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SOLAR AXIONS: theory, production and detection

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Resumo

Esta dissertação de mestrado examina os áxions solares, focando-se na sua base teórica, mecanismos de produção e métodos de detecção. Inicialmente propostos como solução para o problema CP das interações fortes e possíveis candidatos à matéria escura, os áxions potencialmente desempenham um papel fundamental na física de partículas e astrofísica. A exploração teórica esclarece a fundação da cromodinâmica quântica dos áxions e suas características únicas. Passando para a produção, a tese investiga vários mecanismos que contribuem para a geração de áxions solares. A seção subsequente aprofunda os métodos experimentais para a detecção de áxions, examinando os desafios e avanços neste campo. Este trabalho fornece uma síntese atualizada dos fundamentos teóricos, produção e estratégias de detecção dos áxions solares.

Palavras-chave: Áxions, Áxions Solares, Helioscope, Efeito Primakoff, problema CP forte, física de partículas.

Abstract

This master thesis examines solar axions, focusing on their theoretical framework, production mechanisms, and detection methods. Initially proposed as solutions to the strong CP problem and potential candidates for dark matter, axions potentially play a key role in particle physics and astrophysics. The theoretical exploration elucidates the quantum chromodynamics foundation of axions and their unique characteristics. Shifting to production, the thesis investigates various mechanisms contributing to solar axion generation, such as the ABC processes and the Primakoff conversion. The subsequent section delves into experimental methods for axion detection, such as axion helioscopes and underground detectors, examining the challenges and advancements in this field. This work provides an up-to-date synthesis of solar axions' theoretical foundations, production, and detection strategies.

Keywords: Axions, Solar Axions, Helioscope, Primakoff Effect, Strong CP probblem, Particle Physics.

Contents

1 Introduction

The Standard Model (SM) of particle physics is a triumph of human intellect, providing a comprehensive framework to understand the fundamental building units of our universe and the forces that govern their interactions. The SM is very succesfull at explaining and describing numerous phenomens of the nature, which include:

- **Electrowek Unification:** Sheldon Glashow [1], Abdus Salam [2], and Steven Weinberg [3] independently proposed the unification of electromagnetic and weak nuclear forces into a single theoretical framework. They were awarded the 1979 Physics Nobel Prize.
- **Prediction and Discovery of the Higgs Boson:** The Higgs mechanism, proposed by Peter Higgs [4, 5] and others in the 1960s, was a crucial element in the SM. The Large Hadron Collider (LHC) at CERN [6] successfully discovered the Higgs boson in 2012, providing experimental evidence for the mechanism responsible for giving mass to fundamental particles.
- **Accurate Prediction of Particle Properties:** The SM accurately predicts various properties of particles to an extraordinary level of precision. Experimental measurements have consistently validated these predictions, confirming the model's accuracy.
- **Quantum Chromodynamics (QCD):** The theory of QCD, which is part of the SM, successfully describes the strong nuclear force that binds quarks into protons, neutrons, and other hadrons. QCD has been extensively tested through experiments and plays a crucial role in our understanding of the behavior of quarks and gluons.
- **Precision Tests of Quantum Electrodynamics (QED):** The SM incorporates QED, the theory of electromagnetic interactions. Experiments testing QED predictions, such as the anomalous magnetic dipole moment of the electron, have confirmed the accuracy of this aspect of the model.
- **Consistency with Cosmic Microwave Background (CMB):** The SM's predictions for the early universe, including the behavior of particles in the primordial plasma, align with observations of the Cosmic Microwave Background, providing further support for its validity.

Yet, as with any intellectual endeavor, the Standard Model is not without its enigmas and unresolved mysteries. Such puzzles include:

- **Nature of Dark Matter:** The SM does not account for dark matter (DM), which makes up approximately 27% of the universe's mass-energy content. The existence and properties of dark matter remain elusive, prompting searches for new particles and interactions beyond those described by the SM.

- **Neutrino Masses and Mixing:** The Standard Model assumes massless neutrinos, contrary to experimental evidence indicating that neutrinos have non-zero masses and undergo flavor oscillations. The origin of neutrino masses and the nature of neutrino mixing are a very strong research field.

- **Baryon Asymmetry of the Universe:** The observed asymmetry between matter and antimatter (baryon asymmetry) in the universe is not adequately explained by the SM. Understanding the mechanisms responsible for generating this asymmetry, potentially involving processes like baryogenesis, remains an open question.

- **Quantum Gravity:** The SM does not incorporate gravity, as described by general relativity. Developing a consistent quantum theory of gravity and reconciling it with the principles of quantum mechanics is a major challenge, pointing toward the need for a more unified theory of fundamental forces.

- **Cosmic Inflation:** The SM does not provide a mechanism for cosmic inflation, the rapid expansion of the universe in its early stages. Explaining the origins and dynamics of inflation requires extending our understanding beyond the Standard Model (BSM).

- **The Hierarchy Problem:** The hierarchy problem in particle physics is the significant contrast in energy scales between the gravitational and the weak nuclear forces. The mass of the Higgs boson is affected by quantum corrections that involve very high energy scales, such corrections could make its mass much larger than observed. This large discrepancy, spanning 16 orders of magnitude, requires fine-tuning in the SM.

- **Flavor Problem:** The origin and pattern of fermion masses and mixing angles (flavor structure) in the SM are not fully understood. Explaining the hierarchical structure of quark and lepton masses and the observed mixing patterns is an ongoing challenge.

- **Strong CP Problem:** While the strong CP problem is intimately related to the SM, it remains unsolved within its framework. The problem involves explaining why the combination of charge conjugation and parity (CP) symmetry violation in QCD is extremely small.

This master thesis embarks on a compelling exploration into the heart of a persistent puzzle that has thus far resisted a satisfactory resolution, yet promises intriguing consequences — the Strong CP problem.

Within this work, we jump on a journey through the axion theory, tracing its origins to the problems that precipitated its proposal. Our focus centers on solar axions — those generated by the Sun — their production, and the mechanisms for their detection. An intriguing historical note is that the term *axion* was given to this hypothetical particle by Frank Wilczek, inspired by a famous detergent, signifying its virtue in cleansing the Standard Model of its strong CP stain [7].

The initial chapters meticulously delineate the problems that led to the axion proposal. Beginning with the U(1) problem and its resolution, we navigate through the consequences of symmetry breaking, addressing the nature of the $\mathrm{U}(1)_A$ symmetry in QCD. Subsequently, we confront the Strong CP problem, emphasizing its most favorable resolution — the Peccei-Quinn (PQ) mechanism. This leads to the introduction of the axion, and we delve into its properties and dynamics. The subsequent exploration extends to prominent axion models, ranging from the initially proposed PQWW model to the notable KSVZ and DFSZ models. Our primary interest lies in comprehending and calculating axion couplings to SM matter, especially to photons, electrons, and nucleons. With these tools in hand, we elucidate axion production mechanisms within the solar interior, detailing the calculation of the total solar axion flux. Finally, we explore contemporary detection mechanisms, with a special focus on axion helioscopes — an exceptionally promising avenue for solar axion detection.

As following chapters reveals, the axion is very weakly interacting. As a consequence, their production mechanisms in the early Universe are non-thermal. Consequently, these axions are produced with extremely small velocity dispersion and are very cold dark matter (DM), which align seamlessly with the needs of ΛCDM model. However, at the same time, this leads to primordial axions having a really small energy threshold. Notably, solar axions, peaking in the few keV range at production, emerge as an accessible probe of the theory, prompting our exclusive focus on this intriguing subset.

To maintain clarity, this work confines the term axion to refer specifically to the QCD axion. In instances involving other axion-like particles (ALPs), the term ALPs will be employed, ensuring a clear distinction. ALPs encompass any pseudo-Nambu-Goldstone (pNG) boson with low mass and very weak couplings originating from spontaneous symmetry breaking at very high energy scales. The one ALP that solves the strong CP (charge-parity) problem by the PQ mechanism is the QCD axion. Unlike axions, ALPs do not derive their masses from QCD effects and, in general, lack a connection to the PQ mechanism. For QCD axions, the relation $m_a f_a \sim m_\pi f_\pi$ holds, whereas this quantity

$m_a f_a$ may vary for other ALPs.

Over the past two decades, the community's efforts and size have steadily grown, with recent years witnessing a flourishing phase. This surge is attributable to advancements in theoretical and phenomenological aspects of axions, elucidating their roles in astrophysics and cosmology, thereby motivating and guiding detection efforts. Concurrently, advancements in detection technologies allow the exploration of unexplored territories beyond existing constraints. The absence of positive detections of supersymmetry (SUSY) particles at the Large Hadron Collider (LHC) and Weakly Interactive Massive Particles (WIMPs) in underground detectors has heightened interest in axions.

Furthermore, the PQ mechanism can be seamlessly embedded in well-known BSM frameworks such as SUSY [8], Grand Unified Theories (GUTs) [9] and most notably it is built in string theory in a model-independent way [10, 11]. Also, the discovery of axions would signify the identification of a new energy scale in particle physics.

2 U(1) problem and its solution

2.1 U(1) problem

In this section, we will delve into the origin of the U(1) problem [12], also recognized as the missing meson problem. Here we aim to comprehend how the quark mass term breaks the chiral $\mathrm{U}(2)_L \times \mathrm{U}(2)_R$ symmetry and explore the consequences of this breakdown.

In the real world, there are 6 flavors of quarks, among which the up and down quarks are the lightest of them, with masses of $m_u = 2.16^{+0.49}_{-0.26}$ MeV and $m_d = 4.67^{+0.48}_{-0.17}$ MeV [13], respectively. These masses are significantly smaller than the scale of quantum chromodynamics (QCD), denoted as $\Lambda_{QCD} \approx 332 \pm 17$ MeV [14]. Initially, we assume that the up and down quarks are massless. The mass of the strange quark, $m_s = 93.4^{+8.6}_{-3.4}$ MeV [13], is also relatively smaller than Λ_{QCD}, allowing us, in certain contexts, to treat it as massless as well, although this assumption is less justified compared to the case of the up and down quarks.

If we focus on hadron physics below the 1 GeV threshold, we can safely neglect the 3 heaviest quarks, the charm, bottom and top quarks, with masses of $m_c = 1.27 \pm 0.02$ GeV, $m_b = 4.18^{+0.03}_{-0.02}$ GeV and $m_t = 172.69 \pm 0.30$ GeV respectively [13] . For the time being, we will also exclude the strange quark from our analysis. Limiting our focus to two flavors of quarks simplifies the calculations and incorporating the strange quark later on will not pose any problem. Let us consider then, QCD with only 2 flavors of massless quarks. In this context, we have the left-handed Weyl field $\chi_{\alpha i}$, where $\alpha = 1, 2, 3$ represents the color index of the fundamental or 3 representation, and $i = 1, 2$ is the flavor index, with $i = 1$ denoting the up quark and $i = 2$ denoting the down quark. Additionally, we have the left-handed Weyl field $\xi^{\alpha\bar{i}}$, where $\alpha = 1, 2, 3$ represents the color index of the antifundamental or $\bar{3}$ representation, and $\bar{i} = 1, 2$ is the flavor index, with $\bar{i} = 1$ corresponding to the anti-up quark and $\bar{i} = 2$ corresponding to the anti-down quark.

With these considerations, the QCD lagrangian is expressed as follows

$$\mathcal{L} = i\chi^{\dagger\alpha i}\bar{\sigma}^\mu(\mathcal{D}_\mu)_\alpha{}^\beta\chi_{\beta i} + i\xi^\dagger_{\alpha\bar{i}}\bar{\sigma}^\mu(\bar{\mathcal{D}}_\mu)^\alpha{}_\beta\xi^{\beta\bar{i}} - \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} \; , \tag{2.1}$$

where $\mathcal{D}_\mu = \partial_\mu - igT^aA^a_\mu$ and $\bar{\mathcal{D}}_\mu = \partial_\mu - ig\bar{T}^aA^a_\mu$ are the covariant derivatives, with $(\bar{T}^a)^\alpha{}_\beta = -(T^a)_\beta{}^\alpha$. Here the T^a are the SU(2) symmetry generators, $T^a = \frac{1}{2}\sigma^a$ and:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; , \tag{2.2}$$

and the Pauli 4-vector $\bar{\sigma}^\mu$ is defined as

$$\bar{\sigma}^\mu = (I, -\vec{\sigma}) \ , \tag{2.3}$$

where $\vec{\sigma} = \sigma^1\hat{x}_1 + \sigma^2\hat{x}_2 + \sigma^3\hat{x}_3$.

In addition to the SU(3) color gauge symmetry, this lagrangian also possesses a global U(2) $\times$ U(2) flavor symmetry. Specifically, $\mathcal{L}$ remains invariant under these transformations

$$\begin{aligned} \chi_{\alpha i} &\to L_i{}^j \chi_{\alpha j} \ , \\ \xi^{\alpha \bar{i}} &\to (R^*)^{\bar{i}}{}_{\bar{j}} \xi^{\alpha \bar{j}} \ , \end{aligned} \tag{2.4}$$

where L and R^* are independent 2×2 unitary matrices, which can also be represented as $L = e^{i\theta_L T}$ and $R^* = e^{i\theta_R T}$. Applying (2.4) to (2.1), we find that

$$\mathcal{L}' = i(L_i{}^j)^\dagger \chi^{\dagger \alpha j} \bar{\sigma}^\mu (\mathcal{D}_\mu)_\alpha{}^\beta L_i{}^j \chi_{\beta j} + i[(R^*)^{\bar{i}}{}_{\bar{j}}]^\dagger \xi^\dagger_{\alpha \bar{j}} \bar{\sigma}^\mu (\bar{\mathcal{D}}_\mu)^\alpha{}_\beta (R^*)^{\bar{i}}{}_{\bar{j}} \xi^{\beta \bar{j}} - \frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu} \ , \tag{2.5}$$

L and R^* being unitary matrices satisfy the conditions $L^\dagger L = 1$ and $R^{*\dagger} R^* = 1$, ensuring the invariance of the lagrangian. Alternatively, this invariance can be understood by recognizing that any arbitrary rotation of θ_L or θ_R is cancelled out by its conjugate counterpart.

We can express the QCD lagrangian in terms of Dirac fields, as Dirac fields inherently consist of both a left-handed and right-handed component

$$\mathcal{L} = i\overline{\Psi} \not{\mathcal{D}} \Psi - \frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu} \ , \tag{2.6}$$

where

$$\Psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \to \Psi_{\alpha i} = \begin{pmatrix} \chi_{\alpha i} \\ \xi^\dagger_{\alpha \bar{i}} \end{pmatrix} \ . \tag{2.7}$$

Equation (2.4) becomes

$$\begin{aligned} P_L \Psi_{\alpha i} &\to L_i{}^j P_L \Psi_{\alpha i} \ , \\ P_R \Psi_{\alpha i} &\to R_{\bar{i}}{}^{\bar{j}} P_R \Psi_{\alpha i} \ , \end{aligned} \tag{2.8}$$

where $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$ are the projection operators. Consequently, the global flavor symmetry is commonly referred to as $\mathrm{U}(2)_L \times \mathrm{U}(2)_R$, and since it is a symmetry that distinguishes the left- and right-handed components of a Dirac field, it is chiral.

We can decompose $\mathrm{U}(2)_L \times \mathrm{U}(2)_R$ into $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_L \times \mathrm{U}(1)_R$. This decomposition can be further expressed in terms of axial and vectorial terms, where the vectorial term arises when R and L are equal, and the axial term occurs when the R and L are not equal. Consequently, we arrive at $\mathrm{SU}(2)_V \times \mathrm{SU}(2)_A \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A$. Here $\mathrm{U}(1)_V$ represents the vectorial U(1) symmetry corresponding to $L = R = e^{-i\alpha} I$, and $\mathrm{U}(1)_A$ is the axial U(1) symmetry corresponding to $L = R^* = e^{i\alpha} I$. However, it will become clear later on that the apparent $\mathrm{U}(1)_A$ symmetry is, in fact, anomalous and thus not a real symmetry of QCD. Therefore, the non-anomalous global flavor symmetry is, in reality, $\mathrm{SU}(2)_V \times \mathrm{SU}(2)_A \times \mathrm{U}(1)_V$.

According to Noether's Theorem (A), every continuous symmetry corresponds to a conserved charge. In the context of U(1) vectorial symmetry, the corresponding conserved charge is the quark number, representing the difference between the number of quarks and the number of antiquarks. The quark number constitutes one-third of the baryon number. In hadron physics, baryon number holds significant importance as the classification of hadrons relies on their baryon numbers. Baryons, such as protons and neutrons, possess a baryon number equal to 1 and are color singlets, meaning they remain unchanged under any color gauge transformation. In contrast mesons, like pions, possess a baryon number of 0 as they are color singlet bound states comprising a quark-antiquark pair.

Understanding the role of $\mathrm{SU}(2)_V \times \mathrm{SU}(2)_A$ symmetry is a nuanced process. First, it is essential to recognize that this symmetry comprises two SU(2) components: one vectorial, treating left and right components equally, and one axial, which is chiral in nature. Next, examining experimental evidences from nature, we find no indication supporting a hadron classification system that distinguishes between left- and right-handed components. Besides, experiments demonstrate that hadrons appear to be classified by only one SU(2) symmetry group [15]. Thereby, in order to be consistent with experimental observation, $\mathrm{SU}(2)_V \times \mathrm{SU}(2)_A$ invariance is only possible if the axial generators are spontaneously broken. Consequently, the remaining SU(2) symmetry group's classification is of the vectorial kind, and its corresponding conserved charge is referred to as isotopic spin or, more commonly, isospin.

The proton and neutron together form a doublet (fundamental representation) of $\mathrm{SU}(2)_V$, while the pions ($\pi^{\pm,0}$) constitute a triplet (adjoint representation) of $\mathrm{SU}(2)_V$. Due to being in the adjoint representation, pions are expected to mediate interactions between the proton and neutron [16]. This triplet structure of pions arises from the breakdown of $\mathrm{SU}(2)_A$. Since SU(n) has $n^2 - 1$ symmetry generators, SU(2) has three generators.When these generators are spontaneously broken, as described by the Nambu-Goldstone (NG) Theorem, three NG bosons emerge. However, because the up and down quarks are not precisely massless, the symmetries involved are not exact. Therefore, isospin, which is also violated by electromagnetism (EM), is not an exact symmetry. In this context, the pions can be described as pseudo-Nambu-Goldstone (pseudo-NG) bosons.

In order to spontaneously break the axial part of $\mathrm{SU}(2)_V \times \mathrm{SU}(2)_A$, some operator that transforms nontrivially under this group (not a singlet) must acquire a nonzero vacuum expectation value (VEV). However, to preserve Lorentz and color SU(3) invariance, this operator must be a Lorentz scalar and a color singlet. Since there is no fundamental scalar field in pure QCD that could acquire a nonzero VEV, composite fields become necessary.

Note that there is no Higgs field in pure QCD. The simplest candidate for this purpose is a quark-antiquark condensate $\chi^a_{\alpha i}\xi^{\alpha\bar{j}}_a = \bar{\Psi}^{\alpha\bar{j}} P_L \Psi_{\alpha i}$, where a is an undotted spinor index. Assuming this, we have

$$\langle 0| \chi^a_{\alpha i}\xi^{\alpha\bar{j}}_a |0\rangle = -v^3 \delta_i^{\ \bar{j}} \ , \tag{2.9}$$

where v is a paramater with mass dimension, and its specific value depends on the chosen renormalization scheme. The v^3 arises from the fact that fermionic fields are of mass dimension 3/2 in the four-dimensional lagrangian. This condensate effectively breaks the axial generators while preserving the vector generators of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. This phenomenom becomes evident when applying (2.4) in (2.9), revealing the symmetry transformations in the resulting expression

$$\begin{aligned} \langle 0| \chi^a_{\alpha i}\xi^{\alpha\bar{j}}_a |0\rangle &\to \langle 0| L_i^{\ k} \chi^a_{\alpha k} (R^*)^{\bar{j}}_{\ \bar{n}} \xi^{\alpha\bar{n}}_a |0\rangle \ , \\ &\to L_i^{\ k} (R^*)^{\bar{j}}_{\ \bar{n}} \langle 0| \chi^a_{\alpha k}\xi^{\alpha\bar{n}}_a |0\rangle \ , \\ &\to L_i^{\ k} (R^*)^{\bar{j}}_{\ \bar{n}} (-v^3 \delta_k^{\ \bar{n}}) \ , \\ &\to -v^3 (LR^\dagger)_i^{\ \bar{j}} \ . \end{aligned} \tag{2.10}$$

Indeed, the mechanism through which the condensate operates becomes clear when considering how it violates $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The ability to perform independent arbitrary transformations on our left- and right-handed fields is no longer viable due to the presence of the condensate. To maintain the form of (2.9), the only viable option is set $L = R$, leading to $LR^\dagger = 1$, which fulfills the condition for the vectorial part of $\mathrm{SU}(2)_V \times \mathrm{SU}(2)_A$. Consequently, $\mathrm{SU}(2)_V$ (and also $\mathrm{U}(1)_V$) remains unbroken in this scenario.

It is worth noting that if the $\mathrm{U}(1)_A$ symmetry group had not been disregarded due to the anomaly, the theory would have three broken generators from $\mathrm{SU}(2)_A$ and one broken generator from $\mathrm{U}(1)_A$. Consequently, another NG boson, specifically another meson called π^9, would be expected to emerge. This meson, just like the other ones is mass bound by the theory. However, this additional meson has never been observed, giving rise to a missing meson problem. This discrepancy between theoretical expectations and experimental observations is the so called U(1) problem.

In a future section, we will delve into the complexities of its resolution, including the intricacies of the phenomenon known as the chiral anomaly.

2.2 Effective lagrangian for the NG-bosons

It is crucial to emphasize that the expression (2.9) is nonperturbative. This nonperturbative nature arises from the fact that, for chiral QCD with vanishing quark masses, there exists no tree-level potential that could lead to the vacuum expectation value (VEV) of $\langle 0| \chi^a_{\alpha i}\xi^{\alpha\bar{j}}_a |0\rangle$. Consequently, fermionic condensates like this are produced by nonperturbative physics.

Another perspective to grasp this is by considering that $\langle 0| \chi^a_{\alpha i} \xi^{\alpha \bar{j}}_a |0\rangle$ is a dimensional quantity (where v is a paramater with mass dimension, as discussed earilier), and the only scale in our theory is the Λ_{QCD}, which is defined at a point where perturbative physics is not applicable [15]. Therefore, any observable dependant on this scale is inherently linked to nonperturbative phenomena. As a result, we anticipate that $v \sim \Lambda_{QCD}$, and Λ_{QCD} establishes the scale for the masses of all hadrons that are not pseudo-NG bosons, including protons and neutrons. The masses of protons and neutrons are very similiar, where $m_p = 938.27208816 \pm 0.00000029$ MeV and $m_n = 939.56542052 \pm 0.00000054$ MeV [13], respectively. Remember that $\Lambda_{QCD} \approx 332 \pm 17$ MeV [14].

Now, we will construct a low-energy effective lagrangian [17], commonly referred to as the chiral lagrangian, for the pions (the three pseudo-Nambu-Goldstone bosons), and derive their masses. This can be done as follows: we allow the orientation in flavor space of the vacuum expectation value (VEV) of the condensate to vary slowly as a function of spacetime. Consequently, the expression (2.9) transforms into:

$$\langle 0| \chi^a_{\alpha i} \xi^{\alpha \bar{j}}_a |0\rangle = -v^3 U_i{}^{\bar{j}}(x) \ . \tag{2.11}$$

$U(x)$ is a spacetime dependent unitary matrix, represented as

$$U(x) = \exp[2i\pi^a(x) T^a / f_\pi] \ , \tag{2.12}$$

where $T^a = \frac{\sigma^a}{2}$ are the generators of $SU(2)$ symmetry group, $\pi^a(x)$ are the three real scalar fields to be identified with the pions, and f_π is the pion decay constant. f_π is a parameter with mass dimension, and its experimentally measured value is 130.41 ± 0.21 MeV [18]. At first, for simplicity, we will not consider the $\mathrm{U}(1)_A$ NG boson. Later on, we can easily add it by hand. Additionally, we require the condition $\det U(x) = 1$.

$U(x)$ is conceived as a low-energy effective field, and its lagrangian should be the most general one consistent with $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry. Under a general transformation of this group, $U(x)$ transforms as follows:

$$U(x) \to L U(x) R^\dagger \ , \tag{2.13}$$

the terms in the $U(x)$ lagrangian (chiral lagrangian) can be organized based on the number of derivatives they contain. Since $UU^\dagger = 1$ (a consequence of its unitarity), there are no terms with no derivatives. The lagrangian includes one term with two derivatives, which takes the form:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} f_\pi^2 \operatorname{Tr}\Big[\partial^\mu U^\dagger \partial_\mu U\Big] \ . \tag{2.14}$$

Expanding $U(x)$ in inverse powers of f_π in (2.12), we obtain:

Firstly, we can express the exponential function in terms of its Taylor series expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \ , \tag{2.15}$$

applying this taylor series expansion to (2.12), we get:

$$\begin{aligned} U(x) &= 1 + \frac{2i\pi^d(x)T^d}{f_\pi} + \frac{1}{2!}\left(\frac{2i\pi^d(x)T^d}{f_\pi}\right)^2 + \frac{1}{3!}\left(\frac{2i\pi^d(x)T^d}{f_\pi}\right)^3 + \mathcal{O}(f_\pi^{-4}) \ , \\ &= 1 + \frac{2i\pi^d(x)T^d}{f_\pi} - \frac{2\pi^d(x)\pi^e(x)T^dT^e}{f_\pi^2} - \frac{4i\pi^d(x)\pi^e(x)\pi^f(x)T^cT^dT^e}{3f_\pi^3} + \mathcal{O}(f_\pi^{-4}) \ . \end{aligned} \tag{2.16}$$

Indeed, to calculate (2.14), it is also necessary to have the expansion for the hermitian conjugate. Therefore, the expansion for the hermitian conjugate of $U(x)$ is given by:

$$U^\dagger(x) = 1 - \frac{2i\pi^a(x)T^a}{f_\pi} - \frac{2\pi^a(x)\pi^b(x)T^aT^b}{f_\pi^2} + \frac{4i\pi^a(x)\pi^b(x)\pi^c(x)T^aT^bT^c}{3f_\pi^3} + \mathcal{O}(f_\pi^{-4}) \ . \tag{2.17}$$

Certainly, we can now substitute (2.16) and (2.17) into (2.14). To keep the calculations clear, let's first focus on evaluating the derivative terms separately, considering terms up to $\mathcal{O}(f_\pi^{-4})$ when they are multiplied and omitting the dependence of x in U and π^a:

$$\begin{aligned} \partial^\mu U^\dagger = &-\frac{2i(\partial^\mu\pi^a)T^a}{f_\pi} - 2\left[(\partial^\mu\pi^a)\pi^b + \pi^a(\partial^\mu\pi^b)\right]\frac{T^aT^b}{f_\pi^2} \\ &+ \frac{4i}{3}\left[\partial^\mu\pi^a\pi^b\pi^c + \pi^a\partial^\mu\pi^b\pi^c + \pi^a\pi^b\partial^\mu\pi^c\right]\frac{T^aT^bT^c}{f_\pi^3} \ , \end{aligned} \tag{2.18}$$

$$\begin{aligned} \partial_\mu U = &\frac{2i(\partial_\mu\pi^d)T^d}{f_\pi} - 2\left[(\partial_\mu\pi^d)\pi^e + \pi^d(\partial_\mu\pi^e)\right]\frac{T^dT^e}{f_\pi^2} \\ &- \frac{4i}{3}\left[\partial_\mu\pi^d\pi^e\pi^f + \pi^d\partial_\mu\pi^e\pi^f + \pi^d\pi^e\partial_\mu\pi^f\right]\frac{T^dT^eT^f}{f_\pi^3} \ . \end{aligned} \tag{2.19}$$

Using these derivative terms, we can express the chiral lagrangian. For clarity, let's divide each term:

Note that the trace of the sum is equivalent to the sum of the traces, i.e., $\mathrm{Tr}[x+y+z] = \mathrm{Tr}[x] + \mathrm{Tr}[y] + \mathrm{Tr}[z]$.

$$\begin{aligned} \mathcal{L}_1 &= -\frac{1}{4}f_\pi^2\,\mathrm{Tr}\left[\left(-\frac{2i(\partial^\mu\pi^a)T^a}{f_\pi}\right)\left(\frac{2i(\partial_\mu\pi^d)T^d}{f_\pi}\right)\right] \ , \\ &= -\,\mathrm{Tr}\Big[(\partial^\mu\pi^a)T^a(\partial_\mu\pi^d)T^d\Big] \ , \\ &= -\,\mathrm{Tr}\Big[T^aT^d\Big]\partial^\mu\pi^a\partial_\mu\pi^d \ , \\ &= -\frac{1}{2}\delta^{ad}\partial^\mu\pi^a\partial_\mu\pi^d \ , \\ &= -\frac{1}{2}\partial^\mu\pi^a\partial_\mu\pi^a \ , \end{aligned} \tag{2.20}$$

to arrive at this result, we utilized the property of Pauli matrices $\mathrm{Tr}\Big[T^aT^b\Big] = \frac{1}{2^2}\,\mathrm{Tr}\Big[\sigma^a\sigma^b\Big] = \frac{1}{2}\delta^{ab}$. Additionally, the cross-terms are analysed separately for clarity:

$$\begin{aligned} \mathcal{L}_2 = -\frac{1}{4}f_\pi^2\,\mathrm{Tr}\Bigg[&\left(-\frac{2i(\partial^\mu\pi^a)T^a}{f_\pi}\right)\left(-2\left[(\partial_\mu\pi^d)\pi^e + \pi^d(\partial_\mu\pi^e)\right]\frac{T^dT^e}{f_\pi^2}\right) \\ &+ \left(-2\left[(\partial^\mu\pi^a)\pi^b + \pi^a(\partial^\mu\pi^b)\right]\frac{T^aT^b}{f_\pi^2}\right)\left(\frac{2i(\partial_\mu\pi^d)T^d}{f_\pi}\right)\Bigg] \ , \end{aligned} \tag{2.21}$$

The first term of the sum simplifies to

$$
\begin{aligned}
\mathcal{L}_{2_1} &= -\frac{1}{4} f_\pi^2 \operatorname{Tr}\left[\left(-\frac{2i(\partial^\mu \pi^a)T^a}{f_\pi}\right)\left(-2\left[(\partial_\mu \pi^d)\pi^e + \pi^d(\partial_\mu \pi^e)\right]\frac{T^d T^e}{f_\pi^2}\right)\right] \ , \\
&= -\frac{1}{4} f_\pi^2 \left(\frac{4i}{f_\pi^3}\right) \operatorname{Tr}\left[(\partial^\mu \pi^a T^a)\left(\left[(\partial_\mu \pi^d)\pi^e + \pi^d(\partial_\mu \pi^e)\right] T^d T^e\right)\right] \ , \\
&= -\frac{i}{f_\pi} \operatorname{Tr}\left[T^a T^d T^e\right]\left(\partial^\mu \pi^a \partial_\mu \pi^d \pi^e + \partial^\mu \pi^a \pi^d \partial_\mu \pi^e\right) \ , \\
&= -\frac{i}{f_\pi}\frac{1}{4}\epsilon_{ade}\left(\partial^\mu \pi^a \partial_\mu \pi^d \pi^e + \partial^\mu \pi^a \pi^d \partial_\mu \pi^e\right) \ .
\end{aligned}
\tag{2.22}
$$

The second term of the sum simplifies to

$$
\begin{aligned}
\mathcal{L}_{2_2} &= -\frac{1}{4} f_\pi^2 \operatorname{Tr}\left[\left(-2[(\partial^\mu \pi^a)\pi^b + \pi^a(\partial^\mu \pi^b)]\frac{T^a T^b}{f_\pi^2}\right)\left(\frac{2i(\partial_\mu \pi^d)T^d}{f_\pi}\right)\right] \ , \\
&= -\frac{1}{4} f_\pi^2 \left(\frac{-4i}{f_\pi^3}\right) \operatorname{Tr}\left[\left([(\partial^\mu \pi^a)\pi^b + \pi^a(\partial^\mu \pi^b)]T^a T^b\right)\left((\partial_\mu \pi^d)T^d\right)\right] \ , \\
&= \frac{i}{f_\pi} \operatorname{Tr}\left[T^a T^b T^d\right](\partial^\mu \pi^a \pi^b \partial_\mu \pi^d + \pi^a \partial^\mu \pi^b \partial_\mu \pi^d) \ , \\
&= \frac{i}{f_\pi}\frac{1}{4}\epsilon_{abd}(\partial^\mu \pi^a \pi^b \partial_\mu \pi^d + \pi^a \partial^\mu \pi^b \partial_\mu \pi^d) \ ,
\end{aligned}
\tag{2.23}
$$

where we have utilized the property of Pauli matrices $\operatorname{Tr}\left[T^a T^b T^c\right] = \frac{1}{4}\epsilon_{abc}$. We can then change the Levi-Civita indices to α, β, γ so that both terms can be combined, leading to

$$
\begin{aligned}
\mathcal{L}_2 &= \frac{i}{4f\pi}\epsilon_{\alpha\beta\gamma}\left[\partial^\mu \pi^\alpha \pi^\beta \partial_\mu \pi^\gamma + \pi^\alpha \partial^\mu \pi^\beta \partial_\mu \pi^\gamma - \partial^\mu \pi^\alpha \partial_\mu \pi^\beta \pi^\gamma - \partial^\mu \pi^\alpha \pi^\beta \partial_\mu \pi^\gamma\right] \ , \\
&= \frac{i}{4f\pi}\epsilon_{\alpha\beta\gamma}\left[\ \pi^\alpha \partial^\mu \pi^\beta \partial_\mu \pi^\gamma - \partial^\mu \pi^\alpha \partial_\mu \pi^\beta \pi^\gamma\right] \ , \\
&= \frac{i}{4f\pi}\epsilon_{\alpha\beta\gamma}\left[\ \pi^\alpha \partial^\mu \pi^\beta \partial_\mu \pi^\gamma - \pi^\gamma \partial^\mu \pi^\alpha \partial_\mu \pi^\beta\right] = 0 \ ,
\end{aligned}
\tag{2.24}
$$

which evaluates to zero due to the anti-symmetric nature of the Levi-Civita tensor. Taking the first term to demonstrate, we get:

$$
\begin{aligned}
\epsilon_{\alpha\beta\gamma}\pi^\alpha \partial^\mu \pi^\beta \partial_\mu \pi^\gamma &= \epsilon_{\alpha\gamma\beta}\pi^\alpha \partial^\mu \pi^\gamma \partial_\mu \pi^\beta \ , \\
&= \epsilon_{\alpha\gamma\beta}\pi^\alpha \partial_\mu \pi^\beta \partial^\mu \pi^\gamma \ , \\
&= \epsilon_{\alpha\gamma\beta}\pi^\alpha \partial^\mu \pi^\beta \partial_\mu \pi^\gamma \ , \\
&= -\epsilon_{\alpha\beta\gamma}\pi^\alpha \partial^\mu \pi^\beta \partial_\mu \pi^\gamma \ , \\
&= 0 \ .
\end{aligned}
\tag{2.25}
$$

Now, for the third term, we have:

$$
\begin{aligned}
\mathcal{L}_3 &= -\frac{1}{4} f_\pi^2 \,\mathrm{Tr}\left[\left(-2\left[(\partial^\mu \pi^a)\pi^b + \pi^a(\partial^\mu \pi^b)\right]\frac{T^aT^b}{f_\pi^2}\right)\right. \\
&\qquad\qquad\qquad\qquad \left.\left(-2\left[(\partial_\mu \pi^d)\pi^e + \pi^d(\partial_\mu \pi^e)\right]\frac{T^dT^e}{f_\pi^2}\right)\right] \ , \\
&= -\frac{1}{4} f_\pi^2 \frac{4}{f_\pi^4}\,\mathrm{Tr}\Big[T^aT^bT^dT^e\Big]\left[(\partial^\mu \pi^a)\pi^b + \pi^a(\partial^\mu \pi^b)\right]\left[(\partial_\mu \pi^d)\pi^e + \pi^d(\partial_\mu \pi^e)\right] \ , \\
&= -\frac{1}{8 f_\pi^2}(\delta_{ab}\delta_{de} - \delta_{ad}\delta_{be} + \delta_{ae}\delta_{bd}) \\
&\qquad\qquad\qquad\qquad \times \left[\left(\partial^\mu \pi^a \pi^b + \pi^a \partial^\mu \pi^b\right)\left(\partial_\mu \pi^d \pi^e + \pi^d \partial_\mu \pi^e\right)\right] \ ,
\end{aligned}
\tag{2.26}
$$

where we have utilized the property of the Pauli matrices $\mathrm{Tr}\Big[T^aT^bT^dT^e\Big] = \frac{1}{8}(\delta_{ab}\delta_{de} - \delta_{ad}\delta_{be} + \delta_{ae}\delta_{bd})$. The rest of the calculation proceeds as follows:

$$
\begin{aligned}
\mathcal{L}_3 &= -\frac{1}{8 f_\pi^2}\left[(\partial^\mu \pi^a)\pi^a + \pi^a(\partial^\mu \pi^a)\right]\left[(\partial_\mu \pi^d)\pi^d + \pi^d(\partial_\mu \pi^d)\right] \\
&+ \frac{1}{8 f_\pi^2}\left[(\partial^\mu \pi^a)\pi^b + \pi^a(\partial^\mu \pi^b)\right]\left[(\partial_\mu \pi^a)\pi^b + \pi^a(\partial_\mu \pi^b)\right] \\
&- \frac{1}{8 f_\pi^2}\left[(\partial^\mu \pi^a)\pi^b + \pi^a(\partial^\mu \pi^b)\right]\left[(\partial_\mu \pi^b)\pi^a + \pi^b(\partial_\mu \pi^a)\right] \ , \\
&= -\frac{1}{8 f_\pi^2}\left[2\pi^a(\partial^\mu \pi^a)\right]\left[(2\pi^d(\partial_\mu \pi^d)\right] \\
&+ \frac{1}{8 f_\pi^2}\left[\pi^b(\partial^\mu \pi^a) + \pi^a(\partial^\mu \pi^b)\right]\left[\pi^b(\partial_\mu \pi^a) + \pi^a(\partial_\mu \pi^b)\right] \\
&- \frac{1}{8 f_\pi^2}\left[\pi^b(\partial^\mu \pi^a) + \pi^a(\partial^\mu \pi^b)\right]\left[\pi^a(\partial_\mu \pi^b) + \pi^b(\partial_\mu \pi^a)\right] \ , \\
&= -\frac{1}{8 f_\pi^2}\left[4\pi^a\pi^d\partial^\mu\pi^a\partial_\mu\pi^d\right] = -\frac{1}{2 f_\pi^2}\left[\pi^a\pi^d\partial^\mu\pi^a\partial_\mu\pi^d\right] \ .
\end{aligned}
\tag{2.27}
$$

The last 2 terms are symmetrical, only the indices are different, so we are going to calculate one and multiply the result by a factor of 2. The last term reads:

$$
\mathcal{L}_4 = -\frac{1}{4} f_\pi^2 \,\mathrm{Tr}\left[\frac{4i}{3}\left[\partial^\mu \pi^a \pi^b \pi^c + \pi^a \partial^\mu \pi^b \pi^c + \pi^a \pi^b \partial^\mu \pi^c\right]\frac{T^aT^bT^c}{f_\pi^3}\frac{2i(\partial_\mu \pi^d)T^d}{f_\pi}\right] \ , \tag{2.28}
$$

similarly to the third term we utilize the property of Pauli matrices $\mathrm{Tr}\Big[T^aT^bT^cT^d\Big] = \frac{1}{8}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$ and the calculation goes as follows:

$$
\begin{aligned}
\mathcal{L}_4 &= -\frac{1}{4} f_\pi^2 \frac{-8}{3 f_\pi^4}\,\mathrm{Tr}\left[\left(\partial^\mu \pi^a \pi^b \pi^c + \pi^a \partial^\mu \pi^b \pi^c + \pi^a \pi^b \partial^\mu \pi^c\right) T^aT^bT^c(\partial_\mu \pi^d)T^d\right] \ , \\
&= \frac{2}{3 f_\pi^2}\,\mathrm{Tr}\left[T^aT^bT^cT^d\right]\left(\partial^\mu \pi^a \pi^b \pi^c + \pi^a \partial^\mu \pi^b \pi^c + \pi^a \pi^b \partial^\mu \pi^c\right)(\partial_\mu \pi^d) \ , \\
&= \frac{1}{12 f_\pi^2}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\left(\pi^b\pi^c\partial^\mu\pi^a\partial_\mu\pi^d + \pi^a\pi^c\partial^\mu\pi^b\partial_\mu\pi^d\right. \\
&\left. + \pi^a\pi^b\partial^\mu\pi^c\partial_\mu\pi^d\right) \ .
\end{aligned}
\tag{2.29}
$$

Note that similarly to the third term of the lagrangian, the last 2 terms of deltas cancel each other, leaving us with only the first term, so we find:

$$\begin{aligned}
\mathcal{L}_4 &= \frac{1}{12f_\pi^2}(\delta_{ab}\delta_{cd})\left(\pi^b\pi^c\partial^\mu\pi^a\partial_\mu\pi^d + \pi^a\pi^c\partial^\mu\pi^b\partial_\mu\pi^d + \pi^a\pi^b\partial^\mu\pi^c\partial_\mu\pi^d\right) \ , \\
&= \frac{1}{12f_\pi^2}\left(\pi^a\pi^c\partial^\mu\pi^a\partial_\mu\pi^c + \pi^a\pi^c\partial^\mu\pi^a\partial_\mu\pi^c + \pi^a\pi^a\partial^\mu\pi^c\partial_\mu\pi^c\right) \ , \\
&= \frac{1}{12f_\pi^2}\left(\pi^a\pi^a\partial^\mu\pi^c\partial_\mu\pi^c + 2\pi^a\pi^c\partial^\mu\pi^a\partial_\mu\pi^c\right) \ ,
\end{aligned} \tag{2.30}$$

Combining all terms together, we find (remember that we must multiply the last term by 2 and that the indices are mute):

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + 2\mathcal{L}_4 \ , \\
&= -\frac{1}{2}\partial^\mu\pi^a\partial_\mu\pi^a + 0 - \frac{1}{2f_\pi^2}\left(\pi^a\pi^b\partial^\mu\pi^a\partial_\mu\pi^b\right) \\
&+ \frac{1}{6f_\pi^2}\left(\pi^a\pi^a\partial^\mu\pi^b\partial_\mu\pi^b + 2\pi^a\pi^b\partial^\mu\pi^a\partial_\mu\pi^b\right) \ , \\
&= -\frac{1}{2}\partial^\mu\pi^a\partial_\mu\pi^a + \frac{1}{6f_\pi^2}\left(\pi^a\pi^a\partial^\mu\pi^b\partial_\mu\pi^b + 2\pi^a\pi^b\partial^\mu\pi^a\partial_\mu\pi^b - 3\pi^a\pi^b\partial^\mu\pi^a\partial_\mu\pi^b\right) \ .
\end{aligned} \tag{2.31}$$

Finally, we reach the final result for the effective lagrangian:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2}\partial^\mu\pi^a\partial_\mu\pi^a + \frac{1}{6}f_\pi^{-2}\left(\pi^a\pi^a\partial^\mu\pi^b\partial_\mu\pi^b - \pi^a\pi^b\partial^\mu\pi^b\partial_\mu\pi^a\right) + \dots \ . \tag{2.32}$$

Indeed, the pion fields exhibit interactions as described by (2.14). These interactions give rise to Feynman vertices, which contain factors of momenta divided by the pion decay constant, denoted as $\frac{p^2}{f_\pi}$. This plays a fundamental role in understanding the dynamics of pion interactions in low-energy processes.

Now, let's consider the effects of including the small masses for the up and down quarks. With this new consideration, we need to add a new term to our QCD lagrangian (2.1). The most general term we can add is:

$$\begin{aligned}
\mathcal{L}_{\text{mass}} &= -\xi^{\alpha\bar{j}}M_{\bar{j}}{}^{i}\chi_{\alpha i} + \text{h.c.} \ , \\
&= -M_{\bar{j}}{}^{i}\chi_{\alpha i}\xi^{\alpha\bar{j}} + \text{h.c.} \ , \\
&= -\operatorname{Tr}[M\chi_\alpha\xi^\alpha] + \text{h.c.} \ ,
\end{aligned} \tag{2.33}$$

where M is an arbitrary complex 2×2 matrix, and h.c. denotes the hermitian conjugate. By performing a suitable $\text{SU}(2)_L \times \text{SU}(2)_R$ transformation, M can be diagonalized, resulting in positive real entries m_u and m_d representing the up and down masses respectively:

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} e^{-i\theta/2} \ . \tag{2.34}$$

It is important to note that the overall phase factor θ cannot be eliminated without resorting to a prohibited $\text{U}(1)_A$ transformation. Furthermore, θ carries significant physical implications, with experimental observations constraining its value to $|\theta| \lesssim 10^{-10}$ [19–21]. We will delve deeper into the intricacies of this θ term in the Strong CP problem chapter. However, for the purpose of our current discussion, we will adopt the phenomenological approach of setting θ to zero.

Next, we will replace $\chi_\alpha \xi^\alpha$ in (2.33) with its spacetime dependent VEV as depicted in (2.11). This substitution gives rise to a term in the chiral lagrangian that accounts for the effects of quark masses:

$$\begin{aligned}
\mathcal{L}_{\text{mass}} &= -\operatorname{Tr}[M\chi_\alpha\xi^\alpha] + \text{h.c.} \ , \\
&= -\operatorname{Tr}\Big[M(-v^3 U(x))\Big] + \text{h.c.} \ , \\
&= v^3 \operatorname{Tr}[MU] + \text{h.c.} \ , \\
&= v^3 \operatorname{Tr}\Big[MU + M^\dagger U^\dagger\Big] \ .
\end{aligned} \tag{2.35}$$

Note that when $\theta = 0$, M and $M^\dagger$ are the same matrix. Considering that M transforms under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ as $M \to RML^\dagger$ while $U \to LUR^\dagger$, the expression $\operatorname{Tr} MU$ remains formally invariant under this group of symmetry. Consequently, we demand that all terms in the chiral lagrangian exhibit this formal invariance under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.

Expanding $\mathcal{L}_{\text{mass}}$ in a similar manner as we did for (2.14) and considering $M = M^\dagger$, we obtain:

$$\begin{aligned}
\mathcal{L}_{\text{mass}} &= v^3 \operatorname{Tr}\left[M(U + U^\dagger)\right] \ , \\
&= v^3 \operatorname{Tr}\Bigg[M\Bigg(1 + \frac{2i\pi^a(x)T^a}{f_\pi} - \frac{2\pi^a(x)\pi^b(x)T^aT^b}{f_\pi^2} + \mathcal{O}(f_\pi^{-3}) \\
&\qquad +1 - \frac{2i\pi^a(x)T^a}{f_\pi} - \frac{2\pi^a(x)\pi^b(x)T^aT^b}{f_\pi^2} + \mathcal{O}(f_\pi^{-3})\Bigg)\Bigg] \ , \\
&= v^3 \operatorname{Tr}\left[M\left(2 - \frac{4\pi^a\pi^bT^aT^b}{f_\pi^2} + \mathcal{O}(f_\pi^{-3})\right)\right] \ , \\
&= v^3 \operatorname{Tr} 2M - 4\frac{v^3}{f_\pi^2} \operatorname{Tr}\left[MT^aT^b\right]\pi^a\pi^b + \mathcal{O}(f_\pi^{-3}) \ ,
\end{aligned} \tag{2.36}$$

omitting constant terms in the lagrangian, so that $v^3 \operatorname{Tr} 2M$ vanishes, we obtain:

$$\begin{aligned}
\mathcal{L}_{\text{mass}} &= -4\frac{v^3}{f_\pi^2} \operatorname{Tr}\left[MT^aT^b\right]\pi^a\pi^b + \mathcal{O}(f_\pi^{-3}) \ , \\
&= -2\frac{v^3}{f_\pi^2} \operatorname{Tr}\left[M\left\{T^a, T^b\right\}\right]\pi^a\pi^b + \mathcal{O}(f_\pi^{-3}) \ , \\
&= -2\frac{v^3}{f_\pi^2} \operatorname{Tr}\left[M\frac{1}{2}\delta^{ab}I\right]\pi^a\pi^b + \mathcal{O}(f_\pi^{-3}) \ ,
\end{aligned} \tag{2.37}$$

to arrive at this expression, we utilized the relation $\left\{T^a, T^b\right\} = \frac{1}{2}\delta^{ab}$. Continuing the calculation, we obtain:

$$\begin{aligned}
\mathcal{L}_{\text{mass}} &= -\frac{v^3}{f_\pi^2}\,\mathrm{Tr}\,[M]\,\pi^a\pi^a + \mathcal{O}(f_\pi^{-3})\ , \\
&= -\frac{v^3}{f_\pi^2}\,\mathrm{Tr}\,[M]\,\pi^a\pi^a + \mathcal{O}(f_\pi^{-3})\ , \\
&= -\frac{v^3}{f_\pi^2}\,(m_u + m_d)\,\pi^a\pi^a + \mathcal{O}(f_\pi^{-3})\ .
\end{aligned} \tag{2.38}$$

Upon considering (2.37), it becomes evident that all three pions share the same mass, a value dictated by the Gell-Mann-Oakes-Renner relation [22]:

$$m_\pi^2 = 2(m_u + m_d)\frac{v^3}{f_\pi^2}\ . \tag{2.39}$$

It is important to note that both quark masses and v^3 are dependent on the chosen renormalization scheme. However, their product remains invariant across schemes. As mentioned earlier, also due to EM interactions, isospin is not an exact symmetry. This is evident in the slightly elevated mass of charged pions ($\pi^\pm$) in comparison to neutral pions π^0, as confirmed in [13].

This framework can be readily expanded to include the strange quark. Despite being heavier than the up and down quarks, the strange quark remains pertinent to hadron physics below 1 GeV.

With the inclusion of the strange quark, the theory now encompasses three flavors of quarks, necessitating an adaptation of our symmetry framework. Previously, with two flavors, we worked within the framework of $\mathrm{U}(2)_L \times \mathrm{U}(2)_R$. However, with the introduction of the strange quark, the symmetry is extended to $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$. Analogous to the previous case, this can be decomposed into $\mathrm{SU}(3)_L \times \mathrm{U}(1)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_R$, which further decomposes to $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V \times \mathrm{SU}(3)_A \times \mathrm{U}(1)_A$.

