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Critical Curves in Anisotropic Percolation on $\mathbb{Z}^{d+s}$

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Curvas Críticas em Percolação Anisotrópica em Z^(d+s)

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Resumo

O presente trabalho concentra-se no estudo de um modelo de percolação anisotrópica de elos em $\mathbb{Z}^{d+s}$, onde elos de $\mathbb{Z}^d$ estão abertos de forma independente com probabilidade p (menor que o ponto crítico de $\mathbb{Z}^d$), e elos de $\mathbb{Z}^s$ estão abertos com probabilidade q, também de forma independente. Assim, o principal objetivo do trabalho é analisar o comportamento da curva crítica $q_c(p) = \sup\{q; \theta(p,q) = 0\}$, apresentando cotas que garantem a existência ou não existência de um aglomerado aberto infinito quando a curva assume valores de p próximos do ponto crítico de $\mathbb{Z}^d$.

Palavras-chave: Percolação Anisotrópica. Curvas Críticas. Probabilidade.

Abstract

This work concerns the study of an anisotropic bond percolation model on $\mathbb{Z}^{d+s}$, where edges of $\mathbb{Z}^d$ are open independently with probability p (less than the critical threshold of $\mathbb{Z}^d$) and edges of $\mathbb{Z}^s$ are open with probability q, also independently. Thus, the main goal of the work is to analyze the behaviour of the critical curve $q_c(p) = \sup\{q; \theta(p,q) = 0\}$, giving upper and lower bounds that guarantee the existence or non-existence of an infinite open cluster when the curve assumes values of p close to the critical threshold of $\mathbb{Z}^d$.

Keywords: Anisotropic Percolation. Critical Curves. Probability.

Contents

Chapter 1

Introduction

Becoming famous for having problems with simple statements but complex solutions, the mathematical percolation model emerged with [2] by studying the transport of a fluid through a porous medium, where we can cite as an example the phenomenon of the propagation of oil through a porous rock. Thus, the heart of percolation theory is the description of the porous medium, treating it as a network (translated in mathematics as a graph) of small channels (edges/bonds of the graph) and pores (vertices/sites of the graph) that connect in such a way as to allow the passage of the fluid. Keeping in mind the example of oil, it becomes interesting to analyze under which conditions the rock allows the oil to propagate completely, *i.e.* the rock gets all wet. When this occurs, that is, when the porous medium allows the complete passage of the fluid, we say that there exists percolation. Then, to answer the question "when there is percolation in the model?" becomes an object of interest to those who venture to investigate percolation theory.

Over time, derivations of the Broadment and Hammersly model emerged, such as the anisotropic percolation models, which we will treat by associating different probabilities for edges placed in different directions. For instance, we can mention the work of [12] and [9] with results about critical curves, and also the work of [1, 3, 7, 4] regarding critical exponents and phase transition. In the present text, we focus on the anisotropic bond percolation model on $\mathbb{Z}^{d+s}$ presented in the work of [10, 11], where we seek to analyze and understand the behaviour of critical curves in such a percolation model.

In Chapter 2, we define the independent bond percolation model and present essential concepts for a first contact with percolation theory, besides stating some relevant results and theorems, such as the Phase Transition Theorem and the FKG and BK inequalities. We demonstrate the latter, which will be useful for proving a result in the next chapter.

In Chapter 3, we define an anisotropic bond percolation model on $\mathbb{Z}^{d+s}$, where each edge of $\mathbb{Z}^d$ is open independently with probability $p < p_c(\mathbb{Z}^d)$ (the critical threshold of bond percolation), and we declare each edge of $\mathbb{Z}^s$ as open with a probability q, also independently. Thus, we define the critical curve $q_c(p) = \sup\{q; \theta(p, q) = 0\}$ and we study the problem posed by [10, 11] concerning the behaviour of the curve $q_c(p)$ for values of p

close to $p_c(\mathbb{Z}^d)$. Our major goal is to present the following theorems.

Theorem 1. *Consider a bond percolation process on* $\mathbb{Z}^d \times \mathbb{Z}^s$ *with parameters* (p, q), $p < p_c(d)$, *and let* $\chi_d(p)$ *denote the mean size of the open cluster in* $\mathbb{Z}^d$. *If the pair* (p, q) *satisfies*

$$q < \frac{1}{2s\chi_d(p)},$$

then there is a.s. no infinite open cluster in $\mathbb{Z}^{d+s}$.

Theorem 2. *Consider a bond percolation process on* $\mathbb{Z} \times \mathbb{Z}^s$, $s > 1$, *with parameters* (p, q). *Then*

$$\frac{1}{2s\chi_1(p)} \leq q_c(p) \leq \frac{\alpha}{\chi_1(p)},$$

for some $\alpha > 0$ *and* p *sufficiently close to* 1.

Chapter 2

Basic concepts in percolation

In this chapter, we describe the independent percolation model and present some fundamental concepts and results of the percolation theory. We refer the reader to [5] for everything that is presented in the chapter.

Let $G = (\mathcal{V}, \mathcal{E})$ be a infinite graph, locally finite (*i.e.* every vertex has finite degree), where $\mathcal{V}$ and $\mathcal{E}$ denote the set of vertices and edges of G, respectively, and let $p \in [0,1]$. In a bond percolation process, we attribute to each edge $e \in \mathcal{E}$ the probability of being open with parameter p and we say that e is closed with probability $1-p$. In a site percolation model, we consider each edge as open and we say that a vertex $v \in \mathcal{V}$ is open with probability p, and hence v is closed with probability $1-p$. Throughout the text, except where indicated otherwise, we deal with bond percolation models. Also, although we only deal with undirected graphs, we write $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, such that $(x, y) \in \mathcal{E}$ denotes the unique, undirected, edge $\{x, y\}$ between the vertices x and y.

Definition 3. *A* ***configuration*** ω *is a function* $\omega : \mathcal{E} \to \{0,1\}$ *where* $\omega(e) = 1$ *if* e *is open and* $\omega(e) = 0$ *if the edge is closed. We denote by* $\Omega = \{0,1\}^{\mathcal{E}}$ *the set of all possible configurations.*

Definition 4. *Given* $x, y \in \mathcal{V}$*, a* ***path*** *connecting* x *and* y *is a finite sequence of vertices* $\gamma = \langle v_0 = x, v_1, \ldots, v_n = y\rangle$ *such that* $(v_i, v_{i+1}) \in \mathcal{E}$ *for all* i*, that is, there exists an edge between any two consecutive vertices of the sequence. Given a configuration* $\omega \in \Omega$ *we say that the path* γ *is* ***open*** *in* ω *if, for each* $i \in \{1, \ldots, n\}$, $\omega((v_i, v_{i+1})) = 1$*;* i.e *a path is open if all its edges are open. Moreover, we say that* x *is* ***connected*** *to* y *in the configuration* ω *if there exists an open path connecting* x *and* y*, and we denote this event by* $\{x \leftrightarrow y\} = \{\omega \in \Omega; x \text{ is connected to } y \text{ by open paths in } \omega\}$.

Definition 5. *Given a vertex* $x \in \mathcal{V}$ *and a configuration* $\omega \in \Omega$*, the open* ***cluster*** *of* x *in* ω *is the set* $\mathcal{C}_x(\omega) = \{y \in \mathcal{V}; x \text{ is connected to } y \text{ by open paths in } \omega\}$, i.e. *the set of vertices that are connected to* x *by a path of open edges.*

Thus, we can see that the size of the open cluster of a vertex x in a configuration ω, that is, the number of vertices connected to x by open paths, is a discrete random variable since it assumes values in the set of natural numbers or it can be infinite (*e.g* when $p = 1$). Then we write this as $|\mathcal{C}_x| : \Omega \to \mathbb{N} \cup \{+\infty\}$ and investigate the case where $|\mathcal{C}_x| = +\infty$. The event $\{\omega \in \Omega; |\mathcal{C}_x| = +\infty\}$ is what we call **percolation**, meaning that percolation occurs when there exists an infinite open cluster or, equivalently, we can also say that the vertex x is connected to infinity, and then we use the notation $\{x \leftrightarrow \infty\}$.

Now, we consider a particular graph, namely the d-dimensional cubic lattice, denoted by $\mathbb{L}^d = (\mathbb{Z}^d, \mathcal{E})$ with $\mathbb{Z}^d$ being the set of d-dimensional vectors with coordinates in $\mathbb{Z}$ and $\mathcal{E} = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$; where $\exists!\ i \in \{1, \ldots, d\}$ such that $|x_i - y_i| = 1$ while $x_j = y_j$ for all $i \neq j\}$. That is, $\mathcal{E}$ is the set of edges that join two vertices *iff* their coordinates are identical except for a single one, which is different by a single unit. We also call $\mathcal{E}$ the set of *nearest neighbours of* $\mathbb{Z}^d$. Again, we say that edges of $\mathbb{L}^d$ are open with probability p in a configuration ω independently, and we remark that studying the open cluster of the vertex $x = 0$ (the d-dimensional vector of zeros) is equivalent to studying the open cluster of any other vertex, since every edge is equal and it is open with the same probability. Then, in $\mathbb{L}^d$ we bring our attention to the origin of $\mathbb{Z}^d$, denoting by $\mathcal{C}$ the open cluster containing the origin, and we are interested in knowing when $\mathcal{C}$ has infinite size.

At this point, it is important to observe that we cannot consider all the vertices in a general infinite graph as equal. Nevertheless, we verify that in any connected infinite graph G, if a given vertex x is connected to infinity with positive probability, then any other vertex y is also connected to infinity with positive probability (as it is proved in Proposition 10).

