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Local and Global Behavior of Schrödinger-Type Equations

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Local and Global Behavior of Schrödinger-Type Equations

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A mi Madre, Padre
y Hermanos.

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Resumo

Neste trabalho, consideramos o problema de valor inicial para a equação de Schrödinger não linear:

$$i\partial_t u = -\Delta u - \lambda |u|^{\alpha-1}u, \quad x \in \mathbb{R}^N, \quad t > 0. \tag{1}$$

Onde $u = u(t,x)$ é uma função complexa definida em $\mathbb{R} \times \mathbb{R}^N$. Estabelecemos a existência, unicidade e regularidade de soluções locais nos espaços $L^2(\mathbb{R}^N)$ e $H^1(\mathbb{R}^N)$, tanto para o caso subcrítico quanto para o crítico. Estudamos esse problema usando o teorema do ponto fixo de Banach, estimativas de Strichartz e ferramentas da análise harmônica. Além disso, provamos a existência de soluções globais, sob condições que envolvem a não linearidade, o tamanho dos dados iniciais $u_0 \in H^1(\mathbb{R}^N)$ e o sinal de λ. Também estabelecemos um critério de espalhamento (*scattering*).

Finalmente, estabelecemos uma teoria de espalhamento (*scattering*) para dados pequenos da equação não linear de Schrödinger da forma:

$$(i\partial_t + \Delta)u = a(x)|u|^{\alpha-1}u, \tag{2}$$

onde $u\colon \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{C}$, $\alpha > 0$, e $a \in W^{1,\infty}(\mathbb{R}^N)$ para o caso intercrítico, em dimensões $N \geq 3$. Encontramos estimativas de estabilidade para o problema de recuperação de informações sobre a não linearidade no termo não-homogêneo e na potência da não linearidade por meio do mapa de espalhamento (*scattering map*), estendendo os resultados de Chen e Murphy [4].

Palavras-chave: equação de Schrödinger; comportamento local; comportamento global; scattering.

Abstract

In this work, we consider the initial value problem for the nonlinear Schrödinger equation.

$$i\partial_t u = -\Delta u - \lambda |u|^{\alpha-1}u, \quad x \in \mathbb{R}^N, \quad t > 0, \tag{3}$$

where $u = u(t,x)$ is a complex function defined on $\mathbb{R} \times \mathbb{R}^N$. We establish the existence, uniqueness and regularity of local solutions in the spaces $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$, for both subcritical and critical case. We study this problem using the Banach fixed-point theorem, Strichartz estimates and tools from harmonic analysis.

Moreover, we prove the existence of global solutions, under conditions that involve nonlinearity, size of initial data $u_0 \in H^1(\mathbb{R}^N)$ and the sign of λ. Furthermore, we establish a scattering criterion.

Finally, we establish a small data scattering theory for the nonlinear Schrödinger equation of the form

$$(i\partial_t + \Delta)u = a(x)|u|^{\alpha-1}u, \tag{4}$$

where $u\colon \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{C}$, $\alpha > 0$, and $a \in W^{1,\infty}(\mathbb{R}^N)$ for the intercritical case, in dimensions $N \geq 3$. We find stability estimates for the problem of information recovery about nonlinearity in the inhomogeneous term and in the power of nonlinearity through the scattering map, extending the results of Chen and Murphy [4].

Keywords: Schrödinger equation; local behavior; global behavior; scattering.

Contents

Chapter 1

Introduction

In this work, we study the Cauchy problem of the Schrödinger equation

$$\begin{cases} i\partial_t u = -\Delta u - \lambda |u|^{\alpha-1}u, \quad x \in \mathbb{R}^N, \quad t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

where λ and α are real constants with $\alpha > 1$, and $u\colon \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{C}$, $N \geq 3$.

The equation (1.1) is a nonlinear variation of the linear Schrödinger equation. discovered by physicist Erwin Schrödinger (1925). This discovery is of great importance for the development of quantum mechanics. The linear equation describes atoms and particles that move in space, while the nonlinear equation has diverse applications, one of which is in the propagation of light in nonlinear optics (see Boyd [1]).

The Cauchy problem for (1.1), is local well-posed in $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ depending on α, $N \geq 3$. (See Linares-Ponce [15], Tao [20]), and Cazenave [2] using an abstract theory. In other words, if $u_0 \in H^1(\mathbb{R}^N)$ or $L^2(\mathbb{R}^N)$ there exist $T > 0$ and a unique solution $u \in \mathcal{C}([-T,T] : H^1) \cap X$, where X is an auxiliary space solution of (1.1). For the case $N = 1, 2$, we refer to Cazenave [2][Theorem 3.5.1 and Theorem 3.6.1].

Moreover, the NLS equation (1.1) conserves mass $M[u]$ and energy $E[u]$, which are defined by

$$M[u(t)] = \int_{\mathbb{R}^N} |u(x,t)|^2 dx,$$
$$E[u(t)] = \int_{\mathbb{R}^N} \left(|\nabla_x u(x,t)|^2 - \frac{2\lambda}{\alpha+1} |u(x,t)|^{\alpha+1} \right) dx,$$

and play a crucial role in the description of global solutions. Weinstein [22], assuming $\|u_0\|_{H^1} < \|Q\|_{L^2}$, where Q denotes the ground state (positive solution of minimal L^2-norm) of the elliptic equation

$$\Delta Q - Q + |Q|^{\alpha-1}Q = 0,$$

showed global well-posedness in the critical case $\alpha = 1 + \frac{4}{N}$. Holmer and Roudenko [11], for radial initial data $u_0 \in H^1(\mathbb{R}^N)$, satisfying

$$E[u_0]^{s_c} M[u_0]^{s_c} < E[Q]^{s_c} M[Q]^{1-s_c},$$

and

$$\|\nabla_x u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} < \|\nabla_x Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c},$$

with and $s_c = \frac{N}{2} - \frac{2}{\alpha-1}$, proved that the solution exists globally in time and scatters. Duyckaerts, Holmer and Roudenko [7], extend the scattering result [11] to include nonradial H^1 data.

Killip, Murphy, and Visan, in [14], present the method for recovery the nonlinearity of a nonlinear dispersive equation from its small-data scattering behavior, through the scattering map. Watanabe [21] recovery an inhomogeneous coefficient in a nonlinearity of the form $b(x)|u|^{\alpha-1}$. Other works concerning the recovery of nonlinearity from the scattering map can be found in [3] and [16].

We shall find stability estimates for recovery the nonlinearity of inhomogeneus coefficient in a nonlinearity and the power of nonlinearity, for inhomogeneous Schrödinger equation

$$(i\partial_t + \Delta)u = a(x)|u|^{\alpha-1}u, \tag{1.2}$$

where $u\colon \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{C}$, in the intercritical case $1 + \frac{4}{N} \leq \alpha \leq 1 + \frac{4}{N-2}$ and $a \in W^{1,\infty}(\mathbb{R}^N)$.

This text is organized as follows. In Chapter 2, we present properties of the Fourier transform on the line and $\mathbb{R}^N$, along with some spaces of interpolation. In Chapter 3, we describe the solution to the linear Schrödinger equation, presenting fundamental properties for this work, along with Strichartz estimates for the dispersive Schrödinger equation.

Chapter 4 is dedicated to studying the local and global behavior of the nonlinear Schrödinger equation 1.1 in the $L^2(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ spaces (see Linares and Ponce [15]).

Finally, in Chapter 5, we present and prove the principal results of this dissertation.

Chapter 2

The Fourier Transform

In this chapter, we develop a theory of Fourier series, which is used to represent periodic functions through the discrete sum of complex exponentials $x \to e^{2\pi itx}$, and we present some convergence properties. Furthermore, we study the Fourier transform, which extends to non-periodic functions, with some references based on Iorio-Iorio [12], Stein and Weiss [19], and Linares and Ponce [15].

2.1 Fourier Series

This section focuses on representing a 1-periodic function $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ by the series:

$$\frac{a_0}{2} + \sum_{k=0}^{\infty} (a_k \cos(2\pi ky) + b_k \sin(2\pi ky)). \tag{2.1}$$

Since

$$\cos(2\pi ky) = \frac{e^{2\pi iky} + e^{-2\pi iky}}{2} \quad \text{and} \quad \sin(2\pi ky) = \frac{e^{2\pi iky} - e^{-2\pi iky}}{2i}, \quad \text{for all } k = 1, 2, \cdots,$$

we obtain

$$a_k \cos(2\pi ky) + b_k \sin(2\pi ky) = \frac{a_k - ib_k}{2} e^{2\pi iky} + \frac{a_k + ib_k}{2} e^{-2\pi iky}.$$

Therefore (2.1) can be written as

$$\sum_{k=-\infty}^{+\infty} c_k e^{2\pi iky}, \tag{2.2}$$

where

$$c_0 = \frac{a_0}{2}, \quad c_k = \frac{a_k - b_k}{2}, \quad c_{-k} = \frac{a_k + ib_k}{2}, \quad \text{for } k = 1, 2, \cdots.$$

Let us assume that the series (2.2) converges uniformly to the function $f\colon \mathbb{R} \longrightarrow \mathbb{C}$.
If we define

$$\phi_k(x) = e^{2\pi ikx}, \quad k \in \mathbb{Z}, \tag{2.3}$$

we see that $\{\phi_k\}$ satisfies

$$\langle \phi_k, \phi_j \rangle = \int_0^1 \phi_k(x)\overline{\phi_j(x)}dx = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k. \end{cases} \tag{2.4}$$

Thus,

$$\langle f, \phi_k \rangle = \int_0^1 f(x)\overline{\phi(x)}dx = \int_0^1 \left(\sum_{n=-\infty}^{+\infty} c_n e^{2\pi i n x} \right) e^{-2\pi i k x} dx, \tag{2.5}$$

and by uniform convergence of the series (2.2), and (2.4):

$$\langle f, \phi_k \rangle = \sum_{n=-\infty}^{\infty} c_n \int_0^1 e^{2\pi i n x} e^{-2\pi i k x} dx = \sum_{n=-\infty}^{\infty} c_n \langle \phi_n, \phi_k \rangle = c_k.$$

Therefore,

$$c_k = \langle f, \phi_k \rangle = \int_0^1 f(x) e^{-2\pi i k x} dx. \tag{2.6}$$

Definition 2.1.1. *Let $\mathcal{C}^k_{per}([0,1])$, where $k = 0, 1, 2, \ldots$, denote the collection of 1-periodic functions $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ that are of class $\mathcal{C}^k$. If $k = 0$, we denote $\mathcal{C}^0_{per}([0,1])$ by $\mathcal{C}_{per}([0,1])$.*

Definition 2.1.2. *Let f be a function belonging to the set $\mathcal{C}_{per}([0,1])$. The numbers c_k defined by (2.6), are called the Fourier coefficientes of f, and the series*

$$S_f = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$$

is called the Fourier series of f.

The complex sequence $\{\hat{f}(k)\}_{k\in\mathbb{Z}}$, where $\hat{f}(k) = c_k$ is called the Fourier transform of f. The map $f \mapsto \hat{f}$ is linear and satisfies

$$|\hat{f}(k)| = \int_0^1 |f(x)e^{-2\pi i k x}| dx = \|f\|_1 \leq \|f\|_\infty .$$

Now, consider the subspace $V_N = \{\phi_k : -N \leq k \leq N\}$, where $N \in \mathbb{Z}^+$ and let

$$S_N = \sum_{|k|\leq N} \hat{f}(k) e^{2\pi i k x} \tag{2.7}$$

belong to V_N.

Proposition 2.1.1. *Let $V_N = \{\phi_k : -N \leq k \leq N\}$ be an orthonormal subspace in L^2, and $f \in \mathcal{C}_{per}([0,1])$.*

(a) Given $N \in \mathbb{N}$, the best L^2 approximation to f using V_N is given the N-th partial Fourier series S_N. For all $T_N \in V_N$,

$$\|f - S_N\|_2 \leq \|f - T_N\|_2 . \tag{2.8}$$

Proof. Write $g = f - S_N$ and choose

$$T_N = \sum_{|k|\leq N} \alpha_k \phi_k$$

such that

$$f - T_N = f - S_N + (S_N - T_N) = g + (S_N - T_N).$$

Now, using the orthogonality of (2.4) and the Fourier coefficients

$$\begin{aligned}
\langle g, T_N \rangle &= \langle f - S_N, T_N \rangle \\
&= \langle f, \sum_{|k|\le N} \alpha_k \phi_k \rangle - \langle \sum_{|j|\le N} \hat{f}(j)\phi_j, \sum_{|k|\le N} \alpha_k \phi_k \rangle \\
&= \sum_{|k|\le N} \overline{\alpha_k} \langle f, \phi_k \rangle - \sum_{|k|\le N} \sum_{|j|\le N} \overline{\alpha_k} \hat{f}(j) \langle \phi_j, \phi_k \rangle \\
&= \sum_{|k|\le N} (\overline{\alpha_k} \hat{f}(k) - \overline{\alpha_k} \hat{f}(k)) = 0.
\end{aligned}$$

This show that g is orthogonal to V_N. Since $S_N - T_N$ belongs to V_N, It follows from the Pythagorean theorem,

$$\begin{aligned}
\|f - T_N\|_2^2 &= \left\| g + \sum_{|k|\le N} (\hat{f}(k) - \alpha_k)\phi_k \right\|_2^2 \\
&= \|g\|_2^2 + \sum_{|k|\le N} \left| \hat{f}(k) - \alpha(k) \right| \|\phi_k\|_2^2 \\
&\ge \|g\|_2^2 .
\end{aligned}$$

Therefore

$$\|f - S_N\|_2 \le \|f - T_N\| .$$

□

Proposition 2.1.2. *Let $f \in \mathcal{C}_{per}([0,1])$. Then the series*

$$\sum_{k=-\infty}^{\infty} \left| \hat{f}(k) \right|^2 \tag{2.9}$$

converges and satisfies the Bessel's inequality

$$\sum_{k=-\infty}^{+\infty} \left| \hat{f}(k) \right|^2 \le \int_0^1 |f(x)|^2 \, dx = \|f\|_2 . \tag{2.10}$$

In particular, the Riemann-Lebesgue Theorem holds

$$\lim_{|k|\to+\infty} \hat{f}(k) = 0. \tag{2.11}$$

Proof. Let

$$\begin{aligned}
0 \le \|f - S_N\|_2^2 &= \left\| f - \sum_{|k|\le N} \hat{f}(k)\phi_k \right\|_2^2 \\
&= \langle f - \sum_{|k|\le N} \hat{f}(k), f - \sum_{|k|\le N} \hat{f}(k)\rangle \\
&= \langle f, f - \sum_{|k|\le N} \hat{f}(k)\phi_k\rangle - \langle \sum_{|k|\le N} \hat{f}(k)\phi_k, f - \sum_{|k|\le N} \hat{f}(k)\phi_k\rangle \\
&= \|f\|_2^2 - \langle f, \sum_{|k|\le N} \hat{f}(k)\phi_k\rangle \\
&= \|f\|_2^2 - \sum_{|k|\le N} \overline{\hat{f}(k)}\langle f, \phi_k\rangle \\
&= \|f\|_2^2 - \sum_{|k|\le N} |\hat{f}(k)|^2,
\end{aligned}$$

therefore

$$\varphi_N = \sum_{|k|\le N} |\hat{f}(k)|^2 \le \|f\|_2^2, \tag{2.12}$$

for any $N \in \mathbb{N}$. The sequence $\{\varphi_N\}$ is increasing and bounded, therefore it is convergent. So, as $N \to \infty$ we obtain (2.10).
Finally, by the convergence of the series in (2.10):

$$\lim_{|k|\to+\infty} \hat{f}(k) = 0.$$

□

2.1.1 Fourier transform on the line.

We are interested in studying the convergence of the improper integral $I = \int_{-\infty}^{\infty} f(x)dx$, where $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ is a Riemann integrable function on every finite interval $[N, M]$.

We say that the improper integral I converges if

$$\lim_{\substack{M\longrightarrow+\infty \\ N\longrightarrow-\infty}} \int_N^M f(x)dx,$$

exists.

Lemma 2.1.1 (Cauchy Criterion). *$\int_{-\infty}^{\infty} f(x)dx$ converges, if for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that*

$$\left|\int_{-M}^{-N} f(x)dx\right| + \left|\int_N^M f(x)dx\right| < \varepsilon \tag{2.13}$$

for all $M, N \ge M_\varepsilon$.

We say that the improper integral $\int_{-\infty}^{\infty} f(x)dx$ converges absolutely, if $\int_{-\infty}^{\infty} |f(x)|dx$ converges. In other words, the limit

$$\lim_{\substack{M\longrightarrow+\infty\\N\longrightarrow-\infty}}\int_N^M |f(x)|dx,$$

exists. In this case, we say that f is absolutely integrable.

Proposition 2.1.3. *If f is absolutely integrable, then $\int_{-\infty}^{\infty} f(x)dx$ converges.*

Proof. From the hypothesis that f is absolutely integrable and from the Cauchy criterion, given $\varepsilon > 0$, there exist $M_\varepsilon > 0$ such that

$$\left|\int_{-N}^{-M} |f(x)|dx\right| + \left|\int_N^M |f(x)|dx\right| < \varepsilon, \tag{2.14}$$

for all $N, M \geq M_\varepsilon$. Thus

$$\left|\int_{-M}^{-N} f(x)dx\right| + \left|\int_N^M f(x)dx\right| \leq \int_{-N}^{-M} |f(x)|dx + \int_{-M}^{-N} |f(x)|dx < \varepsilon. \tag{2.15}$$

□

Remark 2.1.1. If f is absolutely integrable, according to the Cauchy Criterion, for any given $\varepsilon > 0$ there exists $M'_\varepsilon > 0$ such that

$$\left|\int_{-M}^{-N} |f(x)|dx + \int_N^M |f(x)|dx\right| < \frac{\varepsilon}{2},$$

for all $M > N > M'_\varepsilon$.

Fix $N > M_\varepsilon$. Since $|f| \geq 0$, the map

$$I(M)\colon M \longmapsto \int_{-M}^{-N} |f(x)|dx + \int_N^M |f(x)|dx < \frac{\varepsilon}{2}$$

is non-decreasing and bounded above. Consequently, $\lim_{M\to\infty} I(M)$ exists and

$$\lim_{M\to\infty} I(M) = \int_{|x|>N} |f(x)|dx = \lim_{M\to\infty}\int_{-M}^{-N} |f(x)|dx + \int_N^M |f(x)|dx \leq \frac{\varepsilon}{2} < \varepsilon.$$

Definition 2.1.3. *Let f be an absolutely integrable function. The Fourier transform of f, denoted by $\mathcal{F}f$ or $\widehat{f}$, is defined as the function $\mathcal{F}f\colon \mathbb{R} \longrightarrow \mathbb{C}$ given by*

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\, dx.$$

Proposition 2.1.4. *Let $f, g\colon \mathbb{R} \longrightarrow \mathbb{C}$ absolutely integrable functions, then*

(i) $\widehat{(f+\lambda g)}(\xi) = \widehat{f}(\xi) + \lambda\widehat{g}(\xi)$*;* $\lambda \in \mathbb{C}, \xi \in \mathbb{R}$.

(ii) $\widehat{\overline{f}}(\xi) = \overline{\widehat{f}(-\xi)}$*, for all* $\xi \in \mathbb{R}$.

(iii) If $h \in \mathbb{R}$ *and* $\tau_h(x) = f(x-h)$, $x \in \mathbb{R}$ *then* $\widehat{\tau_h f}(\xi) = e^{-2\pi i h\xi}\widehat{f}(\xi)$.

(iv) $\left|\widehat{f}(\xi)\right| \leq \|f\|_1$.

Proof. (i) By the absolutely convergence and linearity of the integral:

$$\begin{aligned}\widehat{(f+\lambda g)}(\xi) &= \int_{-\infty}^{\infty} [(f+\lambda g)(x)]e^{-2\pi i x\xi}dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi i \xi x}dx + \lambda \int_{-\infty}^{\infty} g(x)e^{-2\pi i x\xi}dx \\ &= \widehat{f}(\xi) + \lambda\widehat{g}(\xi)\end{aligned}$$

(ii) Let $f(x) = u(x) + iv(x)$, then

$$\begin{aligned}\bar{\hat{f}}(-\xi) &= \overline{\int_{-\infty}^{\infty} f(x)e^{2\pi i x\xi}dx} \\ &= \int_{-\infty}^{\infty} (\overline{u(x)e^{2\pi i x\xi}} - i\overline{v(x)e^{2\pi i x\xi}})dx \\ &= \int_{-\infty}^{\infty} \overline{f(x)}e^{-2\pi i x\xi}dx \\ &= \hat{\bar{f}}(\xi)\end{aligned}$$

(iii) Let

$$\widehat{\tau_h f}(\xi) = \int_{-\infty}^{\infty} \tau_h f(x)e^{-2\pi i x\xi}dx = \int_{-\infty}^{\infty} f(x-h)e^{-2\pi i x\xi}dx \tag{2.16}$$

Making a change of variable $w = x - h$ in (2.16)

$$\widehat{\tau_h f}(\xi) = \int_{-\infty}^{\infty} f(w)e^{-2\pi i(w+h)\xi}dw = e^{-2\pi i h\xi}\widehat{f}(\xi).$$

(iv) We have that

$$\left|\widehat{f}(\xi)\right| = \left|\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx\right| \leq \int_{-\infty}^{\infty} |f(x)|\,dx = \|f\|_1 .$$

□

Proposition 2.1.5. *If f is absolutely integrable, then $\widehat{f}$ is uniformly continuous and bounded as $\|\widehat{f}\|_\infty \leq \|f\|_1$.*

Proof. Since f is absolutely integrable, given $\varepsilon > 0$, there exists $M > 0$ such that

$$\int_{|x|>M} |f(x)| < \frac{\varepsilon}{4}. \tag{2.17}$$

Now, we have that

$$
\begin{aligned}
|\hat{f}(\xi+\eta)-\hat{f}(\xi)| &\leq \int_{-\infty}^{+\infty} |f(x)|\left|e^{2\pi i x\eta}-1\right| dx \\
&\leq 2\int_{|x|>M} |f(x)|dx + \int_{|x|\leq M} |f(x)|\left|e^{2\pi i x\eta}-1\right| dx \qquad (2.18) \\
&\leq 2\int_{|x|>M} |f(x)|dx + \sup_{|x|\leq M}\left|e^{2\pi i x\eta}-1\right| \|f\|_1 .
\end{aligned}
$$

By the theorem of calculus

$$
|e^{2\pi i x}-1| = \left|\int_0^x 2\pi i\eta e^{2\pi i t x}dt\right| \leq 2\pi|\eta x|.
$$

Let $\delta = \dfrac{\varepsilon}{4\pi M\|f\|_1} > 0$. If $|\eta|<\delta$, then

$$
\begin{aligned}
|\hat{f}(\xi+\eta)-\hat{f}(\xi)| &\leq \frac{2\varepsilon}{4} + \sup_{|x|\leq M}\left|e^{2\pi i x\eta}-1\right| \|f\|_1 \\
&\leq \frac{\varepsilon}{2} + 2\pi|\eta x|\,\|f\|_1 \\
&< \frac{\varepsilon}{2} + 2\pi M|\eta|\,\|f\|_1 \\
&< \frac{\varepsilon}{2} + 2\pi M\|f\|_1 \frac{\varepsilon}{4\pi M\|f\|_1} = \varepsilon.
\end{aligned}
$$

Due the Proposition 2.1.4, $\widehat{f}$ is bounded with $\|\widehat{f}\|_\infty \leq \|f\|_1$. □

Proposition 2.1.6. *Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be absolutely integrable. Then $\widehat{f}(\xi)\to 0$ when $|\xi|\to 0$.*

Proof. Let $\varepsilon>0$. By Proposition 2.1.5, $\hat{f}$ is uniformly continuous, so there exists a $\delta>0$ such that if $|\xi_1-\xi_2|<\delta$, then $\left|\hat{f}(\xi_1)-\hat{f}(\xi_2)\right|<\varepsilon$.

From the absolute integrability of f, there exists $M>0$ such that

$$
\int_{|x|\geq M} |f(x)|dx < \frac{\varepsilon}{3},
$$

where $M \geq \frac{1}{\delta}$.

Define $g\colon [0,1) \longrightarrow \mathbb{R}$, as

$$
g(y) = 2Mf\left(My\right), \qquad (2.19)
$$

and extend g periodically to the entire real line. Since f is piecewise continuous, g is also piecewise continuous. Let the Fourier coefficients of g be defined by

$$
\hat{g}(k) = \int_0^1 f(x)e^{-2\pi i k x}dx.
$$

Thus, according to the Riemann-Lebesgue Lemma (Lemma 2.1.2), we have:

$$
\lim_{|k|\to+\infty} \hat{g}(k) = 0. \qquad (2.20)
$$

Consequently, there exists $K \in \mathbb{Z}^+$ such that for any $k \in \mathbb{Z}$, if $|k| \geq K$, then $|\hat{g}(k)| < \frac{\varepsilon}{6M}$. If $\xi \in \mathbb{R}$ with

$$|\xi| > \frac{K}{M}, \tag{2.21}$$

Let's consider the ceiling function $\lceil M\xi \rceil$. Choose $K_1 \in \mathbb{Z}$ such that

$$K_1 - 1 < M\xi \leq K_1. \tag{2.22}$$

This implies $|K_1| \geq K$ and

$$\frac{1}{M}(K_1 - 1) < \xi \leq \frac{1}{M}K_1$$
$$\frac{1}{M}(K_1 - 1) - \xi < 0 \leq \frac{1}{M}K_1 - \xi.$$

Therefore

$$0 \leq \frac{1}{M}K_1 - \xi = \frac{1}{M}\left(K_1 - M\xi\right) < \frac{1}{M} \leq \delta.$$

Thus, if $\left|\frac{1}{M}K_1 - \xi\right| < \delta$, by uniform continuity,

$$\left|\hat{f}\left(\frac{K_1}{M}\right) - \widehat{f}(\xi)\right| < \frac{\varepsilon}{3}.$$

Hence,

$$\begin{aligned}
\left|\widehat{f}(\xi)\right| &\leq \left|\widehat{f}(\xi) - \widehat{f}\left(\frac{K_1}{M}\right)\right| + \left|\widehat{f}\left(\frac{K_1}{M}\right)\right| \\
&< \frac{\varepsilon}{3} + \left|\int_{-\infty}^{+\infty} f(x)e^{-2\pi i \frac{K_1}{M}x}dx\right| \\
&< \frac{\varepsilon}{3} + \int_{|x|\geq M} |f(x)|dx + \left|\int_{-M}^{M} f(x)e^{-2\pi i \frac{K_1}{M}x}dx\right|
\end{aligned} \tag{2.23}$$

Through the change of variable $x = My$ and (2.19), we obtain:

$$\begin{aligned}
\left|\widehat{f}(\xi)\right| &< \frac{2\varepsilon}{3} + \left|M\int_{-1}^{1} f\left(My\right)e^{-2\pi i K_1 y}dy\right| \\
&\leq \frac{2\varepsilon}{3} + \left|2M\int_{0}^{1} g(x)e^{-2\pi i K_1 x}dx\right| \\
&= \frac{2\varepsilon}{3} + 2M|\widehat{g}(K_1)| < \varepsilon.
\end{aligned}$$

This completes the proof. □

2.1.2 Schwartz Space

Motivated by analyzing the decay properties of the Fourier transform, we will introduce the Schwartz space on $\mathbb{R}$, which consists of rapidly decreasing functions $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$, which allows us to relate the decay of $\hat{f}(\xi)$ to the continuity and properties of the differentiability

of f, and vice versa.
Defining

$$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}} \left| x^{\alpha}\varphi^{(\beta)}(x) \right| \tag{2.24}$$

and

$$\mathcal{S}(\mathbb{R}) = \left\{ \varphi \in \mathcal{C}^{\infty} : \|\varphi\|_{\alpha,\beta} < +\infty, \text{for all } (\alpha,\beta) \in \mathbb{N}\times\mathbb{N} \right\} \tag{2.25}$$

Consequently, $S(\mathbb{R})$ is a topological vector space over $\mathbb{C}$, with the topology generated by the enumerable intersection of the open sets,

$$B_R(\varphi_0) = \left\{ \varphi \in \mathcal{S}(\mathbb{R}) \colon \ \|\varphi - \varphi_0\|_{\alpha,\beta} < R \right\}.$$

Furthermore,

$$d(\varphi,\eta) = \sum_{\alpha,\beta} \frac{1}{2^{\alpha+\beta}} \frac{\|\varphi-\eta\|_{\alpha,\beta}}{1+\|\varphi-\eta\|_{\alpha,\beta}}. \tag{2.26}$$

defines a metric on $\mathcal{S}(\mathbb{R})$.

Therefore, the Schwartz space $\mathcal{S}(\mathbb{R})$ equipped with metric (2.26) is a complete metric space.

Lemma 2.1.1. *The sequence $(\varphi_n)_{n\geq 1} \subseteq \mathcal{S}(\mathbb{R})$ converges to $\varphi \in \mathcal{S}(\mathbb{R})$ if and only if*

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{\alpha,\beta} = 0$$

for all $\alpha,\beta \in \mathbb{N}$.

Additionally, if $f \in \mathcal{S}(\mathbb{R})$, we have that $f^{(\alpha)}(x)$ and $x^{\alpha}f^{(\beta)}(x)$ belong to Schwartz space. Furthermore, using the fact that the function $\frac{(1+|x|)^2}{1+|x|^2}$ is bounded, there exists $M>0$ such that $(1+|x|)^2 \leq M(1+|x|^2)$. For $\alpha,\beta\in\mathbb{N}$ fixed there exists a constant $C_{\alpha,\beta}\in(0,1)$ such that,

$$\begin{aligned} \left|(1+|x|)^2 x^{\alpha} f^{\beta}(x)\right| &\leq M\left(|x^{\alpha}f^{(\beta)}(x)| + |x^{2+\beta}f^{(\beta)}(x)|\right) \\ &\leq M(C'_{\alpha,\beta} + C''_{2+\alpha,\beta}) \\ &= C_{\alpha,\beta}, \end{aligned}$$

which implies

$$|x^{\alpha}f^{(\beta)(x)}| \leq \frac{C_{\alpha,\beta}}{(1+|x|)^2}.$$

Thus,

$$\lim_{|x|\to\infty} x^{\alpha}f^{(\beta)}(x) = 0.$$

In other words, the Schwartz space consists of smooth functions whose derivatives decay at infinity faster than any power.

