UNIVERSIDADE FEDERAL DE MINAS GERAIS
Instituto de Ciências Exatas
Programa de Pós-graduação em Matemática

Pedro Carlos Barbosa Júnior

THE BRUCE-ROBERTS NUMBER AND THE BRUCE-ROBERTS TJURINA NUMBER FOR HOLOMORPHIC 1-FORMS

Belo Horizonte
2025

Pedro Carlos Barbosa Júnior

THE BRUCE-ROBERTS NUMBER AND THE BRUCE-ROBERTS TJURINA NUMBER FOR HOLOMORPHIC 1-FORMS

Tese apresentada ao Programa de Pós-Graduação em Matemática da Universidade Federal de Minas Gerais, como requisito parcial à obtenção do título de Doutor em Matemática.

Orientador: Arturo Ulises Fernández Pérez

Coorientador: Víctor Arturo Martínez León

Belo Horizonte
2025

Barbosa Júnior, Pedro Carlos.

B238b The Bruce-Roberts number and the Bruce-Roberts Tjurina number for holomorphic 1-forms [recurso eletrônico] / Pedro Carlos Barbosa Júnior. Belo Horizonte — 2025.
1 recurso online (99 f. il.): pdf.

Orientador: Arturo Ulises Fernández Pérez.
Coorientador: Víctor Arturo Martínez León.
Tese (doutorado) - Universidade Federal de Minas Gerais, Instituto de Ciências Exatas, Departamento de Matemática.
Referências: f. 92-99.

1. Matemática – Teses. 2. Variedades (Matemática) – Teses. 3. Variedades complexas – Teses. 4. Folheações (Matemática) – Teses. 5. Singularidades (Matemática) – Teses. I. Fernández Pérez, Arturo Ulises. II. Martínez León, Víctor Arturo. III. Universidade Federal de Minas Gerais, Instituto de Ciências Exatas, Departamento de Matemática. IV. Título.

CDU 51(043)

Ficha catalográfica elaborada pela bibliotecária Irénquer Vismeg Lucas Cruz - CRB 6ª Região - nº 819.

Universidade Federal de Minas Gerais
Departamento de Matemática
Programa de Pós-Graduação em Matemática

FOLHA DE APROVAÇÃO

The Bruce-Roberts number and the Bruce-Roberts Tjurina number for holomorphic 1-forms

PEDRO CARLOS BARBOSA JÚNIOR

Tese defendida e aprovada pela banca examinadora constituída por:

Prof. Arturo Ulises Fernández Pérez
Orientador - UFMG

Prof. Victor Arturo Martinez Leon
Coorientador - UNILA

Prof. Arnulfo Miguel Rodríguez Peña
UFMG

Profa. Ayane Adelina da Silva
UFC

Prof. José Edson Sampaio
UFC

Prof. Márcio Gomes Soares
UFMG

Belo Horizonte, 11 de abril de 2025.

Av. Antônio Carlos, 6627 – Campus Pampulha - Caixa Postal: 702
CEP-31270-901 - Belo Horizonte – Minas Gerais - Fone (31) 3409-5963
e-mail: pgmat@mat.ufmg.br - home page: http://www.mat.ufmg.br/pgmat

Agradecimentos

Inicialmente, quero agradecer a todos que me auxiliaram de alguma forma a concluir esta tese. Quando comecei o doutorado, há quatro anos, não imaginaria quanta coisa estaria por vir, e sou muito grato por cada pequena ajuda que recebi durante esse trajeto.

Agradeço à minha família, pelo auxílio na minha vinda para Belo Horizonte, e por celebrar cada conquista comigo. Em particular, agradeço à minha mãe, Maria Célia, por estar sempre ao meu lado e por todo o carinho.

Agradeço aos meus amigos pela companhia e companheirismo de sempre. Esse processo foi menos solitário graças a vocês, que são a prova de que é possível manter-se presente, mesmo com a barreira da distância.

Quero agradecer ao professor Arturo, meu orientador, por confiar no meu trabalho, e pelo aprendizado que obtive nesses últimos anos. Agradeço ao meu coorientador, professor Víctor León, por todo o auxílio e pela disposição em me ajudar com cada dúvida. Agradeço também aos membros da banca por todas as sugestões, trazendo melhorias para esta tese. Sou muito grato pela contribuição de vocês, que foi fundamental para este trabalho.

Por fim, agradeço também à CAPES pelo apoio financeiro indispensável. O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) – Código de Financiamento 001.

"In order to rise
From its own ashes
A phoenix
First
Must
Burn."

Octavia Butler

Resumo

Neste trabalho, apresentamos dois índices de 1-formas holomorfas. Primeiramente, definimos o número de Bruce-Roberts para 1-formas holomorfas em relação a variedades analíticas complexas, e demonstramos o nosso principal resultado, que mostra que o número de Bruce-Roberts de uma 1-forma ω com respeito a uma hipersuperfície analítica complexa X com singularidade isolada pode ser expresso em função do índice de Ebeling–Gusein-Zade de ω em X, o número de Milnor de ω e o número de Tjurina de X. Esse resultado nos permite obter fórmulas conhecidas para o número de Bruce-Roberts de uma função holomorfa em relação a X, e também estabelecer conexões entre esse número, o índice radial e a obstrução local de Euler de ω ao longo de X. Em seguida, definimos o número de Tjurina de Bruce-Roberts para 1-formas holomorfas com respeito a um par (X, V) de subvariedades analíticas complexas. Quando a dupla (X, V) consiste em hipersuperfícies analíticas complexas isoladas, mostramos que o número de Tjurina de Bruce-Roberts se relaciona com o número de Bruce-Roberts, o número de Tjurina de uma 1-forma com respeito a V, e o número de Tjurina de X, dentre outros invariantes. Mais ainda, exibimos aplicações de ambos os índices para folheações holomorfas globais e locais em dimensão complexa dois.

Palavras-chave: número de Bruce-Roberts; número de Tjurina; número de Milnor; folheações holomorfas; variedades analíticas complexas.

Abstract

In this work, we introduce two indices of holomorphic 1-forms. First, we define the Bruce-Roberts number for holomorphic 1-forms relative to complex analytic varieties, and prove our main result, that shows that the Bruce-Roberts number of a 1-form ω with respect to a complex analytic hypersurface X with an isolated singularity can be expressed in terms of the Ebeling–Gusein-Zade index of ω along X, the Milnor number of ω and the Tjurina number of X. This result allows us to recover known formulas for the Bruce-Roberts number of a holomorhic function along X and to establish connections between this number, the radial index, and the local Euler obstruction of ω along X. After that, we define the Bruce-Roberts Tjurina number for holomorphic 1-forms with respect to a pair (X, V) of complex analytic subvarieties. When the pair (X, V) consists of isolated complex analytic hypersurfaces, we prove that the Bruce-Roberts Tjurina number is related to the Bruce-Roberts number, the Tjurina number of the 1-form with respect to V, and the Tjurina number of X, among other invariants. Moreover, we present applications of both indices to global and local holomorphic foliations in complex dimension two.

Keywords: Bruce-Roberts number; Tjurina number; Milnor number; holomorphic foliations; complex analytic varieties.

Contents

Introduction

When it comes to the theory of complex analytic hypersurfaces, it is possible to find several important indices. In this work, we wanted to establish a connection between known invariants and introduce new indices of 1-forms, which we will call the Bruce-Roberts number and the Bruce-Roberts Tjurina number. As a result, we obtained [6] and [7], which were developed along with this thesis. But first, we start talking about two crucial invariants for us: the Milnor and the Tjurina numbers.

The Milnor number for a hypersurface X with an isolated singularity was first conceived by J. Milnor in [56], and it is exactly the rank of the middle homology group of the Milnor fiber of X. It is denoted by $\mu_0(f)$, where $f \in \mathcal{O}_n$ is the holomorphic function that defines X, with $\mathcal{O}_n$ being the local ring of holomorphic functions from $(\mathbb{C}^n, 0)$ to $\mathbb{C}$. Algebraically, $\mu_0(f)$ is defined as the colength of the Jacobian ideal $J(f)$ in $\mathcal{O}_n$. Other indices with the nomenclature "Milnor number" can be defined for different mathematical objects. When $\mathcal{F}$ is a singular foliation defined by a holomorphic vector field $v = \sum_{i=1}^{n} A_i(z) \frac{\partial}{\partial z_i}$ with an isolated singularity at $0 \in \mathbb{C}^n$, the Milnor number $\mu_0(\mathcal{F})$ of $\mathcal{F}$ can be seen as a colength of an ideal in $\mathcal{O}_n$, but in that case, the ideal in question is defined by the generators of v, or simply $\langle A_1, \dots, A_n \rangle$. As far as we know, the notion of Milnor number for singular foliations by curves (with that name) appears for the first time in the work of C. Camacho, A. Lins Neto and P. Sad ([17]). Both in the case of hypersurfaces and singular foliations by curves, it is known that the Milnor number is a topological invariant. The Milnor number of singular hypersurfaces is often studied, and there are plenty of works regarding its definition and properties. Recently, in particular, A. Fernández-Pérez, G. Costa and R. Rosas studied the Milnor number of foliations by curves with non-isolated singularities [30].

On the other hand, the Tjurina number is an index that measures the dimension of the base space of a semi-universal deformation of the hypersurface X, and it is denoted by $\tau_0(f)$, with f again representing the holomorphic function that defines X. $\tau_0(f)$ can also be seen in an algebraic way, as the colength of the ideal generated by f and its Jacobian ideal $J(f)$ in $\mathcal{O}_n$. The Tjurina number is named after G. Tjurina, who first defined it in [67]. It is also known that, different from the Milnor number, the Tjurina number is not a topological invariant, but an analytic invariant of the singularity. The

Tjurina number $\tau_0(\omega, V)$ of a 1-form $\omega = \sum_{i=1}^{n} A_i dz_i$ with an isolated singularity at $0 \in \mathbb{C}^n$ with respect to a complex analytic hypersurface V is defined in an intuitive way, as the colength of $\langle A_1, \ldots, A_n, f\rangle$ in $\mathcal{O}_n$, with $f \in \mathcal{O}_n$ defining V. In that case, we require that V be a hypersurface invariant by ω, which means that the tangent space of V satisfies $T_pV \subset \ker(\omega_p)$, for all regular points $p \in V$. For a holomorphic vector field v with an isolated singularity on a complex analytic hypersurface V, the Tjurina number of v with respect to V first appears in the work of X. Gómez-Mont [38], though he does not use this terminology. To our knowledge, the first authors to explicitly use the term "Tjurina name" in the context of foliations were F. Cano, N. Corral and R. Mol in [20]. This number also appears in the study of Baum-Bott residues of foliations by S. Licanic [46]. More recently, in [32], A. Fernández-Pérez, E. R. García Barroso and N. Saravia-Molina established a relationship between the Milnor and Tjurina numbers of germs of foliations on $(\mathbb{C}^2, 0)$.

With that in mind, we can talk about the Bruce-Roberts numbers. Denoted most times by μ_{BR} and τ_{BR}, those indices generalize the Milnor and Tjurina numbers described above, and are named after J. Bruce and R. Roberts, who were the first to describe the number now known as the Bruce-Roberts number μ_{BR}.

In 1988, J. Bruce and R. Roberts defined in [13] a number denoted by $\mu_X(f)$, which generalizes the Milnor number of f. This number, called in this work the Milnor number of f on X and also the multiplicity of f on X at 0, was later known as the Bruce-Roberts number of f with respect to X, and it can be defined in the following way: when $(X, 0)$ denotes the germ of a complex analytic variety at $(\mathbb{C}^n, 0)$, and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a germ of a holomorphic function at $(\mathbb{C}^n, 0)$, the Bruce-Roberts number associated with f relative to $(X, 0)$, denoted by $\mu_{BR}(f, X)$, is defined as

$$\mu_{BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)},$$

where $\mathcal{O}_n$ represents the local ring of holomorphic functions from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}, 0)$, df stands for the differential of f, and Θ_X is the $\mathcal{O}_n$-submodule of Θ_n (the $\mathcal{O}_n$-module of germs of holomorphic vector fields) consisting of holomorphic vector fields on $(\mathbb{C}^n, 0)$ that are tangent to $(X, 0)$ over their regular points. If $I_X \subset \mathcal{O}_n$ is the ideal of germs of holomorphic functions vanishing on $(X, 0)$, then

$$\Theta_X = \{\xi \in \Theta_n : \ dh(\xi) \in I_X, \ \forall\ h \in I_X\}.$$

We observe that in [62], K. Saito also presented a definition of Θ_X, calling it the module of logarithmic vector fields and denoting it by $\mathrm{Der}_{S,p}(\log D)$. In particular, when $X = \mathbb{C}^n$, $df(\Theta_n)$ corresponds to the Jacobian ideal $J(f)$ of f which is generated by the partial derivatives of f in $\mathcal{O}_n$. Consequently, $\mu_{BR}(f, \mathbb{C}^n)$ coincides with the Milnor number of f - it is valid to mention that in some works, $\mu_{BR}(f, X)$ is also called the Bruce-Roberts' Milnor number of f with respect to X. Furthermore, if X is the germ of a complex analytic subvariety at $(\mathbb{C}^n, 0)$, then $\mu_{BR}(f, X)$ is finite if, and only if f has an isolated singularity over $(X, 0)$. In [13], the authors also defined the number called the relative Bruce-Roberts number, that can be stated as

$$\mu_{BR}^{-}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X) + I_X}.$$

The Bruce-Roberts number for holomorphic functions has been studied by several authors, who gave it some important properties and characterizations. For example, in 2013, J. Nuño-Ballesteros, B. Oréfice and J. Tomazella established in [58] a relation between the Bruce-Roberts and the Milnor number. When X is a weighted homogeneous hypersurface with isolated singularity and f is $\mathcal{R}_X$-finitely determined, they showed that

$$\mu_{BR}(f, X) = \mu_0(f) + \mu_0(f, X),$$

where $\mu_0(f, X)$ denotes the Milnor number of the fiber $X \cap f^{-1}(0)$, which is an ICIS - an isolated complete intersection singularity. In [59], the same authors (along with B. Lima-Pereira) showed that if $\mu_{BR}(f, X) < \infty$, the Bruce-Roberts number of f with respect to an isolated hypersurface singularity satisfies

$$\mu_{BR}(f, X) = \mu_0(f) + \mu_0(\phi, f) + \mu_0(\phi) - \tau_0(\phi),$$

with ϕ being the function that defines X. In [48], the same authors also present some results regarding the relative Bruce-Roberts number.

For other works concerning the Bruce-Roberts number, we recommend, for instance, [3], [23], [10], [57], [44], [47] and [8].

This thesis has two main goals. The first one was presented by the author, P. Barbosa, along with A. Fernández-Pérez and V. León in [6], and it is to extend the definition of the Bruce-Roberts number to holomorphic 1-forms relative to complex analytic varieties. More precisely, let ω be the germ of a holomorphic 1-form with an isolated singularity at $0 \in \mathbb{C}^n$, $n \geq 2$, and let X be a germ of complex analytic variety with an isolated singularity at $0 \in \mathbb{C}^n$. We define the *Bruce-Roberts number* of the 1-form ω with respect to X as

$$\mu_{BR}(\omega, X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X)}.$$

We observe that $\mu_{BR}(\omega, X)$ is finite if and only if ω is a 1-form on X admitting (at most) an isolated singularity at $0 \in \mathbb{C}^n$, which is equivalent to saying that X is not invariant by ω in the case of a germ of a complex analytic subvariety X. Note also that $\mu_{BR}(\omega, X)$ generalizes the Bruce-Roberts number of a function, since $\mu_{BR}(\omega, X) = \mu_{BR}(f, X)$ when $\omega = df$.

Our main result about the Bruce-Roberts number of a 1-form provides a straightforward method to calculate $\mu_{BR}(\omega, X)$, since the explicit computation of Θ_X can be difficult. With that in mind, our first result establishes that if ω is a germ of a holomorphic 1-form with isolated singularity at $0 \in \mathbb{C}^n$, $n \geq 2$, and X is a germ of a complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$ that is not invariant by ω, then

$$\mu_{BR}(\omega, X) = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) + \mu_0(\omega) - \tau_0(X),$$

where $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0)$ is the GSV-index of ω with respect to X, defined by S. Gusein-Zade and W. Ebeling in [42] (see also [26]). We also observe that, since both $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0)$ and $\mu_0(\omega)$ are topological invariants when $(X, 0)$ is an isolated hypersurface singularity, the equality above implies that $\mu_{BR}(\omega, X)$ is also a topological invariant under homeomorphisms of $(\mathbb{C}^n, 0)$ that fix $(X, 0)$.

Using the definitions and topics explored in [48] as base, we also define the *relative Bruce-Roberts* of the 1-form ω with respect to X as

$$\mu_{BR}^{-}(\omega, X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X) + I_X}.$$

With that, it is also possible to show a relation between $\mu_{BR}(\omega, X)$ and $\mu_{BR}^{-}(\omega, X)$. With ω and X under the same hypothesis shown above, it is possible to show that

$$\mu_{BR}(\omega, X) = \mu_0(\omega) + \mu_{BR}^{-}(\omega, X).$$

A natural application of the Bruce-Roberts that we defined is to the case of foliations. Since a foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ can be defined by a holomorphic 1-form, we set $\mu_{BR}(\mathcal{F}, X) := \mu_{BR}(\omega, X)$, where ω is the 1-form that defines $\mathcal{F}$. In that case, we demonstrate that

$$\mu_{BR}(\mathcal{F}, X) = \mu_0(X) + \mathrm{tang}(\mathcal{F}, X, 0) + \tau_0(X),$$

where $\mathrm{tang}(\mathcal{F}, X, 0)$ is the tangency order of $\mathcal{F}$ to X, as defined in [14] by M. Brunella. Working with foliations, we also managed to give formulas for the Bruce-Roberts number under blow-ups in dimension two, some results in the case of generalized curve foliations in $(\mathbb{C}^2, 0)$, introduced by C. Camacho, A. Lins Neto and P. Sad in [17], and also some applications to global foliations.

On the other hand, in 2011, I. Ahmed and M. Ruas defined in [2] two numbers, called the relative Milnor and Tjurina algebras of a function h on an analytic variety V, denoted by $M_V(h)$ and $T_V(h)$. It is not difficult to see that the definition of this Milnor algebra coincides with the definition of the Bruce-Roberts number $\mu_{BR}(h, V)$. Moreover, the Tjurina algebra seems to be defined intuitively, when comparing its definition with that of the Milnor algebra, and those with the definitions of the classic Milnor and Tjurina numbers.

In 2020, C. Bivià-Ausina and M. Ruas [10] introduced the Bruce-Roberts' Tjurina number of f with respect to X, with an equivalent definition of the Tjurina algebra of a function. With a slightly different notation, here we define the Bruce-Roberts Tjurina number of $f \in \mathcal{O}_n$ with respect to an analytic subvariety X (of $(\mathbb{C}^n, 0)$) by

$$\tau_{BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X) + \langle f \rangle},$$

when the colength on the right side is finite. With that definition, it is not difficult to see that $\tau_{BR}(f, \mathbb{C}^n) = \tau_0(f)$, i.e., the Bruce-Roberts Tjurina number generalizes the classic Tjurina number. It is interesting to note that in [10], the authors also presented a relation between the Bruce-Roberts' Milnor and Tjurina numbers. Denoting by $r_f(I)$ the minimum of $r \in \mathbb{Z}_{\geq 1}$ such that $f^r \in I$, they managed to show that

$$\frac{\mu_{BR}(f, X)}{\tau_{BR}(f, X)} \leq r_f(df(\Theta_X)).$$

For some references who studied the Bruce-Roberts Tjurina number of holomorphic functions, we cite [2], [10], [8], and [9].

The second main goal of this thesis is to introduce the Bruce-Roberts Tjurina number of ω with respect to the pair (X, V). It is defined as follows: let $(X, 0)$ denote the germ of a complex analytic variety at $(\mathbb{C}^n, 0)$, let ω be a germ of a holomorphic 1-form with an isolated singularity at $(\mathbb{C}^n, 0)$, and let V be a germ of a complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$. If X is not invariant by ω, but V is invariant by ω, the *Bruce-Roberts Tjurina number* of ω with respect to the pair (X, V) is given by

$$\tau_{BR}(\omega, X, V) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X) + I_V}.$$

This definition and the consequent results were first introduced, again, by P. Barbosa, A. Fernández-Pérez and V. León in [7]. Note that if $\omega = df$ is an exact holomorphic 1-form with an isolated singularity at $0 \in \mathbb{C}^n$, then $V = \{f = 0\}$ is a complex analytic hypersurface invariant by ω, and consequently, $\tau_{BR}(df, X, V)$ generalizes the Bruce-Roberts Tjurina number of f along X, since $\tau_{BR}(df, X, V) = \tau_{BR}(f, X)$. We observe that it is also a generalization of the classic Tjurina number, as $\tau_{BR}(df, \mathbb{C}^n, V) = \tau_0(V)$.

Our main theorem for this definition says that, with the hypothesis established above, we got

$$\tau_{BR}(\omega, X, V) = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) + \tau_0(\omega, V) - \tau_0(X) + \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V},$$

where Θ_X^T is a submodule of Θ_X, given by $\Theta_X^T = \left\langle \phi \frac{\partial}{\partial x_i}, \frac{\partial \phi}{\partial x_j} \frac{\partial}{\partial x_k} - \frac{\partial \phi}{\partial x_k} \frac{\partial}{\partial x_j} \right\rangle$, with $i, j, k = 1, \ldots, n;\ k \neq j$, and $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0)$ is a GSV-index, also defined in [7], that generalizes $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0)$.

It is natural to compare the Bruce-Roberts and the Bruce-Roberts Tjurina numbers for 1-forms that are defined in this work. Both numbers generalize not only the homonymous indices defined for functions, but also the classic Milnor and Tjurina numbers that we discussed in the beginning. Furthermore, it follows directly from the definitions that

$$\tau_{BR}(\omega, X, V) \leq \mu_{BR}(\omega, X),$$

and then if $\mu_{BR}(\omega, X) < \infty$, then $\tau_{BR}(\omega, X, V) < \infty$.