Now, note that the process of bond percolation on $\mathbb{L}^d$ where each edge is open with probability p in a configuration ω that associates 1 to open edges and 0 to the closed ones can be described by the probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$, where $\Omega = \prod_{e \in \mathcal{E}} \{0, 1\}$, $\mathcal{F}$ is the σ-algebra of subsets of Ω generated by the finite dimensional cylinders, that is, by the events which depend only on edges in finite subsets of $\mathcal{E}$, and $\mathbb{P}_p$ is the product measure $\mathbb{P}_p = \prod_{e \in \mathcal{E}} \mu_e$, where μ_e is the Bernoulli measure in $\{0, 1\}$ defined for every edge $e \in \mathcal{E}$ by $\mu_e(\omega(e) = 1) = p$ and $\mu_e(\omega(e) = 0) = 1 - p$.

2.1 The $\theta(p)$ function

As already mentioned, we are interested in the occurrence of percolation in $\mathbb{L}^d$, *i.e.* in the occurrence of the event that there exists an infinite open cluster, and that motivates the definition of the function $\theta(p)$ given by

$$\theta\colon [0,1] \to [0,1]$$
$$p \mapsto \mathbb{P}_p(0 \leftrightarrow \infty).$$

So given $p \in [0,1]$, $\theta(p)$ is the probability of percolation's occurrence with parameter p. Trivially we have that if $p = 0$ then $\theta(p) = 0$, and if $p = 1$ then $\theta(p) = 1$.

Proposition 6. *θ is a non-decreasing function of parameter p, that is, if $p_1 < p_2$ then $\theta(p_1) \leq \theta(p_2)$.*

Proof. Observe that a bond percolation model with parameter p_1 can be described by the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{p_1})$, as we have already seen. Then if we change the parameter from p_1 to p_2, the model now is described by the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{p_2})$. We use the important space coupling argument, that allows us to compare the parameters p_1 and p_2 in a single probability space. For this, remember that $\Omega = \{0,1\}^{\mathcal{E}}$, that is, Ω is the set of configurations that associate 0 or 1 to each edge $e \in \mathcal{E}$. Now, define the set of configurations $\widetilde{\Omega} = [0,1]^{\mathcal{E}}$ that associate a number between 0 and 1 to each edge $e \in \mathcal{E}$, and let $\widetilde{\mathcal{F}}$ be the smallest σ-algebra that contains all the cylinders in $\widetilde{\mathcal{F}}$. Then, consider $\mathbb{P}$ the probability measure with uniform distribution in $[0,1]$, that is, in which a number belongs to the interval $[\alpha, \beta] \subset [0,1]$ with probability equal to the interval size, *i.e.* $\beta - \alpha$.

Thus, in $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbb{P})$, given $\widetilde{\omega} \in \widetilde{\Omega}$, $e \in \mathcal{E}$ and $p \in [0,1]$, we say that the edge e is **p-open** if $\widetilde{\omega}(e) \leq p$; so an edge is p-open with probability p, since it equals the probability of $\widetilde{\omega}(e) \in [0,p]$. Moreover, note that if we have $p_1 \leq p_2$ then every p_1-open edge is a p_2-open edge: if $e \in \mathcal{E}$ is p_1-open then $\widetilde{\omega}(e) \leq p_1$, and since $p_1 \leq p_2$ we have that $\widetilde{\omega}(e) \leq p_2$, hence e is a p_2-open edge.

Now, let $\mathcal{C}(p, \widetilde{\omega}) = \{v \in \mathcal{V}; 0 \leftrightarrow v \text{ by } p\text{-open paths in } \widetilde{\omega}\}$ be the p-open cluster containing the origin. Then, we have that

$$\theta(p_1) = \mathbb{P}_{p_1}\{\omega; |\mathcal{C}(\omega)| = +\infty\} = \mathbb{P}\{\widetilde{\omega}; |\mathcal{C}(p_1, \widetilde{\omega})| = +\infty\},$$

as every edge is p_1-open with probability p_1. And, since every p_1-open edge is a p_2-open edge, it results that

$$\{\widetilde{\omega}; |\mathcal{C}(p_1, \widetilde{w})| = +\infty\} \subseteq \{\widetilde{\omega}; |\mathcal{C}(p_2, \widetilde{w})| = +\infty\}.$$

Then,

$$\begin{aligned}
\theta(p_1) = \mathbb{P}\{\widetilde{\omega}; |\mathcal{C}(p_1, \widetilde{w})| = +\infty\} &\leq \mathbb{P}\{\widetilde{\omega}; |\mathcal{C}(p_2, \widetilde{w})| = +\infty\} \\
&= \mathbb{P}_{p_2}\{\omega; |\mathcal{C}(\omega)| = +\infty\} \\
&= \theta(p_2).
\end{aligned}$$

Hence, $\theta(p)$ is a non-decreasing function. □

Since θ is a monotone function of parameter p, we bring our attention to the so called *critical threshold of bond percolation on* $\mathbb{Z}^d$, defined by $p_c(d) = \sup\{p; \theta(p) = 0\}$. Moreover, we state the important theorem of Broadment-Hammersly about the existence of the critical threshold in $\mathbb{L}^d$.

Theorem 7 (Phase Transition Theorem). *In $\mathbb{L}^d$, with $d \geq 2$, there exists a critical threshold $p_c(d) \in (0,1)$ such that*

(i) $\theta(p) = 0$ *for all* $p < p_c$,

(ii) $\theta(p) > 0$ *for all* $p > p_c$.

Observe that the above theorem guarantees the existence of a critical point which divides the probability of percolation on $\mathbb{L}^d$ in two phases, namely the *subcritical phase*, where there is no infinite open cluster, and the *supercritical phase*, where there is an infinite open cluster with strictly positive probability. Nevertheless, the theorem says nothing about $\theta(p)$ when evaluated in $p_c(d)$. By the works of [8] and [6], it is known that $\theta(p_c(d)) = 0$ for $d = 2$ and for $d \geq 19$, while determining the value of $\theta(p_c(d))$ for $3 \leq d \leq 18$ might be the most famous open problem in percolation theory.

2.2 The Harris-FKG and BK Inequalities

Let $\omega, \widetilde{\omega} \in \Omega$ be two configurations and define a partial order between them as follows: we say that $\omega \leq \widetilde{\omega}$ if $\omega(e) \leq \widetilde{\omega}(e)$, for all $e \in \mathcal{E}$. Thus, if the smaller configuration maps e to 1 then the larger one also maps e to 1; hence, we can obtain $\widetilde{\omega}$ from ω by simply opening more edges.

Definition 8. *An event $A \in \mathcal{F}$ is said to be **increasing** if, for all $\omega \in A$, we have that $\widetilde{\omega}$ is also in A, for all $\widetilde{\omega} \geq \omega$;* i.e. *an increasing event is favoured by opening more edges. Analogously, we say that A is **decreasing** if A^c is an increasing event. Moreover, we say that a random variable N in $(\Omega, \mathcal{F}, \mathbb{P})$ is increasing if we have $N(\omega) \leq N(\widetilde{\omega})$ whenever $\omega \leq \widetilde{\omega}$.*

The following result is due to Harris, Fortuin, Kasteleyn and Ginibre and gives an important inequality for increasing events.

Theorem 9 (FKG-Inequality). *Considering the probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$, we have*

(i) *If Z and Y are increasing random variables with finite second moments, then*

$$\mathbb{E}_p(ZY) \geq \mathbb{E}_p(Z) \cdot \mathbb{E}_p(Y).$$

(ii) *If A and B are increasing events in $\mathcal{F}$, then*

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A) \cdot \mathbb{P}_p(B).$$

Now, we can use the above theorem in order to prove a statement made at the beginning of the chapter.

Proposition 10. *Given any connected infinite graph G, the existence of the critical threshold does not depend on any vertex, that is, if a vertex x has an infinite open cluster with positive probability, then each vertex of G also has an infinite open cluster with positive probability.*

Proof. Let G be a connected infinite graph where each edge is open with probability $p \in (0, 1)$, let x, y be vertices of G, and suppose that x has an infinite open cluster with positive probability. Considering the events $A = \{x \leftrightarrow y\}$, $B = \{x \leftrightarrow \infty\}$ and $C = \{y \leftrightarrow \infty\}$, we have that $\theta_x(p) = \mathbb{P}_p(B)$ and $\theta_y(p) = \mathbb{P}_p(C)$, and then by hypothesis $\theta_x(p) > 0$. It follows from the connectedness of G that there exists an open path joining the vertices x and y; let d be the distance between them, *i.e.* the number of vertices at the smallest path that connects x and y. Note that A and B are increasing events, so the FKG-inequality implies that

$$\mathbb{P}_p(A \cap B) \;\geq\; \mathbb{P}_p(A) \cdot \mathbb{P}_p(B) \;\geq\; p^d \cdot \; \theta_x(p) > 0,$$

since $\theta_x(p) > 0$.

Now, note that $A \cap B \equiv \{y \leftrightarrow x\} \wedge \{x \leftrightarrow \infty\}$, then the occurrence of $A \cap B$ implies the occurrence of the event $\{y \leftrightarrow \infty\}$, and so we have that $(A \cap B) \subset \{y \leftrightarrow \infty$. Hence,

$$\mathbb{P}(y \leftrightarrow \infty) = \theta_y(p) \geq \mathbb{P}(A \cap B) > 0,$$

i.e. there is an infinite open cluster containing the vertex y with positive probability. □

Let $e_1, \ldots, e_n$ be different edges of $\mathbb{L}^d$, and $\omega = (\omega(e_1), \ldots, \omega(e_n))$ be the vector of the state of the n edges (open or closed). Consider the increasing events A and B, which depend only on ω, and let us describe each ω by the set $K(\omega) = \{e_i \colon \omega(e_i) = 1\}$, *i.e.* $K(\omega)$ is the set of open edges in the configuration ω.

