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Genilson Soares de Santana

# Decay rates of $C_0$-Semigroups on Banach spaces and applications to Spectral Theory

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GENILSON SOARES DE SANTANA

# Decay rates of $C_0$-semigroups on Banach spaces and applications to Spectral Theory

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Orientador: Prof. Dr. Silas Luiz de Carvalho.

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Aos vinte e três dias do mês de fevereiro de 2024, às 09h00, na sala 3060, reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese do aluno **Genilson Soares de Santana**, intitulada: "*Decay rates of CO-semigrups on Banach spaces and applications to spetral theory*" requisito final para obtenção do Grau de Doutor em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Silas Luiz de Carvalho, após dar conhecimento aos presentes do teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se reservadamente, sem a presença do aluno, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 23 de fevereiro de 2024.

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*"A perseverança é a mãe da boa sorte".*
*Miguel de Cervantes*

## Resumo

Estudamos taxas de decaimento de $C_0$-semigrupos, semigrupos auto-adjuntos e grupos unitários de evolução. Para $C_0$-semigrupos em espaços de Banach, obtemos taxas de decaimento sob a suposição de que a norma do resolvente do gerador do semigrupo cresce com $|s|^\beta \log(|s|)^b, \beta, b \geq 0$, com $|s| \to \infty$, e com $|s|^{-\alpha} \log(1/|s|)^a$, $\alpha, a \geq 0$, como $|s| \to 0$. Nossos resultados não supõem que o semigrupo seja limitado. Em particular, para $a = b = 0$, os nossos resultados refinam as taxas envolvendo tipos de Fourier obtidas por Rozendaal e Veraar (J. Funct. Anal. 275(10): 2845-2894, 2018). Quanto aos grupos de evolução unitários, obtemos taxas de decaimento lentas para a média da probabilidade de retorno de um dado inicial no sentido típico (no sentido de Baire), e para os semigrupos auto-adjuntos, obtemos também taxas de decaimento lento para a órbita de um dado inicial.


**Palavras Chaves:** Taxas de decaimentos; $C_0$-semigrupos; semigrupos auto-adjuntos; grupos de evolução unitários.

## Abstract

We study decay rates of $C_0$-semigroups, self-adjoint semigroups and unitary evolution groups. For $C_0$-semigroups in Banach spaces, we obtain decay rates under the assumption that the norm of the resolvent of the semigroup generator grows with $|s|^\beta \log(|s|)^b, \beta, b \geq 0$, with $|s| \to \infty$, and with $|s|^{-\alpha} \log(1/|s|)^a$, $\alpha, a \geq 0$, as $|s| \to 0$. Our results do not assume that the semigroup is bounded. In particular, for $a = b = 0$, our results improve the rates involving Fourier types obtained by Rozendaal and Veraar (J. Funct. Anal. 275(10): 2845-2894, 2018). As for unitary evolution groups, we obtain slow decay rates for the average return probability of a typical (in Baire's sense) initial state, and for self-adjoint semigroups, we also obtain slow decay rates for the orbit of a typical initial state.



**Keywords:** Decay rates; $C_0$-semigroups; self-adjoint semigroups; unitary evolution groups.

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# Selected Notation

$\mathbb{N} := \{1, 2, 3, \cdots\}$ and $\mathbb{C}_\pm := \{z \in \mathbb{C} \mid \mathrm{Re}(\lambda) \gtrless 0\}$

| | |
|---|---|
| $p' = p/(p-1)$ | Hölder conjugate. |
| $X$ | Complex Banach space. |
| $\\|\cdot\\| = \\|\cdot\\|_X$ | Norm in a Banach space $X$ |
| $\mathcal{L}(X, Y)$ | Space of bounded linear operators from $X$ to $Y$. |
| $\mathcal{D}(A)$ | Domain of a linear operator $A$. |
| $\mathrm{Ran}(A)$ | Range of a linear operator $A$. |
| $\sigma(A)$ | Spectrum of a linear operator $A$. |
| $\rho(A)$ | Resolvent set of a linear operator $A$. |
| $R(\lambda, A) := (\lambda - A)^{-1}$ | Resolvent operator of $A$ at $\lambda \in \rho(A)$. |
| $E^A$ | Resolution of the identity of $A$ (self-adjoint). |
| $\mu_x^A$ | Spectral measure of $A$ (self-adjoint) associated with the vector $x$. |
| $(T(t))_{t\geq 0}$ | $C_0$-semigroup. |
| $\mathcal{F}$ | Fourier Transform. |
| $\mathcal{S}(\mathbb{R}; X)$ | Spaces of $X$-valued Schwarz functions. |
| $\mathcal{S}'(\mathbb{R}; X)$ | Spaces of $X$-valued tempered distributions. |
| $C_c^\infty$ | Space of test functions with compact support. |
| $\mathcal{CBF}$ | Set of all Complete Bernstein functions. |
| $f(t) \lesssim g(t)$ | $\exists\ C, t_0 \geq 0$ such that for each $t \geq t_0$, $f(t) \leq Cg(t)$. |
| $S_\omega := \{z \in \mathbb{C} \mid 0 < \lvert\arg(z)\rvert < \omega\}$ | Open sector of angle $\omega$. |
| $B(w, \varepsilon)$ | Open interval $(w - \varepsilon, w + \varepsilon)$ centered at $w \in \mathbb{R}$. |

$H_0^\infty(S_\omega) := \{f : S_\omega \to \mathbb{C} \mid f$ is holomorpic and exist $C \geq 0, s > 0; \lvert f(z)\rvert \leq C \min\{\lvert z\rvert^{-s}, \lvert z\rvert^s\}, \forall z \in S_\omega\}$

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# Contents

# Introduction

This thesis is divided into two parts. In the first part, we discuss results for the decay of $C_0$-semigroups defined in Banach spaces. In the second part, we discuss results for the slow decay of self-adjoint semigroups and unitary $C_0$-groups defined in Hilbert spaces.

## Part I

### Historical background

An important question in the theory of differential equations refers to the asymptotic behavior (in time) of their solutions; more specifically, if they reach an equilibrium and, if so, with which speed. For those linear partial differential equations which can be conveniently analyzed by rewriting them as evolution equations, it is well known that the long-term behavior of the solutions of each one of these equations is related to some spectral properties (and behavior of the resolvent) of the generator of the associated semigroup.

The asymptotic theory of semigroups provides tools for investigating the convergence to zero of mild and classical solutions to the abstract Cauchy problem

$$\begin{cases} u'(t) + Au(t) = 0, & t \geq 0 \\ u(0) = x, \end{cases} \tag{1}$$

We know that (1) has a unique mild solution for every $x \in X$, and that the solution depends continuously on $x$ if, and only if, $-A$ generates a $C_0$-semigroup $(T(t))_{t\geq 0}$ on $X$ (see [7, 29]). In this case, the unique solution $u$ to (1) is given by $u(t) = T(t)x, \ \ \forall\, t \geq 0$, and if $x \in \mathcal{D}(A)$, then $u \in C^1([0,\infty), X) \cap C([0,\infty), X)$ (see [29], Proposition II.6.2).

For the classic theory of ODEs in finite dimension, the Lyapunov stability criterion (see [29], Theorem 2.10) is an excellent tool in the study of the asymptotic behavior of solutions to (1), but this criterion is in general not valid if $X$ has infinite dimension. However, in this case, the asymptotic behavior can be deduced from of the norm of the resolvent of the operator $A$. For example, on a Hilbert space $X$, one has the Gearhart(1978)-Prüss(1984)-Greiner(1985) Theorem.

In what follows, $\rho(A) := \{\lambda \in \mathbb{C} \mid \|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} < \infty\}$ and $\sigma(A) := \mathbb{C} \setminus \rho(A)$ stand, respectively, for the resolvent set and the spectrum of $A$, a densely defined linear operator in a Banach (Hilbert) space $X$.

In order to establish notation and nomenclature within the theory of $C_0$-semigroups, we suggest a quick read of the Appendix A. We also suggest some books for the introductory study of semigroups: [7, 29, 35, 47, 51, 52, 62].

**Theorem 1** (Theorem I1.10 in [29])**.** A $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $X$ is uniformly exponentially stable if, and only if, its generator $-A$ satisfies $\mathbb{C}_- \subset \rho(A)$ and

$$\sup_{\text{Re}\lambda<0} \|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} < \infty.$$

**Remark 1.** A uniform bound for the resolvent is not sufficient to ensure exponential stability on general Banach spaces; see Counterexample IV.2.7 in [29].

The works of Lebeau [39, 40] and Burq [19] raised the question of what is the relation between the growth rates for norm of the resolvent and the decay rates of the norm of semigroup orbits. More precisely, assuming a spectral condition under the generator, $\sigma(A) \subset \mathbb{C}_+$ in (1), and $\|R(is, A)\|_{\mathcal{L}(X)} \to \infty$ as $|s| \to \infty$, then $(T(t))_{t\geq 0}$ is not exponentially stable and one typically obtains other asymptotic behavior.

Until 2010, much attention has been paid to polynomial decay rates of the norm of semigroup orbits. In the work of [9], Bátkai, Engel, Prüss and Schnaubelt proved that for uniformly bounded semigroups, a polynomial growth rate of the norm of the resolvent implies a specific polynomial decay rate for classical solutions to (1).

**Theorem 2** (Theorem 3.5 in [9])**.** Let $(T(t))_{t\geq 0}$ be a *bounded* semigroup on a Banach space $X$ with infinitesimal generator $-A$ such that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Let $s \geq 0$ and set

$$M(s) := \sup_{|\xi|\leq s} \|(i\xi + A)^{-1}\|_{\mathcal{L}(X)}. \tag{2}$$

If there exist constants $C, \beta > 0$ such that $M(s) \leq C(1+s)^{\beta}$, then for each $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ such that for each $t > 0$,

$$\|T(t)(1 + A)^{-1}\|_{\mathcal{L}(X)} \leq C_\varepsilon t^{-\frac{1}{\beta}+\varepsilon}. \tag{3}$$

Liu and Rão obtained in [38] sharper estimates than those given by (3) in case $X$ is a Hilbert space.

**Theorem 3** (Theorem 2.1 in [38])**.** Let $X$ be a Hilbert space, and let $(T(t))_{t\geq 0}$, $A$ and $M$ be as in the statement of Theorem 2. Then, if there exist constants $C, \beta > 0$ such that $M(s) \leq C(1+s)^{\beta}$, then

$$\|T(t)(1 + A)^{-1}\|_{\mathcal{L}(X)} = O\left(\frac{\log(2+t)^{\frac{1}{\beta}+1}}{t^{\frac{1}{\beta}}}\right),\ t \to \infty.$$

In [12], Batty and Duyckaerts extended this correspondence to the case where the resolvent growth is arbitrary; they were also able to reduce the loss $\varepsilon > 0$ (see relation (3)) to a logarithmic scale.

**Theorem 4** (Theorem 1.5 in [12]). **($M_{\log}$-Theorem)** Let $(T(t))_{t\geq 0}$ be a *bounded* semigroup on a Banach space $X$ with infinitesimal generator $-A$ such that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Let $M : (0,\infty) \to (0,\infty)$ be given by (2); then, there exists a positive constant $C$ such that

$$\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} = O\left(\frac{1}{M_{\log}^{-1}(Ct)}\right), \ t \to \infty, \tag{4}$$

where $M_{\log}^{-1}$ is the right inverse of $M_{\log}(s) := M(s)(\log(1+M(s)) + \log(1+s))$. In particular, if $M(s) \leq C(1+s)^{\beta}$ for any $\beta > 0$ and $C > 0$, then

$$\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} = O\left(\frac{\log(t)}{t}\right)^{1/\beta}, \ t \to \infty.$$

**Table 1:** $M_{\log}$

| $M$ | $1/M^{-1}(ct)$ | $1/M_{\log}^{-1}(Ct)$ |
|---|---|---|
| $C$ | | $e^{-Cs}$ |
| $\log s$ | $e^{-ct}$ | $e^{-Ct^{1/2}}$ |
| $(1+s)^{\beta}$ | $t^{-1/\beta}$ | $t^{-1/\beta}\log(t)^{1/\beta}$ |
| $e^{cs}$ | $\dfrac{1}{\log t}$ | $\dfrac{1}{\log t}$ |

Source: Compiled by the author.

Still in [12], Batty and Duyckaerts conjectured that the logarithmic correction may be dropped in the case of Hilbert spaces, but one cannot expect rates better than (4) for general Banach spaces. Then, Borichev and Tomilov partially solved the conjecture in [18]; namely, they have shown that in case of a power-law resolvent growth, the logarithmic correction loss is sharp on general Banach spaces (it is worth noting that this optimality is also valid for sub-polynomial functions, as recently shown by Dubruyne and Seifert in [27]), but that it is not necessarily true on Hilbert spaces.

**Theorem 5** (Theorem 2.4 in [18]). Let $(T(t))_{t\geq 0}$ be a *bounded* $C_0$-semigroup on a Hilbert space $X$ with generator $-A$ so that $i\mathbb{R} \subset \rho(A)$. Then, given $\beta > 0$, the following assertions are equivalent:

1. $\|(is+A)^{-1}\|_{\mathcal{L}(X)} = O(|s|^{\beta})$, $|s| \to \infty$.
2. $\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} = O(t^{-1/\beta})$, $t \to \infty$.

By seeking to answer the conjecture of Batty and Duyckaerts for a larger class of functions than power-law type, Batty, Chill and Tomilov have obtained in [14] the following result.

**Theorem 6** (Theorem 1.1 in [14]). Let $(T(t))_{t\geq 0}$ be a *bounded* $C_0$-semigroup on a Hilbert space $X$ with generator $-A$ so that $i\mathbb{R} \subset \rho(A)$. Let $\beta > 0$ and $b > 0$.

1. The following assertions are equivalent:

(a) $\|(is+A)^{-1}\|_{\mathcal{L}(X)} = O(|s|^{\beta}\log(|s|)^{-b})$, $|s| \to \infty$.

(b) $\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} = O(t^{-1/\beta}\log(t)^{-b/\beta})$, $t \to \infty$.

2. If

$$\|(is+A)^{-1}\|_{\mathcal{L}(X)} = O(|s|^{\beta}\log(|s|)^{b}), \quad |s| \to \infty,$$

then for each $\varepsilon > 0$,

$$\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} = O(t^{-1/\beta}\log(t)^{b/\beta+\varepsilon}), \quad t \to \infty$$

**Remark 2.** Theorem 6 remains valid when replacing $|s|^{\beta}\log(|s|)^{-b}$ with a function of the type $|s|^{\beta}\ell(|s|)^{-1}$, where $\ell$ is an increasing and slowly varying function (see Theorem 5.6 in [14]).

Finally, Rozendaal, Seifert and Stahn in [56] have extended the previous results to a larger class of functions, namely, those of positive increase: a continuous increasing function $M : (0,\infty) \to (0,\infty)$ is said to be of *positive increase* if there exist positive constants $\alpha > 0$, $c \in (0,1]$ and $s_0 > 0$ such that

$$\frac{M(\lambda s)}{M(s)} \geq c\lambda^{\alpha}, \quad \lambda \geq 1, s \geq s_0.$$

**Theorem 7** (Theorem 3.2 in [56])**.** Let $(T(t))_{t\geq 0}$ be a *bounded* $C_0$-semigroup on a Hilbert space $X$, with generator $-A$, and let $M : (0,\infty) \to (0,\infty)$ be a function of positive increase. The following assertions are equivalent:

1. $i\mathbb{R} \subset \rho(A)$ and $\|(is+A)^{-1}\|_{\mathcal{L}(X)} = O(M(|s|))$, $|s| \to \infty$.

2. $\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} = O(M^{-1}(t))$, $t \to \infty$.

So far we have presented a compilation of the main results for the situation in which $A$ has only singularity at infinity, i.e, $\|R(is,A)\|_{\mathcal{L}(X)} \to \infty$ as $|s| \to \infty$; there are also some other works in the literature that study the decay rates of $C_0$-semigroup for this situation, for example [9, 23, 50, 57, 63]. Nevertheless, there are many other works that study decay rates for the situation in which $A$ has a singularity at zero [14, 22, 55, 56], or even when $A$ has singularity at zero and infinity [14, 42, 55, 56]. In the present work, we consider all of these scenarios.

Until this point, we have presented some of the main results of the asymptotic theory of bounded $C_0$-semigroups. Nevertheless, there are many natural classes of examples where the norm of the resolvent of the generator grows with a power-law rate as $|s| \to \infty$, for example, but the semigroup is not uniformly bounded, or where it is unknown whether the semigroup is in fact bounded. For example, this happens with some concrete partial differential equations, like the standard wave equation with periodic boundary conditions; here, uniform boundedness fails (see [55] for a more complete discussion on these examples).

The currently available literature on polynomial or other types of decay deals almost exclusively with uniformly bounded semigroups. To the best of our knowledge, the following result due to Bátkai, Engel, Prüss and Schnaubelt is the first in the literature that proves polynomial decay for not necessarily bounded semigroups. In what follows, $\omega_0(T) := \lim_{t\to\infty}(\log \|T(t)\|_{\mathcal{L}(X)})/t$.

**Theorem 8** (Proposition 3.4 in [9])**.** Let $(T(t))_{t\geq 0}$ be a semigroup defined in a Banach space $X$ with generator $-A$ such that there exists $\beta > 0$ so that the map $\lambda \mapsto (\lambda + A)^{-1}(1 + A)^{-\beta}$, with $\text{Re } \lambda > \omega_0(T)$, has a bounded holomorphic extension to $\text{Re } \lambda \geq 0$. Then, there exists a positive constant $C_{n,\delta}$ such that for each $n \in \mathbb{N}$, $\delta \in (0, 1]$ and $t > 0$,

$$\|T(t)(1 + A)^{-\beta(n+1)-1-\delta}\|_{\mathcal{L}(X)} \leq C_{n,\delta}t^{-n}.$$

Then, by using geometrical properties of the underlying Banach space (like its Fourier type), Rozendaal and Veraar have shown the following result (see Theorem 4.9 in [55]).

**Theorem 9.** Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup with generator $-A$ defined in a Banach space $X$ with Fourier type $p \in [1, 2]$, and let $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). Suppose that $\overline{\mathbb{C}_-} \subset \rho(A)$ and that there exist $\beta, C \geq 0$ such that $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C(1 + |\lambda|)^{\beta}$ for each $\lambda \in \overline{\mathbb{C}_-}$. Let $\tau > \beta + 1/r$; then, for each $\rho \in \left[0, \frac{\tau-1/r}{\beta} - 1\right)$, there exists $C_\rho \geq 0$ such that for each $t \geq 1$,

$$\|T(t)(1 + A)^{-\tau}\|_{\mathcal{L}(X)} \leq C_\rho t^{-\rho}. \tag{5}$$

In case $X$ is a Hilbert space (which corresponds to $p = p' = 2$ and $r = \infty$), they have shown the following result.

**Corollary 1** (Theorem 1.1 in [55])**.** Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup with generator $-A$ defined in a Hilbert space $X$. Suppose that $\overline{\mathbb{C}_-} \subset \rho(A)$ and that there exist $\beta, C \geq 0$ such that $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C(1 + |\lambda|)^{\beta}$ for each $\lambda \in \overline{\mathbb{C}_-}$. Then, for each $\tau \geq \beta$ there exists a positive constant $C_\tau$ such that for each $t \geq 1$,

$$\|T(t)(1 + A)^{-\tau}\|_{\mathcal{L}(X)} \leq C_\tau t^{1-\tau/\beta}.$$

## Main results

By using the techniques developed in [55] that involve Fourier Multipliers and also inspired by the techniques developed by Batty, Chill and Tomilov in [14] that involve functional calculus of sectorial operators, we have obtained decay rates for $C_0$-semigroups as defined in the statement of Theorem 9 by assuming that the norm of the resolvent of the generator behaves as a function of type $|s|^{\beta} \log(|s|)^{b}$ as $|s| \to \infty$ (a particular example of a *regularly varying* function). Under these assumptions on the resolvent and without the assumption of boundedness of the semigroup, to the best knowledge of the authors, these estimates are new and constitute one of the main results in this work.

**Theorem 10.** Let $\beta > 0$, $b \geq 0$ and let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup defined in the Banach space $X$ with Fourier type $p \in [1, 2]$, with $-A$ as its generator. Suppose that $\overline{\mathbb{C}_-} \subset \rho(A)$ and that for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \le 0$,

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim (1+|\lambda|)^{\beta}(\log(2+|\lambda|))^{b}.$$

Let $r \in [1,\infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$, and let $\tau > 0$ be such that $\tau > \beta + \frac{1}{r}$. Then, for each $\delta > 0$, there exist constants $c_{\delta,\tau} \in [0,\infty)$ and $t_0 \ge 1$ such that for each $t \ge t_0$,

$$\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \le c_{\delta,\tau} t^{1-\frac{\tau - r^{-1}}{\beta}} \log(1+t)^{\frac{b(\tau - r^{-1})}{\beta} + \frac{1+\delta}{r}}. \tag{6}$$

The next result is the particular case of Theorem 10 where $X$ is a Hilbert space.

**Corollary 2.** Let $\beta$, $b$, $A$ and $(T(t))_{t\ge 0}$ be as in the statement of Theorem 10 and let $X$ be a Hilbert space. Let $\tau > \beta$. Then, there exist constants $c_\tau \ge 0$ and $t_0 \ge 1$ such that for each $t \ge t_0$,

$$\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \le c_\tau t^{1-\frac{\tau}{\beta}} \log(1+t)^{\frac{b\tau}{\beta}}. \tag{7}$$

Note that in case $b = 0$, one obtains from Theorem 10 the following result.

**Corollary 3.** Let $\beta > 0$ and let $(T(t))_{t\ge 0}$ be a $C_0$-semigroup defined in a Banach space $X$ with Fourier type $p \in [1,2]$, whose generator is given by $-A$. Suppose that $\overline{\mathbb{C}_-} \subset \rho(A)$ and that for each $\lambda \in \overline{\mathbb{C}_-}$, $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim (1+|\lambda|)^{\beta}$. Let $r \in [1,\infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$, and let $\tau > 0$ be such that $\tau > \beta + \frac{1}{r}$. Then, for each $\delta > 0$ and each $\rho \in [0, 1 - (\tau - r^{-1})/\beta]$, there exist constants $c_{\delta,\tau} \in [0,\infty)$ and $t_0 \ge 1$ such that for each $t \ge t_0$,

$$\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \le c_{\delta,\tau} t^{-\rho} \log(1+t)^{\frac{1+\delta}{r}}. \tag{8}$$

**Remark 3.**

1. Note that relation (8) presents a sharper bound to $\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)}$ than the one presented in relation (5); namely, in relation (8), the exponent in $t$ is precisely the unattained upper-bound of $\rho$ in Theorem 9. This partially solves the question posed by Rozendaal and Veraar in [55] if whether (5) is valid for $\rho = \dfrac{\tau - 1/r}{\beta} - 1$ or not, given that the bound presented in (8) has a logarithmic correction. Note that if one lets $b = 0$ in (7), then Corollary 2 coincides with Corollary 1 for $\tau > \beta$.

2. We also note that the power law in the logarithmic factor presented in (8) depends on the geometry of the space (that is, its Fourier type): a greater value of $r$ (which means that the space is "closer" to a Hilbert space) results in a lesser logarithm correction.

3. Furthermore, such logarithm factor is not optimal, even in case $b = 0$. Namely, it is possible to obtain a version of Proposition 2.2.1 and Theorem 2.2.1 (these two results are central in the proof of Theorem 10, which consists of "eliminating" the operator $\log(2+A)^{-b\frac{(\tau - r^{-1})}{\beta} - \frac{1+\delta}{r}}$ from $\|T(t)(1+A)^{-\tau}\log(2+A)^{-b\frac{(\tau-r^{-1})}{\beta} - \frac{1+\delta}{r}}\|_{\mathcal{L}(X)}$), where $\log(1+t)^{\frac{1+\delta}{r}}$ is replaced by $\log(1+t)\log(1+\log(1+t))^{\frac{1+\delta}{r}}$; we do not present a proof of this statement, given that the

techniques discussed here seem to be far from optimal. We just stress that such replacement is possible given that the functions $\log(1+t)$ and $\log(1+\log(1+t))$ are both complete Bernstein functions (see Definition E.1.2).

We have also obtained similar decay rates for the situation in which $0 \in \sigma(A)$. In the following result, as in Theorem 10, let us assume that the norm of the resolvent grows with order $|s|^{-\alpha}\log(1/|s|)^a$ as $|s| \to 0$ and with order $|s|^\beta \log(|s|)^b$ as $|s| \to \infty$.

**Theorem 11.** Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup defined in the Banach space $X$ with Fourier type $p \in [1,2]$, with $-A$ as its generator. Suppose $A$ injective, $\overline{\mathbb{C}_-} \setminus \{0\} \subset \rho(A)$ and that there exist $\alpha \geq 1$, $\beta, a, b > 0$ and positive constants $C_1$ and $C_2$ such that

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq \begin{cases} C_1|\lambda|^{-\alpha}\log(1/|\lambda|)^a, & |\lambda| \leq 1 \\ C_2|\lambda|^\beta \log(|\lambda|)^b, & |\lambda| \geq 1, \end{cases} \tag{9}$$

with $\lambda \in \overline{\mathbb{C}_-} \setminus \{0\}$. Let $\sigma, \tau$ be such that $\sigma > \alpha - 1$ and $\tau > \beta + 1/r$. Then, for each $\rho \in \left[0, \min\left\{\frac{\sigma+1}{\alpha} - 1, \frac{\tau - r^{-1}}{\beta} - 1\right\}\right]$ and each $\delta > 1 - 1/r$, where $r \in [1,\infty]$ is such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$, there exist $C_{\delta,\rho} > 0$ and $t_0 \geq 1$ so that for each $t \geq 1$,

$$\|T(t)A^\sigma(1+A)^{-\sigma-\tau}\|_{\mathcal{L}(X)} \leq C_{\delta,\rho} t^{-\rho}\log(1+t)^{c(\lceil\rho\rceil+1)+1/r+\delta}, \tag{10}$$

with $c = \max\{a, b\}$.

**Remark 4.** By assuming (9), it is natural to let $\alpha \geq 1$. Indeed, suppose that $\alpha \in [0,1)$; then,

$$\frac{1}{\operatorname{dist}(\lambda, \sigma(A))} \leq \|R(\lambda, A)\|_{\mathcal{L}(X)} \lesssim |\lambda|^{-\alpha}(\log(1/|\lambda|))^a.$$

Since $0 \in \sigma(A)$, it follows that $\operatorname{dist}(\lambda, \sigma(A)) \geq |\lambda|$, so $|\lambda|^{\alpha-1}(\log(1/|\lambda|))^{-a} \lesssim C$. On the other hand, since $\alpha \in [0,1)$, it follows that $\lim\limits_{\lambda\to 0}|\lambda|^{\alpha-1}(\log(1/|\lambda|))^{-a} = \infty$, from which follows that $\alpha \geq 1$ if $0 \in \sigma(A)$.

**Remark 5.** In case $X$ is a Hilbert space (that is, when $p = 2$), one has $r = \infty$, and so (10) is just

$$\|T(t)A^\sigma(1+A)^{-\sigma-\tau}\|_{\mathcal{L}(X)} \leq C_{\delta,\rho} t^{-\rho}\log(1+t)^{c(\lceil\rho\rceil+1)+\delta}.$$

In case $a = b = 0$, one has the following result.

**Corollary 4.** Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup defined in the Banach space $X$ with Fourier type $p \in [1,2]$, with $-A$ as its generator. Suppose $A$ injective, $\overline{\mathbb{C}_-} \setminus \{0\} \subset \rho(A)$ and that there exist $\alpha \geq 1$, $\beta > 0$ and positive constants $C_1$ and $C_2$ such that

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq \begin{cases} C_1|\lambda|^{-\alpha}, & |\lambda| \leq 1 \\ C_2|\lambda|^\beta, & |\lambda| \geq 1, \end{cases}$$

with $\lambda \in \overline{\mathbb{C}_-} \setminus \{0\}$. Let $\sigma, \tau$ be such that $\sigma > \alpha - 1$ and $\tau > \beta + 1/r$. Then, for each $\rho \in \left[0, \min\left\{\frac{\sigma+1}{\alpha} - 1, \frac{\tau - r^{-1}}{\beta} - 1\right\}\right]$ and each $\delta > 1 - 1/r$, where $r \in [1,\infty]$ is such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$, there exist $C_{\rho,\delta} > 0$ and $t_0 \geq 1$ so that for each $t \geq 1$,

$$\|T(t)A^{\sigma}(1+A)^{-\sigma-\tau}\|_{\mathcal{L}(X)} \leq C_{\delta,\rho} t^{-\rho} \log(1+t)^{1/r+\delta}. \tag{11}$$

**Remark 6.** 1. Note that relation (11) improves the estimates obtained by Rozendaal and Veraar in [55] (see Theorem 4.9 in [55]). More precisely, we show that it is possible to replace the factor $t^{\varepsilon}$, with $\varepsilon$ any positive number, by $\log(1+t)^{1/r+\delta}$ in their estimate.

2. Note that even in case $X$ is a Hilbert space, the estimate obtained in Corollary 4.11 in [55] still has a factor $t^{\varepsilon}$; Corollary 4 shows that it is possible to replace it by $\log(1+t)^{\delta}$, with $\delta > 1$.

3. Corollary 4 partially solves the question posed by Rozendaal and Veraar in [55], if whether estimate (11) is valid for $\rho = \min\left\{\dfrac{\sigma+1}{\alpha} - 1, \dfrac{\tau - r^{-1}}{\beta} - 1\right\}$ or not, given that the bound presented in (11) has a logarithmic factor.

We also studied the situation in which there is only a singularity at zero (but not at infinity); this situation is also discussed in [14, 22, 55, 56]. As in [22, 55, 56], we suppose that the $C_0$-semigroup is asymptotically analytic on the Banach space $X$ (see Definition 2.4.1 and Section 2.4 for more details).

**Theorem 12.** Let $A$ be an injective sectorial operator defined in the Banach space $X$ such that $-A$ generates $(T(t))_{t\geq 0}$, an asymptotically analytic $C_0$-semigroup on $X$. Suppose that there exist $\alpha \geq 1$ and $a > 0$ such that for each $\lambda \in \overline{\mathbb{C}_-} \setminus \{0\}$,

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim |\lambda|^{-\alpha}(\log(1/|\lambda|))^{a}. \tag{12}$$

Let $\sigma > \alpha - 1$. Then, for each $\delta > 0$ there exists $c_{\delta,\sigma} > 0$ such that for each $t \geq 1$,

$$\|T(t)A^{\sigma}(1+A)^{-\sigma}\|_{\mathcal{L}(X)} \leq c_{\delta,\sigma} t^{1-\frac{\sigma+1}{\alpha}} \log(1+t)^{\frac{a(\sigma+1)}{\alpha}+1+\delta}. \tag{13}$$

In case $a = 0$, the estimate presented in Theorem 12 improves the one presented in Theorem 4.16 in [55]. More precisely, as in the previous cases, we have shown that it is possible to replace the factor $t^{\varepsilon}$ by $\log(1+t)^{1+\varepsilon}$, where $\varepsilon > 0$ (see equation (14)).

**Corollary 5.** Let $A$, $X$ and $(T(t))_{t\geq 0}$ as in Theorem 12. Suppose that $\overline{\mathbb{C}_-} \setminus \{0\} \subset \rho(A)$ and that there exists $\alpha \geq 1$ such that $\|R(\lambda, A)\|_{\mathcal{L}(X)} \lesssim |\lambda|^{-\alpha}$ for each $\lambda \in \overline{\mathbb{C}_-} \setminus \{0\}$. Let $\sigma > \alpha - 1$. Then, for each $\delta > 0$, there exists $c_{\delta,\sigma} > 0$ such that for each $t \geq 1$,

$$\|T(t)A^{\sigma}(1+A)^{-\sigma}\|_{\mathcal{L}(X)} \leq c_{\delta,\sigma} t^{1-\frac{\sigma+1}{\alpha}} \log(1+t)^{1+\delta}. \tag{14}$$

# Part II

The existence of orbits of operator semigroups that converge to zero arbitrarily slowly has been studied by many authors in the last two decades (see [2, 4, 43, 44, 45, 46] and references therein). Pioneering results were established by Müller in the discrete case [43, 44, 45]. Namely, given any $\varepsilon > 0$ and a sequence of real numbers $(a_n)_{n\geq 1}$ satisfying $|a_n| \leq 1$ for all $n \geq 1$ and $\lim_{n\to\infty} a_n = 0$, a known result in [43] states that if $T$ is a bounded operator on a complex Banach space $X$ with spectral radius equal to 1, then there exists a normalized $x \in X$ such that

$$\|T^n x\| \geq (1-\varepsilon)|a_n|, \quad \forall\, n \geq 1.$$

With respect to the continuous case, Müller and Tomilov [46] have established several analogous results.

**Theorem 13** (Theorem 5.3 in [46])**.** Let $(T(t))_{t\geq 0}$ be a weakly stable $C_0$-semigroup on a Hilbert space $X$ (i.e, it converges weakly to zero as $t \to \infty$) such that $\omega_0(T) = 0$. Let $g : \mathbb{R}_+ \longrightarrow (0,\infty)$ be a bounded function such that $\lim_{t\to\infty} g(t) = 0$ and let $\varepsilon > 0$. Then, there exists $x \in X$ so that $\|x\| < \sup_{t\geq 0}\{g(t)\} + \varepsilon$ and

$$|\langle T(t)x, x\rangle| > g(t), \quad \forall t \geq 0. \tag{15}$$

# Contextualition

Let $X$ a separable complex Hilbert space and let $A$ be a pure point negative self-adjoint operator and let $(x_n)_{n\geq 1}$ be the normalized eigenvectors of $A$, say $Ax_n = \lambda_n x_n$, so that $(\lambda_n)_{n\geq 1} \subset (-\infty, 0)$ are the corresponding eigenvalues which satisfy $\limsup_{n\to\infty} \lambda_n = 0$. For $x = \sum_{j=1}^{N} b_j x_j \in X$, one has

$$\|e^{tA}x\| = \left\| \sum_{j=1}^{N} b_j e^{t\lambda_j} x_j \right\| \leq N \max_{1\leq j\leq N} |b_j| e^{\lambda t},$$

with $\lambda = \max_{1\leq j\leq N} \lambda_j < 0$, that is, for these initial conditions, one has that the orbits go to zero exponentially fast. Due to the abstract results by Müller and Tomilov [46] (see also Theorem 13), given $\beta : \mathbb{R}_+ \to (0,\infty)$ with

$$\lim_{t\to\infty} \beta(t) = \infty,$$

there exists $x \in X$ such that

$$\limsup_{t\to\infty} \beta(t)\|e^{tA}x\| = \infty,$$

since $0 \in \sigma(A)$ in this case. In this specific context, in Theorem 14 i), we refine this result in the following sense: we show how it is possible to perturb any initial condition (in terms of the spectral structure of the generator) to explicitly provide a new initial condition whose orbit goes to zero slower than any prescribed speed, at least for a sequence of time going to infinity, and in item ii), we obtain a version of such result in terms of perturbations of the infinitesimal generator.

**Theorem 14.** Let $X$ a Hilbert space and let $\beta : \mathbb{R}_+ \to (0,\infty)$ be a strictly increasing onto function, so

$$\lim_{t\to\infty} \beta(t) = \infty\,.$$

i) Let $A$ be a pure point negative self-adjoint operator whose eigenvalues $(\lambda_n)_{n\geq 1} \subset (-\infty, 0)$ satisfy $\limsup_{n\to\infty} \lambda_n = 0$. For each $x \in X$, there exists a sequence $(x_k)_{k\geq 1} \subset X$ that converges to $x$ such that, for each $k$,

$$\limsup_{t\to\infty} \beta(t)\|e^{tA}x_k\| = \infty\,.$$

ii) Let $A$ be a negative bounded self-adjoint operator. Then, for each $0 \neq x \in X$, there exists a sequence $(A_k)_{k\geq 1}$ of negative bounded pure point self-adjoint operators that strongly converges to $A$ such that, for each $k$,

$$\limsup_{t\to\infty} \beta(t)\|e^{tA_k}x\| = \infty\,.$$

**Remark 7.**

i) Let us describe the vectors $x_k$ in the statement of Theorem 14 i). Write $x = \displaystyle\sum_{l=1}^{\infty} b_l x_l$ and, for each subsequence $(\lambda_{j_l})_{l\geq 1}$ of eigenvalues of $A$ with $\lambda_{j_l} \uparrow 0$ and $\sum_{l=1}^{\infty} \frac{1}{\beta(1/|\lambda_{j_l}|)} < \infty$, one may pick

$$x_k = \sum_{l=1}^{k} b_l x_l + \sum_{l=k+1}^{\infty} \frac{1}{\sqrt{\beta(1/|\lambda_{j_l}|)}}\, x_{j_l}\,.$$

ii) It follows from the Spectral Theorem and dominated convergence that for each $x \in X$,

$$\lim_{t\to\infty} \|e^{tA}x\|^2 = \mu_x^A(\{0\}) + \lim_{t\to\infty} \int_{\mathbb{R}_-\setminus\{0\}} e^{2tw}\mathrm{d}\mu_x^A(w) = \mu_x^A(\{0\}) = \|E^A(\{0\})x\|^2\,.$$

Therefore, $e^{tA}$ is stable (i.e, all the orbits go to zero as $t\to\infty$) if, and only if, $0 \notin \sigma(A)$. Hence, Theorem 14 i) is particularly interesting in this case. Note that a well-known example of injective operator that satisfies the hypotheses of this theorem is the Hydrogen atom model restricted to its point subspace; see Chapter 11 in [26] for details.

iii) If $0 \notin \sigma(A)$, then there exists $\delta > 0$ such that for each $\psi \in X$ and each $t > 0$, by the Spectral Theorem,

$$\|e^{tA}x\|^2 = \int_{-\infty}^{-\delta} e^{2tw}\mathrm{d}\mu_x^A(w) \leq e^{-2\delta t}\|x\|^2,$$

that is, all orbits go to zero exponentially fast as $t \to \infty$.

iv) Given any nonzero initial condition $x \in X$, Theorem 14 ii) says that we may always (strongly) perturb the negative bounded self-adjoint infinitesimal generator $A$ so that the orbit of $\psi$ goes to zero slower than a prescribed speed $\beta(t)$, at least for a sequence of time going to infinity.

Let us now consider unitary evolution groups. Given a self-adjoint operator $A$ in $X$, recall that $\mathbb{R} \ni t \mapsto e^{-itA}$ is a one-parameter strongly continuous unitary evolution group and $(e^{-itA}x)_{t\in\mathbb{R}}$ is the unique solution to the Schrödinger equation

$$\begin{cases} \partial_t x = -iAx, \quad t \in \mathbb{R}, \\ x(0) = x \in \mathcal{D}(A). \end{cases}$$

A standard (and important in quantum mechanics) dynamical quantity that probes the large time behavior of $e^{-itA}x$ is the so-called (time-average) *return probability*, given by the law

$$W_x^A(t) := \frac{1}{t}\int_0^t |\langle e^{-isA}x, x\rangle|^2 \, \mathrm{d}s.$$

By the Spectral Theorem and Wiener's Lemma (Theorem 2.2 in [34]; see also [26, 6]),

$$\lim_{t\to\infty} W_x^A(t) = \sum_{\lambda\in\mathbb{R}} |\mu_x^A(\{\lambda\})|^2;$$

in particular, if $A$ has purely continuous spectrum, then

$$\lim_{t\to\infty} W_x^A(t) = 0.$$

Our next result, Theorem 15, ensures the existence of orbits, under each spectrally continuous unitary evolution group, with arbitrarily slow power-law convergence rates.

**Theorem 15.** Let $A$ be a self-adjoint operator with purely continuous spectrum. Then, there exists $x \in X$ such that for every $\varepsilon > 0$,

$$\limsup_{t\to\infty} t^{\varepsilon}\, W_x^A(t) = \infty.$$

**Remark 8.** Although the existence of orbits of operator semigroups that slowly decay is a subject extensively studied in the literature, to the best of our knowledge, Theorem 15 is the first general result on slow dynamics for (spectrally continuous) unitary evolution groups; see also Example 1.2 and Remark 1.1 in [5].

