, and it is $\mathrm{SL}_n(\mathbb{R})$ *invariant* if

$$Z(u \circ \phi^{-1}) = Z(u)$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\phi \in \mathrm{SL}_n(\mathbb{R})$. We say that $Z : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ is *epi-translation invariant* if it is invariant under translations of the epigraph of u in $\mathbb{R}^{n+1}$, that is, if

$$Z(u \circ \tau^{-1} + c) = Z(u)$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$, translation τ on $\mathbb{R}^n$ and $c \in \mathbb{R}$.

Let $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ denotes the set of super-coercive functions,

$$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) = \left\{ u : \mathbb{R} \to (-\infty, +\infty] : \lim_{|x|_2 \to +\infty} \frac{u(x)}{|x|} = +\infty \right\}.$$

Observe that the property

$$\lim_{|x|_2 \to +\infty} \frac{u(x)}{|x|_2} = +\infty$$

implies that also

$$\lim_{|x|_2 \to +\infty} |\nabla u(x)| = +\infty.$$

We say that a sequence $u_k \in \mathrm{Conv}(\mathbb{R}^n)$ *epi-converges* to $u \in \mathrm{Conv}(\mathbb{R}^n)$ if for all $x \in \mathbb{R}^n$ the following conditions hold:

1. For every sequence x_k that converges to x
$$u(x) \le \liminf_{k\to+\infty} u_k(x_k); \tag{4.4}$$

2. There exists a sequence x_k that converges to x such that
$$u(x) = \lim_{k\to+\infty} u_k(x_k). \tag{4.5}$$

For a convex function u on $\mathbb{R}^n$ the function
$$u^*(x) = \sup_{y\in\mathbb{R}^n} (\langle x, y\rangle - u(y)),\ x \in \mathbb{R}^n, \tag{4.6}$$
denotes the *Legendre transform* or *convex conjugate* of u. By standard properties of the Legendre transform, we have the following result
$$\{u^* : u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)\} = \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}). \tag{4.7}$$
This relation allows us to translate results for valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ easily to results on $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ and vice versa.

The following result is a consequence of Theorem 12.2 and Corollary 12.2.1 in [43].

Proposition 4.1. *If $u \in \mathrm{Conv}(\mathbb{R}^n)$, then $u^* \in \mathrm{Conv}(\mathbb{R}^n)$ and $(u^*)^* = u$.*

Proposition 4.2 ([44], Theorem 11.34)**.** *A sequence u_k of functions in $\mathrm{Conv}(\mathbb{R}^n)$ epi-converges to $u \in \mathrm{Conv}(\mathbb{R}^n)$ if and only if u_k^* epi-converges to u^*.*

For $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, the *subdifferential* of v at $x \in \mathbb{R}^n$ is defined by
$$\partial v(x) = \{y \in \mathbb{R}^n : v(z) \ge v(x) + \langle y, z - x\rangle \text{ for all } \ z \in \mathbb{R}^n\}.$$
Each vector of $\partial v(x)$ is said to be a *subgradient* of v at x. For a convex function $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, differentiability at a point $x \in \mathbb{R}^n$ is equivalent to both $\mathrm{epi}(v)$ having a unique supporting hyperplane at $(x, v(x))$ and the subdifferential of v at x being a singleton. Also, we can establish a relation between the subdifferential at any non-minimizer point of v with the outer normal vectors of the corresponding sublevel set at this point. Given a subset $A \subseteq \mathbb{R}^n$, the image of A through the subdifferential of v is defined as
$$\partial v(A) = \bigcup_{x\in A} \partial v(x).$$
A well-known Radon measure defined on the set of finite convex functions is the *Monge-Ampère measure*. This measure is the content of the following result, which is due to Alexandrov [2]. Let $\mathcal{B}(\Omega)$ be the class of Borel sets in $\Omega \subseteq \mathbb{R}^n$.

Lemma 4.1 ([22], Theorem 2.3)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $v : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ a convex function. If $B \in \mathcal{B}(\Omega)$, then the set $\partial v(B)$ is measurable. Moreover, $\mathrm{MA}(v; \cdot) : \mathcal{B}(\Omega) \to [0, \infty]$,* *defined by*

$$\mathrm{MA}(v;B) := V_n(\partial v(B)),$$

is a Radon measure on $\Omega \subseteq \mathbb{R}^n$.

The measure $\mathrm{MA}(v;\cdot)$ is called the *Monge-Ampère measure* of v.

The following result is very important for us because it guarantees that $\det(\mathrm{D}^2 u(\cdot))$ is measurable and it will help us to prove Theorem 4.7. Items (i) and (ii) are due to Aleksandrov [2] (or see [22, Proposition 2.6 and Theorem A.31]).

Theorem 4.4. *The following properties hold.*

(i) If $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ *and* $v \in C^2(V)$ *on an open set* $V \subset \mathbb{R}^n$*, then* $\mathrm{MA}(v;\cdot)$ *is absolutely continuous on* V *with respect to n-dimensional Lebesgue measure and*

$$d\,\mathrm{MA}(v;x) = \det(D^2 v(x))dx \tag{4.8}$$

for $x \in V$;

(ii) If v_j *is a sequence in* $\mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ *that epi-converges to* $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$*, then the sequence of measures* $\mathrm{MA}(v_j;\cdot)$ *converges weakly to* $\mathrm{MA}(v;\cdot)$.

By [20, Theorem 4.3(a)] there exists the following Steiner formula for the Monge-Ampère measure

$$\mathrm{MA}(v + rh_{B^n};\cdot) = \sum_{j=0}^{n} \binom{n}{j} r^{n-j}\,\mathrm{MA}_j(v;\cdot) \tag{4.9}$$

for $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ and $r \geq 0$. Using the relation (4.7), in [20] the authors define the conjugate Monge–Ampère measure of a function $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ by

$$\mathrm{MA}^*(u;\cdot) := \mathrm{MA}(u^*;\cdot),$$

and the Conjugate Mixed Monge–Ampère Measures is given by

$$\mathrm{MA}_j^*(u;\cdot) = \mathrm{MA}_j(u^*;\cdot)$$

for $0 \leq j \leq n$ (see [20]).

Let $C_c([0,\infty))$ denotes the set of continuous functions with compact support on $[0,\infty)$. A functional version of Hadwiger's Theorem 4.2 is the following. Continuity is understood with respect to epi-convergence.

Theorem 4.5 ([20], Theorem 1.7)**.** *A functional* $Z : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, epi-translation and rotation invariant valuation if and only if there are functions* $\alpha_0, \ldots, \alpha_n \in C_c([0,\infty))$ *such that*

$$Z(u) = \sum_{j=0}^{n} \int_{\mathbb{R}^n} \alpha_j(|y|_2)d\,\mathrm{MA}_j^*(u;y)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Similarly to Theorem 4.5, our goal here is to prove a functional version of Theorem 4.3 for $n = 1$.

4.2 Functional setting and statement of the main result

A function $u \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is called *piecewise affine* if there exist finitely many affine functions $w_1, \dots, w_m : \mathbb{R}^n \to \mathbb{R}$ such that

$$u = \bigvee_{i=1}^{m} w_i.$$

We denote by $\mathrm{Conv}_{\mathrm{p.a}}(\mathbb{R}^n;\mathbb{R})$ the set of piecewise affine functions. Furthermore, we say that $u \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is *piecewise affine outside of a set* $C \subset \mathbb{R}^n$ if there are finitely many affine functions $w_1, \dots, w_m$ on $\mathbb{R}^n$ such that $u = \bigvee_{i=1}^m w_i$ for all $x \in \mathbb{R}^n \setminus C$. Let $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n;\mathbb{R})$ be the following subset of $\mathrm{Conv}_{\mathrm{lip}}(\mathbb{R}^n;\mathbb{R})$:

$$\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n;\mathbb{R}) := \{u \in \mathrm{Conv}_{\mathrm{lip}}(\mathbb{R}^n;\mathbb{R}) : u \text{ is piecewise affine outside of a compact set}\}.$$

We will denote by $\mathrm{dom}_{\mathrm{c}}\, u$ the smallest closed convex subset of $\mathbb{R}^n$, such that u is piecewise affine outside of $\mathrm{dom}_{\mathrm{c}}\, u$.

Unlike Theorem 4.5, we do not use just epi-convergence. We equip $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n;\mathbb{R})$ with the topology induced by the following convergence. We say that $v_j \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n;\mathbb{R})$ *converges* to $v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n;\mathbb{R})$ if the following conditions hold:

1. v_j epi-converges to v;
2. There exists a compact set $C \subset \mathbb{R}^n$ such that $\mathrm{dom}_{\mathrm{c}}\, v_j, \mathrm{dom}_{\mathrm{c}}\, v \subseteq C$.

Note that the first condition implies that the convergence is locally uniform.

We say that $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ is an upper semicontinuous valuation if for every sequence u_k converging to u,

$$Z(u) \geq \limsup_{k \to +\infty} Z(u_k),$$

and it is unchanged by the addition of piecewise affine functions if $Z(u + w) = Z(u)$ for every $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n;\mathbb{R})$ and each piecewise affine function $w \in \mathrm{Conv}_{\mathrm{p.a}}(\mathbb{R}^n;\mathbb{R})$.

Our goal here is to prove the following theorem.

Theorem 4.6. *A functional $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ is an upper semicontinuous and translation invariant valuation which is unchanged by the addition of piecewise affine functions if and only if there is a constant $c_0 \in \mathbb{R}$ and a function $\zeta \in \mathcal{W}$ such that*

$$Z(u) = c_0 + \int_{\mathbb{R}} \zeta(u''(x))dx \tag{4.10}$$

for every $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$.

Recall that $\mathcal{W}$ is given in (4.2). Here we are assuming $u''(x) = 0$ whenever u is not twice differentiable at x. See, for example, [47, 48].

As a consequence of this theorem we have the following result, where continuity is considered with the convergence defined in this section.

Corollary 4.1. *Let* $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ *be a continuous and translation invariant valuation which is unchanged by the addition of piecewise affine functions. Then*

$$Z(u) = c$$

for some constant c.

The following lemma is a simple result that allows us to obtain new valuations from a given one.

Lemma 4.2. *Let* $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ *be a valuation. If* $w \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ *and*

$$Z_w(u) := Z(u + w),$$

for $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$, *then* Z_w *is a valuation on* $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$.