Other properties of Fourier transformation are given in the following proposition.

Proposition 2.1.7. *If $f \in \mathcal{S}(\mathbb{R})$, then*

(i) $\frac{d^{\alpha}}{dx^{\alpha}} f \in \mathcal{S}(\mathbb{R})$ *for all* $\alpha \in \mathbb{N}$ *and*

$$\widehat{\frac{d^{\alpha}}{dx^{\alpha}} f}(\xi) = (2\pi i\xi)^{\alpha}\widehat{f}(\xi), \ \xi\in\mathbb{R}. \tag{2.27}$$

(ii) $\hat{f} \in \mathcal{S}(\mathbb{R})$ *and*

$$\frac{d^\alpha}{d\xi^\alpha}\hat{f}(\xi) = \left[\widehat{(-2\pi i x)^\alpha f}\right](\xi). \tag{2.28}$$

Proof. (i) Since $f \in \mathcal{S}(\mathbb{R})$, its derivatives are rapidly decreasing, so $f^{(\alpha)} \in \mathcal{S}(\mathbb{R})$. Integrating by parts gives

$$\widehat{f'}(\xi) = f(x)e^{-2\pi i x\xi}\Big|_{-\infty}^{+\infty} + 2\pi i\xi\int_{-\infty}^{+\infty} f(x)e^{-2\pi i x\xi}dx = 2\pi i\xi\hat{f}(\xi).$$

Then, using an argument by induction, we have

$$\begin{aligned}\frac{d^{\alpha+1}}{dx^{\alpha+1}}\hat{f}(\xi) &= \frac{d^\alpha}{dx^\alpha}f(x)e^{-2\pi i x\xi}\Big|_{-\infty}^{+\infty} + 2\pi i\xi\int_{-\infty}^{+\infty}\frac{d^\alpha}{dx^\alpha}f(x)e^{-2\pi i\xi x}dx \\ &= (2\pi i\xi)^{\alpha+1}\hat{f}(\xi).\end{aligned}$$

(ii) We must show that $\widehat{f}$ is differentiable and find its derivative. Let

$$\begin{aligned}&\lim_{h\to 0}\left(\frac{\widehat{f}(\xi+h)-\widehat{f}(\xi)}{h} - \widehat{(-2\pi i x f)}(\xi)\right) \\ &= \lim_{h\to 0}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{-2\pi i x(\xi+h)} - e^{-2\pi i\xi x}}{h} + 2\pi i x e^{-2\pi i x\xi}\right)dx.\end{aligned}$$

We consider the function

$$\begin{aligned}g_h(x) &= f(x)\left(\frac{e^{-2\pi i(\xi+h)x} - e^{-2\pi i\xi x}}{h} + 2\pi i x e^{-2\pi i\xi x}\right) \\ &= f(x)e^{-2\pi i\xi x}\left(\frac{e^{-2\pi i h x}-1}{h} + 2\pi i x\right),\end{aligned}$$

where, $g_h \to 0$ almost everywhere, as $h \to 0$.

Now, by the mean value theorem, there exists $\lambda_x \in (0,h)$ such that

$$\begin{aligned}|g_h| &\le |f(x)|\left|\frac{e^{-2\pi i x(\xi+h)} - e^{-2\pi i\xi x}}{h} + 2\pi i x e^{-2\pi i\xi x}\right| \\ &\le |2\pi i x|\,|f(x)|\left|e^{-2\pi i(\xi+\lambda_x)x} - e^{-2\pi i\xi x}\right| \\ &\le 4\pi|x||f(x)| \in L^1.\end{aligned}$$

Thus, by dominated convergence theorem

$$\lim_{h\to 0}\int_{-\infty}^{\infty} g_h(x)dx = \int_{-\infty}^{\infty}\lim_{h\to 0} g_h(x)dx = 0,$$

from this, it follows (2.28).

By (i), the Fourier transform interchanges differentiation and multiplication, that so, for integers α and β

$$\begin{aligned}\xi^\alpha\frac{d^\beta}{dx^\beta}\hat{f}(\xi) &= (-2\pi i)^\beta\xi^\alpha\widehat{x^\beta f}(\xi) \\ &= (-2\pi i)^{\alpha+\beta}\widehat{(x^\beta f)^{(\alpha)}}(\xi).\end{aligned}$$

Furthermore, by (4) $x^\beta f \in \mathcal{S}(\mathbb{R})$, we have

$$\|\hat{f}\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}} \left|\xi^\alpha \hat{f}^{(\alpha}(\xi))\right| < +\infty.$$

□

Proposition 2.1.8 (Fourier Inversion). *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x\xi} d\xi,$$

for all $x \in \mathbb{R}$.

Proof. See Iório and Iório [12, Theorem 3.3]. □

We define the the inverse of Fourier transform, as

$$\overset{\vee}{f}(x) = (\mathcal{F}^{-1} f)(x) = \int_{-\infty}^{\infty} f(\xi) e^{ix\xi} d\xi.$$

The Fourier inversion shows that

$$\mathcal{F}^{-1} \circ \mathcal{F} = I,$$

on $\mathcal{S}(\mathbb{R})$, where I is identity mapping. Furthermore, $\mathcal{F}(f)(y) = \mathcal{F}^{-1} f(-y)$, so we also have $\mathcal{F} \circ \mathcal{F}^{-1} = I$. This justifies that $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$. From this, it is easy to see that $\wedge\colon \mathcal{S}(\mathbb{R}^N) \longrightarrow \mathcal{S}(\mathbb{R}^N)$ is an isomorphism.

The Schwartz space $\mathcal{S}(\mathbb{R})$ can be equipped with the inner product and associated norm:

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx, \text{ and } \|f\|_2 = \langle f, f\rangle^{\frac{1}{2}}.$$

Lemma 2.1.2. *If $f, g \in \mathcal{S}(\mathbb{R})$ then:*

(i) $f * g \in \mathcal{S}(\mathbb{R})$.

(ii) $\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.

Proof. See Stein and Weiss [19, Proposition 1.11]. □

Proposition 2.1.9 (Plancherel). *If $f \in \mathcal{S}(\mathbb{R})$ then $\|\widehat{f}\|_2 = \|f\|_2$.*

Proof. If $f \in \mathcal{S}(\mathbb{R})$, let $g(x) = \overline{f(-x)}$. According to Proposition 2.1.4 item 2, it follows that $\widehat{g}(\xi) = \overline{\widehat{f}(\xi)}$. Furthermore, let $f * g \in \mathcal{S}(\mathbb{R})$ such that

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi) = |\widehat{f}(\xi)|^2,$$

and

$$(f * g)(0) = \int_{-\infty}^{\infty} f(x) g(0 - x) dx = \int_{-\infty}^{\infty} |f(x)|^2 dx. \tag{2.29}$$

Now, the inversion formula applied with $x = 0$

$$(f * g)(0) = \int_{-\infty}^{\infty} \widehat{f * g}(\xi) d\xi = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 d\xi. \tag{2.30}$$

Therefore, (2.29) and (2.30) yield the desired result. □

2.1.3 The Fourier transform on $\mathbb{R}^N$

Definition 2.1.4. *The Fourier Transform of a function $f \in L^1(\mathbb{R}^N)$, denoted by $\hat{f}$, is defined as:*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i x\cdot\xi} dx, \ \text{for } \xi \in \mathbb{R}^N, \tag{2.31}$$

where

$$x \cdot \xi = \sum_{1\leq j \leq N} x_j \xi_j.$$

Proposition 2.1.10. *Let $f \in L^1(\mathbb{R}^N)$, then :*

(i) $f \mapsto \widehat{f}$ *from* $L^1(\mathbb{R}^N)$ *to* $L^\infty(\mathbb{R}^N)$ *with*

$$\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}.$$

(ii) $\widehat{f}$ *is continuous .*

(iii) $\widehat{f}(\xi) \to 0$ *as* $|\xi| \to \infty$.

(iv) If $\tau_h f(x) = f(x-h)$ *denotes the translation by* $h \in \mathbb{R}^N$, *then*

$$\widehat{(\tau_h f)}(\xi) = e^{-2\pi h\cdot x}\widehat{f}(\xi), \text{and } \widehat{(e^{-2\pi i x\cdot h} f)}(\xi) = \tau_{-h}\widehat{f}(\xi).$$

(v) If $\delta_a f(x) = f(ax)$ *denotes the dilation by* $a > 0$, *then*

$$\widehat{(\delta_a f)}(\xi) = \frac{1}{a^N}\widehat{f}\left(\frac{\xi}{a}\right).$$

(vi) Let $g \in L^1(\mathbb{R}^N)$ *and* $f * g$ *be the convolution of* f *and* g. *Then*

$$\widehat{(f*g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

(vii) Let $g \in L^1(\mathbb{R}^N)$. *Then*

$$\int_{\mathbb{R}^N} \widehat{f}(y) g(y) dy = \int_{\mathbb{R}^N} f(y)\widehat{g}(y) dy.$$

The proof of this proposition follows in a manner similar to Theorem 2.1.4.

Proposition 2.1.11. *Let f, $x^\alpha f \in L^1$, with $\alpha \in \mathbb{Z}_+^N$ a multi index. Then $\partial^\alpha \widehat{f}$ is defined and satisfies*

$$\partial^\alpha \widehat{f}(\xi) = \left[\widehat{(-2\pi i x)^\alpha f}\right](\xi).$$

Proof. See Linares-Ponce [15, Proposition 1.1]. □

Definition 2.1.5. *Let $1 \leq p \leq \infty$. A function $f \in L^p(\mathbb{R}^N)$ is differentiable in $L^p(\mathbb{R}^N)$ with respect the kth variable, if there exists $g \in L^p(\mathbb{R}^N)$ such that*

$$\int_{\mathbb{R}^N} \left| \frac{f(x+te_k) - f(x)}{t} - g(x) \right|^p dx \to 0, \ \text{as } t \to 0,$$

where e_k has kth coordinate equals to 1 and 0 in the others.

Proposition 2.1.12. *Let $f \in L^1(\mathbb{R}^N)$ such that $\partial^\alpha f \in L^1(\mathbb{R}^N)$. Then*

$$\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi).$$

Proof. See Linares-Ponce [15, Theorem 1.2]. □

Example 1. Let $N = 1$ and $f(x) = \chi_{[a,b]}$ the characteristic function of interval $[a, b]$. Then

$$\begin{aligned}\widehat{f}(\xi) = \widehat{\chi_{[a,b]}}(\xi) &= \int_{-\infty}^{\infty} \chi_{[0,1]}(x) e^{-2\pi i x \xi} dx = \int_a^b e^{-2\pi i x \xi} dx, \\ &= -\left[\frac{e^{-2\pi i x \xi}}{2\pi i \xi}\right]\Big|_a^b = -e^{-\pi i (a+b)\xi}\ \frac{\sin(\pi(a-b)\xi)}{\pi \xi}.\end{aligned}$$

Notice that $\widehat{\chi_{[a,b]}(\xi)} \notin L^1(\mathbb{R})$.

We have studied some properties of the Fourier transform to find $\widehat{f}$ from f. However, we are interested in how to determine f from $\widehat{f}$. The above example show that $\widehat{f} \notin L^1$ for some $f \in L^1(\mathbb{R}^N)$.

Proposition 2.1.13. *Let $f \in L^1(\mathbb{R}^N)$. Then*

$$f(x) = \lim_{t \to 0^+} \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} e^{-4\pi^2 t |\xi|^2} \widehat{f}(\xi) d\xi,$$

where the limit is taken in the L^1-norm. Moreover, if f is continuous at the point x_0, then the following pointwise equality holds:

$$f(x_0) = \lim_{t \to 0^+} \int_{\mathbb{R}^N} e^{2\pi x_0 \cdot \xi} e^{-4\pi^2 t |\xi|^2} \widehat{f}(\xi) d\xi.$$

Proof. See Linares-Ponce [15, Proposition 1.2]. □

Now, if $f, \widehat{f} \in L^1$, we have the inversion form:

$$f(x) = \int_{\mathbb{R}^N} \widehat{f}(\xi) e^{2\pi i x \cdot \xi},$$

almost everywhere $x \in \mathbb{R}^N$.

2.1.4 The Fourier Transform on L^2

We observe that, while the Fourier transform given by (2.31), makes sense as a convergent integral for functions $f \in L^1(\mathbb{R}^N)$, this integral does not converge absolutely for all functions on L^2. We first define a linear operator on the subset $L^1 \cap L^2$ of $L^2(\mathbb{R}^N)$.

Proposition 2.1.14 (Plancherel). *If $f \in L^1 \cap L^2$. Then $\widehat{f} \in L^2$ and $\|f\|_{L^2} = \|\widehat{f}\|_{L^2}$*

Proof. See Linares-Ponce [15, Theorem 1.3]. □

Thus, the Fourier transform is an L^2 isometry on $L^1 \cap L^2$. By density, there exists a unique bounded extension of Fourier transform $\mathcal{F} \colon L^2 \longrightarrow L^2$.

Proposition 2.1.15. *The inverse of the Fourier transform $\mathcal{F}^{-1}$ can be defined by the formula*

$$\mathcal{F}^{-1}f(x) = \mathcal{F}f(-x),$$

for any $f \in L^2$.

Proof. See Linares-Ponce [15, Theorem 1.5]. □

2.1.5 The Fourier transform in the space of tempered distributions

In this section, we shall prove that any $f \in L^p(\mathbb{R}^N)$ for $p \geq 1$ has a Fourier transform in the distribution sense.

Definition 2.1.6. *We define the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decaying functions as*

$$\mathcal{S}(\mathbb{R}^N) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^N) : \|f\|_{\alpha,\beta} = \sup_x |x^\alpha \partial^\beta f(x)| < \infty \ \forall \alpha, \beta \in \mathbb{Z}^N_{\geq 0} \right\}.$$

This space is a topological vector space, with topology generate by the enumerable intersection of open sets

$$\bigcap_{(\alpha,\beta)\in\mathbb{Z}^N_{\geq 0}} \left\{ f \in \mathcal{S}(\mathbb{R}^N) : \ \|f - g\|_{\alpha,\beta} < R \right\},$$

for some $g \in \mathcal{S}(\mathbb{R}^N)$ and $R > 0$, and is a complete space with the metric:

$$d(f,g) = \sum_{\alpha,\beta} \frac{1}{2^{|\alpha|+|\beta|}} \frac{\|f-g\|_{\alpha,\beta}}{1 + \|f-g\|_{\alpha,\beta}}. \tag{2.32}$$

Definition 2.1.7. *Let $\{f_j\} \subset \mathcal{S}(\mathbb{R}^N)$. Then, $f_j \to 0$ as $j \to \infty$, if for any $(\alpha, \beta) \in \mathbb{Z}^N_{\geq 0}$ one has that $\|f_j\|_{\alpha,\beta} \to 0$ as $j \to \infty$.*

Proposition 2.1.16. *The map $f \mapsto \widehat{f}$ is a isomorphism.*

Proof. Se Linares-Ponce [15, Theorem 1.6] □

Proposition 2.1.17. *The Schwartz space has the following properties :*

(i) *$\mathcal{C}_0^\infty \subset \mathcal{S}(\mathbb{R}^N)$, and injection is continuous.*

(ii) *$\mathcal{C}_0^\infty$ is dense in $\mathcal{S}(\mathbb{R}^N)$.*

(iii) *$\mathcal{S}(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$, and injection is continuous.*

Proof. See Friendlander [8, Theorem 8.2.1]. □

Definition 2.1.8. *We define the space of tempered distribution functions as the linear functionals that satisfy*

$$S'(\mathbb{R}^N) = \left\{ T \in L(\mathcal{S}(\mathbb{R}^N), \mathbb{C}) : \ T(f_n) \to 0 \ \textit{if} \ \|f_n\|_{\alpha,\beta} \to 0, \forall (\alpha,\beta) \in \mathbb{Z}^N_{\geq 0} \right\}.$$

Remark 2.1.2. If is g is any locally integrable function with polynomial growth at infinity, then g defines a tempered distribution T_g, where

$$T_g(f) = \int_{\mathbb{R}^N} g(x)f(x)dx, \text{ for all } f \in \mathcal{S}(\mathbb{R}^N).$$

Definition 2.1.9. *Given $T \in \mathcal{S}'(\mathbb{R}^N)$, its Fourier transform $\widehat{T} \in \mathcal{S}'(\mathbb{R}^N)$ is defined as:*

$$\widehat{T}(f) = T(\widehat{f}),$$

for any $f \in \mathcal{S}(\mathbb{R}^N)$.

It is clear that $\hat{T}$ is linear. To show that $\hat{T} \in \mathcal{S}'(\mathbb{R}^N)$, we need to establish its continuity.

Let $\{f_n\} \subset \mathcal{S}(\mathbb{R}^N)$ be a sequence such that $f_n \to 0$ in $\mathcal{S}(\mathbb{R}^N)$ as $n \to \infty$. Since that the Fourier transform is continuous (see Theorem 2.1.10-((ii))), $\hat{f}_n \to 0$ as $n \to \infty$. It follows that, $\hat{T}(f_n) = T(\hat{f}_n) \to 0$, and consequently $\hat{T} \in \mathcal{S}'(\mathbb{R}^N)$.

Note that, for $g \in L^1(\mathbb{R}^N)$ and $f \in \mathcal{S}(\mathbb{R}^N)$, it holds that

$$\widehat{T_g}(f) = T_g(\widehat{f}) = \int_{\mathbb{R}^N} g(x)\widehat{f}(x)dx = \int_{\mathbb{R}^N} \widehat{g}(x)f(x)dx = T_{\widehat{g}}(f).$$

Definition 2.1.10. *Let $T \in \mathcal{S}'(\mathbb{R}^N)$ and α a multi-index. Define*

$$\partial^\alpha T(f) = (-1)^{|\alpha|} T(\partial^\alpha f).$$

This definition is consistent with the theory of the Fourier transform on $L^1 + L^2$ in the sense of distributions when identifying the function f with a distribution T_f.

Proposition 2.1.18. *The map $\mathcal{F}\colon T \mapsto \widehat{T}$ is an isomorphism from $\mathcal{S}'(\mathbb{R}^N)$.*

Proof. See Linares-Ponce [15]. □

Example 2. The Laplacian Δ, is a partial operator acting on tempered distributions on $\mathbb{R}^N$ as follows:

$$\Delta(T) = \sum_{j=1}^{N} \partial^2_{x_j x_j} T,$$

where $T \in \mathcal{S}'(\mathbb{R}^{\mathbb{N}})$. From Definition 2.1.10

$$\partial^2_{x_j x_j} T(f) = (-1)^2 T(\partial^2_{x_j x_j} f),$$

and from Definition 2.1.9 and the Proposition 2.1.12, it follows that

$$\widehat{\Delta T}(f) = \sum_{j=1}^{N} \widehat{\partial^2_{x_j x_j} T}(f) = \sum_{j=1}^{N} T(\widehat{\partial^2_{x_j x_j} f}(\xi_j)) = -4\pi^2 |\xi|^2 \widehat{T}(f).$$

2.2 Interpolation

Let $(X, \mathcal{M}, \mu)$ be a measure space, with $1 \leq p < \infty$. If $f\colon X \longrightarrow \mathbb{C}$ is a measurable function, we define

$$L^p(X, \mathcal{M}, \mu) := \{f\colon X \longrightarrow \mathbb{C} : \|f\|_{L^p} < \infty\},$$

where the L^p norm of f is defined by

$$\|f\|_{L^p(X,\mathcal{M},\mu)} = \left(\int_X |f|^p d\mu(x)\right)^{\frac{1}{p}}.$$

Furthermore, in the case of limiting value $p = \infty$, we define

$$L^\infty(X, \mathcal{M}, \mu) := \{f\colon X \longrightarrow \mathbb{C} : \|f\|_{L^\infty} < \infty\},$$

where

$$\|f\|_{L^\infty(X,\mathcal{M},\mu)} := \inf\{\lambda \geq 0 : \mu(x : |f(x)| > \lambda) = 0\}.$$

We abbreviate $L^p(X, \mathcal{M}, \mu)$, by $L^p(X)$.

Remark 2.2.1. If $X = \mathbb{R}^N$, we write $\|f\|_{L^p}$ instead of $\|f\|_{L^p(\mathbb{R}^N)}$, for the L^p-norm o f. For a function $g\colon I \times \mathbb{R}^N \to \mathbb{C}$, for some interval $I \subset \mathbb{R}$, we write

$$\|g\|_{L^q_t L^r_x(I\times\mathbb{R}^N)} := \|g\|_{L^q_t L^r_x(I)} = \left\| \|g\|_{L^r_x} \right\|_{L^q(I)}, \tag{2.33}$$

where $1 \leq q, r \leq \infty$.

Next, I will present some classical inequalities that are satisfied in the L^p space.

Lemma 2.2.1. *If $x \geq 0$ and $\lambda \in (0, 1)$, then $x^\lambda \leq (1 - \lambda) + \lambda x$.*

Proof. Let $f(x) = (1 - \lambda) + \lambda x - x^\lambda$, hence $f'(x) = \lambda - \lambda x^{\lambda-1}$. Since $f'(x) = 0$ at $x = 1$, we have that f attains its miminum value at this point. Therefore,

$$f(1) = 0 \leq (1 - \lambda) + \lambda x - x^\lambda.$$

□

By setting $x = \frac{a}{b}$ with $a, b > 0$ and applying the Lemma 2.2.1, we deduce that

$$a^\lambda b^{1-\lambda} \leq (1 - \lambda)b + \lambda a. \tag{2.34}$$

The equality holds if, and only if, $a = b$.

Proposition 2.2.1. *Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. The inequality holds if $\|f\|_{L^p} = 0$ or $\|g\|_{L^q} = 0$, since that $f = 0$ or $g = 0$ almost everywhere.

Now if $\|f\|_{L^p} \neq 0$ and $\|g\|_{L^p} \neq 0$, by homogeneity we may assume that $\|f\|_{L^p} = \|g\|_{L^q} = 1$. We use $\lambda = \frac{1}{p}$ on the inequality (2.34), setting $a = |f(x)|^p$ and $b = |g(x)|^q$, to obtain

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q.$$

Now, integrating this inequality yields

$$\|fg\|_{L^1} \leq \frac{1}{p}\|f\|_{L^p}^p + \frac{1}{q}\|f\|_{L^q}^q = \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_{L^p}\|g\|_{L^q}.$$

□

Collorary 2.2.1 (Hölder Inequality). *Let $1 \leq p, q, r \leq \infty$. If $f \in L^p$ and $g \in L^q$*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

then $fg \in L^r$ and

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}. \tag{2.35}$$

Collorary 2.2.2. *If $f \in L^p \cap L^q$ with $1 \leq p \leq q \leq \infty$, then $f \in L^r$ for all r, such that $p \leq r \leq q$, an the following inequality holds:*

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta}\|g\|_{L^q}^{1-\theta}, \quad \textit{where} \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, 0 \leq \theta \leq 1.$$

Proof. By Holder inequality (2.35), applied to the functions $f = |f|^\theta$ and $g = |f|^{(1-\theta)}$, it is evident that $|f|^\theta \in L^{\frac{p}{\theta}}$ and $|f|^\theta \in L^{\frac{q}{1-\theta}}$, thus

$$\begin{aligned}\int |f|^r = \int |f|^{\theta r}|f|^{(1-\theta)r} &\leq \left\||f|^{\theta r}\right\|_{L^{\frac{p}{\theta r}}}\left\||f|^{(1-\theta)r}\right\|_{L^{\frac{q}{(1-\theta)r}}} \\ &= \left(\int |f|^p\right)^{\frac{\theta r}{p}}\left(\int |f|^q\right)^{\frac{(1-\theta)r}{q}} \\ &= \|f\|_{L^p}^{\theta r}\|f\|_{L^q}^{(1-\theta)r}.\end{aligned}$$

Taking the r-th root, the proof is complete. □

Lemma 2.2.2. *If $1 \leq p \leq \infty$ and $f, g \in L^p$, then*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

Proof. Suposse without loss of generality, that f and g are non-zero. By homogeneity, we can assume $\|f\|_{L^p} + \|g\|_{L^p} = 1$. If $f = (1-\theta)F$ and $g = \theta G$, for some $\theta \in (0, 1)$, again by homogeneity, it is clear that $\|F\|_{L^p} = \|G\|_{L^p} = 1$.

Since the function $z \mapsto |z|^p$ is convex for $p \geq 1$ on $\mathbb{C}$, it follows

$$|(1-\theta)F + \theta G|^p \leq (1-\theta)|F|^p + \theta|G|^p,$$

and by integrating both sides, we deduce that

$$\int_X |(1-\theta)F + \theta G|^p d\mu \leq (1-\theta) \|F\|_{L^p}^p + \theta \|G\|_{L^p}^p = 1,$$

which finishes the proof. □

Lemma 2.2.3. *Suppose that $f \in L^p$ with $1 \leq p, q \leq \infty$, are conjugate exponents, that is*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\|f\|_{L^p} = \sup_{\|g\|_{L^q} \leq 1} \left| \int f\bar{g} \right|$$

Proof. See Stein-Shakarchi [18, Lemma 4.2]. □

2.2.1 Lorentz Space and Real Interpolation

We study a new class of functions space namely, Lorentz space. These spaces are a generalization of L^p spaces.

Let

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^N)}^p &= \int_0^\infty p\lambda^{p-1} \int \chi_{\{|f| \geq \lambda\}} d\mu d\lambda \\ &= \int_0^\infty p \left(\lambda \mu \{|f| \geq \lambda\}^{\frac{1}{p}}\right)^p \frac{d\lambda}{\lambda}, \end{aligned}$$

so

$$\|f\|_{L^p} = p^{\frac{1}{p}} \left\| \lambda \mu(\{|f| \geq \lambda\})^{\frac{1}{p}} \right\|_{L^p\left(\mathbb{R}_+, \frac{d\lambda}{\lambda}\right)}.$$

If we take the norm L^q of this quantity, we define the Lorentz space $L^{p,q}(\mathbb{R}^N)$.

Definition 2.2.1. *If $1 \leq p, q \leq \infty$, the Lorentz space $L^{p,q}(\mathbb{R}^N)$ is the space of measurable functions $f \colon \mathbb{R}^N \longrightarrow \mathbb{C}$ such that*

$$\|f\|_{L^{p,q}(\mathbb{R}^N)}^* = p^{\frac{1}{q}} \left\| \lambda \mu(\{|f| \geq \lambda\})^{\frac{1}{p}} \right\|_{L^q\left(\mathbb{R}_+, \frac{d\lambda}{\lambda}\right)} < +\infty. \tag{2.36}$$

It is evident that, $L^{p,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$. Moreover, we write $\|f\|_{L^{p,\infty}}^* = \sup_{\lambda > 0} \lambda \mu(\{|f| \geq \lambda\})^{\frac{1}{p}}$.

Example 2.2.1. Let $p \geq 1$. Then $L^p(\mathbb{R}^N) \subsetneq L^{p,\infty}(\mathbb{R}^N)$. Indeed:

Set $r = |x|$ and $f(x) = |x|^{-\frac{N}{p}}$, then $dx = r^{N-1} dr$. Thus,

$$\|f\|_{L^p}^p = \int_0^\infty \frac{1}{r^N} r^{N-1} dx = \int_0^\infty \frac{1}{r} dr$$

where this integral is divergent, so $f \notin L^p(\mathbb{R}^N)$.

Now

$$\mu\{x : |f(x)| \geq \lambda\} = \mu\left\{x : |x|^{-\frac{N}{p}} \leq \lambda\right\} = \mu\left\{x : |x| \geq \lambda^{-\frac{p}{N}}\right\}$$

Using the fact that $\mu(cA) = c^N \mu(A)$ for $c > 0$ and each Borel set of $\mathbb{R}^N$, so

$$\mu\left\{x : |x| \leq \lambda^{-\frac{p}{N}}\right\} = \lambda^{-\left(\frac{p}{n}\right)^N} \mu\{x : |x| \leq 1\}, \tag{2.37}$$

hence

$$\mu(\{x : |f(x)| \geq \lambda\})^{\frac{1}{p}} \leq \mu\{x : |x| \leq 1\},$$

therefore taking the supremum and $p-$root, we obtain

$$\|f\|_{L^{p,\infty}(\mathbb{R}^N)} \lesssim \sup_{\lambda>0} \lambda \frac{1}{\lambda} = 1 < \infty,$$

so that $f \in L^{p,\infty}(\mathbb{R}^N)$, proving that $L^p(\mathbb{R}^N) \subsetneq L^{p,\infty}(\mathbb{R}^N)$.

Proposition 2.2.2. *If $1 \leq p < \infty$, then*

1. *If $\|f\|^*_{L^{p,q}} = 0$ then $f = 0$.*
2. *For all $c \in \mathbb{C}$, $\|cf\|^*_{L^{p,q}} = |c|\,\|f\|^*_{L^{p,q}}$.*
3. $\|f + g\|^*_{L^{p,q}} \leq 2(\|f\|^*_{L^{p,q}} + \|g\|^*_{L^{p,q}})$

Proof. Suppose that $\|f\|_{L^{p,q}(\mathbb{R}^N)} = 0$, then

$$p^{\frac{1}{q}} \|\lambda \mu(\{x : |f(x)| > \lambda\})\|_{L^q\left(\mathbb{R}_+, \frac{d\lambda}{\lambda}\right)} = 0,$$

which implies that, for almost $\lambda > 0$, $\mu(\{x : |f(x)| > \lambda\}) = 0$. It follows that $f(x) = 0$ almost everywhere. Thus, $f = 0$.