Stimulated by results due to Müller and Tomilov in [46], our main goal here is to obtain orbits of self-adjoint semigroups and unitary groups (in this case, for the (time-average) return probability) that converge slowly to zero. More precisely, by exploring local dimensional properties of self-adjoint operators, we show explicitly how it is possible to perturb initial conditions, or generators, to obtain orbits of self-adjoint semigroups that converge to zero arbitrarily slowly, at least for a sequence of time going to infinity (see Theorem 14 ahead). We also obtain a result about slow power-law decaying rates of the return probability (see the definition ahead) of unitary evolution groups with purely continuous spectrum (Theorem 15).

As an application of the arguments developed here, we compute (Baire) generically the local dimensions of systems with purely continuous spectrum, as the following theorem shows.

**Theorem 16.** Let $A$ be a bounded self-adjoint operator with purely continuous spectrum. Then, there exists a generic set $\mathcal{M} \subset X$ such that for each $x \in \mathcal{M}$, the set

$$\mathcal{J}_x := \big\{ w \in \sigma(A) \mid d^-_{\mu^A_x}(w) = 0 \ \text{ and } \ d^+_{\mu^A_x}(w) = \infty \big\}$$

is generic in $\sigma(A)$.

Using the Theorem 16 we show that the time-average (quantum) return probability, of (Baire) generic states of systems with purely absolutely continuous spectrum, has an oscillating behavior between a (maximum) fast power-law decay and a (minimum) slow power-law decay .

**Theorem 17.** Let $A$ be a bounded self-adjoint operator with purely absolutely continuous spectrum. Then, the set of $x \in X$ such that for each $k \in \mathbb{N}$,

$$\liminf_{t\to\infty} t^{1-1/k} W^A_x(t) = 0 \quad \textit{and} \quad \limsup_{t\to\infty} t^{1/k} W^A_x(t) = \infty,$$

is generic in $X$, i.e., it contains a dense $G_\delta$ subset of $X$.

## Organization

This thesis is divided into three chapters. In **Chapter 1**, we review elements of operator theory, functional calculus for sectorial operators, and present some properties of the logarithm operator that will help us prove the results of Part I. In **Chapter 2**, we present some aspects of the geometry of Banach spaces: Fourier types, Fourier multipliers, and we prove the estimates obtained for $C_0$-semigroups that were presented in Part I. In **Chapter 3**, we present the results obtained for self-adjoint semigroups and unitary evolution groups presented in Part II. We also provide some appendices composed of basic results from the theory of $C_0$-semigroups (Appendix **A**), spectral theory for self-adjoint operators (Appendix **B**), class of complete Bernstein functions (Appendix **E**) and two appendices (**C** and **D**) with the proofs of some results of **Chapter 2** which were omitted.

# Chapter 1

# Preliminaries

## 1.1 Operator Theory

In this chapter we present some basic results of operator theory on Banach spaces. Throughout this chapter, $X$ always denotes a complex Banach space.

**Definition 1.1.1.** Let $A : \mathcal{D}(A) \to X$ be a linear operator on X. The resolvent set of $A$ is given by

$$\rho(A) := \{\lambda \in \mathbb{C} \mid (\lambda - A)^{-1} \in \mathcal{L}(X)\}$$

and its spectrum by

$$\sigma(A) := \mathbb{C} \setminus \rho(A).$$

**Lemma 1.1.1.** (a) (Resolvent identity) For each $\lambda, \mu \in \rho(A)$, the identity

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A)$$

holds.

(b) For each $\lambda \in \rho(A)$, one has

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \geq \frac{1}{\operatorname{dist}(\lambda, \sigma(A))}.$$

(c) The function $\Psi : \rho(A) \to \mathcal{L}(X, (\mathcal{D}(A), \|\cdot\|_A))$, defined by $\Psi(\lambda) := R(\lambda, A)$ is infinitely differentiable and for each $n \in \mathbb{N}$,

$$\left(\frac{d^n}{d\lambda^n}\Psi\right)(\lambda) = (-1)^n n! \Psi(\lambda)^{n+1},$$

where $\|x\|_A := \|x\| + \|Ax\|$ for each $x \in \mathcal{D}(A)$.

*Proof.* See Theorem 1.13 in [62]. □

## 1.2 Adjoint Operators (Hilbert adjoint)

**Definition 1.2.1.** Let $(X, \langle\cdot,\cdot\rangle)$ a Hilbert space and let $A : \mathcal{D}(A) \subset X \to X$ be a densely defined linear operator. The (Hilbert) adjoint of $A$ is the operator $A^*$ with domain

$$\mathcal{D}(A^*) := \{y \in X \mid \exists\ z \in X; \langle y, Tx\rangle = \langle z, x\rangle,\ \forall\ x \in \mathcal{D}(A)\},$$

with $z = T^*y$.

**Definition 1.2.2.** A densely defined linear operator $A : \mathcal{D}(A) \to X$ on $X$ Hilbert space is called self-adjoint if $A^* = A$ (in particular, $\mathcal{D}(A^*) = \mathcal{D}(A)$).

**Theorem 1.2.1.** Let $X$ be a Hilbert space and let $A$ be densely defined and symmetric. The following assertions are equivalent.

a) $A^* = A$;

b) $\sigma(A) \subset \mathbb{R}$;

c) Let $\rho(A) \cap \mathbb{R} \neq \emptyset$. Then, $A$ is self-adjoint.

d) Let $A$ be self-adjoint. Then we have, for each $\lambda \notin \mathbb{R}$,

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{|\mathrm{Im}\lambda|}.$$

*Proof.* See Theorem 4.7 in [62] or Theorem 2.2.17 in [26]. □

Here we present some examples of self-adjoint operators in the context of differential equations; for more details, see Examples 4.8 a) e 4.8 d) in [62].

**Example 1.2.1.** a) Let $X = L^2(\mathbb{R})$ and define $A : \mathcal{D}(A) \to X$ by the law $Af = i\frac{df}{dx}$, with $\mathcal{D}(A) = W^{1,2}(\mathbb{R})$. Then $A$, is a self-adjoint operator with $\sigma(A) = \mathbb{R}$.

b) Let $X = L^2(\mathbb{R}^n)$ and let $A = \Delta$, with $\mathcal{D}(A) = W^{2,2}(\mathbb{R}^n)$. Then, $A$ is a self-adjoint operator with $\sigma(A) = (-\infty, 0]$.

**Definition 1.2.3.** Let $A$ be a self-adjoint operator. $A$ is called negative if for each $x \in \mathcal{D}(A)$,

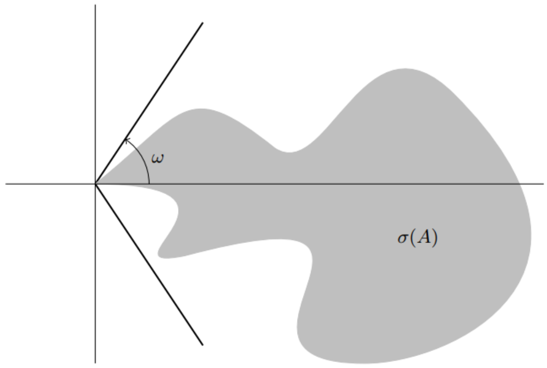
$$\langle x, Ax\rangle \leq 0.$$

## 1.3 Sectorial Operators

For each $\omega \in (0, \pi)$, set $S_\omega := \{z \in \mathbb{C} \mid 0 < |\arg(z)| < \omega\}$; set also $S_0 := (0, \infty)$.

**Definition 1.3.1.** A linear operator $A : \mathcal{D}(A) \subset X \to X$ is called **sectorial** of angle $\omega$ if $\sigma(A) \subset \overline{S_\omega}$ and $M(A, \omega) := \sup\{\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \mid \lambda \in \mathbb{C} \setminus \overline{S_{\omega'}},\ \omega' \in (\omega, \pi)\} < \infty$. One denotes this by $A \in \mathrm{Sect}_X(\omega)$.

**Figure 1.1:** The spectrum of a sectorial operator.




Source: Figure 1.10 in [65]

Set $\omega_A := \min\{\omega \in (0,\pi) \mid A \in \mathrm{Sect}_X(\omega)\}$, which is the minimal angle for which $A$ is sectorial. For the required background on sectoral operators, we refer to [32].

**Remark 1.3.1.** Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator for which $(-\infty, 0) \subset \rho(A)$ and

$$M_A := M(A,\pi) = \sup_{t>0} t\|(t+A)^{-1}\|_{\mathcal{L}(X)} < \infty;$$

then, it follows that $A \in \mathrm{Sect}_X(\pi - \arcsin(1/M_A))$.

**Example 1.3.1.** a) Let $-A$ be the generator of a bounded semigroup $(T(t))_{t\geq 0}$; then, $A$ is a sectorial operator. Indeed, let $M := \sup_{t\geq 0} \|T(t)\|$; then, by Theorem A.0.3 (Hille-Yosida Theorem), one has $\mathbb{C}_- \subset \rho(A)$ and for each $\lambda \in \mathbb{C}_-$,

$$\|R(\lambda, -A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\mathrm{Re}\lambda|}.$$

Now, if $|\arg(\lambda)| > \frac{\pi}{2} + \varepsilon$ with $\varepsilon \in (0,\pi)$, then

$$\|R(\lambda, -A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\mathrm{Re}\lambda|} \leq \frac{M}{|\lambda| \sin\varepsilon},$$

and $A \in \mathrm{Sect}_X(\frac{\pi}{2} + \varepsilon)$, i.e, $\omega_A \leq \pi/2$.

b) Let $p \in [1,\infty)$ and $X = L^p(\mathbb{R}, X)$; the operator $Af = f'$ with $\mathcal{D}(A) = W^{1,p}(\mathbb{R}, X)$ is sectorial with $\omega_A = \frac{\pi}{2}$. Firstly, we shown that $-A$ generates the $C_0$-semigroup $(T(t))_{t\geq 0}$, with $T(t)f(s) = f(s-t)$ for each $f \in \mathcal{D}(A)$. Let $B$ the generator of $(T(t))_{t\geq 0}$. We know that $C_c^1(\mathbb{R}, X) \subset \mathcal{D}(B)$ and it is dense in $L^p(\mathbb{R}, X)$. Since $C_c^1(\mathbb{R}, X)$ is invariant under translations, it follows that $C_c^1(\mathbb{R}, X)$ is dense in $\mathcal{D}(B)$ and still $C_c^1(\mathbb{R}, X)$ is dense in $W^{1,p}(\mathbb{R}, X)$. Since both $A$ and $B$ are closed operators (semigroup generators are always closed, see Proposition G.2.3 in [66]), it follows that $\mathcal{D}(A) = \mathcal{D}(B)$, and so $-A = B$. Therefore, $A \in \mathrm{Sect}_X(\pi/2)$, by item a).

c) Suppose that the resolvent of $A : \mathcal{D}(A) \subset X \to X$, satisfies for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\mathrm{Im}\lambda|}.$$

Then, $A^2$ is a sectorial operator of angle 0. In particular, this is the case if $-iA$ generates a bounded $C_0$-group. See item 4 of example 9.1 in [33].

d) Theorem A.0.2 may be formulated as stating that for a densely defined operator $A$ on a Banach space $X$, $A$ is sectorial of angle $0 < \theta < \frac{\pi}{2}$, if and only if $-A$ generates a bounded analytic $C_0$-semigroup $(T(t))_{t\geq 0}$ on $S_\theta$ for some $0 < \theta < \frac{\pi}{2}$.

The following result presents some useful properties of sectorial operators.

**Lemma 1.3.1.** Let $A \in \mathrm{Sect}_X(\omega_A)$. Then,

(a) $(1+A)^{-1}, A(1+A)^{-1} \in \mathrm{Sect}_X(\omega_A)$. If $A$ is injective, then $A^{-1} \in \mathrm{Sect}_X(\omega_A)$, and the identity

$$\lambda(\lambda + A^{-1})^{-1} = 1 - \frac{1}{\lambda}\left(\frac{1}{\lambda} + A\right)^{-1}, \tag{1.1}$$

holds for each $0 \neq \lambda \in \mathbb{C}$ .

(b) Let $\sigma \in (0,1)$ and set $A_\sigma := (A+\sigma)(1+\sigma A)^{-1} \in \mathcal{L}(X)$; then, $A_\sigma$ is a sectorial operator, $\sup\limits_{\sigma\in(0,1)} M_{A_\sigma} < \infty$, and for each $\lambda \in \rho(A)$, $R(\lambda, A_\sigma)$ converges to $R(\lambda, A)$ in $\mathcal{L}(X)$ as $\sigma \to 0^+$.

*Proof.* We begin proving relation (1.1). Note that for each $0 \neq \lambda \in \mathbb{C}$,

$$\begin{aligned}\lambda(\lambda + A^{-1})^{-1} = 1 - A^{-1}(\lambda + A^{-1})^{-1} &= 1 - ((\lambda + A^{-1})A)^{-1} \\ &= 1 - (\lambda A + 1)^{-1} = 1 - \frac{1}{\lambda}\left(\frac{1}{\lambda} + A\right)^{-1}.\end{aligned}$$

(a) For each $\lambda \notin \overline{S_{\omega_A}}$, it follows from (1.1) that

$$\|\lambda R(\lambda, (1+A)^{-1})\|_{\mathcal{L}(X)} \leq 1 + \frac{1}{|\lambda|}\left\|\left(\frac{1}{\lambda} - 1 - A\right)^{-1}\right\|_{\mathcal{L}(X)}.$$

Moreover, by relation (1.1)

$$\begin{aligned}\lambda R(\lambda, A(1+A)^{-1}) &= \lambda R(\lambda, 1 - (1+A)^{-1}) \\ &= \lambda(\lambda - 1 + (1+A)^{-1})^{-1} = \frac{\lambda}{\lambda - 1} - \frac{\lambda}{(\lambda-1)^2}\left(\frac{1}{\lambda - 1} + (1+A)\right)^{-1} \\ &= \lambda(\lambda - 1 + (1+A)^{-1})^{-1} = \frac{\lambda}{\lambda - 1} - \frac{\lambda}{(\lambda-1)^2}\left(\frac{\lambda}{\lambda - 1} + A\right)^{-1},\end{aligned}$$

and since $A$ is a sectorial operator, one concludes that $(1+A)^{-1}$ and $A(1+A)^{-1}$ are also sectorial operators.

(b) Let $\sigma \in (0,1)$, $\lambda \notin \overline{S_{\omega_A}}$, and note that

$$\begin{aligned}\lambda - (A+\sigma)(1+\sigma A)^{-1} &= (\lambda(1+\delta A) - (A+\sigma))(1+\delta A)^{-1}\\ &= (\lambda - \sigma + (\lambda\sigma - 1)A)(1+\sigma A)^{-1}\\ &= \frac{\lambda\delta - 1}{\sigma}\left(\frac{\lambda-\sigma}{\lambda\sigma-1} + A\right)\left(\frac{1}{\sigma} + A\right)^{-1};\end{aligned}$$

then,

$$\begin{aligned}\lambda(\lambda - (A+\sigma)(1+\sigma A)^{-1})^{-1} &= \frac{\lambda\sigma}{\lambda\sigma-1}\left(\frac{1}{\sigma}+A\right)\left(\frac{\lambda-\sigma}{\lambda\sigma-1}+A\right)^{-1}\\ &= \frac{\lambda}{\lambda\sigma-1}\left(\frac{\lambda-\sigma}{\lambda\sigma-1}+A\right)^{-1} + \frac{\lambda\sigma}{\lambda\sigma-1}A\left(\frac{\lambda-\sigma}{\lambda\sigma-1}+A\right)^{-1}\\ &= \frac{\lambda}{\lambda\sigma-1}\left(\frac{\lambda-\sigma}{\lambda\sigma-1}+A\right)^{-1} + \frac{\lambda\sigma}{\lambda\sigma-1} - \frac{\lambda\sigma(\lambda-\sigma)}{(\lambda\sigma-1)^2}\left(\frac{\lambda-\sigma}{\lambda\sigma-1}+A\right)^{-1}\\ &= \frac{\lambda\sigma}{\lambda\sigma-1} - \frac{\lambda(1-\sigma^2)}{(\lambda\sigma-1)^2}\left(\frac{\lambda-\sigma}{\lambda\sigma-1}+A\right)^{-1}. \qquad (1.2)\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\lambda(\lambda-(A+\sigma)(1+\sigma A)^{-1})^{-1}\right\|_{\mathcal{L}(X)} &\leq \frac{|\lambda|\sigma}{|\lambda\sigma-1|} + \frac{|\lambda(1-\sigma^2)|}{|\lambda\sigma-1|^2}\frac{M_A|\lambda\sigma-1|}{|\lambda-\sigma|}\\ &\leq \frac{|\lambda|\sigma}{|\lambda\sigma-1|} + \frac{M_A|\lambda|(1-\sigma^2)}{|\lambda\sigma-1||\lambda-\sigma|}\\ &\leq \frac{|\lambda|}{|\lambda-1/\sigma|} + \frac{M_A|\lambda|}{|\lambda\sigma-1||\lambda-\sigma|}, \qquad (1.3)\end{aligned}$$

proving that $A_\sigma$ is sectorial. Now, let $\lambda \in \rho(A)$ and by (1.2) and resolvent identity (Lemma 1.1.1-a)) note that,

$$\begin{aligned}(\lambda-(A+\sigma)(1+\sigma A)^{-1})^{-1} - (\lambda-A)^{-1} &= \frac{\sigma}{\lambda\sigma-1} - \frac{1-\sigma^2}{(\lambda\sigma-1)^2}\left(\frac{\lambda-\sigma}{\lambda\sigma-1}+A\right)^{-1}\\ &+ \frac{1-\sigma^2}{(\lambda\sigma-1)^2}(-\lambda+A)^{-1}\\ &- \frac{1-\sigma^2}{(\lambda\sigma-1)^2}(-\lambda+A)^{-1} + (-\lambda+A)^{-1}\\ &= \frac{\sigma}{\lambda\sigma-1}\\ &- \frac{1-\sigma^2}{(\lambda\sigma-1)^2}\left(-\lambda-\frac{\lambda-\sigma}{\lambda\sigma-1}\right)\left(\frac{\lambda-\sigma}{\lambda\sigma-1}+A\right)^{-1}(-\lambda+A)^{-1}\\ &+ \frac{\lambda^2\sigma^2-2\lambda\sigma+\sigma}{(\lambda\sigma-1)^2}(-\lambda+A)^{-1}.\end{aligned}$$

Thus, by (1.3), one has

$$\begin{aligned}\|(\lambda-(A+\sigma)(1+\sigma A)^{-1})^{-1} &- (\lambda-A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{\sigma}{|\lambda\sigma-1|}\\ &+ \left|\frac{\sigma-\lambda^2\sigma}{(\lambda\sigma-1)^3}\right| \left\|\left(\frac{\lambda-\sigma}{\lambda\sigma-1}+A\right)^{-1}\right\|_{\mathcal{L}(X)} \|(-\lambda+A)^{-1}\|_{\mathcal{L}(X)}\\ &+ \left|\frac{\lambda^2\sigma^2-2\lambda\sigma+\sigma}{(\lambda\sigma-1)^2}\right| \|(-\lambda+A)^{-1}\|_{\mathcal{L}(X)}\\ &\leq \frac{\sigma}{|\lambda\sigma-1|} + \|(-\lambda+A)^{-1}\|_{\mathcal{L}(X)}\Big(\left|\frac{\lambda^2\sigma^2-2\lambda\sigma+\sigma}{(\lambda\sigma-1)^2}\right|\\ &+ \left|\frac{\sigma-\lambda^2\sigma}{(\lambda\sigma-1)^2}\right| \left(\frac{|\lambda||\lambda\sigma-1||\lambda-\sigma|+M_A}{|\lambda-1/\sigma||\lambda\sigma-1||\lambda-\delta|}\right)\Big),\end{aligned}$$

which goes to zero as $\sigma \to 0^+$. □

### 1.3.1 Functional Calculus for Sectorial Operators

We begin with the *Riesz-Dunford functional calculus of bounded operators*: for each $A \in \mathcal{L}(X)$, let $U$ be an open connected of $\sigma(A)$, let $\gamma$ be a path in $U$ around $\sigma(A)$ and let $f$ be a complex function whose restriction to $U$ is holomorphic; then, one may define the bounded linear operator $f(A) : X \to X$ by the law

$$f(A) := \frac{1}{2\pi i}\int_\gamma f(z)R(z,A)dz. \tag{1.4}$$

Now, consider the Banach algebra

$$H_0^\infty(S_\omega) := \{f : S_\omega \to \mathbb{C} \mid f \text{ is holomorpic and exist } C \geq 0, s > 0; |f(z)| \leq C\min\{|z|^{-s}, |z|^s\}, \forall\, z \in S_\omega\},$$

endowed with the norm

$$\|f\|_{H_0^\infty(S_\omega)} := \sup\{|f(z)| \mid z \in S_\omega\}.$$

Now, let $A \in \mathrm{Sect}_X(\omega_A)$, $\varphi \in (\omega_A, \pi)$ and $f \in H_0^\infty(S_{\omega_A})$. Define $f(A) \in \mathcal{L}(X)$ by the law

$$f(A) := \frac{1}{2\pi i}\int_{\Gamma_{\omega'}} f(z)R(z,A)dz, \tag{1.5}$$

where $\Gamma_{\omega'}$ stands for the positively oriented boundary of $S_{\omega'}$ for $\omega' \in (\omega_A, \varphi)$. A standard argument using Cauchy's Integral Theorem shows that this definition is actually independent of $\omega'$. An interesting reference for this Functional Calculus and its applications is [32].

**Remark 1.3.2.** Let $\alpha, \beta > 0$, $\upsilon_1, \upsilon_2 \geq 0$, $\varphi \in (0, \pi)$ and

$$f_{\alpha,\beta,\upsilon_1,\upsilon_2}(z) = \frac{z^\alpha}{(1+z)^{\alpha+\beta}\log(2+z)^{\upsilon_1}(2\pi - i\log(z))^{\upsilon_2}}, \qquad z \in S_\varphi;$$

it is straightforward to show that $f_{\alpha,\beta,\upsilon_1,\upsilon_2} \in H_0^\infty(S_\varphi)$. Therefore, by (1.5), one may define

$$\begin{aligned} f_{\alpha,\beta,\upsilon_1,\upsilon_2}(A) &:= \frac{1}{2\pi i}\int_{\Gamma_{\omega'}} f_{\alpha,\beta,\upsilon_1,\upsilon_2}(z)R(z,A)dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_{\omega'}} \frac{z^\alpha}{(1+z)^{\alpha+\beta}\log(2+z)^{\upsilon_1}(2\pi - i\log(z))^{\upsilon_2}}R(z,A)dz, \end{aligned} \tag{1.6}$$

where $\Gamma_{\omega'}$ is the positively oriented boundary of $S_{\omega'}$ for $\omega' \in (\omega_A, \varphi)$. If $A$ is invertible, then one may let $\alpha = 0$ in the expression (1.6). This operator will play an important role in the proofs of Propositions 2.2.2, 2.3.2 and 2.3.3.

**Lemma 1.3.2** (Lemma 2.3.1 in [32])**.** Let $A \in \mathrm{Sect}_X(\omega)$ and let $\varphi \in (\omega, \pi)$. Then, if $B$ is a closed operator which commutes with $R(A,\lambda)$, $\lambda \in \rho(A)$, then $B$ commutes with $f(A)$. In particular, $f(A)$ commutes with $A$ and with $R(\lambda, A)$ for each $\lambda \in \rho(A)$.

The next result is used in the proof of the important Moment Inequality.

**Proposition 1.3.1** (Proposition 2.6.11 in [32])**.** Let $\varphi \in (0,\pi]$ and let $f \in H_0^\infty(S_\varphi)$. Then, there exists a constant $C_f > 0$ such that

$$\sup_{t\geq 0} \|f(tA)\|_{\mathcal{L}(X)} \leq C_f M(A,\varphi)$$

for each sectorial operator $A \in \mathrm{Sect}_X(\omega)$, with $\omega \in (0,\varphi)$. Moreover, given $\theta \in [0, \varphi-\omega)$, one has $\|f(\lambda A)\|_{\mathcal{L}(X)} \leq C_f M(A, \varphi - \theta)$ for each $\lambda \in \mathbb{C}$, $|\arg\lambda| \leq \theta$.

**Proposition 1.3.2** (Moment Inequality)**.** Let $A$ be a sectorial operator on the Banach space $X$. Let $\alpha$, $\beta$, $\gamma \in \mathbb{R}$ such that $\gamma < \beta < \alpha$ and $\gamma > 0$ or $\gamma = 0$. Then, there exists a constant $C > 0$ such that for each $x \in \mathcal{D}(A^\alpha)$,

$$\|A^\beta x\| \leq \frac{C}{\theta(1-\theta)}\|A^\gamma x\|^{1-\theta}\|A^\alpha x\|^\theta,$$

where $\theta := \dfrac{\beta-\gamma}{\alpha-\gamma}$.

*Proof.* Let $\psi(z) := \dfrac{z^\alpha}{(1+z)^{2\alpha}(1+\log(z)^2)} \in H_0^\infty(S_{\omega_A})$ and note that $z^\alpha\psi(z), z^{-\alpha}\psi(z)$ are still bounded functions. Define $h(z) := \dfrac{1}{c}\displaystyle\int_0^1 \frac{\psi(sz)}{s}ds$, $g(z) := \dfrac{1}{c}\displaystyle\int_1^\infty \frac{\psi(sz)}{s}ds$, where $c := \displaystyle\int_0^\infty \frac{\psi(s)}{s}ds$; then, for each $z \in S_{\omega_A}$, $h(z) + g(z) = \displaystyle\int_0^\infty \frac{\psi(s)}{s}ds = 1$.

Now, let $\tilde{h}(z) := z^{-(\alpha-\beta)}h(z)$ and $\tilde{g}(z) := z^{\beta-\gamma}g(z)$, and note that $\tilde{h}, \tilde{g} \in H_0^\infty(S_{\omega_A})$. For each $x \in \mathcal{D}(A^\alpha)$ and $t > 0$, one has

$$A^\beta x = h(tA)A^\beta x + g(tA)A^\beta x = t^{\alpha-\beta}\tilde{h}(tA)A^\alpha x + t^{-(\beta-\gamma)}\tilde{g}(tA)A^\gamma x.$$

Then, it follows from Proposition 1.3.1 that

$$
\begin{aligned}
\|A^\beta x\| &\leq t^{\alpha-\beta}\|\tilde{h}(tA)\|_{\mathcal{L}(X)}\|A^\alpha x\| + t^{-(\beta-\gamma)}\|\tilde{g}(tA)\|_{\mathcal{L}(X)}\|A^\gamma x\| \\
&\leq C_{\tilde{h}} t^{(\alpha-\gamma)(1-\theta)}\|A^\alpha x\| + C_{\tilde{g}} t^{-(\alpha-\gamma)\theta}\|A^\gamma x\|.
\end{aligned}
$$

By taking the infimum with respect to $t > 0$, one gets

$$
\begin{aligned}
\|A^\beta x\| &\leq C_{\tilde{h}}\left(\frac{\theta\|A^\gamma x\|}{(1-\theta)\|A^\alpha x\|}\right)^{1-\theta}\|A^\alpha x\| + C_{\tilde{g}}\left(\frac{(1-\theta)\|A^\alpha x\|}{\theta\|A^\gamma x\|}\right)^{\theta}\|A^\gamma x\| \\
&\leq C_{\tilde{h}}\left(\frac{\theta}{(1-\theta)}\right)^{1-\theta}\|A^\gamma x\|^{1-\theta}\|A^\alpha x\|^{\theta} + C_{\tilde{g}}\left(\frac{1-\theta}{\theta}\right)^{1-\theta}\|A^\gamma x\|^{1-\theta}\|A^\alpha x\|^{\theta} \\
&\leq 2C\left(\left(\frac{\theta}{1-\theta}\right)^{1-\theta} + \left(\frac{1-\theta}{\theta}\right)^{1-\theta}\right)\|A^\gamma x\|^{1-\theta}\|A^\alpha x\|^{\theta},
\end{aligned}
$$

where $C := \max\{C_{\tilde{g}}, C_{\tilde{h}}\}$. □

Now we recall some basic properties of the functional calculus of sectorial operators based on complete Bernstein functions. We use [14] as a reference in our discussion (see also [13, 15, 16, 17]).

**Definition 1.3.2** (Definition 3.3 in [14])**.** Let $A \in \mathrm{Sect}_X(\omega_A)$ be densely defined and let $f \in \mathcal{CBF}$, with Stieltjes representation $(a, b, \mu)$ (see appendix E). One defines the linear operator $f_0(A) : \mathcal{D}(A) \to X$ by the law

$$
f_0(A) = ax + bAx + \int_{0+}^{\infty} A(A+\lambda)^{-1}x\,d\mu(\lambda), \quad x \in \mathcal{D}(A). \tag{1.7}
$$

Set $f(A) := \overline{f_0(A)}$. We call the linear operator $f(A)$ a complete Bernstein function of $A$.

**Theorem 1.3.1** (Theorem 3.6 in [14])**.** Let $A$ be a sectorial operator on a Banach space $X$ and let $f \in \mathcal{CBF}$. Then, $f(A)$ is sectorial.

### 1.3.2 Logarithm operator

Given the nature of our problem, an investigation involving the definition of the logarithm of an injective sectorial operator is required. Such operator was first defined by Nollau [48] and was subsequently studied by Okazawa [49] and Haase [31].

Let $A$ be an injective operator over the Banach space $X$ such that $A \in \mathrm{Sect}_X(\omega_A)$. Let $\varphi \in (\omega_A, \pi)$ and set $\tau(z) := z(1+z)^{-2}$; note that $\tau \in H_0^\infty(S_\varphi)$ and $\tau(A) = A(1+A)^{-2}$, by relation (1.6) (with $\upsilon_1 = \upsilon_2 = 0$, $\alpha = 1$ and $\beta = 1$). Set $\mathcal{B}(S_\varphi) := \{f : S_\varphi \to \mathbb{C} \mid \exists\, n \in \mathbb{N}$ such that $\tau^n f \in H_0^\infty(S_\varphi)\}$. Since $A$ is injective, $\tau(A)$ is also injective, and so one may define for each $f \in \mathcal{B}(S_\varphi)$

$$
f(A) := (\tau(A)^{-1})^n[(\tau^n(z)f(z))](A), \tag{1.8}
$$

with $n$ large enough so that $\tau^n f \in H_0^\infty(S_\varphi)$.

**Remark 1.3.3.** Definition (1.8) is independent of the choice of $n$ (see Proposition 2.1 in [31]). Note that $f(A)$ is a closed operator with domain $\mathcal{D}(f(A)) = \{x \in X \mid (\tau^n(z)f(z))(A)x \in \mathcal{D}(\tau(A)^{-1})^n\}$. We refer to [31] for more details.

**Definition 1.3.3** (**Haase**, [31])**.** Let $A \in \mathrm{Sect}_X(\omega_A)$ and injective. Let $f : S_\pi \to \mathbb{C}$ be given by the law $f(z) = \log(z)$. Since $f \in \mathcal{B}(S_\varphi)$, then

$$\log(A) := f(A). \tag{1.9}$$

**Remark 1.3.4.** Let $A \in \mathrm{Sect}_X(\omega_A)$ be densely defined. It follows from (E.2) and from Definition 1.3.2 that for each $x \in \mathcal{D}(A)$,

$$\log(1 + A)x = \int_1^\infty A(A + t)^{-1}x\frac{dt}{t}. \tag{1.10}$$

This representation for $\log(1 + A)$ was presented for the first time in [48].

**Definition 1.3.4** (**Okazawa**, see [49])**.** Let $A \in \mathrm{Sect}_X(\omega_A)$ and injective. Suppose that $\mathcal{D}(A)$ and $\mathrm{Ran}(A)$ are dense in $X$. Then, $\log(A)$ is defined as the closure of

$$\log(1 + A) - \log(1 + A^{-1}).$$

**Remark 1.3.5.** Naturally, the Definitions 1.3.3 and 1.3.4 for $\log(A)$ when $A$ is an injective operator must coincide when $D(A)$ and $R(A)$ are both dense; for details see [24].

The following result is a direct consequence of Definition 1.3.4.

**Lemma 1.3.3.** *Let $A \in \mathrm{Sect}_X(\omega_A)$ be injective and densely defined (with not necessarily dense range). Then, for each $x \in \mathcal{D}(A) \cap \mathrm{Ran}(A)$,*

$$\log(A)x = \log(1 + A)x - \log(1 + A^{-1})x.$$

*Proof.* Let $\sigma \in (0, 1)$ and set $A_\sigma := (A + \sigma)(1 + \sigma A)^{-1} \in \mathcal{L}(X)$. It follows from Definition 1.3.4 that for each $x \in X$, and in particular, for each $\mathcal{D}(A) \cap \mathrm{Ran}(A)$,

$$\log(A_\sigma)x = \log(1 + A_\sigma)x - \log(1 + A_\sigma^{-1})x. \tag{1.11}$$

Now, it follows from Lemma 3(c) in [48] that for each $x \in \mathcal{D}(A) \cap \mathrm{Ran}(A)$, $\lim_{\sigma\to0^+} \log(A_\sigma)x = \log(A)x$ and $\lim_{\sigma\to0^+} \log(1 + A_\sigma)x = \log(1 + A)x$; thus, by (1.11), one has for each $x \in \mathcal{D}(A) \cap \mathrm{Ran}(A)$ that

$$\lim_{\sigma\to0^+} \log(1 + A_\sigma^{-1})x = \lim_{\sigma\to0^+} \log(1 + A_\sigma)x - \lim_{\sigma\to0^+} \log(A_\sigma)x = \log(1 + A)x - \log(A)x.$$

Since $A$, $A + 1$, $A^{-1} + 1$ are sectorial operators, $\log(1 + A^{-1})$ is well-defined by (1.9); thus, it follows from Proposition 3.1.3 in [24] and Lemma 3.1 in [31] that for each $x \in \mathcal{D}(A) \cap \mathrm{Ran}(A)$,

$$\log(1+A)x - \log(A)x = \log(1+A)x + \log(A^{-1})x = \log((1+A)A^{-1})x = \log(1+A^{-1})x.$$

Then, it follows from the previous relations that for each $x \in \mathcal{D}(A) \cap \mathrm{Ran}(A)$,

$$\lim_{\sigma\to 0^+} \log(1+A_\sigma^{-1})x = \log(1+A^{-1})x,$$

and so, for each $x \in \mathcal{D}(A) \cap \mathrm{Ran}(A)$, one gets

$$\log(A)x = \log(1+A)x - \log(1+A^{-1})x.$$

□

Let us now recall some properties of the logarithm and fractional power.

**Lemma 1.3.4.** Let $A \in \mathrm{Sect}_X(\omega_A)$. Then, the following assertions hold:

(a) $A^\sigma$ is sectorial, with $\sigma \in (0,1)$.

(b) If $A \in \mathcal{L}(X)$, then for each $\sigma > 0$, $A^\sigma \in \mathcal{L}(X)$.

(c) If $A$ is injective, then for each $\sigma \in [0,1]$, $\log(A^\sigma) = \sigma \log(A)$.

(d) Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup on the Banach space $X$, with $-A$ its infinitesimal generator. Let, for each $\varepsilon \in (0,1)$, $f_\varepsilon(A) = (1+A)^\varepsilon - 1$. Then, for each $t, s \geq 0$,

$$T(t)f_\varepsilon(A)(s+f_\varepsilon(A))^{-1} = f_\varepsilon(A)(s+f_\varepsilon(A))^{-1}T(t).$$

*Proof.* (a) Given that for each $\sigma \in (0,1)$, $[s \mapsto s^\sigma] \in \mathcal{CBF}$ (see Example E.1.1-(a)), it follows from Theorem 1.3.1 that the operator $f_\sigma(A) = A^\sigma$ is sectorial. (b) This is Proposition 3.1.1 (a) in [32]. (c) This is Satz 5 in [48]. (d) It follows from Theorem 3.9 (a) in [14] that for each $t \geq 0$, $T(t)f_\varepsilon(A) \subset f_\varepsilon(A)T(t)$, and so, by Proposition B.3 in [7], one has $T(t)(s+f_\varepsilon(A))^{-1} = (s+f_\varepsilon(A))^{-1}T(t)$ for each $s, t \geq 0$. □

# PART I

# Refined Decay of $C_0$-semigroups on Banach spaces

# Chapter 2

# Refined Decay of $C_0$-semigroups on Banach spaces

In this chapter we present the proof of the results presented in Part I of Introduction, that is, our results regarding the decay of $C_0$-semigroups. As mentioned there, we have refined some results in the literature (more specifically, some results presented in [55]) by taking into account only the geometry of the Banach space (i.e., its Fourier type) and the growth of the norm of the resolvent of the generator.

## 2.1 Preliminaries: Fourier Multipliers and Stability for $C_0$-Semigroups

In this section we define Fourier types, Fourier multipliers, and present Theorem 2.1.2 (Theorem 4.6 in [55]), which is very important in the proof of our results.

### 2.1.1 Fourier Types

**Definition 2.1.1.** Let $p \in [1,2]$. A Banach space $X$ is said to have Fourier type $p$ if the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}, X) \to L^{p'}(\mathbb{R}, X)$ extends to a bounded operator in $L^p(\mathbb{R}, X)$.

**Example 2.1.1.** a) Every Banach space $X$ has Fourier type 1. Indeed, for each $f \in L^1(\mathbb{R}, X)$ and $s \in \mathbb{R}$,

$$\mathcal{F}(f)(s) = \int_{\mathbb{R}} e^{-2\pi st} f(t)dt,$$

so $\|\mathcal{F}(f)\|_{L^\infty} \leq \|f\|_{L^1}$.

b) It follows from Plancherel Theorem that every Hilbert space $X$ has Fourier type 2.

c) Let $(S, \mathcal{A}, \mu)$ be a measure space; for each $p \in [1, \infty)$, $L^p(S)$ has Fourier type $\min\{p, p'\}$.

d) Let $X$ be a Banach space, $(S, \mathcal{A}, \mu)$ be a measure space and let $r \in [1, \infty)$. If $X$ has Fourier type $p$, then $L^r(S, X)$ has Fourier type $\min\{p, r, r'\}$.

The next result shows that the converse of item b) in Example 2.1.1 is also valid, that is, any Banach space with Fourier type 2 is isomorphic to a Hilbert space (for more details, see Theorem 2.1.18 in [65]).

**Theorem 2.1.1** (Kwapień)**.** For a Banach space $X$, the following assertions are equivalent.

1. The Fourier–Plancherel transform extends to a bounded operator on $L^2(\mathbb{R}, X)$.

2. $X$ is isomorphic to a Hilbert space.

### 2.1.2 Growth at infinity

Let $X$ and $Y$ be Banach spaces and let $m : \mathbb{R} \to \mathcal{L}(X, Y)$ be a $X$-strongly measurable map (i.e. the map $\xi \mapsto m(\xi)x$ is a strongly measurable $Y$-valued map for every $x \in X$). One says that $m$ is of *moderate growth at infinity* if there exist $\beta \geq 0$ and $g \in L^1(\mathbb{R})$ such that for each $\xi \in \mathbb{R}$,

$$\frac{1}{(1+|\xi|)^{\beta}} \|m(\xi)\|_{\mathcal{L}(X,Y)} \lesssim g(\xi).$$

For such measurable $m$, one defines the *Fourier multiplier operator* associated with $m$, $T_m : \mathcal{S}(\mathbb{R}; X) \to \mathcal{S}'(\mathbb{R}; Y)$, by the law

$$T_m(f) := \mathcal{F}^{-1}(m \cdot \mathcal{F}f), \qquad \forall\, f \in \mathcal{S}(\mathbb{R}; X);$$

$m$ is called the *symbol* of $T_m$. For $p \in [1, \infty)$ and $q \in [1, \infty]$, let $\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(X, Y))$ denote the set of all $X$-strongly measurable maps $m : \mathbb{R} \to \mathcal{L}(X, Y)$ of moderate growth such that $T_m \in \mathcal{L}(L^p(\mathbb{R}; X), L^q(\mathbb{R}; Y))$ and $\|m\|_{\mathcal{M}_{p,q}(\mathbb{R};\mathcal{L}(X,Y))} := \|T_m\|_{\mathcal{L}(L^p(\mathbb{R};X),L^q(\mathbb{R};Y))}$.

