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ESTABILIDADE PARA O OPERADOR LAPLACIANO + MÚLTIPLO DA CURVATURA ESCALAR EM VARIEDADES WARPED PRODUCT

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The mind that opens to new ideas
never returns to its original size.

Albert Einstein

Resumo

Nesse trabalho estudamos a estabilidade da família de operadores $L_a = \Delta - aS$, $a \in \mathbb{R}$, em produtos warpeds de um intervalo infinito da reta real por uma ou mais variedades compactas, onde Δ é o operador de Laplace-Beltrami e S é a curvatura escalar da variedade resultante. Na segunda parte deste trabalho, estudamos esses operadores em superfícies mínimas em $\mathbb{R}^3$, abordando alguns resultados relacionados aos operadores L_a.

Palavras-chave: estabilidade; produto warped; superfície mínima.

Abstract

In this work we studied the stability of the family of operators $L_a = \Delta - aS$, $a \in \mathbb{R}$, in warped products of an infinite interval of the real line by one or more compact manifolds, where Δ is the Laplace-Beltrami operator and S is the scalar curvature of the resulting manifold. In the second part of this work, we study these operators on minimal surfaces in $\mathbb{R}^3$, addressing some results related to the L_a operators.

Keywords: stability; warped product; minimal surface.

Contents

Chapter 1

Introduction

Let M^n be a complete Riemannian manifold of dimension $n \geq 2$, consider the operator $\Delta - q$, where Δ is the rough Laplacian of M and $q : M \to \mathbb{R}$ is a smooth function. D. Fischer-Colbrie and R. Schoen [FS80] did a study about this type of operator and concluded that the existence of a positive function f on M satisfying $\Delta f - qf = 0$ is equivalent to the condition that the first eigenvalue of $\Delta - q$ under the Dirichlet boundary condition is positive in each limited domain of M.

We say that an operator $L = \Delta - q$ is *stable* if L satisfies

$$\int_M -fLf \geq 0 \tag{1.1}$$

for all $f \in C_c^\infty(M)$. L is said to be *unstable* when L is not stable. The inequality (1.1) is equivalent to

$$\int_M |\nabla f|^2 + qf^2 \geq 0 \tag{1.2}$$

for all $f \in C_c^\infty(M)$. Using arguments of approximation in H^1 norm, the space of test functions for (1.2) can be replaced by the space $C_c^{0,1}(M)$ of Lipschitz functions of compact support in M.

Let $D \subset M$ be a bounded domain. According to the theory of elliptical equations, the operator $\Delta - q$ acting on functions with Dirichlet boundary conditions of D has a discrete spectrum $\lambda_1^{(D)} < \lambda_2^{(D)} \leq \lambda_3^{(D)} \leq \ldots$ of eigenvalues. The usual characterization of the first eigenvalue of $\Delta - q$ on D is

$$\lambda_1^{(D)} = \inf\left\{\int_D |\nabla f|^2 + qf^2;\ \ f \in C^\infty(M),\ \ \mathrm{supp} f \subset D, \int_D f^2 = 1\right\}.$$

We can conclude that an operator $L = \Delta - q$ is stable if and only if the first eigenvalue of L is positive on each bounded domain under the Dirichlet boundary condition.

Let S be the scalar curvature of M. We can consider the family of operators $L_a := \Delta - aS$, $a \in \mathbb{R}$. For some values of a, there are interesting geometric properties and results, for example:

(i) When $a = 0$, we have the usual Laplace operator.

(ii) When $a = \frac{n-2}{4(n-1)}$, $n \geq 3$, the operator L_a is related to the Yamabe operator Y of M by the relationship $Y = \frac{1}{a}L_a$. In particular, if the first eigenvalue of L_a is negative, there is a metric in M conformal to the original metric of M that has constant positive scalar curvature. If the first eigenvalue is zero, then the metric will have constant nonnegative scalar curvature [LM23].

(iii) When $a = \frac{1}{4}$, the operator L_a appears in Perelman's work on three dimensional Ricci flow with surgery [Per02; Per03].

(iv) When $a = \frac{1}{2}$, $n \geq 3$ and the first eigenvalue of L_a is negative (resp. nonpositive), there is an isometric immersion of M in a manifold N of positive (resp. nonnegative) scalar curvature such that M becomes a two-sided stable minimal hypersurface. More precisely, N is diffeomorphic to $M \times \mathbb{S}^1$ and L_a is the Jacobi operator, referring to the formula for the second variation of the area of M [LM23].

(v) When $a = 1$, the operator L_a becomes the Jacobi operator of the second variation of the area of a minimal hypersurface of a flat space.

(vi) It follows from the definition that, if the first eigenvalue of L_a is negative (resp. nonpositive) for all $a > 0$, the scalar curvature of M is positive (resp. nonnegative). Similarly, if the first eigenvalue of L_a is negative (resp. nonnegative) for all $a < 0$, the scalar curvature of M is negative (resp. nonpositive).

The main goal of this work is to study the stability of the operator L_a in certain types of Riemannian manifolds M, as well as some properties obtained at M from the results to be addressed. As a convention, we say that a Riemannian manifold M is *a-stable* if the operator L_a is stable.

1.1 Case $n = 2$

The case $n = 2$ is a special case of the general case, as the scalar curvature is twice the sectional curvature, which is simply the Gaussian curvature. Hence, the sectional and scalar curvatures are equivalent objects.

1.1.1 Preliminaries results for case $n = 2$

When $dim(M) = 2$, the family of operators becomes $L_a = \Delta - 2aK$, where K denotes the Gaussian curvature of M. A special case studied by Kawai in [Kaw88] is when M has nonnegative Gaussian curvature. He proved:

Theorem [Kawai 1984] *Let M be an oriented complete noncompact bidimensional Riemannian manifold of nonpositive Gaussian curvature K, where K is not identically zero. Suppose that $a > \frac{1}{8}$, then there is a function f of compact support that satisfies the inequality*

$$\int_M |\nabla f|^2 + 2aKf^2 < 0.$$

In particular, if M has nonpositive Gaussian curvature and is a-stable for some $a > \frac{1}{8}$, then M is flat, that is, its Gaussian curvature is identically zero. Furthermore, under the condition $K \leq 0$, is trivial in that M is a-stable for all $a \leq 0$. With that, if the Gaussian curvature of M is nonpositive, we have an "interval of stability," being it $(-\infty, 0]$ and an "interval of instability," being it $(\frac{1}{8}, \infty)$. Thus, there is an interval of uncertainty regarding the stability of L_a for a two-dimensional manifold of nonpositive Gaussian curvature, being it $(0, \frac{1}{8}]$. In fact, the flat plane is a-stable for all $a > 0$; the minimal two-dimensional catenoid in $\mathbb{R}^3$ is a-unstable for all $a > 0$ and the hyperbolic plane is a-stable for $a \leq \frac{1}{8}$ and a-unstable for $a \geq \frac{1}{8}$, because the hyperbolic space has Gaussian curvature equal to -1 and its first eigenvalue of the Laplacian is $-\frac{1}{4}$. Still in dimension two, let $0 < a \leq \frac{1}{4}$ be a real number, the general idea is that, for a manifold M of negative Gaussian curvature to be a-stable, it is necessary that M has a certain volume growth of geodesic balls. This growth would be of increasing order in values of a. For example, the plane with the metric $dr^2 + r^2(\log(r+e))^2 d\theta^2$ has negative Gaussian curvature, it is a-unstable for all $a > 0$ and the geodesic balls of radio R have area in order of $R^2 \log(R)$. The hyperbolic plane is $\frac{1}{4}$-stable and the geodesic balls of radios R have area $2\pi(\cosh^2(R) - 1)$. Bérard and Castillon [BC14] found the following pattern:

Theorem [Bérard-Castillon 2010] *Let (M, g) be a complete noncompact Riemannian surface and let W be a locally integrable function on M, with $W+$ integrable. Assume that the operator $\Delta + aK + W$ is nonnegative on M and that*

(i) $a \in (\frac{1}{4}, \infty)$, *or*

(ii) $a = \frac{1}{4}$, *and (M, g) has subexponential volume growth, or*

(iii) $a \in (0, \frac{1}{4})$, and (M, g) has k_a*-subpolynomial volume growth, with* $k_a = \frac{2+4a}{1-4a}$.

Then:

(A) The surface (M, g) has finite topology and at most quadratic volume growth. In particular, (M, g) is conformally equivalent to a closed Riemannian surface with finitely many points removed.

(B) The function W is integrable on (M, g), and

$$0 \leq 2\pi a \chi(M) + \int_M W.$$

(C) If $2\pi a\chi(M) + \int_M W = 0$, then (M, g) has subquadratic volume growth, and $aK + W \equiv 0$ a.e. on the surface M.

This theorem guarantee that if $\alpha \geq 1$ and (M^2, g) has an $(\alpha + 1)$ polynomial volume growth of the area, that is, there exists $0 < C_1 \leq C_2 < \infty$ such that $C_1 \leq \frac{|B_R(p)|}{R^{\alpha+1}} \leq C_2$ for all $R > 0$, then M cannot be β-stable for all $\beta > \frac{\alpha-1}{8\alpha}$. On the other hand, there are examples of $\frac{\alpha-1}{8\alpha}$-stable manifolds satisfying $|B_R| = O(R^{\alpha+1})$, as exemplified in work [BC14] by the same author and in Proposition 3.4.1. Another important work to be mentioned regarding these operators is that of Espinar and Rosenberg in [ER11].

Suppose that M is a complete bi-dimensional Riemannian manifold with nonpositive

Gaussian curvature. To check whether M is a-stable, as mentioned in [Kaw88], we only need to check if the universal cover of M is a-stable. Hence, we assume that M is diffeomorphic to $\mathbb{R}^2$. In this case, the metric ds^2 of M can be written as $ds^2 = dr^2 + \rho(r,\theta)^2 d\theta^2$ using polar coordinates centered at any given point.

1.1.2 Application: a-stability of minimal surfaces on $\mathbb{R}^3$

For a given minimal surface M of $\mathbb{R}^3$, the Jacobi operator associated with the second variation of area is $L = \Delta - 2K$, corresponding to L_1 in our family of operators L_a. A minimal surface is said to be stable when L is a stable operator. For an operator L on a Riemannian manifold M, we define the *Index* of M as the dimension of the greatest subspace of $C_c^\infty(M)$ such that L is negatively defined on this subspace, where the index of an operator can be ∞. Hence L is stable if and only if $\text{Index}(L) = 0$ and L has finite index if and only if there exist a compact subset K of M such that L is stable on $M \setminus K$. Naturally for a minimal surface M in $\mathbb{R}^3$ we define the index of M, denoted by $\text{Index}(M)$, as being $\text{Index}(L)$ on M. Many results regarding the index of a minimal surface in $\mathbb{R}^3$ are known, including the following:

- The flat plane has zero index and is characterized by this. [FS80], [CP79] and [Pog81].
- The catenoid and the minimal Enneper surface have index one [Fis85];
- The Costa-Hoffman-Meeks surface family of genus $g \geq 1$ has index $2g + 3$ [Nay92; Mor09];
- The Jorge-Meeks surface with r ends has index $2r - 3$ [MR06; Nay90a].

(1.3)

Naturally, we define the *a-Index* of a minimal surface M of $\mathbb{R}^3$ as the Index of L_a on M. This is one of our objectives of this study, which is covered in Chapter 4. Barbosa and do Carmo [BC76] state the following:

Theorem [Barbosa-do Carmo 1976] *Let M be a minimal surface and g be the Gauss map of M. If the area of the spherical image $g(D) \subset \mathbb{S}^2$ of a domain $D \subset M$ is smaller than 2π, then D is stable.*

For $a \in [0, \infty)$, let z_a is the number in the interval $(-1, 1)$ with the following property: the spherical cap

$$\mathbb{S}^2(-1, z_a) = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1, -1 \leq z \leq z_a\}$$

has $2a$ as the first eigenvalue of the Laplacian. In particular, $z_1 = 0$ and $z_0 = 1$. The area of the spherical caps $\mathbb{S}^2(-1, z_a)$ is $2\pi(z_a + 1)$. As we will see, the area of the Gauss map of the minimal surface being smaller than the area of the spherical cap $\mathbb{S}^2(-1, z_a)$ in $\mathbb{R}^3$ guarantees a-stability. we have:

Proposition 4.2.2: *Let M be a minimal surface such that the area of the image of Gauss map is less than $2\pi(z_a + 1)$, then M is a-stable.*

Another objective of our study is to show that, given a complete minimal surface with finite total curvature, the operator L_a has exactly one negative eigenvalue for all sufficiently small a.

Theorem 4.2.1: *Let M be a complete non flat minimal surface of finite total curvature, then*

$$\lim_{a \to 0^+} Index_a(M) = 1.$$

The Theorem 4.2.1 shows us that for every minimal surface M of finite total curvature in $\mathbb{R}^3$, there is a sufficiently small $a > 0$ such that $\text{Index}_a(M) = 1$. From the work of W. Meeks, J. Pérez and A. Ros [MPR06], the universal covering of a doubly periodic Scherk minimal surface is a-stable for some $a > 0$, which shows that the hypothesis of M having finite total curvature is necessary to guarantee $\lim_{a \to 0^+} \text{Index}_a(M) = 1$. By the same work, the doubly periodic Scherk minimal surface is a-unstable for all $a > 0$. In particular, its a-index is ∞.

A question that arises is: for a given $a > 0$, it is possible to determine the geometric or topological characteristics necessary or sufficient for a minimal surface M of finite total curvature such that, with them, M can or cannot have a-index one. Another way of thinking would be that: given a complete minimal surface M of finite total curvature with some previously defined geometric or topological characteristics, it is possible to obtain $a > 0$ in which the a-index of M is necessarily one or $a > 0$ such that the a-index of M cannot be one. For $a > 1$, there is no complete minimal surface M of finite total curvature such that $\text{Index}_a(M) = 1$, because if M is a non flat complete minimal surface, zero is an eigenvalue of the operator $L_1 = \Delta - 2K$ of multiplicity at least three (see [Nay93]). In particular, if $a > 1$, either M is a plane or $\text{Index}_a(M) \geq 4$. For $a = 1$, by Lopez and Ros [LR89], the catenoid and the minimal Enneper surface are the only complete minimal surfaces in $\mathbb{R}^3$ with index one. By Example 4.2.1, the Jorge-Meeks minimal surface of r ends has a-index one for all $a \leq \frac{r+1}{2r^2}$. Therefore, the number of ends would be a possible topological characteristic for M to characterize a value for a related to it. Another topological characteristic that can be considered naturally is the genus of M. We conjecture:

Conjecture 1.1.1. *Let M be a complete minimal surface in $\mathbb{R}^3$ of genus g and r ends. Given $a > 0$, suppose $Index_a(M) = 1$, then there exists $C > 0$ and $D > 0$ such that $g \leq Ca^{-1}$ and $r \leq Da^{-1}$.*

Another result that we conjecture is similar to that of O. Chodosh and D. Maximo in [CM16]:

Theorem [Chodosh-Máximo 2014] Suppose that $M \to \mathbb{R}^3$ is an immersed complete two-sided minimal surface of genus g and with r ends. Then

$$\text{Index}(M) \geq \frac{2}{3}(g + r) - 1.$$

We conjecture a similar result for a-indexes:

Conjecture 1.1.2. *Let M be a complete minimal surface of finite total curvature in $\mathbb{R}^3$ of genus g and r ends and $a > 0$, then there exist positive constants J and K such that*

$$Index_a(M) \geq (Jg + Kr)a.$$

Remark 1.1.1. *Fischer-Colbrie in [Fis85] demonstrated that a minimal surface in $\mathbb{R}^3$ having a finite index (usual a-index for $a = 1$) is equivalent to having a finite total curvature. For the a-index, she implicitly demonstrated that this property is valid for all $a \geq \frac{1}{2}$, but the previously mentioned examples of minimal surfaces in $\mathbb{R}^3$ a-stable for $a > 0$ are* *indicated by Proposition 4.2.1, which have infinite total curvature. This shows that the finite index property be equivalent the finite total curvature is not valid for all $a > 0$.*

Remark 1.1.2. *The Conjecture 1.1.2 with the Chodosh-Máximo Theorem would imply an interesting property in the set of minimal surfaces of finite total curvature: Let $C > 0$ be an integer, there exists $a > 0$ such that for all minimal surface $M \subset \mathbb{R}^3$ such that $Index_1(M) \leq C$, we have $Index_a(M) = 1$.*

1.2 Case $n \geq 3$ with warped-product metric

A natural question is whether the results of the previous section hold true for higher dimensions. Answering this question, it is not possible to obtain a general result that generalizes these in dimension two. In this work, we show that some of these results can be generalized to a specific class of Riemannian manifolds, namely to warped products of an infinite interval of the line ($[0, \infty)$ or $(-\infty, \infty) = \mathbb{R}$) by one (single warped product case) or more (multiple warped-product case) compact manifolds. Below we define each of these terms, which are discussed in Chapter 2.

Definition 2.0.1. *Let (B^m, g_1) and (F^n, g_2) be Riemannian manifolds, and $\rho : B \to \mathbb{R}$ be a smooth function. We define the* ***warped product of*** B ***and*** F ***with the warping function*** ρ*, denoted by $B \times_\rho F$ as the product manifold $B \times F$ with the metric g defined by*

$$g = g_1 + \rho^2 g_2.$$

Definition 2.2.1 *Let $(B, g_0), (F_1, g_1), \dots, (F_k, g_k)$ be manifolds and $\rho_j : B \mapsto \mathbb{R}^+$, $1 \leq j \leq k$ smooth functions, we can define the manifold $M = B \times_{\rho_1} F_1 \times_{\rho_2} F_2 \times \cdots \times_{\rho_k} F_k$ as being the manifold $B \times F_1 \times \cdots \times F_k$ with the metric g defined by*

$$g = g_0 + \rho_1^2 g_1 + \cdots + \rho_k^2 g_k.$$

We study the stability of the operator $L_a = \Delta - aS$ when M is described as a warped product related to the order of growth of the warping functions. We study for a given growth rate of the warping function, for which the values of a L_a is stable and for which it is unstable. Many results regarding the instability of L_a, as we will see, reduce to analyze the instability of an end E of M, because the growth rate of the warping function determines whether M can or cannot be a-stable. We will also expand these results to include cases involving multiple warped products.

1.2.1 Single warped products

First, we present a result that determines an stability interval of n-dimensional warped products.

Theorem 3.1.1 *Let $M^n = I \times_\rho F^{n-1}$ be a warped product manifold with $n \geq 3$, $I \subset \mathbb{R}$ and F a compact manifold with nonnegative scalar curvature, then M is a-stable for $0 \leq a \leq \frac{n-2}{4(n-1)}$. Consequently, the Yamabe operator $Y := \frac{n-1}{4(n-2)}\Delta - S$ is nonnegative on M, being positive definite if $S(F) > 0$.*

The next theorems states that for certain values of a and under certain conditions of the warping function, in some cases, we can guarantee that a (single) warped product is a-stable or a-unstable. The following theorem resembles the Kawai's Theorem [Kaw88]. More details are provided in Subsection 3.1.1.

Theorem 3.1.2 *Let $M = [0, \infty) \times_\rho F$ be a warped product without boundary such that ρ satisfies $\rho''(r) > 0$ and $\lim_{r\to\infty} \rho'(r) = \infty$. Then the end of M (and therefore, M) is a-unstable for all $a > \frac{n-1}{4n}$. If F has nonpositive total scalar curvature $S(F) = \int_F S_F$ the hypothesis $\lim_{r\to\infty} \rho'(r) = \infty$ can be removed.*

Theorem 3.1.3 *Let $M = I \times_\rho F$ be a smooth warped product such that there exists R_0 such that one of the following two situations occurs:*

(i) $\rho''(r) > 0$ for all $r > R_0$ and $\lim_{r\to\infty} \rho'(r) = \infty$.

(ii) $\rho''(r) > 0$ for all $r < R_0$ and $\lim_{r\to-\infty} \rho'(r) = -\infty$.

Then the corresponding end of M in (i) or (ii) (and therefore, M) is a-unstable for all $a > \frac{n-1}{4n}$.

Theorem 3.1.4 *Let $M = I \times_\rho \mathbb{S}^{n-1}$ be a warped product such that ρ satisfies $|\rho'(r)| \leq C$. Then M is a-stable for all $0 \leq a \leq \frac{(C^2+1)(n-2)}{4C^2(n-1)}$.*

In the case $n = 2$, we saw in Section 1.1 that if in M $C_1 \leq \frac{|B_R|}{R^{\alpha+1}} \leq C_2$, then M is a-unstable for all $a > \frac{\alpha-1}{8\alpha}$. In view of the penultimate paragraph of Subsection 1.1.1, the condition $C_1 \leq \frac{|B_R|}{R^{\alpha+1}} \leq C_2$ can be exchanged, in the case of warped products, as $C_1 r^\alpha \leq \rho(r) \leq C_2 r^\alpha$, where ρ is the warping function. As we will see, in a higher dimension and for warped products, this result follows with the hypothesis $C_1 r^\alpha \leq \rho(r) \leq C_2 r^\alpha$, $\alpha > 1$, where we will conclude that M is a-unstable for all $a > \frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$. Thus obtain an "interval of instability" $\left(\frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}, \infty\right)$. The case covered in the following theorem associates the polynomial growth of ρ with an "instability interval".

Theorem 3.1.5 *Suppose $\alpha > 1$ and $M = I \times_\rho F$ with a metric of the form $g = dr^2 + \rho(r)^2 g_F$. Suppose that there exist positive constants C_1 and C_2 such that $C_1 r^\alpha \leq \rho(r) \leq C_2 r^\alpha$ for all $r > 1$, then M is a-unstable for all $a > \frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$.*

The value $\frac{(n\alpha-\alpha-1)^2)}{4\alpha(n-1)(n\alpha-2)}$ in Theorem 3.1.5 is the best possible because the warped product $M = [0,\infty) \times_\rho \mathbb{S}^{n-1}$ when $\rho(r) = r^\alpha$ is $\frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$-stable (see Proposition 3.4.1). If ρ has α-subpolynomial grown, there is a similar version of the Bérard-Castillon Theorem:

Theorem 3.1.6 *Let $M = I \times_\rho F$ be a warped product, where F is a compact manifold and $\rho(r) = r^\alpha \xi(r)$, where $\alpha \geq 1$ and ξ satisfies $\xi(r) \to 0$ when $r \to \infty$. Suppose that ξ is a non-increasing function and satisfies $r\xi(r)^{-1}\xi'(r) \geq -(\alpha - 1)$ for all r large. Also suppose that M is a-stable for $a = \frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$. Then ρ has linear growth, that is, $\rho(r) \leq Cr$ for some $C \geq 0$. In particular, M has a polynomial volume growth on order of R^n, that is, for each $p \in M$, $Vol(B_R(p)) \leq C_1 R^n$ for some $C_1 > 0$.*

Consider the application

$$\begin{aligned} h : (1,\infty) &\longrightarrow \left(\frac{n-2}{4(n-1)}, \frac{n-1}{4n}\right) \\ \zeta &\longmapsto \frac{(n\zeta-\zeta-1)^2}{4\zeta(n-1)(n\zeta-2)}. \end{aligned}$$

Using elementary analysis tools, it is possible to show that h is an increasing function,

$$\lim_{\zeta\to 1} h(\zeta) = \frac{n-2}{4(n-1)} \quad \text{and} \quad \lim_{\zeta\to\infty} h(\zeta) = \frac{n-1}{4n}.$$

Therefore, there is a continuous and monotonous relationship between the degree of polynomial growth of ρ and the maximum possible value of a for which M can be a-stable.

The next theorem is an improved version of the Theorem 3.1.2 for when we assume that ρ has a polynomial growth.

Theorem 3.1.7 *Let $M = I \times_\rho F$ be a warped product, where F is a compact manifold of dimension at least two and $\rho(r) = r^\alpha \xi(r)$, where α and ξ satisfy*

$$\alpha = \inf \left\{ \gamma; \ \lim_{r\to\infty} \rho(r) r^{-\gamma} = 0 \right\}.$$

Suppose that M is a-stable for some $a > \frac{n-1}{4n}$. Then there exist a positive constant C such that

$$\liminf_{R\to\infty} (\log R)^{-1} \int_1^R r\rho(r)^{-2} dr \geq C.$$

Furthermore, if $S(F) \leq 0$, then $\alpha < \frac{n}{2n-2}$. In particular, M has $(1+\frac{n}{2})$-subpolynomial volume growth, that is,

$$\lim_{r\to\infty} r^{-(1+\frac{n}{2})} Vol(B_r(p)) = 0$$

for all $p \in M$.

The following theorem shows what can happen if the function ξ as in the previous theorem has large variation.

Theorem 3.1.8 *Let $M = I \times_\rho F$, where $\rho(r) = r^\alpha \xi(r)$, where $\alpha \geq 1$ and ξ satisfy (3.6) with the additional condition $C_1 \leq \xi(r) \leq C_2$ for all $r \geq 0$ for some constants $C_1, C_2 > 0$. Suppose that*

$$\lim_{T\to\infty} \frac{logT}{\int_{R_0}^T r\xi(r)^{-2}\xi'(r)^2 dr} = 0,$$

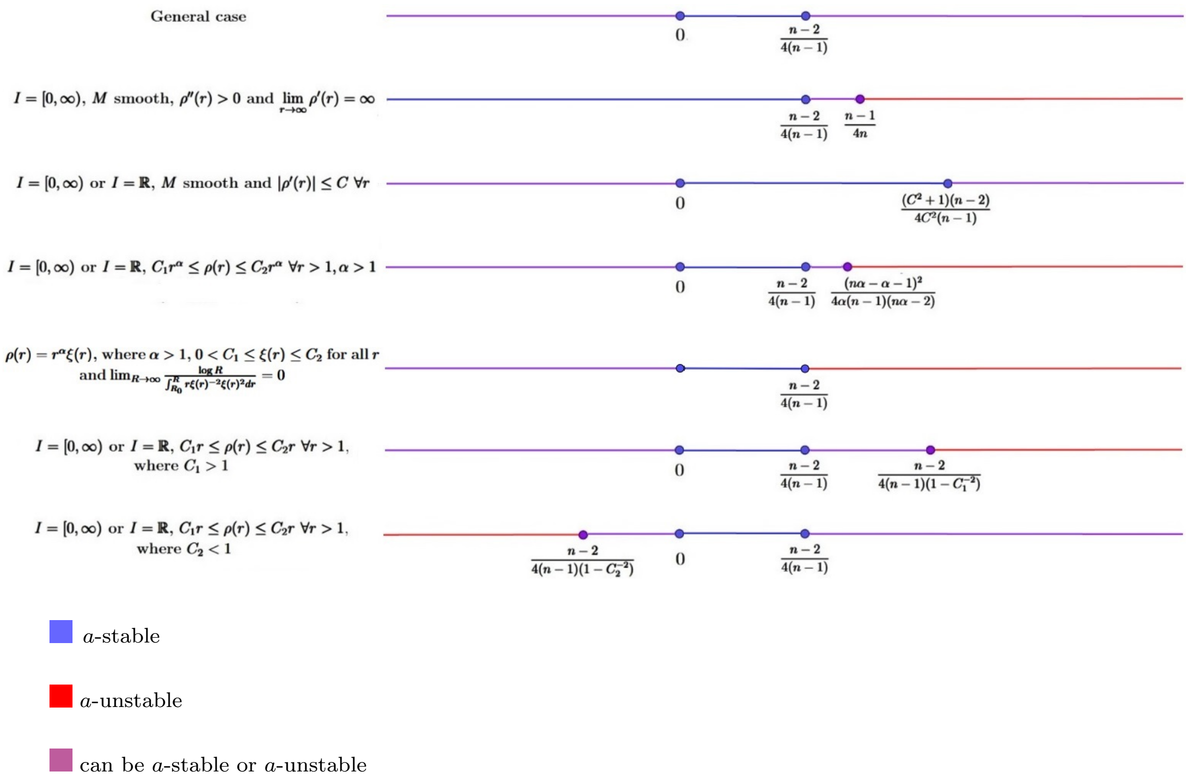
where R_0 is a fixed positive number. Then M is a-unstable for all $a > \frac{n-2}{4(n-1)}$.

In Theorem 3.1.5 we assumed that $\alpha > 1$. The following results exemplify what can happen if $\alpha = 1$, serving as an object of comparison with the previous result in which the assumption $\alpha > 1$ was required.

Theorem 3.1.9 *Let $M = I \times_\rho \mathbb{S}^{n-1}$ be a warped product, where $I = \mathbb{R}$ or $I = [0, \infty)$. Suppose that $C_1 r \leq \rho(r) \leq C_2 r$ $\forall r \geq 1$ for the positive constants C_1, C_2. If $C_1 > 1$, then (the ends of) M is a-unstable for all $a > \frac{n-2}{4(n-1)(1-C_1^{-2})}$.*

Theorem 3.1.10 *Let $M = I \times_\rho \mathbb{S}^{n-1}$ be a warped product, where $I = \mathbb{R}$ or $I = [0, \infty])$. Suppose that $C_1 r \leq \rho(r) \leq C_2 r$ $\forall r \geq 1$ for the positive constants C_1, C_2. If $C_2 < 1$, then (the ends of) M is a-unstable for all $a < \frac{n-2}{4(n-1)(1-C_2^{-2})}$.*

Thus, based on these results, we obtain the following diagram, relating the a-stability of a smooth manifold $M = I \times_\rho \mathbb{S}^{n-1}$ with the possible growth characteristics of the ρ function:



1.2.2 Multiple warped products

For multiple warped products, we have the following results.

(Theorem 3.2.1) Let $M = I \times_{\rho_1} F_1 \times_{\rho_2} \cdots \times_{\rho_k} F_k$ be a multiple warped product, with each F_i compact with nonnegative scalar curvature. Suppose that for any $1 \leq i, j \leq k$ and $r \in I$, $\rho_i'(r)\rho_j'(r) \geq 0$, then M is a-stable for all $0 \leq a \leq \min_{1\leq i\leq k}\{\frac{n_i-1}{4n_i}\}$.

(Theorem 3.2.2) Let $M = \mathbb{R} \times_{\rho_1} F_1 \times_{\rho_2} \cdots \times_{\rho_k} F_k$ be a multiple warped product, n_i the dimension of F_i and $n = n_1 + \cdots + n_k$. Suppose that, for each ρ_i, there exists R_0 such that one of the following two situations occur for:

(i) $\rho_i''(r) > 0$ for all $r > R_0$ and $\lim_{r\to\infty} \rho_i'(r) = \infty$;

(ii) $\rho_i''(r) > 0$ for all $r < R_0$ and $\lim_{r\to-\infty} \rho_i'(r) = -\infty$.

Then the corresponding end of M (and therefore, M) is a-unstable for all $a \geq \frac{n-1}{4n}$.

1.2.3 Applications: a-stability of minimal hypersurfaces on Euclidean spaces

There are three typical examples of minimal hypersurfaces in Euclidean spaces that can be written as a warped product.

The first is any affine hyperplane $H \subset \mathbb{R}^{n+1}$, which geometrically is the warped product $\mathbb{R} \times_\rho \mathbb{S}^{n-1}$ where $\rho(r) = r$, having the topology of $\mathbb{R}^n$, being stable as a minimal hypersurface of $\mathbb{R}^{n+1}$ and, more than it, has the property of minimizing the area for any variation of compact support. When $n \leq 5$, the affine hyperplane is the only minimal hypersurface which is the graph of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ (the problem of determining whether the graph of a function of $\mathbb{R}^n$ is a minimal hypersurface on $\mathbb{R}^{n+1}$ is known as the *Bernstein problem*. For more details, see [Ber17], [Alm66] and [Cho21]). According to our notation, because H is a flat manifold, it is a-stable for all $a \in \mathbb{R}$.

The second example is the minimal cones in $\mathbb{R}^{n+1}$, characterized as being a warped product $[0, \infty) \times_\rho F$, where $\rho(r) = r$ and F is a minimal submanifold of $\mathbb{S}^n$. A particular case is the Simons cones, introduced by J. Simons on [Sim68], and also worked by E. Bombieri, E. De Giorgi, and E. Giusti on [BDG69]. They are characterized as the singular hypersurface of $\mathbb{R}^{2m}$: $x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2$. These cones can be written as the warped product $[0, \infty) \times_\rho \left(\frac{1}{\sqrt{2}}\mathbb{S}^{n-1} \times \frac{1}{\sqrt{2}}\mathbb{S}^{n-1} \right)$, where $\rho(r) = r$ and is a minimal hypersurface of $\mathbb{R}^{2m}$, more precisely, these cones are $\frac{(2m-3)^2}{8(m-1)}$-stable according to our definition of a-stability. Note that $\frac{(2n-3)^2}{8(n-1)} > 1$, if $n \geq 4$. In particular, a Simons cone is a minimal stable hypersurface of $\mathbb{R}^{2n}$ if and only if $2n \geq 8$. In Subsection 3.3.1, we show that if $C = [0, \infty) \times_\rho F$ is a n-dimensional cone in $\mathbb{R}^{n+1}$, where $F = \mathbb{S}^m_{\lambda_1} \times \mathbb{S}^{n-m-1}_{\lambda_2}$ is a minimal hypersurface of $\mathbb{S}^n$, then C is $\frac{(n-2)^2}{4(n-1)}$-stable. In particular, it follows the previous statements about Simons cones, where $n = 2m - 1$. Furthermore, the next proposition shows that when the dimension n is at least seven, the value for $a = \frac{(n-2)^2}{4(n-1)}$ is the best possible value for the existence of a cone a-stable in $\mathbb{R}^{n+1}$.

Theorem 3.3.1 *An a-stable minimal cone C^n of $\mathbb{R}^{n+1}$, with $a > \max\left\{1, \frac{(n-2)^2}{4(n-1)}\right\}$, is flat.*

Simons in [Sim68] has shown that a minimal stable cone $C^n \subset \mathbb{R}^{n+1}$ is flat if $n \leq 6$. The Theorem 3.3.1 and the previous examples of the Simons cone shows that, for $n \geq 7$, the value $\frac{(n-2)^2}{4(n-1)}$ is the greatest possible value for a such that a n-dimensional minimal cone in $\mathbb{R}^{n+1}$ can be a-stable.

The third example is the n-dimensional catenoid in $\mathbb{R}^{n+1}$. In dimension two, the catenoid can be parameterized by $x = c\cosh(v/c)\cos u$, $y = c\cosh(v/c)\sin(u)$, $z = v$, where $c \neq 0$, $v \in \mathbb{R}$ and $y \in [0, 2\pi)$ and is characterized by being a minimal surface in $\mathbb{R}^3$ of revolution (different from plane) and index one (see [TZ09]), which can be written as a warped product $\mathbb{R} \times_\rho \mathbb{S}^1$. In higher dimensions, the catenoid is a minimal hypersurface of $\mathbb{R}^{n+1}$ that inherits certain properties from the two-dimensional catenoid. For example, it is diffeomorphic to the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ and has index one. Perhaps the main difference for dimension two is that in higher dimensions, the catenoid is limited in one of the directions of $\mathbb{R}^{n+1}$. The n-dimensional catenoid is $\frac{n-2}{n}$-stable (see [TZ09]). For more details, see Subsection 3.3.2.