Let $a \in \mathbb{C}$, then

$$\begin{aligned}
\|af\|_{L^{p,q}(\mathbb{R}^N)} &= p^{\frac{1}{q}} \|\lambda \mu(\{x : |af(x)| > \lambda\})\|_{L^q\left(\mathbb{R}_+, \frac{d\lambda}{\lambda}\right)} \\
&= p^{\frac{1}{q}} \left(\int_0^\infty \lambda^q \mu(\{x : |af(x)| > \lambda\})^{\frac{p}{q}} \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}} \\
&= |a| p^{\frac{1}{q}} \left(\int_0^\infty \left(\frac{\lambda}{|a|}\right)^{\frac{1}{q}} \mu\left(\left\{x : |f(x)| > \frac{\lambda}{|a|}\right\}\right)^{\frac{p}{q}} \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}} \\
&= |a| p^{\frac{1}{q}} \left(\int_0^\infty \eta^{\frac{1}{q}} \mu(\{x : |f(x)| > \eta\})^{\frac{p}{q}} \frac{d\eta}{\eta}\right)^{\frac{1}{q}}.
\end{aligned}$$

Above, we used the change of variable $\eta = \dfrac{\lambda}{|a|}$, proving the desired equality.

Now, using the fact that

$$\{x : |f(x) + g(x)| > \lambda\} \subseteq \left\{x : |f(x)| > \frac{\lambda}{2}\right\} \cup \left\{x : |g(x)| > \frac{\lambda}{2}\right\},$$

together with the fact that $x \mapsto x^p$ is subadditive and Lemma 2.2.2

$$\begin{aligned}
\|f+g\|^*_{L^{p,q}(\mathbb{R}^N)} &= p^{\frac{1}{q}} \left\| \lambda \mu \left(\{x : |f(x)+g(x)| > \lambda\} \right)^{\frac{1}{p}} \right\|_{L^q\left(\mathbb{R}_+, \frac{d\lambda}{\lambda}\right)} \\
&\leq p^{\frac{1}{q}} \left\| \lambda \left(\mu\left(\left\{ x : |f(x)| > \frac{\lambda}{2} \right\} \right) + \mu\left(\left\{ x : |g(x)| > \frac{\lambda}{2} \right\} \right) \right)^{\frac{1}{p}} \right\|_{L^q\left(\mathbb{R}_+, \frac{d\lambda}{\lambda}\right)} \\
&\leq 2p^{\frac{1}{q}} \left\| \frac{\lambda}{2} \mu\left(\left\{ x : |f(x)| > \frac{\lambda}{2} \right\} \right)^{\frac{1}{p}} \right\|_{L^q\left(\mathbb{R}_+, \frac{d\lambda}{\lambda}\right)} \\
&+ 2p^{\frac{1}{p}} \left\| \frac{\lambda}{2} \mu\left(\left\{ x : |g(x)| > \frac{\lambda}{2} \right\} \right)^{\frac{1}{p}} \right\|_{L^q\left(\mathbb{R}_+, \frac{d\lambda}{\lambda}\right)} \\
&= 2(\|f\|^*_{L^{p,q}} + \|g\|^*_{L^{p,q}}).
\end{aligned}$$

□

This proposition asserts that the quantity $\|\cdot\|^*_{L^{p,q}}$ is a quasinorm.

Lemma 2.2.4 (Chebyshev's inequality). *If $f \in L^p(\mathbb{R}^{\mathbb{N}})$, then for any $\lambda > 0$ we have*

$$\mu \left\{ x \in \mathbb{R}^N : |f(x)| > \lambda \right\} \leq \frac{1}{\lambda^p} \int |f(x)|^p dx.$$

Proof. If $p > 0$ and $|f(x)| > \lambda$ we have that $|f(x)|^p > \lambda^p$, therefore

$$\begin{aligned}
\int_{\mathbb{R}^N} |f(x)|^p dx &\geq \int_{\{x \in \mathbb{R}^N : |f(x)| > \lambda\}} |f(x)|^p dx \\
&\geq \lambda^p \int_{\{x \in \mathbb{R}^N : |f(x)| > \lambda\}} dx \\
&= \lambda^p \mu \left\{ x \in \mathbb{R}^N : |f(x)| > \lambda \right\}.
\end{aligned}$$

□

Lemma 2.2.5 (The Three Lines Theorem). *Let f be a bounded continuous fuction on the strip $0 \leq Re(z) \leq 1$ that is holomorphic on the interior of strip. If $|f(z)| \leq M_0$ for $Re(z) = 0$ and $|f(z)| \leq M_1$ for $Re(z) = 1$, then $|f(z)| \leq M_0^{1-x} M_1^x$ for $Re(z) = x$, $0 < x < 1$.*

Proof. Define

$$f_\varepsilon(z) = \frac{e^{-\varepsilon z(1-z)} f(z)}{M_0^{1-z} M_1^z}, \tag{2.38}$$

a bounded continuos function and holomorphic on the interior of the strip $0 \leq Re(z) \leq 1$. Let $z = x + iy$, then

$$|f_\varepsilon(z)| \leq \frac{e^{Re(-\varepsilon z(1-z))} f(z)}{M_0^{Re(1-z)} M_1^{Re(z)}} = \frac{e^{-\varepsilon(x(1-x)+y^2)} |f(z)|}{M_0^{1-x} M_1^x},$$

if $Re(z) = 0$, we have

$$|f_\varepsilon(z)| \leq \frac{e^{-\varepsilon y^2} |f(z)|}{M_0} \leq \frac{M_0}{M_0} = 1,$$

and for $Re(z) = 1$,

$$|f_\varepsilon(z)| \leq \frac{e^{-\varepsilon y^2}|f(z)|}{M_1} \leq \frac{M_1}{M_1} = 1.$$

Given $\varepsilon > 0$ there exists $R_\varepsilon > 0$, such that $|y| > R_\varepsilon$, implies that $|f_\varepsilon(z)| \leq 1$. So, by the maximum modulus principle, for any point in the rectangle $\{0 + iA, 1 + iA, 1 - iA, 0 - iA\}$, f attains its maximum on the boundary, that is $|f_\varepsilon(z)| \leq 1$.

Since A is arbitrary, we can let $|A| \to \infty$ such that $|f_\varepsilon(z)| \leq 1$ on the strip $0 \leq Re(z) \leq 1$. Finally, if $Re(z) = t$, we have

$$|e^{-\varepsilon Re(z(1-z))}|f(z)| \leq M_0^t M_1^{1-t},$$

and letting $\varepsilon \to 0$ yields the desired result:

$$|f(z)| \leq M_0^t M_1^{1-t}.$$

□

Theorem 2.2.1 (Riez-Thorin). *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and let $T : L^{p_j}(X, \mu) \longrightarrow L^{q_j}(Y, \nu)$ be a bounded linear operator with norm M_j, where $j = 0, 1$. Then T is bounded from $L^{p_\theta}(X, \mu)$ to $L^{q_\theta}(Y, \nu)$, with a norm denoted as M_θ, such that*

$$M_\theta \leq M_0^{1-\theta} M_1^\theta. \tag{2.39}$$

Furthermore, the operator satisfies the following equations:

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 \leq \theta \leq 1. \tag{2.40}$$

Proof. See [15]. □

Proposition 2.2.3 (Young's Inequality). *Let $1 \leq p \leq \infty$, and $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$. Then $f * g \in L^q(\mathbb{R}^N)$, where $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$. Moreover*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. Let $g \in L^q$, and define the operator

$$Tf = f * g.$$

By Minkoswski's inequality we have

$$\begin{aligned}
\|Tf\|_{L^q} &= \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} f(y)g(x-y)dy\right)^q dx\right)^{\frac{1}{q}} \\
&\leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |f(y)|^q |g(x-y)|^q dx\right)^{\frac{1}{q}} dy \\
&= \int_{\mathbb{R}^N} |f(y)| \left(|\int_{\mathbb{R}^N} |g(x-y)|^q\right)^{\frac{1}{q}} dy \\
&= \|g\|_{L^q} \|f\|_1.
\end{aligned}$$

If $\frac{1}{q} + \frac{1}{q'} = 1$, by Holder's inequality

$$\|Tf\|_{L^\infty} \leq \|g\|_{L^q} \|f\|_{L^{q'}}.$$

Hence Riesz Thorin $T\colon L^{p_\theta} \longrightarrow L^{q_\theta}$ is bounded, satisfying

$$\frac{1}{p_\theta} = \frac{1-\theta}{1} + \frac{\theta}{q'} \quad \text{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q}, \quad 0 \leq \theta \leq 1.$$

Taking θ such that $p_\theta = p$ and $q_\theta = r$, we have that

$$\frac{1}{r} = \frac{1}{q} + \left(1 + \frac{\theta}{q}\right) - 1 = \frac{1}{q} + \frac{1}{p} - 1.$$

□

Proposition 2.2.4. *If $f \in L^p(\mathbb{R}^N)$, $1 \leq p \leq 2$. Then $\widehat{f} \in L^{p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$, and*

$$\|\widehat{f}\|_{L^{p'}} \leq \|f\|_{L^p}.$$

Proof. By Proposition 2.1.5 and Placherel, we have

$$\wedge\colon L^1 \longrightarrow L^\infty$$
$$\wedge\colon L^2 \longrightarrow L^2,$$

with norm 1 . Using Riesz-Thorin theorem $\wedge\colon L^{p_\theta} \longrightarrow L^{q_\theta}$ is bounded. Choosing θ, such that $q_\theta = p'$ and $p_\theta = p$, we conclude that

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}, \quad 0 \leq \theta \leq 1.$$

□

Lemma 2.2.6 (Hardy-Littlewood-Sobolev). *If $1 < p, q, r < \infty$, we have*

$$\|f * g\|_{L^r} \lesssim \|f\|_{L^p} \|g\|^*_{L^{q,\infty}},$$

if $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.

Proof. See Grafakos [9, Theorem 1.4.24]. □

Definition 2.2.2. *Let $1 \leq p, q \leq 1$. A mapping of functions T is of (strong) type (p, q) if*

$$\|Tf\|_{L^q(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)}$$

Similarly, for $q < \infty$, T is of weak type (p, q) if

$$\|Tf\|^*_{L^{q,\infty}(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)}.$$

Theorem 2.2.2 (The Marcinkiewicz Theorem). *Let (X, μ) and (Y, ν) be measure spaces and operator*

$$T \colon L^{p_j}(X, \mu) \longrightarrow L^{q_j,\infty}(Y, \nu),$$

of weak type (p_0, q_0) and weak type (p_1, q_1), with $1 \leq p_j, q_j \leq \infty$, $j = 0, 1$, and with $p_0 \neq p_1$ and $q_0 \neq q_1$. Then, for each $1 \leq r \leq \infty$, we have

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim_{p_0,q_0,p_1,r,\theta} \|f\|^*_{L^{p_\theta,r}} ,$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 < \theta < 1.$$

Proof. See Linares-Ponce [15]. □

2.2.2 Sobolev space $H^{s,p}(\mathbb{R}^N)$

Denote by $g(\Delta) = (1-\Delta)^{\frac{s}{2}}$, for sufficiently regular functions, the Bessel operator

$$\widehat{g(\Delta)f}(\xi) = g(-4\pi^2|\xi|^2)\widehat{f}(\xi) = (1 + 4\pi^2|\xi|^2)^{\frac{s}{2}}\widehat{f}(\xi).$$

In what follows, we denote $\Lambda^s f(x) = ((1 + 4\pi^2|\xi|^2)^2\widehat{f}(\xi))^\vee(x)$. We can define the inhomogeneous Sobolev space.

Definition 2.2.3. *Let $s \in \mathbb{R}$, we define the Sobolev space of order s by*

$$H^{s,p}(\mathbb{R}^N) = \left\{ f \in \mathcal{S}'(\mathbb{R}^N) : \Lambda^s f(x) = ((1 + 4\pi^2|\xi|^2)^{\frac{s}{2}}\widehat{f}(\xi))^\vee(x) \in L^p(\mathbb{R}^N) \right\},$$

with norm $\|\cdot\|_{H^{s,p}}$, defined as:

$$\|f\|_{H^{s,p}} = \|\Lambda^s f\|_{L^p} .$$

When $s = 0$, the Sobolev space $H^{0,p}$ coincides with the Lebesgue space L^p. Additionally, when $p = 2$, we denote $H^{s,2}$ as H^s.

Proposition 2.2.5 ([15]). *Let $s, r \in \mathbb{R}$.*

1. *If $s < r$, then $H^r(\mathbb{R}^N) \subsetneq H^s(\mathbb{R}^N)$.*
2. *$H^s(\mathbb{R}^N)$ is a Hilbert space with respect to the inner product $\langle \cdot \rangle_s$ defined as follows: If $f, g \in H^s(\mathbb{R}^N)$, then $\langle f, g\rangle_s = \int_{\mathbb{R}^N} \Lambda^s f(\xi)\overline{\Lambda^s g(\xi)}d\xi$.*

Theorem 2.2.3 (Sobolev Embedding). *Let $s \geq 0$,*

1. *if $1 \leq p \leq \infty$, then $H^{s,p}(\mathbb{R}^N)$ is continuosly embedded in $L^p(\mathbb{R}^N)$.*
2. *If $s < \frac{N}{p}$ and $1 < p \leq q \leq \frac{Np}{N-sp}$, then $H^{s,p}(\mathbb{R}^N)$ is continuosly embedded in $L^q(\mathbb{R}^N)$.*
3. *If $s = \frac{N}{p}$ and $1 < p \leq q < \infty$, then $H^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$.*
4. *If $s > \frac{N}{2} + k$, then $H^s(\mathbb{R}^N)$ is continuosly embedded in $C^k_\infty(\mathbb{R}^N)$, the space of functions with continuous derivatives vanishing at infinity.*

Proof. See Demengel [6, Proposition 4.18] and Linares-Ponce [15, Theorem 3.2]. □

Chapter 3

The Linear Schrödinger Equation.

In this section, we present the properties of the linear Schrödinger equation:

$$\begin{cases} \partial_t u = i\Delta u \\ u(x,0) = u_0(x) \end{cases} \tag{3.1}$$

By taking Fourier transforms, we observe that

$$\widehat{\partial_t u}(t,\xi) = \widehat{i\Delta u}(\xi) = -4\pi^2|\xi|^2\widehat{u}(\xi),$$

If $w_\xi(t) = \widehat{u}(t,\xi)$, it reduces to solving the following ordinary differential equation

$$e^{4\pi^2 it|\xi|^2}\frac{d}{dt}w_\xi(t) + 4\pi i|\xi|^2 e^{4\pi^2 it|\xi|^2}w_\xi(t) = 0,$$

which is equivalent to

$$\frac{d}{dt}\left(e^{4\pi^2 it|\xi|^2}w_\xi\right) = 0,$$

where the solution is given by

$$w_\xi(t) = c(\xi)e^{-4\pi^2 it|\xi|^2},$$

and with the initial condition, we obtain $\widehat{u_0} = c(\xi)$, so the solution yields

$$\widehat{u}(t,\xi) = e^{-4\pi^2 it|\xi|^2}\widehat{u_0}(\xi).$$

Thus

$$u(t,x) = (e^{-4\pi^2 it|\xi|^2}\widehat{u_0})^\vee = \left(e^{-4\pi^2 it|\xi|^2}\right)^\vee * u_0. \tag{3.2}$$

To calculate the inverse Fourier transform of the exponential function above, we refer to Linares and Ponce [15, Example 1.11], obtaining

$$u(t,x) = \frac{e^{-\frac{|\cdot|^2}{4it}}}{(4\pi it)^{N/2}} * u_0 = \frac{1}{(4\pi it)^{N/2}}\int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4it}}u_0(y)dy.$$

It is common to denote the solution of (3.1) as $u(x,t) = e^{it\Delta}u_0(x)$.

If u is a solution of the equation (3.1), then also are:

(i) $u_1(x,t) = e^{i\theta}u(x,t)$, $\theta \in \mathbb{R}$ fixed,

(ii) $u_2(x,t) = u(x-x_0, t-t_0)$, with $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$ fixed,

(iii) $u_3(x,t) = u(Ax,t)$, with A any orthogonal matrix $N \times N$,

(iv) $u_4(x,t) = u(x-2x_0t, t)e^{i(x\cdot x_0 - |x_0|^2 t)}$ with $x_0 \in \mathbb{R}^N$ fixed,

(v) $u_5(x,t) = \lambda^{N/2}u(\lambda x, \lambda^2 t)$, $\lambda > 0$,

(vi) $u_6(x,t) = \frac{1}{(\alpha+\omega t)^{n/2}} e^{\frac{i\omega|x|^2}{4(\alpha+\omega t)}} u\left(\frac{x}{\alpha+\omega t}, \frac{\gamma+\theta t}{\alpha+\omega t}\right)$, $\alpha\theta - \omega\gamma = 1$,

(vii) $u_7(x,t) = \overline{u(x,-t)}$.

Most of these properties can be verified through straightforward calculations or follow directly from the definition. We explicitly prove the scaling property (v):

$$\partial_t u_5 = \lambda^{\frac{N+4}{2}} \partial_t u(\lambda x, \lambda^2 t),$$

and

$$\Delta u_5 = \sum_{j}^{N} \partial_{x_j}^2 u_5 = \sum_{j=1}^{N} \lambda^{N/2+2} \partial_{x_j}^2 u(\lambda x, \lambda^2) = \lambda^{\frac{N+4}{2}} \Delta u(\lambda x, \lambda^2 t),$$

so,

$$\partial_t u_5 - i\Delta u_5 = \lambda^{N/2+2}\left(\partial_t u(\lambda x, \lambda^2 t) - i\Delta u(\lambda x, \lambda^2, t)\right) = 0.$$

This proves the property (v).

Proposition 3.0.1. *Let* $\left\{e^{it\Delta}\right\}_{t=-\infty}^{\infty}$ *be a family of operators satisfying the following properties:*

(i) For all $t \in \mathbb{R}$, $e^{it\Delta} \colon L^2(\mathbb{R}^N) \longmapsto L^2(\mathbb{R}^N)$ *is an isometry, which implies*

$$\left\|e^{it\Delta} f\right\|_2 = \|f\|_2 .$$

(ii) $(e^{it\Delta})^{-1} = e^{-it\Delta}$, *and* $e^0 = I$.

(iii) Fixing $f \in L^2(\mathbb{R})$, *the function defined as* $\phi_f \colon \mathbb{R} \longmapsto L^2(\mathbb{R}^N)$, *where* $\phi_f(t) = e^{it\Delta} f$ *is a continuous function.*

Proof. To prove (i), we use the Plancherel theorem:

$$\|\widehat{e^{it\Delta} f}\|_{L^2} = \|e^{-4\pi^2 it|\cdot|^2}\widehat{f}\|_{L^2} = \|\widehat{f}\|_{L^2} = \|f\|_{L^2} .$$

Next, for (ii):

$$\begin{aligned} \widehat{e^{it\Delta}(e^{-it\Delta} f)}(\xi) &= e^{-4\pi^2 it|\xi|^2}\widehat{e^{-it\Delta} f}(\xi) \\ &= e^{-4\pi^2 t|\xi|^2} e^{4\pi^2 t|\xi|^2}\widehat{f}(\xi) \\ &= \widehat{f}(\xi), \end{aligned}$$

and it is evident that $e^{i0\Delta} = I$.

Finally, for (iii), let $h \in \mathbb{R}$. We have:

$$\begin{aligned}\|\phi_f(t+h)-\phi(t)\|_{L^2}^2 &= \left\|\widehat{e^{i(t+h)\Delta} f} - \widehat{e^{it\Delta} f}\right\|_{L^2}^2 \\ &= \left\|e^{-4\pi^2 it|\cdot|^2}\widehat{f}\left(e^{-4\pi^2 ih|\cdot|^2}-1\right)\right\|_{L^2}^2 \\ &= \int |\widehat{f}(\xi)|^2\left(e^{-4\pi^2 ih|\cdot|^2}-1\right)^2 d\xi.\end{aligned}$$

Let $g_h(\xi) = |\widehat{f}(\xi)|\left(e^{-4\pi^2 ih|\cdot|^2}-1\right)$. Note that $|g_h(\xi)| \leq 4|f(\xi)|$. As $h \to 0$, $g_h \to 0$ almost everywhere. By the Dominated Convergence Theorem:

$$\lim_{h\to 0}\int g_h(\xi)d\xi = \int \lim_{h\to 0} g_h(\xi)d\xi = 0,$$

thereby proving (iii). □

Theorem 3.0.1 (Stone's Theorem)**.** *The family of operators* $\{T_t\}_{t=-\infty}^{\infty}$ *defined on the Hilbert space* H *is a unitary group of operators if and only if there exists a self-adjoint operator* A *on* H *such that*

$$T_t = e^{itA}$$

in the following sense: $D(A)$ *the domain of the operator* A*, which is a dense subespace of* H*; if* $f \in D(A)$*, then we have*

$$\lim_{t\to 0}\frac{T_t f - f}{t} = iAf.$$

Remark 3.0.1. If the operator A is the Laplacian Δ and $T_t = e^{it\Delta}$ with $D(A) = H^2(\mathbb{R}^N)$, it follows:

$$\begin{aligned}\left(\lim_{t\to 0}\frac{\widehat{e^{it\Delta} f - f}}{t}\right)^{\vee} &= \left(\lim_{t\to 0}\frac{e^{-4\pi^2 it|\cdot|^2}\widehat{f}-\widehat{f}}{t}\right)^{\vee} \\ &= \left(\lim_{t\to 0}\frac{(e^{-4\pi^2 it|\xi|^2}-1)\widehat{f}(\xi)}{t}\right)^{\vee} \\ &= (-4\pi^2 i|\xi|^2\widehat{f}(\xi))^{\vee} = i\Delta f,\end{aligned}$$

almost everywhere. This make sense if $f \in H^2(\mathbb{R}^N)$. The operator $i\Delta$ is called the infinitesimal generator of the unitary group $\left\{e^{it\Delta}\right\}_{t=-\infty}^{\infty}$.

Proposition 3.0.2. *If* $t \neq 0$, $\frac{1}{p'} + \frac{1}{p} = 1$ *and* $p' \in [1,2]$*, then* $e^{it\Delta}\colon L^{p'}(\mathbb{R}^n) \to L^p(\mathbb{R}^N)$ *is continuous and*

$$\left\|e^{it\Delta} f\right\|_p \leq c|t|^{-N/2(1/p'-1/p)}\|f\|_{p'}. \tag{3.3}$$

Proof. By proposition 3.0.1

$$e^{it\Delta}\colon L^2(\mathbb{R}^N) \longmapsto L^2(\mathbb{R}^N)$$

is an isometry, that's

$$\left\|e^{it\Delta} f\right\|_2 = \|f\|_2.$$

Furthermore, Young's inequality:

$$\left\|e^{it\Delta}f\right\|_{\infty} = \left\|\frac{e^{i|\cdot|^2/4t}}{(4\pi it)^{N/2}} * f\right\|_{\infty} \leq \left\|\frac{e^{i|\cdot|^2/4t}}{(4\pi it)^{N/2}}\right\|_{\infty} \|f\|_1 \lesssim |t|^{-N/2}\|f\|_1 .$$

Therefore, using Riesz-Thorin Interpolation, the operator

$$e^{it\Delta}\colon L^{p_\theta}(\mathbb{R}^N) \longmapsto L^{q_\theta}(\mathbb{R}^N)$$

is bounded, it's implies that is continuous. Furthermore, we have the inequality

$$\left\|e^{it\Delta}f\right\|_{L^{q_\theta}} \lesssim 1^{1-\theta}\left(t^{-\frac{N}{2}}\right)^{\theta} \lesssim |t|^{-\frac{N}{2}\left(1-\frac{2}{q_\theta}\right)}\|f\|_{p_\theta} = |t|^{N/2\left(\frac{1}{q_\theta}-\frac{1}{p_\theta}\right)}\|f\|_{p_\theta}. \tag{3.4}$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{2} + \theta = \frac{1+\theta}{2}, \quad and \quad \frac{1}{q_\theta} = \frac{1-\theta}{2},\ 0 \leq \theta \leq 1.$$

Take θ such that $q_\theta = p$, it follows

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

□

This result indicate that if $f \in L^2(\mathbb{R}^N)$ decreases fast enough when $|x| \to \infty$ such that $f \in L^1(\mathbb{R}^N)$, $e^{it\Delta}f$ is bounded.

Proposition 3.0.3. *(i) Given $t_0 \neq 0$ and $p > 2$, there exist $f \in L^2(\mathbb{R}^N)$ such that $e^{it_0\Delta}f \notin L^p(\mathbb{R}^N)$.*

(ii) Let $s > s'$ and $f \in H^s(\mathbb{R}^N)$ such that $f \notin H^{s'}(\mathbb{R}^N)$. Then, for all $t \in \mathbb{R}$, $e^{it\Delta}f \in H^s(\mathbb{R}^N)$ and $e^{it\Delta}f \notin H^{s'}(\mathbb{R}^N)$.

Proof. Consider $g(x) = |x|^{\frac{1}{2}-\varepsilon}\chi_{B(0,1)}$, so that $g \in L^2(\mathbb{R}^N)$ and $g \notin L^p(\mathbb{R}^N)$, thus

$$\|f\|_{L^2} = \left\|e^{-it\Delta}g\right\|_{L^2} = \|g\|_{L^2} < +\infty \quad \text{and}\ \|f\|_{L^p} = \left\|e^{-it\Delta}g\right\|_{L^p} = \|g\|_{L^p} = +\infty,$$

which proves (i).

Now for (ii), using Plancharel

$$\begin{aligned}\left\|e^{it\Delta}f\right\|_{H^s} &= \left\|\Lambda^s(e^{it\Delta}f)\right\|_{L^2} \\ &= \left\|\widehat{\Lambda^s(e^{it\Delta}f)}\right\|_{L^2} \\ &= \left\|(1+4\pi^2|\xi|^2)^{\frac{s}{2}}\widehat{e^{it\Delta}f}\right\|_{L^2} \\ &= \left\|(1+4\pi^2|\xi|^2)^{\frac{s}{2}}(e^{-4\pi^2 it|\xi|^2}\widehat{f})\right\|_{L^2} \\ &= \left\|\widehat{\Lambda^s f}\right\|_{L^2} \\ &= \|f\|_{H^s} .\end{aligned}$$

Consequently, if $e^{it\Delta}f \in H^{s_0}$, then $f = e^{-it\Delta}(e^{it\Delta})f \in H^{s_0}$. Now, assuming $e^{it\Delta}f \in H^{s'}(\mathbb{R}^N)$, implies $f \in H^{s'}(\mathbb{R}^N)$, which is contradiction to the assumption that $f \notin H^{s'}(\mathbb{R}^N)$.

□

3.1 Strichartz Estimates for Schrödinger.

Definition 3.1.1. *If $N \geq 1$ and $s \in (-1,1)$, the pair (q,r) is called H^s admissible if it satisfies*

$$\frac{2}{q} = \frac{N}{2} - \frac{N}{r} - s, \tag{3.5}$$

where $2 \leq q, r < \infty$, and $(q,r,N) \neq (2,\infty,2)$.

If $s = 0$, we say that (q,r) is L^2-admissible.

Definition 3.1.2. *Given $N \geq 3$ and $s \in (0,1)$, consider the set*

$$\begin{aligned}
\mathcal{A}_0 &= \left\{ (q,r) \text{ is } L^2 \text{ admissible} \;\middle|\; 2 \leq r \leq \frac{2N}{N-2} \right\},\\
\mathcal{A}_s &= \left\{ (q,r) \text{ is } \dot{H}^s \text{ admissible} \;\middle|\; \left(\frac{2N}{N-2s}\right)^+ \leq r \leq \left(\frac{2N}{N-2}\right)^- \right\},\\
\mathcal{A}_{-s} &= \left\{ (q,r) \text{ is } \dot{H}^{-s} \text{ admissible} \;\middle|\; \left(\frac{2N}{N-2s}\right)^+ \leq r \leq \left(\frac{2N}{N-2}\right)^- \right\}.
\end{aligned}$$

Here, β^+ is a fixed number defined as $\beta^+ = \beta + \varepsilon$, where $\varepsilon > 0$ is a sufficiently small positive value. Similarly, $\beta^- = \beta - \varepsilon$. We define the following Strichartz norm

$$\|u\|_{S(\dot{H}^s)} = \sup_{(q,r)\in\mathcal{A}_s} \|u\|_{L_t^q L_x^r},$$

and the dual Strichartz norm

$$\|u\|_{S'(\dot{H}^{-s})} = \inf_{(q,r)\in\mathcal{A}_{-s}} \|u\|_{L_t^{q'} L_x^{r'}}.$$

Note that, if $s = 0$ then $S(\dot{H}^0) = S(L^2)$ and $S'(\dot{H}^{-0}) = S'(L^2)$.

Lemma 3.1.1. *Let $\mathcal{H}$ be a Hilbert space, X a Banach space, X^* the dual space of X, and D a vector space densely contained in X. let $T \in \mathcal{L}(\mathcal{H}, D)$ and $T^* \in \mathcal{L}(D^*, \mathcal{H})$ be its adjoint, defined by*

$$\langle T^* f, g\rangle_{\mathcal{H}} = \langle f, Tg\rangle_D \quad \text{for all } f, g \in \mathcal{H}.$$

Then the following three conditions are equivalent.

1. $\|Tf\|_X \lesssim \|f\|_{\mathcal{H}}$.
2. $\|T^* g\|_{\mathcal{H}} \lesssim \|g\|_{X^*}$.
3. $\|TT^* g\|_X \lesssim \|g\|_{X^*}$.