#### Growth at zero and infinity

Let $\dot{\mathcal{S}}(\mathbb{R}, X) := \{f \in \mathcal{S}(\mathbb{R}; X) \mid \hat{f}^{(k)}(0) = 0 \text{ for each } k \in \mathbb{N} \cup \{0\}\}$ and $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ be a $X$-strongly measurable map. One says that $m$ is of *moderate growth at zero and infinity* if there exist $\alpha \geq 0$ and $g \in L^1(\mathbb{R})$ such that for each $\xi \in \mathbb{R}$,

$$\frac{|\xi|^{\alpha}}{(1+|\xi|)^{2\alpha}} \|m(\xi)\|_{\mathcal{L}(X,Y)} \lesssim g(\xi).$$

For such measurable $m$, one defines the *Fourier multiplier operator* associated with $m$, $\dot{T}_m$ : $\dot{\mathcal{S}}(\mathbb{R}; X) \to \dot{\mathcal{S}}'(\mathbb{R}; Y)$, by the law

$$\dot{T}_m(f) := \mathcal{F}^{-1}(m \cdot \mathcal{F}f), \qquad \forall\, f \in \dot{\mathcal{S}}(\mathbb{R}; X);$$

$m$ is called the *symbol* of $T_m$. For $p \in [1, \infty)$ and $q \in [1, \infty]$, let $\mathcal{M}_{p,q}(\mathbb{R}; \mathcal{L}(X, Y))$ denote the set of all $X$-strongly measurable maps $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ of moderate growth such that $T_m \in \mathcal{L}(L^p(\mathbb{R}; X), L^q(\mathbb{R}; Y))$ and $\|m\|_{\mathcal{M}_{p,q}(\mathbb{R};\mathcal{L}(X,Y))} := \|T_m\|_{\mathcal{L}(L^p(\mathbb{R};X),L^q(\mathbb{R};Y))}$. For more details about discussion above, see [54].

The next result will be used in the proofs of Theorems 10 and 11. For more details, see [54].

**Proposition 2.1.1** (Proposition 3.3 in [54])**.** Let $X$ be a Banach space with Fourier type $p \in [1,2]$, let $Y$ be a Banach space with Fourier cotype $q \in [2,\infty]$, and let $r \in [1,\infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Let $m : \mathbb{R} \setminus \{0\} \to \mathcal{L}(X,Y)$ (or $m : \mathbb{R} \to \mathcal{L}(X,Y)$) be an $X$-strongly measurable map such that $\|m(\cdot)\|_{\mathcal{L}(X,Y)} \in L^r(\mathbb{R})$. Then, $m \in \mathcal{M}_{p,q}(\mathbb{R}, \mathcal{L}(X,Y))$.

The theory of $(L^p, L^q)$ Fourier multipliers has proven to be an important tool for the stability theory of $C_0$-semigroups [36, 37, 54, 53, 55, 57, 67, 68]. In particular, by using it, Rozendaal and Veraar have obtained the following result that characterizes polynomial stability. We stress that this result is a necessary tool in our analysis. (see also Theorem 5.1 in [57]).

The following Lemma is used in the proof of Theorem 2.1.2 and intuitively shows us the relation between the resolvent and the Fourier multipliers. For more details, see [55].

**Lemma 2.1.1** (Lemma 3.1 in [55])**.** Let $-A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ defined on a Banach space $X$ and let $n \in \mathbb{N} \cup \{0\}$, $x \in X$ and $\xi \in \mathbb{R}$. Suppose that $-i\xi \in \rho(A)$ and that $[t \mapsto t^n T(t)x] \in L^1([0,\infty), X)$. Then

a)
$$\mathcal{F}([t \mapsto t^n T(t)x])(\xi) = n!(i\xi + A)^{-n-1}x.$$

b) For each $g \in L^1(\mathbb{R})$
$$\mathcal{F}\left(\int_0^\infty t^n T(t)xg(\cdot - t)dt\right)(\xi) = \hat{g}(\xi)n!(i\xi + A)^{-n-1}x.$$

**Theorem 2.1.2** (Theorem 4.6 in [55])**.** Let $-A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ defined on a Banach space $X$ such that $\overline{\mathbb{C}_-} \setminus \{0\} \subset \rho(A)$ and such that there exist $\alpha, \beta \geq 0$ so that
$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim |\lambda|^{-\alpha}(1 + |\lambda|)^{\beta}, \tag{2.1}$$
with $\mathrm{Re}(\lambda) \leq 0$. Let $n \in \mathbb{N} \cup \{0\}$ and $Y$ be a Banach space which is continuously embedded in $X$ and suppose that there exists a constant $C_T \geq 0$ such that, for each $t \geq 0$, $T(t)Y \subset Y$, $\|T(t)|_Y\|_{\mathcal{L}(Y)} \leq C_T \|T(t)\|_{\mathcal{L}(X)}$, and that there exists a dense subspace $Y_0 \subset Y$ such that for each $y \in Y_0$, $[t \mapsto t^n T(t)y] \in L^1([0,\infty), Y)$. Then, the following statements are equivalent:

a) $\sup_{t\geq 0} \left\{t^n \|T(t)\|_{\mathcal{L}(Y,X)}\right\} < \infty$.

b) There exist $\psi \in C_c^\infty(\mathbb{R})$, $p \in [1,\infty)$ and $q \in [p,\infty]$ such that for each $k \in \{n-1, n, n+1\}$,
$$\psi(\cdot)R(i\cdot, A)^k \in \mathcal{M}_{1,\infty}(\mathbb{R}, \mathcal{L}(Y,X)) \quad \text{and} \quad (1 - \psi(\cdot))R(i\cdot, A)^k \in \mathcal{M}_{p,q}(\mathbb{R}, \mathcal{L}(Y,X)).$$

Moreover, if (a) or (b) holds then $R(i\cdot, A)^k \in \mathcal{M}_{p,q}(\mathbb{R}, \mathcal{L}(Y,X))$ for:

(i) $n \geq 2$, $k \in \{1, \cdots, n-1\}$ and $1 \leq p \leq q \leq \infty$;

(ii) $k = n \geq 1$ and $1 \leq p < q \leq \infty$;

(iii) $k = n+1$, $p = 1$ and $q = \infty$.

*Proof.* (b) $\Longrightarrow$ (a) Let $w$, $M_w \geq 1$ be such that $\|T(t)\|_{\mathcal{L}(X)} \leq M_w e^{t(w-1)}$ for each $t \geq 0$ and for $\xi \in \mathbb{R} \setminus \{0\}$

$$m(\xi) := n!(i\xi + A)^{-n}(I_X + w(i\xi + A)^{-1}) \in \mathcal{L}(Y, X)$$

Since, for each $\xi \in \mathbb{R} \setminus \{0\}$, $(i\xi + A)^{-1} = -R(-i\xi, A)$, it follows from Proposition 3.2 in [55] that $T_m : L^p(\mathbb{R}; Y) \cap L^1(Y; \mathbb{R}) \to L^\infty(\mathbb{R}; X)$ is bounded with

$$\|T_m\|_{\mathcal{L}(L^p(\mathbb{R};Y)\cap L^1(Y;\mathbb{R}), L^\infty(\mathbb{R};X))} \lesssim 2Mn!(C_n + wC_{n+1}), \tag{2.2}$$

where $M := \sup_{t\in[0,2]} \|T(t)\|_{\mathcal{L}(X)}$, for each $k \in \mathbb{N}$, $C_k$ is as in Proposition 3.2 [55], and $C_0 := \|I_Y\|_{\mathcal{L}(Y,X)}$. Now let $Y_0 \subset Y$ and fix $x \in Y_0$. By Lemma 2.1.1, one has

$$\mathcal{F}([t \mapsto t^n T(t)x])(\cdot) = n!(i \cdot + A)^{-n-1}x. \tag{2.3}$$

For each $t \geq 0$ define $f(t) := e^{-wt}T(t)x$ and $f \equiv 0$ on $(-\infty, 0)$. Then, for each $t \geq 0$

$$\|f(t)\|_Y \leq \|e^{-wt}T(t)\|_{\mathcal{L}(Y)} \leq C_T \|e^{-wt}T(t)\|_{\mathcal{L}(X)} \|x\|_Y.$$

Hence $f \in L^1(\mathbb{R}, Y) \cap L^\infty(\mathbb{R}, Y)$ and $\|f\|_{L^r(\mathbb{R},Y)} \leq C_T M_w \|x\|_Y$ for each $r \in [1, \infty]$. By Lemma 2.1.1, $\mathcal{F}(f)(\cdot) = (w + i \cdot + A)^{-1}x$. Therefore, by Lemma 1.1.1 item a), for each $\xi \in \mathbb{R} \setminus \{0\}$,

$$m(\xi)\mathcal{F}(f)(\xi) = n!(i\xi + A)^{-n-1}x \tag{2.4}$$

Combining (2.3) and (2.4) with (2.2) yields

$$\sup_{t\geq 0} \|t^n T(t)x\| \leq \|T_m\|_{\mathcal{L}(L^p(\mathbb{R};Y)\cap L^1(Y;\mathbb{R}), L^\infty(\mathbb{R};X))}(\|f\|_{L^p(\mathbb{R},Y)} + \|f\|_{L^1(\mathbb{R},Y)}) \leq C\|x\|_Y$$

where $C := 4Mn!C_T M_w(C_n + wC_{n+1})$. The required result now follows since $Y_0 \subset Y$ is dense.

(a) $\Longrightarrow$ (b) Set $K_n := \sup_{t\geq 0} \|t^n T(t)x\|$ and let $Y_0 \subset Y$ be as in hypothesis. Let $f \in \dot{\mathcal{S}}(\mathbb{R}, X) \bigotimes Y_0$ and set $S_k(f)(s) := \int_0^\infty t^k T(t) f(s-t)dt$ for $s \in \mathbb{R}$ and $k \in \{0, \cdots, n\}$, by Lemma 2.1.1 item b) one has

$$S_k(f) = k!\mathcal{F}^{-1}((i \cdot + A)^{-1}\hat{f}(\cdot)) = k!T_{(i\cdot+A)^{-k-1}}(f). \tag{2.5}$$

Now, for $n \geq 2$, $k \in \{0, \cdots, n-2\}$, and $r \in [1, \infty]$,

$$\|[t \mapsto t^k T(t)]\|_{L^r(\mathbb{R},\mathcal{L}(Y,X))} \leq M + K_n \|[t \mapsto t^{-2}]\|_{L^r(1,\infty)} \leq M + K_n.$$

Similarly, for $n \geq 1$ and $r \in (1, \infty]$,

$$\|[t \mapsto t^{n-1}T(t)]\|_{L^r(\mathbb{R},\mathcal{L}(Y,X))} \leq M + \frac{K_n}{(r-1)^{1/r}}.$$

By combining these estimates with (2.5) and with Young's inequality for operator valued kernels (Proposition 1.3.5 in [7]) one obtains, for $p \in [1, \infty)$ and $q \in [p, \infty]$

$$\|R(i\cdot, A)^k\|_{\mathcal{M}_{p,q}(\mathbb{R},\mathcal{L}(Y,X))} \leq \frac{M + K_n}{(k-1)!} \quad (n \geq 2, k \in \{1, \cdots n-1\}) \tag{2.6}$$

$$\|R(i\cdot, A)^n\|_{\mathcal{M}_{p,q}(\mathbb{R},\mathcal{L}(Y,X))} \leq \frac{M + K_n(r-1)^{-1/r}}{(n-1)!} \quad (n \geq 1, p < q) \tag{2.7}$$

$$\|R(i\cdot, A)^{n+1}\|_{\mathcal{M}_{1,\infty}(\mathbb{R},\mathcal{L}(Y,X))} \leq \frac{K_n}{n!} \tag{2.8}$$

Now (2.5), (2.6), (2.7) and (2.8) yield statements (i)-(iii) for $(i\cdot + A)^{-1}$ and by reflection these statements hold for $R(i\cdot, A)$ as well. Finally, for (b) let $\psi \in C_c^\infty(\mathbb{R})$. Then Young's inequality and relations (2.6), (2.7) and (2.8) yield $\psi(\cdot)R(i\cdot, A)^k \in \mathcal{M}_{1,\infty}(\mathbb{R}, \mathcal{L}(Y, X))$ for each $k \in \{1, \cdots, n+1\}$, and one obtains (2.6), (2.7) and (2.8) for $\psi(\cdot)R(i\cdot, A)$ with an additional multiplicative factor $\|\mathcal{F}^{-1}(\psi)\|_{L^1(\mathbb{R})}$. Similarly, (2.6), (2.7) and (2.8) holds with an additional multiplicative factor $\|\mathcal{F}^{-1}(1-\psi)\|_{L^1(\mathbb{R})}$ upon replacing $R(i\cdot, A)$ by $(1 - \psi(\cdot))R(i\cdot, A)$.

□

**Remark 2.1.1.** The assumption in Theorem 2.1.2 that $\|(\cdot + A)^{-1}\|_{\mathcal{L}(X)}$ satisfies the equation (2.1) for some $\alpha, \beta \geq 0$ is only made to ensure that $T_{R(i\cdot,A)}$ is well-defined, and the specific choice of $\alpha$ and $\beta$ is irrelevant here.

## 2.2 Singularity at Infinity

We begin introducing some notation that will be useful throughout this section.

Let $\nu, \upsilon \geq 0$ and $A \in \mathrm{Sect}_X(\omega_A)$; since $\lambda \mapsto \log(1 + \lambda) \in \mathcal{CBF}$ (see Example E.1.1-(b)), it follows from Theorem 1.3.1 that the operator $\log(2 + A)$ is sectorial, and so $(\log(2 + A))^{-\upsilon}$ is well-defined and bounded (see definition of fractional powers of sectorial operators in [32, 41]). Define the operator

$$\Phi_\nu(\upsilon) = \Phi_\nu(A, \upsilon) := (1 + A)^{-\nu} \log(2 + A)^{-\upsilon} \in \mathcal{L}(X),$$

and set $X_\nu(\upsilon) := \mathrm{Ran}(\Phi_\nu(\upsilon))$. The space $X_\nu(\upsilon)$ is a Banach space with respect to the norm

$$\|x\|_{X_\nu(\upsilon)} = \|x\| + \|\Phi_\nu(\upsilon)^{-1}x\| = \|x\| + \|\log(2 + A)^\upsilon(1 + A)^\nu x\|, \quad x \in X_\nu(\upsilon).$$

Note that $\Phi_\nu(\upsilon) : X \to X_\nu(\upsilon)$ is an isomorphism, so for each $T \in \mathcal{L}(X_\nu(\upsilon), X)$,

$$\|Tx\| = \|T\Phi_\nu(\upsilon)y\| \leq \|T\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)}\|y\| \leq \|T\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)}\|x\|_{X_\nu(\upsilon)}$$

(here, $y := \Phi_\nu(\upsilon)^{-1}x$) and

$$\|T\Phi_\nu(\upsilon)x\| \le \|T\|_{\mathcal{L}(X_\nu(\upsilon),X)}\|\Phi_\nu(\upsilon)x\| \le \|T\|_{\mathcal{L}(X_\nu(\upsilon),X)}\|\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)}\|x\|;$$

therefore, for each $T \in \mathcal{L}(X_\nu(\upsilon), X)$, one has

$$\|T\|_{\mathcal{L}(X_\nu(\upsilon),X)} \le \|T\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)} \le \|\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)}\|T\|_{\mathcal{L}(X_\nu(\upsilon),X)}. \tag{2.1}$$

Note that $\Phi_\nu(0) = \Phi_\nu(A)$ and $X_\nu(0) = X_\nu$, where $\Phi_\nu(A)$ and $X_\nu$ are the objects defined in [55].

In this subsection, we discuss the decay rate of a $C_0$-semigroup whose infinitesimal generator $-A$ is such that $\overline{\mathbb{C}_-} \subset \rho(A)$ and such that there exist $\beta > 0$ and $b \ge 0$ so that $\|(\lambda+A)^{-1}\|_{\mathcal{L}(X)} \lesssim (1+|\lambda|)^\beta \log(2+|\lambda|)^b$, for $\lambda \in \mathbb{C}$ satisfying $\mathrm{Re}(\lambda) \le 0$.

**Theorem 2.2.1.** Let $\beta > 0$, $b \ge 0$ and $(T(t))_{t\ge0}$ be a $C_0$-semigroup defined in the Banach space $X$ with Fourier type $p \in [1, 2]$, with $-A$ as its generator. Suppose $\overline{\mathbb{C}_-} \subset \rho(A)$ and for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \le 0$,

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim (1 + |\lambda|)^\beta \log(2 + |\lambda|)^b. \tag{2.2}$$

Let $r \in [1,\infty]$ be such $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$ and let $\tau$ be such that $\tau > \beta + \frac{1}{r}$. Then, for each $\delta > 0$ and each $\rho \in [0, \frac{\tau-1/r}{\beta} - 1]$, there exists $c_{\rho,\delta} > 0$ such that for each $t \ge 1$,

$$\|T(t)(1 + A)^{-\tau} \log(2 + A)^{-\frac{b}{\beta}(\tau-1/r)-\frac{1+\delta}{r}}\|_{\mathcal{L}(X)} \le c_{\rho,\delta}t^{-\rho}. \tag{2.3}$$

The following results are needed in the proof of Theorem 2.2.1. Note also that the following proposition is a version of Theorem 2.2.1 in case $p = 1$ (that is, in case $X$ is a Banach space with trivial type).

**Proposition 2.2.1.** Let $b \ge 0$, $\beta > 0$ and let $A$ be an injective sectorial operator on a Banach space $X$ such that $-A$ generates a $C_0$-semigroup $(T(t))_{t\ge0}$ on $X$. Suppose $\overline{\mathbb{C}_-} \subset \rho(A)$ and for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \le 0$,

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim (1 + |\lambda|)^\beta \log(2 + |\lambda|)^b. \tag{2.4}$$

Let $\tau \ge \beta + 1$. Then, for each $\delta > 0$ and each $\rho \in [0, \frac{\tau-1}{\beta} - 1]$, there exists $c_{\rho,\delta} > 0$ such that for each $t \ge 1$,

$$\|T(t)(1 + A)^{-\tau} \log(2 + A)^{-\frac{b}{\beta}(\tau-1)-1-\delta}\|_{\mathcal{L}(X)} \le c_{\rho,\delta}t^{-\rho}.$$

*Proof.* We follow the same steps of the proof of Proposition 4.3 in [55]. The proposition is equivalent to the following statement: for each $s \ge 0$ and $\delta > 0$ there exists $C_{s,\delta} > 0$ such that for each $t \ge 1$,

$$\|T(t)(1 + A)^{-\nu} \log(2 + A)^{-\upsilon}\|_{\mathcal{L}(X)} \le C_{s,\delta}t^{-s},$$

where $\upsilon := b(s+1)+1+\delta$, $\nu := (s+1)\beta+1$.

Firstly, we obtain the result for $s = n \in \mathbb{N} \cup \{0\}$ and then for any $s \geq 0$ by an interpolation argument.

So, let $\delta > 0$, $n \in \mathbb{N} \cup \{0\}$, $\upsilon = b(n+1)+1+\delta$, $\nu = (n+1)\beta+1$ and $x \in X_{\nu+1}(\upsilon)$. Set

$$\begin{aligned} y := [\Phi_\nu(\upsilon)]^{-1}x = \log(2+A)^\upsilon(1+A)^\nu x &= \log(2+A)^\upsilon(1+A)^\nu \left((1+A)^{-\nu-1}\log(2+A)^{-\upsilon}z\right) \\ &= \log(2+A)^\upsilon \left((1+A)^{-1}\log(2+A)^{-\upsilon}z\right) \\ &= (1+A)^{-1}z, \end{aligned}$$

with $z \in X$, and note that $(1+A)^{-1}z \in \mathcal{D}(A)$; here, we have used that $\log(2+A)^\upsilon$ commutes with $(1+A)^{-1}$ (for more details, see Proposition 2.3-(d) in [49] and Proposition 3.1.1-(f) in [32]).

Let $g : [0,\infty) \to X$ be given by

$$g(t) = \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{1}{(1+\lambda)^\nu[\log(2+\lambda)]^\upsilon}R(\lambda, A)y d\lambda, \tag{2.5}$$

and note that for each $t \geq 0$, $g(t) \in X$; namely, for each $t \geq 0$, one has

$$\begin{aligned} \|g(t)\| &\leq \left\| \frac{1}{2\pi i}\int_{\mathbb{R}} e^{i\xi t}\frac{1}{(1-i\xi)^\nu[\log(2-i\xi)]^\upsilon}R(-i\xi, A)y d\xi \right\| \\ &\lesssim \left( \int_{\mathbb{R}} \frac{1}{(1+|\xi|)^\nu[\log(2+|\xi|)]^\upsilon}\|(i\xi+A)^{-1}\|_{\mathcal{L}(X)} d\xi \right)\|y\|, \end{aligned}$$

Now, by assuming (2.2), it follows that the integral above is finite.

Moreover, since $y \in \mathcal{D}(A)$, the function $\lambda \mapsto \dfrac{\lambda}{(1+\lambda)^\nu(\log(2+\lambda))^\upsilon}R(\lambda, A)y$ is integrable and by dominated convergence,

$$g'(t) = -\frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{\lambda}{(1+\lambda)^\nu[\log(2+\lambda)]^\upsilon}R(\lambda, A)y d\lambda,$$

which proves that $g$ is differentiable everywhere. Now, by Lemma D.0.1,

$$\begin{aligned} g'(t) &= \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{1}{(1+\lambda)^\nu[\log(2+\lambda)]^\upsilon}(-AR(\lambda, A)y - y)d\lambda \\ &= -A\left(\frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{1}{(1+\lambda)^\nu[\log(2+\lambda)]^\upsilon}R(\lambda, A)y d\lambda\right) - \\ &- \left(\frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{1}{(1+\lambda)^\nu[\log(2+\lambda)]^\upsilon}d\lambda\right)y \\ &= 0 - Ag(t) = -Ag(t), \end{aligned}$$

and $g(0) = \dfrac{1}{2\pi i}\displaystyle\int_{i\infty}^{-i\infty} \frac{1}{(1+\lambda)^\nu[\log(2+\lambda)]^\upsilon}R(\lambda, A)y d\lambda = \Phi_\nu(\upsilon)y = x$, by (1.6). Then, $g'(t) = -Ag(t)$ for each $t \geq 0$, and $g(0) = x$. Therefore, for each $t \geq 0$, $g(t) = T(t)x$, by the uniqueness of the Cauchy problem associated with $-A$.

Integration by parts yields

$$t^n T(t)x = \frac{1}{2\pi i}\int_{i\mathbb{R}} e^{-\lambda t} p(\lambda, A) y d\lambda,$$

where $p(\lambda, A)$ is a finite linear combination of terms of the form

$$\frac{R(\lambda, A)^{n-j+1}}{(1+\lambda)^{\nu+j}(2+\lambda)^i[\log(2+\lambda)]^{\upsilon+i}}, \quad \frac{R(\lambda, A)^{n-j+1}}{(1+\lambda)^{\nu+i}(2+\lambda)^j[\log(2+\lambda)]^{\upsilon+j}},$$

with $0 \leq i < j \leq n$, each one of them being integrable (see the proof of Proposition 4.3 in [55] for details). Then, there exists a positive constant $d_{n,\delta}$ so that for each $t \geq 1$,

$$\|t^n T(t)x\| \leq \left(\frac{1}{2\pi}\int_{i\mathbb{R}} |e^{-\lambda t}| \|p(\lambda, A)\|_{\mathcal{L}(X)} d\lambda\right) \|y\| \leq d_{n,\delta}\|\log(2+A)^{\upsilon}(1+A)^{\nu}x\| \leq d_{n,\delta}\|x\|_{X_\nu(\upsilon)}.$$

Since $X_{\nu+1}(\upsilon)$ is dense in $X_\nu(\upsilon)$, it follows from the previous discussion that for each $t \geq 1$,

$$\|T(t)\|_{\mathcal{L}(X_\nu(\upsilon), X)} \leq d_{n,\delta} t^{-n}. \tag{2.6}$$

It remains to prove the result for any $s \geq 0$. For each $s \geq 0$, let $n \in \mathbb{N} \cup \{0\}$ be such that $n \leq s < n+1$. Let also define $\theta := \theta(s) \in [0, 1)$ by the relation $s = (1-\theta)n + \theta(n+1)$.

Set $a_1 := \frac{\beta}{\beta+b}$ and $a_2 := \frac{b}{\beta+b}$ and note that $a_1 + a_2 = 1$; then, by Proposition E.1.1-(c), $f(\lambda) = (1+\lambda)^{a_1}\log(2+\lambda)^{a_2} \in \mathcal{CBF}$, where $\lambda > 0$. Now, by Lemma 1.3.1, the operator

$$(f(A))^{-1} = (1+A)^{-a_1}\log(2+A)^{-a_2},$$

is sectorial, given that $f(A)$ is sectorial (by Theorem 1.3.1).

Since $(f(A))^{-1}$ is sectorial, it follows from relation (2.6), the moment inequality (see Proposition 1.3.2) and Theorem 2.4.2 in [32] that there exists a positive constant $C_{s,\delta}$ such that for each $t \geq 1$,

$$\begin{aligned}
\|T(t)[(f(A))^{-1}]^{\theta(\beta+b)}\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)} &\lesssim \|T(t)\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)}^{1-\theta}\|T(t)[(f(A))^{-1}]^{\beta+b}\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)}^{\theta} \\
&= \|T(t)\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)}^{1-\theta}\|T(t)(1+A)^{-\beta}\log(2+A)^{-b}\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)}^{\theta} \\
&= \|T(t)\Phi_\nu(\upsilon)\|_{\mathcal{L}(X)}^{1-\theta}\|T(t)\Phi_{\beta(n+2)+1}(b(n+2)+1+\delta)\|_{\mathcal{L}(X)}^{\theta} \\
&\leq (d_{n,\delta}t^{-n})^{1-\theta}(d_{n+1,\delta}t^{-n-1})^{\theta} = C_{s,\delta}t^{-s},
\end{aligned}$$

and we are done. □

Note that for $b = \zeta = 0$, the following result is Proposition 3.4 in [55] (see also Theorem 5.5 in [14]).

**Proposition 2.2.2.** Let $A \in \mathrm{Sect}_X(\omega_A)$ be such that $\overline{\mathbb{C}_-} \subset \rho(A)$, and let $\beta, b, \zeta \geq 0$ and $\beta_0 \in [0, 1)$. If

$$\|(i\xi + A)^{-1}\|_{\mathcal{L}(X)} \lesssim (1+|\xi|)^{\beta}\log(2+|\xi|)^{b}, \tag{2.7}$$

then the family

$$\{|\lambda|^{\beta_0}\log(2+|\lambda|)^{\zeta}\|(\lambda+A)^{-1}\|_{\mathcal{L}(X_{\beta_0+\beta(\zeta+b)},X)} \mid \lambda\in i\mathbb{R}, |\lambda|\geq 1\} \tag{2.8}$$

is uniformly bounded.

*Proof.* We proceed as in the proof of Proposition 3.4-(2) in [55]. Fix $\theta\in(\omega_A,\pi)$ and let the path $\Gamma:=\{re^{i\theta}\mid r\in[0,\infty)\}\cup\{re^{-i\theta}\mid r\in[0,\infty)\}$ be oriented from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$. Set $\tilde{c}:=b+\zeta$; since $A+\frac{1}{2}\in \mathrm{Sect}_X(\omega_A)$, it follows from Remark 1.3.2 (by letting $\alpha=0$, $\upsilon_2=0$) and from Lemma D.0.2 that for each $\lambda\in i\mathbb{R}$, $|\lambda|\geq 1$, and for each $x\in X$

$$\begin{aligned}
(\lambda+A)^{-1}(1+A)^{-\beta-\beta_0}\log(2+A)^{-\tilde{c}}x &= \frac{1}{2\pi i}\int_\Gamma \frac{(\lambda+A)^{-1}}{(\frac{1}{2}+z)^{\beta+\beta_0}\log(\frac{3}{2}+z)^{\tilde{c}}}R\left(z,A+\frac{1}{2}\right)xdz\\
&= \frac{1}{2\pi i}\int_\Gamma \frac{(\lambda+A)^{-1}}{(\frac{1}{2}+z)^{\beta+\beta_0}\log(\frac{3}{2}+z)^{\tilde{c}}(z+\lambda-\frac{1}{2})}xdz\\
&+ \frac{1}{2\pi i}\int_\Gamma \frac{R\left(z,A+\frac{1}{2}\right)}{(\frac{1}{2}+z)^{\beta+\beta_0}\log(\frac{3}{2}+z)^{\tilde{c}}(z+\lambda-\frac{1}{2})}xdz\\
&= \frac{1}{(1-\lambda)^{\beta+\beta_0}\log(2-\lambda)^{\tilde{c}}}(\lambda+A)^{-1}x+T_\lambda x
\end{aligned}$$

with

$$T_\lambda := \frac{1}{2\pi i}\int_\Gamma \frac{1}{(\frac{1}{2}+z)^{\beta+\beta_0}\log(\frac{3}{2}+z)^{\tilde{c}}(z+\lambda-\frac{1}{2})}R(z,A+\frac{1}{2})dz.$$

Let $h_{\beta_0,\zeta}(\lambda):=(1-\lambda)^{\beta_0}\log(2-\lambda)^{\zeta}$, with $\lambda\in i\mathbb{R}$, $|\lambda|\geq 1$; then, for each $x\in X$

$$h_{\beta_0,\zeta}(\lambda)(\lambda+A)^{-1}(1+A)^{-\beta-\beta_0}\log(2+A)^{-\tilde{c}}x = \frac{(\lambda+A)^{-1}}{(1-\lambda)^{\beta}\log(2-\lambda)^{b}}x+h_{\beta_0,\zeta}(\lambda)T_\lambda x. \tag{2.9}$$

Let $\varepsilon\in(0,\beta+\beta_0)$ and note that the function $z\mapsto \frac{1}{(z+1/2)^{\varepsilon}}R(z,A+1/2)$ is integrable on $\Gamma$. Note also that, by Lemma A.1 in [55], for each $z\in\Gamma$ and each $\lambda\in i\mathbb{R}$, one has

$$\frac{1}{|z+\frac{1}{2}|^{\beta+\beta_0-\varepsilon}|\log\left(\frac{3}{2}+z\right)|^{\tilde{c}}|z+\lambda-\frac{1}{2}|}\lesssim\frac{1}{1+|\lambda|}. \tag{2.10}$$

Therefore, by relations (2.9) and (2.10), it follows that

$$\begin{aligned}
\|h_{\beta_0,\zeta}(\lambda)(\lambda+A)^{-1}(1+A)^{-\beta-\beta_0}\log(2+A)^{-\tilde{c}}\|_{\mathcal{L}(X)} &\lesssim \left\|\frac{1}{(1-\lambda)^{\beta}\log(2-\lambda)^{b}}(\lambda+A)^{-1}\right\|_{\mathcal{L}(X)}\\
&+ \frac{|\log(2-\lambda)|^{\zeta}}{(1+|\lambda|)^{1-\beta_0}}.
\end{aligned}$$

By relation (2.7) and since $\lim_{|\lambda|\to\infty}\frac{|\log(2-\lambda)|^{\zeta}}{(1+|\lambda|)^{1-\beta_0}}=0$ (recall that $\beta_0\in[0,1)$), one concludes that

$$\{|\lambda|^{\beta_0}\log(2+|\lambda|)^{\zeta}\|(\lambda+A)^{-1}\|_{\mathcal{L}(X_{\beta_0+\beta(\zeta+b)},X)} \mid \lambda\in i\mathbb{R}, |\lambda|\geq 1\}$$

is uniformly bounded. ∎

**Remark 2.2.1.** Note that by relations (2.9) and (2.10), for each $\lambda \in i\mathbb{R}$,

$$\begin{aligned}\left\| \frac{(\lambda+A)^{-1}}{(1-\lambda)^{\beta}\log(2-\lambda)^{b}} \right\|_{\mathcal{L}(X)} &\lesssim \|(1-\lambda)^{\beta_0}\log(2-\lambda)^{\zeta}(\lambda+A)^{-1}(1+A)^{-\beta-\beta_0}\log(2+A)^{-\tilde{c}}\|_{\mathcal{L}(X)} \\ &+ \|(1-\lambda)^{\beta_0}\log(2-\lambda)^{\zeta}T_\lambda\|_{\mathcal{L}(X)} \\ &\lesssim \|(1-\lambda)^{\beta_0}\log(2-\lambda)^{\zeta}(\lambda+A)^{-1}(1+A)^{-\beta-\beta_0}\log(2+A)^{-\tilde{c}}\|_{\mathcal{L}(X)} \\ &+ \frac{|\log(2-\lambda)|^{\zeta}}{(1+|\lambda|)^{1-\beta_0}};\end{aligned}$$

thus, by assuming that the condition (2.8) is valid, one gets.

$$\left\| \frac{(\lambda+A)^{-1}}{(1-\lambda)^{\beta}\log(2-\lambda)^{b}} \right\|_{\mathcal{L}(X)} \lesssim C.$$

This shows that the converse of Proposition 2.2.2 is also valid.

***Proof of Theorem 2.2.1***. The case $p = 1$ corresponds to Proposition 2.2.1. Let $n \in \mathbb{N}\cup\{0\}$ and set $\nu := (n+1)\beta + \frac{1}{r}$, $\upsilon := b(n+1) + \frac{1+\delta}{r}$ in case $p \in (1,2)$ $(1 < r < \infty)$, and $\nu := (n+1)\beta$, $\upsilon := b(n+1)$ if $p = 2$ (that is, if $r = \infty$). Set also $B := A + 1$. By letting $\beta_0 = \zeta = 0$ in Proposition 2.2.2, it follows that for each $k \in \{1,\dots,n\}$,

$$\sup_{\xi\in\mathbb{R}} \|R(i\xi,A)^k\|_{\mathcal{L}(X_{n\beta}(bn),X)} < \infty. \tag{2.11}$$

Let $\delta > 0$ and let $h_{r,\delta} : \mathbb{R} \to \mathbb{R}$ be given by the law $h_{r,\delta}(\xi) = (1+|\xi|)^{\frac{1}{r}}\log(2+|\xi|)^{\frac{1+\delta}{r}}$; then, it follows from Proposition 2.2.2 (by taking $\beta_0 = 1/r$ and $\zeta = (1+\delta)/r$) that

$$\sup_{\xi\in\mathbb{R}} h_{r,\delta}(\xi)\|R(i\xi,A)B^{-\beta-\frac{1}{r}}\log(1+B)^{-b-\frac{1+\delta}{r}}\|_{\mathcal{L}(X)} < \infty. \tag{2.12}$$

Thus, for each $k \in \{1,\dots,n+1\}$, it follows from relations (2.11) and (2.12) that

$$\begin{aligned}\sup_{\xi\in\mathbb{R}} h_{r,\delta}(\xi)\|R(i\xi,A)^k\|_{\mathcal{L}(X_\nu(\upsilon),X)} &\lesssim \sup_{\xi\in\mathbb{R}}\left(h_{r,\delta}(\xi)\|R(i\xi,A)^kB^{-\beta(n+1)-\frac{1}{r}}\log(1+B)^{-b(n+1)-\frac{1+\delta}{r}}\|_{\mathcal{L}(X)}\right) \\ &\lesssim \sup_{\xi\in\mathbb{R}}\left(h_{r,\delta}(\xi)\|R(i\xi,A)B^{-\beta-\frac{1}{r}}\log(1+B)^{-b-\frac{1+\delta}{r}}\|_{\mathcal{L}(X)}\|R(i\xi,A)^{k-1}B^{-\beta n}\log(1+B)^{-bn}\|_{\mathcal{L}(X)}\right) \\ &\leq \sup_{\xi\in\mathbb{R}}\left(h_{r,\delta}(\xi)\|R(i\xi,A)B^{-\beta-\frac{1}{r}}\log(1+B)^{-b-\frac{1+\delta}{r}}\|_{\mathcal{L}(X)}\right)\sup_{\xi\in\mathbb{R}}\left(\|R(i\xi,A)^{k-1}\|_{\mathcal{L}(X_{\beta n}(bn),X)}\right) < \infty.\end{aligned} \tag{2.13}$$

It follows from Proposition 2.2.1 that the space $X_\nu(\upsilon)$ satisfies the conditions presented in the statement of Theorem 2.1.2. By proceeding as in the proof of this Theorem 9 (see Theorem 4.9 in [55]), let $\psi \in C_c(\mathbb{R})$ be such that $\psi \equiv 1$ on $[-1,1]$. One has, by (2.13), that for each

$k \in \{1, \ldots, n+1\}$,

$$\psi(\cdot)R(i\cdot, A)^k \in L^1(\mathbb{R}, \mathcal{L}(X_\nu(\upsilon), X)) \subset \mathcal{M}_{1,\infty}(\mathbb{R}, \mathcal{L}(X_\nu(\upsilon), X)),$$

and

$$\|(1-\psi(\cdot))R(i\cdot, A)^k\|_{\mathcal{L}(X_\nu(\upsilon),X)} \in L^r(\mathbb{R}). \tag{2.14}$$

Note that $X_\nu(\upsilon)$ has Fourier type $p$, since $X_\nu(\upsilon)$ is isomorphic to $X$. Then, by Proposition 2.1.1 and by (2.14), one concludes that for each $k \in \{1, \ldots, n+1\}$,

$$(1-\psi(\cdot))R(i\cdot, A)^k \in \mathcal{M}_{p,p'}(\mathbb{R}, \mathcal{L}(X_\nu(\upsilon), X)).$$

Now, by Theorem 2.1.2, for each $n \in \mathbb{N} \cup \{0\}$ there exists $c_n \geq 0$ such that for each $t \geq 1$,

$$\|T(t)(1+A)^{-\nu} \log(2+A)^{-b(n+1)-\frac{1+\delta}{r}}\|_{\mathcal{L}(X)} \leq c_n t^{-n}. \tag{2.15}$$

Let $s \geq 0$, $\nu = \beta(s+1) + 1/r$ and let $n \in \mathbb{N} \cup \{0\}$ be such that $n \leq s < n+1$. Let $\theta \in [0, 1)$ be such that $s = (1-\theta)n + \theta(n+1)$. Then, by following the same arguments presented in the proof of Proposition 2.2.1, it follows that for each $t \geq 1$,

$$\|T(t)B^{-\nu} \log(2+A)^{-b(s+1)-\frac{1+\delta}{r}}\|_{\mathcal{L}(X)} \lesssim t^{-s}.$$

□

**Remark 2.2.2.** Let $A$ be a linear operator defined in a Banach space $X$, not necessarily sectorial, such that

1. $-A$ generates a $C_0$-semigroup on $X$;
2. $\overline{\mathbb{C}_-} \subset \rho(A)$ and $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim (1+|\lambda|)^\beta \log(2+|\lambda|)^b$, for each $\lambda \in \overline{\mathbb{C}_-}$.

Under the above assumptions, note that for each $\varepsilon > 0$, $A + \varepsilon$ is sectorial. Then, the operator

$$(2+A)^{-\beta} \log(3+A)^{-b} = (1+1+A)^{-\beta} \log(2+1+A)^{-b}$$

is well-defined through the sectorial functional calculus for $A+1$. Note that previous results are still valid. So, in this context, we are able to remove the hypothesis of sectorially of $A$ (see Theorem 2.2.1).

**Lemma 2.2.1.** Let $-A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$. Suppose that there exist $\beta > 0$, $\delta \in [0, 1)$, $\eta \in \rho(-A)$ that such $1 \notin \sigma(A+\eta)$, and a sequence $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ such that $t_n \to \infty$ and

$$\lim_{n\to\infty} \|T(t_n)(\eta + A)^{-\beta}[\log(A+\eta)]^{-\delta}\|_{\mathcal{L}(X)} = 0. \tag{2.16}$$

Then, $\overline{\mathbb{C}_-} \subset \rho(A)$.