Similar to our previous scenario, $\mathrm{SU}(3)_A$ must undergo spontaneous breaking. However, with the inclusion of the strange quark, there are now eight generators $(3^2 - 1)$, implying the existence of eight NG-bosons, a significant increase from the previous three. If we consider the spontaneous breaking of $\mathrm{U}(1)_A$ symmetry as well, the number of NG-bosons rises to nine. While the mass term in the lagrangian remains applicable, we need to replace the generators and revisit certain steps, as some SU(2) relations used in the previous derivation are no longer valid in this expanded context. Additionally, we can readily incorporate the breakdown of $\mathrm{U}(1)_A$ in the framework as well. With these considerations in mind, we arrive at:

$$\mathcal{L}_{\text{mass}} = -4\frac{v^3}{f_\pi^2}\,\mathrm{Tr}\Big[M\lambda^a\lambda^b\Big]\pi^a\pi^b + -4\frac{v^3}{f_9^2}\,\mathrm{Tr}[M]\pi^9\pi^9 \ldots\ , \tag{2.40}$$

where λ^a are the SU(3) generators, which are given by the Gell-Mann matrices:

$$
\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} ,
\end{aligned}
\tag{2.41}
$$

and f_9 is the π^9 decay constant. Another approach to grasp the distinction between the NG bosons arising from SU(3) and U(1) axial symmetries is to examine the composition of mesons. Each meson is formed by a quark-antiquark pair, where the quarks belong to the fundamental or 3 representation , and the antiquarks come from the anti-fundamental or $\overline{3}$ representation. Consequently, the meson structure can be described as follows:

$$
3 \otimes \overline{3} = 8 \oplus 1 \ . \tag{2.42}
$$

Mesons resulting from the $\mathrm{SU}(3)_A$ symmetry breakdown assemble into an octet, all sharing the same decay constant f_π. In contrast, the meson emerging from $\mathrm{U}(1)_A$ symmetry breakdown constitutes a singlet with its specific decay constant f_9.

The matrix M now incorporates a new diagonal entry to account for the presence of the strange quark and is defined as:

$$
M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} e^{-i\theta/2} \ . \tag{2.43}
$$

To calculate the meson masses, we define a new matrix:

$$
\Pi = \frac{\pi^a \lambda^a}{f_\pi} = \frac{1}{2 f_\pi} \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\ \overline{K}^0 & \frac{-2}{\sqrt{3}}\eta \end{pmatrix} , \tag{2.44}
$$

substituting this into (2.40), we obtain:

$$
\begin{aligned}
\mathcal{L}_{\text{mass}} &= -4v^3 \operatorname{Tr}\Big\{ M\Pi^2 \Big\} - 4v^3 \operatorname{Tr}\bigg\{ \frac{M(\pi^9)^2}{f_9^2} \bigg\} , \\
&= -4v^3 \operatorname{Tr}\bigg\{ M\Pi^2 + \frac{M(\pi^9)^2}{f_9^2} \bigg\}
\end{aligned}
\tag{2.45}
$$

to compute the mass term, we start by finding $M\Pi^2$. Since we are interested only in the main diagonal when taking the trace, and we ignore the phase on M, we obtain:

$$\begin{aligned}
M\Pi^2 + \frac{M(\pi^9)^2}{f_9^2} = \frac{1}{4f_\pi^2}\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \operatorname{diag}\Bigg[\left(\pi^0 + \frac{1}{\sqrt{3}}\eta\right)^2 + 2\pi^+\pi^- \\
+2K^+K^- \;, 2\pi^+\pi^- + \left(-\pi^0 + \frac{1}{\sqrt{3}}\eta\right)^2 + 2K^0\overline{K}^0 \;, 2K^-K^+ + 2\overline{K}^0K^0 + \frac{4}{3}\eta^2\Bigg] \\
+ \frac{1}{4f_9^2}\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \operatorname{diag}\left[(\pi^9)^2,\ (\pi^9)^2,\ (\pi^9)^2\right] \ .
\end{aligned} \tag{2.46}$$

Upon making the product and taking the trace, we find:

$$\begin{aligned}
\mathcal{L}_{\text{mass}} = &-\frac{v^3}{f_\pi^2}\left[2(m_u+m_d)\pi^+\pi^- + 2(m_u+m_s)K^+K^-\right] \\
&-\frac{v^3}{f_\pi^2}\left[2(m_d+m_s)\overline{K}^0K^0 + m_u\left(\frac{1}{\sqrt{3}}\eta + \pi^0 + \frac{f_\pi}{f_9}\pi^9\right)^2\right] \\
&-\frac{v^3}{f_\pi^2}\left[m_d\left(\frac{1}{\sqrt{3}}\eta - \pi^0 + \frac{f_\pi}{f_9}\pi^9\right)^2 + \frac{4}{3}m_s\left(\eta + \frac{f_\pi}{f_9}\pi^9\right)\right] \ .
\end{aligned} \tag{2.47}$$

Finally, the masses of the nine Nambu-Goldstone bosons are as follows:
Firstly, we have the charged NG bosons:

$$\begin{aligned}
m_{\pi^\pm}^2 &= 2\frac{v^3}{f_\pi^2}(m_u+m_d) \ , \\
m_{K^\pm}^2 &= 2\frac{v^3}{f_\pi^2}(m_u+m_s) \ , \\
m_{\overline{K}^0K^0}^2 &= 2\frac{v^3}{f_\pi^2}(m_d+m_s) \ .
\end{aligned} \tag{2.48}$$

Now, for the neutral NG bosons, to simplify the calculation, we set $m_u = m_d = m << m_s$. With this consideration, we obtain:

$$\begin{aligned}
m_{\pi^0}^2 &= 4m\frac{v^3}{f_\pi^2} \ , \\
m_\eta^2 &= \frac{8}{3}m_s\frac{v^3}{f_\pi^2}\left(1 + \frac{3}{4}\frac{f_\pi^2}{f_9^2}\right) \ , \\
m_{\pi^9}^2 &= \frac{9f_\pi^2/f_9^2}{4+3f_\pi^2/f_9^2}m_{\pi^0}^2 \ .
\end{aligned} \tag{2.49}$$

It is evident from (2.49) that if we set $f_9 \to 0$, then the maximum value for the mass of the ninth meson is $m_{\pi^9} = \sqrt{3}m_{\pi^0}$. Consequently, the mass of the ninth meson is bound to the mass of the neutral pion. However, again, the issue arises from the fact that the π^9 meson does not seem to exist in nature; no experimental observation has ever confirmed its presence. This discrepancy is known as the U(1) problem.

2.3 U$(1)_A$ anomaly

In this section, we delve further into the U(1) axial symmetry and its anomalous nature. As previously discussed, if U(1) axial were a true symmetry of QCD, our theory would lack a crucial meson component, given rise to the U(1) problem. This is because the symmetry must be spontaneously broken, giving rise to an expected Nambu-Goldstone (NG) boson. To address this issue, it becomes essential to demonstrate that U(1) axial is not, in fact, a genuine symmetry of our theory.

The most straightforward approach to illustrate this anomalous nature is to start with a U(1) gauge theory featuring massless Dirac fields. The lagrangian for such a theory is given by:

$$\mathcal{L} = i\overline{\Psi}\not{\mathcal{D}}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \ , \tag{2.50}$$

where $\not{\mathcal{D}} = \gamma^\mu \mathcal{D}_\mu$, and $\mathcal{D}_\mu = \partial_\mu - igA_\mu$. Note that this lagrangian bears resemblance to the QCD lagrangian with massless quarks, expressed in terms of the Dirac fields in (2.6). This lagrangian maintains invariance under a global symmetry, wherein our Dirac fields undergo transformations with the same phase as:

$$\begin{aligned} \Psi &\to e^{-i\alpha\gamma_5}\Psi \\ \overline{\Psi} &\to \overline{\Psi}e^{-i\alpha\gamma_5} \ . \end{aligned} \tag{2.51}$$

This global symmetry is our symmetry of interest and it is called axial U(1) symmetry. Calculating the Noether current associated with it we can see why this symmetry is called axial. The associated Noether current is defined by (A.17):

$$j_a^\mu \equiv -\left[\mathcal{L}g_\rho^\mu - \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\partial_\rho\phi\right]\frac{\delta x^\rho}{\delta\omega^a} - \frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]}\frac{\delta\phi}{\delta\omega^a} \ , \tag{2.52}$$

since there is no variation in the spacetime variables, the current simplifies to:

$$j_a^\mu = -\frac{\partial\mathcal{L}}{\partial[\partial_\mu\phi]}\frac{\delta\phi}{\delta\omega^a} \ . \tag{2.53}$$

This simplification can be directly adapted for our specific case, resulting in:

$$j_A^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\frac{\mathrm{d}\Psi}{\mathrm{d}\alpha} \ . \tag{2.54}$$

Applying our global U(1) transformation to (2.50) and subsequently substituting it into (2.54), we obtain:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\frac{\mathrm{d}\Psi}{\mathrm{d}\alpha} &= \frac{\partial}{\partial(\partial_\mu\Psi)}\left[i\overline{\Psi}\not{\mathcal{D}}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right]\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(e^{-i\alpha\gamma_5}\Psi\right) \ , \\ &= \frac{\partial}{\partial(\partial_\mu\Psi)}\left[i\overline{\Psi}\gamma^\mu(\partial_\mu - igA_\mu)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right](-i\gamma_5\Psi) \ , \\ &= \frac{\partial}{\partial(\partial_\mu\Psi)}\left[i\overline{\Psi}\gamma^\mu\partial_\mu\Psi\right](-i\gamma_5\Psi) \ , \\ &= (i\overline{\Psi}\gamma^\mu)(-i\gamma_5\Psi) \ , \\ &= \overline{\Psi}(x)\gamma^\mu\gamma_5\Psi(x) \ . \end{aligned} \tag{2.55}$$

This current represents an axial vector because its spatial component is odd under parity transformation. Because of this axial nature, our global symmetry is termed axial U(1) symmetry. It's important to mention that we have only explicitly shown the transformations for the terms that are somehow affected by this transformation. Overall, the transformations on the fields are going to be canceled out, essentially introducing mess in the calculations without affecting the final results.

Using Noether's Theorem, we expect $\partial_\mu j^\mu_A = 0$. However, we will find that the axial current exhibits an anomalous divergence:

$$\partial_\mu j^\mu_A(x) = -\frac{g^2}{16\pi^2} e^{\mu\nu\rho\sigma} \, \mathrm{Tr}\left[F_{\mu\nu} F_{\rho\sigma}\right] \ . \tag{2.56}$$

One might question the origin of this anomaly. It can be traced back to the functional measure of the fermion field, which, in general, is not gauge invariant.

We will derive (2.56) directly from the path integral using the Fujikawa method [23], and demonstrate its exactness. Initially, we will focus on the path integral over the Dirac field, treating the gauge field as a fixed background to be integrated later in our calculations. This approach leads to the following vacuum-to-vacuum transition amplitude:

$$Z(A) \equiv \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi \ e^{iS(A)} \ , \tag{2.57}$$

where $S(A)$ is the Dirac action, defined by:

$$S(A) \equiv \int d^4x \ \overline{\Psi} i \not{D} \Psi \ . \tag{2.58}$$

We can evaluate (2.57) as a functional determinant:

$$Z(A) = \det\left(i\not{D}\right) \ , \tag{2.59}$$

this equation requires some form of regularization, which will be addressed later. Now, let's transform our axial U(1) parameter α into a spacetime dependent parameter $\alpha(x)$. With this modification, our transformation takes the form:

$$\begin{aligned} \Psi &\to e^{-i\alpha(x)\gamma_5}\Psi \ , \\ \overline{\Psi} &\to \overline{\Psi} e^{-i\alpha(x)\gamma_5} \ . \end{aligned} \tag{2.60}$$

This transformation on the Dirac fields induces a corresponding transformation in the Dirac action and in the integration variables given in (2.57). Analyzing first how it affects the Dirac action:

$$\begin{aligned}
S'(A) &= \int d^4x \ \overline{\Psi} e^{-i\alpha(x)\gamma_5} i\not{D} e^{-i\alpha(x)\gamma_5} \Psi \ , \\
&= \int d^4x \ \overline{\Psi} e^{-i\alpha(x)\gamma_5} i\gamma^\mu \left(\partial_\mu - igA_\mu\right) e^{-i\alpha(x)\gamma_5} \Psi \ , \\
&= \int d^4x \ \overline{\Psi} e^{-i\alpha(x)\gamma_5} i\gamma^\mu \left[\partial_\mu \left(e^{-i\alpha(x)\gamma_5}\Psi\right) - igA_\mu e^{-i\alpha(x)\gamma_5}\Psi\right] \ , \\
&= \int d^4x \ \overline{\Psi} e^{-i\alpha(x)\gamma_5} i\gamma^\mu \left[\left(\partial_\mu e^{-i\alpha(x)\gamma_5}\right)\Psi + e^{-i\alpha(x)\gamma_5}\partial_\mu\Psi - igA_\mu e^{-i\alpha(x)\gamma_5}\Psi\right] \ , \\
&= \int d^4x \ \overline{\Psi} i\gamma^\mu \left[\left(-i\partial_\mu\alpha(x)\gamma_5\right)\Psi + \partial_\mu\Psi - igA_\mu\Psi\right] \ , \\
&= \int d^4x \ \overline{\Psi} i\gamma^\mu \left[-i\gamma_5\partial_\mu\alpha(x)\Psi + \left(\partial_\mu - igA_\mu\right)\Psi\right] \ , \\
&= \int d^4x \ \overline{\Psi} i\not{D}\Psi + \int d^4x \ \overline{\Psi}\gamma^\mu\gamma_5\Psi\partial_\mu\alpha(x) \ , \\
&= S(A) + \int d^4x \ j_A^\mu \partial_\mu\alpha(x) \ .
\end{aligned} \tag{2.61}$$

By integrating the second integral by parts, we can express it as:

$$S(A) \to S(A) - \int d^4x \ \alpha(x)\partial_\mu j_A^\mu \ . \tag{2.62}$$

Finally, we obtain:

$$\delta S = -\int d^4x \ \alpha(x)\partial_\mu j_A^\mu \ . \tag{2.63}$$

In classical physics, to maintain the invariance of the action, $\partial_\mu j_A^\mu = 0$, which implies that the axial current is conserved, and the axial U(1) symmetry is a genuine symmetry of our theory. However, in quantum mechanics, the axial current is not conserved. This was discovered by Adler, Bardeen, Bell, and Jackiw [24–27] through the analysis of the triangle diagram, where an incoming chiral current interacts with two outgoing gluons mediated by three fermions forming a triangle loop.

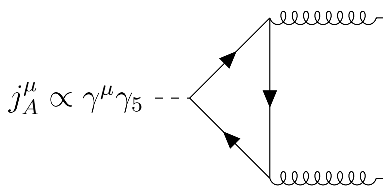


Figure 1 – Anomaly diagram

Fujikawa reinterpreted this phenomenon as a modification in (2.57) under a chiral transformation. To derive (2.56), it is essential to analyze the nature of this modification and how it occurs.

First, let's assume that $\mathcal{D}\overline{\Psi}\mathcal{D}\Psi$ is invariant under axial U(1) transformation. If this assumption is correct, we have:

$$Z'(A) = \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi \ e^{iS(A)} e^{-i\int d^4x \ \alpha(x)\partial_\mu j_A^\mu} \ , \tag{2.64}$$

which must be equal to the original expressions for $Z(A)$. This would imply that $\partial_\mu j_A^\mu = 0$ holds for quantum mechanics. However, the change in the measure $\mathcal{D}\overline{\Psi}\mathcal{D}\Psi$ under a axial U(1) transformation must be analysed thoroughly. The change in our variables is implemented by a functional matrix defined by [15]:

$$J(x, y) = \delta^4(x - y)e^{-i\alpha(x)\gamma_5} \ . \tag{2.65}$$

In a scenario where the path integral is over commuting variables (e.g., bosonic fields), we would obtain a Jacobian factor of $\det(J)$ for each of our transformations. However, in our case, we are integrating over fermionic fields (non-commuting variables). Consequently, we get a Jacobian factor of $\det(J)^{-1}$ for each of our transformations instead of $\det(J)$. Since we have two transformations, we obtain:

$$\mathcal{D}\overline{\Psi}\mathcal{D}\Psi \to (\det J)^{-2}\mathcal{D}\overline{\Psi}\mathcal{D}\Psi \ . \tag{2.66}$$

This implies that our crucial quantity $Z(A)$ under a U(1) transformation is, in fact,

$$Z'(A) = \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi \ (\det J)^{-2}e^{iS(A)}e^{-i\int \ d^4x \ \alpha(x)\partial_\mu j_A^\mu} \ . \tag{2.67}$$

Here, we can use certain identities to calculate $\det J$. First we have:

$$\det e^A = e^{\mathrm{Tr}[A]} \ , \tag{2.68}$$

for any square matrix A. If we denote our Jacobian as $J \equiv e^A$, we can express:

$$\det J = e^{\mathrm{Tr}[\ln J]} \tag{2.69}$$

Utilizing these identities, we obtain:

$$(\det J)^{-2} = \exp\left[2i \int \ d^4x \ \alpha(x) \,\mathrm{Tr}\Big[\delta^4(x - x)\gamma_5\Big]\right] \ , \tag{2.70}$$

where the trace is taken over both spin and group indices. Similar to (2.59), a form of regularization is required in this case as well.

One method to regularize this expression is by replacing the delta function with a Gaussian. However, for the regularization to be consistent with our theory, the Gaussian must be gauge invariant, and our regularization scheme must be compatible with (2.59). Taking these considerations into account, we make the following replacement [15]:

$$\delta^4(x - y) \to e^{(i\not{D}_x)^2/M^2}\delta^4(x - y) \ , \tag{2.71}$$

where M is the regulator mass that we take to infinity at the end the calculations.

To evaluate (2.71), we express the delta function as a Fourier integral:

$$e^{(i\not{D}_x)^2/M^2}\delta^4(x - y) = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \ e^{(i\not{D}_x)^2/M^2}e^{ik(x-y)} \ . \tag{2.72}$$

Here, we can rewrite the Fourier integral more intelligently using the following identity:

$$f(\partial)e^{ikx} = e^{ikx}f(\partial + ik) \ , \tag{2.73}$$

in our case, we have $f(\partial) = e^{i\gamma^\mu \mathcal{D}_\mu}$, so the identity becomes:

$$e^{(i\not{\mathcal{D}}_x)^2/M^2}e^{ik(x-y)} = e^{ik(x-y)}e^{[i\gamma^\mu(\mathcal{D}_\mu + ik)]^2} \ , \tag{2.74}$$

thus, we find:

$$\int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \ e^{(i\not{\mathcal{D}}_x)^2/M^2}e^{ik(x-y)} = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \ e^{ik(x-y)}e^{(i\not{\mathcal{D}} - \not{k})^2/M^2} \ . \tag{2.75}$$

If we continue to explore this expression, we can rewrite the far right exponential argument as:

$$(i\not{\mathcal{D}} - \not{k})^2 = \not{k}^2 - i\Big\{\not{\mathcal{D}}, \not{k}\Big\} - \not{\mathcal{D}}^2 \ , \tag{2.76}$$

where $-i\Big\{\not{\mathcal{D}}, \not{k}\Big\} = -i\not{\mathcal{D}}\not{k} - i\not{k}\not{\mathcal{D}} = -2i\not{\mathcal{D}}\not{k}$, since $\Big[\not{\mathcal{D}}, \not{k}\Big] = 0$.

Continuing, we have:

$$\begin{aligned}
(i\not{\mathcal{D}} - \not{k})^2 &= -k^2 - i\{\gamma^\mu, \gamma^\nu\}k_\mu \mathcal{D}_\nu - \gamma^\mu\gamma^\nu \mathcal{D}_\mu \mathcal{D}_\nu \ , \\
&= -k^2 - i(-2g^{\mu\nu}I_4)k_\mu \mathcal{D}_\nu - (-g^{\mu\nu} - 2iS^{\mu\nu})\mathcal{D}_\mu \mathcal{D}_\nu \ , \\
&= -k^2 + 2ik^\nu \mathcal{D}_\nu + \mathcal{D}^2 + 2iS^{\mu\nu}\mathcal{D}_\mu \mathcal{D}_\nu \ , \\
&= -k^2 + 2ik \cdot \mathcal{D} + \mathcal{D}^2 + 2iS^{\mu\nu}\mathcal{D}_\mu \mathcal{D}_\nu \ ,
\end{aligned} \tag{2.77}$$

here we used (note that the metric here is (-,+,+,+)):

$$\begin{aligned}
S^{\mu\nu} &\equiv \frac{i}{2}\gamma^\mu\gamma^\nu \ , \\
\gamma^\mu\gamma^\nu &= \frac{1}{2}\left[\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]\right] = -g^{\mu\nu} - 2iS^{\mu\nu} \ , \\
\not{k}\not{k} &= k^\mu k^\nu \gamma_\mu \gamma_\nu \ , \\
&= k^\mu k^\nu \frac{1}{2}\left[\{\gamma_\mu, \gamma_\nu\} + [\gamma_\mu, \gamma_\nu]\right] \ , \\
&= k^\mu k^\nu \frac{1}{2}\{\gamma_\mu, \gamma_\nu\} \ , \\
&= k^\mu k^\nu(-g_{\mu\nu}) = -k^2 \ .
\end{aligned} \tag{2.78}$$

In the term $2iS^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu$, we can utilize the antisymmetry of $S^{\mu\nu}$ to make the following substitution:

$$\begin{aligned}
S^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu &= \frac{1}{2}\left[S^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu + S^{\nu\mu}\mathcal{D}_\nu\mathcal{D}_\mu\right] \ , \\
&= \frac{S^{\mu\nu}}{2}\left[\mathcal{D}_\mu\mathcal{D}_\nu - \mathcal{D}_\nu\mathcal{D}_\mu\right] \ , \\
&= \frac{S^{\mu\nu}}{2}[\mathcal{D}_\mu, \mathcal{D}_\nu] \ , \\
&= \frac{S^{\mu\nu}}{2}[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] \ , \\
&= \frac{S^{\mu\nu}}{2}\Big[\partial_\mu\partial_\nu - \partial_\mu(igA_\nu) - igA_\mu\partial_\nu - g^2A_\mu A_\nu \\
&\quad -\partial_\nu\partial_\mu + \partial_\nu(igA_\mu) + igA_\nu\partial_\mu + g^2A_\nu A_\mu\Big] \ , \\
&= \frac{S^{\mu\nu}}{2}\left[-ig\partial_\mu A_\nu - igA_\nu\partial_\mu - igA_\mu\partial_\nu + ig\partial_\nu A_\mu + igA_\mu\partial_\nu + igA_\nu\partial_\mu\right] \ , \\
&= \frac{S^{\mu\nu}}{2}\left[-ig\partial_\mu A_\nu + ig\partial_\nu A_\mu\right] \ , \\
&= \frac{S^{\mu\nu}}{2}\left[-ig(\partial_\mu A_\nu - \partial_\nu A_\mu)\right] \ , \\
&= \frac{S^{\mu\nu}}{2}\left[-igF_{\mu\nu}\right] \ ,
\end{aligned} \tag{2.79}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Applying this result to (2.77), we obtain:

$$(i\not{\mathcal{D}} - \not{k})^2 = -k^2 + 2ik\cdot\mathcal{D} + \mathcal{D}^2 + gS^{\mu\nu}F_{\mu\nu} \ . \tag{2.80}$$

Finally, applying this to (2.75), we find:

$$\begin{aligned}
\delta^4(x-y) &\to \int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\ e^{ik(x-y)}e^{(i\not{\mathcal{D}}-\not{k})^2/M^2} \ , \\
&= \int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\ e^{ik(x-y)}\exp\left[\frac{-k^2 + 2ik\cdot\mathcal{D} + \mathcal{D}^2 + gS^{\mu\nu}F_{\mu\nu}}{M^2}\right] \ .
\end{aligned} \tag{2.81}$$

Next, let's rescale k by M ($k \to Mk$), yielding

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d^4(Mk)}{(2\pi)^4}\ e^{ik(x-y)}\exp\left[\frac{-M^2k^2 + 2i(Mk)\cdot\mathcal{D} + \mathcal{D}^2 + gS^{\mu\nu}F_{\mu\nu}}{M^2}\right] \ , \\
&= M^4\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\ e^{ik(x-y)}e^{-k^2}\exp\left[\frac{2ik\cdot\mathcal{D}}{M} + \frac{\mathcal{D}^2}{M^2} + \frac{gS^{\mu\nu}F_{\mu\nu}}{M^2}\right] \ .
\end{aligned} \tag{2.82}$$

Note that, in (2.70), the delta function is $\delta^4(x-x)$, which is:

$$\delta^4(x-x) \to M^4\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\ e^{-k^2}\exp\left[\frac{2ik\cdot\mathcal{D}}{M} + \frac{\mathcal{D}^2}{M^2} + \frac{gS^{\mu\nu}F_{\mu\nu}}{M^2}\right] \ . \tag{2.83}$$

Therefore, substituting this rescaled expression into the trace of (2.70), we have:

$$\begin{aligned}
\mathrm{Tr}\left[\delta^4(x-x)\gamma_5\right] \to M^4&\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\ e^{-k^2} \\
&\times \mathrm{Tr}\left[\exp\left[\frac{2ik\cdot\mathcal{D}}{M} + \frac{\mathcal{D}^2}{M^2} + \frac{gS^{\mu\nu}F_{\mu\nu}}{M^2}\right]\gamma_5\right] \ .
\end{aligned} \tag{2.84}$$

Now, let's expand the exponential in (2.84) in terms of M up to $\mathcal{O}(M^{-4})$. Since, there is a factor M^4 multiplying the integral, these contributions do not vanish as we take $M \to \infty$:

$$\begin{aligned}\exp\left[\frac{2ik\cdot\mathcal{D}}{M} + \frac{\mathcal{D}^2}{M^2} + \frac{gS^{\mu\nu}F_{\mu\nu}}{M^2}\right] = 1 + \frac{2ik\cdot\mathcal{D}}{M} + \frac{\mathcal{D}^2}{M^2} + \frac{gS^{\mu\nu}F_{\mu\nu}}{M^2} \\ + \frac{1}{2}\left(\frac{2ik\cdot\mathcal{D}}{M} + \frac{\mathcal{D}^2}{M^2} + \frac{gS^{\mu\nu}F_{\mu\nu}}{M^2}\right)^2 \ .\end{aligned} \tag{2.85}$$

The trace over spin indices will vanish unless there are four or more gamma matrices multiplying γ_5. This property arises from the properties of γ_5 :

$$\begin{aligned}&\mathrm{Tr}[\gamma_5] = \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_5] = 0 \ , \\ &\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = -4i\epsilon^{\mu\nu\rho\sigma} \ ,\end{aligned} \tag{2.86}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-civita symbol. With these considerations, the only non-zero contribution from our expansion is of the form:

$$\frac{(gS^{\mu\nu}F_{\mu\nu})^2}{2M^4} \ .$$

Therefore, we have:

$$\mathrm{Tr}\Big[\delta^4(x-x)\gamma_5\Big] \to \frac{1}{2}g^2\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\ e^{-k^2}\,\mathrm{Tr}\left[F_{\mu\nu}F_{\rho\sigma}\right]\mathrm{Tr}\left[S^{\mu\nu}S^{\rho\sigma}\gamma_5\right] \ , \tag{2.87}$$

using the definition of $S^{\mu\nu}$, we have:

$$\begin{aligned}\mathrm{Tr}[S^{\mu\nu}S^{\rho\sigma}\gamma_5] &= \mathrm{Tr}\left[\frac{i}{2}\gamma^\mu\gamma^\nu\frac{i}{2}\gamma^\rho\gamma^\sigma\gamma_5\right] \ , \\ &= -\frac{1}{4}\,\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] \ , \\ &= -\frac{1}{4}(-4i\epsilon^{\mu\nu\rho\sigma}) \ , \\ &= i\epsilon^{\mu\nu\rho\sigma} \ .\end{aligned} \tag{2.88}$$

Finally, we find:

$$\mathrm{Tr}\Big[\delta^4(x-x)\gamma_5\Big] \to \frac{1}{2}g^2\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\ e^{-k^2}(i\epsilon^{\mu\nu\rho\sigma})\,\mathrm{Tr}[F_{\mu\nu}F_{\rho\sigma}] \ . \tag{2.89}$$

At this point, we analytically continue to Euclidean space, introducing a factor of i. Then, upon solving our four Gaussian integrals, each contributes a factor of $\pi^{1/2}$, we obtain a total factor of $(\pi^{1/2})^4 = \pi^2$. Taking these considerations into account, we end up with:

$$\mathrm{Tr}\Big[\delta^4(x-x)\gamma_5\Big] \to -\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[F_{\mu\nu}F_{\rho\sigma}] \ . \tag{2.90}$$

Now, we can apply these results to our original equation (2.70), leading to:

$$(\det J)^{-2} = \exp\left[-\frac{ig^2}{16\pi^2}\int d^4x\ \alpha(x)\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[F_{\mu\nu}(x)F_{\rho\sigma}(x)]\right] \ . \tag{2.91}$$

Finally, substituting this result in (2.67), we obtain::

$$
\begin{aligned}
Z'(A) = &\int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi \; e^{iS(A)} \\
&\times \exp\left[-i\int d^4x \; \alpha(x)\left(\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}[F_{\mu\nu}(x)F_{\rho\sigma}(x)] + \partial_\mu j_A^\mu(x)\right)\right] \; ,
\end{aligned}
\tag{2.92}
$$

which must be equal to the original $Z(A)$ in (2.57). This is because $Z(A)$ is a fundamental quantity in every Quantum Field Theory (QFT); it represents the vacuum-to-vacuum transition amplitude and, as such, cannot be affected by either a real or an anomalous symmetry of the theory.

We can clearly see in (2.92) that our associated Noether current has the same divergence as stated in (2.56):

$$
\partial_\mu j_A^\mu = -\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[F_{\mu\nu}F_{\rho\sigma}\right] \; . \tag{2.93}
$$

Because our derivation did not rely on an expansion in powers of g, our result is exact; there are no higher-order corrections. This result is known as the Adler-Bardeen theorem [28]. The significance of this result lies in its fundamental role in the U(1) problem. Our discussion has revolved around the axial U(1) transformation and whether it is a true or false symmetry of our theory. If U(1) is a true symmetry, it must be spontaneously broken, leading to a missing meson in our experimental results. On the other hand, if U(1) is not a true symmetry, the U(1) problem is solved.

Now, with the expression for the divergence of the associated Noether current of our U(1) transformation, if this expression turns out to be a non-zero contributing term to the action, we have resolved the U(1) problem.

Looking at (2.63), we can see that the chiral anomaly affects the action in the following way:

$$
\begin{aligned}
\delta S &= -\alpha\int d^4x \; \partial_\mu j_A^\mu \; , \\
&= \frac{\alpha g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4x \;\; \mathrm{Tr}\left[F_{\mu\nu}F_{\rho\sigma}\right] \; .
\end{aligned}
\tag{2.94}
$$

Notice that α is a constant again. At first glance, this might seem to prove that U(1) axial is not a true symmetry. However, a thorough analysis is still needed because we can rewrite:

$$
\begin{aligned}
\frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[F_{\mu\nu}F_{\rho\sigma}\right] &= \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\,\mathrm{Tr}\left[A_\nu\partial_\rho A_\sigma - \frac{2}{3}igA_\nu A_\rho A_\sigma\right] \; , \\
&= \frac{g^2}{4\pi^2}\partial_\mu W^\mu \; ,
\end{aligned}
\tag{2.95}
$$

where,

$$
W^\mu = \epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[A_\nu\partial_\rho A_\sigma - \frac{2}{3}igA_\nu A_\rho A_\sigma\right] \; . \tag{2.96}
$$

Applying this in (2.94), we have:

$$\delta S = \frac{\alpha g^2}{4\pi^2} \int d^4x \, \partial_\mu W^\mu \ . \tag{2.97}$$

Now, the variation of the action results in a pure surface integral, and one might think that $\int d^4x \, \partial_\mu W^\mu = 0$. However, in a gauge theory, Gerard 't Hooft [29] proved that this is not true. Gerard 't Hooft did significant work on understanding the QCD vacuum structure and instantons. He discussed this topic in his paper titled "Symmetry Breaking through Bell-Jackiw Anomalies" [29]. In this paper, he addressed the axial U(1) anomaly and its implications in the context of QCD. In the next section, we will delve deeper to understand the intricacies of the QCD vacuum structure.

2.4 QCD vacuum structure

To elucidate 't Hooft's resolution of the U(1) problem, it is imperative to delve into the realms of instantons and θ-vacua. Our journey begins within the framework of SU(2) gauge theory, focusing solely on the gauge fields. It is crucial to recall that within this theory, a gauge transformation assumes the specific form:

$$A_\mu \to U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger \ , \tag{2.98}$$

where g is the gauge coupling. The condition $F^a_{\mu\nu} = 0$ represents the classical field configuration corresponding to the ground (vacuum) state. This condition implies that A_μ is zero at vacuum state, which further implies that, at vacuum state, it is a gauge transformation of zero:

$$\begin{aligned} A_\mu \to A^a_\mu T^a &= U(0)U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger \ , \\ &= \frac{i}{g} U \partial_\mu U^\dagger \end{aligned} \tag{2.99}$$

where T^a are the SU(2) symmetry generators, and U is a 2×2 unitary matrix that is a function of spacetime.

By focusing exclusively on time-independent gauge transformations, we consider U as a function of spatial coordinates, denoted as $U(\vec{x})$, which allows us to establish a temporal gauge condition $A_0 = 0$. We then impose a boundary condition, where $U(\vec{x})$ approaches a specific constant matrix as $|\vec{x}| \to \infty$. This boundary condition effectively introduces a spatial point at infinity, where U attains a definite value. In essence, this mapping of spatial infinity to the vacuum of our theory signifies that the topology of the spatial boundary is intricately linked with the topology of the space of vacuum field configurations. Consequently, both the spatial geometry and the vacuum state exhibit the topology of a three-sphere, denoted as S^3.

The field configurations determined by $U(\vec{x})$ are categorized by an integral quantity known as the winding number, denoted as n. This winding number characterizes the mapping in such a way that it counts how many times we wind the vacuum sphere for every full wind around the spatial sphere. Notably, a negative winding number implies that the vacuum winding occurs in the opposite direction compared to the spatial winding.

To understand this phenomenon, it's essential to recognize that any $SU(2)$ matrix U can be expressed as follows:

$$U = a_0 + i\vec{a} \cdot \vec{\sigma} \ , \tag{2.100}$$

where $a_\mu \equiv (a_0, \vec{a}) \in \mathbb{R}$, and the constraint $a_\mu a_\mu = 1$ defines a point in S^3, known as the vacuum three-sphere. In our specific scenario, the matrix $U(\vec{x})$ serves as a map that connects points on the spatial three-sphere to corresponding points on the vacuum three-sphere.

Similar to our treatment of the vacuum three-sphere, we find it convenient to characterize the spatial three-sphere using a Euclidean four-vector denoted as $z_\mu \equiv (z_0, \vec{z})$, satisfying the unit length condition $z_\mu z_\mu = 1$. Through stereographic projection, we can establish a clear relationship between z_μ and $\vec{x}$ [15]. This construction allows us to:

$$\begin{aligned}
\hat{z} &= \frac{\vec{z}}{|\vec{z}|} = \hat{x} \ , \\
|\vec{z}| &= \frac{2r}{1+r^2} \ , \\
z_0 &= \frac{1-r^2}{1+r^2} \ , \\
r &= |\vec{x}| \ .
\end{aligned} \tag{2.101}$$

We can now construct a specific example of a map from the spatial S^3 to the vacuum S^3 with a winding number n by aligning the polar angles of a_μ with those of z_μ and setting the azimuthal angle of a_μ to be n times the azimuthal angle of z_μ. Here, the polar angles vary within the range $0 \to \pi$, and the azimuthal angles span from $0 \to 2\pi$.

The winding number of a given smooth map $U(x)$ can be expressed as [15]:

$$n = -\frac{1}{24\pi^2} \int d^3x \; \epsilon^{ijk} \, \mathrm{Tr}\Big[(U\partial_i U^\dagger)(U\partial_j U^\dagger)(U\partial_k U^\dagger)\Big] \ , \tag{2.102}$$

which is invariant under change of variables, a property ensured by the cancellation of the Jacobian determinant $J(d^3x)$ with $J(\partial_1\partial_2\partial_3)$.

Now, a fundamental question arises: are these vacuum configurations gauge equivalent? In other words, do they correspond to a single degenerate quantum vacuum state, or are there multiple quantum vacuum states distinguished by the winding number? To answer this question, it is essential to investigate whether the winding number is a topologically invariant quantity. This requires examining its behavior under infinitesimal deformations. If the winding number, denoted as n, proves to be a topological invariant, the theory encompasses more than one quantum vacuum state, each distinguished by n. On the other hand, if n fails to be a topological invariant, the theory possesses only a single quantum vacuum state.

Initially, let's contemplate an infinitesimal deformation of U, represented by $U \to U + \delta U$. To assess the impact of this deformation on (2.102), it becomes necessary to know $\delta U^\dagger$ and $\delta(U\partial_k U^\dagger)$. Notably, due to the unitary nature of $U(x)$, we deduce $\delta(U^\dagger U) = 0$. As a consequence of this property, we obtain:

$$\begin{aligned}\delta(U^\dagger U)U^\dagger &= (\delta U^\dagger U + U^\dagger \delta U)U^\dagger \ ,\\ &= \delta U^\dagger U U^\dagger + U^\dagger \delta U U^\dagger \ ,\\ &= \delta U^\dagger + U^\dagger \delta U U^\dagger = 0 \ .\end{aligned} \tag{2.103}$$

From this obtained expression, we find:

$$\delta U^\dagger = -U^\dagger \delta U U^\dagger \ . \tag{2.104}$$

Hence, we can calculate $\delta(U\partial_k U^\dagger)$:

$$\begin{aligned}\delta(U\partial_k U^\dagger) &= \delta U \partial_k U^\dagger + U\partial_k \delta U^\dagger \ ,\\ &= \delta U \partial_k U^\dagger - U\partial_k (U^\dagger \delta U U^\dagger) \ ,\\ &= \delta U \partial_k U^\dagger - U\partial_k U^\dagger \delta U U^\dagger - U U^\dagger \partial_k \delta U U^\dagger - U U^\dagger \delta U \partial_k U^\dagger \ ,\\ &= \delta U \partial_k U^\dagger - U\partial_k U^\dagger \delta U U^\dagger - U U^\dagger \partial_k \delta U U^\dagger - \delta U \partial_k U^\dagger \ ,\\ &= -U\partial_k U^\dagger \delta U U^\dagger - U U^\dagger \partial_k \delta U U^\dagger \ ,\\ &= -U(\partial_k U^\dagger \delta U + U^\dagger \partial_k \delta U)U^\dagger \ ,\\ &= -U\partial_k (U^\dagger \delta U) U^\dagger \ .\end{aligned} \tag{2.105}$$

Since $\delta(U\partial_i U^\dagger)$ and $\delta(U\partial_j U^\dagger)$ contribute the same manner as $\delta(U\partial_k U^\dagger)$, it suffices to consider only the latter term when multiplying the total variation by 3. Thus, the variation in the winding number can be expressed as follows:

$$\begin{aligned}\delta n &= -\frac{1}{24\pi^2}\int d^3x\ \epsilon^{ijk}\delta\,\mathrm{Tr}\Big[(U\partial_i U^\dagger)(U\partial_j U^\dagger)(U\partial_k U^\dagger)\Big] \ ,\\ &= -\frac{3}{24\pi^2}\int d^3x\ \epsilon^{ijk}\,\mathrm{Tr}\Big[(U\partial_i U^\dagger)(U\partial_j U^\dagger)\delta(U\partial_k U^\dagger)\Big] \ .\end{aligned} \tag{2.106}$$

Continuing the calculations and integrating by parts, we find:

$$\begin{aligned}\delta n &= +\frac{3}{24\pi^2}\int d^3x\ \epsilon^{ijk}\,\mathrm{Tr}\Big[(U\partial_i U^\dagger)(U\partial_j U^\dagger)U\partial_k(U^\dagger \delta U)U^\dagger\Big] \ ,\\ &= +\frac{3}{24\pi^2}\int d^3x\ \epsilon^{ijk}\,\mathrm{Tr}\Big[(\partial_i U^\dagger)(U\partial_j U^\dagger)U\partial_k(U^\dagger \delta U)\Big] \ ,\\ &= +\frac{3}{24\pi^2}\int d^3x\ \epsilon^{ijk}\,\mathrm{Tr}\Big[\partial_k\Big((\partial_i U^\dagger)(U\partial_j U^\dagger)U U^\dagger \delta U\Big)\\ &\qquad\qquad -\partial_k\Big((\partial_i U^\dagger)(U\partial_j U^\dagger)U\Big)U^\dagger \delta U\Big] \ ,\end{aligned} \tag{2.107}$$

In reaching the last line, we employed the cyclical property of traces and applied the chain rule. Now, integrating by parts allows us to eliminate the terms with two derivatives acting on a single U, which vanish upon contraction with ϵ^{ijk}. Furthermore, the surface term becomes zero due to $\delta U = 0$ at the boundary. Consequently, the remaining terms are:

$$\begin{aligned}\delta n = \frac{3}{24\pi^2} \int d^3x \ \epsilon^{ijk} \operatorname{Tr} \Big[&(\partial_i U^\dagger)(\partial_k U)(\partial_j U^\dagger)\delta U \\ &+(\partial_i U^\dagger)(U\partial_j U^\dagger)(\partial_k U U^\dagger)\delta U \Big] \ ,\end{aligned} \tag{2.108}$$

we can rewrite:

$$\begin{aligned}\partial_j(UU^\dagger) &= \partial_j U U^\dagger + U\partial_j U^\dagger = 0 \rightarrow U\partial_j U^\dagger = -\partial_j U U^\dagger \ , \\ \partial_k(UU^\dagger) &= \partial_k U U^\dagger + U\partial_k U^\dagger = 0 \rightarrow \partial_k U U^\dagger = -U\partial_k U^\dagger \ ,\end{aligned} \tag{2.109}$$

and apply on (2.108), we obtain:

$$\begin{aligned}\delta n &= \frac{3}{24\pi^2} \int d^3x \ \epsilon^{ijk} \operatorname{Tr} \Big[(\partial_i U^\dagger)(\partial_k U)(\partial_j U^\dagger)\delta U \\ &\qquad +(\partial_i U^\dagger)(-\partial_j U U^\dagger)(-U\partial_k U^\dagger)\delta U \Big] \ , \\ &= \frac{3}{24\pi^2} \int d^3x \ \epsilon^{ijk} \operatorname{Tr} \Big[(\partial_i U^\dagger)(\partial_k U)(\partial_j U^\dagger)\delta U \\ &\qquad +(\partial_i U^\dagger)(\partial_j U)(\partial_k U^\dagger)\delta U \Big] \ .\end{aligned} \tag{2.110}$$

Due to the symmetry under $j \leftrightarrow k$, this expression vanishes upon contraction with the Levi-Civita tensor ϵ^{ijk}. Consequently, we conclude that:

$$\delta n = 0 \ . \tag{2.111}$$

Hence, the winding number n is indeed a topological invariant. Therefore, as mentioned earlier, the theory encompasses an infinite number of distinct quantum vacuum states, each labeled by n, and these states are separated by energy barriers.

Another perspective to comprehend this phenomenon is as follows: suppose $\tilde{U}(\vec{x})$ and $U(\vec{x})$ cannot be smoothly deformed into each other. Both vector potentials $\tilde{A}_\mu$ and A_μ are gauge transformations of zero, necessitating $\tilde{F}_{\mu\nu}$ and $F_{\mu\nu}$ to vanish. However, any attempt to smoothly deform $\tilde{A}_\mu$ into A_μ requires passing through vector potentials that are not gauge transformations of zero. Consequently, $F'_{\mu\nu} \neq 0$, implying nonzero energy. This discrepancy creates an energy barrier between the field configurations $\tilde{A}_\mu$ and A_μ, establishing them as distinct quantum vacuum states.

Thus far, we have established that SU(2) gauge theory exhibits an infinite set of vacuum states distinguished by an integer n (winding number) and separated by energy barriers. The next crucial inquiry is whether our field can transition between these distinct vacuum states and, if so, whether this transition can occur with a finite amount of energy. To explore this question, we initiate our analysis with the tunneling amplitude between two vacuum states $|n\rangle$ and $|n'\rangle$, where n and n' represent their respective winding numbers. This amplitude is expressed as:

$$\langle n'| H |n\rangle \sim e^{-S} \ , \tag{2.112}$$

where S is the Euclidean action corresponding to a classical field solution that mediates between the two distinct field configurations. In scalar field theory, where the solution is independent of $\vec{x}$, the action scales with the volume of space. Consequently, at spatial infinity, the tunneling amplitude vanishes. However, in the context of our SU(2) gauge theory, constructing a classical solution to the Euclidean field equations becomes more intricate. We need a solution that mediates between vacuum states characterized by different winding numbers while maintaining an action that remains fixed (depending solely on the difference between the winding numbers) and finite even in the infinite-volume limit.

To demonstrate that $\langle n'| H |n\rangle$ depends solely on $n' - n$, we perform a gauge transformation $\mathcal{U}_k$, characterized by a winding number k. The action of $\mathcal{U}_k(x)$ on a field configuration with winding number n transforms it into a configuration with winding number $n + k$. This transformation occurs because the product of a map with winding number n and another map with winding number k is equivalent to a single map with winding number $n + k$. To prove this, it is more convenient to analyze maps that go from $S^1 \to S^1$:

If we consider two maps: $U_n(\phi)$ with winding number n and $U_k(\phi)$ with winding number k. We deform $U_n(\phi)$ to equal one for $0 < \phi < \pi$ and $U_k(\phi)$ to equal one for $\pi < \phi < 2\pi$. The winding numbers for both configurations are then:

$$\begin{aligned} n &= \frac{i}{2\pi}\int_0^{\pi} d\phi \ U_n \partial_\phi U_n^\dagger \ , \\ k &= \frac{i}{2\pi}\int_{\pi}^{2\pi} d\phi \ U_k \partial_\phi U_k^\dagger \ . \end{aligned} \tag{2.113}$$

It is straightforward to observe that if U_n goes to 1 from 0 to π, then $\partial_\phi U_n^\dagger = 0$ within the same region. The same applies for U_k in its respective region, leading to $\partial_\phi U_k^\dagger = 0$ from π to 2π. Consequently, they do not contribute to the integral in their respective region. However, when we make the product $U_n U_k$, it equals U_k from 0 to π and U_n from π to 2π. Therefore, when integrating the winding number for $U_n U_k$, we obtain k from 0 to π and n from π to 2π, resulting in a combined winding number of $n + k$.

In quantum theory, the gauge transformation is implemented by the unitary operator $\mathcal{U}_k$. Having established the property outlined above, we obtain:

$$\mathcal{U}_k |n\rangle = |n + k\rangle \ . \tag{2.114}$$

On the contrary, the Yang-Mills Hamiltonian, constructed from field strengths, must remain invariant under time-independent gauge transformations. Hence,

$$\mathcal{U}_k H \mathcal{U}_k^\dagger = H \ . \tag{2.115}$$

Now, equipped with these tools, we can analyze the impact of this gauge transformation on the tunneling amplitude$\langle n'| H |n\rangle$. As $\mathcal{U}_k$ is unitary, we can introduce identities inside the bracket using $I = \mathcal{U}_k^\dagger \mathcal{U}_k$. Utilizing (2.114) and (2.115), we obtain

$$\begin{aligned} \langle n'| H |n\rangle &= \langle n'| \mathcal{U}_k^\dagger \mathcal{U}_k H \mathcal{U}_k^\dagger \mathcal{U}_k |n\rangle \ , \\ &= \langle n'| \mathcal{U}_k^\dagger H \mathcal{U}_k |n\rangle \ , \\ &= \langle n'+k| H |n+k\rangle \ . \end{aligned} \tag{2.116}$$

Finally, we can conclude that the tunneling amplitude depends solely on $n' - n$.

As the tunneling amplitude $\langle n'| H |n\rangle = f(n'-n) \sim e^{-S}$, θ-vacua of the following form are eigenstates of the Hamiltonian:

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{-in\theta} |n\rangle \ . \tag{2.117}$$

We can observe this by analyzing $\langle n'| H |\theta\rangle$. If $|\theta\rangle$ are eigenstates of H, then we have $\langle n'| H |\theta\rangle = \langle n'|\theta\rangle E_\theta$, where E_θ are the eigenvalues. Therefore, we have:

$$\begin{aligned} \langle n'| H |\theta\rangle &= \sum_n e^{-in\theta} \langle n'| H |n\rangle \ , \\ &= \sum_n e^{-in\theta} f(n'-n) \ , \\ &= \sum_m e^{-i(m+n')\theta} f(-m) \ , \\ &= e^{-in'\theta} \sum_m e^{-im\theta} f(-m) \ , \\ &= \langle n'|\theta\rangle E_\theta \ , \end{aligned} \tag{2.118}$$

where we have replaced the summation variable n with $n' + m$. We find the eigenvalues E_θ to be:

$$E_\theta = \sum_m e^{-im\theta} f(-m) \ . \tag{2.119}$$

If we use Euler's formula on (2.119), we obtain:

$$\begin{aligned} E_\theta &= \sum_m e^{-im\theta} f(-m) \ , \\ &= \sum_m (\cos m\theta - i \sin m\theta) f(-m) \ , \\ &= \sum_m \cos m\theta f(-m) \ , \end{aligned} \tag{2.120}$$

note that the eigenvalues only take real values. Therefore, we have concluded that θ-vacua are eigenstates of H, and the energies must be a periodic, even function of θ. Moreover, the eigenvalues of H should scale with the spatial volume V. In dimensional grounds, we obtain:

$$H\left|\theta\right\rangle = V\Lambda_{QCD}^{4} f(\theta)\left|\theta\right\rangle , \tag{2.121}$$

where Λ_{QCD} is the scale at which the gauge coupling becomes strong. As established earlier, $f(\theta)$ must possess the following properties:

$$\begin{aligned} f(\theta+2\pi) &= f(\theta) \ , \\ f(-\theta) &= f(\theta) \ . \end{aligned} \tag{2.122}$$

Even though we have demonstrated that the energy of the θ-vacuum is proportional to $\cos\theta$, we retain the freedom to add a constant to the Hamiltonian, ensuring that the state with $\theta = 0$ corresponds to zero energy. Consequently, it is anticipated that $f(\theta)$ attains its minimum at $\theta = 0$. It is crucial to emphasize that the topological properties of the gauge fields remain independent of the value of the coupling constant. Therefore, this analysis is applicable across all energy scales.

Now, let's delve into the solutions of the Euclidean field equations:
At euclidean time $x_0 = -T$, we set the vectorial field to:

$$A_\mu(\vec{x}) = \frac{i}{g}U_-(\vec{x})\partial_\mu U_-^\dagger(\vec{x}) \ , \tag{2.123}$$

where $U_-(\vec{x})$ has winding number n_-. Similarly, at $x_0 = +T$, we set:

$$A_\mu(\vec{x}) = \frac{i}{g}U_+(\vec{x})\partial_\mu U_+^\dagger(\vec{x}) \ , \tag{2.124}$$

where $U_+(\vec{x})$ has winding number n_+. We further impose the following boundary condition: for $|\vec{x}| = R$ and $-T \leq x_0 \leq +T$, we set $A_\mu = 0$, which is equivalent to $\partial_\mu U^\dagger = 0$. This condition implies that $U(\vec{x})$ converges to a constant matrix at $|\vec{x}| = R$. Note that, as required by the problem, T and R will be taken to infinity at the end of the calculations.

In the preceding paragraph, we specified $U(x)$ on a cylindrical boundary in 4-dimensional spacetime (2). This boundary topologically forms a 3-sphere, aligning with the problem's context, validating our previous conclusions. Consequently, the winding number of the map on this 3-sphere is $n_+ - n_-$.