Proof. First, we show that 2 implies 1, we have

$$\begin{aligned}
\|Tf\|_X &= \sup_{\|g\|_{X^*}=1} |\langle Tf, g\rangle_X|\\
&= \sup_{\|g\|_{X^*}=1} |\langle f, T^* g\rangle_{\mathcal{H}}|\\
&\leq \sup_{\|g\|_{X^*}=1} \|f\|_{\mathcal{H}} \|T^* g\|_{\mathcal{H}} \qquad (3.6)\\
&\lesssim \sup_{\|g\|_{X^*}=1} \|f\|_{\mathcal{H}} \|g\|_{X^*} \qquad (3.7)\\
&= \|f\|_{\mathcal{H}}.
\end{aligned}$$

1 implies 2

$$\begin{aligned}\|T^*g\|_{\mathcal{H}} &= \sup_{\|f\|_{\mathcal{H}}=1} |\langle T^*g, f\rangle_{\mathcal{H}}| \\ &= \sup_{\|f\|_{\mathcal{H}}=1} |\langle g, Tf\rangle_D| \\ &\leq \|g\|_{X^*} \|Tf\|_X \\ &\lesssim \sup_{\|f\|_{\mathcal{H}}=1} \|g\|_{X^*} \|f\|_{\mathcal{H}} \\ &= \|g\|_{X^*}.\end{aligned}$$

3 implies 2

$$\begin{aligned}\|T^*g\|_{\mathcal{H}}^2 &= |\langle T^*g, T^*g\rangle_{\mathcal{H}}| \\ &= |\langle g, TT^*g\rangle_D| \\ &\leq \|g\|_{X^*} \|TT^*g\|_X \\ &\lesssim \|g\|_{X^*}^2\end{aligned}$$

2 implies 3

$$\begin{aligned}\|TT^*f\|_X &= \sup_{\|g\|_{X^*}=1} |\langle TT^*f, g\rangle_X| \\ &= \sup_{\|g\|_{X^*}=1} |\langle T^*f, T^*g\rangle_{\mathcal{H}}| \\ &\leq \sup_{\|g\|_{X^*}=1} \|T^*f\|_{\mathcal{H}} \|T^*g\|_{\mathcal{H}} \\ &\lesssim \|f\|_{X^*} \|g\|_{X^*} \\ &= \|f\|_{X^*}.\end{aligned}$$

□

Lemma 3.1.2 (Christ-Kiselev lemma, [20]). *Let X, Y be Banach spaces, let I be a time interval, and let $K \in C^0(I \times I \to \mathcal{L}(X, Y))$ be a kernel taking values in the space of bounded operators for $X \times Y$. Suppose that $1 \leq p < q \leq \infty$ is such that*

$$\left\| \int_I K(t,s) f(s) ds \right\|_{L_t^q(I \to Y)} \leq A \|f\|_{L_t^p(I \to X)}$$

for all $f \in L_t^p(I \to X)$ and some $A > 0$. Then one also has

$$\left\| \int_{s \in I : s < t} K(t,s) f(s) ds \right\|_{L_t^p(I \to Y)} \lesssim_{p,q} A \|f\|_{L_t^p(I \to X)}.$$

Theorem 3.1.1 (Strichartz estimates). *The group $\left\{e^{it\Delta}\right\}_{i=1}^n$ satisfies:*

1.
$$\left\| e^{it\Delta} f \right\|_{L_t^q L_x^p} \lesssim \|f\|_{L_x^2} \tag{3.8}$$

2.
$$\left\| \int_{-\infty}^{\infty} e^{it\Delta} g(x,t)dt \right\|_{L^2_x} \lesssim \|g\|_{L_t^{q'} L_x^{p'}} \tag{3.9}$$

3.
$$\left\| \int_{-\infty}^{\infty} e^{i(t-s)\Delta} g(x,s)ds \right\|_{L_t^q L_x^p} \lesssim \|g\|_{L_t^{q'} L_x^{p'}} . \tag{3.10}$$

4.
$$\left\| \int_{s}^{t} e^{i(t-s)\Delta} g(x,s)ds \right\|_{L_t^q L_x^p} \lesssim \|g\|_{L_t^{q'} L_x^{p'}} . \tag{3.11}$$

Proof. Let $T := e^{it\Delta}$. We rely on the fact that $e^{it\Delta}$ forms a unitary group on L^2. Through some straightforward computation

$$\begin{aligned}\langle e^{it\Delta} f, g\rangle_{L_t^\infty L_x^2} &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^N} e^{-4\pi^2 it|\xi|^2} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi dt = \int_{-\infty}^{\infty}\int_{\mathbb{R}^N} \overline{e^{-4\pi^2 it|\xi|^2}\widehat{g}(\xi)}\widehat{f}(\xi)d\xi \\ &= \int_{\mathbb{R}^N} f(x)\overline{\int_{-\infty}^{\infty} e^{-it\Delta} g(x,t)dt}dx = \langle f, T^* g\rangle_{L^2}.\end{aligned}$$

Thus, we find that its adjoint T^* is given by:

$$T^* g = \int_{-\infty}^{\infty} e^{-is\Delta} g(s,x)ds \tag{3.12}$$

In particular

$$TT^* g(t,x) = e^{it\Delta}\int_{-\infty}^{+\infty} e^{-is\Delta} g(s,x)ds = \int_{-\infty}^{+\infty} e^{i(t-s)\Delta} g(s,x)ds.$$

We seek to prove that

$$TT^* \colon L_t^{q'} L_x^{r'} \longrightarrow L_t^q L_x^r,$$

with (p,q) L^2 admissible. We use the dispersive estimate (3.3) and Lemma 2.2.6

$$\begin{aligned}\|TT^*\|_{L_t^q L_x^p} &\le \left\| \int_{-\infty}^{\infty} \left\| e^{i(t-s)\Delta} g(s,x)\right\|_{L_x^p} ds \right\|_{L_t^q} \\ &\lesssim \left\| \int_{-\infty}^{+\infty} |t-s|^{-\frac{N}{2}\left(1-\frac{2}{p}\right)} \|g(s,x)\|_{L_x^{p'}} ds \right\|_{L_t^q} \\ &= \left\| |\cdot|^{-\frac{2}{q}} * \|g(\cdot,x)\|_{L_x^{p'}}(t) \right\|_{L_t^q} \\ &\lesssim \left\| |\cdot|^{-\frac{2}{q}} \right\|_{L_t^{a,\infty}} \left\| \|g(\cdot,x)\|_{L_x^{p'}} \right\|_{L_t^b},\end{aligned}$$

Where

$$1 + \frac{1}{q} = \frac{1}{a} + \frac{1}{b},$$

We have that, $|\cdot|^{-\frac{2}{q}} \in L^{a,\infty}$ if and only if $\frac{2a}{q} = 1$, thus $b = q'$. Thereby

$$\|TT^*\|_{L_t^q L_x^r} \leq \left\| |\cdot|^{-\frac{2}{q}} \right\|_{L_t^{\frac{q}{2},\infty}} \|g\|_{L_t^{q'} L_x^{p'}} \lesssim \|g\|_{L_t^{q'} L_x^{r'}},$$

it proves (3.10). So, by the Lemma 3.1.1, with $X = L_t^q L_x^p$ and $\mathcal{H} = L^2$, we deduce (3.8) and 3.9. If $X = L^{p'}$ and $Y = L^p$, by 3.10 along with Christ Kiselev Lemma 3.1.2, (3.11)follows. □

Collorary 3.1.1. *Let be (p_0, q_0) and (p_1, q_1) be L^2 admissible. Then for all , $T > 0$ we have*

$$\left\| \int_0^t e^{i(t-s)\Delta} g(\cdot, s) dt \right\|_{L^{q_0}_{t([0,T])} L_x^{p_0}} \lesssim \|g(\cdot, t)\|_{L^{q_1'}_{t([0,T])} L_x^{p_1'}}$$

Proof. See Linares-Ponce [15, Corollary 4.1]. □

More generally, [13][Theorem 2.1], shows that

$$\left\| \int_a^b e^{i(t-s)\Delta} g(\cdot, s) ds \right\|_{S(\dot{H}^s, \mathbb{R})} \lesssim \|g\|_{S(\dot{H}^{-s}, [a,b])}. \tag{3.13}$$

Chapter 4

Well-Posedness

4.1 The Nonlinear Schrödinger Equation

In this section, we consider the Cauchy problem for the nonlinear Schrödinger equation (NLS)

$$\begin{cases} i\partial_t u = -\Delta u - \lambda |u|^{\alpha-1}u = 0, & x \in \mathbb{R}^N, \quad t > 0 \\ u(x,0) = u_0(x) \end{cases} \tag{4.1}$$

where λ and α are real constants with $\alpha > 1$, and $u\colon \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{C}$, $N \geq 3$. Based on Chapters 4 and 5, of Linares and Ponces [15], we shall study the local and global well-posedness in L^2 and $H^1(\mathbb{R}^N)$.

If $u \in \mathcal{C}^1([0,T] : \mathcal{C}^2(\mathbb{R}^N))$ is a solution of NLS (4.1) and we define $\varphi(s) = e^{i(t-s)\Delta}u$, then, by differentation and Theorem 3.0.1, we have:

$$\begin{aligned} \frac{d}{ds}\varphi(s) &= \lim_{h\to 0} \frac{e^{i(t-s)\Delta}u(x,s+h) - e^{i(t-s)\Delta}u(x,s)}{h} \\ &= \lim_{h\to 0} \left\{ e^{i(t-(s+h))\Delta} \left(\frac{u(x,s+h) - u(x,s)}{h} - \frac{e^{ih\Delta}u(x,s) - u(x,s)}{h} \right) \right\} \\ &= e^{i(t-s)\Delta}(\partial_s u - i\Delta u) \\ &= i\lambda e^{i(t-s)\Delta}|u|^{\alpha-1}u. \end{aligned}$$

Finally, we have

$$\int_0^t \frac{d}{ds}\varphi(s)ds = u(t) - e^{it\Delta}u_0 = i\lambda \int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds, \tag{4.2}$$

thus

$$u(t) = e^{it\Delta}u_0 + i\lambda \int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds. \tag{4.3}$$

Therefore, the integral equation (4.3) is a solution of (4.1). Now

$$\begin{aligned}
\partial_t u(t) &= \lim_{h\to 0} \frac{u(t+h)-u(t)}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \left\{ (e^{i(t+h)\Delta}u_0 - e^{it\Delta}u_0) + i\lambda \int_0^t (e^{i(t+h-s)\Delta} - e^{i(t-s)\Delta})(|u|^{\alpha-1}u)(s)ds \right\} \\
&+ \lim_{h\to 0} \frac{1}{h} i\lambda \int_t^{t+h} e^{(t+h-s)\Delta}(|u|^{\alpha-1}u)(s)sds \\
&= i\Delta e^{it\Delta}u_0 + i^2\lambda\Delta \int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds + i\lambda(|u|^{\alpha-1}u)(t) \\
&= i\Delta \left(e^{it\Delta}u_0 + i\lambda\Delta \int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds \right) + i\lambda(|u|^{\alpha-1}u)(t) \\
&= i\Delta u(t) + i\lambda(|u|^{\alpha-1}u)(t).
\end{aligned} \tag{4.4}$$

The above shows that the NLS problem is equivalent to integral solution (4.3).

Definition 4.1.1. *We say that the problem* (4.1) *is locally well-posed in X if, for any $u_0 \in X$, there exists $T > 0$ and a unique solution $u \in \mathcal{C}([-T,T] : X) \cap Y$ of* (4.3)*, where Y is an auxiliar space. Furthermore the map $u_0 \mapsto u$ is continuous from X to $\mathcal{C}([-T,T] : X)$.*
If T is arbitrarily large we say that (4.3) *is globally well-posed in X.*

Next, we shall study the local and global well-posedness of the initial value problem (4.1) in $L^2_x(\mathbb{R}^N)$ and $H^1_x(\mathbb{R}^N)$.

Remark 4.1.1. From now on, whenever I is used, it will refer to the interval $[-T,T]$.

4.2 L^2 Theory

In what follows, we shall show the following standard estimates.

Lemma 4.2.1. *(a) Let $F\colon \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$ defined by $F(z) = |z|^{\alpha-1}z$ with $\alpha > 1$. Then,*

$$||z|^{\alpha-1}z - |w|^{\alpha-1}w| \lesssim (|z|^{\alpha-1} + |w|^{\alpha-1})|z-w|. \tag{4.5}$$

(b) Let $N \geq 3$, $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, and $1 < \alpha < 1 + \frac{4}{N}$. Then,

$$\left\| |u|^{\alpha-1}u \right\|_{S'(L^2,[-T,T])} \lesssim T^\theta \left\| u \right\|^\alpha_{S(L^2,[-T,T])},$$

where $\theta = 1 - \frac{N(\alpha-1)}{4}$.

Proof. Straightforward computations reveal that the complex derivative of F is

$$F_z(z) = \frac{\alpha+1}{2}|z|^{\alpha-1} \quad \text{and} \quad F_{\bar{z}}(z) = \frac{\alpha-1}{2}|z|^{\alpha-3}z^2.$$

Moreover, by the fundamental theorem of calculus and the mean value theorem, we obtain

$$\begin{aligned}|F(z)-F(w)| &\leq \left|\int_0^1 F_z(w+t(z-w))(z-w)+F_{\bar{z}}(w+t(z-w))\overline{(z-w)}dt\right| \\ &\lesssim \left[\sup_{t\in[0,1]}\left(|F_z(w+t(z-w))|+|F_{\bar{z}}(w+t(z-w))\right)\right]|z-w| \\ &\lesssim \sup_{t\in[0,1]}|(1-t)w+tz|^{\alpha-1}|z-w| \\ &\lesssim (|w|^{\alpha-1}+|z|^{\alpha-1})|z-w|,\end{aligned}$$

proving (a).

Let $(\hat{q},\hat{r})\in\mathcal{A}_0$. By Hölder, we have

$$\left\||u|^{\alpha-1}u\right\|_{L_x^{\hat{r}'}} \leq \|u\|^{\alpha-1}_{L_x^{(\alpha-1)\hat{r}_1}}\|u\|_{L_x^{\hat{r}}},$$

if $(\alpha-1)\hat{r}_1=\hat{r}$, then

$$\frac{1}{\hat{r}}=\frac{\alpha-1}{\hat{r}}+\frac{1}{\hat{r}},$$

this implies that $\hat{r}=\alpha+1$, and it satisfies the Definition 3.1.1, i.e.,

$$\frac{2}{\hat{q}}=\frac{N}{2}-\frac{N}{\hat{\alpha}+1},$$

this revels that $\hat{q}=\frac{4(\alpha+1)}{N(\alpha-1)}$.

Furthermore, since that $1<\alpha<1+\frac{4}{N}$, we have:

$$2<\alpha+1<\frac{2N+4}{N}<\frac{2N}{N-2}.$$

Therefore, by Hölder inequality on the time variable

$$\begin{aligned}\left\||u|^{\alpha-1}u\right\|_{L^{\hat{q}}L_x^{\hat{r}}([-T,T])} &\leq \left\|\|u\|^{\alpha}_{L_x^{\alpha+1}}\right\|_{L_t^{\hat{q}'}([-T,T])} \\ &= \left\|\|u\|_{L_x^{\alpha+1}}\right\|^{\alpha}_{L_t^{\alpha\hat{q}'}([-T,T])} \\ &\leq \|1\|^{\alpha}_{L_t^{\hat{q}_1}([-T,T])}\left\|\|u\|_{L_x^{\alpha+1}}\right\|^{\alpha}_{L_t^{\hat{q}}([-T,T])} \\ &= T^{\theta}\|u\|^{\alpha}_{L_t^{\hat{q}}L_x^{\hat{r}}},\end{aligned}$$

with $\theta=\frac{\alpha}{\hat{q}_1}$, and

$$\frac{1}{\hat{q}_1}=\frac{1}{\alpha\hat{q}'}-\frac{1}{\hat{q}}=\frac{4-N(\alpha-1)}{4\alpha}>0,$$

if and only if, $1<\alpha<1+\frac{4}{N}$, thus $\alpha\hat{q}'<\hat{q}$.

□

Theorem 4.2.1. *If* $1<\alpha<1+\frac{4}{N}$*, then for each* $u_0\in L^2(\mathbb{R}^N)$ *there exist* $T=T(\|u\|_0,N,\lambda,\alpha)>0$ *and a unique solution* u *of the integral equation* (4.3) *in the time interval* $[-T,T]$ *with*

$$u\in\mathcal{C}\left([-T,T]:L^2\left(\mathbb{R}^N\right)\right)\cap L^r\left([-T,T]:L^{\alpha+1}\left(\mathbb{R}^N\right)\right), \tag{4.6}$$

where $r = \frac{4(\alpha+1)}{N(\alpha-1)}$.

Moreover, for all $T' < T$ *there exists a neighborhood* V *of* u_0 *in* $L^2(\mathbb{R}^N)$ *such that*

$$\mathbb{F}\colon \mapsto \mathcal{C}([-T',T'] : L^2(\mathbb{R}^N)) \cap L^r([-T',T'] : L^{\alpha+1}(\mathbb{R}^N)),$$

$\tilde{u}_0 \mapsto \tilde{u}(t)$*, is Lipschitz.*

Proof. We construct u to satisfy the Duhamel formula

$$\Phi_{u_0}(u)(t) := \phi(u)(t) = e^{it\Delta}u_0 + i\lambda \int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds. \tag{4.7}$$

Let us define the space

$$X := \mathcal{C}([-T,T] : L^2(\mathbb{R}^N)) \cap L^r([-T,T] : L^{\alpha+1}(\mathbb{R}^N)) \tag{4.8}$$

for $(\alpha+1, r)$ L^2-admissible, and

$$E_a(T) = \Big\{ v \in X : \|v\|_T = \|v\|_{L_t^\infty L_x^2(I)} + \|v\|_{L_t^r L_x^{\alpha+1}(I)} \leq a \Big\} \tag{4.9}$$

where $I = [-T,T]$ and a $T > 0$. The space X is complete with the metric

$$d(u,v) = \|u-v\|_X = \|u-v\|_{L_t^\infty L_x^2(I)} + \|u-v\|_{L_t^r L_x^{\alpha+1}(I)}, \tag{4.10}$$

and we shall show that Φ defines a contraction map on $E_a(T)$.

Utilizing Strichartz (3.8) and (3.9), we estimate

$$\begin{aligned} \|\Phi(u)\|_{L_t^\infty L_x^2(I)} &\leq \left\|e^{it\Delta}u_0\right\|_{L_t^\infty L_x^2(I)} + |\lambda| \left\| \int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds \right\|_{L_t^\infty L_x^2(I)} \\ &\leq c\|u_0\|_{L_x^2} + c|\lambda| \left\||u|^{\alpha-1}u\right\|_{L_t^{r'} L_x^{(\alpha+1)'}(I)} \end{aligned} \tag{4.11}$$

Making $r = \hat{q}$ and $\alpha + 1 = \hat{r}$ in the Lemma 4.2.1-(b) , we have

$$\left\||u|^{\alpha-1}u\right\|_{L_t^{r'} L_x^{(\alpha+1)}(I)} \leq T^\theta \|u\|^\alpha_{L_t^r L_x^{\alpha+1}(I)},$$

thus

$$\|\Phi(u)\|_{L_t^\infty L_x^2(I)} \leq c\|u_0\|_{L_x^2(I)} + c|\lambda| T^\theta \|u\|^\alpha_{L_t^r L_x^{\alpha+1}(I)}.$$

Then, if $u \in E_a(T)$ and fix $a = 2c\|u_0\|_{L_x^2}$, we have

$$\|\Phi(u)\|_{L_t^\infty L_x^2(I)} \leq c\|u_0\|_{L_x^2(I)} + 2^\alpha c^{\alpha+1}|\lambda| T^\theta \|u_0\|_{L_x^2(I)}$$

Likewise, if $u \in E_a(T)$ and fix $a = 2c\|u_0\|_{L_x^2}$, we have

$$\begin{aligned} \|\Phi(u)\|_{L_t^r L_x^{\alpha+1}(I)} &\leq \left\|e^{it\Delta}u_0\right\|_{L_t^r L_x^{\alpha+1}(I)} + \left\| \int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds \right\|_{L_t^r L_x^{\alpha+1}(I)} \\ &\leq c\|u_0\|_{L_x^2} + c\left\||u|^{\alpha-1}u\right\|_{L_t^{r'} L_x^{(\alpha+1)'}(I)} \\ &\leq c\|u_0\|_{L_x^2} + c|\lambda| T^\theta \|u\|^\alpha_{L_t L_x^{(\alpha+1)}(I)} \\ &\leq c\|u_0\|_{L_x^2} + 2^\alpha c^{\alpha+1}|\lambda| T^\theta \|u_0\|^\alpha_{L_x^2}. \end{aligned} \tag{4.12}$$

If we choose

$$T = \left(\frac{1}{2^{\alpha+1}\bar{c}|\lambda| \, \|u_0\|_{L_x^2}^{\alpha-1}} \right)^{\frac{1}{\theta}},$$

together (4.11) and (4.12), we have

$$\|\Phi(u)\|_T \le \bar{c}\, \|u_0\|_{L_x^2} + \bar{c} T^\theta a^\alpha \le a \tag{4.13}$$

Therefore $\Phi(u)$ is well defined, that is $\Phi(E_a(T)) \subseteq E_a(T)$.

To see that Φ is a contraction, we use Strichartz, Hölder, and Lemma 4.2.1-(a) to estimate as follows: for $u, v \in E_a(T)$,

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_{L_t^r L_x^{\alpha+1}(I)} &= |\lambda| \left\| \int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u - |v|^{\alpha-1}v)(s)ds \right\|_{L_t^r L_x^{\alpha+1}(I)} \\ &\le c|\lambda| \left\| |u|^{\alpha-1}u - |v|^{\alpha-1}v \right\|_{L_t^{r'} L_x^{(\alpha+1)'}(I)} \\ &\le c|\lambda| \left\| (|u|^{\alpha-1} - |v|^{\alpha-1})|u-v| \right\|_{L_t^{r'} L_x^{(\alpha+1)'}(I)} \\ &\le c|\lambda| \Big(\left\| |u|^{\alpha-1}|u-v| \right\|_{L_t^{r'} L_x^{(\alpha+1)'}(I)} + \left\| |v|^{\alpha-1}|u-v| \right\|_{L_t^{r'} L_x^{(\alpha+1)'}(I)} \Big) \\ &\le c|\lambda| \Big(\left\| \|u\|_{L_x^{\alpha+1}}^{\alpha-1} \|u-v\|_{L_x^{\alpha+1}} \right\|_{L_t^{r'}(I)} + \left\| \|u\|_{L_x^{\alpha+1}}^{\alpha-1} \|u-v\|_{L_x^{\alpha+1}} \right\|_{L_t^{r'}(I)} \Big), \end{aligned} \tag{4.14}$$

where

$$\frac{1}{(\alpha+1)'} = \frac{\alpha}{\alpha+1} = \frac{1}{\alpha+1} + \frac{\alpha-1}{\alpha+1}.$$

We estimate, by Hölder's inequality on the time-space

$$\begin{aligned} \left\| \|u\|_{L_x^{\alpha+1}}^{\alpha-1} \|u-v\|_{L_x^{\alpha+1}} \right\|_{L_t^{r'}(I)} &\le \left\| \|u\|_{L_x^{\alpha+1}}^{\alpha-1} \right\|_{L_t^a(I)} \|u-v\|_{L_t^r L_x^{\alpha+1}(I)} \|1\|_{L_t^q(I)} \\ &= T^{\frac{1}{q}} \|u\|_{L_t^r(I)} \|u-v\|_{L_t^r L_x^{\alpha+1}(I)}, \end{aligned}$$

where we choose

$$\frac{1}{r'} = \frac{1}{a} + \frac{1}{b} + \frac{1}{q}, \quad \text{with} \begin{cases} a &= \frac{r}{\alpha-1} \\ b &= r, \end{cases}$$

therefore

$$\frac{1}{q} = 1 - \frac{\alpha+1}{r} = 1 - \frac{N(\alpha-1)}{4} = \theta > 0,$$

since that $1 < \alpha < 1 + \frac{4}{N}$.

This implies that

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_{L^r_tL^{\alpha+1}_x(I)} &\le c|\lambda|\left(\|u\|^{\alpha-1}_{L^{\alpha+1}_x}+\|v\|^{\alpha-1}_{L^{\alpha+1}_x}\right)T^\theta\|u-v\|_{L^r_tL^{\alpha+1}_x(I)}\\ &\le c|\lambda|2^\alpha c^\alpha\|u_0\|^{\alpha-1}_{L^2_x}T^\theta\|u-v\|_{L^r_tL^{\alpha+1}_x(I)},\end{aligned}$$

analogously for the space $L^\infty_{t([T,T])}L^2_x$. Therefore, if $u\in E_a(T)$ and $a\le 2\bar{c}\|u_0\|_{L^2_x}$, we showed that $\Phi(u)$ is a contraction if

$$\|\Phi(u)-\Phi(v)\|_T\le \bar{c}|\lambda|2^{\alpha+1}\bar{c}^\alpha\|u_0\|^{\alpha-1}_{L^2_x}T^\theta\|u-v\|_T \tag{4.15}$$

assuming $c|\lambda|2^{\alpha+1}\bar{c}^\alpha\|u_0\|^{\alpha-1}_{L^2_x}T^\theta<1$, then by the Banach fixed point theorem there exist an unique solution $u\in E_a$ such that $\Phi(u)=u$.

Let $T'<T$, and define

$$V=\left\{v_0\in L^2(\mathbb{R}^N):\ \|v_0-u_0\|_{L^2_x}<\delta,\ \text{with } 0<\delta^\beta<\frac{1}{2^{\alpha+1}\bar{c}^{\alpha+1}|\lambda|}\left(\frac{1}{T'}-\frac{1}{T}\right)\right\},$$

where $\beta=\frac{4(1-\alpha)}{4-N(\alpha-1)}$.

Thus, if $v_0\in V$, such that v is a solution of (4.1) defined on the time interval $[-T(v_0),T(v_0)]$, with

$$T(v_0)^\theta=\frac{1}{2^{\alpha+1}\bar{c}^{\alpha+1}|\lambda|\,\|v_0\|^{\alpha-1}_{L^2_x}}.$$

Note that,

$$\|v_0\|_{L^2_x}\le\|v_0-u_0\|_{L^2_x}+\|u_0\|_{L^2_x}<\delta+\|u_0\|_{L^2_x}, \tag{4.16}$$

from which

$$\begin{aligned}\frac{1}{T(v_0)}=(2^{\alpha+1}\bar{c}^{\alpha+1}|\lambda|)^{\frac{1}{\theta}}\left[(\delta+\|u_0\|_{L^2_x})^{\alpha-1}\right]^{\frac{1}{\theta}} &\le (2^{\alpha+1}\bar{c}^{\alpha+1}|\lambda|)\left[\delta^\beta+\|u_0\|^\beta_{L^2_x}\right]\\ &\le (2^{\alpha+1}\bar{c}^{\alpha+1}|\lambda|)^{\frac{1}{\theta}}\delta^\beta+\frac{1}{T}<\frac{1}{T'},\end{aligned}$$

this shows that $T(v_0)>T'$.

Let $u(t)$ and $v(t)$ with intial data u_0 and v_0, respectivelty, then

$$u(t)-v(t)=e^{it\Delta}(u_0-v_0)-i\lambda\int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u-|v|^{\alpha-1}v)(s)ds.$$

The same argument used in (4.13) and (4.14) implies

$$\begin{aligned}\|u-v\|_{T'}&=\|\Phi_{u_0}(u)-\Phi_{v_0}(v)\|_{T'}\\ &\le c_1\|u_0-v_0\|_{L^2_x}+2^\alpha\bar{c}^\alpha|\lambda|\left(\|u_0\|^{\alpha-1}_{L^2_x}+\|v\|^{\alpha-1}_{L^2_x}\right)\|u-v\|_{T'}T'^{\frac{1}{q}},\end{aligned}$$

with $2^{\alpha+1}\bar{c}^{\alpha+1}|\lambda|(\|u_0\|^{\alpha-1}_{L^2_x}+\|v_0\|^{\alpha-1}_{L^2_x})T'^{\frac{1}{q}}<1$, we have that

$$\|u-v\|_{T'}\le K\|u_0-v_0\|_{L^2_x}$$

where

$$K:=\frac{c_1}{1-2^\alpha\bar{c}^\alpha|\lambda|(\|u_0\|^{\alpha-1}_{L^2_x}+\|v_0\|^{\alpha-1}_{L^2_x})T'^{\frac{1}{q}}}, \tag{4.17}$$

proving the continuous dependece. □

The existence of a local solution in the subcritical case depends on the size of the norm $\|u_0\|_{L^2_x}$. In Section 4.4, we will prove that, given the mass conservation, that is $\|u\|_{L^2_x} = \|u_0\|_{L^2_x}$, the solution can be extended to the entire interval $[-T, T]$.

Proposition 4.2.1. *Let (p, q) be an admissible pair. Given $u_0 \in L^2(\mathbb{R}^N)$ and $\varepsilon > 0$, there exist $\delta > 0$ and $T > 0$ such that if $\|v_0 - u_0\|_{L^2_x} < \delta$, then*

$$\left\|e^{it\Delta}v_0\right\|_{L^q_t L^p_x(I)} < \varepsilon. \tag{4.18}$$

Proof. Let $\widetilde{u_0} \in \mathcal{S}(\mathbb{R}^N)$ such that

$$\|u_0 - \widetilde{u_0}\|_{L^2_x} < \frac{\varepsilon}{3c}.$$

We choose $\delta < \frac{\varepsilon}{3c}$ and by Sobolev inequality with $s \geq \frac{N}{2} - \frac{N}{p}$, we have

$$\begin{aligned}\left\|e^{it\Delta}v_0\right\|_{L^q_t L^p_x(I)} &\leq \left\|e^{it\Delta}(u_0 - v_0)\right\|_{L^q_t L^p_x(I)} + \left\|e^{it\Delta}(u_0 - \widetilde{u_0})\right\|_{L^q_t L^p_x(I)} + \left\|e^{it\Delta}\widetilde{u_0}\right\|_{L^q_t L^p_x(I)} \\ &\leq c\,\|u_0 - v_0\|_{L^2_x} + c\,\|u_0 - \widetilde{u_0}\|_{L^2_x} + \left\|e^{it\Delta}\widetilde{u_0}\right\|_{L^q_t H^s_x(I)} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + cT^{\frac{1}{q}}\,\|\widetilde{u_0}\|_{H^s_x} < \varepsilon,\end{aligned} \tag{4.19}$$

so we set $cT^{\frac{1}{q}}\,\|u_0\|_{H^s_x} < \frac{\varepsilon}{3}$. □

Theorem 4.2.2. *If $\alpha = 1 + \frac{4}{N}$, then for each $u_0 \in L^2(\mathbb{R}^N)$ there exist $T = T(u_0, \lambda, \alpha) > 0$ and a unique solution u of the integral equation (4.3) in the time interval $[-T, T]$ with*

$$u \in \mathcal{C}\left([-T, T] : L^2\left(\mathbb{R}^N\right)\right) \cap L^\sigma\left([-T, T] : L^\sigma\left(\mathbb{R}^N\right)\right), \tag{4.20}$$

where $\sigma = 2 + \frac{4}{N}$.