1.3 Structure of the thesis

This thesis is organized into four chapters. The first is the introductory chapter. Chapter 2 provides a brief introduction to the theory of warped products, where we will discuss the metric, the Levi-Civita connection and find formulas for the sectional and scalar curvatures in terms of the manifolds involved in the warped product and the warping function. The Chapter 3 is divided into three sections, the main subject of which is the stability of L_a operators in warped products. The Section 3.1 addresses the results related to a simple warped product of a line interval by a compact Riemannian manifold. The Section 3.2 addresses the results related to a multiple warped product of a line interval by a certain number of compact Riemannian manifolds. The Section 3.3 addresses the hypercones and catenoid, which are examples of warped-product minimal hypersurfaces in $\mathbb{R}^{n+1}$. The Section 3.4 deals with the proof of the theorems in Section 3.1 and Section 3.2. Finally, in Chapter 4, we discuss the operators L_a on minimal surfaces of $\mathbb{R}^3$ that have finite total curvature.

Chapter 2

Warped products, an introduction

Let B and F be manifolds. A natural way to construct a manifold from B and F is by the Cartesian product $B \times F$. If B and F are equipped with Riemannian metrics g_B and g_F, the product metric is the metric g given by $g(x, y) = g_B(x) + g_F(y)$, where $x \in B$ and $y \in F$. The product metric is just one example (the most natural) of a metric that we can define in $B \times F$. Another way to build metrics in $B \times F$ is from the combination $\sigma g_B + \rho g_F$, where $\sigma : F \to \mathbb{R}^+$ and $\rho : B \to \mathbb{R}^+$ are positive smooth functions. In particular, the product metric is the metric obtained when the functions σ and ρ are constant and equal to one. When we require that only $\rho : B \to \mathbb{R}$ be the constant function equal to one, as we will see, we will have the class of *warped products* of B by F. Bishop and O'Neill introduced the notion of warped product in 1964 [BO69].

Definition 2.0.1. *Let (B^m, g_1) and (F^n, g_2) be Riemannian manifolds, and $\rho : B \to \mathbb{R}$ be a smooth function. We define the* ***warped product of*** B ***and*** F ***with the warping function*** ρ*, denoted by $B \times_\rho F$ as the product manifold $B \times F$ with the metric g defined by*

$$g = g_1 + \rho^2 g_2.$$

For a warped product $B \times_\rho F$, B is called the *base* of the warped product and F is called the *fiber*. Being $\pi_1 : B \times F \to B$ and $\pi_2 : B \times F \to F$ the natural projections, the leaves $B \times \{q\}$ and the fibers $\{p\} \times F$ are submanifolds of $B \times_\rho F$. Vectors tangent to leaves are called *horizontal vectors* and those tangent to fibers are called *vertical vectors*.

For $p \in B$ and $q \in F$, denoted by $\mathcal{H}$ the orthogonal projection of $T_{(p,q)}(B \times F)$ onto its horizontal subspace $T_{(p,q)}(B \times \{q\})$ and by $\mathcal{V}$ the orthogonal projection onto its vertical subspace $T_{(p,q)}(\{p\} \times F)$. If $v \in T_pB$, $p \in B$ and $q \in F$, we define the *lift* of v to (p,q) as the unique vector $\bar{v}$ in $T_{(p,q)}(\{p\} \times F)$ such that $(\pi_1)_*(\bar{v}) = v$. For a given $X \in \mathfrak{X}(B)$, the lift of X to M is the vector field $\overline{X}$ such that $\overline{X}_{(p,q)}$ is the lift of X_p to (p,q). The set of the horizontal lifts is denoted by $\mathcal{L}(B)$. Similarly, we define the set of vertical lifts by $\mathcal{L}(F)$.

2.1 Connection and curvature in warped products

Being $M = B \times_\rho F$, we denote the metric of M by $\langle \cdot, \cdot \rangle$, the Levi-Civita connection of M by ∇, the Levi-Civita connection of B by ∇^B, the Levi-Civita connection of F by ∇^F, the curvature tensor of B by R_B and the curvature tensor of F by R_F. The next proposition addresses the relationship between the curvature tensor of $M = B \times_\rho F$ and the curvature tensors of the base B and fiber F. It can be seen that the Lie bracket satisfies for $X_1, X_2 \in \mathcal{L}(B)$ and $Y_1, Y_2 \in \mathcal{L}(F)$: $[\overline{X}_1, \overline{X}_2] = \overline{[X_1, X_2]}$, $[\overline{Y}_1, \overline{Y}_2] = \overline{[Y_1, Y_2]}$ and $[\overline{X}_1, \overline{Y}_1] = 0$.

In our calculations, we will consider the function ρ extended on all manifold M, so that ρ is constant on the fibers.

Proposition 2.1.1. *For $X_1, X_2 \in \mathcal{L}(B)$, $Y_1, Y_2 \in \mathcal{L}(F)$, we have:*

(i) $\nabla_{X_1} X_2 \in \mathcal{L}(B)$ is the lift of $\nabla^B_{X_1} X_2$ on M;

(ii) $\nabla_{X_1} Y_1 = \nabla_{Y_1} X_1 = \frac{\langle \nabla\rho, X_1 \rangle}{\rho} Y_1$;

(iii) $(\nabla_{Y_1} Y_2)^\perp = \sigma(Y_1, Y_2) = -\frac{\langle Y_1, Y_2 \rangle}{\rho} \nabla\rho$;

(iv) $(\nabla_{Y_1} Y_2)^T \in \mathcal{L}(F)$ is the lift of $\nabla^F_{Y_1} Y_2$ on M;

Proof. By Koszul formula, for all $X, Y, Z \in (M)$:

$$2\langle \nabla_X Y, Z \rangle = X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle - \langle [X,Y], Z \rangle - \langle [X,Z], Y \rangle - \langle [Y,Z], X \rangle;$$

(i) Using Koszul formula with $[X_1, Y_1] = [X_2, Y_1] = 0$, we obtain

$$2\langle \nabla_{X_1} X_2, Y_1 \rangle = \langle Y_1, [X_1, X_2] \rangle - Y_1 \langle X_1, X_2 \rangle.$$

Since $\langle X_1, X_2 \rangle$ is constant on the fibers, we have $Y_1 \langle X_1, X_2 \rangle = 0$. Since $[X_1, X_2]$ is a horizontal field, we have $\langle Y_1, [X_1, X_2] \rangle = 0$. Therefore, $\langle \nabla_{X_1} X_2, Y_1 \rangle = 0$ for all $Y_1 \in \mathcal{L}(F)$, it shows that $\nabla_{X_1} X_2$ is horizontal. Furthermore, for all $q \in F$, $\pi_1^q : B \times \{q\} \to B$ is an isometry, it implies that $\nabla_{X_1} X_2 \in \mathcal{L}(B)$ is the lift of $\nabla^B_{X_1} X_2$ to M.

(ii) First, $\nabla_{X_1} Y_1 = \nabla_{Y_1} X_1$ because $[X_1, Y_1] = 0$. Using Koszul formula, we have

$$\langle \nabla_{X_1} Y_1, X_2 \rangle = Y_1 \langle X_1, X_2 \rangle - \langle [X_1, X_2], Y_1 \rangle = 0.$$

Therefore, $\nabla_{X_1} Y_1$ is a vertical field. Still using the Koszul formula,

$$2\langle \nabla_{X_1} Y_1, Y_2 \rangle = X_1 \langle Y_1, Y_2 \rangle - \langle [Y_1, Y_2], X_1 \rangle = X_1 \langle Y_1, Y_2 \rangle.$$

Since $\langle Y_1, Y_2 \rangle_{(p,q)} = \rho^2(p) \langle Y_1, Y_2 \rangle_F$, we have

$$2\langle \nabla_{X_1} Y_1, Y_2 \rangle = X_1(\rho^2 \circ \pi_1) \langle Y_1, Y_2 \rangle_F = 2\rho \langle \nabla \rho, X_1 \rangle \langle Y_1, Y_2 \rangle_F = 2 \frac{\langle \nabla \rho, X_1 \rangle}{\rho} \langle Y_1, Y_2 \rangle$$

for all vertical field Y_2. Therefore, $\nabla_{X_1} Y_1 = \frac{\langle \nabla \rho, X_1 \rangle}{\rho} Y_1$.

(iii) By derivative of product and property (ii), we have

$$\langle \nabla_{Y_1} Y_2, X_1 \rangle = -\langle \nabla_{Y_1} X_1, Y_2 \rangle = -\frac{\langle \nabla \rho, X_1 \rangle}{\rho} \langle Y_1, Y_2 \rangle.$$

Being $\{e_1, \ldots, e_l\}$ an orthonormal frame on B and $\{\bar{e}_1, \ldots, \bar{e}_l\}$ our lifting, we have

$$(\nabla_{Y_1} Y_2)^{\perp} = \sum_{i=1}^{l} \langle \nabla_{Y_1} Y_2, \bar{e}_i \rangle \bar{e}_i = -\sum_{i=1}^{l} \frac{\langle \nabla \rho, \bar{e}_i \rangle}{\rho} \langle Y_1, Y_2 \rangle \bar{e}_i = -\frac{\langle Y_1, Y_2 \rangle}{\rho} \nabla \rho.$$

(iv) Since Y_1, Y_2 is tangent to fibers, $\nabla_{Y_1} Y_2$ is the fiber covariant derivative applied to restrictions of Y_1 and Y_2 in that fiber.

■

Remark 2.1.1. *It follows of Proposition 2.1.1 that the geodesics of B will be preserved, in the sense that if $\gamma : I \to B$ is a geodesic of B, then $\gamma_y : I \to B \times_\rho F$ given by $\gamma_y(t) = (\gamma(t), y)$, $y \in F$, is a geodesic in $B \times_\rho F$.*

Proposition 2.1.2. *Let $M = B \times_\rho F$ be a warped product of two Riemannian manifolds and R be the curvature tensor of M. If $X_1, X_2, X_3 \in \mathcal{L}(B)$ and $Y_1, Y_2, Y_3 \in \mathcal{L}(F)$, then:*

(i) $R(X_1, X_2)X_3 = R_B(X_1, X_2)X_3$*;*

(ii) $R(X_1, Y_1)X_2 = \frac{D^2\rho(X_1,X_2)}{\rho}Y_1$*;*

(iii) $R(X_1, X_2)Y_1 = R(Y_1, Y_2)X_1 = 0$*;*

(iv) $R(X_1, Y_1)Y_2 = -\frac{\langle Y_1,Y_2\rangle}{\rho}\nabla_{X_1}\nabla\rho$*;*

(v) $R(Y_1, Y_2)Y_3 = R_F(Y_1, Y_2)Y_3 + \frac{|\nabla\rho|^2}{\rho^2}(\langle Y_1, Y_3\rangle Y_2 - \langle Y_2, Y_3\rangle Y_1)$*.*

Where ∇ is the connection of M and $D^2\rho$ is the Hessian of ρ in B.

Proof. Using the relations of Proposition 2.1.1, we have:

(i)

$$\begin{aligned} R(X_1, X_2)X_3 &= \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1,X_2]}X_3 \\ &= \nabla_{X_1}\nabla^B_{X_2}X_3 - \nabla_{X_2}\nabla^B_{X_1}X_3 - \nabla^B_{[X_1,X_2]}X_3 \\ &= \nabla^B_{X_1}\nabla^B_{X_2}X_3 - \nabla^B_{X_2}\nabla^B_{X_1}X_3 - \nabla^B_{[X_1,X_2]}X_3 \\ &= R_B(X_1, X_2)X_3. \end{aligned}$$

(ii)

$$\begin{aligned} R(X_1, Y_1)X_2 =& \nabla_{X_1}\nabla_{Y_1}X_2 - \nabla_{Y_1}\nabla_{X_1}X_2 - \nabla_{[X_1,Y_1]}X_2 \\ =& \nabla_{X_1}\left(\frac{\langle\nabla\rho, X_2\rangle}{\rho}Y_1\right) - \frac{\langle\nabla\rho, \nabla_{X_1}X_2\rangle}{\rho}Y_1 \\ =& X_1\left(\frac{\langle\nabla\rho, X_2\rangle}{\rho}\right)Y_1 + \frac{\langle\nabla\rho, X_2\rangle}{\rho}\frac{\langle\nabla\rho, X_1\rangle}{\rho}Y_1 - \frac{\langle\nabla_{X_1}X_2, \nabla\rho\rangle}{\rho}Y_1 \\ =& \frac{\rho\langle\nabla_{X_1}\nabla\rho, X_2\rangle + \rho\langle\nabla\rho, \nabla_{X_1}X_2\rangle - \langle\nabla\rho, X_2\rangle\langle\nabla\rho, X_1\rangle}{\rho^2}Y_1 \\ &+ \frac{\langle\nabla\rho, X_2\rangle\langle\nabla\rho, X_1\rangle}{\rho^2}Y_1 - \frac{\langle\nabla_{X_1}X_2, \nabla\rho\rangle}{\rho}Y_1 \\ =& \frac{\langle\nabla_{X_1}\nabla\rho, X_2\rangle}{\rho}Y_1 \\ =& \frac{D^2\rho(X_1, X_2)}{\rho}Y_1. \end{aligned}$$

(iii) Since $D^2\rho$ is symmetric,

$$\begin{aligned}
R(X_1,X_2)Y_1 =& \nabla_{X_1}\nabla_{X_2}Y_1 - \nabla_{X_2}\nabla_{X_1}Y_1 - \nabla_{[X_1,X_2]}Y_1 \\
=& \nabla_{X_1}\left(\frac{\langle\nabla\rho, X_2\rangle}{\rho}Y_1\right) - \nabla_{X_2}\left(\frac{\langle\nabla\rho, X_1\rangle}{\rho}Y_1\right) - \frac{\langle\nabla\rho,[X_1,X_2]\rangle}{\rho}Y_1 \\
=& X_1\left(\frac{\langle\nabla\rho, X_2\rangle}{\rho}\right)Y_1 + \frac{\langle\nabla\rho, X_2\rangle}{\rho}\frac{\langle\nabla\rho, X_1\rangle}{\rho}Y_1 - X_2\left(\frac{\langle\nabla\rho, X_1\rangle}{\rho}\right)Y_1 \\
&- \frac{\langle\nabla\rho, X_1\rangle}{\rho}\frac{\langle\nabla\rho, X_2\rangle}{\rho}Y_1 - \frac{\langle\nabla\rho,[X_1,X_2]\rangle}{\rho}Y_1 \\
=& \frac{\rho\langle\nabla_{X_1}\nabla\rho, X_2\rangle + \rho\langle\nabla\rho, \nabla_{X_1}X_2\rangle - \langle\nabla\rho, X_2\rangle\langle\nabla\rho, X_1\rangle}{\rho^2}Y_1 \\
&- \frac{\rho\langle\nabla_{X_2}\nabla\rho, X_1\rangle + \rho\langle\nabla\rho, \nabla_{X_2}X_1\rangle - \langle\nabla\rho, X_1\rangle\langle\nabla\rho, X_2\rangle}{\rho^2}Y_1 \\
&- \frac{\langle\nabla\rho,[X_1,X_2]\rangle}{\rho}Y_1 \\
=& \frac{\langle\nabla\rho, \nabla_{X_1}X_2\rangle - \langle\nabla\rho, \nabla_{X_2}X_1\rangle - \langle\nabla\rho,[X_1,X_2]\rangle}{\rho}Y_1 \\
=& 0
\end{aligned}$$

and

$$\begin{aligned}
R(Y_1,Y_2)X_1 =& \nabla_{Y_1}\nabla_{Y_2}X_1 - \nabla_{Y_2}\nabla_{Y_1}X_1 - \nabla_{[Y_1,Y_2]}X_1 \\
=& \nabla_{Y_1}\left(\frac{\langle\nabla\rho, X_1\rangle}{\rho}Y_2\right) - \nabla_{Y_2}\left(\frac{\langle\nabla\rho, X_1\rangle}{\rho}Y_1\right) - \frac{\langle\nabla\rho, X_1\rangle}{\rho}[Y_1,Y_2] \\
=& \frac{\rho\langle\nabla_{Y_1}\nabla\rho, X_1\rangle + \rho\langle\nabla\rho, \nabla_{Y_1}X_1\rangle}{\rho^2}Y_2 - \frac{\rho\langle\nabla_{Y_2}\nabla\rho, X_1\rangle + \rho\langle\nabla\rho, \nabla_{Y_2}X_1\rangle}{\rho^2}Y_1 \\
&+ \frac{\langle\nabla\rho, X_1\rangle}{\rho}(\nabla_{Y_1}Y_2 - \nabla_{Y_2}Y_1 - [Y_1,Y_2]) \\
=& 0,
\end{aligned}$$

because $\langle\nabla\rho, Y_i\rangle = 0$ and $\langle\nabla_{Y_i}\nabla\rho, X_1\rangle = \langle\frac{\langle\nabla\rho,\nabla\rho\rangle}{\rho}Y_i, X_1\rangle = 0$.

(iv)

$$
\begin{aligned}
R(X_1,Y_1)Y_2 =&\nabla_{X_1}\nabla_{Y_1}Y_2-\nabla_{Y_1}\nabla_{X_1}Y_2-\nabla_{[X_1,Y_1]}Y_2\\
=&\nabla_{X_1}\left(-\frac{\langle Y_1,Y_2\rangle}{\rho}\nabla\rho\right)+\frac{\langle\nabla\rho,X_1\rangle}{\rho}\nabla^F_{Y_1}Y_2-\nabla_{Y_1}\frac{\langle\nabla\rho,X_1\rangle}{\rho}Y_2\\
=&\frac{\langle Y_1,Y_2\rangle\langle\nabla\rho,X_1\rangle}{\rho^2}\nabla\rho-\frac{\langle\nabla_{X_1}Y_1,Y_2\rangle}{\rho}\nabla\rho-\frac{\langle Y_1,\nabla_{X_1}Y_2\rangle}{\rho}\nabla\rho\\
&-\frac{\langle Y_1,Y_2\rangle}{\rho}\nabla_{X_1}\nabla\rho+\frac{\langle\nabla\rho,X_1\rangle}{\rho}\nabla^F_{Y_1}Y_2-\frac{\langle\nabla\rho,X_1\rangle}{\rho}\nabla_{Y_1}Y_2\\
=&\frac{\langle Y_1,Y_2\rangle\langle\nabla\rho,X_1\rangle}{\rho^2}\nabla\rho-\frac{\langle\nabla\rho,X_1\rangle\langle Y_1,Y_2\rangle}{\rho^2}\nabla\rho\\
&-\frac{\langle Y_1,Y_2\rangle\langle\nabla\rho,X_1\rangle}{\rho^2}\nabla\rho-\frac{\langle Y_1,Y_2\rangle}{\rho}\nabla_{X_1}\nabla\rho+\frac{\langle\nabla\rho,X_1\rangle\langle Y_1,Y_2\rangle}{\rho^2}\nabla\rho\\
=&-\frac{\langle Y_1,Y_2\rangle}{\rho}\nabla_{X_1}\nabla\rho;
\end{aligned}
$$

(v)

$$
\begin{aligned}
R(Y_1,Y_2)Y_3 =&\nabla_{Y_1}\nabla_{Y_2}Y_3-\nabla_{Y_2}\nabla_{Y_1}Y_3-\nabla_{[Y_1,Y_2]}Y_3\\
=&\nabla_{Y_1}\nabla^F_{Y_2}Y_3-\nabla_{Y_2}\nabla^F_{Y_1}Y_3-\nabla^F_{[Y_1,Y_2]}Y_3-\nabla_{Y_1}\left(\frac{\langle Y_2,Y_3\rangle}{\rho}\nabla\rho\right)\\
&+\nabla_{Y_2}\left(\frac{\langle Y_1,Y_3\rangle}{\rho}\nabla\rho\right)+\frac{\langle[Y_1,Y_2],Y_3\rangle}{\rho}\nabla\rho\\
=&R_F(Y_1,Y_2)Y_3-\frac{\langle Y_1,\nabla^F_{Y_2}Y_3\rangle}{\rho}\nabla\rho+\frac{\langle Y_2,\nabla^F_{Y_1}Y_3\rangle}{\rho}\nabla\rho\\
&-\frac{\langle\nabla_{Y_1}Y_2,Y_3\rangle+\langle Y_2,\nabla_{Y_1}Y_3\rangle}{\rho}\nabla\rho-\frac{\langle Y_2,Y_3\rangle}{\rho}\nabla_{Y_1}\nabla\rho\\
&+\frac{\langle\nabla_{Y_2}Y_1,Y_3\rangle+\langle Y_1,\nabla_{Y_2}Y_3\rangle}{\rho}\nabla\rho+\frac{\langle Y_1,Y_3\rangle}{\rho}\nabla_{Y_2}\nabla\rho+\frac{\langle[Y_1,Y_2],Y_3\rangle}{\rho}\nabla\rho\\
=&R_F(Y_1,Y_2)Y_3+\frac{\langle Y_1,(\nabla_{Y_2}Y_3)^\perp\rangle}{\rho}\nabla\rho-\frac{\langle Y_2,(\nabla_{Y_1}Y_3)^\perp\rangle}{\rho}\nabla\rho+\frac{\langle\nabla_{Y_2}Y_1,Y_3\rangle}{\rho}\nabla\rho\\
&-\frac{\langle\nabla_{Y_1}Y_2,Y_3\rangle}{\rho}\nabla\rho-\frac{\langle Y_2,Y_3\rangle}{\rho}\frac{\langle\nabla\rho,\nabla\rho\rangle}{\rho}Y_1+\frac{\langle Y_1,Y_3\rangle}{\rho}\frac{\langle\nabla\rho,\nabla\rho\rangle}{\rho}Y_2\\
&+\frac{\langle[Y_1,Y_2],Y_3\rangle}{\rho}\nabla\rho\\
=&R_F(Y_1,Y_2)Y_3-\frac{\langle Y_2,Y_3\rangle}{\rho}\frac{\langle\nabla\rho,\nabla\rho\rangle}{\rho}Y_1+\frac{\langle Y_1,Y_3\rangle}{\rho}\frac{\langle\nabla\rho,\nabla\rho\rangle}{\rho}Y_2\\
=&R_F(Y_1,Y_2)Y_3+\frac{|\nabla\rho|^2}{\rho^2}(\langle Y_1,Y_3\rangle Y_2-\langle Y_2,Y_3\rangle Y_1).
\end{aligned}
$$

■

As consequence of Proposition 2.1.1 and 2.1.2, the sectional curvature has the relations:

Corollary 2.1.1. *Let $M=B\times_\rho F$ be a warped product of two Riemannian manifolds,*

K the sectional curvature of M, K_B the sectional curvature of B and K_F the sectional curvature of F, then, for $X_1, X_2 \in \mathcal{L}(B)$ and $Y_1, Y_2 \in \mathcal{L}(F)$:

(i) $K(X_1, X_2) = K_B(X_1, X_2)$;

(ii) $K(X_1, Y_1) = -\frac{\langle \nabla_{X_1} \nabla\rho, X_1\rangle}{\rho |X_1|^2}$;

(iii) $K(Y_1, Y_2) = \frac{K_F(Y_1,Y_2)}{\rho^2} - \frac{|\nabla\rho|^2}{\rho^2}$.

Proof. We have:

(i)

$$K(X_1, X_2) = \frac{\langle R(X_1, X_2)X_2, X_1\rangle}{|X_1|^2|X_2|^2 - \langle X_1, X_2\rangle^2} = \frac{\langle R_B(X_1, X_2)X_2, X_1\rangle}{|X_1|^2|X_2|^2 - \langle X_1, X_2\rangle^2} = K_B(X_1, X_2);$$

(ii)

$$K(X_1, Y_1) = \frac{\langle R(X_1, Y_1)Y_1, X_1\rangle}{|X_1|^2|Y_1|^2 - \langle X_1, Y_1\rangle^2} = \frac{-\frac{|Y_1|^2}{\rho}\langle \nabla_{X_1}\nabla\rho, X_1\rangle}{|X_1|^2|Y_1|^2} = -\frac{\langle \nabla_{X_1}\nabla\rho, X_1\rangle}{\rho|X_1|^2};$$

(iii)

$$\begin{aligned} K(Y_1, Y_2) =& \frac{\langle R(Y_1, Y_2)Y_2, Y_1\rangle}{|Y_1|^2|Y_2|^2 - \langle Y_1, Y_2\rangle^2} \\ =& \frac{\langle R_F(Y_1, Y_2)Y_2, Y_1\rangle}{\rho^4(|Y_1|_F^2|Y_2|_F^2 - \langle Y_1, Y_2\rangle_F^2)} + \frac{|\nabla\rho|^2}{\rho^2}\frac{\langle Y_1, Y_2\rangle^2 - |Y_1|^2|Y_2|^2}{|Y_1|^2|Y_2|^2 - \langle Y_1, Y_2\rangle^2} \\ =& \frac{K_F(Y_1, Y_2)}{\rho^2} - \frac{|\nabla\rho|^2}{\rho^2}. \end{aligned}$$

Corollary 2.1.2. *Let $M = B \times_\rho F$ be a warped product of two Riemannian manifolds, denote by S the scalar curvature of M, S_B the scalar curvature of B, S_F the scalar curvature of F and Δ^B the Laplace operator in B, then*

$$S = S_B - 2\frac{\Delta^B\rho}{\rho} + \frac{S_F}{\rho^2} - n(n-1)\frac{|\nabla\rho|^2}{\rho^2}.$$

2.2 Multiple warped product

The concept of warped product can be inductively defined for a product of a finite number of manifolds. In this case, we will have a base and several manifolds representing the fibers.

Definition 2.2.1. *Let* $(B, g_0), (F_1, g_1), \ldots, (F_k, g_k)$ *be manifolds and* $\rho_j : B \mapsto \mathbb{R}^+$, $1 \leq j \leq k$ *smooth functions, we can define the manifold* $M = B \times_{\rho_1} F_1 \times_{\rho_2} F_2 \times \cdots \times_{\rho_k} F_k$ *as being the manifold* $B \times F_1 \times \cdots \times F_k$ *with the metric* g *defined by*

$$g = g_0 + \rho_1^2 g_1 + \cdots + \rho_k^2 g_k.$$

Proposition 2.2.1. *For* $X_1, X_2 \in \mathcal{L}(B)$, $Y_1^i, Y_2^i \in \mathcal{L}(F_i)$, *we have:*

(i) $\nabla_{X_1} X_2 \in \mathcal{L}(B)$ *is the lift of* $\nabla^B_{X_1} X_2$ *on* M*;*

(ii) $\nabla_{X_1} Y_1^i = \nabla_{Y_1^i} X_1 = \frac{\langle \nabla\rho_i, X_1\rangle}{\rho_i} Y_1^i$*;*

(iii) $(\nabla_{Y_1^i} Y_2^i)^\perp = \sigma(Y_1^i, Y_2^i) = -\frac{\langle Y_1^i, Y_2^i\rangle}{\rho_i} \nabla\rho_i$*;*

(iv) $(\nabla_{Y_1^i} Y_2^i)^{T_{F_i}} \in \mathcal{L}(F_i)$ *is the lift of* $\nabla^{(F_i)}_{Y_1^i} Y_2^i$ *on* M*.*

(v) $\nabla_{Y_1^i} Y_1^l = 0$ *if* $i \neq l$*.*

Proof. The proof of (i), (ii), (iii) and (iv) are similar to those of Proposition 2.1.1. (v) is a direct consequence of Koszul formula, where all the terms are null.

Then the curvature tensor satisfies:

Proposition 2.2.2. *Let* $X_1, X_2, X_3 \in \mathcal{L}(B)$ *and* $Y_1^i, Y_2^i, Y_3^i \in \mathcal{L}(F_i)$, $1 \leq i \leq k$, *then:*

(i) $R(X_1, X_2)X_3 = R_B(X_1, X_2)X_3$*;*

(ii) $R(X_1, Y_1^i)X_2 = \frac{D^2\rho(X_1, X_2)}{\rho} Y_1^i$*;*

(iii) $R(X_1, X_2)Y_1^i = 0$*;*

(iv) $R(X_1, Y_1^i)Y_1^l = -\frac{\langle Y_1^i, Y_1^l\rangle}{\rho} \nabla_{X_1}(\nabla f)$ *(is zero if* $i \neq l$*);*

(v) $R(Y_1^i, Y_1^l)X_1 = 0$*;*

(vi) $R(Y_1^i, Y_2^i)Y_1^l = 0$ *if* $i \neq l$*;*

(vii) $R(Y_1^i, Y_1^l)Y_2^i = \langle Y_1^i, Y_2^i\rangle \frac{\langle \nabla\rho_i, \nabla\rho_l\rangle}{\rho_i\rho_l} Y_1^l$ *if* $i \neq l$*;*

(viii) $R(Y_1^i, Y_2^i)Y_3^i = R_{F_i}(Y_1^i, Y_2^i)Y_3^i + \frac{|\nabla\rho_i|^2}{\rho_i^2}(\langle Y_1^i, Y_3^i\rangle Y_2^i - \langle Y_2^i, Y_3^i\rangle Y_1^i)$*;*

(ix) $R(Y_1^i, Y_1^l)Y_1^j = 0$ *if* $i \neq l \neq j \neq i$*.*

Proof. (i), (ii), (iii), (iv) and (viii) are similar to the proofs of Proposition 2.1.2. Let us analyze each of the other items using the Proposition 2.2.1.

(v)

$$\begin{aligned}
R(Y_1^i, Y_1^l)X_1 =& \nabla_{Y_1^i}\nabla_{Y_1^l}X_1 - \nabla_{Y_1^l}\nabla_{Y_1^i}X_1 - \nabla_{[Y_1^i,Y_1^l]}X_1 \\
=& \nabla_{Y_1^i}\left(\frac{\langle \nabla\rho_l, X_1\rangle}{\rho_l}Y_1^l\right) - \nabla_{Y_1^l}\left(\frac{\langle \nabla\rho_i, X_1\rangle}{\rho_i}Y_1^i\right) \\
=& \frac{\langle \nabla\rho_l, X_1\rangle}{\rho_l}\nabla_{Y_1^i}Y_1^l - \frac{\langle \nabla\rho_i, X_1\rangle}{\rho_i}\nabla_{Y_1^l}Y_1^i \\
=& 0
\end{aligned}$$

(vi) It is a direct consequence of item (v) in Proposition 2.2.1.

(vii)

$$\begin{aligned}
R(Y_1^i, Y_1^l)Y_2^i =& \nabla_{Y_1^i}\nabla_{Y_1^l}Y_2^i - \nabla_{Y_1^l}\nabla_{Y_1^i}Y_2^i - \nabla_{[Y_1^i,Y_1^l]}Y_2^i \\
=& -\nabla_{Y_1^l}\nabla_{Y_1^i}Y_2^i \\
=& \nabla_{Y_1^l}\left(\frac{\langle Y_1^i, Y_2^i\rangle}{\rho_i}\nabla\rho_i\right) \\
=& \frac{\langle Y_1^i, Y_2^i\rangle}{\rho_i}\nabla_{Y_1^l}\nabla\rho_i \\
=& \frac{\langle Y_1^i, Y_2^i\rangle}{\rho_i}\frac{\langle \nabla\rho_i, \nabla\rho_l\rangle}{\rho_l}Y_l.
\end{aligned}$$

(ix) It is a consequence of item (v) in Proposition 2.2.1.

As consequence of Proposition 2.2.2, for $X_1, X_2 \in \mathcal{L}(B)$, $Y_1^i, Y_2^i \in \mathcal{L}(F_i)$, $Y_1^l, Y_2^l \in \mathcal{L}(F_l)$, we have

(i) $K(X_1, X_2) = K_B(X_1, X_2)$;

(ii) $K(X_1, Y_1^i) = -\frac{\langle \nabla_{X_1}\nabla\rho_i, X_1\rangle}{\rho_i|X_1|^2}$;

(iii) $K(Y_1^i, Y_2^i) = \frac{K_{F_i}(Y_1^i, Y_2^i)}{\rho_i^2} - \frac{|\nabla\rho_i|^2}{\rho_i^2}$;

(iv) $K(Y_1^i, Y_1^l) = -\frac{\langle \nabla\rho_i, \nabla\rho_l\rangle}{\rho_i\rho_l}$, if $i \neq l$

and

$$S_M = S_B + \sum_{i=1}^{k}\frac{S_{F_i}}{\rho_i^2} - 2\sum_{i=1}^{k}n_i\frac{\Delta^B\rho_i}{\rho_i} - \sum_{i=1}^{k}n_i(n_i-1)\frac{|\nabla\rho_i|^2}{\rho_i^2} - \sum_{i\neq j}n_i n_j\frac{\langle \nabla\rho_i, \nabla\rho_j\rangle}{\rho_i\rho_j},$$

where n_i is the dimension of F_i.

To see more about warped products and their properties or applications, see [Che17].

2.3 Warped products of an interval of $\mathbb{R}$

Suppose $M = I \times_{\rho_1} F_1 \times_{\rho_2} \cdots \times_{\rho_k} F_k$, where $I \subset \mathbb{R}$, the scalar curvature of M is given by

$$S_M = \sum_{i=1}^{k} \frac{S_{F_i}}{\rho_i^2} - 2\sum_{i=1}^{k} n_i \frac{\rho_i''}{\rho_i} - \sum_{i=1}^{k} n_i(n_i - 1)\frac{(\rho_i')^2}{\rho_i^2} - \sum_{i \neq j} n_i n_j \frac{\rho_i' \rho_j'}{\rho_i \rho_j}.$$

In this section, we will suppose that $F_1, \ldots, F_k$ are compacts manifolds, in order to make sense of taking test functions for the operator $L_a = \Delta - aS$ in M that depend only on r in M, the theme of Chapter 3 of this work. The definition of warped products makes sense if the base manifold B has boundary. If B has boundary and each warping function is positive, the resulting manifold also has boundary. If we allow that the warping functions can be zero in the base, that is, each ρ_i is nonnegative, the points p in B such that $\rho_i(p) = 0$ can make their leaves a set of singular points, in the sense that the metric degenerates. It would be complex to analyze these situations for B in general. In this section, we briefly analyze when B is an interval on the line using some examples.

Example 2.3.1. *Let $M = \mathbb{R} \times_{\rho_1} F_1 \times_{\rho_2} \cdots \times_{\rho_k} F_K$ be a multiple warped product, then:*

(i) If $\rho_i > 0$ for all $r \in \mathbb{R}$ and $i = 1, \ldots, k$, then M is a smooth manifold.

(ii) If we only demand $\rho_i \geq 0$, the values r such that $\rho_i(r) = 0$ for some $1 \leq i \leq k$ have degenerated lives.

Analyses (i) and (ii) of Example 2.3.1 also work if we change $\mathbb{R}$ for each other open interval of $\mathbb{R}$, with the contrast that the resulting manifold M will not be complete.