Proof. Let w, u, v be convex functions in $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ such that $u \wedge v, u \vee v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ as well. It is easy to see that

$$\begin{aligned} u \wedge v + w &= (u + w) \wedge (v + w) \\ u \vee v + w &= (u + w) \vee (v + w). \end{aligned}$$

Applying Z in the equations above and using that Z is a valuation, we get

$$\begin{aligned} Z_w(u \wedge v) + Z_w(u \vee v) &= Z(u \wedge v + w) + Z(u \vee v + w) \\ &= Z((u + w) \wedge (v + w)) + Z((u + w) \vee (v + w)) \\ &= Z(u + w) + Z(v + w) \\ &= Z_w(u) + Z_w(v). \end{aligned}$$

Therefore Z_w is a valuation. □

The rest of the chapter is devoted to the proof of our main theorem. This proof is organized as follows. First we prove that if Z is given by (4.10), then Z is an upper semicontinuous and translation invariant valuation which is unchanged by the addition of piecewise affine functions. In order to prove that the integral in (4.10) is upper semicontinous we will introduce the Monge-Ampère measure. Later, we will show that for each functional Z which is unchanged by the addition of piecewise affine functions and is an upper semicontinuous and translation invariant valuation, there is a constant c_0 and $\zeta \in \mathcal{W}$ such that Z can be written as a combination as in (4.10).

4.3 Proof of Theorem 4.6

4.3.1 Sufficiency part

In this subsection we want to show that the functional defined in (4.10) is an upper semicontinuous and translation invariant valuation and it is unchanged by the addition of piecewise affine functions.

Lemma 4.3. *For $\zeta \in \mathcal{W}$, the function $\tilde{Z} : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R}) \to [0,\infty)$ defined by*

$$\tilde{Z}(u) = \int_{\mathbb{R}} \zeta(u''(x))dx,$$

is translation invariant valuation and unchanged by the addition of piecewise affine functions.

Proof. We have

$$\begin{aligned}
\tilde{Z}(u) + \tilde{Z}(v) &= \int_{\mathbb{R}} \zeta(u''(x))dx + \int_{\mathbb{R}} \zeta(v''(x))dx \\
&= \int_{\{u\geq v\}} \zeta(u''(x))dx + \int_{\{u<v\}} \zeta(u''(x))dx + \int_{\{v>u\}} \zeta(v''(x))dx + \int_{\{v\leq u\}} \zeta(v''(x))dx \\
&= \int_{\{u\geq v\}} \zeta((u\vee v)''(x))dx + \int_{\{v>u\}} \zeta((u\vee v)''(x))dx \\
&\quad + \int_{\{u<v\}} \zeta((u\wedge v)''(x))dx + \int_{\{v\leq u\}} \zeta((u\wedge v)''(x))dx \\
&= \int_{\mathbb{R}} \zeta((u\vee v)''(x))dx + \int_{\mathbb{R}} \zeta((u\wedge v)''(x))dx \\
&= \tilde{Z}(u\wedge v) + \tilde{Z}(u\vee v),
\end{aligned}$$

whenever $u, v, u\wedge v, u\vee v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$. Thus, $\tilde{Z}$ is a valuation.

Now let $\tau_y(x) = x + y$, where $y \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} \zeta((u\circ\tau_y^{-1})''(x))dx = \int_{\mathbb{R}} \zeta(u''(x-y))dx = \int_{\mathbb{R}} \zeta(u''(x))dx,$$

and for a piecewise affine function w, we have

$$\int_{\mathbb{R}} \zeta((u+w)''(x))dx = \int_{\mathbb{R}} \zeta(u''(x))dx,$$

as we wanted to prove. □

It remains to show that the valuation

$$\tilde{Z}(u) = \int_{\mathbb{R}} \zeta(u''(x))dx,$$

depends upper semicontinuously on u. That is the content of the following theorem.

Theorem 4.7. *Let $\zeta \in \mathcal{W}$. Then*

$$Z(u) = \int_{\mathbb{R}} \zeta(u''(x))dx \tag{4.11}$$

is finite for every $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ and depends upper semicontinuously on u.

The proof of this result is similar to the proof of the following theorem due to M. Ludwig [38].

Theorem 4.8 ([38], Theorem 2)**.** *Let $f \in \mathcal{W}$. Then for $j = 1, \dots, n-1$*

$$\mu(K) = \int_{S^{n-1}} f(P_j(K,y))d\mathcal{H}^{n-1}(y)$$

is finite for every $K \in \mathcal{K}^n$ and depends upper semicontinuously on f.

Here $P_j(K,y)$ is the j-th elementary symmetric function of the principal radii of curvature at $y \in S^{n-1}$. The strategy of the proof of Theorem 4.8 is to decompose the j-th area measure of the convex body K, namely $S_j(K,\cdot)$, into measures absolutely continuous and singular with respect to the $(n-1)$-dimensional Hausdorff measure on the sphere and use that the absolutely continuous part is given by

$$S_j^a(K,w) = \int_w P_j(K,y)d\mathcal{H}^{n-1}(y).$$

In our approach, we will use the Monge-Ampère measure and decompose it into measures absolutely continuous and singular with respect to the Lebesgue measure on $\mathbb{R}^n$. Its absolutely continuous part is given by

$$\mathrm{MA}^a(v;U) = \int_U \det(\mathrm{D}^2\, v(x))dx,$$

where $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

By Theorem 4.4 we have that $\mathrm{MA}(v_j;\cdot)$ converges weakly to $\mathrm{MA}(v;\cdot)$ whenever $v_j \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n;\mathbb{R})$ converges to $v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n;\mathbb{R})$ and the absolute part of the Monge-Ampère measure vanishes on piecewise affine function. Using this and Lemma 4.1, we have the following result.

Theorem 4.9. *Let $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n;\mathbb{R})$. Then $\mathrm{MA}(u;\mathbb{R}^n) < +\infty$.*

Let $v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$. By Theorem 1.13 the measure $\mathrm{MA}(v;\cdot)$ can be decomposed into measures absolutely continuous and singular with respect to the Lebesgue measure , say,

$$\mathrm{MA}(v;\cdot) = \mathrm{MA}^a(v;\cdot) + \mathrm{MA}^s(v;\cdot),$$

respectively. For the absolutely continuous part, we have

$$\mathrm{MA}^a(v;\beta) = \int_\beta v''(x)dx, \tag{4.12}$$

while for the singular part we only need to remember that it is concentrated on a null set, i.e., there is a set $\beta_0 \subset \mathbb{R}$ such that $V_1(\beta_0) = 0$ and

$$\mathrm{MA}^s(v; \beta \setminus \beta_0) = 0 \tag{4.13}$$

for every Borel set $\beta \subset \mathbb{R}$.

Let $u_k \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}; \mathbb{R})$ be a sequence of functions that converges to $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}; \mathbb{R})$ and $C \subset \mathbb{R}$ be a compact set.

By Theorem 4.4 and Theorem 1.9 we have that

$$\limsup_{k \to +\infty} \mathrm{MA}(u_k; \beta) \leq \mathrm{MA}(u; \beta), \tag{4.14}$$

for every closed set $\beta \subseteq C$.

By A. D. Alexandrov [2] if $v \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}; \mathbb{R})$ then $v''(x)$ exist almost everywhere (a.e) on $\mathbb{R}$ and by (4.12) the function v'' is Lebesgue-measurable.

Now we can start the proof of Theorem 4.7. Recall that we will use the same arguments utilized by M. Ludwig in [38].

4.3.2 Proof of Theorem 4.7

First since $\zeta \in \mathcal{W}$ then ζ is concave and $\lim_{t \to 0} \zeta(t) = 0$. This implies that ζ is a continuous function and $\zeta(0) = 0$. Moreover, since ζ is concave and non-negative on $[0, +\infty)$, ζ is non-decreasing. Using that we get for every $t > 0$ and $0 < \lambda < 1$

$$\zeta(\lambda t + (1 - \lambda)0) \geq \lambda \zeta(t) + (1 - \lambda)\zeta(0).$$

In particular, if we take $s = \lambda t < t$ we obtain

$$\frac{\zeta(s)}{s} \geq \frac{\zeta(t)}{t},$$

and this means that $\frac{\zeta(t)}{t}$ is non-increasing.

Let $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}; \mathbb{R})$ be such that $V_1(\mathrm{dom_c}\, u) > 0$. Since -$\zeta$ is convex, by the Jensen inequality, Theorem 1.12, we obtain

$$\frac{1}{V_1(\mathrm{dom_c}\, u)} \int_{\mathrm{dom_c}\, u} \zeta(u''(x))dx \leq \zeta\left(\frac{1}{V_1(\mathrm{dom_c}\, u)} \int_{\mathrm{dom_c}\, u} u''(x)dx\right).$$

Using that ζ is non-decreasing, $\zeta(0) = 0, \mathrm{MA}(u; \mathrm{dom_c}\, u) < +\infty$ for Lipschitz functions and

(4.12), we arrive at

$$\begin{aligned}
Z(u) &= \int_{\mathrm{dom_c}\, u} \zeta(u''(x))dx \\
&\leq V_1(\mathrm{dom_c}\, u)\zeta\left(\frac{1}{V_1(\mathrm{dom_c}\, u)}\int_{\mathrm{dom_c}\, u} u''(x)dx\right) \\
&\leq V_1(\mathrm{dom_c}\, u)\zeta\left(\frac{1}{V_1(\mathrm{dom_c}\, u)}\,\mathrm{MA}(u;\mathrm{dom_c}\, u)\right).
\end{aligned}$$

We conclude that the valuation (4.11) is finite for every $u \in \mathrm{Conv_{pac}}(\mathbb{R};\mathbb{R})$. The next step is to show that Z depends upper semicontinuously on u.

Let $\varepsilon > 0$ be chosen, $C \subset \mathbb{R}$ be a compact set and $u \in \mathrm{Conv_{pac}}(\mathbb{R};\mathbb{R})$. Let β_0 be the set where the singular part of $\mathrm{MA}(u;\cdot)$ is concentrated. Since $V_1(\beta_0) = 0$ and by (4.12) the function u'' is measurable a.e on C, we can choose by Lusin's Theorem 1.14 a closed set $\beta \subset C$ where u'' is continuous as a function restricted to β, such that

$$\beta \cap \beta_0 = \emptyset \tag{4.15}$$

and

$$V_1(C \setminus \beta) \leq \varepsilon. \tag{4.16}$$

Let u_k be a sequence in $\mathrm{Conv_{pac}}(\mathbb{R};\mathbb{R})$ converging to $u \in \mathrm{Conv_{pac}}(\mathbb{R};\mathbb{R})$. First, we show that

$$\limsup_{k\to+\infty} \int_\beta \zeta(u_k''(x))dx \leq \int_\beta \zeta(u''(x))dx \tag{4.17}$$

holds for $\beta \subseteq C$, where β is the set satisfying (4.15) and (4.16).