We can observe this using a similar reasoning as before when we saw that the product of two maps with different winding numbers results in a map with a winding number equal to the sum of the individual winding numbers. Utilizing (2.102), we deduce that the boundary walls do not contribute since $\partial_\mu U^\dagger = 0$ there. However, each of the caps of the cylinder will contribute with their respective winding numbers. At this point, it might seem like we obtain a winding number equal to $n_+ + n_-$. However, it's crucial to pay attention to the orientation of the caps. The orientation of the lower cap is reversed

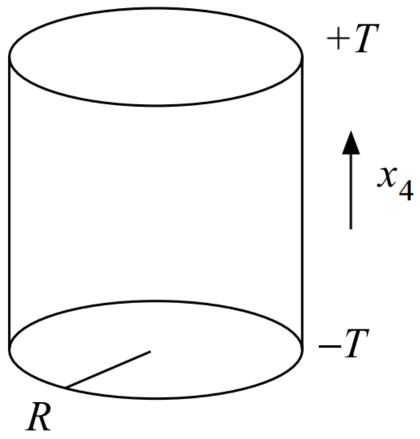


Figure 2 – The boundary in Euclidean spacetime [15]

compared to the upper cap, causing the lower cap to contribute $-n_-$ instead of just n_-. Consequently, our resulting map has a winding number of $n_+ - n_-$.

Finally, as we take $T, R \to \infty$, we consider the boundary to be a 3-sphere at

$$\rho \equiv (x_\mu x_\mu)^{1/2} = \infty \ . \tag{2.125}$$

On this boundary, we have a map $U(\hat{x})$, where $\hat{x} = \frac{x_\mu}{\rho}$. As discussed earlier, $U(\hat{x})$ has winding number $n = n_+ - n_-$.

Now, the subsequent step involves constructing a Bogomolny bound on the Euclidean Yang-Mills action for a field that adheres to the boundary condition we have just established. We begin with the Yang-Mills action:

$$S = \frac{1}{2} \int d^4x \ \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}] \ , \tag{2.126}$$

where the field strength tensor is given by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \ . \tag{2.127}$$

The boundary condition requires:

$$\lim_{\rho \to \infty} A_\mu(x) = \frac{i}{g} U(\hat{x}) \partial_\mu U^\dagger(\hat{x}) \ . \tag{2.128}$$

In order to integrate (2.102) in this context, we begin by defining the angles of the 3-sphere: polar angles χ and ψ, and azimuthal angle ϕ, via

$$\hat{x}_\mu = (\sin\chi \sin\psi \cos\phi, \sin\chi \sin\psi \sin\phi, \sin\chi \cos\psi, \cos\phi) \ . \tag{2.129}$$

Now, let's express n in terms of these angles:

$$n = \frac{-1}{24\pi^2} \int_0^\pi d\chi \int_0^\pi d\psi \int_0^{2\pi} d\phi \ \epsilon^{\alpha\beta\gamma} \, \mathrm{Tr}\Big[(U\partial_\alpha U^\dagger)(U\partial_\beta U^\dagger)(U\partial_\gamma U^\dagger)\Big] \ , \tag{2.130}$$

where α, β, γ run over χ, ψ, ϕ, and $\epsilon^{\chi\psi\phi} = +1$. We can rewrite this expression as a surface integral over a surface at infinity in four dimension Euclidean spacetime

$$n = \frac{1}{24\pi^2} \int dS_\mu \; \epsilon^{\mu\nu\sigma\tau} \, \mathrm{Tr}\Big[(U\partial_\nu U^\dagger)(U\partial_\sigma U^\dagger)(U\partial_\tau U^\dagger)\Big] \; , \tag{2.131}$$

where $\epsilon^{1234} = +1$, and the minus sign disappears when we make the change of variables. Now, utilizing (2.128), we can express this integral in terms of the vector potentials by setting $U\partial_\nu U^\dagger = -igA_\nu$

$$\begin{aligned} n &= \frac{1}{24\pi^2} \int dS_\mu \; \epsilon^{\mu\nu\sigma\tau} \, \mathrm{Tr}\Big[(U\partial_\nu U^\dagger)(U\partial_\sigma U^\dagger)(U\partial_\tau U^\dagger)\Big] \; , \\ &= \frac{1}{24\pi^2} \int dS_\mu \; \epsilon^{\mu\nu\sigma\tau} \, \mathrm{Tr}[(-igA_\nu)(-igA_\sigma)(-igA_\tau)] \; , \\ &= \frac{-i^3 g^3}{24\pi^2} \int dS_\mu \; \epsilon^{\mu\nu\sigma\tau} \, \mathrm{Tr}[A_\nu A_\sigma A_\tau] \; , \\ &= \frac{i g^3}{24\pi^2} \int dS_\mu \; \epsilon^{\mu\nu\sigma\tau} \, \mathrm{Tr}[A_\nu A_\sigma A_\tau] \; . \end{aligned} \tag{2.132}$$

By defining the Chern-Simons current, we can further express this as a volume integral:

$$J^\mu_{CS} \equiv 2\epsilon^{\mu\nu\sigma\tau} \, \mathrm{Tr}\Big[A_\nu F_{\sigma\tau} + \frac{2}{3} i g A_\nu A_\sigma A_\tau\Big] \; , \tag{2.133}$$

we can easily understand why this is possible, as on the surface at infinity, A_μ is a gauge transformation of zero, leading to $F_{\mu\nu}$ vanishing. Therefore, we can express the winding number n as:

$$n = \frac{g^2}{32\pi^2} \int dS_\mu \; J^\mu_{CS} \; , \tag{2.134}$$

applying Gauss's theorem, we obtain:

$$n = \frac{g^2}{32\pi^2} \int d^4x \; \partial_\mu J^\mu_{CS} \; . \tag{2.135}$$

The Chern-Simons current is not gauge invariant; however, the relative coefficient of its two terms has been chosen in such a way that its divergence becomes gauge invariant. Hence, there is no inconsistency in the results. To demonstrate this, let's calculate its divergence:

$$\begin{aligned} \partial_\mu J^\mu_{CS} &= \epsilon^{\mu\nu\sigma\tau} \, \mathrm{Tr}[F_{\mu\nu} F_{\sigma\tau}] \; , \\ &= 2 \, \mathrm{Tr}\Big[\tilde{F}^{\mu\nu} F_{\mu\nu}\Big] \; , \end{aligned} \tag{2.136}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\tau} F_{\sigma\tau}$ is the dual field strength tensor. Hence, we get:

$$n = \frac{g^2}{16\pi^2} \int d^4x \; \mathrm{Tr}\Big[\tilde{F}^{\mu\nu} F_{\mu\nu}\Big] \; . \tag{2.137}$$

At this point, we have n expressed as a four-dimensional volume integral of a gauge invariant quantity. Now, we can construct the Bogomolny bound. We start with the fact that

$$\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} = F^{\mu\nu} F_{\mu\nu} \; , \tag{2.138}$$

thus, we find by using $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$:

$$\begin{aligned}\frac{1}{2}\,\mathrm{Tr}\Big[\tilde{F}^{\mu\nu} \pm F_{\mu\nu}\Big]^2 &= \frac{1}{2}\,\mathrm{Tr}\Big[\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} \pm \tilde{F}^{\mu\nu}F_{\mu\nu} \pm F_{\mu\nu}\tilde{F}^{\mu\nu} + F^{\mu\nu}F_{\mu\nu}\Big]\ , \\ &= \frac{1}{2}\,\mathrm{Tr}\Big[2F^{\mu\nu}F_{\mu\nu} \pm 2\tilde{F}^{\mu\nu}F_{\mu\nu}\Big]\ , \\ &= \mathrm{Tr}[F^{\mu\nu}F_{\mu\nu}] \pm \mathrm{Tr}\Big[\tilde{F}^{\mu\nu}F_{\mu\nu}\Big]\ ,\end{aligned} \tag{2.139}$$

which is nonnegative. Consequently, we have:

$$\int d^4x\ \ \mathrm{Tr}[F^{\mu\nu}F_{\mu\nu}] \geq |\int d^4x\ \ \mathrm{Tr}\Big[\tilde{F}^{\mu\nu}F_{\mu\nu}\Big]|\ . \tag{2.140}$$

The left-hand side of this equation is twice the Euclidean Yang-Mills action, as evident from (2.126), while the right-hand side involves a portion of the expression derived for the winding number (2.137). Finally, we arrive at the following conclusion:

$$S \geq \frac{8\pi^2|n|}{g^2}\ . \tag{2.141}$$

This provides us with the minimum value of the Yang-Mills Euclidean action for a solution to the Euclidean field equations, which mediates between a vacuum state with winding number n_- at $x_0 = -\infty$ and a vacuum state with winding number n_+ at $x_0 = +\infty$. For $n_+ - n_- = 1$, this solution is referred to as an *instanton*. As demonstrated above, the instanton is localized within all four Euclidean dimensions. For $n_+ - n_- = -1$, the solution is the *anti-instanton*.

For cases where $|n_+ - n_-| \geq 1$, the solution is a dilute gas of $|n_+ - n_-|$ instantons or anti-instantons distributed throughout Euclidean spacetime.

We can saturate the bound in (2.141) if and only if we, by (2.140), make:

$$\tilde{F}_{\mu\nu} = (\mathrm{sign}\ n)F_{\mu\nu}\ . \tag{2.142}$$

Therefore, we obtain:

$$S = \frac{8\pi^2|n|}{g^2}\ . \tag{2.143}$$

Furthermore, it is evident that for the instanton solution, the action is given by

$$S = \frac{8\pi^2}{g^2}\ , \tag{2.144}$$

and

$$\tilde{F}_{\mu\nu} = F_{\mu\nu} \tag{2.145}$$

In conclusion, we have demonstrated that the field can transition between these vacuum states, achieving this interpolation with a finite amount of energy through the instanton solution. While constructing this proof, our focus was on utilizing the θ-vacua to describe the theory's vacuum states. However, we did not delve deeply into understanding the significance of the θ parameter.

To shed light on this, let's consider the Euclidean path integral with the same boundary conditions. We begin with a vacuum state characterized by n_- at $x_0 = -\infty$ and transition to a vacuum state with n_+ at $x_0 = +\infty$. As established earlier, the transition amplitude $\langle n_+| H |n_-\rangle$ solely depends on the difference between the winding numbers. Consequently, only field configurations with a winding number of $n_+ - n_-$ contribute to the path integral. Therefore, the path integral can be expressed as follows:

$$Z_{n_+\leftarrow n_-}(J) = \int dA_{n_+-n_-}\ e^{-S+JA}\ , \tag{2.146}$$

where

$$JA = \int d^4x\ \operatorname{Tr}[J^\mu A_\mu]\ . \tag{2.147}$$

Now, suppose we initiate the process with a θ-vacuum state characterized by n_- and conclude with a θ'-vacuum state characterized by n_+. Utilizing equation (2.117), we can express this transition as follows:

$$Z_{\theta'\leftarrow\theta}(J) = \sum_{n_-,n_+} e^{i(n_+\theta' - n_-\theta)} Z_{n_+\leftarrow n_-}(J)\ , \tag{2.148}$$

let, as previously used, $n_+ - n_- = n$, so that $n_+\theta' - n_-\theta = n_-(\theta' - \theta) + n\theta'$, we obtain:

$$\begin{aligned} Z_{\theta'\leftarrow\theta}(J) &= \sum_{n_-,n_+} e^{i(n_-(\theta'-\theta)+n\theta')} \int dA_n\ e^{-S+JA}\ , \\ &= \sum_{n_-} e^{in_-(\theta'-\theta)} \sum_n e^{in\theta'} \int dA_n\ e^{-S+JA}\ , \\ &= \delta(\theta'-\theta) \sum_n e^{in\theta'} \int dA_n\ e^{-S+JA}\ , \\ &= \sum_n e^{in\theta} \int dA_n\ e^{-S+JA}\ . \end{aligned} \tag{2.149}$$

Next, we combine the sum over n and the integral over A_n, into an integral over all A. Also, utilizing (2.137), we find:

$$Z_\theta(J) = \int dA\ \exp\left[\int d^4x\ \operatorname{Tr}\left[-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{ig^2\theta}{16\pi^2}\tilde{F}^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu\right]\right]\ . \tag{2.150}$$

The vacuum angle θ appears as the coefficient of an additional term in the Yang-Mills lagrangian. To transition to Minkowski space, we substitute $x_0 = it$. Since the new term contains one derivative with respect to x_0, it picks up a factor of $-i$. Combining these considerations, we obtain:

$$Z_\theta(J) = \int dA\ \exp\left[i\int d^4x\ \operatorname{Tr}\left[-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{g^2\theta}{16\pi^2}\tilde{F}^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu\right]\right]\ . \tag{2.151}$$

The additional term is gauge invariant, Lorentz invariant, Hermitian, and possesses a dimensionless coefficient. In principle, we could have included this term when initially formulating Yang-Mills theory. However, since it is a total divergence, it was omitted. Despite being a total divergence, we have demonstrated that it does have a significant impact on the physics of the problem.

At this stage, the ability to transition between different vacua with distinct winding numbers via instantons leads us to crucial conclusions regarding the U(1) problem. Firstly, we establish a direct connection between the new term in the lagrangian and the term derived in the chiral anomaly section. In the latter, we demonstrated that for the U(1) axial symmetry to be a genuine symmetry of the QCD lagrangian, the variation of the action in equation (2.94) must be zero. However, our analysis involving instantons revealed that despite the term being a total divergence (and thus expected to integrate to zero), it significantly influences the physics of the problem.

This connection becomes evident when reintroducing the massless quark fields into the theory. By incorporating the results of this section into equation (2.50), we gain deeper insights into the interplay between the U(1) problem, chiral anomalies, and the behavior of massless quarks within the QCD framework.

$$\mathcal{L} = i\overline{\Psi}\not{D}\Psi - \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} - \frac{g^2\theta}{32\pi^2}\tilde{G}^{a\mu\nu}G^a_{\mu\nu}\ . \tag{2.152}$$

The path integral for this theory is

$$\begin{aligned} Z = \int & \mathcal{D}A\mathcal{D}\Psi\mathcal{D}\bar{\Psi} \\ & \times \exp\left[i\int d^4x\ \left[i\overline{\Psi}\not{D}\Psi - \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} - \frac{g^2\theta}{32\pi^2}\tilde{G}^{a\mu\nu}G^a_{\mu\nu} + J^{a\mu}A^a_\mu\right]\right]\ . \end{aligned} \tag{2.153}$$

From (2.94) and employing the definition of the dual field strength tensor, we know that a $U(1)_A$ transformation modifies the integration measure by a factor of

$$\mathcal{D}\Psi\mathcal{D}\overline{\Psi} \to \exp\left[-i\int d^4x\ \frac{g^2\alpha}{16\pi^2}\tilde{G}^{a\mu\nu}G^a_{\mu\nu}\right]\mathcal{D}\Psi\mathcal{D}\overline{\Psi}\ , \tag{2.154}$$

and we find:

$$\begin{aligned} Z = \int & \mathcal{D}A\mathcal{D}\Psi\mathcal{D}\overline{\Psi} \\ & \times \exp\left[i\int d^4x\ \left[i\overline{\Psi}\not{D}\Psi - \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} - \frac{g^2(\theta+2\alpha)}{32\pi^2}\tilde{G}^{a\mu\nu}G^a_{\mu\nu} + J^{a\mu}A^a_\mu\right]\right]\ . \end{aligned} \tag{2.155}$$

Hence, we observe that the impact of the $U(1)_A$ transformation is to shift the value of the vacuum angle from θ to $\theta + 2\alpha$. As mentioned earlier, the vacuum angle influences the physics, leading to the realization that the U(1) axial symmetry cannot be a genuine symmetry of QCD. Consequently, the long-standing U(1) problem no longer persists.

The second significant implication is that the QCD Lagrangian incorporates a new term, thereby affecting the physics. Within this term, a new parameter θ emerges. The consequences of this parameter will be explored in the Strong CP Problem chapter.

3 Strong CP problem and its solution

As mentioned earlier, the QCD Lagrangian has a new term, θ, which exerts a notable influence on the dynamics of strong interactions. Specifically, it leads to the violation of both P and CP symmetries and induces an electric dipole moment (EDM) in the neutron. Notably, while CP violation is theoretically expected through the θ term, experimental observations indicate that CP is conserved and $|\theta| \lesssim 10^{-10}$ [19–21]. This fine-tuning of the vacuum angle θ is recognized as the strong CP problem.

3.1 Neutron electric dipole moment (nEDM)

The electric dipole moment (EDM) is the measure of the electric charge separation within a system, such as a molecule, an atom or, in certain instances like ours, subatomic particles. It is classically defined as the product of the charge magnitude and the displacement vector between the positive and negative charges, standing as one of the fundamental properties characterizing charged particles.

In the case of a neutron, being a neutral particle with zero net charge might raise the question of how it could possess an EDM. The key lies in recognizing that the neutron is not a fundamental particle but rather a composite particle composed of quarks, which carry fractional electric charges. Consequently, even though the neutron itself has a net charge of zero, it exhibits an electric charge distribution within its composite structure.

The neutron EDM (nEDM) is, thus, the measure of the separation of the charges within the neutron, offering a fundamental measure that can provide insights into beyond standard model (BSM) physics. The computation of the nEDM involves a blend of theoretical and experimental techniques, presenting an ongoing challenge for physicists.

The main insight that the nEDM provides is the establishment of one of the 26 free paramaters within the Standard Model (SM). Its measured valued, $d_n = (0.0 + 1.1_{\text{stat}} + 0.2_{\text{sys}}) \times 10^{-26}$ e.cm [30], imposes a stringent constraint on the θ parameter, requiring it to be exceedingly small, specifically $\theta < 10^{-10}$. The challenge arises from the absence of a fundamental justification within the SM for θ parameter to be this small.

One might argue that this CP violation from the nEDM could potencially arise from the weak interactions, as it is already CP violating in the SM. However, the level of CP violation achievable through weak interactions is $|d_n| \approx 10^{-32}$ e.cm [31], which is many orders of magnitude smaller than the experimental limit. Therefore, additional contributions are essential to account for the observed experimental results.

While the detailed derivation of the QCD effective theory for hadrons lies beyond the scope of our current discussion, we can employ it as a tool to compute the nEDM. Within this framework, the CP-violating process involves contributions from the following diagrams:

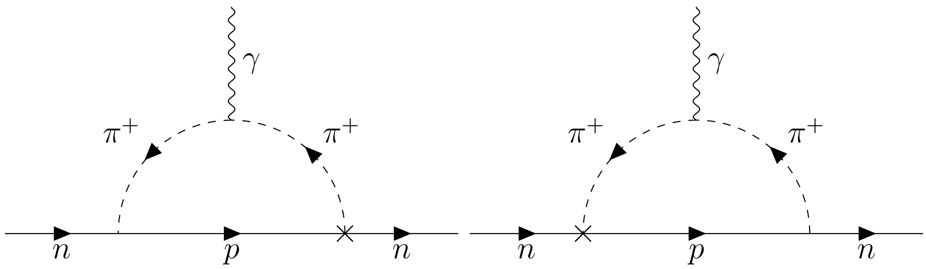


Figure 3 – Diagrams contributing to the nEDM. The CP violating vertex is denoted with a cross.

The diagrams above contribute to an amplitude of the form

$$\mathcal{M} = -2iD(q^2)\epsilon^*_\mu(q)\overline{u}'_s(p')S^{\mu\nu}q_\nu i\gamma_5 u_s(p) \ , \tag{3.1}$$

where $q = p' - p$ and $D(q^2)$ represents the nEDM. In the $q \to 0$ limit, this corresponds to a term within the effective lagrangian of

$$\mathcal{L} = D(0)F_{\mu\nu}\overline{n}S^{\mu\nu}i\gamma_5 n \ . \tag{3.2}$$

To account for $i\gamma_5$, we employ:

$$S^{\mu\nu}i\gamma_5 = -\frac{1}{2}e^{\mu\nu\rho\sigma}S_{\rho\sigma} \ , \tag{3.3}$$

which leads to

$$\mathcal{L} = -D(0)\tilde{F}_{\mu\nu}\overline{n}S^{\mu\nu}n \ . \tag{3.4}$$

It results in a contribution of $D(0)$ to the nEDM. To compute this amplitude, we require the vertices, which we obtain from the theory as [15]:

$$\begin{aligned}\mathcal{L}_{\pi\overline{N}N} &= -i\sqrt{2}\left(\frac{g_A m_N}{f_\pi}\right)(\pi^+\overline{p}\gamma_5 n + \pi^-\overline{n}\gamma_5 p) \ , \\ \mathcal{L}_{\theta\pi\overline{N}N} &= -\sqrt{2}\left(\frac{\theta c_+\tilde{m}}{f_\pi}\right)(\pi^+\overline{p}n + \pi^-\overline{n}p) \ ,\end{aligned} \tag{3.5}$$

where $\tilde{m} = \frac{m_u m_d}{m_u + m_d}$ is the reduced mass of the up and down quarks, $g_A = 1.27$ is the axial vector coupling, and $c_+ = 1.7$ is a coefficient that appears in the effective lagrangian. The amplitude for (4) then reads:

$$\begin{aligned}i\mathcal{M} = &\left(\frac{1}{i}\right)^3(ie)\left(\frac{\sqrt{2}g_A m_N}{f_\pi}\right)\left(\frac{-i\sqrt{2}\theta c_+\tilde{m}}{f_\pi}\right)\epsilon^*_\mu \\ &\times \int_0^\Lambda \frac{d^4l}{(2\pi)^4}\,\frac{(2l^\mu)\overline{u}'[(-\not{l} - \not{\overline{p}} + m_N)\gamma_5 + \gamma_5(-\not{l} - \not{\overline{p}} + m_N)]u}{[(l+\overline{p})^2 + m_N^2][(l+\frac{1}{2}q)^2 + m_\pi^2][(l-\frac{1}{2}q)^2 + m_\pi^2]} \ ,\end{aligned} \tag{3.6}$$

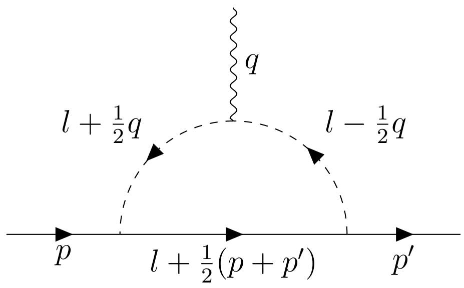


Figure 4 – Momenta labeled diagram

where $\Lambda \sim 4\pi f_\pi$ is the ultraviolet cutoff in the effective theory, $\overline{p} = \frac{1}{2}(p' + p)$. Now we can use $\{\gamma^\mu, \gamma_5\} = 0$ to simplify the spinor factors in the numerator. With these considerations, it is easy to see that the slashed terms vanish, resulting in the simplification of the amplitude to:

$$\begin{aligned}\mathcal{M} = &\left(\frac{2e\theta g_A c_+ m_N \tilde{m}}{f_\pi^2}\right)\epsilon_\mu^* \\ &\times \int_0^\Lambda \frac{d^4 l}{(2\pi)^4}\frac{(2l^\mu)(2m_N \overline{u}'\gamma_5 u)}{[(l+\overline{p})^2 + m_N^2][(l+\frac{1}{2}q)^2 + m_\pi^2][(l-\frac{1}{2}q)^2 + m_\pi^2]} \ .\end{aligned} \tag{3.7}$$

Next, we observe that, due to the properties of spinors, the term $\overline{u}'\gamma_5 u$ vanishes when $p' = p$. Consequently, it must be of order $\mathcal{O}(q)$, allowing us to set $q = 0$ everywhere else. Considering $p >> l$, we can further simplify by setting:

$$\begin{aligned}(l+\overline{p})^2 + m_N^2 &= l^2 + l\cdot(p'+p) + \frac{1}{4}(p'+p)^2 + m_N^2 \ , \\ &= 2l\cdot p + p^2 + m_N^2 \ , \\ &= 2l\cdot p - m_N^2 + m_N^2 \ , \\ &= 2l\cdot p \ .\end{aligned} \tag{3.8}$$

Substituting on the main relation, we obtain:

$$\mathcal{M} = \left(\frac{4e\theta g_A c_+ m_N \tilde{m}}{f_\pi^2}\right)\epsilon_\mu^* \int_0^\Lambda \frac{d^4 l}{(2\pi)^4}\frac{(l^\mu)(m_N \overline{u}'\gamma_5 u)}{(l\cdot p)(l^2 + m_\pi^2)^2} \ , \tag{3.9}$$

integrating over the direction of l results in

$$\frac{l^\mu}{p\cdot l} \to \frac{p^\mu}{p^2} = -\frac{p^\mu}{m_N^2} \ , \tag{3.10}$$

which leads us to:

$$\mathcal{M} = -\left(\frac{4e\theta g_A c_+ \tilde{m}}{f_\pi^2}\right)\epsilon_\mu^* \int_0^\Lambda \frac{d^4 l}{(2\pi)^4}\frac{(p^\mu)(\overline{u}'\gamma_5 u)}{(l^2 + m_\pi^2)^2} \ . \tag{3.11}$$

Next, we utilize the Gordon identity [32]:

$$\overline{u}'(p')[(p'+p)^\mu - 2iS^{\mu\nu}(p'-p)_\nu]\gamma_5 u(p) = 0 \ . \tag{3.12}$$

To derive the Gordon identity, we begin with

$$\begin{aligned} \gamma^\mu \not{p} &= \frac{1}{2}\Big\{\gamma^\mu, \not{p}\Big\} + \frac{1}{2}\Big[\gamma^\mu, \not{p}\Big] = -p^\mu - 2iS^{\mu\nu}p_\nu \ , \\ \not{p}'\gamma^\mu &= \frac{1}{2}\Big\{\gamma^\mu, \not{p}\Big\} - \frac{1}{2}\Big[\gamma^\mu, \not{p}\Big] = -p'^\mu + 2iS^{\mu\nu}p'_\nu \ , \end{aligned} \tag{3.13}$$

adding both expressions, we obtain:

$$\gamma^\mu \not{p} + \not{p}'\gamma^\mu = -p^\mu - p'^\mu - 2iS^{\mu\nu}p_\nu + 2iS^{\mu\nu}p'_\nu \ . \tag{3.14}$$

Now, we multiply on the right by γ_5 and sandwich the expression between spinors

$$\overline{u}'(p')[\gamma^\mu \not{p} + \not{p}'\gamma^\mu]\gamma_5 u(p) = \overline{u}'(p')[-p^\mu - p'^\mu - 2iS^{\mu\nu}p_\nu + 2iS^{\mu\nu}p'_\nu]\gamma_5 u(p) \ . \tag{3.15}$$

Using the Dirac equation:

$$\overline{u}'(p')(\not{p}' + m_N) = 0 \ , \tag{3.16}$$

we find

$$\overline{u}'(p')[\gamma^\mu \not{p} - m_N\gamma^\mu]\gamma_5 u(p) = -\overline{u}'(p')[p^\mu + p'^\mu + 2iS^{\mu\nu}(p_\nu - p'_\nu)]\gamma_5 u(p) \ . \tag{3.17}$$

Focusing on the left side of the previous equation, we utilize the property $\{\gamma^\mu, \gamma_5\} = 0$ and the Dirac equation to express

$$\begin{aligned} \overline{u}'(p')[\gamma^\mu \not{p} - m_N\gamma^\mu]\gamma_5 u(p) &= \overline{u}'(p')[\gamma^\mu p_\nu \gamma^\nu \gamma_5 - m_N\gamma^\mu\gamma_5]u(p) \ , \\ &= \overline{u}'(p')[-\gamma^\mu\gamma_5 p_\nu \gamma^\nu - m_N\gamma^\mu\gamma_5]u(p) \ , \\ &= \overline{u}'(p')[-\gamma^\mu\gamma_5 \not{p} - m_N\gamma^\mu\gamma_5]u(p) \ , \\ &= \overline{u}'(p')[m_N\gamma^\mu\gamma_5 - m_N\gamma^\mu\gamma_5]u(p) \ , \\ &= 0 \ . \end{aligned} \tag{3.18}$$

Finally, we reached the Gordon identity expression:

$$\overline{u}'(p')[(p' + p)^\mu - 2iS^{\mu\nu}(p' - p)_\nu]\gamma_5 u(p) = 0 \ . \tag{3.19}$$

In the case under consideration, this expression becomes:

$$\overline{u}'(p')[2p^\mu - 2iS^{\mu\nu}q_\nu]\gamma_5 u(p) = 0 \ , \tag{3.20}$$

which leads to

$$p^\mu \overline{u}'(p')\gamma_5 u(p) = \overline{u}'(p')S^{\mu\nu}q_\nu i\gamma_5 u(p) \ . \tag{3.21}$$

Finally, applying the last equation to the transition amplitude, we obtain:

$$\mathcal{M} = -\left(\frac{4e\theta g_A c_+ \tilde{m}}{f_\pi^2}\right)\epsilon^*_\mu \overline{u}' S^{\mu\nu} q_\nu i\gamma_5 u \int_0^\Lambda \frac{d^4 l}{(2\pi)^4}\frac{1}{(l^2 + m_\pi^2)^2} \ . \tag{3.22}$$

In the limit as $m_\pi \to 0$, the integral exhibits divergence at small l, giving rise to a chiral log. Upon performing a Wick rotation, the integral evaluates to:

$$\int_0^\Lambda \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2+m_\pi^2)^2} = \frac{i}{16\pi^2}\ln\frac{\Lambda^2}{m_\pi^2} \ . \tag{3.23}$$

The ultimate transition amplitude finally becomes:

$$\mathcal{M} = -2i\left(\frac{e\theta g_A c_+ \tilde{m}}{8\pi^2 f_\pi^2}\ln\frac{\Lambda^2}{m_\pi^2}\right)\epsilon_\mu^* \overline{u}' S^{\mu\nu} q_\nu i\gamma_5 u \ . \tag{3.24}$$

By comparing this result with (3.1), we can deduce that the neutron Electric Dipole Moment (nEDM) is:

$$d_n = \frac{e\theta g_A c_+ \tilde{m}}{8\pi^2 f_\pi^2}\left[\ln\frac{\Lambda^2}{m_\pi^2} + \mathcal{O}(1)\right] \ . \tag{3.25}$$

Substituting the constant values ($g_A = 1.27,\ c_+ = 1.7,\ \tilde{m} = 1.2$ MeV), we find:

$$d_n = 3.2\times 10^{-16}\theta \text{ e.cm} \ . \tag{3.26}$$

As mentioned earlier, the experimental measure is $d_n = (0.0 + 1.1_{\text{stat}} + 0.2_{\text{sys}})\times 10^{-26}$ e.cm [30]. Consequently, the θ parameter acquires an upper bound of

$$|\theta| < 0.4\times 10^{-10} \ . \tag{3.27}$$

3.2 Solution of the Strong CP problem

Introducing an additional chiral symmetry presents a compelling solution to the strong CP problem, effectively rotating the θ-vacua away. Two prominent suggestions have been proposed for this chiral symmetry:

i) The assertion that the up quark possesses no mass [33] (contradicted by [34]).

ii) The proposition that the Standard Model (SM) incorporates an extra global U(1) chiral symmetry [35, 36].

This study focuses on exploring the Peccei-Quinn approach to addressing the strong CP problem. The forthcoming section will demonstrate that CP invariance in Quantum Chromodynamics (QCD) naturally arises when at least one quark flavor obtains its mass through a Yukawa coupling to a scalar field with a non-zero vacuum expectation value (VEV). Additionally, the lagrangian must possess a U(1) symmetry involving all Yukawa couplings. 't Hooft's work establishes that the parameter θ defines the selection of vacuum from an infinite array of distinct and generally inequivalent vacua, signifying that each θ represents a potential true vacuum.

In the scenario where all fermions coupling with the non-Abelian gauge field are massless, each θ yields an equivalent theory. This equivalence arises because the $\mathrm{U}(1)_A$

transformation shifts $\theta \to \theta + 2\alpha$, and since the value of θ can be altered by a simple change in the integration variable within the path integral, we must infer that the vacuum-to-vacuum transition amplitude Z does not depend on θ. This assertion, however, contradicts our earlier conclusion regarding the U(1) problem. The crucial question arises: How do we reconcile this apparent contradiction with the previous conclusion that instanton-mediated tunneling amplitudes make the vacuum energy density dependent on θ? The answer lies in the integration over the quark field in

$$Z = \int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}\bar{\Psi} \ \exp\left[i\int d^4x \ \left[i\overline{\Psi}\not{D}\Psi - \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} - \frac{g^2\theta}{32\pi^2}\tilde{G}^{a\mu\nu}G^a_{\mu\nu}\right]\right] \ , \tag{3.28}$$

we obtain:

$$Z = \int \mathcal{D}A_n \det\left(i\not{D}\right)e^{S_{YM}}e^{in\theta} \ , \tag{3.29}$$

where S_{YM} is the Yang-Mills actions and n is the winding number. From 3.29, it becomes evident that Z would be independent of θ only in the absence of contributions from gauge fields with nonzero winding number. This condition holds true if the determinant of the operator $\det\left(i\not{D}\right)$ vanishes for gauge fields with a nonzero winding number.

However, in our pursuit of theories involving massive quarks, we need to introduce a mass term for the quarks. The mass term to be incorporated is expressed as follows:

$$\mathcal{L}_{\text{mass}} = -m\chi\xi - m^*\xi^\dagger\chi^\dagger \ , \tag{3.30}$$

expressing it in terms of the Dirac field Ψ, defined as:

$$\Psi = \begin{pmatrix} \chi \\ \xi^\dagger \end{pmatrix} \ , \tag{3.31}$$

and allowing the mass to be complex, denoted as $m = |m|e^{i\phi}$, the mass term becomes:

$$\mathcal{L}_{\text{mass}} = -|m|\overline{\Psi}e^{-i\phi\gamma_5}\Psi \ . \tag{3.32}$$

Now, under the U$(1)_A$ transformation, the phase ϕ changes to $\phi + 2\alpha$. Simultaneously, θ changes by the same parameter, ensuring that $\phi - \theta$ remains unchanged. Equivalently, the product $me^{-i\theta}$ is also invariant, as demonstrated by:

$$\begin{aligned} me^{-i\theta} &= |m|e^{i\phi}e^{-i\theta} \ , \\ &= |m|e^{i(\phi-\theta)} \ . \end{aligned} \tag{3.33}$$

Consequently, the path integral does exhibit dependence on $me^{-i\theta}$ but not on θ and ϕ individually. Thus, one can define inequivalent theories that share the same mass term but have various choices of θ. This reconciliation of conclusions from the previous section establishes that there is no straightforward solution to the strong CP problem. A simple U(1) transformation cannot merely shift the θ parameter away.

The crucial question then arises: under what conditions does a theory like QCD describe a world such as the one in which we live, which is absent of strong CP and P violations? Only theories in which $\theta \to 0$ after all fermion masses have been rendered real through an appropriate U(1) transformation yield a CP- and P-invariant theory of strong interactions. To address this issue, the Peccei-Quinn mechanism [35, 36] will be presented.

3.2.1 The Peccei-Quinn mechanism

The Peccei-Quinn solution posits that $\mathcal{L}_{\mathrm{SM}}$ must exhibit a new chiral U(1) invariance, where alterations in θ are equivalent to changes in the definitions of the various fields within $\mathcal{L}_{\mathrm{SM}}$ and bear no physical consequences. Such a theory is effectively equivalent to a $\theta = 0$ theory, consequently lacking strong P and CP violation.

To attain this, consider theories in which at least one fermion flavor obtains its mass from a Yukawa coupling to a color-singlet scalar field with a nonzero vacuum expectation value (VEV). The effective potential for the scalar fields is θ-dependent and does not share the same symmetry as the scalar polynomial $U(\phi)$ appearing in the forthcoming $\mathcal{L}$. In practical terms, this implies that the minimum of the potential corresponds to a specific choice of phases for the various scalar VEVs. These phases manifest in the fermion mass terms and are consistently arranged such that, upon making all fermion masses real through $\mathrm{U}(1)_A$ rotations of the fermion fields, the resulting θ is zero.

To illustrate this mechanism, we will employ a toy model for strong interactions. In this simplified model, we consider only one flavor of fermion and one color-singlet complex scalar field. This simplification is justified since introducing additional fermion flavors and scalar multiplets will not alter the ensuing results. The lagrangian for such a toy model is presented below, with the note that we are working in Euclidean space.

$$
\begin{aligned}
\mathcal{L}_E = &-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + i\overline{\Psi}\not{D}\Psi \\
&+ \overline{\Psi}\left[G\phi\left(\frac{1+\gamma_5}{2}\right) + G^*\phi^*\left(\frac{1-\gamma_5}{2}\right)\right]\Psi \\
&- |\partial_\mu\phi|^2 - \mu^2|\phi|^2 - h|\phi|^4 \ ,
\end{aligned}
\tag{3.34}
$$

where $(1 \pm \gamma_5)/2 = P_{R/L}$, and with $\mu^2 < 0$, the lagrangian is formally invariant under $\mathrm{U}(1)_A$ transformation, which induces changes as follows:

$$
\begin{aligned}
\Psi &\to e^{i\eta\gamma_5}\Psi \ , \\
\phi &\to e^{-2i\eta}\phi \ .
\end{aligned}
\tag{3.35}
$$

Additionally, the condition $\mu^2 < 0$ ensures that the potential takes on the requisite form for spontaneous symmetry breaking to occur.

Nevertheless, this theory still exhibits the previously discussed chiral anomaly, which redefines θ to $\theta - 2\eta$, resulting in an effective lagrangian given by:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_E + i\theta \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \ , \tag{3.36}$$

this leads to the following effective action:

$$S_{\text{eff}} = \int d^4x \ \mathcal{L}_E + in\theta \ , \tag{3.37}$$

where n is the winding number defined by

$$n = \frac{g^2}{32\pi^2} \int d^4x \ F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \ . \tag{3.38}$$

To demonstrate that these theories conserve P and CP, a specific condition is required:

$$\Delta = \arg(e^{i\theta} G \langle \phi \rangle) = 0 \ . \tag{3.39}$$

This condition is fulfilled when $\mu^2 > 0$, ensuring that ϕ possesses a vanishing vacuum expectation value (VEV). However, our interest lies in scalar fields with a non-zero VEV. In such cases, the condition translates to requiring that the fermion mass $G\langle \phi \rangle$ be real when the fields are defined, leading to $\theta = 0$.

To establish the satisfaction of (3.39), we examine the generating functional of the scalar Green's functions:

$$Z_\theta(J, J^*) = \sum_n \int (dA_\mu)_n \ e^{i\theta n} \int d\Psi \ \int d\overline{\Psi} \ \int d\phi \ \int d\phi^* \ \exp[\mathcal{L}_E + J\phi + J^*\phi^*] \ . \tag{3.40}$$

The scalar VEV is, then, defined as:

$$\frac{1}{Z_\theta} \frac{\delta Z_\theta}{\delta J} \bigg|_{J=J^*=0} \equiv \langle \phi \rangle = \lambda e^{i\beta} \ , \tag{3.41}$$

where λ and β are real constants.
The generating functional (3.40) can be rewritten as:

$$\begin{aligned} Z_\theta(J, J^*) &= \sum_n \int (dA_\mu)_n \ e^{i\theta n} \int d\Psi \ \int d\overline{\Psi} \ \int d\phi \ \int d\phi^* \ \exp\Big[\int d^4x \ \mathcal{L}(\phi\phi^*)\Big] \\ &\times \exp\Big[-\frac{1}{4} FF + i\overline{\Psi} \not{D} \Psi\Big] \exp\Big[\int d^4x \ \overline{\Psi} \left(G\phi P_R + G^*\phi^* P_L\right) \Psi\Big] \\ &\times \exp\Big[\int d^4x \ J\phi + J^*\phi^*\Big] \ , \\ &= \sum_n \int (dA_\mu)_n \ e^{i\theta n} \int d\Psi \ \int d\overline{\Psi} \ \int d\phi \ \int d\phi^* \ \exp\Big[\int d^4x \ \mathcal{L}(\phi\phi^*)\Big] \\ &\times \exp\Big[-\frac{1}{4} FF + i\overline{\Psi} \not{D} \Psi\Big] \sum_{k,m} \frac{1}{k!m!} \left[\int d^4x \ \overline{\Psi} G\phi P_R \Psi\right]^k \\ &\times \left[\int d^4x' \ \overline{\Psi} G^*\phi^* P_L \Psi\right]^m \exp\Big[\int d^4x \ J\phi + J^*\phi^*\Big] \ , \end{aligned} \tag{3.42}$$

where $\mathcal{L}(\phi\phi^*)$ is the complex scalar lagrangian defined by:

$$\mathcal{L}(\phi\phi^*) = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \ . \tag{3.43}$$

In this expression, the terms depending on both fermion and scalar fields were expanded using the identity:

$$e^x = \sum_k \frac{x^k}{k!} \ , \tag{3.44}$$

applied to the present case:

$$\begin{aligned}
\exp\Big[\int d^4x \ \overline{\Psi}\left(G\phi P_R + G^*\phi^* P_L\right)\Psi\Big] &= e^{\int d^4x \ \overline{\Psi}G\phi P_R\Psi} e^{\int d^4y \ \overline{\Psi}G^*\phi^* P_L\Psi} \ , \\
&= \sum_{k,m} \frac{[\int d^4x \ \overline{\Psi}G\phi P_R\Psi]^k}{k!}\frac{[\int d^4y \ \overline{\Psi}G^*\phi^* P_L\Psi]^m}{m!} \ .
\end{aligned} \tag{3.45}$$

Considering the argument from the instanton section, it is crucial to note that transitions between vacua are only possible when the winding numbers are different and integers. Therefore, in each n sector, only the terms with $k-m=n$ will contribute to the generating functional [37].

With this consideration, the integration of the vector and fermion fields can be performed. For this purpose, the following notation is used:

$$\begin{aligned}
Z_\theta(J,J^*) &= \sum_n \int (dA_\mu)_n \ e^{i\theta n} \int d\Psi \int d\overline{\Psi} \int d\phi \int d\phi^* \ \exp\Big[\int d^4x \ \mathcal{L}(\phi\phi^*)\Big] \\
&\times \exp\Big[-\frac{1}{4}FF + i\overline{\Psi}\not{D}\Psi\Big]\sum_m \frac{1}{(m+n)!m!}\Big[\int d^4x \ \overline{\Psi}G\phi P_R\Psi\Big]^{m+n} \\
&\times \Big[\int d^4y \ \overline{\Psi}G^*\phi^* P_L\Psi\Big]^m \exp\Big[\int d^4x \ J\phi + J^*\phi^*\Big] \ , \\
&= \sum_n \int d\phi \int d\phi^* \ e^{i\theta n}\exp\Big[\int d^4x \ \mathcal{L}(\phi\phi^*)\Big]\exp\Big[\int d^4x \ J\phi + J^*\phi^*\Big] \\
&\times \int (dA_\mu)_n \int d\Psi \int d\overline{\Psi} \ \exp\Big[-\frac{1}{4}FF + i\overline{\Psi}\not{D}\Psi\Big] \\
&\times \sum_m \frac{1}{(m+n)!m!}\Big[\int d^4x \ G\phi\left(\overline{\Psi}P_R\Psi\right)\Big]^{m+n}\Big[\int d^4y \ G^*\phi^*\left(\overline{\Psi}P_L\Psi\right)\Big]^m \ , \\
&= \sum_n \int d\phi \int d\phi^* \ e^{i\theta n}\exp\Big[\int d^4x \ \mathcal{L}(\phi\phi^*)\Big]\exp\Big[\int d^4x \ J\phi + J^*\phi^*\Big] \\
&\times \int (dA_\mu)_n \int d\Psi \int d\overline{\Psi} \ \exp\Big[-\frac{1}{4}FF + i\overline{\Psi}\not{D}\Psi\Big] \\
\times \sum_m &\frac{1}{(m+n)!m!}\prod_{a=1}^{m+n}\prod_{b=1}^{m}\int d^4x_a \ G\phi_a\left(\overline{\Psi}_a P_R\Psi_a\right)\int d^4y_b \ G^*\phi_b^*\left(\overline{\Psi}_b P_L\Psi_b\right) \ .
\end{aligned} \tag{3.46}$$

Note that the removal of the term $e^{i\theta n}$ from the integral over the vector field is possible because the integration is over vector fields with a specific winding number n. Therefore, for each specific winding number, $e^{i\theta n}$ is not dependent on the A_μ's.

Also, it's important to emphasize that this last step is only possible because the integral as a whole is included in the exponentiation. To illustrate, consider the following example:

$$\begin{aligned}\left[\int d^4x\ \overline{\Psi}G\phi P_R\Psi\right]^z &= \left[\int d^4x_1\ \overline{\Psi}_1 G\phi_1 P_R\Psi_1\right]\left[\int d^4x_2\ \overline{\Psi}_2 G\phi_2 P_R\Psi_2\right]\dots\ ,\\ &= \prod_{i=1}^{z}\int d^4x_i\ \overline{\Psi}_i G\phi_i P_R\Psi_i\ .\end{aligned} \tag{3.47}$$

Integrating (3.46) over the vector and fermion fields, we can rewrite it as follows:

$$\begin{aligned}Z_\theta(J,J^*) = \sum_n \int d\phi \int d\phi^*\ e^{i\theta n}\exp\Big[\int d^4x\ \mathcal{L}(\phi\phi^*)\Big]\exp\Big[\int d^4x\ J\phi + J^*\phi^*\Big]\\ \sum_m \prod_{a=1}^{m+n}\prod_{b=1}^{m}\int d^4x_a\ [G\phi(x_a)]\int d^4y_b\ [G^*\phi^*(y_b)]\frac{C_m^n(x_a,y_b)}{m!(m+n)!}\ ,\\ \equiv \sum_n \int d\phi \int d\phi^*\ e^{i\theta n}A_n(\phi\phi^*)(G\phi)^n\exp\Big[\int d^4x\ J\phi + J^*\phi^*\Big]\ .\end{aligned} \tag{3.48}$$

Note that:

$$\sum_m \prod_{a=1}^{m+n}\prod_{b=1}^{m}\int d^4x_a\ [G\phi(x_a)]\int d^4y_b\ [G^*\phi^*(y_b)] = (G\phi)^n\ , \tag{3.49}$$

we can think that each product after integration of $(G\phi)(G^*\phi^*) = 1$ until $a, b = m$, then we are left only with $(G\phi)$ for the n extra multiplications. Also, we have:

$$A_n(\phi\phi^*) = \exp\Big[\int d^4x\ \mathcal{L}(\phi\phi^*)\Big]\sum_m \prod_{a=1}^{m+n}\prod_{b=1}^{m}\int d^4x_a \int d^4y_b\ \frac{C_m^n(x_a,y_b)}{m!(m+n)!}\ . \tag{3.50}$$

Even though we cannot determine the specific form of the coefficients C_m^n, we know two important properties:

1) They are real, because C_m^n are integrals over bilinear forms of fermion fields.