Example 2.3.2. *Suppose that $M = [0, \infty) \times_\rho \mathbb{S}^{n-1}$, with $\rho(r) > 0$ for all $r > 0$, $\rho(r) = 0$ and $\rho'(0) = 1$. Then M is a smooth manifold.*

In the case of Example 2.3.2, M becomes diffeomorphic to $\mathbb{R}^n$ and the natural identification $\Phi : \mathbb{R}^n \to M$ given by $\Phi(r, x) = (r, x)$, $x \in \mathbb{S}^{n-1}$, is the exponential application around the point $r = 0$ in M. If we set $M = [0, \infty) \times_\rho F$, where F is a compact Riemannian manifold different from the round sphere, then the leaf $r = 0$ is a singular point of M. If we only demand $\rho(r) > 0$ for all r, then M is a differentiable manifold with a boundary. If we consider M under all conditions of Example 2.3.2, except $\rho'(0) = 1$, then M will be topologically $\mathbb{R}^n$, but the leaf $r = 0$ would still be singular.

Example 2.3.3. *A cone is a warped product $M = I \times_\rho F$, where $I = [0, \infty)$, $\rho(0) = 0$ and F compact. The only regular cone is the hyperplane.*

Example 2.3.4. *If $I = [a, b] \subset \mathbb{R}$ and $M = [a, b] \times_\rho F$ is smooth without boundary, then $\rho(a) = \rho(b) = 0$, F is a round sphere and M is topologically a sphere.*

Example 2.3.5. *The sphere $\mathbb{S}^{2n+1}$ can be written as a double-warped product, where it*

$$\left[0, \frac{\pi}{2}\right] \times_{\rho_1} \mathbb{S}^n \times_{\rho_2} \mathbb{S}^n,$$

where $\rho_1(r) = \cos(r)$ and $\rho_2(r) = \sin(r)$. A point (r, q_1, q_2) in this degenerate coordinate system corresponds to the point $(\cos(r)q_1, \sin(r)q_2) \in \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2}$, where q_1 and q_2 are identified by their respective coordinates in $\mathbb{R}^{n+1}$. The leaf r is isometric to the torus $(\cos r)\mathbb{S}^n \times (\sin r)\mathbb{S}^n$, and when $r = 0$ or $r = \frac{\pi}{2}$, the leaf r becomes an $\mathbb{S}^n$.

Chapter 3

Operator $\Delta - aS$ on Warped Products

When we make a warped product of two smooth manifolds with a positive warping function ρ, the resulting manifold is necessarily a smooth manifold. If we require ρ to be nonnegative, the points in the base such that ρ vanishes represent a degenerate fiber, in the sense that becomes one point. The Examples 2.3.4 and 2.3.5 in the previous chapter shows that the resulting manifold of a warped product of a closed interval of $\mathbb{R}$ with a manifold F can result in a smooth manifold.

Consider an elliptic operator of the form $L = \Delta - q$ on a warped product $M = I \times_\rho F$, where Δ is the Laplace-Beltrami operator of M, $q : M \to \mathbb{R}$ is a smooth function, I is a closed interval of $\mathbb{R}$ (which we can essentially reduce to three cases: $\mathbb{R}$, $[0, \infty)$ and $[0, 1]$), F is a compact manifold and ρ is a nonnegative warping function. Unless otherwise mentioned, we will always assume that M is without boundary, which for us will be the same as requiring that ρ vanishes in ∂I. The operator L is said to be **stable** if

$$\int_M -fLf \geq 0$$

for all $f \in C_c^\infty(U)$, where U is the set of regular points of M, that is, the largest subset of M which is a manifold. In particular, if $V = \{r \in I; \rho(r) > 0\}$, then $V \times_\rho F \subset U$. The previous inequality is equivalent to

$$\int_M ||\nabla f||^2 + qf^2 \geq 0$$

for all $f \in C_c^\infty(U)$. Similarly, this analysis extends to multiple warped products, where each warping function in the multiple warped product must satisfy the same properties that the function ρ mentioned above.

3.1 Case $M = I \times_\rho F$, F compact

First, consider an $(n-1)$-dimensional compact manifold F with $n \geq 3$ and $M = I \times_\rho F$, such that the metric of M is expressed in the form

$$g = dr^2 + \rho(r)^2 g_F,$$

where ρ is a positive real function in the interior of I. We have M compact if and only if I is a finite closed interval. M there will be no boundary if ρ is zero at the endpoints of I. M will be complete and noncompact if and only if I is an unbounded interval. Up to a linear variable change in I, there are two types of closed and unbounded intervals in $\mathbb{R}$: $[0, \infty)$ and $\mathbb{R}$ itself.

Consider the case in which $I = [0, \infty)$ and $\rho > 0$ in $(0, \infty)$. Under this condition, M has one end, M is a manifold with boundary if $\rho(0) > 0$ and M is smooth without boundary if and only if $\rho(0) = 0$, F is a round sphere of radius R and $\rho'(0) = \frac{1}{R}$, $R > 0$. In the case $I = \mathbb{R}$, M will have two ends and will be smooth if and only if ρ is positive everywhere. In all the cases, the volume element of M is $\rho(r)^{n-1} dr dA$, where dA is the area element of $\mathbb{S}^{n-1}$.

We studied the stability of the operator $L = \Delta - aS$ on M and sometimes at an end of M. We say that L is *stable on an end E of M* if the operator L restricted to a representative of E of the form $((R, \infty) \times_\rho F, g)$ (or $((-\infty, -R) \times F, g)$) for all R sufficiently large is stable. If E is not stable, we say that E is *unstable*. It is obvious that an end E be unstable implies that M is unstable.

For this type of warped product, the scalar curvature can be written as

$$S_M = \frac{S_F}{\rho(r)^2} - (n-1)\left(2\frac{\rho''(r)}{\rho(r)} + (n-2)\frac{\rho'(r)^2}{\rho(r)^2}\right).$$

For f of compact support in M, only dependent on r and support contained in the interval $[b, c]$, we have:

(i) if $I = \mathbb{R}$,then $f(b) = f(c) = 0$;

(ii) if $I = [0, \infty)$ and $b \neq 0$, then $f(b) = 0$. If $b = 0$, we only guarantee $f(c) = 0$, but if we demand M without boundary, then $\rho(b) = \rho(0) = 0$.

Hence, in the case M without boundary and f of compact support, we have

$$\rho(b)^{n-2} f(b) = \rho(c)^{n-2} f(c) = 0.$$

Therefore, we have:

$$\begin{aligned}\int_M -f L_a f &= \int_M |\nabla f|^2 + aSf^2 \\ &= \int_b^c \int_F \left\{ |\nabla f|^2 + a \left[-(n-1) \left(2\frac{\rho''(r)}{\rho(r)} + (n-2)\frac{\rho'(r)^2}{\rho(r)^2} \right) + \frac{S_F}{\rho(r)^2} \right] f^2 \right\} dA dr.\end{aligned} \tag{3.1}$$

First, let us look at a result that will make our calculations of whether M is or is not stable. We will use principally to prove stability in a warped product of an interval with round spheres and an interval with compact manifolds of null scalar curvature.

Proposition 3.1.1. *Let $M^n = I \times_\rho F^{n-1}$ be a warped product, with $n \geq 3$. Suppose that F has constant scalar curvature and*

$$-\int_M f L_a f \geq 0$$

for all f dependent only on I, that is, for all f constants on the fibers. Then M is a-stable.

Proof. If f is a function of compact support in M, then $\pi_1(supp(f))$ has compact support in I. If f depends only on I, by (3.1), we have

$$\int_I \left\{ f_r^2 + a \left[-(n-1) \left(2\frac{\rho''(r)}{\rho(r)} + (n-2)\frac{\rho'(r)^2}{\rho(r)^2} \right) + \frac{S(F)}{A(F)\rho(r)^2} \right] f^2 \right\} dr \geq 0, \tag{3.2}$$

where $S(F)$ is the total scalar curvature of F and $A(F)$ is the area of F. Let f be any function of compact support on M, using Fubini's theorem and the fact that $|\nabla f|^2 \geq f_r^2$, can be obtained by (3.1):

$$\int_M -f L_a f$$

$$= \int_F \int_b^c \left\{ |\nabla f|^2 + a \left[-(n-1) \left(2\frac{\rho''(r)}{\rho(r)} + (n-2)\frac{\rho'(r)^2}{\rho(r)^2} \right) + \frac{S_F(q)}{\rho(r)^2} \right] f(r,q)^2 \right\} drdA$$

$$\geq \int_F \int_b^c \left\{ f_r(r,q)^2 + a \left[-(n-1) \left(2\frac{\rho''(r)}{\rho(r)} + (n-2)\frac{\rho'(r)^2}{\rho(r)^2} \right) + \frac{S_F(q)}{\rho(r)^2} \right] f(r,q)^2 \right\} drdA,$$

where $[b,c] = \pi_1(supp(f))$. Since $f_r(r,q)$ and $f(r,q)$ depend only of f on r in each leaf $I \times \{q\}$ and $\frac{S_F}{\rho(r)^2}$ is constant on F, the last expression is equal to

$$\int_F \int_b^c \left\{ f_r(r,q)^2 + a \left[-(n-1) \left(2\frac{\rho''(r)}{\rho(r)} + (n-2)\frac{\rho'(r)^2}{\rho(r)^2} \right) + \frac{S(F)}{A(F)\rho(r)^2} \right] f(r,q)^2 \right\} drdA,$$

that is nonnegative because (3.2).

■

The Proposition 3.1.1 shows that, with the hypothesis that F has constant scalar curvature, we simply take real functions f as a test function (that is, dependent only on the first coordinate) to demonstrate whether M is stable.

Continuing the calculation in (3.1), we have, for f of compact support dependent only on r:

$$\begin{aligned}
\int_M -fL_a f =& A(F) \int_b^c f_r^2 \rho(r)^{n-1} dr - 2a(n-1) \int_b^c \rho''(r)\rho(r)^{n-2} dr \\
&+ (n-2) \int_b^c \rho'(r)^2 \rho(r)^{n-3}) f^2 dr + aS(F) \int_b^c \rho(r)^{n-3} f^2 dr \\
=& A(F) \int_b^c f_r^2 \rho(r)^{n-1} dr + a(n-1)(n-2)A(F) \int_b^c \rho'(r)^2 \rho(r)^{n-3} f^2 dr \\
&+ 4a(n-1)A(F) \int_b^c \rho'(r)\rho(r)^{n-2} f f_r dr + aS(F) \int_b^c \rho(r)^{n-3} f^2 dr,
\end{aligned}$$

where $A(F)$ is the area of F and $S(F)$ is the total scalar curvature of F. Therefore

$$\begin{aligned}
\frac{1}{A(F)} \int_M -fL_a f =& \int_b^c f_r^2 \rho(r)^{n-1} dr + a(n-1)(n-2) \int_b^c \rho'(r)^2 \rho(r)^{n-3} f^2 dr + \\
&+ 4a(n-1) \int_b^c \rho'(r)\rho(r)^{n-2} f f_r dr + a\frac{S(F)}{A(F)} \int_b^c \rho(r)^{n-3} f^2 dr.
\end{aligned} \tag{3.3}$$

From equation (3.3), we can find an interval on the line for which the operator L_a is always stable in any warped product of form $I \times_\rho F$, whenever F has nonnegative total scalar curvature.

Theorem 3.1.1. *Let $M^n = I \times_\rho F^{n-1}$ be a warped product manifold without boundary, where $n \geq 3$, $I \subset \mathbb{R}$ and F a compact manifold with nonnegative scalar curvature, then*

M is a-stable for $0 \le a \le \frac{n-2}{4(n-1)}$. Consequently, the Yamabe operator $Y = \frac{n-1}{4(n-2)}\Delta - S$ is nonnegative on M, being positive if (i) $S(F) > 0$ or (ii) I is unbounded.

Proof. By (3.1) taking $f : M \to \mathbb{R}$ of compact support and $\pi_1(Suppf) \subset [b, c]$, using Fubini's theorem, integration by parts, that $S_F \ge 0$ and the fact

$$\rho(b)^{n-2}f(b) = \rho(c)^{n-2}f(c) = 0,$$

we obtain:

$$\begin{aligned}\int_M -fL_a f &= \int_M |\nabla f|^2 + \int_F \Big[a(n-1)(n-2)\int_b^c \rho'(r)^2\rho(r)^{n-3}f(r,q)^2 dr \\ &\quad +4a(n-1)\int_b^c \rho(r)^{n-2}\rho'(r)f(r,q)f_r(r,q)dr + \int_b^c \frac{S_F(q)}{\rho(r)^2}dr\Big]\,dA \\ &\ge \int_F \Big[\int_b^c f_r(r,q)^2\rho(r)^{n-1}dr + a(n-1)(n-2)\int_b^c \rho(r)^{n-3}\rho'(r)^2 f(r,q)^2 dr \\ &\quad +4a(n-1)\int_b^c \rho(r)^{n-2}\rho'(r)f(r,q)f_r(r,q)dr\Big]\,dA.\end{aligned} \tag{3.4}$$

Using the AM-GM inequality, we have, for each $q \in F$:

$$\begin{aligned}&-4a(n-1)\int_b^c \rho(r)^{n-2}\rho'(r)f(r,q)f_r(r,q)dr \\ &= -4a(n-1)\int_b^c f_r(r,q)\rho(r)^{\frac{n-1}{2}}\rho(r)^{\frac{n-3}{2}}\rho'(r)f(r,q)dr \\ &\le \int_b^c f_r(r,q)^2\rho(r)^{n-1}dr + 4a^2(n-1)^2\int_b^c \rho(r)^{n-3}\rho'(r)^2 f(r,q)^2 dr.\end{aligned}$$

Then

$$\begin{aligned}&\int_b^c f_r(r,q)^2\rho(r)^{n-1}dr + 4a^2(n-1)^2\int_b^c \rho(r)^{n-3}\rho'(r)^2 f(r,q)^2 dr \\ &+ 4a(n-1)\int_b^c \rho(r)^{n-2}\rho'(r)f(r,q)f_r(r,q)dr \ge 0.\end{aligned} \tag{3.5}$$

So to conclude the proof, we just need to show that

$$\begin{aligned}&a(n-1)(n-2) \ge 4a^2(n-1)^2 \\ \Longleftrightarrow\, &a \ge 0 \;\text{ and }\; n-2 \ge 4a(n-1) \\ \Longleftrightarrow\, &a \ge 0 \;\text{ and }\; a \le \frac{n-2}{4(n-1)}.\end{aligned}$$

This ensures the a-stability of M for all $a \in [0, \frac{n-2}{4(n-1)}]$. Taking $a = \frac{n-2}{4(n-1)}$, then the Yamabe operator Y is nonnegative. If $S(F) > 0$, then the existence of the term $\int_b^c \frac{S_F(q)}{\rho(r)^2} dr$ in the calculation (3.4) implies the positively of Y. If I is unbounded and $S(F) = 0$, which in our hypotheses means that the scalar curvature of F is identically zero, suppose that $f \in C_c^\infty(M)$ satisfies

$$-\int_M fY(f) = 0,$$

then the equality in (3.5) occurs, it implies

$$\int_b^c \left[f_r(r,q)\rho(r)^{\frac{n-1}{2}} + \frac{n-2}{2}\rho(r)^{\frac{n-3}{2}}\rho'(r)f(r,q) \right]^2 dr = 0$$

for all $q \in F$, this implies

$$\begin{aligned} f_r(r,q) = -\frac{n-2}{2}\rho(r)^{-1}\rho'(r)f(r,q) &\Longrightarrow \log(f(r,q)) = -\frac{n-2}{2}\log(\rho(r)) + C(q) \\ &\Longrightarrow f(r,q) = C(q)\rho(r)^{-\frac{n-2}{2}}, \end{aligned}$$

that has compact support if and only if $C(q) = 0$ for all $q \in F$, that is, if and only if $f \equiv 0$.

■

We study the a-stability based on the order of growth of ρ.

3.1.1 a-stability results from the growth of ρ

Kawai [Kaw88] showed that if a complete two-dimensional manifold has nonpositive Gaussian curvature and is a-stable for $a > \frac{1}{8}$, then the manifold is flat. A natural way to find a similar result for higher-dimensional complete manifold M is if there exists a_0 such that $L_a = \Delta - aS$ is a-unstable for all $a > a_0$, where S is its scalar curvature, assuming, for example, $S \leq 0$. In fact the answer is no. For each $a > 0$, there are examples of warped product metrics of negative scalar curvature, which, according to our definition, is a-stable (see Example 3.1.2). In this section, we will analyze special cases when the manifold M in question is a warped product, which is possible because we can calculate the scalar curvature in a relatively simple way from the coefficients of the metric and the warping function.

In dimension two, the curvature of $M = I \times_\rho \mathcal{C}$, where $\mathcal{C}$ is a closed curve, being negative is equivalent to having positive the function ρ''. In a higher dimension, this, as we will see, is a sufficient but not necessary condition and the following theorem generalizes this result by taking as a hypothesis the condition that ρ'' is a positive function.

Let $M = [0, \infty) \times_\rho F$, where $\rho(0) = 0$ (M without boundary) and ρ'' is a positive function, there are two possibilities for ρ': ρ' limited (with $\lim_{r\to\infty} \rho'(r) = C$ for some $C > 0$) or $\lim_{r\to\infty} \rho'(r) = \infty$. First, we have the following theorem.

Theorem 3.1.2. *Let $M = [0, \infty) \times_\rho F$ be a warped product without boundary such that ρ satisfies $\rho''(r) > 0$ and $\lim_{r\to\infty} \rho'(r) = \infty$. Then, the end of M (and therefore, M) is a-unstable for all $a > \frac{n-1}{4n}$. If F has nonpositive total scalar curvature, the hypothesis $\lim_{r\to\infty} \rho'(r) = \infty$ can be removed.*

Under the conditions of Theorem 3.1.2, if M is smooth, then the scalar curvature is a negative function, because M smooth implies that there exists $R > 0$ such that $F = R\mathbb{S}^{n-1}$, $\rho'(0) = \frac{1}{R}$ and, with the hypothesis $\rho''(r) > 0$, the expression of S_M becomes negative. Therefore, the Theorem 3.1.2, in a certain way, generalizes Kawai's result [Kaw88] in manifolds M under the conditions of this theorem.

In Theorem 3.1.2, the base of the warped product M is $I = [0, \infty)$. Note that $\rho''(r) > 0$, together with (i) $\lim_{r\to\infty} \rho'(r) = \infty$ or (ii) $S(F) \leq 0$, guarantees the negativity of S on the end E of M. When M is of the form $\mathbb{R} \times_\rho F$, we only need to adapt the Theorem 3.1.2, obtaining a similar result:

Theorem 3.1.3. *Let $M = \mathbb{R} \times_\rho F$ be a warped product such that there exists R_0 such that one of the following occurs:*

(i) $\rho''(r) > 0$ for all $r > R_0$ and $\lim_{r\to\infty} \rho'(r) = \infty$.

(ii) $\rho''(r) > 0$ for all $r < R_0$ and $\lim_{r\to-\infty} \rho'(r) = -\infty$.

Then the corresponding end of M in (i) or (ii) (and therefore, M) is a-unstable for all $a > \frac{n-1}{4n}$.

Note that, in the case where $S_F \equiv C \geq 0$, the Theorem 3.1.1 implies that the warped product $M = I \times_\rho F$ is a-stable for all $0 \leq a \leq \frac{n-2}{4(n-1)}$, where $I = [0, \infty)$ or $\mathbb{R}$. On the other hand, the Theorem 3.1.2 and Theorem 3.1.3 obtain sufficient conditions in M to characterize M as $\frac{n-1}{4n}$-unstable.

As examples that fit the hypotheses of Theorem 3.1.1 and Theorem 3.1.2, we have the warped product $M = [0, \infty) \times_\rho F$ with the warping function given by $\rho(r) = r^\alpha$ with $\alpha > 1$. Similarly, the warped product $\mathbb{R} \times_\rho F$ with warping function $\rho(r) = r^{|\alpha|}$ for $\alpha > 1$ satisfies the hypothesis of Theorem 3.1.1 and Theorem 3.1.3.

Another important example is that when $F = \mathbb{S}^{n-1}$ and $\rho(r) = \sinh(r)$. We have $M = \mathbb{H}^n$, whose first eigenvalue for the Laplacian is $-\frac{(n-1)^2}{4}$ and the scalar curvature is $-n(n-1)$, which implies that $\mathbb{H}^n$ is $\frac{n-1}{4n}$-stable.

The next theorem addresses the case when we have $F = \mathbb{S}^{n-1}$ of constant scalar curvature equal to $(n-1)(n-2)$ and $\lim_{r\to\infty} \rho'(r) = C$.

Theorem 3.1.4. *Let $M = I \times_\rho \mathbb{S}^{n-1}$ be a warped product such that ρ satisfies $|\rho'(r)| \leq C$. Then M is a-stable for all $0 \leq a \leq \frac{(C^2+1)(n-2)}{4C^2(n-1)}$.*

Remark 3.1.1. *Note that, in Theorem 3.1.4, for C large,*

$$\frac{(C^2+1)(n-2)}{4C^2(n-1)} \approx \frac{n-2}{4(n-1)}$$

and the interval of the result of Theorem 3.1.4 is approximately the interval of a in Theorem 3.1.1. Furthermore, for $n \geq 3$, $\frac{(C^2+1)(n-2)}{4C^2(n-1)} \to \infty$ when $C \to 0$. It shows that the hypothesis $\lim_{r\to\infty} \rho'(r) = \infty$ in Theorem 3.1.2 cannot be dropped. In the case M smooth, where $F = \mathbb{S}^{n-1}$ and $C \geq 1$ (is necessary that $\rho'(0) = 1$), for C close to one,

$$\frac{(C^2+1)(n-2)}{4C^2(n-1)} \approx \frac{n-2}{2(n-1)} > \frac{n-1}{4n},$$

in which it is possible to have a smooth manifold M with negative curvature and is a-stable for some $a > \frac{n-1}{4n}$.

Remark 3.1.2. *Taking $M = [0, \infty) \times_\rho \mathbb{S}^{n-1}$ with $\rho(r) = C(r+1) - (C-1)\log(r+1) - C$, we have $\rho(0) = 0$ and $\rho'(0) = 1$, then M is smooth. Furthermore, $\rho'(r) = C - \frac{C-1}{r+1} \to C$ when $r \to \infty$ and $\rho''(r) = \frac{1}{(r+1)^2} > 0$ for all r, then M satisfies all the hypotheses of Theorem 3.1.4 and and build an example for the conclusion obtained in the Remark 3.1.1.*

Consider the warped product $M = I \times_\rho F$ such that $S_F \geq 0$, $S_M < 0$ and, furthermore, satisfies the hypotheses of theorem 3.1.2 or of Theorem 3.1.3. Looking from now on only the case of Theorem 3.1.2 and Theorem 3.1.3, when $S_F \geq 0$ and ρ' is unbounded. Using Theorem 3.1.1 and the negativity of S we have an "interval of stability," being it $(-\infty, \frac{n-2}{4(n-1)}]$ and an "interval of instability," being it $(\frac{n-1}{4n}, \infty)$. For a in the interval $(\frac{n-2}{4(n-1)}, \frac{n-1}{4n}]$ and M under these conditions, the stability of L_a in M is unknown. In the following theorems, we will study the cases where ρ has a growth of polynomial order, focusing on the values of a in the interval $(\frac{n-2}{4(n-1)}, \frac{n-1}{4n}]$ to obtain a relationship between a-stability and the "degree" of a polynomial which has a growth comparable to the growth of ρ.

First, we assume that ρ has a polynomial growth greater than the Euclidean, that is, there exists $1 < \zeta < \infty$ such that $\lim_{r\to\infty} \rho(r)r^{-\zeta} = 0$ and $\lim_{r\to\infty} \rho(r)r^{-1} = +\infty$. Here, assuming $M = I \times_\rho F$ with $I = [0, \infty)$ or $I = \mathbb{R}$, we are analyzing the end of M corresponding to the positive end of the real line $\mathbb{R}$. The analysis of the negative end of $\mathbb{R}$, when $I = \mathbb{R}$, is analogous, taking the application $\tilde{\rho}$ defined by $\tilde{\rho}(r) = \rho(-r)$ and analyzing $\tilde{\rho}$ on the positive end of $\mathbb{R}$. Under these conditions, there exists a smooth nonnegative real function $\xi : M \to \mathbb{R}$ and a real number $\alpha \geq 1$ such that for $r \geq 1$,

$$\rho(r) = r^\alpha \xi(r), \text{ where } \alpha = \inf\left\{\gamma;\ \lim_{r\to\infty} \rho(r)r^{-\gamma} = 0\right\}. \tag{3.6}$$

Many situations in relation to ξ can occur, among these, we can have $\lim_{r\to\infty} \xi(r) = \infty$ (exemplified by $\xi(r) = \log(r^2+2)^k$, $k > 0$), we can have $\lim_{r\to\infty} \xi(r) = 0$ (exemplified by $\xi(r) = \log(r^2+2)^k$, $k < 0$) and the case where there exist positive constants C_1 and C_2 such that $C_1 \leq \xi(r) \leq C_2$ as when ξ is constant. If $\lim_{r\to\infty} \xi(r) = \infty$, we have $\lim_{r\to\infty} \xi(r)r^{-\gamma} = 0$ for all $\gamma > 0$ and if $\lim_{r\to\infty} \xi(r) = 0$, we have $\lim_{r\to\infty} \xi(r)r^{\gamma} = \infty$ for all $\gamma > 0$. Being f of compact support only dependent on r, assume that the support of f is of form $[b, c] \times F$, with $b, c \neq 0$. Being $\rho(r) = r^\alpha \xi(r)$, we have, for $r \neq 0$, $\rho'(r) = \alpha r^{\alpha-1}\xi(r) + r^\alpha \xi'(r)$ and

$$\rho'(r)^2 = \alpha^2 r^{2\alpha-2}\xi(r)^2 + 2\alpha r^{2\alpha-1}\xi(r)\xi'(r) + r^{2\alpha}\xi'(r)^2.$$

Then (3.3) becomes

$$\begin{aligned}\frac{1}{A(F)}\int_M -fL_a f = \int_b^c f_r^2 r^{n\alpha-\alpha}\xi(r)^{n-1}dr + a\alpha^2(n-1)(n-2)\int_b^c r^{n\alpha-\alpha-2}\xi(r)^{n-1}f^2 dr \\ + 2a\alpha(n-1)(n-2)\int_b^c r^{n\alpha-\alpha-1}\xi(r)^{n-2}\xi'(r)f^2 dr\end{aligned}$$

$$+ a(n-1)(n-2)\int_b^c r^{n\alpha-\alpha}\xi(r)^{n-3}\xi'(r)^2 f^2 dr$$
$$+ 4a\alpha(n-1)\int_b^c r^{n\alpha-\alpha-1}\xi(r)^{n-1} f f_r dr$$
$$+ 4a(n-1)\int_b^c r^{n\alpha-\alpha}\xi(r)^{n-2}\xi'(r) f f_r dr$$
$$+ a\frac{S(F)}{A(F)}\int_b^c r^{n\alpha-3\alpha}\xi(r)^{n-3} f^2 dr.$$

(3.7)

The expression (3.7) can be written in other way as

$$\frac{1}{A(F)}\int_M -fL_a f = \int_b^c f_r^2 r^{n\alpha-\alpha}\xi(r)^{n-1} dr + a\alpha^2(n-1)(n-2)\int_b^c r^{n\alpha-\alpha-2}\xi(r)^{n-1} f^2 dr$$
$$+ 2a\alpha(n-1)(n-2)\int_b^c r^{n\alpha-\alpha-1}\xi(r)^{n-2}\xi'(r) f^2 dr$$
$$+ a(n-1)(n-2)\int_b^c r^{n\alpha-\alpha}\xi(r)^{n-3}\xi'(r)^2 f^2 dr$$
$$+ 4a\alpha(n-1)\Big[-\frac{n\alpha-\alpha-1}{2}\int_b^c r^{n\alpha-\alpha-2}\xi(r)^{n-1} f^2 dr$$
$$-\frac{n-1}{2}\int_b^c r^{n\alpha-\alpha-1}\xi(r)^{n-2}\xi'(r) f^2 dr\Big]$$
$$+ 4a(n-1)\Big[-\frac{n\alpha-\alpha}{2}\int_b^c r^{n\alpha-\alpha-1}\xi(r)^{n-2}\xi'(r) f^2 dr$$
$$-\frac{n-2}{2}\int_b^c r^{n\alpha-\alpha}\xi(r)^{n-3}\xi'(r)^2 dr \; -\frac{1}{2}\int_b^c r^{n\alpha-\alpha}\xi(r)^{n-2}\xi''(r) f^2 dr\Big]$$
$$+ a\frac{S(F)}{A(F)}\int_b^c r^{n\alpha-3\alpha}\xi(r)^{n-3} f^2 dr$$
$$= \int_b^c f_r^2 r^{n\alpha-\alpha}\xi(r)^{n-1} dr - a\alpha(n-1)(n\alpha-2)\int_b^c r^{n\alpha-\alpha-2}\xi(r)^{n-1} f^2 dr$$
$$- 2a\alpha n(n-1)\int_b^c r^{n\alpha-\alpha-1}\xi(r)^{n-2}\xi'(r) f^2 dr$$
$$- a(n-1)(n-2)\int_b^c r^{n\alpha-\alpha}\xi(r)^{n-3}\xi'(r)^2 f^2 dr$$
$$- 2a(n-1)\int_b^c r^{n\alpha-\alpha}\xi(r)^{n-2}\xi''(r) f^2 dr$$
$$+ a\frac{S(F)}{A(F)}\int_b^c r^{n\alpha-3\alpha}\xi(r)^{n-3} f^2 dr.$$

(3.8)

The next two theorems relate a-instability to the order of the degree of the polynomial which has growth comparable to the growth of ρ.

Theorem 3.1.5. *Suppose $\alpha > 1$ and $M = I \times_\rho F$ with a metric of the form $g = dr^2 + \rho(r)^2 g_F$, with α and ρ under the conditions (3.6). Suppose that there exist positive constants C_1 and C_2 such that $C_1 r^\alpha \leq \rho(r) \leq C_2 r^\alpha$ for all $r > 1$. Then the corresponding end (and therefore M) is a-unstable for all $a > \frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$.*

Remark 3.1.3. *No hypotheses about ρ', ρ'', the curvature of F and the curvature of M in Theorem 3.1.5.*

Remark 3.1.4. *In Theorem 3.1.5 the constant $\frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$ is the best possible in the following sense: there exists an example of M, satisfying all the hypotheses of Theorem 3.1.5, which is $\frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$-stable. One example is $M = [0,\infty) \times_\rho \mathbb{S}^2$ where $\rho(r) = \frac{1}{\alpha+1} r^\alpha$. For more details, see Proposition 3.4.1 in Section 3.4.*

The next theorem characterizes a warped product as $\frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$-unstable, occurring when ρ has a α-subpolynomial growth, that is, $\lim_{r\to\infty} \rho(r) r^{-\alpha} = 0$. It is very similar to the Bérard-Castilon Theorem in dimension $n = 2$.

Theorem 3.1.6. *Let $M = I \times_\rho F$ be a warped product, where F is a compact manifold and $\rho(r) = r^\alpha \xi(r)$, where $\alpha \geq 1$ and ξ satisfies $\xi(r) \to 0$ when $r \to \infty$. Suppose that ξ is a non-increasing function and satisfies $r\xi(r)^{-1}\xi'(r) \geq -(\alpha - 1)$ for all r large. Also suppose that M is a-stable for $a = \frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$. Then ρ has linear growth, that is, $\rho(r) \leq Cr$ for some $C \geq 0$. In particular, M has a polynomial volume growth on order of R^n, that is, for each $p \in M$, $Vol(B_R(p)) \leq C_1 R^n$ for some $C_1 > 0$.*

Remark 3.1.5. *Note that on Theorem 3.1.6, we are not requiring that ξ and α satisfy the condition (3.6).*

The Theorems 3.1.5 and 3.1.6 determine an instability interval relative to the exponent α related to the growth of ρ. We can observe that

$$h : \left(\frac{2}{n}, \infty\right) \longrightarrow \mathbb{R}$$
$$\alpha \longmapsto \frac{(n\alpha - \alpha - 1)^2}{4\alpha(n-1)(n\alpha-2)}$$

is decreasing in $(\frac{2}{n}, 1]$, increasing in $[1, \infty)$, satisfies $h(1) = \frac{n-2}{4(n-1)}$ and $\lim_{\alpha\to\infty} h(r) = \frac{n-1}{4n}$.

The next theorem characterize a warped product $M^n = I \times_\rho F^{n-1}$ with polynomial volume growth, that is, such that ρ has a polynomial growth and is a-stable for $a \geq \frac{n-1}{4n}$.

Theorem 3.1.7. *Let $M = I \times_\rho F$ be a warped product, where F is a compact manifold of dimension at least two and $\rho(r) = r^\alpha \xi(r)$, where α and ξ satisfy (3.6). Suppose that M is a-stable for some $a > \frac{n-1}{4n}$. Then there exist a positive constant C such that*

$$\liminf_{R\to\infty} (\log R)^{-1} \int_1^R r\rho(r)^{-2} dr \geq C.$$

Furthermore, if $S(F) \leq 0$, then $\alpha < \frac{n}{2n-2}$. In particular, M has $(1+\frac{n}{2})$-subpolynomial volume growth, that is,

$$\lim_{r\to\infty} r^{-(1+\frac{n}{2})} Vol(B_r(p)) = 0$$

for all $p \in M$.

The next theorem shows what can happen if the function ξ in (3.6) has a large variation.

Theorem 3.1.8. *Let $M = I \times_\rho F$, where $\rho(r) = r^\alpha \xi(r)$, where $\alpha > 1$ and ξ satisfy (3.6) with the additional condition $C_1 \leq \xi(r) \leq C_2$ for all $r \geq 0$ for some constants $C_1, C_2 > 0$. Suppose that*

$$\lim_{T\to\infty} \frac{logT}{\int_{R_0}^T r\xi(r)^{-2}\xi'(r)^2 dr} = 0,$$

where R_0 is a fixed positive number. Then M is a-unstable for all $a > \frac{n-2}{4(n-1)}$.

Example 3.1.1. *Let $\xi : \mathbb{R} \to \mathbb{R}$ defined by $\xi(r) = 2 + \cos(e^r)$. It is possible to show that the application*

$$R \longmapsto \int_{R_0}^R r\xi(r)^{-2}\xi'(r)^2 dr$$

has a exponential increasing grown. Note that $1 \leq \xi(r) \leq 3$ for all $r \in \mathbb{R}$. Therefore, ξ satisfies the hypothesis of Theorem 3.1.8.