Set

$$\begin{aligned}
a &= \inf\{\zeta(u''(x)) : x \in \beta\} \\
b &= \sup\{\zeta(u''(x)) : x \in \beta\}.
\end{aligned}$$

Note that $b < +\infty$ because u'' is continuous on β and β is compact. Therefore, ζ is uniformly continuous on $[a, b]$. Let $\eta > 0$ be arbitrarily given. Then, there exists a number $\delta > 0$ such that

$$|\zeta(s) - \zeta(t)| \leq \eta \tag{4.18}$$

whenever $s, t \in [a, b]$ are such that $|s - t| \leq \delta$.

Consider a subdivision $a = t_1 \leq t_2 \leq \cdots \leq t_{m+1} = b$ of $[a, b]$, such that

$$\max_{i=1,\ldots,m} \{t_{i+1} - t_i\} \leq \delta \tag{4.19}$$

and such that

$$V_1(\{x \in \beta : u''(x) = t_i\}) = 0$$

for $i = 2, \ldots, m$. This last condition is possible since $V_1(\{x \in \beta : u''(x) = t_i\}) > 0$ holds only for countably many t.

Now consider the subsets of β given by

$$\beta_i = \{x \in \beta : t_i \leq u''(x) \leq t_{i+1}\},$$

for $i = 1, \ldots, m$.

By linearity of the integral and since ζ is non-decreasing, we have that

$$\begin{aligned} \int_\beta \zeta(u''(x))dx &= \sum_{i=1}^m \int_{\beta_i} \zeta(u''(x))dx \\ &\geq \sum_{i=1}^m \zeta(t_i) V_1(\beta_i), \end{aligned} \tag{4.20}$$

and by (4.12)

$$\int_{\beta_i} u_k''(x)dx \leq \mathrm{MA}(u_k; \beta_i).$$

Consider $J \subseteq \{1, \ldots, m\}$ such that $V_1(\beta_j) = 0$ whenever $j \notin J$. By the Jensen inequality, Theorem 1.12, the following inequality

$$\frac{1}{V_1(\beta_i)} \int_{\beta_i} \zeta(u_k''(x))dx \leq \zeta\left(\frac{1}{V_1(\beta_i)} \int_{\beta_i} u_k''(x)dx\right)$$

holds for each $i \in J$.

Using these inequalities and the monotonicity of ζ, we obtain

$$\begin{aligned} \int_\beta \zeta(u_k''(x))dx &\leq \sum_{i=1}^m \int_{\beta_i} \zeta(u_k''(x))dx \\ &= \sum_{i \in J} \int_{\beta_i} \zeta(u_k''(x))dx \\ &\leq \sum_{i \in J} \zeta\left(\frac{1}{V_1(\beta_i)} \mathrm{MA}(u_k; \beta_i)\right) V_1(\beta_i). \end{aligned}$$

Since u'' is continuous on β and β is closed, the sets β_i are closed for $i = 1, \ldots, m$. This implies by (4.14) that

$$\limsup_{k \to +\infty} \mathrm{MA}(u_k; \beta_i) \leq \mathrm{MA}(u; \beta_i).$$

By continuity and monotonicity of ζ

$$\limsup_{k \to +\infty} \int_\beta \zeta(u_k''(x))dx \leq \sum_{i \in J} \zeta\left(\frac{1}{V_1(\beta_i)} \mathrm{MA}(u; \beta_i)\right) V_1(\beta_i). \tag{4.21}$$

By (4.15) and (4.13),

$$\mathrm{MA}(u; \beta_i) = \mathrm{MA}^a(u; \beta_i)$$

and by (4.12) and the definition of β_i,

$$\mathrm{MA}^a(u;\beta_i) \le t_{i+1} V_1(\beta_i).$$

Hence,

$$\begin{aligned}\limsup_{k\to+\infty} \int_\beta \zeta(u_k''(x))dx &\le \sum_{i\in J} \zeta\left(\frac{1}{V_1(\beta_i)}\,\mathrm{MA}(u;\beta_i)\right) V_1(\beta_i) \\ &\le \sum_{i=1}^m \zeta(t_{i+1}) V_1(\beta_i).\end{aligned}$$

Now using (4.20), (4.19) and (4.18), we conclude that

$$\limsup_{k\to+\infty} \int_\beta \zeta(u_k''(x))dx \le \int_\beta \zeta(u''(x))dx + \eta V_1(\beta).$$

Since $\eta > 0$ is arbitrary, this proves (4.17).

The second step is to show that

$$\limsup_{k\to+\infty} \int_C \zeta(u_k''(x))dx \le \int_C \zeta(u''(x))dx. \tag{4.22}$$

Since ζ is non-decreasing and $\zeta(t)/t$ is non-increasing, using (4.12) we see that for every $t > 0$,

$$\begin{aligned}\int_{C\setminus\beta} \zeta(u_k''(x))dx &= \int_{\{x\in C\setminus\beta:\ u_k''(x)\le t\}} \zeta(u_k''(x))dx + \int_{\{x\in C\setminus\beta:\ u_k''(x)>t\}} \zeta(u_k''(x))dx \\ &\le \zeta(t) V_1(C\setminus\beta) + \frac{\zeta(t)}{t}\,\mathrm{MA}(u_k;C).\end{aligned}$$

This implies, combined with (4.17), (4.16) and (4.14) that

$$\begin{aligned}\limsup_{k\to+\infty} \int_C \zeta(u_k''(x))dx &= \limsup_{k\to+\infty} \int_\beta \zeta(u_k''(x))dx + \limsup_{k\to+\infty} \int_{C\setminus\beta} \zeta(u_k''(x))dx \\ &\le \int_C \zeta(u''(x))dx + \zeta(t)\varepsilon + \frac{\zeta(t)}{t}\,\mathrm{MA}(u;C),\end{aligned}$$

for every $t > 0$. Since $\varepsilon > 0$ is arbitrary and since t does not depend on ε, it follows that for every $t > 0$

$$\limsup_{k\to+\infty} \int_C \zeta(u_k''(x))dx \le \int_C \zeta(u''(x))dx + \frac{\zeta(t)}{t}\,\mathrm{MA}(u;C). \tag{4.23}$$

Using the fact that $\zeta(t)/t$ is continuous and that $\lim_{t\to+\infty} \zeta(t)/t = 0$, we now can make $\zeta(t)/t$ arbitrarily small by choosing t suitably large. Therefore (4.23) proves (4.22).

To finish, recall that $\zeta(0) = 0$ and since u_k converges to u as $k \to +\infty$, then there is a compact set $K \subset \mathbb{R}$ such that $\mathrm{dom_c}\, u_k, \mathrm{dom_c}\, u \subseteq K$. Hence

$$\int_{\mathbb{R}} \zeta(u_k''(x))dx = \int_K \zeta(u_k''(x))dx$$

and

$$\int_{\mathbb{R}} \zeta(u''(x))dx = \int_{K} \zeta(u''(x))dx,$$

and this concludes the proof.

4.3.3 Necessity part

In this subsection we want to show that if $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ is an upper semicontinuous and translation invariant valuation which is unchanged by the addition of piecewise affine functions, then there is a constant $c_0 \in \mathbb{R}$ and $\zeta \in \mathcal{W}$ such that

$$Z(u) = c_0 + \int_{\mathbb{R}} \zeta(u''(x))dx$$

for every $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$. We will first prove the following particular case where the valuation is *simple*, i.e., when $\tilde{Z}(w) = 0$ whenever $V_1(\mathrm{dom_c}\, w) = 0$.

Proposition 4.3. *Let* $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ *be a simple, upper semicontinuous and translation invariant valuation which is unchanged by the addition of piecewise affine functions. Then there is a function* $\zeta \in \mathcal{W}$ *such that*

$$Z(u) = \int_{\mathbb{R}} \zeta(u''(x))dx \tag{4.24}$$

for every $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$.

Throughout this section $m > 0$ is fixed and $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ has the same properties as in Proposition 4.3. Since every function in $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ can be approximate by piecewise affine functions and Z is upper semicontinuous, we have

$$Z(u) \geq 0 \tag{4.25}$$

for every $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$.

Let $f \in \mathrm{Conv}(\mathbb{R};\mathbb{R})$ and consider the closed interval $J = [a,b] \subset \mathbb{R}$. Define $f \dotplus L_J$ by

$$(f \dotplus L_J)(x) = \begin{cases} f'_+(a)x + (f(a) - f'_+(a)a), & \text{if} \quad x < a \\ f(x), & \text{if} \quad x \in J\ , \\ f'_-(b)x + (f(b) - f'_-(b)b), & \text{if} \quad x > b \end{cases} \tag{4.26}$$

where $f'_+(x)$ and $f'_-(x)$ denote the one-sided derivatives of f in x. Note that $f \dotplus L_J \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ whenever $f \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$.

Given $a > 0$ define the function $\zeta : [0,+\infty) \to \mathbb{R}$ by

$$\zeta(a) = \frac{1}{2m} Z(f \dotplus L_{[-m,m]}), \tag{4.27}$$

where $f(x) = \frac{a}{2}x^2$. Also define the function $g_a : [0, +\infty) \to \mathbb{R}$ by

$$Z(f \dotplus L_J) = g_a(V_1(J)), \tag{4.28}$$

where $J \subset \mathbb{R}$ is a closed interval. Note that in this case the function g_a depends on a and it is well-defined, since Z is simple and unchanged by the addition of piecewise affine functions. By Lemma 4.2 and once that Z is simple, we have that

$$g_a(s_1 + s_2) = g_a(s_1) + g_a(s_2),$$

whenever $s_1, s_2 \geq 0$. Since Z is upper semicontinuous, so is g_a. Therefore g_a is a solution of Cauchy's functional equation and there is a constant $p = p(a)$ such that

$$g_a(s) = ps.$$

For $J = [-m, m]$, we have

$$Z(f \dotplus L_J) = pV_1(J)$$

which shows that

$$Z(f \dotplus L_{[x_1,x_2]}) = g_a(x_2 - x_1) = p(x_2 - x_1) = \frac{Z(f \dotplus L_{[-m,m]})}{2m}(x_2 - x_1) \tag{4.29}$$

$$= \zeta(a)(x_2 - x_1). \tag{4.30}$$

Lemma 4.4. *ζ is a non-negative function.*

Proof. This is a consequence of (4.25). □

Proposition 4.3 is a consequence of the two next results.

Lemma 4.5. $\zeta \in \mathcal{W}$.