In this work, another aim is to establish connections between the Bruce-Roberts Tjurina number and other indices of 1-forms. As an application, we can obtain again some results for holomorphic foliations in complex dimension two, defining $\tau_{BR}(\mathcal{F}, X, V) := \tau_{BR}(\omega, X, V)$, with ω being the 1-form defining $\mathcal{F}$. In fact, we prove a quasihomogeneity result for germs of non-dicritical generalized curve holomorphic foliations with respect to a non-invariant complex analytic curve X: when $\mathcal{F}$ is a germ of a non-dicritical curve generalized foliation at $(\mathbb{C}^2, 0)$ defined by a 1-form ω, X a germ of a reduced curve at $(\mathbb{C}^2, 0)$ not invariant by $\mathcal{F}$, and $V = \{f = 0\}$ is the reduced equation of the total set of separatrices of $\mathcal{F}$, if $\mu_{BR}(\mathcal{F}, X) = \tau_{BR}(\mathcal{F}, X, V)$, then there exist coordinates $(u, v) \in \mathbb{C}^2$, $g, h \in \mathcal{O}_2$, with $u(0) = 0$, $v(0) = 0$, $g(0) \neq 0$ and integers $\alpha, \beta, \zeta \in \mathbb{N}$ such that

$$f(u, v) = \sum_{\alpha i + \beta j = \zeta} P_{i,j} u^i v^j, \quad P_{i,j} \in \mathbb{C},$$

$$g\omega = df + h(\beta v du - \alpha u dv),$$

and the pair (f, X) is relatively quasihomogeneous in these coordinates. This result is obtained by combining a J.-F. Mattei's theorem [55], with a corollary of C. Bivià-Ausina, K. Kourliouros and M. Ruas given in [8].

This text is organized as follows: In Chapter 1, we present some important definitions and already known results that will be fundamental to the development of this work. In the first part of the chapter, we present the concept of holomorphic foliations, and in the second one, we define and talk about the properties of several indices that will appear in our main results: the Milnor numbers, the Tjurina numbers, the GSV-index, and the Bruce-Roberts numbers.

In Chapter 2, we define the Bruce-Roberts number for holomorphic 1-forms, prove our main result (namely, Theorem 2.1.3), and give some examples. Then, we present a definition of the relative Bruce-Roberts number of a holomorphic 1-form, and apply our results to germs of holomorphic foliations on $(\mathbb{C}^2, 0)$, also showing blow-up formulas for $\mu_{BR}(\omega, X)$ and $\mu_{BR}^{-}(\omega, X)$ and providing more examples. Additionally, we establish a characterization of a non-dicritical foliation $\mathcal{F}$ when it is a generalized curve foliation, and finish the chapter with a formula for the sum of the Bruce-Roberts numbers of a global foliation on a compact complex surface. As a consequence, we obtain an upper bound to the global Tjurina number of X.

In Chapter 3, we define the Bruce-Roberts Tjurina number for holomorphic 1-forms, prove the main theorem of this chapter (Theorem 3.1.1), and also provide examples. Then, we present a formula that relates the Bruce-Roberts Tjurina number and the Bruce-Roberts number, and other relations between them. At last, we work again with foliations on $(\mathbb{C}^2, 0)$, and use the Bruce-Roberts Tjurina number of a foliation to prove the quasihomogeneity result mentioned earlier.

Finally, in Chapter 4, we conclude the thesis and present some problems that we intend to work on in the future.

Chapter 1

Preliminaries

In this first chapter, we start by presenting some definitions and results that are the basic principles to comprehend this study. The works used as reference to this chapter are [14], [17], [19], [22], [39], [50], [62] and others that will appear throughout the text. Initially, we present some notations and definitions.

Let $(X,0)$ denote the germ of a complex analytic variety at $(\mathbb{C}^n,0)$. We consider Θ_n the $\mathcal{O}_n$-module of germs of vector fields, where $\mathcal{O}_n$ represents the local ring of holomorphic functions from $(\mathbb{C}^n,0)$ to $(\mathbb{C},0)$. We write as Θ_X the module of the so-called *logarithmic vector fields*, as defined by K. Saito [62, Definition 1.4]:

$$\Theta_X = \{\xi \in \Theta_n : \xi(I_X) \subseteq I_X\},$$

where I_X is the ideal of germs of holomorphic functions vanishing on $(X,0)$. It is also possible to write $I_X = \langle \phi \rangle$, with $\phi : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ being the holomorphic function that defines X. We highlight that Θ_X defines a coherent sheaf of modules in a small neighborhood U of $0 \in \mathbb{C}^n$ ([10, p. 2]). Observe that, geometrically, Θ_X represents the $\mathcal{O}_n$-submodule of Θ_n of vector fields that are tangent to $(X,0)$ along the regular points of X. With that, we can also write

$$\Theta_X = \{\xi \in \Theta_n :\ dh(\xi) \in I_X,\ \forall\ h \in I_X\}. \tag{1.1}$$

Now, we can present the notion of the $\mathcal{O}_n$-submodule of Θ_X generated by the trivial vector fields, denoted by Θ_X^T and defined in [47, Section 2]. Thus, we write

$$\Theta_X^T = \left\langle \phi\frac{\partial}{\partial x_i},\ \frac{\partial \phi}{\partial x_j}\frac{\partial}{\partial x_k} - \frac{\partial \phi}{\partial x_k}\frac{\partial}{\partial x_j} \right\rangle,\ \text{with } i,j,k = 1,\dots,n;\ k \neq j. \tag{1.2}$$

Through the concept of flatness, we can define the concept of complete intersection. First, we need a definition, given in [40, Definition 7.2.1].

Definition 1.0.1. Let (X,x) and (S,s) be complex space germs. A *deformation of* (X,x) *over* (S,s) consists of a flat morphism $\phi : (\mathscr{X},x) \to (S,s)$ of complex germs together with an isomorphism $(X,x)\xrightarrow{\cong}(\mathscr{X}_s,x)$. $(\mathscr{X},x)$ is called the *total space*, (S,s) the *base space*, and $(\mathscr{X}_s,x) := (\phi^{-1}(s),x)$ or (X,x) the *special fibre* of the deformation.

We can write a deformation as a Cartesian diagram

$$\begin{array}{ccc} (X,x) & \overset{i}{\hookrightarrow} & (\mathscr{X},x) \\ \downarrow & & \downarrow \phi \text{ flat} \\ \{p\} & \hookrightarrow & (S,s) \end{array},$$

where i is a closed embedding mapping (X, x) isomorphically onto $(\mathscr{X}_s, x)$ and $\{p\}$ is the reduced point considered as a complex space germ with local ring $\mathbb{C}$. A deformation can also be denoted by

$$(i, \phi) : (X, x) \overset{i}{\hookrightarrow} (\mathscr{X}, x) \overset{\phi}{\to} (S, s),$$

or simply by $\phi : (\mathscr{X}, x) \to (S, s)$ in order to shorten notation. Now, we refer to [40, Remark 7.2.3] to the following properties:

1. $f = (f_1, \ldots, f_k) : (\mathscr{X}, x) \to (\mathbb{C}^k, 0)$ is flat if, and only if, $f_1, \ldots, f_k$ is a regular sequence;

2. If $(\mathscr{X}, x)$ is Cohen-Macaulay, then $f_1, \ldots, f_k \in \mathfrak{m} \subset \mathcal{O}_{\mathscr{X},x}$ is a regular sequence if, and only if
$$\dim \frac{\mathcal{O}_{\mathscr{X},x}}{\langle f_1, \ldots, f_k \rangle} = \dim(\mathscr{X}, x) - k;$$

3. In particular, $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^k, 0)$ is flat if, and only if, $\dim(f^{-1}(0), 0) = m - k$. If this holds, $(X, 0) := (f^{-1}(0), 0)$ is called a *complete intersection* and $(i, f) : (X, 0) \subset (\mathbb{C}^m, 0) \to (\mathbb{C}^k, 0)$ is a deformation of $(X, 0)$ over $(\mathbb{C}^k, 0)$. If $k = 1$, then $(X, 0)$ is called a hypersurface singularity.

We recall that if $f_1, \ldots, f_k \in A$ is a *regular sequence* (on a A-module M, where A is a ring) if, and only if,

(i) $\langle f_1, \ldots, f_k \rangle M \neq M$,

(ii) for $i = 1, \ldots, k$, f_i is a non-zero divisor of $M / \langle f_1, \ldots, f_{k-1} \rangle M$,

by [39, Definition B.6.2]. Also, if A is a Noetherian ring with maximal ideal $\mathfrak{m}$, a A-module M is called *Cohen-Macaulay* if $\operatorname{depth}(M) = \dim(M)$, with $\operatorname{depth}(M)$ being the maximal length of an M-regular sequence contained in $\mathfrak{m}$ ([39, Definition B.8.1]). With that, we say that A is a Cohen-Macaulay ring if it is a Cohen-Macaulay A-module.

The definition of a Cohen-Macaulay ring is also related to the concept of a complete intersection ring. We refer to [39, Definition B.8.9] to present the following definition:

Definition 1.0.2. Let A be a regular local ring and $I \subset A$ an ideal. Then A/I is called a *complete intersection ring* if I is generated by $\dim(A) - \dim(A/I)$ elements.

Consequently, any minimal set of generators $x_1, \dots, x_k$ of I is an A-regular sequence. Hence, $\operatorname{depth}(A/I) = \dim(A) - k = \dim(A/I)$, and then any complete intersection ring is a Cohen-Macaulay ring.

Finally, we say that the germ (X, x) is a *complete intersection singularity* if the local ring $\mathcal{O}_{X,x}$ is a complete intersection. In this work, we sometimes refer to an isolated complete intersection singularity as an *ICIS*.

The following proposition is a result from [47, Proposition 2.1], and gives us a characterization of Θ_X^T when $(X, 0)$ is an ICIS.

Proposition 1.0.3. If $(X, 0)$ is the ICIS determined by $\phi = (\phi_1, \dots, \phi_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$, then

$$\Theta_X^T = I_{k+1}\begin{pmatrix} \dfrac{\partial}{\partial x_1} & \cdots & \dfrac{\partial}{\partial x_n} \\ \dfrac{\partial \phi_1}{\partial x_1} & \cdots & \dfrac{\partial \phi_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial \phi_k}{\partial x_1} & \cdots & \dfrac{\partial \phi_k}{\partial x_n} \end{pmatrix} + \left\langle \phi_i \frac{\partial}{\partial x_j},\ i = 1, \dots, k,\ j = 1, \dots, n \right\rangle,$$

where $I_{k+1}(A)$ is the submodule of Θ_X generated by the $(k+1)$-minors of a matrix A.

Lastly, let X be a germ of a reduced curve at $(\mathbb{C}^2, 0)$. We set

$$\Omega_X := \Omega^1_{\mathbb{C}^2,0} / I_X \Omega^1_{\mathbb{C}^2,0} + \mathcal{O}_2 dI_X$$

as the set of holomorphic 1-forms on $(X, 0)$, as given in [25, p. 1].

1.1 Holomorphic Foliations

In this section, we give some definitions, examples and known results about holomorphic foliations. In particular, the case when a foliation is defined by a holomorphic 1-form, an object that will appear many times in the next chapters. For a general overview of Foliation theory, we refer the reader to [50], from which the next two results were extracted (Definition 1.1 and Remark 1.1, respectively).

Definition 1.1.1. Let M be a complex manifold of complex dimension n. A *holomorphic foliation of dimension k, or codimension $n-k$,* $1 \leq k \leq n-1$ is a decomposition $\mathcal{F}$ on M in disjoint complex submanifolds (called *leaves* of the foliation $\mathcal{F}$) of complex dimension k, bijective immersed, having the following properties:

(i) for all $p \in M$, there exists a unique submanifold L_p of the decomposition that contains the point p (called the *leaf* by p);

(ii) for all $p \in M$, there exists a holomorphic chart of M (called a *distinguished chart* of $\mathcal{F}$), (φ, U), $p \in U$, $\varphi : U \to \varphi(U) \subset \mathbb{C}^n$, such that $\varphi(U) = P \times Q$, where P and Q are open polydisc subsets in $\mathbb{C}^k$ and $\mathbb{C}^{n-k}$, respectively;

(iii) if L is a leaf of $\mathcal{F}$ such that $L \cap U \neq \phi$, then $L \cap U = \bigcup_{q \in D_{L,U}} \varphi^{-1}(P \times \{q\})$, where $D_{L,U}$ is a countable subset of Q. The subsets of U of the form $\varphi^{-1}(P \times \{q\})$ are called *plaques* of the distinguished chart (φ, U). A foliation of dimension one is also called *foliation by curves*. In this case, the leaves are Riemann surfaces bijectively immersed in the ambient manifold.

Remark 1.1.2. A dimension k foliation $\mathcal{F}$ in M, induces in M a *distribution* of planes of dimension k, denoted by $T\mathcal{F}$, which is defined by

$$T_p\mathcal{F} = T_p(L_p) = \text{tangent plane at } p,$$
$$\text{of the leaf } \mathcal{F} \text{ passing by } p.$$

From Definition 1.1.1 (iii), this distribution is holomorphic. It defines a holomorphic vector sub-bundle of the tangent bundle TM, which will also be denoted by $T\mathcal{F}$.

In [50, Proposition 1.1], we have another two definitions of a foliation, which are equivalent as the definition above. We enunciate them here as Definition 1.1.3:

Definition 1.1.3. A dimension k *foliation* $\mathcal{F}$ of M can be set equivalent to the following modes:

(I) Description by distinguished charts: $\mathcal{F}$ is given by an atlas of M, $\{(\varphi_\alpha, U_\alpha)/\alpha \in A\}$ where:

(I.1) $\varphi_\alpha(U_\alpha) = P_\alpha \times Q_\alpha$, where P_α, Q_α are polydiscs of dimension k and $n-k$, respectively;

(I.2) if $U_\alpha \cap U_\beta \neq \emptyset$, then the change of charts $\varphi_\beta \circ \varphi_\alpha^{-1}$ is locally of the form

$$\varphi_\beta \circ \varphi_\alpha^{-1}(x_\alpha, y_\alpha) = (h_{\alpha\beta}(x_\alpha, y_\alpha), g_{\alpha\beta}(y_\alpha)).$$

In this case, the plaques of $\mathcal{F}$ in U_α are the sets of the form $\varphi_\alpha^{-1}(P_\alpha \times \{q\})$.

(II) Description by local submersions: $\mathcal{F}$ is given by an open cover $M = \bigcup_{\alpha \in A} U_\alpha$ and by collections $\{y_\alpha\}_{\alpha \in A}$ and $\{g_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \emptyset}$, that satisfy:

(II.1) for all $\alpha \in A$, $y_\alpha : U_\alpha \to \mathbb{C}^{n-k}$ is a submersion;

(II.2) if $U_\alpha \cap U_\beta \neq \emptyset$, then $y_\alpha = g_{\alpha\beta}(y_\beta)$, where $g_{\alpha\beta} : y_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^k \to y_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^k$ is a local holomorphic diffeomorfism. In this case, the plaques of $\mathcal{F}$ in U_α are the sets of the form $y_\alpha^{-1}(q)$, $q \in V_\alpha$.

Example 1.1.4. If we consider any decomposition of $\mathbb{C}^n$ in the form of $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$, that decomposition defines a foliation $\mathcal{F}$ of dimension k in $\mathbb{C}^n$. The leaves of $\mathcal{F}$ are the affine subspaces $\mathbb{C}^k \times \{q\}$, with $q \in \mathbb{C}^{n-k}$.

Example 1.1.5 (Pull-back or inverse image of a foliation)**.** For the next example, first, we need to define the pullback of a foliation.

Definition 1.1.6. Let M and N be complex varieties, $f : M \to N$ a holomorphic application and $\mathcal{F}$ a foliation of N of codimension k. We say that f *is transversal to* $\mathcal{F}$ if for every point $p \in N$, the subspace $df_q(T_qM)$ and $T_p\mathcal{F}$ generate the tangent space T_pN, where $p = f(q)$. When that happens, there is a foliation in M, denoted by $f^*(\mathcal{F})$, of the same codimension k, whose leaves are the inverse images by f of the leaves of $\mathcal{F}$ in N. The foliation $f^*(\mathcal{F})$ is called the *pull-back* or *inverse image* of $\mathcal{F}$ by f.

The construction of the foliation $f^*(\mathcal{F})$ is given by item (II) of Definition 1.1.3.

Example 1.1.7 (Foliations generated by differential 1-forms)**.** Let M be a complex variety of dimension n and ω a 1-form not identically zero holomorphic in M. Let $S = \{p \in M; \omega_p \equiv 0\}$ be the singular set of ω. In this case, ω induces a distribution of hyperplanes Ω in open $N = M \setminus S$ defined by

$$\Omega_p = \ker \omega_p = \{v \in T_pM;\ \omega_p(v) = 0\}.$$

Now, we say that ω (or Ω) is *integrable* if there is a holomorphic foliation $\mathcal{F}$ in N such that $T\mathcal{F} = \Omega$ (which means that $T_p\mathcal{F} = \Omega_p$). It is an established result - known as the Theorem of Frobenius - that ω is integrable if, and only if, $\omega \wedge d\omega = 0$. It is often said that the foliation $\mathcal{F}$ is defined by the differential equation $\omega = 0$ and that the leaves of $\mathcal{F}$ are integral submanifolds of this equation.

The foliations generated by 1-forms given in the example above are a fundamental part of this work. Regarding that, [19, Proposition 2.21] gives us the following result:

Proposition 1.1.8. A foliation $\mathcal{F}$ over $\mathbb{C}^n$ can be defined by a global integrable holomorphic 1-form.

Note that if η is a 1-form such that $\eta = f\omega$, where f is a holomorphic function that does not vanish in N, then the hyperplane distribution induced by η coincides with Ω. In particular, η is also integrable and the foliations defined by $\eta = 0$ and $\omega = 0$ coincide.

Example 1.1.9. Let $f : \mathbb{C}^n \to \mathbb{C}$ a holomorphic function. We can associate the differential equation $y' = f(x, y)$ to a foliation $\mathcal{F}$ over $\mathbb{C}^n$ defined by the 1-form

$$\omega = dy - f(x, y)dx.$$

In that case, the leaves of $\mathcal{F}$ are precisely the graphs of the solutions of the equation. It is worth noting that, even when f is a polynomial, little is known about such foliations.

Back to Example 1.1.7, it shows us that a holomorphic differential 1-form integrable ω, defined in a complex manifold M, defines a foliation of codimension one in $M \setminus S$, where S is the singular set of ω. With that in mind, we refer to [50, Section 1.4] for our next results regarding a *singular foliation of codimension one* that, in a rough way, can be seen locally as an object defined by an integrable one-form.

Definition 1.1.10. Let M be a complex manifold of dimension $n \geq 2$. A *holomorphic singular foliation of codimension one in* M is an object $\mathcal{F}$ given by collections $\{\omega_\alpha\}_{\alpha \in A}$, $\{U_\alpha\}_{\alpha \in A}$ and $\{g_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \emptyset}$, such that:

(i) $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M.

(ii) ω_α is a holomorphic differential one-form integrable non-identically zero in U_α.

(iii) $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$.

(iv) if $U_\alpha \cap U_\beta \neq \emptyset$ then $\omega_\alpha = g_{\alpha\beta} \cdot \omega_\beta$ in $U_\alpha \cap U_\beta$. For each ω_α, we consider its singular set given by

$$\text{sing}(\omega_\alpha) = \{p \in U_\alpha \mid \omega_\alpha(p) = 0\} =: S_\alpha.$$

We observe that S_α is an analytic subset of U_α. From (iii) and (iv) it follows that $S_\alpha \cap U_\alpha \cap U_\beta = S_\beta \cap U_\alpha \cap U_\beta$. Thus, the union of these S_α defines an analytic subset S of M. This set, denoted by $\text{sing}(\mathcal{F})$, is called the *singular set* of $\mathcal{F}$. We say that two foliations $\mathcal{F}$ and $\mathcal{F}_1$ coincide if $\text{sing}(\mathcal{F}) = \text{sing}(\mathcal{F}_1)$ and $\mathcal{F}|_{M \setminus \text{sing}(\mathcal{F})} = \mathcal{F}_1|_{M \setminus \text{sing}(\mathcal{F}_1)}$. In the case where $\text{sing}(\mathcal{F}) = \emptyset$, we see that $\mathcal{F}$ is a foliation of codimension one, which we call a *regular foliation*. The next proposition gives us some properties of singular foliations.

Proposition 1.1.11. Let $\mathcal{F}$ be a singular foliation of codimension one in M. There exists a foliation $\mathcal{F}_1$ in M with the following properties:

(a) the irreducible components of $\text{sing}(\mathcal{F}_1)$ are of codimension ≥ 2, where $\text{sing}(\mathcal{F}_1) \subset \text{sing}(\mathcal{F})$.

(b) $\mathcal{F}_1$ coincides with $\mathcal{F}$ in $M \setminus \text{sing}(\mathcal{F})$.

(c) $\mathcal{F}_1$ is maximal, that is, if $\mathcal{F}_2$ is the foliation in M satisfying (a) and (b), then $\mathcal{F}_2 = \mathcal{F}_1$.

1.1.1 Foliations in $(\mathbb{C}^2, 0)$

In this subsection, we present some definitions and results of holomorphic foliations on $(\mathbb{C}^2, 0)$.

If $\mathcal{F}$ is a foliation on $(\mathbb{C}^2, 0)$, Proposition 1.1.8 gives us that $\mathcal{F}$ is defined by a global integrable holomorphic 1-form ω. Then, consider $\mathcal{F}$ being defined by

$$\omega = A(x, y)dx + B(x, y)dy, \tag{1.3}$$

where $A, B \in \mathbb{C}\{x, y\}$ are relatively prime, with $\mathbb{C}\{x, y\}$ denoting the ring of complex convergent power series in two variables. Then, we define *the order* ν_m of the foliation $\mathcal{F}$ at the point m as

$$\nu_m = \min\{\nu_m(A), \nu_m(B)\},$$

where $\nu_m(X)$ is the order of the first term of the first jet in the Taylor series of X. In that way, m is a non-singular point of $\mathcal{F}$ if, and only if, $\nu_m(\mathcal{F}) = 0$.

If a holomorphic foliation $\mathcal{F}$ is defined by a holomorphic 1-form $\omega = A(x,y)dx + B(x,y)dy$, it is equivalent to say that $\mathcal{F}$ is defined by its dual vector field, given by

$$v = -B(x,y)\frac{\partial}{\partial x} + A(x,y)\frac{\partial}{\partial y}. \tag{1.4}$$

Now, the concept of non-invariancy of a hypersurface over a 1-form can be given by the next definition:

Definition 1.1.12. We say that $X \subset (\mathbb{C}^2, 0)$ is *invariant* by an 1-form ω, if $T_pX \subset \ker(\omega_p)$ for all regular points p on X.

We can define then the invariancy of a curve over a foliation in a similar way of Definition 1.1.12. Let $f(x,y) \in \mathbb{C}\{x,y\}$. We say that $C : f(x,y) = 0$ is *invariant* by $\mathcal{F}$ if $T_pC \subset \ker(\omega_p)$ for all regular points p on C. That statement is equivalent to saying that

$$\omega \wedge df = (f \cdot h)dx \wedge dy,$$

for some $h \in \mathbb{C}\{x,y\}$. If C is irreducible, then we will say that C is a *separatrix* of $\mathcal{F}$. We say that the separatrix C is *analytical* if f is convergent. We denote by $Sep_0(\mathcal{F})$ the set of all separatrices of $\mathcal{F}$. When $Sep_0(\mathcal{F})$ is a finite set, we will say that the foliation $\mathcal{F}$ is *non-dicritical*. Otherwise, we will say that $\mathcal{F}$ is *dicritical*.