Inspired on the Theorems 3.1.5, 3.1.6, 3.1.7 and 3.1.8, we conjecture:

Conjecture 3.1.1. *Let $M = I \times_\rho F$, where $\rho(r) = r^\alpha \xi(r)$, where $\alpha > 1$ and ξ satisfy (3.6). Suppose that, for all $\gamma > 0$, $\lim_{r\to\infty} \xi(r) r^\gamma = \infty$ and $\lim_{r\to\infty} \xi(r) r^{-\gamma} = 0$. Then M is a-unstable for all $a > \frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$. Furthermore, if $\lim_{r\to\infty} \xi(r) = 0$, then M is a-unstable.*

Under these conditions, ξ becomes, for values large of r, a positive function, which neither increases fast to ∞ nor decreases fast to zero, in the sense that it increases and decreases more slowly than any polynomial function $r \mapsto r^\gamma$, $\gamma > 0$. The natural examples are logarithmic functions. In the case where $I = \mathbb{R}$ in Conjecture 3.1.1, it is enough that the end corresponding to the positive direction of M satisfies their hypotheses, where by a simple change the variable of r, we guarantee an analogous result if we take the end corresponding to the negative direction of $\mathbb{R}$.

We can study the $\alpha = 1$ case separately from the others. This case includes the case corresponding to Theorem 3.1.4. In this case, the scalar curvature of M can be strictly positive or strictly negative. For example, if M is of the singular type and $F = \mathbb{S}^{n-1}$, taking $\rho(r) = cr$, we have $S > 0$ if $c < 1$, $S < 0$ if $c > 1$ and the case $c = 1$ gives us exactly the n-dimensional Euclidean space. The crucial difference is that the last term of (3.8) is relevant in relation to the other terms, which does not occur if $\alpha > 1$. Here, we present two cases related to the case of Theorem 3.1.5 in two different theorems.

Theorem 3.1.9. *Let $M = I \times_\rho \mathbb{S}^{n-1}$ be a warped product, with $I = \mathbb{R}$ or $I = [0, \infty)$. Suppose that $C_1 r \leq \rho(r) \leq C_2 r$ $\forall r \geq 1$ for some positive constants C_1, C_2. If $C_1 > 1$, then (the corresponding end of) M is a-unstable for all $a > \frac{n-2}{4(n-1)(1-C_1^{-2})}$.*

Theorem 3.1.10. *Let $M = I \times_\rho \mathbb{S}^{n-1}$ be a warped product, with $I = \mathbb{R}$ or $I = [0, \infty])$. Suppose that $C_1 r \leq \rho(r) \leq C_2 r$ $\forall r \geq 1$ for some positive constants C_1, C_2. If $C_2 < 1$, then (the corresponding end of) M is a-unstable for all $a < \frac{n-2}{4(n-1)(1-C_2^{-2})}$.*

Remark 3.1.6. *The condition $C_1 r \leq \rho(r) \leq C_2 r$ $\forall r \geq 1$ in Theorem 3.1.9 and in Theorem 3.1.10 can be dropped by $C_1 r \leq \rho(r) \leq C_2 r$ for r sufficiently large. This can "improve" the constants $\frac{n-2}{4(n-1)(1-C_1^{-2})}$ of Theorem 3.1.9 (respectively, $\frac{n-2}{4(n-1)(1-C_2^{-2})}$ of Theorem 3.1.10), because the Theorems 3.1.9 and 3.1.10 with this modified hypothesis could cause the new value of C_1 to increase (respectively, C_2 to decrease), causing the increasing of the interval $(\frac{n-2}{4(n-1)(1-C_1^{-2})}, \infty)$ (respectively, $(-\infty, \frac{n-2}{4(n-1)(1-C_2^{-2})})$). In the proof in the next section, we only need $C_1 r \leq \rho(r) \leq C_2 r$ for r sufficiently large, proving the instability of the end. In the case $I = \mathbb{R}$, we can analyze the end of M relative to the negative direction of $\mathbb{R}$ changing ρ by $\tilde{\rho} = \rho \circ A$, being A the application determined by $A(x) = -x$.*

Example 3.1.2. *Let $M = [0, \infty] \times_\rho \mathbb{S}^{n-1}$ with $\rho(r) = Cr$ is a (geometric) cone, singular in the origin $r = 0$. The operator L becomes $\Delta - \frac{a(n-1)(n-2)(1-C^2)}{C^2 r^2}$. Note that $S > 0$ if $C < 1$, $S = 0$ if $C = 1$ and $S < 0$ if $C > 1$. Suppose f dependent only on r, then (3.7)* *described as (3.8) becomes:*

$$\frac{1}{A(F)}\int_M -fL_af = C^{n-1}\int_b^c f_r^2 r^{n-1}dr + a(n-1)(n-2)(1-C^2)C^{n-3}\int_b^c r^{n-3}f^2dr,$$

where b *and* c *are such that* $\pi_1(suppf)\subset[b,c]$. *By Cauchy-Schwartz inequality, we have:*

$$\begin{aligned}\int_b^c r^{n-3}f^2dr &= -\frac{2}{n-2}\int_b^c r^{n-2}ff_rdr\\ &= -\frac{2}{n-2}\int_b^c r^{\frac{n-1}{2}}fr^{\frac{n-3}{2}}f_rdr\\ &\leq \frac{2}{n-2}\left(\int_b^c r^{n-1}f_r^2dr\right)^{\frac{1}{2}}\left(\int_b^c r^{n-3}f^2dr\right)^{\frac{1}{2}}.\end{aligned}$$

Then

$$\int_b^c r^{n-3}f^2dr \leq \frac{4}{(n-2)^2}\int_b^c r^{n-1}f_r^2dr.$$

Therefore,

$$C^{n-1}\int_b^c f_r^2r^{n-1}dr + a(n-1)(n-2)(1-C^2)C^{n-3}\int_b^c r^{n-3}f^2dr$$

is:

(1) $\geq \left(\frac{(n-2)^2}{4}C^2 + a(n-1)(n-2)(1-C^2)\right)C^{n-3}\int_b^c r^{n-3}f^2dr$, *if* $C<1$, *and*

(2) $\geq \left(C^2 + \frac{4a(n-1)(1-C^2)}{n-2}\right)C^{n-3}\int_b^c f_r^2r^{n-1}dr$, *if* $C>1$.

Using the Proposition 3.1.1, we conclude that M *is a-stable, if:*

$$\begin{cases} C<1 \quad and \quad a\geq \frac{n-2}{4(n-1)(1-C^{-2})};\\ C>1 \quad and \quad a\leq \frac{n-2}{4(n-1)(1-C^{-2})}.\end{cases}$$

3.2 Case $M = \mathbb{R}\times_{\rho_1} F_1 \times_{\rho_2} F_2 \times_{\rho_3} \cdots \times_{\rho_k} F_k$

The expression of the scalar curvature in M is

$$S = \sum_{i=1}^k \frac{S_{F_i}}{\rho_i^2} - 2\sum_{i=1}^k n_i\frac{\rho_i''}{\rho_i} - \sum_{i=1}^k n_i(n_i-1)\frac{\rho_i'^2}{\rho_i^2} - \sum_{i\neq j} n_in_j\frac{\rho_i'\rho_j'}{\rho_i\rho_j},$$

where n_i is the dimension of F_i, $i = 1, 2, \dots k$. In this section we always assume that each $n_i \geq 2$. The element of volume of M is expressed by

$$dV_M = \left(\prod_{i=1}^{k} \rho_i(r)^{n_i}\right) dV_{F_1} dV_{F_2} \dots dV_{F_k} dr.$$

We remember that we are working only with the cases in which each F_i is a compact manifold. Thus, we always assume that each F_i is compact. So we can continue using, as test functions, functions that only depend on r. Therefore, if $f \in C_c^\infty(M)$ is a function that is only dependent on r with support in the real line contained in the interval $[b, c]$, then

$$\begin{aligned}
\int_M -f L_a f &= \int_M |\nabla f|^2 + aSf^2 \\
&= \int_{-\infty}^{\infty} \int_{F_1 \times \dots \times F_n} \left[f_r^2 + \sum_{i=1}^{k} \frac{S_{F_i}}{\rho_i^2} f^2 - 2a \sum_{i=1}^{k} n_i \frac{\rho_i''(r)}{\rho_i(r)} f^2 \right. \\
&\quad -a \sum_{i=1}^{k} n_i(n_i - 1) \frac{\rho_i'(r)^2}{\rho_i(r)^2} f^2 \\
&\quad \left. -2a \sum_{i<j} n_i n_j \frac{\rho_i'(r)\rho_j'(r)}{\rho_i(r)\rho_j(r)} f^2 \right] \left(\prod_{i=1}^{k} \rho_i^{n_i}\right) dV_{F_1} dV_{F_2} \dots dV_{F_k} dr \\
&= \left(\prod_{i=1}^{k} Vol(F_i)\right) \left[\int_b^c f_r^2 \left(\prod_{i=1}^{k} \rho_i(r)^{n_i}\right) dr \right. \\
&\quad -2a \sum_{i=1}^{k} n_i \int_b^c \rho_i(r)^{n_i - 1} \left(\prod_{j \neq i} \rho_j(r)^{n_j}\right) \rho_i''(r) f^2 dr \\
&\quad -a \sum_{i=1}^{k} n_i(n_i - 1) \int_b^c \rho_i(r)^{n_i - 2} \left(\prod_{j \neq i} \rho_j(r)^{n_j}\right) \rho_i'(r)^2 f^2 dr \\
&\quad \left. -2a \sum_{i<j} n_i n_j \int_b^c \rho_i(r)^{n_i - 1} \rho_j(r)^{n_j - 1} \left(\prod_{i \neq l \neq j} \rho_l(r)^{n_l}\right) \rho_i'(r)\rho_j'(r) f^2 dr \right] \\
&\quad +a \sum_{i=1}^{k} S(F_i) \prod_{i \neq j} Vol(F_j) \int_b^c \rho_i^{n_i - 2} \left(\prod_{j \neq i} \rho_j(r)^{n_j}\right) f^2 dr \\
&= \left(\prod_{i=1}^{k} Vol(F_i)\right) \left[\int_b^c f_r^2 \left(\prod_{i=1}^{k} \rho_i(r)^{n_i}\right) dr \right.
\end{aligned}$$

$$+2a\sum_{i=1}^{k} n_i(n_i-1)\int_b^c \rho_i(r)^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)^2 f^2 dr$$

$$+2a\sum_{j\neq i} n_i n_j \int_b^c \rho_i(r)^{n_i-1}\rho_j(r)^{n_j-1}\rho_i'(j)\rho_j'(r)\left(\prod_{l\neq i, l\neq j}\rho_l(r)^{n_l}\right) f^2 dr$$

$$+4a\sum_{i=1}^{k} n_i \int_b^c \rho_i(r)^{n_i-1}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r) f f_r dr$$

$$-a\sum_{i=1}^{k} n_i(n_i-1)\int_b^c \rho_i(r)^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)^2 f^2 dr$$

$$\left.-2a\sum_{i<j} n_i n_j \int_b^c \rho_i(r)^{n_i-1}\rho_j(r)^{n_j-1}\left(\prod_{i\neq l\neq j}\rho_l(r)^{n_l}\right)\rho_i'(r)\rho_j'(r) f^2 dr\right]$$

$$+a\sum_{i=1}^{k} S(F_i)\prod_{i\neq j} Vol(F_j)\int_b^c \rho_i^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right) f^2 dr$$

$$=\left(\prod_{i=1}^{k} Vol(F_i)\right)\left[\int_b^c f_r^2\left(\prod_{i=1}^{k}\rho_i(r)^{n_i}\right) dr\right.$$

$$+a\sum_{i=1}^{k} n_i(n_i-1)\int_b^c \rho_i(r)^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)^2 f^2 dr$$

$$+2a\sum_{i<j} n_i n_j \int_b^c \rho_i(r)^{n_i-1}\rho_j(r)^{n_j-1}\rho_i'(j)\rho_j'(r)\left(\prod_{l\neq i, l\neq j}\rho_l(r)^{n_l}\right) f^2 dr$$

$$\left.+4a\sum_{i=1}^{k} n_i \int_b^c \rho_i(r)^{n_i-1}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r) f f_r dr\right]$$

$$+a\sum_{i=1}^{k} S(F_i)\prod_{i\neq j} Vol(F_j)\int_b^c \rho_i^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right) f^2 dr, \tag{3.9}$$

where $S(F_i)$ denotes the total scalar curvature of F_i. Thus, we have:

Theorem 3.2.1. *Let $M = I \times_{\rho_1} F_1 \times_{\rho_2} \cdots \times_{\rho_k} F_k$ be a multiple warped product, where each F_i is compact and with nonnegative scalar curvature. Suppose that for any $1 \le i, j \le k$ and $r \in I$, $\rho_i'(r)\rho_j'(r) \ge 0$. Then M is a-stable for all $0 \le a \le \min_{1\le i\le k}\{\frac{n_i-1}{4n_i}\}$.*

This theorem generalizes the Theorem 3.1.1. The following examples discuss the results of Theorem 3.2.1.

Example 3.2.1. *One of the simplest examples of a multiple warped product satisfying the hypotheses of the Theorem 3.2.1 is when* $\rho_1 = \ldots \rho_k = \rho$ *and the warped product can be written "in evidence" as a simple warped product given by* $M = I \times_\rho (F_1 \times \cdots \times F_k)$*, thus obtaining a "stability interval" greater than that of the conclusion of the theorem, being it* $[0, \frac{n-1}{4n}]$*, where* $n = n_1 + \cdots + n_k$*.*

Example 3.2.2. *Consider the warped product given by* $M = [0, \infty) \times_{\rho_1} F_1 \times_{\rho_2} F_2$ *where* F_1 *and* F_2 *are n-dimensional Riemannian manifolds with scalar curvature identically null,* $\rho_1(r) = r$ *for all* $r \geq 1$*,* $\rho_2(r) = r^{-1}$ *for all* $r \geq 1$*,* $\rho_1(0) = \rho_2(0) = 0$*, where each* ρ_i *is defined in* $(0, 1)$ *so as to become a smooth function. Then, in the submanifold of* M *formed by the fibers* $r \geq 1$ *we have* $S = -2nr^{-2}$ *and* $L_a = \Delta + 2anr^{-2}$*. Taking* f *of compact support only dependent on* r *and such that* $supp f \subset [b, c] \subset [1, \infty)$*, we have*

$$\int_M -f L_a f = Vol(F_1) Vol(F_2) \int_b^c (f_r^2 - 2anr^{-2} f^2) dr.$$

Taking $b = 1$*,* $c = 2R$ *and*

$$f(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq 1; \\ r - 1 & \text{if } 1 \leq r \leq 2; \\ \frac{1}{\sqrt{2}} r^{\frac{1}{2}} & \text{if } 2 \leq r \leq R; \\ -\frac{1}{\sqrt{2}} R^{-\frac{1}{2}} r + \sqrt{2} R^{\frac{1}{2}} & \text{if } R \leq r \leq 2R; \\ 0 & \text{if } r > 2R, \end{cases}$$

then

$$\begin{aligned} \int_b^c (f_r^2 - 2anr^{-2} f^2) dr) =& C + \int_2^R \left(\frac{1}{8} r^{-1} - anr^{-1} \right) dr \\ &+ \underbrace{\int_R^{2R} \left[\frac{1}{2} R^{-1} - anR^{-1} r^{-2} (-r + 2R)^2 \right] dr}_{O(1)} \\ \leq & C_2 + \left(\frac{1}{8} - an \right) \log R. \end{aligned}$$

Therefore, if $a > \frac{1}{8n}$*,* M *will be a-unstable, because we can choose* R *so large as desired. Note that* $\frac{n-1}{4n} > \frac{1}{8} n$*. This example shows the importance of the hypothesis* $\rho_i'(r) \rho_j'(r) \geq 0$ *in Theorem 3.2.1.*

Example 3.2.3. *Let $M = \mathbb{R} \times_{\rho_1} F_1 \times_{\rho_2} \cdots \times_{\rho_k} F_k$ be a multiple warped product, where each F_i is a copy of an n-dimensional compact manifold F with a metric g of null total scalar curvature and volume one, $\rho_i(r) = 1$ for $1 \leq i \leq k-1$ and $\rho_k(r) = r^\alpha$, $\alpha > 1$. The expression of (3.9) reduces to*

$$\int_M -fLf = \int_b^c f_r^2 r^{n\alpha} dr + an(n-1)\alpha^2 \int_b^c r^{n\alpha-2} f^2 dr + 4an\alpha \int_b^c r^{n\alpha-1} f f_r dr. \tag{3.10}$$

Take as a test function the function f defined by

$$f(r) = f_R(r) = \begin{cases} 0 & \text{if } r \leq 0,; \\ r & \text{if } 0 \leq r \leq 1 \\ r^{\frac{1-n\alpha}{2}} & \text{if } 1 \leq r \leq R; \\ -R^{-\frac{1+n\alpha}{2}} r + 2R^{\frac{1-n\alpha}{2}} & \text{if } R \leq r \leq 2R; \\ 0 & \text{if } r > 2R. \end{cases}$$

Then

$$\begin{aligned} \int_M -fL_a f =& C + \left[\left(\frac{1-n\alpha}{2} \right)^2 + an(n-1)\alpha^2 + 2an\alpha(1-n\alpha) \right] \int_1^R r^{-1} dr \\ &+ \underbrace{R^{-1-n\alpha} \int_R^{2R} r^{n\alpha} dr}_{O(1)} + \underbrace{an(n-1)\alpha^2 \int_R^{2R} r^{n\alpha-2} (-R^{-\frac{1+n\alpha}{2}} r + 2R^{\frac{1-n\alpha}{2}})^2 dr}_{O(1)} \\ &+ \underbrace{4an\alpha \int_R^{2R} -r^{n\alpha-1} R^{-\frac{1+n\alpha}{2}} (-R^{-\frac{1+n\alpha}{2}} r + 2R^{\frac{1-n\alpha}{2}}) dr}_{O(1)}, \end{aligned}$$

where C is a constant relative to the integration of (3.10) in the interval $[0,1]$. Then M is a-unstable, if

$$\left(\frac{1-n\alpha}{2} \right)^2 + an(n-1)\alpha^2 + 2an\alpha(1-n\alpha) < 0 \Longleftrightarrow a > \frac{(1-n\alpha)^2}{4n\alpha(n\alpha+\alpha-2)}.$$

Therefore, M is a-unstable for $a > \frac{(1-n\alpha)^2}{4n\alpha(n\alpha+\alpha-2)}$. Since

$$\lim_{\alpha \to 1} \frac{(1-n\alpha)^2}{4n\alpha(n\alpha+\alpha-2)} = \frac{n-1}{4n},$$

we can find an example of manifold M, under the conditions of the Theorem 3.2.1, which is a-unstable for each $a > \min_{1 \leq i \leq k} \{ \frac{n_i - 1}{4n_i} \}$.

Example 3.2.4. *Suppose that $M = \mathbb{R} \times_{\rho_1} F$, where F^n is a compact manifold of constant scalar curvature equal to $-S_0$ and $\rho(r) \equiv 1$. Then M is exactly the standard product*

$\mathbb{R} \times F$, $S_M \equiv -S_0$ and for f only dependent on r and of compact support contained in $[b, c]$:

$$\int_M -fL_a f = Vol(F) \int_b^c (f_r^2 - aS_0 f^2) dr.$$

Taking the sequence of functions f_N given by

$$f_N(r) = \begin{cases} 0 & \text{if } |r| \geq N+1; \\ r+N+1 & \text{if } -N-1 \leq r \leq N; \\ -r+N+1 & \text{if } N \leq r \leq N+1; \\ 1 & \text{if } |r| \leq N. \end{cases}$$

If $a > 0$, then, for N large, it is easy to see that $\int_M -fL_a f < 0$. Therefore, M is a-unstable. This example shows the importance of the nonnegativity of the scalar curvature of F.

The following theorem is a similar for the Theorem 3.1.3, for the multiple warped product case.

Theorem 3.2.2. *Let $M = \mathbb{R} \times_{\rho_1} F_1 \times_{\rho_2} \cdots \times_{\rho_k} F_k$ be a multiple warped product, n_i the dimension of F_i and $n = n_1 + \cdots + n_k$. Suppose that for each ρ_i, there exists R_0 such that one of the following two situations occurs for:*

(i) $\rho_i''(r) > 0$ for all $r > R_0$ and $\lim_{r \to \infty} \rho_i'(r) = \infty$;

(ii) $\rho_i''(r) > 0$ for all $r < R_0$ and $\lim_{r \to -\infty} \rho_i'(r) = -\infty$.

Then the corresponding end of M (and therefore, M) is a-unstable for all $a > \frac{n}{4(n+1)}$.

3.3 Applications: minimal immersions on $\mathbb{R}^n$

In this section, we present examples of minimal submanifolds in Euclidean space in the form of warped products.

3.3.1 Cones

C^n is said to be a cone if C can be written as a warped product $[0,\infty)\times_\rho F^{n-1}$, with $\rho(r)=r$. In a cone, there is a vertex, corresponding to the leaf $r=0$. We have

$$S=\frac{S_F-(n-1)(n-2)}{r^2}$$

and the expression of a-stability becomes

$$Vol(F)\left[\int_b^c f_r^2 r^{n-1}dr-a(n-1)(n-2)\int_b^c r^{n-3}f^2dr\right]+a\int_C S_F r^{n-3}f^2\geq 0. \tag{3.11}$$

Suppose that $F=\mathbb{S}^{n_1}_{r_1}\times\cdots\times\mathbb{S}^{n_k}_{r_k}$ is a product of k round spheres, with $n_1+\cdots+n_k=n-k+1$. Then F is a minimal submanifold of $\mathbb{S}^n\subset\mathbb{R}^{n+1}$, from the natural inclusion, if and only if $r_i=\sqrt{\frac{n_i}{n-k+1}}$ (see [Net14], Proposition 2.6). It is a known fact that, under these conditions, the singular cone $C=[0,\infty)\times_\rho F$, where $\rho(r)=r$, is a minimal cone in $\mathbb{R}^{n+1}$, invariant under dilatation. The scalar curvature of F is

$$S_F=\sum_{i=1}^k\frac{n-k+1}{n_i}n_i(n_i-1)=\sum_{i=1}^k(n-k+1)(n_i-1)=(n-k+1)(n-2k+1).$$

Since F has constant scalar curvature, to analyze if C is a-stable, we only need to analyze the set of test functions f only dependent on r (see Proposition 3.1.1). Suppose f is dependent only on r, then the expression of the a-stability of C (3.11) becomes:

$$\prod_{i=1}^k\left[Vol(\mathbb{S}^{n+i)}_{r_i})\right]\left[\int_b^c f_r^2r^{n-k+1}dr-a(n-k+1)(n-k)\int_b^c r^{n-k-1}f^2dr\right.$$

$$\left.+a(n-k+1)(n-2k+1)\int_b^c r^{n-k-1}f^2dr\right]\geq 0$$

$$\iff\int_b^c f_r^2r^{n-k+1}dr-a(n-k+1)(k-1)\int_b^c r^{n-k-1}f^2dr\geq 0. \tag{3.12}$$

Integrating by parts and using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\int_b^c r^{n-k-1}f^2dr&=-\frac{2}{n-k}\int_b^c r^{n-k}ff_rdr\\&\leq\frac{2}{n-k}\left(\int_b^c r^{n-k-1}f^2dr\right)^{\frac12}\left(\int_b^c f_r^2r^{n-k+1}dr\right)^{\frac12}\\\Longrightarrow\int_b^c f_r^2r^{n-k+1}dr&\geq\frac{(n-k)^2}{4}\int_b^c r^{n-k-1}f^2dr.\end{aligned}$$

Then the expression (3.12) is positive, if

$$\frac{(n-k)^2}{4} - a(n-k+1)(k-1) \geq 0 \Longleftrightarrow a \leq \frac{(n-k)^2}{4(n-k+1)(k-1)},$$

whenever $k > 1$.

a-stability of minimal hypercones

Let us consider the special case where $k = 2$ in the above situation. Under this condition, F becomes a minimal hypersurface of $\mathbb{S}^n$ and

$$S_F = (n-1)(n-3).$$

Therefore, $C = [0, \infty) \times_\rho F$, $\rho(r) = r$, is a n-dimensional minimal cone of $\mathbb{R}^{n+1}$, its scalar curvature is

$$S = \frac{S_F - (n-1)(n-2)}{r^2} = -\frac{n-1}{r^2}$$

and C is $\frac{(n-2)^2}{4(n-1)}$-stable.

The next proposition shows that this value of a is the highest possible positive value of a such that an n-dimensional cone can be a-stable, when $n \geq 7$.

Theorem 3.3.1. *An a-stable minimal cone C^n of $\mathbb{R}^{n+1}$, with $a > \max\left\{1, \frac{(n-2)^2}{4(n-1)}\right\}$, is flat.*

Before proving this result, we need of a result.

Proposition 3.3.1. *(Simons [Sim68]) Suppose that $M \to \mathbb{R}^n$ is a minimal immersion. Then, the second fundamental for A satisfies*

$$\frac{1}{2}\Delta|A|^2 + |A|^4 = |\nabla A|^2$$

on M.

For a demonstration, see ([Cho21], Proposition 8.13). It implies

$$|A|\Delta|A| + |A|^4 = |\nabla A|^2 - |\nabla|A||^2. \tag{3.13}$$

Proof of Theorem 3.3.1. Let A_F be the second fundamental form of the immersion $F \to \mathbb{S}^n$ and A be the second fundamental form of the immersion $C \to \mathbb{R}^{n+1}$. The cone C can be seen as the set

$$C = rF = \{rx, x \in F, r \geq 0\}.$$

Let $rx \in C$, $r \geq 0$, $x \in F$, by a direct calculus, $A(\partial r, \cdot) = 0$ and for $X, Y \in T_xF$, $A(X,Y) = r^{-1}A_F(X,Y)$. By Proposition 2.1.1,

$$\begin{aligned}(\nabla_{\partial r}A)(X,Y) =& \nabla_{\partial r}(A(X,Y)) - A(\nabla_{\partial r}X, Y) - A(X, \nabla_{\partial r}Y)\\ =& -r^{-2}A_F(X,Y)\\ =& -r^{-1}A(X,Y),\end{aligned}$$

where ∇ is the Levi-Civita connection of C. We can then consider $A = r^{-1}A_F$ and $\nabla_{\partial r}A = -r^{-1}A$. Being $\{e_1, \dots, e_{n-1}\}$ an orthonormal frame of F and $e_n = \partial r$ such that $\{e_1, \dots, e_n\}$ on a given point $p = (r, x)$ is geodesic and diagonalizing A, then

$$|\nabla A|^2 = \sum_{i,j,k=1}^{n} [(\nabla_{e_i}A)(e_j, e_k)]^2$$

and

$$\begin{aligned}|\nabla|A||^2 =& \sum_{i=1}^{n} \langle \nabla|A|, e_i\rangle^2\\ =& \sum_{i=1}^{n} \left\langle \nabla \left(\sum_{j,k=1}^{n} A(e_j, e_k)^2 \right)^{\frac{1}{2}}, e_i \right\rangle^2\\ =& |A|^{-2} \sum_{i,j,k=1}^{n} A(e_j, e_k)^2 [e_i(A(e_j, e_k))]^2\\ =& |A|^{-2} \sum_{i,j=1}^{n} A(e_j, e_j)^2 [e_i(A(e_j, e_j))]^2\\ =& |A|^{-2} \sum_{i,j-1}^{n} A(e_j, e_j)^2 [(\nabla_{e_i}A)(e_j, e_j)]^2\\ \leq& |A|^{-2} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A(e_j, e_j)^2 \right) \left(\sum_{j=1}^{n} [(\nabla_{e_i}A)(e_j, e_j)]^2 \right)\\ =& \sum_{i,j=1}^{n} [\nabla_{e_i}A(e_j, e_j)]^2.\end{aligned}$$

Hence

$$|\nabla A|^2 - |\nabla |A||^2 \geq 2\sum_{i=1}^{n} \sum_{1\leq j<k\leq n} [(\nabla_{e_i} A)(e_j, e_k)]^2$$
$$\geq 2\sum_{i=1}^{n}\sum_{j=1}^{n-1}[(\nabla e_i A)(e_j, e_n)]^2.$$

By Codazzi equation, $(\nabla_{e_i} A)(e_j, e_n) = (\nabla_{e_n} A)(e_i, e_j)$. Therefore,

$$|\nabla A|^2 - |\nabla |A||^2 \geq 2\sum_{i=1}^{n}\sum_{j=1}^{n-1}[(\nabla e_n A)(e_i, e_j)]^2$$
$$= 2r^{-2}|A|^2.$$

It and (3.13) imply

$$|A|\Delta|A| + |A|^4 \geq 2r^{-2}|A|^2.$$

For $\varphi \in C_c^\infty(C \setminus \{0\})$, multiplying the inequality above by $|A|^{\frac{1-a}{a}}\varphi^2$ and integrating on C, we have

$$\begin{aligned} 2\int_C |A|^{\frac{1+a}{a}} r^{-2}\varphi^2 &\leq \int_C |A|^{\frac{1}{a}}\varphi^2\Delta|A| + |A|^{\frac{1+3a}{a}}\varphi^2 \\ &= \int_C |A|^{\frac{1+3a}{a}}\varphi^2 - \frac{1}{a}|A|^{\frac{1-a}{a}}\varphi^2|\nabla|A||^2 - 2|A|^{\frac{1}{a}}\varphi\langle\nabla\varphi, \nabla|A|\rangle. \end{aligned} \tag{3.14}$$

Substituting $|A|^{\frac{a+1}{2a}}\varphi$ on a-stability inequality (recall that $|A|^2 = -S$ on minimal hypersurfaces of $\mathbb{R}^n$), we have:

$$\begin{aligned} a\int_C |A|^{\frac{1+3a}{a}}\varphi^2 &\leq \int_C |\nabla(|A|^{\frac{a+1}{2a}}\varphi)|^2 \\ &= \int_C \left(\frac{a+1}{2a}\right)^2 |A|^{\frac{1-a}{a}}\varphi^2|\nabla|A||^2 + |A|^{\frac{a+1}{a}}|\nabla\varphi|^2 + \frac{a+1}{a}|A|^{\frac{1}{a}}\varphi\langle\nabla|A|, \nabla\varphi\rangle \end{aligned}$$

(3.15)

Multiplying (3.14) by a and adding to (3.15), we find

$$\begin{aligned} 2a\int_C |A|^{\frac{a+1}{a}} r^{-2}\varphi^2 &\leq \int_C \left[\left(\frac{a+1}{2a}\right)^2 - 1\right]|A|^{\frac{1-a}{a}}\varphi^2|\nabla|A||^2 \\ &\quad + \frac{-2a^2+a+1}{a}|A|^{\frac{1}{a}}\varphi\langle\nabla|A|, \nabla\varphi\rangle + |A|^{\frac{a+1}{a}}|\nabla\varphi|^2 \\ &= \int_C \frac{(-3a-1)(a-1)}{4a^2}|A|^{\frac{1-a}{a}}\varphi^2|\nabla|A||^2 \\ &\quad + \frac{(-2a-1)(a-1)}{a}|A|^{\frac{1}{a}}\varphi\langle\nabla|A|, \nabla\varphi\rangle + |A|^{\frac{a+1}{a}}|\nabla\varphi|^2. \end{aligned} \tag{3.16}$$

Take $\varphi = \varphi(r)$ only dependent of r. Since $|A| = r^{-1}|A_F|$, then

$$\nabla\varphi = \varphi'(r)\partial_r, \quad \langle \nabla|A|, \nabla\varphi\rangle = -r^{-2}\phi'(r)|A_F| \quad \text{and} \quad |\nabla|A||^2 \geq r^{-4}|A_F|^2.$$

Hence (3.16) with the hypothesis $a > 1$ implies

$$2a\int_C |A_F|^{\frac{a+1}{a}} r^{-\frac{3a+1}{a}}\varphi^2 \leq \int_C -\frac{3a^2-2a-1}{4a^2} r^{-\frac{3a+1}{a}}|A_F|^{\frac{a+1}{a}}\varphi^2 + \frac{2a^2-a-1}{a}|A_F|^{\frac{a+1}{a}} r^{-\frac{2a+1}{a}}\varphi\varphi' + r^{-\frac{a+1}{a}}|A_F|^{\frac{a+1}{a}}\varphi'^2.$$

Since $dV_C = r^{n-1}dV_F dr$, the last inequality is equivalent to:

$$0 \leq \int_F |A_F|^{\frac{a+1}{a}} \int_0^\infty -\frac{8a^3+3a^2-2a-1}{4a^2} r^{\frac{na-4a-1}{a}}\varphi^2 + \frac{2a^2-a-1}{a} r^{\frac{na-3a-1}{a}}\varphi\varphi' + r^{\frac{na-2a-1}{a}}\varphi'^2.$$

If $\int_F |A_F|^{\frac{a+1}{a}} = 0$, then $|A_F| \equiv 0$ and $|A| \equiv 0$, therefore, C is flat. Thus, suppose that C is non flat, then

$$0 \leq \int_0^\infty -\frac{8a^3+3a^2-2a-1}{4a^2} r^{\frac{na-4a-1}{a}}\varphi^2 + \frac{2a^2-a-1}{a} r^{\frac{na-3a-1}{a}}\varphi\varphi' + r^{\frac{na-2a-1}{a}}\varphi'^2. \quad (3.17)$$

By an argument of approximation, we can take $\varphi \in C_c^{0,1}(C)$ defined by:

$$\varphi(r) = \varphi_R(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq \frac{1}{2}; \\ 2r & \text{if } \frac{1}{2} \leq r \leq 1; \\ r^{\frac{-na+3a+1}{2a}} & \text{if } 1 \leq r \leq R; \\ -R^{\frac{-na+a+1}{2a}} r + 2R^{\frac{-na+3a+1}{2a}} & \text{if } R \leq r \leq 2R; \\ 0 & \text{if } r > 2R. \end{cases}$$

The integral (3.17) on the interval $[0,1]$ is a constant independent of R. The integral on the interval $[1,R]$ becomes

$$\int_1^R \left[\frac{-8a^3-3a^2+2a+1}{4a^2} + \frac{2a^2-a-1}{a}\frac{-na+3a+1}{2a} + \frac{(na-3a-1)^2}{4a^2}\right] r^{-1}dr$$

$$= \left(\frac{-8a^3-3a^2+2a+1}{4a^2} + \frac{-4na^3+12a^3+4a^2+2na^2-6a^2-2a+2na-6a-2}{4a^2} + \frac{n^2a^2+9a^2+1-6na^2-2na+6a}{4a^2}\right)\log R$$

$$= \frac{-4na^3+4a^3+n^2a^2-4na^2+4a^2}{4a^2}\log R$$

$$= \left(-na + a + \frac{n^2}{4} - n + 1\right) \log R.$$

Hence, the integral of the expression (3.17) on interval $[1, R]$ will be negative and a multiple of $\log R$, if

$$-na + a + \frac{n^2}{4} - n + 1 < 0 \Longleftrightarrow a(1-n) < -\frac{(n-2)^2}{4} \Longleftrightarrow a > \frac{(n-2)^2}{4(n-1)}.$$

Let us check that the value of expression (3.17) on the interval $[R, 2R]$ is constant and independent of R. Note that:

$$\int_R^{2R} r^{\frac{na-4a-1}{a}} \varphi(r)^2 dr = R^{\frac{-na+a+1}{a}} \int_R^{2R} \left(r^{\frac{na-2a-1}{a}} - 4Rr^{\frac{na-3a-1}{a}} + 4R^2 r^{\frac{na-4a-1}{a}}\right) dr = C_1;$$

$$\int_R^{2R} r^{\frac{na-3a-1}{a}} \varphi(r)\varphi'(r) dr = R^{\frac{-na+a+1}{a}} \int_R^{2R} \left(-r^{\frac{na-2a-1}{a}} + 2Rr^{\frac{na-3a-1}{a}}\right) dr = C_2;$$

$$\int_R^{2R} r^{\frac{na-2a-1}{a}} \varphi'(r)^2 dr = R^{\frac{-na+a+1}{a}} \int_R^{2R} r^{\frac{na-2a-1}{a}} dr = C_3,$$

where C_1, C_2 and C_3 are independent of R. Hence, the expression of (3.17) on the interval $[R, 2R]$ is constant and independent of R.