Definition 11. *We define the event* ***disjoint occurrence*** *of A and B by $A \circ B = \{\omega \in \Omega : \exists H \subseteq K(\omega)$ such that $\omega' \in A$, and $\omega'' \in B$, with ω' determined by $K(\omega') = H$, and ω'' determined by $K(\omega'') = K(w) \setminus H\}$. That is, the disjoint occurrence of A and B is the set of configurations ω for which there exist disjoint sets of open edges where the first set ensures the occurrence of A and the second one ensures the occurrence of B.*

Now, let m be a positive integer number and consider the space $(\Gamma, \mathcal{G}, \mathbb{P})$ where $\Gamma = \prod_{i=1}^{m}\{0,1\}$, $\mathcal{G}$ is the set of all subsets of Γ, and $\mathbb{P}$ is the product measure defined by $\mathbb{P} = \prod_{i=1}^{m} \mu_i$, where $\mu_i(0) = 1 - p(i)$ and $\mu_i(1) = p(i)\cdot$ Then we proceed with the famous inequality of [13] concerning disjoint occurrences.

Theorem 12 (The BK Inequality). *Considering the space* $(\Gamma, \mathcal{G}, \mathbb{P})$*, if* A *and* B *are increasing events, then*

$$\mathbb{P}(A \circ B) \;\leq\; \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Proof. Let $(\Gamma_1, \mathcal{G}_1, \mathbb{P}_1)$ and $(\Gamma_2, \mathcal{G}_2, \mathbb{P}_2)$ be two copies of $(\Gamma, \mathcal{G}, \mathbb{P})$, and consider the product space $(\Gamma_1 \times \Gamma_2, \mathcal{G}_1 \times \mathcal{G}_2, \mathbb{P}_{12})$, where $\mathbb{P}_{12} = \mathbb{P}_1\mathbb{P}_2$. We will write (x, y) for a point in $\Gamma_1 \times \Gamma_2$, with $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, where each x_i, y_i equals 0 or 1.

Define the events A' and B'_k by

$$\begin{aligned} A' &= \{(x, y) \in \Gamma_1 \times \Gamma_2 \colon x \in A\}, \\ B'_k &= \{(x, y) \in \Gamma_1 \times \Gamma_2 \colon (y_1, \ldots, y_k, x_{k+1}, \ldots, x_m) \in B, \text{ for } 0 \leq k \leq m\}. \end{aligned}$$

Note that A' and B'_k are increasing events in $\Gamma_1 \times \Gamma_2$: if $\omega, \widetilde{\omega} \in \Gamma_1 \times \Gamma_2$ are configurations such that $\omega \leq \widetilde{\omega}$ and $\omega \in A'$, then we have that $(x_1, \ldots, x_m) \in A$, and then $(\widetilde{x}_1, \ldots, \widetilde{x}_m) \in A$ since A is an increasing event; therefore $\widetilde{\omega} \in A'$, which implies that A' is increasing. By an analogous argument we have that B'_k is also an increasing event in the product space.

For each point $(x, y) \in \Gamma_1 \times \Gamma_2$ we say a subset I of $\{1, \ldots, m\}$ forces A' if $(u, v) \in A'$ whenever $(u, v) \in \Gamma_1 \times \Gamma_2$ and $u_i = x_i$ for all $i \in I$. We say that I forces B'_k if $(u, v) \in B'_k$ whenever $u_i = y_i$ for each $i \in I$ with $i \leq k$, and $v_i = x_i$ for each $i \in I$ with $i > k$.

We verify next that $\mathbb{P}(A \circ B) = \mathbb{P}_{12}(A' \circ B'_0)$. Since B'_0 is the set of all points in $\Gamma_1 \times \Gamma_2$ for which the vector $(x_1, \ldots, x_m) \in B$, we have that

$$\begin{aligned} A' \circ B'_0 = \{\omega \in \Gamma_1 \times \Gamma_2 \colon \exists S_1, S_2 \subseteq K(\omega) \text{ such that } \omega' \in A' \text{ and } \omega'' \in B'_0, \\ \text{where } \omega' \text{ and } \omega'' \text{ are such that } K(\omega') = S_1 \text{ and } K(\omega'') = S_2\}, \end{aligned}$$

with $\omega = (x_1, \ldots, x_m, y_1, \ldots, y_m)$. But $(x_1, \ldots, x_m) \in \Gamma_1$, then

$$\begin{aligned} \mathbb{P}_{12}(A' \circ B'_0) &= \mathbb{P}_{12}\{\omega \in \Gamma_1 \times \Gamma_2 \colon \exists S_1, S_2 \text{ disjoint, such that } \omega' \in A' \text{ and } \omega'' \in B'_0\} \\ &= \mathbb{P}_1\{\omega \in \Gamma_1 \colon \exists S_1, S_2 \text{ disjoint, such that } \omega' \in A \text{ and } \omega'' \in B\} \\ &= \mathbb{P}(A \circ B). \end{aligned}$$

On the other hand, note that the events A' and B'_m are defined for disjoint coordinate sets of $\Gamma_1 \times \Gamma_2$, since A' is the set of all points (x, y) for which $(x_1, \ldots, x_m) \in A$ and B'_m is the set of all points (x, y) for which the vector $(y_1, \ldots, y_m) \in B$. That is,

A' depends on $(x_1, \dots, x_m)$ and B'_m depends on $(y_1, \dots, y_m)$. Hence,

$$\begin{aligned}\mathbb{P}_{12}(A' \circ B'_m) &= \mathbb{P}_{12}(A' \cap B'_m)\\ &= \mathbb{P}_{12}(A') \cdot \mathbb{P}_{12}(B'_m)\\ &= \mathbb{P}_1(A) \cdot \mathbb{P}_2(B)\\ &= \mathbb{P}(A) \cdot \mathbb{P}(B).\end{aligned}$$

Thus, we can rewrite $\mathbb{P}(A \circ B) \leq \mathbb{P}(A) \cdot \mathbb{P}(B)$ as $\mathbb{P}_{12}(A' \circ B'_0) \leq \mathbb{P}_{12}(A' \circ B'_m)$, and to show that the last inequality is true we prove that $\mathbb{P}_{12}(A' \circ B'_{k-1}) \leq \mathbb{P}_{12}(A' \circ B'_k)$ for all $1 \leq k \leq m$.

We first divide the event $A' \circ B'_{k-1}$ in two other events, $A' \circ B'_{k-1} = C_1 \cup C_2$, where

$$\begin{aligned}C_1 &= \{(x, y) \colon A' \circ B'_{k-1} \text{ occurs independently on the value of } x_k\},\\ C_2 &= \{(x, y) \colon A' \circ B'_{k-1} \text{ occurs if and only if } x_k = 1\} \cap \{x_k = 1\}.\end{aligned}$$

Then we divide the event C_2 in two other ones, $C_2 = C'_2 \cup C''_2$, where

$$\begin{aligned}C'_2 &= C_2 \cap \{(x, y) \colon \exists I \subset 1, \dots, m \text{ such that } k \in I, I \text{ forces } A', I^c \text{ forces } B'_{k-1}\},\\ C''_2 &= C_2 \setminus C'_2, \text{ with } I^c = \{1, \dots, m\} \setminus I.\end{aligned}$$

Then we can see C'_2 as the sub-event of C_2 where x_k contributes essentially with A'.

Now we construct an injective map φ from $A' \circ B'_{k-1}$ to $A' \circ B'_k$: for $(x, y) \in \Gamma_1 \times \Gamma_2$, let (x', y') be the point in $\Gamma_1 \times \Gamma_2$ obtained from (x, y) as follows

$$\begin{cases} x'_i = x_i, \text{ for } i \neq k, \text{ and } x'_k = y_k\\ y'_i = y_i, \text{ for } i \neq k, \text{ and } y'_k = x_k \end{cases}$$

and let φ be defined on $A' \circ B'_{k-1}$ by

$$\varphi(x, y) = \begin{cases} (x, y), & \text{if } (x, y) \in C_1 \cup C'_2\\ (x', y'), & \text{if } (x, y) \in C''_2. \end{cases}$$

Note that $C_1 \subseteq A' \circ B'_k$. If $(x, y) \in C_1$ then $A' \circ B'_{k-1}$ occurs independently of the value of x_k and y_k: if $A' \circ B'_{k-1}$ occurs, then the occurrence of the event B'_{k-1} is guaranteed, *i.e.* the vector $(y_1, \dots, y_{k-1}, x_k, x_{k+1}, \dots, x_m) \in B$ no matter the value of x_k. Thus, if we take $x_k = y_k$ then the vector $(y_1, \dots, y_{k-1}, y_k, x_{k+1}, \dots, x_m)$ still belongs to B, that is, the occurrence of B'_k is guaranteed, so we have that the event $A' \circ B'_k$ also occurs.