Definition 1.1.13. Let X be a curve not invariant by $\mathcal{F}$. Following [14, Chapter 2, Section 2], we can consider the *tangency order of $\mathcal{F}$ to X at* $0 \in \mathbb{C}^2$:

$$\operatorname{tang}(\mathcal{F}, X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \phi, v(\phi)\rangle},$$

where $\{\phi = 0\}$ is the local (reduced) equation of X around 0, and v is a local holomorphic vector field generating $\mathcal{F}$ around 0.

Example 1.1.14. Consider the foliation $\mathcal{F}$, defined by the vector field

$$v = -2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

and consider X the curve defined by $\phi(x,y) = y^3 - x^2$. Note that X is not invariant by $\mathcal{F}$, since $\mathcal{F}$ is also defined by $\omega = 2xdy + ydx$ and

$$\begin{aligned}\omega \wedge d\phi &= (2xdy + ydx) \wedge (-2xdx + 3y^2dy)\\ &= (3y^3 + 4x^2)dx \wedge dy.\end{aligned}$$

In that case, we have

$$\langle \phi, v(\phi)\rangle = \langle y^3 - x^2, 4x^2 + 3y^3\rangle = \langle x^2, y^3\rangle,$$

and with that

$$\operatorname{tang}(\mathcal{F}, X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle x^2, y^3\rangle} = 6.$$

Later, we will see that Example 1.1.14 is a particular case of the tangency order of a foliation defined by an 1-form $\omega = \lambda x dy + y dx$, with $\lambda \neq 1$ to a curve X defined by $\phi = y^p - x^q$ (see Example 2.3.2).

Next, we define reduced singularities of a foliation, an important definition that we will work in the next topic of this section.

Definition 1.1.15. We say that $0 \in \mathbb{C}^2$ is a *reduced* singularity for $\mathcal{F}$ if the linear part $Dv(0)$ of the vector field v in (1.4) is non-zero and has eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ fitting in one of the cases:

(i) $\lambda_1\lambda_2 \neq 0$ and $\frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}^+$ (*non-degenerate*);

(ii) $\lambda_1 \neq 0$ and $\lambda_2 = 0$ (or vice-versa) (*saddle-node singularity*).

This is a known definition for reduced singularities, which can be seen in [14, Definition 1.1] or [50, Definition 1.17], for example. In the case (i), there is a system of coordinates (x, y) in which $\mathcal{F}$ is defined by the equation

$$\omega = x(\lambda_1 + a(x, y))dy - y(\lambda_2 + b(x, y))dx,$$

where $a(x, y), b(x, y) \in \mathbb{C}\{x, y\}$ are non-units, so that $Sep_0(\mathcal{F})$ is formed by two transversal analytic branches, given by $\{x = 0\}$ and $\{y = 0\}$. In the case (ii), up to a formal change of coordinates, the saddle-node singularity is given by a 1-form of the type

$$\omega = x^{k+1}dy - y(1 + \lambda x^k)dx,$$

where $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}^+$ are invariants after formal changes of coordinates (see [53, Proposition 4.3]). The curve $\{x = 0\}$ is an analytic separatrix, called *strong* separatrix, while $\{y = 0\}$ corresponds to a possibly formal separatrix, called *weak* separatrix. The integer $k + 1 > 1$ is called *tangency index* of $\mathcal{F}$ with respect to the weak separatrix.

1.1.2 Blow-Ups

To talk about reduction of singularities, we have to define the process of blow-up (or explosion), that we shall describe in this section. Our main reference is [50, p. 37].

In this work, we are focusing in defining the blow-up of $\mathbb{C}^2$ in 0. Consider two copies of $\mathbb{C}^2$, say U and V, with coordinates (t, x) and (s, y), respectively. We define a complex manifold $\tilde{\mathbb{C}}^2$, identifying the point $(t, x) \in U \setminus \{t = 0\}$ with the point $(s, y) = \alpha(t, x) = (1/t, tx) \in V \setminus \{s = 0\}$.

A divisor of $\tilde{\mathbb{C}}^2$ is, by definition, a submanifold D of $\tilde{\mathbb{C}}^2$ such that $U \cap D = \{x = 0\}$ and $V \cap D = \{y = 0\}$. Note that $y = tx$, D is well-defined and is biholomorphic to $\overline{\mathbb{C}} = \mathbb{C}P(1)$.

Consider now a holomophic map $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ defined by

$$\pi|_U(t, x) = (x, tx) \text{ and } \pi|_V(s, y) = (sy, y).$$

Note that π is well-defined, given that in $U \cap V$ we have $y = tx$ and $x = sy$. Moreover, π has the following properties:

- $\pi^{-1}(0) = D$;
- $\pi|_{\tilde{\mathbb{C}}^2 \setminus D} : \tilde{\mathbb{C}}^2 \setminus D \to \mathbb{C}^2 \setminus \{0\}$ is a biholomorphism;
- π is proper.

With all that, we say that $\tilde{\mathbb{C}}^2$ is the *blow-up* or *explosion* of $\tilde{\mathbb{C}}^2$ in 0, with the map of a blow-down π.

Now, if $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ is a blow-down of $\mathbb{C}^2$ in 0, we can take the expression of π for the chart $((t, x), U)$ of $\mathbb{C}^2$, and obtain

$$\begin{aligned} f \circ \pi(t, x) = f(x, tx) &= \sum_{j=k}^{\infty} f_j(x, tx) \\ &= x^k \sum_{j=k}^{\infty} x^{j-k} f_j(1, t) \\ &= x^k f_U(t, x), \end{aligned}$$

such that

$$\pi^{-1}(C) \cap U = \{x = 0\} \cup \{f_U(t, x) = 0\}.$$

In an analogous way, we obtain in the chart $((s, y), V)$ that

$$\pi^{-1}(C) \cap V = \{y = 0\} \cup \{f_V(s, y) = 0\},$$

where $f_V(s, y) = \sum_{j=k}^{\infty} y^{j-k} f_j(s, 1)$. Thus, we have that $\pi^{-1}(C) = D \cup \tilde{C}$, where

$$\tilde{C} = \{f_U = 0\} \cup \{f_V = 0\}.$$

The curve $\tilde{C}$ defined above is called the *strict transform of* C.

Example 1.1.16. Consider a singular curve C, in $\mathbb{C}^2$ given by $f(x, y) = y^2 - x^3 = 0$, and let $\pi : \tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ be the blow-down for $\mathbb{C}^2$ in 0. Taking the expression of π in the chart $((t, x), U)$, we obtain

$$f \circ \pi = f \circ \pi|_U(t, x) = f(x, tx) = x^2(t^2 - x).$$

Thus, $\pi^{-1}(C) \cap U$ consists of the divisor $\{x = 0\}$ and the strict transform C_1 of C, with equation $x - t^2 = 0$. It is also not difficult to see that $\pi^{-1}(C) \subset U$, such that it is not necessary to consider another chart.

The next result was given in [50, Theorem 1.4] (in which they refer to [18]), and its proof is obtained by making use of blow-ups:

Theorem 1.1.17 (Theorem of the separatrix of Camacho–Sad)**.** Let $\mathcal{F}$ be a holomorphic foliation of dimension one in a complex manifold M of dimension two with an isolated singularity $q \in M$. Then $\mathcal{F}$ has a separatrix in q.

Now, consider a holomorphic foliation $\mathcal{F}$ in a neighborhood of $0 \in \mathbb{C}^2$, with isolated singularity at 0, and suppose that $\mathcal{F}$ is defined by $\omega = A(x,y)dx + B(x,y)dy$, as seen in (1.3). We denote by $\mathcal{F}^*$ a foliation with isolated singularities obtained from $\pi^*(\omega)$. If we write the Taylor expansion of ω in 0 as

$$\omega = \sum_{j=k}^{\infty}(A_j dx + B_j dy),$$

where A_j and B_j are homogeneous polynomials of degree j, with $A_k \not\equiv 0$ or $B_k \not\equiv 0$, a form $\pi^*(\omega)$ can be written in the chart $((t,x),U)$ as

$$\begin{aligned}\pi^*(\omega) &= \sum_{j=k}^{\infty}(A_j(x,tx)dx + B_j(x,tx)d(tx))\\ &= x^k \cdot \sum_{j=k}^{\infty} x^{j-k}[(A_j(1,t) + tB_j(1,t))dx - xB_j(1,t)dt].\end{aligned}$$

Dividing the formula above by x^k we obtain

$$x^{-k}\pi^*(\omega) = (A_k(1,t) + tB_k(1,t))dx + xB_k(1,t)dt + x \cdot \alpha, \tag{1.5}$$

where $\alpha = \sum_{j=k+1}^{\infty} x^{j-k-1}[(A_j(1,t) + tB_j(1,t))dx - xB_j(1,t)dt]$. Similarly, calculating $\pi^*(\omega)$ in the chart $((s,y),V)$, we obtain

$$y^{-k} \cdot \pi^*(\omega) = (sA_k(s,1) + B_k(s,1))dy + yA_k(s,1)ds + y \cdot \beta. \tag{1.6}$$

When we divide (1.5) by a convenient power of x, we obtain a foliation ω_1. For example, if $A_k(1,t) + tB_k(1,t) \equiv 0$, we have that $\omega_1 = B_k(1,t)dt + \alpha$. In a similar way, we can obtain ω_2 by dividing (1.6) by y. Hence, the foliation $\mathcal{F}^*$ will be represented in the first chart by ω_1, and in the other chart by ω_2. For more details of this process, see [50, Section 1.6, p.41].

When we deal with a simple blow-up π at $0 \in \mathbb{C}^2$, it may occur that $\pi^{-1}(p)$ is invariant or not. We say that π is *dicritical* with respect to a holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$, if the exceptional divisor $\pi^{-1}(0)$ is not $\mathcal{F}^*$-invariant. Otherwise, π is called *non-dicritical.*

In [19, Théorème 4.6], it is shown that the blow-ups play a fundamental role in the process of reduction of singularities.

Theorem 1.1.18. Let $\mathcal{F}$ be a germ of a foliation over $(\mathbb{C}^2, 0)$. Then, there exists a morphism $\pi : \mathcal{E} \to (\mathbb{C}^2, 0)$ (with $\mathcal{E} = \pi^{-1}(0)$), given by a finite number of blow-ups of points, such that every singularity of $\pi^*\mathcal{F}$ is a reduced singularity.

With that in mind, we can define generalized curve foliations:

Definition 1.1.19. A *generalized curve foliation* is a foliation $\mathcal{F}$ defined by a vector field v whose reduction of singularities admits only non-degenerate singularities with non-vanishing eigenvalues.

The concept of a generalized curve foliation was defined by Camacho-Lins Neto-Sad, in [17, p. 144]. The reduction of singularities is the process described in Theorem 1.1.18, first achieved by Seidenberg ([64]) and also by Van den Essen ([29]) using the notion of intersection multiplicity.

To present a property of generalized curve foliations, we say that a *formal first integral* for a germ of holomorphic foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is a formal series

$$\hat{f}(x, y) = \sum_{i,j=0}^{\infty} f_{ij} x^i y^j \in \mathbb{C}\{x, y\}$$

such that $d\hat{f} \wedge \omega = 0$ as a formal expression, where ω is the holomorphic 1-form defining $\mathcal{F}$. It can be shown that, if $\mathcal{F}$ admits a formal first integral, then it is a generalized curve foliation ([63, p. 83]).

1.1.3 Global Foliations

To end this section, we present some notions on global foliations. The main reference to this section is [14, Chapter 2]. Let S be a compact complex surface and let $\{U_j\}_{j \in I}$ be an open covering of S. A *holomorphic foliation* $\mathcal{F}$ on S can be described by a collection of holomorphic 1-forms $\omega_j \in \Omega^1_S(U_j)$ with isolated zeros such that

$$\omega_i = g_{ij}\omega_j \text{ on } U_i \cap U_j, \quad g_{ij} \in \mathcal{O}^*_S(U_i \cap U_j),$$

which means that each g_{ij} is a nowhere vanishing holomorphic function. The *singular set* $\mathsf{Sing}(\mathcal{F})$ of $\mathcal{F}$ is the finite subset of S defined by

$$\mathsf{Sing}(\mathcal{F}) \cap U_j = \text{zeros of } \omega_j, \ \forall j \in I.$$

We say that a point $q \in S$ is a regular point if $q \notin \mathsf{Sing}(\mathcal{F})$. The functions $\{g_{ij}\}$ form a multiplicative cocycle - i.e., g_{ii} is the identity and $g_{ij} \circ g_{jk} = g_{ik}$ for all different i, j, k - that defines a line bundle (a complex vector bundle of rank 1) $N_{\mathcal{F}}$ on S, called the *normal bundle* of $\mathcal{F}$, and its dual $N^*_{\mathcal{F}}$ is called the *conormal bundle* of $\mathcal{F}$. The foliation gives rise to a global holomorphic section of $N_{\mathcal{F}} \otimes T^*S$, with finite zero set and modulo multiplication by $\mathcal{O}^*_S(S)$, and to an exact sequence

$$0 \to N^*_{\mathcal{F}} \to T^*S \to \mathcal{I}_Z \cdot T^*_{\mathcal{F}} \to 0,$$

where $T^*_{\mathcal{F}}$ is a line bundle on S, called the *canonical bundle* of $\mathcal{F}$, and $\mathcal{I}_Z$ is an ideal sheaf supported on $\mathsf{Sing}(\mathcal{F})$. The dual $T_{\mathcal{F}}$ of $T^*_{\mathcal{F}}$ is called *the tangent bundle* of $\mathcal{F}$. These line bundles are related to each other in the following way

$$K_S = T^*_{\mathcal{F}} \otimes N^*_{\mathcal{F}},$$

where K_S is the canonical bundle S, i.e., the line bundle of S whose sections are the 2-forms on S:

$$K_S = T^*S \wedge T^*S.$$

Finally, we present the next two propositions, according to [14, Propositions 2.1 and 2.2]. In the first one, the indice $\mu_p(\mathcal{F})$, which represents the *Milnor number* of the foliation $\mathcal{F}$ and that appears in Proposition 1.1.20, will be defined in the next section of this chapter.

Proposition 1.1.20. Let $\mathcal{F}$ be a foliation on a compact surface X. Then

$$\sum_{p \in \mathsf{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = T_{\mathcal{F}} \cdot T_{\mathcal{F}} + T_{\mathcal{F}} \cdot K_S + c_2(X),$$

where $c_2(X)$ represents the second Chern class of X.

Proposition 1.1.21. Let $\mathcal{F}$ be a foliation on a complex surface S, and let $X \subset S$ be a compact curve, each component of which is not invariant by $\mathcal{F}$. Then

$$\sum_{p \in \mathsf{Sing}(\mathcal{F}) \cap X} \operatorname{tang}(\mathcal{F}, X, p) = N_{\mathcal{F}} \cdot X - \chi(X),$$

where $\chi(X) = -K_S \cdot X - X \cdot X$ is the virtual Euler characteristic of X.

Remark 1.1.22. In Proposition 1.1.20, we mentioned the (second) Chern class of X, a subject that comes from the study of the Chern-Weil theory of characteristic classes that, due to its extension, will not be covered here. In a simpler way, we define a *Chern class* $c_i(E)$ of a complex vector bundle E in a manifold M by

$$c_i(E) = \left[P^i \left(\frac{\sqrt{-1}}{2\pi} \Theta \right) \right] \in H^{2i}_{DR}(M),$$

where P is an invariant polynomial of degree k and Θ is a curvature operator. For a more detailed definition of Chern classes, we refer the reader [41, Chapter 3, Section 3].

1.2 Indices

In this section, we give the definition and a few properties of important indices that will be fundamental to comprehend this work: the Milnor and Tjurina numbers, the GSV-Index, and the Bruce-Roberts numbers.

1.2.1 Milnor numbers

The terminology "Milnor number" can refer to foliations, hypersurfaces, and many other mathematical objects. Here, we first present the classical *Milnor number* of a function f defined in [56, Section 7].

Definition 1.2.1. The *Milnor number* $\mu_m(f)$ *of the isolated zero* m *is the degree of the mapping*

$$z \mapsto \frac{\nabla f(z)}{||\nabla f(z)||},$$

from a small sphere S_ϵ *centered at* m *to a small sphere of the unit sphere of* $\mathbb{C}^n$*, where* $f : \mathbb{C}^n \to \mathbb{C}$.

In the same work ([56, Appendix B]), it is also shown that the Milnor number can be computed as the length of the quotient of $\mathbb{C}\{x_1, \dots, x_n\}$ by the Jacobian ideal of f. In other words,

$$\mu_m(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_1, \dots, x_n\}}{\left\langle \dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n} \right\rangle}.$$

In particular, by [22, Theorem 6.6.3], if $n \geq 1$ and f have an isolated critical point at 0, then the Milnor number of f can be written as

$$\mu_0(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle \dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n} \right\rangle}, \tag{1.7}$$

where $\mathcal{O}_n$ represents the local ring of holomorphic functions from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}, 0)$. The name "Milnor number" is given after the author of [56], although in that work this indice is simply called the multiplicity μ.

As a consequence of this result, we have that the Milnor number is an invariant of a hypersurface with isolated singularity of the form $(X, 0) = (f^{-1}(0), 0)$. The next result, proven in [22, Lemma 6.6.4], states that formally:

Lemma 1.2.2. Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, with $n \geq 1$, be holomorphic functions with isolated singularity such that $f^{-1}(0) = g^{-1}(0)$. Then, $\mu_0(f) = \mu_0(g)$.

With that, we can present the definition of the Milnor number of a hypersurface.

Definition 1.2.3. Let $(X, 0)$ be a hypersurface with isolated singularity at $(\mathbb{C}^n, 0)$, $n \geq 1$. The *Milnor number of* $(X, 0)$ is defined as

$$\mu_0(X) = \mu_0(f),$$

where $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is any reduced holomorphic function such that $f^{-1}(0) = X$.

Namely, the Milnor number of a hypersurface is the Milnor number of the function that defines it, and that does not depend on the choice of the function. Moreover, in this work, we will also write $f^{-1}(0) = X$ as $X = \{f = 0\}$.

An important property of the Milnor number is its topological invariance. For that, we present that definition formally (along with the definition of analytical invariance, which we are going to use later), as given in [39, Definition 3.30]:

Definition 1.2.4. Let $(X, z) \subset (\mathbb{C}^n, z)$ and $(Y, w) \subset (\mathbb{C}^n, w)$ be two germs of isolated hypersurface singularities.

1. (X, z) and (Y, w) (or any defining power series) are said to be *analytically equivalent* (or contact equivalent) if there exists a local analytic isomorphism $(\mathbb{C}^n, z) \to (\mathbb{C}^n, w)$ mapping (X, z) to (Y, w). The corresponding equivalence classes are called analytic types.

2. (X, z) and (Y, w) (or any defining power series) are said to be *topologically equivalent* if there exists a homeomorphism $(\mathbb{C}^n, z) \to (\mathbb{C}^n, w)$ mapping (X, z) to (Y, w). The corresponding equivalence classes are called topological types.

3. A number (or a set, or a group, ...) associated to a singularity is called an *analytic* (respectively *topological*) invariant if it does not change its value within an analytic (respectively topological) equivalence class.

With that, we observe that the Milnor number is a topological invariant of complex hypersurfaces with isolated singularities, as proved in [69, Theorem 2.1].

Now, consider V a hypersurface that has an isolated complete intersection singularity (an ICIS) at 0, defined by a germ

$$h = (g_1, \ldots, g_k) : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0). \tag{1.8}$$

Then, the Milnor number of an ICIS can be defined in a similar way to Definition 1.2.1, as established in [60, Section 1]: for a small enough ε, the fiber

$$X_t = f^{-1}(t) \cap \mathbb{B}_\varepsilon,$$

where $0 < |t| < \varepsilon$, and $\mathbb{B}_\varepsilon$ is the ball centered in 0 and radius ε, is called the *Milnor fiber* of $(V, 0)$. In this case, Hamm [43] establishes that the Milnor fiber of $(V, 0)$ possesses the homotopy type of a bouquet of spheres of the same dimension $\dim(V, 0) = n$. We also denote by $b_n(X_t)$ the middle Betti number of the Milnor fiber.

Definition 1.2.5. The Milnor number of the ICIS $(V, 0)$, denoted as $\mu_0(V)$ (or $\mu_0(h)$ and $\mu_0(g_1, \ldots, g_k)$, according to (1.8), is defined by

$$\mu_0(V) := b_n(X_t),$$

representing the number of these spheres.

With that, we can present the *Lê-Greuel formula* for the Milnor number of an ICIS, that generalizes (1.7), as seen in [12], [68] or [22, Theorem 6.6.8]:

Theorem 1.2.6 (Lê-Greuel formula)**.** If $g_1, \dots, g_k$ and f are holomorphic map germs $(\mathbb{C}^{n+k+1}, 0) \to (\mathbb{C}, 0)$ such that $h = (g_1, \dots, g_k)$ and (h, f) define isolated complete intersection germs, then their Milnor numbers are related by

$$\mu_0(h) + \mu_0(h, f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+k,0}}{(h, \mathrm{J}_{k+1}(h, f))},$$

where $\mathrm{J}_{k+1}(h, f)$ denotes the ideal generated by the determinants of all $(k+1)$-minors of

$$\begin{pmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_{n+k+1}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_k}{\partial x_1} & \cdots & \dfrac{\partial g_k}{\partial x_{n+k+1}} \\ \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_{n+k+1}} \end{pmatrix}.$$

In dimension two, the definition of the Milnor number is equivalent to the definition of the *intersection multiplicity.* By introducing this definition, our main goal is to make easier the calculation of the Milnor number through the properties of the intersection multiplicity. We refer to [19, Section 4.2.2] to the next results, but also to [33, Section 3.3] to some properties.