Proof. Consider the following sequence of functions

$$u_k(x) = \frac{1}{k}x^2.$$

Note that $u_k \dotplus \mathrm{L}_{[-m,m]} \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ converges to the function $l \equiv 0$ as $k \to +\infty$. Since by Lemma 4.4 ζ is non-negative, then

$$\limsup_{a \to 0^+} \zeta(a) = \limsup_{k \to +\infty} \frac{Z(u_k \dotplus \mathrm{L}_{[-m,m]})}{2m} = Z(l) = 0.$$

To prove that ζ is concave we will use a geometric construction. Let $0 \leq r < a < s$ and consider the points

$$p_i = -m + \left(\frac{2m}{n}\right) i,$$

where $i = 0, \ldots, n$. Note that $p_0 = -m, p_n = m$ and $p_i \in (-m, m)$ for every $i = 1, \ldots, n-1$.

Consider the functions

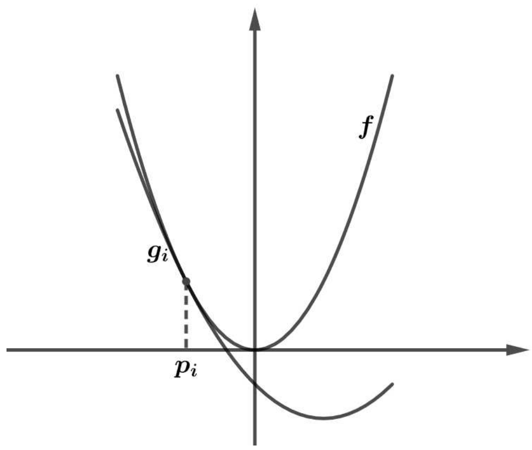
$$\begin{aligned} g_i(x) &= rx^2 + (2a-2r)\left(-m + \left(\frac{2m}{n}\right)i\right)x - (a-r)\left(-m + \left(\frac{2m}{n}\right)i\right)^2 \\ &= rx^2 + (2a-2r)p_i x - (a-r)p_i^2, \end{aligned}$$

for $i = 0, \ldots, n-1$. We have that

$$g_i(p_i) = f(p_i), \quad g_i'(p_i) = f'(p_i) \quad \text{and} \quad \mathrm{epi}(f \dotplus \mathrm{L}_{[-m,m]}) \subset \mathrm{epi}(g_i \dotplus \mathrm{L}_{[-m,m]}),$$

for every $i = 0, \ldots, n-1$. Recall that m is fixed.

Figure 4.1: $f \dotplus \mathrm{L}_{[-m,m]}$ and g_i.



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Now we want to find for every $i = 1, \ldots, n$, the function

$$h_i(x) = sx^2 + b_i x + c_i$$

such that

$$\begin{cases} h_i(x_i) &= g_{i-1}(x_i) \\ h_i'(x_i) &= g_{i-1}'(x_i) \end{cases} \tag{4.31}$$

for some $x_i \in (p_{i-1}, p_i)$ and epi $(h_i \dotplus \mathrm{L}_{[-m,m]}) \subset \mathrm{epi}(g_{i-1} \dotplus \mathrm{L}_{[-m,m]})$, and

$$\begin{cases} h_i(y_i) &= g_i(y_i) \\ h_i'(y_i) &= g_i'(y_i) \end{cases} \tag{4.32}$$

for some $y_i \in (x_i, p_i)$ and epi $(h_i \dotplus \mathrm{L}_{[-m,m]}) \subset \mathrm{epi}(g_i \dotplus \mathrm{L}_{[-m,m]})$.

By second equations in (4.31) and (4.32) we have, respectively,

$$x_i = \left(\frac{a-r}{s-r}\right) p_{i-1} - \frac{b_i}{2s-2r} \tag{4.33}$$

$$y_i = \left(\frac{a-r}{s-r}\right) p_i - \frac{b_i}{2s-2r}. \tag{4.34}$$

Thus

$$y_i - x_i = \left(\frac{a-r}{s-r}\right) \frac{2m}{n}, \tag{4.35}$$

for every $i = 1, \ldots, n$, that is, $(y_i - x_i)$ is a constant that does not depend on i. Moreover

$$x_{i+1} - y_i = \frac{b_i}{2s-2r} - \frac{b_{i+1}}{2s-2r}.$$

By first equation in (4.31) and (4.33), we must have

$$(b_i - (2a-2r)p_{i-1})^2 - 4(s-r)\left(c_i + (a-r)p_{i-1}^2\right) = 0 \tag{4.36}$$

and by first equation in (4.32) and (4.34), we get

$$(b_i - (2a-2r)p_i)^2 - 4(s-r)\left(c_i + (a-r)p_i^2\right) = 0. \tag{4.37}$$

Using (4.36) and (4.37) we obtain that

$$b_i - b_{i+1} = \frac{4m}{n}((s-r) - (a-r))$$

and therefore

$$\begin{aligned} x_{i+1} - y_i &= \frac{1}{2s-2r}\left[\frac{4m}{n}((s-r)-(a-r))\right] \\ &= \frac{2m}{n}\left(1 - \left(\frac{a-r}{s-r}\right)\right), \end{aligned} \tag{4.38}$$

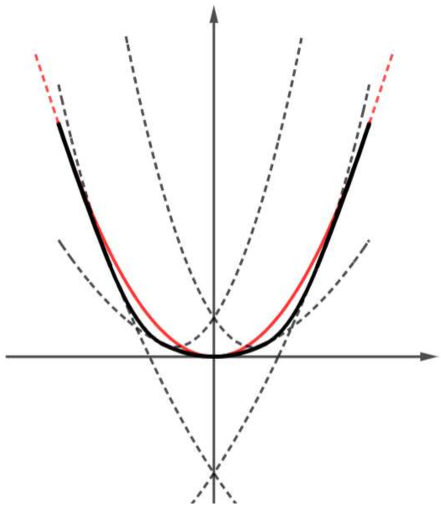
for every $i = 1, \ldots, n-1$. This means that $(x_{i+1} - y_i)$ also does not depend on i.

Note that if $\varphi(n) = \frac{2m}{n}$, then $(y_i - x_i) + (x_{i+1} - y_i) = \varphi(n)$.

We approximate the function $f \dotplus \mathrm{L}_{[-m,m]}$ by a convex function $u_n \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ constructed in the following way

$$u_n = (g_0 \dotplus \mathrm{L}_{[-m,x_1]}) \vee (h_1 \dotplus \mathrm{L}_{[x_1,y_1]}) \vee (g_1 \dotplus \mathrm{L}_{[y_1,x_2]}) \vee (h_2 \dotplus \mathrm{L}_{[x_2,y_2]}) \vee \cdots \vee (g_n \dotplus \mathrm{L}_{[y_n,m]}). \tag{4.39}$$

Figure 4.2: Functions f (red) and u_n (black).



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Since $(y_i - x_i)$ and $(x_{i+1} - y_i)$ do not depend on i and since Z is a simple valuation and unchanged by the addition of piecewise affine functions, we have

$$\begin{aligned}Z(u_n) &= Z((g_0 \dotplus \mathrm{L}_{[-m,x_1]}) \vee (h_1 \dotplus \mathrm{L}_{[x_1,y_1]}) \vee (g_1 \dotplus \mathrm{L}_{[y_1,x_2]}) \vee (h_2 \dotplus \mathrm{L}_{[x_2,y_2]}) \vee \cdots \vee (g_n \dotplus \mathrm{L}_{[y_n,m]}))\\ &= Z(g_0 \dotplus \mathrm{L}_{[-m,x_1]}) + Z(g_n \dotplus \mathrm{L}_{[y_n,m]}) + \sum_{i=1}^{n-1} Z(g_i \dotplus \mathrm{L}_{[y_i,x_{i+1}]}) + \sum_{i=1}^{n} Z(h_i \dotplus \mathrm{L}_{[x_i,y_i]})\\ &= nZ(h_1 \dotplus \mathrm{L}_{[x_1,y_1]}) + nZ(g_1 \dotplus \mathrm{L}_{[y_1,x_2]}),\end{aligned}$$

and by (4.29), (4.35) and (4.38), we get

$$\begin{aligned}Z(u_n) &= n(y_1 - x_1)\frac{Z(h_1 \dotplus \mathrm{L}_{[-m,m]})}{2m} + n(x_2 - y_1)\frac{Z(g_1 \dotplus \mathrm{L}_{[-m,m]})}{2m}\\ &= \frac{n}{2m}\left(\frac{a-r}{s-r}\right)\frac{2m}{n}Z(h_1 \dotplus \mathrm{L}_{[-m,m]}) + \frac{n}{2m}\left(1 - \left(\frac{a-r}{s-r}\right)\right)\frac{2m}{n}Z(g_1 \dotplus \mathrm{L}_{[-m,m]})\\ &= \left(\frac{a-r}{s-r}\right)Z(h_1 \dotplus \mathrm{L}_{[-m,m]}) + \left(1 - \left(\frac{a-r}{s-r}\right)\right)Z(g_1 \dotplus \mathrm{L}_{[-m,m]}).\end{aligned} \tag{4.40}$$

Note that $Z(u_n)$ does not depend on n. To finish the proof that ζ is concave we use that u_n converges to $f \dotplus \mathrm{L}_{[-m,m]}$ as $n \to +\infty$, that Z is upper semicontinuous and use equation (4.40) such that

$$\begin{aligned}2m\zeta(2a) &= Z(f \dotplus \mathrm{L}_{[-m,m]})\\ &\geq \limsup_{n\to+\infty} Z(u_n)\\ &= \limsup_{n\to+\infty}\left(\frac{a-r}{s-r}\right)Z(h_1 \dotplus \mathrm{L}_{[-m,m]}) + \left(1 - \frac{a-r}{s-r}\right)Z(g_1 \dotplus \mathrm{L}_{[-m,m]})\\ &= 2m\left(\left(\frac{a-r}{s-r}\right)\zeta(2s) + \left(1 - \frac{a-r}{s-r}\right)\zeta(2r)\right).\end{aligned}$$

Setting $\lambda = \frac{a-r}{s-r}$, since $0 \leq r < a < s$, we have that $0 < \lambda < 1$ and

$$\zeta(\lambda 2s + (1-\lambda)2r) \geq \lambda\zeta(2s) + (1-\lambda)\zeta(2r).$$

Once that r and s were arbitrarily taken, we arrive at

$$\zeta(\lambda s + (1-\lambda)r) \geq \lambda\zeta(s) + (1-\lambda)\zeta(r)$$

for every $\lambda \in (0,1)$.