*Proof.* We begin with the following remarks:

- One may let $\eta \in \mathbb{R}$ be such that $\eta - 1 > \omega_0(T)$; namely, it follows from the definition of $\omega_0(T)$ that $1-\eta \in \rho(A)$ (see also [47]).
- Since $\eta - 1 > \omega_0(T)$, $\eta + A$ is sectorial, and so $(\eta + A)^{-\beta}$, $\log(A+\eta)^{-\delta}$ are well-defined.
- One has for each $t > 0$,

$$\|T(t)(\eta+A)^{-\beta}[\log(A+\eta)]^{-2}\|_{\mathcal{L}(X)} \leq \|[\log(A+\eta)]^{-2+\delta}\|_{\mathcal{L}(X)}\|T(t)(\eta+A)^{-\beta}[\log(A+\eta)]^{-\delta}\|_{\mathcal{L}(X)}, \tag{2.17}$$

and so it is sufficient to assume that there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ such that $t_n \to \infty$ and

$$\lim_{t_n\to\infty} \|T(t_n)(\eta + A)^{-\beta}[\log(A+\eta)]^{-2}\|_{\mathcal{L}(X)} = 0.$$

For each $\lambda \in \mathbb{C}$, with $\mathrm{Re}\lambda > -\eta$, and each $a > 0$, set $f_{t,a}(\lambda) := e^{-\lambda t}(\eta+\lambda)^{-a}[\log(\lambda+\eta)]^{-2}$. Let $a \geq 0$ and for each $\xi \in \mathbb{R}$, one has $\xi^{-a}\log(\xi)^{-1} = \mathcal{L}(v(\cdot, a-1))(\xi)$ (see Table 5.7 in [8]), where

$$v(x,a) = \int_0^\infty \frac{x^{s+a}}{\Gamma(s+a+1)}ds.$$

Now, for $a > 1$, the inverse Laplace transform of $\xi^{-a}\log(\xi)^{-2}$ reads

$$\begin{aligned}
\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{\xi\lambda}\frac{1}{\lambda^a \log^2(\lambda)}d\lambda &= -\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{\xi\lambda}\lambda^{-a+1}\frac{d}{d\lambda}\left(\frac{1}{\log(\lambda)}\right)d\lambda \\
&= -\lim_{r\to\infty}\frac{1}{2\pi i}e^{\xi\lambda}\lambda^{-a+1}\frac{1}{\log(\lambda)}\Big]_{b-ir}^{b+ir} + \frac{(-a+1)}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{\xi\lambda}\frac{1}{\lambda^a\log(\lambda)}d\lambda \\
&+ \frac{\xi}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{\xi\lambda}\frac{1}{\lambda^{a-1}\log(\lambda)}d\lambda \\
&= (-a+1)v(\xi, a-1) + \xi v(\xi, a-2)
\end{aligned}$$

Next, set

$$k_a(\xi) := \begin{cases} [(-a+1)v(\xi,a-1) + \xi v(\xi,a-2)]e^{-\eta\xi}, & \xi > 0 \\ 0, \quad \xi \leq 0, \end{cases}$$

and note that for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > -\eta$, one has

$$\begin{aligned}
\mathcal{L}(\delta_t * k_a)(\lambda) &= e^{-\lambda t}\int_0^\infty e^{-\lambda s}k_a(s)ds \\
&= e^{-\lambda t}\int_0^\infty e^{-(\lambda+\eta)s}(-a+1)v(s,a-1) + sv(s,a-2)]ds \\
&= e^{-\lambda t}(\lambda+\eta)^{-a}(\log(\lambda+\eta))^{-2} = f_{t,a}(\lambda).
\end{aligned}$$

Now, by Hille-Phillips Functional Calculus, one has

$$f_{t,a}(A) := T(t)(\eta + A)^{-a}\log(A+\eta)^{-2}.$$

**Case $\beta > 1$:**

Let $a = \beta$; by the Spectral Mapping Theorem (which is a consequence of Hille-Phillips Functional Calculus; see [35, 32]), one has $f_t(\sigma(A)) \subset \sigma(f_t(A))$ for each $t > 0$. Let $\lambda \in \sigma(A)$; then, $f_{t_n}(\lambda) \in \sigma(f_{t_n}(A))$ for each $t_n$ and

$$\frac{e^{-\mathrm{Re}\lambda t_n}}{|(\eta+\lambda)|^{\beta}|\log(\lambda+\eta))|^2} = |f_{t_n}(\lambda)| \leq \|T(t_n)(\eta+\lambda)^{-\beta}[\log(A+\eta)]^{-2}\|.$$

It follows from relation (2.17) that $\lim_{t_n\to\infty} e^{-(\mathrm{Re}\lambda)t_n} = 0$, which is only possible if $-\mathrm{Re}\lambda < 0$.

**Case $\beta \leq 1$:**

Let $a > 1$, and note that

$$\begin{aligned}\|T(t_n)(\eta+A)^{-a}(\log(\eta+A))^{-1-\delta}\|_{\mathcal{L}(X)} &\leq \|T(t_n)(\eta+A)^{-(\beta+(a-\beta))}(\log(\eta+A))^{-1-\delta}\|_{\mathcal{L}(X)}\\ &\leq \|(\eta+A)^{-(a-\beta)}\|_{\mathcal{L}(X)}\|T(t_n)(\eta+A)^{-\beta}(\log(\eta+A))^{-1-\delta}\|_{\mathcal{L}(X)}\end{aligned}$$

□

### 2.2.1 Proof of Theorem 10

***Proof of Theorem 10.*** The result is equivalent to the following statement: for each $s > 0$ and each $\delta > 0$, there exists $C_{\delta,s} > 0$ such that for each $t \geq 1$,

$$\|T(t)(1+A)^{-\nu}\|_{\mathcal{L}(X)} \leq C_{\delta,s}t^{-s}\log(1+t)^{\upsilon},$$

where $\nu := \beta(s+1) + 1/r$ and $\upsilon := b(s+1) + \dfrac{1+\delta}{r}$ for $p \neq 2$, $\nu := \beta(s+1)$ and $\upsilon := b(s+1)$ otherwise. Set $m := \lfloor \upsilon \rfloor$ and $\eta := \{\upsilon\} \in (0,1)$. We divide the proof into the cases where $\eta = 0$ and $\eta > 0$. In both of them, we proceed recursively over $m$.

**Case $\eta > 0$.**

**Step 1: removing $\eta > 0$.** Since $(0,\infty) \ni \tau \mapsto \log(1+\tau)^{\eta} \in \mathcal{CBF}$ (by Proposition E.1.1), it follows that

$$\log(1+\tau)^{\eta} = \int_{0+}^{\infty} \frac{\tau}{\tau+\lambda}d\mu(\lambda).$$

Let $\theta = \theta(s) \in (0,1)$ be such that $s \geq \theta > 0$. Let, for each $\sigma \in (0,1)$, $f_\sigma : [0,\infty) \to \mathbb{R}$ be given by the law $f_\sigma(\xi) = (1+\xi)^{\sigma} - 1$; it is a complete Bernstein function (see Example E.1.1). Then, for $\sigma = \varepsilon := \frac{\min\{1,\beta\}\theta}{2}$, $f_\varepsilon(B)$ is a sectorial operator, by Theorem 1.3.1, where $B := A + 1$.

Therefore, by Lemma 1.3.4,

$$\log(1+B)^{\eta} = \frac{1}{\varepsilon^{\eta}} \log(1+f_{\varepsilon}(B))^{\eta}.$$

By the choice of $\varepsilon > 0$, $\mathcal{D}(B^{\beta s}) \subset \mathcal{D}(f_{\varepsilon}(B))$ (see Proposition 3.1.1 (c) in [32]). It follows from equation (1.7) (with $a = b = 0$; see (E.2)) and from the previous facts that for each $x \in X$,

$$\begin{aligned} T(t)B^{-\nu}\log(1+B)^{-m}x &= \frac{1}{\varepsilon^{\eta}}T(t)\log(1+f_{\varepsilon}(B))^{\eta}\log(1+B)^{-\eta}B^{-\nu}\log(1+B)^{-m}x \\ &= \frac{1}{\varepsilon^{\eta}}T(t)\log(1+f_{\varepsilon}(B))^{\eta}B^{-\nu}\log(1+B)^{-m-\eta}x \\ &= \frac{1}{\varepsilon^{\eta}}T(t)\int_{0+}^{\infty} f_{\varepsilon}(B)(\lambda+f_{\varepsilon}(B))^{-1}B^{-\nu}\log(1+B)^{-m-\eta}x\,d\mu(\lambda). \end{aligned}$$

Let $\phi \in [0,\nu)$, $\zeta > 0$, set $P_{\zeta}(B_{\phi}) := B^{-\nu+\phi}\log(1+B)^{-\zeta} \in \mathcal{L}(X)$ and

$$\tau := \frac{\|T(t)P_{m+\eta}(B_{\varepsilon})\|_{\mathcal{L}(X)}}{\|T(t)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}} > 0.$$

Since $f_{\varepsilon}(B)$ is a sectorial operator and since for each $t \geq 0$, $T(t)$ commutes with $f_{\varepsilon}(B)$ (see Lemma 1.3.4), it follows that for each $t \geq 0$,

$$\begin{aligned} &\left\| T(t)\int_{0+}^{\tau} f_{\varepsilon}(B)(\lambda+f_{\varepsilon}(B))^{-1}P_{m+\eta}(B_0)x\,d\mu(\lambda)\right\| \\ &\leq \|T(t)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}(M_{f_{\varepsilon}(B)}+1)\int_{0+}^{\tau} d\mu(\lambda)\|x\| \\ &\leq 4\|T(t)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}M_{f_{\varepsilon}(B)}\int_{0+}^{\tau}\frac{\tau}{\tau+\lambda}d\mu(\lambda)\|x\| \end{aligned} \tag{2.18}$$

where $M_{f_{\varepsilon}(B)} := \sup_{\lambda>0}\|\lambda(\lambda+f_{\varepsilon}(B))^{-1}\|_{\mathcal{L}(X)} \geq 1$. Moreover, by Lemma 1.3.4,

$$\begin{aligned} &\left\| T(t)\int_{\tau}^{\infty} f_{\varepsilon}(B)(\lambda+f_{\varepsilon}(B))^{-1}P_{m+\eta}(B_0)x\,d\mu(\lambda)\right\| \\ &\leq \|T(t)f_{\varepsilon}(B)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}\int_{\tau}^{\infty}\|(\lambda+f_{\varepsilon}(B))^{-1}\|_{\mathcal{L}(X)}d\mu(\lambda)\|x\| \\ &\leq \|T(t)f_{\varepsilon}(B)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}M_{f_{\varepsilon}(B)}\int_{\tau}^{\infty}\frac{1}{\lambda}d\mu(\lambda)\|x\| \\ &\leq 2\|T(t)f_{\varepsilon}(B)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}M_{f_{\varepsilon}(B)}\int_{\tau}^{\infty}\frac{1}{\lambda+\tau}d\mu(\lambda)\|x\|. \end{aligned} \tag{2.19}$$

By combining relations (2.18) and (2.19), one gets, for each $t \geq 0$,

$$\begin{aligned} \|T(t)P_m(B_0)x\| &\leq C'_{\varepsilon}\|T(t)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}\left(\int_{0+}^{\tau}\frac{\tau}{\tau+\lambda}d\mu(\lambda)\right. \\ &\quad + \left.\frac{\|T(t)f_{\varepsilon}(B)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}}{\|T(t)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}}\int_{\tau}^{\infty}\frac{1}{\lambda+\tau}d\mu(\lambda)\right)\|x\|, \end{aligned} \tag{2.20}$$

where $C'_{\varepsilon} := 4M_{f_{\varepsilon}(B)}$.

Note that, for each $t \geq 0$, $T(t)$ commutes with $(1+B)^{\varepsilon}B^{-\varepsilon} - B^{-\varepsilon}$, hence for each $x \in X$,

$$\begin{aligned}
\|T(t)f_{\varepsilon}(B)P_{m+\eta}(B_0)x\| &= \|T(t)((1+B)^{\varepsilon}-1)P_{m+\eta}(B_0)x\| \\
&= \|T(t)((1+B)^{\varepsilon} - B^{-\varepsilon}B^{\varepsilon})B^{-\nu}\log(1+B)^{-m-\eta}x\| \\
&= \|T(t)((1+B)^{\varepsilon}B^{-\varepsilon} - B^{-\varepsilon})B^{-\nu+\varepsilon}\log(1+B)^{-m-\eta}x\| \\
&= \|((1+B)^{\varepsilon}B^{-\varepsilon} - B^{-\varepsilon})T(t)P_{m+\eta}(B_{\varepsilon})x\| \\
&\leq \|((1+B)^{\varepsilon}B^{-\varepsilon} - B^{-\varepsilon})\|_{\mathcal{L}(X)}\|T(t)P_{m+\eta}(B_{\varepsilon})\|_{\mathcal{L}(X)}\|x\|,
\end{aligned}$$

and so

$$\|T(t)f_{\varepsilon}(B)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)} \leq C_{\varepsilon}\|T(t)P_{m+\eta}(B_{\varepsilon})\|_{\mathcal{L}(X)}, \tag{2.21}$$

where $C_{\varepsilon} := \|(1+B)^{\varepsilon}B^{-\varepsilon} - B^{-\varepsilon}\|_{\mathcal{L}(X)}$.

Therefore, it follows from relations (2.20) and (2.21) that for each $t \geq 0$,

$$\begin{aligned}
\|T(t)B^{-\nu}\log(1+B)^{-m}\|_{\mathcal{L}(X)} &\leq C'_{\varepsilon}(1+C_{\varepsilon})\|T(t)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}\left(\int_{0+}^{\tau}\frac{\tau}{\tau+\lambda}d\mu(\lambda) + \tau\int_{\tau}^{\infty}\frac{1}{\lambda+\tau}d\mu(\lambda)\right) \\
&= C'_{\varepsilon}(1+C_{\varepsilon})\|T(t)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}\log(1+\tau)^{\eta}.
\end{aligned} \tag{2.22}$$

On the other hand, by the definition $\varepsilon = \frac{\min\{1,\beta\}\theta}{2}$, one has $\beta(s+1) + 1/r - \varepsilon > \beta + 1/r$; then, it follows from Theorem 2.2.1 that there exists a positive constant $C_{s,\varepsilon,\eta}$ such that for each $t \geq 1$,

$$\|T(t)P_{m+\eta}(B_{\varepsilon})\|_{\mathcal{L}(X)} \leq C_{s,\varepsilon,\eta}t^{-s+\frac{\varepsilon}{\beta}}. \tag{2.23}$$

One also has from Theorem 2.2.1 that there exists a positive constant $C_{s,\eta}$ so that for each $t \geq 1$,

$$\|T(t)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)} \leq C_{s,\eta}t^{-s}. \tag{2.24}$$

Now, set $k_{m,\eta}(t) := \|T(t)P_{m+\eta}(B_0)\|_{\mathcal{L}(X)}$; by (2.24), one has for each $t \geq 1$,

$$\frac{C_{s,\varepsilon,\eta}t^{-s+\frac{\varepsilon}{\beta}}}{k_{m,\eta}(t)} \geq \tilde{C}_{s,\eta}C_{s,\varepsilon,\eta}t^{\frac{\varepsilon}{\beta}}, \tag{2.25}$$

with $\tilde{C}_{s,\eta} := (C_{s,\eta})^{-1}$. It follows from relations (2.23), (2.24) and by letting $\gamma = 1$, $\ell(w) = \log(1+w)^{\eta}$, $s = \tilde{C}_{s,\eta}C_{s,\varepsilon,\eta}t^{\frac{\varepsilon}{\beta}}$ and $w = \frac{C_{s,\varepsilon,\eta}t^{-s+\frac{\varepsilon}{\beta}}}{k_{m,\eta}(t)}$ in Proposition E.2.1 (note that $w \geq s$, by (2.25)) that there exists $C > 0$ (which depends only on the function log) such that for each sufficiently large $t$,

$$\begin{aligned}
\log(1+\tau)^{\eta} \leq \log\left(1 + \frac{C_{s,\varepsilon,\eta}t^{-s+\frac{\varepsilon}{\beta}}}{k_{m,\eta}(t)}\right)^{\eta} &\leq C\frac{C_{s,\varepsilon,\eta}t^{-s+\frac{\varepsilon}{\beta}}}{\tilde{C}_{s,\eta}C_{s,\varepsilon,\eta}t^{\frac{\varepsilon}{\beta}}k_{m,\eta}(t)}\log\left(1+\tilde{C}_{s,\delta}C_{s,\varepsilon,\eta}t^{\frac{\varepsilon}{\beta}}\right)^{\eta} \\
&= \frac{C}{k_{m,\eta}(t)\tilde{C}_{s,\eta}t^{s}}\log\left(1+\tilde{C}_{s,\eta}C_{s,\varepsilon,\eta}t^{\frac{\varepsilon}{\beta}}\right)^{\eta} \\
&= C_1(s,\varepsilon,\eta)\frac{1}{k_{m,\eta}(t)t^{s}}\log\left(1+C_2(s,\varepsilon,\eta)t^{\frac{\varepsilon}{\beta}}\right)^{\eta}
\end{aligned} \tag{2.26}$$

with $C_1(s,\varepsilon,\eta) := C/\tilde{C}_{s,\eta}$ and $C_2(s,\varepsilon,\eta) := \tilde{C}_{s,\eta}C_{s,\varepsilon,\eta}$.

Then, one concludes from (2.22) and (2.26) that for each sufficiently large $t$,

$$\begin{aligned} \|T(t)B^{-\nu}\log(1+B)^{-m}\|_{\mathcal{L}(X)} &\leq \frac{C'_\varepsilon(1+C_\varepsilon)C_1(s,\varepsilon,\eta)}{\varepsilon}\frac{k_{m,\eta}(t)}{k_{m,\eta}(t)t^s}\log\left(1+C_2(s,\varepsilon,\eta)t^{\frac{\varepsilon}{\beta}}\right)^\eta \\ &= \tilde{C}_1(s,\varepsilon,\eta)t^{-s}\log\left(1+C_2(s,\varepsilon,\eta)t^{\frac{\varepsilon}{\beta}}\right)^\eta, \end{aligned} \tag{2.27}$$

with $\tilde{C}_1(s,\varepsilon,\eta) := \dfrac{C'_\varepsilon(1+C_\varepsilon)C_1(s,\varepsilon,\eta)}{\varepsilon}$.

**Step 2: removing** $m$. If $m=0$, there is nothing to be done. So, let $m \in \mathbb{N}$. It follows from the discussion presented in the beginning of **Step 1** that for each $x \in X$,

$$\begin{aligned} T(t)B^{-\nu}\log(1+B)^{-m+1}x &= \frac{1}{\varepsilon}T(t)\log(1+f_\varepsilon(B))B^{-\nu}\log(1+B)^{-m}x \\ &= \frac{1}{\varepsilon}T(t)\int_{0+}^\infty f_\varepsilon(B)(\lambda+f_\varepsilon(B))^{-1}P_m(B_0)x\,d\mu(\lambda), \end{aligned}$$

where $\mu$ now stands for the Borel measure related to the integral representation of $\log(1+\lambda)$ (which is a complete Bernstein function).

Let $\tau := \dfrac{\|T(t)P_m(B_\varepsilon)\|_{\mathcal{L}(X)}}{\|T(t)P_m(B_0)\|_{\mathcal{L}(X)}} > 0$. By proceeding as in **Step 1**, one gets from (2.27) that for each $t \geq 1$,

$$\|T(t)P_{m-1}(B_\varepsilon)\|_{\mathcal{L}(X)} \lesssim t^{-s+\varepsilon/\beta}\log\left(1+t\right)^\eta, \tag{2.28}$$

and

$$\|T(t)P_{m-1}(B_0)\|_{\mathcal{L}(X)} \lesssim \|T(t)P_m(B_0)\|_{\mathcal{L}(X)}\log(1+\tau). \tag{2.29}$$

Now, let $n_m(t) := \|T(t)P_m(B_0)\|_{\mathcal{L}(X)}$; it follows from (2.29) and (2.28) that there exists a positive constant $\tilde{c}_{s,\varepsilon}$ so that for each $t \geq 1$,

$$\frac{t^{-s+\frac{\varepsilon}{\beta}}\log\left(1+t\right)^\eta}{n_m(t)} \geq \tilde{c}_{s,\varepsilon}t^{\frac{\varepsilon}{\beta}}. \tag{2.30}$$

By letting $\gamma=1$, $\ell(w)=\log(1+w)$, $s=\tilde{c}_{s,\varepsilon}t^{\frac{\varepsilon}{\beta}}$ and $w=\frac{t^{-s+\frac{\varepsilon}{\beta}}\log(1+t)^\eta}{n_m(t)}$ in Proposition E.2.1 (note that $w \geq s$, by (2.30)), it follows from relation (2.30) that there exists $\tilde{c} > 0$ such that for each sufficiently large $t$,

$$\begin{aligned} \log(1+\tau) &\leq \log\left(1+\frac{\tilde{C}_{\varepsilon,s}t^{-s+\frac{\varepsilon}{\beta}}\log\left(1+t\right)^\eta}{n_m(t)}\right) \\ &\leq \tilde{c}\frac{\tilde{C}_{\varepsilon,s}t^{-s+\frac{\varepsilon}{\beta}}\log\left(1+t\right)^\eta}{\tilde{C}_{\varepsilon,s}\tilde{c}_{s,\varepsilon}t^{\frac{\varepsilon}{\beta}}n_m(t)}\log\left(1+C_{\varepsilon,s,\delta}\tilde{c}_{s,\varepsilon}t^{\frac{\varepsilon}{\beta}}\right) \\ &= \frac{\tilde{c}\log\left(1+t\right)^\eta}{n_m(t)\tilde{c}_{s,\varepsilon}t^s}\log\left(1+\tilde{C}_{\varepsilon,s}\tilde{c}_{s,\varepsilon}t^{\frac{\varepsilon}{\beta}}\right). \end{aligned} \tag{2.31}$$

Then, by (2.29) and (2.31), one has for each $t \geq 1$,

$$\begin{aligned} \|T(t)P_{m-1}(B_0)\|_{\mathcal{L}(X)} &\lesssim t^{-s}\log(1+t)^{\eta}\log\left(1+\tilde{C}_{\varepsilon,s}\tilde{c}_s t^{\frac{\varepsilon}{\beta}}\right) \\ &\lesssim t^{-s}\log(1+t)^{1+\eta}. \end{aligned}$$

By proceeding recursively over $m$, it follows from the previous discussion that for each $t \geq 1$,

$$\|T(t)B^{-\nu}\|_{\mathcal{L}(X)} \lesssim t^{-s}\log(1+t)^{m+\eta}.$$

**Case** $\eta = 0$. Since in this case $m \in \mathbb{N}$, one just needs to proceed as in **Step 2** of the case $\eta > 0$ in order to obtain, for each $t \geq 1$,

$$\|T(t)B^{-\nu}\|_{\mathcal{L}(X)} \lesssim t^{-s}\log(1+t)^{m}.$$

Hence, in both cases, relation (6) follows, and we are done. □

### 2.2.2 Resolvent growth slower than $\log(|\xi|)^b$

- **Case** $p \neq 2$

By using the same strategy presented in the proof of Theorem 2.2.1, we conclude that for $\beta = 0 < b$ (that is, for $\|(i\xi + A)^{-1}\|_{\mathcal{L}(X)} \lesssim \log(2+|\xi|)^b$, $\xi \in \mathbb{R}$), for each $s \geq 0$ and each $\delta > 0$, there exists $c_{s,\delta} > 0$ such that for each $t \geq 1$,

$$\|T(t)(1+A)^{-1/r}\log(1+B)^{-b(s+1)-\frac{1+\delta}{r}}\|_{\mathcal{L}(X)} \leq c_{s,\delta}t^{-s}.$$

Actually, it is possible to obtain in this setting a better estimate than the previous one. Namely, note that for each $x \in \mathcal{D}(A)$,

$$T(t)x = x + \int_0^t T(w)Axdw.$$

Let $x \in X_{1+1/r}(\upsilon)$, with $\upsilon := b(s+1) + \frac{1+\delta}{r}$. We argue that $[t \mapsto T(t)x]$ is a Lipschitz continuous function: it follows from the previous identity that for each $t, u \geq 0$, $\|T(t)x - T(u)x\| \lesssim |t - u|\|x\|_{X_{1/r}(\upsilon)}$, and since $X_{1+1/r}(\upsilon)$ is dense $X_{1/r}(\upsilon)$, one concludes that $\|T(t) - T(u)\|_{\mathcal{L}(X_1(\upsilon),X)} \lesssim |t-u|$.

Now, note that for each $x \in X$ and each $t > 0$, $f_x(t) = T(t)(1+A)^{-1/r}\log(1+B)^{-\upsilon}x$ satisfies the assumptions of Theorem 2.1 in [22] (with $F_x(s) = R(is, A)(1+A)^{-1/r}\log(1+B)^{-\upsilon}x$ and $M(s) = \log(2+|s|)^b$), so for any $c \in (0, 1/2)$ and $t_0$ such that for each $t \geq t_0$ and each $x \in X$ with $\|x\| = 1$, one has

$$\|T(t)(1+A)^{-1/r}\log(1+B)^{-\upsilon}x\| \lesssim \frac{1}{M^{-1}_{\log(ct)}} \lesssim e^{-ct^{\frac{1}{b+1}}}.$$

Therefore, for each $t \geq t_0$,

$$\|T(t)(1+A)^{-1/r}\log(1+B)^{-\upsilon}\|_{\mathcal{L}(X)} \leq e^{-ct^{\frac{1}{b+1}}}.$$

By proceeding as in the proof of Lemma 4.1 in [55], one can show that for each $\tau > 1/r$,

$$\|T(t)(1+A)^{-\tau}\log(1+B)^{-\upsilon\tau r}\|_{\mathcal{L}(X)} \leq e^{-c\tau r t^{\frac{1}{b+1}}}. \tag{2.32}$$

**Theorem 2.2.2.** Let $b \geq 0$ and let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup on a Banach space $X$ whose generator $-A$ satisfies $\overline{\mathbb{C}_-} \subset \rho(A)$. Suppose that $X$ has Fourier type $p \in [1,2)$ and that for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \leq 0$,

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim \log(2+|\lambda|)^b.$$

Let $r \in [1,\infty)$ be such that $1/r = 1/p - 1/p'$. Then, for each $\delta, \varepsilon > 0$ and each $\tau > 1/r$, there exists $c_{\tau,\delta,r,\varepsilon} \geq 0$ such that for each $t \geq 1$,

$$\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \leq c_{\tau,\delta,\varepsilon,r} e^{-c\tau r t^{\frac{1}{b+1}}} t^{\frac{\tau r}{b+1}(b(\varepsilon+1)+\frac{1+\delta}{r})}.$$

*Proof.* We proceed as in the proof of Theorem 10, by replacing relation (2.3) by relation (2.32). □

- **Case** $p = 2$

As in the case above, by using the same strategy presented in the proof of Theorem 2.2.1, we conclude that for $\beta = 0 < b$, for each $s \geq 0$ and each $\delta \in (0, 1/2)$, there exists $c_{s,\delta} > 0$ such that for each $t \geq 1$,

$$\|T(t)(1+A)^{-\delta}\log(1+B)^{-b(s+1)}\|_{\mathcal{L}(X)} \leq c_{s,\delta} t^{-s}.$$

Actually, it is possible to obtain in this setting a better estimate than the previous one. Namely, note that for each $x \in \mathcal{D}(A)$,

$$T(t)x = x + \int_0^t T(w)Axdw.$$

Let $x \in X_{1+\delta}(\upsilon)$, with $\upsilon := b(s+1)$. We argue that $[t \mapsto T(t)x]$ is a Lipschitz continuous function: it follows from the previous identity that for each $t, u \geq 0$, $\|T(t)x - T(u)x\| \lesssim |t-u|\|x\|_{X_\delta(\upsilon)}$, and since $X_{1+\delta}(\upsilon)$ is dense $X_\delta(\upsilon)$, one concludes that $\|T(t) - T(u)\|_{\mathcal{L}(X_\delta(\upsilon),X)} \lesssim |t-u|$.

Now, note that for each $x \in X$ and each $t > 0$, $f_x(t) = T(t)(1+A)^{-\delta}\log(1+B)^{-\upsilon}x$ satisfies the assumptions of Theorem 2.1 in [22] (with $F_x(s) = R(is,A)(1+A)^{-\delta}\log(1+B)^{-\upsilon}x$ and $M(s) = \log(2+|s|)^b$), so there exists $t_0(\delta) \geq 1$ such that for each $t \geq t_0$ and each $x \in X$ with $\|x\| = 1$, one has

$$\|T(t)(1+A)^{-\delta}\log(1+B)^{-\upsilon}x\| \lesssim \frac{1}{M^{-1}_{\log}(\delta t)} \lesssim e^{-\delta t^{\frac{1}{b+1}}}.$$

Therefore, for each $t \geq t_0$,

$$\|T(t)(1+A)^{-\delta}\log(1+B)^{-\upsilon}\|_{\mathcal{L}(X)} \lesssim e^{-\delta t^{\frac{1}{b+1}}}.$$

Let $\tau > 0$ and $\delta \in (0, 1/2)$ be such that $\tau > \delta > 0$, then by proceeding as in the proof of Lemma 4.1 in [55], one can show ,

$$\|T(t)(1+A)^{-\tau}\log(1+B)^{-\upsilon\tau/\delta}\|_{\mathcal{L}(X)} \leq e^{-\tau t^{\frac{1}{b+1}}}.$$

**Theorem 2.2.3.** Let $b \geq 0$ and let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup on a Hilbert space $X$ whose generator $-A$ satisfies $\overline{\mathbb{C}_-} \subset \rho(A)$. Suppose that for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \leq 0$,

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim \log(2 + |\lambda|)^b.$$

Let $\varepsilon, \tau > 0$ and $\delta \in (0, 1/2)$ be such that $\tau > \delta > 0$. Then, there exists $c_{\tau,\varepsilon,\delta} \geq 0$ such that for each $t \geq 1$,

$$\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \leq c_{\tau,\varepsilon,\delta} e^{-\tau t^{\frac{1}{b+1}}} t^{\frac{\tau b(\varepsilon+1)}{\delta(b+1)}}.$$

*Proof.* We proceed as in the proof of Theorem 10, by replacing relation (2.3) by relation (2.32). □

**Example 2.2.1.** Let $\mu \geq 0$ and suppose that for each $t \geq 1$, $\|T(t)\|_{\mathcal{L}(X)} \lesssim t^{\mu}$. Now, let $a > 0$ and define, for each $t \geq 0$, $S_a(t) := e^{-at}T(t)$. Then,

$$\sup_{t\geq 0} \|S_a(t)\|_{\mathcal{L}(X)} \lesssim \sup_{t\geq 0}\{e^{-at}t^{\mu}\} \lesssim a^{-\mu}.$$

It follows from Theorem 4 that for each $\tau > 0$ and $t \geq 0$,

$$\|S_a(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \lesssim \sup_{t\geq 0} \|S_a(t)\|_{\mathcal{L}(X)} t^{-\tau/\beta}\log(1+t)^{\tau/\beta} \lesssim a^{-\mu}t^{-\tau/\beta}\log(1+t)^{\tau/\beta},$$

so, by setting $a := 1/t$, one has

$$\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \lesssim t^{\mu-\tau/\beta}\log(1+t)^{\tau/\beta}. \tag{2.33}$$

Note that for each $\tau > \beta + 1$, it follows from Corollary 3 that

$$\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \lesssim t^{1-\frac{\tau-1}{\beta}}\log(1+t)^{\tau/\beta}, \tag{2.34}$$

with $\delta = \tau/\beta - 1 > 0$. The estimate (2.34) improves (2.33) if, and only if, $\mu \geq 1 + \frac{1}{\beta}$.

The next example shows that one cannot expect to improve the power-law presente in Corollary 2 (see Exemple 4.20 in [55]).

**Example 2.2.2.** We show that for each $\delta, \gamma \in (0, 1)$, there exists an operator $A$ be as in the statement of Corollary 2 such that $\|T(\cdot)\|_{\mathcal{L}(X_\tau, X)}$ is unbounded for each $\tau \in \left[0, \frac{1-\gamma}{\log(1/\delta)}\right)$. Set

$\beta_0 := \frac{\log(1/\gamma)}{\log(1/\delta)}$ and for $n \in \mathbb{N}$, let the $n \times n$ matrix $B_n$ be given by

$$B_n(k,l) := \begin{cases} 1 & l = k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $m(n) := \left\lfloor \dfrac{\log(n)}{\log(1/\delta)} \right\rfloor$, let $n_0 \in \mathbb{N}$ be such that $m(n_0) \geq 2$ and let the Hilbert space $X = \bigoplus_{n \geq n_0} \ell^2_{m(n)}$ be the $\ell^2$ direct sum of the $m(n)$-dimensional $\ell^2_{m(n)}$ spaces for $n \geq n_0$. Let, for each $n \geq n_0$,

$$A_n := -i\frac{n}{\log(n)} + \gamma - B_{m(n)}$$

and define the operator $A$ by the law $Ax := (A_n x_n)_{n \geq m(n_0)}$, with

$$\mathcal{D}(A) := \left\{ x = (x_n)_{n \geq m(n_0)} \mid \sum_{n \geq m(n_0)} \frac{n^2}{\log(n)^2} |x_n|^2 < \infty \right\}.$$

Let also define the family of bounded linear operators on $X$, $(T(t))_{t \geq 0}$, by the law $T(t)x := (e^{-tA_n} x_n)_{n \geq m(n_0)}$, and note that it is a $C_0$-semigroup (namely, $-A$ generate $(T(t))_{t \geq 0}$).

Now, we show that $A$ satisfies the hypotheses of Corollary 2.

**Claim:** $\overline{\mathbb{C}_-} \subset \rho(A)$ and that there exists a $C > 0$ such that, for each $\eta \geq 0$ and $\xi \in \mathbb{R}$

$$\|(\eta + i\xi + A)^{-1}\|_{\mathcal{L}(X)} \leq C(|\xi|^{\beta_0} \log(2 + |\xi|)^{\beta_0} + 1). \tag{2.35}$$

Namely, for each $\eta \geq 0$ and $\xi \in \mathbb{R}$, set $z := \eta + i\xi$ and note that $B_{m(n)}^{m(n)} = 0 \in \mathcal{L}(\ell^2_{m(n)})$ and $\|B_{m(n)}\|_{\mathcal{L}(\ell^2_{m(n)})} = 1$, for each $n \geq n_0$. Moreover,

$$\left\| \left( z - i\frac{n}{\log(n)} + \gamma - B_{m(n)} \right)^{-1} \right\|_{\mathcal{L}(\ell^2_{m(n)})} \leq \sum_{k=0}^{m(n)-1} \frac{\|B^k_{m(n)}\|_{\mathcal{L}(\ell^2_{m(n)})}}{|z - i(n/\log(n)) + \gamma|^{k+1}}.$$

Let $n_1 \in \mathbb{N}$ be such that $n_1 \geq n_0$ and $\left| \frac{n_1}{\log(n_1)} - \xi \right| = \min\{|n/\log(n) - \xi| \mid n \in \mathbb{N}, n \geq n_0\}$. Note that $|z - in/\log(n) + \gamma| \geq \gamma$, for each $n \in \mathbb{N}$. Hence, in case $\xi \geq 0$ and $n \in \{n_0, \ldots, n_1 + 1\}$, one has

$$\begin{aligned} \|(z - i(n/\log(n)) + \gamma - B_{m(n)})^{-1}\|_{\mathcal{L}(\ell^2_{m(n)})} &\leq \sum_{k=0}^{m(n)-1} \frac{1}{\gamma^{k+1}} = \frac{\gamma^{-m(n)} - 1}{1 - \gamma} \\ &\leq \frac{\gamma^{-m(n_1+1)}}{1-\gamma} \lesssim (n_1 + 1)^{\beta_0} \\ &\leq (f^{-1}(\xi))^{\beta_0} + 1 \leq (|\xi| \log(2 + |\xi|)^{\beta_0} + 1, \end{aligned}$$

with $f(s) = s/\log(s+2)$, where we have used that $f(n_1) \leq \xi + 2$.

Now, in case $\xi < 0$ or $n \geq n_1 + 2$, then $|z - in/\log(n) + \gamma| \geq c_\gamma := \sqrt{1+\gamma^2} > 1$. Therefore,

$$\|(z - i(n\log(n)) + \gamma - B_{m(n)})^{-1}\|_{\mathcal{L}(\ell^2_{m(n)})} \leq \sum_{k=0}^{\infty} \frac{1}{c_\gamma^{k+1}} < \infty.$$

and now (2.35) follows.

We now show that $\|T(t)\|_{\mathcal{L}(X_\tau,X)}$ is unbounded for $\tau \in \left[0, \frac{1-\gamma}{\log(1/\delta)}\right)$. First note that $\|T(t)\|_{\mathcal{L}(X_\tau,X)} \geq \frac{\|T(t)x\|}{\|(1+A)^\tau x\|}$ for each $x \in X_\tau$ with $\|(1+A)^\tau x\| \leq \|x\|_{X_\tau} = 1$. Let $n \geq n_0$ and let $x := (x_k)_{k\geq n_0} \in X$ be such that $x_k = 0$ for each $k \neq m(n)$ and $x_{m(n)} = (0,0,0,\ldots,0,1)$. Then, for each $\tau \in \mathbb{N}\cup\{0\}$, Newton's Binomial Formula yields

$$\|(1+A)^\tau x\| = \|(1+\gamma - in/\log(n) - B_{m(n)})^\tau x\| \lesssim \left(\frac{n}{\log(n)}\right)^\tau. \tag{2.36}$$

Then, it follows from Proposition 1.3.2 that (2.36) follows for each $\tau \geq 0$. Now, set $t := m(n) - 1 \geq 1$. Then, by Lemma A.2 in [55],

$$\|T(t)x\| = e^{-\gamma t}\|e^{tB_{m(n)}}x_n\|_{\mathcal{L}(\ell^2_{m(n)})} = e^{-\gamma t}\left(\sum_{k=0}^{m(n)-1}\left(\frac{t^k}{k!}\right)^2\right)^{1/2} \gtrsim \frac{e^{(1-\gamma)m(n)}}{m(n)^{1/4}} \gtrsim \frac{n^{\frac{1-\gamma}{\log(1/\delta)}}}{m(n)^{1/4}}.$$

By combining this with relation (2.36), one gets

$$\frac{\|T(t)x\|}{\|(1+A)^\tau x\|} \gtrsim \frac{n^\nu}{\log(n)^{1/4-\tau}} = ct^{\tau-1/4}e^{\nu t}$$

with $\nu := \dfrac{1-\gamma}{\log(1/\delta)} - \tau$, where the implicit constant does not depend on $n \geq n_0$ and $t \geq 1$.

## 2.3 Singularity at infinity and zero

Let $\mu, \nu, \upsilon \geq 0$ and $A \in \mathrm{Sect}_X(\omega_A)$; it is known that $2\pi - i\log(A)$ is sectorial (see page 92 in [32]), and so $(2\pi - i\log(A))^{-\upsilon}$ is well-defined, by the functional calculus of fractional powers (see [32, 41]). Define the operator

$$\Phi^\mu_\nu(\upsilon) = \Phi^\mu_\nu(A,\upsilon) := A^\mu(1+A)^{-\mu-\nu}(2\pi - i\log(A))^{-\upsilon} \in \mathcal{L}(X),$$

and the space $X^\mu_\nu(\upsilon) := \mathrm{Ran}(\Phi^\mu_\nu(\upsilon))$. If $A$ is injective, then the space $X^\mu_\nu(\upsilon)$ is a Banach space with the norm

$$\|x\|_{X^\mu_\nu(\upsilon)} = \|x\| + \|\Phi^\mu_\nu(\upsilon)^{-1}x\| = \|x\| + \|(2\pi - i\log(A))^\upsilon(1+A)^\nu A^\mu x\|, \quad \forall\, x \in X^\mu_\nu(\upsilon).$$

Moreover, $\Phi^\mu_\nu(\upsilon) : X \to X^\mu_\nu(\upsilon)$ is an isomorphism and so

$$\|T\|_{\mathcal{L}(X^\mu_\nu(\upsilon),X)} \leq \|T\Phi^\mu_\nu(\upsilon)\|_{\mathcal{L}(X)} \leq \|\Phi^\mu_\nu(\upsilon)\|_{\mathcal{L}(X)}\|T\|_{\mathcal{L}(X^\mu_\nu(\upsilon),X)}, \quad T \in \mathcal{L}(X^\mu_\nu(\upsilon), X). \tag{2.1}$$

Note that $\Phi_\nu^\mu(0) = \Phi_\nu^\mu(A)$ and $X_\nu^\mu(0) = X_\nu^\mu$, where $\Phi_\nu^\mu(A)$ and $X_\nu^\mu$ are the objects defined in [55].