Definition 1.2.7. Let $a, b \in \mathcal{O}_2$, or two principal ideals $a\mathcal{O}_2$ and $b\mathcal{O}_2$. The *intersection multiplicity* $i_0(a, b)$ is, by definition

$$i_0(a, b) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle a, b \rangle}.$$

Although the intersection multiplicity can be defined on the ring of holomorphic functions around a point m, our focus here will be on holomorphic functions on $(\mathbb{C}^2, 0)$. The intersection multiplicity has the following properties:

1. $i_0(a, b) = i_0(b, a)$;
2. $i_0(aa', b) = i_0(a, b) + i_0(a', b)$;
3. $i_0(a, b) < \infty$ if, and only if a and b don't have common factors;
4. $i_0(a, b) = 0$ if, and only if $a(0) \neq 0$ or $b(0) \neq 0$;
5. $i_0(a, b) = i_0(a, b + k_1 a)$ and $i_0(a, b) = i_0(a + k_2 b, b)$, with $k_1, k_2 \in \mathbb{C}\{x, y\}$;
6. If $a = c_m x^m$ and $b = c_n y^n$, with $m, n \geq 1$ and $c_m, c_n \in \mathbb{C}$, then $i_0(a, b) = mn$.

With that definition, we have that $\mu_0(f) = i_0(f_x, f_y)$, when $f \in \mathcal{O}_2$.

Example 1.2.8. Consider the polynomial $f(x,y) = x^5 + x^2y + y^4$. We are going to compute the Milnor number of f using the properties of the intersection multiplicity:

$$\begin{aligned}
\mu_0(f) &= i_0(f_x, f_y) \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle 5x^4 + 2xy, x^2 + 4y^3\rangle} \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle x(5x^3 + 2y), x^2 + 4y^3\rangle} \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle x, x^2 + 4y^3\rangle} + \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle 5x^3 + 2y, x^2 + 4y^3\rangle} \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle x, x^2 + 4y^3 - x\cdot x\rangle} + \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle 5x^3 + 2y - 5x(x^2+4y^3), x^2 + 4y^3\rangle} \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle x, 4y^3\rangle} + \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle 2y - 20xy^3, x^2 + 4y^3\rangle} \\
&= 3 + \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle 2 - 20xy^2, x^2 + 4y^3\rangle} + \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle y, x^2 + 4y^3\rangle} \\
&= 3 + 0 + \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle y, x^2\rangle} \\
&= 3 + 2 = 5.
\end{aligned}$$

As an application of the Milnor number and the intersection multiplicity in dimension two, we refer to [34, p. 3] to present the following definition:

Definition 1.2.9. Let $\phi \in \mathcal{O}_2$, and consider $l = ay - bx \in \mathbb{C}\{x,y\}$ a regular parameter that does not divide ϕ, for $(a,b) \neq (0,0)$. We call the *polar of ϕ with respect to l*

$$\mathcal{P}_l(\phi) = \frac{\partial \phi}{\partial x}\frac{\partial l}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial l}{\partial x} = a\frac{\partial \phi}{\partial x} + b\frac{\partial \phi}{\partial y}.$$

With that, we refer to [34, Corollary 3.4] to enunciate the next proposition, that is a result first proved in [66, Chapter II, Proposition 1.2].

Proposition 1.2.10. (Teissier's Lemma) Let $\phi \in \mathcal{O}_2$, and consider $l = ay - bx \in \mathbb{C}\{x,y\}$ a regular parameter that does not divide f, for $(a,b) \neq (0,0)$. Then, for $X = \{\phi = 0\}$,

$$\mu_0(X) = i_0(\phi, \mathcal{P}_l(\phi)) - i_0(\phi, l) + 1.$$

Moreover, we observe that, in [71, Theorem 6.5.1], it is shown that the Milnor number of a product of two functions $f, g \in \mathcal{O}_2$ satisfies

$$\mu_0(fg) = \mu_0(f) + \mu_0(g) + 2i_0(f,g) - 1. \tag{1.9}$$

Our next step is to define the Milnor number for holomorphic 1-forms. Based on [17, Section 2], we present the following:

Definition 1.2.11. The *Milnor number* of a holomorphic 1-form ω, defined by $\omega = \sum_{j=1}^{n} A_j(z)dz_j$, with an isolated singularity at $0 \in \mathbb{C}^n$, $n \geq 2$, is defined as follows:

$$\mu_0(\omega) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle A_1, \ldots, A_n \rangle}.$$

Thus, the *Milnor number of a foliation* $\mathcal{F}$ is given by $\mu_0(\mathcal{F}) = \mu_0(\omega)$, where $\mathcal{F}$ is a foliation by curves in dimension two defined by an integrable holomorphic 1-form ω. Moreover, an important property of the Milnor number of a foliation by curves is that $\mu_p(\mathcal{F})$ is a topological invariant when $(X, 0)$ is an isolated hypersurface singularity, and $n \geq 2$, as shown in [17, Theorem A]. In that case, that means that, if $\mathcal{F}$ and $\mathcal{F}'$ are one-dimensional holomorphic foliations locally topologically equivalent at p and p', respectively, i.e., there is a homeomorphism ϕ between neighborhoods of p and p taking leaves of $\mathcal{F}$ to leaves of $\mathcal{F}'$, with $\phi(p) = p'$, then

$$\mu_p(\mathcal{F}) = \mu_{p'}(\mathcal{F}').$$

In the literature, the Milnor number of ω at $0 \in \mathbb{C}^n$ can sometimes be denoted as $\mathrm{ind}(\omega; \mathbb{C}^n, 0)$, and referred to as the *Poincaré-Hopf index* of ω at $0 \in \mathbb{C}^n$; see, for instance, [28, Definition 5.2.3].

The Milnor number defined above has some properties. In particular, let's work with the case where $\mathcal{F}$ is a holomorphic foliation in $(\mathbb{C}^2, 0)$. Then, we can define the Milnor number of $\mathcal{F}$ using the 1-form ω that defines $\mathcal{F}$, but also notice that $\mathcal{F}$ is defined by a vector field $v = -B(x,y)\frac{\partial}{\partial x} + A(x,y)\frac{\partial}{\partial y}$, as shown in (1.4). Then, we have

1. $\mu_0(\mathcal{F}) = 0$ if, and only if, $v(0) \neq 0$;
2. $0 < \mu_0(\mathcal{F}) < \infty$ if, and only if, 0 is an isolated singularity for v;
3. $\mu_0(\mathcal{F}) = 1$ if, and only if,

$$\det \begin{pmatrix} \dfrac{-\partial B(0,0)}{\partial x} & \dfrac{-\partial B(0,0)}{\partial y} \\ \dfrac{\partial A(0,0)}{\partial x} & \dfrac{\partial A(0,0)}{\partial y} \end{pmatrix} \neq 0.$$

We also remark that in [17, p. 146] it is proved that the properties above are valid when $\mathcal{F}$ is defined by a holomorphic vector field $v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial z_i}$ defined in an open set of $\mathbb{C}^n$, $n \geq 2$. In that case, (3) can be rewritten as $\mu_0(\mathcal{F}) = 1 \Leftrightarrow \det \left(\frac{\partial v_i(0)}{\partial z_j} \right)_{1 \leq i,j \leq n} \neq 0.$

Another property of the Milnor number of a foliation $\mathcal{F}$, is given in [36, p. 1440], and gives us a lower bound of $\mu_0(\mathcal{F})$ in terms of the algebraic multiplicity of $\mathcal{F}$. Writing $\nu = \nu_0(\mathcal{F})$, it is given by

$$\mu_0(\mathcal{F}) \geq \frac{\nu(\nu+1)}{2}. \tag{1.10}$$

Example 1.2.12. The Milnor number $\mu_0(\mathcal{F})$ of a foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$, defined by the 1-form $\omega = A(x,y)dx + B(x,y)dy$, is given by

$$\mu_0(\mathcal{F}) = \mu_0(\omega) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle A, B \rangle}.$$

In [19, Definition 4.11], the definition above is given making use of the intersection multiplicity, as

$$\mu_0(\mathcal{F}) = i_0(A, B).$$

An important result about generalized curve foliations, defined in Definition 1.1.19, can be given by making use of the Milnor number of a foliation. Assuming that the foliation $\mathcal{F}$ is non-dicritical at $0 \in \mathbb{C}^2$, and denoting by $C = Sep_0(\mathcal{F})$ the union of the separatrices of $\mathcal{F}$, in [17, Theorem 4] the following theorem is proved:

Theorem 1.2.13. Given a germ of a non-dicritical holomorphic foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$, one has $\mu_0(\mathcal{F}) \geq \mu_0(C)$. The equality holds if, and only if $\mathcal{F}$ is a generalized curve foliation.

Lastly, we have a way to describe how the Milnor number of a foliation of dimension two behaves under a blow-up. The next proposition was first demonstrated by Mattei-Moussu [54], but here we follow the approach given in [19, Proposition 4.13]:

Proposition 1.2.14. Let $\pi : (\tilde{\mathbb{C}^2}, \mathcal{E}) \to (\mathbb{C}^2, 0)$, be the blow-up of center q, where $\mathcal{E} = \pi^{-1}(0)$. Writing $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$, and by $\nu = \nu_0(\mathcal{F})$ the algebraic multiplicity of $\mathcal{F}$, where $\mathcal{F}$ is a foliation on $(\mathbb{C}^2, 0)$, we have that

$$\mu_0(\mathcal{F}) = \begin{cases} \nu^2 - (\nu+1) + \sum\limits_{p \in \pi^{-1}(0)} \mu_p(\tilde{\mathcal{F}}) & \text{if } \pi \text{ is non-dicritical;} \\ (\nu+1)^2 - (\nu+2) + \sum\limits_{p \in \pi^{-1}(0)} \mu_p(\tilde{\mathcal{F}}) & \text{if } \pi \text{ is dicritical.} \end{cases}$$

Speaking of blow-ups, we observe that a relation between the Milnor number of an irreducible curve and its blow-up is also known. From [72, p. 4], we have

$$\mu_0(X) - \mu_q(\tilde{X}) = m(m-1), \tag{1.11}$$

where m is the multiplicity of X, sometimes denoted by $\nu_0(X)$ and $\tilde{X}$ stands for the strict transform of X at $q \in \pi^{-1}(0)$.

1.2.2 Tjurina numbers

In this section, our main subject is the *Tjurina number*. It was first defined in [67], as the dimension of the base space of a semi-universal deformation of a hypersurface. Despite not being called "Tjurina number" in that work, the name came after the author, G. Tjurina. Here, we are going to focus on an equivalent definition of this index in the case of a hypersurface. For that, we refer to [39, Definition 2.1]:

Definition 1.2.15. Let $C = \{f = 0\}$ be a germ of a reduced hypersurface, with $f \in \mathbb{C}\{x_1, \ldots, x_n\}$. Then, the *Tjurina number* of $(C, 0)$ is defined as

$$\tau_0(C) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_1, \ldots, x_n\}}{\left\langle f, \dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n} \right\rangle}.$$

While the Milnor number is a topological invariant, a known property about the Tjurina number is that it is an analytical invariant. We give an example that shows that the Tjurina number τ is not a topological invariant.

Example 1.2.16. Consider the holomorphic functions $f = y^3 - x^7$ and $g = y^3 - x^7 + x^5 y$. In [39, Example 3.43.1(a)], it is shown that f and g are topologically equivalent. As expected, we have $\mu_0(f) = \mu_0(g) = 12$. However, f and g are not analytically equivalent, since $\tau_0(f) = 12$ and $\tau_0(g) = 11$.

Remark 1.2.17. The computations above were made using the software *Singular* ([24]). *Singular* will be referred to more times in this work, since it is a very important tool in our algebraic calculations. The codes used to compute Example 1.2.16 are shown below:

```
> ring r=0,(x,y),ds; // local ring
> poly f=y3-x7; // f
> poly g=f+x5y; // g
> LIB "sing.lib"; // package with commands "milnor" and "tjurina"

> milnor(f); // Milnor number of f
12

> milnor(g); // Milnor number of g
12

> tjurina(f); // Tjurina number of f
12

> tjurina(g); // Tjurina number of g
11
```

In particular, when $(X, 0) \subset (\mathbb{C}^n, 0)$ is an isolated singularity hypersurface defined by $f \in \mathcal{O}_n$, we have

$$\tau_0(X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle f, \dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n} \right\rangle}.$$

A characterization of the Tjurina number for an ICIS was given in [47, Proposition 4.6] (and also demonstrated in [65, Theorem 2.3] by Tajima, but with a different proof). When $(X, 0)$ is an ICIS, then

$$\tau_0(X) = \dim_{\mathbb{C}} \frac{\Theta_X}{\Theta_X^T}. \tag{1.12}$$

Now, observe that when X is an irreducible curve, there is a relation of the Tjurina numbers through a blow-up (defined in Section 1.1.2). As the one shown in (1.11) for Milnor numbers, [72, p. 4] gives us that

$$\tau_0(X) - \tau_q(\tilde{X}) = \frac{m(m-1)}{2} + \mathcal{D}, \tag{1.13}$$

where, again, m is the multiplicity of X and $\mathcal{D} = \dim_{\mathbb{C}} \frac{\tilde{\sigma}^*\Omega_{\tilde{X}}}{\sigma^*\Omega_X}$, where $\tilde{\sigma} : (\overline{X}, 0) \to (\tilde{X}, 0)$ is the normalization, and $\sigma = \pi \circ \tilde{\sigma}$, with π being the strict transform of X. A better study of these concepts is given in [72, Section 2]. In this work, we highlight that the construction of [72, Section 2] reveals that such $\mathcal{D}$ can be seen in terms of Milnor and Tjurina numbers, as

$$\mathcal{D} = \left(\tau_0(X) - \frac{\mu_0(X)}{2}\right) - \left(\tau_0(\tilde{X}) - \frac{\mu_0(\tilde{X})}{2}\right). \tag{1.14}$$

Now, we present a definition of the Tjurina number of a 1-form with respect to a curve, in a definition that is similar to the one that appears in [32, p. 24]. In that case, we request a curve that is invariant by the 1-form.

Definition 1.2.18. Let $V = \{f = 0\}$ be a germ of complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$. The *Tjurina number* of $\omega = \sum_{j=1}^{n} A_j(z)dz_j$ with respect to V, when V is invariant by ω, is defined as follows:

$$\tau_0(\omega, V) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle A_1, \ldots, A_n, f\rangle}. \tag{1.15}$$

By the definitions above, it is easy to see that $\tau_0(X) \leq \mu_0(X)$ and $\tau_0(\omega, V) \leq \mu_0(\omega)$.

We observe that the Tjurina number $\tau_0(v, V)$ of a holomorphic vector field v with an isolated singularity on a complex analytic hypersurface V first appears in [38, Theorem 2], but is referred to as $h_0(\Omega_{V,0}, X)$. We believe that the term "Tjurina number" first appeared in the context of foliations in [20, Remark 7]. To end this section, we refer to [32, Example 6.9] to present an example that calculates $\tau_0(\mathcal{F}, V)$.

Example 1.2.19. Consider $\omega = 4xydx + (y - 2x^2)dy$ a 1-form defining a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$. We consider the curve $C = \{y = 0\}$, which is the unique separatrix of $\mathcal{F}$. With that, we get

$$\tau_0(\mathcal{F}, C) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle 4xy, y - 2x^2, y\rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle -2x^2, y\rangle} = 2.$$

1.2.3 GSV-Index

In this section, we define several numbers that can be named *GSV-index*, named after X. Goméz-Mont, J. Seade e A. Verjovsky [37]. The ideia of its definition is to generalize the Poincaré-Hopf index and inherit its stability under perturbations - which means that, if we approximate a given vector field by another vector field that only has Morse singularities, then the local index of the initial vector field equals the number of singularities of its Morsification (counted with signs).

In our first definition of this section, we will consider $f = (f_1, \ldots, f_k) : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ a holomorphic mapping germ defining an ICIS $(X, 0) = (f^{-1}(0), 0)$. Since 0 is an isolated singularity of X, the complex conjugate gradient vector fields $\overline{\nabla} f_1, \ldots, \overline{\nabla} f_k$ are linearly independent away from the origin, and

$$\overline{\nabla} f_i = \left(\overline{\frac{\partial f_i}{\partial x_1}}, \ldots, \overline{\frac{\partial f_i}{\partial x_n}} \right).$$

We will also need the following definition, given in [11, Definition 1.3.1]:

Definition 1.2.20. An *r-field* on a subset A of M is a set $v^{(r)} = \{v_1, \ldots, v_r\}$ of r continuous vector fields defined on A. A singular point of $v^{(r)}$ is a point where the vectors (v_i) fail to be linearly independent. A nonsingular r-field is also called an *r-frame*.

We denote by $W^{r,m}$ the Stiefel manifold of complex r-frames in $\mathbb{C}^m$. In the case where these frames are orthonormal, we denote the Stiefel manifold by $W^r(m)$.

On an ICIS, [11, Section 3.2] define the GSV-index as follows:

Definition 1.2.21. Consider the set $\{v, \overline{\nabla} f_1, \ldots, \overline{\nabla} f_k\}$, which forms a $(k+1)$-frame on $X \setminus \{0\}$, where v is a continuous vector field on X, singular only at the origin. The *GSV-index* of v at $0 \in X$, denoted by $\mathrm{ind}_{\mathrm{GSV}}(v, X, 0)$ is the degree of the map

$$\varphi_v = (v, \overline{\nabla} f_1, \ldots, \overline{\nabla} f_k) : K \to W^{k+1}(n+k),$$

where $K = X \cap \mathbb{S}_\varepsilon$ represents the link of 0 in X, and $W^{k+1}(n+k)$ denotes the Stiefel manifold of complex $(k+1)$-frames in $\mathbb{C}^{n+k}$.

In [37, Theorem 4.5], the authors managed to show that the GSV-index is a topological invariant, in the case where $(X, 0)$ is an isolated hypersurface singularity in $\mathbb{C}^{n+1}, n > 2$. Also, as stated above, the GSV-index generalizes the Poincaré-Hopf index, as shown in [11, Theorem 3.2.1]:

Theorem 1.2.22. The GSV-index of v at 0 is equal to the Poincaré-Hopf index of v in the Milnor fiber M_f of $X = \{f = 0\}$:

$$\mathrm{ind}_{\mathrm{GSV}}(v, X, 0) = \mathrm{ind}_{\mathrm{PH}}(v, M_f).$$

Using [11, p. 47] as a reference, the Milnor fiber M_f can be regarded as a compact $2n$-manifold with boundary $\partial M_f = \mathrm{K}$.

In [42], [26, Section 2], Ebeling–Gusein-Zade introduced the notion of the GSV-index for a 1-form ω with respect to X, in a similar way to Definition 1.2.21:

Definition 1.2.23. The index $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0)$ of the 1-form ω on the ICIS X at the origin is the degree of the map

$$\Psi = (\omega, df_1, \ldots, df_k) : K \to W^{k+1}(n).$$

We observe that $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0)$ is finite if and only if the 1-forms $\omega, df_1, \ldots, df_k$ are linearly independent for all points over K. According to [42] or [28, Theorem 5.3.35], the index $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0)$ can also be defined as

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I},$$

where I is the ideal generated by $f_1, \ldots, f_k$ and the $(k+1) \times (k+1)$-minors of the matrix

$$\begin{pmatrix} \omega \\ d\phi \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_N} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial f_k}{\partial x_1} & \cdots & \dfrac{\partial f_k}{\partial x_N} \\ A_1 & \cdots & A_N \end{pmatrix},$$

where $\phi = (f_1, \ldots, f_k)$. The definition of this GSV-index can also be stated as

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I_X + I_{k+1}\begin{pmatrix} \omega \\ d\phi \end{pmatrix}}, \tag{1.16}$$

where I_X denotes the ideal of germs of holomorphic functions vanishing on $(X, 0)$ and $I_{k+1}\begin{pmatrix} \omega \\ d\phi \end{pmatrix}$ represents the ideal generated by the $(k+1) \times (k+1)$-minors of the matrix $\begin{pmatrix} \omega \\ d\phi \end{pmatrix}$.

Example 1.2.24. Set $k = 1$, $n = 1$ and $N = 2$ in the definition above. Then, X is a hypersurface, and we can write $\omega = Adx + Bdy$. We also consider $X = \{\phi = 0\}$. Then, I is generated by ϕ and the 2×2 minors of

$$\begin{pmatrix} \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} \\ A & B \end{pmatrix},$$

i.e., $I = \langle \phi, \frac{\partial \phi}{\partial x}B - \frac{\partial \phi}{\partial y}A\rangle$. Then, we have

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \phi, \frac{\partial \phi}{\partial x}B - \frac{\partial \phi}{\partial y}A\rangle}. \tag{1.17}$$

In that example, we can see that in dimension two, the GSV-index is equivalent to the tangency order of $\mathcal{F}$ to X. In fact, according to Definition 1.1.13, we have

$$\mathrm{tang}(\mathcal{F}, X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \phi, v(\phi)\rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\left\langle \phi, -B\frac{\partial \phi}{\partial x} + A\frac{\partial \phi}{\partial y}\right\rangle} = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0).$$

Example 1.2.25. Let $n = 2, k = 1, f(x, y) = x^2 + y^3$ and consider the 1-form

$$\omega = 3y^2dx - 2xdy.$$

The form ω on $X = \{f = 0\}$ has an isolated zero at the origin. In [26, p. 6], it is shown that the degree of the map

$$(\omega, df) : K \to W^2(2)$$

is 6, and then, $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) = 6$. Here, we are going to show that by computing the right side of (1.16). With that, we have

$$\begin{aligned}
\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) &= \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{I_X + I_2\begin{pmatrix}\omega \\ d\phi\end{pmatrix}} \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle x^2 + y^3, -4x^2 - 9y^4\rangle} \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle x^2, y^3\rangle} \\
&= 6.
\end{aligned}$$

Now, consider the case where X is an isolated singularity hypersurface defined by $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. If $\omega = \sum_{j=1}^{n} A_j dz_j$, then we can write

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi, \frac{\partial \phi}{\partial x_j}A_k - \frac{\partial \phi}{\partial x_k}A_j\rangle_{(j,k)\in\Lambda}} \tag{1.18}$$

where $\Lambda = \{(j, k);\ j, k = 1, \ldots, n,\ j \neq k\}$. Moreover, according to Lê-Greuel formula (Theorem 1.2.6), when $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a holomorphic function with an isolated singularity on $(X, 0)$, we have

$$\mathrm{Ind}_{\mathrm{GSV}}(df; X, 0) = \mu_0(X) + \mu_0(\phi, f). \tag{1.19}$$

For another application of that GSV Index, we present some definitions and known results about the radial index and the local Euler obstruction of a 1-form. We refer to [28, Sections 5.3.1 and 5.3.5] for the following propositions and definitions. Here, our focus is on the case of complex hypersurfaces with an isolated singularity, but a more detailed study of these subjects can also be seen in that reference. But first, we recur to [70, Sections 4.1 and 4.2] to define the Whitney stratification of a subset of a differentiable manifold.