The last property needed to conclude that $\zeta \in \mathcal{W}$ is

$$\lim_{a\to+\infty} \frac{\zeta(a)}{a} = 0.$$

Consider the family of functions $f_k(x) = kx^2$, $k \in \mathbb{N}$. We have

$$\left(f_k \dotplus \mathrm{L}_{[-\frac{1}{k},\frac{1}{k}]}\right)(x) = \begin{cases} -2x - \frac{1}{k}, & \text{if} \quad x < -\frac{1}{k} \\ kx^2, & \text{if} \quad x \in \left[-\frac{1}{k}, \frac{1}{k}\right] \\ 2x - \frac{1}{k}, & \text{if} \quad x > \frac{1}{k} \end{cases}.$$

Note that $f_k \dotplus \mathrm{L}_{[-\frac{1}{k},\frac{1}{k}]}$ converges to $l = |2x|$. Since Z is upper semicontinuous and simple, we obtain

$$\limsup_{k\to+\infty} Z(f_k \dotplus \mathrm{L}_{[-\frac{1}{k},\frac{1}{k}]}) = 0. \tag{4.41}$$

By (4.30), we have that

$$\lim_{a\to+\infty} \frac{\zeta(a)}{a} = 0.$$

□

The second result necessary to prove Proposition 4.3 is the following.

Proposition 4.4. *For a given $\zeta \in \mathcal{W}$, there is a unique $Z : \mathrm{Conv_{pac}}(\mathbb{R};\mathbb{R}) \to \mathbb{R}$ with the following properties:*

(i) Z is upper semicontinuous;

(ii) Z is a simple and translation invariant valuation, unchanged by the addition of piecewise affine functions;

(iii) $Z(f \dotplus \mathrm{L}_{[-m,m]}) = 2m\zeta(a)$, where $f(x) = \frac{a}{2}x^2$.

A function $v \in \mathrm{Conv_{pac}}(\mathbb{R};\mathbb{R})$ is called *piecewise linear-quadratic* if $\mathrm{dom_c}\, v$ can be expressed as the union of finitely many intervals $J_i, i = 1,\dots,l$, such that the restriction of v to J_i is a quadratic or affine function. The set of piecewise linear-quadratic functions will be denoted by $P_{\,\mathrm{l.q}}(\mathbb{R})$. Note that piecewise linear functions belong to $P_{\,\mathrm{l.q}}(\mathbb{R})$ and since $P_{\,\mathrm{l.q}}(\mathbb{R})$ is dense in $\mathrm{Conv_{pac}}(\mathbb{R};\mathbb{R})$, one can approximate every $u \in \mathrm{Conv_{pac}}(\mathbb{R};\mathbb{R})$ by elements of $P_{\,\mathrm{l.q}}(\mathbb{R})$. The upper semicontinuity of Z implies that for every sequence $u_k \in P_{\,1.\mathrm{q}}(\mathbb{R})$ such that $u_k \to u$

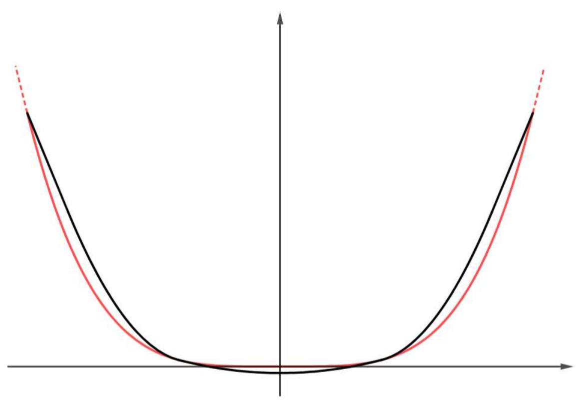
$$Z(u) \geq \limsup_{k\to+\infty} Z(u_k). \tag{4.42}$$

We will prove that for every $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ there is a sequence $u_k \in P_{\,1.\mathrm{q}}(\mathbb{R})$ such that equality holds in (4.42), that is,

$$Z(u) = \sup\left\{\limsup_{k\to+\infty} Z(u_k) : u_k \in P_{\,1.\mathrm{q}}(\mathbb{R}), u_k \to u\right\}. \tag{4.43}$$

Proving this result we have that Z is uniquely determined by ζ and therefore proves Proposition 4.4.

Figure 4.3: Function u (red) and a piecewise linear-quadratic function u_k (black).



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We call a closed triangle $T = T(x, y)$ a *support triangle* of a convex function u with endpoints $(x, u(x))$ and $(y, u(y))$ if $x, y \in \operatorname{dom} u$ and T is bounded by support lines (that is, 1-dimensional support hyperplanes) to u at x and y and the chord connecting $(x, u(x))$ and $(y, u(y))$. Using suitable support triangles of $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ we will construct an $v \in P_{\,1.\mathrm{q}}(\mathbb{R})$ such that

$$Z(u) \leq Z(v) + \rho V_1(\operatorname{dom_c} u), \tag{4.44}$$

where $\rho > 0$ is given.

One of the most useful tools of geometric measure theory is the Vitali covering theorem. Given a 'sufficiently large' collection of sets that cover some set S, the Vitali theorem selects a disjoint subcollection that covers almost all of S. A collection of sets $\mathcal{C}$ is called a Vitali class for S if for each $x \in S$ and $\delta > 0$ there exists $U \in \mathcal{C}$ with $x \in U$ and $0 < V_n(U) \leq \delta$.

Theorem 4.10 ([21], Theorem 1.10)**.** *(Vitali covering theorem)*

1. *Let S be an $\mathcal{H}^n$-measurable subset of $\mathbb{R}^n$ and let $\mathcal{C}$ be a Vitali class of closed sets for S. Then we may select a (finite or countable) disjoint sequence $\{U_i\}_i$ of $\mathcal{C}$ such that either $\sum_i V_n(U) = +\infty$ or $\mathcal{H}^n(S \setminus \cup_i U_i) = 0$;*

2. If $\mathcal{H}^n(S) < +\infty$, then given $\varepsilon > 0$, we may also require that

$$\mathcal{H}^n(S) \leq \sum_i V_n(U) + \varepsilon. \tag{4.45}$$

What we will do is to show that for the set

$$S = \{x \in \mathrm{dom_c}\, u;\ u''(x) \text{ is well-defined}\}$$

there is a suitable Vitali class defined with the help of support triangles of u. Since Z is a translation invariant valuation, we can consider without loss of generality that $u \in \mathrm{Conv_{pac}}(\mathbb{R};\mathbb{R})$ is such that $\mathrm{dom_c}\, u = [-m, m]$.

Lemma 4.6. *Let $x_0 \in S$ such that in a neighborhood of x_0 the function u is not linear. For every $\rho, \tau > 0$ there is a support triangle $T = T(x, y)$ of u, a convex body $A_T \subset \mathbb{R}^2$ and a function $v_T \in P_{\,\mathrm{l.q}}(\mathbb{R})$ such that:*

(i) $(x_0, u(x_0)) \in T$ and $0 < y - x < \tau$;

(ii) $A_T \subset T$ and A_T is a support triangle of v_T;

(iii) $Z(u \dotplus \mathrm{L}_{[x,y]}) \leq Z(v_T) + \frac{\rho}{2}(y - x)$.

Proof. First since u is a convex function in particular u is twice differentiable almost everywhere. Then, by Taylor expansion 1.18, u can be represented locally around x_0 by

$$u(x) = u(x_0) + u'(x_0)(x - x_0) + \frac{1}{2}u''(x_0)(x - x_0)^2 + o((x - x_0)^2). \tag{4.46}$$

To prove this lemma we will consider two cases. The first case is when $u''(x_0) > 0$. Take $P_\varepsilon = (x_0 - \varepsilon, u(x_0 - \varepsilon))$ and $Q_\varepsilon = (x_0 + \varepsilon, u(x_0 + \varepsilon))$ points on the graph of u and let $T_\varepsilon = T_\varepsilon(x_0 - \varepsilon, x_0 + \varepsilon)$ be the support triangle of u with endpoints P_ε and Q_ε. By (4.46), we have

$$Q_\varepsilon - P_\varepsilon = \left(2\varepsilon, 2\varepsilon u'(x_0)\right).$$

Hence for ε sufficiently small we have that $0 < 2\varepsilon < \tau$ and (i) happens. To prove (ii), consider $H(P_\varepsilon)$ and $H(Q_\varepsilon)$ the support lines of u at $x_0 - \varepsilon$ and $x_0 + \varepsilon$, respectively, and let $W_\varepsilon = (w_1, w_2)$ be the point where $H(P_\varepsilon)$ and $H(Q_\varepsilon)$ intersect. Without loss of generality, assume that

$$(x_0 + \varepsilon) - w_1 \leq w_1 - (x_0 - \varepsilon).$$

Consider $Q_1^\varepsilon = (q_1^\varepsilon, h_1^\varepsilon)$ as the point on $H(Q_\varepsilon)$ such that $q_1^\varepsilon - w_1 = w_1 - (x_0 - \varepsilon)$. Note that $Q_\varepsilon \in [W_\varepsilon, Q_1^\varepsilon]$, where $[W_\varepsilon, Q_1^\varepsilon]$ is the closed line segment with endpoints W_ε and Q_1^ε. Thus, there exists a quadratic function

$$f_\varepsilon(x) = ax^2 + bx + c = a(\varepsilon)x^2 + b(\varepsilon)x + c(\varepsilon)$$

such that $H(P_\varepsilon)$ is tangent to f at P_ε and $H(Q_\varepsilon)$ is tangent to f at Q_1^ε (see Lemma 5.1 in Appendix and Figure 4.4). Note that $h_1^\epsilon = f_\varepsilon(q_1^\varepsilon)$.