Moreover, note that $C'_2 \subset A' \circ B'_k$. If $(x, y) \in C'_2$ then A' and B'_{k-1} occur disjointly and there exists such a disjoint occurrence where x_k contributes essentially with A' and not with B'_{k-1}. Now, since the event B'_{k-1} occurs, the vector $(y_1, \dots, y_{k-1}, x_k, x_{k+1}, \dots, x_m) \in B$. Thus, we have that the value of x_k does not change the occurrence of B'_{k-1}; then, if we take $x_k = y_k$, the vector $(y_1, \ldots, y_{k-1}, y_k, x_{k+1}, \ldots, x_m)$ still belongs to B, that is, the occurrence of B'_k is guaranteed, so we have that the event $A' \circ B'_k$ also occurs.

Now, take a point in C''_2. If $(x, y) \in C''_2$ then $\varphi(x, y) = (x', y') \in A' \circ B'_k$, since in this case $x_k = 1$ and there exists $I \subseteq \{1, \ldots, m\} \setminus \{k\}$ such that I forces A', and I^c forces B'_{k-1}; *i.e.* x_k contributes essentially with B'_{k-1}. So we have that $\varphi(x, y) = (x', y') = (x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_m, y_1, \ldots, y_{k-1}, x_k, x_{k+1}, \ldots, x_m)$, and from the point of view of the configuration (x', y'), I forces A' and I^c forces B'_k which implies that $(x', y') \in A' \circ B'_k$, and we have that $\varphi(C''_2) \subseteq A' \circ B'_k$ □

Another result that will be useful in a proof in the next chapter provides a relation between the critical threshold of bond percolation and that of site percolation. A more general version of the following result is proved in [5, Theorem 1.33].

Theorem 13. *Let G be a connected infinite graph. Let $p^s_c(G)$ be the critical threshold of independent site percolation in G, and $p^b_c(G)$ be the critical threshold of independent bond percolation in G. Then,*

$$p^s_c(G) \geq p^b_c(G).$$

Proof. The idea of the proof is as follows. Given a configuration of the vertices of G, we order the vertices so that we start from a vertex v_0, which we ask to be open. Then, looking at the nearest neighbours of v_0, we take the vertex with smallest index (according to our order) and we check its state: if it is open, call it an infected vertex; if it is closed, call it a dead vertex. We repeat this step, labelling the vertices as infected or dead, and we induce a bond percolation process by opening the edges of G according to the state of the vertices. We repeat this until we get to a final step or we "reach" infinity - that is, the process does not stop, in which case we have percolation.

To formalize the argument, take an arbitrary order on the vertices $\mathcal{V}(G)$ of G and on the set of edges $\mathcal{E}(G)$ of G, and let $\{\chi_v\}_{v \in \mathcal{V}(G)}$ be *i.i.d.* random variables with $\chi_v \sim Ber(p)$. Consider the sets

$I_0 = \{v_0\}$ (infected vertices);

$S_0 = \mathcal{V}(G) \setminus \{v_0\}$ (non explored vertices);

$D_0 = \emptyset$ (dead vertices);

$E_0 = \emptyset$ (explored edges).

To determine the iteration completely, suppose that I_n, D_n, S_n and E_n are defined, take $u = \min\{v \in S_n \setminus D_n \colon v \in \Gamma(t)$, for some $t \in I_n\}$, where $\Gamma(t)$ is the neighbourhood of t, and let f be the edge with endpoints u and t.

Now, define $Y_f = \chi_u$.

If $\chi_u = 1$, then we define
$$I_{n+1} = I_n \cup \{u\};$$
$$S_{n+1} = S_n \backslash \{u\};$$
$$D_{n+1} = D_n;$$
$$E_{n+1} = E_n \cup \{f\}.$$

If $\chi_u = 0$, then we define
$$I_{n+1} = I_n;$$
$$S_{n+1} = S_n \backslash \{u\};$$
$$D_{n+1} = D_n \cup \{u\};$$
$$E_{n+1} = E_n \cup \{f\}.$$

Thus, we can see that the process ends if $\{v \in S_n \backslash D_n : v \in \Gamma(t), t \in I_n\} = \emptyset$.

Then,
$$\mathbb{P}_p(\text{the process does not end}) = p \cdot \theta^s(v_0, p),$$
where $\theta^s(v_0, p)$ is the percolation probability for the independent site percolation process with parameter p.

Moreover, we note that the random variables $\{Y_f\}_{f \in E_n}$ are $i.i.d.$ with $Y_f \sim Ber(p)$, and we have that $I_n \subset C_p^b(v_0)$.

Therefore, if we take $p > p_c^s(G)$, then $|I_n| \to \infty$ (when n goes to infinity) with probability $p \cdot \theta^s(v_0, p) > 0$, which implies that $\theta^s(v_0, p) > 0$. Finally, by inclusion of sets, we have that $\mathbb{P}_p\left(|C_p^b(v_0)| = \infty\right) > 0$. Thus, $p_c^b(G) \leq p_c^s(G)$. $\square$

Chapter 3

Anisotropic percolation in $\mathbb{Z}^{d+s}$

In the previous chapter, we presented fundamental definitions and results from percolation theory, always dealing with isotropic bond percolation process. In this chapter, that is based in [10, 11] we consider an anisotropic bond percolation process on the graph $G = (\mathbb{Z}^{d+s}, \mathcal{E}(\mathbb{Z}^{d+s}))$, where $\mathcal{E} = \mathcal{E}(\mathbb{Z}^{d+s})$ denotes the set of edges between nearest neighbours of $\mathbb{Z}^d \times \mathbb{Z}^s$, and we associate different probabilities for a bond if it is placed in $\mathbb{Z}^d$ or in $\mathbb{Z}^s$

Definition 14. *An edge of $\mathcal{E}(\mathbb{Z}^{d+s})$ is called a $\mathbb{Z}^d$**-edge** if it connects two vertices which are different only in their $\mathbb{Z}^d$ components (in a single coordinate, necessarily); similarly, we call the edge a $\mathbb{Z}^s$**-edge** if it connects two vertices which differ only in their $\mathbb{Z}^s$ components.*

Given two parameters $p, q \in [0,1]$, we say that each $\mathbb{Z}^d$-edge is open with probability p independently of the others, and each $\mathbb{Z}^s$-edge is open with probability q, also independently of the others. One notes readily that this can be described by the probability space $(\Omega, \mathcal{F}, \mathbb{P}_{p,q})$, where $\Omega = [0,1]^{\mathcal{E}}$, $\mathcal{F}$ is the σ-algebra generated by the cylindrical sets in Ω, that is, those which depend only on finite subsets of edges, and

$$\mathbb{P}_{p,q} = \prod_{e \in \mathcal{E}} \mu(e),$$

where $\mu(e)$ is the Bernoulli measure with parameter p if e is a $\mathbb{Z}^d$-edge or q if it is a $\mathbb{Z}^s$-edge.

Now, for u and v vertices in $\mathbb{Z}^{d+s}$, let $\{u \leftrightarrow v\}$ be the event where u and v are connected, and $\mathcal{C}(\omega) = \{u \in \mathbb{Z}^d \times \mathbb{Z}^s; u \leftrightarrow 0\}$ denote the open cluster containing the origin. Again, we are interested in the probability that the open cluster of the origin is infinite. Hence we define the function θ on $[0,1] \times [0,1]$ by $\theta(p,q) = \mathbb{P}_{p,q}\{\omega \in \Omega; |C(\omega)| = \infty\}$, that is, $\theta(p,q)$ is the probability of percolation on $\mathbb{Z}^{d+s}$ with parameters p and q. Moreover, define $\chi(p,q) = \mathbb{E}_{p,q}(|C(\omega)|)$, so that $\chi(p,q)$ denotes the mean size of the open cluster in $\mathbb{Z}^{d+s}$.

Proposition 15. *$\theta(p,q)$ is a non-decreasing function of parameters p and q.*

Proof. We simply observe that

$$\begin{cases} (p_1, q) < (p_2, q), & \text{if } p_1 < p_2 \\ (p, q_1) < (p, q_2), & \text{if } q_1 < q_2 \end{cases}$$

So, if $p_1 < p_2$ and $q_1 < q_2$, then

$$\begin{aligned}\theta(p_1, q_1) &\leq \theta(p_2, q_1), \text{ since } \theta \text{ is non-decreasing for a single variable} \\ &\leq \theta(p_2, q_2).\end{aligned}$$

□

Now, by the above proposition, we can define the function $q_c(p) = \sup\{q : \theta(p, q) = 0\}$ for $p \in [0, 1]$. The function $q_c(p)$ is what we call our **critical curve** in $\mathbb{Z}^{d+s}$, and the following proposition makes some relevant observations about this curve. Recall from Theorem 7 in Chapter 2 that $p_c(d) = \sup\{p; \theta(p) = 0\}$ denotes the critical threshold of bond percolation on $\mathbb{Z}^d$.

Proposition 16. *If $q_c(p) : [0, 1] \to [0, 1]$ is defined by $q_c(p) = \sup\{q : \theta(p, q) = 0\}$, then*

(i) $q_c(0) = p_c(s)$*;*

(ii) $q_c(p) = 0$ *for* $p > p_c(d)$*;*

(iii) $q_c(p)$ *is a non-increasing function of parameter* p*.*

Proof.

(i) By definition of $q_c(p)$, we have that

$$q_c(0) = \sup\{q : \theta(0, q) = 0\} = \sup\{q : \theta(q) = 0\} = p_c(s).$$

That is, if every edge of $\mathbb{Z}^d$ is closed, then the probability of percolation on $\mathbb{Z}^{d+s}$ depends only on values of q, so $q_c(0)$ is equal the critical threshold on $\mathbb{Z}^s$.