Moreover, for all $T' < T$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^N)$ such that

$$\mathbb{F} \colon V \mapsto \mathcal{C}([-T, T] : L^2(\mathbb{R}^N)) \cap L^\sigma([-T, T] : L^\sigma(\mathbb{R}^N)),$$

$\widetilde{u}_0 \mapsto \widehat{u}, (t)$ is Lipschitz.

Proof. Let

$$X := \mathcal{C}\left([-T, T] : L^2\left(\mathbb{R}^N\right)\right) \cap L^\sigma\left([-T, T] : L^\sigma\left(\mathbb{R}^N\right)\right),$$

and define the set

$$\widetilde{E}_a(T) := \left\{v \in X \colon \|v\|_T = \left\|v - e^{it\Delta}u_0\right\|_{L^\infty_t L^2_x} + \|v\|_{L^\sigma_t L^\sigma} \leq a\right\}$$

equipped with the metric

$$d(u, v) = \|u - v\|_X = \|u - v\|_{L^\infty_t L^2_x(I)} + \|u - v\|_{L^r_t L^{\alpha+1}_x(I)}. \tag{4.21}$$

We see that $(\widetilde{E}_a(T), d)$ is a complete metric space. We shall show that $\Phi \colon \widetilde{E}_a(T) \longrightarrow \widetilde{E}_a(T)$ as in (4.7) defines a contraction.

Firstly, let us observe that (σ, σ) is admissible, since $2 \leq \sigma \leq \frac{2N}{N-2}$ and

$$\frac{N}{2} - \frac{N}{\sigma} = \frac{N}{2} - \frac{N^2}{2N + 4} = \frac{2}{\sigma},$$

then applying Strichartz estimate (3.8) and (3.9) together with Hölder's inequality, we have

$$\begin{aligned} \|\Phi(u)\|_{L_t^\sigma L_x^\sigma(I)} &\leq \left\|e^{it\Delta}u_0\right\|_{L_t^\sigma L_x^\sigma(I)} + |\lambda| \left\|\int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds\right\|_{L_t^\sigma L_x^\sigma(I)} \\ &\leq C\varepsilon + |\lambda| \, \||u|^\alpha\|_{L_t^{\sigma'} L_x^{\sigma'}(I)} \\ &= C\varepsilon + |\lambda| \, \|u\|^\alpha_{L_t^{\alpha\sigma'} L_x^{\alpha\sigma'}(I)} \\ &= C\varepsilon + c|\lambda| \, \|u\|^\alpha_{L_t^\sigma L_x^\sigma(I)} \,, \end{aligned} \tag{4.22}$$

where

$$\alpha\sigma' = \left(\frac{N+4}{N}\right)\left(\frac{2N+4}{N+4}\right) = 2 + \frac{4}{N} = \sigma.$$

In an analogously way

$$\begin{aligned} \left\|\Phi(u) - e^{it\Delta}u_0\right\|_{L_t^\infty L_x^2(I)} &= |\lambda| \left\|\int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds\right\|_{L_t^\infty L_x^2(I)} \\ &\leq c|\lambda| \, \|u\|^\alpha_{L_t^{\alpha\sigma'} L_x^{\alpha\sigma'}(I)} \\ &\leq c|\lambda| \, \|u\|^\alpha_{L_t^\sigma L_x^\sigma(I)} \,. \end{aligned} \tag{4.23}$$

By Proposition 4.2.1, inequalities (4.22) and 4.23, we obtain that given $\varepsilon > 0$, there exist $T > 0$ such that if $u \in \widetilde{E}_a(T)$, satisfy

$$\|\Phi(u)\|_T \leq c\varepsilon + c|\lambda|a^\alpha.$$

Therefore $c\varepsilon < \frac{a}{2}$ and $a < \left(\frac{1}{2|\lambda|c}\right)^{\alpha-1}$ imply that $\Phi(\widetilde{E}_a(T)) \subseteq \widetilde{E}_a(T)$. To show that $\Phi(\cdot)$ is a contraction, using the argument employed to show theorem 4.2.1, yields:

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_{L_t^\infty L_x^2(I)} &\leq c|\lambda| \left\|(|u|^{\alpha-1} + |v|^{\alpha-1})|u-v|\right\|_{L_t^{\sigma'} L_x^{\sigma'}(I)} \\ &\leq 2c|\lambda|a^{\alpha-1}| \, \|u-v\|_{L_t^\sigma L_x^\sigma(I)} \,, \end{aligned}$$

therefore

$$\|\Phi(u) - \Phi(v)\|_X \leq 4c|\lambda|a^{\alpha-1} \, \|u-v\|_X \,,$$

with $a < \left(\frac{1}{4c|\lambda|}\right)^{\alpha-1}$, this implies that $\phi(\cdot)$ is a contraction and by the Banach fixed point theorem there exists a unique $u \in X$, such that $\phi(u) = u$.

Similar argument to the $L^2(\mathbb{R}^N)$ subcritical case, show that for all $T' < T$ there exists a neighborhood of $u_0 \in L^2(\mathbb{R}^N)$, such that:

$$\|u - v\|_{T'} \leq K \, \|u_0 - v_0\|_{L_x^2} \,,$$

where

$$K = \frac{1}{1 - 2^\alpha C^\alpha |\lambda| (\|u_0\|^{\alpha-1}_{L_x^2} + \|v_0\|^{\alpha-1}_{L_x^2})},$$

and $1 - 2^\alpha C^\alpha |\lambda| (\|u_0\|^{\alpha-1}_{L_x^2} + \|v_0\|^{\alpha-1}_{L_x^2}) < 1$. This prove of continuous dependence. $\square$

4.3 H^1-Theory

Theorem 4.3.1. *Let $N \geq 3$, and $1 < \alpha < \frac{N+2}{N-2}$. Then for all $u_0 \in H^1(\mathbb{R}^N)$ there exist $T = T(\|u_0\|_{H^1_x}, N, \lambda, \alpha) > 0$ and a unique solution u of the integral equation* (4.3) *in the time interval $[-T, T]$ with*

$$u \in \mathcal{C}\left([-T,T] : H^1\left(\mathbb{R}^N\right)\right) \cap L^r\left([-T,T] : H^{1,\rho}\left(\mathbb{R}^N\right)\right), \tag{4.24}$$

where

$$(r, \rho) = \left(\frac{4(\alpha+1)}{(N-2)(\alpha-1)}, \frac{N(\alpha+1)}{N+\alpha-1}\right) \tag{4.25}$$

an admissible pair.

Lemma 4.3.1. *If $f(u) = |u|^{\alpha-1}u$, the following estimate it holds:*

$$|\nabla f(u)| \lesssim \left||u|^{\alpha-1}\nabla u\right|.$$

Proof. Let us the consider expression

$$\begin{aligned}
\partial_{x_j}(|u|^{\alpha-1}u) &= \partial_{x_j}(u\bar{u})^{\frac{\alpha-1}{2}}u + |u|^{\alpha-1}\partial_{x_j}u \\
&= \frac{\alpha-1}{2}|u|^{\alpha-3}(\partial_{x_j}u\bar{u} + u\partial_{x_j}\bar{u})u + |u|^{\alpha-1}\partial_{x_j}u \\
&= \frac{\alpha-1}{2}|u|^{\alpha-3}|u|^2\partial_{x_j}u + \frac{\alpha-1}{2}|u|^{\alpha-3}u^2\partial_{x_j}\bar{u} + |u|^{\alpha-1}\partial_{x_j}u,
\end{aligned}$$

by triangle inequality, we obtain:

$$\begin{aligned}
\left|\partial_{x_j}(|u|^{\alpha-1}u)\right| &\leq (\alpha-1)\left||u|^{\alpha-1}|\partial_{x_j}u|\right| + \left||u|^{\alpha-1}|\partial_{x_j}u|\right| \\
&\lesssim \left||u|^{\alpha-1}\partial_{x_j}u\right|.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 4.3.1. Let

$$X := \mathcal{C}([-T,T] : H^1(\mathbb{R}^N)) \cap L^r([-T,T] : H^{1,\rho}(\mathbb{R}^N))$$

and define the set

$$E^1(T,a) = \left\{v \in X : \|v\|_T = \|v\|_{L^\infty_t H^1_x([-T,T])} + \|v\|_{L^r_t H^{1,\rho}_x([-T,T])} \leq a\right\},$$

with the metric

$$\rho(u,v) = \|u-v\|_{L^\infty_t L^2_x([-T,T])} + \|u-v\|_{L^r_t L^\rho_x([-T,T])}.$$

Note that

$$\|\Phi(u)\|_{L^\infty_t H^1_x(I)} \lesssim \|\Phi(u)\|_{L^\infty_t L^2_x(I)} + \|\nabla\Phi(u)\|_{L^\infty_t L^2_x(I)}.$$

Let C the implicit constants in inequalities such as Strichartz and Sobolev embedding, so

$$\begin{aligned}
\|\Phi(u)\|_{L^\infty_t L^2_x(I)} &\leq \left\|e^{it\Delta}u_0\right\|_{L^\infty_t L^2_x(I)} + |\lambda|\left\|\int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds\right\|_{L^\infty_t L^2_x(I)} \\
&\leq C\|u_0\|_{L^2_x} + C|\lambda|\,\||u|^\alpha\|_{L^{r'}_t L^{\rho'}_x(I)} \\
&= C\|u_0\|_{L^2_x} + C|\lambda|\left\|\|u\|^\alpha_{L^{\alpha\rho'}_x}\right\|_{L^{r'}_t(I)} && (4.26) \\
C &\leq \|u\|_{L^2_x} + C|\lambda|\left\|\|u\|_{H^{1,\rho}_x}\right\|^\alpha_{L^{\alpha r'}(I)} && (4.27) \\
&\leq C\|u_0\|_{L^2_x} + CT^{\frac{\alpha}{q}}|\lambda|\,\|u\|^\alpha_{L^r_t H^{1,\rho}_x(I)}, && (4.28)
\end{aligned}$$

by Sobolev embedding $\|u\|_{L^{\alpha\rho'}} \lesssim \|f\|_{H^{1,\rho}_x}$, if $\frac{N}{\rho} - \frac{N}{\alpha\rho'} \leq 1$, since

$$\frac{\alpha+1}{\rho} = \frac{N+\alpha-1}{N} \leq \frac{\alpha}{N} + 1.$$

Furthermore, $\alpha r' < r$ if $\alpha < r-1$, so $\alpha + 1 < \dfrac{4(\alpha+1)}{(N-2)(\alpha-1)}$ if and only if $\alpha < \dfrac{N+2}{N-2}$ satisfying the initial condition. Therefore:

$$\frac{1}{q} = \frac{1}{\alpha r'} - \frac{1}{r} = \frac{4-(N-2)(\alpha-1)}{4\alpha} > 0,$$

This implies that

$$\delta := \frac{\alpha}{q} = 1 - \frac{(N-2)(\alpha-1)}{4}$$

Now

$$\begin{aligned}
\|\nabla\Phi(u)\|_{L^\infty_t L^2_x(I)} &\leq \left\|e^{it\Delta}\nabla u_0\right\|_{L^\infty_t L^2_x(I)} + |\lambda| \left\|\int_0^t e^{it\Delta}\nabla(|u|^{\alpha-1}u)(s)ds\right\|_{L^\infty_t L^2_x(I)} \\
&\leq C\|\nabla u_0\|_{L^2_x} + C|\lambda| \left\|\nabla(|u|^{\alpha-1}u)\right\|_{L^{r'}_t L^{\rho'}_x(I)} \\
&\leq C\|\nabla u_0\|_{L^2_x} + C|\lambda| \left\|\left\||u|^{\alpha-1}\nabla u\right\|_{L^{\rho'}_x}\right\|_{L^{r'}_t(I)} \\
&\leq C\|\nabla u_0\|_{L^2_x} + C|\lambda| \left\|\|u\|^{\alpha-1}_{L^{l(\alpha-1)}_x}\|\nabla u\|_{L^\rho_x}\right\|_{L^{r'}_t(I)}
\end{aligned} \tag{4.29}$$

Above, we used the Lemma 4.3.1 and Hölder's inequality to show

$$\left\||u|^{\alpha-1}\nabla u\right\|_{L^{\rho'}_x} \lesssim \left\||u|^{\alpha-1}\right\|_{L^l_x}\|\nabla u\|_{L^\rho_x}.$$

Indeed, $\frac{1}{\rho'} = \frac{1}{l} + \frac{1}{\rho}$, which implies

$$\frac{1}{l} = 1 - \frac{2}{\rho}, \tag{4.30}$$

and by the Sobolev embedding $\|u\|_{L^{l(\alpha-1)}_x} \lesssim \|\nabla u\|_{L^\rho_x}$, with

$$\frac{1}{N} = \frac{1}{\rho} - \frac{1}{(\alpha-1)l},$$

that is,

$$\frac{1}{l} = \frac{\alpha-1}{\rho} - \frac{\alpha-1}{N}.$$

Therefore, (4.30) implies

$$\frac{1}{\rho} = \frac{N+\alpha-1}{N(\alpha+1)}. \tag{4.31}$$

Since (r,ρ) is L^2-admissible, it satisfies $\frac{2}{r} = \frac{N}{2} - \frac{N}{\rho}$. Thus by (4.31)

$$\begin{aligned}\frac{2}{r} &= \frac{N}{2} - \frac{N+\alpha-1}{\alpha+1} \\ &= \frac{(N-2)\alpha-(N-2)}{2(\alpha+1)} \\ &= \frac{(\alpha-1)(N-2)}{2(\alpha+1)}.\end{aligned}$$

Hence,

$$\begin{aligned}C\left\|\nabla u_0\right\|_{L^2_x} + C|\lambda| \left\|\|u\|^{\alpha-1}_{L^{l(\alpha-1)}_x} \|\nabla u\|_{L^\rho_x}\right\|_{L^{r'}_x(I)} &\le C\left\|\nabla u_0\right\|_{L^2_x} + C|\lambda| \left\|u\right\|^{\alpha}_{L^{\alpha r'}_t H^{1,\rho}_x(I)} \\ &\le C\left\|\nabla u_0\right\|_{L^2_x} + C|\lambda| T^\delta \left\|u\right\|^{\alpha}_{L^r_t H^{1,\rho}_x(I)}\end{aligned}$$

By a similar argument we show the same estimate in the space $L^r_t H^{1,\rho}_x(I)$. Thus choosing $a = 2c\left\|u_0\right\|_{H^1_x}$, then

$$\left\|\Phi(u)\right\|_T \le c\left\|u_0\right\|_{H^1_x} + C|\lambda| T^\delta \left\|u\right\|^\alpha_T \le a,$$

with

$$T \le \left(\frac{1}{2^\alpha |\lambda| C^\alpha \left\|u_0\right\|^{\alpha-1}_{H^1_x}}\right)^{\frac{1}{\delta}}. \tag{4.32}$$

Let $u, v \in E^1_a(T)$, such that:

$$\begin{aligned}\left\|\Phi(u)-\Phi(v)\right\|_{L^\infty_t L^2_x(I)} &\le c|\lambda| \left(\left\|\left\||u|^{\alpha-1}|u-v|\right\|_{L^{\rho'}_x}\right\|_{L^{r'}_t(I)} + \left\|\left\||v|^{\alpha-1}|u-v|\right\|_{L^{\rho'}_x}\right\|_{L^{r'}_t(I)}\right) \\ &\le C|\lambda| \left(\left\|u\right\|^{\alpha-1}_{L^r_t L^\rho_x(I)} + \left\|v\right\|^{\alpha-1}_{L^r_t L^\rho_x(I)}\right) \left\|u-v\right\|_{L^r_t L^\rho_x(I)} T^\delta\end{aligned}$$

Analogously for the space $L^r_t L^\rho_x$, thus

$$\rho(u,v) \le 4c|\lambda| a^{\alpha-1} T^\delta \left\|u-v\right\|_T$$

where

$$T < \left(\frac{1}{4c|\lambda| a^{\alpha-1}}\right)^{\frac{1}{\delta}}$$

this implies that $\phi(\cdot)$ is a contraction and by fixed point Banach theorem there exists a unique $u \in X$, such that $\phi(u) = u$.

This complete the proof. □

Theorem 4.3.2 (Critical case H^1 [15]). *Let $N \ge 3$ and $\alpha = \frac{N+2}{N-2}$. Given $u_0 \in H^1(\mathbb{R}^N)$, there exist $T = T(u_0, N, \lambda, \alpha) > 0$ and a unique solution u of the integral equation* (4.3) *in the time interval $[-T, T]$ with*

$$u \in \mathcal{C}([-T,T] : H^1(\mathbb{R}^N)) \cap L^r([-T,T] : H^{1,\rho}(\mathbb{R}^N)),$$

where

$$(r, \rho) = \left(\frac{2N}{N-2}, \frac{2N^2}{N^2-2N+4}\right).$$

Proof. Let

$$X := \mathcal{C}([-T,T]:H^1(\mathbb{R}^N)) \cap L^r([-T,T]:H^{1,\rho}(\mathbb{R}^N))$$

and define the set

$$\widetilde{E}_a^1(T) = \left\{ v \in X : \|v\|_T = \left\| v - e^{it\Delta}u_0 \right\|_{L_t^\infty H_x^1([-T,T])} + \|v\|_{L_t^r H_x^{1,\rho}([-T,T])} \leq a \right\}.$$

This space forms a complete metric space equipped with the following metric:

$$d(u,v) = \|u-v\|_{L_t^\infty L_x^2(I)} + \|u-v\|_{L_t^r L_x^\rho(I)}, \tag{4.33}$$

Note that

$$\|\phi(u)\|_{L_t^r H_x^{1,\rho}(I)} \leq \|\phi(u)\|_{L_t^r L_x^\rho(I)} + \|\nabla\phi(u)\|_{L_t^r L_x^\rho(I)}.$$

Firstly, by Propisition 4.2.1 given $\varepsilon > 0$

$$\begin{aligned}
\|\phi(u)\|_{L_t^r L_x^\rho} &\leq \left\| e^{it\Delta}u_0 \right\|_{L_t^r L_x^\rho(I)} + |\lambda| \left\| \int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds \right\|_{L_t^r L_x^\rho(I)} \\
&\leq \varepsilon + c|\lambda| \left\| |u|^{\alpha-1}u \right\|_{L_t^{r'} L_x^{\rho'}(I)} \\
&= \varepsilon + c|\lambda| \left\| \|u\|_{L_x^{\alpha\rho'}}^\alpha \right\|_{L_t^{r'}(I)} \\
&\leq \varepsilon + c|\lambda| \left\| \|u\|_{H_x^{1,\rho}} \right\|_{L_t^{\alpha r'}(I)}^\alpha,
\end{aligned} \tag{4.34}$$

where

$$\alpha r' = \left(\frac{2N}{N+2}\right)\left(\frac{N+2}{N-2}\right) = \frac{2N}{N-2}.$$

$$\begin{aligned}
\|\nabla\phi(u)\|_{L_t L_x^\rho(I)} &\leq \left\| e^{it\Delta}\nabla u_0 \right\|_{L_t^r L_x^\rho(I)} + |\lambda| \left\| \int_0^t e^{it\Delta}\nabla(|u|^{\alpha-1}u)(s)ds \right\|_{L_t^r L_x^\rho(I)} \\
&\leq \varepsilon + c|\lambda| \left\| \nabla(|u|^{\alpha-1}u) \right\|_{L_t^{r'} L_x^{\rho'}(I)} \\
&\leq \varepsilon + c|\lambda| \left\| \|u\|_{L_x^{q(\alpha-1)}}^{\alpha-1} \|\nabla u\|_{L_x^\rho} \right\|_{L_t^{r'}(I)} \\
&\leq \varepsilon + c|\lambda| \|\nabla u\|_{L^r L_x^\rho(I)}^\alpha \\
&\leq \varepsilon + c|\lambda| a^\alpha.
\end{aligned}$$

Therefore, there exists T such that

$$\|\phi(u)\|_T \leq \varepsilon + c|\lambda|a^\alpha \leq a \tag{4.35}$$

with $\varepsilon < \frac{a}{2}$ and $a < \left(\frac{1}{c|\lambda|}\right)^{\frac{1}{\alpha-1}}$.

The proof of the contraction follows the same ideas as the previous theorems. □

4.3.1 Conservation laws for the Schrödinger equation.

After analyzing the local theory of the Schrödinger equation, we will observe that it preserves important quantities, such as mass $M[u]$, energy $E[u]$ and momentum $L[u]$, defined by

$$M[u(t)] = \int_{\mathbb{R}^N} |u(x,t)|^2 dx, \tag{4.36}$$

$$E[u(t)] = \int_{\mathbb{R}^N} \left(|\nabla_x u(x,t)|^2 - \frac{2\lambda}{\alpha+1} |u(x,t)|^{\alpha+1} \right) dx \tag{4.37}$$

$$L_j[u(t)] = \mathrm{Im} \int_{\mathbb{R}^N} \partial_{x_j} u(x,t) \bar{u}(x,t) dx \tag{4.38}$$

These conservation laws are particularly useful for controlling long-time dispersive behavior.

We will also have the following identity.

Proposition 4.3.1 (Virial identity). *Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$, and let $u(x,t)$ be a solution of IVP* (4.1) *in $H^1(\mathbb{R}^N)$. For all $t \in [0,T]$ define*

$$M_\varphi(u) = \int_{\mathbb{R}^N} \varphi(x) |u(x,t)|^2 dx.$$

Then we have

$$\frac{d}{dt} M_\varphi(u) = 2\mathrm{Im} \int_{\mathbb{R}^N} \nabla\varphi \cdot \nabla u(x,t) \bar{u}(x,t) dx \tag{4.39}$$

and

$$\begin{aligned} \frac{d^2}{dt^2} M_\varphi(u) = \frac{4}{\alpha+1} \int_{\mathbb{R}^N} \lambda |u|^{\alpha+1} \Delta\varphi dx - \int_{\mathbb{R}^N} \lambda |u|^{\alpha+1} \Delta\varphi dx \\ + \int_{\mathbb{R}^N} |u|^2 \Delta^2 \varphi dx + 4\mathrm{Re} \sum_{k \neq j} \int_{\mathbb{R}^N} \partial_{x_k} \bar{u} \partial_{x_j} u \partial^2_{x_j x_k} \varphi dx \end{aligned} \tag{4.40}$$

Proof. Note that $\partial_t |u|^2 = \partial_t u \bar{u} + u \partial_t \bar{u} = \mathrm{Re}(2\partial_t u \bar{u})$, thus

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} \varphi(x) |u(x,t)|^2 dx &= \int_{\mathbb{R}^N} \varphi(x) \partial_t |u|^2 dx \\ &= 2\mathrm{Re} \int_{\mathbb{R}^N} \varphi(x) \partial_t u \bar{u} dx \\ &= 2\mathrm{Im} \int_{\mathbb{R}^N} \varphi(x) i \partial_t u \bar{u} dx \\ &= 2\mathrm{Im} \int_{\mathbb{R}^N} \varphi(x) (-\Delta u - \lambda |u|^{\alpha-1} u) \bar{u} dx \\ &= -2\mathrm{Im} \int_{\mathbb{R}^N} \varphi(x) \Delta u \bar{u} dx - 2\mathrm{Im} \int_{\mathbb{R}^N} \lambda \varphi(x) |u|^{\alpha+1} dx \\ &= 2\mathrm{Im} \int_{\mathbb{R}^N} \nabla\varphi(x) \cdot \nabla u \bar{u} dx. \end{aligned} \tag{4.41}$$

Now, for (4.40), integrating by parts and using the fact that $\mathrm{Im}(i\mathrm{Im}z) = \mathrm{Im}z$, we obtain

$$\begin{aligned}
\partial_t^2 M_{\varphi(u)} &= 2\mathrm{Im}\int_{\mathbb{R}^N} \nabla\varphi(x)\cdot(\partial_t \nabla u\bar{u} + \nabla u\partial_t\bar{u})dx \\
&= 2\mathrm{Im}\sum_k \int_{\mathbb{R}^N} \partial_{x_j}\varphi\partial^2_{x_i t}u\bar{u} + \partial_{x_j}\varphi\partial u\partial_t\bar{u}dx \\
&= 2\mathrm{Im}\sum_k \int_{\mathbb{R}^N} -\partial_t u\partial_{x_j}(\bar{u}\partial_{x_j}\varphi) + \partial_{x_j}\varphi\partial_{x_j}u\partial_t\bar{u}dx \\
&= 2\mathrm{Im}\sum_k \int_{\mathbb{R}^N} -\partial_t u\partial_{x_j}\bar{u}\partial_{x_j}\varphi - \partial_t u\bar{u}\partial^2_{x_j}\varphi + \partial_{x_j}\varphi\partial_{x_j}u\partial_t\bar{u}dx \\
&= 4\mathrm{Im}\sum_k \int_{\mathbb{R}^N} i\mathfrak{I}\partial_t\bar{u}\partial_{x_j}u\partial_{x_j}\varphi dx - 2\mathrm{Im}\sum_k \int_{\mathbb{R}^N} \partial_t u\bar{u}\partial^2_{x_j}\varphi dx \\
&= \underbrace{4\mathrm{Im}\sum_k \int_{\mathbb{R}^N} \partial_t\bar{u}\partial_{x_j}u\partial_{x_j}\varphi dx}_{I} - \underbrace{2\mathrm{Im}\sum_k \int_{\mathbb{R}^N} \partial_t u\bar{u}\partial^2_{x_j}\varphi dx}_{II}
\end{aligned} \tag{4.42}$$

Then

$$\begin{aligned}
I &= 4\mathrm{Im}\sum_k \int_{\mathbb{R}^N} \partial_t\bar{u}\partial_{x_j}u\partial_{x_j}\varphi dx \\
&= 4\mathrm{Im}\sum_k \int_{\mathbb{R}^N} (-i\Delta\bar{u} - i\lambda|u|^{\alpha-1}\bar{u})\partial_{x_j}u\partial_{x_j}\varphi dx \\
&= 4\mathrm{Im}\sum_k \int_{\mathbb{R}^N} -i\Delta\bar{u}\partial_{x_j}u\partial_{x_j}\varphi - i\lambda|u|^{\alpha-1}\bar{u}\partial_{x_j}u\partial_{x_j}\varphi dx \\
&= -4\mathrm{Im}\ i\sum_{j\neq k} \int_{\mathbb{R}^N} \partial^2_{x_k}\bar{u}\partial_{x_j}u\partial_{x_j}\varphi dx - 4\mathrm{Im}\ i\sum_j \int_{\mathbb{R}^N} \lambda|u|^{\alpha-1}\bar{u}\partial_{x_j}u\partial_{x_j}\varphi dx
\end{aligned} \tag{4.43}$$

We now develop each term of (4.43). For the second term, we use the fact that

$$\partial_{x_j}|u|^{\alpha+1} = \partial_{x_j}((u\bar{u})^{\frac{\alpha+1}{2}}) = (\alpha+1)\mathrm{Re}(\lambda|u|^{\alpha-1}\partial_{x_j}u\bar{u}), \tag{4.44}$$

therefore,

$$\begin{aligned}
-4\mathrm{Im}\ i\sum_j \int_{\mathbb{R}^N} \lambda|u|^{\alpha-1}\bar{u}\partial_{x_j}u\partial_{x_j}\varphi dx &= -4\sum_j \int_{\mathbb{R}^N} \mathrm{Re}(\lambda|u|^{\alpha-1}\partial_{x_j}u\bar{u})\partial\varphi_{x_j} \\
&= -\frac{4}{\alpha+1}\sum_j \int_{\mathbb{R}^N} \lambda\partial_{x_j}|u|^{\alpha+1}\partial_{x_j}\varphi dx \\
&= \frac{4}{\alpha+1}\sum_j \int_{\mathbb{R}^N} \lambda|u|^{\alpha+1}\partial^2_{x_j}\varphi dx \\
&= \frac{4}{\alpha+1}\int_{\mathbb{R}^N} \lambda|u|^{\alpha+1}\Delta\varphi dx.
\end{aligned} \tag{4.45}$$

For the first term of (4.43) and using, $\partial_{x_j}|\partial_{x_k}u|^2 = 2\mathrm{Re}(\partial^2_{x_jx_k}u\partial_{x_k}\bar{u})$,

$$\begin{aligned}
-4\mathrm{Im}\ i\sum_{j\neq k}\int_{\mathbb{R}^N}\partial^2_{x_k}\bar{u}\partial_{x_j}u\partial_{x_j}\varphi dx &= 4\mathrm{Im}\ i\sum_{j\neq k}\int_{\mathbb{R}^N}\partial_{x_k}\bar{u}\partial^2_{x_jx_k}u\partial_{x_j}\varphi + \partial_{x_k}\bar{u}\partial_{x_j}u\partial_{x_jx_k}\varphi dx\\
&= 4\sum_{j\neq k}\int_{\mathbb{R}^N}\mathrm{Re}(\partial_{x_k}\bar{u}\partial^2_{x_jx_k}u)\partial_{x_j}\varphi + \mathrm{Re}(\partial_{x_k}\bar{u}\partial_{x_j}u\partial_{x_jx_k}\varphi)dx\\
&= 2\sum_{j\neq k}\int_{\mathbb{R}^N}\partial_{x_j}|\partial_{x_k}\bar{u}|^2\partial_{x_j}\varphi dx + 4\mathrm{Re}\sum_{k\neq j}\int_{\mathbb{R}^N}\partial_{x_k}\bar{u}\partial_{x_j}u\partial^2_{x_jx_k}\varphi dx\\
&= -2\int_{\mathbb{R}^N}|\nabla u|^2\Delta\varphi + 4\mathrm{Re}\sum_{k\neq j}\int_{\mathbb{R}^N}\partial_{x_k}\bar{u}\partial_{x_j}u\partial^2_{x_jx_k}\varphi dx
\end{aligned}$$

Therefore,

$$I = -2\int_{\mathbb{R}^N}|\nabla u|^2\Delta\varphi + 4\mathrm{Re}\sum_{k\neq j}\int_{\mathbb{R}^N}\partial_{x_k}\bar{u}\partial_{x_j}u\partial^2_{x_jx_k}\varphi dx + \frac{4}{\alpha+1}\int_{\mathbb{R}^N}\lambda|u|^{\alpha+1}\Delta\varphi dx \quad (4.46)$$