A simple calculation using (4.46) shows that as $\varepsilon \to 0$

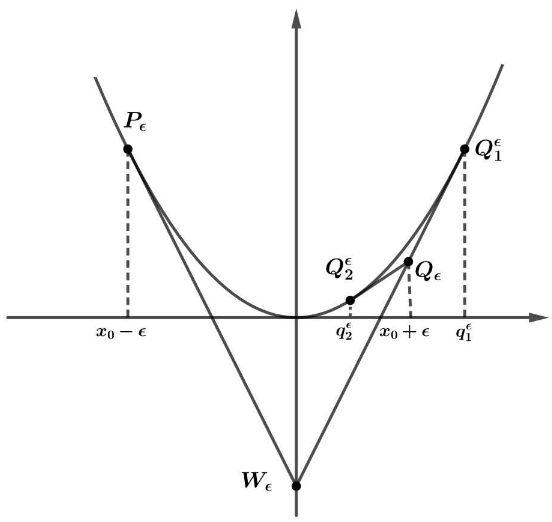
$$f_\varepsilon(x) \to \frac{u''(x_0)}{2}x^2 + (u'(x_0) - x_0u''(x_0))x + (u(x_0) - x_0u'(x_0) + \frac{x_0^2}{2}u''(x_0)) = f_{x_0}(x). \quad (4.47)$$

Since f_ε is a convex function, Q_ε does not lie in the interior of $\mathrm{epi}(f_\varepsilon)$ and $[Q_\varepsilon, Q_1^\varepsilon]$ is tangent to f_ε. Let $Q_2^\varepsilon = (q_2^\varepsilon, f_\varepsilon(q_2^\varepsilon))$ be the second point on boundary of $\mathrm{epi}(f_\varepsilon)$ such that $[Q_2^\varepsilon, Q_\varepsilon]$ is tangent to f_ε, and let

$$T_1^\varepsilon = \mathrm{conv}\{P_\varepsilon, W_\varepsilon, Q_1^\varepsilon\} \quad \text{and} \quad T_2^\varepsilon = \mathrm{conv}\{P_\varepsilon, W_\varepsilon, Q_2^\varepsilon\}.$$

See Figure 4.4.

Figure 4.4: Case $u''(x_0) > 0$.



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Now it is sufficient to define

$$A_T^\varepsilon = (\mathrm{epi}(f_\varepsilon) \cap T_2^\varepsilon) \cup \mathrm{conv}\{P_\varepsilon, Q_2^\varepsilon, Q_\varepsilon\}$$

and

$$v_T^\varepsilon = (f_\varepsilon \dotplus \mathrm{L}_{[x_0-\varepsilon, q_2^\varepsilon]}) \vee (w \dotplus \mathrm{L}_{[q_2^\varepsilon, q_1^\varepsilon]}), \quad (4.48)$$

where $w = l([Q_2^\varepsilon, Q_\varepsilon]) \vee l([Q_\varepsilon, Q_1^\varepsilon])$ and $l([Q_2^\varepsilon, Q_\varepsilon])$ is the affine function that contains the segment $[Q_2^\varepsilon, Q_\varepsilon]$ and $l([Q_\varepsilon, Q_1^\varepsilon])$ is the affine function that contains the segment $[Q_\varepsilon, Q_1^\varepsilon]$.

Therefore, $A_T^\varepsilon \subset T_\varepsilon$, A_T^ε is a support triangle of v_T^ε and $v_T^\varepsilon \in P_{\,\mathrm{l.q}}(\mathbb{R})$.

It remains to show the item (iii). Using that $[Q_2^\varepsilon, Q_\varepsilon]$ and that $[Q_\varepsilon, Q_1^\varepsilon]$ are tangent to f_ε, we obtain

$$
\begin{aligned}
q_2^\varepsilon &= (x_0+\varepsilon) - \sqrt{(x_0+\varepsilon)^2 - a^{-1}(u(x_0+\varepsilon) - b(x_0+\varepsilon) - c)} \\
q_1^\varepsilon &= (x_0+\varepsilon) + \sqrt{(x_0+\varepsilon)^2 - a^{-1}(u(x_0+\varepsilon) - b(x_0+\varepsilon) - c)}
\end{aligned}
$$

and since $u(x_0+\varepsilon) \to f_{x_0}(x_0)$ as $\varepsilon \to 0$, we get

$$
\begin{aligned}
0 &= \lim_{\varepsilon\to 0} 2\frac{\sqrt{(x_0+\varepsilon)^2 - a^{-1}(u(x_0+\varepsilon) - b(x_0+\varepsilon) - c)}}{q_1^\varepsilon + \varepsilon - x_0} \\
&= \lim_{\varepsilon\to 0} \frac{q_1^\varepsilon - q_2^\varepsilon}{q_1^\varepsilon + \varepsilon - x_0} \\
&= \lim_{\varepsilon\to 0} \frac{1}{2}\frac{x_0+\varepsilon - q_2^\varepsilon}{q_1^\varepsilon + \varepsilon - x_0}.
\end{aligned}
$$

By (4.48), and using that Z vanishes on piecewise affine functions, we have

$$
Z(v_T^\varepsilon) = Z(f_\varepsilon \dotplus \mathrm{L}_{[x_0-\varepsilon, q_2^\varepsilon]}).
$$

Therefore, by (4.30)

$$
\begin{aligned}
Z(v_T^\varepsilon) &= \frac{q_2^\varepsilon + \varepsilon - x_0}{2m} Z(f_\varepsilon \dotplus \mathrm{L}_{[-m,m]}) \\
&= \frac{q_1^\varepsilon + \varepsilon - x_0}{2m}\left(Z(f_\varepsilon \dotplus \mathrm{L}_{[-m,m]}) - \frac{q_1^\varepsilon - q_2^\varepsilon}{q_1^\varepsilon + \varepsilon - x_0} Z(f_\varepsilon \dotplus \mathrm{L}_{[-m,m]})\right).
\end{aligned}
$$

Then, for every $\eta > 0$ and for ε sufficiently small

$$
\frac{q_1^\varepsilon + \varepsilon - x_0}{2m}\big(Z(f_\varepsilon \dotplus \mathrm{L}_{[-m,m]}) - \eta\big) \le Z(v_T^\varepsilon). \tag{4.49}
$$

Now observe that the triangle T_1^ε is a support triangle of the function

$$
(u \dotplus \mathrm{L}_{[x_0-\varepsilon, x_0+\varepsilon]}) \vee (l([Q_\varepsilon, Q_1^\varepsilon]) \dotplus \mathrm{L}_{[x_0+\varepsilon, q_1^\varepsilon]})
$$

which is a function in $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ and T_1^ε is also a support triangle of the quadratic function $f_\varepsilon(x) = ax^2 + bx + c$. Thus there are translations $\tau_{y_1}, \dots, \tau_{y_n}$ with

$$
|y_i| = q_1^\varepsilon + \varepsilon - x_0 \quad \text{and} \quad n \le \frac{2m}{q_1^\varepsilon + \varepsilon - x_0} < n+1
$$

such that $T_i = T(\tau_{y_i}^{-1}(x_0-\varepsilon), \tau_{y_i}^{-1}(q_1^\varepsilon))$ is a support triangle of $f_\varepsilon(x) = ax^2 + bx + c$ for every $i = 1, \dots, n$, and $T_i^\prime s$ have pairwise disjoint interiors (see Lemma 5.2 in Appendix).

Define

$$
v_\varepsilon = \bigvee_{i=1}^n ((u \dotplus \mathrm{L}_{[x_0-\varepsilon, x_0+\varepsilon]}) \vee (l([Q_\varepsilon, Q_1^\varepsilon]) \dotplus \mathrm{L}_{[x_0+\varepsilon, q_1^\varepsilon]})) \circ \tau_{y_i}^{-1} \vee (f_\varepsilon \dotplus \mathrm{L}_{[-m,m]\setminus I}), \tag{4.50}
$$

where $I = \bigcup_{i=1}^n \tau_{y_i}([x_0-\epsilon, q_1^\epsilon])$. Note that $[-m,m]\setminus I$ is not necessarily an interval, it can be union of intervals, and we used an abuse of notation.

By construction, v_ε is a convex function in $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ and since Z is a non-negative, simple and translation invariant valuation,

$$Z(v_\varepsilon) \geq nZ(u \dotplus \mathrm{L}_{[x_0-\varepsilon,x_0+\varepsilon]})$$

and by (4.47), v_ε converges to $f_{x_0} \dotplus \mathrm{L}_{[-m,m]}$ as $\varepsilon \to 0$.

By upper semicontinuity of Z, we obtain

$$\begin{aligned} Z(f_{x_0} \dotplus \mathrm{L}_{[-m,m]}) &\geq \limsup_{\varepsilon\to 0} Z(v_\varepsilon) \\ &\geq \limsup_{\varepsilon\to 0} \frac{2m}{q_1^\varepsilon + \varepsilon - x_0} Z(u \dotplus \mathrm{L}_{[x_0-\varepsilon,x_0+\varepsilon]}). \end{aligned}$$

Since the function $\zeta(2a) = \frac{1}{2m} Z(ax^2 \dotplus \mathrm{L}_{[-m,m]}(x))$ is contained in $\mathcal{W}$, in particular, it is continuous and by (4.47), we get for every $\eta > 0$

$$Z(u \dotplus \mathrm{L}_{[x_0-\varepsilon,x_0+\varepsilon]}) \leq \frac{q_1^\varepsilon + \varepsilon - x_0}{2m} \big(Z(f_\varepsilon \dotplus \mathrm{L}_{[-m,m]}) + \eta\big) \tag{4.51}$$

for ε sufficiently small. Furthermore,

$$q_1^\varepsilon - (x_0+\varepsilon) \leq \frac{q_1^\varepsilon - (x_0 - \varepsilon)}{2} = \varepsilon + \frac{q_1^\varepsilon - (x_0+\varepsilon)}{2}.$$

These last inequalities and (4.49) now imply that

$$\begin{aligned} Z(u \dotplus \mathrm{L}_{[x_0-\varepsilon,x_0+\varepsilon]}) &\leq Z(v_T^\varepsilon) + \frac{q_1^\varepsilon + \varepsilon - x_0}{2m} 2\eta \\ &\leq Z(v_T^\varepsilon) + \frac{2\varepsilon + \sqrt{(x_0+\varepsilon)^2 - a^{-1}(u(x_0+\varepsilon) - b(x_0+\varepsilon) - c)}}{2m} 2\eta \\ &\leq Z(v_T^\varepsilon) + \frac{4\varepsilon}{2m} 2\eta \end{aligned}$$

for ε sufficiently small. In the last inequality we used the simple estimate

$$\sqrt{(x_0+\varepsilon)^2 - a^{-1}(u(x_0+\varepsilon) - b(x_0+\varepsilon) - c)} \leq 2\varepsilon.$$

Setting $\eta = \frac{\rho m}{4}$ now shows that (iii) holds for $\varepsilon > 0$ sufficiently small.

Now let $u''(x_0) = 0$. Let T_ε be the support triangle of u with endpoints $P = (x_0, u(x_0))$ and $Q_\varepsilon = (x_0+\varepsilon, u(x_0+\varepsilon))$ and $A_T^\varepsilon = T_\varepsilon$. Then (i) holds for ε sufficiently small. For every $a > 0$ the quadratic function

$$f(x) = ax^2 - 2ax_0 x + (u(x_0) + ax_0^2)$$

is such that P is a point of the graph of f and whose epigraph is locally contained in $\mathrm{epi}(u)$.