(ii) If $p > p_c(d)$, then by the Phase Transition Theorem we have that the probability of percolation on $\mathbb{Z}^d$ is strictly positive, and so $\theta(p, q) > 0$, which give us that $q_c(p) = \sup\{q : \theta(p, q) = 0\} = 0$.

(iii) We want to show that for $p_1, p_2 \in [0, 1]$ if $p_1 < p_2$, then $q_c(p_1) \geq q_c(p_2)$. By definition of $q_c(p)$, we have that

$$q_c(p_1) = \sup\{q : \theta(p_1, q) = 0\}, \qquad q_c(p_2) = \sup\{q : \theta(p_2, q) = 0\}.$$

Consider the sets $\mathcal{Q}_1 = \{q; \theta(p_1, q) = 0\}$, $\mathcal{Q}_2 = \{q; \theta(p_2, q) = 0\}$, and let $q' \in \mathcal{Q}_2$. Since $\theta(p, q)$ is a non-decreasing function, if $p_1 < p_2$ then $\theta(p_1, q) \leq \theta(p_2, q)$. Hence, we have that

$$\theta(p_1, q') \leq \theta(p_2, q') = 0,$$

that is, $\theta(p_1, q') = 0$. Then $q' \in \mathcal{Q}_1$, and we have that $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$, which implies that $\sup \mathcal{Q}_2 \leq \sup \mathcal{Q}_1$. Thus, $q_c(p_2) \leq q_c(p_1)$, as desired. □

Now, looking at the curve $q_c(p)$, we aim to analyze its critical behaviour for values of p less than or equal to $p_c(d)$, specially when $p \uparrow p_c(d)$. In addition, as the study of *critical exponents* is part of the interest of percolation theory, we turn our attention to quantities such as $\chi_d(p)$, since it is believed that there exists $\gamma(d) > 0$ such that

$$\chi_d(p) \approx |p - p_c(d)|^{-\gamma}, \tag{3.1}$$

when $p \uparrow p_c(d)$ and the relation $a(p) \approx b(p)$ means that $\frac{\log a(p)}{\log b(p)} \longrightarrow 1$ when $p \uparrow p_c(d)$; in this case, we say γ is a critical exponent.

Then, we would like to obtain an answer for the following question: is it true that there exists a constant ψ such that $q_c(p) \approx |p - p_c(d)|^{\psi}$ when $p \uparrow p_c(d)$? We will call ψ the *crossover exponent*, and state that the answer for the above question is partially positive.

Conjecture 17. *There exists a critical exponent $\psi = \psi(d) > 0$ such that*

$$q_c(p) \approx |p - p_c(d)|^{\psi}.$$

Moreover, if $\gamma(d)$ exists, then $\psi(d) = \gamma(d)$.

We shall prove that the above conjecture holds for $d = 1$, and establish a relation between $\psi(d)$ and $\gamma(d)$ for general d. We begin with some necessary results.

Theorem 18. *Consider a bond percolation process on $\mathbb{Z}^d \times \mathbb{Z}^s$ with parameters (p, q), $p < p_c(d)$. If the pair (p, q) satisfies*

$$q < \frac{1}{2s\chi_d(p)},$$

then there is a.s. no infinite open cluster in $\mathbb{Z}^{d+s}$.

Proof. First, we put forward some relevant notation.

- A point in $\mathbb{Z}^{d+s}$ will be written as (u, t), where u and t denote its $\mathbb{Z}^d$ and $\mathbb{Z}^s$ components, respectively.
- $\{u_n, n \in \mathbb{Z}_+\}$ and $\{t_n, n \in \mathbb{Z}_+\}$ denote sequences of points in $\mathbb{Z}^d$ and $\mathbb{Z}^s$, respectively.
- O_d denotes the d-dimensional vector of zeros.
- The distance between two points $x, y \in \mathbb{Z}^s$ is defined by

$$\delta(x, y) = \sum_{i=1}^{s} |x_i - y_i|.$$

- Given $u \in \mathbb{Z}^d$ and $s, t \in \mathbb{Z}^s$ such that $\delta(s, t) = 1$, write $e_{(u,t),(u,s)} \in \mathcal{E}(\mathbb{Z}^s)$ for the edge with end-vertices (u, t) and (u, s).

Thus, to see that there is no infinite open cluster in $\mathbb{Z}^{d+s}$, we will show that the mean size of the open cluster, $\chi(p,q)$, is bounded.

We have that

$$\chi(p,q) = \mathbb{E}_{p,q} \sum_{(u,t)\in\mathbb{Z}^{d+s}} \mathbb{1}_{(0\leftrightarrow(u,t))}$$
$$= \sum_{(u,t)\in\mathbb{Z}^{d+s}} \mathbb{P}_{p,q}(0_{d+s} \leftrightarrow (u,t)). \tag{3.2}$$

Now, observe that the event $\{0_{d+s} \leftrightarrow (u,t)\}$ occurs if and only if there exist sequences of points $\hat{u}_n = (u_0,\ldots,u_n)$ and $\hat{t}_n = (t_0,\ldots,t_n)$ such that $u_j \in \mathbb{Z}^d$ for all $0 \leq j \leq n$, and $t_i \in \mathbb{Z}^s$ for all $1 \leq i \leq n$, with $\delta(t_i,t_{i+1}) = 1$, which we construct according to the following steps:

(1) We start at the point $(0_d,t_0) = (0_d,0_s)$ and then we connect it to the point (u_0,t_0) using only $\mathbb{Z}^d \times \{t_0\}$-edges, that is, using $\mathbb{Z}^d$-edges with t_0 in their $\mathbb{Z}^s$ component.

(2) Move from (u_0,t_0) to the point (u_0,t_1) by a connection that uses a single open $\mathbb{Z}^s$-edge.

(3) Connect the point (u_0,t_1) to (u_1,t_1) using only $\mathbb{Z}^d \times \{t_1\}$-edges not used to connect $(0_d,t_0)$ to (u_0,t_0).

⋮

(m) Move from (u_{m-1},t_{m-1}) to the point (u_{m-1},t_m) by a connection that uses a single open $\mathbb{Z}^s$-edge not used in any previous step.

(m+1) Connect the point (u_{m-1},t_m) to (u_m,t_m) using only $\mathbb{Z}^d \times \{t_m\}$-edges which do not use $\mathbb{Z}^d$-edges used to connect (u_{l-1},t_l) at (u_l,t_l) for any $1 \leq l \leq m-1$, neither those that were used to connect $(0_d,t_0)$ to (u_0,t_0).

⋮

We repeat these steps until we reach the point $(u_n,t_n) = (u,t)$.

Note that, in this construction, the distance between any two points $t_i,t_{i+1} \in \mathbb{Z}^s$ is equal to 1, but we do not make any restriction about how many points in $\mathbb{Z}^d$ are there between any two points $u_j,u_{j+1} \in \mathbb{Z}^d$, we can even have none, *i.e.* it is possible that $u_j = u_{j+1}$.

Now, given $\hat{t}_n = (t_0,\ldots,t_n)$ and $\hat{u}_n = (u_0,\ldots,u_n)$, consider the sequence of events $\{A_i\}_{i=0}^n$ such that, for $0 \leq i \leq n-1$,

$$A_i = \big\{\{(u_{i-1},t_i) \leftrightarrow (u_i,t_i) \text{ in } \mathbb{Z}^d \times \{t_i\}\} \cap \{e_{(u_i,t_i),(u_i,t_{i+1})} \text{ is open }\}\big\},$$

where we define $u_{-1} = 0_d$, and

$$A_n = \{(u_{n-1}, t_n) \leftrightarrow (u_n, t_n) \text{ in } \mathbb{Z}^d \times \{t_n\}\}.$$

Note that $\{A_i\}_{i=0}^n$ is a sequence of increasing events, since every A_i connects two vertices and hence opening up edges benefits the events. This enables us to use the BK inequality later.

Thus, by the above construction we have

$$\{0_{d+s} \leftrightarrow (u,t)\} \;=\; \bigcup_{n\geq 0} \bigcup_{\substack{\hat{u}_n : u_n = u \\ \hat{t}_n : t_n = t}} \{A_0 \circ A_1 \circ \cdots \circ A_n\}.$$

Then,

$$\begin{aligned}
\chi(p,q) &= \sum_{(u,t)\in\mathbb{Z}^{d+s}} \mathbb{P}_{p,q}(0_{d+s} \leftrightarrow (u,t)), \quad \text{by Eq. (3.2)} \\
&= \sum_{(u,t)} \mathbb{P}_{p,q}\Big(\bigcup_{n\geq 0} \bigcup_{\substack{\hat{u}_n \\ \hat{t}_n}} \{A_0 \circ A_1 \cdots \circ A_n\} \Big) \\
&\leq \sum_{(u,t)} \sum_{n\geq 0} \sum_{\substack{\hat{u}_n \\ \hat{t}_n}} \mathbb{P}_{p,q}(A_0 \circ A_1 \circ \cdots \circ A_n) \\
&\leq \sum_{(u,t)} \sum_{n\geq 0} \sum_{\substack{\hat{u}_n \\ \hat{t}_n}} \prod_{i=0}^{n} \mathbb{P}_{p,q}(A_i), \quad \text{by the BK inequality} \\
&= \sum_{n\geq 0} \sum_{\hat{t}_n} \sum_{\hat{u}_n} \prod_{i=0}^{n} \mathbb{P}_{p,q}(A_i), \qquad (3.3)
\end{aligned}$$

where the last two summations in Inequality (3.3) are taken over all sequences $\hat{u}_n$ and $\hat{t}_n$ such that $\hat{u}_n = (u_0, \ldots, u_n)$ and $\hat{t}_n = (t_0, \ldots, t_n)$, and $(u_0, t_0) = 0_{d+s}$. That is, $\hat{u}_n$ and $\hat{t}_n$ are sequences of points in $\mathbb{Z}^d$ and $\mathbb{Z}^s$, respectively, that start at the origin of $\mathbb{Z}^d$ and $\mathbb{Z}^s$, respectively.