**Theorem 2.3.1.** Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup defined in the Banach space $X$ with Fourier type $p \in [1,2]$, with $-A$ as its generator. Suppose $A$ injective, $\overline{\mathbb{C}_-} \setminus \{0\} \subset \rho(A)$ and that there exist $\alpha \geq 1$, $\beta > 0$, $a, b \geq 0$ such that, for each $\lambda \in \overline{\mathbb{C}_-} \setminus \{0\}$,

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim \begin{cases} |\lambda|^{-\alpha} \log(1/|\lambda|)^a, & |\lambda| \leq 1 \\ |\lambda|^{\beta} \log(|\lambda|)^b, & |\lambda| \geq 1. \end{cases}$$

Let $\sigma, \tau$ be such that $\sigma \geq \alpha - 1$ and $\tau \geq \beta + 1/r$. Then, for each $\rho \in \left[0, \min\left\{\frac{\sigma+1}{\alpha} - 1, \frac{\tau - r^{-1}}{\beta} - 1\right\}\right]$ and each $\delta > 1 - 1/r$, where $r \in [1, \infty]$ is such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$, there exist $C_{\rho,\delta} > 0$ and $t_0 > 1$ so that for each $t \geq 1$,

$$\|T(t)A^\sigma(1+A)^{-\sigma-\tau}(2\pi - i\log(A))^{-c(\lceil\rho\rceil+1)-1/r-\delta}\|_{\mathcal{L}(X)} \leq C_{\rho,\delta} t^{-\rho}, \tag{2.2}$$

with $c = \max\{a, b\}$.

In order to prove Theorem 2.3.1, some preparation is required. The next result is the version of Proposition 2.2.1 in this setting, and consists in the result stated in Theorem 2.3.1 in case $p = 1$.

**Proposition 2.3.1.** Let $A$ be an injective sectorial operator defined in the Banach space $X$ such that $-A$ generates the $C_0$-semigroup $(T(t))_{t\geq 0}$ on $X$. Suppose that there exist $\alpha \geq 1$, $\beta > 0$, $a, b \geq 0$ such that, for each $\lambda \in \overline{\mathbb{C}_-} \setminus \{0\}$

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim \begin{cases} |\lambda|^{-\alpha} (\log(1/|\lambda|))^a; & |\lambda| \leq 1 \\ |\lambda|^{\beta} \log(|\lambda|)^b; & |\lambda| \geq 1. \end{cases} \tag{2.3}$$

Let $\sigma \geq \alpha - 1$ and $\tau \geq \beta + 1$. Then, for each $\delta > 0$ and each $\rho \in \left[0, \min\left\{\frac{\sigma+1}{\alpha} - 1, \frac{\tau-1}{\beta} - 1\right\}\right]$, there exists $c_{\rho,\delta} > 0$ such that for each $t \geq 1$,

$$\|T(t)A^\sigma(1+A)^{-\sigma-\tau}(2\pi - i\log(A))^{-c(\lceil\rho\rceil+1)-1-\delta)}\|_{\mathcal{L}(X)} \leq c_{\rho,\delta} t^{-\rho}, \tag{2.4}$$

where $c = \max\{a, b\}$.

*Proof.* Let $n \in \mathbb{N} \cup \{0\}$ and set $\mu := \alpha(n+1) - 1$, $\nu := (n+1)\beta + 1$, $\upsilon := c(n+1) + 1 + \delta$. For each $x \in X_{\nu+1}^\mu(\upsilon)$, let

$$\begin{aligned} y := (\Phi_\nu^\mu(\upsilon))^{-1}x &= (2\pi - i\log(A))^\upsilon (1+A)^{\nu+\mu} A^{-\mu} x \\ &= (2\pi - i\log(A))^\upsilon (1+A)^{\nu+\mu} A^{-\mu} \left(A^\mu (1+A)^{-\mu-\nu-1} (2\pi - i\log(A))^{-\upsilon} z\right) \\ &= B^{-1} z \in \mathcal{D}(A), \end{aligned}$$

where $z := (\Phi^{\mu}_{\nu+1}(\upsilon))^{-1}x$. Let $g : [0, \infty) \to X$ be defined by the law

$$g(t) := \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{\lambda^{\mu}}{(1+\lambda)^{\nu+\mu}(2\pi - i\log(\lambda))^{\upsilon}}R(\lambda, A)yd\lambda.$$

Note that for each $t \geq 0$, $g(t)$ is indeed an element of $X$ and it is differentiable. Namely, since $y \in \mathcal{D}(A)$, then $\lambda \mapsto \dfrac{\lambda^{\mu+1}}{(1+\lambda)^{\nu+\mu}(2\pi - i\log(\lambda))^{\upsilon}}R(\lambda, A)y$ is integrable in $i\mathbb{R}$. Therefore, by dominated convergence, one gets

$$g'(t) = -\frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{\lambda^{\mu+1}}{(1+\lambda)^{\nu+\mu}(2\pi - i\log(\lambda))^{\upsilon}}R(\lambda, A)yd\lambda.$$

Moreover, by Lemma D.0.1,

$$\begin{aligned}
g'(t) &= \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{\lambda^{\mu}}{(1+\lambda)^{\nu+\mu}(2\pi - i\log(\lambda))^{\upsilon}}(-AR(\lambda, A)y - y)d\lambda \\
&= -A\frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{\lambda^{\mu}}{(1+\lambda)^{\nu+\mu}(2\pi - i\log(\lambda))^{\upsilon}}R(\lambda, A)yd\lambda - \\
&- \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{\lambda^{\mu}}{(1+\lambda)^{\nu+\mu}(2\pi - i\log(\lambda))^{\upsilon}}yd\lambda \\
&= -Ag(t)
\end{aligned}$$

Then, $g'(t) = -Ag(t)$ for each $t \geq 0$, and $g(0) = x$. Therefore, $g(t) = T(t)x$, by the uniqueness of the solution to the Cauchy problem associated with $-A$.

Integration by parts yields

$$t^n T(t) = \frac{1}{2\pi i}\int_{i\mathbb{R}} e^{-\lambda t}p(\lambda, A)yd\lambda,$$

where $p(\lambda, A)$ is a finite linear combination of terms of the form

$$\frac{\lambda^{\mu-k}R(\lambda, A)^{n-l+1}}{(1+\lambda)^{\mu+\nu+(l-k)}(2\pi - i\log(\lambda))^{\upsilon+j}} \quad \text{and} \quad \frac{\lambda^{\mu-k-m}R(\lambda, A)^{n-l+1}}{(1+\lambda)^{\mu+\nu+j}(2\pi - i\log(\lambda))^{\upsilon+(l-k)}},$$

where $0 \leq j \leq k \leq l \leq n$ and $l - k \leq m \leq l - k + 1$.

Then, for each $t > 0$,

$$\begin{aligned}
\|t^n T(t)x\| &\leq \frac{1}{2\pi}\int_{i\mathbb{R}} |e^{-\lambda t}|\|p(\lambda, A)y\|d\lambda \\
&\leq \frac{1}{2\pi}\int_{i\mathbb{R}} \|p(\lambda, A)\|_{\mathcal{L}(X)}d\lambda\|y\| \leq C\|(\Phi^{\mu}_{\nu}(\upsilon))^{-1}x\| \lesssim \|x\|_{X^{\mu}_{\nu}(\upsilon)}.
\end{aligned}$$

Since $X^{\mu}_{\nu+1}(\upsilon)$ is dense in $X^{\mu}_{\nu}(\upsilon)$, it follows from the previous discussion that for each $t \geq 1$,

$$\|T(t)\|_{\mathcal{L}(X^{\mu}_{\nu}(\upsilon), X)} \lesssim t^{-n}.$$

In general, for each $s \geq 0$, let $n \in \mathbb{N} \cup \{0\}$ be such that $n < s < n + 1$; then, there exists $\theta = \theta(s) \in (0, 1)$ so that $s = (1 - \theta)n + \theta(n + 1)$. Let $\alpha_1 := \alpha(n + 1) - 1$, $\alpha_2 := \alpha(n + 2) - 1$,

$\beta_1 := \beta(n+1)+1$ and $\beta_2 := \beta(n+2)+1$, then $\alpha(s+1)-1 = (1-\theta)\alpha_1 + \theta\alpha_2$ and $\beta(s+1)+1 = (1-\theta)\beta_1 + \theta\beta_2$. Set $\tilde{v} := c(\lceil s \rceil + 1) + 1 + \delta$. Then, by a moment-like inequality (Lemma 4.2 in [55]), it follows that

$$\begin{aligned}\|T(t)\Phi^{\mu}_{\nu}(A)(2\pi - i\log(A))^{-\tilde{v}}\|_{\mathcal{L}(X)} &\lesssim \|T(t)\Phi^{\alpha_1}_{\beta_1}(A)(2\pi - i\log(A))^{-\tilde{v}}\|^{1-\theta}_{\mathcal{L}(X)} \cdot \\ &\quad \|T(t)\Phi^{\alpha_2}_{\beta_2}(A)(2\pi - i\log(A))^{-\tilde{v}}\|^{1-\theta}_{\mathcal{L}(X)} \lesssim t^{-s}.\end{aligned}$$

□

The following result is analogous to Proposition 2.2.2; its proof is presented in Appendix C.

**Proposition 2.3.2.** Let $A \in \mathrm{Sect}_X(\omega_A)$, with $\overline{\mathbb{C}_-} \setminus \{0\} \subset \rho(A)$. The following statements hold:

(a) Let $\alpha \geq 1$ and $a \geq 0$ be such that, for each $\lambda \in \overline{\mathbb{C}_-} \setminus \{0\}$ with $|\lambda| < 1$, one has

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim |\lambda|^{-\alpha}(\log(1/|\lambda|))^{a}; \tag{2.5}$$

then, for each $\zeta > 1$,

$$\left\{\lambda(2\pi - i\log(\lambda))^{\zeta}(\lambda + A)^{-1} \mid \lambda \in i\mathbb{R} \setminus \{0\}, |\lambda| < 1\right\} \subset \mathcal{L}(X^{\alpha-1}(a+\zeta), X)$$

is uniformly bounded.

(b) Let $\alpha \geq 1$, $\beta \geq 0$, $\beta_0 \in [0,1)$ and $b \geq 0$. If

$$\sup\{|\lambda|^{-\beta}\log(1+|\lambda|)^{-b}\|(\lambda + A)^{-1}\| \mid \lambda \in \overline{\mathbb{C}_-} \setminus \{0\}, |\lambda| \geq 1\} < \infty, \tag{2.6}$$

then for each $\zeta > 1$,

$$\left\{\lambda^{\beta_0}(2\pi - i\log(\lambda))^{\zeta}(\lambda + A)^{-1} \mid \lambda \in i\mathbb{R}, |\lambda| \geq 1\right\} \subset \mathcal{L}(X^{\alpha}_{\beta+\beta_0}(\zeta + b), X) \tag{2.7}$$

is uniformly bounded.

**Proposition 2.3.3.** Let $A \in \mathrm{Sect}_X(\omega_A)$ be such that $\overline{\mathbb{C}_-} \setminus \{0\} \subset \rho(A)$ and let $\alpha \geq 1$, $\beta, a, b \geq 0$. Then,

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim \begin{cases} |\lambda|^{-\alpha}\log(1/|\lambda|)^{a}, & |\lambda| \leq 1 \\ |\lambda|^{\beta}\log(|\lambda|)^{b}, & |\lambda| \geq 1 \end{cases} \tag{2.8}$$

implies

$$\sup\{\|(\lambda + A)^{-1}\|_{\mathcal{L}(X^{\alpha n}_{\beta n}(cn), X)} \mid \lambda \in i\mathbb{R} \setminus \{0\}\} < \infty,$$

where $c = \max\{a, b\}$ and $n \in \mathbb{N}$.

*Proof.* We consider the following cases.

- **Case 1:** $\alpha = 1$.

**Case 1(a):** $c \in (0,1)$. Note that for each $\lambda \in \rho(-A)$,

$$(\lambda+A)^{-1}A(1+A)^{-1-\beta}(2\pi-i\log(A))^{-c} = \frac{1}{1+\lambda}\left((1+A)^{-1-\beta} - \lambda(\lambda+A)^{-1}(1+A)^{-\beta}\right)(2\pi-i\log(A))^{-c}. \tag{2.9}$$

By the moment inequality (recall that $(2\pi - i\log(A))^{-1}$ is sectorial) and by (C.3), it follows from (2.8) that for each $|\lambda| < 1$,

$$\begin{aligned}
&\|\lambda(\lambda+A)^{-1}(1+A)^{-\beta}(2\pi - i\log(A))^{-c}\|_{\mathcal{L}(X)} \lesssim \|\lambda(\lambda+A)^{-1}(1+A)^{-\beta}\|_{\mathcal{L}(X)}^{1-c} \\
\cdot\ &\|\lambda(\lambda+A)^{-1}(1+A)^{-\beta}(2\pi - \log(A))^{-1}\|_{\mathcal{L}(X)}^{c} \\
&\lesssim \log(1/|\lambda|)^{c(1-c)}\left(\left\|\frac{\lambda(\lambda+A)^{-1}}{(2\pi - i\log(-\lambda))}\right\|_{\mathcal{L}(X)} + \frac{|\lambda|-1}{\log(|\lambda|)}\right)^{c} = \left(\frac{\left\|\frac{\lambda\log(|\lambda|)(\lambda+A)^{-1}}{(2\pi-i\log(-\lambda))}\right\|_{\mathcal{L}(X)}}{|\log(|\lambda|)|^{c}} + \frac{1-|\lambda|}{|\log(|\lambda|)|^{c}}\right)^{c} \\
&\lesssim \left(\left\|\frac{\lambda(\lambda+A)^{-1}}{\log(|\lambda|)^{c}}\right\|_{\mathcal{L}(X)} + \frac{1-|\lambda|}{|\log(|\lambda|)|^{c}}\right)^{c}
\end{aligned} \tag{2.10}$$

Then, it follows from relations (2.9) and (2.10) that for each $\lambda \in i\mathbb{R} \setminus \{0\}$,

$$\sup\{\|(\lambda+A)^{-1}A(1+A)^{-1-\beta}(2\pi - i\log(A))^{-c}\|_{\mathcal{L}(X)} \mid \lambda \in i\mathbb{R}\setminus\{0\}, |\lambda| < 1\} < \infty.$$

Now, note that $A(1+A)^{-1}$ commutes with $(2\pi - i\log(A))^{-1}$, and by Closed Graph Theorem, $\log(2+A)^{c}(2\pi - i\log(A))^{-c} \in \mathcal{L}(X)$; thus,

$$\begin{aligned}
&\frac{|\lambda|}{|1+\lambda|}\|(\lambda+A)^{-1}A(1+A)^{-1-\beta}(2\pi - i\log(A))^{-c}\|_{\mathcal{L}(X)} \\
&\lesssim \frac{|\lambda|}{|1+\lambda|}\|(\lambda+A)^{-1}(1+A)^{-\beta}(\log(2+A))^{-c}\|_{\mathcal{L}(X)},
\end{aligned}$$

and so, it follows from Proposition 2.2.2 that

$$\sup_{\lambda\in i\mathbb{R}, |\lambda|\geq 1}\left\{\frac{|\lambda|}{|1+\lambda|}\|(\lambda+A)^{-1}A(1+A)^{-1-\beta}(2\pi - i\log(A))^{-c}\|_{\mathcal{L}(X)}\right\} < \infty.$$

**Case 1(b):** $c = 1$. It follows from (2.8) and (C.3) that for each $\lambda \in i\mathbb{R}$ with $|\lambda| \leq 1$,

$$\|\lambda(\lambda+A)^{-1}A(1+A)^{-1-\beta}(2\pi - i\log(A))^{-1}\|_{\mathcal{L}(X)} \lesssim \left\|\frac{\lambda(\lambda+A)^{-1}}{(2\pi - i\log(-\lambda))}\right\|_{\mathcal{L}(X)} + \frac{|\lambda|-1}{\log(|\lambda|)} \lesssim 1.$$

For $\lambda \in i\mathbb{R}$ with $|\lambda| \geq 1$, one just proceeds as in **Case 1(a)**.

**Case 1(c):** $c > 1$. Note that for each $\lambda \in \rho(-A)$,

$$\begin{aligned}(\lambda+A)^{-1}A(1+A)^{-1-\beta}(2\pi - i\log(A))^{-c} &= (1-\lambda(\lambda+A)^{-1})A(1+A)^{-2-\beta}(2\pi - i\log(A))^{-c} \\ &+ (\lambda+A)^{-1}A(1+A)^{-2-\beta}(2\pi - i\log(A))^{-c} \\ &= A(1+A)^{-2-\beta}(2\pi - i\log(A))^{-c} \\ &+ (1-\lambda)(\lambda+A)^{-1}A(1+A)^{-2-\beta}(2\pi - i\log(A))^{-c}.\end{aligned} \tag{2.11}$$

Now, by Remark 1.3.2, one has

$$\begin{aligned}(\lambda+A)^{-1}A(1+A)^{-2-\beta}(2\pi - i\log(A))^{-c} &= \frac{1}{2\pi i}\int_\Gamma \frac{zR(z,A)}{(1+z)^{2+\beta}(2\pi - i\log(z))^c}dz(\lambda+A)^{-1} \\ &= \frac{(-\lambda)}{(2\pi - i\log(-\lambda))^c(1-\lambda)^{2+\beta}}(\lambda+A)^{-1} \\ &+ \frac{1}{2\pi i}\int_\Gamma \frac{zR(z,A)}{(1+z)^{2+\beta}(2\pi - i\log(z))^c(\lambda+z)}dz,\end{aligned} \tag{2.12}$$

where $\Gamma$ is given as in the proof of Proposition 2.2.2. Since $A$ is sectorial, one can replace $\|R(z,A)\|_{\mathcal{L}(X)}$ by $1/|z|$, and so the function $z \mapsto (2\pi - i\log(z))^{-c}R(z,A)$ is integrable on $\Gamma$ (recall that $c > 1$). Now, by letting $\gamma = \delta = 1$ in Lemma A.1 in [55], it follows that for each $z \in \Gamma$ and $\lambda \in i\mathbb{R}\setminus\{0\}$,

$$\left|\frac{z(1-\lambda)}{(1+z)^{2+\beta}(z+\lambda)}\right| \lesssim 1. \tag{2.13}$$

Now, by (2.8) and relations (2.11), (2.12) and (2.13), it follows that

$$\begin{aligned}\|(\lambda+A)^{-1}A(1+A)^{-1-\beta}(2\pi - i\log(A))^{-c}\|_{\mathcal{L}(X)} &\leq \|A(1+A)^{-2-\beta}(2\pi - i\log(A))^{-c}\|_{\mathcal{L}(X)} \\ &+ |1-\lambda|\|(\lambda+A)^{-1}A(1+A)^{-2-\beta}(2\pi - i\log(A))^{-c}\|_{\mathcal{L}(X)} \\ &\lesssim \left\|\frac{(-\lambda)(\lambda+A)^{-1}}{(2\pi - i\log(-\lambda))^c(1-\lambda)^{1+\beta}}\right\|_{\mathcal{L}(X)} + 1 \lesssim 1.\end{aligned}$$

- **Case 2:** $\alpha > 1$. Let $c > 0$, and notice that

$$\begin{aligned}(\lambda+A)^{-1}A^\alpha(1+A)^{-\alpha-\beta}(2\pi - i\log(A))^{-c} &= (\lambda+A)^{-1}(A+1)A^\alpha(1+A)^{-\alpha-\beta-1}(2\pi - i\log(A))^{-c} \\ &= A^\alpha(1+A)^{-\alpha-\beta-1}(2\pi - i\log(A))^{-c} \\ &+ (1-\lambda)A^\alpha(1+A)^{-\alpha-\beta-1}(2\pi - i\log(A))^{-c}\end{aligned}$$

Hence, by Remark 1.3.2, one has

$$
\begin{aligned}
(\lambda+A)^{-1}A^{\alpha}(1+A)^{-\alpha-\beta-1}(2\pi-i\log(A))^{-c} &= \frac{1}{2\pi i}\int_{\Gamma}\frac{z^{\alpha}R(z,A)}{(1+z)^{\alpha+\beta+1}(2\pi-i\log(z))^{c}}dz(\lambda+A)^{-1} \\
&= \frac{(-\lambda)^{\alpha}}{(2\pi-i\log(-\lambda))^{c}(1-\lambda)^{\alpha+\beta+1}} \\
&+ \frac{1}{2\pi i}\int_{\Gamma}\frac{z^{\alpha}R(z,A)}{(1+z)^{\alpha+\beta+1}(2\pi-i\log(z))^{c}(\lambda+z)}dz.
\end{aligned}
$$

It follows from (2.8) that the first term in the right-hand side of the previous relation is bounded. As for the second term, let $\varepsilon\in(0,\min\{\alpha-1,1\})$ and consider the map $z\mapsto\frac{z^{\varepsilon}}{(1+z)^{2\varepsilon}}R(z,A)$, which in integrable over $\Gamma$; then, by letting $\gamma=\alpha-\varepsilon$ and $\delta=\beta+1-\varepsilon$ in Lemma A.1 in [55], one gets

$$
\sup\left\{\frac{|z|^{\alpha-\varepsilon}|1-\lambda|}{|(1+z)^{\alpha+\beta+1-2\varepsilon}(2\pi-i\log(z))^{c}(\lambda+z)|}\mid z\in\Gamma,\lambda\in i\mathbb{R}\setminus\{0\}\right\}<\infty
$$

(note that $(2\pi-i\log(z))^{-c}$ is uniformly bounded over $\Gamma$). Therefore,

$$
\sup_{\lambda\in i\mathbb{R}\setminus\{0\}}\|(\lambda+A)^{-1}A^{\alpha}(1+A)^{-\alpha-\beta}(2\pi-i\log(A))^{-c}\|_{\mathcal{L}(X)}<\infty.
$$

□

***Proof of Theorem 2.3.1***. We follow the same arguments presented in the proof of Theorem 2.2.1. Let $n\in\mathbb{N}\cup\{0\}$ and set $\mu:=(n+1)\alpha-1$, $\nu:=(n+1)\beta+\frac{1}{r}$ and $\upsilon:=c(n+1)+1/r+\delta$. By Proposition 2.3.3 one has, for each $k\in\{1,\dots,n\}$,

$$
\sup_{\xi\in\mathbb{R}}\|R(i\xi,A)^{k}\|_{\mathcal{L}(X^{\alpha n}_{\beta n}(cn),X)}<\infty. \tag{2.14}
$$

Let $h_{r,\delta}:\mathbb{R}\to\mathbb{R}$ be given by the law $h_{r,\delta}(\xi)=\frac{|\xi|}{(1+|\xi|)^{1-\frac{1}{r}}}(2\pi+|\log(|\xi|)|)^{\frac{1}{r}+\delta}$, and note that for each $\xi\in\mathbb{R}$,

$$
\begin{aligned}
h_{r,\delta}(\xi)\|R(i\xi,A)^{k}\|_{\mathcal{L}(X^{\mu}_{\nu}(\upsilon),X)} &\lesssim h_{r,\delta}(\xi)\|R(i\xi,A)^{k}B^{-(\alpha(n+1)-1+\beta(n+1)+\frac{1}{r})}(2\pi-i\log(A))^{-\upsilon}\|_{\mathcal{L}(X)} \\
&\leq h_{r,\delta}(\xi)\|R(i\xi,A)A^{\alpha-1}B^{-(\alpha-1+\beta+\frac{1}{r})}(2\pi-i\log(A))^{-c-\frac{1}{r}-\delta}\|_{\mathcal{L}(X)} \\
&\cdot \|R(i\xi,A)^{k-1}A^{\alpha n}B^{-(\alpha+\beta)n}(2\pi-i\log(A))^{-cn}\|_{\mathcal{L}(X)},
\end{aligned}
$$

where $B:=A+1$. It follows from Proposition 2.3.2 and relation (2.14) that for each $k\in\{1,\dots,n+1\}$,

$$
\sup_{\xi\in\mathbb{R}}h_{r,\delta}(\xi)\|R(i\xi,A)^{k}\|_{\mathcal{L}(X^{\mu}_{\nu}(\upsilon),X)}<\infty. \tag{2.15}
$$

As in the proof of Theorem 2.2.1, let $\psi\in C_c(\mathbb{R})$ with be such that $\psi\equiv 1$ on $[-1,1]$. It follows from (2.15) that for each $k \in \{1, \dots, n\}$,

$$\psi(\cdot)R(i\cdot, A)^k \in L^1(\mathbb{R}, \mathcal{L}(X_\nu^\mu(\upsilon), X)) \subset \mathcal{M}_{1,\infty}(\mathbb{R}, \mathcal{L}(X_\nu^\mu(\upsilon), X)),$$

and $\|(1-\psi(\cdot))R(i\cdot, A)^k\|_{\mathcal{L}(X_\nu^\mu(\upsilon),X)} \in L^r(\mathbb{R})$. Note that $X_\nu^\mu(\upsilon)$ has Fourier type $p$, since $X_\nu^\mu(\upsilon)$ is isomorphic to $X$. Then,

$$(1-\psi(\cdot))R(i\cdot, A)^k \in \mathcal{M}_{p,p'}\left(\mathbb{R}, \mathcal{L}(X_\nu^\mu(\upsilon), X)\right),$$

and by Theorem 2.1.2, there exists $c_n \geq 0$ such that for each $t \geq 1$,

$$\|T(t)\|_{\mathcal{L}(X_\nu^\mu(\upsilon),X)} \leq c_n t^{-n}.$$

In general, for each $s \geq 0$, set $\mu := \alpha(s+1)-1$, $\nu := \beta(s+1)+1/r$ and $\tilde{\upsilon} := c(\lceil s \rceil+1)+1/r+\delta$; by following the same argument presented in the proof of Proposition 2.3.1, one concludes that for each $t \geq 1$,

$$\|T(t)A^\mu B^{-\mu-\nu}(2\pi - i\log(A))^{-\tilde{\upsilon}}\|_{\mathcal{L}(X)} \quad \lesssim \quad t^{-s}.$$

□

### 2.3.1 Proof of Theorem 11

**Lemma 2.3.1.** Let $A$ be an injective sectorial operator defined in a Banach space $X$. Let $\alpha \geq 1$, $\beta, s \geq 0$, $r \in [1, \infty]$ and $x \in \mathcal{D}(A) \subset X$. Then,

$$A^{\alpha(s+1)-1}(1+A)^{-(\alpha+\beta)(s+1)+1-\frac{1}{r}}x \in \mathcal{D}(A) \cap \mathrm{Ran}(A).$$

*Proof.* Let $x \in X$ and set $\mu := \alpha(s+1)-1$, $\nu := \beta(s+1)+\frac{1}{r}$ and $B := 1+A$; it follows from Proposition 3.1.1 in [32] (see items (c) and (f)) that

$$\begin{aligned} A^{\mu-1}B^{-(\mu+\nu)}x &= A^{\mu-1}B^{-(\mu-1+1+\nu)}x \\ &= B^{-1}\left(A^{\mu-1}B^{-(\mu-1+\nu)}x\right) \in \mathcal{D}(A). \end{aligned}$$

Thus, it follows that for each $x \in X$,

$$A^\mu B^{-(\mu+\nu)}x = A\left(A^{\mu-1}B^{-(\mu+\nu)}x\right) \in \mathrm{Ran}(A). \tag{2.16}$$

Now one has, for each $x \in \mathcal{D}(A)$,

$$\begin{aligned} A^\mu B^{-(\mu+\nu)}x &= A^\mu B^{-(\mu+\nu)}B^{-1}Bx \\ &= B^{-1}\left(A^\mu B^{-(\mu+\nu)}\right)Bx \in \mathcal{D}(A). \end{aligned}$$

□

***Proof of Theorem 11***. Let $\delta > 1 - 1/r$, set $B := A + 1$ and for each $\varepsilon, s > 0$, set $\mu := \alpha(s+1) - 1$, $\nu := \beta(s+1) + 1/r$, $\upsilon := c(n+1) + 1/r + \delta$ (with $n = \lceil s \rceil$) and

$$Q_\upsilon(A_\varepsilon, B_\varepsilon) := A^{\mu-\varepsilon} B^{-\mu-\nu+\varepsilon} (2\pi - i\log(A))^{-\upsilon} \in \mathcal{L}(X).$$

Set $m := \lceil \upsilon \rceil$, so $m \in \mathbb{N} \setminus \{1\}$ and $m - 1 < \upsilon \leq m$. We divide the proof into the cases $\upsilon = m$ and $\upsilon \in (m-1, m)$.

- **Case** $\upsilon = m$**.**

**Step 1:** estimating $\|T(t)\log(A) Q_\upsilon(A_0, B_0)\|$. For each $s > 0$, let $\varepsilon = \dfrac{\min\{\alpha, \beta, 1\}\theta}{2} > 0$, where $\theta \in (0, \min\{1, s\})$; then, one has for each $x \in \mathcal{D}(A)$,

$$\begin{aligned} T(t)\log(A) Q_\upsilon(A_0, B_0)x &= T(t)\log(1+A) Q_\upsilon(A_0, B_0)x - T(t)\log(1+A^{-1}) Q_\upsilon(A_0, B_0)x \\ &= \frac{T(t)}{\varepsilon} \log((1+A)^\varepsilon) Q_\upsilon(A_0, B_0)x - \frac{T(t)}{\varepsilon} \log((1+A^{-1})^\varepsilon) Q_\upsilon(A_0, B_0)x \\ &= \frac{T(t)}{\varepsilon} \int_{0+}^{\infty} f_\varepsilon(A)(\lambda + f_\varepsilon(A))^{-1} Q_\upsilon(A_0, B_0)x d\mu(\lambda) - \\ &- \frac{T(t)}{\varepsilon} \int_{0+}^{\infty} f_\varepsilon(A^{-1})(\lambda + f_\varepsilon(A^{-1}))^{-1} Q_\upsilon(A_0, B_0)x d\mu(\lambda) \\ &= I_1 - I_2, \end{aligned} \tag{2.17}$$

with $f_\varepsilon(\lambda) = (1+\lambda)^\varepsilon - 1$, where we have applied Lemmas 1.3.3 and 2.3.1 in the first identity, Lemma 1.3.4 in the second identity and relation 1.7 in the third identity.

**Estimating** $I_1$. Let $\tau := \dfrac{\|T(t) Q_\upsilon(A_0, B_\varepsilon)\|_{\mathcal{L}(X)}}{\|T(t) Q_\upsilon(A_0, B_0)\|_{\mathcal{L}(X)}}$; by following the same arguments presented in the proof of Theorem 2.2.1, one gets

$$\left\| T(t) \int_{0+}^{\tau} f_\varepsilon(A)(\lambda + f_\varepsilon(A))^{-1} Q_\upsilon(A_0, B_0)x d\mu(\lambda) \right\| \leq 4\|T(t) Q_\upsilon(A_0, B_0)\|_{\mathcal{L}(X)} M_{f_\varepsilon(A)} \int_{0+}^{\tau} \frac{\tau}{\lambda + \tau} d\mu(\lambda) \|x\|,$$

and

$$\left\| T(t) \int_{\tau}^{\infty} f_\varepsilon(A)(\lambda + f_\varepsilon(A))^{-1} Q_\upsilon(A_0, B_0)x d\mu(\lambda) \right\| \leq 2\|T(t) f_\varepsilon(A) Q_\upsilon(A_0, B_0)\|_{\mathcal{L}(X)} M_{f_\varepsilon(A)} \int_{\tau}^{\infty} \frac{1}{\lambda + \tau} d\mu(\lambda)$$

Note that for each $t \geq 0$ and each $x \in X$,

$$\begin{aligned} \|T(t) f_\varepsilon(A) Q_\upsilon(A_0, B_0)x\| &= \|(1 - B^{-\varepsilon}) T(t) B^\varepsilon Q_\upsilon(A_0, B_0)x\| \\ &\leq \|1 - B^{-\varepsilon}\|_{\mathcal{L}(X)} \|T(t) B^\varepsilon Q_\upsilon(A_0, B_0)x\| \\ &\leq C_\varepsilon \|T(t) Q_\upsilon(A_0, B_\varepsilon)\|_{\mathcal{L}(X)} \|x\|. \end{aligned} \tag{2.18}$$

Then, by (2.18), it follows that

$$\begin{aligned} \|I_1\| &\lesssim \varepsilon^{-1}\|T(t)Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \left(\int_{0+}^{\tau} \frac{\tau}{\tau+\lambda} d\mu(\lambda) + C_\varepsilon \tau \int_{\tau}^{\infty} \frac{1}{\tau+\lambda} d\mu(\lambda)\right) \|x\| \\ &\lesssim \varepsilon^{-1}\|T(t)Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \log(1+\tau)\|x\|. \end{aligned} \tag{2.19}$$

**Estimating** $I_2$.

Let $\sigma := \dfrac{\|T(t)Q_\upsilon(A_\varepsilon,B_\varepsilon)\|_{\mathcal{L}(X)}}{\|T(t)Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)}}$, so

$$\left\|T(t)\int_{0+}^{\sigma} f_\varepsilon(A^{-1})(\lambda+f_\varepsilon(A^{-1}))^{-1}Q_\upsilon(A_0,B_0)x\,d\mu(\lambda)\right\| \lesssim \|T(t)Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \int_{0+}^{\sigma} \frac{\sigma}{\lambda+\sigma} d\mu(\lambda)\|x\|,$$

and

$$\left\|T(t)\int_{\sigma}^{\infty} f_\varepsilon(A^{-1})(\lambda+f_\varepsilon(A))^{-1}Q_\upsilon(A_0,B_0)x\,d\mu(\lambda)\right\| \lesssim \|T(t)f_\varepsilon(A^{-1})Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \int_{\sigma}^{\infty} \frac{\|x\|}{\lambda+\sigma} d\mu(\lambda).$$

Note that for each $x \in X$,

$$\begin{aligned} \|T(t)f_\varepsilon(A^{-1})Q_\upsilon(A_0,B_0)x\| &= \|(1-(1+A^{-1})^{-\varepsilon})T(t)(1+A^{-1})^{\varepsilon}A^{\mu}B^{-(\mu+\nu)}(2\pi - i\log(A))^{-\upsilon}x\| \\ &\le \tilde{C}_\varepsilon\|T(t)(1+A^{-1})^{\varepsilon}A^{\mu}B^{-(\mu+\nu)}(2\pi - i\log(A))^{-\upsilon}x\| \end{aligned}$$

Now, by relation (1.1) one has $(1+A^{-1})^{-1} = 1-(1+A)^{-1} = A(1+A)^{-1}$, so it follows from Propositions 3.1.1 (e) and 3.1.9 (b) in [32] that

$$[(1+A^{-1})^{\varepsilon}]^{-1} = (A(1+A)^{-1})^{\varepsilon} = A^{\varepsilon}((1+A)^{\varepsilon})^{-1}.$$

Then, $(1+A^{-1})^{\varepsilon} = (1+A)^{\varepsilon}(A^{\varepsilon})^{-1} = (1+A)^{\varepsilon}A^{-\varepsilon}$ (see Proposition 3.2.1 (a) in [32]). Therefore, by the previous discussion,

$$\begin{aligned} \|T(t)f_\varepsilon(A^{-1})Q_\upsilon(A_0,B_0)x\| &\le \tilde{C}_\varepsilon\|T(t)A^{\mu-\varepsilon}B^{-(\mu+\nu)+\varepsilon}(2\pi - i\log(A))^{-\upsilon}x\| \\ &\le \tilde{C}_\varepsilon\|T(t)Q_\upsilon(A_\varepsilon,B_\varepsilon)\|_{\mathcal{L}(X)}\|x\|. \end{aligned} \tag{2.20}$$

Thus, by (2.20),

$$\begin{aligned} \|I_2\| &\lesssim \varepsilon^{-1}\|T(t)Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \left(\int_{0+}^{\sigma} \frac{\sigma}{\sigma+\lambda} d\mu(\lambda) + \tilde{C}_\varepsilon \sigma \int_{\sigma}^{\infty} \frac{1}{\sigma+\lambda} d\mu(\lambda)\right) \|x\| \\ &\lesssim \varepsilon^{-1}\|T(t)Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \log(1+\sigma)\|x\|. \end{aligned} \tag{2.21}$$

Finally, by combining relations (2.17), (2.19) and (2.21), and by the density of $\mathcal{D}(A)$, one gets for each sufficiently large $t$,

$$\begin{aligned}\|T(t)\log(A)Q_v(A_0,B_0)\|_{\mathcal{L}(X)} &\leq C_\varepsilon\|T(t)Q_v(A_0,B_0)\|_{\mathcal{L}(X)}\log\left(1+\frac{C_\varepsilon\|T(t)Q_v(A_0,B_\varepsilon)\|_{\mathcal{L}(X)}}{\|T(t)Q_v(A_0,B_0)\|_{\mathcal{L}(X)}}\right)\\ &+ \tilde{C}_\varepsilon\|T(t)Q_v(A_0,B_0)\|_{\mathcal{L}(X)}\log\left(1+\frac{\tilde{C}_\varepsilon\|T(t)Q_v(A_\varepsilon,B_\varepsilon)\|_{\mathcal{L}(X)}}{\|T(t)Q_v(A_0,B_0)\|_{\mathcal{L}(X)}}\right)\\ &\leq 2C_{\varepsilon,s}t^{-s}\log(1+c_st^s),\end{aligned}$$

with $C_{\varepsilon,s}$ and $c_s$ positive constants, where in the last inequality we have applied Proposition E.2.1 to $\log(1+\lambda)$ (see the proof of Theorem 10 for details).

**Step 2:** removing $m$. The idea is to apply **Step 1** recursively in order to obtain an estimate for $\|T(t)A^\mu B^{-\mu-\nu}\|_{\mathcal{L}(X)}$.

First of all, note that for each $k\in\mathbb{N}$ and each $y\in\mathcal{D}(\log(A)^k)$, one has

$$(2\pi - i\log(A))^k y = \sum_{n=0}^{k}\binom{k}{j}(2\pi)^{k-j}(i\log(A))^j y. \tag{2.22}$$

Now, note that for each $n\in\{1,\ldots,m\}$ and each $x\in X$, $(2\pi-i\log(A))^{-m}A^\mu B^{-(\mu+\nu)}x \in \mathcal{D}((2\pi-i\log(A))^m)\subset\mathcal{D}(\log(A)^n)$, and so by (2.16), for each $x\in\mathcal{D}(A)$, one has

$$\begin{aligned}\mathcal{D}(A) &\ni B^{-1}(2\pi-i\log(A))^n Q_v(A_0,B_0)Bx\\ &= A^\mu B^{-(\mu+\nu)}(2\pi-i\log(A))^n(2\pi-i\log(A))^{-m}x\in\mathrm{Ran}(A).\end{aligned}$$

Therefore, it follows from relation (2.22) that for each $x\in\mathcal{D}(A)$,

$$T(t)(2\pi-i\log(A))^m Q_v(A_0,B_0)x = \sum_{n=0}^{m}\binom{m}{n}(-i)^n(2\pi)^{m-n}T(t)(\log(A))^n Q_v(A_0,B_0)x. \tag{2.23}$$

The next step consists in estimating the norm of each one of the terms presented in relation (2.23).