Since Z is an upper semicontinuous, non-negative and simple valuation, we obtain

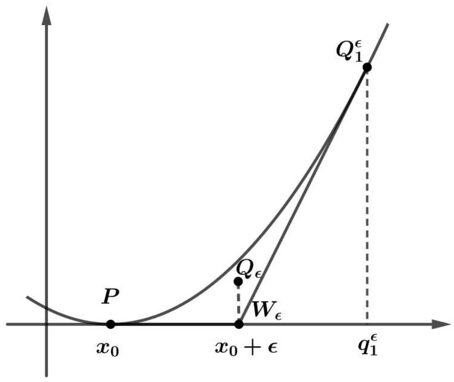
$$\limsup_{a\to 0} Z(f \dotplus \mathrm{L}_{[-m,m]}) = 0.$$

Therefore, for a sufficiently small

$$Z(f \dotplus \mathrm{L}_{[-m,m]}) \leq \frac{\rho m}{4}. \tag{4.52}$$

Let W_ε be the point on the support line to u at x_0 such that the first coordinate is $x_0+\varepsilon$ and let $Q_1^\varepsilon = (q_1^\varepsilon, f(q_1^\varepsilon))$ be the point on graphic of f such that $[W_\varepsilon, Q_1^\varepsilon]$ is tangent to f.

Figure 4.5: Case $u''(x_0) = 0$.



Source: Compiled by the author.

Then the triangle

$$T_1^\varepsilon = \mathrm{conv}\{P, W_\varepsilon, Q_1^\varepsilon\}$$

is a support triangle of f. Since epi(f) is locally contained in epi(u) the support line of u at Q_ε does not intersect f for $\varepsilon > 0$ sufficiently small. Therefore

$$(u \dotplus \mathrm{L}_{[x_0,x_0+\varepsilon]}) \vee (w \vee l([Q_\varepsilon, Q_1^\varepsilon])),$$

where w is the tangente line to u at Q_ϵ, is a convex function in $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ and T_1^ε is also a support triangle of this function. Using the same idea of (4.50), define v_ε by

$$v_\varepsilon = \bigvee_{i=1}^{n} ((u \dotplus \mathrm{L}_{[x_0,x_0+\varepsilon]}) \vee (w \vee l([Q_\varepsilon, Q_1^\varepsilon]))) \circ \tau_{y_i}^{-1} \vee (f_\varepsilon \dotplus \mathrm{L}_{[-m,m]\setminus I}),$$

where $I = \bigcup_{i=1}^{n} \tau_{y_i}([x_0, q_1^\epsilon])$ and $n \leq \frac{2m}{q_1^\varepsilon - x_0} < n+1$. Then v_ε converges to $f \dotplus \mathrm{L}_{[-m,m]}$ as $\varepsilon \to 0$ and

$$Z(v_\varepsilon) \geq nZ(u \dotplus \mathrm{L}_{[x_0,x_0+\varepsilon]}).$$

Since Z is upper semicontinuous, we have for every $\eta > 0$

$$\frac{2m}{q_1^\varepsilon - x_0} Z(u \dotplus \mathrm{L}_{[x_0, x_0+\varepsilon]}) \leq Z(f \dotplus \mathrm{L}_{[-m,m]}) + \eta \tag{4.53}$$

for ε sufficiently small.

Since W_ε belongs to a support line to f that contains P and the support line that contains Q_1^ε, we have that $q_1^\varepsilon = x_0 + 2\varepsilon$. Thus, replacing q_1^ε in the inequality (4.53) we obtain

$$Z(u \dotplus \mathrm{L}_{[x_0, x_0+\varepsilon]}) \leq \frac{2\varepsilon}{2m}\eta + \frac{2\varepsilon}{2m} Z(f \dotplus \mathrm{L}_{[-m,m]}).$$

By (4.52), it follows that

$$Z(u \dotplus \mathrm{L}_{[x_0, x_0+\varepsilon]}) \leq \varepsilon \left(\frac{\eta}{m} + \frac{\rho}{4}\right).$$

Taking $\eta = \dfrac{\rho m}{4}$, we obtain

$$Z(u \dotplus \mathrm{L}_{[x_0, x_0+\varepsilon]}) \leq \frac{\rho}{2}\varepsilon.$$

Thus, since

$$v_T^\epsilon = (u \dotplus \mathrm{L}_{[x_0, x_0+\varepsilon]}) \vee l([P, Q_\epsilon])$$

is a piecewise affine function, (ii) and (iii) hold. ☐

Since Z is a simple valuation, we have the immediately result.

Lemma 4.7. *Let $J \subset \mathrm{dom_c}\, u$ be a closed interval such that $u|_J$ is a piecewise affine function. Then for every $\rho > 0$ it holds*

$$Z(u \dotplus \mathrm{L}_J) \leq \frac{\rho}{2} V_1(J). \tag{4.54}$$

Lemma 4.8. *There is a constant c_M such that*

$$Z(u \dotplus \mathrm{L}_J) \leq c_M V_1(J)$$

for every closed interval $J \subseteq \mathrm{dom_c}\, u$, where M is the Lipschitz constant of u.

Proof. Since M is the Lipschitz constant of u, then there exists a quadratic function

$$f_{x_0}(x) = Mx^2 + b(x_0)x + c(x_0)$$

such that $f_{x_0}(x_0) = u(x_0)$ and $f'_{x_0}(x_0) = u'_+(x_0)$ and this means that $\mathrm{epi}(f_{x_0})$ is locally contained in $\mathrm{epi}(u)$. Let T_ϵ be the support triangle of u with endpoints $P = (x_0, u(x_0))$ and $Q_\epsilon(x_0+\epsilon, u(x_0+\epsilon))$ for $\varepsilon > 0$ sufficiently small. As in Lemma 4.6, for the case where $u''(x_0) = 0$, define

$$v_\varepsilon = \bigvee_{i=1}^{n} ((u \dotplus \mathrm{L}_{[x_0, x_0+\varepsilon]}) \vee (w \vee l([Q_\varepsilon, Q_1^\varepsilon]))) \circ \tau_{y_i}^{-1} \vee (f_{x_0} \dotplus \mathrm{L}_{[-m,m] \setminus I}),$$

where $I = \bigcup_{i=1}^{n} \tau_{y_i}([x_0, q_1^\epsilon])$. Then v_ϵ converges to $f_{x_0} \dotplus \mathrm{L}_{[-m,m]}$ as $\epsilon \to 0$ and for $\eta = 1$

$$Z(u \dotplus \mathrm{L}_{[x_0,x_0+\epsilon]}) \leq \left(\frac{1}{m} + \frac{Z(f_{x_0} \dotplus \mathrm{L}_{[-m,m]})}{m}\right)\epsilon \tag{4.55}$$

for ϵ sufficiently small. Note that $Z(f_{x_0} \dotplus \mathrm{L}_{[-m,m]})$ just depends on M, since Z is invariant by affine functions. We can therefore dissect J into finitely many intervals for which (4.55) holds. □

Proposition 4.5. *If* $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$ *is a simple, upper semicontinuous and translation invariant valuation which is unchanged by the addition of piecewise affine functions, then*

$$Z(u) = \sup\left\{\limsup_{n\to+\infty} Z(v_n) : v_n \in P_{1.\mathrm{q}}(\mathbb{R}), v_n \to u\right\}.$$

Proof. Since $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$, the set $S \subseteq \mathrm{dom_c}\, u$ of points where u is twice differentiable is such that

$$V_1(S) = V_1(\mathrm{dom_c}\, u).$$

By Lemmas 4.6 and 4.7 the sets

$$\{x \in \mathrm{dom_c}\, u : (x, u(x)) \in T\} \text{ and } \mathrm{cl}(\{x \in \mathrm{dom_c}\, u : u \text{ is linear in a neighborhood } J \text{ of } x\})$$

are a Vitali class for S and this remains true if we only take T and J with $V_1(P_{e_1}T), V_1(J) \leq \delta$ for $\delta > 0$, where $P_{e_1}T$ denotes the projection of the set T over the space generated by the canonical vector $e_1 \in \mathbb{R}^2$. Take $\eta > 0$ such that $\eta \leq \delta$ and $\eta \leq \frac{\rho}{2c_M} V_1(\mathrm{dom_c}\, u)$.

Then we can choose by Vitali's Theorem 4.10 support triangles $T_1, \dots, T_n$ and closed intervals $J_1, \dots, J_l$ such that

$$V_1(\mathrm{dom_c}\, u) = V_1(S) \leq \sum_{i=1}^{n} V_1(P_{e_1}T_i) + \sum_{j=1}^{l} V_1(J_j) + \eta$$

and such that the intervals $P_{e_1}T_1, \dots, P_{e_1}T_n, J_1, \dots, J_l$ are pairwise disjoint.

We choose closed intervals $K_1, \dots, K_k$ such that $P_{e_1}T_1, \dots, P_{e_1}T_n, J_1, \dots, J_l, K_1, \dots, K_k$, have pairwise disjoint interiors and such that $\mathrm{dom_c}\, u$ can be decomposed as

$$P_{e_1}T_1 \cup \dots \cup P_{e_1}T_n \cup J_1 \cup \dots \cup J_l \cup K_1 \cup \dots \cup K_k.$$

Define

$$v_T = v_{T_1} \vee \dots \vee v_{T_n} \vee (u \dotplus \mathrm{L}_{J_1}) \vee \dots \vee (u \dotplus \mathrm{L}_{J_l}) \vee (l_1 \dotplus \mathrm{L}_{K_1}) \vee \dots \vee (l_k \dotplus \mathrm{L}_{K_k}),$$

where l_i are piecewise affine functions such that v_T is a convex function in $\mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$, v_T

coincides with u in $\mathbb{R} \setminus \mathrm{dom}_c\, u$, and

$$Z(v_T) = \sum_{i=1}^{n} Z(v_{T_i}).$$

Therefore, since Z is a simple valuation and by Lemmas 4.6, 4.7 and 4.8 we conclude that for every $\rho > 0$

$$\begin{aligned}
Z(u) &= Z((u \dotplus \mathrm{L}_{P_{e_1}T_1}) \vee \cdots \vee (u \dotplus \mathrm{L}_{P_{e_1}T_n}) \vee (u \dotplus \mathrm{L}_{J_1}) \vee \cdots \vee (u \dotplus \mathrm{L}_{J_l}) \vee (u \dotplus \mathrm{L}_{K_1}) \vee \cdots \vee (u \dotplus \mathrm{L}_{K_k}))\\
&= \sum_{i=1}^{n} Z(u \dotplus \mathrm{L}_{P_{e_1}T_i}) + \sum_{j=1}^{l} Z(u \dotplus \mathrm{L}_{J_j}) + \sum_{s=1}^{k} Z(u \dotplus \mathrm{L}_{K_s})\\
&\le \sum_{i=1}^{n} \Big(Z(v_{T_i}) + \frac{\rho}{2} V_1(P_{e_1}T_i) \Big) + \sum_{j=1}^{l} \frac{\rho}{2} V_1(J_j) + c_M \sum_{s=1}^{k} V_1(\mathrm{dom}\, u \cap K_s)\\
&\le Z(v_T) + \frac{\rho}{2} \left(\sum_{i=1}^{n} V_1(P_{e_1}T_i) + \sum_{j=1}^{l} V_1(J_j) \right) + c_M \eta\\
&\le Z(v_T) + \frac{\rho}{2} \left(\sum_{i=1}^{n} V_1(P_{e_1}T_i) + \sum_{j=1}^{l} V_1(J_j) \right) + \frac{\rho}{2} V_1(\mathrm{dom_c}\, u)\\
&= Z(v_T) + \rho V_1(\mathrm{dom_c}\, u).
\end{aligned}$$