2) They satisfy the relation

$$C_m^n(x_a,y_b) = C_m^{-n}(y_b,x_a)\ . \tag{3.51}$$

This comes from the instantons' properties and is easier to understand if we consider a tunneling process from point x_a with winding number k to point y_b with winding number m. This process has energy dictated by $n = k - m$, and we can make the same process but exchange the start and finish, resulting in a process from y_b with m and x_a with k, which gives us a $-n = m - k$ energy. Using this result leads to the following property:

$$A_{-q}(\phi\phi^*) = A_q^*(\phi\phi^*)\ . \tag{3.52}$$

Making use of (3.52) in (3.48), we express:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} e^{i\theta n} A_n(\phi\phi^*)(G\phi)^n &= \sum_{n=-\infty}^{\infty} A_n(\phi\phi^*)(Ge^{i\theta}\phi)^n \ , \\
&= A_0(\phi\phi^*) + \sum_{n=-\infty}^{-1} A_n(\phi\phi^*)(Ge^{i\theta}\phi)^n + \sum_{n=1}^{\infty} A_n(\phi\phi^*)(Ge^{i\theta}\phi)^n \ , \\
&= A_0(\phi\phi^*) + \sum_{n'=1}^{\infty} A_{-n'}(\phi\phi^*)(Ge^{i\theta}\phi)^{-n'} + \sum_{n=1}^{\infty} A_n(\phi\phi^*)(Ge^{i\theta}\phi)^n \ , \\
&= A_0(\phi\phi^*) + \sum_{n=1}^{\infty} A_n^*(\phi\phi^*)(G^*e^{-i\theta}\phi^*)^n + \sum_{n=1}^{\infty} A_n(\phi\phi^*)(Ge^{i\theta}\phi)^n \ ,
\end{aligned} \tag{3.53}$$

this leads us to:

$$\begin{aligned}
Z_\theta(J, J^*) = \int d\phi \int d\phi^* \ \Big[A_0(\phi\phi^*) + \sum_{n=1}^{\infty} A_n(\phi\phi^*)(Ge^{i\theta}\phi)^n \\
+ A_n^*(\phi\phi^*)(G^*e^{-i\theta}\phi^*)^n \Big] \times \exp\Big[\int d^4x \ J\phi + J^*\phi^* \Big] \ .
\end{aligned} \tag{3.54}$$

Now we must employ some complex algebra to derive the CP and P conserving condition (3.39). Remember the CP condition reads:

$$\Delta = \arg(e^{i\theta} G \langle\phi\rangle) = 0 \ . \tag{3.55}$$

Note that in (3.54) there are terms that resemble this condition. Therefore, we are now going to explore these terms to see if we can reach the desired condition. We begin with:

$$A_n(\phi\phi^*)(Ge^{i\theta}\phi)^n + A_n^*(\phi\phi^*)(G^*e^{-i\theta}\phi^*)^n = 2\,\mathrm{Re}\, A_n(Ge^{i\theta}\phi)^n \ , \tag{3.56}$$

next, we separate the product into two parts:

$$\begin{aligned}
Z_1 &\equiv A_n \ , \\
Z_2 &\equiv Ge^{i\theta}\phi \ .
\end{aligned} \tag{3.57}$$

We can write Z_2 as a general complex number in the form of:

$$\begin{aligned}
Z_2 &= |G||\phi||e^{i\theta}|e^{i\chi} \ , \\
&= |G||\phi|e^{i\chi} \ ,
\end{aligned} \tag{3.58}$$

where χ is the angle between the positive real axis and the line joining the origin and Z_2, and is defined by:

$$\chi \equiv \arg(e^{i\theta} G\phi) \ . \tag{3.59}$$

Note that analyzing χ with respect to the ϕ VEV leads to the CP and P conserving condition. To understand how this condition is incorporated into the generating functional, we proceed with the calculations from (3.56)

$$
\begin{aligned}
\operatorname{Re} A_n(Ge^{i\theta}\phi)^n &= \operatorname{Re}[Z_1 Z_2^n] \ , \\
&= \operatorname{Re}[|Z_1||Z_2|^n e^{i(\xi+n\chi)}] \ , \\
&= |Z_1||Z_2|^n \cos(\xi + n\chi) \ , \\
&= |Z_1||Z_2|^n \left[\cos\xi\cos n\chi - \sin\xi \sin n\chi\right] \ , \\
&= \operatorname{Re}[Z_1]|G\phi|^n \cos n\chi - \operatorname{Im}[Z_1]|G\phi|^n \sin n\chi \ , \\
&= \operatorname{Re}[A_n]|G\phi|^n \cos n\chi - \operatorname{Im}[A_n]|G\phi|^n \sin n\chi \ .
\end{aligned}
\tag{3.60}
$$

Now, we can substitute this into (3.54) and obtain:

$$
\begin{aligned}
Z_\theta(J, J^*) = \int d\phi \int d\phi^* \ \Bigg[& A_0(\phi\phi^*) + 2\sum_{n=1}^{\infty} \operatorname{Re}[A_n]|G\phi|^n \cos n\chi \\
& - \operatorname{Im}[A_n]|G\phi|^n \sin n\chi \Bigg] \exp\left[\int d^4x \ J\phi + J^*\phi^*\right] \ .
\end{aligned}
\tag{3.61}
$$

At this point, we make a change of variables

$$
\phi = e^{i\beta}(\lambda + \rho + i\sigma) \ , \tag{3.62}
$$

where ρ and σ are real scalar fields with vanishing VEVs. With these considerations, we find:

$$
\begin{aligned}
Z_\theta(J, J^*) = \int d\rho \int d\sigma \ \Bigg[& A_0(\rho, \sigma^2) + 2\sum_{n=1}^{\infty} \operatorname{Re}[A_n]|G(\lambda + \rho + i\sigma)|^n \cos n\chi \\
& - \operatorname{Im}[A_n]|G(\lambda + \rho + i\sigma)|^n \sin n\chi \Bigg] \\
& \times \exp\left[\int d^4x \ Je^{i\beta}(\lambda + \rho + i\sigma) + J^* e^{-i\beta}(\lambda + \rho - i\sigma)\right] \ , \\
= \int d\rho \int d\sigma \ \Bigg[& A_0(\rho, \sigma^2) + 2\sum_{n=1}^{\infty} F_n(\rho, \sigma^2)\cos n\chi - G_n(\rho, \sigma^2)\sin n\chi \Bigg] \\
& \times \exp\left[\int d^4x \ Je^{i\beta}(\lambda + \rho + i\sigma) + J^* e^{-i\beta}(\lambda + \rho - i\sigma)\right] \ ,
\end{aligned}
\tag{3.63}
$$

where F_n and G_n are given by:

$$
\begin{aligned}
F_n &= \operatorname{Re}[A_n]|G(\lambda + \rho + i\sigma)|^n \ , \\
G_n &= \operatorname{Im}[A_n]|G(\lambda + \rho + i\sigma)|^n \ .
\end{aligned}
\tag{3.64}
$$

Finally, we assess the previous result in the ϕ VEV regime, where the parameter χ precisely represents the CP and P conserving condition Δ as defined in (3.39). Additionally, we enforce the constraints on the scalar fields ρ and σ, where $\langle\rho\rangle = 0$ and $\langle\sigma\rangle = 0$. We also incorporate the definition of the ϕ VEV as given in (3.41). With all these conditions taken into account, we obtain:

$$
\begin{aligned}
\langle\phi\rangle &= \frac{1}{Z_\theta}\frac{\delta Z_\theta}{\delta J}\bigg|_{J=J^*=0} , \\
&= \frac{1}{Z_\theta}\frac{\delta}{\delta J}\int d\rho \int d\sigma \ \left[A_0(\rho,\sigma^2) + 2\sum_{n=1}^{\infty} F_n(\rho,\sigma^2)\cos n\Delta - G_n(\rho,\sigma^2)\sin n\Delta\right] \\
&\times \exp\left[\int d^4x \ Je^{i\beta}(\lambda+\rho+i\sigma) + J^* e^{-i\beta}(\lambda+\rho-i\sigma)\right]\bigg|_{J=J^*=0} , \\
&= \frac{1}{Z_\theta}\int d\rho \int d\sigma \ \left[A_0(\rho,\sigma^2) + 2\sum_{n=1}^{\infty} F_n(\rho,\sigma^2)\cos n\Delta - G_n(\rho,\sigma^2)\sin n\Delta\right] \\
&\times (e^{i\beta}(\lambda+\rho+i\sigma)) , \\
&= \lambda e^{i\beta} .
\end{aligned} \tag{3.65}
$$

It's important to note that the portion of the product proportional to λ sreadily satisfies this equation. However, simultaneously, it establishes conditions for the ρ and λ VEVs to ensure compliance with the imposed constraints. The condition for the ρ VEV is given by:

$$
\begin{aligned}
0 &= \frac{1}{Z_\theta}\int d\rho \int d\sigma \ \left[A_0(\rho,\sigma^2) + 2\sum_{n=1}^{\infty} F_n(\rho,\sigma^2)\cos n\Delta - G_n(\rho,\sigma^2)\sin n\Delta\right]\rho e^{i\beta} , \\
&= \int d\rho \int d\sigma \ \rho\left[A_0(\rho,\sigma^2) + 2\sum_{n=1}^{\infty} F_n(\rho,\sigma^2)\cos n\Delta - G_n(\rho,\sigma^2)\sin n\Delta\right] , \\
&= \int d\rho \int d\sigma \ \rho\left[A_0(\rho,\sigma^2) + 2\sum_{n=1}^{\infty} F_n(\rho,\sigma^2)\cos n\Delta\right] , \\
&= \langle\rho\rangle .
\end{aligned} \tag{3.66}
$$

We exclusively focus on the F_n component since ρ is a real term in the variable change we implemented. Similarly, the condition for the σ VEV is expressed as follows (notably, σ incorporates an i factor, thus combining with G_n):

$$
\begin{aligned}
0 &= \int d\rho \int d\sigma \ i\sigma e^{i\beta}\left[A_0(\rho,\sigma^2) + 2\sum_{n=1}^{\infty} F_n(\rho,\sigma^2)\cos n\Delta - G_n(\rho,\sigma^2)\sin n\Delta\right] , \\
&= \int d\rho \int d\sigma \ i\sigma\left[A_0(\rho,\sigma^2) + 2\sum_{n=1}^{\infty} F_n(\rho,\sigma^2)\cos n\Delta - G_n(\rho,\sigma^2)\sin n\Delta\right] , \\
&= \int d\rho \int d\sigma \ \sigma\sum_{n=1}^{\infty} G_n(\rho,\sigma^2)\sin n\Delta , \\
&= \langle\sigma\rangle .
\end{aligned} \tag{3.67}
$$

Combining both expressions to present the conditions more explicitly, we have:

$$
\begin{aligned}
\langle\rho\rangle &= \int d\rho \int d\sigma \ \rho\left[A_0(\rho,\sigma^2) + 2\sum_{n=1}^{\infty} F_n(\rho,\sigma^2)\cos n\Delta\right] = 0 , \\
\langle\sigma\rangle &= \int d\rho \int d\sigma \ \sigma\sum_{n=1}^{\infty} G_n(\rho,\sigma^2)\sin n\Delta = 0 .
\end{aligned} \tag{3.68}
$$

Finally, we can compare these conditions with the CP and P condition for Δ in (3.39). These conditions will be satisfied only for $\Delta = 0, \pi$, and these are stationary points of the scalar potential. To determine which is the true minimum, we must examine the potential itself, a task we cannot perform with the generality of the previous argument. However, in the leading-order approximation, we find that the minimum indeed occurs at $\Delta = 0$, resulting in a CP-conserving theory [35, 36].

In this section, we have demonstrated that, in the absence of an additional U(1) chiral symmetry (referred to as U(1) Peccei-Quinn) at the lagrangian level, we cannot construct a theory for strong interactions where all fermion masses are real, and the CP-violating θ parameter is zero. It is important to note that this new $\mathrm{U}(1)_{PQ}$ chiral symmetry is distinct from the U(1) chiral symmetry we previously established as not being a true symmetry of the theory. Additionally, we imposed that the new $\mathrm{U}(1)_{PQ}$ is spontaneously broken by instantons (subject to the same anomaly condition), leading to the emergence of a new Nambu-Goldstone (NG) boson. Certainly, it does not lead to the same strong coupled and mass bound NG boson of the U(1) problem.

While the scalar fields we are dealing with do not directly interact with instantons, their interaction occurs only at the quantum level through the coupling (proportional to G) they have with quarks. Consequently, the particle that emerges from this mechanism is anticipated to be a very light pseudo-Nambu-Goldstone (pNG) boson, named the *axion* by Franz Wilczek and Steven Weinberg [38, 39]. In the subsequent chapter, we will delve into the properties and dynamics of axions, exploring the most prominent axion models.

4 Axion

4.1 Axion fundamentals

As illustrated earlier, the new U(1) Peccei-Quinn (PQ) symmetry eliminates the CP-violating parameter θ by introducing a dynamical CP-conserving field ($a(x)$) - the axion. Under a PQ transformation, this field changes as:

$$a(x) \to a(x) + \alpha f_a \ , \tag{4.1}$$

where f_a is the axion decay constant, associated with the breaking of $\mathrm{U(1)_{PQ}}$, and $\alpha = -\theta/\xi$. To maintain $\mathrm{U(1)_{PQ}}$ invariance within the Standard Model, the $\mathcal{L}_{\mathrm{SM}}$ must be extended to include the axion's interactions. The lagrangian for this interaction is expressed as:

$$\mathcal{L}_{\mathrm{TOTAL}} = \mathcal{L}_{\mathrm{SM}} + \theta \frac{g_s^2}{32\pi^2} G_b^{\mu\nu} \tilde{G}_{b\mu\nu} - \frac{1}{2}\partial_\mu a \partial^\mu a + \mathcal{L}_{\mathrm{Int}}[\partial^\mu a/f_a, \Psi] + \xi \frac{a}{f_a}\frac{g_s^2}{32\pi^2} G_b^{\mu\nu}\tilde{G}_{b\mu\nu} \ . \tag{4.2}$$

Here, the last term is designed to ensure the vanishing of the CP-invariant term after a PQ transformation. Additionally, it introduces a chiral anomaly for the $\mathrm{U(1)_{PQ}}$ current:

$$\partial_\mu j^\mu_{PQ} = \xi \frac{g_s^2}{32\pi^2} G_b^{\mu\nu}\tilde{G}_{b\mu\nu} \ . \tag{4.3}$$

Moreover, this term serves as an effective potential for the axion field, which, as previously derived, exhibits a minimum at $\langle a \rangle = -\theta f_a/\xi$

$$\left\langle \frac{\partial V_{\mathrm{eff}}}{\partial a} \right\rangle = -\frac{\xi}{f_a}\frac{g_s^2}{32\pi^2} \left\langle G_b^{\mu\nu}\tilde{G}_{b\mu\nu} \right\rangle \Big|_{\langle a \rangle = -\frac{\theta}{\xi} f_a} = 0 \ . \tag{4.4}$$

It's worth noting that the Lagrangian can now be expressed in terms of $a_{\mathrm{phys}} = a - \langle a \rangle$, eliminating the CP-violating term. The expansion of the effective potential around the minimum yields the axion mass:

$$m_a^2 = \left\langle \frac{\partial^2 V_{\mathrm{eff}}}{\partial a^2} \right\rangle = -\frac{\xi}{f_a}\frac{g_s^2}{32\pi^2}\frac{\partial}{\partial a} \left\langle G_b^{\mu\nu}\tilde{G}_{b\mu\nu} \right\rangle \Big|_{\langle a \rangle = -\frac{\theta}{\xi} f_a} \ . \tag{4.5}$$

4.2 Peccei-Quinn-Weinberg-Wilczek (PQWW) axion

In the original PQ model, the symmetry breaking coincided with the electroweak (EW) breaking parameter $v_{\mathrm{SM}} = 246.22$ GeV [13], implying $f_a = v_{\mathrm{SM}}$. However, it is not a requirement of the axion theory; in principle, f_a can take any value. For instance, if the axion decay constant is much larger than the EW scale ($f_a \gg v_{\mathrm{SM}}$), then the axion becomes very light, weakly coupled, and exceptionally long-lived [40]. Naturally, if f_a were much smaller than the EW scale, the axion would strongly couple with Standard Model (SM) matter, leading to its detection or exclusion. However, the theory does not prescribe a fixed value for f_a. Despite the PQWW axion being long ago excluded by experiment [41], it remains useful to first derive the properties of the PQWW axion and then generalize it, as more modern models share the same theoretical basis.

To maintain $\mathrm{U}(1)_{\mathrm{PQ}}$ invariance within the SM, two Higgs fields are introduced to absorb independent chiral transformations of the u and d quarks, as well as leptons. The relevant Yukawa terms involving these new fields are given by:

$$\mathcal{L}_{\mathrm{Yukawa}} = \Gamma^u_{ij}\overline{Q}_{Li}\Phi_1 u_{Rj} + \Gamma^d_{ij}\overline{Q}_{Li}\Phi_2 d_{Rj} + \Gamma^l_{ij}\overline{L}_{Li}\Phi_2 l_{Rj} + \mathrm{h.c}\ . \tag{4.6}$$

The axion serves as the common phase field in Φ_1 and Φ_2. Defining $x \equiv v_2/v_1$ and $v_{\mathrm{SM}} = \sqrt{v_1^2 + v_2^2}$, the two Higgs fields can be expressed as:

$$\begin{aligned} \Phi_1 &= \frac{v_1}{\sqrt{2}} e^{iax/v_{\mathrm{SM}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ , \\ \Phi_2 &= \frac{v_2}{\sqrt{2}} e^{ia/xv_{\mathrm{SM}}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ . \end{aligned} \tag{4.7}$$

It's important to note that the x transitions from the numerator in the exponential of the first field to the denominator in the exponential of the second field. The Yukawa lagrangian is symmetric under the $\mathrm{U}(1)_{\mathrm{PQ}}$ transformation specified as:

$$\begin{aligned} a &\to a + \alpha v_{\mathrm{SM}}\ , \\ u_{Rj} &\to e^{-i\alpha x} u_{Rj}\ , \\ d_{Rj} &\to e^{-i\alpha/x} d_{Rj}\ , \\ l_{Rj} &\to e^{-i\alpha/x} l_{Rj}\ , \end{aligned} \tag{4.8}$$

the last 3 can be written as infinitesimal transformations as:

$$\begin{aligned} u_{Rj} &\to (1 - i\alpha x) u_{Rj}\ , \\ d_{Rj} &\to (1 - i\alpha/x) d_{Rj}\ , \\ l_{Rj} &\to (1 - i\alpha/x) l_{Rj}\ . \end{aligned} \tag{4.9}$$

This symmetry can be readily verified by applying the transformation to (4.6)

$$\begin{aligned}
\mathcal{L}'_{\text{Yukawa}} &= \Gamma^u_{ij}\overline{Q}_{Li}\frac{v_1}{\sqrt{2}}e^{i(a+\alpha v_{\text{SM}})x/v_{\text{SM}}}\begin{pmatrix}1\\0\end{pmatrix}e^{-i\alpha x}u_{Rj} \\
&+ \Gamma^d_{ij}\overline{Q}_{Li}\frac{v_2}{\sqrt{2}}e^{i(a+\alpha v_{\text{SM}})/xv_{\text{SM}}}\begin{pmatrix}0\\1\end{pmatrix}e^{-i\alpha/x}d_{Rj} \\
&+ \Gamma^l_{ij}\overline{L}_{Li}\frac{v_2}{\sqrt{2}}e^{i(a+\alpha v_{\text{SM}})/xv_{\text{SM}}}\begin{pmatrix}0\\1\end{pmatrix}e^{-i\alpha/x}l_{Rj} + h.c \ , \\
&= \Gamma^u_{ij}\overline{Q}_{Li}\Phi_1 e^{i\alpha v_{\text{SM}}x/v_{\text{SM}}}e^{-i\alpha x}u_{Rj} + \Gamma^d_{ij}\overline{Q}_{Li}\Phi_2 e^{i\alpha v_{\text{SM}}/xv_{\text{SM}}}e^{-i\alpha/x}d_{Rj} \\
&+ \Gamma^l_{ij}\overline{L}_{Li}\Phi_2 e^{i\alpha v_{\text{SM}}/xv_{\text{SM}}}e^{-i\alpha/x}l_{Rj} + h.c \ , \\
&= \Gamma^u_{ij}\overline{Q}_{Li}\Phi_1 u_{Rj} + \Gamma^d_{ij}\overline{Q}_{Li}\Phi_2 d_{Rj} + \Gamma^l_{ij}\overline{L}_{Li}\Phi_2 l_{Rj} + h.c \ , \\
&= \mathcal{L}_{\text{Yukawa}} \ .
\end{aligned} \tag{4.10}$$

We will focus on the quark components. The symmetry current for $\text{U}(1)_{\text{PQ}}$ is calculated using the general equation for conserved currents (A.17) [42]:

$$j^\mu_a = -\left[\mathcal{L}g^\mu_\rho - \frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\partial_\rho\Phi\right]\frac{\delta x^\rho}{\delta\omega^a} - \frac{\partial\mathcal{L}}{\partial[\partial_\mu\Phi]}\frac{\delta\Phi}{\delta\omega^a} \ , \tag{4.11}$$

where ω^a is the transformation parameter and Φ represents any field that is affected by the transformation. Since we are dealing with a gauge symmetry, there is no variation in the space-time variable, and therefore, the first term vanishes, simplifying the calculation. In our case, the calculation proceeds as follows:

$$j^\mu_{PQ} = -\frac{\partial\mathcal{L}}{\partial[\partial_\mu a]}\frac{\delta a}{\delta\alpha} - \sum_i\frac{\partial\mathcal{L}}{\partial[\partial_\mu u_{iR}]}\frac{\delta u_{iR}}{\delta\alpha} - \sum_i\frac{\partial\mathcal{L}}{\partial[\partial_\mu d_{iR}]}\frac{\delta d_{iR}}{\delta\alpha} \ . \tag{4.12}$$

The lagrangian needed for this calculation is the $\text{U}(1)_{\text{PQ}}$-invariant SM lagrangian with Yukawa terms. However, due to the structure of the current equation, only the kinetic terms are required. These terms involve the axion and up and down quarks, given by:

$$\mathcal{L}_{\text{kinetic}} = -\frac{1}{2}\partial_\mu a\partial^\mu a + i\overline{u}_{iR}\partial\!\!\!/ u_{iR} + i\overline{d}_{iR}\partial\!\!\!/ d_{iR} \ . \tag{4.13}$$

Calculating each field separately for clarity. The axion term is:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial[\partial_\mu a]} &= -\partial_\mu a \ , \\
\frac{\delta a}{\delta\alpha} &= \frac{\delta(a+\alpha v_{\text{SM}})}{\delta\alpha} = v_{\text{SM}} \ .
\end{aligned} \tag{4.14}$$

The up quark term is:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial[\partial_\mu u]} &= \frac{\partial[i\overline{u}\partial\!\!\!/ u]}{\partial[\partial_\mu u]} = i\overline{u}\gamma^\mu \ , \\
\frac{\delta u}{\delta\alpha} &= \frac{\delta[(1-i\alpha x)u]}{\delta\alpha} = -ixu \ .
\end{aligned} \tag{4.15}$$

The down quark term is:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial[\partial_\mu d]} &= \frac{\partial[i\bar{d}\partial\!\!\!/ d]}{\partial[\partial_\mu d]} = i\bar{d}\gamma^\mu \ , \\ \frac{\delta d}{\delta\alpha} &= \frac{\delta[(1-i\alpha/x)d]}{\delta\alpha} = -\frac{i}{x}d \ . \end{aligned} \tag{4.16}$$

Putting it all together, we have:

$$\begin{aligned} j^\mu_{PQ} &= -\frac{\partial\mathcal{L}}{\partial[\partial_\mu a]}\frac{\delta a}{\delta\alpha} - \sum_i \frac{\partial\mathcal{L}}{\partial[\partial_\mu u_{iR}]}\frac{\delta u_{iR}}{\delta\alpha} - \sum_i \frac{\partial\mathcal{L}}{\partial[\partial_\mu d_{iR}]}\frac{\delta d_{iR}}{\delta\alpha} \ , \\ &= -(-v_{\mathrm{SM}}\partial_\mu a) - (\sum_i x\bar{u}_{iR}\gamma^\mu u_{iR}) - (\sum_i \frac{1}{x}\bar{d}_{iR}\gamma^\mu d_{iR}) \ , \\ &= v_{\mathrm{SM}}\partial_\mu a - x\sum_i \bar{u}_{iR}\gamma^\mu u_{iR} - \frac{1}{x}\sum_i \bar{d}_{iR}\gamma^\mu d_{iR} \ . \end{aligned} \tag{4.17}$$

The anomaly coefficient ξ in (4.2) is identified as

$$\xi = \frac{N}{2}\left(x + \frac{1}{x}\right) \equiv N_g\left(x + \frac{1}{x}\right) \ , \tag{4.18}$$

N is more explicitly defined in (4.48). To compute the axion mass m_a, it is beneficial to separate the effects of interactions with the light quarks from the rest. These interactions can be inferred from the theory by constructing an appropriate chiral effective lagrangian, while the effects of heavy quarks can be accounted for through their contribution to the current chiral anomaly.

As done in the U(1) problem chapter, for two light quarks, one introduces a 2×2 matrix of Nambu-Goldstone fields:

$$\Sigma = \exp\left(i\frac{\vec{\tau}\cdot\vec{\pi} + \eta}{f_\pi}\right) \ . \tag{4.19}$$

The meson sector of the light quark theory, ignoring the Yukawa interactions, is embodied in the $\mathrm{U}(2)_V \times \mathrm{U}(2)_A$ invariant effective lagrangian, the same as the one discussed in the U(1) problem chapter. Therefore, we will not delve into the details here.

The chiral lagrangian is written as:

$$\mathcal{L}_{\text{chiral}} = -\frac{f_\pi^2}{4}\,\mathrm{Tr}\Big[\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\Big] \ , \tag{4.20}$$

we must add $\mathrm{U}(2)_V \times \mathrm{U}(2)_A$ breaking terms to (4.20), which function similarly to the $\mathrm{U}(1)_{\mathrm{PQ}}$ invariant Yukawa interaction for the up and down quarks. With this consideration, we have:

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}(f_\pi m_\pi^0)^2\,\mathrm{Tr}\Big[\Sigma AM + (\Sigma AM)^\dagger\Big] \ , \tag{4.21}$$

where

$$\begin{aligned} A &= \begin{pmatrix} e^{-iax/v_{\mathrm{SM}}} & 0 \\ 0 & e^{-ia/xv_{\mathrm{SM}}} \end{pmatrix} \ , \\ M &= \begin{pmatrix} \frac{m_u}{m_u+m_d} & 0 \\ 0 & \frac{m_d}{m_u+m_d} \end{pmatrix} \ . \end{aligned} \tag{4.22}$$

Note the resemblance of equations (2.35)(2.34). Also note that the invariance of $\mathcal{L}_{\text{mass}}$ under PQ symmetry requires:

$$\Sigma \to \Sigma \begin{pmatrix} e^{i\alpha x} & 0 \\ 0 & e^{i\alpha/x} \end{pmatrix} \ . \tag{4.23}$$

However, this mass lagrangian does not capture the complete physics associated with the breakdown of $\mathrm{U}(2)_A$. In fact, the quadratic terms in the mass lagrangian involving neutral fields are given by:

$$\mathcal{L}^{(2)}_{\text{mass}} = -\frac{(m^0_\pi)^2}{2(m_u+m_d)}\left[m_u\left(\pi^0+\eta-\frac{xf_\pi}{v_{\text{SM}}}a\right)^2 + m_d\left(\eta-\pi^0-\frac{f_\pi}{xv_{\text{SM}}}a\right)^2\right] \ , \tag{4.24}$$

which results in:

$$\frac{m^2_\eta}{m^2_\pi} = \frac{m_d}{m_u} \approx 1.6 \ , \tag{4.25}$$

which contradicts experiment. This discrepancy with experimental observations becomes apparent when considering the most recent data from [13], where $m_\eta = 547.862 \pm 0.017$ MeV, $m^\pm_\pi = 139.57039 \pm 0.00018$ MeV, and $m^0_\pi = 134.9768 \pm 0.0005$ MeV. This results in a ratio of

$$\frac{m^2_\eta}{m^2_{\pi^\pm}} = 15.41 \ . \tag{4.26}$$

If $\mathcal{L}_{\text{mass}}$ was the sole factor, we would essentially be reformulating the U(1) problem in a different theoretical language, using the effective lagrangian approach. Furthermore, the axion in this specific context remains massless. This underscores the need for additional considerations or modifications within the theoretical framework to align with experimental observations.

The solution to this issue involves incorporating an additional mass term into the effective lagrangian that accounts for the anomalies in both $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_{\text{PQ}}$ symmetries. This term is expressed as follows [40]:

$$\mathcal{L}_{\text{anomaly}} = -\frac{(m^0_\eta)^2}{2}\left[\eta + \frac{f_\pi}{v_{\text{SM}}}\frac{(N_g-1)(x+1/x)}{2}a\right]^2 \ , \tag{4.27}$$

where

$$(m^0_\eta)^2 \approx m^2_\eta \gg m^2_\pi \ . \tag{4.28}$$

This addition not only provides the correct mass for the η but also produces a mass for the axion.

The coefficient attached to the axion field in (4.27) signifies the relative strength of the couplings of the axion and η to the field strengths $G\tilde{G}$ due to the anomalies in $\mathrm{U}(1)_{\text{PQ}}$ and $\mathrm{U}(1)_A$. It may raise the question of why this ratio is proportional to (N_g-1) rather than solely N_g, which would make this ratio proportional to the anomaly coefficient ξ.

The reason for this lies in the fact that the mass lagrangian already encompasses the light quark interactions of axions. Therefore, without the -1 factor, these interactions would be included twice. This way, only the heavy quarks' contribution to the PQ anomaly is considered in the anomaly lagrangian.

Diagonalizing the quadratic terms in the mass and anomaly lagrangians provides us with both the axion mass and the parameters for axion-pion and axion-η mixing for the PQ model.

It is convenient to define

$$\overline{m}_a = m_\pi \frac{f_\pi}{v_{\rm SM}} \frac{\sqrt{m_u m_d}}{m_u + m_d} \approx 25 \text{ keV} . \tag{4.29}$$

One finds

$$m_a = \lambda_m \overline{m}_a \ , \tag{4.30}$$

$$\xi_{a\pi} = \lambda_3 \frac{f_\pi}{v_{\rm SM}} \ , \tag{4.31}$$

$$\xi_{a\eta} = \lambda_0 \frac{f_\pi}{v_{\rm SM}} \ , \tag{4.32}$$

where

$$\begin{aligned} \lambda_m &= N_g \left(x + \frac{1}{x} \right) \ , \\ \lambda_3 &= \frac{1}{2} \left[\left(x - \frac{1}{x} \right) - N_g \left(x + \frac{1}{x} \right) \frac{m_d - m_u}{m_u + m_d} \right] \ , \\ \lambda_0 &= \frac{1}{2} (1 - N_g) \left(x + \frac{1}{x} \right) \ . \end{aligned} \tag{4.33}$$

In addition to these three parameters, every axion model is characterized by how it couples to ordinary matter, especially the axion coupling to two photons, the coupling to electrons, and the coupling to nucleons. The lagrangian that describes the axion-photon interaction is:

$$\mathcal{L}_{a\gamma\gamma} = \frac{\alpha}{4\pi} g_{a\gamma\gamma} \frac{a_{\rm phys}}{f_a} F^{\mu\nu} \tilde{F}_{\mu\nu} \ , \tag{4.34}$$

where $g_{a\gamma\gamma}$ is the coupling of this interaction. We are going to find it for the PQ model. The $g_{a\gamma\gamma}$ follows from the electromagnetic anomaly of the PQ current, which is analogous to (4.3):

$$\partial_\mu j^\mu_{PQ} = \frac{\alpha}{4\pi} \xi_\gamma F_{\mu\nu} \tilde{F}^{\mu\nu} \ , \tag{4.35}$$

where ξ_γ receives contributions from both quarks and leptons, and we obtain:

$$\begin{aligned} \xi_\gamma &= N_g \left[3 \left(\frac{2}{3} \right)^2 x + \left[3 \left(\frac{1}{3} \right)^2 + (-1)^2 \right] \frac{1}{x} \right] \ , \\ &= \frac{4}{3} N_g \left(x + \frac{1}{x} \right) \ . \end{aligned} \tag{4.36}$$

As done previously, we must remove the light quark contribution of the axion in the anomaly. In doing so, we find:

$$\xi_\gamma^{\text{eff}} = \frac{4}{3}N_g\left(x+\frac{1}{x}\right) - \frac{4}{3}x - \frac{1}{3}\frac{1}{x} \ , \tag{4.37}$$

and adding back the contribution arising from the coupling of the π^0 and η to two photons via axion-pion and axion-η mixing $\lambda_3 + \frac{5}{3}\lambda_0$, we get:

$$\begin{aligned}
g_{a\gamma\gamma} &= \frac{4}{3}N_g\left(x+\frac{1}{x}\right) - \frac{4}{3}x - \frac{1}{3}\frac{1}{x} + \lambda_3 + \frac{5}{3}\lambda_0 \ , \\
&= \frac{4}{3}N_g\left(x+\frac{1}{x}\right) - \frac{4}{3}x - \frac{1}{3}\frac{1}{x} \\
&+ \frac{1}{2}\left[\left(x-\frac{1}{x}\right) - N_g\left(x+\frac{1}{x}\right)\frac{m_d-m_u}{m_u+m_d}\right] + \frac{5}{3}\left[\frac{1}{2}(1-N_g)\left(x+\frac{1}{x}\right)\right] \ , \\
&= \frac{4}{3}N_g\left(x+\frac{1}{x}\right) - \frac{1}{2}\left[N_g\left(x+\frac{1}{x}\right)\frac{m_d-m_u}{m_u+m_d}\right] - \frac{5}{3}\left[\frac{1}{2}N_g\left(x+\frac{1}{x}\right)\right] \\
&- \frac{4}{3}x - \frac{1}{3}\frac{1}{x} + \frac{1}{2}\left(x-\frac{1}{x}\right) + \frac{5}{3}\left[\frac{1}{2}\left(x+\frac{1}{x}\right)\right] \ , \\
&= \left(\frac{8}{6}-\frac{5}{6}\right)N_g\left(x+\frac{1}{x}\right) - \frac{1}{2}\left[N_g\left(x+\frac{1}{x}\right)\frac{m_d-m_u}{m_u+m_d}\right] \\
&+ \frac{-8+3+5}{6}x + \frac{-2-3+5}{6}\frac{1}{x} \ , \\
&= \frac{1}{2}N_g\left(x+\frac{1}{x}\right)\left(1-\frac{m_d-m_u}{m_u+m_d}\right) \ , \\
&= \frac{1}{2}N_g\left(x+\frac{1}{x}\right)\left(\frac{m_u+m_d-m_d+m_u}{m_u+m_d}\right) \ , \\
&= \frac{1}{2}N_g\left(x+\frac{1}{x}\right)\left(\frac{2m_u}{m_u+m_d}\right) \ , \\
&= N_g\left(x+\frac{1}{x}\right)\left(\frac{m_u}{m_u+m_d}\right) \ .
\end{aligned} \tag{4.38}$$

Writing it separately for clarity, we find the axion-photon coupling to be:

$$g_{a\gamma\gamma} = N_g\left(x+\frac{1}{x}\right)\left(\frac{m_u}{m_u+m_d}\right) \ . \tag{4.39}$$

In fact, the original PQ model, with $f_a = v_{\text{SM}}$, was ruled out by experiment long ago. For example, the branching ratio measurement of $K^+ \to \pi^+ + a$ obtained at the Japanese National Laboratory for High Energy Physics (KEK) [43] had a theoretical estimate of [41]:

$$\text{BR}(K^+ \to \pi^+ + a) \approx 3\times 10^{-5}\lambda_0 = 3\times 10^{-5}\left(x+\frac{1}{x}\right)^2 \ , \tag{4.40}$$

which significantly exceeded the bounds measured at KEK:

$$\text{BR}(K^+ \to \pi^+ + \text{invisible}) \leq 3.8\times 10^{-8} \ . \tag{4.41}$$

This, however, does not exclude models in which f_a is much greater than the electroweak scale v_{SM}; these models are known as invisible axion models.

4.3 Invisible axion models

The distinct feature of invisible axion models is the introduction of new scalar fields that carry PQ charge but are SU(2)×U(1) singlets. Consequently, the VEVs of these fields can have a much larger scale than the one set by the weak interactions.

Two most famous types of models have been proposed:

1. The Kim [44] and Shifman, Vainshtein, Zakharov [45], known as the KSVZ model, introduces a scalar field σ with $f_a = \langle\sigma\rangle \gg v_{\mathrm{SM}}$ and a super heavy quark Q with $m_Q \approx f_a$ as the only field carrying PQ charge.

2. The Dine, Fischler, Srednicki [46] and Zhitnisky [47], known as the DFSZ model, introduces to the original PQ model a scalar field ϕ that carries PQ charge, and the axion decay constant is $f_a = \langle\phi\rangle \gg v_{\mathrm{SM}}$. However, this new scalar is not the only field that carries PQ charge; ordinary quarks, leptons, and Higgs fields also carry it.

Initially, we will construct a generic invisible axion model and then explore the specific features of the KSVZ and DFSZ models.

The key element in any axion model is the PQ symmetry. PQ symmetry is global and remains valid only at the classical level, it also must be affected by a color anomaly. Additionally, it may or may not exhibit an electromagnetic (EM) anomaly.

This symmetry gives rise to a current expressed as follows:

$$\partial_\mu j^\mu_{PQ} = \frac{Ng^2}{16\pi^2}(G\tilde{G}) + \frac{Ee^2}{16\pi^2}(F\tilde{F}) \ . \tag{4.42}$$

Here,

$$(G\tilde{G}) = \frac{1}{2}\epsilon^{\mu\nu\sigma\tau}G^b_{\mu\nu}G^b_{\sigma\tau} \ , \tag{4.43}$$

where $G^b_{\mu\nu}$ is the color field strength, and

$$(F\tilde{F}) = \frac{1}{2}\epsilon^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau} = -4\vec{E}\cdot\vec{B} \ , \tag{4.44}$$

in Heaviside-Lorentz units, with $F_{\mu\nu}$ denoting the field strength of the EM interactions. The current will be normalized with the condition that N is an integer distinct from zero. If color triality is correlated with electric charge in the standard manner for all fermions undergoing non-trivial transformations under PQ, then E must be one-third of an integer, though it can also be zero.

To explicitly normalize the current, we begin by representing all fermion fields as left-handed, two-component Weyl spinors ψ, each belonging to a specific representation of $\mathrm{SU}(3)_C\times\mathrm{U}(1)_{\mathrm{EM}}$. We assign a PQ charge X_i to each fermion field ψ_i, choosing these X_i values to be integers with the least common denominator of 1 [48]. The lagrangian for these fermion fields incorporates Yukawa couplings to the PQ complex scalar field ϕ in the form:

$$\mathcal{L} \supset \lambda_{nij}\phi_n\psi_i\psi_j + \kappa_{nij}\phi_n^\dagger\psi_i\psi_j + \text{h.c} \ . \tag{4.45}$$

If $\lambda_{nij} \neq 0$ for specific values of n, i, and j, then ϕ_n is assigned a PQ charge $Z_n = -X_i - X_j$. Conversely, if $\kappa_{nij} \neq 0$, then $Z_n = X_i + X_j$. The corresponding PQ current is expressed as:

$$j_\mu^{PQ} = \sum_n \frac{1}{i} Z_n \phi_n^\dagger \overleftrightarrow{\partial}_\mu \phi_n + \sum_i X_i \psi_i^\dagger \sigma_\mu \psi_i \ , \tag{4.46}$$

where σ_μ is the Pauli 4-vector defined as:

$$\sigma_\mu = (I, \vec{\sigma}) \ . \tag{4.47}$$

The anomaly coefficients N and E can be expressed as follows:

$$\begin{aligned} N &= \sum_i X_i T(R_i) \ , \\ E &= \sum_i X_i Q_i^2 D(R_i) \ , \end{aligned} \tag{4.48}$$

where $T(R_i)$ is the index of $\mathrm{SU}(3)_C$ representation of ψ_i, with tabulated values available in [49](for example, $T(3) = 1/2$,$T(8) = 3$). $D(R_i)$ denotes the dimension of the representation (e.g, $D(3) = 3$), and Q_i is the electric charge of ψ_i. Given that the number of color triplets must equal the number of anti-triplets, N is necessarily an integer.

A noteworthy observation is that in any Grand Unified Theory (GUT) where color and electromagnetism are embedded in the same manner as in Georgi-Glashow SU(5), N and E exhibit a relation [50]:

$$\frac{E}{N} = \frac{8}{3} \quad (\text{GUT}) \ . \tag{4.49}$$

Performing a PQ transformation of the form:

$$\begin{aligned} \phi_n &\to e^{iZ_n\alpha}\phi_n \ , \\ \psi_i &\to e^{iX_i\alpha}\psi_i \ , \end{aligned} \tag{4.50}$$

where the transformation is periodic in α with a period of 2π, induces a change in the QCD vacuum angle:

$$\theta \to \theta + 2N\alpha \ , \tag{4.51}$$

where N is the anomaly coefficient present in (4.42). Consequently, the PQ symmetry is explicitly broken, as different values of θ correspond to distinct physics. An unbroken Z(2N) subgroup is identified, corresponding to $\alpha = 2\pi n/2N$, where it can be readily observed that the vacuum angle remains invariant, with $n = 0, 1, ..., 2N - 1$.

Spontaneous breaking of the PQ symmetry is a crucial requirement. If this symmetry breaking is achieved through the vacuum expectation values (VEVs) of the new scalar field

ϕ_n, we can determine the axion decay constant f_a in terms of these VEVs. We express ϕ_n as:

$$\phi_n = 2^{-1/2}(V_n + \rho_n)e^{i\chi_n/V_n} \ , \tag{4.52}$$

where V_n is the VEV of ϕ_n, and ρ_n and χ_n are fields. The mass of ρ_n is on the order of $\sqrt{\lambda}V_n$, where λ is a coupling in the Higgs potentials. Consequently, we neglect the ρ_n field in our analysis. With these considerations, the PQ current takes the form:

$$\begin{aligned} j_\mu^{PQ} &= \sum_n \frac{1}{i} Z_n \phi_n^\dagger \overleftrightarrow{\partial}_\mu \phi_n + \sum_i X_i \psi_i^\dagger \sigma_\mu \psi_i \ , \\ &= \sum_n \frac{1}{i} Z_n \phi_n^\dagger \partial_\mu \phi_n - \frac{1}{i} Z_n \partial_\mu \phi_n^\dagger \phi_n + \sum_i X_i \psi_i^\dagger \sigma_\mu \psi_i \ , \\ &= \sum_n \frac{1}{i} Z_n \phi_n^\dagger \phi_n \frac{i}{V_n} \partial_\mu \chi_n - \frac{1}{i} Z_n \frac{-i}{V_n} \partial_\mu \chi_n \phi_n^\dagger \phi_n + \sum_i X_i \psi_i^\dagger \sigma_\mu \psi_i \ , \\ &= \sum_n Z_n \left(\frac{V_n^2}{2}\right) \frac{1}{V_n} \partial_\mu \chi_n + Z_n \frac{1}{V_n} \partial_\mu \chi_n \left(\frac{V_n^2}{2}\right) + \sum_i X_i \psi_i^\dagger \sigma_\mu \psi_i \ , \\ &= \sum_n Z_n V_n \partial_\mu \chi_n + \sum_i X_i \psi_i^\dagger \sigma_\mu \psi_i \ . \end{aligned} \tag{4.53}$$

The axion field, denoted as a, is defined by the expression:

$$a = f_a^{-1} \sum_n Z_n V_n \chi_n \ , \tag{4.54}$$

where the axion decay constant is determined by:

$$f_a^2 = \sum_m Z_n^2 V_n^2 \ . \tag{4.55}$$

The PQ current becomes:

$$j_\mu^{PQ} = f_a \partial_\mu a + \sum_i X_i \psi_i^\dagger \vec{\sigma}^\mu \psi_i \ . \tag{4.56}$$

Under the PQ transformation described in (4.50), the axion field undergoes the following change:

$$a \to a + f_a \alpha \ . \tag{4.57}$$

Neglecting explicit breaking, this implies that the current can annihilate a Goldstone boson with the following amplitude:

$$\langle 0| \, j_\mu^{PQ}(k) \, |a(k)\rangle = f_a k_\mu e^{ikx} \ . \tag{4.58}$$

The expression can be easily obtained through a Fourier transformation on the axion field. The normalization of the axion state $|a(k)\rangle$ is given by:

$$\langle a(k')|a(k)\rangle = (2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{k'}) \ , \tag{4.59}$$

with this normalization, the current, denoted as:

$$j^3_\mu = \frac{1}{2}\left(\overline{u}\gamma_\mu\gamma_5 u - \overline{d}\gamma_\mu\gamma_5 d\right) \ , \tag{4.60}$$

satisfies the relation:

$$\langle 0|\, j^3_\mu(x) \left|\pi^0(k)\right\rangle = f_\pi k_\mu e^{ikx} \ , \tag{4.61}$$

with $f_\pi = 130.41 \pm 0.21$ MeV [18].

The PQ charges are intentionally selected to be orthogonal, meaning they are uncorrelated, with all anomaly-free, conserved charges—whether they are gauged or ungauged, spontaneously broken or unbroken. The normalization scheme specified in (4.42) sets the framework for this choice of PQ charges. This particular selection is unique, and it is also assumed that, in addition to the axion, there are no other Nambu-Goldstone bosons present.

The spontaneous breaking of the PQ symmetry may lead to the spontaneous breaking of some or all of the Z(2N) subgroup of $\mathrm{U(1)_{PQ}}$ that is not explicitly broken [51,52]. However, it is also possible that the spontaneous breaking does not affect any part of this subgroup. The spontaneous breaking of Z(2N), even if only partial, poses a theoretical challenge. This is because the broken generators of this subgroup are not incorporated into a gauged group like $\mathrm{U(1)_{PQ}}$ [53]. As a result, there emerges a discrete set of degenerate, inequivalent vacuum states, potentially leading to a cosmological domain wall problem [51].

Now, with a well-defined PQ current, the next step in this generic model involves computing the mass of the axion. To achieve this, current algebra is employed, and an axion current is defined as:

$$j^a_\mu = j^{PQ}_\mu - N(1+z+w)^{-1}[\overline{u}\gamma_\mu\gamma_5 u + z\overline{d}\gamma_\mu\gamma_5 d + w\overline{s}\gamma_\mu\gamma_5 s] \ , \tag{4.62}$$

where z, w are arbitrary numbers [54]. Notably, this current incorporates the form of the current that annihilates/creates a Goldstone boson, as seen in (4.60). The axion current satisfies:

$$\begin{aligned} \partial^\mu j^a_\mu = & \frac{Ne^2}{16\pi^2}(F\tilde{F})_{EM}\left[\frac{E}{N} - \frac{2}{3}\frac{4+z+w}{1+z+w}\right] \\ & - \frac{2N}{1+z+w}[m_u\overline{u}i\gamma_5 u + zm_d\overline{d}i\gamma_5 d + wm_s\overline{s}i\gamma_5 s] \ , \end{aligned} \tag{4.63}$$

the first term originates from (4.42). To derive the second term, the Dirac equation is applied:

$$\begin{aligned} (i\partial\!\!\!/ - m)\psi = 0 \ , \\ \overline{\psi}(i\partial\!\!\!/ + m) = 0 \ , \end{aligned} \tag{4.64}$$

utilizing the anticommutation of the gamma matrices $\{\gamma_\mu, \gamma_5\} = 0$, the calculation proceeds as follows:

$$
\begin{aligned}
\partial^\mu j_\mu^{a(2nd)} &= -\partial^\mu \left[\frac{N}{1+z+w} [\overline{u}\gamma_\mu\gamma_5 u + z\overline{d}\gamma_\mu\gamma_5 d + w\overline{s}\gamma_\mu\gamma_5 s] \right] , \\
&= -\frac{N}{1+z+w} [\partial^\mu \overline{u}\gamma_\mu\gamma_5 u + \overline{u}\gamma_\mu\gamma_5 \partial^\mu u + z\partial^\mu \overline{d}\gamma_\mu\gamma_5 d + z\overline{d}\gamma_\mu\gamma_5 \partial^\mu d \\
&+ w\partial^\mu \overline{s}\gamma_\mu\gamma_5 s + w\overline{s}\gamma_\mu\gamma_5 \partial^\mu s] , \\
&= -\frac{N}{1+z+w} [\not{\partial}\overline{u}\gamma_5 u - \overline{u}\gamma_5\gamma_\mu \partial^\mu u + z\not{\partial}\overline{d}\gamma_5 d - z\overline{d}\gamma_5\gamma_\mu \partial^\mu d \\
&+ w\not{\partial}\overline{s}\gamma_5 s - w\overline{s}\gamma_5\gamma_\mu \partial^\mu s] , \\
&= -\frac{N}{1+z+w} [\not{\partial}\overline{u}\gamma_5 u - \overline{u}\gamma_5 \not{\partial} u + z\not{\partial}\overline{d}\gamma_5 d - z\overline{d}\gamma_5 \not{\partial} d \\
&+ w\not{\partial}\overline{s}\gamma_5 s - w\overline{s}\gamma_5 \not{\partial} s] , \\
&= -\frac{N}{1+z+w} [(im_u\overline{u})\gamma_5 u - \overline{u}\gamma_5(-im_u u) + z(im_d\overline{d})\gamma_5 d - z\overline{d}\gamma_5(-im_d d) \\
&+ w(im_s\overline{s})\gamma_5 s - w\overline{s}\gamma_5(-im_s s)] , \\
&= -\frac{N}{1+z+w} [2m_u\overline{u}i\gamma_5 u + 2zm_d\overline{d}i\gamma_5 d + 2wm_s\overline{s}i\gamma_5 s] , \\
&= -\frac{2N}{1+z+w} [m_u\overline{u}i\gamma_5 u + zm_d\overline{d}i\gamma_5 d + wm_s\overline{s}i\gamma_5 s] .
\end{aligned}
\tag{4.65}
$$

The axion current can be compared to the pion current (4.60), where

$$
\partial^\mu j_\mu^3 = \frac{e^2}{16\pi^2}(F\tilde{F})_{EM} + m_u\overline{u}i\gamma_5 u - m_d\overline{d}i\gamma_5 d, \tag{4.66}
$$

this observation suggests that the axion current j_μ^a shares the same divergence structure as the ordinary pion current. Consequently, it is reasonable to perform standard current algebra manipulations with the axion current.

Since z and w are arbitrary numbers, we can conveniently choose them as $z = m_u/m_d$ and $w = m_u/m_s$ to eliminate the off-diagonal terms in Dashen's Formula [55] for the mass-squared matrix of a, π^0, and η. Dashen's Formula for this case is given by [56]:

$$
(m^2)_{12} = \frac{1}{f_1 f_2} \langle 0| \left[Q_5^1, \left[Q_5^2, \mathcal{L} \right] \right] |0\rangle \ , \tag{4.67}
$$

where $Q_5^{1,2}$ are the generators of the chiral symmetries.

The axion mass, then, is given by:

$$
\begin{aligned}
m_a &= \frac{f_\pi m_\pi}{f_a/N} \left[\frac{4z}{(1+z+w)(1+z)} \right]^{1/2} , \\
&= (1.2 \times 10^5 \text{eV}) \left(\frac{10^{12}\text{GeV}}{f_a/N} \right) ,
\end{aligned}
\tag{4.68}
$$

where the numeric values for z and w were obtained from [13].

4.4 Axion coupling to photons

The results from the previous section imply the possibility of writing an effective lagrangian for the interaction between axions and photons:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2}\partial_\mu a \partial^\mu a - \frac{1}{2}m_a^2 a^2 + \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \frac{\alpha N C}{\pi f_a} a\vec{E}\cdot\vec{B} \ , \tag{4.69}$$

where

$$\begin{aligned} \alpha &= \frac{e^2}{4\pi} \ , \\ (F\tilde{F}) &= -4\vec{E}\cdot\vec{B} \ , \\ C &= \frac{E}{N} - \frac{2}{3}\frac{4+z+w}{1+z+w} = \frac{E}{N} - 1.92 \approx 0.75 \quad (\text{GUT}) \ . \end{aligned} \tag{4.70}$$

Note that $\mathcal{L}_{\text{eff}}$ involves N only in ratios E/N and f_a/N.

An important feature of this result is that any axion in any standard Grand Unified Theory (GUT) has a specified coupling to photons. Specifically, the coefficient of $a\vec{E}\cdot\vec{B}$, which represents the coupling of axions to two photons, has a model-independent value in units of the axion mass:

$$g_{a\gamma\gamma} = \frac{\alpha N C}{\pi f_a} \approx (1.45 \times 10^{-15}\ \text{GeV}^{-1})(m_a/10^{-15}\ \text{eV}) \ . \tag{4.71}$$

4.5 Tree-level coupling to electrons

To compute g_{ae} at classical level, it is necessary to modify the axion current to account for the spontaneous breakdown of the electroweak symmetry $\text{SU}(2)_L \times \text{U}(1)_Y$.

Care must be taken to define the axion in a way that it does not mix with the Nambu-Goldstone (NG) boson that becomes the longitudinal Z^0 boson. This NG field, denoted as y, is given by:

$$y = v_{\text{SM}}^{-1} \sum_n Y_n V_n \chi_n \ , \tag{4.72}$$

where the NG boson decay constant f is:

$$v_{\text{SM}}^2 = \sum_n Y_n^2 V_n^2 = (246\ \text{GeV})^2 \ , \tag{4.73}$$

and Y_n are the hypercharges of ϕ_n. The constraint $Y_n = \pm 1$ in (4.72) and(4.73) is imposed to ensure the validity of the relation between the W and Z bosons, $m_W = m_Z \cos\theta_W$, where θ_W is the Weinberg angle.

To ensure that the new axion field a' does not overlap with y, we define a' as:

$$a' = a - \left[(f_a v_{\text{SM}})^{-1} \sum_n Z_n Y_n V_n^2\right] y \ , \tag{4.74}$$

where a is given by (4.54). This expression can be simplified through algebraic manipulations:

$$\begin{aligned} a' &= f_a^{-1}\sum_n Z_nV_n\chi_n - \left[(f_a v_{\rm SM})^{-1}\sum_m Z_mY_mV_m^2\right] y \ , \\ &= f_a^{-1}\sum_n Z_nV_n\chi_n - \left[(f_a v_{\rm SM})^{-1}\sum_m Z_mY_mV_m^2\right] v_{\rm SM}^{-1}\sum_n Y_nV_n\chi_n \ , \\ &= f_a^{-1}\sum_n Z_nV_n\chi_n - f_a^{-1}\sum_n\left[v_{\rm SM}^{-2}\sum_m Z_mY_mV_m^2\right] Y_nV_n\chi_n \ , \\ &= f_a^{-1}\sum_n\left[Z_n - \left(v_{\rm SM}^{-2}\sum_m Z_mY_mV_m^2\right)Y_n\right]V_n\chi_n \ , \\ &= f_a^{-1}\sum_n Z_n'V_n\chi_n \ , \end{aligned} \tag{4.75}$$

where we introduce the definition:

$$Z_n' \equiv Z_n - \left[v_{\rm SM}^{-2}\sum_m Z_mY_mV_m^2\right]Y_n \ . \tag{4.76}$$

Therefore, as demonstrated earlier, the new axion field is expressed as:

$$a' = f_a^{-1}\sum_n Z_n'V_n\chi_n \ . \tag{4.77}$$

Associated with this change in the a field is a corresponding adjustment in the axion decay constant f_a. However, as this change is tied to $v_{\rm SM}$ and considering that $f_a \gg v_{\rm SM}$, it can be safely neglected. It is also imperative to modify the PQ charges assigned to fermions X_i to X_i' to maintain the properties of the axion current divergence. For instance, consider the electron e^-; if the left-handed e^- has a PQ charge X_e, this charge must be modified to:

$$X_e' = X_e - \frac{1}{2}\left[v_{\rm SM}^{-2}\sum_m Z_mY_mV_m^2\right] \ , \tag{4.78}$$

where $Y_e = -1$ and $Z_e' = -2X_e'$ for the Higgs field coupling to the left-handed e^- were used to derive this expression. A similar formula applies to the PQ charge of any charged lepton or down quark. As for the up quark, the modification is given by:

$$Z_u' = Z_u + \frac{1}{2}\left[v_{\rm SM}^{-2}\sum_m Z_mY_mV_m^2\right] \ . \tag{4.79}$$

With these new considerations, (4.56) transforms into:

$$j_\mu'^{PQ} = f_a\partial_\mu a' + \sum_i X_i'\overline{\psi}_i\gamma_\mu\gamma_5\psi_i + \dots \ , \tag{4.80}$$

where the sum in i extends over every charged lepton (e,μ, τ) and every SM quark (u, d, s, c, b, t). The indication of extra terms accounts for possible contributions of neutrinos and any other fermions, including supermassive ones.