Therefore, for R large, φ_R negatives (3.17) and C cannot be a-stable for $a > \frac{(n-2)^2}{4(n-1)}$, if C is non flat.

■

3.3.2 Minimal catenoid

In $\mathbb{R}^3$ the catenoid

$$(u, v) \longmapsto (c\cosh(v/c)\cos u, c\cosh(v/c)\sin u, v)$$

is such that is the union of two graphs of radial functions defined on the plane (x, y) minus the disk of radius c centered on the origin and these two functions are additively opposite. With this in mind, let us find a nonnegative radial function $\phi : \mathbb{R}^n \setminus B_{s_0}(0) \to \mathbb{R}$, where $s_0 > 0$ is a real number such that its graph: $x \mapsto (x, \phi(x))$ satisfies

$$\operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\right) = 0, \tag{3.18}$$

where is the equation of the minimal graph. Consider $\mathbb{R}^n \setminus B_{s_0}(0)$ as the warped product $[s_0, \infty) \times_\xi \mathbb{S}^{n-1}$, where $\xi(s) = s$. Let $\{e_1, \dots e_{n-1}\}$ be a local orthonormal frame of $\mathbb{S}^{n-1}$, using that $\nabla_{e_i}\partial s = \frac{1}{s}e_i$ (see Proposition 2.1.1, item (ii)), we have

$$\begin{aligned}
\operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\right) =& div\left(\frac{\phi'(s)\partial s}{\sqrt{1+\phi'(s)^2}}\right) \\
=& \frac{\phi'(s)}{\sqrt{1+\phi'(s)^2}}\sum_{i=1}^{n-1}\langle\nabla_{e_i}\partial_s, e_i\rangle \\
&+ \frac{\phi''(s)(1+\phi'(s)^2)^{\frac{1}{2}} - (1+\phi'(s)^2)^{-\frac{1}{2}}\phi'(s)^2\phi''(s)}{1+\phi'(s)^2} \\
=& \frac{(n-1)\phi'(s)}{s\sqrt{1+\phi'(s)^2}} + \frac{\phi''(s)(1+\phi'(s)^2)^{\frac{1}{2}} - (1+\phi'(s)^2)^{-\frac{1}{2}}\phi'(s)^2\phi''(s)}{1+\phi'(s)^2}.
\end{aligned}$$

Therefore, (3.18) is equivalent to

$$(n-1)s^{-1}\phi'(s) + \phi''(s) - \frac{\phi'(s)^2\phi''(s)}{1+\phi'(s)^2} = 0.$$

Because the idea is to generalize the properties of the two-dimensional catenoid, our problem becomes

$$\begin{cases} (n-1)s^{-1}\phi'(s) + \phi''(s) - (1+\phi'(s)^2)^{-1}\phi'(s)^2\phi''(s) = 0 \\ \phi(s_0) = 0 \\ \lim_{r\to s_0^+}\phi'(s) = \infty. \end{cases} \tag{3.19}$$

Let us do the change of variable $u = \phi'(s)^{-2} + 1$, then

$$du = -2\phi'(s)^{-3}d(\phi')|_s = -2(u-1)^{\frac{3}{2}}d(\phi')|_s \Longrightarrow \phi'' = -\frac{1}{2}(u-1)^{-\frac{3}{2}}u'.$$

Substituting in (3.19), we obtain

$$(n-1)s^{-1}(u-1)^{-\frac{1}{2}} - \frac{1}{1+\frac{1}{u-1}}\frac{1}{2}(u-1)^{-\frac{3}{2}}u' = 0 \Leftrightarrow \frac{u'}{u} = 2(n-1)s^{-1}.$$

Integrating and applying the exponential application on both sides of the last equality, we obtain $u = C_0 s^{2(n-1)}$, where $C_0 > 0$. Since $u = (\phi')^{-2} + 1$, we find

$$\phi'(s) = \frac{1}{\sqrt{C_0 s^{2(n-1)} - 1}}.$$

Then

$$\phi(s) = C_1 + \int_{s_0}^{s} \frac{dt}{\sqrt{C_0 t^{2(n-1)} - 1}}.$$

By the initial conditions, we have $C_1 = 0$ and $C_0 = s_0^{-2(n-1)}$, therefore,

$$\phi(s) = \int_{s_0}^{s} \frac{dt}{\sqrt{s_0^{-2(n-1)} t^{2(n-1)} - 1}}$$

is the solution to the problem (3.19). The graph of ϕ represents half of the catenoid. The catenoid becomes the union of the graphs of functions ϕ and $-\phi$.

We wish to write the catenoid as $\mathbb{R} \times_{\rho} \mathbb{S}^{n-1}$. Under the previous conditions, the catenoid metric is $(1+\phi'^2)ds^2 + s^2 g_{\mathbb{S}^{n-1}}$. We must find the variable r such that $dr^2 = (1+\phi'(s)^2)ds^2$, which is equivalent to

$$\frac{dr}{ds} = (1+\phi'^2)^{\frac{1}{2}} = \frac{s_0^{-n+1} s^{n-1}}{\sqrt{s_0^{-2(n-1)} s^{2(n-1)} - 1}}.$$

Because $r = 0$ when $s = s_0$, we have, for $r > 0$:

$$r = \int_{s_0}^{s} \frac{s_0^{-n+1} t^{n-1} dt}{\sqrt{s_0^{-2(n-1)} t^{2(n-1)} - 1}}. \tag{3.20}$$

Using the symmetry of the catenoid relative to the hyperplane $x_{n+1} = 0$, then $\rho(r) = \rho(-r) = s$ for all $r \geq 0$, where s satisfies (3.20).

We can find ρ explicitly when $n = 2$, in this case, we have

$$r = \int_{s_0}^{s} \frac{s_0^{-1} t dt}{\sqrt{s_0^{-2} t^2 - 1}} = \sqrt{s^2 - s_0^2},$$

since $s = \sqrt{r^2 + s_0^2}$, then

$$\rho(r) = \sqrt{r^2 + s_0^2}.$$

When $n \geq 3$, the solution ϕ of (3.19) is limited by a real positive number T. Therefore, the catenoid is completely contained in the set

$$\{x = (x_1, \ldots, x_{n+1}); -T < x_{n+1} < T\} \subset \mathbb{R}^{n+1}.$$

According to [TZ09], the catenoid is $\frac{n-2}{n}$-stable. Since the catenoid has two ends, it is a-unstable for all $a > (\frac{n-1}{n})^2$ (see [YS76], [CSZ97] and an adaptation of Theorem 8.7 in [Cho21]). Therefore, the catenoid is a-stable for all $a \leq \frac{n-2}{n}$ and a-unstable for all $a > (\frac{n-1}{n})^2$.

3.4 Proofs of the theorems

3.4.1 Theorems of Section 2.1

We start by proving the Theorem 3.1.2.

Proof of Theorem 3.1.2. First, since M is without boundary, $\rho(0) = 0$ and by hypothesis $\rho''(r) > 0$ for all r, we have $r\rho'(r) > \rho(r)$ for $r > 0$, because for $r = 0$ the equality holds and the application $r \mapsto r\rho'(r) - \rho(r)$ has as derivative the application $r \mapsto r\rho''(r) > 0$. Therefore, $r\rho(r)^{-1}\rho'(r) > 1$ for all $r > 0$. Furthermore, the application $r \mapsto r\rho(r)^{-1}$ is decreasing because its derivative is

$$r \mapsto \rho(r)^{-1} - r\rho(r)^{-2}\rho'(r) = \rho(r)^{-1}(1 - r\rho(r)^{-1}\rho'(r)) < 0.$$

Rewriting (3.3) we have, for $f = f(r)$ of compact support contained in $[b, c]$:

$$\begin{aligned}\frac{1}{A(F)}\int_M -fL_a f = \int_b^c f_r^2\rho(r)^{n-1}dr + a(n-1)(n-2)\int_b^c \rho'(r)^2\rho(r)^{n-3}f^2dr \\ + 4a(n-1)\int_b^c \rho'(r)\rho(r)^{n-2}ff_rdr + a\frac{S(F)}{A(F)}\int_b^c \rho(r)^{n-3}f^2dr.\end{aligned} \tag{3.21}$$

Take

$$f(r) = f_{Q,R}(r) = \begin{cases} 0 & \text{if } \; 0 \geq \frac{Q}{2} \\ 2\rho(r)^{-\frac{1}{2}(n-1)}Q^{-\frac{1}{2}}r - \rho(r)^{-\frac{1}{2}(n-1)}Q^{\frac{1}{2}} & \text{if } \; \frac{Q}{2} \leq r \leq Q; \\ \rho(r)^{-\frac{1}{2}(n-1)}r^{\frac{1}{2}}, & \text{if } \; Q \leq r \leq R; \\ C(\rho(r)^{-\frac{1}{2}(n-1)}r^{\frac{1}{2}} - \rho(T)^{-\frac{1}{2}(n-1)}T^{\frac{1}{2}}) & \text{if } \; R \leq r \leq T; \\ 0 & \text{if } \; r \geq T, \end{cases}$$

where T is such that $\rho(T) = 2\rho(R)$ and C is such that becomes f a continuous function. Assume Q is large such that $r\rho(r)^{-1}\rho'(r) > 1$ for all $r > Q$. We fix Q, then the integration of the expression (3.21) in the interval $[\frac{Q}{2}, Q]$ is fixed. In the interval $[Q, R]$ it becomes, using that $f_r = -\frac{1}{2}(n-1)\rho(r)^{-\frac{1}{2}(n-1)-1}\rho'(r)r^{\frac{1}{2}} + \frac{1}{2}\rho(r)^{-\frac{1}{2}(n-1)}r^{-\frac{1}{2}}$:

$$\frac{1}{4}(n-1)^2\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr - \frac{1}{2}(n-1)\int_Q^R \rho(r)^{-1}\rho'(r)dr + \frac{1}{4}\int_Q^R r^{-1}dr$$

$$+a(n-1)(n-2)\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr + 4a(n-1)\int_Q^R -\frac{1}{2}(n-1)r\rho(r)^{-2}\rho'(r)^2dr$$

$$+4a(n-1)\int_Q^R \frac{1}{2}\rho(r)^{-1}\rho'(r)dr + a\frac{S(F)}{A(F)}\int_Q^R r\rho(r)^{-2}dr$$

$$= \frac{1}{4}(n-1)(n-1-4an)\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr + \left(2a-\frac{1}{2}\right)(n-1)\int_Q^R \rho(r)^{-1}\rho'(r)dr$$

$$+\frac{1}{4}\int_Q^R r^{-1}dr + a\frac{S(F)}{A(F)}\int_Q^R r\rho(r)^{-2}dr. \tag{3.22}$$

Since $\rho'(r) \to \infty$ when $r \to \infty$, the last term of (3.22) is arbitrary small comparing to $\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr$ if we choose Q large. Since $r\rho(r)^{-1}\rho'(r) > 1$, we have $\rho(r)^{-1}\rho'(r) > \frac{1}{r}$, implying $r\rho(r)^{-2}\rho'(r)^2 > \rho(r)^{-1}\rho'(r) > \frac{1}{r}$. We wish to show that (3.22) is negative and upper limited by a negative constant C_1 times $\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr$. We have $(2a-\frac{1}{2})(n-1)+\frac{1}{4} < 0$ if and only if $a < \frac{2n-3}{8(n-1)}$, then this statement is true for $\frac{n-1}{4n} < a \leq \frac{2n-3}{8(n-1)}$ taking $C_1 = \frac{1}{4}(n-1)(n-1-4an)$. For $a > \frac{2n-3}{8(n-1)}$, we only have to show that

$$\frac{1}{4}(n-1)(n-1-4an) + \left(2a-\frac{1}{2}\right)(n-1) + \frac{1}{4} < 0,$$

that is equivalent to

$$(n-1)\left(-(n-2)a + \frac{n}{4} - \frac{3}{4}\right) < -\frac{1}{4}. \tag{3.23}$$

Since $n \geq 3$, the left-hand side of the previous inequality decreases as a function of a. At $a = \frac{2n-3}{8(n-1)}$ the left-hand side of (3.23) becomes

$$(n-1)\left(\frac{n}{4} - \frac{(2n-3)(n-2)}{8(n-1)} - \frac{3}{4}\right) = \frac{2n^2-2n-2n^2+7n-6-6n+6}{8} = -\frac{n}{8} < -\frac{1}{4}.$$

Hence, it follows from the decreasing property that the inequality (3.23) holds for all $a > \frac{2n-3}{8(n-1)}$. Note that $\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr \to \infty$ when $R \to \infty$, because $r\rho(r)^{-2}\rho'(r)^2 > \frac{1}{r}$ for $r > Q$. Let us now analyze the expression of f in (3.21) in the interval $[R,T]$. We have

$$f_r = -\frac{1}{2}(n-1)\rho(r)^{-\frac{1}{2}(n+1)}\rho'(r)r^{\frac{1}{2}} + \frac{1}{2}\rho(r)^{-\frac{1}{2}(n-1)}r^{-\frac{1}{2}}.$$

Then the expression of f in (3.21) in interval $[R,T]$ is C^2 times the expression

$$\frac{1}{4}(n-1)^2\int_R^T r\rho(r)^{-2}\rho'(r)^2dr - \frac{1}{2}(n-1)\int_R^T \rho(r)^{-1}\rho'(r)dr + \frac{1}{4}\int_R^T r^{-1}dr$$

$$+a(n-1)(n-2)\int_R^T r\rho(r)^{-2}\rho'(r)^2dr-2a(n-1)(n-2)\rho(T)^{-\frac{1}{2}(n-1)}T^{\frac{1}{2}}\int_R^T \rho(r)^{\frac{n-5}{2}}\rho'(r)^2r^{\frac{1}{2}}dr$$

$$+a(n-1)(n-2)\rho(T)^{-n+1}T\int_R^T \rho(r)^{n-3}\rho'(r)^2dr - 2a(n-1)^2\int_R^T r\rho(r)^{-2}\rho'(r)^2dr$$

$$+2a(n-1)^2\rho(T)^{-\frac{1}{2}(n-1)}T^{\frac{1}{2}}\int_R^T r^{\frac{1}{2}}\rho(r)^{\frac{n-5}{2}}\rho'(r)^2dr + 2a(n-1)\int_R^T \rho(r)^{-1}\rho'(r)dr$$

$$-2a(n-1)\rho(T)^{-\frac{1}{2}(n-1)}T^{\frac{1}{2}}\int_R^T r^{-\frac{1}{2}}\rho(r)^{\frac{n-3}{2}}\rho'(r)dr + a\frac{S(F)}{A(F)}\int_R^T r\rho(r)^{-2}dr.$$

(3.24)

Being $\rho(T) = 2\rho(R)$, by the mean value theorem and using that $\rho''(r) > 0$ and

$$r\rho(r)^{-1}\rho'(r) > 1 \Longleftrightarrow r\rho'(r) > \rho(r)$$

for all $r > Q$, there exists $c \in (R, 2R)$ such that

$$\rho(2R) - \rho(R) = R\rho'(c) > R\rho'(R) > \rho(R) \Longrightarrow \rho(2R) > 2\rho(R).$$

Hence $R < T < 2R$. The relation $\rho(T) = 2\rho(R)$ with $R < T < 2R$ implies that all the terms of (3.24) are upper limited by a multiple of

$$\int_R^T r\rho(r)^{-2}\rho'(r)^2dr,$$

because all the expressions are limited by a constant times $\int_R^T r\rho(r)^{-2}\rho'(r)^2dr$. We observe the expression $\int_R^T r\rho(r)^{-2}\rho'(r)^2dr$, as a consequence of integration by parts and using that $\rho(T) = 2\rho(R)$, $r\rho(r)^{-1}$ is decreasing and $\rho''(r) \geq 0$ for $r > Q$, we have

$$\begin{aligned}\int_R^T r\rho(r)^{-2}\rho'(r)^2dr &= \int_R^T \rho(r)^{-1}\rho'(r)dr + \int_R^T r\rho(r)^{-1}\rho''(r)dr - r\rho(r)^{-1}\rho'(r))|_R^T \\ &\leq \log(\rho(T)) - \log(\rho(R)) + R\rho(R)^{-1}(\rho'(T) - \rho'(R)) - T\rho(T)^{-1}\rho'(T) \\ &\quad + R\rho(R)^{-1}\rho'(R) \\ &= \log 2 + 2R\rho(T)^{-1}\rho'(T) - T\rho(T)^{-1}\rho'(T) \\ &\leq \log 2 + T\rho(T)^{-1}\rho'(T).\end{aligned}$$

(3.25)

Now, our objective is to show that the expression of the stability operator in $[Q, R]$ can be arbitrarily larger than the expression of the stability operator in $[R, T]$ for R and T appropriated. Let $g(r) = \rho(r)^{-1}\rho'(r)$. We assert that for all $C > 0$, there exists V sufficiently large such that

$$\int_Q^V rg(r)^2 dr > CVg(V). \tag{3.26}$$

For this, suppose that $\int_Q^V rg(r)^2 dr \leq CVg(V)$ for some constant C and for all $V > Q$, let $h(V) = \int_Q^V rg(r)^2 dr$, then

$$h(V) \leq C(Vh'(V))^{\frac{1}{2}} \Longrightarrow h(V)^2 \leq C^2 V h'(V) \Longrightarrow \frac{h'(V)}{h(V)^2} \geq \frac{1}{C^2 V}.$$

Integrating the last inequality from Q to V, we obtain

$$h(Q)^{-1} - h(V)^{-1} \geq \frac{1}{C^2}(\log V - \log Q).$$

This is a contradiction because we saw that $h(V) = \int_Q^V r\rho(r)^{-2}\rho'(r)^2 dr$ tends to infinity when V tends to infinity, the left-hand side of the last inequality is upper limited by $h(Q)^{-1}$ and the right-hand side tends to infinity when V tends to infinity.

That said, combining (3.25) and (3.26), there is an increasing and unbounded sequence $\{T_1, T_2, \dots\}$ such that

$$\lim_{j\to\infty} \frac{\int_Q^{T_j} r\rho(r)^{-2}\rho'(r)^2 dr}{\int_{R_j}^{T_j} r\rho(r)^{-2}\rho'(r)^2 dr} = \infty,$$

where R_j is such that $\rho(T_j) = 2\rho(R_j)$. It implies

$$\lim_{j\to\infty} \frac{\int_Q^{R_j} r\rho(r)^{-2}\rho'(r)^2 dr}{\int_{R_j}^{T_j} r\rho(r)^{-2}\rho'(r)^2 dr} = \infty.$$

Therefore, the expression of the stability operator in $[Q, R_j]$ is arbitrarily larger than the expression of the stability operator in $[R_j, T_j]$ when J increases to infinity. It finishes the proof.

Note that if F has nonpositive total scalar curvature, the last term of (3.22) and (3.24) will be negative and can be disregarded in the respective calculus of instability and we do not need of $\rho'(r) \to \infty$ when $r \to \infty$.

■

Proof of Theorem 3.1.3: Suppose that (i) occurs. The proof is similar to Theorem 3.1.2, with some modifications. We affirm that it continues to be valid that $r\rho'(r) > \rho(r)$ for r sufficiently large (larger than R_0). To verify this, because $(r\rho'(r) - \rho(r))' = r\rho''(r) > 0$, it follows that $r\rho'(r) - \rho(r)$ increases and we only have to prove that the inequality $r\rho'(r) > \rho(r)$ holds for one value of r. Suppose that $r\rho'(r) \leq \rho(r)$ for all $r > R_0$, then for $r > R_0$:

$$\begin{aligned}\frac{\rho'(r)}{\rho(r)} \leq \frac{1}{r} \Longrightarrow & \int_{R_0}^{R} \frac{\rho'(r)}{\rho(r)} \leq \log R - \log R_0 \\ \Longrightarrow & \log(\rho(R)) - \log(\rho(R_0)) \leq \log R - \log R_0 \\ \Longrightarrow & \rho(R) \leq CR,\end{aligned}$$

where C is a constant, expressed in terms of R_0. It is a contradiction, because

$$\rho(r) \leq CR \Longrightarrow \liminf_{r\to\infty} \rho'(r) \leq C$$

and, by hypothesis, $\lim_{r\to\infty} \rho'(r) = \infty$. Therefore, for r large, $r\rho(r)^{-1}\rho'(r) > 1$ and everything else follows in a manner analogous to the proof of Theorem 3.1.2. The proof of (ii) follows from (i) taking the analysis of $\tilde{\rho}$ defined by $\tilde{\rho}(r) = \rho(-r)$.

■

Proof of Theorem 3.1.4. Using the Proposition 3.1.1, we only have to prove that for r of compact support only dependent on r and b, c such that $\pi_1(suppf) \subset [b, c]$:

$$\int_b^c f_r^2 \rho(r)^{n-1} dr + a(n-1)(n-2)\int_b^c (\rho'(r)^2+1)\rho(r)^{n-3} f^2 dr$$

$$+4a(n-1)\int_b^c \rho'(r)\rho(r)^{n-2} f f_r dr \geq 0$$

$$\Longleftrightarrow \int_b^c f_r^2 \rho(r)^{n-1} dr + 4a(n-1)\int_b^c \rho'(r)\rho(r)^{n-2} f f_r dr + 4a^2(n-1)^2 \int_b^c \rho'(r)^2 \rho(r)^{n-3} f^2 dr$$

$$-4a^2(n-1)^2\int_b^c \rho'(r)^2\rho(r)^{n-3} f^2 dr + a(n-1)(n-2)\int_b^c (\rho'(r)^2+1)\rho(r)^{n-3} f^2 dr \geq 0$$

$$\Longleftrightarrow \int_b^c (f_r \rho(r)^{\frac{n-1}{2}} + 2a(n-1)\rho'(r)\rho(r)^{\frac{n-3}{2}} f)^2 dr$$

$$+\int_b^c \big(-4a(n-1)\rho'(r)^2 + (n-2)(\rho'(r)^2+1)\big)\, a(n-1)\rho(r)^{n-3} f^2 dr \geq 0.$$

Because the first term of the least inequality is nonnegative, the inequality holds if

$$-4a(n-1)\rho'(r)^2 + (n-2)(\rho'(r)^2+1) \geq 0 \Longleftrightarrow a \leq \frac{(\rho'(r)^2+1)(n-2)}{4\rho'(r)^2(n-1)}.$$

Since the application $r \mapsto \frac{1+r}{r}$ is decreasing and $\rho'(r) \leq C$ for all $r > 0$, we conclude that M is a-stable for all $0 \leq a \leq \frac{(C^2+1)(n-2)}{4C^2(n-1)}$.

■

Proof of Theorem 3.1.5. Taking $\rho(r) = r^{\alpha}\xi(r)$, then $C_1 \leq \xi(r) \leq C_2$ for $r \geq 1$. Let $R > 0$ be large and the family $f_{R,\beta}$ of functions defined by:

$$f(r) = f_{R,\beta}(r) = \begin{cases} 0 & \text{if } \ 0 \leq r \leq \frac{m-1}{m}R; \\ (2R^{-\frac{n\alpha-\alpha+1}{2}}r - R^{-\frac{n\alpha-\alpha-1}{2}})\xi(R)^{-2a(n-1)} & \text{if } \ \frac{R}{2} \leq r \leq R; \\ r^{-\frac{n\alpha-\alpha-1}{2}}\xi(r)^{-2a(n-1)} & \text{if } \ R \leq r \leq R^{\beta}; \\ (-2R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}r + 3R^{-\frac{n\alpha\beta-\alpha\beta-\beta}{2}})\xi(r)^{-2a(n-1)} & \text{if } \ R^{\beta} \leq r \leq \frac{3}{2}R^{\beta}; \\ 0 & \text{if } \ r \geq \frac{3}{2}R^{\beta}. \end{cases}$$

Let us analyze the value of the expression obtained in (3.7) in each one of the intervals $[\frac{R}{2}, R]$, $[R, R^{\beta}]$, $[R^{\beta}, \frac{3}{2}R^{\beta}]$ separately.

For interval $[\frac{R}{2}, R]$, we will only define the value of (3.7) by K_R. Let us now analyze the expression (3.7) in the interval $[R, R^{\beta}]$. In this interval, we have:

(i) $f(r) = r^{-\frac{n\alpha-\alpha-1}{2}}\xi(r)^{-2a(n-1)}$;

(ii) $f_r = -\dfrac{n\alpha-\alpha-1}{2}r^{-\frac{n\alpha-\alpha+1}{2}}\xi(r)^{-2a(n-1)} - 2a(n-1)r^{-\frac{n\alpha-\alpha-1}{2}}\xi(r)^{-2a(n-1)-1}\xi'(r)$;

(iii) $f_r^2 = \dfrac{(n\alpha-\alpha-1)^2}{4}r^{-n\alpha+\alpha-1}\xi(r)^{-4a(n-1)}$

$\quad +2a(n-1)(n\alpha-\alpha-1)r^{-n\alpha+\alpha}\xi(r)^{-4a(n-1)-1}\xi'(r)$

$\quad +4a^2(n-1)^2r^{-n\alpha+\alpha+1}\xi(r)^{-4a(n-1)-2}\xi'(r)^2$;

(iv) $ff_r = -\dfrac{n\alpha-\alpha-1}{2}r^{-n\alpha+\alpha}\xi(r)^{-4a(n-1)} - 2a(n-1)r^{-n\alpha+\alpha+1}\xi(r)^{-4a(n-1)-1}\xi'(r)$.

Substituting the expression of $f = f_{R,\beta}$ in (3.7) in the interval $[R, R^{\beta}]$ and using (i), (ii), (iii) and (iv), we obtain

$$\frac{(n\alpha-\alpha-1)^2}{4}\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr+2a(n-1)(n\alpha-\alpha-1)\int_R^{R^\beta}\xi(r)^{(1-4a)(n-1)-1}\xi'(r)dr$$

$$+4a^2(n-1)^2\int_R^{R^\beta} r\xi(r)^{(1-4a)(n-1)-2}\xi'(r)^2dr+a\alpha^2(n-1)(n-2)\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr$$

$$+2a\alpha(n-1)(n-2)\int_R^{R^\beta}\xi(r)^{(1-4a)(n-1)-1}\xi'(r)dr$$

$$+a(n-1)(n-2)\int_R^{R^\beta} r\xi(r)^{(1-4a)(n-1)-2}\xi'(r)^2dr$$

$$-2a\alpha(n-1)(n\alpha-\alpha-1)\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr$$

$$-8a^2\alpha(n-1)^2\int_R^{R^\beta}\xi(r)^{(1-4a)(n-1)-1}\xi'(r)dr$$

$$-2a(n-1)(n\alpha-\alpha-1)\int_R^{R^\beta}\xi(r)^{(1-4a)(n-1)-1}\xi'(r)dr$$

$$-8a^2(n-1)^2\int_R^{R^\beta} r\xi(r)^{(1-4a)(n-1)-2}\xi'(r)^2dr$$

$$+a\frac{S(F)}{A(F)}\int_R^{R^\beta} r^{-2\alpha+1}\xi(r)^{(1-4a)(n-1)-2}dr$$

$$=\left(\frac{(n\alpha-\alpha-1)^2}{4}-a\alpha(n-1)(n\alpha-2)\right)\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr$$

$$+2a\alpha(n-1)[(1-4a)(n-1)-1]\int_R^{R^\beta}\xi(r)^{(1-4a)(n-1)-1}\xi'(r)dr$$

$$+a(n-1)[(1-4a)(n-1)-1]\int_R^{R^\beta} r\xi(r)^{(1-4a)(n-1)-2}\xi'(r)^2dr$$

$$+a\frac{S(F)}{A(F)}\int_R^{R^\beta} r^{-2\alpha+1}\xi(r)^{(1-4a)(n-1)-2}dr. \tag{3.27}$$

Note that, since $a>\frac{n-2}{4(n-1)}$,

$$(1-4a)(n-1)-1<\left(1-\frac{n-2}{n-1}\right)(n-1)-1=0,$$

$(1-4a)(n-1)-1$ is negative. By hypothesis, $a>\frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$, therefore, the first and third terms of (3.27) are negatives. The fourth term of (3.27) is significantly small when

R is large because $-2\alpha+1 < -1$, implying that this term is less than $\frac{S(F)}{(2\alpha-2)A(F)} R^{2-2\alpha} \to 0$ when $R \to \infty$. The second term of (3.27) is equal to

$$\frac{2a\alpha[(1-4a)(n-1)-1]}{1-4a}(\xi(R^\beta)^{(1-4a)(n-1)} - \xi(R)^{(1-4a)(n-1)}),$$

if $a \neq \frac{1}{4}$, and equal to

$$2a\alpha[(1-4a)(n-1)-1][\log(\xi(R^\beta)) - \log(\xi(R))],$$

if $a = \frac{1}{4}$.

In both cases, it is limited because in $[R, R^\beta]$, $C_1 \leq \xi(r) \leq C_2$. Since there exists $C_3 > 0$ such that $\xi(r)^{(1-4a)(n-1)} \geq C_3$ for all $r \in [R, R^\beta]$, where we can take

$$C_3 = \min\{C_1^{(1-4a)(n-1)}, C_2^{(1-4a)(n-1)}\},$$

then

$$\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)} dr \geq C_3(\beta-1)\log R$$

can be so large as we want. Therefore, the expression in (3.7) in the interval $[R, R^\beta]$ can be so (negatively) large as we want, based on the choice of β.

Because of the previous conclusion, on interval $[R^\beta, \frac{3}{2}R^\beta]$, we only have to prove that the expression (3.7) in this interval is upper limited for a constant that is independent of β. We have

$$f(r) = (-2R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}} r + 3R^{-\frac{n\alpha\beta-\alpha\beta-\beta}{2}})\xi(r)^{-2a(n-1)},$$

then

$$f_r(r,\cdot) = -2R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\xi(r)^{-2a(n-1)} - 2a(n-1)f\xi(r)^{-1}\xi'(r).$$

Substituting in (3.7) and using that there is a constant $C_2 > 0$ such that $\xi(r) \leq C_2$ for all $r \geq R^\beta$, we obtain that the expression (3.7) in interval $[R^\beta, \frac{3}{2}R^\beta]$, when $\beta \to \infty$, is:

$$\underbrace{4R^{-n\alpha\beta+\alpha\beta-\beta}\int_{R^\beta}^{R^{\frac{3}{2}\beta}} r^{n\alpha-\alpha}\xi(r)^{(1-4a)(n-1)}dr}_{O(1)} + 4a^2(n-1)^2\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2 dr$$

$$+8a(n-1)R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f\xi(r)^{(1-2a)(n-1)-1}\xi'(r)dr$$

$$+\underbrace{4a\alpha^2(n-1)(n-2)R^{-n\alpha\beta+\alpha\beta-\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}\xi(r)^{(1-4a)(n-1)}dr}_{O(1)}$$

$$
\begin{aligned}
&\underbrace{-12a\alpha^2(n-1)(n-2)R^{-n\alpha\beta+\alpha\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}\xi(r)^{(1-4a)(n-1)}dr}_{O(1)}\\
&\underbrace{+9a\alpha^2(n-1)(n-2)R^{-n\alpha\beta+\alpha\beta+\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-2}\xi(r)^{(1-4a)(n-1)}dr}_{O(1)}\\
&+2a\alpha(n-1)(n-2)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r)dr\\
&+a(n-1)(n-2)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2dr\\
&\underbrace{-8a\alpha(n-1)R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}\xi(r)^{(1-2a)(n-1)}f dr}_{O(1)}\\
&-8a^2\alpha(n-1)^2\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r)dr\\
&-8a(n-1)R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f\xi(r)^{(1-2a)(n-1)-1}\xi'(r)dr\\
&-8a^2(n-1)^2\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2dr+\underbrace{a\frac{S(F)}{A(F)}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-3\alpha}\xi(r)^{n-3}f^2dr}_{o(1)}\\
&\le C+a(n-1)\underbrace{(n-2-4(n-1))}_{\le 0}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2dr\\
&+2a\alpha(n-1)(n-2-4a(n-1))\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r)dr.
\end{aligned}
$$

It remains to be shown that the last term is limited by a constant independent of β. If $a \neq \frac{1}{4}$, we have:

$$
\begin{aligned}
\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r)dr =&4R^{-n\alpha\beta+\alpha\beta-\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha+1}\xi(r)^{(1-4a)(n-1)-1}\xi'(r)dr\\
&-12R^{-n\alpha\beta+\alpha\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}\xi(r)^{(1-4a)(n-1)-1}\xi'(r)dr\\
&+9R^{-n\alpha\beta+\alpha\beta+\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}\xi(r)^{(1-4a)(n-1)-1}\xi'(r)dr
\end{aligned}
$$

$$= \frac{4R^{-n\alpha\beta+\alpha\beta-\beta}}{(1-4a)(n-1)} \left[r^{n\alpha-\alpha+1}\xi(r)^{(1-4a)(n-1)} \,\Big|_{R^\beta}^{\frac{3}{2}R^\beta} \right.$$

$$\left. -(n\alpha-\alpha+1)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}\xi(r)^{(1-4a)(n-1)}dr \right]$$

$$- \frac{12R^{-n\alpha\beta+\alpha\beta}}{(1-4a)(n-1)} \left[r^{n\alpha-\alpha}\xi(r)^{(1-4a)(n-1)} \,\Big|_{R^\beta}^{\frac{3}{2}R^\beta} \right.$$

$$\left. -(n\alpha-\alpha)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}\xi(r)^{(1-4a)(n-1)}dr \right]$$

$$+ \frac{9R^{-n\alpha\beta+\alpha\beta+\beta}}{(1-4a)(n-1)} \left[r^{n\alpha-\alpha-1}\xi(r)^{(1-4a)(n-1)} \,\Big|_{R^\beta}^{\frac{3}{2}R^\beta} \right.$$

$$\left. -(n\alpha-\alpha-1)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-2}\xi(r)^{(1-4a)(n-1)}dr \right].$$

Using that $C_1 \leq \xi(r) \leq C_2$, we can conclude that the last expression is limited by a constant independent of β. This ends the proof for the case $a \neq \frac{1}{4}$. If $a = \frac{1}{4}$, we have

$$\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r)dr = 4R^{-n\alpha\beta+\alpha\beta-\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha+1}\xi(r)^{-1}\xi'(r)dr$$

$$-12R^{-n\alpha\beta+\alpha\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}\xi(r)^{-1}\xi'(r)dr$$

$$+9R^{-n\alpha\beta+\alpha\beta+\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}\xi(r)^{-1}\xi'(r)dr$$

$$= 4R^{-n\alpha\beta+\alpha\beta-\beta}\left[r^{n\alpha-\alpha-1}\log(\xi(r)) \,\Big|_{R^\beta}^{\frac{3}{2}R^\beta} \right.$$

$$\left. -(n\alpha-\alpha+1)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}\log(\xi(r))dr \right]$$

$$-12R^{-n\alpha\beta+\alpha\beta}\left[r^{n\alpha-\alpha}\log(\xi(r)) \,\Big|_{R^\beta}^{\frac{3}{2}R^\beta} \right.$$

$$\left. -(n\alpha-\alpha)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}\log(\xi(r))dr \right]$$

$$+9R^{-n\alpha\beta+\alpha\beta+\beta}\left[r^{n\alpha-\alpha-1}\log(\xi(r)) \,\Big|_{R^\beta}^{\frac{3}{2}R^\beta} \right.$$

$$-(n\alpha-\alpha-1)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-2}\log(\xi(r))dr\Bigg].$$

As in the previous case, the limitation $C_1 \leq \xi(r) \leq C_2$ ensures that the expression is limited by a constant independent of β. This completes the proof.