Definition 1.2.26. Let Z be a closed subset of a differentiable manifold M of class C^k. A C^k *stratification* of Z is a filtration by closed subsets

$$Z = Z_d \supset Z_{d-1} \supseteq \cdots \supseteq Z_1 \supseteq Z_0$$

such that each difference $Z_i - Z_{i-1}$ is a differentiable submanifold of M of class C^k and dimension i, or is empty. Each connected component of $Z_i - Z_{i-1}$ is called a *stratum* of dimension i. Thus, Z is a disjoint union of the *strata*, denoted $\{X_\alpha\}_{\alpha\in A}$, and Z is a *stratified set*.

Definition 1.2.27. The pair (X, Y) is said to satisfy Whitney's condition (b) at $y \in Y$, or to be (b)-regular at y if: for all sequences $\{x_i\} \in X$ and $\{y_i\} \in Y$ with limit y such that, in a local chart at y, $\{T_{x_i}X\}$ tends to τ and the lines $\overline{x_i y_i}$ tend to λ, one has $\lambda \in \tau$.

When every pair of adjacent strata of a stratification is (b)-regular (at each point) then we say that the stratification is (b)-regular. Hence, we have the next definition:

Definition 1.2.28. Let Z be a closed subset of a differentiable manifold M of class C^1. When $Z = \bigcup_{\alpha\in A} X_\alpha$ is a locally finite (b)-regular stratification satisfying the frontier condition, we say we have a *Whitney stratification* of Z.

Now, let V be a closed (real) subanalytic variety in a smooth manifold M, where M is equipped with a (smooth) Riemannian metric. Consider also $V = \cup_{i=1}^{q} V_i$ be a subanalytic Whitney stratification of V, and ω be the germ at $p \in \mathbb{C}^N$ of a (continuous) 1-form on (V, p). For each $p \in V$, set $V_{(p)} = V_i$ be the stratum containing p. Then, a point $p \in V$ is a singular point of ω if the restriction of ω to the stratum $V_{(p)}$ containing p vanishes at the point p. With that, we present the following definition:

Definition 1.2.29. The germ ω of a 1-form at the point p is called *radial* if, for all $\epsilon > 0$ small enough, the 1-form is positive on the outward normals to the boundary of the ϵ-neighborhood of the point p.

Let $p \in V_i = V_{(p)}$, $\dim V_{(p)} = k$, and let η be a 1-form defined in a neighborhood of the point p. As above, let N_i be a normal slice (with respect to the Riemannian metric) of M to the stratum V_i at the point p and h a diffeomorphism from a neighborhood of p in M to the product $U_i(p) \times N_i$, where $U_i(p)$ is an ϵ-neighborhood of p in V_i, which is the identity on $U_i(p)$. A 1-form η is called a *radial extension* of the 1-form $\eta|_{V_{(p)}}$ if there exists such a diffeomorphism h which identifies η with the restriction to V of the 1-form

$\pi_1^*\eta|_{V_{(p)}} + \pi_2^*\eta_{N_i}^{\text{rad}}$, where π_1 and π_2 are the projections from a neighborhood of p in M to $V_{(p)}$ and N_i respectively, and $\eta_{N_i}^{\text{rad}}$ is a radial 1-form on N_i.

For a 1-form ω on (V, p) with an isolated singular point at the point p there exists a 1-form $\tilde{\omega}$ on V such that

1. $\tilde{\omega}$ coincides with ω on a neighborhood of the intersection of V with the boundary ∂B_ϵ of the ϵ-neighborhood around the point p;

2. the 1-form $\tilde{\omega}$ has a finite number of singular points (zeros);

3. in a neighborhood of each singular point $q \in V \cap B_\epsilon$, $q \in V_i$, the 1-form $\tilde{\omega}$ is a radial extension of its restriction to the stratum V_i.

Definition 1.2.30. The *radial index* $\text{Ind}_{\text{rad}}(\omega; V, p)$ of the 1-form ω at the point p is

$$\text{Ind}_{\text{rad}}(\omega; V, p) = \sum_{q \in \mathsf{Sing}(\tilde{\omega})} \text{ind}(\tilde{\omega}|_{V_{(q)}}; V_{(q)}, q),$$

where $\text{ind}(\tilde{\omega}|_{V_{(q)}}; V_{(q)}, q)$ is the usual index of the restriction of the 1-form $\tilde{\omega}$ to the stratum $V_{(q)}$, see [28, Definition 5.2.3].

By [28, Proposition 5.3.9], we can state that the radial index $\text{Ind}_{\text{rad}}(\omega; V, p)$ is well-defined. Also, it follows from the definition that $\text{Ind}_{\text{rad}}(\omega; V, p)$ satisfies the law of conservation of number, i.e., if a 1-form ω' with isolated singular points on X is close to the 1-form ω, then

$$\text{Ind}_{\text{rad}}(\omega; V, 0) = \sum_{q \in Sing(\omega')} \text{Ind}_{\text{rad}}(\omega'; V, q),$$

where the sum on the right-hand side is over all singular points q of the 1-form ω' on V in a neighbourhood of the origin. Moreover, it is valid a Poincaré-Hopf type theorem [28, Theorem 5.3.10]: for a compact real subanalytic variety V and a 1-form ω with isolated singular points on V, we have

$$\sum_{q \in \mathsf{Sing}(\omega)} \text{Ind}_{\text{rad}}(\omega; V, q) = \chi(V),$$

where $\chi(V)$ denotes the *Euler characteristic* of the space (variety) V.

Finally, we can prove a relation between $\text{Ind}(\omega; X, 0)$ and $\text{Ind}_{\text{rad}}(\omega; V, p)$. For a 1-form ω, we have the following proposition, proved in [27, Proposition 2.8].

Proposition 1.2.31. For a 1-form ω on an ICIS $(V, 0)$ we have

$$\text{Ind}_{\text{GSV}}(\omega; V, 0) = \text{Ind}_{\text{rad}}(\omega; V, 0) + \mu_0(V).$$

Now, we define the *local Euler obstruction* of a singular point of a 1-form following [28, Section 5.3.5]. The initial idea was given by MacPherson [52], who defined the *Euler obstruction* of a singular point of a complex analytic variety. For a recent reference where the reader can find an explicit definition, we refer [16].

Definition 1.2.32. The *local Euler obstruction* $\mathrm{Eu}(\omega; V, 0)$ of the 1-form ω on V at $0 \in \mathbb{C}^N$ is the obstruction to extend the non-zero section $\widehat{\omega}$ from the preimage of a neighborhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ to the preimage of its interior. More precisely, its value (as an element of the cohomology group $H^{2n}(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon), \mathbb{Z})$) on the fundamental class of the pair $[(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon))]$.

The Euler obstruction of a 1-form can be considered as an index. In particular, it satisfies the law of conservation of number - just as the radial index - and, on a smooth variety, the Euler obstruction and the radial index coincide. We set $\bar{\chi}(Z) := \chi(Z) - 1$ and call it the *reduced* (modulo a point) Euler characteristic of the topological space Z (though, strictly speaking, this name is only correct for a non-empty space Z). Hence, we have the following result from [28, Proposition 5.3.32]:

Proposition 1.2.33. Let $(V, 0) \subset (\mathbb{C}^N, 0)$ have an isolated singularity at $0 \in \mathbb{C}^N$ and let $\ell : \mathbb{C}^N \to \mathbb{C}$ be a generic linear function. Then

$$\mathrm{Ind}_{\mathrm{rad}}(\omega; V, 0) - \mathrm{Eu}(\omega; V, 0) = \mathrm{Ind}_{\mathrm{rad}}(d\ell; V, 0) = (-1)^{n-1}\bar{\chi}(M_\ell),$$

where M_ℓ is the Milnor fiber of the linear function ℓ on V. In particular, if f is a germ of a holomorphic function with an isolated critical point on $(V, 0)$, then

$$\mathrm{Eu}(df; V, 0) = (-1)^n(\chi(M_\ell) - \chi(M_f)),$$

where M_f is the Milnor fiber of f.

Back to foliations, let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $0 \in \mathbb{C}^2$ defined by the holomorphic 1-form ω, and let $V = \{f = 0\}$ be an $\mathcal{F}$-invariant curve, where $f \in \mathcal{O}_2$. Then, we can establish our next result (we are enunciating the result in [50, Lemma 3.1], but it is also presented in [62, (1.1)]):

Lemma 1.2.34. There exist holomorphic functions g, h and a holomorphic 1-form η, defined in a neighborhood of 0, such that

$$g\omega = hdf + f\eta, \tag{1.20}$$

where $h \not\equiv 0 \not\equiv g$ on each branch of V.

We remark that, in that lemma, f is relatively prime to $g \in \mathcal{O}_2$ and $h \in \mathcal{O}_2$. With this context, we can define our next index. It was introduced in [37], but here we follow the presentation of [15, Section 3].

Definition 1.2.35. The *Gómez-Mont–Seade–Verjovsky index* of the foliation $\mathcal{F}$ at the origin with respect to V is given by

$$\mathrm{GSV}_0(\mathcal{F}, V) = \frac{1}{2\pi i}\int_{\partial V} \frac{g}{h} d\left(\frac{h}{g}\right),$$

where ∂V is the link of V at $0 \in \mathbb{C}^2$ and g, h are given by (1.20).

In the next proposition, we enunciate a characterization of the index defined above that can be seen at [32, Proposition 6.2]. We also remark that this result is a particular case of [38, Corollary 2.7].

Proposition 1.2.36. Let $\mathcal{F}$ be a singular foliation at $(\mathbb{C}^2, 0)$ and V be a reduced curve of separatrices of $\mathcal{F}$. Then

$$\mathrm{GSV}_0(\mathcal{F}, V) = \tau_0(\mathcal{F}, V) - \tau_0(V). \tag{1.21}$$

We also have the following results by [15, Propositions 6 and 7]:

Proposition 1.2.37. If S is a non-dicritical separatrix of the foliation $\mathcal{F}$, then

$$\mathrm{GSV}_0(\mathcal{F}, S) \geq 0.$$

Proposition 1.2.38. If the singularity of $\mathcal{F}$ at 0 is a generalized curve and if S is the union of all separatrices of $\mathcal{F}$ then

$$\mathrm{GSV}_0(\mathcal{F}, S) = 0.$$

Example 1.2.39. We use the Example of [15, p. 538] to illustrate Proposition 1.2.36.

Let $\mathcal{F}$ be generated by the normal form of a saddle-node, i. e.,

$$\omega = z^{p+1}dw - w(1 + \lambda z^p)dz, \; p \geq 1, \lambda \in \mathbb{C}.$$

Then, we have $d\omega = \lambda \wedge \omega$, with $\lambda = (p+1)\dfrac{dz}{z} + \dfrac{dw}{w}$. If $S_1 = \{z = 0\}$ and $S_2 = \{w = 0\}$, we have, again by [15, p. 538],

$$\mathrm{GSV}_0(\mathcal{F}, S_1) = 1, \; \mathrm{GSV}_0(\mathcal{F}, S_2) = p + 1.$$

Now, we compute the Tjurina numbers. It is not difficult to see that $\tau_0(S_1) = \tau_0(S_2) = 0$, and for the Tjurina numbers of ω with respect to S_1 and S_2, we have

$$\tau_0(\mathcal{F}, S_1) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle z^{p+1}, w(1 + \lambda z^p), z\rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle w, z\rangle} = 1$$

and

$$\tau_0(\mathcal{F}, S_2) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle z^{p+1}, w(1 + \lambda z^p), w\rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle z^{p+1}, w\rangle} = p + 1.$$

Thus,

$$\begin{aligned} \tau_0(\mathcal{F}, S_1) - \tau_0(S_1) &= 1 = \mathrm{GSV}_0(\mathcal{F}, S_1) \\ \tau_0(\mathcal{F}, S_2) - \tau_0(S_2) &= p + 1 = \mathrm{GSV}_0(\mathcal{F}, S_2), \end{aligned}$$

and then (1.21) is satisfied.

1.2.4 Bruce-Roberts numbers

The *Bruce-Roberts number* associated with f relative to $(X, 0)$ was originally introduced by J. W. Bruce and R. M. Roberts in [13, Definition 2.4] (but named in this work as *the multiplicity of f on X*), and studied by many authors - see, for example, [3], [8], [10], [23], [44], [47], [48], [57], [58] and [59]. This number, denoted by $\mu_{BR}(f, X)$, is defined as follows:

Definition 1.2.40. Let $f \in \mathcal{O}_n$ and $(X, 0)$ be a germ of a complex analytic variety at $(\mathbb{C}^n, 0)$. The *Bruce-Roberts number* of f with respect to X is given by

$$\mu_{BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)},$$

where Θ_X is the $\mathcal{O}_n$-submodule of Θ_n defined in (1.1).

We remark that the definition above is the same as in [13, Definition 2.4], but with a different notation, similar to the definition of the Bruce-Roberts number in [58, Definition 2.1], for example. It is worth noting that some authors refer to $\mu_{BR}(f, X)$ as the *Bruce-Roberts' Milnor number* of f with respect to X. We also remark that, if $df(\Theta_X)$ has finite colength, then $J(f)$ - the Jacobian ideal of f - also has finite colength, and then $\mu_{BR}(f, X) \geq \mu_0(f)$, since $df(\Theta_X) \subseteq J(f)$. In particular, when $X = \mathbb{C}^n$, $df(\Theta_n)$ corresponds to the Jacobian ideal of f which is generated by the partial derivatives of f in $\mathcal{O}_n$. Consequently,

$$\mu_{BR}(f, \mathbb{C}^n) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_n)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle \dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n} \right\rangle} = \mu_0(f).$$

In other words, $\mu_{BR}(f, \mathbb{C}^n)$ coincides with the classical *Milnor number* $\mu_0(f)$ of f. Moreover, if X is the germ of a complex analytic subvariety at $(\mathbb{C}^n, 0)$, then $\mu_{BR}(f, X)$ is finite if, and only if f has an isolated singularity over $(X, 0)$. The following example is given in [10, Example 2.4].

Example 1.2.41. Let $X = \{(x, y, z) \in \mathbb{C}^3 : xyz = 0\}$ and let $f \in \mathcal{O}_3$ be given by $f(x, y, z) = xy + xz + yz$. We observe that

$$\Theta_X = \left\langle x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z} \right\rangle,$$

and then $df(\Theta_X) = \langle xy + xz, xy + yz, xz + yz \rangle$. In that case, $\mu_{BR}(f, X)$ is not finite, while f has an isolated singularity at the origin.

Example 1.2.42. Consider $f, h : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ given by $f(x, y) = x^2 + y^5$ and $h(x, y) = x^3 + x^2y^2 + y^7$. Let X be the plane curve defined by $X = h^{-1}(0)$. In [58, Example 3.5], it is shown that

$$df(\Theta_X) = \langle 882x^3 - 80x^2y + 42x^2y^2 - 8xy^4 + 945xy^5 - 80y^6, -8x^3 + 42x^2y^3 - 8xy^5 + 45y^8 \rangle.$$

With that, using *Singular* ([24]), we can compute $\mu_{BR}(f, X) = 15$.

In [10, Theorem 2.3], it is shown that $\mu_{BR}(f,X)$ is finite if, and only if, $V(df(\Theta_X)) \subseteq \{0\}$, with $V(df(\Theta_X))$ being the variety of zeros of the ideal $df(\Theta_X)$. We may also state a result that shows that, when the Bruce-Roberts number of f with respect to X is finite, the index defined as $c(f,h)$ is also finite.

Proposition 1.2.43. ([10, Proposition 2.8]) Let $h = (h_1, \ldots, h_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an ICIS, where $p \leq n-1$, and let $f \in \mathcal{O}_n$. Let $X = h^{-1}(0)$. If $\mu_{BR}(f,X) < \infty$, then $c(f,h) < \infty$, where

$$c(f,h) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle h_1, \ldots, h_p\rangle + J(f, h_1, \ldots, h_p)}.$$

Consider now $(X,0)$ a germ of a hypersurface with isolated singularity in $(\mathbb{C}^n, 0)$. We say that two germs $f, g : (\mathbb{C}^n, 0) \to \mathbb{C}$ are $\mathcal{R}_X$-equivalent (respectively, C^0-$\mathcal{R}_X$-equivalent) if there exists a germ of diffeomorphism (respectively homeomorphism) $\psi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that $\psi(X) = X$ and $f = g \circ \psi$. With that characterization, we have that $\mu_{BR}(f,X)$ is finite if, and only if, f is finitely $\mathcal{R}_X$-determined - i.e., $\mathcal{R}_X$ is the subgroup of $\mathcal{R}$ of coordinate changes that preserve X (see [59, p. 1051] or [13, Section 2]).

Remark 1.2.44. We can extend the equality shown in (1.12) using the concept of $\mathcal{R}_X$-equivalence. In [59, Corollary 3.4] it is shown that, if $(X,0)$ is an isolated hypersurface singularity and $f \in \mathcal{O}_n$ is finitely $\mathcal{R}_X$-determined, then

$$\tau_0(X) = \dim \frac{df(\Theta_X)}{df(\Theta_X^T)} = \dim \frac{\Theta_X}{\Theta_X^T}.$$

For our next results, we need to define the concept of weighted homogeneous functions. The next definition comes from [1, Section 2].

Definition 1.2.45. A holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a *weighted homogeneous function* of degree δ with weights $w_1, \ldots, w_n$ if

$$f(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^{\delta} f(x_1, \ldots, x_n),\ \forall \lambda > 0.$$

In the terms of the Taylor series $\sum f_{\underline{k}} x^{\underline{k}}$ of f, the weighted homogeneity condition means that the exponents of the nonzero terms of the series lie in the hyperplane

$$L = \{\underline{k} : w_1 k_1 + \ldots + w_n k_n = \delta\}.$$

Example 1.2.46. Consider the polynomial $f = x^2 y + z^2$. For the terms $x^2 y$ and z^2, we can write two equalities regarding the degree and the weights of f:

$$2w_1 + w_2 + 0 \cdot w_3 = \delta$$
$$0 \cdot w_1 + 0 \cdot w_2 + 2w_3 = \delta.$$

With that, we have $2w_1 + w_2 = 2w_3 = \delta$. Then, we can say that f is a weighted homogeneous polynomial of degree 6 with respect to weights $(2,2,3)$.

Given a vector of weights $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n_{\geq 1}$, if the coordinates $x_1, \ldots, x_n$ are fixed, then we define the *Euler vector field associated to* w as

$$\theta_w = w_1 x_1 \frac{\partial}{\partial x_1} + \ldots + w_n x_n \frac{\partial}{\partial x_n}.$$

With that, we can enunciate [10, Theorem 2.6], which gives us a way to compute Θ_X when X is defined by a weighted homogeneous function.

Theorem 1.2.47. Let $w \in \mathbb{Z}^n_{\geq 1}$ and let $h \in \mathcal{O}_n, n \geq 2$ such that h is weighted homogeneous with respect to w and h has an isolated singularity at the origin. Let $X = h^{-1}(0)$. Then Θ_X is generated by θ_w and the derivations $\theta_{ij} = \dfrac{\partial h}{\partial x_j}\dfrac{\partial}{\partial x_i} - \dfrac{\partial h}{\partial x_i}\dfrac{\partial}{\partial x_j}$, for $1 \leq i < j \leq n$. Hence, for all $f \in \mathcal{O}_n$, we have

$$df(\Theta_X) = \langle \theta_w(f) \rangle + J(f, h).$$

Example 1.2.48. Let's consider $X = h^{-1}(0)$, with $h(x, y, z) = xy + z^4$. We observe that h is weighted homogeneous, and also that $w_1 = 1$, $w_2 = 3$ and $w_3 = 1$. With that, it is possible to compute Θ_X as

$$\Theta_X = \left\langle y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} - 4z^3\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - 4z^3\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right\rangle.$$

Still working with weighted homogeneity, we enunciate [10, Theorem 2.13], that gives a relation of the Bruce-Roberts number and two Milnor numbers in that case. We also remark that this result can be seen in [58, Theorem 3.1], with the hypothesis of f being a $\mathcal{R}_X$-finitely determined germ instead of $\mu_{BR}(f, X) < \infty$ (which we saw that are equivalent conditions).

Theorem 1.2.49. Let $w \in \mathbb{Z}^n_{\geq 1}$, $n \geq 2$. Let $h \in \mathbb{C}[x_1, \ldots, x_n]$ be weighted homogeneous with respect to w with isolated singularity and let $X = h^{-1}(0)$. Let $f \in \mathcal{O}_n$ such that $\mu_{BR}(f, X) < \infty$. Then $(f, h) : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ is an ICIS whose Milnor number satisfies the relation

$$\mu_{BR}(f, X) = \mu_0(f) + \mu_0(f, h).$$

And, as a direct application of the theorem above, we have [10, Corollary 2.16]:

Corollary 1.2.50. Let $f, h \in \mathbb{C}[x_1, \ldots, x_n]$ be weighted homogeneous polynomials, not necessarily with respect to the same vector of weights, and $n \geq 2$. Let $X = h^{-1}(0)$ and $Y = f^{-1}(0)$. Suppose also that $\mu_{BR}(f, X) < \infty$ and $\mu_{BR}(h, Y) < \infty$. Then

$$\mu_{BR}(f, X) - \mu_{BR}(h, Y) = \mu_0(f) - \mu_0(h).$$

Example 1.2.51. In Example 1.2.42, we considered $f, h : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ given by $f(x, y) = x^2 + y^5$ and $h(x, y) = x^3 + x^2y^2 + y^7$, and $X = h^{-1}(0)$. That example, originally from [58, Example 3.5], shows us that Theorem 1.2.49 is not valid when X is not weighted homogeneous. In fact, they showed that $\mu_0(f) = 4$ and $\mu_0(f, h) = 14$, and then

$$\mu_{BR}(f, X) = 15 \neq \mu_0(f) + \mu_0(f, h) = 18.$$

One interesting result regarding the Bruce-Roberts number for functions is given in [59, Corollary 4.1], and it also relates it with the Tjurina number, along with the Milnor numbers as seen above. The idea, as mentioned in [59], is to extend the formula of Theorem 1.2.49 to the general case that $(X, 0)$ is a hypersurface with isolated singularity (and not necessarily weighted homogeneous).

Corollary 1.2.52. Let X be an isolated hypersurface singularity defined by $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and consider $f \in \mathcal{O}_n$, that has an isolated singularity over $(X, 0)$. Then

$$\mu_{BR}(f, X) = \mu_0(f) + \mu_0(X) + \mu_0(\phi, f) - \tau_0(X).$$

Two consequent results of Corollary 1.2.52 are shown below. For them, we refer to [59, Corollaries 4.2 and 4.3]. In particular, Corollary 1.2.54 shows that the Bruce-Roberts number is a topological invariant when $(X, 0)$ is an isolated hypersurface singularity.