Note that

$$\sum_{u_k\in\mathbb{Z}^d} \mathbb{P}_{p,q}(A_n) \;\leq\; \chi_d(p) \;=\; \sum_{u\in\mathbb{Z}^d} \mathbb{P}_p(0_d \leftrightarrow u), \qquad (3.4)$$

because

$$\{A_n \text{ occurs given } \hat{u}_n = (u_0, \ldots, u_n = u)\} \subseteq \{0_d \leftrightarrow u, u \in \mathbb{Z}^d\},$$

since A_n is the event where, given a sequence of points in $\mathbb{Z}^d$, with the first one being the origin and the last being any point u, we connect the last but one point of this sequence to the point u; hence A_n is contained in the event where the origin is connected to u by any open paths.

In addition we have that the events

$$\big\{(u_{i-1}, t_i) \leftrightarrow (u_i, t_i) \text{ in } \mathbb{Z}^d \times \{t_i\}\big\}$$

and

$$\left\{e_{(u_i,t_i),(u_i,t_{i+1})} \text{ is open}\right\}$$

are independent, since in the first one we connect points in $\mathbb{Z}^d$, while in the second we do it in $\mathbb{Z}^s$. Thus, for all $k = 0, 1, \ldots, n-1$, we verify

$$\begin{aligned}
\sum_{\substack{\hat{t}_k \\ \hat{u}_k}} \mathbb{P}_{p,q}(A_k) &= \sum_{\substack{\hat{t}_k \\ \hat{u}_k}} \mathbb{P}_{p,q}\left\{(u_{i-1},t_i) \leftrightarrow (u_i,t_i) \text{ in } \mathbb{Z}^d \times \{t_i\}\right\} \sum_{\substack{\hat{t}_k \\ \hat{u}_k}} \mathbb{P}_{p,q}\left\{e_{(u_i,t_i),(u_i,t_{i+1})} \text{ is open}\right\} \\
&= q \sum_{\substack{\hat{t}_k \\ \hat{u}_k}} \mathbb{P}_{p,q}\left\{\{(u_{i-1},t_i) \leftrightarrow (u_i,t_i) \text{ in } \mathbb{Z}^d \times \{t_i\}\right\} \\
&= q \sum_{t_k} \sum_{u_k} \mathbb{P}_{p,q}\left\{(u_{i-1},t_i) \leftrightarrow (u_i,t_i) \text{ in } \mathbb{Z}^d \times \{t_i\}\right\} \\
&\leq \sum_{t_k} \chi_d(p), \quad \text{by the same argument in (3.2)} \\
&= q\chi_d(p)2s
\end{aligned} \tag{3.5}$$

Thus, going back to (3.1) and using Equation (3.2) and Inequality (3.3) we have

$$\begin{aligned}
\chi(p,q) &\leq \sum_{n\geq 0} \sum_{\hat{t}_n} \sum_{\hat{u}_n} \prod_{i=0}^{n} \mathbb{P}_{p,q}(A_i) \\
&= \sum_{n\geq 0} \sum_{t_k} \sum_{u_k} \sum_{t_n} \sum_{u_n} \prod_{i=0}^{n} \mathbb{P}_{p,q}(A_i) \\
&= \sum_{n\geq 0} \sum_{t_k} \sum_{u_k} \sum_{t_n} \sum_{u_n} \mathbb{P}_{p,q}(A_n) \prod_{i=0}^{n-1} \mathbb{P}_{p,q}(A_i) \\
&= \sum_{n\geq 0} \sum_{t_n} \sum_{u_n} \mathbb{P}_{p,q}(A_n) \sum_{t_k} \sum_{u_k} \prod_{i=0}^{n-1} \mathbb{P}_{p,q}(A_i) \\
&\leq \sum_{n\geq 0} 2s\chi_d(p) \sum_{t_k} \sum_{u_k} \prod_{i=0}^{n-1} \mathbb{P}_{p,q}(A_i) \\
&= \sum_{n\geq 0} 2s\chi_d(p) \sum_{t_k} \sum_{u_k} \sum_{t_{n-1}} \sum_{u_{n-1}} \prod_{i=0}^{n-1} \mathbb{P}_{p,q}(A_i) \\
&= \sum_{n\geq 0} 2s\chi_d(p) \sum_{t_k} \sum_{u_k} \sum_{t_{n-1}} \sum_{u_{n-1}} \mathbb{P}_{p,q}(A_{n-1}) \prod_{i=0}^{n-2} \mathbb{P}_{p,q}(A_i) \\
&= \sum_{n\geq 0} 2s\chi_d(p) \sum_{t_{n-1}} \sum_{u_{n-1}} \mathbb{P}_{p,q}(A_{n-1}) \sum_{t_k} \sum_{u_k} \prod_{i=0}^{n-2} \mathbb{P}_{p,q}(A_i) \\
&\leq \sum_{n\geq 0} 2s\chi_d(p) \cdot [2sq\chi_d(p)] \sum_{t_k} \sum_{u_k} \prod_{i=0}^{n-2} \mathbb{P}_{p,q}(A_i) \\
&= \sum_{n\geq 0} (2s)^2 \cdot q\chi_d(p)^2 \sum_{t_{n-2}} \sum_{u_{n-2}} \mathbb{P}_{p,q}(A_{n-2}) \sum_{t_k} \sum_{u_k} \prod_{i=0}^{n-3} \mathbb{P}_{p,q}(A_i)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n\geq 0}(2s)^2 q\chi_d(p)^2[2sq\chi_d(p)]\sum_{t_k}\sum_{u_k}\prod_{i=0}^{n-3}\mathbb{P}_{p,q}(A_i)\\
&= \sum_{n\geq 0}(2s)^3 q^2\chi_d(p)^3\sum_{t_k}\sum_{u_k}\prod_{i=0}^{n-3}\mathbb{P}_{p,q}(A_i)\\
&\vdots\\
&\leq \sum_{n\geq 0}(2s)^{n+1}q^n\chi_d(p)^{n+1}\\
&= 2s\chi_d(p)\sum_{n=0}^{\infty}[2sq\chi_d(p)]^n, \;\; \text{since if } p<p_c(d) \text{ then } \chi_d(p)<\infty \text{ a.s.}\\
&< +\infty \;\; \text{iff} \;\; 2sq\chi_d(p)<1, \;\; \textit{i.e.}, \text{ if } q<\frac{1}{2s\chi_d(p)},
\end{aligned}
$$

since $\sum_{n=0}^{\infty}[2sq\chi_d(p)]^n$ is a geometric series with ratio $2sq\chi_d(p)$, which completes the proof.

□

Theorem 19. *Consider a bond percolation process on $\mathbb{Z}\times\mathbb{Z}^s$, $s>1$, with parameters (p,q). There exists $\alpha>0$ such that if p is sufficiently close to 1 and $q>\alpha\frac{1+p}{1-p}$, then there is a.s. an infinite open cluster in $\mathbb{Z}^{1+s}$.*

Proof. First we shall construct an independent site percolation process in $\mathbb{Z}^s$ induced by the bond percolation process in $\mathbb{Z}^{1+s}$. Then we will show that, under some hypothesis, site percolation will occur (*i.e.* there exists an infinite open cluster) in $\mathbb{Z}^s$, and so will the bond percolation in $\mathbb{Z}^{1+s}$.

Let $u=(\bar{u},x_1,\dots,x_s)$ and $z=(\bar{z},x_1,\dots,x_s)\in\mathbb{Z}^{1+s}$. So given a configuration $w\in\Omega$, we say u and z are *updownwards connected* in ω if every $\mathbb{Z}\times\{x_1,\dots,x_s\}$-edge between u and z is open in ω, and we denote this event by $\{\omega\in\Omega; u\updownarrow z \text{ in } \omega\}$.

For $u\in\{0\}\times\mathbb{Z}^s$, let $W_u(\omega)=\{z\in\mathbb{Z}^{1+s}; z\updownarrow u \text{ in } \omega\}$. Thus, we're going to construct a site percolation process of *good* vertices on $\{0\}\times\mathbb{Z}^s$ as follows. Given $\epsilon>0$ and a configuration ω, we declare each vertex $u\in\{0\}\times\mathbb{Z}^s$ as ϵ-*good* if

(i) $|W_u(\omega)|>\dfrac{1+p}{1-p}\cdot\epsilon$

(ii) in each of the s possible directions of increasing coordinates, there is at least one open edge with exactly one end-vertex in W_u.

Now, consider the sequence of events $\{A_u(\omega)\}_{u\in\{0\}\times\mathbb{Z}^s}$, where

$$A_u(\omega)=\{\omega\in\Omega; u \text{ is } \epsilon\text{-good in } \omega\}, \;\text{ for }\; u\in\{0\}\times\mathbb{Z}^s.$$

Note that the events are independent of each other: since we consider directions of increasing coordinate-values, a vertex u is ϵ-good independently of any other vertex, as they depend on disjoint sets of edges.