In the next proposition, we show that the constant $\frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$ which lowers the constant a is the best possible in the general case of Theorem 3.1.5.

Proposition 3.4.1. *Let $n \geq 2$ and M be the n-dimensional plane with the metric $dr^2 + \rho(r)^2 g_{\mathbb{S}^{n-1}}$, when $\rho(r) = \frac{1}{\alpha+1}r^\alpha$, with $\alpha > 1$, then L is a-stable for all $a \leq \frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$.*

Proof. Note that, for all real function f of compact support:

$$\begin{aligned}\int_0^\infty r^{n\alpha-\alpha-2}f^2dr &= -\frac{2}{n\alpha-\alpha-1}\int_0^\infty r^{n\alpha-\alpha-1}ff'dr\\ &= -\frac{2}{n\alpha-\alpha-1}\int_0^\infty r^{\frac{n\alpha-\alpha-2}{2}}fr^{\frac{n\alpha-\alpha}{2}}f'dr\\ &\leq \frac{2}{n\alpha-\alpha-1}\left(\int_0^\infty r^{n\alpha-\alpha-2}f^2dr\right)^{\frac{1}{2}}\left(\int_0^\infty r^{n\alpha-\alpha}(f')^2dr\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\int_0^\infty r^{n\alpha-\alpha-2}f^2dr \leq \frac{4}{(n\alpha-\alpha-1)^2}\int_0^\infty r^{n\alpha-\alpha}(f')^2dr.$$

By Proposition 3.1.1, we can reduce the space of test functions to the space of smooth compact support functions that depend only on r. Suppose f is dependent only on r, substituting in (3.8) (note that, in this case, $\xi(r) = \frac{1}{\alpha+1}$), we have

$$\begin{aligned}\frac{1}{(\alpha+1)^{n-1}A(F)}\int_M -fL_af &= \int_0^\infty f_r^2 r^{n\alpha-\alpha}dr - a\alpha(n-1)(n\alpha-2)\int_0^\infty r^{n\alpha-\alpha-2}f^2dr\\ &\quad + a(n-1)(n-2)(\alpha+1)^2\int_0^\infty r^{n\alpha-3\alpha}f^2dr\\ &\geq \left(\frac{(n\alpha-\alpha-1)^2}{4} - a\alpha(n-1)(n\alpha-2)\right)\int_0^\infty r^{n\alpha-\alpha-2}f^2dr,\end{aligned}$$

the last term is nonnegative for all $a \leq \frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$.

■

Proof of Theorem 3.1.6. Since $\xi(r) \to 0$ when $r \to \infty$ and $r\xi(r)^{-1}\xi'(r) \geq -(\alpha - 1)$, we have

$$\begin{aligned}
\xi(r)^{-1}\xi'(r) &\geq -\frac{\alpha - 1}{r} \\
\Longrightarrow \int_1^r \xi(s)^{-1}\xi'(s)ds &\geq -(\alpha - 1)\int_1^r s^{-1}ds \\
\Longrightarrow \log(\xi(r)) - \log(\xi(1)) &\geq -(\alpha - 1)\log r \\
\Longrightarrow \xi(r) &\geq C_1 r^{1-\alpha},
\end{aligned}$$

where $C_1 = \xi(1)^{-1}$. In order to prove by contradiction, let us suppose that

$$\limsup_{r\to\infty} \xi(r)r^{\alpha-1} = \infty.$$

Since

$$(\xi(r)r^{\alpha-1})' = \xi'(r)r^{\alpha-1} + (\alpha-1)\xi(r)r^{\alpha-2} \geq -(\alpha-1)r^{-1}\xi(r)r^{\alpha-1} + (\alpha-1)\xi(r)r^{\alpha-2} = 0,$$

the application $r \mapsto \xi(r)r^{\alpha-1}$ is nondecreasing and we can assume that

$$\lim_{r\to\infty} \xi(r)r^{\alpha-1} = \infty.$$

Take as a test function

$$f(r) = f_{R,\beta}(r,\cdot) = \begin{cases} 0 & \text{if } 0 \leq r \leq \frac{R}{2}; \\ (2R^{-\frac{n\alpha-\alpha+1}{2}}r - R^{-\frac{n\alpha-\alpha-1}{2}})\xi(R)^{-\frac{1}{2}(n-1)} & \text{if } \frac{R}{2} \leq r \leq R; \\ r^{-\frac{n\alpha-\alpha-1}{2}}\xi(r)^{-\frac{1}{2}(n-1)} & \text{if } R \leq r \leq R^\beta; \\ (-2R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}r + 3R^{-\frac{n\alpha\beta-\alpha\beta-\beta}{2}})\xi(r)^{-\frac{1}{2}(n-1)} & \text{if } R^\beta \leq r \leq \frac{3}{2}R^\beta; \\ 0 & \text{if } r \geq \frac{3}{2}R^\beta. \end{cases}$$

As in the proof of the previous theorem, let us analyze the value of the expression obtained in (3.7) in each of the intervals: $[\frac{R}{2}, R]$, $[R, R^\beta]$, $[R^\beta$ and $\frac{3}{2}R^\beta]$ separately.

On interval $[\frac{R}{2}, R]$, we will only define the value of (3.7) on interval $[\frac{R}{2}, R]$ by K_R. Let us now analyze the expression (3.7) in the interval $[R, R^\beta]$. In this interval, we have

(i) $f(r) = r^{-\frac{n\alpha-\alpha-1}{2}}\xi(r)^{-\frac{1}{2}(n-1)}$;

(ii) $f_r = -\dfrac{n\alpha-\alpha-1}{2}r^{-\frac{n\alpha-\alpha+1}{2}}\xi(r)^{-\frac{1}{2}(n-1)} - \dfrac{1}{2}(n-1)r^{-\frac{n\alpha-\alpha-1}{2}}\xi(r)^{-\frac{1}{2}(n-1)-1}\xi'(r)$;

(iii) $f_r^2 = \dfrac{(n\alpha-\alpha-1)^2}{4}r^{-n\alpha+\alpha-1}\xi(r)^{-n+1} + \dfrac{1}{2}(n-1)(n\alpha-\alpha-1)r^{-n\alpha+\alpha}\xi(r)^{-n}\xi'(r)$

$$+\frac{1}{4}(n-1)^2 r^{-n\alpha+\alpha+1}\xi(r)^{-n-1}\xi'(r)^2;$$

(iv) $ff_r = -\frac{n\alpha-\alpha-1}{2} r^{-n\alpha+\alpha}\xi(r)^{-n+1} - \frac{1}{2}(n-1)r^{-n\alpha+\alpha+1}\xi(r)^{-n}\xi'(r).$

Substituting the expression of $f = f_{R,\beta}$ in (3.7) in the interval $[R, R^\beta]$ and using (i), (ii), (iii) and (iv), we obtain

$$\frac{(n\alpha-\alpha-1)^2}{4}\int_R^{R^\beta} r^{-1}dr + \frac{1}{2}(n-1)(n\alpha-\alpha-1)\int_R^{R^\beta}\xi(r)^{-1}\xi'(r)dr$$
$$+\frac{1}{4}(n-1)^2\int_R^{R^\beta} r\xi(r)^{-2}\xi'(r)^2dr + a\alpha^2(n-1)(n-2)\int_R^{R^\beta} r^{-1}dr$$
$$+2a\alpha(n-1)(n-2)\int_R^{R^\beta}\xi(r)^{-1}\xi'(r)dr + a(n-1)(n-2)\int_R^{R^\beta} r\xi(r)^{-2}\xi'(r)^2dr$$
$$-2a\alpha(n-1)(n\alpha-\alpha-1)\int_R^{R^\beta} r^{-1}dr - 2a\alpha(n-1)^2\int_R^{R^\beta}\xi(r)^{-1}\xi'(r)dr$$
$$-2a(n-1)(n\alpha-\alpha-1)\int_R^{R^\beta}\xi(r)^{-1}\xi'(r)dr - 2a(n-1)^2\int_R^{R^\beta} r\xi(r)^{-2}\xi'(r)^2dr$$
$$+a\frac{S(F)}{A(F)}\int_R^{R^\beta} r^{-2\alpha+1}\xi(r)^{-2}dr$$
$$=\left(\frac{(n\alpha-\alpha-1)^2}{4} - a\alpha(n-1)(n\alpha-2)\right)\int_R^{R^\beta} r^{-1}dr$$
$$+\frac{1}{2}(n-1)(n\alpha-\alpha-4a\alpha n+4a-1)\int_R^{R^\beta}\xi(r)^{-1}\xi'(r)dr$$
$$+(n-1)\left(\frac{1}{4}n - an - \frac{1}{4}\right)\int_R^{R^\beta} r\xi(r)^{-2}\xi'(r)^2dr$$
$$+a\frac{S(F)}{A(F)}\int_R^{R^\beta} r^{-2\alpha+1}\xi(r)^{-2}dr. \qquad (3.28)$$

By hypothesis, $a \geq \frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$, then the first term of (3.28) is nonpositive. Let us show that the second term of (3.28) increases arbitrarily more than the fourth term of (3.28) (in module), if we choose R and β appropriately. Note that

$$\int_R^{R^\beta}\xi(r)^{-1}\xi'(r)dr = \log(\xi(R^\beta)) - \log(\xi(r)) \to \infty$$

when $R \to \infty$, because $\xi(r) \to 0$ when $r \to \infty$. If

$$\int_R^\infty r^{-2\alpha+1}\xi(r)^{-2}dr < \infty,$$

there is nothing to do. Suppose that

$$\int_R^\infty r^{-2\alpha+1}\xi(r)^{-2}dr = \infty,$$

then for all $\delta > 0$,

$$\liminf_{r\to\infty} \xi(r)r^{\alpha-1-\delta} = 0,$$

because otherwise,

$$\liminf_{r\to\infty} \xi(r)r^{\alpha-1-\delta} = C > 0,$$

where C can be ∞. It implies

$$\limsup_{r\to\infty} \xi(r)^{-2}r^{-2\alpha+2+2\delta} = C^{-2}.$$

Then for R sufficiently large,

$$\int_R^\infty r^{-2\alpha+1}\xi(r)^{-2}dr = \int_R^\infty r^{-1-2\delta}\xi(r)^{-2}r^{-2\alpha+2+2\delta} \leq (C^{-2}+1)\int_R^\infty r^{-1-2\delta} < \infty.$$

It is a contradiction. Hence, there exists a sequence $\{R_1, R_2, \dots\}$ such that

$$\xi(R_i) < R_i^{-\alpha+1+\frac{\alpha-1}{2}} \Longrightarrow \xi(R_i) < R_i^{-\frac{\alpha-1}{2}} \Longrightarrow \log(\xi(R_i)) < -\frac{\alpha-1}{2}\log R_i$$

for all positive integer i. Therefore, it is possible to choose β large and appropriately such that

$$\int_R^{R^\beta} \xi(r)^{-1}\xi'(r)dr < C_1\log(R^\beta) = C_1\beta\log R$$

for some negative constant C_1 (we can choose β such that $R^\beta \in \{R_1, R_2, \dots\}$, where $\{R_i\}$ is the sequence defined above and C_1 a little bigger than $-\frac{\alpha-1}{2}$). On the other hand,

$$\int_r^{R^\beta} r^{-2\alpha+1}\xi(r)^{-2}dr = \int_R^{R^\beta} (\xi(r)^{-2}r^{-2\alpha+2})r^{-1}dr.$$

Since we are assuming that $\lim_{r\to\infty}\xi(r)r^{\alpha-1} = \infty$, taking R large, then

$$\int_R^{R^\beta} r^{-2\alpha+1}\xi(r)^{-2}dr \ll \beta\log R.$$

It shows that the fourth term of (3.28) is arbitrarily less than the second term of (3.28) (in module). Intending to compare the second and the third term, we have

$$\int_R^{R^\beta} r\xi(r)^{-2}\xi'(r)^2dr \leq -(\alpha-1)\int_R^{R^\beta} \xi(r)^{-1}\xi'(r)dr,$$

then the combination between the second and the third term of (3.28) is less than

$$\int_R^{R^\beta} \xi(r)^{-1}\xi'(r)dr$$

times the constant

$$\frac{1}{2}(n-1)(n\alpha-\alpha-4a\alpha n+4a-1)-(\alpha-1)(n-1)\left(\frac{1}{4}n-an-\frac{1}{4}\right)$$

$$=\frac{1}{4}(n-1)(n\alpha-\alpha-4a\alpha n+8a+n-4an-3). \qquad (3.29)$$

Substituting $a=\frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$ on (3.29), we have

$$\frac{1}{4}(n-1)\left[\frac{n\alpha^2(n-1)(n\alpha-2)}{\alpha(n-1)(n\alpha-2)}-\frac{\alpha^2(n-1)(n\alpha-2)}{\alpha(n-1)(n\alpha-2)}-\frac{\alpha n(n\alpha-\alpha-1)^2}{\alpha(n-1)(n\alpha-2)}\right.$$
$$+\frac{2(n\alpha-\alpha-1)^2}{\alpha(n-1)(n\alpha-2)}+\frac{n\alpha(n-1)(n\alpha-2)}{\alpha(n-1)(n\alpha-2)}-\frac{n(n\alpha-\alpha-1)^2}{\alpha(n-1)(n\alpha-2)}$$
$$\left.-\frac{3\alpha(n-1)(n\alpha-2)}{\alpha(n-1)(n\alpha-2)}\right]$$
$$=\frac{1}{4\alpha(n\alpha-2)}(n^3\alpha^3-2n^2\alpha^2-n^2\alpha^3+2n\alpha^2-n^2\alpha^3+2n\alpha^2+n\alpha^3-2\alpha^2-n^3\alpha^3-n\alpha^3-n\alpha$$
$$+2n^2\alpha^3+2n^2\alpha^2-2n\alpha^2+2n^2\alpha^2+2\alpha^2+2-4n\alpha^2-4n\alpha+4\alpha+n^3\alpha^2-2n^2\alpha$$
$$-n^2\alpha^2+2n\alpha-n^3\alpha^2-n\alpha^2-n+2n^2\alpha^2+2n^2\alpha-2n\alpha-3n^2\alpha^2+6n\alpha+3n\alpha^2-6\alpha)$$
$$=\frac{n\alpha-2\alpha-n+2}{4\alpha(n\alpha-2)}=\frac{(n-2)(\alpha-1)}{4\alpha(n\alpha-2)}>0;$$

because $\alpha>1$ and $n\geq 3$. Therefore, the combination of the second and the third term of (3.28) is less than

$$\frac{(n-2)(\alpha-1)}{4\alpha(n\alpha-2)}\int_R^{R^\beta} \xi(r)^{-1}\xi'(r)dr.$$

Since the first term of (3.28) is zero and the fourth term is irrelevant to the second term for R large, then, for R sufficiently large, the expression (3.28) is less than

$$\frac{(n-2)(\alpha-1)}{8\alpha(n\alpha-2)}\int_R^{R^\beta} \xi(r)^{-1}\xi'(r)dr=\frac{(n-2)(\alpha-1)}{8\alpha(n\alpha-2)}(\log(\xi(R^\beta)-\xi(R))).$$

Using that $\lim_{r\to\infty}\xi(r)=0$, we conclude that (3.28) can be (negatively) so large as we want, taking β large.

Based on the previous conclusion, on interval $[R^\beta, \frac{3}{2}R^\beta]$, we only have to prove that the substitution of (3.7) by f is limited and independent of β. We have

$$f(r) = (-2R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}r + 3R^{-\frac{n\alpha\beta-\alpha\beta-\beta}{2}})\xi(r)^{-\frac{1}{2}(n-1)},$$

then

$$f_r = -2R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\xi(r)^{-\frac{1}{2}(n-1)} - \frac{1}{2}(n-1)f\xi(r)^{-1}\xi'(r).$$

Substituting in (3.7), we obtain when $\beta \to \infty$:

$$\underbrace{4R^{-n\alpha\beta+\alpha\beta-\beta}\int_{R^\beta}^{R^{\frac{3}{2}\beta}} r^{n\alpha-\alpha}dr}_{O(1)} + \frac{1}{4}(n-1)^2\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2dr$$

$$+2(n-1)R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f\xi(r)^{\frac{1}{2}(n-1)-1}\xi'(r)dr$$

$$+\underbrace{4a\alpha^2(n-1)(n-2)R^{-n\alpha\beta+\alpha\beta-\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}dr}_{O(1)}$$

$$-\underbrace{12a\alpha^2(n-1)(n-2)R^{-n\alpha\beta+\alpha\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}dr}_{O(1)}$$

$$+\underbrace{9a\alpha^2(n-1)(n-2)R^{-n\alpha\beta+\alpha\beta+\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-2}dr}_{O(1)}$$

$$+2a\alpha(n-1)(n-2)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r)dr$$

$$+a(n-1)(n-2)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2dr$$

$$-\underbrace{8a\alpha(n-1)R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}\xi(r)^{\frac{1}{2}(n-1)}fdr}_{O(1)}$$

$$-2a\alpha(n-1)^2\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r)dr$$

$$-8a(n-1)R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f\xi(r)^{\frac{1}{2}(n-1)-1}\xi'(r)dr$$

$$-2a(n-1)^2\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2dr + \underbrace{a\frac{S(F)}{A(F)}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-3\alpha}\xi(r)^{n-3}f^2dr}_{O(1)}$$

$$\leq C + (n-1)\left(\frac{1}{4}n - an - \frac{1}{4}\right)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2dr$$

$$-2a\alpha(n-1)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r)dr$$

$$+(2-8\alpha)(n-1)R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f\xi(r)^{\frac{1}{2}(n-1)-1}\xi'(r)dr. \tag{3.30}$$

We must show that all these terms is upper limited by a constant independent of β. We can note that, on the interval $[R^\beta, \frac{3}{2}R^\beta]$,

$$0 \leq f(r) \leq R^{-\frac{n\alpha\beta-\alpha\beta-\beta}{2}}\xi(r)^{-\frac{1}{2}(n-1)}.$$

Furthermore, $-(\alpha+1) \leq r\xi(r)^{-1}\xi'(r) \leq 0$. Hence,

$$\begin{aligned}
0 \leq \int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2dr &\leq \int_{R^\beta}^{\frac{3}{2}R^\beta}\left(\frac{r}{R^\beta}\right)^{n\alpha-\alpha-1} r\xi(r)^{n-2}\xi'(r)^2dr\\
&\leq \left(\frac{3}{2}\right)^{n\alpha-\alpha-1}(\alpha-1)^2\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{-1}dr\\
&= \left(\frac{3}{2}\right)^{n\alpha-\alpha-1}(\alpha-1)^2\log\left(\frac{3}{2}\right);
\end{aligned}$$

$$\begin{aligned}
0 \geq \int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r)dr &\geq \int_{R^\beta}^{\frac{3}{2}R^\beta}\left(\frac{r}{R^\beta}\right)^{n\alpha-\alpha-1}\xi(r)^{-1}\xi'(r)dr\\
&\geq -\left(\frac{3}{2}\right)^{n\alpha-\alpha-1}(\alpha-1)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{-1}dr\\
&= -\left(\frac{3}{2}\right)^{n\alpha-\alpha-1}(\alpha-1)\log\left(\frac{3}{2}\right);
\end{aligned}$$

$$\begin{aligned}
0 \geq R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f\xi(r)^{\frac{1}{2}(n-1)-1}\xi'(r)dr &\geq \int_{R^\beta}^{\frac{3}{2}R^\beta}\left(\frac{r}{R^\beta}\right)^{n\alpha-\alpha-1}\xi(r)^{-1}\xi'(r)dr\\
&\geq -\left(\frac{3}{2}\right)^{n\alpha-\alpha-1}(\alpha-1)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{-1}dr\\
&= -\left(\frac{3}{2}\right)^{n\alpha-\alpha-1}(\alpha-1)\log\left(\frac{3}{2}\right).
\end{aligned}$$

It shows that (3.30) is upper limited by a constant independent of β and shows that M is a-unstable for $a = \frac{(n\alpha-\alpha-1)^2}{4\alpha(n-1)(n\alpha-2)}$.

Therefore, $r \mapsto \xi(r) r^{\alpha-1} \leq C$ for some $C > 0$ and

$$\rho(r) = r^{\alpha}\xi(r) \leq Cr.$$

■

Proof of Theorem 3.1.7 Let us remember (3.6), that

$$\alpha = \inf\{\gamma; \lim_{r\to\infty} \rho(r) r^{-\gamma} = 0\} \tag{3.31}$$

and $\xi(r) = \rho(r) r^{-\alpha}$. We affirm that

$$\limsup_{r\to\infty} \frac{\log(\xi(r))}{\log r} = 0.$$

To verify this, let

$$C = \limsup_{r\to\infty} \frac{\log(\xi(r))}{\log r}.$$

If $C > 0$, then there exists an unlimited sequence $\{R_1, R_2, \dots\}$ such that

$$\frac{\log(\xi(R_i))}{\log R_i} > \frac{C}{2} \Longrightarrow \xi(R_i) > R_i^{\frac{C}{2}} \quad \forall i \in \mathbb{N}.$$

It contradicts the characterization of α in (3.31), because it would imply

$$\limsup_{r\to\infty} \rho(r) r^{-(\alpha+\frac{C}{4})} = \limsup_{r\to\infty} \xi(r) r^{-\frac{C}{4}} \geq \limsup_{i\to\infty} \xi(R_i) R_i^{-\frac{C}{4}} = +\infty.$$

On the other hand, if $C < 0$, there exists R_0 such that for all $r > R_0$,

$$\frac{\log(\xi(r))}{\log r} < \frac{C}{2} \Longrightarrow \xi(r) < r^{\frac{C}{2}}.$$

It also contradicts the characterization of α in (3.31), because it would imply

$$\lim_{r\to\infty} \rho(r) r^{-(\alpha+\frac{C}{4})} = \lim_{r\to\infty} \xi(r) r^{-\frac{C}{4}} < \lim_{r\to\infty} r^{\frac{C}{4}} = 0.$$

Note that $\alpha + \frac{C}{4} < \alpha$, contracting the characterization of α in (3.31). Therefore, fixed R, we can choose β appropriately large such that

$$\frac{\int_R^{R^\beta} \xi(r)^{-1}\xi'(r)dr}{\beta\log(R)} \approx 0. \tag{3.32}$$

Our test function will be the same of the proof of Theorem 3.1.6, that is

$$f(r) = f_{R,\beta}(r,\cdot) = \begin{cases} 0 & \text{if} \quad 0 \leq r \leq \frac{R}{2}; \\ (2R^{-\frac{n\alpha-\alpha+1}{2}} r - R^{-\frac{n\alpha-\alpha-1}{2}})\xi(R)^{-\frac{1}{2}(n-1)} & \text{if} \quad \frac{R}{2} \leq r \leq R; \\ r^{-\frac{n\alpha-\alpha-1}{2}}\xi(r)^{-\frac{1}{2}(n-1)} & \text{if} \quad R \leq r \leq R^\beta; \\ (-2R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}} r + 3R^{-\frac{n\alpha\beta-\alpha\beta-\beta}{2}})\xi(r)^{-\frac{1}{2}(n-1)} & \text{if} \quad R^\beta \leq r \leq \frac{3}{2}R^\beta; \\ 0 & \text{if} \quad r \geq \frac{3}{2}R^\beta. \end{cases}$$

Our strategy will be similar to that of the proofs of the previous theorems. We will use the expression (3.7) again. Fixed R, let K_R be the value of the expression (3.7) on interval $[0, R]$, then K_R is a constant. Let us analyze the expression (3.7) on the interval $[R, R^\beta]$. The expression is the same of (3.28), that is

$$= \left(\frac{(n\alpha-\alpha-1)^2}{4} - a\alpha(n-1)(n\alpha-2)\right)\int_R^{R^\beta} r^{-1}dr$$

$$+\frac{1}{2}(n-1)(n\alpha-\alpha-4a\alpha n+4a-1)\int_R^{R^\beta} \xi(r)^{-1}\xi'(r)dr$$

$$+(n-1)\left(\frac{1}{4}n-an-\frac{1}{4}\right)\int_R^{R^\beta} r\xi(r)^{-2}\xi'(r)^2dr$$

$$+a\frac{S(F)}{A(F)}\int_R^{R^\beta} r^{-2\alpha+1}\xi(r)^{-2}dr. \tag{3.33}$$

the relation (3.32) implies that the second term of (3.33) can be arbitrarily small than the first term of (3.33) from an appropriate choice for β large. Since $a > \frac{n-1}{4n}$, the third term of (3.33) is nonpositive and the fourth term is equal to

$$a\frac{S(F)}{A(F)}\int_R^{R^\beta} r\rho(r)^{-2}dr.$$

Suppose that

$$\lim_{r\to\infty}\int_R^{R^\beta} r\rho(r)^{-2}dr = 0,$$

then this term is irrelevant in relation to the first, hence, the expression (3.33) can be negatively so large as we want.

It remains to show that the expression (3.7) in the interval $[R^\beta, \frac{3}{2}R^\beta]$ is upper bounded by a constant independent of β. The expression, by (3.30), is upper limited by

$$C+(n-1)\left(\frac{1}{4}n-an-\frac{1}{4}\right)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2dr$$

$$+a\frac{S(F)}{A(F)}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-3\alpha}\xi(r)^{n-3}f^2dr$$

$$-2a\alpha(n-1)\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r)dr$$

$$+(2-8\alpha)(n-1)R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}f\xi(r)^{\frac{1}{2}(n-1)-1}\xi'(r)dr, \tag{3.34}$$

where C is a constant independent of β. Our objective is to show that (3.34) increases arbitrary less than $\log R$. Since $a > \frac{n-1}{4n}$, we have $\frac{1}{4}n - an - \frac{1}{4} < 0$. Let D and H be negative fixed constants such that

$$D + H = (n-1)\left(\frac{1}{4}n - an - \frac{1}{4}\right),$$

$E = -2a\alpha(n-1)$ and $J = (2-8a)(n-1)$. Since

$$\left(\sqrt{-D}r^{\frac{n\alpha-\alpha}{2}} f\xi(r)^{\frac{n-3}{2}}\xi'(r) - \frac{E}{2\sqrt{-D}}r^{\frac{n\alpha-\alpha-2}{2}} f\xi(r)^{\frac{n-1}{2}}\right)^2 \geq 0,$$

we have

$$Dr^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2 + Er^{n\alpha-\alpha-1}f^2\xi(r)^{n-2}\xi'(r) + \frac{E^2}{4D}r^{n\alpha-\alpha-2}f^2\xi(r)^{n-1} \leq 0 \quad (3.35)$$

and since

$$\left(\sqrt{-H}r^{\frac{n\alpha-\alpha}{2}} f\xi(r)^{\frac{n-3}{2}}\xi'(r) - \frac{J}{2\sqrt{-H}}R^{-\frac{n\alpha\beta-\alpha\beta-\beta}{2}} r^{\frac{n\alpha-\alpha}{2}}\right)^2 \geq 0,$$

we have

$$Hr^{n\alpha-\alpha}f^2\xi(r)^{n-3}\xi'(r)^2 + JR^{-\frac{n\alpha\beta-\alpha\beta-\beta}{2}}r^{n\alpha-\alpha}f\xi(r)^{\frac{1}{2}(n-1)-1}\xi'(r) + \frac{J^2}{4H}R^{-n\alpha\beta-\alpha\beta-\beta}r^{n\alpha-\alpha} \leq 0 \quad (3.36)$$

Integrating (3.35) and (3.36) from R^β to $\frac{3}{2}R^\beta$ ans addicting this two inequalities, we have that the expression (3.34) is upper limited by

$$C - \frac{E^2}{4D}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha-2}f^2\xi(r)^{n-1}dr - \frac{J^2}{4H}R^{-n\alpha\beta-\alpha\beta-\beta}\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-\alpha}dr,$$

which is upper limited by a constant because $f(r) \leq R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\xi(r)^{-\frac{1}{2}(n-1)}$ on $[R^\beta, \frac{3}{2}R^\beta]$. Therefore, the combination of the first, third and fourth term of (3.34) is upper limited by a constant. Using that $f(r) \leq R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}\xi(r)^{-\frac{1}{2}(n-1)}$ on $[R^\beta, \frac{3}{2}R^\beta]$, we have that

$$\int_{R^\beta}^{\frac{3}{2}R^\beta} r^{n\alpha-3\alpha}\xi(r)^{n-3}f^2dr \leq D\int_{R^\beta}^{\frac{3}{2}R^\beta} r\rho(r)^{-2}dr$$

for a determined constant D, where by our previous assumption, the right-hand side of the inequality grows arbitrarily less than $\log r$. Therefore, under this conditions, M can not be a-stable for $a > \frac{n-1}{4n}$. Therefore,

$$\limsup_{r\to\infty}(\log R)^{-1}\int_1^R r\rho(r)^{-2}dr \geq C$$

for some $C > 0$.

If $S(F) \leq 0$, the fourth term of (3.33) and the second term of (3.34) is nonpositive and, assuming that $\alpha > \frac{2}{n}$, to M be a-stable, we must to have

$$\begin{aligned}
&\frac{(n\alpha-\alpha-1)^2}{4} - a\alpha(n-1)(n\alpha-2) \geq 0\\
\Longrightarrow\ &\frac{(n\alpha-\alpha-1)^2}{4} - \frac{n-1}{4n}\alpha(n-1)(n\alpha-2) > 0\\
\Longrightarrow\ &n(n^2\alpha^2+\alpha^2+1-2n\alpha^2-2n\alpha+2\alpha) - (n\alpha^2-2\alpha)(n^2-2n+1) > 0\\
\Longrightarrow\ &-2\alpha n + 2\alpha + n > 0\\
\Longrightarrow\ &\alpha < \frac{n}{2n-2},
\end{aligned}$$

Therefore, $\lim_{r\to\infty}\rho(r)r^{-\frac{n}{2n-2}} = 0$. It finishes the proof.

■

Remark 3.4.1. *In the case $n = 2$, where F has dimension one, we have $\frac{2}{n} = 1$ and $n\alpha - 2 = 0$ when $\alpha = 1$. The conclusion will be that the volume growth of M satisfies*

$$\lim_{R\to\infty} r^{-2-\delta}Vol(B_r(p)) = 0,$$

for all $\delta > 0$ and $p \in M$.

Proof of Theorem 3.1.8. Take as test function the same of the proof of Theorem 3.1.5, that is,

$$f(r) = f_{R,\beta}(r) = \begin{cases} 0 & \text{if } \ 0 \leq r \leq \frac{m-1}{m}R;\\ (2R^{-\frac{n\alpha-\alpha+1}{2}}r - R^{-\frac{n\alpha-\alpha-1}{2}})\xi(R)^{-2a(n-1)} & \text{if } \ \frac{R}{2} \leq r \leq R;\\ r^{-\frac{n\alpha-\alpha-1}{2}}\xi(r)^{-2a(n-1)} & \text{if } \ R \leq r \leq R^\beta;\\ (-2R^{-\frac{n\alpha\beta-\alpha\beta+\beta}{2}}r + 3R^{-\frac{n\alpha\beta-\alpha\beta-\beta}{2}})\xi(r)^{-2a(n-1)} & \text{if } \ R^\beta \leq r \leq \frac{3}{2}R^\beta;\\ 0 & \text{if } \ r \geq \frac{3}{2}R^\beta. \end{cases}$$

Then the expression (3.7) on interval $[R, R^\beta]$ is, by (3.27),

$$\left(\frac{(n\alpha-\alpha-1)^2}{4} - a\alpha(n-1)(n\alpha-2)\right)\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr$$

$$+2a\alpha(n-1)[(1-4a)(n-1)-1]\int_R^{R^\beta} \xi(r)^{(1-4a)(n-1)-1}\xi'(r)dr$$

$$+a(n-1)[(1-4a)(n-1)-1]\int_R^{R^\beta} r\xi(r)^{(1-4a)(n-1)-2}\xi'(r)^2dr$$

$$+a\frac{S(F)}{A(F)}\int_R^{R^\beta} r^{-2\alpha+1}\xi(r)^{(1-4a)(n-1)-2}dr$$

$$=\left(\frac{(n\alpha-\alpha-1)^2}{4}-a\alpha(n-1)(n\alpha-2)\right)C_3\int_R^{R^\beta} r^{-1}dr$$

$$+2a\alpha(n-1)[(1-4a)(n-1)-1]C_4\int_R^{R^\beta}\xi(r)^{-1}\xi'(r)dr$$

$$+a(n-1)[(1-4a)(n-1)-1]C_5\int_R^{R^\beta} r\xi(r)^{-2}\xi'(r)^2dr$$

$$+a\frac{S(F)}{A(F)}C_6\int_R^{R^\beta} r^{-2\alpha+1}dr, \tag{3.37}$$

where C_3, C_4, C_5 and C_6 are positive constants, that are dependent of C_1, C_2 and a. Note that in (3.37), the first term is arbitrary less than the third term, the second and the fourth term are limited. Therefore, if $(1-4a)(n-1)-1<0 \Leftrightarrow a>\frac{n-2}{4(n-1)}$, (3.37) is negative and, in module, so arbitrary large as we want. On the expression (3.7) on the interval $[R^\beta, \frac{3}{2}R^\beta]$, the argument is the same of in the proof of Theorem 3.1.5, where we can conclude that the expression (3.7) on the interval $[R^\beta, \frac{3}{2}R^\beta]$ is upper limited by a constant. Therefore, L_a for $a>\frac{n-2}{4(n-1)}$ is unstable. It finishes the proof.