The computation of the axion-electron coupling, g_{ae}, from (4.80) involves first considering the current divergence. Given the validity of the PQ symmetry at the classical level, the expression becomes:

$$\begin{aligned}\partial^\mu j'^{PQ}_\mu &= f_a \partial^\mu \partial_\mu a' + X'_e \partial^\mu (\bar{e}\gamma_\mu \gamma_5 e) + \dots \ , \\ &= f_a \Box a' + 2X'_e m_e \bar{e} i \gamma_5 e + \dots \ , \\ &= 0 \ ,\end{aligned} \tag{4.81}$$

where the Dirac equation was employed to derive the second line of the second term. This equation leads directly to:

$$-\Box a' = 2X'_e (m_e/f_a) a \bar{e} i \gamma_5 e + \dots \ . \tag{4.82}$$

Hence, we find that the effective lagrangian for the axion-electron interaction incorporates the term

$$\mathcal{L} \supset 2X'_e (m_e/f_a) a \bar{e} i \gamma_5 e \ . \tag{4.83}$$

Consequently, the coefficient of $a\bar{e}i\gamma_5 e$ represents the axion-electron coupling:

$$g_{ae} = 2X'_e (m_e/f_a) \ , \tag{4.84}$$

which, unlike $g_{a\gamma\gamma}$, due to the presence of X'_e, is model dependent. It is worth noting that g_{ae} involves solely f_a and not the fraction f_a/N with the anomaly coefficient N. Now, we can proceed to evaluate the coupling in the two distinct invisible axion models under consideration—the KSVZ and DFSZ models.

In the KSVZ model, conventional quarks, leptons, or a Higgs field carrying PQ charge are absent. Consequently, the electron PQ charge within this model is given by:

$$X'_e = 0 \quad \text{(KSVZ)} \ . \tag{4.85}$$

However, despite the absence of direct coupling to electrons, the axion in the KSVZ model exhibits a radiatively induced g_{ae} at the 1-loop (quantum) level, a topic we will delve into later. This implies that, within the KSVZ framework, the axion-electron coupling is inherently suppressed when compared to models where the tree-level contribution is non-zero.

In the DFSZ model, where $X_e = +1$ and $N = 6$ for three generations, two scalar fields, ϕ_u and ϕ_d, contribute to the y field. Specifically, ϕ_u has $Z_u = -2$ and $Y_u = +1$, while ϕ_d has $Z_d = -2$ and $Y_d = -1$. Additionally, a crucial relationship is established:

$v_{\mathrm{SM}}^2 = V_u^2 + V_d^2$. Therefore, we can determine X'_e as follows:

$$\begin{aligned}
X'_e &= X_e - \frac{1}{2}\left[v_{\mathrm{SM}}^{-2}\sum_m Z_m Y_m V_m^2\right] \ , \\
&= 1 - \frac{1}{2}v_{\mathrm{SM}}^{-2} Z_u Y_u V_u^2 - \frac{1}{2}v_{\mathrm{SM}}^{-2} Z_d Y_d V_d^2 \ , \\
&= 1 - \frac{1}{2}v_{\mathrm{SM}}^{-2}(-2)(+1)V_u^2 - \frac{1}{2}v_{\mathrm{SM}}^{-2}(-2)(-1)V_d^2 \ , \\
&= 1 + v_{\mathrm{SM}}^{-2}V_u^2 - v_{\mathrm{SM}}^{-2}V_d^2 \ , \\
&= 1 + \frac{V_u^2}{V_u^2+V_d^2} - \frac{V_d^2}{V_u^2+V_d^2} \ , \\
&= \frac{V_u^2+V_d^2+V_u^2-V_d^2}{V_u^2+V_d^2} \ , \\
&= 2\frac{V_u^2}{V_u^2+V_d^2} \ , \\
&= 2\frac{V_u^2}{v_{\mathrm{SM}}^2} \ .
\end{aligned} \tag{4.86}$$

Parametrizing $\cos\beta = V_u/v_{\mathrm{SM}}$, the effective coupling becomes:

$$g_{ae} = 4\cos^2\beta\frac{m_e}{f_a} \quad \text{(DFSZ)} \ . \tag{4.87}$$

To facilitate a more straightforward comparison with $g_{a\gamma\gamma}$, introducing the factor $N = 6$ to the coupling yields:

$$g_{ae} = \frac{2}{3}\cos^2\beta\frac{m_e N}{f_a} \quad \text{(DFSZ)} \ . \tag{4.88}$$

The model, however, does not prescribe the value of β. It is reasonable to conjecture that $V_u > V_d$ given the significantly larger masses of up quarks compared to down quarks in the second and third generations. This inference leads us to:

$$2\cos^2\beta > 1 \ . \tag{4.89}$$

4.6 One-loop induced coupling to electrons

As mentioned earlier, despite $X'_e = 0$ and the absence of a tree-level coupling in the KSVZ model, there persists a one-loop level contribution to g_{ae} arising from the interaction depicted in Figure 5.

To calculate this diagram, we must initially derive the axion-photon vertex, which is proportional to the axion-photon coupling $g_{a\gamma\gamma}$. We commence with the interaction term from (4.69):

$$\mathcal{L}_{\mathrm{int}} = g_{a\gamma\gamma} a\vec{E}\cdot\vec{B} \ . \tag{4.90}$$

However, in this form, deriving the vertex is not feasible. To proceed, it is necessary to express it in terms of the gauge fields A^μ. Utilizing the relation (4.70), we obtain:

$$\mathcal{L}_{\mathrm{int}} = -\frac{1}{4}g_{a\gamma\gamma} aF^{\mu\nu}\tilde{F}_{\mu\nu} \ . \tag{4.91}$$

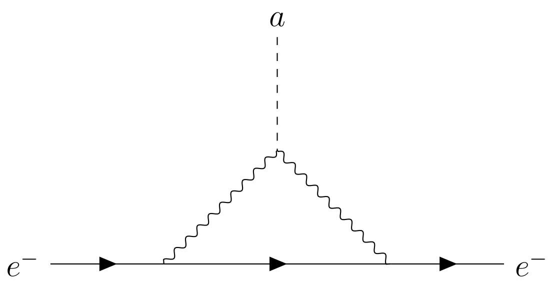


Figure 5 – 1-loop contribution to g_{ae}

The next step involves explicitly expressing the gauge fields in the field strength tensor, yielding:

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \ , \\ \tilde{F}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho A^\sigma - \partial^\sigma A^\rho) \ . \end{aligned} \tag{4.92}$$

This brings us to the multiplication of field strengths, resulting in:

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial^\rho A^\sigma - \partial^\sigma A^\rho) \ . \tag{4.93}$$

Here, we take the explicit route to obtain the result of the multiplication:

$$\begin{aligned} F^{\mu\nu}\tilde{F}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial^\rho A^\sigma - \partial^\sigma A^\rho) \ , \\ &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu \partial^\rho A^\sigma - \partial^\mu A^\nu \partial^\sigma A^\rho - \partial^\nu A^\mu \partial^\rho A^\sigma + \partial^\nu A^\mu \partial^\sigma A^\rho) \ , \end{aligned} \tag{4.94}$$

next, we apply the chain rule in each term individually, for example:

$$\partial^\mu(A^\nu \partial^\rho A^\sigma) = \partial^\mu A^\nu \partial^\rho A^\sigma + A^\nu \partial^\mu \partial^\rho A^\sigma, \tag{4.95}$$

which leads us to:

$$\partial^\mu A^\nu \partial^\rho A^\sigma = -A^\nu \partial^\mu \partial^\rho A^\sigma + S.T. \ , \tag{4.96}$$

where $S.T.$ denotes a surface term taken to be 0 when integrated. With this, the multiplication reads:

$$\begin{aligned} F^{\mu\nu}\tilde{F}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(-A^\nu\partial^\mu\partial^\rho A^\sigma + A^\nu\partial^\mu\partial^\sigma A^\rho + A^\mu\partial^\nu\partial^\rho A^\sigma - A^\mu\partial^\nu\partial^\sigma A^\rho) \ , \\ &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(-A^\nu\partial^\mu(\partial^\rho A^\sigma - \partial^\sigma A^\rho) + A^\mu\partial^\nu(\partial^\rho A^\sigma - \partial^\sigma A^\rho)) \ , \\ &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(-A^\nu\partial^\mu F^{\rho\sigma} + A^\mu\partial^\nu F^{\rho\sigma}) \ , \\ &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}(A^\mu\partial^\nu - A^\nu\partial^\mu)F^{\rho\sigma} \ . \end{aligned} \tag{4.97}$$

Here, we express the term in the parenthesis as:

$$A^\mu\partial^\nu - A^\nu\partial^\mu = (\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta)A^\alpha\partial^\beta \ , \tag{4.98}$$

and utilize the property of the Levi-Civita tensor in Minkowski space:

$$\epsilon^{\mu\nu\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -2(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}) \ , \tag{4.99}$$

yielding:

$$A^{\mu}\partial^{\nu} - A^{\nu}\partial^{\mu} = -\frac{1}{2}\epsilon^{\mu\nu\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}A^{\alpha}\partial^{\beta} \ . \tag{4.100}$$

With this term, we proceed with the previous calculation:

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = -\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}A^{\alpha}\partial^{\beta}F^{\rho\sigma} \ , \tag{4.101}$$

now, using the same reversed Levi-Civita property, we get:

$$\begin{aligned} F^{\mu\nu}\tilde{F}_{\mu\nu} &= \frac{1}{2}(\delta^{\gamma}_{\rho}\delta^{\delta}_{\sigma} - \delta^{\delta}_{\rho}\delta^{\gamma}_{\sigma})\epsilon_{\alpha\beta\gamma\delta}A^{\alpha}\partial^{\beta}F^{\rho\sigma} \ , \\ &= \frac{1}{2}(\epsilon_{\alpha\beta\rho\sigma} - \epsilon_{\alpha\beta\sigma\rho})A^{\alpha}\partial^{\beta}F^{\rho\sigma} \ , \\ &= \epsilon_{\alpha\beta\rho\sigma}A^{\alpha}\partial^{\beta}F^{\rho\sigma} \ . \end{aligned} \tag{4.102}$$

Next, let's focus on the $\partial^{\beta}F^{\rho\sigma}$ part:

$$\begin{aligned} \partial^{\beta}F^{\rho\sigma} &= \partial^{\beta}(\partial^{\rho}A^{\sigma} - \partial^{\sigma}A^{\rho}) \ , \\ &= \partial^{\beta}(\delta^{\rho}_{\gamma}\delta^{\sigma}_{\delta} - \delta^{\sigma}_{\gamma}\delta^{\rho}_{\delta})\partial^{\gamma}A^{\delta} \ , \\ &= \partial^{\beta}(-\frac{1}{2}\epsilon^{\rho\sigma\xi\chi}\epsilon_{\rho\sigma\xi\chi})\partial^{\gamma}A^{\delta} \ , \\ &= -\frac{1}{2}\epsilon^{\rho\sigma\xi\chi}\epsilon_{\rho\sigma\xi\chi}\partial^{\beta}\partial^{\gamma}A^{\delta} \ . \end{aligned} \tag{4.103}$$

Continuing, we obtain:

$$\begin{aligned} F^{\mu\nu}\tilde{F}_{\mu\nu} &= -\frac{1}{2}\epsilon_{\alpha\beta\rho\sigma}\epsilon^{\rho\sigma\xi\chi}\epsilon_{\rho\sigma\xi\chi}A^{\alpha}\partial^{\beta}\partial^{\gamma}A^{\delta} \ , \\ &= (\delta^{\xi}_{\alpha}\delta^{\chi}_{\beta} - \delta^{\chi}_{\alpha}\delta^{\xi}_{\beta})\epsilon_{\rho\sigma\xi\chi}A^{\alpha}\partial^{\beta}\partial^{\gamma}A^{\delta} \ , \\ &= (\epsilon_{\gamma\delta\alpha\beta} - \epsilon_{\gamma\delta\beta\alpha})A^{\alpha}\partial^{\beta}\partial^{\gamma}A^{\delta} \ , \\ &= 2\epsilon_{\gamma\delta\alpha\beta}A^{\alpha}\partial^{\beta}\partial^{\gamma}A^{\delta} \ . \end{aligned} \tag{4.104}$$

Here, we rewrite it in terms of $\mu\nu\rho\sigma$ indexes and apply the chain rule to get:

$$F^{\mu\nu}\tilde{F}_{\mu\nu} = -2\epsilon_{\mu\nu\rho\sigma}\partial^{\mu}A^{\nu}\partial^{\rho}A^{\sigma} \ . \tag{4.105}$$

Finally, the interaction lagrangian (4.91) can be expressed as:

$$i\mathcal{L}_{\text{int}} = \frac{i}{2}g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma}a\partial^{\mu}A^{\nu}\partial^{\rho}A^{\sigma} \ . \tag{4.106}$$

To derive the vertex, we apply a technique presented in Chapter 4 of Quigg's textbook [57], where we first express the interaction lagrangian in momentum space:

$$\begin{aligned} a(x) &\to a(k)e^{ikx} \ , \\ A^{\mu}(x) &\to A^{\mu}(q)e^{iqx} \ , \end{aligned} \tag{4.107}$$

this transformation yields:

$$
\begin{aligned}
i\mathcal{L}_{\text{int}} &\to \frac{i}{2} g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma} a (i q_1^\mu A^\nu)(i q_2^\rho A^\sigma) e^{i(k+q_1+q_2)x} \ , \\
&\to -\frac{i}{2} g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma} a q_1^\mu q_2^\rho A^\nu A^\sigma \ ,
\end{aligned}
\tag{4.108}
$$

next, we derive the interaction Lagrangian in terms of the fields:

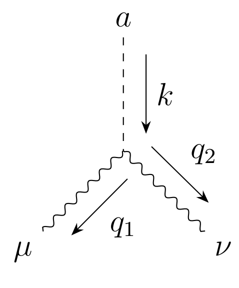
$$
\begin{aligned}
\frac{\delta^2 (A^\nu A^\sigma)}{\delta A^\alpha \delta A^\beta} &= \frac{\delta}{\delta A^\alpha}(\delta^\nu_\beta A^\sigma + A^\nu \delta^\sigma_\beta) \ , \\
&= \delta^\nu_\beta \delta^\sigma_\alpha + \delta^\nu_\alpha \delta^\sigma_\beta
\end{aligned}
\tag{4.109}
$$

$$
\begin{aligned}
\frac{\delta^3}{\delta a \delta A^\alpha \delta A^\beta}(i\mathcal{L}_{\text{int}}) &= -\frac{i}{2} g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma} q_1^\mu q_2^\rho (\delta^\nu_\beta \delta^\sigma_\alpha + \delta^\nu_\alpha \delta^\sigma_\beta) \ , \\
&= -i g_{a\gamma\gamma}\epsilon_{\mu\beta\rho\alpha} q_1^\mu q_2^\rho \ .
\end{aligned}
\tag{4.110}
$$

This is already a valid expression for the axion-photon vertex. However, we can write it in terms of the symmetric and anti-symmetric parts of $q_1^\mu q_2^\rho$ and combine it with the anti-symmetry of the Levi-Civita tensor to eliminate the symmetric part. Besides, as the indices β, α are mute, we can change it to the standard μ, ν, ρ, σ of the levi-Civita. With these considerations, we finally obtain:

$$
\begin{aligned}
-i g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma} q_1^\mu q_2^\rho &= -i g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma}\left[\frac{1}{2}(q_1^\mu q_2^\rho + q_1^\rho q_2^\mu) + \frac{1}{2}(q_1^\mu q_2^\rho - q_1^\rho q_2^\mu)\right] \ , \\
&= -\frac{i}{2} g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma}(q_1^\mu q_2^\rho - q_1^\rho q_2^\mu) \ .
\end{aligned}
\tag{4.111}
$$

We can make a slight change in the indices so that the contracted ones are the last 2 in the Levi-Civita tensor, thus:



$$
= -\frac{i}{2} g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma}(q_1^\rho q_2^\sigma - q_1^\sigma q_2^\rho) \ . \tag{4.112}
$$

To generate the diagram depicted in (5), it is essential to couple it to two electron-photon vertices, as illustrated in (6).

Finally, the transition amplitude is expressed as:

$$
\begin{aligned}
i\mathcal{M} = \overline{u}(p_3)(-ie\gamma^\beta) i \frac{\not{p}_2 + m_e}{p_2^2 - m_e^2 + i\epsilon}(-ie\gamma^\alpha)\left(\frac{i g_{\mu\alpha}}{q_1^2}\right)\left(\frac{i g_{\nu\beta}}{q_2^2}\right) \\
\left(-\frac{i}{2} g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma}(q_1^\rho q_2^\sigma - q_1^\sigma q_2^\rho)\right) u(p_1) \ ,
\end{aligned}
\tag{4.113}
$$

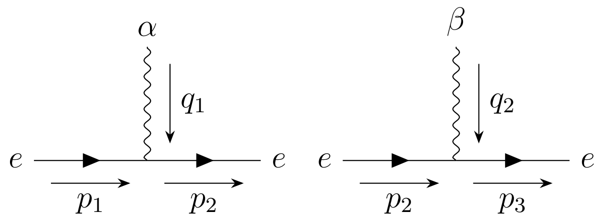


Figure 6 – electron-photon vertices

which needs to be integrated over the virtual electron momentum p_2. We can rewrite the transition amplitude as:

$$i\mathcal{M} = \frac{e^2}{2}\overline{u}(p_3)\int \frac{d^4p_2}{(2\pi)^4}\,\frac{\gamma^\nu(\not{p}_2 + m_e)\gamma^\mu g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma}(q_1^\rho q_2^\sigma - q_1^\sigma q_2^\rho)}{[p_2^2 - m_e^2 + i\epsilon][q_1^2 + i\epsilon][q_2^2 + i\epsilon]}u(p_1)\ . \tag{4.114}$$

Now, utilizing the momentum conservation of each vertex, we express the integral in terms of p_2, with the momentum relations given by:

$$\begin{aligned} k &= q_1 + q_2\ , \\ q_1 &= p_2 - p_1\ , \\ q_2 &= p_3 - p_2\ . \end{aligned} \tag{4.115}$$

First, we substitute the terms in the denominator:

$$i\mathcal{M} = \frac{e^2}{2}\overline{u}(p_3)\int \frac{d^4p_2}{(2\pi)^4}\,\frac{\gamma^\nu(\not{p}_2 + m_e)\gamma^\mu g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma}(q_1^\rho q_2^\sigma - q_1^\sigma q_2^\rho)}{[p_2^2 - m_e^2 + i\epsilon][(p_2 - p_1)^2 + i\epsilon][(p_3 - p_2)^2 + i\epsilon]}u(p_1)\ . \tag{4.116}$$

Our aim is to simplify the integral to make it more manageable. We start by introducing the Feynman parameters:

$$\frac{1}{ABC} = 2\int_0^1 dxdydz\ \delta(x + y + z - 1)\frac{1}{[xA + yB + zC]^3}\ , \tag{4.117}$$

where A, B and C are the terms in the denominator:

$$\begin{aligned} A &= p_2^2 - m_e^2 + i\epsilon\ , \\ B &= (p_2 - p_1)^2 + i\epsilon\ , \\ C &= (p_3 - p_2)^2 + i\epsilon\ . \end{aligned} \tag{4.118}$$

The denominator becomes:

$$\begin{aligned} xA + yB + zC &= x(p_2^2 - m_e^2 + i\epsilon) + y((p_2 - p_1)^2 + i\epsilon) + z((p_3 - p_2)^2 + i\epsilon)\ , \\ &= (x + y + z)p_2^2 - xm_e^2 - 2yp_2p_1 + yp_1^2 + zp_3^2 - 2zp_3p_2 \\ &+ (x + y + z)i\epsilon\ , \\ &= p_2^2 - 2yp_2p_1 - 2zp_3p_2 + yp_1^2 + zp_3^2 - xm_e^2 + i\epsilon\ , \end{aligned} \tag{4.119}$$

we can further simplify using the relation:

$$(p_2 - yp_1 - zp_3)^2 = p_2^2 - 2yp_2p_1 - 2zp_2p_3 + 2yzp_1p_3 + y^2p_1^2 + z^2p_3^2\ . \tag{4.120}$$

Comparing it to (4.119), we get:

$$xA + yB + zC = (p_2 - yp_1 - zp_3)^2 - \Delta + i\epsilon \tag{4.121}$$

where

$$\Delta = (x+z)yp_1^2 + (x+y)zp_3^2 - 2yzp_1p_3 + xm_e^2 \ . \tag{4.122}$$

Now it is possible to use some momentum relations to rewrite Δ. The following relations hold:

$$\begin{aligned} p_3^2 &= m_e^2 \ , \\ p_1^2 &= m_e^2 \ , \\ p_1 &= p_3 - k \ , \\ p_3 k &= -\frac{p_1^2}{2} + m_e^2 \ , \end{aligned} \tag{4.123}$$

we find:

$$\begin{aligned} \Delta &= (x+z)yp_1^2 + (x+y)zp_3^2 - 2yzp_1p_3 + xm_e^2 \ , \\ &= (x+z)ym_e^2 + (x+y)zm_e^2 - 2yz(p_3-k)p_3 + xm_e^2 \ , \\ &= (1-y)ym_e^2 + (1-z)zm_e^2 - 2yzp_3^2 + 2yzp_3k + xm_e^2 \ , \\ &= (1-y)ym_e^2 + (1-z)zm_e^2 - 2yzm_e^2 + 2yz\left(-\frac{p_1^2}{2} + m_e^2\right) + xm_e^2 \ , \\ &= (1-y)ym_e^2 + (1-z)zm_e^2 - 2yzm_e^2 - yzm_e^2 + 2yzm_e^2 + xm_e^2 \ , \\ &= (1-y)ym_e^2 + (1-z)zm_e^2 - yzm_e^2 + xm_e^2 \ , \\ &= m_e^2[(1-y)y + (1-z)z - yz + x] \ , \\ &= m_e^2[(1-y-z)y + (1-z)z + x] \ , \\ &= m_e^2[xy + (1-z)z + x] \ , \\ &= m_e^2[(1+y)x + (1-z)z] \ . \end{aligned} \tag{4.124}$$

We can also effectuate a shift in the momentum p_2 using the following expression:

$$p_2^\mu \to p_2^\mu - yp_1^\mu - zp_3^\mu \ . \tag{4.125}$$

Consequently, we express the denominator as:

$$xA + yB + zC = (p_2^2 - \Delta + i\epsilon) \ . \tag{4.126}$$

In conclusion, we determine:

$$\frac{1}{ABC} = 2\int_0^1 dxdydz \ \delta(x+y+z-1)\frac{1}{(p_2^2 - \Delta + i\epsilon)^3} \ , \tag{4.127}$$

which, in turn, leads us to the following transition amplitude:

$$\begin{aligned} i\mathcal{M} &= e^2\overline{u}(p_3)\int_0^1 dxdydz \ \delta(x+y+z-1) \\ &\times \int \frac{d^4p_2}{(2\pi)^4}\frac{\gamma^\nu(\not{p}_2 + m_e)\gamma^\mu g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma}(q_1^\rho q_2^\sigma - q_1^\sigma q_2^\rho)}{(p_2^2 - \Delta + i\epsilon)^3}u(p_1) \ , \end{aligned} \tag{4.128}$$

where $\Delta = m_e^2[(1+y)x + (1-z)z]$.

Now, our focus shifts to the numerator, denoted as:

$$N = \overline{u}(p_3)\gamma^\nu(\not{p}_2 + m_e)\gamma^\mu g_{a\gamma\gamma}\epsilon_{\mu\nu\rho\sigma}(q_1^\rho q_2^\sigma - q_1^\sigma q_2^\rho)u(p_1) \ . \tag{4.129}$$

To initiate the analysis, we substitute the photon momenta q_1 and q_2 utilizing (4.115), resulting in:

$$\begin{aligned} q_1^\rho q_2^\sigma - q_1^\sigma q_2^\rho = & \ p_2^\rho p_3^\sigma - p_2^\rho p_2^\sigma - p_1^\rho p_3^\sigma + p_1^\rho p_2^\sigma \\ & - p_2^\sigma p_3^\rho + p_2^\sigma p_2^\rho + p_1^\sigma p_3^\rho - p_1^\sigma p_2^\rho \ , \end{aligned} \tag{4.130}$$

it is essential to note that these terms are contracted with the Levi-Civita tensor. Consequently, any symmetric term becomes zero due to the anti-symmetric characteristic of the Levi-Civita. Given this consideration, the product is:

$$\epsilon_{\mu\nu\rho\sigma}(q_1^\rho q_2^\sigma - q_1^\sigma q_2^\rho) = \epsilon_{\mu\nu\rho\sigma}[(p_1^\rho p_2^\sigma - p_1^\sigma p_2^\rho) + (p_2^\rho p_3^\sigma - p_2^\sigma p_3^\rho) - (p_1^\rho p_3^\sigma - p_1^\sigma p_3^\rho)] \ . \tag{4.131}$$

Following this, we apply the same momentum shift executed in the denominator, where:

$$p_2^\mu \to p_2^\mu - y p_1^\mu - z p_3^\mu \ , \tag{4.132}$$

this transformation yields (for clarity, the Levi-Civita symbol will be omitted):

$$\begin{aligned} (q_1^\rho q_2^\sigma - q_1^\sigma q_2^\rho) = & \ [p_1^\rho(p_2^\sigma - y p_1^\sigma - z p_3^\sigma) - p_1^\sigma(p_2^\rho - y p_1^\rho - z p_3^\rho)] \\ & + [(p_2^\rho - y p_1^\rho - z p_3^\rho)p_3^\sigma - (p_2^\sigma - y p_1^\sigma - z p_3^\sigma)p_3^\rho] \\ & - [p_1^\sigma p_3^\rho - p_1^\rho p_3^\sigma] \ , \\ = & \ [p_1^\rho p_2^\sigma - y p_1^\rho p_1^\sigma - z p_1^\rho p_3^\sigma - p_1^\sigma p_2^\rho + y p_1^\sigma p_1^\rho + z p_1^\sigma p_3^\rho] \\ & + [p_2^\rho p_3^\sigma - y p_1^\rho p_3^\sigma - z p_3^\rho p_3^\sigma - p_2^\sigma p_3^\rho + y p_1^\sigma p_3^\rho + z p_3^\sigma p_3^\rho] \\ & - [p_1^\sigma p_3^\rho - p_1^\rho p_3^\sigma] \ , \\ = & \ [(p_1^\rho p_2^\sigma - p_1^\sigma p_2^\rho) - y(p_1^\rho p_1^\sigma - p_1^\sigma p_1^\rho) - z(p_1^\rho p_3^\sigma - p_1^\sigma p_3^\rho)] \\ & + [(p_2^\rho p_3^\sigma - p_2^\sigma p_3^\rho) - y(p_1^\rho p_3^\sigma - p_1^\sigma p_3^\rho) - z(p_3^\rho p_3^\sigma - p_3^\sigma p_3^\rho)] \\ & - [p_1^\sigma p_3^\rho - p_1^\rho p_3^\sigma] \ , \\ = & \ (p_1^\rho p_2^\sigma - p_1^\sigma p_2^\rho) - y(p_1^\rho p_1^\sigma - p_1^\sigma p_1^\rho) - (1+y+z)(p_1^\rho p_3^\sigma - p_1^\sigma p_3^\rho) \\ & + (p_2^\rho p_3^\sigma - p_2^\sigma p_3^\rho) - z(p_3^\rho p_3^\sigma - p_3^\sigma p_3^\rho) \ , \end{aligned} \tag{4.133}$$

once again, recognizing that symmetric terms can be canceled, and incorporating the constraint $\delta(x+y+z-1)$, the expression simplifies to:

$$(q_1^\rho q_2^\sigma - q_1^\sigma q_2^\rho) = (p_1^\rho p_2^\sigma - p_1^\sigma p_2^\rho) + (p_2^\rho p_3^\sigma - p_2^\sigma p_3^\rho) + (x-2)(p_1^\rho p_3^\sigma - p_1^\sigma p_3^\rho) \ . \tag{4.134}$$

Substituting this into the numerator expression, we obtain:

$$\begin{aligned} N = & \ \overline{u}(p_3)\gamma^\nu(\not{p}_2 - y\not{p}_1 - z\not{p}_3 + m_e)\gamma^\mu g_{a\gamma\gamma} \\ & \times \epsilon_{\mu\nu\rho\sigma}[(p_1^\rho p_2^\sigma - p_1^\sigma p_2^\rho) + (p_2^\rho p_3^\sigma - p_2^\sigma p_3^\rho) + (x-2)(p_1^\rho p_3^\sigma - p_1^\sigma p_3^\rho)]u(p_1) \ . \end{aligned} \tag{4.135}$$

We will now employ an algebraic manipulation involving the Levi-Civita tensor, a technique also found in Quigg's book in Appendix B [57]:

$$\gamma_5\gamma_\sigma = \frac{i}{3!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho \ , \tag{4.136}$$

this relation can be logically extended to:

$$\gamma_5\gamma_\sigma\gamma_\rho = \frac{4i}{3!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu \ , \tag{4.137}$$

and further expressed as:

$$\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu = -\frac{3}{2}i\gamma_5\gamma_\sigma\gamma_\rho \ . \tag{4.138}$$

Substituting the derived expression into (4.135), we arrive at:

$$\begin{aligned} N = &-\frac{3}{2}g_{a\gamma\gamma}\overline{u}(p_3)(\not{p}_2 - y\not{p}_1 - z\not{p}_3 + m_e)i\gamma_5\gamma_\sigma\gamma_\rho \\ &\times [(p_1^\rho p_2^\sigma - p_1^\sigma p_2^\rho) + (p_2^\rho p_3^\sigma - p_2^\sigma p_3^\rho) + (x-2)(p_1^\rho p_3^\sigma - p_1^\sigma p_3^\rho)]u(p_1) \ . \end{aligned} \tag{4.139}$$

Now, considering terms with at least two occurrences of p_2, as any odd number of p_2 will integrate to zero and focusing solely on terms that can exhibit logarithmic divergence, we simplify the expression:

$$\begin{aligned} N &= -\frac{3}{2}g_{a\gamma\gamma}\overline{u}(p_3)\not{p}_2 i\gamma_5\gamma_\sigma\gamma_\rho[(p_1^\rho p_2^\sigma - p_1^\sigma p_2^\rho) + (p_2^\rho p_3^\sigma - p_2^\sigma p_3^\rho)]u(p_1) \ , \\ &= -\frac{3}{2}g_{a\gamma\gamma}\overline{u}(p_3)\not{p}_2 i\gamma_5[(\not{p}_2\not{p}_1 - \not{p}_1\not{p}_2) + (\not{p}_3\not{p}_2 - \not{p}_2\not{p}_3)]u(p_1) \ . \end{aligned} \tag{4.140}$$

To further simplify the numerator, we utilize a set of relations given by:

$$\begin{aligned} &\not{p}\not{p} = p^2 \ , \\ &\not{p}u(p) = mu(p) \ , \\ &\overline{u}(p)\not{p} = \overline{u}(p)m \ , \\ &\not{p}_1\not{p}_2 = 2p_1^\mu p_{2\mu} - \not{p}_2\not{p}_1 \ , \\ &\{\gamma_5, \gamma_\mu\} = 0 \ . \end{aligned} \tag{4.141}$$

Therefore, the first term becomes:

$$\begin{aligned} N_1 &= -\frac{3}{2}g_{a\gamma\gamma}\overline{u}(p_3)\not{p}_2 i\gamma_5\not{p}_2\not{p}_1 u(p_1) \ , \\ &= -\frac{3}{2}g_{a\gamma\gamma}\overline{u}(p_3)\not{p}_2 i\gamma_5\not{p}_2 m_e u(p_1) \ , \\ &= \frac{3}{2}m_e g_{a\gamma\gamma}\overline{u}(p_3)\not{p}_2\not{p}_2 i\gamma_5 u(p_1) \ , \\ &= \frac{3}{2}m_e g_{a\gamma\gamma}\overline{u}(p_3)p_2^2 i\gamma_5 u(p_1) \ . \end{aligned} \tag{4.142}$$

The second term becomes:

$$\begin{aligned}
N_2 &= \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) \not{p}_2 i\gamma_5 \not{p}_1 \not{p}_2 u(p_1) \ , \\
&= \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) \not{p}_2 i\gamma_5 (2p_1^\mu p_{2\mu} - \not{p}_2 \not{p}_1) u(p_1) \ , \\
&= \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) \not{p}_2 i\gamma_5 2p_1^\mu p_{2\mu} u(p_1) - \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) \not{p}_2 i\gamma_5 \not{p}_2 \not{p}_1 u(p_1) \\
&= 3g_{a\gamma\gamma} \overline{u}(p_3) \gamma^\beta p_{2\beta} i\gamma_5 p_1^\mu p_{2\mu} u(p_1) + \frac{3}{2} m_e g_{a\gamma\gamma} \overline{u}(p_3) p_2^2 i\gamma_5 u(p_1) \ , \\
&= 3g_{a\gamma\gamma} \overline{u}(p_3) \gamma^\beta i\gamma_5 p_1^\mu p_{2\beta} p_{2\mu} u(p_1) + \frac{3}{2} m_e g_{a\gamma\gamma} \overline{u}(p_3) p_2^2 i\gamma_5 u(p_1) \ .
\end{aligned} \tag{4.143}$$

The third term becomes:

$$\begin{aligned}
N_3 &= -\frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) \not{p}_2 i\gamma_5 \not{p}_3 \not{p}_2 u(p_1) \ , \\
&= -\frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) \not{p}_2 \not{p}_3 \not{p}_2 i\gamma_5 u(p_1) \ , \\
&= -\frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) (2p_2^\mu p_{3\mu} - \not{p}_3 \not{p}_2) \not{p}_2 i\gamma_5 u(p_1) \ , \\
&= -3g_{a\gamma\gamma} \overline{u}(p_3) p_2^\mu p_{3\mu} \not{p}_2 i\gamma_5 u(p_1) + \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) \not{p}_3 \not{p}_2 \not{p}_2 i\gamma_5 u(p_1) \ , \\
&= -3g_{a\gamma\gamma} \overline{u}(p_3) p_2^\mu p_{3\mu} \gamma_\beta p_2^\beta i\gamma_5 u(p_1) + \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) m_e p_2^2 i\gamma_5 u(p_1) \ , \\
&= -3g_{a\gamma\gamma} \overline{u}(p_3) \gamma_\beta i\gamma_5 p_{3\mu} p_2^\mu p_2^\beta u(p_1) + \frac{3}{2} m_e g_{a\gamma\gamma} \overline{u}(p_3) p_2^2 i\gamma_5 u(p_1) \ .
\end{aligned} \tag{4.144}$$

The fourth term becomes

$$\begin{aligned}
N_4 &= \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) \not{p}_2 i\gamma_5 \not{p}_2 \not{p}_3 u(p_1) \ , \\
&= \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) \not{p}_2 \not{p}_2 \not{p}_3 i\gamma_5 u(p_1) \ , \\
&= \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) p_2^2 \not{p}_3 i\gamma_5 u(p_1) \ , \\
&= \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) \not{p}_3 p_2^2 i\gamma_5 u(p_1) \ , \\
&= \frac{3}{2} g_{a\gamma\gamma} \overline{u}(p_3) m_e p_2^2 i\gamma_5 u(p_1) \ , \\
&= \frac{3}{2} m_e g_{a\gamma\gamma} \overline{u}(p_3) p_2^2 i\gamma_5 u(p_1) \ .
\end{aligned} \tag{4.145}$$

Combining the terms N_1, N_2, N_3, and N_4 in the numerator $N = N_1 + N_2 + N_3 + N_4$, we arrive at:

$$\begin{aligned}
N &= \frac{3}{2} m_e g_{a\gamma\gamma} \overline{u}(p_3) p_2^2 i\gamma_5 u(p_1) + 3g_{a\gamma\gamma} \overline{u}(p_3) \gamma^\beta i\gamma_5 p_1^\mu p_{2\beta} p_{2\mu} u(p_1) \\
&+ \frac{3}{2} m_e g_{a\gamma\gamma} \overline{u}(p_3) p_2^2 i\gamma_5 u(p_1) - 3g_{a\gamma\gamma} \overline{u}(p_3) \gamma_\beta i\gamma_5 p_{3\mu} p_2^\mu p_2^\beta u(p_1) \\
&+ \frac{3}{2} m_e g_{a\gamma\gamma} \overline{u}(p_3) p_2^2 i\gamma_5 u(p_1) + \frac{3}{2} m_e g_{a\gamma\gamma} \overline{u}(p_3) p_2^2 i\gamma_5 u(p_1) \ , \\
&= 6m_e g_{a\gamma\gamma} \overline{u}(p_3) p_2^2 i\gamma_5 u(p_1) \\
&+ 3g_{a\gamma\gamma} \overline{u}(p_3) (\gamma^\beta i\gamma_5 p_1^\mu p_{2\beta} p_{2\mu} - \gamma_\beta i\gamma_5 p_{3\mu} p_2^\mu p_2^\beta) u(p_1) \ .
\end{aligned} \tag{4.146}$$

Finally, substituting this numerator into 4.128 yields:

$$
\begin{aligned}
i\mathcal{M} &= 6e^2 m_e g_{a\gamma\gamma}\overline{u}(p_3)\int_0^1 dxdydz\ \delta(x+y+z-1) \\
&\times \int \frac{d^4p_2}{(2\pi)^4}\ \frac{p_2^2}{(p_2^2-\Delta+i\epsilon)^3} i\gamma_5 u(p_1) \\
&+ 3e^2 g_{a\gamma\gamma}\overline{u}(p_3)\int_0^1 dxdydz\ \delta(x+y+z-1) \\
&\times \int \frac{d^4p_2}{(2\pi)^4}\ \frac{(\gamma^\beta p_1^\mu p_{2\beta}p_{2\mu} - \gamma_\beta p_{3\mu}p_2^\mu p_2^\beta)}{(p_2^2-\Delta+i\epsilon)^3} i\gamma_5 u(p_1)\ .
\end{aligned}
\tag{4.147}
$$

Given the symmetry of the second term in p_1 and p_3, it integrates to zero, simplifying the expression to:

$$
\begin{aligned}
i\mathcal{M} &= 6e^2 m_e g_{a\gamma\gamma}\overline{u}(p_3)\int_0^1 dxdydz\ \delta(x+y+z-1) \\
&\times \int \frac{d^4p_2}{(2\pi)^4}\ \frac{p_2^2}{(p_2^2-\Delta+i\epsilon)^3} i\gamma_5 u(p_1)\ .
\end{aligned}
\tag{4.148}
$$

Now, the focus shifts to solving the integral over the free internal momentum using dimensional regularization. We start with the following integral:

$$
\int \frac{d^4p_2}{(2\pi)^4}\ \frac{p_2^2}{(p_2^2-\Delta+i\epsilon)^3}\ , \tag{4.149}
$$

and perform a Wick rotation on it [58]:

$$
\begin{aligned}
\int \frac{d^4p_2}{(2\pi)^4}\ \frac{p_2^2}{(p_2^2-\Delta+i\epsilon)^3} &= -\frac{i}{16\pi^4}\int d\Omega_4 \int dp_{2E}\ p_{2E}^3 \frac{p_{2E}^2}{(p_{2E}^2+\Delta)^3}\ , \\
&= -\frac{i}{8\pi^2}\int dp_{2E}\ \frac{p_{2E}^5}{(p_{2E}^2+\Delta)^3}\ ,
\end{aligned}
\tag{4.150}
$$

where $\int d\Omega_4 = 2\pi^2$ is the integration over the solid angle in 4 dimensions. Next, we integrate using substitution and renormalize introducing a cutoff, yielding:

$$
\begin{aligned}
-\frac{i}{8\pi^2}\int_{m_e}^{\Lambda} dp_{2E}\ \frac{p_{2E}^5}{(p_{2E}^2+\Delta)^3} &= -\frac{i}{8\pi^2}\int_{m_e^2}^{\Lambda^2} \frac{du}{2p_{2E}}\ p_{2E}\frac{u^2}{(u+\Delta)^3}\ , \\
&= -\frac{i}{16\pi^2}\int_{m_e^2}^{\Lambda^2} du\ \frac{u^2}{(u+\Delta)^3}\ ,
\end{aligned}
\tag{4.151}
$$

where $u = p_{2E}^2$. It is important to note that the cutoff integral does not start at zero but rather at the electron mass. Additionally, we use an arbitrary cutoff for now; later, when we substitute the expression for $g_{a\gamma\gamma}$, we will establish the exact cutoffs. This arises from our decision to integrate over the internal virtual electron momentum.

Continuing with the integration, we make another substitution $v = u + \Delta$:

$$\begin{aligned}
&-\frac{i}{16\pi^2}\int_{m_e^2}^{\Lambda^2} du\ \frac{u^2}{(u+\Delta)^3} = -\frac{i}{16\pi^2}\int_{m_e^2+\Delta}^{\Lambda^2+\Delta} dv\ \frac{(v-\Delta)^2}{v^3}\ ,\\
&= -\frac{i}{16\pi^2}\int_{m_e^2+\Delta}^{\Lambda^2+\Delta} dv\ \frac{v^2-2\Delta v+\Delta^2}{v^3}\ ,\\
&= -\frac{i}{16\pi^2}\int_{m_e^2+\Delta}^{\Lambda^2+\Delta} dv\ \left(\frac{1}{v}-\frac{2\Delta}{v^2}+\frac{\Delta^2}{v^3}\right),\\
&= -\frac{i}{16\pi^2}\left(\ln v\Big|_{m_e^2+\Delta}^{\Lambda^2+\Delta} - 2\Delta\frac{v^{-1}}{-1}\Big|_{m_e^2+\Delta}^{\Lambda^2+\Delta} + \Delta^2\frac{v^{-2}}{-2}\Big|_{m_e^2+\Delta}^{\Lambda^2+\Delta}\right)\ ,\\
&= -\frac{i}{16\pi^2}\left[\ln\left(\frac{\Lambda^2+\Delta}{m_e^2+\Delta}\right)+\left(\frac{2\Delta}{\Lambda^2+\Delta}-\frac{2\Delta}{m_e^2+\Delta}\right)\right.\\
&\left.-\frac{\Delta^2}{2}\left(\frac{1}{(\Lambda^2+\Delta)^2}-\frac{1}{(m_e^2+\Delta)^2}\right)\right]\ .
\end{aligned} \tag{4.152}$$

Since we are interested in the logarithmic part of the diagram, we obtain:

$$\begin{aligned}
i\mathcal{M} &= 6e^2 m_e g_{a\gamma\gamma}\overline{u}(p_3)\int_0^1 dxdydz\ \delta(x+y+z-1)\\
&\times\int\frac{d^4p_2}{(2\pi)^4}\ \frac{p_2^2}{(p_2^2-\Delta+i\epsilon)^3} i\gamma_5 u(p_1)\ ,\\
&= 6e^2 m_e g_{a\gamma\gamma}\overline{u}(p_3)\int_0^1 dxdydz\ \delta(x+y+z-1)\\
&\times\left[-\frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2+\Delta}{m_e^2+\Delta}\right)\right] i\gamma_5 u(p_1)\ ,\\
&= -\frac{3ie^2 m_e g_{a\gamma\gamma}}{8\pi^2}\overline{u}(p_3)\int_0^1 dxdydz\ \delta(x+y+z-1)\ln\left(\frac{\Lambda^2+\Delta}{m_e^2+\Delta}\right) i\gamma_5 u(p_1)\ .
\end{aligned} \tag{4.153}$$

Now, it is convenient to substitute the arbitrary cutoff and use the definition of $g_{a\gamma\gamma}$ given by:

$$g_{a\gamma\gamma} = \frac{\alpha NC}{\pi f_a} = \frac{\alpha N}{\pi f_a}\left[\frac{E}{N}-\frac{2}{3}\frac{4+z'+w'}{1+z'+w'}\right]\ , \tag{4.154}$$

where z' is used to avoid confusion with the integration variable z. Note that $\frac{E}{N}$ represents the part associated with QED, while $\frac{2}{3}\frac{4+z'+w'}{1+z'+w'}$ represents the part associated with QCD. As a result, the QED part is cutoff at a scale of order f_a, and the QCD part is cutoff at the QCD confinement scale Λ_{QCD}. This is because for loop momenta larger than Λ_{QCD}, the effects of the color anomaly become negligible. Therefore, in this regime, we should describe the axion with the PQ current of (4.56) instead of (4.62). The transition amplitude is then:

$$
\begin{aligned}
i\mathcal{M} &= -\frac{3ie^2 m_e g_{a\gamma\gamma}}{8\pi^2}\overline{u}(p_3)\int_0^1 dxdydz\ \delta(x+y+z-1)\ln\left(\frac{\Lambda^2+\Delta}{m_e^2+\Delta}\right) i\gamma_5 u(p_1)\ , \\
&= -\frac{3i\alpha^2 m_e NC}{2\pi^2 f_a}\overline{u}(p_3)\int_0^1 dxdydz\ \delta(x+y+z-1)\ln\left(\frac{\Lambda^2+\Delta}{m_e^2+\Delta}\right) i\gamma_5 u(p_1)\ , \\
&= -\frac{3i\alpha^2 N m_e}{2\pi^2 f_a}\overline{u}(p_3)\int_0^1 dxdydz\ \delta(x+y+z-1)\left[\frac{E}{N}\ln\left(\frac{f_a^2+\Delta}{m_e^2+\Delta}\right)\right. \\
&\left. -\frac{2}{3}\frac{4+z'+w'}{1+z'+w'}\ln\left(\frac{\Lambda_{QCD}^2+\Delta}{m_e^2+\Delta}\right)\right] i\gamma_5 u(p_1)\ .
\end{aligned}
\tag{4.155}
$$

Now, let's recover the definition of Δ so that we can integrate over x, y and z:

$$
\Delta = m_e^2[(1+y)x + (1-z)z]\ , \tag{4.156}
$$

comparing Δ, which is proportional to the squared electron mass, to the cut-offs that we used (f_a, Λ_{QCD}), where in the invisible axion models $f_a \gg v_{\text{SM}} = 246.22$ GeV [13] and $\Lambda_{QCD} \sim 1$ GeV, we can make the approximation $f_a^2 + \Delta \approx f_a^2$ and $\Lambda_{QCD}^2 + \Delta \approx \Lambda_{QCD}^2$. With these approximations, we find:

$$
\begin{aligned}
i\mathcal{M} = &-\frac{3i\alpha^2 N m_e}{2\pi^2 f_a}\overline{u}(p_3)\int_0^1 dxdydz\ \delta(x+y+z-1)\left[\frac{E}{N}\ln\left(\frac{f_a^2}{m_e^2+\Delta}\right)\right. \\
&\left. -\frac{2}{3}\frac{4+z'+w'}{1+z'+w'}\ln\left(\frac{\Lambda_{QCD}^2}{m_e^2+\Delta}\right)\right] i\gamma_5 u(p_1)\ .
\end{aligned}
\tag{4.157}
$$

We can also rewrite the logarithmic terms more intelligently as follows:

$$
\begin{aligned}
\ln\left(\frac{f_a^2}{m_e^2+\Delta}\right) &= \ln\left(\frac{f_a^2}{m_e^2 + m_e^2[(1+y)x+(1-z)z]}\right)\ , \\
&= \ln\left(\frac{f_a^2}{m_e^2[(1+y)x+(1-z)z+1]}\right)\ , \\
&= \ln\left(\frac{f_a^2}{m_e^2}\frac{1}{[(1+y)x+(1-z)z+1]}\right)\ , \\
&= \ln\left(\frac{f_a^2}{m_e^2}\right) + \ln\left(\frac{1}{[(1+y)x+(1-z)z+1]}\right)\ , \\
&= 2\ln\left(\frac{f_a}{m_e}\right) + \ln(1) - \ln\left[(1+y)x+(1-z)z+1\right]\ , \\
&= 2\ln\left(\frac{f_a}{m_e}\right) - \ln\left[(1+y)x+(1-z)z+1\right]\ ,
\end{aligned}
\tag{4.158}
$$

this rewriting is analogous for the Λ_{QCD} term. For clarity, let's focus on the integration of the first term:

$$
\begin{aligned}
&\frac{E}{N}\int_0^1 dxdydz\ \delta(x+y+z-1)\left[2\ln\left(\frac{f_a}{m_e}\right)-\ln\left[(1+y)x+(1-z)z+1\right]\right]\\
=&\frac{E}{N}\int_0^1 dxdy\ \left[2\ln\left(\frac{f_a}{m_e}\right)-\ln\left[(1+y)x+(1-(1-x-y))(1-x-y)+1\right]\right]\ ,\\
=&\frac{E}{N}\int_0^1 dxdy\ \left[2\ln\left(\frac{f_a}{m_e}\right)-\ln\left[x+xy+(x+y)(1-x-y)+1\right]\right]\ ,\\
=&\frac{E}{N}\int_0^1 dxdy\ \left[2\ln\left(\frac{f_a}{m_e}\right)-\ln\left[x+xy+x+y-x^2-xy-xy-y^2+1\right]\right]\ ,\\
=&\frac{E}{N}\int_0^1 dxdy\ \left[2\ln\left(\frac{f_a}{m_e}\right)-\ln\left[-x^2-y^2-xy+2x+y+1\right]\right]\ ,
\end{aligned}
\tag{4.159}
$$

since the integration is from 0 to 1, it is clear that the second log term is negligible in comparison to the other. Therefore, we get:

$$
\begin{aligned}
\frac{2E}{N}\left[\ln\left(\frac{f_a}{m_e}\right)\right]\int_0^1 dxdy\ &=\frac{2E}{N}\left[\ln\left(\frac{f_a}{m_e}\right)\right]\int_0^1 dx\ \int_0^{1-x} dy\ ,\\
&=\frac{2E}{N}\left[\ln\left(\frac{f_a}{m_e}\right)\right]\int_0^1 dx\ y\Big|_0^{1-x}\ ,\\
&=\frac{2E}{N}\left[\ln\left(\frac{f_a}{m_e}\right)\right]\int_0^1 dx\ 1-x\ ,\\
&=\frac{2E}{N}\left[\ln\left(\frac{f_a}{m_e}\right)\right]\left(x\Big|_0^1-\frac{x^2}{2}\Big|_0^1\right)\ ,\\
&=\frac{2E}{N}\left[\ln\left(\frac{f_a}{m_e}\right)\right]\left(1-\frac{1}{2}\right)\ ,\\
&=\frac{E}{N}\ln\left(\frac{f_a}{m_e}\right)\ ,
\end{aligned}
\tag{4.160}
$$

Finally, including both cutoff logs, we reach the final result:

$$
i\mathcal{M}=-\frac{3i\alpha^2Nm_e}{2\pi^2f_a}\overline{u}(p_3)\left[\frac{E}{N}\ln\frac{f_a}{m_e}-\frac{2}{3}\frac{4+z+w}{1+z+w}\ln\frac{\Lambda_{QCD}}{m_e}\right]i\gamma_5u(p_1)\ , \tag{4.161}
$$

Note that we have returned to the original z, w notation, since the integration over x, y, z is already performed.