Thus, we aim to obtain an estimate for the probability of A_u. If we denote the event of the condition (ii) by F_u, then we verify for each $u \in \{0\} \times \mathbb{Z}^s$ that

$$\begin{aligned}\mathbb{P}_{p,q}(u \text{ is } \epsilon\text{-good}) &= \mathbb{P}_{p,q}\left(|W_u| > \frac{1+p}{1-p} \cdot \epsilon \,; F_u\right) \\ &= \mathbb{P}_{p,q}\left(|W_u| > \frac{1+p}{1-p} \cdot \epsilon\right) \mathbb{P}_{p,q}\left(F_u \;\middle|\; |W_u| > \frac{1+p}{1-p} \cdot \epsilon\right).\end{aligned}$$

Now for $\epsilon > 0$ sufficiently small and p close to 1,

$$\begin{aligned}\mathbb{P}_{p,q}\left(|W_u| > \frac{1+p}{1-p}\epsilon\right) &= p^{\frac{1+p}{1-p}\epsilon} \\ &\geq 1 - 2\epsilon, \end{aligned} \tag{3.6}$$

where the last inequality can be proven as follows. We'd like to know the value of $p^{\frac{1+p}{1-p}\epsilon}$ when $p \to 1$, that is,

$$\lim_{p\to 1} p^{\frac{1+p}{1-p}\epsilon} = \left(\lim_{p\to 1} p^{\frac{1+p}{1-p}}\right)^{\epsilon}.$$

Defining $y = \lim_{p\to 1} p^{\frac{1+p}{1-p}}$, we have

$$\begin{aligned}\log y &= \log\left(\lim_{p\to 1} p^{\frac{1+p}{1-p}}\right) \\ &= \lim_{p\to 1} \log p^{\frac{1+p}{1-p}} \\ &= \lim_{p\to 1} \frac{1+p}{1-p} \log p \\ &= \lim_{p\to 1} \left(1 + \frac{2p}{1-p}\right) \log p \\ &= \lim_{p\to 1} \frac{2p}{1-p} \log p \\ &= \lim_{p\to 1} \frac{\log p}{\frac{1-p}{2p}} \\ &= \lim_{p\to 1} \frac{\frac{1}{p}}{\frac{-1}{2p^2}} \qquad\qquad \text{(by L'Hôpital's rule)} \\ &= \lim_{p\to 1} -\frac{1}{p} 2p^2 = -2.\end{aligned}$$

Thus, $\log y = -2$ implies that $y = e^{-2}$ and hence

$$\begin{aligned}\left(\lim_{p\to 1} p^{\frac{1+p}{1-p}}\right)^{\epsilon} &= e^{-2\epsilon} \\ &\geq 1 - 2\epsilon, \text{ since } e^{-x} \geq 1 - x, \text{ for } x \geq 0.\end{aligned}$$

This proves the inequality in (3.6).

Now, define $F_{u,j}(\omega) = \{$there is at least one open edge with an end-vertex in W_u in direction j in configuration $\omega\}$, for $j = 1, \ldots, s$. Note that $\{F_{u,j}\}_{j=1}^{s}$ is a collection of independent events with equal probability, since the edges are open independently. Hence, we have

$$\begin{aligned}
\mathbb{P}_{p,q}\left(F_u \;\middle|\; |W_u| > \frac{1+p}{1-p}\epsilon\right) &= \mathbb{P}_{p,q}\left(\bigcap_{j=i}^{s} F_{u,j} \;\middle|\; |W_u| > \frac{1+p}{1-p}\epsilon\right) \\
&= \frac{\mathbb{P}_{p,q}\left[\left(\bigcap_{j=1}^{s} F_{u,j}\right) \cap \left(|W_u| > \frac{1+p}{1-p}\epsilon\right)\right]}{\mathbb{P}_{p,q}\left(|W_u| > \frac{1+p}{1-p}\epsilon\right)} \\
&= \frac{\mathbb{P}_{p,q}\left[\bigcap_{j=1}^{s}\left(F_{u,j} \cap \left(|W_u| > \frac{1+p}{1-p}\epsilon\right)\right)\right]}{\mathbb{P}_{p,q}\left(|W_u| > \frac{1+p}{1-p}\epsilon\right)} \\
&= \frac{\prod_{j=1}^{s}\mathbb{P}_{p,q}\left[F_{u,j} \cap \left(|W_u| > \frac{1+p}{1-p}\epsilon\right)\right]}{\mathbb{P}_{p,q}\left(|W_u| > \frac{1+p}{1-p}\epsilon\right)} \quad \text{(by independence)} \\
&= \prod_{j=1}^{s}\frac{\mathbb{P}_{p,q}\left[F_{u,j} \cap \left(|W_u| > \frac{1+p}{1-p}\epsilon\right)\right]}{\mathbb{P}_{p,q}\left(|W_u| > \frac{1+p}{1-p}\epsilon\right)} \\
&= \prod_{j=1}^{s}\mathbb{P}_{p,q}\left[F_{u,j} \;\middle|\; \left(|W_u| > \frac{1+p}{1-p}\epsilon\right)\right] \\
&= \left[\mathbb{P}_{p,q}\left(F_{u,1} \;\middle|\; \left(|W_u| > \frac{1+p}{1-p}\epsilon\right)\right)\right]^{s},
\end{aligned}$$

where the last equality holds due to the fact that every $F_{u,j}$ has the same probability.

Note that

$$\begin{aligned}
\mathbb{P}_{p,q}\left[F_{u,1} \;\middle|\; \left(|W_u| > \frac{1+p}{1-p}\epsilon\right)\right] &= 1 - \mathbb{P}_{p,q}\left((F_{u,1})^{\mathsf{c}} \;\middle|\; \left(|W_u| > \frac{1+p}{1-p}\epsilon\right)\right) \\
&= 1 - \mathbb{P}_{p,q}\Big(\text{there is no open edge in direction } j = 1 \text{ with an} \\
&\qquad \text{end-vertex in } W_u, \text{given that } |W_u| > \frac{1+p}{1-p}\epsilon\Big) \\
&= 1 - (1-q)^{|W_u|} \\
&\geq 1 - (1-q)^{\frac{1+p}{1-p}\epsilon}, \text{ since } |W_u| > \frac{1+p}{1-p}\epsilon\ .
\end{aligned}$$

Thus we have

$$\mathbb{P}_{p,q}\left(F_u \;\middle|\; |W_u| > \frac{1+p}{1-p}\epsilon\right) \geq \left[1 - (1-q)^{\frac{1+p}{1-p}\epsilon}\right]^{s}.$$

Therefore, if for some $\alpha > 0$ we choose $q > \alpha \dfrac{1-p}{1+p}$, then

$$\mathbb{P}_{p,q}\left(F_u \,\middle|\, |W_u| > \frac{1+p}{1-p}\epsilon\right) \geq \left[1 - \left(1 - \alpha\frac{1-p}{1+p}\right)^{\frac{1+p}{1-p}\epsilon}\right]^s$$
$$\geq \left(1 - e^{-\alpha\epsilon}\right)^s, \tag{3.7}$$

and we can see that the last inequality holds thanks to

$$\left[\left(1 - \alpha\frac{1-p}{1+p}\right)^{\frac{1+p}{1-p}}\right]^\epsilon \leq \left[\left(e^{-\alpha\frac{1-p}{1+p}}\right)^{\frac{1+p}{1-p}}\right]^\epsilon = e^{-\alpha\epsilon},$$

since $1 - x \leq e^{-x}$, when $x \geq 0$.

Hence by (3.6) and (3.7),

$$\overline{p} := \mathbb{P}_{p,q}\,(u \text{ is } \epsilon\text{-good}) \geq (1-2\epsilon)(1-e^{-\alpha\epsilon})^s.$$

We can take $\alpha = \alpha(\epsilon)$ sufficiently large such that $\overline{p}$ is strictly larger than the critical threshold of site percolation on $\mathbb{Z}^s$. This, in turn, implies that our ϵ-good site percolation on $\mathbb{Z}^s$ does occur, and since the anisotropic bond percolation process with parameters (p,q) stochastically dominates the isotropic site percolation process with parameter $\overline{p}$, we have that

$$\mathbb{P}_{p,q}\left(\text{bond percolation occurs in } \mathbb{Z}^{1+s}\right) \geq \mathbb{P}_{\overline{p}}\,(\text{site percolation occurs in } \mathbb{Z}^s) > 0,$$

i.e., there is an infinite open cluster in $\mathbb{Z}^{1+s}$. □

Corollary 20. *Consider a bond percolation process on $\mathbb{Z} \times \mathbb{Z}^s$, $s > 1$, with parameters (p,q). Then,*

$$\frac{1}{2s\chi_1(p)} \leq q_c(p) \leq \frac{\alpha}{\chi_1(p)},$$

for some $\alpha > 0$ and p sufficiently close to 1.