Proof of Theorem 3.1.9. When $\alpha=1$, we have $n\alpha-3\alpha=n\alpha-\alpha-2=n-3$. Redoing all calculations and steps in the proof of Theorem 3.1.5, in the analysis in the interval $[R, R^\beta]$ taking $\alpha=1$, the expression (3.27) becomes, for $a\neq\frac{1}{4}$:

$$\frac{n-2}{4}(n-2-4a(n-1))\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr$$

$$+\frac{2a(n-1)[(1-4a)(n-1)-1]}{(1-4a)(n-1)}[\xi(R^\beta)^{(1-4a)(n-1)}-\xi(R)^{(1-4a)(n-1)}]$$

$$+a(n-1)[(1-4a)(n-1)-1]\int_R^{R^\beta} r\xi(r)^{(1-4a)(n-1)-2}\xi'(r)^2dr$$

$$+a(n-1)(n-2)\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)-2}dr. \tag{3.38}$$

It is easy to see that, just as in the case $\alpha>1$ (case of Theorem 3.1.5), the part of integration in the interval $[R^\beta, \frac{3}{2}R^\beta]$ in is limited by a constant that independent of β, the only difference is that the last term that is $o(1)$ becomes a term $O(1)$ (see the proof of Theorem 3.1.5). Returning to the analysis in the interval $[R, R^\beta]$, under the hypothesis $0 < C_1 \leq \xi(r) \leq C_2 < \infty$, we only have to prove that the expression in (3.38) is negative and large in module for β large. In (3.38), we have four terms; the second is limited because $C_1 \leq \xi(r) \leq C_2(r)$ and the third is nonpositive for all $a \geq \frac{n-2}{4(n-1)}$. Then, we only have to prove that the combination of the first and fourth terms is negative and, in module, so large as we want. We have:

$$\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)-2}dr \leq C_1^{-2}\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr,$$

then

$$\begin{aligned}
&\frac{n-2}{4}(n-2-4a(n-1))\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr \\
&+a(n-1)(n-2)\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)-2}dr \\
&\leq \left(\frac{(n-2)^2}{4} + a(n-1)(n-2)(-1+C_1^{-2})\right)\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr.
\end{aligned}$$

Since $a > \frac{n-2}{4(n-1)(1-C_1^{-2})}$ and $C_1 > 1$, the last term above is negative. When $\beta \to \infty$, the integral $\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr$ tends to ∞. It ends the proof for $a \neq \frac{1}{4}$. When $a = \frac{1}{4}$, (3.27) becomes

$$\begin{aligned}
&-\frac{n-2}{4}\int_R^{R^\beta} r^{-1}dr - \frac{n-1}{2}\int_R^{R^\beta} \xi(r)^{-1}\xi'(r)dr - \frac{n-1}{4}\int_R^{R^\beta} r\xi(r)^{-2}\xi'(r)^2dr \\
&+\frac{(n-1)(n-2)}{4}\int_R^{R^\beta} r^{-1}\xi(r)^{-2}dr \\
&\leq \frac{n-2}{4}(-1+(n-1)C_1^{-2})(\beta-1)\log R - \frac{n-1}{2}(\log(\xi(R^\beta)) - \log(\xi(R))),
\end{aligned}$$

in which will be negative and large in module, if $-1 + (n-1)C_1^{-2} < 0 \Leftrightarrow C_1 > (n-1)^{\frac{1}{2}}$. In this conditions,

$$\frac{n-2}{4(n-1)(1-C_1^{-2})} < \frac{n-2}{4(n-1)}\frac{1}{1-\frac{1}{n-1}} = \frac{1}{4}.$$

Hence, the same conclusions follow for $a = \frac{1}{4}$.

■

Proof of Theorem 3.1.10.: Similarly to the proof of the previous theorem, we have that

$$\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)-2}dr \geq C_2^{-2}\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr$$

and

$$\frac{n-2}{4}(n-2-4a(n-1))\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr$$

$$+a(n-1)(n-2)\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)-2}dr$$

$$\leq\left(\frac{(n-2)^2}{4}C_2^2+a(n-1)(n-2)(1-C_2^2)\right)\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)-2}dr.$$

Since $a<\frac{n-2}{4(n-1)(1-C_2^{-2})}$ and $C_2<1$, the the last term above is negative. When $\beta\to\infty$, the integral $\int_R^{R^\beta} r^{-1}\xi(r)^{(1-4a)(n-1)}dr$ tends to ∞. It ends the proof.

■

3.4.2 Theorems of Section 2.2

Proof ofTheorem 3.2.1. First let us proof that

$$\int_M -fL_af\geq 0$$

for all f dependent only on I and $0\leq a\leq \min_{1\leq i\leq k}\{\frac{n_i-1}{4n_i}\}$. Let $a\geq 0$ and $A_i: I\to\mathbb{R}$, $0\leq i\leq k$ defined by $A_0(r)=\left(\prod_{i=1}^k\rho_i(r)^{\frac{n_i}{2}}\right)f_r$ and $A_i(r)=2an_i\rho_i(r)^{\frac{n_i-2}{2}}\prod_{j\neq i}\rho_j(r)^{\frac{n_j}{2}}\rho_i'(r)f$. For $1\leq i\leq k$, we have

$$\begin{aligned}(A_0+A_1+\cdots+A_k)^2=&f_r^2\left(\prod_{i=1}^k\rho_i(r)^{n_i}\right)+4a^2\sum_{i=1}^k n_i^2\rho_i(r)^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)^2f^2\\&+4a^2\sum_{i<j}n_in_j\rho_i(r)^{n_i-1}\rho_j(r)^{n_j-1}\left(\prod_{l\neq i,l\neq j}\rho_l(r)^{n_l}\right)\rho_i'(r)\rho_j'(r)f^2\\&+4a\sum_{i=1}^k n_i\rho_i(r)^{n_i-1}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)ff_r.\end{aligned}$$

Using (3.9) and the fact that S has nonnegative scalar curvature, we have

$$\left(\prod_{i=1}^{k} Vol(F_i)\right)^{-1} \int_M -fL_af = \int_b^c \left(\sum_{i=0}^{k} A_i\right)^2 dr$$
$$+\sum_{i=1}^{k}(an_i(n_i-1)-4a^2n_i^2)\int_b^c \rho_i(r)^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)^2 f^2 dr$$
$$+\sum_{i<j}(2a-4a^2)n_in_j\int_b^c \rho_i(r)^{n_i-1}\rho_j(r)^{n_j-1}\left(\prod_{l\neq i,l\neq j}\rho_l(r)^{n_l}\right)\rho_i'(r)\rho_j'(r)f^2dr$$
$$+a\sum_{i=1}^{k} S(F_i)\prod_{i\neq j} Vol(F_j)\int_b^c \rho_i(r)^{n-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right) f^2 dr.$$
(3.39)

The expression (3.39) is a sum of four terms, where the first and fourth are nonnegative (the fourth because the hypothesis of the scalar curvature of each F_i is nonnegative). Regarding the second term, note that if $0 \leq a \leq \frac{n_i-1}{4n_i}$, then

$$4an_i \leq n_i - 1 \Longrightarrow 4a^2n_i^2 \leq an_i(n_i-1),$$

because $0 \leq a \leq \min_{1\leq i\leq k} \frac{n_i-1}{4n_i}$. Hence, the second term of (3.39) is nonnegative. Furthermore, under these conditions, $0 \leq a < \frac{1}{4}$, $2a - 4a^2 \geq 0$ and the third term of (3.39) is also nonnegative. This completes the proof for f only dependent on I.

Consider an arbitrary $f \in C_c^\infty$. Let $dV(r)$ be the volume element of the fiber r, then $dV(r) = \prod_{i=1}^{k}\rho_i(r)^{n_i}dV_{F_1}\dots dV_{F_k}$. Denote $q \in F_1 \times \cdots \times F_k$ by $q = (q_1, \dots q_k)$ where $q_i \in F_i$. Using Fubini's theorem, our strategy is to integrate f in each leaf and to conclude the result proving that the value of the integral of f in each leaf is nonnegative. Hence, we just need to prove this. Based on the previous case, let $A_i^q(r) = 2an_i\rho_i(r)^{\frac{n_i-2}{2}}\prod_{j\neq i}\rho_j(r)^{\frac{n_j}{2}}\rho_i'(r)f(r,q)$ for $1 \leq i \leq q$ and $A_0^q(r) = f_r(r,q)\prod_{i-1}^{k}\rho_j(r)^{\frac{n_j}{2}}$, we have, integrating and using (3.9) on the leaf $I \times \{q\}$:

$$\int_{I\times\{q\}} -fL_af = \int_b^c |\nabla f|^2_{(r,q)}\left(\prod_{i=1}^{k}\rho_i(r)^{n_i}\right) dr$$
$$+a\sum_{i=1}^{k} n_i(n_i-1)\int_b^c \rho_i(r)^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)^2 f(r,q)^2 dr$$

$$+2a\sum_{i<j} n_in_j \int_b^c \rho_i(r)^{n_i-1}\rho_j(r)^{n_j-1}\rho_i'(j)\rho_j'(r)\left(\prod_{l\neq i, l\neq j}\rho_l(r)^{n_l}\right) f(r,q)^2dr$$

$$+4a\sum_{i=1}^k n_i \int_b^c \rho_i(r)^{n_i-1}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)f(r,q)f_r(r,q)dr$$

$$+a\sum_{i=1}^k S_{F_i}(q_i)\int_b^c \rho_i^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right) f(r,q)^2dr$$

$$\geq \int_b^c f_r(r,q)^2\left(\prod_{i=1}^k \rho_i(r)^{n_i}\right)dr$$

$$+a\sum_{i=1}^k n_i(n_i-1)\int_b^c \rho_i(r)^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)^2 f(r,q)^2dr$$

$$+2a\sum_{i<j} n_in_j \int_b^c \rho_i(r)^{n_i-1}\rho_j(r)^{n_j-1}\rho_i'(j)\rho_j'(r)\left(\prod_{l\neq i, l\neq j}\rho_l(r)^{n_l}\right) f(r,q)^2dr$$

$$+4a\sum_{i=1}^k n_i \int_b^c \rho_i(r)^{n_i-1}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)f(r,q)f_r(r,q)dr.$$

By a calculus similar to the case when f is dependent only on I (using the expressions $A_i^q(r)$), the last expression is nonnegative for all $a \leq \min_{1\leq i\leq k}\{\frac{n_i-1}{4n_i}\}$. It finishes the proof.

Proof of Theorem 3.2.2. Similar to the problem, the solution is also similar to the solution of Theorem 3.1.2 and Theorem 3.1.3, but some important details need to be addressed. The conditions (i) and (ii) are equivalent by a change of variables on each ρ_i. We only have to prove (i). Suppose that (i) holds for all i for a given R_0. Being $\rho(r) = \rho_1(r)^{n_1}\dots\rho_k(r)^{n_k}$, then $\rho'(r) = \sum_{i=1}^k n_i\rho(r)\rho_i(r)^{-1}\rho_i'(r)$ and

$$\rho''(r) = \sum_{i=1}^k n_i(n_i-1)\rho_i(r)^{n_i-2}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i'(r)^2 + \sum_{i=1}^k n_i\rho_i(r)^{n_i-1}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i''(r)$$
$$+ 2\sum_{i<j} n_in_j\rho_i(r)^{n_i-1}\rho_j(r)^{n_j-1}\left(\prod_{l\neq i, l\neq j}\rho_l(r)^{n_l}\right)\rho_i'(r)\rho_j'(r).$$

Substituting on (3.9), then, for $b \geq R_0$ and f such that $\pi_1(suppf) \subset [b,c]$:

$$\int_M -fL_af = \left(\prod_{i=1}^k Vol(F_i)\right)\left\{\int_b^c f_r^2\rho(r)dr\right.$$

$$+a\int_b^c \left[\rho''(r) - \sum_{i=1}^{k} n_i\rho_i(r)^{n_i-1}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i''(r)\right] f^2dr$$

$$+4a\int_b^c \rho'(r)ff_rdr\Bigg\} + a\sum_{i=1}^{k} S(F_i)\left(\prod_{i\neq j} Vol(F_j)\right)\int_b^c \rho(r)\rho_i(r)^{-2}f^2dr$$

$$= \left(\prod_{i=1}^{k} Vol(F_i)\right)\Bigg\{\int_b^c f_r^2\rho(r)dr + a\int_b^c \rho''(r)f^2dr + 4a\int_b^c \rho'(r)ff_rdr$$

$$-a\int_b^c \sum_{i=1}^{k} n_i\rho_i(r)^{n_i-1}\left(\prod_{j\neq i}\rho_j(r)^{n_j}\right)\rho_i''(r)f^2dr\Bigg\}$$

$$+a\sum_{i=1}^{k} S(F_i)\left(\prod_{i\neq j} Vol(F_j)\right)\int_b^c \rho(r)\rho_i(r)^{-2}f^2dr$$

$$= \left(\prod_{i=1}^{k} Vol(F_i)\right)\Bigg\{\int_b^c f_r^2\rho(r)dr + 2a\int_b^c \rho'(r)ff_rdr$$

$$-a\int_b^c \sum_{i=1}^{k} n_i\rho(r)\rho_i(r)^{-1}\rho_i''(r)f^2dr\Bigg\}$$

$$+a\sum_{i=1}^{k} S(F_i)\left(\prod_{i\neq j} Vol(F_j)\right)\int_b^c \rho(r)\rho_i(r)^{-2}f^2dr$$

$$= \left(\prod_{i=1}^{k} Vol(F_i)\right)\Bigg\{\int_b^c f_r^2\rho(r)dr + 2a\int_b^c \rho'(r)ff_rdr$$

$$+a\int_b^c \sum_{i=1}^{k} n_i\rho'(r)\rho_i(r)^{-1}\rho_i'(r)f^2dr - a\int_b^c \sum_{i=1}^{k} n_i\rho(r)\rho_i(r)^{-2}\rho_i'(r)^2f^2dr$$

$$+2a\int_b^c \sum_{i=1}^{k} n_i\rho(r)\rho_i(r)^{-1}\rho_i'(r)ff_rdr\Bigg\}$$

$$+a\sum_{i=1}^{k} S(F_i)\left(\prod_{j\neq i} Vol(F_j)\right)\int_b^c \rho(r)\rho_i(r)^{-2}f^2dr.$$

$$= \left(\prod_{i=1}^{k} Vol(F_i)\right)\Bigg\{\int_b^c f_r^2\rho(r)dr + 4a\int_b^c \rho'(r)ff_rdr$$

$$+a\int_b^c \rho(r)^{-1}\rho'(r)^2f^2dr - a\int_b^c \sum_{i=1}^{k} n_i\rho(r)\rho_i(r)^{-2}\rho_i'(r)^2f^2dr\Bigg\}$$

$$+a\sum_{i=1}^{k} S(F_i)\left(\prod_{j\neq i} Vol(F_j)\right)\int_b^c \rho(r)\rho_i(r)^{-2}f^2dr, \tag{3.40}$$

where in the last equality we use that $\rho(r)^{-1}\rho'(r) = \sum_{i=1}^{k} n_i\rho_i(r)^{-1}\rho_i'(r)$. Take as a test function the function f defined by

$$f = f_{Q,R}(r) = \begin{cases} 0 & \text{if } \ r \leq \frac{Q}{2}; \\ 2\rho(r)^{-\frac{1}{2}}Q^{-\frac{1}{2}}r - \rho(r)^{-\frac{1}{2}}Q^{\frac{1}{2}} & \text{if } \ \frac{Q}{2} \leq r \leq Q; \\ \rho(r)^{-\frac{1}{2}}r^{\frac{1}{2}} & \text{if } \ Q \leq r \leq R; \\ C(\rho(r)^{-\frac{1}{2}}r^{\frac{1}{2}} - \rho(T)^{-\frac{1}{2}}T^{\frac{1}{2}}) & \text{if } \ R \leq r \leq T; \\ 0 & \text{if } \ r \geq T, \end{cases}$$

where $\frac{Q}{2} \geq R_0$, R is larger than Q, T is such that $\rho(T) = 2\rho(R)$ and C is defined such that f becomes a continuous function. Substituting in (3.40) integrating on the interval $[Q, R]$, we have

$$\begin{aligned}
&\left(\prod_{i=1}^{k} Vol(F_i)\right)\left\{\frac{1}{4}\int_Q^R r^{-1}dr - \frac{1}{2}\int_Q^R \rho(r)^{-1}\rho'(r)dr\right. \\
&+\frac{1}{4}\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr + 2a\int_Q^R \rho(r)^{-1}\rho'(r)dr - 2a\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr \\
&+a\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr - a\int_Q^R \sum_{i=1}^{k} n_i r\rho_i(r)^{-2}\rho_i'(r)^2dr \\
&+a\sum_{i=1}^{k} S(F_i)\left(\prod_{i\neq j} Vol(F_j)\right)\int_Q^R r\rho_i(r)^{-2}dr \\
&=\left(\prod_{i=1}^{k} Vol(F_i)\right)\left\{\frac{1}{4}\int_Q^R r^{-1}dr + \left(2a - \frac{1}{2}\right)\int_Q^R \rho(r)^{-1}\rho'(r)dr\right. \\
&\left.+\left(\frac{1}{4} - a\right)\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr - a\int_Q^R \sum_{i=1}^{k} n_i r\rho_i(r)^{-2}\rho_i'(r)^2dr\right\} \\
&+a\sum_{i=1}^{k} S(F_i)\left(\prod_{i\neq j} Vol(F_j)\right)\int_Q^R r\rho_i(r)^{-2}dr. \qquad (3.41)
\end{aligned}$$

Using that $\rho(r)^{-1}\rho'(r) = \sum_{i=1}^{k} n_i\rho_i(r)^{-1}\rho_i'(r)$ and using the QM-AM inequality, then

$$\rho(r)^{-2}\rho'(r)^2 = \left(\sum_{i=1}^{k} n_i\rho_i(r)^{-1}\rho_i'(r)\right)^2 \leq n\sum_{i=1}^{k} n_i\rho_i(r)^{-2}\rho_i'(r)^2.$$

Using (3.41), we obtain

$$\int_M -fL_a f \leq \left(\prod_{i=1}^k Vol(F_i)\right)\left\{\frac{1}{4}\int_Q^R r^{-1}dr + \left(2a-\frac{1}{2}\right)\int_Q^R \rho(r)^{-1}\rho'(r)dr\right.$$
$$\left. + \left(\frac{1}{4}-a-\frac{a}{n}\right)\int_Q^R r\rho(r)^{-2}\rho'(r)^2 dr\right\}$$
$$+a\sum_{i=1}^k S(F_i)\left(\prod_{i\neq j} Vol(F_j)\right)\int_Q^R r\rho_i(r)^{-2}dr.$$

Note that since $\rho'(r)^2 \to \infty$ when $r \to \infty$, then the last term of the right-hand side of the inequality above is arbitrary lesser than $\int_Q^R r\rho(r)^{-2}\rho'(r)^2 dr$ when $r \to \infty$. Let us prove that for Q large and $a > \frac{n}{4(n+1)}$, the expression above is less than a negative constant times $\int_Q^R r\rho(r)^{-2}\rho'(r)^2 dr$. Because $r\rho_i(r)^{-1}\rho_i'(r) > 1$ for all $r > R_0$ and $1 \leq i \leq k$, then

$$\rho(r)^{-1}\rho'(r) = \sum_{i=1}^k n_i\rho_i(r)^{-1}\rho_i'(r)dr > \sum_{i=1}^k n_i\frac{1}{r} = \frac{n}{r},$$

that is, $r\rho(r)^{-1}\rho'(r) > n$. Note that

$$\frac{1}{4}\int_Q^R r^{-1}dr + \left(2a-\frac{1}{2}\right)\int_Q^R \rho(r)^{-1}\rho'(r)dr \leq \left(\frac{1}{4n}+2a-\frac{1}{2}\right)\int_Q^R \rho(r)^{-1}\rho'(r)dr$$

and

$$\frac{1}{4n}+2a-\frac{1}{2} \leq 0 \Longleftrightarrow a \leq \frac{2n-1}{8n}.$$

This implies that for $\frac{n}{4(n+1)} < a \leq \frac{2n-1}{8n}$ and Q large, (3.41) is lesser than a negative constant $(\frac{1}{4}-a-\frac{a}{n})$ times $\int_Q^R r\rho(r)^{-2}\rho'(r)^2 dr$. To obtain the same conclusion for $\frac{2n-1}{8n} < a \leq \frac{1}{4}$, we note that $2a-\frac{1}{2} \leq 0$,

$$\frac{1}{4}\int_Q^R r^{-1}dr + \left(\frac{1}{4}-a-\frac{a}{n}\right)\int_Q^R r\rho(r)^{-2}\rho'(r)^2 dr$$
$$\leq \left(\frac{1}{4n^2}+\frac{1}{4}-a-\frac{a}{n}\right)\int_Q^R r\rho(r)^{-2}\rho'(r)^2 dr$$

and

$$\frac{1}{4n^2}+\frac{1}{4}-a-\frac{a}{n} < 0 \Longleftrightarrow a > \frac{n^2+1}{4n^2+4n}.$$

Because $\frac{2n-1}{8n} \geq \frac{n^2+1}{4n^2+4n}$ for $n \geq 3$, the above conclusion regarding a extends to $\frac{n}{4(n+1)} < a \leq \frac{1}{4}$. For $a > \frac{1}{4}$, we have

$$\frac{1}{4}\int_Q^R r^{-1}dr + \left(2a-\frac{1}{2}\right)\int_Q^R \rho(r)^{-1}\rho'(r)dr + \left(\frac{1}{4}-a-\frac{a}{n}\right)\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr$$

$$\leq \left(\frac{1}{4n^2}+\frac{2a}{n}-\frac{1}{2n}+\frac{1}{4}-a-\frac{a}{n}\right)\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr$$

and

$$\frac{1}{4n^2}+\frac{2a}{n}-\frac{1}{2n}+\frac{1}{4}-a-\frac{a}{n}<0 \Longleftrightarrow a>\frac{n-1}{4n}.$$

Since $\frac{1}{4} > \frac{n-1}{4n}$, the conclusion follows. It shows that (3.41) is less than a negative constant times $\int_Q^R r\rho(r)^{-2}\rho'(r)^2dr$ for all $a > \frac{n}{4(n+1)}$.

Let us now analyze the expression of f in (3.40) on the interval $[R,T]$. We have $f_r = C(-\frac{1}{2}\rho(r)^{-\frac{3}{2}}\rho'(r)r^{\frac{1}{2}}+\frac{1}{2}\rho(r)^{-\frac{1}{2}}r^{-\frac{1}{2}})$. Then the expression of f in (3.40) on the interval $[R,T]$ is

$$C^2\left(\prod_{i=1}^k Vol(F_i)\right)\left[\frac{1}{4}\int_R^T r\rho(r)^{-2}\rho'(r)^2dr - \frac{1}{2}\int_R^T \rho(r)^{-1}\rho'(r)dr + \frac{1}{4}\int_R^T r^{-1}dr\right.$$

$$-2a\int_R^T r\rho(r)^{-2}\rho'(r)^2dr + 2a\int_R^T \rho(r)^{-1}\rho'(r)dr + 2a\rho(T)^{-\frac{1}{2}}T^{\frac{1}{2}}\int_R^T r^{\frac{1}{2}}\rho(r)^{-\frac{3}{2}}\rho'(r)^2dr$$

$$-2a\rho(T)^{-\frac{1}{2}}T^{\frac{1}{2}}\int_R^T r^{-\frac{1}{2}}\rho(r)^{-\frac{1}{2}}\rho'(r)dr + a\int_R^T r\rho(r)^{-2}\rho'(r)^2dr$$

$$-2a\rho(T)^{-\frac{1}{2}}T^{\frac{1}{2}}\int_R^T r^{\frac{1}{2}}\rho(r)^{-\frac{3}{2}}\rho'(r)^2dr$$

$$+a\rho(T)^{-1}T\int_R^T \rho(r)^{-1}\rho'(r)^2dr - a\int_R^T\left(\sum_{i=1}^k n_i\rho_i(r)^{-2}\rho_i'(r)^2\right)rdr$$

$$+2a\rho(T)^{-\frac{1}{2}}T^{\frac{1}{2}}\int_R^T\left(\sum_{i=1}^k n_i\rho_i(r)^{-2}\rho_i'(r)^2\right)r^{\frac{1}{2}}\rho(r)^{\frac{1}{2}}$$

$$\left.-a\rho(T)^{-1}T\int_R^T\left(\sum_{i=1}^k n_i\rho_i(r)^{-2}\rho_i'(r)^2\right)\rho(r)dr\right]$$

$$+a\sum_{i=1}^k S(F_i)\left(\prod_{j\neq i} Vol(F_j)\right)\int_R^T \rho(r)\rho_i(r)^{-2}f^2dr.$$

Using that, on $[R,T]$ $f(r,\cdot)^2 \leq C^2\rho(r)^{-1}r$, $\int_R^T r\rho_i(r)^{-2}dr \to 0$ when $R\to\infty$, $\rho(T) = 2\rho(R) \Rightarrow T < 2R$ (it is analogous to in the proof of Theorem 3.1.2), then the last term grows in a order lesser than the rest and $R < T < 2R$. It is easy to see that all terms in expression above is upper limited by a multiple of $\int_R^T r\rho(r)^{-2}\rho'(r)^2dr$, then all of expression is limited by a constant times it. Note that ρ'' is a positive function because is a product of positive functions with positive first fist and second derivative (note that $(gh)'' = g''h + 2g'h' + gh''$, and the conclusion follows by induction). From the calculus made at the end of the proof of Theorem 3.1.2, we have that

$$\int_R^T r\rho(r)^{-2}\rho'(r)^2dr \leq \log 2 + T\rho(T)^{-1}\rho'(T)$$

and there exists a sequence (R_j, T_j), with $\rho(T_j) = 2\rho(R_j)$ such that

$$\lim_{j\to\infty} \frac{\int_Q^{R_j} r\rho(r)^{-2}\rho'(r)^2dr}{\int_{R_j}^{T_j} r\rho(r)^{-2}\rho'(r)^2dr} = \infty.$$

Therefore, the expression of the stability operator in $[Q, R_j]$ is arbitrarily larger than the expression of the stability operator in $[R_j, T_j]$ when J increases to infinity. It finishes the proof.

■

Chapter 4

a-Index of Minimal Surfaces

In this chapter we will analyze the operator from the previous chapter on minimal surfaces in $\mathbb{R}^3$. Let M be a complete minimal surface in $\mathbb{R}^3$, denote the Laplacian in M by Δ and the Gaussian curvature in M by K. The operator $L_a = \Delta - as$ becomes $\Delta - 2aK$, acting on $C_c^\infty(M)$, where $a \in \mathbb{R}$. D. Fischer-Colbrie and R. Schoen [FS80] observed that, given $q \in C^\infty(M)$, the existence of a positive function f in M satisfying $\Delta f - qf = 0$ is equivalent to the condition that the first eigenvalue of $\Delta - q$ is positive in each bounded domain of M. This fact has many interesting applications for minimal immersion and constant mean curvature, especially when the immersion occurs in a three-dimensional manifold of identically zero Gaussian curvature (in particular, on $\mathbb{R}^3$), where the Jacobi operator related to the second variation of the area becomes $L = \Delta - 2K$. When M is and has finite total curvature, Osserman [Oss13] shows that M is conformally equivalent to a compact surface Σ with finitely many points removed, each one corresponding to an end of M.

Let M be a minimal surface in $\mathbb{R}^3$, we define $Index_a(M)$ as the number of negative eigenvalues (counted with multiplicities) of L_a. It can also be defined as the maximal subspace of $H^1(M)$ such that L_a is negative definite. Being $G : M \to \mathbb{R}^3$ a local Gauss map of M, then $|dG|^2 = -2K$ and $L_a = \Delta + a|dG|^2$. Fischer-Colbrie [Fis85] proved that for the usual Jacobi operator $L = L_1 = \Delta - 2K$ in a minimal surface M in $\mathbb{R}^3$, the index only depends of the Gauss map in the conformal class of M and coincides with the index of the operator L in the compactification Σ of M. With a similar proof, this result also holds for L_a by a similar reasoning. More precisely:

Proposition 4.0.1. *Let M be a complete minimal surface in $\mathbb{R}^3$ with metric ds^2 and L_a be the corresponding operator defined above. Let $\widetilde{ds}^2 = \mu ds^2$ be another metric on M, where μ is a positive function on M and $\tilde{L}_a$ is the operator L_a in the metric $\tilde{ds}^2$, then*

$$Index_{L_a}(M) = Index_{\tilde{L_a}}(M).$$

Corollary 4.0.1. *Let $ds^2 = \mu|dz|^2$ be the metric of M, Σ be its compactification and $\widetilde{\mu}|dz|^2$ be a smooth metric on Σ, where μ and $\tilde{\mu}$ are positive functions obtained from a local parameterization of M and Σ, respectively. Then*

$$Index_a(M) = Index_{\tilde{L_a}}(\Sigma).$$

With the objective of studying a-index of minimal surfaces of finite total curvature, we consider a compact Riemann surface Σ and $G : \Sigma \to \mathbb{S}^2$ as a nonconstant holomorphic map, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ endowed with the complex structure induced by the stereographic projection from the north pole. This is inspired by the fact that the Gauss normal map $G : M \to \mathbb{S}^2$ of a minimal surface is, in a sense, a holomorphic map such that $|dG|^2 = -2K$. Fix a conformal metric ds^2 in Σ and consider the operator $L_a = \Delta + a|dG|^2$, with quadratic form associated

$$Q_a(f, f) = \int_\Sigma |\nabla f|^2 - a|dG|^2 f^2,$$

where $f \in C^\infty(\Sigma)$. We note that by the invariance of L^2norm of one-forms in conformal class in dimension two, Q_a is independent of the particular choice of the metric on Σ.

In a similar way to the work of Shin Nayatani in [Nay93], we now consider on Σ the metric ds_G^2 induced by G of $\mathbb{S}^2$. Thus $ds_G^2 = \frac{1}{2}|dG|^2 ds^2$. This metric is singular at the ramifications points of G, that is, at the points where $dG = 0$. By the choice of the metric, we have $L_{G,a} = \Delta^G + 2a$, where Δ^G is the Laplacian with respect to the metric ds_G^2. Let $\lambda^{(a)}$ be an eigenvalue of $L_{G,a}$, the corresponding eigenspace is

$$V_\lambda^a(G) = \left\{ u \in H^1(\Sigma); Q_a(u, v) = \lambda^{(a)} \int_\Sigma uv dA_G \quad \forall v \in H^1(\Sigma) \right\},$$

where $dA_G = \frac{1}{2}|dG|^2 dA$ is the area element of the metric ds_G^2. It follows from the regularity of the eigenfunctions that $V_\lambda(G) \subset C^\infty(\Sigma)$. For $u \neq 0$, we define

$$R_G^a(u) = \frac{Q_a(u, u)}{\int_\Sigma u^2 dA_G}.$$

The k-th eigenvalue (counted with multiplicity) $\lambda_k^{(a)}(G)$ is characterized as:

$$\lambda_k^{(a)}(G) = \inf_{V \in G_k} \sup\{R_G^a(u); u \in V, u \neq 0\} \tag{4.1}$$

where G_k is the set of all k-dimensional subspaces of $H^1(\Sigma)$. $Index_a(\Sigma)$, as we can see, will coincide with the number of negative eigenvalues of $L_{G,a}$.

In this chapter we will also analyze cases in which M is not complete, or "pieces" of complete minimal surfaces. First, we performed a quantitative analysis of the Laplacian in some spherical domains.

4.1 First eigenvalue of Laplacian in spherical domains

Let $a > 0$ be a real positive number, we wish to find numbers $z_1 < z_2$ such that $2a$ is the first eigenvalue of Laplacian in the set $\mathbb{S}^2(z_1, z_2) = \{(x, y, z) \in \mathbb{S}^2; z_1 \leq z \leq z_2\}$ in Dirichlet boundary conditions. We have that, on spherical domains of the form $S^2(z_1, z_2)$, the eigenvalues of the Laplacian are nonnegatives and is zero if and only if the domain is $\mathbb{S}^2 = S^2(-1, 1)$. Because the first eigenspace is simple and by the symmetry of $\mathbb{S}^2(z_1, z_2)$, an eigenfunction associated to the first eigenvalue, being it $2a$, is of form $f = f(z)$.

If Δ and Δ^S denote the Laplacians of $\mathbb{R}^3$ and $\mathbb{S}^2$, respectively, the relation between these two Laplacians is

$$\Delta f = \Delta^S f + 2\langle \nabla f, N \rangle + D^2 f(N, N),$$

where N is the unity normal vector field in $\mathbb{S}^2$ pointing outside the unitary ball and $D^2 f$ is the quadratic Hessian form of f in $\mathbb{R}^3$. Assuming $\Delta^S f = -2af$, $f = f(z)$ dependent only on z and using the relationship $(\nabla f)^\perp = zf'$, the last equation is equivalent to the ODE:

$$(1 - z^2)f'' - 2zf' + 2af = 0. \tag{4.2}$$

4.1.1 Numerical quantities

Assume that the solution of (4.2) has Taylor series centered at 0 in the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

substituting in (4.2), we have:

$$\sum_{n=0}^{\infty} a_n n(n-1) z^{n-2} - \sum_{n=0}^{\infty} a_n n(n-1) z^n - 2\sum_{n=0}^{\infty} n a_n z^n + 2a \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) z^n - \sum_{n=0}^{\infty} a_n n(n-1) z^n - 2\sum_{n=0}^{\infty} n a_n z^n + 2a \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (-n(n-1) - 2n + 2a)a_n] z^n = 0.$$

The last equality implies

$$a_{n+2} = \frac{n^2 + n - 2a}{(n+2)(n+1)} a_n. \tag{4.3}$$

Therefore, the function f is completely determined by a_0 and a_1. Since $\lim_{n\to\infty} \frac{a_{n+2}}{a_n} = 1$, is easy to see that $f(z)$ exists for all $z \in (-1, 1)$. In particular:

(i) When $a_0 = 0$, f is an odd function. If $m > 0$ is an odd number and we take $a = \frac{m(m+1)}{2}$, we have $a_{m+2} = 0$. Thus, $a_k = 0$ for $k \geq m$, the series is a polynomial of degree m and $f(1)$ is defined in all $\mathbb{S}^2$. Furthermore, $f(-1) = -f(1)$ (this values can be $\pm\infty$), which implies that the polynomial f has a largest zero in $(-1, 1)$. Suppose that z_0 is the largest root of f less than one, we have that $\mathbb{S}^2(z_0, 1)$ is such that the signal of f does not change. Therefore, is a domain such that the first eigenvalue of Laplacian is $2a = m(m+1)$. This implies that we can take a spherical domain such that the first eigenvalue of the Laplacian is so large as we want (this is consistent with the fact that smaller domain implies a larger first eigenvalue of Laplacian). The intuitive relationship is the fact that in plane domains there are no nonzero eigenvalues for Laplacian and very small spherical cap is approximately a flat domain.