The divergent part of the diagram corresponds to a term in the low-energy effective lagrangian:

$$
\mathcal{L}\supset\frac{3\alpha^2Nm_e}{2\pi^2f_a}\left[\frac{E}{N}\ln\frac{f_a}{m_e}-\frac{2}{3}\frac{4+z+w}{1+z+w}\ln\frac{\Lambda}{m_e}\right]a\bar{e}i\gamma_5e\ , \tag{4.162}
$$

this expression is valid only for on-shell electrons.

$$
g_{ae}=\frac{3\alpha^2Nm_e}{2\pi^2f_a}\left[\frac{E}{N}\ln\frac{f_a}{m_e}-\frac{2}{3}\frac{4+z+w}{1+z+w}\ln\frac{\Lambda}{m_e}\right] \tag{4.163}
$$

Numerically, for $f_a \approx 10^{12}$ GeV, $\Lambda = 1$ GeV, $E/N = 8/3$ and $N = 1$, it corresponds to

$$|X'_e| = 10^{-3} \ , \tag{4.164}$$

in (4.83).

4.7 Coupling to nucleons

Consider the axion current given by (4.62) but in a form where the axion does not mix with the NG boson y:

$$j^{a'}_\mu = j'^{PQ}_\mu - N(1+z+w)^{-1}(\overline{u}\gamma_\mu\gamma_5 u + z\overline{d}\gamma_\mu\gamma_5 d + w\overline{s}\gamma_\mu\gamma_5 s) \ , \tag{4.165}$$

where j'^{PQ}_μ is defined in (4.80). Explicitly writing this expression, we obtain:

$$\begin{aligned} j^{a'}_\mu &= f_a\partial_\mu a' + \sum_i X'_i\overline{\psi}_i\gamma_\mu\gamma_5\psi_i \\ &- N(1+z+w)^{-1}(\overline{u}\gamma_\mu\gamma_5 u + z\overline{d}\gamma_\mu\gamma_5 d + w\overline{s}\gamma_\mu\gamma_5 s) \ , \\ &= f_a\partial_\mu a' + X'_u\overline{u}\gamma_\mu\gamma_5 u + X'_d\overline{d}\gamma_\mu\gamma_5 d + X'_s\overline{s}\gamma_\mu\gamma_5 s \\ &- N(1+z+w)^{-1}(\overline{u}\gamma_\mu\gamma_5 u + z\overline{d}\gamma_\mu\gamma_5 d + w\overline{s}\gamma_\mu\gamma_5 s) \ , \\ &= f_a\partial_\mu a' + \left(X'_u - \frac{N}{1+z+w}\right)\overline{u}\gamma_\mu\gamma_5 u + \left(X'_d - z\frac{N}{1+z+w}\right)\overline{d}\gamma_\mu\gamma_5 d \\ &+ \left(X'_s - w\frac{N}{1+z+w}\right)\overline{s}\gamma_\mu\gamma_5 s \ . \end{aligned} \tag{4.166}$$

Given that the coupling is to nucleons and not any strange quark-composed baryon, it is reasonable to eliminate the last term above. This simplifies the expression to:

$$j^{a'}_\mu = f_a\partial_\mu a' + \left(X'_u - \frac{N}{1+z+w}\right)\overline{u}\gamma_\mu\gamma_5 u + \left(X'_d - z\frac{N}{1+z+w}\right)\overline{d}\gamma_\mu\gamma_5 d \ . \tag{4.167}$$

Now, let's express it in terms of the isoscalar and isovector components, j^0_μ and j^3_μ, defined as:

$$\begin{aligned} j^0_\mu &= \frac{1}{2}\overline{u}\gamma_\mu\gamma_5 u + \frac{1}{2}\overline{d}\gamma_\mu\gamma_5 d \ , \\ j^3_\mu &= \frac{1}{2}\overline{u}\gamma_\mu\gamma_5 u - \frac{1}{2}\overline{d}\gamma_\mu\gamma_5 d \ . \end{aligned} \tag{4.168}$$

The process unfolds as follows:

$$\begin{aligned} j^{a'}_\mu &= f_a\partial_\mu a' + \left(X'_u - \frac{N}{1+z+w}\right)(j^0_\mu + j^3_\mu) + \left(X'_d - z\frac{N}{1+z+w}\right)(j^0_\mu - j^3_\mu) \ , \\ &= f_a\partial_\mu a' + \left(X'_u - \frac{N}{1+z+w}\right)j^0_\mu + \left(X'_d - z\frac{N}{1+z+w}\right)j^0_\mu \\ &+ \left(X'_u - \frac{N}{1+z+w}\right)j^3_\mu - \left(X'_d - z\frac{N}{1+z+w}\right)j^3_\mu \ , \\ &= f_a\partial_\mu a' + \left(X'_u + X'_d - \frac{1+z}{1+z+w}N\right)j^0_\mu + \left(X'_u - X'_d - \frac{1-z}{1+z+w}N\right)j^3_\mu \ . \end{aligned} \tag{4.169}$$

To translate these coefficients of j^0_μ and j^3_μ into the couplings, one must utilize the Goldberger-Treiman relation [59]:

$$F(0) = -\frac{m_N}{2\pi^2}\sqrt{2}g_{\pi\overline{N}N}g_A\frac{J}{1+(g^2_{\pi\overline{N}N}/4\pi)(2J/\pi)} \ , \tag{4.170}$$

where J is the following integral:

$$J = \int_0^\infty dk \ \frac{k^2}{(k^2+m_N^2)^{3/2}}\cos\phi(k)\exp\left\{\frac{2}{\pi}\int_0^\infty dk' \ k'\phi(k')\left(\frac{1}{k'^2-k^2}-\frac{1}{k'^2-m_N^2}\right)\right\} \ , \tag{4.171}$$

where the specific details are not crucial for our purposes, as a more modern notation for this relation is available [60]:

$$g_{A\pi\overline{N}N}m_N = f_a g_{a\overline{N}N} \ . \tag{4.172}$$

This relation connects four phenomenological/non-perturbative QCD+axion constants: the pion-nucleon axial coupling $g_{A\pi\overline{N}N}$, the nucleon mass m_N, the axion decay constant f_a, and $g_{a\overline{N}N}$, the axion-nucleon coupling.

First, let's define the nucleon doublet as:

$$N = \begin{pmatrix} p \\ n \end{pmatrix} \ . \tag{4.173}$$

The effective lagrangian $\mathcal{L}_{\text{eff}}$ is given by:

$$\mathcal{L} \supset a\overline{N}(g_0 + g_3\tau_3)i\gamma_5 N \ , \tag{4.174}$$

where τ_3 is a Pauli matrix in isospin space, and the coefficients g_0 and g_3 are given by:

$$\begin{aligned} g_0 &= \left(X'_u + X'_d - \frac{1+z}{1+z+w}N\right)(-g_{A0\pi\overline{N}N})(m_N/f_a) \ , \\ g_3 &= \left(X'_u - X'_d - \frac{1-z}{1+z+w}N\right)(-g_{A3\pi\overline{N}N})(m_N/f_a) \ , \end{aligned} \tag{4.175}$$

where $g_{A3\pi\overline{N}N}$ is the axial isovector pion-nucleon coupling, which is experimentally measured to be -1.25 and quark models arguments [61] suggest $g_{A0\pi\overline{N}N} = \frac{3}{5}g_{A3\pi\overline{N}N}$.

The axion-nucleon coupling is given by:

$$g_{a\overline{N}N} = g_0 + g_3\tau_3 \ . \tag{4.176}$$

In the KSVZ model, where no regular Standard Model particle has PQ charge ($X'_u = X'_d = 0$), the axion still exhibits a substantial coupling to nucleons. In the DFSZ model, we have $N = 6$, $X'_u + X'_d = 2$, and $X'_u - X'_d = 2\cos 2\beta$, which also results in a substantial coupling to nucleons.

5 Solar Axion Production

Now that we have established the two crucial parameters governing axion production, namely g_{ae} and $g_{a\gamma\gamma}$, we are prepared to delve into the examination of axion production. Our primary focus will be on elucidating the Sun's role as a potent source of axions.

Acquiring a precise understanding of the solar axion flux is crucial for any experiment with the goal of detecting solar axions. This knowledge is paramount in designing experiments that are not only sensitive but also finely tuned to detect solar axion particles.

The fundamental concept revolves around stars being powerful sources of weakly interacting particles, such as neutrinos, gravitons, hypothetical axions, and other new particles that can be produced by nuclear reactions or by thermal processes in the stellar interior. Even when this particle flux cannot be directly measured, the properties of stars would undergo changes if they were to lose too much energy into a new channel. Additionally, this excess of energy loss would directly impact stellar evolution. This "energy-loss argument" has been extensively employed to constrain a diverse array of particle properties. In our case, we focus on the resulting constraints for invisible axions, and these constraints are summarized in Table.1.

Couplings bounds		
Coupling	Bound	Observable
$g_{a\gamma\gamma}$	$< 0.65 \times 10^{-10}$ GeV^{-1} (95% C.L.)	HB/RG stars in 39 GCs [62]
g_{ae}	$< 2.6 \times 10^{-13}$ (95% C.L.)	WD cooling + RGB tip M5 + HB/RG in GCs [63]
g_{ap}	$< 0.9 \times 10^{-9}$	SN1987A ν-pulse duration [64]
g_{an}	$< 0.8 \times 10^{-9}$	Neutron star cooling [65]
g_{an}	$< 0.5 \times 10^{-9}$	CAS A NS cooling [63, 66]
$g_{a\gamma N}$	$< 3 \times 10^{-9}$ GeV^{-2}	SN1987A ν-pulse duration [67]

Table 1 – Main astrophysical bounds on ALPs coupled to photons, electrons, protons and neutrons. HB (Horizontal Branch) and RG (Red Giant) bounds are valid for masses $m_a \lesssim 10$ keV, WD (White Dwarf) for $m_a \lesssim 1$ keV, SN (Solar Neutrino) and NS (Neutron Star) require $m_a \lesssim 1$ MeV [68].

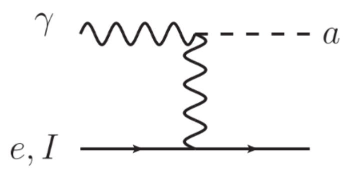


Figure 7 – Primakoff Conversion [72]

5.1 Axion production from $g_{a\gamma\gamma}$

5.1.1 Axion-photon oscillation

One of the pivotal processes in axion production is the axion-photon oscillation, commonly referred to as Primakoff conversion. In this mechanism, a photon undergoes conversion into an axion, a phenomenon facilitated by the presence of a strong external magnetic field.

The Primakoff conversion is governed by the axion-two-photon coupling, denoted as $g_{a\gamma\gamma}$. This process takes precedence in hadronic axion models, such as the KSVZ model, where the axion-electron coupling g_{ae} is absent at the tree-level. Due to this characteristic and its simplicity as a mode of axion production, Primakoff conversion historically emerged as the foremost process and has been extensively explored in previous studies (see, for example, [69–71]).

This process is represented by the Feynman diagram shown in Figure 7. To initiate the investigation of this mechanism, we start with the axion-photon effective lagrangian previously employed in this work:

$$\mathcal{L}_{a\gamma\gamma} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu a\partial^\mu a - \frac{1}{2}m_a^2 a^2 - \frac{1}{4}g_{a\gamma\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu} \ . \tag{5.1}$$

To calculate the oscillation probability, we begin by deriving the classical equations of motion for the axion and vector fields:

$$\begin{aligned} \frac{\partial\mathcal{L}_{a\gamma\gamma}}{\partial a} - \partial_\mu\frac{\partial\mathcal{L}_{a\gamma\gamma}}{\partial(\partial_\mu a)} &= 0 \ , \\ \frac{\partial\mathcal{L}_{a\gamma\gamma}}{\partial A_\nu} - \partial_\mu\frac{\partial\mathcal{L}_{a\gamma\gamma}}{\partial(\partial_\mu A_\nu)} &= 0 \ . \end{aligned} \tag{5.2}$$

Solving these equations yields:

$$(\partial_\mu\partial^\mu + m_a^2)a = -\frac{1}{4}g_{a\gamma\gamma}F_{\mu\nu}\tilde{F}^{\mu\nu} = g_{a\gamma\gamma}\vec{E}\cdot\vec{B} \ , \tag{5.3}$$

$$\partial_\mu F^{\mu\nu} = -g_{a\gamma\gamma}\tilde{F}^{\mu\nu}\partial_\mu a \ , \tag{5.4}$$

where $\vec{E}$ is the electric field generated by the photon, and $\vec{B}$ is the external magnetic field oriented in the $\hat{y}$ direction, i.e., $\vec{B} = B\hat{y}$.

Considering that the photon is polarized in the $\hat{y}$ direction and propagates in the $\hat{z}$ direction, the total magnetic field is a combination of the photon's magnetic field and the external magnetic field. Thus, the dual field strength tensor can be expressed as [73, 74]:

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -(B_y + B_{ext}) & -B_z \\ B_x & 0 & E_z & -E_y \\ (B_y + B_{ext}) & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} . \tag{5.5}$$

If we consider B_{ext} to be much stronger than the photon magnetic field, the expression in (5.5) can be approximated as [75–78]:

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & 0 & -B & 0 \\ 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{5.6}$$

where the external magnetic field B_{ext} is simply denoted as B for clarity. Expressing the electric field in terms of the vector potential $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$ and considering the established direction for the external magnetic field, (5.3) becomes:

$$(\partial_\mu \partial^\mu + m_a^2)a = -g_{a\gamma\gamma} B \frac{\partial A_y}{\partial t} . \tag{5.7}$$

Now, using (5.6), it is evident that the only non-zero component is $\tilde{F}^{02}$. Therefore, the right-hand side of (5.4) becomes $-g_{a\gamma\gamma} B \frac{\partial a}{\partial t}$.

On the other hand, for $\nu = 2$, the left-hand side of (5.4) is $\partial_\mu F^{\mu 2}$, where $F^{\mu 2} = \partial^\mu A^2 - \partial^2 A^\mu$. Using the Coulomb gauge, $\partial_\mu A^\mu = 0$, we obtain:

$$\begin{aligned} \partial_\mu F^{\mu 2} &= \partial_\mu (\partial^\mu A^2 - \partial^2 A^\mu) , \\ &= \partial_\mu \partial^\mu A^2 . \end{aligned} \tag{5.8}$$

Thus, (5.4) can be expressed as:

$$\partial_\mu \partial^\mu A_y = g_{a\gamma\gamma} B \frac{\partial a}{\partial t} . \tag{5.9}$$

Equations (5.7) and (5.9) describe the behavior of the axion and of the potencial vector on the inside of the system.

Given that the photon's propagation direction aligns with $\hat{z}$ and the magnetic field variations exhibit higher-order magnitudes compared to the photon wavelength, the solutions to these equations adopt a specific form [75–78]

$$\vec{A}(z,t) = A_x(z,t)\hat{x} + A_y(z,t)\hat{y} , \tag{5.10}$$

$$a(z,t) = a(z)e^{i(kz-\omega t)} \ , \tag{5.11}$$

where,

$$A_x(z,t) = A_\perp(z)e^{i(kz-\omega t)} \ , \tag{5.12}$$

$$A_y(z,t) = A_\parallel(z)e^{i(kz-\omega t)} \ , \tag{5.13}$$

depicting the components of $\vec{A}$ that are parallel and perpendicular to the external magnetic field, k is the wave vector, and ω is the wave frequency.

Substituting (5.11) and (5.13) into (5.7), we arrive at the following coupled equations:

$$\begin{aligned} (\partial_\mu\partial^\mu + m_a^2)a(z)e^{i(kz-\omega t)} &= -g_{a\gamma\gamma}B\frac{\partial A_\parallel(z)e^{i(kz-\omega t)}}{\partial t} \ , \\ (-\omega^2 - \partial_z^2 + m_a^2)a(z)e^{i(kz-\omega t)} &= ig_{a\gamma\gamma}\omega BA_\parallel(z)e^{i(kz-\omega t)} \ , \end{aligned} \tag{5.14}$$

leading to the derived equation:

$$(\omega^2 + \partial_z^2 - m_a^2)a(z,t) + ig_{a\gamma\gamma}\omega BA_y(z,t) = 0 \ . \tag{5.15}$$

Applying a similar procedure as in (5.9), we obtain:

$$(\omega^2 + \partial_z^2)A_y(z,t) - ig_{a\gamma\gamma}\omega Ba(z,t) = 0 \ , \tag{5.16}$$

both equations (5.15) and (5.16) can be compactly expressed as:

$$\left[(\omega^2 + \partial_z^2)\mathbf{I} + \begin{pmatrix} 0 & -ig_{a\gamma\gamma}\omega B \\ ig_{a\gamma\gamma}\omega B & -m_a^2 \end{pmatrix}\right]\begin{pmatrix} A_y(z,t) \\ a(z,t) \end{pmatrix} = 0 \ . \tag{5.17}$$

Establishing $w^2 + \partial_z^2 = (\omega + i\partial_z)(\omega - i\partial_z)$, and utilizing the expression

$$(\omega - i\partial_z)\begin{pmatrix} A_y(z,t) \\ a(z,t) \end{pmatrix} = (\omega + k)\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix} \cong 2\omega\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix} \ , \tag{5.18}$$

where we made the approximation $\omega + k \cong 2\omega$ [75], we can apply it to (5.17):

$$\left[2\omega(\omega + i\partial_z)\mathbf{I} + \begin{pmatrix} 0 & -ig_{a\gamma\gamma}\omega B \\ ig_{a\gamma\gamma}\omega B & -m_a^2 \end{pmatrix}\right]\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix} = 0 \ , \tag{5.19}$$

which further simplifies to:

$$\left[i\partial_z\mathbf{I} + \begin{pmatrix} \omega & -\frac{1}{2}ig_{a\gamma\gamma}B \\ \frac{1}{2}ig_{a\gamma\gamma}B & \omega - \frac{m_a^2}{2\omega} \end{pmatrix}\right]\begin{pmatrix} A_\parallel(z) \\ a(z,) \end{pmatrix} = 0 \ . \tag{5.20}$$

This yields a Schrödinger-like equation for a wave-function represented as:

$$|\Psi(z)\rangle = \begin{pmatrix} A_\parallel(z) \\ a(z,) \end{pmatrix} \ , \tag{5.21}$$

which evolves over a distance z through the evolution operator $U(z)$, described by the equation:

$$|\Psi(z)\rangle = U(z)\,|\psi(0)\rangle \ , \tag{5.22}$$

where $|\psi(0)\rangle$ is the initial state of the photon, $U(z) = e^{iKz}$, and K is given by:

$$K = \begin{pmatrix} \omega & -\frac{1}{2}ig_{a\gamma\gamma}B \\ \frac{1}{2}ig_{a\gamma\gamma}B & \omega - \frac{m_a^2}{2\omega} \end{pmatrix} \ . \tag{5.23}$$

With equation (5.20) at our disposal, we can compute the photon-to-axion oscillation probability using the expression:

$$P_{\gamma\to a} = \langle\Psi(z)|a\rangle\,\langle a|\Psi(z)\rangle \ . \tag{5.24}$$

Given that the initial photon state is represented by $|\psi(0)\rangle = \left|A_{\parallel}\right\rangle$, such that $|\Psi(z)\rangle = U(z)\left|A_{\parallel}\right\rangle$, (5.24) can be expressed as:

$$P_{\gamma\to a} = |\,\langle a|\,U(z)\left|A_{\parallel}\right\rangle|^2 \ . \tag{5.25}$$

To determine the probability, we need the evolution operator in terms of the eigenstates $\left|A_{\parallel}\right\rangle$ and $|a\rangle$. This involves obtaining the eigenvalues and eigenstates of the K matrix (5.23) by diagonalizing it. The procedure begins with solving the characteristic equation:

$$\det|K - \lambda I| = 0 \ , \tag{5.26}$$

where λ are the eigenvalues of K. This leads to the following:

$$\begin{aligned} &\det\begin{pmatrix} \omega-\lambda & -\frac{1}{2}ig_{a\gamma\gamma}B \\ \frac{1}{2}ig_{a\gamma\gamma}B & \omega - \frac{m_a^2}{2\omega} - \lambda \end{pmatrix} \\ &= (\omega-\lambda)\left(\omega - \frac{m_a^2}{2\omega} - \lambda\right) - \left(\frac{1}{2}ig_{a\gamma\gamma}B\right)\left(-\frac{1}{2}ig_{a\gamma\gamma}B\right) \ , \\ &= \omega^2 - \omega\frac{m_a^2}{2\omega} - \omega\lambda - \omega\lambda + \frac{m_a^2}{2\omega}\lambda + \lambda^2 - \frac{1}{4}g_{a\gamma\gamma}^2B^2 \ , \\ &= \lambda^2 - 2\omega\lambda + \frac{m_a^2}{2\omega}\lambda + \omega^2 - \frac{m_a^2}{2} - \frac{1}{4}g_{a\gamma\gamma}^2B^2 \ , \\ &= \lambda^2 + \lambda\left(\frac{m_a^2}{2\omega} - 2\omega\right) + \omega^2 - \frac{m_a^2}{2} - \frac{1}{4}g_{a\gamma\gamma}^2B^2 \ . \\ &= 0 \end{aligned} \tag{5.27}$$

Now, we solve the quadratic equation for λ, and the solution is:

$$\begin{aligned} \Delta &= \left(\frac{m_a^2}{2\omega} - 2\omega\right)^2 - 4(1)\left(\omega^2 - \frac{m_a^2}{2} - \frac{1}{4}g_{a\gamma\gamma}^2B^2\right) \ , \\ &= \frac{m_a^4}{4\omega^2} - 2m_a^2\frac{2\omega}{2\omega} + 4\omega^2 - 4\omega^2 + 2m_a^2 + g_{a\gamma\gamma}^2B^2 \ , \\ &= \frac{m_a^4}{4\omega^2} + g_{a\gamma\gamma}^2B^2 \ , \end{aligned} \tag{5.28}$$

Let's redefine Δ for convenience as:

$$\begin{aligned} \Delta' &= 4\omega^2 \Delta \ , \\ &= m_a^4 + 4\omega^2 g_{a\gamma\gamma}^2 B^2 \ . \end{aligned} \tag{5.29}$$

Solving it for λ gives us:

$$\begin{aligned} \lambda_\pm &= \frac{-\left(\frac{m_a^2}{2\omega} - 2\omega\right) \pm \frac{\sqrt{\Delta'}}{2\omega}}{2} \ , \\ &= \omega - \frac{m_a^2}{4\omega} \pm \frac{\sqrt{\Delta'}}{4\omega} \ . \end{aligned} \tag{5.30}$$

After calculating the previous determinant and solving the characteristic equation, we find the eigenvalues of K which are given by:

$$\lambda_\pm = \omega - \frac{m_a^2}{4\omega} \pm \frac{\sqrt{\Delta'}}{4\omega} \ . \tag{5.31}$$

Now, let's proceed to find the corresponding eigenstates:

- For $\lambda_+ = \omega - \frac{m_a^2}{4\omega} + \frac{\sqrt{\Delta'}}{4\omega}$

$$\begin{aligned} &\begin{pmatrix} \omega - \omega + \frac{m_a^2}{4\omega} + \frac{\sqrt{\Delta'}}{4\omega} & -\frac{1}{2} i g_{a\gamma\gamma} B \\ \frac{1}{2} i g_{a\gamma\gamma} B & \omega - \frac{m_a^2}{2\omega} - \omega + \frac{m_a^2}{4\omega} + \frac{\sqrt{\Delta'}}{4\omega} \end{pmatrix} \begin{pmatrix} \left|A_\parallel\right\rangle \\ |a\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ , \\ &\begin{pmatrix} \frac{m_a^2}{4\omega} + \frac{\sqrt{\Delta'}}{4\omega} & -\frac{1}{2} i g_{a\gamma\gamma} B \\ \frac{1}{2} i g_{a\gamma\gamma} B & -\frac{m_a^2}{4\omega} + \frac{\sqrt{\Delta'}}{4\omega} \end{pmatrix} \begin{pmatrix} \left|A_\parallel\right\rangle \\ |a\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ , \end{aligned} \tag{5.32}$$

which gives us two equations:

$$\begin{aligned} &\left(\frac{m_a^2}{4\omega} + \frac{\sqrt{\Delta'}}{4\omega}\right) \left|A_\parallel\right\rangle - \frac{1}{2} i g_{a\gamma\gamma} B \,|a\rangle = 0 \ , \\ &\frac{1}{2} i g_{a\gamma\gamma} \left|A_\parallel\right\rangle + \left(-\frac{m_a^2}{4\omega} + \frac{\sqrt{\Delta'}}{4\omega}\right) |a\rangle = 0 \ , \end{aligned} \tag{5.33}$$

multiplying both by 4ω gives:

$$\begin{aligned} &\left(m_a^2 + \sqrt{\Delta'}\right) \left|A_\parallel\right\rangle - 2i\omega g_{a\gamma\gamma} B \,|a\rangle = 0 \ , \\ &2i\omega g_{a\gamma\gamma} \left|A_\parallel\right\rangle + \left(-m_a^2 + \sqrt{\Delta'}\right) |a\rangle = 0 \ . \end{aligned} \tag{5.34}$$

- For $\lambda_- = \omega - \frac{m_a^2}{4\omega} - \frac{\sqrt{\Delta'}}{4\omega}$, similarly we find:

$$\begin{aligned} &\left(m_a^2 - \sqrt{\Delta'}\right) \left|A_\parallel\right\rangle - 2i\omega g_{a\gamma\gamma} B \,|a\rangle = 0 \ , \\ &2i\omega g_{a\gamma\gamma} \left|A_\parallel\right\rangle + \left(-m_a^2 - \sqrt{\Delta'}\right) |a\rangle = 0 \ . \end{aligned} \tag{5.35}$$

We obtain the non-normalized eigenvectors as:

$$\begin{aligned} |\lambda_+\rangle &= \left(m_a^2 + \sqrt{\Delta'}\right) \left|A_\parallel\right\rangle + 2i\omega g_{a\gamma\gamma} B \,|a\rangle \ , \\ |\lambda_-\rangle &= \left(m_a^2 - \sqrt{\Delta'}\right) \left|A_\parallel\right\rangle + 2i\omega g_{a\gamma\gamma} B \,|a\rangle \ , \end{aligned} \tag{5.36}$$

The normalization proceeds as follows, taking into account the complex nature of the vectors:

$$\begin{aligned}
|| \, |\lambda_+\rangle \, || &= \left[(m_a^2 + \sqrt{\Delta'})^2 + (-2\omega g_{a\gamma\gamma} B)^2\right]^{1/2} \ , \\
&= \left[m_a^4 + 2m_a^2\sqrt{\Delta'} + \Delta' + 4\omega^2 g_{a\gamma\gamma}^2 B^2\right]^{1/2} \ , \\
&= \left[m_a^4 + 2m_a^2\sqrt{\Delta'} + \Delta' + \Delta' - m_a^4\right]^{1/2} \ , \\
&= \left[2\left(\Delta' + m_a^2\sqrt{\Delta'}\right)\right]^{1/2} \ ,
\end{aligned} \tag{5.37}$$

note that we used (5.29) in the third line. The normalization for $|\lambda_-\rangle$ is analogous. Therefore, we finally have the normalized eigenvectors:

$$\begin{aligned}
|\lambda_+\rangle &= \frac{1}{\left[2\left(\Delta' + m_a^2\sqrt{\Delta'}\right)\right]^{1/2}} \left[\left(m_a^2 + \sqrt{\Delta'}\right)\left|A_\parallel\right\rangle + 2i\omega g_{a\gamma\gamma} B \,|a\rangle\right] \ , \\
|\lambda_-\rangle &= \frac{1}{\left[2\left(\Delta' - m_a^2\sqrt{\Delta'}\right)\right]^{1/2}} \left[\left(m_a^2 - \sqrt{\Delta'}\right)\left|A_\parallel\right\rangle + 2i\omega g_{a\gamma\gamma} B \,|a\rangle\right] \ ,
\end{aligned} \tag{5.38}$$

which correspond to the eigenvectors that compose the matrix that diagonalizes K.

Recognizing that the evolution operator is diagonal in this basis, we can express it in terms of the eigenstates of the matrix K as:

$$\begin{aligned}
U(z) = e^{iKz} &= e^{iKz}(|\lambda_+\rangle\langle\lambda_+| + |\lambda_-\rangle\langle\lambda_-|) \ , \\
&= e^{i\lambda_+ z}\,|\lambda_+\rangle\langle\lambda_+| + e^{i\lambda_- z}\,|\lambda_-\rangle\langle\lambda_-| \ ,
\end{aligned} \tag{5.39}$$

where we used the fact that $|\lambda_+\rangle\langle\lambda_+| + |\lambda_-\rangle\langle\lambda_-| = 1$ and $K\,|\lambda_\pm\rangle = \lambda_\pm\,|\lambda_\pm\rangle$.

Substituting (5.38) into (5.39), the operator can be expressed as:

$$\begin{aligned}
U(z) = &\frac{e^{i\lambda_+ z}}{2\left(\Delta' + m_a^2\sqrt{\Delta'}\right)} \left[(m_a^2 + \sqrt{\Delta'})^2 \left|A_\parallel\right\rangle\left\langle A_\parallel\right| \right. \\
&- 2i\omega g_{a\gamma\gamma} B(m_a^2 + \sqrt{\Delta'})\left|A_\parallel\right\rangle\langle a| \\
&\left. + 2i\omega g_{a\gamma\gamma} B(m_a^2 + \sqrt{\Delta'})\,|a\rangle\left\langle A_\parallel\right| + 4\omega^2 g_{a\gamma\gamma}^2 B^2\,|a\rangle\langle a|\right] \\
&+ \frac{e^{i\lambda_- z}}{2\left(\Delta' - m_a^2\sqrt{\Delta'}\right)} \left[(m_a^2 - \sqrt{\Delta'})^2 \left|A_\parallel\right\rangle\left\langle A_\parallel\right| \right. \\
&- 2i\omega g_{a\gamma\gamma} B(m_a^2 - \sqrt{\Delta'})\left|A_\parallel\right\rangle\langle a| \\
&\left. + 2i\omega g_{a\gamma\gamma} B(m_a^2 - \sqrt{\Delta'})\,|a\rangle\left\langle A_\parallel\right| + 4\omega^2 g_{a\gamma\gamma}^2 B^2\,|a\rangle\langle a|\right] \ ,
\end{aligned} \tag{5.40}$$

thus, the oscillation probability given in (5.25) becomes (note that $\left|A_\parallel\right\rangle$ and $|a\rangle$ are orthogonal):

$$
\begin{aligned}
P_{\gamma\to a} &= |\left\langle a\right| U(z) \left|A_{\parallel}\right\rangle|^2 \ , \\
&= \left|\frac{e^{i\lambda_+ z}}{2\left(\Delta' + m_a^2\sqrt{\Delta'}\right)} 2i\omega g_{a\gamma\gamma} B(m_a^2 + \sqrt{\Delta'}) \right. \\
&\left. + \frac{e^{i\lambda_- z}}{2\left(\Delta' - m_a^2\sqrt{\Delta'}\right)} 2i\omega g_{a\gamma\gamma} B(m_a^2 - \sqrt{\Delta'}) \right|^2 \ , \\
&= \omega^2 g_{a\gamma\gamma}^2 B^2 \left| \frac{1}{\sqrt{\Delta'}} \left[\left(\frac{m_a^2 + \sqrt{\Delta'}}{\sqrt{\Delta'} + m_a^2}\right) e^{i\lambda_+ z} + \left(\frac{m_a^2 - \sqrt{\Delta'}}{\sqrt{\Delta'} - m_a^2}\right) e^{i\lambda_- z} \right] \right|^2 \ , \\
&= \frac{\omega^2 g_{a\gamma\gamma}^2 B^2}{\Delta'} \left| e^{i\lambda_+ z} - e^{i\lambda_- z} \right|^2 \ , \\
&= \frac{2\omega^2 g_{a\gamma\gamma}^2 B^2}{\Delta'} [1 - \cos\left[(\lambda_+ - \lambda_-) z\right]] \ , \\
&= \frac{4\omega^2 g_{a\gamma\gamma}^2 B^2}{\Delta'} \sin^2 \left[\left(\frac{\lambda_+ - \lambda_-}{2}\right) z \right] \ .
\end{aligned} \tag{5.41}
$$

Using (5.31), we have:

$$
\lambda_+ - \lambda_- = \frac{\sqrt{\Delta'}}{2\omega} \ , \tag{5.42}
$$

thus, the oscillation probability becomes:

$$
P_{\gamma\to a} = \frac{4\omega^2 g_{a\gamma\gamma}^2 B^2}{\Delta'} \sin^2 \left(\frac{\sqrt{\Delta'}}{4\omega} z \right) \ , \tag{5.43}
$$

which can be further rewritten using the definition of $\Delta' = m_a^4 + 4\omega^2 g_{a\gamma\gamma}^2 B^2$, leading to:

$$
\begin{aligned}
P_{\gamma\to a} &= \frac{4\omega^2 g_{a\gamma\gamma}^2 B^2}{m_a^4 + 4\omega^2 g_{a\gamma\gamma}^2 B^2} \sin^2 \left(\frac{\sqrt{m_a^4 + 4\omega^2 g_{a\gamma\gamma}^2 B^2}}{4\omega} z \right) \ , \\
&= \left(\frac{m_a^4}{4\omega^2 g_{a\gamma\gamma}^2 B^2} + 1 \right)^{-1} \sin^2 \left[\frac{g_{a\gamma\gamma} B z}{2} \left(\frac{m_a^4}{4\omega^2 g_{a\gamma\gamma}^2 B^2} + 1 \right)^{1/2} \right] \ ,
\end{aligned} \tag{5.44}
$$

in such a way that for $\frac{m_a^2}{2\omega g_{a\gamma\gamma} B} \ll 1$, an approximation that deals with the fact that the external magnetic field is much stronger than the parameters of the fields a and $\vec{A}$, the term $\left(\frac{m_a^4}{4\omega^2 g_{a\gamma\gamma}^2 B^2} + 1 \right) \sim 1$. Thus, the oscillation probability can be approximated as:

$$
P_{\gamma\to a} \cong \sin^2 \left(\frac{g_{a\gamma\gamma} B}{2} z \right) \ . \tag{5.45}
$$

Given the significant suppression of axion production via $g_{a\gamma\gamma}$ compared to production via g_{ae} (see Fig.13), our focus lies in approximating the oscillation using experimental parameters. Helioscopes, being the foremost detection technique for solar axions, often provide these parameters. For instance, assuming the axion-photon coupling is approximately $g_{a\gamma\gamma} \sim 10^{-13}$ GeV^{-1}, the external magnetic field of order $B \sim 10$ T and the path distance of order $z \sim 15$ m, which are values commonly present in axion detection experiments like helioscopes and haloscopes [79–84], we can further approximate with $\sin\theta \sim \theta$ and find:

$$P_{\gamma \to a} = \frac{g_{a\gamma\gamma}^2 B^2}{4} z^2 \ . \tag{5.46}$$

Note that the probability depends on the axion-photon coupling, the intensity of the external magnetic field, and the path distance submerged in the magnetic field.

5.2 Axion production from g_{ae}

The axion-electron coupling, distinct from the axion-two-photons coupling, governs a series of reactions, which hold significant importance and completely overshadow the Primakoff flux in non-hadronic models like the DFSZ. Among these, the most notable are the ABC reactions: **A**tomic axio-recombination [85–87] and **A**tomic axio-deexcitation, axio-**B**remsstrahlung in electron-Ion [70, 88, 89] or electron-electron collisions [70] and **C**ompton scattering [90–92]. These reactions play a dominant role in the solar axion flux in non-hadronic models such as the DFSZ.

As mentioned earlier, the historical emphasis on the Primakoff reaction can be attributed in part to the formidable challenge posed by the comprehensive calculation of contributions to the ABC reactions. Unlike the Primakoff effect, the solar composition now emerges as a crucial factor influencing the axion flux, adding more complexity to the calculations.

Initially, it might seem evident that $e^- + I$ and e^-e^- bremsstrahlung processes dominate the flux from g_{ae}, given that hydrogen constitutes 74% and helium 24% of the photospheric mass fraction of the Sun [93]. This conclusion is supported by previous studies [70], emphasizing the prevalence of these processes over others. Additionally, the A processes are primarily relevant for metallic ions, which are much less abundant than hydrogen, helium, and electrons [93].

However, a 1986 study by Dimopoulos [86] revealed that axio-recombination cross-sections are, in fact, much larger than those for Compton scattering and bremsstrahlung, partially compensating for the lower abundance of metal ions. Initial estimates, nonetheless, deemed the axio-recombination flux to be largely subdominant [86]. A subsequent work by Pospelov [87] highlighted that the axio-recombination had been underestimated by a factor of 1/2. This indicates that there were indications of an underestimation of the axio-recombination flux, and, furthermore, axio-deexcitation had not even been considered at that time.

Confronting this challenge, Redondo [72] took on the task of reevaluating the A processes flux of solar axions, providing a new estimate for the total solar axion flux. In the following pages, we will delve into the logic and physics employed by Redondo in this endeavor.

The calculation of the A processes, axio-deexcitation and recombination, which are the atomic reactions, involves a comprehensive understanding of the energy levels and occupation probabilities of numerous atomic states for each nucleus inside the Sun, and then computing cross-sections and transition probabilities. This is a whole field of research in astrophysics, considering analogous processes for photons. Directly addressing this problem may seem like a Herculean task.

Redondo's paper, however, provides immense value by asserting that this monumental effort is not necessary. Instead, he demonstrates that we already possess all the necessary ingredients to calculate the total solar axion flux. These crucial ingredients include:

- **Analogous Cross Section:** The spin-averaged differential cross section for axion emission in an atomic transition is proportional to the photon analogous cross-section in the same transition. This establishes a direct relation between axion and photon production rates as functions of energy (ω):

$$\Gamma_a^P(\omega) \propto \Gamma_\gamma^P(\omega) \ , \tag{5.47}$$

  where $\Gamma^P(\omega)$ are the production rates.

- **Detailed Balance in Thermal Equilibrium:** In a plasma in thermal equilibrium, the rates of photon absorption (Γ_γ^A) and production (Γ_γ^P) are related by detailed balance.

- **Photon Absorption Coefficients:** Extensively researched due to their central role in stellar evolution and plasma diagnosis, photon absorption coefficients (radiative opacities) have available libraries of monochromatic opacities for various nuclei across a wide range of temperatures and densities. These opacities facilitate the understanding of photon behavior in different astrophysical environments.

The procedural sequence unfolds as follows:

1. **Calculate Photon Absorption Rates:** Employ the data from the radiative opacity libraries to compute photon absorption rates for any specific location within the Sun. This step provides insights into how photons are absorbed at different depths and temperatures.

2. **Determine Photon Production Rates:** With the correlated relation between photon production and absorption, extract the photon production rates as a function of energy.

3. **Apply Proportionality to Axion Production Rates:** Exploit the proportionality relationship between photon and axion production rates. This allows the derivation of axion production rates based on the established photon production rates.

4. **Obtain Axion Flux:** Through the above steps, calculate the axion flux—quantifying the abundance of axions produced within the Sun.

The next section delves deeper into the theoretical framework that underpins each of these logical steps, offering a more comprehensive understanding of the entire process.

5.2.1 Calculating ABC process from radiative opacities

In this section, we delve into the theoretical framework enabling us to leverage monochromatic opacities for the calculation of the total ABC axion flux. Our exploration commences with the interaction lagrangians governing the interactions of axions and photons with electrons:

$$\begin{aligned} \mathcal{L}_{aee} &= g_{ae}\frac{\partial_\mu a}{2m_e}\overline{\psi}_e\gamma^\mu\gamma_5\psi_e \equiv -ig_{ae}a\overline{\psi}_e\gamma_5\psi_e \ , \\ \mathcal{L}_{\gamma ee} &= eA_\mu\overline{\psi}_e\gamma^\mu\psi_e \ , \end{aligned} \tag{5.48}$$

where g_{ae} is the Yukawa axion-electron coupling, a parameter quantifying the strength of the axion-electron interaction, which we calculated at tree and one-loop level on the last chapter.

The axion production rate is expressed through the equation:

$$\begin{aligned} \frac{dN_a}{dV\,dt} &= \int \frac{d^3k}{(2\pi)^3}\ \Gamma_a^P(\omega) \ , \\ &= \int_0^\infty \frac{\omega^2 d\omega}{(2\pi)^2}\ \Gamma_a^P(\omega) \ , \end{aligned} \tag{5.49}$$

where N_a is the number of axions, V is the volume, t is time, and $\Gamma_a^P(\omega)$ is the sum of the axion production rate resulting from each ABC process. The distinct contributions include (I represents ions, the I^* indicates that its electron is excited):

- A processes:

  (bb) Atomic axio-deexcitation, or bound-bound electron transition

$$I^* \to I + a$$

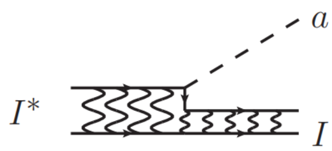


Figure 8 – Atomic axio-deexcitation [72]

(fb) Atomic axio-recombination, also know as electron capture or free-bound electron transition

$$e + I \rightarrow I^- + a$$

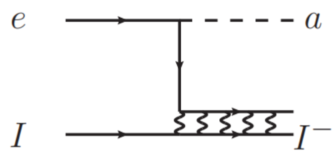


Figure 9 – Atomic axio-recombination [72]

- B processes:

  (ee) electron-electron bremsstrahlung

$$e + e \rightarrow e + e + a$$

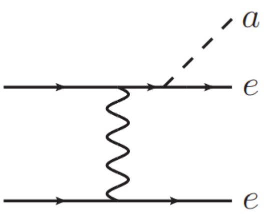


Figure 10 – e-e bremsstrahlung [72]

(ff) electron-Ion bremsstrahlung, also known as free-free electron transition

$$e + I \rightarrow e + I + a$$

- C process:

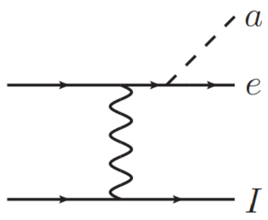


Figure 11 – e-I bremsstrahlung [72]

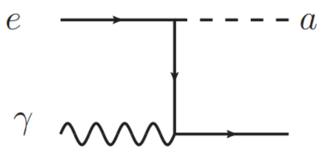


Figure 12 – Compton [72]

(C) Compton-like scattering

$$e + \gamma \to e + a$$

Hence, the total axion production rate is given by:

$$\Gamma_a^P(\omega) = \Gamma_a^{P,\mathrm{ff}}(\omega) + \Gamma_a^{P,\mathrm{fb}}(\omega) + \Gamma_a^{P,\mathrm{bb}}(\omega) + \Gamma_a^{P,\mathrm{C}}(\omega) + \Gamma_a^{P,\mathrm{ee}}(\omega) \ . \tag{5.50}$$

When discussing photon opacities, the specific absorption coefficient $k(\omega)$ is determined by the energy transport equation:

$$\frac{\mathrm{d}I_\omega}{\mathrm{d}s} = -k(\omega)I_\omega + j(\omega) \ , \tag{5.51}$$

this equation governs the sourcing and damping of radiation with specific intensity I_ω along a line of sight within the plasma, where $j(\omega)$ is the source of the radiation [94]. The radiative opacity $\kappa(\omega)$ is then defined as the absorption coefficient per unit mass of target:

$$\kappa(\omega) = \frac{k(\omega)}{\rho} \ , \tag{5.52}$$

with ρ being the density of the medium.

In the solar interior, the absorption coefficient $k(\omega)$ receives contributions analogous to the ABC processes but time-reversed and involving photons instead of axions: free-free, free-bound, and bound-bound electronic atomic transitions, which are the true absorption processes, and Compton scattering. Notably, electron-electron bremsstrahlung is not considered in the absorption coefficient [72]. The equation for $k(\omega)$ is thus:

$$k(\omega) = \left(\Gamma_\gamma^{A,\mathrm{ff}}(\omega) + \Gamma_\gamma^{A,\mathrm{bf}}(\omega) + \Gamma_\gamma^{A,\mathrm{bb}}(\omega)\right)(1 - e^{-\omega/T}) + \Gamma_\gamma^{A,\mathrm{C}}(\omega) \ , \tag{5.53}$$

where $(1 - e^{-\omega/T})$ corrects for stimulated emissions in thermal equilibrium and applies only to the true absorption processes.

Each Γ_γ^A depends on the density of different atomic species n_Z and temperature T and requires a substantial numerical calculation for its computation. For example, the absorption rate for the bound-free transition can be calculated as:

$$\Gamma_\gamma^{A,\mathrm{bf}} = \sum_Z n_Z \sum_s r_s \sigma(\gamma + Z_s \to Z_{s'} + e^-) \ , \tag{5.54}$$

where n_Z are the densities of atoms of nuclear charge Z, r_s the fraction of these atoms in state s and $\sigma(\gamma + Z_s \to Z_{s'} + e^-)$ is the total cross-section for this ionization process.

The calculation of r_s is rather complicated, involving the solution of the atomic structure in a relatively dense medium. It requires solving the Saha equation, which is an expression that relates the ionization state of a gas in thermal equilibrium to the temperature and pressure, for the ionization fraction and computing the partition functions for the probability of initial states for each atom. In addition, the cross section must account for the non-trivial atomic structure, electrostatic screening, Coulomb wave functions in the final states when applicable, and other complexities. The remarkable achievement in Redondo's work [72] was avoiding this formidable task for axions.

As we are not required to perform these intricate calculations directly for axions, we can leverage photon opacities available in scientific databases. Photon opacities play a crucial role in stellar evolution and plasma physics, leading to their routine calculation and refinement. Several opacity databases are publicly accessible, but only a few provide the monochromatic opacities (opacities specified by frequency), with a notable mention of the Opacity Project (OP) (https://cds.unistra.fr//topbase/) and the Los Alamos Light Element Detailed Configuration OPacity code (LEDCOP) (https://aphysics2.lanl.gov/apps/).

The subsequent step involves establishing a relation between photon absorption rates and photon production rates, enabling the incorporation of photon opacities into our calculation. In thermal equilibrium, the rates of reactions and their inverses are linked by detailed balance, as expressed by the equation:

$$\Gamma_\gamma^{A,p}(\omega) = e^{\omega/T}\Gamma_\gamma^{P,p}(\omega) \ . \tag{5.55}$$

A critical aspect for the overall argument and calculation is that this relationship holds for each process individually. It's essential to note that $\Gamma_\gamma^{P,i}(\omega)$ is the production rate of photons per phase-space volume averaged over polarizations. This implies that the total rate is $2\times\ \Gamma_\gamma^{P,i}(\omega)$, accounting for each of the two photon polarization states.

5.2.2 Relations between photon and axion emission processes

To achieve the goal of expressing Γ_a^P in terms of Γ_γ^P, specific relations need to be established for each process under consideration. This task requires a case-by-case analysis, as the dependencies vary depending on the particular process involved.

1. **ff, fb, bb**

   To establish the relationship between the matrix elements for processes involving the emission of an axion/photon during an atomic transition $e_i \to e_f$, a key proportionality can be defined [85,87,95]:

$$\frac{\sum_{s_i,s_f} |\mathcal{M}(e_i \to e_f + a)|^2}{\frac{1}{2}\sum_\epsilon \sum_{s_i,s_f} |\mathcal{M}(e_i \to e_f + \gamma)|^2} = \frac{1}{2}\frac{g_{ae}^2}{e^2}\frac{\omega^2}{m_e^2} \ . \tag{5.56}$$

   This equality is contingent on three fundamental approximations: 1) nonrelativistic expansion of the interaction hamiltonians. 2) The separability of initial and final states in spatial and spin wave functions. 3) The use of a multipole expansion for $\mathcal{M}$.

   The photon emission can be well described by the electric dipole approximation ($e^{i\hat{k}\cdot\hat{X}} \approx 1$):

$$\begin{aligned} \langle f| H_I^\gamma |i\rangle &\sim -2e \langle f| e^{i\vec{k}\cdot\hat{X}} \left(\vec{\epsilon}\cdot\hat{P} + i\hat{S}\cdot(\vec{k}\times\vec{\epsilon})\right) |i\rangle \ , \\ &\sim -2e \langle f| \left(\vec{\epsilon}\cdot\hat{P}\right) |i\rangle \ , \\ &= -2iem_e\omega \langle f| \left(\vec{\epsilon}\cdot\hat{X}\right) |i\rangle \ , \end{aligned} \tag{5.57}$$

   where $\hat{X}, \hat{P}, \hat{S}$ are the electron's position, momentum, and spin operators, respectively.

   For the axion emission, one needs to retain one more order:

$$\begin{aligned} \langle f| H_I^a |i\rangle &\sim -2g_{ae} \langle f| e^{i\vec{k}\cdot\hat{X}} \left(\vec{k}\cdot\hat{P} - \frac{\omega}{m_e}\hat{P}\times\hat{S}\right) |i\rangle \ , \\ &\sim -2g_{ae}\omega^2 i \langle f| \left((\hat{n}\cdot\hat{X})(\hat{n}\cdot\hat{S}) - \hat{X}\cdot\hat{S}\right) |i\rangle \ , \end{aligned} \tag{5.58}$$

   where $\hat{n}$ is a unit vector in the direction of the axion momentum $\vec{k}$.

   The integration over the phase space of final states finds the same proportionality as in (5.56), which applies to the emission cross-sections as well:

$$\frac{\sigma(e_i + I \to e_f + I + a)}{\frac{1}{2}\sigma(e_i + I \to e_f + I + \gamma)} = \frac{1}{2}\frac{g_{ae}^2}{e^2}\frac{\omega^2}{m_e^2} \ . \tag{5.59}$$

   After convolving these with the appropriate densities of final states, the same proportionality holds for thermal axion/photon emission rates:

$$\frac{\Gamma_a^{P,i}(\omega)}{\Gamma_\gamma^{P,i}(\omega)} = \frac{1}{2}\frac{g_{ae}^2}{e^2}\frac{\omega^2}{m_e^2} \ ; \quad \text{i= ff, fb, bb} \tag{5.60}$$

   because the kinematics are the same for axions and photon in the massless limit.