- $n=0$. It follows from Theorem 2.3.1 that $\|T(t)Q_v(A_0,B_0)\|_{\mathcal{L}(X)}\lesssim t^{-s}$.
- $1\leq n\leq m$. We proceed by induction over $n$. Case $n=1$ is just Step 1. If $n>1$, for each $0\leq\varepsilon<1$ let
$$\tau(t,A_\varepsilon,B_\varepsilon)=\frac{\|T(t)(\log(A))^{n-1}Q_v(A_\varepsilon,B_\varepsilon)\|_{\mathcal{L}(X)}}{\|T(t)(\log(A))^{n-1}Q_v(A_0,B_0)\|_{\mathcal{L}(X)}}$$
and note that, by proceeding as in **Step 1**, one gets

$$
\begin{aligned}
\|T(t)(\log(A))^n Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} &= \|T(t)\log(A)(\log(A))^{n-1}Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \\
&\lesssim \varepsilon^{-1}\|T(t)(\log((1+A)^\varepsilon)(\log(A))^{n-1}Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \\
&+ \varepsilon^{-1}\|T(t)(\log((1+A^{-1})^\varepsilon)(\log(A))^{n-1}Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \\
&\lesssim \|T(t)(\log(A))^{n-1}Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)}\log\left(1+\tau(t,A_0,B_\varepsilon)\right) \\
&+ \|T(t)(\log(A))^{n-1}Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)}\log\left(1+\tau(t,A_\varepsilon,B_\varepsilon)\right);
\end{aligned}
$$

then, by the inductive hypothesis, it follows that for each $t \geq 1$,

$$\|T(t)(\log(A))^{n-1}Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \lesssim t^{-s}\log(1+t)^{n-1}.$$

Now, by replacing the previous estimates on (2.23), it follows that for each $t \geq 1$,

$$
\begin{aligned}
\|T(t)A^\mu B^{-(\mu+\nu)}\|_{\mathcal{L}(X)} &= \|T(t)(2\pi - i\log(A))^m Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \\
&\lesssim t^{-s}\log(1+t)^m.
\end{aligned}
$$

**Case $\upsilon \neq m$.**

Since $\upsilon \in (m-1,m)$, it follows from the moment inequality (recall that $(2\pi - i\log(A))^{-1}$ is a sectorial operator) and from the previous case applied to $\upsilon = m$ and $\upsilon = m-1$ (recall that $m \geq 2$) that for each $t \geq 1$,

$$
\begin{aligned}
\|T(t)A^\mu B^{-(\nu+\mu)}\|_{\mathcal{L}(X)} &= \|T(t)(2\pi - i\log(A))^\upsilon Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)} \\
&\lesssim \|T(t)(2\pi - i\log(A))^{m-1} Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)}^{m-\upsilon} \\
&\cdot \|T(t)(2\pi - i\log(A))^{m} Q_\upsilon(A_0,B_0)\|_{\mathcal{L}(X)}^{\upsilon-m+1} \\
&\lesssim (t^{-s}\log(1+t)^{m-1})^{m-\upsilon}(t^{-s}\log(1+t)^m)^{\upsilon-m+1} \\
&\lesssim t^{-s}\log(1+t)^\upsilon.
\end{aligned}
$$

□

## 2.4 Singularity at zero

Let $\mu,\upsilon \geq 0$ and let $A \in \mathrm{Sect}(\omega_A)$ be an injective operator over the Banach space $X$ (by Lemma 1.3.1, $A^{-1}$ is a sectorial operator); since $\lambda \mapsto \log(1+\lambda) \in \mathcal{CBF}$ (see Example E.1.1-(b)), it follows from Theorem 1.3.1 that the operator $\log(2+A^{-1})$ is sectorial, hence $(\log(2+A^{-1}))^{-\upsilon} \in \mathcal{L}(X)$ is well-defined. Define the bounded operator

$$\Phi^\mu(\upsilon) = \Phi^\mu(A,\upsilon) := A^\mu(1+A)^{-\mu}\log(2+A^{-1})^{-\upsilon}$$

and set $X^\mu(\upsilon) := \mathrm{Ran}(\Phi^\mu(\upsilon))$. The space $X^\mu(\upsilon)$ is a Banach space with respect to the norm

$$\|x\|_{X^\mu(\upsilon)} = \|x\| + \|\Phi^\mu(\upsilon)^{-1}x\| = \|x\| + \|\log(2+A^{-1})^\upsilon(1+A)^\mu A^{-\mu}x\|, \quad x \in X^\mu(\upsilon).$$

Note that $\Phi^\mu(\upsilon) : X \to X^\mu(\upsilon)$ is an isomorphism, so for each $T \in \mathcal{L}(X^\mu(\upsilon), X)$,

$$\|T\|_{\mathcal{L}(X^\mu(\upsilon),X)} \leq \|T\Phi^\mu(\upsilon)\|_{\mathcal{L}(X)} \leq \|\Phi^\mu(\upsilon)\|_{\mathcal{L}(X)}\|T\|_{\mathcal{L}(X^\mu(\upsilon),X)}. \tag{2.1}$$

**Definition 2.4.1.** Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ with generator $-A$. One defines the non-analytic growth bound $\zeta(T)$ of $(T(t))_{t\geq 0}$ as

$$\zeta(T) := \inf\{w \in \mathbb{R} \mid \sup_{t>0} e^{-tw}\|T(t) - S(t)\|_{\mathcal{L}(X)} < \infty \text{ for some } S \in \mathcal{H}(\mathcal{L}(X))\},$$

where $\mathcal{H}(\mathcal{L}(X))$ is the set of the operators $S : (0,\infty) \to \mathcal{L}(X)$ having an exponentially bounded analytic extension to some sector containing $(0,\infty)$. One says that $(T(t))_{t\geq 0}$ is asymptotically analytic if $\zeta(T) < 0$.

**Remark 2.4.1.** Let

$$s_0^\infty(-A) := \inf\Bigg\{w \in \mathbb{R} \mid \exists\, R \geq 0 \text{ such that } \{\mathrm{Re}(\lambda) \geq w \text{ and } |\mathrm{Im}(\lambda)| \geq R\} \subset \rho(-A) \text{ and } \sup_{\mathrm{Re}(\lambda)\geq w, |\mathrm{Im}(\lambda)|\geq R} \|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} < \infty\Bigg\}.$$

It is shown in [10] (Proposition 2.4) that $\zeta(T) \geq s_0^\infty(-A)$. So, if $(T(t))_{t\geq 0}$ is asymptotically analytic, then $s_0^\infty(-A) < 0$; more generally, Theorem 3.6 in [11] states that $\zeta(T) < 0$ if, and only if, $s_0^\infty(-A) < 0$. In our strategy, we use the fact that $s_0^\infty(-A) < 0$.

### 2.4.1 Proof of Theorem 12

***Proof of Theorem 12.*** **Step 1:** Here, we use the same ideas presented in the proof of Theorem 10. Let $n \in \mathbb{N}$ and set $\mu := \alpha(n+1) - 1$, $\upsilon := a(n+1) + 1 + \delta$. For each $x \in X^\mu(\zeta)$, let

$$\begin{aligned} y := (\Phi_1^\mu(\upsilon))^{-1}x &= \log(1+A^{-1})^\upsilon(1+A)^\mu A^{-\mu}x \\ &= \log(2+A^{-1})^\upsilon(1+A)^\mu A^{-\mu}\left(A^\mu(1+A)^{-\mu-1}\log(2+A^{-1})^{-\upsilon}z\right) \\ &= B^{-1}z \in \mathcal{D}(A), \end{aligned}$$

where $z := (\Phi^\mu(\upsilon))^{-1}x$. Let $g : [0,\infty) \to X$ be defined by the law

$$g(t) := \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{\lambda^\mu}{(1+\lambda)^\mu \log(2+\lambda^{-1})^\upsilon} R(\lambda, A)y d\lambda.$$

Note that for each $t \geq 0$, $g(t)$ is indeed an element of $X$ (which follows from relation (12) and from $s_0^\infty(-A) < 0$) and it is differentiable. Namely, since $y \in \mathcal{D}(A)$, then $\lambda \mapsto$

$\dfrac{\lambda^{\mu+1}}{(1+\lambda)^{\mu}\log(2+\lambda^{-1})^{\upsilon}}R(\lambda, A)y$ is integrable in $i\mathbb{R}$. Therefore, by dominated convergence,

$$g'(t) = -\frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t}\frac{\lambda^{\mu+1}}{(1+\lambda)^{\mu}\log(2+\lambda^{-1})^{\upsilon}}R(\lambda, A)yd\lambda.$$

Moreover, by Lemma D.0.1, $g'(t) = -Ag(t)$ for each $t \geq 0$, and $g(0) = x$. Therefore, $g(t) = T(t)x$, by the uniqueness of the solution to the Cauchy problem associated with $-A$.

Now, integration by parts yields

$$t^n T(t) = \frac{1}{2\pi i}\int_{i\mathbb{R}} e^{-\lambda t} q(\lambda, A)yd\lambda,$$

where $q(\lambda, A)$ is a finite linear combination of terms of the form

$$\frac{\lambda^{\mu-j}R(\lambda, A)^{n-k+1}}{(1+\lambda)^{\mu+k-j}(2\lambda+1)^{i}\log(2+\lambda^{-1})^{\upsilon+i}} \quad \text{and} \quad \frac{\lambda^{\mu-j}R(\lambda, A)^{n-k+1}}{(1+\lambda)^{\mu+i}(2\lambda+1)^{l}\log(2+\lambda^{-1})^{\upsilon+j}},$$

where $0 \leq i \leq j \leq k \leq n$ and $k - j \leq l \leq k - j + 1$.

Then, for each $t > 0$,

$$\begin{aligned}\|t^n T(t)x\| &\leq \frac{1}{2\pi}\int_{i\mathbb{R}} |e^{-\lambda t}|\|q(\lambda, A)y\| d\lambda \\ &\leq \frac{1}{2\pi}\int_{i\mathbb{R}} \|q(\lambda, A)\|_{\mathcal{L}(X)} d\lambda \|y\| \leq C\|(\Phi^{\mu}(\upsilon))^{-1}x\| \lesssim \|x\|_{X^{\mu}(\upsilon)}.\end{aligned}$$

Since $X_1^{\mu}(\upsilon)$ is dense in $X^{\mu}(\upsilon)$, it follows from the previous discussion that for each $t \geq 1$,

$$\|T(t)\|_{\mathcal{L}(X^{\mu}(\upsilon), X)} \lesssim t^{-n}.$$

It remains to prove the result for any $s > 0$. So, for each fixed $s > 0$, let $n \in \mathbb{N}$ be such that $n \leq s < n+1$. Let also define $\theta := \theta(s) \in [0, 1)$ by the relation $s = (1-\theta)n + \theta(n+1)$.

Set $a_1 := \frac{\alpha}{\alpha + a}$ and $a_2 := \frac{a}{\alpha + a}$ and note that $a_1 + a_2 = 1$; then, by Proposition E.1.1-(c), $f(\lambda) = (1+\lambda)^{a_1}\log(2+\lambda)^{a_2} \in \mathcal{CBF}$, where $\lambda > 0$. Now, by Lemma 3.2 in [14], the operator

$$\begin{aligned}(f(A^{-1}))^{-1} &= (1 + A^{-1})^{-a_1}\log(2 + A^{-1})^{-a_2} \\ &= A^{-a_1}(1 + A)^{-a_1}\log(2 + A^{-1})^{-a_2}\end{aligned}$$

is sectorial, given that $f(A^{-1})$ is sectorial, by Theorem 1.3.1.

Since $(f(A^{-1}))^{-1}$ is sectorial, it follows from the moment inequality (see Proposition 4.6 in

[32]) and Theorem 2.4.2 in [32] that

$$\begin{aligned}
\|T(t)[(f(A^{-1}))^{-1}]^{\theta(\alpha+a)}\Phi^{\mu}(\upsilon)\|_{\mathcal{L}(X)} &\lesssim \|T(t)\Phi^{\mu}(\upsilon)\|_{\mathcal{L}(X)}^{1-\theta}\|T(t)[(f(A))^{-1}]^{\alpha+a}\Phi^{\mu}(\upsilon)\|_{\mathcal{L}(X)}^{\theta} \\
&= \|T(t)\Phi^{\mu}(\upsilon)\|_{\mathcal{L}(X)}^{1-\theta}\|T(t)A^{\alpha}(1+A)^{-\alpha}\log(2+A^{-1})^{-a}\Phi_{\nu}(a)\|_{\mathcal{L}(X)}^{\theta} \\
&= \|T(t)\Phi^{\mu}(\upsilon)\|_{\mathcal{L}(X)}^{1-\theta}\|T(t)\Phi^{\alpha(n+2)-1}(a(n+2)+1+\delta)\|_{\mathcal{L}(X)}^{\theta} \\
&\lesssim t^{-n(1-\theta)}t^{-\theta(n+1)} = t^{-s}. \qquad (2.2)
\end{aligned}$$

**Step 2.** For each $\varepsilon > 0$, set

$$W_{\upsilon}(A_{\varepsilon}, B_{\varepsilon}) := A^{\mu-\varepsilon}B^{-\mu+\varepsilon}\log(1+A^{-1})^{-\upsilon} \in \mathcal{L}(X).$$

Set $m := \lfloor \upsilon \rfloor$ and $\eta := \{\upsilon\}$. As in the proof of Theorem 10, we divide the proof into the cases where $\eta = 0$ and $\eta > 0$. In both of them, we proceed recursively over $m \in \mathbb{N}$.

**Case $\eta > 0$.**

- **Removing** $\eta$.

Let $\varepsilon = \frac{\alpha\theta}{2} > 0$, where $\theta \in (0, \min\{1, s\})$. Note that for each $x \in \mathcal{D}(A)$, one has

$$\begin{aligned}
T(t)W_m(A_0, B_0)x &= \frac{1}{\varepsilon^{\eta}}T(t)\log(1+f_{\varepsilon}(A^{-1}))^{\eta}W_{m+\eta}(A_0, B_0))x \\
&= \frac{1}{\varepsilon^{\eta}}\int_0^{\infty} f_{\varepsilon}(A^{-1})(s+f_{\varepsilon}(A^{-1}))^{-1}T(t)W_{m+\eta}(A_0, B_0)x\,d\mu(s).
\end{aligned}$$

Let

$$\tau := \frac{\|T(t)W_{m+\eta}(A_{\varepsilon}, B_{\varepsilon})\|_{\mathcal{L}(X)}}{\|T(t)W_{m+\eta}(A_0, B_0)\|_{\mathcal{L}(X)}};$$

then, by proceeding as in the proof of Theorem 10, one gets

$$\|T(t)W_m(A_0, B_0)\|_{\mathcal{L}(X)} \lesssim \|T(t)W_{m+\eta}(A_0, B_0)\|_{\mathcal{L}(X)}\log(1+\tau)^{\eta}. \qquad (2.3)$$

Again, by combining the estimates (2.2) and (2.3) with the arguments presented in the proof of Theorem 10, it follows that for each $t \geq 1$,

$$\|T(t)W_m(A_0, B_0)\|_{\mathcal{L}(X)} \lesssim t^{-s}\log(1+t)^{\eta}. \qquad (2.4)$$

- **Removing** $m$. It follows from the discussion presented in the previous item that for each $x \in X$,

$$\begin{aligned}
T(t)A^{\mu}B^{-\mu}\log(1+A^{-1})^{-m+1}x &= \frac{1}{\varepsilon}T(t)\log(1+f_{\varepsilon}(A^{-1}))A^{\mu}B^{-\mu}\log(1+A^{-1})^{-m}x \\
&= \frac{1}{\varepsilon}T(t)\int_{0+}^{\infty} f_{\varepsilon}(B)(\lambda+f_{\varepsilon}(B))^{-1}W_m(A_0, B_0)x\,d\mu(\lambda).
\end{aligned}$$

Let $\tau := \dfrac{\|T(t)W_m(A_\varepsilon, B_\varepsilon)\|_{\mathcal{L}(X)}}{\|T(t)W_m(A_0, B_0)\|_{\mathcal{L}(X)}}$; then, by proceeding as in the proof of Theorem 10, it follows from relation (2.4) that for each $t \geq 1$,

$$\|T(t)W_{m-1}(A_0, B_0)\|_{\mathcal{L}(X)} \lesssim t^{-s} \log(1+t)^{1+\eta}.$$

By proceeding recursively over $m$ (see the proof of Theorem 10 for details), it follows from the previous discussion that for each $t \geq 1$,

$$\|T(t)A^\mu B^{-\mu}\|_{\mathcal{L}(X)} \lesssim t^{-s} \log{(1+t)}^{m+\eta}.$$

**Case $\eta = 0$.** Since in this case $\upsilon = m \in \mathbb{N}$, one just needs to proceed as in the previous item in order to conclude that for each $t \geq 1$,

$$\|T(t)A^\mu B^{-\mu}\|_{\mathcal{L}(X)} \lesssim t^{-s} \log{(1+t)}^{m}.$$

□

**Remark 2.4.2.** Suppose that $\alpha = 1$ in the statement of Corollary 5, so $\|R(\lambda, A)\|_{\mathcal{L}(X)} \lesssim |\lambda|^{-1}$ with $\mathrm{Re}(\lambda) \leq 0$. Then, by relation (13), for each $\sigma > 0$ there exists $C_{\delta,\sigma} > 0$ so that for each $t \geq 1$,

$$\|T(t)A^\sigma(1+A)^{-\sigma}\|_{\mathcal{L}(X)} \leq C_{\sigma,\delta} t^{-\sigma} \log(1+t)^{1+\delta}. \tag{2.5}$$

Now, for each $\sigma > 1$, take $\delta = \sigma - 1 > 0$, and so for each $t \geq 1$,

$$\|T(t)A^\sigma(1+A)^{-\sigma}\|_{\mathcal{L}(X)} \leq C_{\sigma,\delta} t^{-\sigma} \log(1+t)^{\sigma} = O\left(\left(\frac{1}{M_{\log}^{-1}(t)}\right)^{\sigma}\right).$$

This shows that in case $\alpha = 1$ and $\sigma > 1$, one gets the same estimate as in Corollary 2.12 in [22] for bounded $C_0$-semigroups, and so for these particular parameters, the result is optimal.

# PART II

## Slow dynamics for self-adjoint semigroups and unitary evolution groups

# Chapter 3

# Slow dynamics for self-adjoint semigroups and unitary evolution groups

In this chapter, we discuss the proofs of the results presented in Part II of Introduction, with an application to almost-Mathieu operators. For the definitions and notations regarding spectral measures, spectral types and the Spectral Theorem, we refer to Appendix B.

## 3.1 Proofs of Theorems 14 and 15

### 3.1.1 Proof of Theorem 14

Recall that $A$ is a pure point negative self-adjoint operator, whose eigenvalues $(\lambda_n)_{n\geq 1} \subset (-\infty, 0)$ satisfy $\limsup_{n\to\infty} \lambda_n = 0$. Let $(x_n)_{n\geq 1}$ be the corresponding normalized vectors of $A$, that is, $Ax_n = \lambda_n x_n$ for each $n \geq 1$.

The main ingredient in the proof of this theorem is the well-known expression of the resolution of the identity of a pure point self-adjoint operator. Namely, since $A$ is a pure point operator, each $x \in X$ can be written as $\sum_{j=1}^{\infty} b_j x_j$ for some square-summable sequence $(b_j)_j$ of complex numbers, and, for each Borel set $\Lambda \subset \mathbb{R}$, the corresponding spectral measure is $\mu_x^A(\Lambda) = \sum_{\lambda_j\in\Lambda} |b_j|^2 \delta_{\lambda_j}$; in particular, for every $n \geq 1$, $\mu_x^A(\{\lambda_n\}) = |b_n|^2$. See Appendix B for more details.

**i)** Let $(\lambda_{j_l})_{l\geq 1}$ be a subsequence of eigenvalues of $A$, with the corresponding orthonormal eigenvectors $(x_{j_l})_{l\geq 1}$, so that $\lambda_{j_l} \uparrow 0$ and $\displaystyle\sum_{l=1}^{\infty} \frac{1}{\beta(1/|\lambda_{j_l}|)} < \infty$. For each $x \in X$, write $x = \displaystyle\sum_{j=1}^{\infty} b_j x_j$ and set

$$x_k := \sum_{l=1}^{k} b_l x_l + \sum_{l=k+1}^{\infty} \frac{1}{\sqrt{\beta(1/|\lambda_{j_l}|)}}\, x_{j_l}.$$

It follows that for each $l \geq k+1$,

$$\mu_{x_k}^A([\lambda_{j_l}, 0]) \geq \mu_{x_k}^A(\{\lambda_{j_l}\}) = \frac{1}{\beta(1/|\lambda_{j_l}|)},$$

and therefore

$$\begin{aligned}
\beta(1/|\lambda_{j_l}|)\,\|e^{(1/|\lambda_{j_l}|)A}x_k\| &= \beta(1/|\lambda_{j_l}|)\left(\int_{-\infty}^{0} e^{2(1/|\lambda_{j_l}|)w}\mathrm{d}\mu^A_{x_k}(w)\right)^{1/2}\\
&\geq \beta(1/|\lambda_{j_l}|)\left(\int_{\lambda_{j_l}}^{0} e^{2(1/|\lambda_{j_l}|)w}\mathrm{d}\mu^A_{x_k}(w)\right)^{1/2}\\
&\geq e^{-1}\beta(1/|\lambda_{j_l}|)\,(\mu^A_{x_k}([\lambda_{j_l},0]))^{1/2}\\
&\geq e^{-1}\sqrt{\beta(1/|\lambda_{j_l}|)}\,,
\end{aligned}$$

which implies

$$\limsup_{t\to\infty}\beta(t)\|e^{tA}x_k\| = \infty\,.$$

**ii)** Let $x \in X$ and let $\{e_j\}_{j\geq 1}$ be an orthonormal basis of $X$ such that $x = \sum_{j=1}^{\infty} a_j e_j$, with $a_j \neq 0$ for infinitely many $j$'s. Let $(a_{j_l})_{l\geq 1}$ be a subsequence of $(a_j)_{j\geq 1}$ with $|a_{j_l}| \downarrow 0$. Let $(t_l)_{l\geq 1}$ be a positive sequence such that $t_l \to \infty$ and $\beta(t_l) = |a_{j_l}|^{-2}$.

For each $k \geq 1$, set

$$A_k := P_{l\leq k}AP_{l\leq k} - \sum_{l=k+1}^{\infty}\frac{1}{t_l}\langle e_{j_l},\cdot\rangle e_{j_l},$$

where $P_{l\leq k}$ is the projection onto the subspace generated by $\{e_l\}_{l\leq k}$. It is clear that $A_k \to A$ as $k \to \infty$ in the strong sense. The operator $P_{l\leq k}AP_{l\leq k}$ is pure point and negative. Note that for large enough $l$, $A_k(e_{j_l}) = -\frac{1}{t_l}e_{j_l}$.

Fix $k$; for large enough $l$, one has

$$\mu^{A_k}_x([-1/t_l,0]) \geq \mu^{A_k}_x(\{-1/t_l\}) = |a_{j_l}|^2 = \frac{1}{\beta(t_l)},$$

and therefore

$$\begin{aligned}
\beta(t_l)\|e^{t_lA_k}x\| &= \beta(t_l)\left(\int_{-\infty}^{0} e^{2t_lw}\mathrm{d}\mu^{A_k}_x(w)\right)^{1/2}\\
&\geq \beta(t_l)\left(\int_{-1/t_l}^{0} e^{2t_lw}\mathrm{d}\mu^{A_k}_x(w)\right)^{1/2}\\
&\geq e^{-1}\beta(t_l)(\mu^{A_k}_x([-1/t_l,0]))^{1/2}\\
&\geq e^{-1}\sqrt{\beta(t_l)},
\end{aligned}$$

which results in

$$\limsup_{t\to\infty}\beta(t)\|e^{tA_k}x\| = \infty\,.$$

### 3.1.2 Proof of Theorem 15

Let $\alpha \in [0,1]$. Recall that a finite positive Borel measure $\mu$ on $\mathbb{R}$ is uniformly $\alpha$-Hölder continuous (denoted U$\alpha$H) if there exists a constant $C>0$ such that for each interval $I$ with $\mathcal{L}(I)<1$, $\mu(I)\le C\,\mathcal{L}(I)^\alpha$. Theorem 3.1.1 i) is, indeed, a particular case of a well-known theorem by Strichartz [64].

**Theorem 3.1.1** (Theorems 2.5 and 3.1 in [34])**.** Let $\mu$ be a finite Borel measure on $\mathbb{R}$ and $\alpha\in[0,1]$.

i) If $\mu$ is U$\alpha$H, then there exists $C_\mu>0$, depending only on $\mu$, such that for every $f\in \mathrm{L}^2(\mathbb{R},d\mu)$ and every $t>0$,

$$\frac{1}{t}\int_0^t \left|\int_{\mathbb{R}} e^{-isw} f(w)\,\mathrm{d}\mu(w)\right|^2 \mathrm{d}s < C_\mu \|f\|^2_{\mathrm{L}^2(\mathbb{R},d\mu)}\, t^{-\alpha}.$$

ii) If there exists $C_\mu>0$ such that for every $t>0$,

$$\frac{1}{t}\int_0^t \left|\int_{\mathbb{R}} e^{-isw}\,\mathrm{d}\mu(w)\right|^2 \mathrm{d}s < C_\mu\, t^{-\alpha},$$

then $\mu$ is U$\frac{\alpha}{2}$H.

**Lemma 3.1.1** (Lemma 2.1 in [4])**.** Let $A$ be a negative self-adjoint operator with $0\in\sigma(A)$ and let $\alpha:\mathbb{R}_+\longrightarrow(0,\infty)$ be such that

$$\lim_{t\to\infty}\alpha(t)=\infty.$$

Then, there exist $x\in X$ and a sequence $t_j\to\infty$ such that, for sufficiently large $j$,

$$\mu_x^A\big(B(0;1/t_j)\big)\ge \frac{1}{\alpha(t_j)}.$$

***Proof of Theorem 15****.* Let $w\in\sigma(A)$ and set $L_w=(-\infty,w]\cap\sigma(A)$, $A_w=AE^A(L_w)$ and also $A_w^0=A_w-wI$. So, by Lemma 3.1.1, there exist $x\in X$ and $\varepsilon_j\to 0$ such that, for sufficiently large $j$,

$$\begin{aligned}\mu_x^{A_w^0}(B(0;\varepsilon_j))\ge \frac{1}{-\ln(\varepsilon_j)} \ &\Rightarrow\ \mu_x^{A_w}(B(w;\varepsilon_j))\ge\frac{1}{-\ln(\varepsilon_j)}\\ \Rightarrow\ \mu_x^{A}(B(w;\varepsilon_j))\ge \mu_x^{A}(B(w;\varepsilon_j)\cap L_w) &= \mu_x^{A_w}(B(w;\varepsilon_j))\ge\frac{1}{-\ln(\varepsilon_j)}.\end{aligned}$$

Hence, $\mu_x^A$ is not U$\alpha$H for all $0<\alpha\le 1$. Thus, by Theorem 3.1.1, for every $\varepsilon>0$,

$$\limsup_{t\to\infty} t^{\varepsilon}\,\frac{1}{t}\int_0^t\left|\int_{\mathbb{R}} e^{-isw}\,\mathrm{d}\mu_x^A(w)\right|^2\mathrm{d}s=\infty.$$

Since one has, by the Spectral Theorem, that for every $s\in\mathbb{R}$

$$\langle e^{-isA}x,x\rangle=\int_{\mathbb{R}} e^{-isw}\,\mathrm{d}\mu_x^A(w),$$

the result follows. □

## 3.2 Generic spectral properties and proofs of Theorems 16 and 17

In this section we compute (Baire) generically the local dimensions of systems with purely continuous spectrum (Theorem 16) in order to prove Theorem 17.

Note that, for each $w \in \mathbb{R}$ and each $\varepsilon > 0$,

$$\int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu(s) \geq \int_{B(w;1/t)} e^{-2t|w-s|}\mathrm{d}\mu(s) \geq e^{-2}\mu(B(w;1/t)).$$

On the other hand, for each $0 < \delta < 1$ and each $t > 0$,

$$\begin{aligned} \int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu(s) &= \int_{B(w;\frac{1}{t^{1-\delta}})} e^{-2t|w-s|}\mathrm{d}\mu(s) + \int_{B(w;\frac{1}{t^{1-\delta}})^c} e^{-2t|w-s|}\mathrm{d}\mu(s) \\ &\leq \mu\big(B\big(w, 1/t^{1-\delta}\big)\big) + e^{-t^{\delta}}\mu(\mathbb{R}). \end{aligned} \tag{3.1}$$

Thus, at least when $\mu$ has a certain local regularity (with respect to the Lebesgue measure), we expect that $\int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu(s)$ and $\mu(B(w;1/t))$ are asymptotically comparable as $t \to \infty$. In this sense, the following identities are expected:

$$\liminf_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu(s)]}{\ln t} = -d_\mu^+(w), \tag{3.2}$$

$$\limsup_{t\to\infty} \frac{\ln[\int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu(s)]}{\ln t} = -d_\mu^-(w). \tag{3.3}$$

Indeed, these identities were proven in [4] (note that since it is not possible to compare directly the two terms on the right-hand side of (3.1), some caution should be exercised when checking (3.2) and (3.3)). We use them in the proof of Theorem 16 below.

***Proof of Theorem 16.*** Note that it is enough to show that for each $w \in \sigma(A)$, the set

$$\mathcal{G}(w) := \big\{x \in X \mid d^-_{\mu_x^A}(w) = 0 \;\text{ and }\; d^+_{\mu_x^A}(w) = \infty\big\} \tag{3.4}$$

is generic in $X$. Namely, given $0 \in X$, set

$$\Omega_- := \bigcap_{l\geq 1}\bigcap_{n\geq 1}\bigcap_{k\geq 1}\bigcup_{t\geq k}\Big\{w \in \sigma(A) \mid t^l \int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu_x^A(s) < \frac{1}{n}\Big\}$$

and

$$\Omega_+ := \bigcap_{l\geq 1}\bigcap_{n\geq 1}\bigcap_{k\geq 1}\bigcup_{t\geq k}\Big\{w \in \sigma(A) \mid t^{1/l} \int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu_x^A(s) > n\Big\}.$$

Note that

$$\mathcal{J}_x = \Omega_- \cap \Omega_+.$$

So, since the mapping

$$\sigma(A) \ni w \mapsto \int_{\mathbb{R}} e^{-2t|w-s|} \mathrm{d}\mu_x^A(s)$$

is continuous for each $t > 0$ (by dominated convergence), it follows that $\Omega_-$ and $\Omega_+$ are $G_\delta$ sets in $\sigma(A)$; consequently, $\mathcal{J}_x$ is a $G_\delta$ set in $\sigma(A)$.

Now, let $(w_n)_{n\geq 1} \subset \sigma(A)$ be a dense sequence in $\sigma(A)$. So, if

$$x \in \mathcal{M} = \bigcap_{n\geq 1} \mathcal{G}(w_n) = \left\{\varphi \in X \mid d^-_{\mu_\varphi^A}(w_n) = 0 \text{ and } d^+_{\mu_\varphi^A}(w_n) = \infty, \text{ for each } n \in \mathbb{N}\right\},$$

it follows that $\mathcal{J}_x$ is generic in $\sigma(A)$.

After such preliminaries, we divide the proof of Theorem 16 into 4 steps.

**Step 1.** Let us show that for each $\rho > 0$ and each $w \in \sigma(A)$,

$$\left\{x \in X \mid d^+_{\mu_x^A}(w) \geq d^-_{\mu_x^A}(w) \geq \rho\right\}$$

is dense in $X$. Namely, let for each $n \in \mathbb{N}$ and each $s \in \mathbb{R}$,

$$f_{n,\rho}(w,s) := \left(1 - e^{-n|w-s|^\rho}\right)^{1/2},$$

and for each $x \neq 0$, let $x_n := f_{n,\rho}(w,A)x$, where $f_{n,\rho}(w,A) := E^A(f_{n,\rho}(w,\cdot))$. Since $\mu_x^A$ is purely continuous, one gets, by the Spectral Theorem and dominated convergence, that

$$\begin{aligned}
\|x_n - x\|^2 &= \|f_{n,\rho}(w,A)x - x\|^2 \\
&= \|(f_{n,\rho}(w,A) - 1)x\|^2 \\
&= \int_{\mathbb{R}} \left|\left(1 - e^{-n|w-s|^\rho}\right)^{1/2} - 1\right|^2 \mathrm{d}\mu_x^A(s) \\
&= \mu_x^A(\{w\}) + \int_{\mathbb{R}\setminus\{w\}} \left|\left(1 - e^{-n|w-s|^\rho}\right)^{1/2} - 1\right|^2 \mathrm{d}\mu_x^A(s) \\
&= \int_{\mathbb{R}\setminus\{w\}} \left|\left(1 - e^{-n|w-s|^\rho}\right)^{1/2} - 1\right|^2 \mathrm{d}\mu_x^A(s) \longrightarrow 0
\end{aligned}$$

as $n \to \infty$, that is, $x_n \to x$ in $X$.

Now, by Fubini's Theorem,

$$\begin{aligned}\int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu^A_{x_n}(s) &= \int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu^A_{f_{n,\rho}(w,A)x}(s)\\ &= \int_{\mathbb{R}} e^{-2t|w-s|}|f_{n,\rho}(w,s)|^2\mathrm{d}\mu^A_x(s)\\ &= \int_{\mathbb{R}} e^{-2t|w-s|}(1-e^{-n|x-s|^\rho})\mathrm{d}\mu^A_x(s)\\ &= \int_{\mathbb{R}} e^{-2t|w-s|}|w-s|^\rho\frac{(1-e^{-n|w-s|^\rho})}{|w-s|^\rho}\mathrm{d}\mu^A_x(s)\\ &\le \frac{e^{-\rho}\rho^\rho}{2^\rho t^\rho}\int_{\mathbb{R}}\frac{(1-e^{-n|w-s|^\rho})}{|w-s|^\rho}\mathrm{d}\mu^A_x(s)\\ &= \frac{e^{-\rho}\rho^\rho}{2^\rho t^\rho}\int_{\mathbb{R}}\int_0^n e^{-r|w-s|^\rho}\mathrm{d}r\,\mathrm{d}\mu^A_x(s)\\ &\le \frac{ne^{-\rho}\rho^\rho}{2^\rho t^\rho}\|x\|^2;\end{aligned}$$

we have used the fact that $\max_{u\ge 0}(e^{-u}u^\rho) \le e^{-\rho}\rho^\rho$. Thus, it follows from identity (3.3) that for each $n \ge 1$, $d^-_{\mu^A_{x_n}}(w) \ge \rho$, and so

$$\{x \in X \mid d^+_{\mu^A_x}(w) \ge d^-_{\mu^A_x}(w) \ge \rho\}$$

is dense in $X$.

**Step 2.** Let us show that, for every $w \in \sigma(A)$, there exists $x \in X$ such that $d^-_{\mu^A_x}(w) = 0$. Set $L_w = (-\infty, w] \cap \sigma(A)$, $A_w = AE^A(L_w)$ and $A^0_w = A_w - wI$. So, by Lemma 3.1.1, there exist $x \in X$ and $\varepsilon_j \to 0$ such that for sufficiently large $j$,

$$\begin{aligned}\mu^{A^0_w}_x(B(0;\varepsilon_j)) \ge \frac{1}{-\ln(\varepsilon_j)} \;\Rightarrow\; \mu^{A_w}_x(B(w;\varepsilon_j)) \ge \frac{1}{-\ln(\varepsilon_j)} \;&\Rightarrow\; \ln\left(\mu^A_x(B(w;\varepsilon_j))\right) \ge \ln\left(\mu^A_x(B(w;\varepsilon_j)\cap L_w)\right)\\ &= \;\ln\left(\mu^{A_w}_x(B(w;\varepsilon_j))\right) \ge \ln\left(\frac{1}{-\ln(\varepsilon_j)}\right),\end{aligned}$$

then

$$\frac{\ln\left(\mu^A_x(B(w;\varepsilon_j))\right)}{\ln\varepsilon_j} \le \frac{\ln\left(\frac{1}{-\ln(\varepsilon_j)}\right)}{\ln\varepsilon_j} \Rightarrow d^-_{\mu^A_x}(w) = 0.$$

**Step 3.** Let us show that for every $w \in \sigma(A)$,

$$\{x \in X \mid d^-_{\mu^A_x}(w) = 0\}$$

is dense in $X$. Namely, let $w \in \sigma(A)$ and set, for every $n \ge 1$,

$$S_n := \left(-\infty, w - \frac{1}{n}\right) \cup \{w\} \cup \left(w + \frac{1}{n}, \infty\right).$$

Set also, for each $x \in X$ and each $n \geq 1$,

$$x_n := E^A(S_n)x + \frac{1}{n}x,$$

where $x$ is given by **Step 2**. One has that $x_n \to x$ in $X$, since $E^A(S_n) \to I$ in the strong sense. Moreover, for each $n \geq 1$ and each $0 < \varepsilon < \frac{1}{n}$, one has

$$\begin{aligned}
\mu^A_{x_n}(B(w;\varepsilon)) &= \langle E^A(B(w;\varepsilon))x_n, x_n\rangle \\
&= \langle E^T(B(w;\varepsilon))E^A(S_n)x, x_n\rangle + \frac{1}{n}\langle E^A(B(w;\varepsilon))x, x_n\rangle \\
&= \langle E^A(B(w;\varepsilon)\cap S_n)x, x_n\rangle + \frac{1}{n}\langle E^A(B(w;\varepsilon))x, x_n\rangle \\
&= \langle E^A(\{w\})x, x_n\rangle + \frac{1}{n}\langle E^A(B(w;\varepsilon))x, E^A(S_n)x\rangle \\
&+ \frac{1}{n^2}\langle E^A(B(w;\varepsilon))x, x\rangle \\
&= \langle E^A(\{w\})x, x_n\rangle + \frac{1}{n}\langle E^A(\{w\})x, x\rangle + \frac{1}{n^2}\langle E^A(B(w;\varepsilon))x, x\rangle \\
&= \frac{1}{n^2}\langle E^A(B(w;\varepsilon))x, x\rangle \\
&= \frac{1}{n^2}\,\mu^A_x(B(w;\varepsilon))\,,
\end{aligned}$$

and so

$$d^-_{x^A_{x_n}}(w) = \liminf_{\varepsilon\downarrow 0}\frac{\ln(\mu^A_{x_n}(B(w;\varepsilon)))}{\ln\varepsilon} = \liminf_{\varepsilon\downarrow 0}\frac{\ln(\mu^A_x(B(w;\varepsilon))}{\ln\varepsilon} = d^-_{x^A_x}(w) = 0.$$

Hence,

$$\left\{x \in X \mid d^-_{\mu^A_x}(w) = 0\right\}$$

is dense in $X$.

**Step 4.** Finally, in this step, we finish the proof of the theorem. Since, for each $w \in \mathbb{R}$ and each $t > 0$, the mapping

$$X \ni x \mapsto \int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu^A_x(s) = \langle g_t(A,w)x, x\rangle\,,$$

with $g_t(s,w) = e^{-2t|w-s|}$, is continuous, it follows that for every $w \in \mathbb{R}$, each one of the sets

$$\begin{aligned}
B_-(w) &:= \left\{x \in X \mid d^+_{\mu^A_x}(w) = \infty\right\} \\
&= \bigcap_{l\geq 1}\bigcap_{n\geq 1}\bigcap_{k\geq 1}\bigcup_{t\geq k}\left\{x \in X \mid t^l\int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu^A_x(s) < 1/n\right\}
\end{aligned}$$

and

$$\begin{aligned} B_+(w) &:= \left\{x \in X \mid d^-_{\mu_x^A}(w) = 0\right\} \\ &= \bigcap_{l\geq 1}\bigcap_{n\geq 1}\bigcap_{k\geq 1}\bigcup_{t\geq k}\left\{x \in X \mid t^{\frac{1}{l}} \int_{\mathbb{R}} e^{-2t|w-s|}\mathrm{d}\mu_x^A(s) > n\right\}, \end{aligned}$$

is a $G_\delta$ set in $X$. Thus, it follows from **Steps 1 and 3** that for each $w \in \sigma(A)$, both $B_-(w)$ and $B_+(w)$ are generic sets in $X$, and so

$$\mathcal{G}(w) = \bigcap_{n\geq 1}\left\{x \in X \mid d^-_{\mu_x^A}(w) = 0 \ \text{ and } \ d^+_{\mu_x^A}(w) \geq n\right\} \tag{3.5}$$

is also generic in $X$. □

**Remark 3.2.1.** Let $x$ and $A$ be as in the statement of Theorem 16. Consider also $w = \min_{\lambda\in\sigma(A)} \lambda$. Note that $A_w = wI - A$ is a bounded negative self-adjoint operator and, by the Spectral Theorem, we may rewrite (3.2) and (3.3) by using the norms of semigroup orbits as

$$d^+_{\mu_x^A}(w) = d^+_{\mu_x^{A_w}}(0) = -\liminf_{t\to\infty} \frac{\ln \|e^{t(wI-A)}x\|^2}{\ln t};$$

$$d^-_{\mu_x^A}(w) = d^-_{\mu_x^{A_w}}(0) = -\limsup_{t\to\infty} \frac{\ln \|e^{t(wI-A)}x\|^2}{\ln t}.$$

In this case, Theorem 16 has a clear asymptotic meaning.

**Remark 3.2.2.** Theorem 16 is particularly interesting when $A$ has purely absolutely continuous spectrum, since it shows the striking difference between the typical behaviour of $d^\pm_{\mu_x^A}$ from the topological and measure points of view; namely, if $\mu_x^A$ is purely absolutely continuous, then it is well known that there is a Borel set $\Lambda \subset \mathbb{R}$ such that $\mu_x^A(\Lambda) = \mu_x^A(\mathbb{R}) = \|x\|^2$ and, for every $w \in \Lambda$, $d^\mp_{\mu_x^A}(w) = 1$ (see [30] for details).