(ii) Even in the case $a_0 = 0$, assuming that $a < 1$, we have $a_n = 0$ for n even and for n odd, by (4.3):

$$a_{n+2} = \frac{n^2 + n - 2a}{(n+2)(n+1)} a_n = \left(\frac{n}{n+2} - \frac{2a}{(n+2)(n+1)} \right) a_n,$$

and, considering without loss of generality $a_1 > 0$ (if no, multiply f by -1):

$$a_{n+2} > \left(\frac{n}{n+2} - \frac{2}{(n+2)(n+1)}\right) a_n = \frac{n-1}{n+1} a_n.$$

Therefore, $a_n > \frac{1}{n+1} a_1$ for n odd. It implies that $f(1) = \infty$ and f is positive in $(0, 1)$. Therefore, $2a$ is not an eigenvalue of a domain in a half-sphere. Therefore, in the case $a < 1$, necessarily $a_0 \neq 0$.

Still in the case $f = f(z)$, fixing $a_0 = f(0)$ and $a_1 = f'(0)$, by the above formula, $a_2 = -aa_0$, $a_4 = (-\frac{1}{2}a + \frac{a^2}{6})a_0$, $a_6 = (-\frac{1}{6}a + \frac{13}{90}a^2 - \frac{a^3}{90})a_0$; $a_3 = (\frac{1}{3} - \frac{a}{3})a_1$, $a_5 = (\frac{1}{5} - \frac{7}{30}a + \frac{a^2}{30})a_1$, $a_7 = (\frac{1}{7} - \frac{37}{210}a + \frac{11}{315}a^2 - \frac{a^3}{630})a_1$, etc.

Let us exemplify the case $a = 1$. We have

$$a_{n+2} = \frac{n-1}{n+1} a_n$$

it implies $a_n = 0$ for $n \geq 3$ odd and $a_n = -\frac{a_0}{n}$, for n even. Therefore,

$$f(z) = a_0 + a_1 z - a_0 \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n}.$$

Note that the natural logarithmic function satisfies:

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{and} \quad \log(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n},$$

for all $-1 < z < 1$. Therefore,

$$f(z) = a_0 + a_1 z - a_0 z(\log(1+z) - \log(1-z)).$$

This family of functions is determined by a_0 and a_1. By the homogeneity of (4.2), if $a_0 \neq 0$, we can fix $a_0 > 0$ to obtain a family of lineament independent solutions of (4.2), which is null in two circles in $\mathbb{S}^2$, contained in horizontal planes. Thus, the circle of $\mathbb{S}^2$ $z = 0$ is contained in the domain of $\mathbb{S}^2$ between these two planes, which is a spherical segment (which naturally becomes a domain whose first Laplacian eigenvalue is 2) that f is positive in it. When we vary a_1 to ∞ those domains tend to the northern hemisphere of $\mathbb{S}^2$ minus the north pole and when we vary a_1 to $-\infty$ those domains tend to the southern hemisphere without the south pole.

4.1.2 First eigenvalue of Laplacian on spherical caps

It is known that, among all spherical domains with the same first eigenvalue of Laplacian, the spherical cap has the smaller area (see [Pee57], pg 19). We denote by z_a the number $z \in (-1, 1)$ with the following property: the spherical cap $\mathbb{S}^2(-1, z_a)$ has $2a$ as the first eigenvalue of the Laplacian. Note that if a cap has λ as an eigenfunction of the Laplacian, the complementary cap has λ as an eigenfunction of the Laplacian. One way to verify if an eigenvalue of the Laplacian in a spherical domain is the first is verify if the corresponding eigenspace is of functions that does not change the signal. As example, we have $z_1 = 0$, $z_a \to -1$ when $a \to \infty$ and $z_0 = 1$. The area of the spherical cap $\mathbb{S}^2(-1, z)$ is $2\pi(z + 1)$.

Therefore, it is convenient to find the Taylor expansion of a solution f of (4.2) around -1. Taking $z = w - 1$, we can consider the Taylor series on the variable w around zero. First, (4.2) becomes

$$(1 - (w-1)^2)f'' - 2(w-1)f' + 2af = 0$$

$$\iff (-w^2 + 2w)f'' - 2(w-1)f' + 2af = 0.$$

Being $f(w) = \sum_{n=0}^{\infty} a_n w^n$, the expression above becomes

$$-(w^2 - 2w)\sum_{n=0}^{\infty} n(n-1)a_n w^{n-2} - 2(w-1)\sum_{n=0}^{\infty} n a_n w^{n-1} + 2a\sum_{n=0}^{\infty} a_n w^n = 0$$

$$\implies -\sum_{n=0}^{\infty} n(n-1)a_n w^n + 2\sum_{n=0}^{\infty} n(n-1)a_n w^{n-1} - 2\sum_{n=0}^{\infty} n a_n w^n + 2\sum_{n=0}^{\infty} n a_n w^{n-1} + 2a\sum_{n=0}^{\infty} a_n w^n = 0$$

$$\implies -\sum_{n=0}^{\infty} n(n-1)a_n w^n + 2\sum_{n=0}^{\infty}(n+1)n a_{n+1} w^n - 2\sum_{n=0}^{\infty} n a_n w^n + 2\sum_{n=0}^{\infty}(n+1)a_{n+1}w^n + 2a\sum_{n=0}^{\infty} a_n w^n = 0$$

$$\implies \sum_{n=0}^{\infty}\left[-(n^2 + n - 2a)a_n + 2(n+1)^2 a_{n+1}\right] y^n = 0.$$

Therefore,

$$a_{n+1} = \frac{n^2 + n - 2a}{2(n+1)^2} a_n$$

and f where completely determined by a_0. Backing to variable z, we found the Taylor series of f around -1:

$$f(z) = \sum_{n=0}^{\infty} a_n(z+1)^n.$$

z_a is the smallest value of z in $(-1, 1)$ such that $f(z) = 0$. By homogeneity, we always assume $a_0 = f(-1) = 1$. As known values, we have $z_1 = 0$ and $z_0 = 1$.

Example 4.1.1. *(z_3) To find z_3, we have $a = 3$ and*

$$a_{n+1} = \frac{n^2+n-6}{2(n+1)^2} a_n$$

is easy to see that $a_n = 0$ for all $n \geq 3$, $a_0 = 1$, $a_1 = -3$, $a_2 = \frac{3}{2}$ and

$$f(z) = 1 - 3(z+1) + \frac{3}{2}(z+1)^2 = \frac{3}{2}z^2 - \frac{1}{2}.$$

The smallest root of $\frac{3}{2}z^2 - \frac{1}{2}$ is $-\sqrt{\frac{1}{3}}$. Therefore, $z_3 = -\sqrt{\frac{1}{3}}$ and $\mathbb{S}^2\left(-1, -\sqrt{\frac{1}{3}}\right)$ has 6 as the first eigenvalue of the Laplacian. Note that $\mathbb{S}^2\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$ and $\mathbb{S}^2\left(\sqrt{\frac{1}{3}}, 1\right)$ also have 6 as the first eigenvalue of Laplacian, because f is null on the boundary and does not change the signal in those domains.

Example 4.1.2. *(z_6) To find z_6, we have $a = 6$ and*

$$a_{n+1} = \frac{n^2+n-12}{2(n+1)^2} a_n$$

is easy to see that $a_n = 0$ for all $n \geq 4$, $a_0 = 1$, $a_1 = -6$, $a_2 = \frac{15}{2}$, $a_3 = -\frac{5}{2}$ and

$$f(z) = 1 - 6(z+1) + \frac{15}{2}(z+1)^2 - \frac{5}{2}(z+1)^3 = -\frac{5}{2}z^3 + \frac{3}{2}z.$$

The roots of $-\frac{5}{2}z^3 - \frac{3}{2}z$ are $-\sqrt{\frac{3}{5}}, 0$ and $\sqrt{\frac{3}{5}}$. Therefore, $z_4 = -\sqrt{\frac{3}{5}}$ and $\mathbb{S}^2\left(-1, -\sqrt{\frac{3}{5}}\right)$ has 12 as the first eigenvalue of the Laplacian. Note that we determine four domains that have 12 as the first eigenvalue of the Laplacian: $\mathbb{S}^2\left(-1, -\sqrt{\frac{3}{5}}\right)$, $\mathbb{S}^2\left(-\sqrt{\frac{3}{5}}, 0\right)$, $\mathbb{S}^2\left(0, \sqrt{\frac{3}{5}}\right)$ and $\mathbb{S}^2\left(\sqrt{\frac{3}{5}}, 1\right)$.

Example 4.1.3. *In general, if $a = \frac{1}{2}k(k+1)$ for some natural number k, then*

$$a_{n+1} = \frac{n^2+n-k(k+1)}{2(n+1)^2} a_n$$

and $a_n = 0$ *for all* $n \geq k+1$*, therefore,* f *will be a polynomial of degree* k*. By induction, we deduce that all roots of that polynomial are distinct and are in the interval* $(-1, 1)$*, where we can find* $k+1$ *domains such that have* $k(k+1)$ *as the first eigenvalue of the Laplacian.*

So far, we have analyzed eigenfunctions of the Laplacian only in domains of $\mathbb{S}^2$ that depend only on the z coordinate, that is, such that is invariant under horizontal rotations. But we can perform this entire analysis for functions that depend only on the x coordinate or the y coordinate in $\mathbb{R}^3$ where, in these cases, the corresponding domains of $\mathbb{S}^2$ will be symmetric with respect to the x axis (of the form $\mathbb{S}^2(x_1, x_2)$) and y axis (of the form $\mathbb{S}^2(y_1, y_2)$), respectively. Considering that all the solutions found before, we can find, from a linear combinations of functions that satisfy the condition that they depend only on one of the coordinates, many functions that satisfy $\Delta f = -2af$, defined on $\mathbb{S}^2$ minus a finite number of points (let us remember that the solutions obtained by Taylor series around zero only dependent on z were well defined throughout $\mathbb{S}^2$ with the possible exception of the north and south poles). We also find a vast number of domains of $\mathbb{S}^2$ that have $2a$ as the first eigenvalue of the Laplacian, where it is sufficient for such a domain to have as its boundary a set in which one of these obtained functions vanishes identically and so that do not change of signal in its interior. If the domain is such that one of this obtained functions vanishes identically in the boundary, but without the requirement of do not change the signal in interior, we have that $2a$ will be an eigenvalue, but not necessarily the first.

4.2 Results about a-index of minimal surfaces

First a result that characterize the plane as the unique minimal surface in $\mathbb{R}^3$ with finite total curvature and a stable for all $a > 0$.

Proposition 4.2.1. *Let* M *be a complete minimal surface in* $\mathbb{R}^3$ *with finite total curvature and* $a > 0$*, then or* Σ *is a plane or* $Index_a(M) \geq 1$*.*

Proof. By [Oss13], M having finite total curvature implies M conformal to a compact Riemann's surface Σ punctured at a finite number of points $\{p_1, \ldots, p_m\}$. Let $\varphi_i : D_1(0) \subset$

$\mathbb{R}^2 \to \Sigma$, $i = 1, \dots, m$, be a conformal parameterization of a neighborhood of p_i in Σ with $\varphi_i(0) = p_i$. Let $0 < \varepsilon < 1$ and $\phi_\varepsilon : D_1(0) \to \mathbb{R}$ defined by

$$\phi_\varepsilon(x) \begin{cases} = & 1 & \text{if } |x| > \varepsilon \\ & 2 - \frac{\log|x|}{\log\varepsilon} & \text{if } \varepsilon^2 \le |x| \le \varepsilon; \\ & 0 & \text{if } |x| < \varepsilon^2. \end{cases}$$

The family ϕ_ε satisfies $\phi_\varepsilon \to 1$ and $|\nabla\phi_\varepsilon| \to 0$ in $L^2(D_1(0))$ when $\varepsilon \to 0$. Let $\psi_{i,\varepsilon} = \phi_\varepsilon \circ \varphi_i^{-1} : \Sigma \to \mathbb{R}$, for each i and ε, extend $\psi_{i,\varepsilon}$ to Σ defining $\psi_{i,\varepsilon} = 1$ for all $q \in \Sigma \backslash \varphi_i(D_1(0))$, then $\psi_{i,\varepsilon} \to 1$ in $L^2(\Sigma)$ when $\varepsilon \to 0$ and by conformal invariance of the norm L^2 of one forms in dimension two, $|\nabla^\Sigma \psi_{i,\varepsilon}| \to 0$ in $L^2(\Sigma)$ when $\varepsilon \to 0$. Define $\psi_\varepsilon = \prod_{i=1}^{m} \psi_{i,\varepsilon}$, let $\eta : \Sigma \setminus \{p_1, \dots, p_m\} \to M$ a conformal diffeomorphism, then $\psi_\epsilon \circ \eta^{-1} : M \to \mathbb{R}$ is such that $\psi_\epsilon \circ \eta^{-1} \to 1$ pointwise and $|\nabla^M(\psi_\epsilon \circ \eta^{-1})| \to 0$ in $L^2(M)$, then:

$$\lim_{\varepsilon \to 0} \int_M |\nabla(\psi_\varepsilon \circ \eta^{-1})|^2 + 2aK(\psi_\varepsilon \circ \eta^{-1})^2 = 2a \int_M K.$$

Since for a complete minimal surface M, or M is a plane or M has negative total curvature, then when M is not a plane, there exist $f = \psi_\varepsilon \circ \eta^{-1}$ for ε sufficiently small such that $Q_a(f, f) < 0$. Therefore, in this case, $Index_a(M) \ge 1$.

■

4.2.1 a-stability based on the size of the spherical image

We thus have a theorem related to the main theorem in [BC76]:

Proposition 4.2.2. *Let M be a minimal surface such that the area of the image of Gauss normal map is less than $2\pi(z_a + 1)$, where z_a as above, then M is a-stable.*

Proof: Assume that M is connected (because the property of, if M is unstable, then some connected component is unstable) and is a minimal surface a-unstable with image of Gauss map G less than $2\pi(z_a - 1)$, then M is not a part of a plane and there exists a compact set $C \subset M$ of smooth boundary such that there exist a function $f : C \to \mathbb{R}$, positive in $int(C)$ and null in ∂C such that $L_a f = -\lambda f$ for some $\lambda < 0$ (under these conditions, λ is the first eigenvalue of L_a in C). Then $\Delta^M f = 2aKf - \lambda f$ on C. Under these conditions,

$f \in C_0^\infty(C)$. The Gauss map is a holomorphic map from M to $\mathbb{S}^2$, in particular, since G is not a constant function, then G is an open function, $\partial[G(C)] \subset G(\partial C)$ and the ramifications points are isolated. In particular, the pre-images of a fixed point of $\mathbb{S}^2$ by G are isolated points. Let us prove that $\tilde{L}_a = \Delta^{\mathbb{S}^2} + 2a$ is unstable in $G(C) \subset \mathbb{S}^2$ building an appropriated test function. Let $g : G(C) \to \mathbb{R}$ be defined as follows: if $q \in G(C)$ and $\{p_1, \ldots, p_k\} = G^{-1}(q)$, then $g(q) = \sum_{i=1}^k r_{p_i} f(p_i)$, where r_{p_i} is the ramification index of p_i by G (in particular, if p_i is a regular point, $r_{p_i} = 1$). Since $\partial[G(C)] \subset G(\partial C)$, then $G^{-1}(\partial G(C)) \subset \partial C$ and $g(q) = 0$ for all $q \in \partial[G(C)]$, therefore, $g \equiv 0$ on $\partial[G(C)]$. We will verify the following property of g:

(i) g is smooth almost everywhere and

$$\int_D -g\Delta^{\mathbb{S}^2} g - 2ag^2 < 0,$$

where $D \subset \{q \in G(C); g \text{ is smooth on } q\}$ is an open domain.

Let q be a regular value of G in interior of $G(C)$ such that $G^{-1}(q) = \{p_1, \ldots, p_k\} \subset int(C)$, then there exists a neighborhood $V \subset int(G(C))$ of q such that each connected component of $(G|_C)^{-1}(V)$ is diffeomorphic to V by G. We will call this components $U_1, \ldots, U_k$, such that $p_i \in U_i$, $G_i : U_i \to V$ those diffeomorphisms, where $G_i = G$ in U_i and $g_i \coloneqq f \circ G_i^{-1}$. Using the relationship $\Delta^C = -K\Delta^{\mathbb{S}^2}$, we have

$$(\Delta^{\mathbb{S}^2} g_i)(G_i(q)) = \Delta^{\mathbb{S}^2}(f \circ G_i^{-1})(G_i(q)) = -\frac{1}{K}(\Delta^C f)(q) = -2af(q) + \frac{\lambda}{K} f(q).$$

Since $|detJ(G)| = -K$ on minimal surfaces, we have:

$$\int_V -g_i\Delta^{\mathbb{S}^2} g_i - 2ag_i^2 = \int_{U_i} -2aKf^2 + \lambda f^2 + 2aKf^2 = \int_{U_i} \lambda f^2 < 0;$$

$$\begin{aligned}\int_V -g_i\Delta^{\mathbb{S}^2} g_j - 2ag_i g_j &= \int_V -g_i(\Delta^{\mathbb{S}^2} g_j - 2ag_j) \\ &= \int_{U_j} (f \circ G_i^{-1} \circ G_j)(-2aKf + \lambda f + 2aKf) \\ &= \int_{U_j} \lambda f (f \circ G_i^{-1} \circ G_j) < 0;\end{aligned}$$

Therefore, since $g = g_1 + \cdots + g_k$ in V:

$$\int_V -g\Delta^{\mathbb{S}^2} g - 2ag^2 = \sum_{i,j=1}^k \int_V -g_i\Delta^{\mathbb{S}^2} g_j - 2ag_i g_j < 0$$

Therefore,

$$\int_V -g\Delta g - 2ag^2 < 0. \tag{4.4}$$

This property occurs in sufficiently small neighborhood V of regular values q of G such that its pre-images are interior points of C. Note that using Sard's theorem and the fact that G is an open function, $G(\partial C)$ has null measure in $G(C)$ and the set

$$W = \{q \in G(C); G^{-1}(q) \subset int(C) \text{ and } g \text{ is smooth in } q\}$$

has full measure in $G(C)$. It shows (i).

(ii) g is locally Lipschitz in

$$X = G(C) \setminus R,$$

where R is the set of ramification values of $G|_C$.

Note that since the branch points of G are isolated, we can assume that G has no branch values in $G(\partial C)$, doing a small modification on C, if necessary, in order to keep $L_a = \Delta - 2aK$ unstable, still have a smooth boundary, the first eigenvalue is negative and has an eigenfunction f associated with the first eigenvalue. If $q \in Y$ is such that $q \notin G(\partial C)$, g is Lipschitz in a neighborhood of q because g is smooth in q. we will analyse when $q \in G(\partial C)$, where by the form that we define C, q is a regular value of G. Let $(G|_C^{-1})(q) = \{p_1, \dots p_k, r_1, \dots, r_l\} = G^{-1}(q) \cap C$, where $p_i \in int(C)$ and $r_i \in \partial C$. Let V be a small neighborhood of q such that

$$(G|_C^{-1})(V) = U_1 \cup \dots \cup U_k \cup Y_1 \cup \dots \cup Y_l,$$

where $p_i \in U_i$, $r_j \in Y_j$, each U_i and Y_j is diffeomorphic to V by G and each U_i is a subset of $int(G(C))$. We note that, for $y \in V$,

$$g(y) = \sum_{i=1}^{k} f(x_i) + \sum_{j=1}^{l} \tilde{f}(z_j),$$

where $\{x_1, \dots x_k, z_1, \dots, z_l\} = (G|_C^{-1})(y)$ and $\tilde{f}(z)$ is defined as $f(z)$, if $z \in C$, and 0, if $z \notin C$. Note that $f|_{U_i} : U_i \to \mathbb{R}$ is continuous and Lipschitz, with Lipschitz constant $M_i = \max\{|df_p|, p \in U_i\}$ and, since $f \equiv 0$ in ∂C, $\tilde{f}|_{Y_j} : Y_j \to \mathbb{R}$ is continuous and Lipschitz, with Lipschitz constant $N_j = \max\{|df_p|, p \in Y_j \cap C\}$. Since $det(J(G^{-1})) = -\frac{1}{K}$, we have that g is Lipschitz in a neighborhood of y, with Lipschitz constant

$$M = -\frac{1}{K}\left(\sum_{i=1}^{k} M_i + \sum_{j=1}^{l} N_j\right).$$

It shows that g is locally Lipschitz in X.

(iii) g is continuous.

By (ii), g is continuous on X, that is, in the set of regular values of G in $G(C)$. If q is not a regular value, there exists $p_1 \in G^{-1}(q)$ that is an isolated critical point of G and there exists a neighborhood U_1 of p_1 such that

$$G_1 = G|_{U_1 \setminus \{p_1\}} : U_1 \setminus \{p_1\} \longrightarrow U \setminus \{q\}$$

is an r_{p_1}-to-one function, where r_{p_1} is the ramification index of p_1. Since $g_1(p_1) = r_{p_1} f(q)$, then $g_1(p) = r_{p_1} f(p_1)$ is continuous in p by an argument of limit. Using this at all points in the pre-image of q by G, we conclude that g is continuous in the ramification values of G. Hence g is continuous.

Since f is limited, g is also limited. Let $0 < \varepsilon < 1$ and $\varphi = \varphi_\varepsilon : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\varphi(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq \varepsilon^2; \\ 2 - \frac{\log r}{\log \varepsilon} & \text{if } \varepsilon^2 \leq r \leq \varepsilon; \\ 1 & \text{if } r \geq \varepsilon. \end{cases}$$

If N is a bidimensional manifold and $q \in N$, the family of functions $\phi_{q,\varepsilon} : N \longrightarrow \mathbb{R}$ defined by $\phi_{q,\varepsilon}(p) = \varphi_\varepsilon(d(p,q))$ is called logarithmic cutoff functions, they have the properties $\lim_{\varepsilon \to 0} \phi_{q,\varepsilon} = 1$ pointwise and $\lim_{\varepsilon \to 0} \int_N |\nabla \phi_{q,\varepsilon}|^2 = 0$.

Returning to our problem, being $q_1, \ldots, q_k$ the ramification values of G in $G(C)$ and $\phi_{q,\varepsilon}$ defined as above for $N = G(C)$, define

$$\begin{aligned} g_\varepsilon : G(C) &\longrightarrow \mathbb{R} \\ q &\longmapsto g(q) \prod_{i=1}^{k} \phi_{q_i,\varepsilon}. \end{aligned}$$

Then $g_\varepsilon \to g$ pointwise, $g = g_\varepsilon$ except in small balls of radios ε centered in ramifications values of G in $G(C)$ and g_ε is Lipschitz. Calling $\phi_\varepsilon = \prod_{i=1}^{k} \phi_{q_i,\varepsilon}$, then:

$$\begin{aligned}
\int_{G(C)} |\nabla g_\varepsilon|^2 - 2ag_\varepsilon^2 &= -\int_{G(C)} \phi_\varepsilon g \Delta(\phi_\varepsilon g) + 2ag^2\phi_\varepsilon^2 \\
&= -\int_{G(C)} \phi_\varepsilon g(\phi_\varepsilon \Delta g + 2\langle \nabla\phi_\varepsilon, \nabla g\rangle + g\Delta\phi_\varepsilon) + 2ag^2\phi_\varepsilon^2 \\
&= -\int_{G(C)} \phi_\varepsilon^2 g\Delta g + 2ag^2\phi_\varepsilon^2 - \int_{G(C)} \frac{1}{2}\langle \nabla\phi_\varepsilon^2, \nabla g^2\rangle + g^2\phi_\varepsilon\Delta\phi_\varepsilon \\
&= -\int_{G(C)} \phi_\varepsilon^2 g\Delta g + 2ag^2\phi_\varepsilon^2 + \int_{G(C)} \frac{1}{2} g^2\Delta\phi_\varepsilon^2 - g^2\phi_\varepsilon\Delta\phi_\varepsilon \\
&= -\int_{G(C)} \phi_\varepsilon^2 g\Delta g + 2ag^2\phi_\varepsilon^2 + \int_{G(C)} g^2|\nabla\phi_\varepsilon|^2.
\end{aligned}$$

The first term is negative because (4.4) and the second term is close to zero when ε is small. Thus, we can find a test function that guarantees that the operator $\Delta + 2a$ is unstable in $G(C)$. This is a contradiction, because $G(C)$ has a smaller area than the spherical cap with $2a$ as the first eigenvalue of the Laplacian.

■

4.2.2 a-index of complete minimal surface with finite total curvature

We now address the a-index of complete minimal surfaces with finite total curvature. The usual index of some minimal surfaces in $\mathbb{R}^3$ (which corresponds to our 1-index) is known, as shown in (1.3).

If M is the catenoid or the Enneper surface, then the Gauss normal map of M extends to a holomorphic application $G : \mathbb{S}^2 \to \mathbb{S}^2$ that is bijective. Therefore, by Corollary 3.0.2, the index of L_a on M is the index of $\Delta - 2a$ on $\mathbb{S}^2$. It is known that the eigenvalues of the Laplacian in $\mathbb{S}^2$ are of the form $n(n+1)$ with a multiplicity $2n+1$. That said, by a simple calculus, the a-index of the catenoid and the minimal Enneper surface are k^2 if k is the natural number such that $\frac{1}{2}k(k-1) < a \leq \frac{1}{2}k(k+1)$, given according to the table:

a	a-index
$0 < a \leq 1$	1
$1 < a \leq 3$	4
$3 < a \leq 6$	9
$6 < a \leq 10$	16
$10 < a \leq 15$	25
$\vdots$	$\vdots$

Other examples of minimal surface conformal to $\mathbb{S}^2$ with a finite number of punctures are the Jorge-Meeks family of minimal surfaces, where the Gauss map extends to $G : \overline{\mathbb{C}} \to \mathbb{S}^2$, where $\overline{\mathbb{C}}$ is the Riemann's sphere. Therefore, if $\Pi : \mathbb{S}^2 \to \overline{\mathbb{C}}$ is the stereographic projection relative to the north pole and $g = \Pi \circ G$, then $g(z) = z^n$. When $n = 1$, we obtain the catenoid. The Jorge-Meeks surface with Gauss map related to this relation with the application $g(z) = z^n$ has $n+1$ ends. Let us call the n-ended Jorge-Meeks surface By M_n. According to [Nay90a], on $\mathbb{S}^2$, the eigenvalue problem

$$\Delta u = -\lambda |dG|^2 u \tag{4.5}$$

has eigenvalues of the form

$$\lambda_i = \frac{i}{2n}\left(\frac{i}{n}+1\right), \text{ with multiplicity } \begin{cases} 2p+2 & \text{if } i = pn+q \\ 2p+1 & \text{if } i = pn, \end{cases} \tag{4.6}$$

where $p, q \in \mathbb{N}, q \leq n-1$. Therefore, if we want to find the a-Index of M_n, we have to find the number of negative eigenvalues of $L_a = \Delta u + a|dG|^2 u$, which will be reduced to having how many eigenvalues λ_i (counted with multiplicity) of (4.5) are less than a. For example, for $a = 1$, we have $\frac{i}{2n}(\frac{i}{n}+1) < 1 \Rightarrow i < n$, for $i = 0$ we have the pair $(p, q) = (0, 0)$ and for $i = 1, \ldots, n-1$, we have the pair $(p, q) = (0, i)$. Hence, $Index_a(M_n) = 1+2(n-1) = 2n-1$, which is the index usual of the Jorge-Meeks surfaces.

The next results are only adaptations of results in Section 3 of [Nay93]. Assume that $G : \Sigma \to \mathbb{S}^2$ is a nonconstant holomorphic map. Let us define the set of conformal diffeomorphisms formed by the elements $\mathcal{A}_t$, $0 < t < \infty$, of $\mathbb{S}^2$ given by $\Pi \circ \mathcal{A}_t \circ \Pi^{-1}(w) =$

tw, $w \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

$$
\begin{array}{ccc}
\mathbb{S}^2 & \xrightarrow{\mathcal{A}_t} & \mathbb{S}^2 \\
{\scriptstyle \Pi^{-1}}\big\uparrow & & \big\downarrow{\scriptstyle \Pi} \\
\overline{\mathbb{C}} & \xrightarrow{w \mapsto tw} & \overline{\mathbb{C}}
\end{array}
$$

Let $G_t = \mathcal{A}_t \circ G$. If g is the meromorphic map associated with G, that is, $g = \Pi \circ G$, then we have $\Pi \circ G_t = tg$.

$$
\begin{array}{ccc}
\mathbb{S}^2 & \xrightarrow{\mathcal{A}_t} & \mathbb{S}^2 \\
{\scriptstyle G}\big\uparrow & \overset{G_t}{\nearrow} & \big\downarrow{\scriptstyle \Pi} \\
\Sigma & \xrightarrow{\Pi \circ G_t} & \overline{\mathbb{C}}
\end{array}
$$

As the correspondence $t \to G_t$ is continuous with respect to the usual topology C^1, it is possible to show, using (4.1), that $\lambda_k(G_t)$ is also continuous in t.

Let $P(G) = m_1 p_1 + \cdots + m_\nu p_\nu$ be the polar divisor of g, where $p_i, i = 1, \ldots, \nu$ are distinct. Note that $m_1 + \cdots + m_\nu = d$, where d is the degree of the application G.

For each $i = 1, \ldots, \nu$, we define the holomorphic map $\widetilde{G}_i : \overline{\mathbb{C}}_i \to \mathbb{S}^2$ given by $\Pi \circ \widetilde{G}_i(z) = z^{m_i}$, where $\overline{C}_i$ is a copy of $\overline{C}$. Let $\widetilde{\Sigma}$ be the disjoint union of the $\overline{C}_i$'s, $i = 1, \ldots, \nu$, and $\widetilde{G} : \widetilde{\Sigma} \to \mathbb{S}^2$ be the holomorphic map defined by $\widetilde{G}(z) = \widetilde{G}_i(z)$, if $z \in \overline{C}_i$.

Proposition 4.2.3. *Let $G_t : \Sigma \to \mathbb{S}^2$, $t \in (0, \infty)$ and $\widetilde{G} : \widetilde{\Sigma} \to \mathbb{S}^2$ be as before , then for $k = 1, 2, \ldots$:*

$$\lim_{t \to 0} \lambda_k^{(a)}(G_t) = \lambda_k^{(a)}(\widetilde{G}).$$

For a proof, see [Nay93].

Remark 4.2.1. *Let $G^* : \Sigma \to \mathbb{S}^2$ be the holomorphic map defined by $\Pi \circ G^* = \frac{1}{g}$. It is easy to see that $G^* = PG$, where*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

As $P \in O(3)$, we have $ds^2_{G^*} = ds^2_G$ and therefore, the eigenvalues of $L_{G^*,a}$ coincide with those of $L_{G,a}$. In particular, $Index_a(G^*) = Index_a(G)$ and $Nul_a(G^*) = Nul_a(G)$, where $Nul_a(G) = \#\{$eigenspace dimension of eigenvalue zero$\}$.

This observation allows us to conclude the same result of Proposition 4.2.3 when $t \to \infty$ instead of $t \to 0$.

By (4.6) we conclude that $Index_a(\widetilde{G}_i) = 1 \; \forall a \leq \frac{m_i+1}{2m_i^2}$. In particular, we have proved:

Corollary 4.2.1. *Let $G : \Sigma \to \mathbb{S}^2$ be a holomorphic nonconstant map of degree d and $G_t = \mathcal{A} \circ G$, $t \in (0, \infty)$. Let ν be the number of distinct poles of $g = \Pi \circ G$. Fixed $a > 0$, then for all sufficiently small t, $Index_a(G_t) \geq \nu$.*

Example 4.2.1. *: For the Jorge-Meeks family of minimal surfaces with the usual Gauss map $G : \overline{\mathbb{C}} \to \mathbb{S}^2$ satisfying $\Pi \circ G(z) = z^{r-1}$, we have, for $a \leq \frac{r+1}{2r^2}$ fixed, $Index_a(G_t) = 1$. If we apply a rotation T such that ∞ becomes a regular value of $T \circ G$, for t sufficiently small, $Index_a((T \circ G)_t) = r - 1$. For this, we can find (i) a minimal surface of finite total curvature and an a-index so large as we wish and (ii) a surface with number of ends so large as we wish and an a-index equal to one for all a sufficiently small.*

Finally, by the Proposition 4.2.1 we have $Index_a(M) \geq 1$ for all complete minimal surfaces M with finite total curvature and $a > 0$. The next theorem shows that, for values of a positive and close to zero, $Index_a(M) = 1$.

Theorem 4.2.1. *Let M be a complete non flat minimal surface of finite total curvature, then*

$$\lim_{a \to 0^+} Index_a(M) = 1.$$

Proof. Let Σ be the Riemann's surface which is the compactification of M. Consider the operator $\Delta - a|dG|^2$ on Σ. We have to show that

$$\lim_{a \to 0^+} Index_a(G) = 1.$$

Since M is non flat, M is not a plane, it implies that G is nonconstant and for all a, $Index_a(G) \geq 1$. Note that $Q_a(f, f) < 0$, if f is constant. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of Laplacian in Σ. Suppose that there exists a sequence $\{a_1, a_2, \ldots\}$ such that each $a_i > 0$, $\lim_{i \to 0} a_i = 0$ and $Index_{a_i}(G) \geq 2$, then there exist two sequences of functions $\{f_{1,1}, f_{1,2}, f_{1,3}, \ldots\}$ and $\{f_{2,1}, f_{2,2}, f_{2,3} \ldots\}$ such that $||f_{i,j}||_{L^2(\Sigma)} = 1$, $Q_{a_i}(f_{i,j}, f_{i,j}) < 0$, $i = 1, 2$, $j \in \mathbb{N}$ and

$$\int_{\Sigma} f_{1,j} f_{2,j} = 0.$$

For $i = 1, 2$, we have $\lim_{j\to\infty} Q_{a_i}(f_{i,j}, f_{i,j}) = 0$, because $|dG|$ is limited and

$$0 \geq \lim_{j\to\infty} Q_{a_i}(f_{i,j}, f_{i,j}) = \lim_{j\to\infty} \int_\Sigma |\nabla f_{i,j}|^2 - a_i|dG|^2 f_{i,j}^2 \geq -a_i vol(\Sigma)|dG|_{L^\infty(\Sigma)} \to 0$$

when $i \to \infty$. Furthermore, this argument guarantees us $\lim_{j\to\infty} |\nabla f_{i,j}| = 0$, then

$$0 = \lim_{j\to\infty} \int_\Sigma |\nabla f_{i,j}|^2 = -\lim_{j\to\infty} \int_\Sigma f_{i,j} \Delta f_{i,j}$$

for $i = 1, 2$. Hence, for j large, $\{f_{1,j}, f_{2,j}\}$ generate, in $C^\infty(\Sigma)$, a subspace of dimension two in $C^\infty(\Sigma)$ such that the Laplacian in a function g in this subspace satisfies

$$-\int_\Sigma g\Delta g < \lambda_1 ||g||_{L^2(\Sigma)},$$

which is a contradiction because the characterization of λ_1 in (4.1).

■

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