Corollary 1.2.53. Let $f, \phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be function germs with isolated singularity, and let $(X, 0)$ and $(Y, 0)$ be the hypersurfaces determined by ϕ and f, respectively. If $\mu_{BR}(f, X) < \infty$ and $\mu_{BR}(\phi, Y) < \infty$, then

$$\mu_{BR}(f, X) - \mu_{BR}(\phi, Y) = \tau_0(Y) - \tau_0(X).$$

Corollary 1.2.54. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an isolated hypersurface singularity. Let $f, g \in \mathcal{O}_n$ be finitely $\mathcal{R}_X$-determined function germs such that f is $C^0 - \mathcal{R}_X$-equivalent to g. Then $\mu_{BR}(f, X) = \mu_{BR}(g, X)$.

In [13, p. 71], the notion of the relative Bruce-Roberts number is also defined, although they do not use that denomination. We present the definition of that number with a notation similar to the one used in Definition 1.2.40:

Definition 1.2.55. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a function germ in $\mathcal{O}_n$. The *relative Bruce-Roberts number* of f with respect to $(X, 0)$ is defined by

$$\mu_{BR}^-(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X) + I_X}.$$

When $(X, 0)$ is an isolated hypersurface singularity, [48, p. 6] shows a relation between the Bruce-Roberts number and the relative Bruce-Roberts number of a function. In that case, when $f \in \mathcal{O}_n$ is $\mathcal{R}_X$-finitely determined, we have

$$\mu_{BR}(f, X) = \mu_0(f) + \mu_{BR}^-(f, X). \tag{1.22}$$

In addition, [47, Theorem 2.2] presents another relation between the Bruce-Roberts number and the relative Bruce-Roberts number of a function, but now on an ICIS:

Theorem 1.2.56. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be an ICIS and $f \in \mathcal{O}_n$ such that $\mu_{BR}^-(f, X) < \infty$. Then, $(X \cap f^{-1}(0), 0)$ defines an ICIS and

$$\mu_0(X \cap f^{-1}(0)) = \mu_{BR}^-(f, X) - \mu_0(X) + \tau_0(X).$$

We remark a result equivalent to Theorem 1.2.56 is shown in [48, Theorem 2.5].

One interesting corollary of this result, shown in [47, Corollary 2.3], is that when $(X, 0)$ is a weighted homogeneous ICIS, then the Milnor and the Tjurina numbers of X are the same. Beyond that, another consequence of Theorem 1.2.56, regarding the topological invariance of the relative Bruce-Roberts number, is given in [47, Corollary 2.4]:

Corollary 1.2.57. The relative Bruce-Roberts number is a topological invariant for a family of functions over a fixed ICIS.

The Milnor and Tjurina numbers are two indices with a similar definition. With that in mind, if the Bruce-Roberts number defined above generalizes the Milnor number of a function, it is possible to think of an index that generalizes the Tjurina number of a function. We refer to [10, Definition 3.1] to present the following definition, but other works regarding the Bruce-Roberts Tjurina number of holomorphic functions can be found in [2], [8], and [9].

Definition 1.2.58. Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. We define the *Bruce-Roberts Tjurina number* of f with respect to X as

$$\tau_{BR}(f, X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X) + \langle f \rangle}, \tag{1.23}$$

when the colength on the right side of (1.23) is finite.

A definition similar to that can be found on [2, p. 2], but with the name of Tjurina algebras of a function on an analytic variety. It follows from the definition that when $X = \mathbb{C}^n$, $\tau_{BR}(f, \mathbb{C}^n) = \tau_0(f)$, which means that the Bruce-Roberts Tjurina number generalizes the Tjurina number of a function. Moreover, comparing Definitions 1.2.40 and 1.2.58, it is immediate that

$$\tau_{BR}(f, X) \leq \mu_{BR}(f, X).$$

It is interesting to note that in [10], the authors presented a relation between the Bruce-Robert's Milnor and Tjurina numbers. We will construct that relation below, but first, we refer to [10, p. 9] to present the following definition:

Definition 1.2.59. Let R be a ring and let I be an ideal of R, with $f \in R$. The number $r_f(I)$ is the minimum of $r \in \mathbb{Z}_{\geq 1}$ such that $f^r \in I$. If such r does not exist, we set $r_f(I) = \infty$.

With that in mind, let us also denote by $\varphi_{f,I}$ the morphism $R/I \to R/I$ defined by $g + I \mapsto fg + I$, for all $g \in R$. If M is a R-module, denote by $\ell(M)$ the length of M and $\ell(R/I)$ the colength of I. Now, we can enunciate [10, Theorem 3.2]:

Theorem 1.2.60. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let I be an ideal of R of finite colength and let $f \in R$ such that $r_f(I) < \infty$. Then

$$\frac{\ell\left(\dfrac{R}{I}\right)}{\ell\left(\dfrac{R}{\langle f\rangle + I}\right)} \leq r_f(I),$$

and the equality holds if, and only if

$$\ker(\varphi_{f,I}) = \frac{\langle f^{r-1}\rangle + I}{I},$$

where $r = r_f(I)$.

A direct corollary of Theorem 1.2.60, given in [10, Corollary 3.3], provides the following comparison between the Bruce-Roberts and the Bruce-Roberts Tjurina number:

Corollary 1.2.61. Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$. Let $f \in \mathcal{O}_n$ such that $\mu_{BR}(f, X) < \infty$. Then

$$\frac{\mu_{BR}(f, X)}{\tau_{BR}(f, X)} \leq r_f(df(\Theta_X)),$$

and the equality holds if, and only if

$$\ker(\varphi_{f,df(\Theta_X)}) = \frac{\langle f^{r-1}\rangle + df(\Theta_X)}{df(\Theta_X)},$$

where $r = r_f(df(\Theta_X))$.

Example 1.2.62. We refer to [10, Example 3.5] to provide an example where Corollary 1.2.61 is valid. Consider $h \in \mathcal{O}_2$ the polynomial given by $h(x, y) = xy^6 + x^4y^4 + x^{10}$, and let $X = h^{-1}(0)$. We have that Θ_X is given by

$$\Theta_X = \left\langle -2x^4y^3\frac{\partial}{\partial x} + (5y^6 + 2x^3y^4 + 5x^9)\frac{\partial}{\partial y},\ 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}\right\rangle.$$

Now, consider the function $f(x, y) = x + y$. With that, we have $\mu_{BR}(f, X) = 6$, $\tau_{BR}(f, X) = 1$ and

$$r_f(df(\Theta_X)) = r_f(-2x^4y^3 + 5y^6 + 2x^3y^4 + 5x^9, 2x + 3y) = 6.$$

With that, Corollary 1.2.61 is satisfied.

To end this section, we present an equivalence relating the Bruce-Roberts numbers. We refer to [8, Corollary 2.2] to enunciate the following proposition:

Proposition 1.2.63. Let (f, X) be a pair with $\mu_{BR}(f, X) < \infty$. Then the following conditions are equivalent

1. $\mu_{BR}(f, X) = \tau_{BR}(f, X)$;

2. $\mu_0(f) = \tau_0(f)$ and $\Theta_Y = \Theta_X \cap \Theta_Y + H_Y$,

where $Y = f^{-1}(0)$ - making Θ_Y the submodule of Θ_n of vector fields that are tangent to $(Y, 0)$ - and $H_Y = \{\eta \in \Theta_n : df(\eta) = 0\}$.

Remark 1.2.64. To obtain the results of Proposition 1.2.63, [8, Section 2] there is a construction that uses the definition of two numbers $\overline{\mu}_X(f)$ and $\overline{\tau}_X(f)$, given by

$$\overline{\mu}_X(f) := \dim_{\mathbb{C}} \frac{\Theta_n}{\Theta_X + H_Y}$$

and

$$\overline{\tau}_X(f) = \dim_{\mathbb{C}} \frac{\Theta_n}{\Theta_X + \Theta_Y}$$

in the case where this numbers are finite. Although it is a bit of an extensive work to be presented here, we would like to note that a similar construction will be made in Section 3.2, and that will be made using the work done in [8, Section 2] as a base.

Chapter 2

The Bruce-Roberts Number for Holomorphic 1-Forms

In this chapter, our main goal is to present the notions of the *Bruce-Roberts number* for holomorphic 1-forms relative to complex analytic varieties, first presented in [6]. We will also establish a theorem that relates this number with other important indices. With that, we can define the *relative Bruce-Roberts number* and, in $(\mathbb{C}^2, 0)$, it is possible to define the Bruce-Roberts number for foliations defined by 1-forms in the form of (1.3). In that case, it is also possible to see how the Bruce-Roberts number behaves under blow-ups, and finish the study of this chapter working with generalized curve foliations.

Definition 2.0.1. Let ω be the germ of a holomorphic 1-form with an isolated singularity at $0 \in \mathbb{C}^n$, $n \geq 2$, and let X be a germ of a complex analytic variety with an isolated singularity at $0 \in \mathbb{C}^n$. We define the *Bruce-Roberts number* of the 1-form ω with respect to X as

$$\mu_{BR}(\omega, X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X)}. \tag{2.1}$$

Note that $\mu_{BR}(\omega, X)$ is finite if, and only if ω is a 1-form on X admitting (at most) an isolated singularity at $0 \in \mathbb{C}^n$. In the case of a germ of a complex analytic subvariety X, this condition is equivalent to saying that X is not *invariant* by ω, as stated in Definition 1.1.12.

Note also that, if $\omega = df$, for some $f \in \mathcal{O}_n$, then

$$\mu_{BR}(\omega, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)} = \mu_{BR}(f, X),$$

which means that Definition 2.0.1 generalizes the Bruce-Roberts number for functions defined in Definition 1.2.40. Also, it generalizes the Milnor number of ω when $X = \mathbb{C}^n$, since

$$\mu_{BR}(\omega, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega\left(\left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle\right)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle A_1, \dots, A_n \rangle} = \mu_0(\omega),$$

with ω being defined by $\omega = \sum_{i=1}^{n} A_i(x)dx_i$. Thus, when $\omega = df$, it is easy to see that

$$\mu_{BR}(\omega, X) = \mu_0(X),$$

showing that it is a generalization of the classical *Milnor number* of f as well.

2.1 Main Results

First, we state a technical lemma on commutative algebra inspired by [32, Lemma 6.1]:

Lemma 2.1.1. Let $f_1, \dots, f_m, g, p_1, \dots, p_n \in \mathcal{O}_n$, where f_i and g are relatively prime, for any $i \in \{1, \dots, m\}$. Then

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f_1, \dots, f_m, gp_1, \dots, gp_n \rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f_1, \dots, f_m, p_1, \dots, p_n \rangle} + \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f_1, \dots, f_m, g \rangle}.$$

Proof. First, we write $\mathcal{O} = \dfrac{\mathcal{O}_n}{\langle f_1, \dots, f_m \rangle}$ and, for every $k \in \mathcal{O}_n$, we consider $k' = k + \langle f_1, \dots, f_m \rangle$. With that, when $r_i' = r_i + \langle f_1, \dots, f_m \rangle$, for any $i \in \{1, \dots, n\}$ and any $r_i \in \mathcal{O}_n$, observe that

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f_1, \dots, f_m, r_1, \dots, r_n \rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}}{\langle r_1', \dots, r_n' \rangle}.$$

Now, consider the following sequence:

$$0 \longrightarrow \frac{\mathcal{O}}{\langle p_1', \dots, p_n' \rangle} \xrightarrow{\sigma} \frac{\mathcal{O}}{\langle g'p_1', \dots, g'p_n' \rangle} \xrightarrow{\delta} \frac{\mathcal{O}}{\langle g' \rangle} \longrightarrow 0,$$

where $\sigma(z' + \langle p_1', \dots, p_n' \rangle) = g'z' + \langle g'p_1', \dots, g'p_n' \rangle$ and $\delta(z' + \langle g'p_1', \dots, g'p_n' \rangle) = z' + \langle g' \rangle$ for any $z' \in \mathcal{O}$.

We shall prove that the sequence is exact:

- Consider $h' \in \ker \sigma$. With that, $\sigma(h' + \langle p_1', \dots, p_n' \rangle) \in \langle g'p_1', \dots, g'p_n' \rangle$, which means

  $$h'g' + \langle g'p_1', \dots, g'p_n' \rangle = \sigma(h' + \langle p_1', \dots, p_n' \rangle) \in \langle g'p_1', \dots, g'p_n' \rangle.$$

  Hence, $h'g' \in \langle g'p_1', \dots, g'p_n' \rangle$. That is equivalent to

  $$hg \in \langle gp_1, \dots, gp_n, f_1, \dots, f_m \rangle.$$

  From the hypothesis, that means that either $h \in \langle f_1, \dots, f_m \rangle$, or $h \in \langle p_1, \dots, p_n \rangle$. In each case, $h' \in \langle g'p_1', \dots, g'p_n' \rangle$, and then σ is injective.

- Consider $h' \in \dfrac{\mathcal{O}}{\langle g' \rangle}$. Then, we can write $h' = z' + \langle g' \rangle$, with $z' \in \mathcal{O}'$. With that, $z' + \langle g'p_1', \dots, g'p_n' \rangle \in \dfrac{\mathcal{O}}{\langle g'p_1', \dots, g'p_n' \rangle}$, and then

  $$\delta(z' + \langle g'p_1', \dots, g'p_n' \rangle) = z' + \langle g' \rangle = h',$$

  giving us that δ is surjective.

- Let $h' \in \operatorname{Im} \sigma$. Hence, there exists $y' + \langle p'_1, \dots, p'_n \rangle \in \dfrac{\mathcal{O}}{\langle p'_1, \dots, p'_n \rangle}$ such that $h' = \sigma(y' + \langle p'_1, \dots, p'_n \rangle) = g'y' + \langle g'p'_1, \dots, g'p'_n \rangle$. With that,

$$\delta(h') = g'y' + \langle g' \rangle \in \langle g' \rangle,$$

giving us $\operatorname{Im} \sigma \subseteq \ker \delta$.

- Now, let $h' \in \ker \delta$. Then, writing $h' = z' + \langle g'p'_1, \dots, g'p'_n \rangle$, we have

$$\delta(h') = z' + \langle g' \rangle \in \langle g' \rangle \;\Rightarrow\; z' \in \langle g' \rangle.$$

Then, there exists $y' \in \mathcal{O}$ such that $z' = y'g'$. Then, considering $y' + \langle p'_1, \dots, p'_n \rangle \in \dfrac{\mathcal{O}}{\langle p'_1, \dots, p'_n \rangle}$, we have

$$\sigma(y' + \langle p'_1, \dots, p'_n \rangle) = g'y' + \langle g'p'_1, \dots, g'p'_n \rangle = z' + \langle g'p'_1, \dots, g'p'_n \rangle = h'.$$

Thus, $\ker \delta \subseteq \operatorname{Im} \sigma$, which implies $\ker \delta = \operatorname{Im} \sigma$.

The exactness of the sequence concludes the proof:

$$\begin{aligned}
\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f_1, \dots, f_m, gp_1, \dots, gp_n \rangle} &= \dim_{\mathbb{C}} \frac{\mathcal{O}}{\langle g'p'_1, \dots, g'p'_n \rangle} \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}}{\langle p'_1, \dots, p'_n \rangle} + \dim_{\mathbb{C}} \frac{\mathcal{O}}{\langle g' \rangle} \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f_1, \dots, f_m, p_1, \dots, p_n \rangle} + \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f_1, \dots, f_m, g \rangle}.
\end{aligned}$$

□

Before we enunciate our main result, let's prove the following lemma:

Lemma 2.1.2. The sequence

$$0 \longrightarrow \frac{\Theta_X}{\Theta_X^T} \xrightarrow{\cdot\omega} \frac{\mathcal{O}_n}{\omega(\Theta_X^T)} \xrightarrow{\beta} \frac{\mathcal{O}_n}{\omega(\Theta_X)} \longrightarrow 0,$$

of $\mathbb{C}$-vector spaces is exact, where β is induced by the inclusion $\omega(\Theta_X^T) \subset \omega(\Theta_X)$, and $\cdot\omega$ is the evaluation map.

Proof. First, it is not difficult to see that the sequence is well-defined. Let us now prove that it is an exact sequence. Suppose $f \in \operatorname{Im}(\omega)$. Then, there exists $\eta \in \Theta_X$ such that $f = \omega(\eta + \Theta_X^T) = \omega(\eta) + \omega(\Theta_X^T)$. Hence,

$$\beta(f) = \beta(\omega(\eta) + \omega(\Theta_X^T)) = \beta(\omega(\eta)) + \omega(\Theta_X).$$

Since $\eta \in \Theta_X$, we have $\omega(\eta) \in \omega(\Theta_X)$, and thus, $\beta(\omega(\eta)) = \omega(\eta) \in \omega(\Theta_X)$. Consequently, $\beta(f) \in \omega(\Theta_X)$, and then $f \in \ker(\beta)$.

On the other hand, consider $g \in \ker(\beta)$. Then, $g = f + \omega(\Theta_X^T)$, where $f \in \mathcal{O}_n$ and $\beta(g) \in \omega(\Theta_X)$. This implies

$$f + \omega(\Theta_X) = \beta(f + \omega(\Theta_X^T)) = \beta(g) \in \omega(\Theta_X).$$

Therefore, $f \in \omega(\Theta_X)$, i.e., $f = \omega(\xi)$, where $\xi \in \Theta_X$. Thus, we have $g = f + \omega(\Theta_X) = \omega(\xi + \Theta_X^T)$, which gives us $g \in \mathrm{Im}(\omega)$.

Consequently, $\ker(\beta) = \mathrm{Im}(\omega)$, establishing the exactness of the sequence and concluding the proof. □

Theorem 2.1.3. Let ω be the germ of a holomorphic 1-form with an isolated singularity at $0 \in \mathbb{C}^n$, $n \geq 2$. Let X be the germ of a complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$. Assume that X is not invariant by ω. Then

$$\mu_{BR}(\omega, X) = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) + \mu_0(\omega) - \tau_0(X).$$

Proof. From Lemma 2.1.2, we obtain

$$\mu_{BR}(\omega, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T)} - \dim_{\mathbb{C}} \frac{\Theta_X}{\Theta_X^T}. \tag{2.2}$$

Note that, by the definition of Θ_X^T given in (1.2) and Lemma 2.1.1, we have

$$\begin{aligned}\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T)} &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi A_1, \ldots, \phi A_n, \frac{\partial \phi}{\partial x_j} A_k - \frac{\partial \phi}{\partial x_k} A_j \rangle_{(j,k)\in\Lambda}} \\ &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi, \frac{\partial \phi}{\partial x_j} A_k - \frac{\partial \phi}{\partial x_k} A_j \rangle_{(j,k)\in\Lambda}} \\ &\quad + \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle A_1, \ldots, A_n, \frac{\partial \phi}{\partial x_j} A_k - \frac{\partial \phi}{\partial x_k} A_j \rangle_{(j,k)\in\Lambda}},\end{aligned}$$

since the components $\dfrac{\partial \phi}{\partial x_j} A_k - \dfrac{\partial \phi}{\partial x_k} A_j$ and ϕ are relatively prime, by the non-invariance hypothesis. Note also that

$$\frac{\partial \phi}{\partial x_j} A_k - \frac{\partial \phi}{\partial x_k} A_j \in \langle A_1, \ldots, A_n \rangle,$$

for all $(j,k) \in \Lambda$. In that way, we get

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle A_1, \ldots, A_n, \frac{\partial \phi}{\partial x_j} A_k - \frac{\partial \phi}{\partial x_k} A_j \rangle_{(j,k)\in\Lambda}} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle A_1, \ldots, A_n \rangle} = \mu_0(\omega).$$

Thus, we deduce

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi, \frac{\partial \phi}{\partial x_j} A_k - \frac{\partial \phi}{\partial x_k} A_j \rangle_{(j,k)\in\Lambda}} + \mu_0(\omega). \tag{2.3}$$

Finally, (1.12) gives us that $\dim_{\mathbb{C}} \dfrac{\Theta_X}{\Theta_X^T} = \tau_0(X)$. The proof is concluded by substituting (2.3) in (2.2), and considering the definition of $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0)$ given by (1.18). □

With that, since both $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0)$ and $\mu_0(\omega)$ are topological invariants when $(X, 0)$ is an isolated hypersurface singularity, Theorem 2.1.3 implies that $\mu_{BR}(\omega, X)$ is also a topological invariant under homeomorphisms of $(\mathbb{C}^n, 0)$ that fix $(X, 0)$.

As a consequence, we can use our tools to recover [59, Corollary 4.1], which we enunciated as Corollary 1.2.52:

Corollary 2.1.4. Let X be an isolated hypersurface singularity defined by $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and let $f \in \mathcal{O}_n$ with an isolated singularity over $(X, 0)$. Then,

$$\mu_{BR}(f, X) = \mu_0(f) + \mu_0(\phi, f) + \mu_0(X) - \tau_0(X).$$

Proof. Consider the 1-form ω given by $\omega = df$. As we observed in previous results, we have

$$\mu_{BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X)} = \mu_{BR}(df, X) \text{ and}$$
$$\mu_0(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle} = \mu_0(df).$$

By Theorem 2.1.3, we have

$$\mu_{BR}(f, X) = \mu_{BR}(df, X) = \mathrm{Ind}_{\mathrm{GSV}}(df; X, 0) + \mu_0(df) - \tau_0(X). \tag{2.4}$$

Since $\mu_0(f) = \mu_0(df)$, the proof concludes by rewriting $\mathrm{Ind}_{\mathrm{GSV}}$ in equation (2.4) using the Lê-Greuel formula (1.19), given by

$$\mathrm{Ind}_{\mathrm{GSV}}(df; X, 0) = \mu_0(X) + \mu_0(\phi, f).$$

□

As a second corollary, we establish a connection between the Bruce-Roberts number $\mu_{BR}(\omega, X)$ and other indices of 1-forms along X presented in our first chapter.

Corollary 2.1.5. Let ω be a germ of a holomorphic 1-form with isolated singularity at $0 \in \mathbb{C}^n$, $n \geq 2$. Let X be a germ of a complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$. Assume that X is not invariant by ω. Then

$$\mu_{BR}(\omega, X) - \mu_0(\omega) + \tau_0(X) - \mu_0(X) - \mathrm{Eu}(\omega; X, 0) = (-1)^{n-2}\bar{\chi}(M_\ell),$$

where M_ℓ is the Milnor fiber of the generic linear function $\ell : \mathbb{C}^n \to \mathbb{C}$ on X and $\bar{\chi}(M_\ell) = \chi(M_\ell) - 1$.

Proof. We have $\mu_{BR}(\omega, X) = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) + \mu_0(\omega) - \tau_0(X)$ by Theorem 2.1.3. On the other hand, Proposition 1.2.31 gives us that $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) = \mathrm{Ind}_{\mathrm{rad}}(\omega; X, 0) + \mu_0(X)$. Therefore,

$$\mu_{BR}(\omega, X) = \mathrm{Ind}_{\mathrm{rad}}(\omega; X, 0) + \mu_0(X) + \mu_0(\omega) - \tau_0(X). \tag{2.5}$$

The proof concludes by applying Proposition 1.2.33 to $(X, 0)$ and substituting into (2.5).