Since ρ is arbitrary we conclude the proof of this proposition. ☐

Proof of Proposition 4.4. Let $Z : \mathrm{Conv_{pac}}(\mathbb{R}; \mathbb{R}) \to [0, +\infty)$ be a valuation that satisfies the conditions $(i) - (iii)$. By Lemma 4.5, $\zeta \in \mathcal{W}$. Since Z is simple and by (4.27) determined by ζ on piecewise linear-quadratic functions, then $Z(u)$ is determined by ζ for every $u \in P_{\mathrm{l.q}}(\mathbb{R})$. And by Proposition 4.5 we conclude that Z is uniquely determined by ζ. ☐

Proof of Proposition 4.3. Let $\zeta \in \mathcal{W}$ be given by (4.27). By Lemma 4.3 and by Theorem 4.7, the functional $Z_\zeta : \mathrm{Conv_{pac}}(\mathbb{R}; \mathbb{R}) \to [0, +\infty)$ defined by

$$Z_\zeta(u) = \int_{\mathbb{R}} \zeta(u''(x))dx$$

is a simple, upper semicontinuous and translation invariant valuation, unchanged by the addition of piecewise affine functions and for $f(x) = \frac{a}{2}x^2 \dotplus \mathrm{L}_{[-m,m]}$ satisfies

$$Z_\zeta(f \dotplus \mathrm{L}_{[-m,m]}) = 2m\zeta(a).$$

Therefore by Proposition 4.4 we conclude the proof. ☐

Proof of Theorem 4.6. Let w be a piecewise affine function on $\mathbb{R}$. Since Z is unchanged by the addition of piecewise affine functions, we have $Z(w) = c_0$ for some constant that does not depend on w. Define

$$Z_0(u) = Z(u) - c_0.$$

Then Z_0 is a simple, upper semicontinuous, translation invariant valuation and unchanged by the addition of piecewise affine functions. The proof now follows from Proposition 4.3 that guarantees the existence of a function $\zeta \in \mathcal{W}$ such that

$$Z_0(u) = \int_{\mathbb{R}} \zeta(u''(x))dx$$

for every $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R};\mathbb{R})$. □

Chapter 5

Conclusion and further research

This thesis compiles works on convex geometry in different themes. We use this last chapter to reinforce some conclusions and pose some questions that we encountered during this study.

In Chapter 2 we gave a constructive proof for the existence of isotropic measures for the John and positive John positions. It would be nice to know if there is a possibility of weakening the hypothesis about f and g requested in this chapter. Other question is about the uniqueness of the position. For example, for two different convex bodies, is it possible to give on explicit representation of an isotropic measure? Recall that we answered just for the positive John position because for this position there is uniqueness.

In Chapter 3 we gave a constructive proof for s-isotropic measures in s- John position. One question about the functional Löwner ellipsoids is if it is possible to give an explicit representation of the "John decomposition" using the approach in [31].

Finally, regarding the Chapter 4, we would like to generalize the Theorem 4.6 for $n \in \mathbb{N}$, i.e., to resolve the following problem.

Problem: Let $Z : \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ be an upper semicontinuous, $\mathrm{SL}_n(\mathbb{R})$, translation invariant valuation and unchanged by the addition of piecewise affine functions. Then there is a constant $c_0 \in \mathbb{R}$ and a function $\zeta \in \mathcal{W}$ such that

$$Z(u) = c_0 + \int_{\mathbb{R}^n} \zeta(\det \mathrm{D}^2 u(x)) dx$$

for every $u \in \mathrm{Conv}_{\mathrm{pac}}(\mathbb{R}^n; \mathbb{R})$.

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Appendix

The objective of this appendix is to prove two simple results that were used in Chapter 4, more specifically in Lemma 4.6.

Let $g, h : \mathbb{R} \to \mathbb{R}$ be two affine functions given by

$$\begin{aligned} g(x) &= l_1 x + l_2 \\ h(x) &= m_1 x + m_2, \end{aligned}$$

respectively, where $l_1, l_2, m_1, m_2 \in \mathbb{R}$ and $l_1 \neq m_1$. Assume that $g(p_0) = h(p_0)$ for some $p_0 \in \mathbb{R}$ and take $x_0 < p_0$. We want to find a quadratic function

$$f(x) = ax^2 + bx + c$$

such that

$$\begin{cases} f(x_0) & = g(x_0) \\ f'(x_0) & = g'(x_0) \end{cases} \tag{5.1}$$

and

$$\begin{cases} f(p_0 + (p_0 - x_0)) & = g(p_0 + (p_0 - x_0)) \\ f'(p_0 + (p_0 - x_0)) & = g'(p_0 + (p_0 - x_0)) \end{cases}. \tag{5.2}$$

First note that $g(p_0) = h(p_0)$ implies that

$$p_0 = -\frac{m_2 - l_2}{m_1 - l_1}$$

and by second equations in (5.1) and (5.2), respectively, we get

$$\begin{aligned} 2ax_0 + b &= l_1 \\ 2a(2p_0 - x_0) + b &= m_1. \end{aligned}$$

Hence,

$$a = \frac{1}{4}\left(\frac{m_1 - l_1}{p_0 - x_0}\right) \quad \text{and} \quad b = \left(l_1 - \frac{x_0}{2}\left(\frac{m_1 - l_1}{p_0 - x_0}\right)\right).$$

Now using the first equation in (5.1) we obtain

$$c = l_2 + \frac{x_0^2}{4}\left(\frac{m_1 - l_1}{p_0 - x_0}\right).$$

From this we obtain the following result.

Lemma 5.1. *Consider the affine functions $g, h : \mathbb{R} \to \mathbb{R}$ given by*

$$\begin{aligned} g(x) &= l_1 x + l_2 \\ h(x) &= m_1 x + m_2, \end{aligned}$$

respectively, where $l_1, l_2, m_1, m_2 \in \mathbb{R}, l_1 \neq m_1$ and let p_0 be a point at which the functions coincide. If $x_0 < p_0$, then the quadratic function

$$f(x) = \frac{1}{4}\left(\frac{m_1 - l_1}{p_0 - x_0}\right)x^2 + \left(l_1 - \frac{x_0}{2}\left(\frac{m_1 - l_1}{p_0 - x_0}\right)\right)x + l_2 + \frac{x_0^2}{4}\left(\frac{m_1 - l_1}{p_0 - x_0}\right)$$

is such that

$$f(x_0) = g(x_0),\ f'(x_0) = g'(x_0) \quad \textit{and} \quad f(2p_0 - x_0) = g(2p_0 - x_0),\ f'(2p_0 - x_0) = g'(2p_0 - x_0).$$

Lemma 5.2. *If $T = T(x_1, x_2)$ is a support triangle of $f(x) = ax^2 + bx + c$, where $a > 0, b, c \in \mathbb{R}$, then there exists a rotation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(T)$ is a support triangle of f as well. Moreover if $\varphi(x) = (\varphi_1(x), \varphi_2(x))$, then*

$$\varphi(T) = T(\varphi_1(x_1), \varphi_1(x_2)).$$

Proof. Define the function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ by

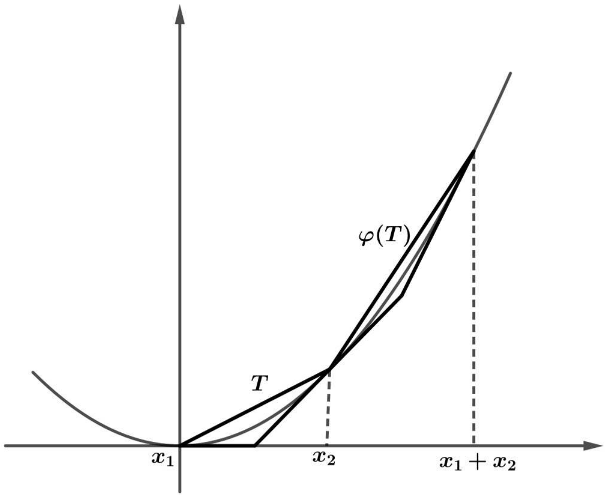
$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ a(x_2 - x_1)^2 + b(x_2 - x_1) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2a(x_2 - x_1) & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Observe that

$$\begin{aligned} \varphi\begin{pmatrix} x \\ f(x) \end{pmatrix} &= \begin{pmatrix} x_2 - x_1 \\ a(x_2 - x_1)^2 + b(x_2 - x_1) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2a(x_2 - x_1) & 1 \end{pmatrix}\begin{pmatrix} x \\ ax^2 + bx + c \end{pmatrix} \\ &= \begin{pmatrix} x + (x_2 - x_1) \\ a(x + (x_2 - x_1))^2 + b(x_2 - x_1) + c \end{pmatrix} \\ &= \begin{pmatrix} x + (x_2 - x_1) \\ f(x + (x_2 - x_1)) \end{pmatrix}. \end{aligned}$$

Since φ is a C^1 function, then $\varphi(T(x_1, x_2))$ is also a support triangle of f. Moreover, if $\varphi(x) = (\varphi_1(x), \varphi_2(x))$, then $\varphi_1(x) = x + (x_2 - x_1)$ and

$$\varphi(T(x_1, x_2)) = T(x_2, x_2 + (x_2 - x_1)) = T(\varphi_1(x_1), \varphi_1(x_2)).$$



Source: Compiled by the author.

$\square$