Finally, we can explicitly cross-check this formula against the calculations present in the literature for the case of free-free transitions. The photon production rate in electron collisions with ionized nuclei of electric charge Z_e and number density n_Z, including Debye screening in the Born approximation [96], is given by:

$$\Gamma_\gamma^{P,\text{ff}}(\omega) = \alpha^3 Z^2 \frac{64\pi^2}{3\sqrt{2\pi}} \frac{n_Z n_e}{\sqrt{T} m_e^{3/2} \omega^3} e^{-\omega/T} F(w,y) \ , \tag{5.61}$$

where

$$F(w,y) = \int_0^\infty dx\, x e^{-x^2} \int_{\sqrt{x^2+w}-x}^{\sqrt{x^2+w}+x} \frac{t^3 dt}{(t^2+y^2)^2} \ , \tag{5.62}$$

with $y = k_s/\sqrt{2m_e T}$, where k_s is the Debye screening scale. The axion production rate can then be translated from (5.60). We obtain:

$$\Gamma_a^{P,\text{ff}}(\omega) = \alpha^2 g_{ae}^2 Z^2 \frac{8\pi}{3\sqrt{2\pi}} \frac{n_Z n_e}{\sqrt{T} m_e^{7/2} \omega} e^{-\omega/T} F(w,y) \ . \tag{5.63}$$

2. **e-e bremsstrahlung**

The cross-section of photon bremsstrahlung in e-e collisions is zero in the electric dipole approximation because the electric dipole moment of two colliding electrons in the center of mass is zero. Photon emission occurs at the 4-pole level, which is much more suppressed than the e-I bremsstrahlung (free-free electron transition) [97]. As a result, $\Gamma_\gamma^{P,\text{ee}}$ is often neglected for the conditions of opacities in the solar interior, or included as an $\mathcal{O}(10^{-3})$ correction to $\Gamma_\gamma^{P,\text{ff}}$ [98]. However, for the axion case, emissions in e-e collisions are of the same order than in e-I collision [70] and thus has to be included. The emission rate was computed by Raffelt [70, 99]:

$$\Gamma_a^{P,\text{ee}} = \alpha^2 g_{ae}^2 \frac{4\sqrt{\pi}}{3} \frac{n_e^2}{\sqrt{T} m_e^{7/2} \omega} e^{-\omega/T} F(w, \sqrt{2}y) \ , \tag{5.64}$$

where $F(w,\sqrt{2}y)$ is given by (5.62).

3. **Compton scattering**

In the non-relativistic limit, the cross section for the photo-production of axions in Compton-like scattering is given by [90, 100]:

$$\sigma_{\text{C},a} = \frac{\alpha g_{ae}^2 \omega^2}{3m_e^4} \ . \tag{5.65}$$

Consequently, the production rate is:

$$\Gamma_a^{P,\text{C}}(\omega) = \frac{\alpha g_{ae}^2 \omega^2}{3m_e^2} \frac{n_e}{e^{\omega/T}-1} \ . \tag{5.66}$$

It's worth noting that the cross-section for normal Compton scattering in the non-relativistic limit is the Thomson cross-section:

$$\sigma_{\text{C},\gamma} = \frac{8\pi\alpha^2}{3m_e^2} \ . \tag{5.67}$$

Therefore, the production ratio for Compton-like scattering processes is given by:

$$\begin{aligned}
\frac{\Gamma_a^{P,\mathrm{C}}(\omega)}{\Gamma_\gamma^{P,\mathrm{C}}(\omega)} &= \frac{f(\omega)n_e\sigma_{\mathrm{C},a}}{f(\omega)n_e\frac{1}{2}\sigma_{\mathrm{C},\gamma}}\ , \\
&= \frac{\sigma_{\mathrm{C},a}}{\frac{1}{2}\sigma_{\mathrm{C},\gamma}}\ , \\
&= \frac{\alpha g_{ae}^2\omega^2}{3m_e^4}\frac{6m_e^2}{8\pi\alpha^2}\ , \\
&= \frac{g_{ae}^2\omega^2}{m_e^2}\frac{1}{4\pi\alpha}\ , \\
&= \frac{g_{ae}^2\omega^2}{m_e^2}\frac{4\pi}{4\pi e^2}\ , \\
&= \frac{g_{ae}^2\omega^2}{e^2m_e^2}\ .
\end{aligned} \tag{5.68}$$

It's important to note the factor of 2 difference between the Compton and (5.60) production rates, indicating that Compton processes are relatively more efficient than other processes (free-free, bound-bound, and free-bound) for emitting axions compared to the photon emission rate.

5.2.3 Total axion production rate

Finally, with all the ingredients and their expressions at our disposal, we can derive the total axion production rate step by step. To simplify the notation, we omit the explicit dependence on ω for clarity:

$$\begin{aligned}
\Gamma_a^P &= \Gamma_a^{P,\mathrm{ff}} + \Gamma_a^{P,\mathrm{fb}} + \Gamma_a^{P,\mathrm{bb}} + \Gamma_a^{P,\mathrm{C}} + \Gamma_a^{P,\mathrm{ee}}\ , \\
&= \frac{1}{2}\frac{g_{ae}^2\omega^2}{e^2m_e^2}\left(\Gamma_\gamma^{P,\mathrm{ff}} + \Gamma_\gamma^{P,\mathrm{fb}} + \Gamma_\gamma^{P,\mathrm{bb}}\right) + \Gamma_a^{P,\mathrm{C}} + \Gamma_a^{P,\mathrm{ee}}\ ,
\end{aligned} \tag{5.69}$$

note that we utilized (5.60) to arrive at the second line. Continuing the calculations, we obtain:

$$\begin{aligned}
\Gamma_a^P &= \frac{1}{2}\frac{g_{ae}^2\omega^2}{e^2m_e^2}\left(\Gamma_\gamma^{P,\mathrm{ff}} + \Gamma_\gamma^{P,\mathrm{fb}} + \Gamma_\gamma^{P,\mathrm{bb}}\right) + \frac{1}{2}\Gamma_a^{P,\mathrm{C}} + \frac{1}{2}\Gamma_a^{P,\mathrm{C}} + \Gamma_a^{P,\mathrm{ee}}\ , \\
&= \frac{1}{2}\frac{g_{ae}^2\omega^2}{e^2m_e^2}\left(\Gamma_\gamma^{P,\mathrm{ff}} + \Gamma_\gamma^{P,\mathrm{fb}} + \Gamma_\gamma^{P,\mathrm{bb}} + \Gamma_\gamma^{P,\mathrm{C}}\right) + \frac{1}{2}\Gamma_a^{P,\mathrm{C}} + \Gamma_a^{P,\mathrm{ee}}\ ,
\end{aligned} \tag{5.70}$$

where we have utilized (5.68) in the second line. Now, using the relation for photon production and absorption of (5.55), we find:

$$\Gamma_a^P = \frac{1}{2}\frac{g_{ae}^2\omega^2}{e^2m_e^2}\left(\Gamma_\gamma^{A,\mathrm{ff}} + \Gamma_\gamma^{A,\mathrm{fb}} + \Gamma_\gamma^{A,\mathrm{bb}} + \Gamma_\gamma^{A,\mathrm{C}}\right)e^{-\omega/T} + \frac{1}{2}\Gamma_a^{P,\mathrm{C}} + \Gamma_a^{P,\mathrm{ee}} \tag{5.71}$$

Now, to consult the photon opacities libraries, we express the total production rate in a form that evokes (5.53). For this purpose, we obtain:

$$
\begin{aligned}
\Gamma_a^P &= \frac{1}{2}\frac{g_{ae}^2\omega^2}{e^2 m_e^2}\left(\Gamma_\gamma^{A,\mathrm{ff}} + \Gamma_\gamma^{A,\mathrm{fb}} + \Gamma_\gamma^{A,\mathrm{bb}} + \Gamma_\gamma^{A,\mathrm{C}}\frac{1-e^{-\omega/T}}{1-e^{-\omega/T}}\right) e^{-\omega/T} + \frac{1}{2}\Gamma_a^{P,\mathrm{C}} + \Gamma_a^{P,\mathrm{ee}} \ , \\
&= \frac{1}{2}\frac{g_{ae}^2\omega^2}{e^2 m_e^2}\left(\Gamma_\gamma^{A,\mathrm{ff}} + \Gamma_\gamma^{A,\mathrm{fb}} + \Gamma_\gamma^{A,\mathrm{bb}} + \frac{\Gamma_\gamma^{A,\mathrm{C}}}{1-e^{-\omega/T}}\right) e^{-\omega/T} \\
&+ \frac{1}{2}\left(1 - \frac{e^{-\omega/T}}{1-e^{-\omega/T}}\right)\Gamma_a^{P,\mathrm{C}} + \Gamma_a^{P,\mathrm{ee}} \ , \\
&= \frac{1}{2}\frac{g_{ae}^2\omega^2}{e^2 m_e^2}\frac{k(\omega)}{1-e^{-\omega/T}} e^{-\omega/T} + \frac{1}{2}\left(1 - \frac{1}{e^{\omega/T}-1}\right)\Gamma_a^{P,\mathrm{C}} + \Gamma_a^{P,\mathrm{ee}} \ , \\
&= \frac{1}{2}\frac{g_{ae}^2\omega^2}{e^2 m_e^2}\frac{k(\omega)}{e^{\omega/T}-1} + \frac{1}{2}\left(\frac{e^{\omega/T}-2}{e^{\omega/T}-1}\right)\Gamma_a^{P,\mathrm{C}} + \Gamma_a^{P,\mathrm{ee}} \ .
\end{aligned}
\tag{5.72}
$$

The total solar axion flux from the Sun is obtained by integrating the total axion production rate Γ_a^P times the phase space density over the volume of the Sun. This results in the expression:

$$
\frac{\mathrm{d}\Phi_a}{\mathrm{d}\omega} = \frac{1}{4\pi R_{\mathrm{Earth}}^2}\int_{\mathrm{Sun}} dV \ \frac{4\pi\omega}{(2\pi)^3}\Gamma_a^P(\omega) \ , \tag{5.73}
$$

where $\Gamma_a^P(\omega)$ depends on the position in the Sun due to its reliance on local plasma characteristics such as temperature (T), mass density (ρ), and the mass-fraction of the chemical element Z (X_Z). The latter is taken to be dependent solely on the radial position within the Sun and provided by a solar model.

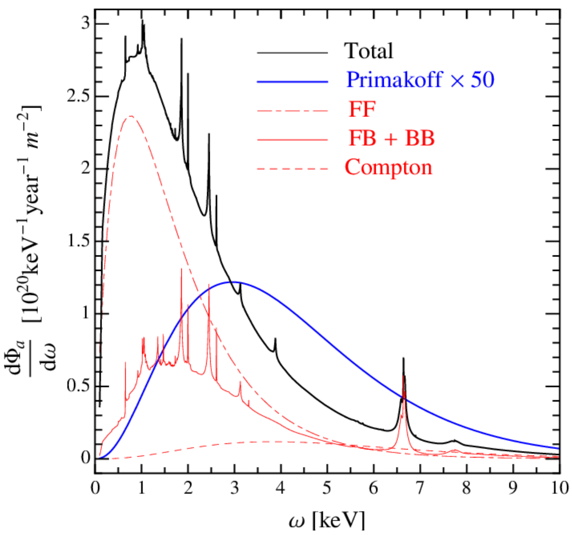


Figure 13 – Total solar axion flux due to ABC reactions (for $g_{ae} = 10^{-13}$) and to Primakoff conversion (for $g_{a\gamma\gamma} = 10^{-12}$) scaled up by a factor 50 to make it visible. The different contributions are shown as red lines: Atomic recombination and deexcitation (FB+BB, solid), Bremsstrahlung (FF, dot-dashed) and Compton (dashed). [72]

6 Solar Axion Detection

In contrast to conventional particle physics experiments, the detection of axions necessitates unique and specialized techniques. In 1983, Sikivie [101] proposed two highly effective methods for searching invisible axions:

- The axion heliscope: employed for the detection of solar axions.
- The axion haloscope: employed for the detection of axions from hypothetical dark matter (DM) halos.

Despite our specific interest in solar axions, both of these experimental approaches share a common principle: leveraging coherent effects across macroscopic distances and/or extended durations to enhance axion detection.

Sikivie's proposal introduced a crucial concept that, in retrospect, may seem evident, owing partly to the manner in which this work has been presented. During that period, the prevailing notion was that axion detection experiments entailed the production of axions within a laboratory setting. However, Sikivie showed that we could use natural sources of axions — namely, the Sun and the Big Bang — exploiting their inherent efficiency and concentrating the search efforts solely on the detection phase.

Due to the exceptionally high fluxes of natural axions, helioscopes and haloscopes typically exhibit much greater sensitivity than their purely laboratory-based counterparts. However, it is crucial to note that their luminosities are also susceptible to larger uncertainties, especially in the case of DM.

Many experiments designed to investigate the effects of axion coupling to SM particles also exhibit sensitivity to ALPs. This characteristic enables axion experiments to uncover ALPs beyond the conventional axion, which arise quite naturally in extensions of the SM. Unfortunately, the potential overlap in signals makes it challenging to unequivocally attribute a discovery in one experiment solely to the existence of purely QCD axions. In such instances, the necessity arises for a multitude of distinct signals to discriminate between axions and ALPs. Despite this complication, the diverse array of ongoing experiments and the continual emergence of new ones suggest that, in certain scenarios, it may be feasible to distinguish and identify a discovery specifically linked to QCD axions.

Moreover, there exists a category of low-mass, very weakly coupled particles that share numerous theoretical and phenomenological aspects with axions and ALPs. These particles, such as dark photons and other Weakly Interacting Slim Particles (WISPs), can often be investigated using the same experimental setups. It is noteworthy that the search for ALPs through conventional High-Energy Physics (HEP) tools, such as accelerators, is suitable only for very high masses (on the order of MeV or more), a range largely excluded for QCD axions.

Both primary mechanisms of axion production within the solar interior, namely Primakoff conversion and ABC reactions, yield distinct peaks in the keV range (see Fig. 13). Specifically, Primakoff conversion exhibits a peak around 3 keV, while ABC reactions manifest a peak at approximately 1 keV.

6.1 Axion Helioscopes

The axion helioscope stands out as the primary technique for the exploration of solar axions, relying on the conversion of axions into X-ray photons within powerful laboratory magnets. This conversion process, known as the inverse Primakoff conversion or axion-photon oscillation, represents the reverse of the process outlined in (5.1.1). Consequently, the detection hinges on the coupling parameter $g_{a\gamma\gamma}$.

In helioscopes, the standard approach involves considering only the Primakoff component of the solar axion flux. This choice maintains broad generality across models and yields significant constraints on $g_{a\gamma\gamma}$ across a wide range of masses. It is noteworthy that all helioscope experiments conducted to date were conceived prior to Redondo's work on the ABC flux [72]. By focusing solely on the Primakoff component, the total solar axion flux can be found in Fig.14

It is crucial to note that signals from non-Primakoff axions in helioscopes depend on the corresponding product of couplings, and typically, they do not pose a challenge to astrophysical limits. However, this scenario may undergo a transformation in the near future, as anticipated improvements in experimental sensitivities, as seen in upcoming helioscopes like IAXO, are expected to surpass astrophysical limits on g_{ae}. This opens up the exciting prospect of probing a compelling set of non-hadronic models, such as the DFSZ model.

6.1.1 Intricacies of the axion helioscope

Utilizing the $a\gamma\gamma$ vertex once again, solar axions can efficiently convert back into photons in the presence of a strong magnetic field. When the background field is static, the energy of the reconverted photon matches that of the incoming axion, resulting in an anticipated flux of detectable X-rays with energies in the range of a few keV. In (5.1.1), we computed the probability of photon-to-axion conversion in a transverse magnetic field B over a length z. It is crucial to recognize that, via crossing symmetry, the probability

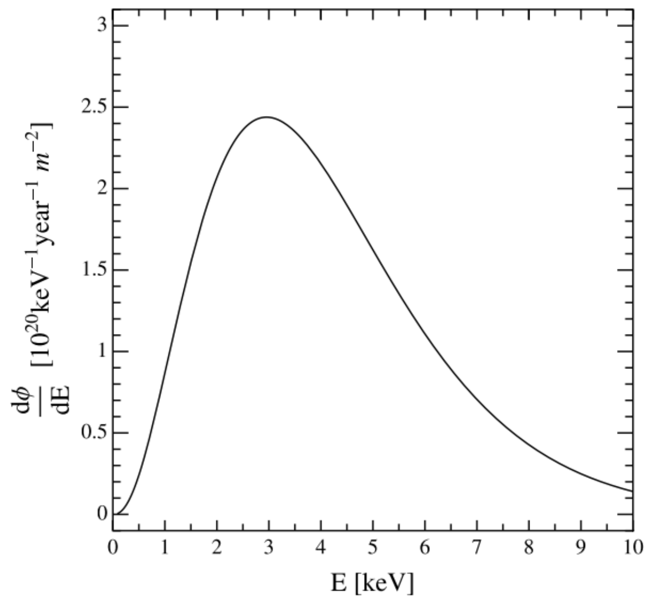


Figure 14 – Solar axion flux spectra at Earth by the most generic situation in which only the Primakoff conversion of plasma photons into axions is assumed (for $g_{a\gamma\gamma} = 10^{-12}$) [102].

of axion-to-photon conversion is the same as photon-to-axion:

$$P(a \to \gamma) = P(\gamma \to a) \ . \tag{6.1}$$

A commonly referenced formula for helioscopes is provided in [101, 103, 104]:

$$P(a \to \gamma) = 2.6 \times 10^{-17} \left(\frac{g_{a\gamma\gamma}}{10^{-10}\text{GeV}^{-1}}\right)^2 \left(\frac{B}{10\text{T}}\right)^2 \left(\frac{L}{10\text{m}}\right)^2 \mathcal{F}(q, L) \ , \tag{6.2}$$

where $\mathcal{F}(q, L)$ is the homogenous B-field form factor defined as:

$$\mathcal{F}(q, L) = \left(\frac{2}{qL}\right)^2 \sin^2 \frac{qL}{2} \ , \tag{6.3}$$

and q is the difference between the photon and axion wave numbers, which in the relativistic limit and in vacuum can be written as:

$$q = k_\gamma - k_a \approx \frac{m_a^2}{2\omega} \ . \tag{6.4}$$

Coherent conversion along the whole length gives $\mathcal{F} = 1$, a condition met when $qL \ll 1$. Utilizing the q approximation, the coherence condition for solar axion energies and a magnet length of approximately 10 m is satisfied for axion masses $m_a \lesssim 10^{-2}$ eV.

However, for higher masses, $\mathcal{F}$ decreses as $(2/qL)^2 \propto 1/m_a^4$, resulting in a decrease in the conversion probability and, consequently, a reduction in the experiment's sensitivity. To counteract the loss of coherence, a buffer gas can be introduced into the magnet beam pipes [105, 106] to impart an effective mass to the photons:

$$m_\gamma = \omega_p \ , \tag{6.5}$$

where ω_p is the plasma frequency of the gas and is defined by:

$$\omega_p^2 = \frac{4\pi\alpha n_e}{m_e} \ , \tag{6.6}$$

being n_e and m_e the electron density and mass, respectively.

When the axion mass matches the photon mass, $q = 0$ and the coherence is restored. By systematically adjusting the pressure of the gas within the pipe, the photon mass can be controlled, allowing for an increase of sensitivity to higher axion masses.

In this setup, in the event of a positive detection, helioscopes have the capability to determine the value of m_a. Even in a vacuum, m_a can be determined from the spectral distortion caused by the initiation of ALP-photon oscillations in the helioscope, particularly in the low-energy part of the spectrum. This phenomenon can be detectable for masses down to 10^{-3} eV, contingent upon the intensity of the signal [107].

The configuration of a helioscope entails a robust magnet linked to one or more X-ray detectors. In contemporary iterations of the concept (see Fig.15), an extra focusing stage is incorporated at the conclusion of the magnet. This stage serves to concentrate the signal photons, thereby enhancing the signal-to-noise ratio. When the magnet is aligned with the Sun, an excess of X-rays at the detector is expected, contrasting with the background measurements taken during non-alignment periods.

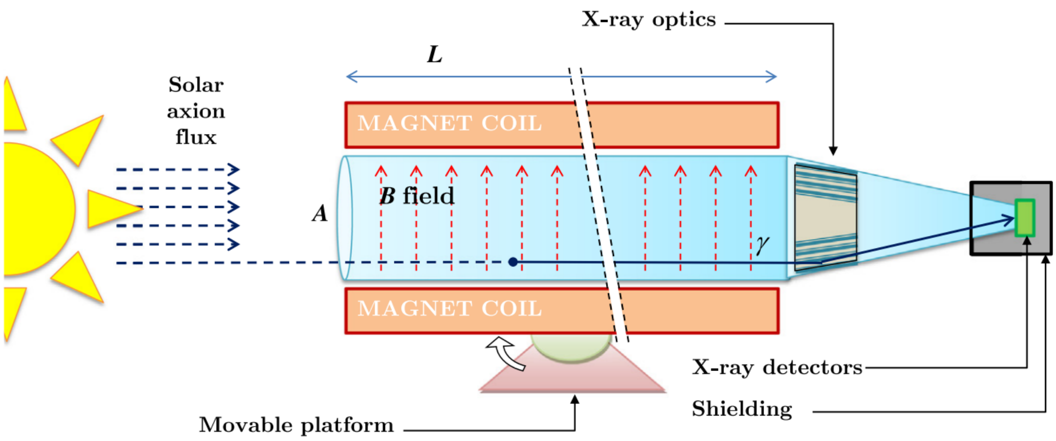


Figure 15 – Conceptual layout of a modern helioscope [108].

6.1.2 Experimental History and Development

The first experimental realization of the axion helioscope took place at BNL (Brookhaven National Laboratory) in 1992. This first-generation experiment featured a stationary dipole magnet with a magnetic field strength of $B = 2.2$ T and length of $L = 1.8$ m. Positioned towards the setting Sun [109], this configuration established the initial upper limit:

$$g_{a\gamma\gamma} < 3.6 \times 10^{-9}\ \text{GeV}^{-1}\ , \tag{6.7}$$

for $m_a < 0.03$ eV at 99% C.L. (Confidence Level).

In 1998, the University of Tokyo introduced a second-generation experiment known as the Sumico axion helioscope. This configuration incorporated dynamic tracking of the Sun, a more potent and larger magnet with $B = 4$ T and $L = 2.3$ m. Technical enhancements included evacuating the bore, the region between the two coils of the magnet, and installing higher-performance detectors [83, 84, 110]. These improvements substantially augmented the experiment's sensitivity, establishing an improved upper limit:

$$g_{a\gamma\gamma} < 6.0 \times 10^{-10}\ \text{GeV}^{-1}\ , \tag{6.8}$$

for $m_a < 0.03$ eV at 95% C.L.. Subsequent advancements involved the introduction of a buffer gas to enhance sensitivity at higher masses.

The most important and restrictive helioscope to date is a third-generation experiment: the CERN Axion Solar Telescope (CAST), which started data collection in 2003. In its configuration, a dipole prototype magnet from the LHC with B up to 9 T and $L = 9.3$ m was employed [111]. The magnet, equipped with elevation and azimuth drives, enables solar tracking for several hours per day (see to Fig.16).

Figure 16 – Picture of the CAST experiment setup. Credit: M. Rosu/CAST collaboration, CERN.

The CAST experiment stands out as the first to incorporate X-ray focusing optics for one of its fours detector lines [112], along with low background techniques from detectors in underground laboratories [113].

Its main observational program happened in the period of 2003-2011:

- (2003-2004): Operated first with the magnet bores in vacuum and obtained:

$$g_{a\gamma\gamma} < 8.8 \times 10^{-11}\ \mathrm{GeV}^{-1}\ , \tag{6.9}$$

  for $m_a < 0.02$ eV at 95% C.L. [103, 104].

- (2005-2011): Upgraded to be operated with buffer gases of ^{4}He (2005-2006) and ^{3}He (2008-2011) to obtain continuous, high sensitivity up to $m_a = 1.17$ eV, during this period the experiment obtained an average upper limit:

$$g_{a\gamma\gamma} \lesssim 2.3 \times 10^{-10}\ \mathrm{GeV}^{-1}\ , \tag{6.10}$$

  for 0.02 eV $< m_a <$ 0.64 eV at 95% C.L. [106, 114] and

$$g_{a\gamma\gamma} \lesssim 3.3 \times 10^{-10}\ \mathrm{GeV}^{-1}\ , \tag{6.11}$$

  for 0.64 eV $< m_a <$ 1.17 eV at 95% C.L. [115].

In the period from 2013 to 2015, CAST revisited the vacuum phase, implementing improved detectors and novel X-ray optics. These enhancements resulted from extensive research and development efforts conducted in preparation for the next-generation axion helioscope, the IAXO (International Axion Observatory). Notably, one of the detection lines, named the IAXO pathfinder system [116], marks the first instance where both low-background techniques and a newly designed X-ray optics, purpose-built for this endeavor, are combined. This system enjoys an effective background count rate of 0.003 counts per hour in the signal region.

The outcome of this phase represents the most stringent experimental limit on $g_{a\gamma\gamma}$:

$$g_{a\gamma\gamma} < 0.66 \times 10^{-10}\ \mathrm{GeV}^{-1}\ , \tag{6.12}$$

for $m_a < 0.02$ eV at 95% C.L [117].

CAST has marked a significant milestone as the first helioscope with sensitivities to $g_{a\gamma\gamma}$ values belows 10^{-10} GeV^{-1} and has rivaled the strongest limits from astrophysics on this coupling (see Table.1). This limit is derived from the properties and evolution of horizontal branch (HB) and red giants (RG) stars in 39 globular clusters (GCs) [62]:

$$g_{a\gamma\gamma} < 0.65 \times 10^{-10}\ \mathrm{GeV}^{-1}\ . \tag{6.13}$$

This outcome leads to a highly promising and intriguing conclusion: the long-called invisible axion may not be too far from becoming visible. For a more direct overview of the experimental parameters and their evolution across generations, see Table 2 and the limits on $g_{a\gamma\gamma}$ are meticulously illustrated in Fig.17.

Helioscope Experiments						
Experiment	Status	B (T)	L (m)	A (cm^2)	Focusing	g_{10}
Brookhaven	Past	2.2	1.8	130	No	36
SUMICO	Past	4	2.5	18	No	6
CAST	Ongoing	9	9.3	30	Partially	0.66
TASTE [118]	Concept	3.5	12	2.8×10^3	Yes	0.2
BabyIAXO	In construction	~ 2.5	10	2.8×10^3	Yes	0.15
IAXO [108, 119]	In design	~ 2.5	22	2.3×10^4	Yes	0.04

Table 2 – Evolution of Heliscopes. The last column represents the sensitivity achieved in $g_{a\gamma\gamma}$ in terms of an upper limit on $g_{10} = 10^{10} \times g_{a\gamma\gamma}$ [68].

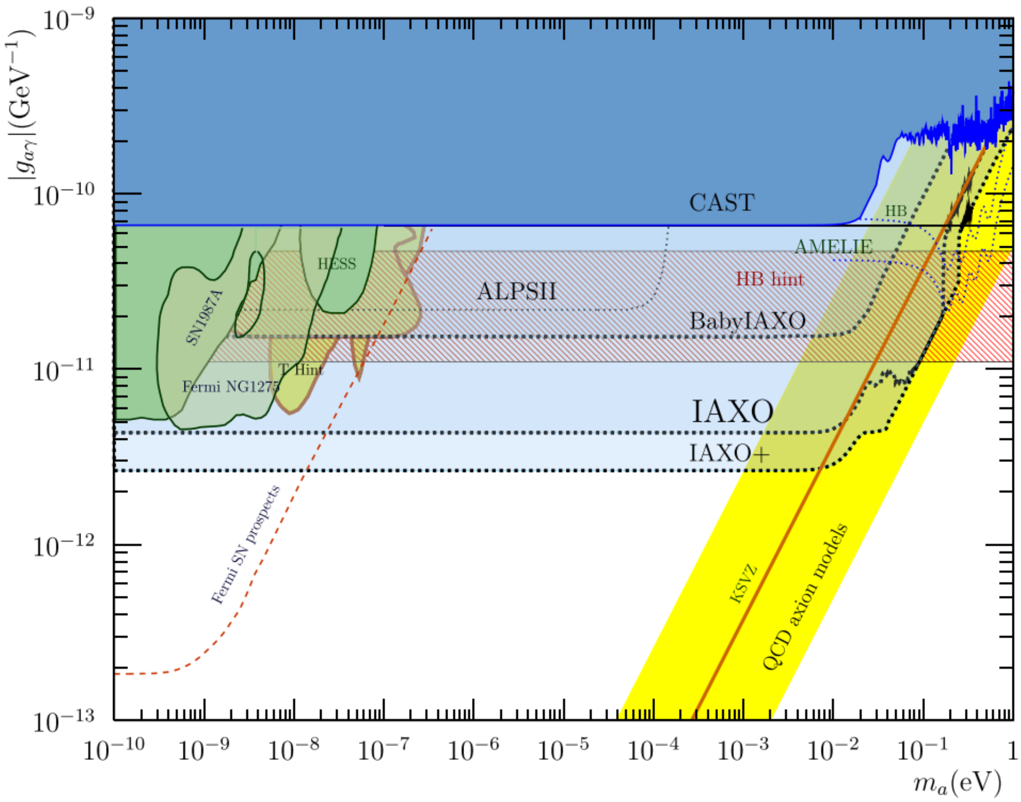


Figure 17 – The most stringent upper limit established by CAST and future experimental prospects [68].

As illustrated in Fig.17, in the region of higher axion masses ($m_a \gtrsim 0.1$ eV), the experiment has ventured into the band of QCD axions and excluded KSVZ axions for those specific values. However, this result is not surprising, given that the $m_a \sim$ eV region had already been excluded by astrophysical bounds.

Beyond this main result, CAST has also explored other axion production channels in the Sun, enabled by the g_{ae} and g_{aN}. As previously stated, in these scenarios, helioscopes provide limits to the product of $g_{a\gamma\gamma}$ and the corresponding coupling. A more detailed view of these limits is provided in Fig.23.

All helioscopes thus far have primarily relied on repurposing existing equipment, particularly the magnet. CAST, in particular, benefited from the availability of the first-class LHC test magnet, even though it was not specifically designed for CAST itself. There appears to be substantial room for improvement in sensitivity beyond CAST by designing a dedicated magnet optimized to maximize a parameter known as the helioscope magnet's figure of merit, defined as:

$$f_M = B^2 L^2 A \ , \tag{6.14}$$

where A is the cross-sectional aperture area, as detailed in Table 2.

Enhancing the value of f_M obtained by CAST is only feasible [108] through a radically different magnet configuration with a much larger A, which in the CAST magnet is only 3×10^{-3} m^2. However, for f_M to directly translate into an improved signal-to-noise ratio of the overall experiment, the entire cross-sectional area of the magnet must be equipped with X-ray focusing optics. This forms the foundational layout for the IAXO.

6.1.3 Future of Helioscopes

IAXO, currently in the design stage, represents the next generation of axion helioscopes. The objective of the experiment is to search for axions with a signal-to-noise ratio some 10^5 times better than CAST, aiming to reach $g_{a\gamma\gamma} \sim 10^{-12}$ GeV^{-1}. Building upon the experience gained from CAST, IAXO plans to construct a new large-scale magnet optimized for axion searches, incorporating extensive focusing and low-background techniques.

The central component of IAXO is a new superconducting magnet that, in contrast to previous helioscopes, adopts a toroidal multibore configuration [120] to effectively generate an intense magnetic field over a large volume. The design envisions a 25 m long and 5.2 m diameter toroid assembled from 8 coils, effectively generating an average (peak) magnetic field of 2.5 T (5.1 T) in 8 bores of 600 mm diameter. This represents a remarkable 300-fold improvement in f_M compared to the CAST magnet, with the toroid's stored energy amounting to 500 MJ. The design draws inspiration from the ATLAS barrel and end-cap toroids [121, 122], the largest superconducting toroids built and presently in operation at CERN.

Beyond the magnet, several improvements are anticipated in optics and detector parameters. However, the intricate details of these technologies are beyond the scope of this work.

The initial phase towards IAXO, already under construction, is the BabyIAXO experiment. This scaled-down version of IAXO features a magnet only 10 m long with a single bore of 600 mm diameter. The expected sensitivity throughout the parameter space is illustrated in Fig.17.

6.2 Underground detectors

Axions can also interact with matter through their couplings with electrons or nucleons. One notable mechanism is the axioelectric effect [123–126], where solar axions could generate visible signals in ionization detectors. Utilizing this technique to search for solar axions produced by g_{ae} processes is particularly attractive, as the final signal depends solely on g_{ae}, enabling robust limits to be set on this coupling.

Several large liquid xenon detectors designed for dark matter WIMP detection, such as XMASS [127], XENON [128–130], PANDAX-II [131] and LUX [132], have conducted searches for solar axions as a byproduct of their experiments. Despite the most competitive result to date achieved by the XENONnT experiment, which has not yet surpassed 10^{-13} in g_{ae}, the result value is still considerably larger than the astrophysical limits presented in Table.1. Due to the mild dependency on exposure $g_{ae} \propto (MT)^{-1/4}$ (being M the mass and T the exposure time), even the future DARWIN detector, with a target mass of 50 ton of liquid xenon, is expected to fall short of reaching these limits [133].

While these conventional detection techniques may still be a bit distant from achieving the necessary astrophysical upper limits, understanding how these experiments work and exploring the latest results is crucial. Our focus will be on the results from the XENON collaboration, specifically the XENON1T and XENONnT experiments.

6.2.1 XENON experiment

The XENON experiment features a time projection chamber filled with liquid xenon (LXe TPC), as illustrated in Fig.18. A time projection chamber is a detector in particle physics that combines electric and magnetic fields with a sensible gas or liquid to reconstruct particle trajectories or interactions in three dimensions. Originally designed to detect DM WIMPs, the XENON experiment, due to its low background rate, substantial target mass, and low energy threshold, is also sensitive to solar axions. Positioned under a mountain, as is common in particle physics experiments, the XENON experiment benefits from background shielding.

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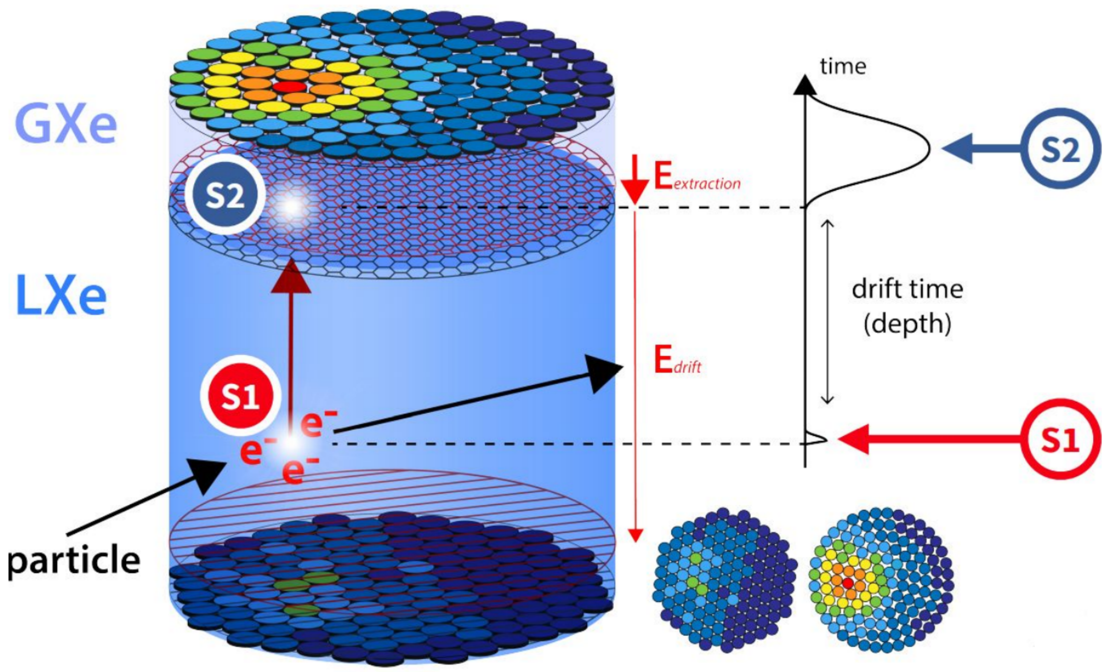

Figure 18 – Illustration of a particle interaction in a LXe TPC. Found in: Sid El Moctar AHMED MAOULOUD / LPNHE GDR, June 1st, 2021

Currently in its fourth generation, the XENON experiment has undergone significant improvements over the course of its development. Notably, these improvements include a larger target mass and lower background rates, as detailed in Table 3. We will delve into the results and details of last two generations, the XENON1T and XENONnT.

XENON generations				
Parameters	XENON10	XENON100	XENON1T	XENONnT
Operation	2005-2007	2008-2016	2012-2019	2020-2026
Xe mass target	14 kg	62 kg	2t	5.9t
Cross-section sensitivity	$\sim 10^{-43}$ cm^2	$\sim 10^{-45}$ cm^2	4×10^{-47} cm^2	1.4×10^{-48} cm^2
Background ER events/keV.t.y	$\sim 2 \times 10^6$	1.8×10^3	82	16.1

Table 3 – The evolution of XENON experiment

In the XENON experiment, a particle interaction within the detector generates two distinct signals: a prompt scintillation signal (S1) and a delayed electroluminescence signal (S2). These light signals are detected by arrays of photomultiplier tubes (PMTs) positioned on the top and bottom of the active volume. They play a crucial role in determining the deposited energy and the interaction position of an event. Accurate determination of the event position is essential for eliminating background events occurring near the edges of the target volume, such as those originating from radioactivity in detector materials, through a process known as fiducialization.

The ratio of S2 to S1 (S2/S1) is employed to differentiate electronic recoils (ERs), produced by sources like gamma rays or beta electrons, from nuclear recoils (NRs), generated by particles such as neutrons or WIMPs. This ratio allows for a degree of particle identification. The capability to determine the scatter multiplicity further aids in reducing background events, as signals are expected to exhibit only a single energy deposition. Since most background events are ERs, the primary objective of the experiment, when investigating the hypothesis of solar axions, is to detect an excess of events over the known background.

As their mass is significantly below the keV range, dark matter axions from the primordial Universe are not observable by any XENON experiment. However, as mentioned earlier, solar axions have production rates that peak at low keV energies, precisely within the sensitivity range of both XENON1T and XENONnT.

Both ABC and Primakoff axions could be detected in XENON via the axioelectric effect, which is the axion analog to the photoelectric effect. The cross-section for this effect scales with the axion-electron coupling g_{ae} and is given by [85, 87, 134]:

$$\sigma_{ae} = \sigma_{pe} \frac{g_{ae}^2}{\beta} \frac{3E_a^2}{16\pi\alpha m_e^2} \left(1 - \frac{\beta^{2/3}}{3}\right) \ , \tag{6.15}$$

where β and E_a are the velocity and energy of the axion, respectively, α is the fine structure constant, and m_e is the mass of the electron.

6.2.1.1 XENON1T results

In 2020, the XENON collaboration reported [129] an excess over known background at low energies (see Fig.19), most notably between $2-3$ keV, in the XENON1T experiment.

The solar axion model showed a significance of $3,4\sigma$ over the background, and a three-dimensional 90% confidence surface was presented for axion couplings to electrons, photons, and nucleons. This surface is enclosed within the cuboid defined by:

$$\begin{aligned} g_{ae} &< 3.8 \times 10^{-12} \ , \\ g_{ae} g_{an}^{\text{eff}} &< 4.8 \times 10^{-18} \ , \\ g_{ae} g_{a\gamma\gamma} &< 7.7 \times 10^{-22} \text{ GeV}^{-1} \ . \end{aligned} \tag{6.16}$$

Another plausible explanation for the reported excess was a potential tritium (^{3}H) contamination, favored at 3.2σ when compared to background. The amount of tritium needed to account for such an excess is exceptionally small, corresponding to a tritium-xenon

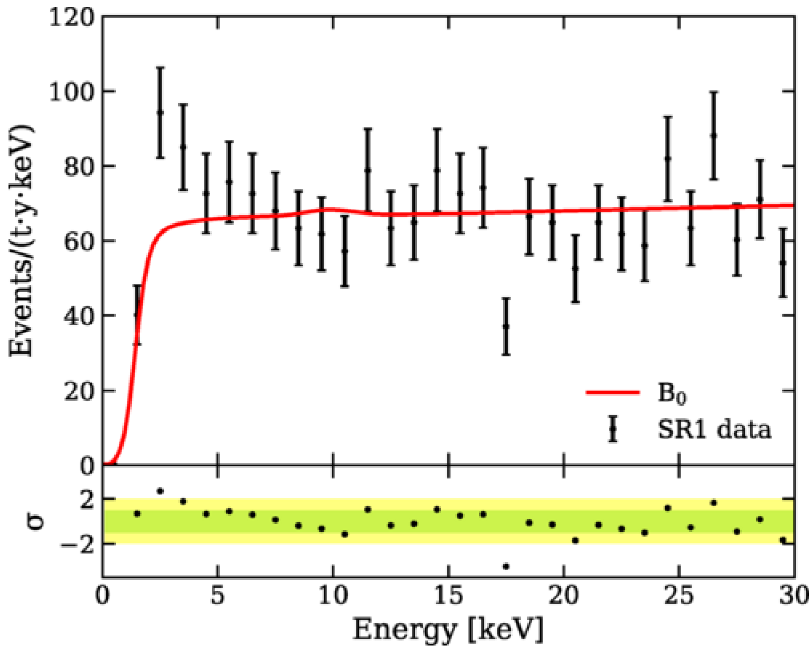

Figure 19 – Detected events in XENON1T. Red line represents the expected background events [129].

ratio of:

$$\frac{^3\mathrm{H}}{\mathrm{Xe}} = (6.2 \pm 2.0) \times 10^{-25}\mathrm{mol/mol}\ . \tag{6.17}$$

Both fits can be visualized in Fig.20.

As evident in the Fig.21, XENON1T did not solely establish an upper limit on solar axion couplings but rather defined a region where solar axions would need to exist to explain the observed excess. While this may seem intriguing initially, it is crucial to note the simultaneous presence of astrophysical upper limits in the same figure, which directly conflict with this hypothesis. The figure also displays upper limits from other collaborations and experiments, including CAST.

This conflict with astrophysical bounds largely excludes the possibility of solar axions being the explanation for this result. Accepting such an explanation would imply inconsistencies in stellar evolution, a field with robust and well-established results. Therefore, the most plausible explanation remains the tritium contamination, a hypothesis that was further tested by the XENONnT experiment.

6.2.1.2 XENONnT results

In 2022, the XENON collaboration reported the XENONnT first science run [130]. The excess observed in the XENON1T experiment, when modeled as a 2.3 keV monoenergetic peak, is excluded with a statistical significance of $\sim 4\sigma$, as depicted in Fig.22.This outcome aligns with expectations given the astrophysical conflict and further supports

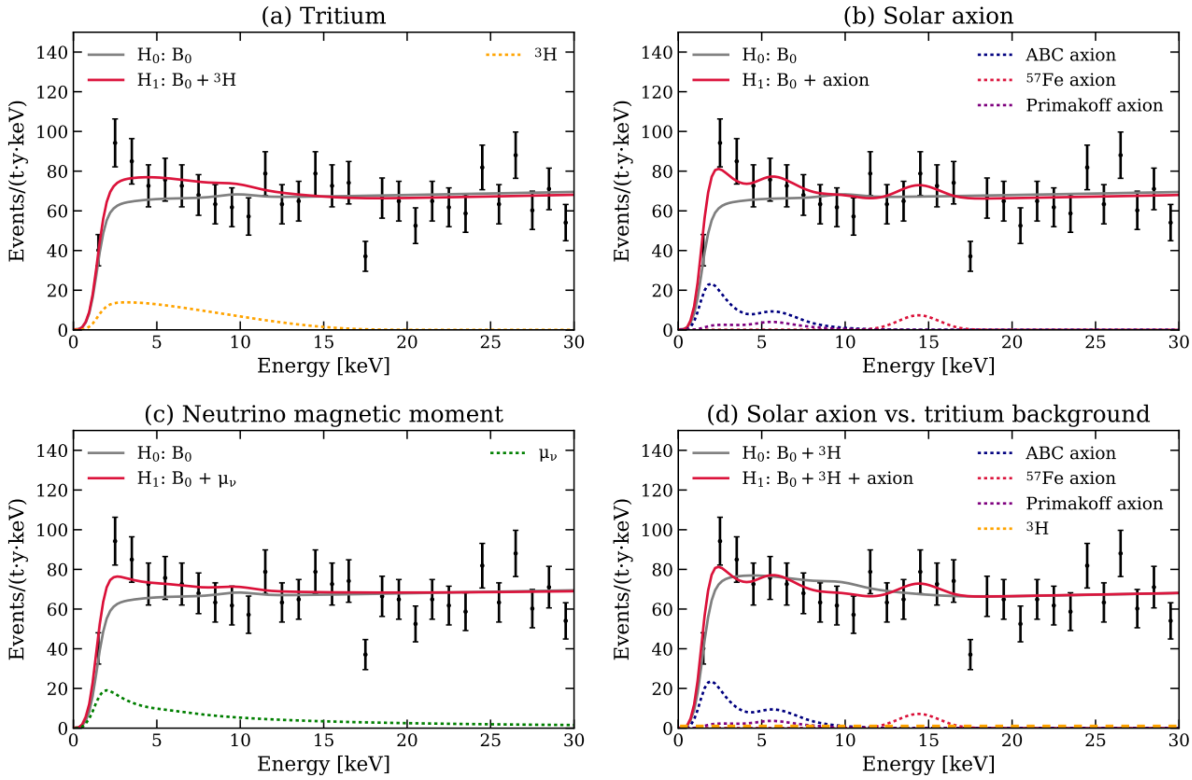

Figure 20 – Detected events in XENON1T. Red line represents the expected background events [129].

the hypothesis of tritium contamination, which at XENONnT was confirmed to not be present.

The new upper limits on g_{ae} and $g_{a\gamma\gamma}$ are presented in Fig.23. In this figure, it is evident that there is no longer an astrophysical conflict, and it is also apparent that these limits have not yet been surpassed. However, as indicated in Table 3, the XENONnT experiment is still in progress and is expected to conclude in 2026.

6.2.2 Primakoff-Bragg conversion in crystalline detectors

Axion-photon conversion can also occur in the atomic eletromagnetic field of materials, particularly in crystalline media where the periodic nature of the electronic structure imposes a Bragg condition. This condition implies that the conversion is coherently enhanced if the momentum of the incoming particles matches one of the Bragg angles [135, 136]. This concept has been applied to search for solar axions with crystalline detectors [137, 138].

The Earth's rotation induces a continuous variation of the angle between the incoming direction of axions and the crystal plane, resulting in distinct and sharp energy- and time-dependent patterns in the expected signal in the detector. This characteristic

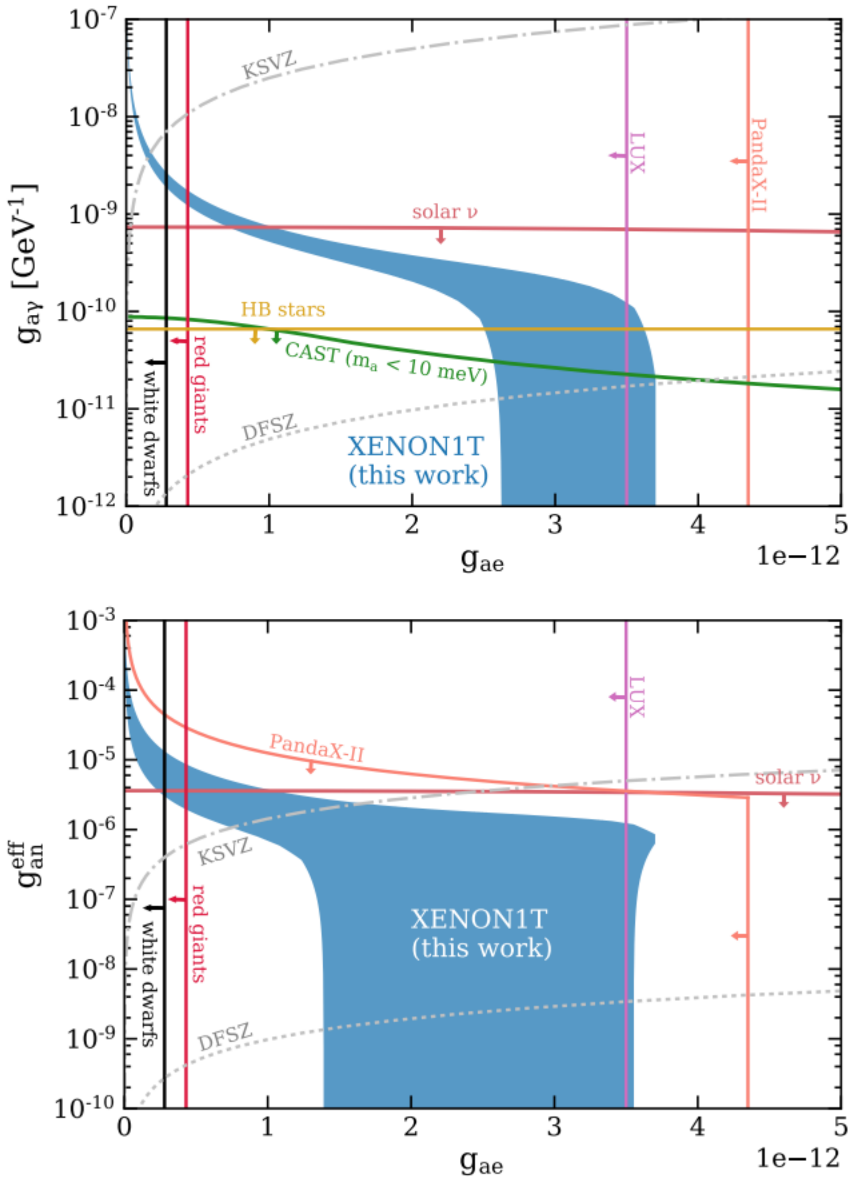

Figure 21 – Parameter region for axion couplings constraints in XENON1T experiment. [129].

pattern can effectively help identify a potential signal over the detector background. One advantage of this technique is its ability to be utilized as a byproduct of ongoing low-background underground detectors, provided they have a low enough threshold and the orientation of the crystal plane is (at least partially) known.

After its initial application with small germanium (Ge) detectors by the SOLAX [139] and COSME [140] experiments, this technique has also been employed as a byproduct in experiments such as DAMA [141], CDMS [142], and EDELWEISS [143]. It is also foreseen as part of the physics program of experiments like CUORE [144], GERDA, and MAJORANA [145]. However, in the mass range where helioscopes achieve full coherent conversion of axions, this technique's prospects are not competitive [146, 147]. The most

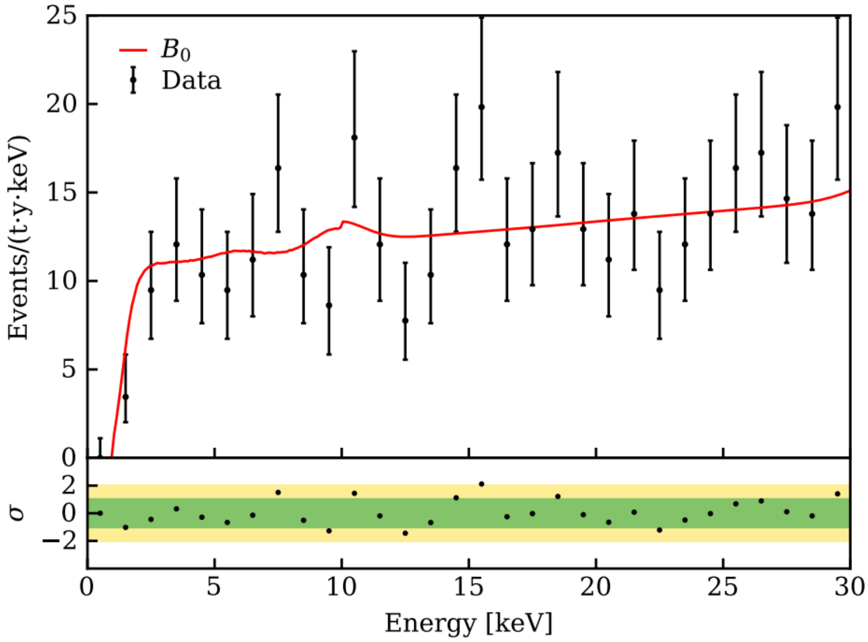

Figure 22 – Detected events in XENONnT. Red line represents the expected background events [130].