□

We finish this section with two examples where Theorem 2.1.3 is verified.

Example 2.1.6. Let $X = \{\phi = 0\}$, where $\phi : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is given by $\phi(x, y, z) = x^3 + yz^2 + y^3 + xy^4$. Using [57, Example 9], we have

$$\begin{aligned}\Theta_X = \Big\langle &(3xy^2 + 3y^2 + z^2)\frac{\partial}{\partial x} + (-3x^2 - y^3)\frac{\partial}{\partial y},\\ &(2z^3)\frac{\partial}{\partial x} + (-6x^2 z)\frac{\partial}{\partial y} + (9x^3 y + 9x^2 y - y^2 z^2)\frac{\partial}{\partial z},\\ &(2yz)\frac{\partial}{\partial x} + (-3x^2 - y^3)\frac{\partial}{\partial z},\\ &(2yz)\frac{\partial}{\partial y} + (-3xy^2 - 3y^2 - z^2)\frac{\partial}{\partial z},\\ &\left(-\frac{4}{3}x^2 y - x\right)\frac{\partial}{\partial x} + \left(-\frac{2}{3}xy^2 - y\right)\frac{\partial}{\partial y} + \left(-\frac{5}{3}xyz - z\right)\frac{\partial}{\partial z}\Big\rangle.\end{aligned}$$

Let $\omega = zdx + xdy + ydz$ a germ of a holomorphic 1-form with isolated singularity at $0 \in \mathbb{C}^3$. Since

$$\begin{aligned}\omega \wedge d\phi =&(z^3 + 3y^2 z + 4xy^3 z - 3x^3 - xy^4)dx \wedge dy +\\ &(2yz^2 - 3x^2 y - y^5)dx \wedge dz + (2xyz - yz^2 - 3y^3 - 4xy^4)dy \wedge dz,\end{aligned}$$

then X is not invariant by ω. Using *Singular* ([24]) to compute the indices, we verify that Theorem 2.1.3 holds, since $\mu_{BR}(\omega, X) = 14$, $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) = 21$, $\mu_0(\omega) = 1$ and $\tau_0(X) = 8$.

Example 2.1.7. Let $X = \{\phi(x, y, z) = xy + z^4 = 0\}$ be a germ of an isolated complex hypersurface on $(\mathbb{C}^3, 0)$, and let f be the polynomial $f = x^2 + y^2 + z^2$. With that, consider

$$\omega = (2x + zf)dx + (2y + xf)dy + (2z + yf)dz.$$

Since

$$\begin{aligned}\omega \wedge d\phi &= [((2x + zf)dx + (2y + xf)dy + (2z + yf)dz)] \wedge (ydx + xdy + 4z^3 dz)\\ &= (2x^2 - 2y^2 + (xz - xy)f)dx \wedge dy + (8yz^3 - 2xz + (4xz^3 - xy)f)dy \wedge dz\\ &+ (8xz^3 - 2yz + (4z^4 - y^2)f)dx \wedge dz,\end{aligned}$$

we have that X is not invariant by ω. In Example 1.2.48, we showed that

$$\Theta_X = \Big\langle y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} - 4z^3\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - 4z^3\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\Big\rangle.$$

With the help of *Singular* ([24]), we computed $\mu_{BR}(\omega, X) = 6$, $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) = 8$, $\mu_0(\omega) = 1$ and $\tau_0(X) = 3$, and with that

$$\begin{aligned}6 = \mu_{BR}(\omega, X) &= \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) + \mu_0(\omega) - \tau_0(X)\\ &= 8 + 1 - 3.\end{aligned}$$

Thus, Theorem 2.1.3 is verified.

2.2 Relative Bruce-Roberts number for 1-forms

The definition of the relative Bruce-Roberts number of a function, given in Definition 1.2.55, comes straightforward from the definition of the Bruce-Roberts number of a function. Likewise, in this section we define the relative Bruce-Roberts number of a foliation, getting some similar results to the ones verified for $\mu_{BR}^{-}(f, X)$. We also provide some examples of these results.

Definition 2.2.1. Let ω be a germ of a holomorphic 1-form with isolated singularity at $(\mathbb{C}^n, 0)$ and let X be a complex analytic germ with isolated singularity at $(\mathbb{C}^n, 0)$. The *relative Bruce-Roberts number* of the 1-form ω with respect to X is defined as

$$\mu_{BR}^{-}(\omega, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X) + I_X}.$$

It is evident that, when $\omega = df$, $\mu_{BR}^{-}(\omega, X) = \mu_{BR}^{-}(f, X)$. In other words, the relative Bruce-Roberts number of ω with respect to X generalizes the relative Bruce-Roberts number of f with respect to X. Now, our main goal is to establish a relation between $\mu_{BR}(\omega, X)$ and $\mu_{BR}^{-}(\omega, X)$. For that, we present the following Lemma, which is analogous to [48, Lemma 2.2].

Lemma 2.2.2. Let $\omega = \sum_{i=1}^{n} A_i dx_i \in \Omega^1(\mathbb{C}^n, 0)$ be a germ of 1-form and let $g \in \mathcal{O}_n$ be such that $\dim V((\omega, dg)) = 1$ and $V(\omega) = \{0\}$, where $(\omega) = \langle A_1, \ldots, A_n \rangle$ and (ω, dg) is the ideal in $\mathcal{O}_n$ generated by the maximal minors of the matrix of $\left(A_i, \frac{\partial g}{\partial x_i}\right)$. Consider the matrices

$$A = \begin{pmatrix} A_1 & \cdots & A_n \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix}, \quad A' = \begin{pmatrix} \mu & A_1 & \cdots & A_n \\ \lambda & \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix},$$

where $\lambda, \mu \in \mathcal{O}_n$. Let M and M' be the submodules of $\mathcal{O}_n^2$ generated by the columns of A and A', respectively. If $I_2(A) = I_2(A')$ then $M = M'$, where $I_2(A)$ the ideal in R generated by the 2×2 minors of the matrix A.

Proof. The proof follows the same approach as the proof of [48, Lemma 2.2], so we shall only give an idea of the demonstration. First, considering $R = \mathcal{O}_n$, we see A and A' as homomorphisms of modules over R:

$$A : R^n \to R^2, \ A' : R^{n+1} \to R^2.$$

Then, it is possible to show that the R-module R^2/M is Cohen-Macaulay, and if its submodule M'/M is not equal to 0, then $\dim(M'/M) = 1$. Now, let U be a neighborhood of $0 \in \mathbb{C}^n$ such that 0 is the only point where $A_1(0) = \ldots = A_n(0) = 0$. Then, for all $x \in U \setminus \{0\}$, there exist $i_0 \in \{1, \ldots, n\}$ such that $A_{i_0}(x) \neq 0$. We can suppose that $i_0 = 1$, and then obtain

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ c_1 & c_2 & \ldots & c_n \end{pmatrix}, \quad B' = \begin{pmatrix} \mu & 1 & 0 & \cdots & 0 \\ \lambda & c_1 & c_2 & \ldots & c_n \end{pmatrix}$$

making elementary operations on the columns of A and A', respectively, such that $I_2(A) = I_2(B)$, $I_2(A') = I_2(B')$, $\mathrm{Im}(A) = \mathrm{Im}(B)$ and $\mathrm{Im}(A') = \mathrm{Im}(B')$. Then, by the hypothesis, $I_2(B) = I_2(B')$, and this implies $\langle c_2, \ldots, c_n\rangle = \langle \mu c_1 - \lambda, c_2, \ldots, c_n\rangle$. With that, we can write

$$\begin{pmatrix} \mu \\ \lambda \end{pmatrix} = \mu \begin{pmatrix} 1 \\ c_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ c_2 \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} 0 \\ c_n \end{pmatrix},$$

for some $\alpha_i \in R$, $i = 2, \ldots, n$. This gives us that $\mathrm{Supp}(M'/M) \subset \{0\}$, and then $M' = M$. □

We can now prove a theorem similar to [48, Theorem 2.3].

Theorem 2.2.3. Let $\omega = \sum_{i=1}^{n} A_i(x)dx_i$ be a germ of a holomorphic 1-form with isolated singularity at $0 \in \mathbb{C}^n$, $n \geq 2$, where $A_i \in \mathcal{O}_n$. Let $X = \{\phi = 0\}$ be a germ of a complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$. Assume that X is not invariant by ω. Then

(i) $\dfrac{\Theta_X}{\Theta_X^T} \cong \dfrac{\omega(\Theta_X) + I_X}{\omega(\Theta_X^T) + I_X}$;

(ii) $\omega(\Theta_X) \cap I_X = (\omega) \cap I_X$;

(iii) $\dfrac{\mathcal{O}_n}{(\omega)} \cong \dfrac{\omega(\Theta_X) + I_X}{\omega(\Theta_X)}$,

where $(\omega) = \langle A_1, \ldots, A_n\rangle$ and $I_X = \langle \phi\rangle$.

Proof.

(i) Let's define the map $\Psi : \Theta_X \to \omega(\Theta_X) + I_X$ by $\Psi(\xi) = \omega(\xi)$. It's easy to see that Ψ is an homomorphism. Note that Ψ induces the isomorphism

$$\overline{\Psi} : \frac{\Theta_X}{\Theta_X^T} \to \frac{\omega(\Theta_X) + I_X}{\omega(\Theta_X^T) + I_X}.$$

To see that, it is enough to show that $\ker(\overline{\Psi}) = \Theta_X^T$. Let $\xi + \Theta_X^T \in \ker(\overline{\Psi})$. Then, $\omega(\xi) \in \omega(\Theta_X^T) + I_X$, that is, there exist $\eta \in \Theta_X^T$ and $\mu, \lambda \in \mathcal{O}_n$, such that

$$\omega(\xi - \eta) = \mu\phi \quad \text{and} \quad d\phi(\xi - \eta) = \lambda\phi,$$

since $\phi \in I_X$ and from the definition of Θ_X showed in (1.1). Then, we have

$$\begin{pmatrix} \mu\phi \\ \lambda\phi \end{pmatrix} \in \left\langle \begin{pmatrix} A_i \\ \frac{\partial\phi}{\partial x_i} \end{pmatrix} : \ i = 1, \ldots, n \right\rangle$$

and

$$I_2 \begin{pmatrix} \mu\phi & A_1 & \cdots & A_n \\ \lambda\phi & \frac{\partial\phi}{\partial x_1} & \cdots & \frac{\partial\phi}{\partial x_n} \end{pmatrix} = I_2 \begin{pmatrix} A_1 & \cdots & A_n \\ \frac{\partial\phi}{\partial x_1} & \cdots & \frac{\partial\phi}{\partial x_n} \end{pmatrix} = (\omega, d\phi),$$

where $I_2(B)$ is denoted as the ideal in the ring $\mathcal{O}_n$ generated by the 2×2 minors of the matrix B. Therefore,

$$\begin{vmatrix} \mu & A_i \\ \lambda & \dfrac{\partial \phi}{\partial x_i} \end{vmatrix} \phi \in (\omega, d\phi).$$

Now, from the non-invariance hypothesis, we have that ϕ is regular in $\dfrac{\mathcal{O}_n}{(\omega, d\phi)}$, which means ϕ is not a zero divisor in $\dfrac{\mathcal{O}_n}{(\omega, d\phi)}$. Thus,

$$\begin{vmatrix} \mu & A_i \\ \lambda & \dfrac{\partial \phi}{\partial x_i} \end{vmatrix} \in (\omega, d\phi), \ i = 1, \ldots, n.$$

By Lemma 2.2.2, $\lambda \in J(\phi)$, since $\lambda \in M$ and $M' = J(\phi)$. We finish the proof using [59, Lemma 3.1], that states the following:

Lemma 2.2.4. Let $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic germ with isolated singularity and let $X = \phi^{-1}(0)$. Assume that $\eta \in \Theta_X$ and $d\phi(\eta) = \lambda\phi$, for some $\phi \in \mathcal{O}_n$. Then $\eta \in \Theta_X^T$ if, and only if $\lambda \in J(\phi)$.

Then, $\xi \in \Theta_X^T$, and then $\xi + \Theta_X^T \in \Theta_X^T$. Hence, $\ker(\overline{\Psi}) = \Theta_X^T$.

(ii) Let $\xi \in \omega(\Theta_X) \cap I_X$. Then, there exist $\eta \in \Theta_X$ and $\mu, \lambda \in \mathcal{O}_n$ such that

$$\xi = \omega(\eta) = \mu\phi \quad \text{and} \quad d\phi(\eta) = \lambda\phi,$$

again by the definition of Θ_X. Using a similar argument as in the proof of (i), we have that $\mu \in (\omega)$. Hence, $\xi \in (\omega) \cap I_X$. Conversely, let $\xi \in (\omega) \cap I_X$. Then, there exist $\alpha_1, \ldots, \alpha_n, \mu \in \mathcal{O}_n$ such that

$$\xi = \sum_{i=1}^{n} \mu\phi\alpha_i A_i,$$

since $(\omega) = \langle A_1, \ldots, A_n \rangle$ and $I_X = \langle \phi \rangle$. Taking

$$\eta = \sum_{i=1}^{n} \mu\phi\alpha_i \frac{\partial}{\partial x_i} \in \Theta_X,$$

we obtain $\omega(\eta) = \xi$, which means that $\xi \in \omega(\Theta_X)$, and then, $\xi \in \omega(\Theta_X) \cap I_X$.

(iii) Consider the following module isomorphism, from [5, Proposition 2.1 (ii)]:

$$\frac{M_1 + M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}.$$

Then, if $M_1 = \omega(\Theta_X)$ and $M_2 = I_X$, and using item (ii), we have

$$\frac{\omega(\Theta_X) + I_X}{\omega(\Theta_X)} \cong \frac{I_X}{\omega(\Theta_X) \cap I_X} = \frac{I_X}{(\omega) \cap I_X} \cong \frac{\mathcal{O}_n}{(\omega)},$$

and the last step follows from the isomorphism

$$\begin{array}{cccc} \varphi : & \dfrac{\langle \phi \rangle}{(\omega) \cap \langle \phi \rangle} & \longrightarrow & \dfrac{\mathcal{O}_n}{(\omega)} \\ & s\phi + (\omega) \cap \langle \phi \rangle & \longmapsto & s + (\omega) \end{array}.$$

It is not difficult to see that φ is well-defined. Moreover, consider $g \in \ker \varphi$. Then, $g = g_0\phi + (\omega) \cap \langle \phi \rangle \in \dfrac{\langle \phi \rangle}{(\omega) \cap \langle \phi \rangle}$, and $\varphi(g) = 0$ in $\dfrac{\mathcal{O}_n}{(\omega)}$, i. e., $g_0 \in (\omega)$. With that, $g_0\phi \in (\omega) \cap \langle \phi \rangle$, and then $g \equiv 0$ in $\dfrac{\langle \phi \rangle}{(\omega) \cap \langle \phi \rangle}$, which means that φ in injective. On the other hand, consider $f \in \dfrac{\mathcal{O}_n}{(\omega)}$. Taking $f\phi + (\omega) \cap \langle \phi \rangle \in \dfrac{\langle \phi \rangle}{(\omega) \cap \langle \phi \rangle}$, we have

$$\varphi(f\phi + (\omega) \cap \langle \phi \rangle) = f + (\omega),$$

giving us that φ is surjective.

□

Remark 2.2.5. Let $X = \{\phi = 0\}$ be a germ of a complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$. By (1.12), we have

$$\dim_{\mathbb{C}} \frac{\Theta_X}{\Theta_X^T} = \tau_0(X).$$

Therefore, by Theorem 2.2.3 (i), we have

$$\dim_{\mathbb{C}} \frac{\omega(\Theta_X) + I_X}{\omega(\Theta_X^T) + I_X} = \dim_{\mathbb{C}} \frac{\Theta_X}{\Theta_X^T} = \tau_0(X).$$

With two similar definitions, a natural question arises: is it possible to show a relation connecting the indices $\mu_{BR}(\omega, X)$ and $\mu_{BR}^-(\omega, X)$? With that in mind, we enunciate the following theorem:

Theorem 2.2.6. Let ω be a germ of a holomorphic 1-form with an isolated singularity at $0 \in \mathbb{C}^n$, $n \geq 2$, and let X be a germ of a complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$. Assume that X is not invariant by ω. Then

$$\mu_{BR}(\omega, X) = \mu_0(\omega) + \mu_{BR}^-(\omega, X).$$

Proof. From Theorem 2.2.3 (iii), we have

$$\mu_0(\omega) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\omega)} = \dim_{\mathbb{C}} \frac{\omega(\Theta_X) + I_X}{\omega(\Theta_X)}.$$

Hence, by the following sequence

$$0 \longrightarrow \frac{\omega(\Theta_X) + I_X}{\omega(\Theta_X)} \longrightarrow \frac{\mathcal{O}_n}{\omega(\Theta_X)} \longrightarrow \frac{\mathcal{O}_n}{\omega(\Theta_X) + I_X} \longrightarrow 0,$$

which is easy to see that it is an exact sequence (this will also be shown in Lemma 3.0.2), we conclude the proof of the theorem, since

$$\begin{aligned}\mu_{BR}(\omega, X) &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X)} \\ &= \dim_{\mathbb{C}} \frac{\omega(\Theta_X) + I_X}{\omega(\Theta_X)} + \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X) + I_X} \\ &= \mu_0(\omega) + \mu_{BR}^{-}(\omega, X).\end{aligned}$$

□

Example 2.2.7. In Example 2.1.6, we have $\omega = zdx + xdy + ydz$, and $X = \{\phi = x^3 + yz^2 + y^3 + xy^4 = 0\}$, with $\mu_{BR}(\omega, X) = 14$ and $\mu_0(\omega) = 1$. Computing the relative Bruce-Roberts number using *Singular* [24], we have

$$\mu_{BR}^{-}(\omega, X) = \frac{\mathcal{O}_3}{\omega(\Theta_X) + (x^3 + yz^2 + y^3 + xy^4)} = 13.$$

Therefore, note that $\mu_{BR}^{-}(\omega, X) = \mu_{BR}(\omega, X) - \mu_0(\omega)$, that indeed satisfies Theorem 2.2.6.

Remark 2.2.8. If we take $\omega = df$ in the Theorem 2.2.6, we recover the formula

$$\mu_{BR}(f, X) = \mu_0(f) + \mu_{BR}^{-}(f, X)$$

of [48, Section 3], that we showed in (1.22).

With that remark, we have a corollary that gives a relation between the Milnor numbers of two functions and the Bruce-Roberts numbers of the product of these functions in dimension two.

Corollary 2.2.9. Let X be an isolated hypersurface singularity, and let $f, g \in \mathcal{O}_2$ such that fg has an isolated singularity over $(X, 0)$. Then

$$\mu_{BR}(fg, X) = \mu_0(f) + \mu_0(g) + 2i_0(f, g) + \mu_{BR}^{-}(fg, X) - 1,$$

where $i_0(f, g)$ is the intersection multiplicity of the pair (f, g).

Proof. Consider Y defined by $Y = (fg)^{-1}(0)$. Taking $\omega = d(fg)$ in Theorem 2.2.6, we have

$$\mu_{BR}(fg, X) = \mu_0(fg) + \mu_{BR}^{-}(fg, X).$$

The proof is concluded by rewriting the Milnor number $\mu_0(fg)$ as

$$\mu_0(fg) = \mu_0(f) + \mu_0(g) + 2i_0(f, g) - 1,$$

as seen in (1.9). □

Corollary 2.2.10. Let ω be a germ of a holomorphic 1-form with an isolated singularity at $0 \in \mathbb{C}^n$, $n \geq 2$, and let X be a germ of a complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$. Assume that X is not invariant by ω. Then

$$\mu_{BR}^-(\omega, X) = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) - \tau_0(X).$$

Proof. The proof follows from Theorem 2.1.3 and Theorem 2.2.6. □

In our last result evolving the relative Bruce-Roberts number of a foliation, we recover [48, Theorem 2.5]. We remark that an equivalent result was enounced in Theorem 1.2.56.

Corollary 2.2.11. Let X be a germ of a complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$ defined by $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $f \in \mathcal{O}_n$ be a function germ such that $\mu_{BR}(f, X) < \infty$. Then (ϕ, f) defines an ICIS and

$$\mu_0(\phi, f) = \mu_{BR}^-(f, X) + \tau_0(X) - \mu_0(X).$$

Proof. Consider the 1-form ω given by $\omega = df$. By Corollary 2.2.10, we have

$$\mu_{BR}^-(df, X) = \mathrm{Ind}_{\mathrm{GSV}}(df; X, 0) - \tau_0(X)$$

The proof follows by applying the Lê-Greuel formula (see equality (1.19)), and then

$$\mu_0(\phi, f) = \mathrm{Ind}_{\mathrm{GSV}}(df; X, 0) - \mu_0(X) = \mu_{BR}^-(f, X) + \tau_0(X) - \mu_0(X).$$

□

2.3 The Bruce-Roberts Number for Foliations on $(\mathbb{C}^2, 0)$

In this section, our main goal is to show how Theorem 2.1.3 and some results of Section 2.2 work in dimension two. Since a holomorphic 1-form in $(\mathbb{C}^2, 0)$ can be seen as a foliation, we may be able to see, for example, what happens with the Bruce-Roberts number of a foliation after a blow-up. Intuitively, the Bruce-Roberts number of a singular foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$ over a germ of a reduced curve X is defined by

$$\mu_{BR}(\mathcal{F}, X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\omega(\Theta_X)},$$

where ω is the germ of the 1-form defining $\mathcal{F}$. It is worth noting that this definition does not depend on the choice of the 1-form ω.