Again, for the (II) term of the equation (4.42), we have

$$\begin{aligned}
II &= -2\mathrm{Im}\sum_{j}\int_{\mathbb{R}^N}\partial_t u\bar{u}\partial^2_{x_j}\varphi dx\\
&= -2\mathrm{Im}\sum_{j}\int_{\mathbb{R}^N}(i\Delta u + i|u|^{\alpha-1}u)\bar{u}\partial^2_{x_j}\varphi dx\\
&= -2\mathrm{Im}\ i\sum_{j}\int_{\mathbb{R}^N}\Delta u\bar{u}\partial^2_{x_j}\varphi + |u|^{\alpha+1}\partial^2_{x_j}\varphi dx\\
&= 2\mathrm{Re}\sum_{j\neq k}\int_{\mathbb{R}^N}\partial^2_{x_k}u\bar{u}\partial^2_{x_j}\varphi + \lambda|u|^{\alpha+1}\partial^2_{x_j}\varphi dx\\
&= 2\mathrm{Re}\sum_{j\neq k}\int_{\mathbb{R}^N}\partial_{x_k}u\partial_{x_k}\bar{u}\partial^2_{x_j}\varphi + \partial_{x_k}u\bar{u}\partial^3_{x_j^2x_k}\varphi dx - 2\mathrm{Re}\sum_{j\neq k}\int_{\mathbb{R}^N}\lambda|u|^{\alpha+1}\partial^2_{x_j}\varphi dx \quad (4.47)\\
&= 2\int_{\mathbb{R}^N}(|\nabla u|^2 - \lambda|u|^{\alpha+1})\Delta\varphi dx + 2\sum_{k\neq j}\int_{\mathbb{R}^N}\mathrm{Re}(\partial_{x_k}u\bar{u})\partial^3_{x_j^2x_k}\varphi dx\\
&= 2\int_{\mathbb{R}^N}(|\nabla u|^2 + \lambda|u|^{\alpha+1})\Delta\varphi dx + \sum_{k\neq j}\int_{\mathbb{R}^N}\partial_{x_k}|u|^2\partial^3_{x_j^2x_k}dx\\
&= 2\int_{\mathbb{R}^N}(|\nabla u|^2 - \lambda|u|^{\alpha+1})\Delta\varphi dx - \sum_{k\neq j}\int_{\mathbb{R}^N}|u|^2\partial^4_{x_j^2x_k^2}\varphi dx\\
&= 2\int_{\mathbb{R}^N}(|\nabla u|^2 - \lambda|u|^{\alpha+1})\Delta\varphi dx - \int_{\mathbb{R}^N}|u|^2\Delta^2\varphi dx.
\end{aligned}$$

Next, collecting (4.46) and (4.47) we have

$$\begin{aligned}
\frac{d^2}{dt^2}\int_{\mathbb{R}^N}\varphi(x)\left|u(x,t)\right|^2dx = 4\left(\frac{1}{\alpha+1} - 2\right)\int_{\mathbb{R}^N}\lambda|u|^{\alpha+1}\Delta\varphi dx - \int_{\mathbb{R}^N}|u|^2\Delta^2\varphi dx\\
+ 4\mathrm{Re}\sum_{k\neq j}\int_{\mathbb{R}^N}\partial_{x_k}\bar{u}\partial_{x_j}u\partial^2_{x_jx_k}\varphi dx.
\end{aligned} \quad (4.48)$$

□

Collorary 4.3.1. *Let $u(x,t)$ be a solution of* (4.1)*, then*

$$\begin{aligned} M[u(t)] &= M[u_0] \\ L[u(t)] &= L[u_0]. \end{aligned} \tag{4.49}$$

Proof. Let us consider the function

$$\varphi_R(x) = \exp\left\{-\frac{|x|^2}{R^2}\right\}.$$

We observe that $\varphi_R \to 1$ almost everywhere as $R \to \infty$. If

$$M_{\varphi_R}(t) = \int \varphi_R(x)|u|^2 dx,$$

then, by the dominated convergence theorem $M_{\varphi_R}(t) \to M(t)$ and $M_{\varphi_R}(0) \to M(0)$. It follows from Proposition 4.3.1 that

$$\frac{d}{dt}M_R(t) = 2\mathrm{Im}\int \nabla\varphi_R \cdot \nabla u\ \bar{u}dx,$$

moreover

$$M_R(t) - M_R(0) = 2\int_0^t \int \mathrm{Im}\ \nabla\varphi_R \cdot \nabla u\ \bar{u}dxds.$$

Since $|\nabla\varphi_R| \le \frac{1}{R}\left|\nabla\varphi\left(\frac{|x|}{R}\right)\right|$, by the dominated convergence theorem, and Cauchy-Schwarz

$$M[t] = M[0].$$

Now, consider the function

$$\varphi_R(x) = x_j \exp\left\{-\frac{|x|^2}{R^2}\right\}$$

and

$$\begin{aligned} L_{\varphi_R}(x) &= 2\mathrm{Im}\int \nabla\left(x_j \exp\left\{-\frac{|x|^2}{R^2}\right\}\right)\cdot\nabla u\ \bar{u} \\ &= 2\mathrm{Im}\int \exp\left\{-\frac{|x|^2}{R^2}\right\}\partial_j u\ \bar{u} + 2\mathrm{Im}\int x_j\nabla\left(\exp\left\{-\frac{|x|^2}{R^2}\right\}\right)\cdot\nabla u\ \bar{u}, \end{aligned}$$

thus, $L_{\varphi_R} \to L(t)$ as $R \to \infty$.

It is a consequence of Proposition 4.3.1

$$\begin{aligned} \frac{d}{dt}L_{\varphi_R}(t) = &\frac{4}{\alpha+1}\int \lambda|u|^{\alpha+1}\Delta\varphi_R dx - \int \lambda|u|^{\alpha+1}\Delta\varphi_R \\ &+ \int |u|^2\Delta^2\varphi_R dx + 4\mathrm{Re}\sum_{k\neq j}\int \partial_{x_k}\bar{u}\partial_{x_j}u\partial^2_{x_jx_k}\varphi_R dx. \end{aligned}$$

We have that $\left\|\partial_{x_{jk}}\varphi_R\right\|_\infty \to 0$ as $R \to \infty$. Moreover,

$$L_{\varphi_R}(t) - L_{\varphi_R}(0) = \int_0^t \frac{d}{dt}L_{\varphi_R}(t)ds.$$

From the dominated convergence theorem and Cauchy-Schwarz it follows that $L[t] = L[0]$. □

Proposition 4.3.2. *If $u(x,t)$ is a solution for* (4.1)*, then the energy equation* (4.37) *is conserved.*

Proof. Let

$$K[u] = \int_{\mathbb{R}^{\mathbb{N}}} |\nabla_x u|^2 dx, \quad \text{and} \quad P[u] = \frac{\lambda}{\alpha+1} \int_{\mathbb{R}^{\mathbb{N}}} |u|^{\alpha+1} dx. \tag{4.50}$$

Differentiating and integrating by parts, it follows that

$$\partial_t K[u] = \int_{\mathbb{R}^N} \partial_t (\nabla u \nabla \bar{u}) dx = -2\mathrm{Re} \int_{R^N} \Delta \bar{u} \partial_t u dx,$$

and

$$\begin{aligned} \partial_t P[u] = \frac{\lambda}{\alpha+1} \int_{\mathbb{R}^N} \partial_t |u|^{\alpha+1} dx &= \frac{\lambda}{\alpha+1} \int_{\mathbb{R}^N} \partial_t (u\bar{u})^{\frac{\alpha+1}{2}} dx \\ &= -\lambda \mathrm{Re} \int_{\mathbb{R}^N} (u\bar{u})^{\frac{\alpha-1}{2}} u \partial_t \bar{u} dx \\ &= -\lambda \mathrm{Re} \int_{\mathbb{R}^N} |u|^{\alpha-1} u \partial_t \bar{u} dx. \end{aligned} \tag{4.51}$$

Now

$$\begin{aligned} \partial_t (K[u] - P[u]) &= -\mathrm{Re} \int_{\mathbb{R}^N} (\Delta \bar{u} + \lambda |u|^{\alpha-1} \bar{u}) \partial_t u dx \\ &= -\mathrm{Re} \int_{\mathbb{R}^N} i \partial_t \bar{u} \partial_t u dx = 0, \end{aligned}$$

which implies that $E[u]$ is constant, in particular $E[u] = E[u_0]$. □

4.4 Global well-posedness

In this chapter, we shall study the long time behavior of the local solutions of NLS (4.1).

Theorem 4.4.1. *If the nonlinear power $1 < \alpha < 1 + \frac{4}{N}$, then for any $u_0 \in L^2(\mathbb{R}^N)$ the local solution $u = u(x,t)$ of the initial value problem* (4.1) *extends globally.*

Proof. From the Theorem 4.2.1, there exists $T = T(\|u_0\|_{L^2_x}) > 0$, such that the NLS (4.1) is locally well-posed. By iteratively applying the conservation of mass many times, preserving the length of the time interval we get a global solution.

□

Theorem 4.4.2. *If $\lambda < 0$, then all solutions of NLS* (4.1) *with $u_0 \in H^1(\mathbb{R}^N)$ provided by Theorem 4.3.1 extends globally in time.*

Proof. Since energy is conserved and $\lambda < 0$, it follows that

$$\|\nabla_x u(t)\|^2_{L^2_x} \leq E[u_0],$$

and due to the conservation of mass

$$\|u(t)\|^2_{L^\infty_{t([-T,T])} H^1_x} \leq E[u_0] + \|u_0\|^2_{L^2_x}, \tag{4.52}$$

combining (4.32) in the proof of Theorem 4.3.1 with (4.52), we deduce that

$$T^{\theta} = \frac{c'}{c|\lambda|(2c\,\|u_0\|_{H_x^1})^{\alpha-1}} \gtrsim \frac{c'}{(E[u_0] + \|u_0\|_{L_x^2})^{\frac{\alpha-1}{2}}}$$

uniformly, to extend the local solution u in any time interval. □

Theorem 4.4.3. *Let $\lambda > 0$ and $1 < \alpha < 1 + \frac{4}{N}$. If $u_0 \in H^1(\mathbb{R}^N)$, then the local solution extends globally.*

Proof. Using Gagliardo-Nirenberg inequality,

$$\|u(t)\|_{L_x^{\alpha+1}}^{\alpha+1} \leq c\,\|\nabla_x u(t)\|_{L_x^2}^{\theta}\,\|u(t)\|_{L_x^2}^{1-\theta}, \quad \text{with} \quad \frac{1}{\alpha+1} = \theta\left(\frac{1}{2} - \frac{1}{N}\right) + \frac{1-\theta}{2} \tag{4.53}$$

indeed

$$\theta = \frac{N(\alpha-1)}{2(\alpha+1)},$$

Furthermore, regarding the energy,

$$\begin{aligned}\|\nabla_x u(t)\|_{L_x^2}^2 &= E[u_0] + \frac{2\lambda}{\alpha+1}\,\|u(t)\|_{L_x^{\alpha+1}}^{\alpha+1} \\ &\leq E[u_0] + \frac{c_{\lambda,\alpha}}{\varepsilon}\,\|u_0\|_{L_x^2}^{\left((\alpha+1)-\frac{N(\alpha-1)}{2}\right)}\,\varepsilon\,\|\nabla_x u(t)\|_{L_x^2}^{\frac{N(\alpha-1)}{2}}\end{aligned}$$

Since $1 < \alpha < 1 + \frac{4}{N}$ we have $\frac{N(\alpha-1)}{2} < 2$. Consequently, by the Young's inequality

$$\|\nabla_x u(t)\|_{L_x^2}^2 \leq E[u_0] + \left(\frac{c_{\lambda,\alpha}}{\varepsilon}\,\|u_0\|_{L_x^2}^{\left((\alpha+1)-\frac{N(\alpha-1)}{2}\right)}\right)^{\frac{N(\alpha-1)}{N(\alpha-1)-4}} + \frac{N(\alpha-1)}{4}\varepsilon^{\frac{4}{N(\alpha-1)}}\,\|\nabla_x u(t)\|_{L_x^2}^2.$$

Therefore,

$$\|\nabla_x u(t)\|_{L_x^2}^2 \lesssim \frac{C_{\varepsilon}(E[u_0], M[u_0])}{1-\varepsilon}.$$

This allows us to extend the solution to any interval. □

If $f \in H^1(\mathbb{R}^N)$, *Weinstein* [22] showed that the optimal constant C_{GN} for the Gagliardo-Nirenberg inequality (4.53),

$$\|f\|_{L_x^{\alpha+1}}^{\alpha+1} \leq C_{GN}^{\alpha+1}\,\|f\|_{L_x^2}^{\frac{N(\alpha-1)}{2}}\,\|\nabla_x f\|_{L_x^2}^{\left[(\alpha+1)-\frac{N(\alpha-1)}{2}\right]} \tag{4.54}$$

is given by

$$C_{GN} = \left(\frac{\alpha+1}{2\,\|Q\|_{L_x^2}^{\alpha-1}}\right)^{\frac{1}{\alpha+1}}, \tag{4.55}$$

and Q is the ground state (positive solution of minimal L^2-norm) of the elliptic equation

$$\Delta Q - Q + |Q|^{\alpha-1}Q = 0. \tag{4.56}$$

Theorem 4.4.4. *Let $\lambda > 0$, $\alpha = 1 + \frac{4}{N}$ and Q be the spherically symmetric, positive ground state of* (4.56). *If $u_0 \in H^1(\mathbb{R}^N)$ is such that $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution $u \in H^1(\mathbb{R}^N)$ to the problem* (4.1) *is global.*

Proof. By Gagliardo Nireberg's inequality and Conservation of energy, we have

$$\|\nabla_x u(t)\|_{L_x^2}^2 \leq E[u_0] + \frac{C_{\alpha,N}^{\alpha+1}}{\alpha+1} \|u_0\|_{L_x^2}^{\frac{4}{N}} \|\nabla_x u(t)\|_{L_x^2}^2$$

where

$$C_{\alpha,N} = \left(\frac{1+\frac{2}{N}}{\|Q\|_{L_x^2}} \right)^{\frac{1}{2+\frac{4}{N}}}$$

this implies that

$$\left[1 - \left(\frac{\|u_0\|_{L_2^x}}{\|Q\|_{L_x^2}} \right)^{\frac{4}{N}} \right] \|\nabla_x u(t)\|_{L_x^2}^2 \leq E[u_0],$$

now, from the hypothesis that $\|u_0\|_{L_x^2} < \|Q\|_{L_x^2}$,

$$\|u_0\|_{L_x^2} + \left[E[u_0] \left(1 - \left(\frac{\|u_0\|_{L_2^x}}{\|Q\|_{L_x^2}} \right)^{\frac{4}{N}} \right)^{-1} \right] \geq \|\nabla_x u(t)\|_{L_x^2} + \|u(t)\|_{L_x^2} = \|u(t)\|_{H_x^1},$$

therefore $\sup_{t\in I_{max}} \|u(t)\|_{H_x^1} < +\infty$ and the solution may be globally extended. □

Theorem 4.4.5. *Let $1 + \frac{4}{N} < \alpha < \frac{N+2}{N-2}$ and set $s_c = \frac{N}{2} - \frac{2}{\alpha-1}$ with $s_c \in (0,1)$. Suppose that $u(t)$ is the solution of with initial condition $u_0 \in H^1(\mathbb{R}^N)$ satisfying*

$$E[u_0]^{s_c} M[u_0]^{1-s_c} < E[Q]^{s_\alpha} M[Q]^{1-s_c} \tag{4.57}$$

and

$$\|\nabla u_0\|_{L_x^2}^{s_c} \|u_0\|_{L_x^2}^{1-s_c} < \|\nabla Q\|_{L_x^2}^{s_c} \|Q\|_{L_x^2}^{1-s_c}. \tag{4.58}$$

Then $u(t)$ is a global solution in $H^1(\mathbb{R}^N)$.

Proof. Multiplying $E[u]$ by $M[u]^{\frac{1-s_c}{s_c}}$, and Sharp Gagliardo Nirenberg (4.54), we obtain

$$\begin{aligned} E[u]M[u]^{\frac{1-s_c}{s_c}} &= M[u]^{\frac{1-s_c}{s_c}} \|\nabla_x u(t)\|_{L_x^2}^2 - \frac{2}{\alpha+1} M[u]^{\frac{1-s_c}{s_c}} \|u(t)\|_{L_x^{\alpha+1}}^{\alpha+1} \\ &\geq \left(\|u_0\|_{L_x^2}^{\frac{1-s_c}{s_c}} \|\nabla_x u(t)\|_{L_x^2} \right)^2 - \frac{2C_{GN}}{\alpha+1} \left[\left(\|u_0\|_{L_x^2}^{\frac{1-s_c}{s_c}} \|\nabla_x u(t)\|_{L_x^2} \right)^2 \right]^{\frac{N(\alpha-1)}{4}} \end{aligned}$$

Since

$$2\left(\frac{1-s_c}{s_c}\right) + \left((\alpha+1) - \frac{N(\alpha-1)}{2}\right) = 2\left(\frac{1-s_c}{s_c}\right) \cdot \frac{N(\alpha-1)}{4}$$

Therefore

$$E[u]M[u]^{\frac{1-s_c}{s_c}} \geq f\left(\|u_0\|^{\frac{2(1-s_c)}{s_c}} \|\nabla_x u(t)\|_{L_x^2}^2 \right) \tag{4.59}$$

where $f(x) = x - Bx^{\frac{N(\alpha-1)}{4}}$, with

$$x(t) = \|u_0\|_{L_x^2}^{\frac{2(1-s_c)}{s_c}} \|\nabla_x u(t)\|_{L_x^2}^2 \quad \text{and} \quad B = \frac{2C_{GN}}{\alpha+1}.$$

If $x^* = \|Q\|_{L_x^2}^{\frac{2(1-s_c)}{s_c}} \|\nabla_x Q\|_{L_x^2}^2$, By Gagaliardo Nirenberg Sharp, it holds that

$$f(x^*) = M[Q]^{\frac{1-s_c}{s_c}} E[Q].$$

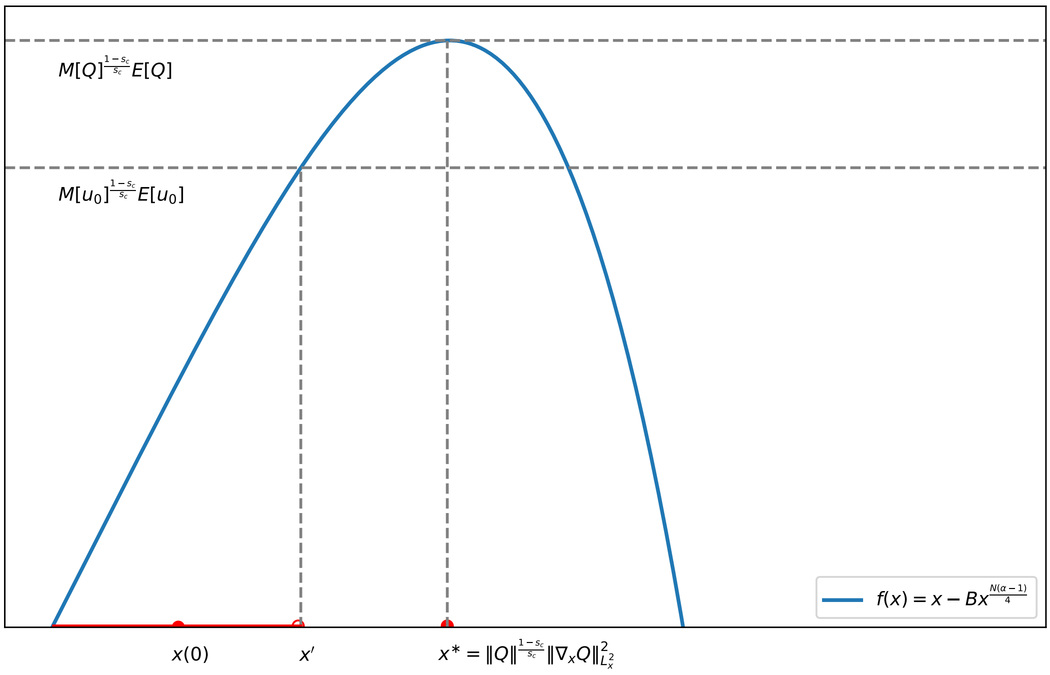

Figure 4.1: $x(t)$ is trapped on the interval $[0, x')$.

From the computation above, it follows for any $x(t)$, that

$$f(x(t)) \le M[u_0]^{\frac{1-s_c}{s_c}} E[u_0] < f(x^*).$$

From the hypothesis, $x(0) < x^*$ and the continuity of $x(t)$, we have that

$$\sup_t x(t) \le x' < x^*.$$

(See the Figure 4.1). This proves the global existence.

$\square$

The following Lemma provides estimates for the nonlinearity of the NLS in the Strichartz spaces.

Lemma 4.4.1. *Let $N \geq 3$, $u, v \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, $1 + \frac{4}{N} \leq \alpha \leq 1 + \frac{4}{N-2}$, then*

$$\left\| |u|^{\alpha-1} u \right\|_{L_I^\infty L_x^r} \lesssim \|u\|_{L_I^\infty H_x^1}^\alpha, \quad \frac{2N}{N+4} < r < \frac{2N}{N+2}. \tag{4.60}$$

$$\left\| |u|^{\alpha-1} u \right\|_{S'(L^2,I)} \lesssim \|u\|_{S(\dot{H}^{s_c},I)}^{\alpha-1} \|u\|_{S(L^2,I)} \tag{4.61}$$

$$\left\| |u|^{\alpha-1} v \right\|_{S'(\dot{H}^{-s_c},I)} \lesssim \|u\|_{S(\dot{H}^{s_c})}^{\alpha-1} \|v\|_{S(L^2,I)} \tag{4.62}$$

$$\left\| \nabla(|u|^{\alpha-1} u) \right\|_{S'(L^2,I)} \lesssim \|u\|_{S(\dot{H}^{s_c},I)}^{\alpha-1} \|\nabla u\|_{S(L^2,I)} \tag{4.63}$$

Proof. From Sobolev's inequality, $\dot{H}^s \hookrightarrow L^{\alpha r}$ if $s = \frac{N}{2} - \frac{N}{\alpha r}$, and the conditions given on α and r, ensures that $s \in (0,1)$. Thus

$$\left\| |u|^{\alpha-1} u \right\|_{L_x^r} = \|u\|_{L_x^{\alpha r}}^\alpha \lesssim \|u\|_{\dot{H}^s}^\alpha,$$

and by interpolation and Young's inequality

$$\|u\|_{\dot{H}_x^s}^\alpha \lesssim (\|u\|_{\dot{H}_x^0}^{1-\theta} \|u\|_{\dot{H}_x^1}^\theta)^\alpha \lesssim (\|u\|_{\dot{H}^0} + \|u\|_{\dot{H}^1})^\alpha = \|u\|_{H_x^1}^\alpha, \text{ with } s = (1-\theta)0 + \theta$$

Hence, (4.60) is proved.

Let $(\hat{q}, \hat{r}) \in \mathcal{A}_0$ and $(\hat{a}, \hat{r}) \in \mathcal{A}_{s_\alpha}$, by Hölder inequality, we obtain

$$\left\| |u|^{\alpha-1} u \right\|_{L_t^{\hat{q}'} L_x^{\hat{r}'}} \lesssim \|u\|_{L_t^{(\alpha-1)\hat{a}_1} L_x^{(\alpha-1)\hat{r}_1}}^{\alpha-1} \|u\|_{L_t^{\hat{q}} L_x^{\hat{r}}},$$

satisfying

$$\frac{1}{\hat{r}'} = \frac{1}{\hat{r}_1} + \frac{1}{\hat{r}} \tag{4.64}$$

$$\frac{1}{\hat{q}'} = \frac{1}{\hat{a}_1} + \frac{1}{\hat{q}}, \tag{4.65}$$

if $(\alpha-1)\hat{r}_1 = \hat{r}$ and $(\alpha-1)\hat{a}_1 = \hat{a}$, we get $\hat{r} = \alpha + 1$. Furthermore,

$$\frac{1}{\hat{q}'} = \frac{\alpha-1}{\hat{a}} + \frac{1}{\hat{q}} \tag{4.66}$$

$$\frac{2}{\hat{q}} = \frac{N}{2} - \frac{2}{\hat{r}} \tag{4.67}$$

$$\frac{2}{\hat{a}} = \frac{N}{2} - \frac{2}{\hat{r}} - s_c, \tag{4.68}$$

from the equations (4.67) and (4.68)

$$\frac{2}{\hat{q}} - \frac{2}{\hat{a}} = \frac{N}{2} - \frac{2}{\alpha-1}$$

now, by (4.66)

$$\left(1 - \frac{\alpha-1}{\hat{a}}\right) - \frac{2}{\hat{a}} = \frac{N(\alpha-1)-4}{2(\alpha-1)},$$

this implies that

$$\hat{a} = \frac{2(\alpha-1)(\alpha+1)}{4-(N-2)(\alpha-1)}, \quad \hat{q} = \frac{4(\alpha+1)}{N(\alpha-1)},$$

this prove (4.61).

Now estimate (4.62), choose $(\tilde{a}, \hat{r}) \in \mathcal{A}_{-s_c}$ and $(\hat{a}, \hat{r}) \in \mathcal{A}_{s_c}$ as above, By a similar argument to the previous one, if

$$\left\| |u|^{\alpha-1} v \right\|_{L_t^{\tilde{a}'} L_x^{\hat{r}}} \lesssim \|u\|_{L_t^{\hat{a}} L_x^{\hat{r}}}^{\alpha-1} \|v\|_{L_t^{\hat{a}} L_x^{\hat{r}}},$$

from Holder's inequality, we derive

$$\tilde{a} = \frac{2(\alpha-1)}{(\alpha-1)(N\alpha-2)-4},$$

which subsequently implies (4.62).

Finally, leveraging Lemma 4.3.1 and following a similar approach as in (4.62), we establish (4.63).

□

Lemma 4.4.2 (Small Data Theory). *Let $N \geq 3$, $1 + \frac{4}{N} \leq \alpha \leq 1 + \frac{4}{N-2}$. Then there exists $\delta_{sd} > 0$ such that if*

$$\left\| e^{it\Delta} u_0 \right\|_{S(\dot{H}^{s_c}, [0,+\infty))} \leq \delta_{sd}.$$

Then the solution to (4.1) with condition $u_0 \in H^1(\mathbb{R}^N)$ is globally defined on $[0, +\infty)$. Moreover,

$$\|u\|_{S(\dot{H}^{s_c}, [0,+\infty))} \leq 2 \left\| e^{it\Delta} u_0 \right\|_{S(\dot{H}^{s_c}, [0,+\infty))},$$

and

$$\|u\|_{S(L^2, [0,+\infty))} + \|\nabla u\|_{S(L^2, [0,+\infty))} \leq 2 \|u_0\|_{H_1}.$$

Proof. Let $c > 0$ the implicit constant of Strichartz and define the space

$$E = \left\{ u : \|u\|_{S(\dot{H}^{s_c}, [0,+\infty))} \leq 2 \left\| e^{it\Delta} u_0 \right\|_{S(\dot{H}^{s_c}, [0,+\infty))} \quad \text{and} \quad \|u\|_{S(H^1, [0,+\infty))} \leq 2c \|u_0\|_{H_1} \right\}$$

equipped with the metric

$$vd(u, v) = \|u - v\|_{S(\dot{H}^{s_c})}.$$

We observe that (E, d) is a metric complete space, and let the Duhamel's operator

$$\phi u(t) := e^{it\Delta} u_0 + i \int_0^t e^{i((t-s)\Delta}(|u|^{\alpha-1} u)(s) ds.$$

It sufficiently proves that ϕ: $E \longrightarrow E$ is a contraction. Applying the Strichartz estimates, and (4.4.1) for some $C > 0$,

$$\begin{aligned} \|\phi(u)\|_{S(\dot{H}^{s_\alpha}, [0,+\infty))} &\leq \left\| e^{it\Delta} u_0 \right\|_{S(\dot{H}^{s_\alpha}, [0,+\infty))} + \left\| |u|^{\alpha-1} u \right\|_{S'(\dot{H}^{-s_\alpha}, [0,+\infty))} \\ &\leq \left\| e^{it\Delta} u_0 \right\|_{S(\dot{H}^{s_\alpha}, [0,+\infty))} + C \|u\|_{S(\dot{H}^{s_\alpha})}^{\alpha-1} \|u\|_{S(\dot{H}^{s_\alpha}, [0,+\infty))} \\ &\leq (1 + 2^\alpha C \delta^{\alpha-1}) \left\| e^{it\Delta} u_0 \right\|_{S(\dot{H}^{s_\alpha}, [0,+\infty))}, \end{aligned}$$

If $u \in E$, then $\|\phi(u)\|_{S(\dot{H}^{s_\alpha},[0,+\infty))} \leq 2 \left\|e^{it\Delta}u_0\right\|_{S(\dot{H}^{s_\alpha},[0,+\infty))}$, this implies that $\delta \leq \left(\frac{1}{2^\alpha C}\right)^{\frac{1}{\delta-1}}$. Analogously

$$\begin{aligned}\|\phi(u)\|_{S(L^2,[0,+\infty))} &\leq \left\|e^{it\Delta}u_0\right\| + \left\||u|^{\alpha-1}u\right\|_{S'(L^2,[0,+\infty))}\\ &\leq C\|u_0\|_{L^2} + C\|u\|^{\alpha-1}_{S(\dot{H}^s,[0,+\infty))}\|u\|_{S(L^2,[0,+\infty))}\\ &\leq (1+2^\alpha C\delta^{\alpha-1})c\|u_0\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}\|\nabla\phi(u)\|_{S(L^2,[0,+\infty))} &\leq \left\|\nabla e^{it\Delta}u_0\right\|_{S(L^2,[0,+\infty))} + c\left\|\nabla(|u|^{\alpha-1}u)\right\|_{S'(L^2,[0,+\infty))}\\ &\leq C\|\nabla u_0\|_{L^2} + C\|u\|^{\alpha-1}_{S(\dot{H}^{s_\alpha})}\|\nabla u\|_{S(L^2,[0,+\infty))}\\ &\leq (1+2^\alpha C\delta^{\alpha-1})c\|\nabla u_0\|_{L^2}.\end{aligned}$$

This is

$$\|\phi(u)\|_{S(H^1,[0,+\infty))} \leq (1+2^\alpha C\delta^{\alpha-1})c\|u_0\|_{H_1}$$

It follows that for $\delta \leq \left(\frac{1}{2^\alpha C}\right)^{\frac{1}{\alpha-1}}$, ϕ is well defined.