Proof. The left inequality follows directly from Theorem 18 since it guarantees that, if $q < \dfrac{1}{2s\chi(p)}$, then there is a.s. no infinite open cluster in $\mathbb{Z}^{1+s}$, while by definition we have $q_c(p) = \sup\{q : \theta(p,q) = 0\}$; then

$$q_c(p) \geq \frac{1}{2s\chi_1(p)}.$$

On the other hand, by Theorem 19 we have that if $q > \alpha\dfrac{1-p}{1+p}$, then $\theta(p,q) > 0$; since $\theta(p,q) > 0$ for all $q > q_c(p)$, by definition of $q_c(p)$, it follows that

$$q_c(p) \leq \alpha\frac{1-p}{1+p}.$$

Since the mean size of the open cluster containing the origin in $\mathbb{Z}$ is defined as $\chi_1(p) = \mathbb{E}_p\left(|\mathcal{C}(\omega)|\right)$, note that

$$\begin{aligned}
\mathbb{E}_p|\mathcal{C}| &= 1 + \mathbb{E}_p \sum_{x\in\mathbb{Z}} \mathbb{1}_{(0\leftrightarrow x)} \\
&= 1 + \mathbb{E}_p \sum_{x\in\mathbb{Z}^+} \mathbb{1}_{(0\leftrightarrow x)} + \mathbb{E}_p \sum_{x\in\mathbb{Z}^-} \mathbb{1}_{(0\leftrightarrow x)} \\
&= 1 + \sum_{x=1}^{\infty} 2\mathbb{P}_p(0 \leftrightarrow x) \\
&= 1 + \sum_{x=1}^{\infty} 2p^x \\
&= 1 + \left(\sum_{x=0}^{\infty} 2p^x\right) - 2 \\
&= 1 + \frac{2}{1-p} - 2, \text{ since } \sum_{x=0}^{\infty} 2p^x \text{ is a geometric series with ratio } p < 1 \\
&= \frac{1+p}{1-p}.
\end{aligned}$$

Hence, $q_c(p) \leq \dfrac{\alpha}{\chi_1(p)}$, as desired. □

Corollary 21. *The critical exponents $\gamma(d)$ and $\psi(d)$, defined in (3.1), are related as follows:*

(i) $\psi(1)$ exists and is equal to $\gamma(1) = 1$. Hence Conjecture 17 holds in the case $d = 1$.

(ii) If $\gamma(d)$ and $\psi(d)$ exist, then $\psi(d) \leq \gamma(d)$ for all d.

Proof. *(i)* Corollary 20 states that

$$\frac{1}{2s\chi_1(p)} \leq q_c(p) \leq \frac{\alpha}{\chi_1(p)}.$$

Then,

$$\begin{aligned}
\frac{1}{2s\chi_1(p)} \leq q_c(p) \quad &\Rightarrow \quad \frac{1}{2s} \leq q_c(p)\chi_1(p) \\
&\Rightarrow \quad \log(2s)^{-1} \leq \log\left[q_c(p)\chi_1(p)\right] \\
&\Rightarrow \quad -\log(2s) \leq \log q_c(p) + \log \chi_1(p). \qquad (3.8)
\end{aligned}$$

We also have that

$$\begin{aligned}
q_c(p) \leq \frac{\alpha}{\chi_1(p)} \quad &\Rightarrow \quad q_c(p)\chi_1(p) \leq \alpha \\
&\Rightarrow \quad \log[q_c(p)\chi_1(p)] \leq \log \alpha \\
&\Rightarrow \quad \log q_c(p) + \log \chi_1(p) \leq \log \alpha. \qquad (3.9)
\end{aligned}$$

Thus, it follows from (3.8) and (3.9) that

$$-\log(2s) \;\le\; \log q_c(p) + \log \chi_1(p) \;\le\; \log \alpha,$$

which implies that

$$-\frac{\log(2s)}{\log|1-p|} \;\ge\; \frac{\log q_c(p) + \log \chi_1(p)}{\log|1-p|} \;\ge\; \frac{\log \alpha}{\log|1-p|},$$

since $\log|1-p|$ is negative (as $|1-p|$ is between 0 and 1).

Now, taking the limit when $p \uparrow 1$, we have

$$\lim_{p\uparrow 1} -\frac{\log(2s)}{\log|1-p|} \;\ge\; \lim_{p\uparrow 1} \frac{\log q_c(p) + \log \chi_1(p)}{\log|1-p|} \;\ge\; \lim_{p\uparrow 1} \frac{\log \alpha}{\log|1-p|}.$$

Hence,

$$\begin{aligned}
\lim_{p\uparrow 1} \frac{\log q_c(p)}{\log|1-p|} &= -\lim_{p\uparrow 1} \frac{\log \chi_1(p)}{\log|1-p|} \\
&= -\lim_{p\uparrow 1} \frac{\log \frac{1+p}{1-p}}{\log|1-p|} \\
&= -\lim_{p\uparrow 1} \frac{\log(1+p) - \log(1-p)}{\log|1-p|} \\
&= \lim_{p\uparrow 1} \frac{\log(1-p)}{\log|1-p|} - \lim_{p\uparrow 1} \frac{\log(1+p)}{\log|1-p|} \\
&= 1.
\end{aligned}$$

That is, $q_c(p) \approx |1-p|$. Since $p_c(1) = 1$, we have that $q_c(p) \approx |p_c(1) - p|$, and then $\psi = 1$.

Moreover,

$$\begin{aligned}
-\lim_{p\uparrow 1} \frac{\log \chi_1(p)}{\log|1-p|} = 1 \quad &\Rightarrow \quad \lim_{p\uparrow 1} \frac{\log \chi_1(p)}{\log|1-p|^{-1}} = 1 \\
&\Rightarrow \quad \chi_1(p) \approx |1-p|^{-1} \\
&\Rightarrow \quad \chi_1(p) \approx |p_c(1) - p|^{-1} \\
&\Rightarrow \quad \gamma = 1.
\end{aligned}$$

Therefore, Conjecture 17 holds for $d = 1$.

(ii) From Theorem 18, we have

$$q_c(p) \;\ge\; \frac{1}{2s\chi_d(p)}\;.$$

Hence,

$$\begin{aligned}
\log q_c(p) \;\ge\; \log(2s\chi_d(p))^{-1} \quad &\Rightarrow \quad \log q_c(p) \;\ge\; -\log(2s\chi_d(p)) \\
&\Rightarrow \quad \log q_c(p) \;\ge\; -\log 2s - \log \chi_d(p) \\
&\Rightarrow \quad \frac{\log q_c(p)}{\log|p - p_c(d)|} \;\le\; -\frac{\log 2s}{\log|p - p_c(d)|} - \frac{\log \chi_d(p)}{\log|p - p_c(d)|}
\end{aligned}$$

Taking the limit when $p \uparrow p_c(d)$, we obtain

$$\lim_{p\uparrow 1} \frac{\log q_c(p)}{\log |p-p_c(d)|} \;\leq\; -\lim_{p\uparrow 1} \frac{\log \chi_d(p)}{\log |p-p_c(d)|}.$$

So, if the above limits exist, the fact that $q_c(p) \approx |p-p_c(d)|^{\psi}$ and $\chi_d(p) \approx |p-p_c(d)|^{-\gamma}$ yields $\psi(d) \leq \gamma(d)$ for all d. □

Finally, if the critical exponent $\gamma(d) > 0$ exists in the manner of (3.1) we can make use of Theorem 3.2 and the following statement (a Conjecture presented by [10, 11]) to show that the Conjecture 17 holds for an arbitrary d.

Conjecture 22. *Consider a bond percolation process on* $\mathbb{Z}^d \times \mathbb{Z}^s$ *with parameters* (p, q)*,* $p < p_c(d)$*. Then,*

$$q_c(p) \leq \frac{\beta}{\chi_d(p)},$$

for some $\beta > 0$.

References

[1] D Arovas, RN Bhatt, and B Shapiro. Anisotropic bond percolation in two dimensions. *Physical Review B*, 28(3):1433–1437, 1983.

[2] S. R. Broadbent and J. M. Hammersley. Percolation processes: I. Crystals and mazes. *Mathematical Proceedings of the Cambridge Philosophical Society*, 53(3):629–641, 1957.

[3] CM Chaves, P Murilo Oliveira, Sergio LA de Queiroz, and Rosane Riera. Remarks on the percolation problem in anisotropic systems. *Progress of Theoretical Physics*, 62(6):1550–1555, 1979.

[4] Aglae CN de Magalhães, Constantino Tsallis, and Georges Schwachheim. Probability renormalisation group treatment of bond percolation in square, cubic and hypercubic lattices. *Journal of Physics C: Solid State Physics*, 13(3):321–330, 1980.

[5] Geoffrey Grimmett. *Percolation*. Springer Verlag, 2nd edition, 1999.

[6] Takashi Hara and Gordon Slade. Mean-field critical behaviour for percolation in high dimensions. *Communications in Mathematical Physics*, 128(2):333–391, 1990.

[7] Hiroshi Ikeda. Percolation in Anisotropic Systems: Real-Space Renormalization Group. *Progress of Theoretical Physics*, 61(3):842–849, 1979.

[8] Harry Kesten. The critical probability of bond percolation on the square lattice equals 1/2. *Communications in mathematical physics*, 74(1):41–59, 1980.

[9] Harry Kesten. *Percolation theory for mathematicians*, volume 423. Springer, 1982.

[10] Rémy Sanchis and Roger W. C. Silva. Dimensional crossover in anisotropic percolation on $\mathbb{Z}^{d+s}$. *Journal of Statistical Physics*, 169(5):981–988, 2017.

[11] Rémy Sanchis and Roger W. C. Silva. Correction to: Dimensional crossover in anisotropic percolation on $\mathbb{Z}^{d+s}$. *Journal of Statistical Physics*, 172(3):904–904, Aug 2018.

[12] M. F. Sykes and J. W. Essam. Exact critical percolation probabilities for site and bond problems in two dimensions. *Journal of Mathematical Physics*, 5(8):1117–1127, 1964.

[13] Jacob van den Berg and Harry Kesten. Inequalities with applications to percolation and reliability. *Journal of applied probability*, 22(3):556–569, 1985.