### 3.2.1 Proof Theorem 17

We will also need the following result.

**Claim**. If $w \in \sigma(A)$, then $\mathcal{G}(w) \subset \left\{x \in X \mid \mu_x^A \text{ is not U}\beta\text{H}, \forall\ \beta > 0\right\}$. Indeed, it is enough to note that given $\alpha > 0$, if $\mu_x^A$ is U$\alpha$H, then for each $w \in \sigma(A)$, $d^-_{\mu_x^A}(w) \geq \alpha$.

For $w \in \sigma(A)$, if $x \in \mathcal{G}(w)$, then one has from the Claim that for each $\beta = \frac{1}{k}$, $k \geq 1$, $\mu_x^A$ is not U$\beta$H. Thus, it follows from Theorem 3.1.1 ii) that for each $k \geq 1$,

$$\limsup_{t\to\infty} t^{1/k}\, W_x^A(t) = \infty,$$

and then one has from the proof of Theorem 16 (recall (3.5)) that for each $k \geq 1$, the set

$$\big\{x \in X \mid \limsup_{t\to\infty} t^{1/k}\, W_x^A(t) = \infty\big\} \supset \mathcal{G}(w)$$

is generic in $X$.

It remains to prove that for each $k \geq 1$, the set

$$X_k := \big\{x \in X \mid \liminf_{t\to\infty} t^{1-1/k}\, W_x^A(t) = 0\big\}$$

is generic in $X$. The proof that for each $k \geq 1$, $X_k$ is a $G_\delta$ subset of $X$, follows from the same arguments presented in the proof of Lemma 3.3.1 and Theorem 16. On the other hand, it follows from Theorem 3.1.1 i) that for each $k \geq 1$,

$$\{x \in X \mid \mu_x^A \text{ is uniformly 1-Hölder continuous}\} =: X_{\mathrm{UH}}(1) \subset X_k.$$

Finally, since by Theorem 5.2 in [34] (by taking $\alpha = 1$) $X_{\mathrm{UH}}(1)$ is dense in $X$, it follows that for each $k \geq 1$, $X_k$ is a dense $G_\delta$ subset of $X$ (recall that $T$ has purely absolutely continuous spectrum, by hypothesis).

## 3.3 Application to the Almost Mathieu Operator

We recall that the Almost Mathieu Operator $H_\omega^{\lambda,\alpha}$ is a bounded operator defined on $\ell^2(\mathbb{Z})$ by the law

$$(H_\omega^{\lambda,\alpha} u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos(2\pi(\omega + n\alpha))u_n, \tag{3.6}$$

with $\alpha, \omega \in \mathbb{T} = \mathbb{R}\setminus\mathbb{Z}$, $\alpha$ irrational. It is well known that for every $0 < \lambda < 1$, $H_{\omega,\alpha}^\lambda$ has purely absolutely continuous spectrum. For more details, see [25].

**Theorem 3.3.1.** Let $\alpha$ and $\omega$ be as before, and let $0 < \lambda_1 < \lambda_2 < 1$. Then, there exists a generic set $\mathcal{M} \subset \ell^2(\mathbb{Z})$ so that, for each $x \in \mathcal{M}$, the set of $\lambda \in [\lambda_1, \lambda_2]$ such that for each $k \geq 1$,

$$\liminf_{t\to\infty} t^{1-1/k}\, W_x^{H_{\omega,\alpha}^\lambda}(t) = 0 \quad \textit{and} \quad \limsup_{t\to\infty} t^{1/k}\, W_x^{H_{\omega,\alpha}^\lambda}(t) = \infty,$$

is generic in $[\lambda_1, \lambda_2]$.

**Remark 3.3.1.** It is worth underlying that such phenomenon has been shown, for some singular continuous systems, by Aloísio, Carvalho and de Oliveira in [1, 20] (see also [21]) by exploring the density of pure point operators in appropriate spaces, a quite different setting from this work. In this case of purely absolutely continuous spectrum, this phenomenon is, in some sense, the counterpart of the situation of an operator with pure point spectrum and quasiballistic transport [3, 28].

Let us proceed to the proof of Theorem 3.3.1. We begin with some preparation.

**Lemma 3.3.1.** *Let $\alpha$ and $\omega$ be as before, and let $0 < \lambda_1 < \lambda_2 < 1$. Then, for every $x \in \ell^2(\mathbb{Z})$ and $k \geq 1$,*

$$\{\lambda \mid \liminf_{t\to\infty} t^{1-1/k}\, W_x^{H^\lambda_{\omega,\alpha}}(t) = 0 \quad \textit{and} \quad \limsup_{t\to\infty} t^{1/k}\, W_x^{H^\lambda_{\omega,\alpha}}(t) = \infty\}$$

*is a $G_\delta$ set in $[\lambda_1, \lambda_2]$.*

*Proof.* Note that this proof is based on the same arguments presented in the proof of Proposition 3.1 in [3]. Let $\lambda \in [\lambda_1, \lambda_2]$; if $\lambda_n \to \lambda$, then $H^{\lambda_n}_{\omega,\alpha}$ converges strongly to $H^\lambda_{\omega,\alpha}$ as $n \to \infty$. Thus, it follows from Propositions 10.1.8 and 10.1.13 in [26] that for every $t \in \mathbb{R}$, $e^{-itH^{\lambda_n}_{\omega,\alpha}}$ converges strongly to $e^{-itH^\lambda_{\omega,\alpha}}$, as $n \to \infty$, and so it follows from dominated convergence that for each $x \in \ell^2(\mathbb{Z})$, $t \in \mathbb{R}$ and $k \geq 1$, the map

$$[\lambda_1, \lambda_2] \ni \lambda \longmapsto t^{1-1/k}\, W_x^{H^\lambda_{\omega,\alpha}}(t) = \frac{t^{1-1/k}}{t} \int_0^t |\langle e^{-isH^\lambda_{\omega,\alpha}} x, x\rangle|^2\, \mathrm{d}s$$

is continuous. Since

$$\left\{\lambda \mid \liminf_{t\to\infty} t^{1-1/k}\, W_x^{H^\lambda_{\omega,\alpha}}(t) = 0\right\} = \bigcap_{n\geq 1}\bigcap_{l\geq 1}\bigcup_{t\geq l}\left\{\lambda \mid t^{1-1/k}\, W_x^{H^\lambda_{\omega,\alpha}}(t) < n\right\},$$

$$\left\{\lambda \mid \limsup_{t\to\infty} t^{1-1/k}\, W_x^{H^\lambda_{\omega,\alpha}}(t) = \infty\right\} = \bigcap_{n\geq 1}\bigcap_{l\geq 1}\bigcup_{t\geq l}\left\{\lambda \mid t^{1-1/k}\, W_x^{H^\lambda_{\omega,\alpha}}(t) > n\right\},$$

the result follows. □

***Proof of Theorem 3.3.1.*** The result is a direct consequence of Theorem 17 and of an argument involving separability (see [1, 5]). Let $(\lambda_j)_{j\geq 1}$ be a dense sequence in $[\lambda_1, \lambda_2]$ and let $H_j = H^{\lambda_j}_{\omega,\alpha}$ be the corresponding operators. If $\mu^j_x$ denotes the spectral measure of the pair $(H_j, x)$, it follows from Theorem 17 that

$$\mathcal{M} = \bigcap_{j\geq 1} \{x \in \ell^2(\mathbb{Z}) \mid \liminf_{t\to\infty} t^{1-1/k}\, W_x^{H_j}(t) = 0 \quad \textit{and} \quad \limsup_{t\to\infty} t^{1/k}\, W_x^{H_j}(t) = \infty\}$$

is generic in $\ell^2(\mathbb{Z})$. Since, by Lemma 3.3.1, for every $x \in \mathcal{M}$ and $k \geq 1$,

$$\{\lambda \mid \liminf_{t\to\infty} t^{1-1/k}\, W_x^{H^\lambda_{\omega,\alpha}}(t) = 0 \quad \textit{and} \quad \limsup_{t\to\infty} t^{1/k}\, W_x^{H^\lambda_{\omega,\alpha}}(t) = \infty\} \supset \{\lambda_j\}$$

is a $G_\delta$ set in $[\lambda_1, \lambda_2]$, the result follows. □

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# Appendix A

# $C_0$-semigroups

Here, we present the basic definitions and results regarding $C_0$-semigroups. For a real exposition of the theme, see [7, 29, 35, 47, 51, 52, 62].

**Definition A.0.1.** Let $X$ be a Banach space and let $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ be a one parameter family of linear operators satisfying the following properties:

1. $T(0) = I_X = 1$;
2. for each $s, t \geq 0$, $T(t+s) = T(t) \circ T(s)$ (**semigroup property**).

Then, $(T(t))_{t\geq 0}$ is called an **operator semiproup**. The linear operator $A$, given by

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t},$$

where

$$\mathcal{D}(A) := \left\{ x \in X \mid \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\},$$

is the so-called **infinitesimal generator** of the semigroup $(T(t))_{t\geq 0}$ and $\mathcal{D}(A)$ is the domain of $A$.

**Lemma A.0.1** (Theorem 1.4, Chapter II in [29])**.** The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.

**Definition A.0.2.** A semigroup $(T(t))_{t\geq 0}$ defined on $X$ is called a $C_0$**-semigroup** (or strongly continuous semigroup) if for each $x \in X$,

$$\lim_{t\to 0^+} T(t)x = x.$$

A $(T(t))_{t\geq 0}$ $C_0$-semigroup is bounded (uniformly bounded) if

$$\sup_{t\geq 0} \|T(t)\|_{\mathcal{L}(X)} < \infty.$$

**Example A.0.1** (Proposition 4.11 in [29])**.** Let $(T(t))_{t\geq 0}$ be the multiplication semigroup generated by a measurable function $\varphi : \mathbb{R} \to \mathbb{C}$ satisfying

$$\operatorname*{ess.\,sup}_{s\in\mathbb{R}} \mathrm{Re}\varphi(s) < \infty,$$

that is, let for each $t \geq 0$ and each $f \in L^p(\mathbb{R}, d\mu)$,

$$(T(t)f)(s) := e^{t\varphi(s)} f(s), \quad \forall\, t \geq 0.$$

Then, the mappings

$$\mathbb{R}_+ \ni t \mapsto T(t)f = e^{t\varphi} f \in L^p(\mathbb{R}, d\mu)$$

are continuous for every $f \in L^p(\mathbb{R}, d\mu)$. Moreover, the semigroup $(T(t))_{t\geq 0}$ is uniformly continuous if, and only if, $\varphi$ is essentially bounded.

**Example A.0.2.** Let $X = W^{1,2}(1, \infty)$ and let $\varphi_b(s) = 1/s + is^b$ $(s \geq 1)$, with $b \in (0, 1)$. Define

$$(T(t)f)(s) := e^{-t\varphi_b(s)} f(s).$$

For each $t > 0$, there exist positive constants $C$ and $\tilde{C}$ such that

$$\|T(t)\|_{\mathcal{L}(X)} = C \sup_{k\in\{0,1\}} \sup_{s\geq 1} \left| \frac{d^k}{ds^k} e^{-t\varphi(s)} \right| = \tilde{C} t^b.$$

Therefore, $(T(t))_{t\geq 0}$ is an unbounded $C_0$-semigroup.

**Proposition A.0.1** (Proposition G.2.2 in [66])**.** Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup on the Banach space $X$. There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for each $t \geq 0$.

**Theorem A.0.1** (Theorem 1.10, Chapter II in [29])**.** Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup on the Banach space $X$ and suppose that there exist constants $\omega \in \mathbb{R}$, $M \geq 1$ such that for each $t \geq 0$,

$$\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}.$$

For the generator $-A$ of $(T(t))_{t\geq 0}$, the following properties hold.

a) If $\lambda \in \mathbb{C}$ is such that $R(\lambda)x := \int_0^\infty e^{-\lambda s} T(s)x ds$ exists for all $x \in X$, then $\lambda \in \rho(-A)$ and $R(\lambda, -A) = R(\lambda)$.

b) If $\mathrm{Re}\lambda > \omega$, then $\lambda \in \rho(-A)$, the resolvent is given by the integral expression in $a$) and

$$\|R(\lambda, -A)\|_{\mathcal{L}(X)} \leq \frac{M}{\mathrm{Re}\lambda - \omega}.$$

**Definition A.0.3** (Analytic Semigroups)**.** A $C_0$-semigroup$(T(t))_{t\geq 0}$ on a Banach space $X$ is called analytic on $S_\omega$ if for all $x \in X$, the function $[0, \infty) \ni t \mapsto T(t)x$ extends analytically to

$S_\omega$ and satisfies

$$\lim_{z \in S_\omega, z \to 0} S(z)x = x.$$

We call $(T(t))_{t\geq 0}$ an analytic $C_0$-semigroup if $(T(t))_{t\geq 0}$ is analytic on $S_\omega$ for some $\omega \in (0, \pi)$.

**Theorem A.0.2** (Theorem G.5.2 in [65])**.** For a closed and densely defined operator $A$ on a Banach space $X$ the following assertions are equivalent:

a) there exists $\varphi \in \left(0, \frac{\pi}{2}\right)$ such that $A$ generates a bounded analytic $C_0$- semigroup on $S_\varphi$;

b) there exists $\theta \in \left(\frac{\pi}{2}, \pi\right)$ such that $S_\theta \subset \rho(A)$ and

$$\sup_{\lambda \in S_\theta} \|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} < \infty$$

The following are some results for the characterization of strongly continuous groups and semigroups, for more details we suggest [29, 65, 26].

**Theorem A.0.3** (Hille-Yosida)**.** For a densely defined operator $A$ on a Banach space $X$ and constants $M \geq 1$ and $w \in \mathbb{R}$, the following assertions are equivalent:

(a) $A$ generates a $C_0$-semigroup on $X$ satisfying $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{wt}$ for each $t \geq 0$.

(b) $\{\lambda \in \mathbb{C} \mid \text{Re}\lambda > w\} \subset \rho(A)$ and $\|R(\lambda, A)^n\|_{\mathcal{L}(X)} \leq \dfrac{M}{(\text{Re}\lambda - w)^n}$ for each $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > w$ and $n \in \mathbb{N}$.

**Theorem A.0.4** (Stone)**.** An operator $-iA$ on a Hilbert space $X$ generates a $C_0$-group of unitary operators if, and only if, $A$ is self-adjoint.

# Appendix B

# Spectral Theorem, spectral resolution, spectral measures and spectral types

**Definition B.0.1.** Let $\mathcal{A}$ the Borel $\sigma$-algebra in $\Omega \subset \mathbb{C}$. A resolution of the identity is a mapping

$$\mathcal{A} \ni \Lambda \ \mapsto E(\Lambda) \in \mathcal{L}(X)$$

with the following properties:

1. $E(\emptyset) = 0_{\mathcal{L}(X)}$ and $E(\Omega) = 1$;
2. for each $\Lambda \in \mathcal{A}$; $E(\Lambda)$ is is an orthogonal projection;
3. if $\Lambda_1 \cap \Lambda_2 = \emptyset$, then $E(\Lambda_1 \cup \Lambda_2) = E(\Lambda_1) + E(\Lambda_2)$;
4. $E(\Lambda_1 \cap \Lambda_2) = E(\Lambda_1) \circ E(\Lambda_2)$;
5. to each pair $x, y \in X$, one associates the complex Borel measure

$$\mathcal{A} \ni \Lambda \mapsto \mu_{x,y}(\Lambda) = \langle x, E(\Lambda)y \rangle.$$

   The measure $\mu_{x,y}$ is called the spectral measure of the resolution of the identity $E$ associated with the pair $x, y \in X$.

We note that for $x = y$, the spectral measure of $E$ with respect to $X$ is always a real-valued measure, since $E(\Lambda) = \langle x, E(\Lambda)x \rangle \geq 0$; we denote it by $\mu_x$.

**Example B.0.1.** Let $X = L^2(1, \infty)$, let $\varphi : (1, \infty) \to \mathbb{R}$ be a measurable function and set $\mathcal{D}(A) := \{f \in L^2(1, \infty) \mid \varphi f \in L^2(1, \infty)\}$. Define the linear operator $A : \mathcal{D}(A) \subset X \to X$ by the law

$$(Af)(s) = \varphi(s)f(s).$$

In fact, $A$ is a self-adjoint operator and the map

$$\mathcal{A} \ni \Lambda \mapsto E(\Lambda) := \chi_{\varphi^{-1}(\Lambda)}$$

is a resolution of the identity.

**Theorem B.0.1** (Spectral Theorem)**.** Every self-adjoint operator $A$ defined on a Hilbert space corresponds to a unique resolution $E^A$ of the identity such that

$$A = \int_{\sigma(A)} \lambda dE^A(\lambda).$$

**Theorem B.0.2** (Functional Calculus I)**.** Let $f : \mathbb{R} \to \mathbb{C}$ be measurable Borel function and let $A$ be a self-adjoint operator defined in the Hilbert space $X$. Then,

1. $$f(A) := \int_{\sigma(A)} f(\lambda)dE^A(\lambda)$$

   is a well-defined linear operator, whose domain

   $$\mathcal{D}(f(A)) := \left\{x \in X; \int_{\sigma(A)} |f(\lambda)|^2 d\mu_x^A(\lambda) < \infty\right\}$$

   is dense in $X$.

2. $\langle f(A)x, y\rangle = \int_{\sigma(A)} f(\lambda)d\mu_{x,y}^A(\lambda)$ and $\|f(A)x\|^2 = \int_{\sigma(A)} |f(\lambda)|^2 d\mu_x^A(\lambda)$.

**Example B.0.2.** Let $(\lambda_n)_{n\in\mathbb{N}}$ be a real sequence and let $(P_n)_{n\in\mathbb{N}}$ be a sequence of pairwise orthogonal projections on $X$ such that $\sum_n P_n = I$. Let $A$ be a self-adjoint defined by the law

$$\mathcal{D}(A) = \left\{x \in X \mid \sum_n |\lambda_n|^2 \|P_n x\| < \infty\right\}, \quad Ax = \sum_n \lambda_n P_n x, \ \forall\, x \in \mathcal{D}(A).$$

Then, for each $\Lambda \in \mathcal{A}$, one has

$$E^A(\Lambda) = \sum_{\lambda_n \in \Lambda} P_n.$$

It follows from Theorem B.0.2 that

$$\mathcal{D}(f(A)) = \left\{x \in X \mid \sum_n |f(\lambda_n)|^2 \|P_n x\| < \infty\right\}, \quad f(A)x = \sum_n f(\lambda_n)P_n x, \quad \forall\, x \in \mathcal{D}(f(A)).$$

Let $\mathcal{L}$ denote the Lebesgue measure over the Borel sets $A \subset \mathbb{R}$. Recall that, by Lebesgue Decomposition Theorem (see Theorem 6.10 in [59]), a Borel measure $\mu$ over $\mathbb{R}$ can be (uniquely) decomposed as $\mu = \mu_p + \mu_c$, with $\mu_c$ and $\mu_p$ denoting its continuous part (that is, $\mu_c(\{a\}) = 0$, for each $a \in \mathbb{R}$) and point part (that is, there is a countable set $\Lambda \subset \mathbb{R}$ so that $\mu_p(\mathbb{R} \setminus \Lambda) = 0$), respectively. Observe that $\mathcal{L}$ and $\mu_p$ are mutually singular measures. Again by Lebesgue Decomposition Theorem, one has (uniquely) $\mu_c = \mu_{ac} + \mu_{sc}$, where $\mu_{sc}$ and $\mathcal{L}$ are mutually singular measures and $\mu_{ac}$ is absolutely continuous with respect to $\mathcal{L}$. Then, $\mu = \mu_p + \mu_{ac} + \mu_{sc}$, where $\mu_{ac}$ is called the absolutely continuous component of $\mu$, while $\mu_{sc}$ is the singular continuous component of $\mu$. We will study this decomposition in the context of spectral measures.

**Definition B.0.2.** Let $X$ be a Hilbert space and let $A$ be a self-adjoint operator. The point subspace of $A$ is $X_p = X_p(A) \subset X$ given by the closure of the linear subspace spanned by the eigenvectors of $A$. Its orthogonal complement $X_c = X_p(A) := X_p^{\perp}$ is the continuous subspace of $A$.

**Theorem B.0.3** (Theorem 12.1.2 in [26])**.** Let $A$ be a self-adjoint operator and let $\mu_x^A$ the spectral measure of $A$ at $x \in X$. Then,

(a) there exists a countable set $\Lambda \subset \mathbb{R}$ so that

$$X_p = \{x \in X \mid \mu_x^A(\mathbb{R} \setminus \Lambda) = 0\}.$$

$\Lambda \subset \mathbb{R}$ can be taken as the set of eigenvalues of $A$;

(b) $X_c = \{x \in X \mid \mu_x^A(\{a\}) = 0, \ \forall\ a \in \mathbb{R}\}$, that is, the function $\xi \mapsto \|E^A((-\infty, \xi])x\|$ is continuous;

(c) $X = X_p \oplus X_c$.

Let $E_p^A := E^A\Big|_{X_p}$, $E_c^A := E^A\Big|_{X_c}$, $A_p := AE_p^A$ and $A_c := AE_c^A$. Then, the decomposition $A := A_p + A_c$ is valid (see Theorem 9.8.3 in [26]).

**Definition B.0.3** (Definition 12.1.3 in [26])**.** The point spectrum of $A$ is $\sigma_p(A) := \sigma(A_p)$, and the continuous spectrum of $A$ is $\sigma_c(A) := \sigma(A_c)$.

**Definition B.0.4** (Definition 12.1.5 in [26])**.** Let $A$ be a self-adjoint operator and let $\mu_x^A$ be the spectral measures of $A$ at $x \in X$.

(a) The singular subspace of $A$ is

$$X_s(A) := \{x \in X \mid \mu_x^A \perp \mathcal{L}\}.$$

($\mu_x^A \perp \mathcal{L}$ *indicates that* $\mu_x^A$ *and* $\mathcal{L}$ *are mutually singular*). So, $X_p(A) \subset X_s(A)$.

(b) The absolutely continuous subspace of $A$ is

$$X_{ac}(A) = \{x \in X \mid \mu_x^A \ll \mathcal{L}\}.$$

($\mu_x^A \ll \mathcal{L}$ *indicates that* $\mu_x^A$ *is absolutely continuous with respect to* $\mathcal{L}$). So, $X_{ac}(A) \subset Xc(A)$.

(c) The singular continuous subspace of $A$, denoted by $X_{sc}(A)$, is the set of $x \in X$ so that $\mu_x^A(\mathbb{R} \setminus \Lambda_1) = 0$ for some Borel set $\Lambda_{\subset}\mathbb{R}$ with $\mathcal{L}(\Lambda_1) = 0$ and $\mu_x^A(\Lambda) = 0$ for each countable sets $\Lambda \subset \mathbb{R}$. Hence, $\mu_x^A$ is a singular continuous measure. So, $X_{sc}(A) \subset X_c(A) \cap X_s(A)$.

**Definition B.0.5** (Definition 12.1.10 in [26])**.** The **absolutely continuous spectrum** of $A$ is $\sigma_{ac}(A) := \sigma(A_{ac})$ and the **singular continuous spectrum** of $A$ is $\sigma_{sc}(A) := \sigma(A_{sc})$. The operator $A$ has **purely point spectrum** if $\sigma_{ac}(A) = \emptyset = \sigma_{sc}(A)$; **purely absolutely**

**continuous spectrum** if $\sigma_p(A) = \emptyset = \sigma_{sc}(A)$; **purely singular continuous spectrum** if $\sigma_{ac}(A) = \emptyset = \sigma_p(A)$. It is also common to say that $A$ is pure point, and so on.

**Example B.0.3.** If $A$ is a self-adjoint and compact operator, then $A$ is pure point.

**Definition B.0.6.** Let $\mu$ be a finite (positive) Borel measure on $\mathbb{R}$. The pointwise lower and upper local scaling exponents of $\mu$ at $w \in \mathbb{R}$ are defined, respectively, by

$$d_\mu^-(w) := \liminf_{\varepsilon \downarrow 0} \frac{\ln \mu(B(w,\epsilon))}{\ln \varepsilon} \quad \text{and} \quad d_\mu^+(w) := \limsup_{\epsilon \downarrow 0} \frac{\ln \mu(B(w,\varepsilon))}{\ln \varepsilon},$$

if, for all $\epsilon > 0$, $\mu(B(w,\varepsilon)) > 0$; $d_\mu^\mp(w) := \infty$, otherwise.

**Proposition B.0.1** (Proposition 2.2 in [4])**.** Let $A$ be a negative self-adjoint operator and let $x \in X$, with $x \neq 0$. Then,

$$d_{\mu_x^A}^+(0) = -\liminf_{t\to\infty} \frac{\ln \|e^{tA}x\|^2}{\ln t} \quad \text{and} \quad d_{\mu_x^A}^-(0) = -\limsup_{t\to\infty} \frac{\ln \|e^{tA}x\|^2}{\ln t},$$

where $\mu_x^A$ is the spectral measure of $A$ associated with the vector $x$.

Note that Proposition B.0.1 indicates that the power-law decaying rates of an orbit $(e^{tA}x)_{t\geq 0}$ may depend on sequences of time going to infinity; i.e., if $d_{\mu_x^A}^-(0) < d_{\mu_x^A}^+(0)$ (see [4] for more details).

# Appendix C

# Proof of Proposition 2.3.2

**Item (a).** Let $\zeta > 1$ and set $\tilde{c} := \zeta + a$.

- **Case 1:** $\alpha = 1$.

**Case 1(a):** $\tilde{c} \in (1, 2]$. Note that in this case, $a \in [0, 1)$. Set $h_{\alpha,\zeta}(\lambda) = \lambda^{\alpha}(2\pi - i\log(\lambda))^{\zeta}$, with $\lambda \in i\mathbb{R} \setminus \{0\}$, and define the operator $L_{\nu,\tilde{c}}(A) := (1 + A)^{-\nu}(2\pi - i\log(A))^{-\tilde{c}} \in \mathcal{L}(X)$. Since $(\lambda + A)^{-1}$ commutes with $L_{\nu,\tilde{c}}(A)$, it follows from the Moment Inequality (see Proposition 1.3.2) that

$$\|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}L_{\nu,\tilde{c}}(A)\|_{\mathcal{L}(X)} \lesssim \|h_{1,1-a}(\lambda)(\lambda+A)^{-1}L_{\nu,1}(A)\|^{2-\tilde{c}}_{\mathcal{L}(X)}\|h_{1,2-a}(\lambda)(\lambda+A)^{-1}L_{\nu,2}(A)\|^{\tilde{c}-1}_{\mathcal{L}(X)}. \tag{C.1}$$

Let $\varepsilon > 0$, set $A_\varepsilon := (A + \varepsilon)(1 + \varepsilon A)^{-1}$ and note that $A_\varepsilon^{-1} \in \mathcal{L}(X)$. For each $\lambda \in i\mathbb{R} \setminus \{0\}$, let $r \in (0, |\lambda|/2]$ and $R \geq 2|\lambda| + 2$ be such that $\sigma(A_\varepsilon) \subset \{z \in \mathbb{C} \mid r < |z| < R\}$, let $\theta \in (\pi/2, \pi)$ and set $\gamma_+ = \{se^{i\theta} \mid s \in [r, R]\}$, $\gamma_- = \{te^{-i\theta} \mid t \in [r, R]\}$, $\gamma_r = \{re^{is} \mid s \in [-\theta, \theta]\}$, $\gamma_R = \{Re^{is} \mid s \in [-\theta, \theta]\}$ and $\gamma := \gamma_+ \cup \gamma_- \cup \gamma_r \cup \gamma_R$. Then, by the Riesz-Dunford functional calculus (see (1.4)), for each $x \in X$ (here, $y := (1 + A)^{-\nu}x$),

$$\begin{aligned}
h_{1,1-a}(\lambda)(\lambda + A_\varepsilon)^{-1}(2\pi - i\log(A_\varepsilon))^{-1}y &= \frac{h_{1,1-a}(\lambda)}{2\pi i}\int_\gamma \frac{1}{(2\pi - i\log(z))}R(z, A_\varepsilon)(\lambda + A_\varepsilon)^{-1}y\,dz \\
&= \frac{h_{1,1-a}(\lambda)}{2\pi i}\int_\gamma \frac{1}{(2\pi - i\log(z))(\lambda + z)}dz(\lambda + A_\varepsilon)^{-1}y + \\
&+ \frac{h_{1,1-a}(\lambda)}{2\pi i}\int_\gamma \frac{1}{(2\pi - i\log(z))(\lambda + z)}R(z, A_\varepsilon)y\,dz
\end{aligned}$$

$$
\begin{aligned}
&= \frac{h_{1,1-a}(\lambda)(\lambda+A_\varepsilon)^{-1}y}{2\pi - i\log(-\lambda)} + \frac{1}{2\pi i}\int_r^R \frac{h_{1,1-a}(\lambda)e^{-i\theta}R(te^{-i\theta},A_\varepsilon)y}{(2\pi-\theta-i\log(t))(\lambda+te^{-i\theta})}dt \\
&- \frac{h_{1,1-a}(\lambda)}{2\pi i}\int_r^R \frac{e^{i\theta}}{(2\pi+\theta-i\log(t))(\lambda+te^{i\theta})}R(te^{i\theta},A_\varepsilon)ydt \\
&+ \frac{h_{1,1-a}(\lambda)}{2\pi i}\int_{-\theta}^{\theta} \frac{iRe^{is}}{(2\pi-s+i\log(R))(\lambda+Re^{is})}R(Re^{is},A_\varepsilon)yds \\
&- \frac{h_{1,1-a}(\lambda)}{2\pi i}\int_{-\theta}^{\theta} \frac{ire^{is}}{(2\pi-s+i\log(r))(\lambda+re^{is})}R(re^{is},A_\varepsilon)yds,
\end{aligned}
$$

where we have used the residue theorem in the third identity. By taking the limit $\theta \to \pi$ on both sides of the identity above, one gets

$$
\begin{aligned}
h_{1,1-a}(\lambda)(\lambda+A_\varepsilon)^{-1}(2\pi - i\log(A_\varepsilon))^{-1}y &= \frac{h_{1,1-a}(\lambda)}{2\pi - i\log(-\lambda)}(\lambda+A_\varepsilon)^{-1}y \\
&+ \frac{1}{2\pi i}\int_r^R \frac{h_{1,1-a}(\lambda)}{(\pi - i\log(t))(\lambda - t)}(t+A_\varepsilon)^{-1}ydt \\
&- \frac{1}{2\pi i}\int_r^R \frac{h_{1,1-a}(\lambda)(t+A_\varepsilon)^{-1}y}{(3\pi - i\log(t))(\lambda-t)}dt \\
&+ \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{ih_{1,1-a}(\lambda)Re^{is}R(Re^{is},A_\varepsilon)y}{(2\pi - s - i\log(R))(\lambda+Re^{is})}ds \\
&- \frac{h_{1,1-a}(\lambda)}{2\pi i}\int_{-\pi}^{\pi} \frac{ire^{is}}{(2\pi+s-i\log(r))(\lambda+re^{is})}R(re^{is},A_\varepsilon)yds
\end{aligned}
$$

Now, by taking the limits $r \to 0$ and $R \to \infty$ on both sides of the last identity, one gets for each $x \in X$,

$$
\begin{aligned}
h_{1,1-a}(\lambda)(\lambda+A_\varepsilon)^{-1}(2\pi - i\log(A_\varepsilon))^{-1}y &= \frac{h_{1,1-a}(\lambda)(\lambda+A_\varepsilon)^{-1}y}{2\pi - i\log(-\lambda)} \\
&+ \int_0^\infty \frac{ih_{1,1-a}(\lambda)}{(3\pi^2 - 4\pi i\log(t) - \log(t)^2)(\lambda-t)}(t+A_\varepsilon)^{-1}y\,dt.
\end{aligned}
$$

Finally, by taking the limit $\varepsilon \to 0^+$ on both hands of the identity above, one gets

$$
\begin{aligned}
h_{1,1-a}(\lambda)(\lambda+A)^{-1}(2\pi - i\log(A))^{-1}y &= \frac{h_{1,1-a}(\lambda)(\lambda+A)^{-1}y}{2\pi - i\log(-\lambda)} \\
&+ \int_0^\infty \frac{ih_{1,1-a}(\lambda)(t+A)^{-1}y}{(3\pi^2 - 4\pi i\log(t) - \log(t)^2)(\lambda-t)}dt, \quad \text{(C.2)}
\end{aligned}
$$

where we have used on the left-hand side that $(\lambda + A_\varepsilon)^{-1} \to (\lambda + A)^{-1}$ uniformly (by Lemma 1.3.1), $(2\pi - i\log(A_\varepsilon))^{-1} \to (2\pi - i\log(A))^{-1}$ strongly (see the proof of Lemma 3.5.1 [32]), and on the right-hand side dominated convergence.

Then, by (C.2), one gets

$$
\begin{aligned}
&|h_{1,1-a}(\lambda)| \left\|(\lambda+A)^{-1}(2\pi - i\log(A))^{-1}(1+A)^{-\nu}\right\|_{\mathcal{L}(X)} \\
&\lesssim \left\|\frac{h_{1,1-a}(\lambda)(\lambda+A)^{-1}}{2\pi - i\log(-\lambda)}\right\|_{\mathcal{L}(X)} + \int_0^\infty \frac{|h_{1,1-a}(\lambda)|}{(\pi^2+\log(t)^2)|\lambda - t|}\|(t+A)^{-1}\|_{\mathcal{L}(X)}dt \\
&\lesssim \left\|\frac{h_{1,1-a}(\lambda)(\lambda+A)^{-1}}{2\pi - i\log(-\lambda)}\right\|_{\mathcal{L}(X)} + \int_0^\infty \frac{|h_{1,1-a}(\lambda)|}{t(\pi^2+\log(t)^2)|(\lambda|+t)}dt \\
&\lesssim \left\|\frac{h_{1,1-a}(\lambda)(\lambda+A)^{-1}}{2\pi - i\log(-\lambda)}\right\|_{\mathcal{L}(X)} + \int_0^\infty \frac{|h_{1,1-a}(\lambda)|(t+1)}{t(\pi^2+\log(t)^2)(|\lambda|+t)}dt \\
&= \left\|\frac{h_{1,1-a}(\lambda)(\lambda+A)^{-1}}{2\pi - i\log(-\lambda)}\right\|_{\mathcal{L}(X)} + \frac{|h_{1,1-a}(\lambda)|(|\lambda|-1)}{|\lambda|\log(|\lambda|)},
\end{aligned}
\tag{C.3}
$$

where we have used relation (E.3) in the last identity.

Note that for each $\lambda \in i\mathbb{R}\setminus\{0\}$ with $|\lambda| \leq 1$, it follows from (2.5) that

$$
\left\|\frac{h_{1,1-a}(\lambda)(\lambda+A)^{-1}}{2\pi - i\log(-\lambda)}\right\|_{\mathcal{L}(X)} \lesssim 1,
$$

and since for each $\eta > 0$, $\lim_{|\lambda|\to 0^+} |\lambda|\log(|\lambda|)^\eta = 0$, one gets

$$
\frac{|h_{1,1-a}(\lambda)|(|\lambda|-1)}{|\lambda|\log(|\lambda|)} \leq \frac{(2\pi + |\log(|\lambda|)|)^{1-a}(|\lambda|-1)}{\log(|\lambda|)} \lesssim |\lambda||\log(|\lambda|)|^{-a} \overset{|\lambda|\to 0^+}{\longrightarrow} 0
$$

and $|h_{1,1-a}(\lambda)| \to 0$ as $|\lambda| \to 0^+$. Hence, one concludes that

$$
\sup\{\left\|h_{1,1-a}(\lambda)(\lambda+A)^{-1}(1+A)^{-\nu}(2\pi - i\log(A))^{-1}\right\|_{\mathcal{L}(X)} \mid \lambda \in i\mathbb{R}\setminus\{0\}, |\lambda|\leq 1\} < \infty. \tag{C.4}
$$

Now, by using the same ideas as before, one has for each $\varepsilon > 0$ and each $x \in X$,

$$
\begin{aligned}
h_{1,2-a}(\lambda)(\lambda+A_\varepsilon)^{-1}(2\pi - i\log(A_\varepsilon))^{-2}(1+A)^{-\nu}x &= \frac{h_{1,2-a}(\lambda)(\lambda+A_\varepsilon)^{-1}(1+A)^{-\nu}x}{(2\pi - i\log(\lambda))^2} \\
&- \int_0^\infty \frac{2ih_{1,2-a}(\lambda)(2\pi - i\log(t))(t+A_\varepsilon)^{-1}(1+A)^{-\nu}x}{(3\pi^2 - 4\pi i\log(t) - \log(t)^2)^2(\lambda - t)}dt.
\end{aligned}
$$

So, by taking the limit $\varepsilon \to 0^+$ on both sides of the identity, one gets

$$
\begin{aligned}
h_{1,2-a}(\lambda)(\lambda+A)^{-1}(2\pi - i\log(A))^{-2}(1+A)^{-\nu}x &= \frac{h_{1,2-a}(\lambda)(\lambda+A)^{-1}(1+A)^{-\nu}x}{(2\pi - i\log(\lambda))^2} \\
&- \int_0^\infty \frac{2ih_{1,2-a}(\lambda)(2\pi - i\log(t))(t+A)^{-1}(1+A)^{-\nu}x}{(3\pi^2 - 4\pi i\log(t) - \log(t)^2)^2(\lambda - t)}dt.
\end{aligned}
$$

Then,

$$\left\|h_{1,2-a}(\lambda)(\lambda+A)^{-1}(2\pi-i\log(A))^{-2}\right\|_{\mathcal{L}(X)} \lesssim \left\|\frac{h_{1,2-a}(\lambda)(\lambda+A)^{-1}}{(2\pi-i\log(\lambda))^2}\right\|_{\mathcal{L}(X)}$$
$$+\int_0^{e^{-2\pi}} \frac{|h_{1,2-a}(\lambda)||\log(t)|}{t(\pi^2+\log(t)^2)^2(|\lambda|+t)}dt + \int_{e^{-2\pi}}^{e^{2\pi}} \frac{|h_{1,2-a}(\lambda)|(|\log(t)|+2\pi)}{|(3\pi^2-4\pi i\log(t)-\log(t)^2)^2|}\frac{dt}{t}$$
$$+\int_{e^{2\pi}}^{\infty} \frac{|h_{1,2-a}(\lambda)||\log(t)|}{t(\pi^2+\log(t)^2)^2(|\lambda|+t)}dt$$

$$\begin{aligned}
&\lesssim \left\|\frac{h_{1,2-a}(\lambda)(\lambda+A)^{-1}}{(2\pi-i\log(\lambda))^2}\right\|_{\mathcal{L}(X)} + |h_{1,2-a}(\lambda)|\int_0^\infty \frac{\pi^2-2(1+1/t)\log(t)+\log(t)^2}{(\pi^2+(\log(t))^2)^2(|\lambda|+t)}dt + |h_{1,2-a}(\lambda)| \\
&= \left\|\frac{h_{1,2-a}(\lambda)(\lambda+A)^{-1}}{(2\pi-i\log(\lambda))^2}\right\|_{\mathcal{L}(X)} + \frac{|h_{1,2-a}(\lambda)|(|\lambda|\log(|\lambda|)-|\lambda|+1+|\lambda|\log(|\lambda|)^2)}{|\lambda|\log(|\lambda|)^2},
\end{aligned}$$

where we have used relation (E.4) in the last identity.