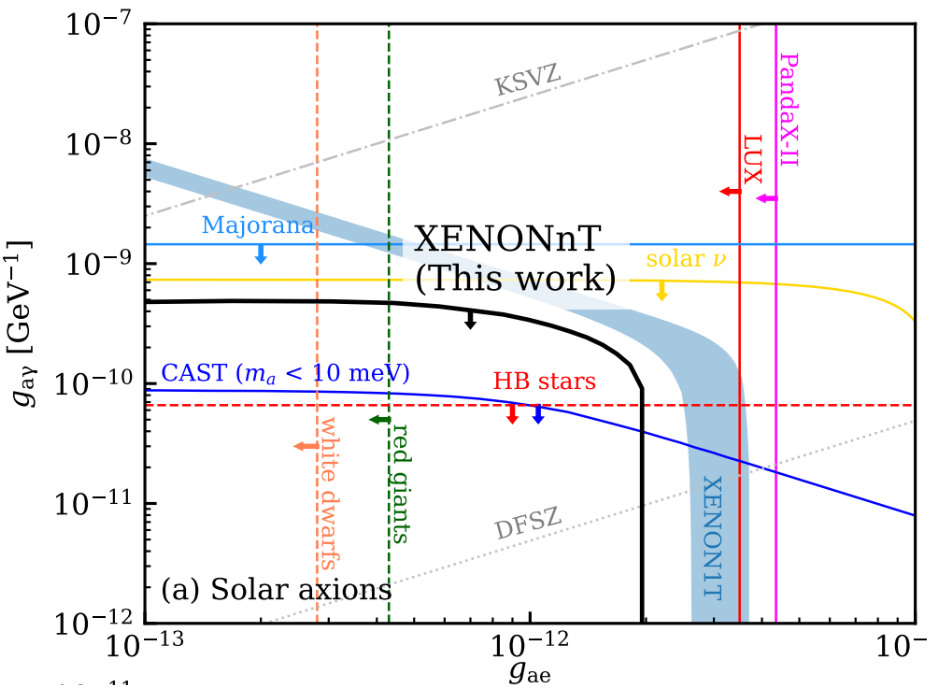

Figure 23 – Parameter region for axion couplings constraints in XENONnT experiment. [130].

stringent result to date comes from DAMA, which sets a limit at 90% C.L.:

$$g_{a\gamma\gamma} < 1.7 \times 10^{-9}\ \mathrm{GeV}^{-1}. \tag{6.18}$$

It's important to note that this limit is less stringent than the limit derived from solar physics itself, indicating that these bounds are not yet self-consistent. While future results, such as those from [144], may improve upon this, even with future multiton target masses, this technique may not reach sensitivity to $g_{a\gamma\gamma}$ similar to current helioscopes [146]. For higher masses above ~ 1 eV, where the sensitivity of helioscopes diminishes, this technique surpasses the former; however, this parameter region is disfavored by astrophysics and cosmology, as explained earlier.

7 Conclusion

After reviewing the theoretical foundations, production mechanisms, and detection strategies for solar axions, it is evident that these particles have been increasingly gaining focus in particle physics through the course of the last four decades. As the favored resolution to the strong CP problem, axions (and more broadly, ALPs) are not only driven by theoretical considerations but are also supported by a diverse array of astrophysical and cosmological arguments, which have seen substantial development in recent years. These arguments extend to models involving inflation, dark radiation, and dark energy. Crucially, axions and ALPs emerge as compelling candidates for constituting a fraction or all of the elusive dark matter in the Universe, especially given the absence of positive results in searches for WIMPs, making axions increasingly pivotal in unraveling the mystery of dark matter.

In the realm of astrophysics, axions and ALPs, in contrast to WIMPs, assume significant roles. Certain astrophysical observations already suggest the possible existence of these particles. Furthermore, the detection possibilities offered by axions produced in stellar interiors, such as solar axions, present unique detection opportunities, exemplified by axion helioscopes, which lack analogs for WIMPs.

The advantageous aspect of having numerous plausible BSM theoretical frameworks that naturally predict axions or ALPs is underscored. The theoretical framework explored in this study readily finds applications in diverse fields such as SUSY, GUTs, and string theory.

While axions were traditionally considered invisible due to their extraordinarily low couplings, recent developments suggest that we are approaching a juncture where axions may no longer remain elusive. The surge in the intensity and diversity of experimental efforts to detect axions is noteworthy. Established detection techniques are now being challenged by next-generation experiments with ambitious sensitivity goals, presenting substantial prospects for transformative impacts on the field. We are on the brink of delving into the intriguing region below astrophysical constraints.

In conclusion, we argue that the robustness of the axion-ALP paradigm today underscores the imperative to pursue their experimental detection as a major goal in particle physics. Should axions indeed exist, there is a realistic chance of their positive detection in the near future. Such a discovery would represent a groundbreaking moment, reshaping the trajectories of particle physics, cosmology, and astrophysics on a grand scale.

Bibliography

[1] Glashow, Sheldon L: *Partial-symmetries of weak interactions.* Nuclear physics, 22(4):579–588, 1961. Mentioned on page 10.

[2] Salam, Abdus and John Clive Ward: *Electromagnetic and weak interactions.* Physics Letters, 13(2):168–171, 1964. Mentioned on page 10.

[3] Weinberg, Steven: *A model of leptons.* Physical review letters, 19(21):1264, 1967. Mentioned on page 10.

[4] Higgs, Peter W: *Broken symmetries and the masses of gauge bosons.* Physical review letters, 13(16):508, 1964. Mentioned on page 10.

[5] Higgs, Peter W: *Spontaneous symmetry breakdown without massless bosons.* Physical review, 145(4):1156, 1966. Mentioned on page 10.

[6] Aad, Georges, Tatevik Abajyan, B Abbott, J Abdallah, S Abdel Khalek, Ahmed Ali Abdelalim, R Aben, B Abi, M Abolins, OS AbouZeid, *et al.*: *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC.* Physics Letters B, 716(1):1–29, 2012. Mentioned on page 10.

[7] Wilczek, Frank: *THE BIRTH OF AXIONS-A CITATION-CLASSIC COMMENTARY ON PROBLEM OF STRONG P AND T INVARIANCE IN THE PRESENCE OF INSTANTONS BY WILCZEK, F.* CURRENT CONTENTS/PHYSICAL CHEMICAL & EARTH SCIENCES, (16):8–9, 1991. Mentioned on page 12.

[8] Nilles, H Peter and Stuart Raby: *Supersymmetry and the strong CP problem.* Nuclear Physics B, 198(1):102–112, 1982. Mentioned on page 13.

[9] Wise, Mark B, Howard Georgi, and Sheldon L Glashow: *SU(5) and the Invisible Axion.* Physical Review Letters, 47(6):402, 1981. Mentioned on page 13.

[10] Witten, Edward: *Some properties of O(32) superstrings.* Physics Letters B, 149(4-5):351–356, 1984. Mentioned on page 13.

[11] Conlon, Joseph P: *The QCD axion and moduli stabilisation.* Journal of High Energy Physics, 2006(05):078, 2006. Mentioned on page 13.

[12] Weinberg, Steven: *The U(1) problem.* Phys. Rev. D, 11:3583–3593, Jun 1975. `https://link.aps.org/doi/10.1103/PhysRevD.11.3583`. Mentioned on page 14.

[13] Workman, R. L. and Others: *Review of Particle Physics*. PTEP, 2022:083C01, 2022. Mentioned 7 times on page 14, 18, 24, 61, 65, 72 e 87.

[14] Tanabashi, Masaharu, K Hagiwara, K Hikasa, Katsumasa Nakamura, Y Sumino, F Takahashi, J Tanaka, K Agashe, G Aielli, Claude Amsler, *et al.*: *Review of Particle Physics: particle data groups*. Physical Review D, 98(3):1–1898, 2018. Mentioned 2 times on page 14 e 18.

[15] Srednicki, Mark: *Quantum field theory*. Cambridge University Press, 2007. Mentioned 6 times on page 16, 18, 30, 36, 42 e 48.

[16] Gell-Mann, Murray and Kenneth M Watson: *The Interactions Between pi-Mesons and Nucleons*. Annual review of nuclear science, 4:219–270, 1954. Mentioned on page 16.

[17] Weinberg, S: *The Quantum Theory of Fields, Volume II, Chapter 23*, 1996. Mentioned on page 18.

[18] Olive, Keith A, Kaustubh Agashe, Claude Amsler, Mario Antonelli, Jean Francois Arguin, David M Asner, H Baer, Henry R Band, RM Barnett, Tullio Basaglia, *et al.*: *Review of particle physics*. Chinese physics C, 38(9):090001, 2014. Mentioned 2 times on page 18 e 71.

[19] Baker, CA, DD Doyle, P Geltenbort, K Green, MGD Van der Grinten, PG Harris, P Iaydjiev, SN Ivanov, DJR May, JM Pendlebury, *et al.*: *Improved experimental limit on the electric dipole moment of the neutron*. Physical Review Letters, 97(13):131801, 2006. Mentioned 2 times on page 23 e 47.

[20] Pendlebury, JM, Samer Afach, Nicholas J Ayres, Charles A Baker, Gilles Ban, Georg Bison, Kazimierz Bodek, Martin Burghoff, Peter Geltenbort, Katie Green, *et al.*: *Revised experimental upper limit on the electric dipole moment of the neutron*. Physical Review D, 92(9):092003, 2015. Mentioned 2 times on page 23 e 47.

[21] Graner, Brent, Y Chen, EG Lindahl, BR Heckel, *et al.*: *Reduced limit on the permanent electric dipole moment of hg 199*. Physical review letters, 116(16):161601, 2016. Mentioned 2 times on page 23 e 47.

[22] Gell-Mann, Murray, Robert J Oakes, and Brian Renner: *Behavior of current divergences under SU(3)× SU(3)*. Physical Review, 175(5):2195, 1968. Mentioned on page 24.

[23] Fujikawa, Kazuo: *Path-integral measure for gauge-invariant fermion theories*. Physical Review Letters, 42(18):1195, 1979. Mentioned on page 28.

[24] Adler, Stephen L: *Axial-vector vertex in spinor electrodynamics*. Physical Review, 177(5):2426, 1969. Mentioned on page 29.

[25] Bardeen, William A: *Anomalous Ward identities in spinor field theories*. Physical Review, 184(5):1848, 1969. Mentioned on page 29.

[26] Bell, John S and Roman Jackiw: *A PCAC puzzle: $\pi^0\gamma\gamma$ in the σ-model*. Il Nuovo Cimento A (1965-1970), 60(1):47–61, 1969. Mentioned on page 29.

[27] Bardeen, William A: *Anomalous currents in gauge field theories*. Nuclear Physics B, 75(2):246–258, 1974. Mentioned on page 29.

[28] Adler, Stephen L and William A Bardeen: *Absence of higher-order corrections in the anomalous axial-vector divergence equation*. Physical Review, 182(5):1517, 1969. Mentioned on page 34.

[29] Hooft, Gerard t: *Symmetry breaking through Bell-Jackiw anomalies*. Physical Review Letters, 37(1):8–11, 1976. Mentioned on page 35.

[30] Abel, Christopher, Samer Afach, Nicholas J Ayres, Colin A Baker, Gilles Ban, Georg Bison, Kazimierz Bodek, Vira Bondar, Martin Burghoff, Estelle Chanel, *et al.*: *Measurement of the permanent electric dipole moment of the neutron*. Physical Review Letters, 124(8):081803, 2020. Mentioned 2 times on page 47 e 51.

[31] Dar, Shahida: *The neutron EDM in the SM: a review*. arXiv preprint hep-ph/0008248, 2000. Mentioned on page 47.

[32] Gordon, Walter: *Der comptoneffekt nach der schrödingerschen theorie*. Zeitschrift für Physik, 40(1-2):117–133, 1926. Mentioned on page 49.

[33] Kaplan, David B. and Aneesh V. Manohar: *Current-Mass Ratios of the Light Quarks*. Phys. Rev. Lett., 56:2004–2007, May 1986. `https://link.aps.org/doi/10.1103/PhysRevLett.56.2004`. Mentioned on page 51.

[34] Gasser, J. and H. Leutwyler: *Quark Masses*. Phys. Rept., 87:77–169, 1982. Mentioned on page 51.

[35] Peccei, R. D. and Helen R. Quinn: *Constraints imposed by* CP *conservation in the presence of pseudoparticles*. Phys. Rev. D, 16:1791–1797, Sep 1977. `https://link.aps.org/doi/10.1103/PhysRevD.16.1791`. Mentioned 3 times on page 51, 53 e 60.

[36] Peccei, R. D. and Helen R. Quinn: CP *Conservation in the Presence of Pseudoparticles*. Phys. Rev. Lett., 38:1440–1443, Jun 1977. `https://link.aps.org/doi/10.1103/PhysRevLett.38.1440`. Mentioned 3 times on page 51, 53 e 60.

[37] Callan, Jr., Curtis G., R. F. Dashen, and David J. Gross: *The Structure of the Gauge Theory Vacuum*. Phys. Lett. B, 63:334–340, 1976. Mentioned on page 55.

[38] Wilczek, F.: *Problem of Strong P and T Invariance in the Presence of Instantons*. Phys. Rev. Lett., 40:279–282, Jan 1978. `https://link.aps.org/doi/10.1103/PhysRevLett.40.279`. Mentioned on page 60.

[39] Weinberg, Steven: *A New Light Boson?* Phys. Rev. Lett., 40:223–226, Jan 1978. `https://link.aps.org/doi/10.1103/PhysRevLett.40.223`. Mentioned on page 60.

[40] Kuster, Markus, Georg Raffelt, and Berta Beltrán: *Axions: Theory, cosmology, and experimental searches*, volume 741. Springer, 2007. Mentioned 2 times on page 62 e 65.

[41] Bardeen, William A, Roberto D Peccei, and Tsutomu Yanagida: *Constraints on variant axion models*. Nuclear Physics B, 279(3-4):401–428, 1987. Mentioned 2 times on page 62 e 67.

[42] Ramond, Pierre: *Field theory: a modern primer*. Routledge, 2020. Mentioned on page 63.

[43] Asano, Y, E Kikutani, S Kurokawa, T Miyachi, M Miyajima, Y Nagashima, T Shinkawa, S Sugimoto, and Y Yoshimura: *Search for a rare decay mode K+ π+ νv and axion*. Physics Letters B, 107(1-2):159–162, 1981. Mentioned on page 67.

[44] Kim, Jihn E: *Weak interaction singlet and strong CP invariance*. Physical Review Letters, 43(2):103, 1979. Mentioned on page 68.

[45] Shifman, Mikhail A, AI Vainshtein, and Valentin I Zakharov: *Can confinement ensure natural CP invariance of strong interactions?* Nuclear Physics B, 166(3):493–506, 1980. Mentioned on page 68.

[46] Dine, Michael, Willy Fischler, and Mark Srednicki: *A simple solution to the strong CP problem with a harmless axion*. Physics letters B, 104(3):199–202, 1981. Mentioned on page 68.

[47] Zhitnitskii, AP: *Possible suppression of axion-hadron interactions*. Sov. J. Nucl. Phys.(Engl. Transl.);(United States), 31(2), 1980. Mentioned on page 68.

[48] Dimopoulos, Savas, Paul H Frampton, Howard Georgi, and Mark B Wise: *Automatic invisible axion without domain walls*. Physics Letters B, 117(3-4):185–188, 1982. Mentioned on page 68.

[49] Slansky, Richard: *Group theory for unified model building.* Physics reports, 79(1):1–128, 1981. Mentioned on page 69.

[50] Georgi, Howard and Sheldon L Glashow: *Unity of all elementary-particle forces.* Physical Review Letters, 32(8):438, 1974. Mentioned on page 69.

[51] Sikivie, P: *Axions, domain walls, and the early universe.* Physical Review Letters, 48(17):1156, 1982. Mentioned on page 71.

[52] Vilenkin, Alexander and Allen E Everett: *Cosmic strings and domain walls in models with Goldstone and pseudo-Goldstone bosons.* Physical Review Letters, 48(26):1867, 1982. Mentioned on page 71.

[53] Lazarides, G and Qaisar Shafi: *Axions and the primordial monopole problem.* Physics Letters B, 124(1-2):26–28, 1983. Mentioned on page 71.

[54] Srednicki, Mark: *Axion couplings to matter:(I). CP-conserving parts.* Nuclear Physics B, 260(3-4):689–700, 1985. Mentioned on page 71.

[55] Dashen, RP: *private communication RP Dashen and M. Weinstein.* Phys. Rev, 183:1261, 1969. Mentioned on page 72.

[56] Shore, Graham M: *Generalizations of Dashen's formula in supersymmetric QCD.* Nuclear Physics B, 231(1):139–156, 1984. Mentioned on page 72.

[57] Quigg, Chris: *Gauge theories of strong, weak, and electromagnetic interactions.* CRC Press, 2021. Mentioned 2 times on page 78 e 83.

[58] Schwartz, Matthew D: *Quantum field theory and the standard model.* Cambridge university press, 2014. Mentioned on page 85.

[59] Goldberger, ML and SB Treiman: *Conserved currents in the theory of Fermi interactions.* Physical Review, 110(6):1478, 1958. Mentioned on page 90.

[60] Donoghue, John F., Eugene Golowich, and Barry R. Holstein: *Dynamics of the Standard Model.* Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology. Cambridge University Press, 2nd edition, 2014. Mentioned on page 90.

[61] Adler, Stephen L., E. W. Colglazier, J. B. Healy, Inga Karliner, Judy Lieberman, Yee Jack Ng, and Hung Sheng Tsao: *Renormalization constants for scalar, pseudoscalar, and tensor currents.* Phys. Rev. D, 11:3309–3318, Jun 1975. `https://link.aps.org/doi/10.1103/PhysRevD.11.3309`. Mentioned on page 90.

[62] Straniero, Oscar, Adrian Ayala, Maurizio Giannotti, Alessandro Mirizzi, and Inma Dominguez: *Axion-Photon Coupling: Astrophysical Constraints.* In *11th Patras*

Workshop on Axions, WIMPs and WISPs, pages 77–81, 2015. Mentioned 2 times on page 91 e 114.

[63] Giannotti, Maurizio, Igor G Irastorza, Javier Redondo, Andreas Ringwald, and Ken'ichi Saikawa: *Stellar recipes for axion hunters*. Journal of Cosmology and Astroparticle Physics, 2017(10):010, 2017. Mentioned on page 91.

[64] Raffelt, Georg G: *Astrophysical axion bounds*. In *Axions: Theory, Cosmology, and Experimental Searches*, pages 51–71. Springer, 2008. Mentioned on page 91.

[65] Sedrakian, Armen: *Axion cooling of neutron stars*. Physical Review D, 93(6):065044, 2016. Mentioned on page 91.

[66] Leinson, Lev B: *Axion mass limit from observations of the neutron star in Cassiopeia A*. Journal of Cosmology and Astroparticle Physics, 2014(08):031, 2014. Mentioned on page 91.

[67] Graham, Peter W and Surjeet Rajendran: *New observables for direct detection of axion dark matter*. Physical Review D, 88(3):035023, 2013. Mentioned on page 91.

[68] Irastorza, Igor G and Javier Redondo: *New experimental approaches in the search for axion-like particles*. Progress in Particle and Nuclear Physics, 102:89–159, 2018. Mentioned 2 times on page 91 e 115.

[69] Dicus, Duane A, Edward W Kolb, Vigdor L Teplitz, and Robert V Wagoner: *Astrophysical bounds on the masses of axions and Higgs particles*. Physical Review D, 18(6):1829, 1978. Mentioned on page 92.

[70] Raffelt, Georg G: *Astrophysical axion bounds diminished by screening effects*. Physical Review D, 33(4):897, 1986. Mentioned 3 times on page 92, 99 e 106.

[71] Raffelt, Georg G: *Plasmon decay into low-mass bosons in stars*. Physical Review D, 37(6):1356, 1988. Mentioned on page 92.

[72] Redondo, Javier: *Solar axion flux from the axion-electron coupling*. Journal of Cosmology and Astroparticle Physics, 2013(12):008, 2013. Mentioned 7 times on page 92, 99, 102, 103, 104, 108 e 110.

[73] Torre, Lorenzo De la: *Elementos de relatividad*. Universidad de Antioquia, 2008. Mentioned on page 93.

[74] Ohanian, Hans C: *Special relativity: a modern introduction*. Physics Curriculum & Instruction, 2001. Mentioned on page 93.

[75] Galanti, G, M Roncadelli, A De Angelis, and GF Bignami: *Axion-like particles explain the unphysical redshift-dependence of AGN gamma-ray spectra.* ArXiv e-prints, 1503, 2015. Mentioned 2 times on page 93 e 94.

[76] Mirizzi, Alessandro, Georg G Raffelt, and Pasquale D Serpico: *Signatures of axionlike particles in the spectra of TeV gamma-ray sources.* Physical Review D, 76(2):023001, 2007. Mentioned on page 93.

[77] Meyer, Manuel, Dieter Horns, and Martin Raue: *First lower limits on the photon-axion-like particle coupling from very high energy gamma-ray observations.* Physical Review D, 87(3):035027, 2013. Mentioned on page 93.

[78] Sánchez-Conde, MA, D Paneque, E Bloom, F Prada, and A Dominguez: *Hints of the existence of axionlike particles from the gamma-ray spectra of cosmological sources.* Physical Review D, 79(12):123511, 2009. Mentioned on page 93.

[79] Jaeckel, Joerg and Andreas Ringwald: *The low-energy frontier of particle physics.* Annual Review of Nuclear and Particle Science, 60:405–437, 2010. Mentioned on page 99.

[80] Redondo, Javier and Andreas Ringwald: *Light shining through walls.* Contemporary Physics, 52(3):211–236, 2011. Mentioned on page 99.

[81] Jaeckel, Joerg, Eduard Masso, Javier Redondo, Andreas Ringwald, and Fuminobu Takahashi: *Need for purely laboratory-based axionlike particle searches.* Physical Review D, 75(1):013004, 2007. Mentioned on page 99.

[82] Dias, Alex G and Germán Lugones: *Probing light pseudoscalar particles using synchrotron light.* Physics Letters B, 673(1):101–105, 2009. Mentioned on page 99.

[83] Moriyama, Shigetaka, Makoto Minowa, Toshio Namba, Yoshizumi Inoue, Yuko Takasu, and Akira Yamamoto: *Direct search for solar axions by using strong magnetic field and X-ray detectors.* Physics Letters B, 434(1-2):147–152, 1998. Mentioned 2 times on page 99 e 113.

[84] Inoue, Y, Y Akimoto, R Ohta, T Mizumoto, A Yamamoto, and M Minowa: *Search for solar axions with mass around 1 eV using coherent conversion of axions into photons.* Physics Letters B, 668(2):93–97, 2008. Mentioned 2 times on page 99 e 113.

[85] Dimopoulos, Savas, Glenn D Starkman, and Bryan W Lynn: *Atomic enhancements in the detection of axions.* Modern Physics Letters A, 1(08):491–500, 1986. Mentioned 3 times on page 99, 105 e 119.

[86] Dimopoulos, Savas, Joshua Frieman, Bryan W Lynn, and Glenn D Starkman: *Axiorecombination: a new mechanism for stellar axion production*. Physics Letters B, 179(3):223–227, 1986. Mentioned on page 99.

[87] Pospelov, Maxim, Adam Ritz, and Mikhail Voloshin: *Bosonic super-WIMPs as keV-scale dark matter*. Physical Review D, 78(11):115012, 2008. Mentioned 3 times on page 99, 105 e 119.

[88] Zhitnitskij, AR and Yu I Skovpen: *On production and detecting of axions at transmission of electrons through matter*. Yadernaya Fizika, 29(4):995–1000, 1979. Mentioned on page 99.

[89] Krauss, Lawrence M, John E Moody, and Frank Wilczek: *A stellar energy loss mechanism involving axions*. Physics Letters B, 144(5-6):391–394, 1984. Mentioned on page 99.

[90] Mikaelian, Karnig O: *Astrophysical implications of new light Higgs bosons*. Physical Review D, 18(10):3605, 1978. Mentioned 2 times on page 99 e 106.

[91] Fukugita, M, S Watamura, and M Yoshimura: *Light pseudoscalar particle and stellar energy loss*. Physical Review Letters, 48(22):1522, 1982. Mentioned on page 99.

[92] Fukugita, M, S Watamura, and M Yoshimura: *Astrophysical constraints on a new light axion and other weakly interacting particles*. Physical Review D, 26(8):1840, 1982. Mentioned on page 99.

[93] Asplund, M, AM Amarsi, and Nicolas Grevesse: *The chemical make-up of the Sun: A 2020 vision*. Astronomy & Astrophysics, 653:A141, 2021. Mentioned on page 99.

[94] Rutten, Robert J *et al.*: *Radiative transfer in stellar atmospheres*. Sterrekundig Instituut Utrecht, 1995. Mentioned on page 103.

[95] Derevianko, A, VA Dzuba, VV Flambaum, and M Pospelov: *Axio-electric effect*. Physical Review D, 82(6):065006, 2010. Mentioned on page 105.

[96] Redondo, Javier and Georg Raffelt: *Solar constraints on hidden photons re-visited*. Journal of Cosmology and Astroparticle Physics, 2013(08):034, 2013. Mentioned on page 106.

[97] Maxon, M Stephen and E Gary Corman: *Electron-electron bremsstrahlung from a quantum plasma (Z= 1)*. Physical Review, 163(1):156, 1967. Mentioned on page 106.

[98] Iglesias, Carlos A and Steve J Rose: *Corrections to bremsstrahlung and thomson scattering at the solar center.* The Astrophysical Journal, 466(2):L115, 1996. Mentioned on page 106.

[99] Raffelt, Georg Gottfried: *Astrophysikalische aspekte eines speziellen problems der teilchenphysik.* na, 1986. Mentioned on page 106.

[100] Raffelt, Georg G: *Stars as laboratories for fundamental physics: The astrophysics of neutrinos, axions, and other weakly interacting particles.* University of Chicago press, 1996. Mentioned on page 106.

[101] Sikivie, Pierre: *Experimental tests of the"invisible"axion.* Physical Review Letters, 51(16):1415, 1983. Mentioned 2 times on page 109 e 111.

[102] Irastorza, Igor G, FT Avignone, A Liolios, S Russenschuck, H ten Kate, K van Bibber, T Geralis, O Limousin, JA Villar, H Gomez, *et al.*: *The international axion observatory iaxo. letter of intent to the cern sps committee.* Technical report, 2013. Mentioned on page 111.

[103] Andriamonje, S, S Aune, D Autiero, K Barth, A Belov, B Beltrán, H Bräuninger, JM Carmona, S Cebrián, JI Collar, *et al.*: *An improved limit on the axion–photon coupling from the CAST experiment.* Journal of Cosmology and Astroparticle Physics, 2007(04):010, 2007. Mentioned 2 times on page 111 e 114.

[104] collaboration, CAST *et al.*: *First results from the CERN Axion Solar Telescope (CAST).* arXiv preprint hep-ex/0411033, 2004. Mentioned 2 times on page 111 e 114.

[105] Van Bibber, K, PM McIntyre, DE Morris, and GG Raffelt: *Design for a practical laboratory detector for solar axions.* Physical Review D, 39(8):2089, 1989. Mentioned on page 111.

[106] Arik, E, S Aune, D Autiero, K Barth, A Belov, B Beltrán, S Borghi, G Bourlis, FS Boydag, H Bräuninger, *et al.*: *Probing eV-scale axions with CAST.* Journal of Cosmology and Astroparticle Physics, 2009(02):008, 2009. Mentioned 2 times on page 111 e 114.

[107] Dafni, Theopisti, Ciaran AJ O'Hare, Biljana Lakić, Javier Galán, Francisco J Iguaz, Igor G Irastorza, Krešimir Jakovčić, Gloria Luzón, Javier Redondo, and Elisa Ruiz Chóliz: *Weighing the solar axion.* Physical Review D, 99(3):035037, 2019. Mentioned on page 112.

[108] Irastorza, Igor G, FT Avignone, S Caspi, JM Carmona, T Dafni, M Davenport, A Dudarev, G Fanourakis, E Ferrer-Ribas, J Galán, *et al.*: *Towards a new generation*

axion helioscope. Journal of Cosmology and Astroparticle Physics, 2011(06):013, 2011. Mentioned 3 times on page 112, 115 e 116.

[109] Lazarus, DM, GC Smith, R Cameron, AC Melissinos, G Ruoso, Yannis K Semertzidis, and FA Nezrick: *Search for solar axions.* Physical Review Letters, 69(16):2333, 1992. Mentioned on page 112.

[110] Inoue, Yoshizumi, Toshio Namba, Shigetaka Moriyama, Makoto Minowa, Yuko Takasu, Takashi Horiuchi, and Akira Yamamoto: *Search for sub-electronvolt solar axions using coherent conversion of axions into photons in magnetic field and gas helium.* Physics Letters B, 536(1-2):18–23, 2002. Mentioned on page 113.

[111] Zioutas, Konstantinos, CE Aalseth, D Abriola, FT Avignone Iii, RL Brodzinski, JI Collar, R Creswick, DE Di Gregorio, H Farach, AO Gattone, *et al.*: *A decommissioned LHC model magnet as an axion telescope.* Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment, 425(3):480–487, 1999. Mentioned on page 113.

[112] Kuster, M, H Bräuninger, S Cebrián, M Davenport, C Eleftheriadis, J Englhauser, H Fischer, J Franz, P Friedrich, R Hartmann, *et al.*: *The x-ray telescope of CAST.* New Journal of Physics, 9(6):169, 2007. Mentioned on page 114.

[113] Abbon, P, S Andriamonje, S Aune, T Dafni, M Davenport, E Delagnes, R de Oliveira, G Fanourakis, E Ferrer Ribas, J Franz, *et al.*: *The Micromegas detector of the CAST experiment.* New Journal of Physics, 9(6):170, 2007. Mentioned on page 114.

[114] Arik, M, S Aune, K Barth, A Belov, S Borghi, H Bräuninger, Giovanni Cantatore, JM Carmona, SA Cetin, JI Collar, *et al.*: *Search for Sub-eV Mass Solar Axions by the CERN Axion Solar Telescope with He 3 Buffer Gas.* Physical Review Letters, 107(26):261302, 2011. Mentioned on page 114.

[115] Arik, M, S Aune, K Barth, A Belov, S Borghi, H Bräuninger, Giovanni Cantatore, JM Carmona, SA Cetin, JI Collar, *et al.*: *Search for solar axions by the CERN axion solar telescope with He 3 buffer gas: Closing the hot dark matter gap.* Physical review letters, 112(9):091302, 2014. Mentioned on page 114.

[116] Aznar, F, J Castel, FE Christensen, T Dafni, TA Decker, E Ferrer-Ribas, JA Garcia, I Giomataris, JG Garza, CJ Hailey, *et al.*: *A Micromegas-based low-background x-ray detector coupled to a slumped-glass telescope for axion research.* Journal of Cosmology and Astroparticle Physics, 2015(12):008, 2015. Mentioned on page 114.

[117] Anastassopoulos, V, S Aune, K Barth, A Belov, G Cantatore, JM Carmona, JF Castel, SA Cetin, F Christensen, JI Collar, *et al.*: *New CAST limit on the axion-*

photon interaction. arXiv preprint arXiv:1705.02290, 2017. Mentioned on page 114.

[118] Anastassopoulos, Vassilis, F Avignone, A Bykov, G Cantatore, SA Cetin, A Derbin, I Drachnev, R Djilkibaev, V Eremin, H Fischer, *et al.*: *Towards a medium-scale axion helioscope and haloscope.* Journal of Instrumentation, 12(11):P11019, 2017. Mentioned on page 115.

[119] Armengaud, E, FT Avignone, M Betz, P Brax, Pierre Brun, Giovanni Cantatore, JM Carmona, GP Carosi, Fritz Caspers, S Caspi, *et al.*: *Conceptual design of the international axion observatory (IAXO).* Journal of Instrumentation, 9(05):T05002, 2014. Mentioned on page 115.

[120] Shilon, I, A Dudarev, H Silva, and HHJ Ten Kate: *Conceptual design of a new large superconducting toroid for IAXO, the new International AXion Observatory.* IEEE transactions on applied superconductivity, 23(3):4500604–4500604, 2013. Mentioned on page 116.

[121] Ten Kate, HHJ: *The ATLAS superconducting magnet system at the Large Hadron Collider.* Physica C: Superconductivity, 468(15-20):2137–2142, 2008. Mentioned on page 116.

[122] Ten Kate, Herman HJ: *ATLAS magnet system nearing completion.* IEEE transactions on applied superconductivity, 18(2):352–355, 2008. Mentioned on page 116.

[123] Ljubičić, Ante, Dalibor Kekez, Zvonko Krečak, and T Ljubičić: *Search for hadronic axions using axioelectric effect.* Physics Letters B, 599(3-4):143–147, 2004. Mentioned on page 117.

[124] Derbin, AV, AS Kayunov, VV Muratova, DA Semenov, and EV Unzhakov: *Constraints on the axion-electron coupling for solar axions produced by a Compton process and bremsstrahlung.* Physical Review D, 83(2):023505, 2011. Mentioned on page 117.

[125] Derbin, AV, IS Drachnev, AS Kayunov, and VN Muratova: *Search for solar axions produced by Compton process and bremsstrahlung using axioelectric effect.* arXiv preprint arXiv:1206.4142, 2012. Mentioned on page 117.

[126] Bellini, G, J Benziger, D Bick, G Bonfini, D Bravo, M Buizza Avanzini, B Caccianiga, L Cadonati, Frank Calaprice, C Carraro, *et al.*: *Search for solar axions produced in the p (d, He 3) A reaction with Borexino detector.* Physical Review D, 85(9):092003, 2012. Mentioned on page 117.

[127] Abe, K, K Hieda, K Hiraide, S Hirano, Y Kishimoto, K Kobayashi, S Moriyama, K Nakagawa, M Nakahata, H Ogawa, *et al.*: *Search for solar axions in XMASS, a large liquid-xenon detector.* Physics Letters B, 724(1-3):46–50, 2013. Mentioned on page 117.

[128] Aprile, E, F Agostini, M Alfonsi, K Arisaka, F Arneodo, M Auger, C Balan, P Barrow, L Baudis, B Bauermeister, *et al.*: *First axion results from the XENON100 experiment.* Physical Review D, 90(6):062009, 2014. Mentioned on page 117.

[129] Aprile, Elena, Jelle Aalbers, Fa Agostini, M Alfonsi, L Althueser, FD Amaro, Vasile C Antochi, E Angelino, JR Angevaare, F Arneodo, *et al.*: *Excess electronic recoil events in XENON1T.* Physical Review D, 102(7):072004, 2020. Mentioned 5 times on page 117, 119, 120, 121 e 122.

[130] Aprile, E, K Abe, F Agostini, S Ahmed Maouloud, L Althueser, B Andrieu, E Angelino, JR Angevaare, VC Antochi, D Antón Martin, *et al.*: *Search for new physics in electronic recoil data from XENONnT.* Physical Review Letters, 129(16):161805, 2022. Mentioned 3 times on page 117, 120 e 123.

[131] Fu, Changbo, Xiaopeng Zhou, Xun Chen, Yunhua Chen, Xiangyi Cui, Deqing Fang, Karl Giboni, Franco Giuliani, Ke Han, Xingtao Huang, *et al.*: *Limits on axion couplings from the first 80 days of data of the pandax-ii experiment.* Physical review letters, 119(18):181806, 2017. Mentioned on page 117.

[132] Akerib, DS, S Alsum, C Aquino, HM Araújo, X Bai, AJ Bailey, J Balajthy, P Beltrame, EP Bernard, A Bernstein, *et al.*: *First searches for axions and axionlike particles with the LUX experiment.* Physical review letters, 118(26):261301, 2017. Mentioned on page 117.

[133] Aalbers, J, F Agostini, M Alfonsi, FD Amaro, C Amsler, E Aprile, L Arazi, F Arneodo, P Barrow, L Baudis, *et al.*: *DARWIN: towards the ultimate dark matter detector.* Journal of Cosmology and Astroparticle Physics, 2016(11):017, 2016. Mentioned on page 117.

[134] Alessandria, F, RAFFAELE Ardito, DR Artusa, FT Avignone III, O Azzolini, M Balata, TI Banks, G Bari, J Beeman, F Bellini, *et al.*: *Search for 14.4 keV solar axions from M1 transition of Fe-57 with CUORE crystals.* arXiv preprint arXiv:1209.2800, 2012. Mentioned on page 119.

[135] Buchmüller, Wilfried and F Hoogeveen: *Coherent production of light scalar or pseudoscalar particles in Bragg scattering.* Physics Letters B, 237(2):278–283, 1990. Mentioned on page 121.

[136] Liao, Wei: *Generation and search of axion-like light particle using intense crystalline field.* Physics Letters B, 702(1):55–58, 2011. Mentioned on page 121.

[137] Yamaji, T, T Yamazaki, K Tamasaku, and T Namba: *Theoretical Calculation of coherent Laue-case conversion between X rays and ALPs for an X-ray LSW experiment.* arXiv preprint arXiv:1709.03299, 2017. Mentioned on page 121.

[138] Paschos, EA and Konstantin Zioutas: *A Proposal for solar axion detection via Bragg scattering.* Physics Letters B, 323(3-4):367–372, 1994. Mentioned on page 121.

[139] Avignone III, FT, D Abriola, RL Brodzinski, JI Collar, RJ Creswick, DE DiGregorio, HA Farach, AO Gattone, CK Guérard, F Hasenbalg, *et al.*: *Experimental search for solar axions via coherent Primakoff conversion in a germanium spectrometer.* Physical Review Letters, 81(23):5068, 1998. Mentioned on page 122.

[140] Morales, A, FT Avignone Iii, RL Brodzinski, S Cebrián, E Garcıa, D González, IG Irastorza, HS Miley, J Morales, A Ortiz de Solórzano, *et al.*: *Particle dark matter and solar axion searches with a small germanium detector at the Canfranc Underground Laboratory.* Astroparticle Physics, 16(3):325–332, 2002. Mentioned on page 122.

[141] Bernabei, R, P Belli, R Cerulli, F Montecchia, F Nozzoli, A Incicchitti, D Prosperi, CJ Dai, HL He, HH Kuang, *et al.*: *Search for solar axions by Primakoff effect in NaI crystals.* Physics Letters B, 515(1-2):6–12, 2001. Mentioned on page 122.

[142] Ahmed, Z, DS Akerib, S Arrenberg, CN Bailey, D Balakishiyeva, L Baudis, DA Bauer, J Beaty, PL Brink, T Bruch, *et al.*: *Search for Axions with the CDMS Experiment.* Physical review letters, 103(14):141802, 2009. Mentioned on page 122.

[143] Armengaud, E, Q Arnaud, C Augier, Angélique Benoit, L Bergé, T Bergmann, J Blümer, A Broniatowski, V Brudanin, P Camus, *et al.*: *Axion searches with the EDELWEISS-II experiment.* Journal of Cosmology and Astroparticle Physics, 2013(11):067, 2013. Mentioned on page 122.

[144] Li, Dawei, Richard J Creswick, Frank T Avignone III, and Yuanxu Wang: *Theoretical estimate of the sensitivity of the CUORE detector to solar axions.* Journal of Cosmology and Astroparticle Physics, 2015(10):065, 2015. Mentioned 2 times on page 122 e 124.

[145] Xu, Wenqin and Steven R Elliott: *Solar axion search technique with correlated signals from multiple detectors.* Astroparticle Physics, 89:39–50, 2017. Mentioned on page 122.

[146] Cebrian, S, E Garcia, D Gonzalez, IG Irastorza, A Morales, J Morales, A Ortiz de Solórzano, J Puimedón, A Salinas, ML Sarsa, *et al.*: *Prospects of solar axion searches with crystal detectors*. Astroparticle Physics, 10(4):397–404, 1999. Mentioned 2 times on page 122 e 124.

[147] Avignone III, FT, RJ Creswick, and S Nussinov: *The experimental challenge of detecting solar axion-like particles to test the cosmological ALP-photon oscillation hypotheses*. Astroparticle Physics, 34(8):640–642, 2011. Mentioned on page 122.

Appendix

APPENDIX A – Noether's Theorem

Noether's theorem is a fundamental principle in physics that establishes a profound connection between symmetries in a physical system and the corresponding conservation laws. Named after the pioneering mathematician Emmy Noether, who formulated it in 1915, the theorem has become a powerful guide in various branches of physics, including quantum mechanics and general relativity. Its implications have significantly improved our understanding of the universe.

At its core, Noether's theorem provides a powerful method for identifying conserved quantities, such as energy, momentum, and angular momentum, in diverse physical systems. This theorem highlights the deep relationship between the symmetries inherent in a system's laws and the unchanging quantities that emerge from those symmetries. Consequently, Noether's theorem stands as a fundamental tool, shedding light on the underlying laws that govern the behavior of the universe.

Noether's theorem is derived from the general properties of the action, as well as its invariance. To begin, let's define the functional action as follows:

$$S\left(\tau_1, \tau_2, [\phi]\right) \equiv \int_{\tau_1}^{\tau_2} d^4x \; \mathcal{L}\left(\phi, \partial_\mu \phi\right) \ , \tag{A.1}$$

where the action must be real to conserve total probability. It provides the classical equations of motion, in which are at most 2nd order in derivatives. The action is invariant under Poincaré group, a fundamental symmetry group in physics encompassing translations and Lorentz transformations. Additional symmetries, such as gauge symmetries, can be imposed if necessary, although this may not always be required. Despite the action having units of angular momentum, in natural units where $\hbar = 1$, the action S is dimensionless. Making a small variation $\delta\phi$ on the field, the action varies as follows:

$$\delta S = \int_{\tau_1}^{\tau_2} d^4x \; \delta\mathcal{L} = \int_{\tau_1}^{\tau_2} d^4x \; \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\delta(\partial_\mu\phi)\right] \ . \tag{A.2}$$

In this case, there are no variations on the space-time variables x^μ, so that the variation $\delta(\partial_\mu\phi)$ simplifies to:

$$\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi) \ , \tag{A.3}$$

utilizing this relation, we obtain:

$$\begin{aligned}\delta S &= \int_{\tau_1}^{\tau_2} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\partial_\mu(\delta\phi)\right] ,\\ &= \int_{\tau_1}^{\tau_2} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\right]\delta\phi + \int_{\tau_1}^{\tau_2} dx^4 \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\delta\phi\right] ,\\ &= \int_{\tau_1}^{\tau_2} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\right]\delta\phi + \int_\sigma d\sigma_\mu \left[\frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\delta\phi\right] ,\\ &= \int_{\tau_1}^{\tau_2} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\right]\delta\phi \ .\end{aligned} \tag{A.4}$$

If $\delta\phi(\sigma) = 0$, the surface term vanishes. Utilizing the principle of minimum action $\delta S = 0$, we arrive at the Euler-Lagrange equation:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \ . \tag{A.5}$$

We have reached a significant result, but this is not enough to derive the theorem. To do so, we must also consider variations in x (δx) and in the integration variable d^4x. The variation of the integration variable is given by the Jacobi formula:

$$\delta\left(d^4x\right) = d^4x\left(\partial_\mu \delta x^\mu\right) \ , \tag{A.6}$$

with these considerations, the variation of the action becomes

$$\delta S = \int_{\tau_1}^{\tau_2} d^4x \left[\partial_\mu\left(\delta x^4 \mathcal{L} + \delta\mathcal{L}\right)\right] \ , \tag{A.7}$$

where

$$\begin{aligned}\delta\mathcal{L} &= \delta x^\mu \partial_\mu \mathcal{L} + \delta_0 \mathcal{L} \ ,\\ &= \delta x^\mu \partial_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi}\delta_0\phi + \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]}\delta_0 \partial_\mu \phi \ ,\end{aligned} \tag{A.8}$$

and δ_0 is the functional variation

$$\begin{aligned}\delta_0\left[\partial_\mu \phi\right] &= [\delta_0, \partial_\mu]\phi + \partial_\mu \delta_0 \phi \ ,\\ &= \partial_\mu \delta_0 \phi \ .\end{aligned} \tag{A.9}$$

Putting it all together and utilizing the chain rule, we obtain:

$$\delta\mathcal{L} = \delta x^\rho \partial_\rho \mathcal{L} + \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial \mu \phi]}\right]\delta_0\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\delta_0\phi\right) \ , \tag{A.10}$$

note that the second term is equivalent to the equation of motion found in A.5. Consequently, this term vanishes as

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]} = 0 \ . \tag{A.11}$$

Therefore, we obtain:

$$\begin{aligned}\delta S &= \int_{\tau_1}^{\tau_2} d^4x \left[\mathcal{L}\partial_\mu \delta x^\mu + \delta x^\mu \partial_\mu \mathcal{L} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\delta_0 \phi\right)\right] \ , \\ &= \int_{\tau_1}^{\tau_2} d^4x \ \partial_\mu \left[\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]}\delta_0 \phi\right] \ .\end{aligned} \tag{A.12}$$

Utilizing

$$\delta = \delta_0 + \delta x^\mu \partial_\mu \mapsto \delta_0 = \delta - \delta x^\mu \partial_\mu \ , \tag{A.13}$$

we reach:

$$\begin{aligned}\delta S &= \int_{\tau_1}^{\tau_2} d^4x \ \partial_\mu \left[\mathcal{L}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]}\left[\delta\phi - \delta x^\rho \partial_\rho \phi\right]\right] \ , \\ &= \int_{\tau_1}^{\tau_2} d^4x \ \partial_\mu \left[\left(\mathcal{L}g^\mu_\rho - \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\partial_\rho \phi\right)\delta x^\rho + \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]}\delta\phi\right] \ .\end{aligned} \tag{A.14}$$

Writing the variations of the coordinates and the fields in terms of the global parameters

$$\begin{aligned}\delta x^\rho &= \frac{\delta x^\rho}{\delta w^a}\delta w^a \ , \\ \delta\phi &= \frac{\delta\phi}{\delta\omega^a}\delta\omega^a \ ,\end{aligned} \tag{A.15}$$

the variation of the action can be expressed as:

$$\delta S = \int_{\tau_1}^{\tau_2} d^4x \ \partial_\mu \left[\left(\mathcal{L}g^\mu_\rho - \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\partial_\rho \phi\right)\frac{\delta x^\rho}{\delta\omega^a} + \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]}\frac{\delta\phi}{\delta w^a}\right]\delta w^a \ . \tag{A.16}$$

At this point, we can define the Noether current j^μ as:

$$j^\mu_a \equiv -\left[\mathcal{L}g^\mu_\rho - \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\partial_\rho \phi\right]\frac{\delta x^\rho}{\delta\omega^a} - \frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]}\frac{\delta\phi}{\delta\omega^a} \ . \tag{A.17}$$

If the action is invariant under the small variation $\delta\phi$, i.e., $\delta S = 0$, then the Noether current is conserved. This implies:

$$\partial_\mu j^\mu_a = 0 \ . \tag{A.18}$$

In the case where $\delta S \neq 0$, the action is not symmetric under the variation, and consequently, the corresponding Noether current is not conserved

$$\partial_\mu j^\mu_a \neq 0 \ . \tag{A.19}$$

In the case where there is no variation in the space-time variable, $\delta x^\rho = 0$, the Noether current simplifies to:

$$j^\mu_a = -\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]}\frac{\delta\phi}{\delta\omega^a} \ , \tag{A.20}$$

which leads to:

$$\begin{aligned}\partial_\mu j^\mu_a &= -\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\frac{\delta\phi}{\delta\omega^a}\right) \ , \\ &= -\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial[\partial_\mu \phi]}\right)\frac{\delta\phi}{\delta\omega^\alpha} - \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi\right]}\partial_\mu \frac{\delta\phi}{\delta w^a} \ .\end{aligned} \tag{A.21}$$

Since $\delta x^\rho = 0$, accordingly to A.13, the variation operator simplifies to $\delta = \delta_0$, and we obtain:

$$\begin{aligned}\partial_\mu j_a^\mu &= -\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\right)\frac{\delta_0 \phi}{\delta_0 w^a} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\partial_\mu \frac{\delta_0 \phi}{\delta_0 w^a} \ , \\ &= -\left[\frac{\partial \mathcal{L}}{\partial \phi}\frac{\delta_0 \phi}{\delta_0 \omega^a} + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\frac{\delta_0}{\delta_0 \omega^a}\left(\partial_\mu \phi\right)\right] \ ,\end{aligned} \tag{A.22}$$

where we utilized the equation of motion to get to the second line. Therefore for $\delta S \neq 0$, $\delta\phi \neq 0$ and $\delta x^\mu = 0$, we reach:

$$\partial_\mu j_a^\mu = -\frac{\delta \mathcal{L}}{\delta \omega^a} \ . \tag{A.23}$$

Now, assuming that we have identified a set of transformations that maintain the invariance of the action, this leads to:

$$\partial_\mu j_a^\mu = 0 \ . \tag{A.24}$$

Expressing this in terms of the complete variation of the action, we obtain:

$$\begin{aligned} 0 &= \int_{\tau_1}^{\tau_2} d\tau \ \int_{-\infty}^{+\infty} dx^3 \ \partial_\mu j_a^\mu \ , \\ &= \int_{\tau_1}^{\tau_2} dx^0 \ \frac{\partial}{\partial x^0}\int_{-\infty}^{+\infty} d^3x \ j_a^0 + \int_{\tau_1}^{\tau_2} dx^0 \ \int_{-\infty}^{+\infty} d^3x \ \partial_i j_a^i(x) \ , \\ &= \int_{\tau_1}^{\tau_2} dx^0 \ \frac{\partial}{\partial x^0}\int_{-\infty}^{+\infty} d^3x \ j_a^0 + \int_{\tau_1}^{\tau_2} dx^0 \ \oint_\sigma j_a^i(x) d\sigma_i \ , \end{aligned} \tag{A.25}$$

if we choose the suitable boundary conditions, the surface term vanishes

$$\oint_\sigma j_a^i(x) d\sigma_i = 0 \ . \tag{A.26}$$

Continuing the calculations, we obtain:

$$\begin{aligned} 0 &= \int_{\tau_1}^{\tau_2} dx^0 \ \frac{\partial}{\partial x^0}\int_{-\infty}^{+\infty} d^3x \ j_a^0 \ , \\ &= \int_{-\infty}^{+\infty} d^3x \ j_a^0\left(\tau_2, x\right) - \int_{-\infty}^{+\infty} d^3x \ j_a^0\left(\tau_1, x\right) \ . \end{aligned} \tag{A.27}$$

Finally, we can define an arbitrary charge

$$Q_a(\tau) \equiv \int_{-\infty}^{+\infty} d^3x j_a^0(\tau, \vec{x}) \ , \tag{A.28}$$

which is conserved

$$\frac{dQ_a}{dt} = 0 \ . \tag{A.29}$$

At last, from $\delta S = 0$, we have derived the existence of conserved charges.

In the realm of quantum theory, the direct reliance on classical equations of motion fades away. Instead, the concept of current conservation transitions into the profound framework of Ward's identities, offering a nuanced perspective in quantum dynamics.

Noether's theorem, despite this shift, retains its elegance and power. It seamlessly links the inherent symmetries of a physical system to its conserved quantities. This fundamental principle finds wide applications in contemporary physics, spanning from the intricate realms of particle physics to the vast cosmic landscapes explored in cosmology.