Furthermore, for $u, v \in X$

$$\begin{aligned}\|\phi(u)-\phi(v)\|_{S(\dot{H}^{s_c},[0,+\infty))} &\leq \left\||u|^{\alpha-1}u - |v|^{\alpha-1}v\right\|_{S(\dot{H}^{s_c},[0,+\infty))}\\ &\leq (\|u\|^{\alpha-1}_{S(\dot{H}^{s_\alpha},[0,+\infty))} + \|v\|^{\alpha-1}_{S(\dot{H}^{s_\alpha},[0,+\infty))})\|u-v\|_{S'(\dot{H}^{s_\alpha},[0,+\infty))}\\ &\leq 2C\delta^{\alpha-1}c\|u-v\|_{S'(\dot{H}^{s_\alpha},[0,+\infty))},\end{aligned}$$

if $2C\delta^{\alpha-1} < 1$, the result follow by the contraction principle. □

Theorem 4.4.6 (Scattering Criterion.). *Let $N \geq 3$, $1+\frac{4}{N} \leq \alpha \leq 1+\frac{4}{N-2}$. If $u(t)$ is a global solution to* (4.1) *in H^1, with $\|u\|_{S(\dot{H}^{s_c},[0,+\infty))} < +\infty$, there exists a function $u_+ \in H^1(\mathbb{R}^N)$ such that*

$$\lim_{t\to+\infty}\left\|u(t) - e^{it\Delta}u_+\right\|_{H^1} = 0.$$

Proof. Let $u(t)$ be the solution to the integral equation

$$u(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}(|u|^{\alpha-1}u)(s)ds.$$

Since that $\|u\|_{S(\dot{H}^{s_c},[0,+\infty))} < \infty$, we can partition $[0,+\infty)$ into N intervals $I_j = [t_j, t_{j+1}]$, such that for each j,

$$\|u\|_{S(\dot{H}^{s_c},I_j)} \leq \delta.$$

From (4.61) and (4.63), we have that

$$\|\langle\nabla\rangle u\|_{S(L^2,I_j)} \lesssim_\delta \|u\|_{L^\infty_t H^1_x(I_j\times\mathbb{R}^N)},$$

which implies, $\|\langle\nabla\rangle u\|_{S(L^2,[0,+\infty))} \lesssim_\delta \|u\|_{L^\infty_t H^1_x}$.

Now, we show that $\left\{e^{-it\Delta}u(t)\right\}$ is Cauchy in H^1 as $t \to \infty$. We fix $t > s > 0$ such that

$$\begin{aligned}
\left\|e^{-it\Delta}u(t) - e^{-is\Delta}u(s)\right\|_{H^1_x} &= \left\|\int_s^t e^{-i\tau\Delta}|u|^{\alpha-1}u(\tau)d\tau\right\|_{H^1_x} \\
&\lesssim \left\||u|^{\alpha-1}u\right\|_{S'(L^2,(s,t))} + \left\|\nabla[|u|^{\alpha-1}u]\right\|_{S'(L^2,(s,t))} \\
&\lesssim \|u\|^{\alpha-1}_{S(\dot{H}^{s_c},(s,t))} \|\langle\nabla\rangle u\|_{S(L^2,(s,t))} \\
&\to 0,
\end{aligned}$$

as $s, t \to \infty$.

Thus, $\left\{e^{-it\Delta}u(t)\right\}$ converges in H^1 as $t \to \infty$, and therefore has a unique limit $u_+ \in H^1$. Notice that

$$\begin{aligned}
\left\|e^{-it\Delta}u(t) - u_+\right\|_{H^1} &= \left\|e^{-it\Delta}(u(t) - e^{it\Delta}u_+)\right\|_{H^1} \\
&= \left\|u(t) - e^{it\Delta}u_+\right\|_{H^1} \to 0,
\end{aligned}$$

as $t \to +\infty$. In fact,

$$u_+ = \lim_{t\to+\infty} e^{-it\Delta}u(t) = u_0 - i\int_0^\infty e^{-is\Delta}|u|^{\alpha-1}u(s)ds.$$

where

$$u_+ = u_0 + i\int_0^\infty e^{-is\Delta}|u|^{\alpha-1}u(s)ds$$

as desired. □

Chapter 5

Recovery of the nonlinear from scattering data

In this chapter, we consider the nonlinear Schrödinger equation of the form

$$(i\partial_t + \Delta)u = a(x)|u|^{\alpha-1}u, \tag{5.1}$$

where $u\colon \mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{C}$, $\alpha > 0$ and $a \in W^{1,\infty}(\mathbb{R}^N)$.

The integral equation, is given by the Duhamel's operator:

$$\phi(u)(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}a(x)|u|^{\alpha-1}(s)ds. \tag{5.2}$$

Due to the pointwise estimates:

$$|a(x)|u|^{\alpha-1}v| \leq \|a\|_{L^\infty} ||u|^{\alpha-1}v|,$$

and

$$|\nabla(a(x)|u|^{\alpha-1}v)| \leq \|\nabla a\|_{L^\infty} ||u|^{\alpha-1}v| + \|a\|_{L^\infty} |\nabla(|u|^{\alpha-1}v)|,$$

the Local theory in $H^1(\mathbb{R}^N)$ studied in the Chapter 4 holds for the nonlinear Schrödinger equation (5.1):

Theorem 5.0.1 ($H^1(\mathbb{R}^N)$ subcritical)**.** *Let $N \geq 3$, $a \in W^{1,\infty}(\mathbb{R}^N)$, $1 < \alpha < 1 + \frac{4}{N-2}$ and $a \in W^{1,\infty}(\mathbb{R}^N)$. If $u_0 \in H^1(\mathbb{R}^N)$ then there exist $T = T(\|u_0\|_{H^1}, \|a\|_{W^{1,\infty}}, N, \alpha) > 0$ and a unique solution u of the integral equation* (5.2)*, such that*

$$u \in \mathcal{C}([-T,T] : H^1(\mathbb{R}^N)) \cap L^r([-T,T] : H^{1,\rho}(\mathbb{R}^N)),$$

for

$$(r,\rho) = \left(\frac{4(\alpha+1)}{(N-2)(\alpha-1)}, \frac{N(\alpha+1)}{N+\alpha-1}\right),$$

an L^2 admissible pair.

Theorem 5.0.2 ($H^1(\mathbb{R}^N)$ critical)**.** *Let $N \geq 3$, $a \in W^{1,\infty}(\mathbb{R}^N)$ and $\alpha = \frac{N+2}{N-2}$. Given $u_0 \in H^1(\mathbb{R}^N)$, there exists $T = T(u_0, \|a\|_{W^{1,\infty}}, N, \alpha) > 0$ and unique solution u of the the integral equation* (5.2)*, such that*

$$u \in \mathcal{C}([-T,T] : H^1(\mathbb{R}^N)) \cap L^r([-T,T] : H^{1,\rho}(\mathbb{R}^N)),$$

for

$$(r,\rho)=\left(\frac{2N}{N-2},\frac{2N^2}{N^2-2N+4}\right),$$

an L^2-admissible.

We shall prove that the equation (5.1) admits small-data scattering theory in H^1 if the nonlinear exponent α is chosen such that $1+\frac{4}{N}\le\alpha\le 1+\frac{4}{N-2}$, and

$$0\le s_c\le 1,\quad \text{where}\quad s_c=\frac{N}{2}-\frac{2}{\alpha-1}.$$

We prove stability estimates for the problem of recovering the nonlinearity from scattering data, addressing both in the nonlinearity power and in the inhomogeneous term (See Theorems 5.0.5 and 5.0.4 respectively). This dissertation extends the results of Chen and Murphy [4] (2023) to the N-dimensional intercritical case, $N\ge 3$.

Using Strichartz's estimates, Holder's inequality, we prove the following standard estimate.

Lemma 5.0.1. *Let $N\ge 3$, $a\in W^{1,\infty}(\mathbb{R}^N)$, $u\colon\mathbb{R}\times\mathbb{R}^N\to\mathbb{C}$ and $1+\frac{4}{N}\le\alpha\le 1+\frac{4}{N-2}$. Then there exist a (q,r) L^2-admissible, such that the following inequality holds*

$$\left\|a|u|^{\alpha-1}u\right\|_{L_t^{q'}L_x^{r'}}\lesssim\|a\|_{L^\infty}\|u\|_{L_t^qL_x^{r_c}}^{\alpha-1}\|u\|_{L_t^qL_x^r}, \tag{5.3}$$

where

$$r_c=\frac{N(\alpha-1)(\alpha+1)}{4}.$$

Proof. Let

$$\begin{aligned}\left\|a|u|^{\alpha-1}u\right\|_{L_t^{q'}L_x^{r'}}&\le\|a\|_{L^\infty}\left\|\|u\|_{L_x^{(\alpha-1)r_1}}^{\alpha-1}\|u\|_{L_x^r}\right\|_{L_t^{q'}}\\&\le\|a\|_{L^\infty}\|u\|_{L_t^qL_x^{r_c}}^{\alpha-1}\|u\|_{L_t^qL_x^r}.\end{aligned}$$

with

$$\frac{1}{q'}=\frac{1}{q_1}+\frac{1}{q} \tag{5.4}$$

$$\frac{1}{r'}=\frac{1}{r_1}+\frac{1}{r} \tag{5.5}$$

$$\frac{2}{q}=\frac{N}{2}-\frac{N}{r}. \tag{5.6}$$

If $q_1(\alpha-1)=q$ in the equation (5.4), we have $q=\alpha+1$. In the equation (5.6), we find that

$$\frac{1}{r}=\frac{N(\alpha+1)-4}{2N(\alpha+1)}.$$

For $(\alpha-1)r_1=r_c$ and equation (5.5)

$$\frac{1}{r_c}=\frac{4}{N(\alpha-1)(\alpha+1)}. \tag{5.7}$$

With condition $1+\frac{4}{N}\le\alpha\le 1+\frac{4}{N-2}$ it is clear that

$$2\le r\le\frac{2N}{N-2}.$$

We finish the proof of the lemma. □

Theorem 5.0.3 (Small data Scattering). *Let $a \in W^{1,\infty}$ and $1+\frac{4}{N} \leq \alpha \leq 1+\frac{4}{N-2}$. Define*

$$(q,r) = \left(\alpha+1, \frac{2N(\alpha+1)}{N(\alpha+1)-4}\right) \tag{5.8}$$

There exists $\eta_{sd} > 0$ sufficiently small so that for any $u_- \in H^1$ satisfying $\|u_-\|_{H^1} < \eta_{sd}$, there exists a unique global solution u to (5.1) *and $u_+ \in H^1$ satisfying the following:*

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^N)} \lesssim \|u_-\|_{L^2},$$
$$\|\nabla u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^N)} \lesssim \|u_-\|_{H^1},$$
$$\||\nabla|^{s_c} u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^N)} \lesssim \|u_-\|_{\dot{H}^{s_c}},$$

and

$$\lim_{t\to\pm\infty} \left\|u(t) - e^{it\Delta}u_\pm\right\|_{H^1} = 0 \tag{5.9}$$

Proof. We will show that the map

$$u(t) \to \phi(u)(t) := e^{it\Delta}u_- - i\int_{-\infty}^{t} e^{i(t-s)\Delta}a(x)|u|^{\alpha-1}u(s)ds \tag{5.10}$$

is a contraction on a suitable complete metric space (E,d), whenever $u_- \in H^1$ and $\|u_-\|_{H^1} \leq \eta_{sd} \ll 1$, where

$$E := \Big\{u\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}\colon \ \|u\|_{L_t^q L_x^r} \leq 4C\|u_-\|_{L^2}, \|\nabla u\|_{L_t^q L_x^r} \leq 4C\|u_-\|_{H^1},$$
$$\||\nabla|^{s_c}u\|_{L_t^q L_x^r} \leq 4C\|u_-\|_{\dot{H}^{s_c}}\Big\},$$

equipped with the metric

$$d(u,v)\colon = \|u-v\|_{L_t^q L_x^r}.$$

The constant $C>0$ in the definition of E, encodes the implicit constants appearing in the Strichartz and Sobolev embedding.

Using Sobolev embedding, Strichartz estimates, and Hölder's inequality and Lemma 5.0.1 for $u \in E$, we estimate

$$\begin{aligned}
\|\phi(u)\|_{L_t^q L_x^r} &\leq \left\|e^{it\Delta}u_-\right\|_{L_t^q L_x^r} + \left\|i\int_{-\infty}^{t} e^{i(t-s)\Delta}a(x)|u|^{\alpha-1}u(s)ds\right\|_{L_t^q L_x^r} \\
&\leq C\|u_-\|_{L^2} + C\left\|a|u|^{p-1}u\right\|_{L_t^{q'}L_x^{r'}} \\
&\leq C\|u_-\|_{L^2} + C\|a\|_{L^\infty}\|u\|_{L_t^q L_x^{r_c}}^{\alpha-1}\|u\|_{L_t^q L_x^r} \\
&\leq C\|u_-\|_{L^2} + C^2\|a\|_{L^\infty}\||\nabla|^{s_c}\|_{L_t^q L_x^r}^{\alpha-1}\|u\|_{L_t^q L_x^r} \\
&\leq C\|u_-\|_{L^2} + 4^\alpha C^{2\alpha+1}\|a\|_{L^\infty}\eta_{sd_1}^{\alpha-1}\|u_-\|_{L^2}.
\end{aligned}$$

Above, we have used the fact that if $r_c = \frac{N(\alpha-1)(\alpha+1)}{4}$ as in the Lemma 5.0.1 and observe that $\dot{H}^{s_c,r} \hookrightarrow L^{r_c}$, since

$$\begin{aligned}
\frac{N}{r} - \frac{N}{r_c} &= \frac{N(\alpha+1)-4}{2(\alpha+1)} - \frac{4}{(\alpha-1)(\alpha+1)} \\
&= \frac{\alpha\left[N(\alpha-1)-4\right]}{2\alpha(\alpha-1)} \\
&= s_c,
\end{aligned}$$

and with the condition $1+\frac{4}{N} \leq \alpha \leq 1+\frac{4}{N-2}$, it follows that $0 \leq s_c \leq 1$. In particular, for

$$\eta_{sd_1} < \left(\frac{3}{4^{\alpha} C^{2\alpha} \|a\|_{L^\infty}}\right)^{\frac{1}{\alpha-1}}$$

is satisfied

$$\|\phi(u)\|_{L_t^q L_x^r} \leq 4C \|u_-\|_{L^2} .$$

Similarly, using the product rule, Sobolev embedding and Lemma 4.3.1

$$\begin{aligned}
\|\nabla \phi(u)\|_{L_t^q L_x^r} &\leq \|\nabla u_-\|_{L^2} + \left\|\nabla (a|u|^{\alpha-1}u)\right\|_{L_t^{q'} L_x^{r'}} \\
&\leq C \|u_-\|_{\dot{H}^1} + \|\nabla a\|_{L^\infty} \left\||u|^{\alpha-1}u\right\|_{L_t^{q'} L_x^{r'}} + C\|a\|_{L^\infty} \left\|\nabla (|u|^{\alpha-1}u)\right\|_{L_t^{q'} L_x^{r'}} \\
&\leq C \|u_-\|_{\dot{H}^1} + C\left\{\|\nabla a\|_{L^\infty} \|u\|_{L_t^q L_x^{r_c}}^{\alpha-2} \|u\|_{L_t^q L_x^r} \|u\|_{L_t^q L_x^{r_c}} + \|a\|_{L^\infty} \|u\|_{L_t^q L_x^{r_c}}^{\alpha-1} \|u\|_{L_t^q L_x^r}\right\} \\
&\leq C \|u_-\|_{\dot{H}^1} + 4^{\alpha-1} C^{3\alpha-3} \eta_{sd_2}^{\alpha-1} \|\nabla a\|_{L^\infty} \||\nabla|^{s_c}\|_{L_t^q L_x^r} + 4^{\alpha} C^{3\alpha-1} \eta_{sd_2}^{\alpha-1} \|a\|_{L^\infty} \|u_-\|_{H^1} \\
&\leq C \|u_-\|_{\dot{H}^1} + 4^{\alpha} C^{3\alpha-1} \eta_{sd_2}^{\alpha-1} \|a\|_{W^{1,\infty}} \|u_-\|_{H^1} .
\end{aligned}$$

Provided η_{sd_2} is chosen small enough, such that

$$\eta_{sd_2} < \left(\frac{3}{4^{\alpha} C^{3\alpha-2} \|a\|_{W^{1,\infty}}}\right)^{\frac{1}{\alpha-1}},$$

so that

$$\|\nabla \phi(u)\|_{L_t^q L_x^r} \leq 4C \|u_-\|_{H^1} .$$

Finally,

$$\||\nabla|^{s_c} \phi(u)\|_{L_t^q L_x^r} \leq C \|u_-\|_{\dot{H}^{s_c}} + C \left\|a|u|^{\alpha-1}u\right\|_{L_t^{q'} \dot{H}_x^{s_c, r'}} .$$

Since that $0 \leq s_c \leq 1$, by interpolation

$$\begin{aligned}
\left\|a|u|^{\alpha-1}u\right\|_{\dot{H}_x^{s_c,r'}} &\leq \left\|a|u|^{\alpha-1}u\right\|_{\dot{H}_x^{0,r'}}^{1-s_c} \left\|a|u|^{\alpha-1}u\right\|_{\dot{H}^{1,r'}}^{s_c} \\
&\lesssim \left\|a|u|^{\alpha-1}u\right\|_{\dot{H}_x^{0,r'}} + \left\|a|u|^{\alpha-1}u\right\|_{\dot{H}^{1,r'}} = \left\|a|u|^{\alpha-1}u\right\|_{H_x^{1,r'}} ,
\end{aligned}$$

therefore,

$$\begin{aligned}
\||\nabla|^{s_c} \phi(u)\|_{L_t^q L_x^r} &\leq C \|u_-\|_{\dot{H}^{s_c}} + C \left\|a|u|^{\alpha-1}u\right\|_{L_t^{q'} H_x^{1,r'}} \\
&\leq C \|u_-\|_{\dot{H}^{s_c}} + C \|a\|_{L^\infty} \|u\|_{L_t^q L_x^{r_c}}^{\alpha-2} \|u\|_{L_t^q L_x^{r_c}} \left(\|u\|_{L_t^q L_x^r} + \|\nabla u\|_{L_t^q L_x^r}\right) \\
&+ C \|\nabla a\|_{L^\infty} \|u\|_{L_t^q L_x^{r_c}}^{\alpha-2} \|u\|_{L_t^q L_x^r} \|u\|_{L_t^q L_x^{r_c}} \\
&\leq C \|u_-\|_{\dot{H}^{s_c}} + C \|a\|_{W^{1,\infty}} \|u\|_{L_t^q L_x^{r_c}}^{\alpha-2} \|u\|_{L_t^q H_x^{1,r}} \|u\|_{L_t^q L_x^{r_c}} \\
&\leq C \|u\|_{\dot{H}^{s_c}} + 4^{\alpha} C^{\alpha+3} \eta_{sd_3}^{\alpha-1} \|a\|_{W^{1,\infty}} \|u_-\|_{\dot{H}^{s_c}} ,
\end{aligned}$$

it follows that for

$$\eta_{sd_3} < \left(\frac{3}{4^{\alpha} C^{\alpha+2} \|a\|_{W^{1,\infty}}}\right)^{\frac{1}{\alpha-1}},$$

this guarantees that

$$\||\nabla|^{s_c}\phi(u)\|_{L_t^q L_x^r} \leq 4C\,\|u_-\|_{\dot{H}^{s_c}}.$$

Finally, by choosing $\eta_{sd} \leq \min\{\eta_{sd_1}, \eta_{sd_2}, \eta_{sd_3}\}$, we conclude that $\phi\colon E \to E$ is well defined.

Now to prove that ϕ is a contraction, for $u, v \in E$:

$$\begin{aligned}\|\phi(u)-\phi(v)\|_{L_t^q L_x^r} &\leq C\left\|a(|u|^{\alpha-1}u - |v|^{\alpha-1}v)\right\|_{L_t^{q'} L_x^{r'}}\\ &C\,\|a\|_{L^\infty}\,(\|u\|_{L_t^q L^{r_c}}^{\alpha-1} + \|v\|_{L_t^q L^{r_c}}^{\alpha-1})\,\|u-v\|_{L_t^q L_x^r}\\ &\leq 2(4^{\alpha-1}C^{2\alpha}\eta_{sd}^{\alpha-1})\,\|u-v\|_{L_t^q L_x^r}\end{aligned}$$

We show that ϕ is a contraction if:

$$\eta_{sd} < \left(\frac{1}{2(4^{\alpha-1}C^{2\alpha})\,\|a\|_{L^\infty}}\right)^{\frac{1}{\alpha-1}}.$$

By the Banach fixed point theorem, it follows that ϕ has a unique fixed point $u \in E$, which is our desired solution.

Finally, for (5.9) we will prove that $\{e^{-it\Delta}u(t)\}$ is Cauchy in H^1 as $t \to \infty$. We fixed $t > s > 0$ and used the estimates above to obtain

$$\begin{aligned}\left\|e^{-it\Delta}u(t) - e^{-is\Delta}u(s)\right\|_{H_x^1} &= \left\|\int_s^t e^{-i\tau\Delta}a(x)|u|^{\alpha-1}u(\tau)d\tau\right\|_{H_x^1}\\ &\leq C\left\|a|u|^{\alpha-1}u\right\|_{L_t^{q'} H_x^{1,r'}((s,t)\in\mathbb{R}\times\mathbb{R}^N)}\\ &\leq C\,\|a\|_{W^{1,\infty}}\,\|u\|_{L_t^q L_x^{r_c}((s,t)\times\mathbb{R}^N)}^{\alpha-1}\,\|u\|_{L_t^q H_x^{1,r}((s,t)\times\mathbb{R}^N)}\end{aligned}$$

which converges to zero as $s, t \to \infty$, and hence has a unique limit $u_+ \in H^1$ as $t \to +\infty$.

Thus

$$\begin{aligned}\left\|u(t) - e^{it\Delta}u_+\right\|_{H^1} &= \left\|e^{it\Delta}(e^{-it\Delta}u(t) - u_+)\right\|_{H^1}\\ &= \left\|e^{-it\Delta}u(t) - u_+\right\|_{H^1} \to 0,\end{aligned}$$

as $t \to \infty$. From the Duhamel formula (5.10) can we obtain the implicit formula

$$u_+ = \lim_{t\to\infty} e^{-it\Delta}u(t) = u_- - i\int_{\mathbb{R}} e^{-is\Delta}a|u|^{\alpha-1}u(s)ds.$$

for the final state u_+. □

We define the scattering map $S_a\colon B \to H^1$ via $S_a(u_-) = u_+$, where B is a suitabily small ball in H^1. This map encodes of all information about the nonlinearity in (5.1), in the sense that the map is injective (See Murphy [16], Corollary 4.2).

Definition 5.0.1. *The norm between two scattering maps S_a, S_b for* (5.1) *is*

$$\|S_a - S_b\| := \sup\left\{\frac{\|S_a(\varphi) - S_b(\varphi)\|}{\|\varphi\|_{H^1}} : \varphi \in B\setminus\{0\}\right\},$$

where $B \subset H^1$ is the common domain of S_a and S_b.

Now, we present an explicit formula for the solution to the linear Schröndinger equation with Gaussian data.

Lemma 5.0.2. *Let*

$$\varphi_{\sigma,x_0}(x) = \exp\left\{-\frac{|x-x_0|^2}{4\sigma^2}\right\}, \tag{5.11}$$

with $x_0 \in \mathbb{R}^N$ *and* $\sigma > 0$. *Then*

$$e^{it\Delta}\varphi_{\sigma,x_0}(x) = \left(\frac{\sigma^2}{\sigma^2+it}\right)^{\frac{N}{2}} exp\left\{-\frac{|x-x_0|^2}{4(\sigma^2+it)}\right\}. \tag{5.12}$$

Proof. Consider $u_0(x) = \varphi_{\sigma,x_0}(x)$ in (3.2). We find that the solution of initial value problem (3.1) is given by

$$\begin{aligned} e^{it\Delta}\varphi_{\sigma,x_0} &= (e^{-4\pi^2 it|\xi|^2}\widehat{\varphi_{\sigma,x_0}}(\xi))^{\vee} \\ &= \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi - 4\pi^2 it|\xi|^2}\widehat{\varphi_{\sigma,x_0}}(\xi)d\xi. \end{aligned}$$

Using the proposition 2.1.4, we have that

$$\begin{aligned} \widehat{\varphi_{\sigma,x_0}}(\xi) &= e^{-2\pi i x_0\cdot\xi}\widehat{e^{-\frac{|\cdot|^2}{4\sigma^2}}}(\xi) \\ &= (4\pi\sigma^2)^{\frac{N}{2}}e^{-2\pi i x_0\cdot\xi}\widehat{\left(\frac{e^{-\frac{|\cdot|^2}{4\sigma^2}}}{(4\pi\sigma^2)^{\frac{N}{2}}}\right)}(\xi) \\ &= (4\pi\sigma^2)^{\frac{N}{2}}e^{-2\pi i x_0\cdot\xi}e^{-4\pi^2\sigma^2|\xi|^2}, \end{aligned} \tag{5.13}$$

thus

$$\begin{aligned} e^{it\Delta}\varphi_{\sigma,x_0}(x) &= (4\pi\sigma^2)^{\frac{N}{2}}\prod_{j=1}^{N}\int_{\mathbb{R}} e^{-4\pi^2\xi_j^2(\sigma^2+it)+2\pi\xi_j(x_j-x_{0_j})}d\xi_j \\ &= (4\pi\sigma^2)^{\frac{N}{2}}\prod_{j=1}^{N}\int_{\mathbb{R}} e^{-4\pi^2(\sigma^2+it)\left[\xi_j^2 - i\frac{\xi_j\left(x_j-x_{0_j}\right)}{2\pi(\sigma^2+it)}\right]}d\xi_j \\ &= (4\pi\sigma^2)^{\frac{N}{2}}\prod_{j=1}^{N}\int_{\mathbb{R}} e^{-4\pi^2(\sigma^2+it)\left[\left(\xi_j - i\frac{\left(x_j-x_{0_j}\right)}{4\pi(\sigma^2+it)}\right)^2 + \frac{\left(x_j-x_{0_j}\right)^2}{16\pi^2(\sigma^2+it)^2}\right]}d\xi_j \\ &= (4\pi\sigma^2)^{\frac{N}{2}}\prod_{j=1}^{N}e^{-\frac{(x_j-x_{0_j})^2}{4(\sigma^2+it)}}\int_{\mathbb{R}} e^{-4\pi^2(\sigma^2+it)\left[\xi_j - i\frac{\left(x_j-x_{0_j}\right)}{4\pi(\sigma^2+it)}\right]^2}d\xi_j. \end{aligned} \tag{5.14}$$

Since that $f(z) = e^{-z^2}$ is a entire function, by the Cauchy's Theorem

$$\int_{\gamma} f(z)dz = 0,$$

where $\gamma := \gamma_1 \cup \gamma_2\gamma_3 \cup \gamma_4$, as the Figure 5.1.

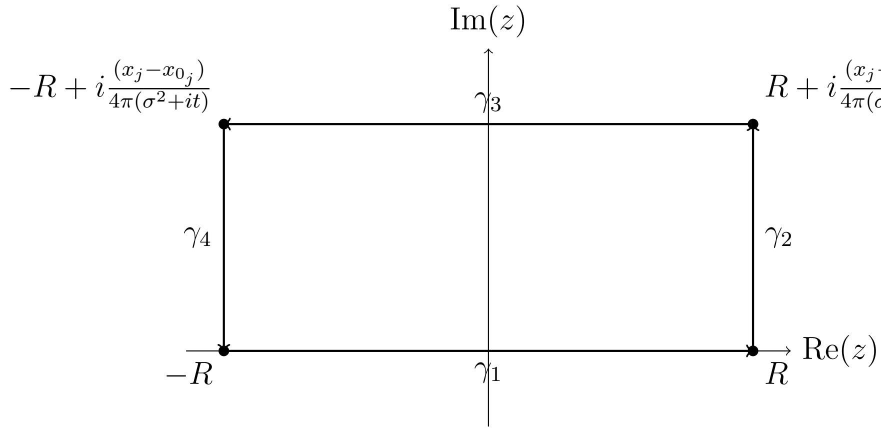

Figure 5.1:

It is clear that along the paths γ_4 and γ_2 the integral is zero. Indeed, one these segments are parametrised by $z = \pm R + i\xi_j$, with $|\xi_j| \leq \left|\frac{(x_j - x_{0_j})}{4\pi(\sigma^2+it)}\right|$, thus

$$\left|e^{-z^2}\right| = e^{\mathrm{Re}(-z^2)} \leq e^{\left(\frac{(x_j-x_{0_j})}{4\pi(\sigma^2+it)}\right)^2} e^{-R^2} \to 0,$$

as $R \to \pm\infty$.