By using the same reasoning as before, one concludes that

$$\sup\{\left\|h_{1,2-a}(\lambda)(\lambda+A)^{-1}(2\pi-i\log(A))^{-2}(1+A)^{-\nu}\right\|_{\mathcal{L}(X)} \mid \lambda\in i\mathbb{R}\setminus\{0\}, |\lambda|\le 1\}<\infty. \quad \text{(C.5)}$$

Finally, by combining (C.1), (C.4) and (C.5), it follows that

$$\sup\{\|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}L_{\nu,\tilde{c}}(A)\|_{\mathcal{L}(X)} \mid \lambda\in i\mathbb{R}\setminus\{0\}, |\lambda|\le 1\}<\infty.$$

**Case 1(b):** $\tilde{c}\in(2,3]$. In this case, $a\in[1,2)$; then, by Propostion 1.3.2, one gets for each $\lambda\in i\mathbb{R}\setminus\{0\}$,

$$\|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}L_{\nu,\tilde{c}}(A)\|_{\mathcal{L}(X)} \lesssim \|h_{1,2-a}(\lambda)(\lambda+A)^{-1}L_{\nu,2}(A)\|_{\mathcal{L}(X)}^{2-\tilde{c}}\|h_{1,3-a}(\lambda)(\lambda+A)^{-1}L_{\nu,3}(A)\|_{\mathcal{L}(X)}^{\tilde{c}-1},$$

and it remains to estimate $\|h_{1,3-a}(\lambda)(\lambda+A)^{-1}L_{\nu,3}(A)\|_{\mathcal{L}(X)}^{\tilde{c}-1}$. Note that for each $\lambda\in i\mathbb{R}\setminus\{0\}$, $\varepsilon>0$ and each $x\in X$, one has (here, $y=(1+A)^{-\nu}x$)

$$\begin{aligned}
h_{1,3-a}(\lambda)(\lambda+A_\varepsilon)^{-1}(2\pi-i\log(A_\varepsilon))^{-3}y &= \frac{h_{1,3-a}(\lambda)(\lambda+A_\varepsilon)^{-1}y}{(2\pi-i\log(-\lambda))^3} \\
&+ ih_{1,3-a}(\lambda)\int_0^\infty \frac{(26\pi^3-24\pi^2 i\log(t)+6\pi\log(t))(t+A_\varepsilon)^{-1}y}{(3\pi^2-4\pi i\log(t)-\log(t)^2)^3(\lambda-t)}dt,
\end{aligned}$$

and then, by taking the limit $\varepsilon\to 0^+$ on both sides of the last identity, one gets

$$\begin{aligned}
h_{1,3-a}(\lambda)(\lambda+A)^{-1}L_{\nu,3}(A)x &= \frac{h_{1,3-a}(\lambda)(\lambda+A)^{-1}y}{(2\pi-i\log(-\lambda))^3}+ \\
&+ \int_0^\infty \frac{ih_{1,3-a}(\lambda)(26\pi^3-24\pi^2 i\log(t)+6\pi\log(t)}{(3\pi^2-4\pi i\log(t)-\log(t)^2)(\lambda-t)}(t+A)^{-1}y\,dt.
\end{aligned}$$

Thus, by relation (E.5),

$$\begin{aligned}
&\left\|h_{1,3-a}(\lambda)(\lambda+A)^{-1}L_{\nu,3}(A)\right\|_{\mathcal{L}(X)} \lesssim \left\|\frac{h_{1,3-a}(\lambda)(\lambda+A)^{-1}}{(2\pi - i\log(-\lambda))^3}\right\|_{\mathcal{L}(X)} \\
&+\int_0^{e^{-\sqrt{3}\pi}} \frac{|h_{1,3-a}(\lambda)|(26\pi^3+24\pi^2|\log(t)|+6\pi\log(t)^2)\|(t+A)^{-1}\|_{\mathcal{L}(X)}}{(\pi^2+\log(t)^2)^3|\lambda-t|}dt \\
&+\int_{e^{-\sqrt{3}\pi}}^{e^{\sqrt{3}\pi}} \frac{|h_{1,3-a}(\lambda)|}{|3\pi^2-4\pi i\log(t)-\log(t)^2||\lambda-t|}\|(t+A)^{-1}\|_{\mathcal{L}(X)}dt
\end{aligned}$$

$$\begin{aligned}
&+|h_{1,3-a}(\lambda)|\int_{e^{\sqrt{3}\pi}}^{\infty} \frac{26\pi^3+24\pi^2|\log(t)|+6\pi\log(t)^2}{(\pi^2+\log(t)^2)^3|\lambda-t|}\|(t+A)^{-1}\|_{\mathcal{L}(X)}dt \\
&\lesssim \left\|\frac{h_{1,3-a}(\lambda)}{(2\pi - i\log(-\lambda))^3}(\lambda+A)^{-1}\right\|_{\mathcal{L}(X)} + \int_0^\infty \frac{|h_{1,3-a}(\lambda)|f(t)}{t(\pi^2+\log(t)^2)^3(|\lambda|+t)}dt \\
&+\int_{e^{-\sqrt{3}\pi}}^{e^{\sqrt{3}\pi}} \frac{|h_{1,3-a}(\lambda)|}{t^2|3\pi^2-4\pi i\log(t)-\log(t)^2|}dt \\
&\lesssim \left\|\frac{h_{1,3-a}(\lambda)}{(2\pi - i\log(-\lambda))^3}(\lambda+A)^{-1}\right\|_{\mathcal{L}(X)} + |h_{1,3-a}(\lambda)|\frac{(|\lambda|\log(|\lambda|)^2-2(|\lambda|\log(|\lambda|)-|\lambda|+1)}{|\lambda|\log(|\lambda|)^3} \\
&+|h_{1,3-a}(\lambda)|,
\end{aligned}$$

where for each $t > 0$,

$$f(t) = \pi^2((-2+\pi^2)t-2) + t\log(t)^4 - 4t\log(t)^3 + 2((3+\pi^2)t+3)\log(t)^2 - 4\pi^2 t\log(t).$$

By proceeding as in **Case 1(a)**, one concludes that

$$\sup\{\|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}L_{\nu,\tilde{c}}(A)\|_{\mathcal{L}(X)} \mid \lambda \in i\mathbb{R}\setminus\{0\}, |\lambda| \le 1\} < \infty.$$

**Case 1(c):** $\tilde{c} > 3$. In this case, $a \ge 2$. Let $\zeta = \zeta_1 + \zeta_2$, with $\zeta_2 \in (1,2)$. Again, by applying the Moment Inequality (see Proposition 1.3.2) over $\zeta_2$, one gets

$$\begin{aligned}
&\|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}L_{\nu,a+\zeta_1+\zeta_2}(A)\|_{\mathcal{L}(X)} \lesssim \|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}L_{\nu,a+\zeta_2}(A)\|_{\mathcal{L}(X)} \\
&\lesssim \|h_{1,1}(\lambda)(\lambda+A)^{-1}L_{\nu,1+a}(A)\|_{\mathcal{L}(X)}^{2-\zeta_2}\|h_{1,2}(\lambda)(\lambda+A)^{-1}L_{\nu,2+a}(A)\|_{\mathcal{L}(X)}^{\zeta_2-1}.
\end{aligned}$$

Let $\gamma$ be the same path as presented in **Case 1(a)**. Then, for each $\varepsilon > 0$ and each $x \in X$,

$$\begin{aligned}
&h_{1,1}(\lambda)(\lambda+A_\varepsilon)^{-1}(2\pi - i\log(A_\varepsilon))^{-(1+a)}x = \frac{h_{1,1}(\lambda)}{2\pi i}\int_\gamma \frac{1}{(2\pi - i\log(z))^{1+a}} R(z,A_\varepsilon)(\lambda+A_\varepsilon)^{-1}x\,dz \\
&\overset{\theta\to\pi}{\longrightarrow} \frac{h_{1,1}(\lambda)(\lambda+A_\varepsilon)^{-1}x}{(2\pi - i\log(-\lambda))^{1+a}} + \frac{1}{2\pi i}\int_r^R \frac{h_{1,1}(\lambda)(t+A_\varepsilon)^{-1}}{(2\pi - i\log(t))^{1+a}(\lambda - t)} x\,dt \\
&-\frac{1}{2\pi i}\int_r^R \frac{h_{1,1}(\lambda)}{(3\pi - i\log(t))^{1+a}(\lambda - t)}(t+A_\varepsilon)^{-1}x\,dt \\
&+\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{i h_{1,1}(\lambda) R e^{is}}{(2\pi - s - i\log(R))^{1+a}(\lambda + Re^{is})} R(Re^{is},A_\varepsilon)x\,ds \\
&-\frac{h_{1,1}(\lambda)}{2\pi i}\int_{-\pi}^{\pi} \frac{i r e^{is}}{(2\pi + s - i\log(r))^{1+a}(\lambda + re^{is})} R(re^{is},A_\varepsilon)x\,ds \\
&\overset{r\to0,R\to\infty}{\longrightarrow} \frac{h_{1,1}(\lambda)}{(2\pi - i\log(-\lambda))^{1+a}}(\lambda+A_\varepsilon)^{-1}x + \frac{1}{2\pi i}\int_0^\infty \frac{h_{1,1}(\lambda)(t+A_\varepsilon)^{-1}}{(\pi - i\log(t))^{1+a}(\lambda - t)}x\,dt \\
&-\frac{1}{2\pi i}\int_0^\infty \frac{h_{1,1}(\lambda)(t+A_\varepsilon)^{-1}}{(3\pi - i\log(t))^{1+a}(\lambda - t)}x\,dt.
\end{aligned}$$

Now, it follows from dominated convergence that for each $x \in X$,

$$\begin{aligned}
&h_{1,1}(\lambda)(\lambda+A)^{-1}(2\pi - i\log(A))^{-(1+a)}x = \frac{h_{1,1}(\lambda)(\lambda+A)^{-1}x}{(2\pi - i\log(-\lambda))^{1+a}} \\
&+ \quad \frac{1}{2\pi i}\int_0^\infty \frac{h_{1,1}(\lambda)}{(\pi - i\log(t))^{1+a}(\lambda - t)}(t+A)^{-1}x\,dt - \frac{1}{2\pi i}\int_0^\infty \frac{h_{1,1}(\lambda)(t+A)^{-1}}{(3\pi - i\log(t))^{1+a}(\lambda - t)}x\,dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|h_{1,1}(\lambda)(\lambda+A)^{-1}(2\pi - i\log(A))^{-(1+a)}\|_{\mathcal{L}(X)} \le \left\| \frac{h_{1,1}(\lambda)(\lambda+A)^{-1}}{(2\pi - i\log(-\lambda))^{1+a}} \right\|_{\mathcal{L}(X)} \\
&+\frac{1}{2\pi}\int_0^\infty \frac{|h_{1,1}(\lambda)|\|(t+A)^{-1}\|_{\mathcal{L}(X)}}{(\pi^2+\log(t)^2)(|\lambda|+t)}dt + \frac{1}{2\pi}\int_0^\infty \frac{|h_{1,1}(\lambda)|\|(t+A)^{-1}\|_{\mathcal{L}(X)}}{(\pi^2+\log(t)^2)(|\lambda|+t)}dt \\
&= \left\| \frac{h_{1,1}(\lambda)(\lambda+A)^{-1}}{(2\pi - i\log(-\lambda))^{1+a}} \right\|_{\mathcal{L}(X)} + 2\frac{|h_{1,1}(\lambda)|(|\lambda|-1)}{|\lambda|\log(|\lambda|)}.
\end{aligned}$$

Now, by the same reasoning as before, one gets

$$\begin{aligned}
&h_{1,2}(\lambda)(\lambda+A)^{-1}(2\pi - i\log(A))^{-(2+a)} = \frac{h_{1,2}(\lambda)(\lambda+A)^{-1}}{(2\pi - i\log(-\lambda))^{2+a}} \\
&+\frac{1}{2\pi i}\int_0^\infty \frac{h_{1,2}(\lambda)}{(\pi - i\log(t))^{2+a}(\lambda - t)}(t+A)^{-1}dt - \frac{1}{2\pi i}\int_0^\infty \frac{h_{1,2}(\lambda)(t+A)^{-1}}{(3\pi - i\log(t))^{2+a}(\lambda - t)}dt,
\end{aligned}$$

so

$$\begin{aligned}
\|h_{1,2}(\lambda)(\lambda+A)^{-1}L_{\nu,2+a}(A)\|_{\mathcal{L}(X)} \quad &\lesssim \quad \left\| \frac{h_{1,2}(\lambda)(\lambda+A)^{-1}}{(2\pi - i\log(-\lambda))^{2+a}} \right\|_{\mathcal{L}(X)} + \\
&+ \quad \frac{|h_{1,2}(\lambda)|}{\pi}\int_0^\infty \frac{(\pi^2 - 2(1+1/t)\log(t)+\log(t)^2)}{(\pi^2+\log(t)^2)^2(|\lambda|+t)}dt \\
&= \quad \left\| \frac{h_{1,2}(\lambda)(\lambda+A)^{-1}}{(2\pi - i\log(-\lambda))^{2+a}} \right\|_{\mathcal{L}(X)} + \frac{|h_{1,2}(\lambda)|(|\lambda|\log(|\lambda|) - |\lambda| + 1)}{\pi|\lambda|\log(|\lambda|)^2}.
\end{aligned}$$

Again, by proceeding as in **Case 1(a)**, one concludes that

$$\sup\{\|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}L_{\nu,a+\zeta_1+\zeta_2}(A)\|_{\mathcal{L}(X)} \mid \lambda\in i\mathbb{R}\setminus\{0\}, |\lambda|\leq 1\}<\infty.$$

• **Case 2:** $\alpha \geq 2$**.** By using the functional calculus for $H_0^\infty$ functions (see Remark 1.3.2), one gets for each $x\in X$,

$$\begin{aligned} h_{1,\zeta}(\lambda)(\lambda+A)^{-1}A^{\alpha-1}(1+A)^{-(\alpha-1)}(2\pi-i\log(A))^{-\tilde{c}}x &= \frac{h_{1,\zeta}(\lambda)}{2\pi i}\int_\Gamma \frac{z^{\alpha-1}(\lambda+A)^{-1}}{(1+z)^{\alpha-1}h_{0,\tilde{c}}(z)}R(z,A)x\,dz \\ &= \frac{h_{1,\zeta}(\lambda)(-\lambda)^{\alpha-1}}{(1-\lambda)^{\alpha-1}(2\pi-i\log(-\lambda))^{\tilde{c}}}(\lambda+A)^{-1}x \\ &+ h_{1,\zeta}(\lambda)S''_\lambda x, \end{aligned}$$

where

$$S''_\lambda := \frac{1}{2\pi i}\int_\Gamma \frac{z^{\alpha-1}}{(1+z)^{\alpha-1}h_{0,\tilde{c}}(z)(z+\lambda)}R(z,A)\,dz.$$

The function $z\mapsto(2\pi-i\log(z))^{-\tilde{c}}R(z,A)$ is integrable on $\Gamma$ and by Lemma 5.9 in [55], for $z\in\Gamma$ and $|\lambda|\leq 1$, one has

$$\left|\frac{z^{\alpha-1}h_{1,\zeta}(\lambda)}{(1+z)^{\alpha-1}(z+\lambda)}\right| \leq \frac{C}{|1-\lambda|}\leq C;$$

hence, $\sup\{\|h_{1,\zeta}(\lambda)S''_\lambda\|_{\mathcal{L}(X)} \mid \lambda \in i\mathbb{R}\setminus\{0\}, |\lambda|\leq 1\}<\infty$, and since $\left\|\frac{h_{1,\zeta}(\lambda)(-\lambda)^{\alpha-1}(\lambda+A)^{-1}}{(1-\lambda)^{\alpha-1}(2\pi-i\log(-\lambda))^{\tilde{c}}}\right\|_{\mathcal{L}(X)}$ is also bounded (by hypothesis), then

$$\sup\{\|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}A^{\alpha-1}(1+A)^{-(\alpha-1)}(2\pi-i\log(A))^{-\tilde{c}}\|_{\mathcal{L}(X)} \mid \lambda\in i\mathbb{R}\setminus\{0\}, |\lambda|\leq 1\}<\infty.$$

• **Case 3:** $\alpha\in(1,2)$**.** By Proposition 1.3.2 (applied over $\alpha-1\in(0,1)$), one gets

$$\begin{aligned} &\|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}(A(1+A)^{-1})^{\alpha-1}L_{\nu,\tilde{c}}(A)\|_{\mathcal{L}(X)} \\ &\lesssim \|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}L_{\nu,\tilde{c}}(A)\|^{2-\alpha}_{\mathcal{L}(X)}\|h_{1,\zeta}(\lambda)(\lambda+A)^{-1}A(1+A)^{-1}L_{\nu,\tilde{c}}(A)\|^{\alpha-1}_{\mathcal{L}(X)}. \end{aligned}$$

The first factor is treated as in **Case 1**, and the second factor is treated as in **Case 2**.

**Item (b)**

• **Case 1:** $\alpha=1$. Let $\zeta>1$ and set $\tilde{c}:=\zeta+a>1$.

Given that the operator $(\log(2+A))^{\tilde{c}}(2\pi-i\log(A))^{-\tilde{c}}$ is closed, it follows from the Closed Graph Theorem that it is bounded; hence,

$$\|(\lambda+A)^{-1}(1+A)^{-1}(2\pi-i\log(A))^{-\tilde{c}}\|_{\mathcal{L}(X)} \lesssim \|(\lambda+A)^{-1}(1+A)^{-1}\log(A+2)^{-\tilde{c}}\|_{\mathcal{L}(X)}.$$

Now, by Proposition 2.2.2, one gets

$$\sup\left\{\frac{|\lambda|}{(1+|\lambda|)^{1-\beta_0}}|(2\pi-\log(\lambda))|^{\zeta}\|(\lambda+A)^{-1}(1+A)^{-1}\log(A+2)^{-\tilde{c}}\|_{\mathcal{L}(X)} \mid \lambda\in i\mathbb{R}, |\lambda|\geq 1\right\}<\infty.$$

• **Case 2:** $\alpha \geq 2$**.** Let $g_{\alpha,\zeta}(\lambda)=\dfrac{\lambda^{\alpha}}{(1-\lambda)^{1-\beta_0}}(2\pi-i\log(\lambda))^{\zeta}$, with $\lambda\in i\mathbb{R}\setminus\{0\}$; then, by the functional calculus for $H_0^{\infty}$ functions (see Remark 1.3.2), for each $x\in X$, one has

$$\begin{aligned}
&g_{1,\zeta}(\lambda)(\lambda+A)^{-1}A^{\alpha-1}(1+A)^{-(\alpha+\beta+\beta_0-1)}(2\pi-i\log(A))^{-\tilde{c}}x\\
&=\frac{g_{1,\zeta}(\lambda)}{2\pi i}\int_{\Gamma}\frac{z^{\alpha-1}(\lambda+A)^{-1}}{(1+z)^{\alpha+\beta+\beta_0-1}(2\pi-i\log(z))^{\tilde{c}}}R(z,A)xdz\\
&=\frac{g_{1,\zeta}(\lambda)(-\lambda)^{\alpha-1}}{(1-\lambda)^{\alpha+\beta+\beta_0-1}(2\pi-i\log(-\lambda))^{\tilde{c}}}(\lambda+A)^{-1}x+g_{1,\zeta}(\lambda)T_{\lambda}^{''}x,
\end{aligned}$$

where

$$T_{\lambda}^{''}:=\frac{1}{2\pi i}\int_{\Gamma}\frac{z^{\alpha-1}}{(1+z)^{\alpha+\beta+\beta_0-1}(2\pi-i\log(z))^{\tilde{c}}(z+\lambda)}R(z,A)dz,$$

with $\Gamma$ the path defined in the proof of Proposition 2.2.2. The function $z\mapsto(2\pi-i\log(z))^{-\tilde{c}}R(z,A)$ is integrable on $\Gamma$ and by Lemma 5.9 in [55], for $z\in\Gamma$ and $|\lambda|\geq 1$, one has

$$\left|\frac{z^{\alpha-1}g_{1,\zeta}(\lambda)}{(1+z)^{\alpha+\beta+\beta_0-1}(z+\lambda)}\right|\lesssim\frac{|g_{1,\zeta}(\lambda)|}{|1-\lambda|}\leq C;$$

thus, $\sup\{\|g_{1,\zeta}(\lambda)T_{\lambda}^{''}\|_{\mathcal{L}(X)} \mid \lambda\in i\mathbb{R},\ |\lambda|\geq 1\}<\infty$, and since

$$\sup\left\{\left\|\frac{g_{1,\zeta}(\lambda)(-\lambda)^{\alpha-1}}{(1-\lambda)^{\alpha+\beta+\beta_0-1}(2\pi-i\log(-\lambda))^{\tilde{c}}}(\lambda+A)^{-1}\right\|_{\mathcal{L}(X)} \mid \lambda\in i\mathbb{R}, |\lambda|\geq 1\right\}<\infty,$$

by hypothesis, it follows that

$$\sup\left\{|g_{\alpha,\zeta}(\lambda)|\|(\lambda+A)^{-1}A^{\alpha-1}(1+A)^{-\beta-\beta_0-\alpha+1}(2\pi-i\log(A))^{-\tilde{c}}\|_{\mathcal{L}(X)} \mid \lambda\in i\mathbb{R}, |\lambda|\geq 1\right\}<\infty.$$

• **Case 3:** $\alpha\in(1,2)$**.** It follows from Propositon 1.3.2 (applied to $\alpha-1\in(0,1)$) that

$$\begin{aligned}
&\|g_{1,\tilde{c}}(\lambda)(\lambda+A)^{-1}(A(1+A)^{-1})^{\alpha-1}L_{\beta+\beta_0,\tilde{c}}(A)\|_{\mathcal{L}(X)}\\
&\lesssim\|g_{1,\tilde{c}}(\lambda)(\lambda+A)^{-1}L_{\beta+\beta_0,\tilde{c}}(A)\|_{\mathcal{L}(X)}^{2-\alpha}\|g_{1,\tilde{c}}(\lambda)(\lambda+A)^{-1}A(1+A)^{-1}L_{\beta+\beta_0,\tilde{c}}(A)\|_{\mathcal{L}(X)}^{\alpha-1}.
\end{aligned}$$

The first factor must be treated as in **Case 1**, and the second one as in **Case 2**.

# Appendix D

# Estimates

**Lemma D.0.1.** Let $\mu, \zeta \geq 0$ and $\nu \geq 1$; then, for each $t \geq 0$,

1. $$\int_{i\infty}^{-i\infty} e^{-\lambda t} \frac{1}{(1+\lambda)^{\nu}(\log(2+\lambda))^{\zeta}} d\lambda = 0.$$

2. $$\int_{i\infty}^{-i\infty} e^{-\lambda t} \frac{\lambda^{\mu}}{(1+\lambda)^{\nu+\mu}(2\pi - i\log(\lambda))^{\zeta}} d\lambda = 0.$$

*Proof.* We just present the proof of the first equality, since the proof of the other one is analogous. Let us first show the following statement.

**Claim:**

$$\frac{1}{2\pi i}\int_{i\infty}^{-i\infty} e^{-\lambda t} \frac{1}{(1+\lambda)^{\nu}(\log(2+\lambda))^{\zeta}} d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{\varphi}} e^{-\lambda t} \frac{1}{(1+\lambda)^{\nu}(\log(2+\lambda))^{\zeta}} d\lambda, \tag{D.1}$$

where $\Gamma_{\varphi} = \{re^{i\varphi} \mid r \in [0,\infty)\} \cup \{re^{-i\varphi} \mid r \in [0,\infty)\}$ and $0 < \varphi < \frac{\pi}{2}$.

Namely, for $t \geq 0$, set $i\mathbb{R} \ni \lambda \mapsto h_t(\lambda) := e^{-\lambda t}\dfrac{1}{(1+\lambda)^{\nu}(\log(2+\lambda))^{\zeta}}$, and for each $R, r > 0$ and each $\eta \in [\varphi, \pi/2]$, set $\Gamma^{+}_{R,\varphi} = \{Re^{i\theta} \mid \theta \in (\varphi, \frac{\pi}{2})\}$, $\Gamma^{+}_{r,\varphi} = \{re^{i\theta} \mid \theta \in (\varphi, \frac{\pi}{2})\}$, $\Gamma^{-}_{R,\varphi} = \{Re^{-i\theta} \mid \theta \in (\varphi, \frac{\pi}{2})\}$, $\Gamma^{-}_{r,\varphi} = \{re^{-i\theta} \mid \theta \in (\varphi, \frac{\pi}{2})\}$, $\gamma^{+}_{\eta} = \{se^{i\eta} \mid s \in [r,R]\}$ and $\gamma^{-}_{\eta} = \{se^{-i\eta} \mid s \in [r,R]\}$. By Cauchy's Integral Theorem,

$$-\int_{\Gamma^{+}_{R,\varphi}} h_t(\lambda)d\lambda + \int_{\gamma^{+}_{\frac{\pi}{2}}} h_t(\lambda)d\lambda + \int_{\Gamma^{+}_{r,\varphi}} h_t(\lambda)d\lambda - \int_{\gamma^{+}_{\varphi}} h_t(\lambda)d\lambda = 0, \tag{D.2}$$

and

$$\int_{\Gamma^{-}_{R,\varphi}} h_t(\lambda)d\lambda - \int_{\gamma^{-}_{\frac{\pi}{2}}} h_t(\lambda)d\lambda - \int_{\Gamma^{-}_{r,\varphi}} h_t(\lambda)d\lambda + \int_{\gamma^{-}_{\varphi}} h_t(\lambda)d\lambda = 0. \tag{D.3}$$

Note that, by Lemma 5.2.2 in [24],

$$\begin{aligned}\left|\int_{\Gamma^{\pm}_{R,\varphi}} h_t(\lambda)d\lambda\right| &\leq \int_{\varphi}^{\frac{\pi}{2}} \frac{Re^{-t\cos\theta}}{|(1+Re^{\pm i\theta})|^{\nu}|\log(2+Re^{\pm i\theta})|^{\zeta}}d\theta \\ &\leq 2^{\nu/2}\int_{\varphi}^{\frac{\pi}{2}} \frac{Re^{-t\cos\theta}}{(1+R)^{\nu}(1+\cos(\theta))^{\nu/2}\left(\log(2+R)+\frac{1}{2}\log\left(\frac{1+\cos(\theta)}{2}\right)\right)^{\zeta}}d\theta \\ &\lesssim \frac{R^{1-\nu}}{\log(2+R)^{\zeta}}\end{aligned}$$

and

$$\left|\int_{\Gamma^{\pm}_{r,\varphi}} h_t(\lambda)d\lambda\right| \lesssim r$$

By adding the equations (D.2) and (D.3), and by taking the limits $R\to\infty$, $r\to 0$, one gets (D.1).

By Claim, it suffices to prove that

$$\frac{1}{2\pi i}\int_{\Gamma_{\varphi}} e^{-\lambda t}\frac{1}{(1+\lambda)^{\nu}\log(2+\lambda)^{\zeta}}d\lambda = 0.$$

It follows from Cauchy's Integral Theorem that for each $0<r<R$,

$$\frac{1}{2\pi i}\int_{\Gamma_{\varphi}} h_t(\lambda)d\lambda + \frac{1}{2\pi i}\int_{\gamma_{R,\varphi}} h_t(\lambda)d\lambda + \frac{1}{2\pi i}\int_{\gamma_{r,\varphi}} h_t(\lambda)d\lambda = 0, \tag{D.4}$$

with $\gamma_{R,\varphi} := \{Re^{i\theta} \mid \theta\in[-\varphi,\varphi]\}$ and $\gamma_{r,\varphi} := \{re^{-i\theta} \mid \theta\in[-\varphi,\varphi]\}$.

Note that for each sufficiently large $R$,

$$\left|\int_{\gamma_{R,\varphi}} e^{-\lambda t}\frac{1}{(1+\lambda)^{\nu}\log(2+\lambda)^{\zeta}}d\lambda\right| \lesssim \frac{R^{1-\nu}}{\log(2+R)^{\zeta}},$$

and for each sufficiently small $r$,

$$\left|\int_{\gamma_{r,\varphi}} e^{-\lambda t}\frac{1}{(1+\lambda)^{\nu}\log(2+\lambda)^{\zeta}}d\lambda\right| \lesssim r.$$

The result follows by taking the limits $r\to 0$ and $R\to\infty$ in relation (D.4). □

**Lemma D.0.2.** Let $\varphi\in(0,\frac{\pi}{2}]$ and $\theta\in(\pi-\varphi,\pi)$. Set $\Omega := \overline{\mathbb{C}_+}\setminus(S_{\varphi}\cup\{0\})$ and let $\Gamma := \{re^{i\theta} \mid r\in[0,\infty)\}\cup\{re^{i\theta} \mid r\in[0,\infty)\}$ be oriented from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$. Then, for each $\alpha\in[0,\infty)$, $\beta\in(0,\infty)$, $\eta\in(0,1]$ and each $\lambda\in\Omega$, one has

a) $\displaystyle\int_{\Gamma}\frac{1}{(\eta+z)^{\beta}(\log(1+\eta+z))^{\zeta}(z+\lambda+\eta-1)}dz = \frac{1}{(1-\lambda)^{\beta}(\log(2-\lambda))^{\zeta}}.$

b) $\displaystyle\int_{\Gamma}\frac{z^{\alpha}}{(\eta+z)^{\alpha+\beta}(2\pi-i\log(-1+\eta+z))^{\zeta}(z+\lambda+\eta-1)}dz = \frac{(1-\lambda-\eta)^{\alpha}}{(1-\lambda)(2\pi-i\log(-\lambda))^{\zeta}}.$

*Proof.* We just present the proof of item a). Let $\lambda \in \Omega$. For each $r \in (0, \mathrm{Im}(\lambda)/2]$ and each $R \geq 2|\lambda| + 2$, set $\gamma_+ := \{se^{i\theta} \mid s \in [r,R]\}$, $\gamma_- := \{se^{-i\theta} \mid s \in [r,R]\}$, $\gamma_r := \{re^{i\nu} \mid \nu \in [-\theta,\theta]\}$, $\gamma_R := \{Re^{i\nu} \mid \nu \in [-\theta,\theta]\}$ and $\gamma_{r,R} := (-\gamma_+) \cup \gamma_- \cup (-\gamma_r) \cup \gamma_R$. Let $f_{\beta,\zeta,\lambda} : \overline{\mathbb{C}_+} \to \mathbb{C}$ be given by the law $f_{\beta,\zeta,\lambda}(z) = \dfrac{1}{(\eta+z)^\beta(\log(1+\eta+z))^\zeta(z+\lambda+\eta-1)}$; then,

$$\begin{aligned}\left|\int_{\gamma_R} f_{\beta,\zeta,\lambda}(z)dz\right| &\leq \int_{-\theta}^{\theta} \frac{R}{|\eta+Re^{i\nu}|^\beta \log(|1+\eta+Re^{i\nu}|)^\zeta |Re^{i\nu}+\lambda+\eta-1|}d\nu \\ &\lesssim \frac{R^{-\beta}}{\log(1+R)^\zeta},\end{aligned}$$

which goes to zero as $R \to \infty$. Similarly, one can show that

$$\lim_{r\to 0}\left|\int_{\gamma_r} f_{\beta,\zeta,\lambda}(z)dz\right| = 0.$$

On the other hand, by the Residue Theorem, one has

$$\int_{\gamma_{r,R}} \frac{1}{(\eta+z)^\beta(\log(1+\eta+z))^\zeta(z+\lambda+\eta-1)}dz = \frac{1}{(1-\lambda)^\beta \log(2-\lambda)^\zeta}.$$

Thus, it follows that

$$\begin{aligned}&\int_\Gamma \frac{1}{(\eta+z)^\beta(\log(1+\eta+z))^\zeta(z+\lambda+\eta-1)}dz \\ &= \lim_{r\to 0, R\to\infty} \int_{\gamma_{r,R}} \frac{1}{(\eta+z)^\beta(\log(1+\eta+z))^\zeta(z+\lambda+\eta-1)}dz = \frac{1}{(1-\lambda)^\beta \log(2-\lambda)^\zeta}.\end{aligned}$$

□

# Appendix E

# Some important classes of functions

## E.1 Complete Bernstein functions

In this section, we recall the definitions and some properties of some special functions that appear throughout the text. We refer to [60] for details (see also [14]).

**Definition E.1.1** (Definition 1.3 in [60]). A function $f \in C^\infty(0,\infty)$ is called **completely monotone** if

$$(-1)^n f^{(n)}(\lambda) \geq 0 \text{ for each } n \in \mathbb{N} \cup \{0\} \text{ and each } \lambda > 0.$$

By Theorem 1.4 in [60], which is known as Bernstein's Theorem, every completely monotone function $f$ is the Laplace transform of a positive Radon measure on $\mathbb{R}_+$. Recall that $f \in C^\infty(0,\infty)$ is called a **Bernstein function** if

$$f \geq 0 \text{ and } f' \text{ is completely monotone.}$$

It is easy to see from this definition that the fractional powers $\lambda \mapsto \lambda^\alpha$, with $0 \leq \alpha \leq 1$, and $\lambda \mapsto \log(1+\lambda)$, are Bernstein functions. By Lévy-Khintchine Representation Theorem (see Theorem 3.2 in [60]), a function $f$ is a Bernstein function if, and only if, there exist constants $a, b \geq 0$ and a positive Radon measure $\mu_{LK}$ (this notation is used in [14]) defined over the Borel subsets of $(0,\infty)$ such that for each $\lambda > 0$,

$$f(\lambda) = a + b\lambda + \int_{0^+}^{\infty} (1 - e^{-\lambda s}) d\mu_{LK}(s),$$

with

$$\int_{0^+}^{\infty} \frac{s}{s+1} d\mu_{LK}(s) < \infty.$$

The triple $(a, b, \mu_{LK})$ determines $f$ uniquely and vice versa (see Theorem 3.2 in [60]), and it is called the Lévy-Khintchine triple of $f$. Every Bernstein function can also be extended to a holomorphic function in $\mathbb{C}_+$ (this is Proposition 3.6 in [60]).

Now we consider a subclass of the Bernstein functions, the so-called complete Bernstein functions.

**Definition E.1.2** (Definition 6.1 in [60]). A function $f \in C^{\infty}(0,\infty)$ is called a **complete Bernstein function** if it is a Bernstein function and the measure $\mu_{LK}$ in the Lévy-Khintchine triple has a completely monotone density with respect to Lebesgue measure. The set of all complete Bernstein functions is denoted by $\mathcal{CBF}$.

By Theorem 6.2-(vi) in [60], every $f \in \mathcal{CBF}$ admits a representation of the form

$$f(\lambda) = a + b\lambda + \int_{0+}^{\infty} \frac{\lambda}{\lambda + s} d\mu(s), \quad \lambda > 0, \tag{E.1}$$

with $a, b \geq 0$ constants and $\mu$ a positive Radon measure defined over the Borel subsets of $(0,\infty)$ that satisfies

$$\int_{0+}^{\infty} \frac{1}{s+1} d\mu(s) < \infty.$$

As discussed in Remark 2.1 in [14], complete Bernstein functions admit other representations than the one given by (E.1). In particular, one has

$$f(\lambda) = a + \int_{0}^{\infty} \frac{\lambda}{1+\lambda t} d\nu(t) = a + \nu(\{0\})\lambda + \int_{0+}^{\infty} \frac{\lambda}{1+\lambda t} d\nu(t),$$

where $\nu$ is a positive Radon measure defined over the Borel subsets of $(0,\infty)$ that satisfies

$$\int_{0}^{\infty} \frac{1}{1+t} d\nu(t) < \infty,$$

and the pair $(a,\nu)$ is unique.

The representation formula (E.1) is unique (that is, the triple $(a,b,\mu)$ is unique), and it is called the Stieltjes representation for $f$ (see Chapter 6 in [60] for details). Note that

$$a = \lim_{\lambda \to 0^+} f(\lambda) \quad \text{and} \quad b = \lim_{\lambda \to 0^+} \frac{f(\lambda)}{\lambda}.$$

**Example E.1.1.** (a) The function $f : (0,\infty) \to \mathbb{R}$ given by $f(\lambda) = \lambda^{\alpha}$, with $\alpha \in [0,1]$, is a complete Bernstein function whose Stieltjes representation is given by

$$f(\lambda) = \frac{\sin(\alpha\pi)}{\pi} \int_{0+}^{\infty} s^{\alpha} \frac{\lambda}{s+\lambda} \frac{ds}{s}, \qquad \lambda > 0.$$

(b) The function $\lambda \mapsto (1+\lambda)^{\alpha} - 1$, with $\alpha \in (0,1)$, is a complete Bernstein function whose Stieltjes representation is given by

$$(1+\lambda)^{\alpha} - 1 = \frac{\sin(\alpha\pi)}{\pi} \int_{0+}^{\infty} (s-1)^{\alpha} \chi_{(1,\infty)} \frac{\lambda}{s+\lambda} \frac{ds}{s}, \qquad \lambda > 0.$$

(c) The function $\lambda \mapsto \log(1+\lambda)$ is a complete Bernstein function whose Stieltjes representation is given by

$$\log(1+\lambda) = \int_{0+}^{\infty} \chi_{(1,\infty)}(s) \frac{\lambda}{\lambda+s} \frac{ds}{s}, \qquad \lambda > 0. \tag{E.2}$$

(d) The function $\lambda \mapsto \dfrac{\lambda-1}{\log(\lambda)}$ is a complete Bernstein function whose Stieltjes representation is given by

$$\frac{\lambda-1}{\log(\lambda)} = \int_{0+}^{\infty} \frac{s+1}{s(\pi^2+\log(s)^2)} \frac{\lambda}{\lambda+s} ds, \qquad \lambda > 0. \tag{E.3}$$

(e) The function $\lambda \mapsto \dfrac{\lambda\log(\lambda)-\lambda+1}{\log(\lambda)^2}$ is a complete Bernstein function whose Stieltjes representation is given by

$$\frac{\lambda\log(\lambda)-\lambda+1}{\log(\lambda)^2} = \int_{0+}^{\infty} \frac{\pi^2-2(1+1/s)\log(s)+\log(s)^2}{(\pi^2+\log(s)^2)^2} \frac{\lambda}{\lambda+s} ds, \qquad \lambda > 0. \tag{E.4}$$

(f) The function $\lambda \mapsto \dfrac{(-2+2\lambda-2\lambda\log(\lambda)+\lambda\log(\lambda)^2)}{\log(\lambda)^3}$ is a complete Bernstein function whose Stieltjes representation is given by

$$\frac{(-2+2\lambda-2\lambda\log(\lambda)+\lambda\log(\lambda)^2)}{\log(\lambda)^3} = \int_{0+}^{\infty} \frac{f(s)}{s(\pi^2+\log(s)^2)^3} \frac{\lambda}{(\lambda+s)} ds, \qquad \lambda > 0, \tag{E.5}$$

where $f(s) = \pi^2((-2+\pi^2)s-2)+s\log(s)^4-4s\log(s)^3+2((3+\pi^2)s+3)\log(s)^2-4\pi^2 s\log(s)$.

The next results play an important role in the proofs of Theorems 10 and 11; items (a) and (b) are Theorem 2.2 in [14], and item (c) is Proposition 7.13 in [60].

**Proposition E.1.1.** Let $f, g : (0,\infty) \to \mathbb{R}$ be non-zero functions.

(a) If $f \in \mathcal{CBF}$, then $\dfrac{\lambda}{f(\lambda)}, \lambda f\left(\dfrac{1}{\lambda}\right) \in \mathcal{CBF}$. Conversely, if $\dfrac{\lambda}{f(\lambda)} \in \mathcal{CBF}$ or $\lambda f\left(\dfrac{1}{\lambda}\right) \in \mathcal{CBF}$, then $f \in \mathcal{CBF}$.

(b) If $f, g \in \mathcal{CBF}$, then $g \circ f \in \mathcal{CBF}$.

(c) Let $a_1, a_2 \in (0,1)$ be such that $a_1 + a_2 \leq 1$. Then, for each $f, g \in \mathcal{CBF}$, one has $f^{a_1} \cdot g^{a_2} \in \mathcal{CBF}$.

## E.2 Slowly varying functions

**Definition E.2.1.** Let $a \in \mathbb{R}$ and let $\ell : [a,\infty) \to \mathbb{R}$ be a strictly positive measurable function such that for each $\lambda > 0$,

$$\lim_{s\to\infty} \frac{\ell(\lambda s)}{\ell(s)} = 1.$$

Then, $\ell$ is said to be **slowly varying**.

**Example E.2.1.** (a) The function $s \mapsto \log(1+s)$ is a slowly varying function.

(b) If $\ell$ is a slowly varying function, then the following ones are also slowly varying functions: $s \mapsto \ell(s)^\alpha$, with $\alpha \in \mathbb{R}$; $s \mapsto \ell(s)\log(s)$.

The next result also plays an important role in the proof of Theorems 10, 11 and 12.

**Proposition E.2.1** (Corollary 2.8-(a) in [14])**.** Let $\ell$ be a slowly varying function and let $\gamma > 0$. Then, there are positive constants $C, c$ such that for each sufficiently large $s, t$ with $t \geq s$,

$$c\left(\frac{s}{t}\right)^\gamma \leq \frac{\ell(t)}{\ell(s)} \leq C\left(\frac{t}{s}\right)^\gamma.$$