Therefore

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-\left(\xi_j - i\frac{(x_j-x_{0_j})}{4\pi(\sigma^2+it)}\right)^2} d\xi_j = \int_{-\infty}^{\infty} e^{-\xi_j^2} d\xi_j = \pi^{\frac{1}{2}}.$$

Again in the equation (3.3)

$$\begin{aligned} e^{it\Delta}\varphi_{\sigma,x_0} &= (4\pi\sigma^2)^{\frac{N}{2}} \prod_{j=1}^{N} \left[e^{-\frac{(x_j-x_{0_j})^2}{4(\sigma^2+it)}} \left(\frac{\pi}{4\pi^2(\sigma^2+it)}\right)^{\frac{1}{2}} \right] \\ &= \left(\frac{\sigma^2}{\sigma^2+it}\right)^{\frac{N}{2}} e^{-\frac{|x-x_0|^2}{4(\sigma^2+it)}}. \end{aligned}$$

□

Proposition 5.0.1 (Approximate identity estimates). *Let $N \geq 1$ and $p > 1 + \frac{2}{N}$. Given $x_0 \in \mathbb{R}^N$ and $\sigma > 0$, define*

$$\varphi_{\sigma,x_0} = \exp\left\{-\frac{|x-x_0|^2}{4\sigma^2}\right\} \tag{5.15}$$

and

$$\lambda(N,p) := \pi^{\frac{N}{2}+\frac{1}{2}} \left(\frac{4}{p+1}\right)^{\frac{N}{2}} \frac{\Gamma\left(\frac{N(p-1)}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{N(p-1)}{4}\right)},$$

where

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \; x > 0$$

is the Gamma function.

Given $a \in W^{1,\infty}(\mathbb{R}^N)$, we have

$$\left| \int \int_{\mathbb{R}\times\mathbb{R}^N} \left| e^{it\Delta} \varphi_{\sigma,x_0}(x) \right|^{p+1} a(x) dx dt - \sigma^{N+2} \lambda(N,p) a(x_0) \right| \leq \sigma^{\frac{s}{1+s}} \sigma^{N+2} \|a\|_{W^{1,\infty}}$$

for any $0 < s < \frac{N(p-1)}{2} - 1$.

Proof. As shown in the Lemma 5.0.2,

$$e^{it\Delta} \varphi_{\sigma,x_0}(x) = \left(\frac{\sigma^2}{\sigma^2 + it} \right)^{\frac{N}{2}} exp \left\{ -\frac{|x - x_0|^2}{4(\sigma^2 + it)} \right\}.$$

Now, if we let $z_1 = \left(\frac{\sigma^2}{\sigma^2 + it} \right)^{\frac{N}{2}}$ we obtain

$$\begin{aligned} |z_1|^{p+1} &= \left\{ \left[\left(\frac{\sigma^2}{\sigma^2 + it} \right) \left(\frac{\sigma^2}{\sigma^2 - it} \right) \right]^{\frac{N}{2}} \right\}^{\frac{p+1}{2}} \\ &= \left(\frac{\sigma^4}{\sigma^4 + t^2} \right)^{\frac{N(p+1)}{4}}, \end{aligned}$$

and for $z_2 = exp \left\{ -\frac{|x - x_0|^2}{4(\sigma^2 + it)} \right\}$, we have

$$\begin{aligned} |z_2|^{p+1} &= exp \left\{ (p+1) \mathrm{Re} \left(-\frac{|x - x_0|^2}{4(\sigma^2 + it)} \right) \right\} \\ &= exp \left\{ -\frac{\sigma^2 |x - x_0|^2 (p+1)}{4(\sigma^4 + t^2)} \right\}, \end{aligned}$$

where

$$\begin{aligned} \left| e^{it\Delta} \varphi_{\sigma,x_0} \right|^{p+1} &= \left(\frac{\sigma^4}{\sigma^4 + t^2} \right)^{\frac{N(p+1)}{4}} exp \left\{ -\frac{\sigma^2 |x - x_0|^2 (p+1)}{4(\sigma^4 + t^2)} \right\} \\ &= K \left(\frac{t}{\sigma^2}, \frac{x - x_0}{\sigma} \right), \end{aligned}$$

here

$$K(t,x) := \left(\frac{1}{1 + t^2} \right)^{\frac{N(p+1)}{4}} exp \left\{ -\frac{|x|^2 (p+1)}{4(1 + t^2)} \right\}.$$

We now show that $\int K(t,x) dx dt = \lambda(N,p)$. For that, we consider the changes of variables

$$u = \left(\frac{p+1}{4(1 + t^2)} \right)^{\frac{1}{2}} |x|, \quad \text{and} \quad v = \frac{1}{1 + t^2}.$$

Recall the Gaussian integral and Gamma function to obtain

$$\begin{aligned} \int_{\mathbb{R}\times\mathbb{R}^N} K(t,x) dx dt &= \left(\frac{4(1 + t^2)}{p+1} \right)^{\frac{N}{2}} \int_{\mathbb{R}^N} e^{-|u|^2} du \int_{\mathbb{R}} \left(\frac{1}{1 + t^2} \right)^{\frac{N(p+1)}{4}} dt \\ &= \pi^{\frac{N}{2}} \left(\frac{4}{p+1} \right)^{\frac{N}{2}} \int_{\mathbb{R}} (1 + t^2)^{-\frac{N(p-1)}{4}} dt \end{aligned}$$

Let $c = \frac{N(p-1)}{4}$, and we use the change of variables $u = (1+t^2)^{-1}$ to obtain the integral

$$\begin{aligned}\int_{\mathbb{R}} (1+t^2)^{-c} dt &= 2\int_0^\infty (1+t^2)^{-\frac{Np}{4}} dt \\ &= -\int_1^0 u^{c-2+\frac{1}{2}}(1-u)^{-\frac{1}{2}} du \\ &= \int_0^1 u^{\left(c-\frac{1}{2}\right)-1}(1-u)^{\frac{1}{2}-1} du \\ &= B\left(\frac{1}{2}, c-\frac{1}{2}\right) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\right)}{\Gamma(c)} = \pi^{\frac{1}{2}}\frac{\Gamma\left(c-\frac{1}{2}\right)}{\Gamma(c)}.\end{aligned}$$

The convergence of the Gamma function requires that $c > \frac{1}{2}$. The Euler function Beta, B, is defined as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

for $x, y > 0$. For details, see Conway [5, p. 186]. Thus

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}^N} K(t,x) dx dt &= \pi^{\frac{N}{2}+\frac{1}{2}}\left(\frac{4}{p+1}\right)^{\frac{N}{2}} \frac{\Gamma\left(\frac{N(p-1)}{4}-\frac{1}{2}\right)}{\Gamma\left(\frac{N(p-1)}{4}\right)} \\ &= \lambda(N,p),\end{aligned}$$

where $p > 1 + \frac{2}{N}$.

We note that for any $R > 0$, by Chevyshev's inequality (Lemma 2.2.4), and employing a change of variables similar to the one given above

$$\begin{aligned}\int_{\mathbb{R}}\int_{|x|>R} K(t,x) dx dt &\lesssim R^{-s}\int\int |x|^s K(t,x) dx dt \\ &= \left(R^{-s}\int (1+t^2)^{-\frac{N(p+1)}{4}} dt\right)\left(\int\left(\frac{4(1+t^2)}{p+2}\right)^{\frac{s}{2}+\frac{N}{2}} |u|^s e^{-|u|^2} du\right) \\ &\lesssim R^{-s}\int (1+t^2)^{-\frac{N(p-1)}{4}+\frac{s}{2}} dt \\ &= CR^{-s},\end{aligned} \tag{5.16}$$

where

$$C := \pi^{\frac{1}{2}}\frac{\Gamma\left(\left(\frac{N(p-1)}{4}-\frac{s}{2}\right)-\frac{1}{2}\right)}{\Gamma\left(\frac{N(p-1)}{4}-\frac{s}{2}\right)},$$

usig the fact that $0 < s < \frac{N(p-1)}{2} - 1$.

By another change of variables, we have

$$\int\int_{\mathbb{R}\times\mathbb{R}^N} K\left(\frac{t}{\sigma^2}, \frac{x}{\sigma}\right) dx dt = \sigma^{N+2}\lambda(N,p). \tag{5.17}$$

Thus we can write

$$\begin{aligned}
&\left|\int\int_{\mathbb{R}\times\mathbb{R}^N} \left|e^{it\Delta}\varphi_{\sigma,x_0}\right|^{p+1} a(x)dxdt - \sigma^{N+2}\lambda(N,p)a(x_0)\right| \\
&= \left|\int\int_{\mathbb{R}\times\mathbb{R}^N} K\left(\frac{t}{\sigma^2},\frac{x}{\sigma}\right)(a(x+x_0)-a(x_0))\,dxdt\right| \\
&\leq \int_{\mathbb{R}}\int_{|x|\leq\delta} K\left(\frac{t}{\sigma^2},\frac{x}{\sigma}\right)|a(x+x_0)-a(x_0)|\,dxdt \qquad (5.18)\\
&+ \int_{\mathbb{R}}\int_{|x|>\delta} K\left(\frac{t}{\sigma^2},\frac{x}{\sigma}\right)|a(x+x_0)-a(x_0)|\,dxdt, \qquad (5.19)
\end{aligned}$$

where $\delta > 0$ will be chosen below.

By the fundamental theorem of calculus and (5.17), we first estimate

$$(5.18) \lesssim \delta\sigma^{N+2}\left\|\nabla a\right\|_{L^\infty},$$

and by (5.16), we obtain

$$(5.19) \lesssim \sigma^{N+2}\left\|a\right\|_{L^\infty}\int_{\mathbb{R}}\int_{|y|>\frac{\delta}{\sigma}} K(t,y)dydt \lesssim_s \left(\frac{\delta}{\sigma}\right)^{-s}\sigma^{N+2}\left\|a\right\|_{L^\infty},$$

so,

$$(5.18) + (5.19) \lesssim \sigma^{N+2}\left\|\nabla a\right\|_{W^{1,\infty}}\left(\delta+\sigma^s\delta^{-s}\right).$$

Define $f(\delta) = \delta + \sigma^s\delta^{-s}$ and finding the critical point by setting $\delta = \sigma^{\frac{s}{1+s}}$ leads to

$$\left|\int\int_{\mathbb{R}\times\mathbb{R}^N} \left|e^{it\Delta}\varphi_{\sigma,x_0}(x)\right|^{p+1} a(x)dxdt - \sigma^{N+2}\lambda(d,p)a(x_0)\right| \lesssim_s \sigma^{\frac{s}{1+s}}\sigma^{N+2}\left\|a\right\|_{W^{1,\infty}}$$

for any $0 < s < \frac{N(p-1)}{2} - 1$.

□

Theorem 5.0.4. *Suppose* $1+\frac{4}{N} \leq \alpha \leq 1+\frac{4}{N-2}$ *and let* $a, b \in W^{1,\infty}(\mathbb{R}^N)$. *Let* S_a, S_b *denote the corresponding scattering maps for* (5.1) *with nonlinearities* $a|u|^{\alpha-1}u$ *and* $b|u|^{\alpha-1}u$, *respectively. Then*

$$\begin{aligned}
\left\|a-b\right\|_{L^\infty} &\lesssim \left(\left\|a\right\|_{W^{1,\infty}}+\left\|b\right\|_{W^{1,\infty}}\right)^{\frac{10}{11}}\left\|S_a-S_b\right\|^{\frac{1}{11}} \\
&+ \left(\left\|a\right\|_{W^{1,\infty}}+\left\|b\right\|_{W^{1,\infty}}\right)^{\frac{12}{11}}\left\|S_a-S_b\right\|^{\frac{10}{11}}.
\end{aligned} \qquad (5.20)$$

Proof. Given $\sigma > 0$ and $x_0 \in \mathbb{R}^N$, let φ_{σ,x_0} be the Gaussian data defined as in (5.15). From (5.13)

$$\widehat{\varphi_{\sigma,x_0}}(\xi) = (4\pi\sigma^2)^{\frac{N}{2}}e^{-2\pi i x_0\cdot\xi}e^{-4\pi^2\sigma^2|\xi|^2}.$$

Let $s > -\frac{N}{2}$,

$$\begin{aligned}\|\varphi_{\sigma,x_0}\|^2_{\dot{H}^s(\mathbb{R}^N)} &= \left\|\widehat{\nabla\varphi_{\sigma,x_0}}(\xi)\right\|^2_{L^2(\mathbb{R}^N)}\\ &= (4\pi)^N\sigma^{2N}\int_{\mathbb{R}^N}|\xi|^{2s}e^{-|2\pi\sigma|\xi||^4}d\xi,\end{aligned}$$

Through the change of variable $y = 2\pi\sigma|\xi|$, we obtain:

$$(4\pi)^N\frac{\sigma^{2N}}{(2\pi\sigma)^N}\int_{\mathbb{R}^N}\left|\frac{y}{2\pi\sigma}\right|^{2s}e^{-|y|^4}dy,$$

which leads to

$$\|\varphi_{\sigma,x_0}\|_{\dot{H}^s(\mathbb{R}^N)} \lesssim \sigma^{\frac{N}{2}-s}. \tag{5.21}$$

We then have that φ_{σ,x_0} belongs to the common domain of S_a and S_b for all $\sigma > 0$ sufficiently small. Recall the Scattering map

$$S_a(\varphi_{\sigma,x_0}) = \varphi_{\sigma,x_0} - i\int_{\mathbb{R}} e^{-it\Delta}a|u|^{\alpha-1}u(t)dt,$$

where u is the solution to (5.1) that scatters to φ_{σ,x_0} as $t \to -\infty$. We can write it as

$$\begin{aligned}S_a(\varphi_{\sigma,x_0}) =& \varphi_{\sigma,x_0} - i\int_{\mathbb{R}}\left\{a|e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}e^{it\Delta}\varphi_{\sigma,x_0}\right\}dt\\ &- i\int_{\mathbb{R}} e^{-it\Delta}\left\{a\left(|u|^{\alpha-1}u - |e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}\right)e^{it\Delta}\varphi_{\sigma,x_0}\right\},\end{aligned}$$

and

$$\begin{aligned}S_b(\varphi_{\sigma,x_0}) =& \varphi_{\sigma,x_0} - i\int_{\mathbb{R}}\left\{b|e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}e^{it\Delta}\varphi_{\sigma,x_0}\right\}dt\\ &- i\int_{\mathbb{R}} e^{-it\Delta}\left\{b\left(|v|^{\alpha-1}v - |e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}\right)e^{it\Delta}\varphi_{\sigma,x_0}\right\}dt,\end{aligned}$$

where v is the solution to (5.1), with nonlinearity $b|v|^{\alpha-1}v$ that scatters to φ_{σ,x_0} as $t \to -\infty$. Thus,

$$\langle S_a(\varphi_{\sigma,x_0}) - S_b(\varphi_{\sigma,x_0}), \varphi_{\sigma,x_0}\rangle \tag{5.22}$$

$$= -i\int\int_{\mathbb{R}\times\mathbb{R}^3}(a(x) - b(x))|e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha+1}dxdt \tag{5.23}$$

$$- i\int\int_{\mathbb{R}\times\mathbb{R}^3} a(x)(|u|^{\alpha-1}u - |e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}e^{it\Delta}\varphi_{\sigma,x_0})\overline{e^{it\Delta}\varphi_{\sigma,x_0}}dxdt \tag{5.24}$$

$$- i\int\int_{\mathbb{R}\times\mathbb{R}^3} b(x)(|v|^{\alpha-1}u - |e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}e^{it\Delta}\varphi_{\sigma,x_0})\overline{e^{it\Delta}\varphi_{\sigma,x_0}}dxdt. \tag{5.25}$$

Considering q, r and r_c as defined in Theorem 5.0.3 and the embedding $\dot{H}^{s_c,r} \hookrightarrow L^{r_c}$, we shall estimate each term in the expression above. We have:

$$\begin{aligned}&\left|-i\int\int_{\mathbb{R}\times\mathbb{R}^3} a(x)(|u|^{\alpha-1}u - |e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}e^{it\Delta}\varphi_{\sigma,x_0})\overline{e^{it\Delta}\varphi_{\sigma,x_0}}dxdt\right|\\ &\leq \left\|a(|u|^{\alpha-1}u - |e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}e^{it\Delta}\varphi_{\sigma,x_0})e^{it\Delta}\varphi_{\sigma,x_0}\right\|_{L^1_{t,x}}\\ &\lesssim \|a\|_{L^\infty}\left\|e^{it\Delta}\varphi_{\sigma,x_0}\right\|_{L^q_tL^r_x}\left\||u|^{\alpha-1} + |e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}\right\|_{L^{\hat{q}}_tL^{\hat{r}}_x}\left\|u(t) - e^{it\Delta}\varphi_{\sigma,x_0}\right\|_{L^q_tL^r_x},\end{aligned}$$

where

$$\frac{2}{q}+\frac{1}{\hat{q}}=1=\frac{2}{r}+\frac{1}{\hat{r}},$$

which leads to the relations $\frac{1}{\hat{q}}=\frac{\alpha-1}{\alpha+1}$ and $\frac{1}{\hat{r}}=\frac{4}{N(\alpha+1)}$.

By the Theorem 5.0.3, $\|u\|_{L_t^q\dot{H}^{s_c,r}}\lesssim\|\varphi_{\sigma,x_0}\|_{\dot{H}^{s_c}}$, therefore

$$\begin{aligned}
&\|a\|_{L^\infty}\left\|e^{it\Delta}\varphi_{\sigma,x_0}\right\|_{L_t^qL_x^r}\left\||u|^{\alpha-1}+|e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}\right\|_{L_t^{\frac{q}{\alpha-1}}L_x^{\frac{r_c}{\alpha-1}}}\left\|u(t)-e^{it\Delta}\varphi_{\sigma,x_0}\right\|_{L_t^qL_x^r}\\
&\lesssim\|a\|_{L^\infty}\|\varphi_{\sigma,x_0}\|_{L^2}\left\{\|u\|^{\alpha-1}_{L_t^qL_x^{r_c}}+\left\|e^{it\Delta}\varphi_{\sigma,x_0}\right\|^{\alpha-1}_{L_t^qL_x^{r_c}}\right\}\left\|\int_{-\infty}^t e^{i(t-s)\Delta}a(x)|u|^{\alpha-1}u(s)ds\right\|_{L_t^qL_x^r}\\
&\lesssim\|a\|_{L^\infty}\|\varphi_{\sigma,x_0}\|_{L^2}\left\{\|u\|_{L_t^q\dot{H}^{s_c,r}}+\|\varphi_{\sigma,x_0}\|^{\alpha-1}_{\dot{H}^{s_c}}\right\}\left\|a|u|^{\alpha-1}u\right\|_{L_t^{q'}L_x^{r'}}\\
&\lesssim\|a\|^2_{L^\infty}\|\varphi_{\sigma,x_0}\|_{L^2}\|\varphi_{\sigma,x_0}\|^{\alpha-1}_{\dot{H}^{s_c}}\|u\|^{\alpha-1}_{L_t^qL_x^{r_c}}\|u\|_{L_t^qL_x^r}\\
&\lesssim\|a\|^2_{L^\infty}\|\varphi_{\sigma,x_0}\|^2_{L^2}\|\varphi_{\sigma,x_0}\|^{2\alpha-2}_{\dot{H}^{s_c}}\\
&\lesssim\sigma^{N+4}\|a\|^2_{L^\infty},
\end{aligned}$$

since $\|\varphi_{\sigma,x_0}\|_{L^2}\lesssim\sigma^{\frac{N}{2}}$ and $\|\varphi_{\sigma,x_0}\|_{\dot{H}^{s_c}}\lesssim\sigma^{\frac{2}{\alpha-1}}$, where we use (5.21) for $s=0$ and $s=s_c$ respectively.

Analogously,

$$\left\|b\left(|v|^{\alpha-1}v-|e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha-1}e^{it\Delta}\varphi_{\sigma,x_0}\right)\right\|_{L^1_{t,x}}\lesssim\sigma^{N+4}\|b\|^2_{L^\infty}.$$

Now , where we use Cauchy-Schwarz, and (5.21) with $s=1$ and $s=-1$

$$\begin{aligned}
|\langle S_a(\varphi_{\sigma,x_0})-S_b(\varphi_{\sigma,x_0}),\varphi_{\sigma,x_0}\rangle|&\le\|S_a(\varphi)-S_b(\varphi)\|_{\dot{H}^1}\|\varphi_{\sigma,x_0}\|_{\dot{H}^{-1}}\\
&\lesssim\|S_a-S_b\|\|\varphi_{\sigma,x_0}\|_{\dot{H}^1}\|\varphi_{\sigma,x_0}\|_{\dot{H}^{-1}}\\
&\lesssim\sigma^N\|S_a-S_b\|.
\end{aligned}$$

Finally, in the Proposition 5.0.1, $0<s<\frac{N(\alpha-1)}{2}-1$, together with the condition that $\alpha\ge1+\frac{4}{N}$, we can choose $s=\frac{1}{4}$. Thus,

$$\begin{aligned}
&\left|\int\int(a(x)-b(x))|e^{it\Delta}\varphi_{\sigma,x_0}|^{\alpha+1}dxdt-\sigma^{N+2}\lambda(N,\alpha)(a(x_0)-b(x_0))\right|\\
&\qquad\lesssim\sigma^{N+2+\frac{1}{5}}(\|a\|_{W^{1,\infty}}+\|b\|_{W^{1,\infty}}).
\end{aligned}$$

Hence, based on the estimates found above, we have

$$\begin{aligned}
\lambda(N,\alpha)\sigma^{N+2}|a(x_0)-b(x_0)|&\lesssim\sigma^N\|S_a-S_b\|+\sigma^{N+4}\left(\|a\|^2_{L^\infty}+\|a\|^2_{L^\infty}\right)\\
&\quad+\sigma^{N+\frac{1}{5}}\left(\|a\|_{W^{1,\infty}}+\|b\|_{W^{1,\infty}}\right)
\end{aligned}$$

to obtain

$$|a(x_0) - b(x_0)| \lesssim \sigma^{-2} \|S_a - S_b\| + \sigma^2 \left(\|a\|_{L^\infty}^2 + \|b\|_{L^\infty}^2\right) \tag{5.26}$$

$$+ \sigma^{\frac{1}{5}} \left(\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}\right). \tag{5.27}$$

Now considering the function

$$f(\sigma) := \sigma^{-2} \|S_a - S_b\| + \sigma^{\frac{1}{5}} \left(\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}\right), \tag{5.28}$$

we find that its critical point is

$$\sigma = (10)^{\frac{5}{11}} \left(\frac{\|S_a - S_b\|}{\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}}\right)^{\frac{5}{11}}.$$

Optimizing (5.26), we obtain

$$|a(x_0) - b(x_0)| \lesssim \left(\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}\right)^{\frac{10}{11}} \|S_a - S_b\|^{\frac{1}{11}} + \left(\|a\|_{W^{1,\infty}} + \|b\|_{W^{1,\infty}}\right)^{\frac{12}{11}} \|S_a - S_b\|^{\frac{10}{11}}.$$

Taking the supremum over $x_0 \in \mathbb{R}^N$ yields (5.20). □

Theorem 5.0.5. *Suppose $\alpha, \beta \in \left[1 + \frac{4}{N}, 1 + \frac{4}{N-2}\right]$. Let S_α and S_β denote the scattering maps for* (5.1) *corresponding to nonlinearities $|u|^{\alpha-1}u$ and $|u|^\beta u$, with $a(x) \equiv 1$. Then*

$$|\alpha - \beta| \lesssim \|S_\alpha - S_\beta\|^{\frac{1}{11}} + \|S_\alpha - S_\beta\|^{\frac{10}{11}}$$

Proof. Let

$$S_\alpha(\varphi_\sigma) = \varphi_\sigma - i \int_{\mathbb{R}} e^{-it\Delta}(|e^{it\Delta}\varphi_\sigma|^{\alpha-1} e^{it\Delta}\varphi_\sigma)dt - i \int_{\mathbb{R}} e^{-it\Delta}(|u|^{\alpha-1}u - |e^{it\Delta}\varphi_\sigma|^{\alpha-1} e^{it\Delta}\varphi_\sigma)dt,$$

where u is the solution to (5.1) with nonlinearity $|u|^{\beta-1}u$ that scatters to φ_σ as $t \to -\infty$.

Similarly,

$$S_\beta(\varphi_\sigma) = \varphi_\sigma - i \int_{\mathbb{R}} e^{-it\Delta}(|e^{it\Delta}\varphi_\sigma|^{\beta-1} e^{it\Delta}\varphi_\sigma)dt - i \int_{\mathbb{R}} e^{-it\Delta}(|u|^{\beta-1}u - |e^{it\Delta}\varphi_\sigma|^{\beta-1} e^{it\Delta}\varphi_\sigma)dt,$$

where v is the solution to (5.1) with nonlinearity $|v|^{\beta-1}v$ that scatters to φ_σ as $t \to -\infty$. Thus

$$\int\int_{\mathbb{R}\times\mathbb{R}^N} (|e^{it\Delta}\varphi_\sigma|^{\alpha+1} - |e^{it\Delta}\varphi_\sigma|^{\beta+1})dxdt \tag{5.29}$$

$$= i\langle S_\alpha(\varphi_\sigma) - S_\beta(\varphi_\sigma), \varphi_\sigma\rangle \tag{5.30}$$

$$+ \int\int_{\mathbb{R}\times\mathbb{R}^N} (|u|^{\alpha-1}u - |e^{it\Delta}\varphi_\sigma| e^{it\Delta}\varphi_\sigma)\overline{e^{it\Delta}\varphi_\sigma} dxdt \tag{5.31}$$

$$+ \int\int_{\mathbb{R}\times\mathbb{R}^N} (|v|^{\beta-1}v - |e^{it\Delta}\varphi_\sigma| e^{it\Delta}\varphi_\sigma)\overline{e^{it\Delta}\varphi_\sigma} dxdt, \tag{5.32}$$

Note that to estimate (5.31) and (5.32), we will make a development similar to the estimates found for (5.24) and (5.25) with $a(x) \equiv 1$, so that

$$|(5.31)| + |(5.32)| \lesssim \sigma^{N+4}. \tag{5.33}$$

For (5.30), we use Cauchy- Schwartz and (5.21) to obtain

$$\begin{aligned}|\langle S_\alpha(\varphi_\sigma)-S_\beta(\varphi_\sigma),\varphi_\sigma\rangle| &\le \|S_\alpha(\varphi)-S_\beta(\varphi_\sigma)\|_{H^1}\|\varphi_\sigma\|_{H^{-1}}\\ &\lesssim \sigma^N\|S_\alpha-S_\beta\|.\end{aligned}$$

By the proposition 5.0.1 with $s=\frac{1}{4}$. Using the fact that $\alpha\ge 1+\frac{4}{N}$, this proposition implies that

$$\left|\int\int_{\mathbb{R}\times\mathbb{R}^N}(|e^{it\Delta}\varphi_\sigma|^{\alpha+1}-|e^{it\Delta}\varphi_\sigma|^{\beta+1})dxdt-\sigma^{N+2}(\lambda(N,\alpha)-\lambda(N,\beta))\right|\lesssim\sigma^{N+2+\frac{1}{5}},$$

combining this with (5.33), we deduce

$$|\lambda(N,\alpha)-\lambda(N,\beta)|\lesssim\sigma^2\|S_\alpha-S_\beta\|+\sigma^{-2}+\sigma^{\frac{1}{4}},$$

optimizing the function $f(\sigma)=\sigma^{-2}\|S_\alpha-S_\beta\|+\sigma^{\frac{1}{5}}$, we have that

$$\sigma=(10)^{\frac{5}{11}}\|S_\alpha-S_\beta\|^{\frac{5}{11}},$$

so that

$$|\lambda(N,\alpha)-\lambda(N,\beta)|\lesssim\|S_\alpha-S_\beta\|^{\frac{1}{11}}+\|S_\alpha-S_\beta\|^{\frac{10}{11}}. \tag{5.34}$$

Now it only remains to prove that

$$|\lambda(N,\alpha)-\lambda(N,\beta)|\gtrsim|\alpha-\beta|. \tag{5.35}$$

Recalling the definition of λ

$$\begin{aligned}\lambda'(\alpha)=&-\pi^{\frac{N+1}{2}}\frac{2^{N-1}N}{(\alpha+1)^{\frac{N+2}{2}}}\frac{\Gamma\left(\frac{N(\alpha-1)}{4}-\frac{1}{2}\right)}{\Gamma\left(\frac{N(\alpha-1)}{4}\right)}-\frac{N}{4}\pi^{\frac{N+1}{2}}\left(\frac{4}{\alpha+1}\right)^{\frac{N}{2}}\frac{\Gamma\left(\frac{N(\alpha-1)}{4}-\frac{1}{2}\right)}{\Gamma\left(\frac{N(\alpha-1)}{4}\right)}A\\ =&-\pi^{\frac{N+1}{2}}N\frac{\Gamma\left(\frac{N(\alpha-1)}{4}-\frac{1}{2}\right)}{\Gamma\left(\frac{N(\alpha-1)}{4}\right)}\left\{\frac{2^{N-1}}{(\alpha+1)^{\frac{N+2}{2}}}+\frac{4^{\frac{N-2}{2}}}{(\alpha+1)^{\frac{2}{N}}}A\right\}\end{aligned}$$

where

$$A:=\psi\left(\frac{N(\alpha-1)}{4}\right)-\psi\left(\frac{N(\alpha-1)}{4}-\frac{1}{2}\right),$$

and $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function.
By Gautschi's inequality (see [17, Theorem A]), we have

$$\left|\frac{\Gamma\left(\frac{N(\alpha-1)}{4}-\frac{1}{2}\right)}{\Gamma\left(\frac{N(\alpha-1)}{4}\right)}\right|\le\left|\left(\frac{N(\alpha-1)}{4}\right)\right|^{-\frac{1}{2}}.$$

Using the fact that ψ is increasing on $(0,\infty)$, it follows that

$$-\pi^{\frac{N+1}{2}}N\left(\frac{3(\alpha-1)}{4}\right)^{-\frac{1}{2}}\left[\psi\left(\frac{3(\alpha-1)}{4}\right)-\psi\left(\frac{3(\alpha-1)}{4}-\frac{1}{2}\right)\right]\ge 0,$$

therefore

$$|\lambda'(\alpha)| \geq \pi^{\frac{N+1}{2}} 2^{N-1} N \left(\frac{3(\alpha-1)}{4}\right)^{-\frac{1}{2}} \gtrsim 1,$$

uniformly for $1+\frac{4}{N} \leq \alpha \leq 1+\frac{4}{N-2}$.

By the mean value theorem, we obtain $|\lambda(\alpha)-\lambda(\beta)| \gtrsim |\alpha-\beta|$. □

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