The first result of this section is a version of Theorem 2.1.3 when $n = 2$, and it is established as follows:

Corollary 2.3.1. Let $\mathcal{F}$ be a germ of a singular foliation at $(\mathbb{C}^2, 0)$, and let X be a germ of a reduced curve at $(\mathbb{C}^2, 0)$. Assume that X is not invariant by $\mathcal{F}$. Then

$$\mu_{BR}(\mathcal{F}, X) = \mu_0(\mathcal{F}) + \mathrm{tang}(\mathcal{F}, X, 0) - \tau_0(X). \tag{2.6}$$

Proof. Consider the foliation $\mathcal{F}$ defined by the 1-form $\omega = A\,dx + B\,dy$, where $A, B \in \mathcal{O}_2$. Let X be the complex analytic curve defined by the function $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. From equation (1.17) it follows that

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\left\langle \phi, B\frac{\partial \phi}{\partial x} - A\frac{\partial \phi}{\partial y} \right\rangle}.$$

The vector field v generating $\mathcal{F}$ is given by $v = -B\frac{\partial}{\partial x} + A\frac{\partial}{\partial y}$. Therefore, by definition,

$$\mathrm{tang}(\mathcal{F}, X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \phi, v(\phi) \rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\left\langle \phi, -B\frac{\partial \phi}{\partial x} + A\frac{\partial \phi}{\partial y} \right\rangle} = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0).$$

Hence, by Theorem 2.1.3,

$$\mu_{BR}(\mathcal{F}, X) = \mathrm{tang}(\mathcal{F}, X, 0) + \mu_0(\mathcal{F}) - \tau_0(X).$$

□

To proceed, we present some illustrative examples:

Example 2.3.2. Consider the curve X given by $X = \{\phi = y^p - x^q = 0\}$, and let $\mathcal{F}$ be a foliation defined by the 1-form $\omega = \lambda x dy + y dx$, with $\lambda \neq -\frac{p}{q}$. Then, by [35, Example 1], we have:

$$\Theta_X = \left\langle qy\frac{\partial}{\partial y} + px\frac{\partial}{\partial x}, py^{p-1}\frac{\partial}{\partial x} + qx^{q-1}\frac{\partial}{\partial y} \right\rangle.$$

Note that X is not invariant by $\mathcal{F}$, since

$$\omega \wedge d\phi = (\lambda x dy + y dx) \wedge (py^{p-1}dy - qx^{q-1}dx) = (py^p + \lambda q x^q)\ dx \wedge dy$$

and $\lambda \neq -\frac{p}{q}$. Thus, we have

- $\mu_0(\mathcal{F}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle A, B \rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle y, \lambda x \rangle} = 1;$
- $\mathrm{tang}(\mathcal{F}, X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \phi, \phi_x B - \phi_y A \rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle y^p - x^q, -q\lambda x^q - py^p \rangle} = pq;$
- $\tau_0(X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \phi, \phi_x, \phi_y \rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle y^p - x^q, -qx^{q-1}, py^{p-1} \rangle} = (p-1)(q-1).$

Computing the Bruce-Roberts number in this case, we find

$$\mu_{BR}(\mathcal{F}, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\omega(\Theta_X)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \lambda q x y + pxy, py^p + q\lambda x^q \rangle} = p + q,$$

which indeed satisfies equation (2.6).

Example 2.3.3. Let X be a curve given by $X = \{\phi = y^5 - x^6 + x^4y^3\}$ (see [35, Example 2]) and $\mathcal{F}$ be a foliation defined by the 1-form $\omega = xdy + ydx$. Since

$$\begin{aligned} \omega \wedge d\phi &= (xdy + ydx) \wedge ((-6x^5 + 4x^3y^3)dx + (5y^4 + 3x^4y^2)dy) \\ &= (5y^5 + 6x^6 - x^4y^3)\ dx \wedge dy, \end{aligned}$$

it follows that X is not $\mathcal{F}$-invariant. Again, note that (2.6) applies, since computations (using *Singular* [24]) show us that $\mu_0(\mathcal{F}) = 1$, $\text{tang}(\mathcal{F}, X, 0) = 30$, $\tau_0(X) = 19$ and $\mu_{BR}(\mathcal{F}, X) = 12$.

Example 2.3.4 (Topologically Conjugate Foliations)**.** In [19, Remarque 4.22], it is defined that two foliations are *topologically conjugate* if there exists a homeomorphism sending leaves from one foliation to another (which also means the topological invariance of these two foliations, as stated in Subsection 1.2.1). In this example, we are going to compute the Bruce-Roberts number of two topologically conjugate foliations.

First, let's consider two holomorphic vector fields in $(\mathbb{C}^2, 0)$ given by

$$V_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},\ V_2 = (x - 4y^3)\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{2.7}$$

Now, consider $\mathcal{F}_1$ and $\mathcal{F}_2$ the two foliations defined by V_1 and V_2, respectively (as shown in (1.4)). We recur to [19, Section 2.7.4] for a way to describe the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$, according to its vector fields. Thus, the leaves of $\mathcal{F}_1$ passing through (x_0, y_0) can be described by the parametrization

$$\varphi(t, (x_0, y_0)) = (x_0e^t, y_0e^{-t}),$$

and the leaves of $\mathcal{F}_2$ passing through (x_0, y_0) can be described by

$$\psi(t, (x_0, y_0)) = (y_0^3e^{-3t} + (x_0 - y_0^3)e^t, y_0e^{-t}).$$

Now, consider the homeomorphism

$$\begin{array}{cccc} h: & \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ & (x,y) & \longmapsto & (x + y^3, y) \end{array}.$$

With that, we have

$$\begin{aligned} h(\varphi(t, (x_0, y_0))) &= h(x_0e^t, y_0e^{-t}) \\ &= (x_0e^t + y_0^3e^{-3t}, y_0e^{-t}) \end{aligned}$$

and

$$\begin{aligned} \psi(t, h(x_0, y_0)) &= \psi(t, (x_0 + y_0^3, y_0)) \\ &= (y_0^3e^{-3t} + ((x_0 + y_0^3) - y_0^3)e^t, y_0e^{-t}) \\ &= (y_0^3e^{-3t} + x_0e^t, y_0e^{-t}). \end{aligned}$$

Thus, $h(\varphi(t,(x_0,y_0))) = \psi(t,h(x_0,y_0))$ and then $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologically conjugate. From (2.7), we have that $\mathcal{F}_1$ and $\mathcal{F}_2$ are also defined by

$$\omega_1 = ydx + xdy,\ \omega_2 = ydx + (x - 4y^3)dy,$$

respectively. Now, consider the curve X given by $X = \{\phi = y^3 - x^2 = 0\}$, that is not invariant by $\mathcal{F}_1$ or $\mathcal{F}_2$ since

$$\omega_1 \wedge d\phi = (3y^3 + 2x^2)dx \wedge dy,$$
$$\omega_2 \wedge d\phi = (3y^3 + 2x^2 - 8xy^3)dx \wedge dy.$$

Computing the Bruce-Roberts number for both foliations, we have that $\mu_{BR}(\mathcal{F}_1, X) = \mu_{BR}(\mathcal{F}_2, X) = 5$.

Example 2.3.5. Consider the foliation $\mathcal{F}_\omega$ (Suzuki's foliation) defined by the 1-form

$$\omega = (y^3 + y^2 - xy)dx - (2xy^2 + xy - x^2)dy,$$

and the foliation $\mathcal{F}_\eta$ defined by the 1-form

$$\eta = (2y^2 + x^3)dx - 2xydy.$$

The foliations $\mathcal{F}_\omega$ and $\mathcal{F}_\eta$ are topologically conjugate ([21, Part 3, Chapter II]). Using the curve X from Example 2.3.2, with $p = 7$ and $q = 3$, i.e., $X = \{\phi = 0\}$, with $\phi = y^7 - x^3$, we have $\mu_{BR}(\mathcal{F}_\omega, X) = \mu_{BR}(\mathcal{F}_\eta, X) = 17$, $\mu_0(\mathcal{F}_\omega) = \mu_0(\mathcal{F}_\eta) = 5$, $\operatorname{tang}(\mathcal{F}_\omega, X, 0) = \operatorname{tang}(\mathcal{F}_\eta, X, 0) = 24$, and $\tau_0(X) = 12$. Thus, for both foliations, (2.6) is satisfied.

Remark 2.3.6. The calculations that we could not do by hand were made by making use of the software *Singular*, which we refer to in [24]. We are going to take the codes used in Example 2.3.3 to illustrate how we managed to calculate the index used in this chapter:

```
> ring r=0,(x,y),ds; // local ring
> poly f=y5-x6+x4y3; // polynomial that defines X
> ideal I1=(x,y); // defining the ideals that appear in each index
> ideal I2=(f,-6x6+x4y3-5y5);
> ideal I3=(f,-6x5+4x3y3,5y4+3x4y2);
> ideal I4=(165x2y+20y4+36x2y5-16x4y2,55xy2+18x3y3-8x5);

> size(kbase(groebner(I1)));
1 // Milnor number of the foliation

> size(kbase(groebner(I2)));
30 // tangency order of the folation
```

```
> size(kbase(groebner(I3)));
19 // Tjurina number of X

> size(kbase(groebner(I4)));
12 // Bruce-Roberts number of the foliation
```

2.3.1 The Bruce-Roberts number of a foliation and blow-ups

When it comes to foliations in dimension two, it's possible to think about what happens when a blow-up is applied. In this part of our work, the idea is to use the definitions of Subsection 1.1.2 and see what happens when we apply Corollary 2.3.1. In that case, we are going to work with irreducible curves X, that can be parametrized by a germ of a non-constant morphism $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$, called a *Puiseux parametrization of* X. In short terms, a Puiseux parametrization is a germ of a non-constant morphism $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ that parameterizes a curve X. To make sure that X has such a parametrization, it is demanded that X is an irreducible curve (see [19, Section 1.4.3] for more details). This concept will be very useful from now on, as seen in the next proposition:

Proposition 2.3.7. Let $\mathcal{F}$ be a germ of a singular foliation at $(\mathbb{C}^2, 0)$, and let X be a germ of a reduced curve at $(\mathbb{C}^2, 0)$ defined by $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Suppose that X is irreducible and not invariant by $\mathcal{F}$. Then

$$\operatorname{tang}(\mathcal{F}, X, 0) = \operatorname{ord}_{t=0} \varphi^*\omega + \mu_0(X),$$

where φ is a Puiseux parametrization of X.

Proof. Suppose that $\mathcal{F}$ is defined by the vector field $v = -B(x, y)\frac{\partial}{\partial x} + A(x, y)\frac{\partial}{\partial y}$. Then, it follows from Definition 1.1.13 that

$$\operatorname{tang}(\mathcal{F}, X, 0) = \operatorname{ord}_{t=0} \varphi^*(A\phi_y - B\phi_x),$$

where $\varphi(t) = (x(t), y(t))$ is a Puiseux parametrization of X. Without loss of generality, we can suppose that $y(t) \neq 0$. Since $\phi(x(t), y(t)) = 0$, we obtain

$$\begin{aligned} & x'(t)\ \phi_x(x(t), y(t)) + y'(t)\ \phi_y(x(t), y(t)) = 0 \\ \Rightarrow\ & \phi_y(x(t), y(t)) = -\frac{x'(t)\phi_x(x(t), y(t))}{y'(t)}. \end{aligned}$$

Hence,

$$\begin{aligned}
\varphi^*(A\phi_y - B\phi_x) &= (A\phi_y - B\phi_x)(x(t), y(t)) \\
&= A(x(t), y(t))\phi_y(x(t), y(t)) - B(x(t), y(t))\phi_x(x(t), y(t)) \\
&= A(x(t), y(t))\left(-\frac{x'(t)\phi_x(x(t), y(t))}{y'(t)}\right) - B(x(t), y(t))\phi_x(x(t), y(t)) \\
&= -\frac{\phi_x(x(t), y(t))}{y'(t)}\Big(x'(t)A(x(t), y(t)) + y'(t)B(x(t), y(t))\Big) \\
&= -\frac{\phi_x(x(t), y(t))}{y'(t)}[\varphi^*\omega](t) \ = \frac{-\varphi^*(\phi_x)(t)\cdot\varphi^*\omega(t)}{y'(t)} \\
&= -\frac{\varphi^*(\phi_x \cdot \omega)(t)}{y'(t)}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\operatorname{tang}(\mathcal{F}, X, 0) &= \operatorname{ord}_{t=0}\ \varphi^*(A\phi_y - B\phi_x) \\
&= \operatorname{ord}_{t=0}\ (A\phi_y - B\phi_x)(x(t), y(t)) \\
&= \operatorname{ord}_{t=0} -\frac{\varphi^*(\phi_x \cdot \omega)(t)}{y'(t)} \\
&= \operatorname{ord}_{t=0}\ \varphi^*(\phi_x \cdot \omega) - \operatorname{ord}_{t=0}\ y'(t) \\
&= \operatorname{ord}_{t=0}\ \varphi^*(\phi_x \cdot \omega) - \operatorname{ord}_{t=0}\ y(t) + 1 \\
&= \operatorname{ord}_{t=0}\ \varphi^*\phi_x + \operatorname{ord}_{t=0}\ \varphi^*\omega - \operatorname{ord}_{t=0}\ y(t) + 1.
\end{aligned} \tag{2.8}$$

Now, from Proposition 1.2.10, we obtain

$$\mu_0(X) = i_0(\phi, \mathcal{P}_l(\phi)) - i_0(\phi, l) + 1, \tag{2.9}$$

where $\mathcal{P}_l(\phi) = \dfrac{\partial\phi}{\partial x}\dfrac{\partial l}{\partial y} - \dfrac{\partial\phi}{\partial y}\dfrac{\partial l}{\partial x}$ is the polar of X with respect to $l = ay - bx \in \mathbb{C}\{x, y\}$, assuming that l does not divide f, as defined in Definition 1.2.9. Let $l = ay - bx$. Computing the items on the right side of (2.9), we have

$$\begin{aligned}
i_0(\phi, \mathcal{P}_l(\phi)) &= \operatorname{ord}_{t=0}\ \varphi^*(a\phi_x + b\phi_y) \\
&= \operatorname{ord}_{t=0}\ (a\phi_x(x(t), y(t)) + b\phi_y(x(t), y(t))) \\
&= \operatorname{ord}_{t=0}\ \left(a\phi_x(x(t), y(t)) - \frac{b\phi_x(x(t), y(t))x'(t)}{y'(t)}\right) \\
&= \operatorname{ord}_{t=0}\ \left(\frac{a\phi_x(x(t), y(t))y'(t) - b\phi_x(x(t), y(t))x'(t)}{y'(t)}\right) \\
&= \operatorname{ord}_{t=0}\ \left(\frac{\phi_x(x(t), y(t))}{y'(t)}(ay'(t) - bx'(t))\right) \\
&= \operatorname{ord}_{t=0}\ \varphi^*\phi_x + \operatorname{ord}_{t=0}\ (ay'(t) - bx'(t)) - \operatorname{ord}_{t=0}\ y'(t) \\
&= \operatorname{ord}_{t=0}\ \varphi^*\phi_x + \min\{\operatorname{ord}_{t=0}\ y(t), \operatorname{ord}_{t=0}\ x(t)\} - \operatorname{ord}_{t=0}\ y(t),
\end{aligned}$$

and

$$\begin{aligned} i_0(\phi, l) &= \operatorname{ord}_{t=0} \varphi^*(ay - bx) \\ &= \operatorname{ord}_{t=0} (ay(t) - bx(t)) \\ &= \min\{\operatorname{ord}_{t=0} y(t), \operatorname{ord}_{t=0} x(t)\}. \end{aligned}$$

Finally, by rewriting (2.8) using that and (2.9), we get

$$\begin{aligned} \operatorname{tang}(\mathcal{F}, X, 0) &= \operatorname{ord}_{t=0} \varphi^*\omega + \operatorname{ord}_{t=0} \varphi^*\phi_x - \operatorname{ord}_{t=0} y(t) + 1 \\ &= \operatorname{ord}_{t=0} \varphi^*\omega + i_0(\phi, \mathcal{P}_l(\phi)) - i_0(\phi, l) + 1 \\ &= \operatorname{ord}_{t=0} \varphi^*\omega + \mu_0(X). \end{aligned}$$

□

Remark 2.3.8. Let $X = \{\phi = 0\}$ be a germ of an irreducible reduced curve and $f \in \mathcal{O}_2$ is a germ with an isolated singularity over $(X, 0)$. Note that applying Proposition 2.3.7 to foliation $\mathcal{F} : \omega = df$, we obtain an expression to compute the Milnor number of the isolated complete intersection singularity defined by (ϕ, f). Indeed, it follows from (1.19) that

$$\operatorname{ord}_{t=0} \varphi^*\omega + \mu_0(X) = \operatorname{tang}(\mathcal{F}, X, 0) = \operatorname{Ind}_{\mathrm{GSV}}(df; X, 0) = \mu_0(X) + \mu_0(\phi, f).$$

Hence, $\mu_0(\phi, f) = \operatorname{ord}_{t=0} \varphi^*(df)$, where φ is a Puiseux parametrization of X.

On the following proposition, we show how the Bruce-Roberts number in dimension two behaves under a blow-up. We denote by $\mu_{BR}(\mathcal{F}, X, p)$ the Bruce-Roberts number of $\mathcal{F}$ along X around a neighborhood of the point p (in Corollary 2.3.1, for example, it is defined around $0 \in \mathbb{C}^2$).

Proposition 2.3.9. Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, 0)$, let X be a germ of an irreducible reduced curve and let $\pi : \tilde{\mathbb{C}}^2 \to (\mathbb{C}^2, 0)$ be the blow-up at $(\mathbb{C}^2, 0)$. Assume that X is not invariant by $\mathcal{F}$, $\tilde{\mathcal{F}} := \pi^*\mathcal{F}$, and $q \in \pi^{-1}(0) \cap \tilde{X}$, where $\tilde{\mathcal{F}}$ and $\tilde{X}$ are the strict transforms of $\mathcal{F}$ and X respectively. Denote by m the multiplicity of X and by ν the algebraic multiplicity of the foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$. Then, we have the following statements:

(a) If π is non-dicritical, then

$$\begin{aligned} \mu_{BR}(\mathcal{F}, X, 0) = \mu_{BR}(\tilde{\mathcal{F}}, \tilde{X}, q) + \nu^2 - \nu - 1 + \nu m \\ + \sum_{\substack{p \in \pi^{-1}(0) \\ p \neq q}} \mu_p(\tilde{\mathcal{F}}) + \frac{m(m-1)}{2} - \mathcal{D}. \end{aligned}$$

(b) If π is dicritical, then

$$\begin{aligned} \mu_{BR}(\mathcal{F}, X, 0) = \mu_{BR}(\tilde{\mathcal{F}}, \tilde{X}, q) + \nu^2 + \nu - 1 + (\nu + 1)m \\ + \sum_{\substack{p \in \pi^{-1}(0) \\ p \neq q}} \mu_p(\tilde{\mathcal{F}}) + \frac{m(m-1)}{2} - \mathcal{D}. \end{aligned}$$

We write $\mathcal{D} = \dim_{\mathbb{C}} \dfrac{\tilde{\sigma}^*\Omega_{\tilde{X}}}{\sigma^*\Omega_X}$, where $\tilde{\sigma} : (\overline{X}, 0) \to (\tilde{X}, 0)$ is the normalization, and $\sigma = \pi \circ \tilde{\sigma}$.

Proof. Let ω be the 1-form defining $\mathcal{F}$ and let $\tilde{\omega}$ be the 1-form defining $\tilde{\mathcal{F}}$. Applying Proposition 2.3.7 to the Corollary 2.3.1, we have

$$\mu_{BR}(\mathcal{F}, X, 0) = \mu_0(\mathcal{F}) + \mathrm{ord}_{t=0}\, \varphi^*\omega + \mu_0(X) - \tau_0(X) \tag{2.10}$$

and similarly,

$$\mu_{BR}(\tilde{\mathcal{F}}, \tilde{X}, q) = \mu_q(\tilde{\mathcal{F}}) + \mathrm{ord}_{t=q}\, \tilde{\varphi}^*\tilde{\omega} + \mu_q(\tilde{X}) - \tau_q(\tilde{X}),$$

where φ and $\tilde{\varphi}$ are the Puiseux parametrization of X and $\tilde{X}$, respectively.

The proof is obtained using results that show how the indexes on the right side of equation (2.10) change through a blow-up. By Proposition 1.2.14, we have

$$\mu_0(\mathcal{F}) = \begin{cases} \nu^2 - (\nu+1) + \displaystyle\sum_{p \in \pi^{-1}(0)} \mu_p(\tilde{\mathcal{F}}) & \text{if } \pi \text{ is non-dicritical;} \\ (\nu+1)^2 - (\nu+2) + \displaystyle\sum_{p \in \pi^{-1}(0)} \mu_p(\tilde{\mathcal{F}}) & \text{if } \pi \text{ is dicritical.} \end{cases}$$

and, from (1.11) and (1.13), we have

$$\mu_0(X) - \mu_q(\tilde{X}) = m(m-1) \text{ and } \tau_0(X) - \tau_q(\tilde{X}) = \frac{m(m-1)}{2} + \mathcal{D}.$$

Now, from [32, Section 2], evaluating the 1-form that defines $\tilde{\mathcal{F}}$ in $\tilde{\varphi}$ and taking orders, we get

$$\mathrm{ord}_{t=0}\, \varphi^*\omega = \begin{cases} \nu m + \mathrm{ord}_{t=q}\, \tilde{\varphi}^*\tilde{\omega} & \text{if } \pi \text{ is non-dicritical;} \\ (\nu+1)m + \mathrm{ord}_{t=q}\, \tilde{\varphi}^*\tilde{\omega} & \text{if } \pi \text{ is dicritical.} \end{cases}$$

Substituting the above formulas into (2.10), we conclude the proof of the Proposition. □

The following examples illustrate Proposition 2.3.9 in both the critical and dicritical cases.

Example 2.3.10. Let $\mathcal{F}$ be a foliation defined by the 1-form $\omega = 2xdy - 3ydx$, and let X be the curve given by $X = \{\phi = y^2 - x^5 = 0\}$. Observe that X is not invariant by $\mathcal{F}$. If π is a blow-up at $0 \in \mathbb{C}^2$, in local coordinates, we have

$$\tilde{\mathcal{F}}_1 = \pi^*\mathcal{F} = x(-tdx + 2xdt) \text{ and}$$
$$\tilde{\mathcal{F}}_2 = \pi^*\mathcal{F} = y(-3ydu - udy),$$

where $\tilde{\mathcal{F}}_1$ is obtained using the local chart $\pi(x,t) = (x, tx)$, and $\tilde{\mathcal{F}}_2$ is obtained using the local chart $\pi(u,y) = (uy, y)$. Now, we consider $\tilde{X} = \{t^2 - x^3 = 0\}$, since

$$\phi \circ \pi(x,t) = \phi(x, tx) = x^2(t^2 - x^3).$$

Since $\tilde{X}$ is not invariant by $\tilde{\mathcal{F}}_1$, set $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_1$. Thus, $\tilde{\mathcal{F}}$ is defined by the 1-form $\tilde{\omega}$, given by

$$\tilde{\omega} = -tdx + 2xdt.$$

Using *Singular* ([24]), we get $\mu_{BR}(\mathcal{F}, X, 0) = 7$, $\mu_{BR}(\tilde{\mathcal{F}}, \tilde{X}, q) = 5$, $\nu = 1$, $m = 2$, $\sum_{\substack{p\in\pi^{-1}(0)\\ p\neq q}} \mu_p(\tilde{\mathcal{F}}) = \mu_0(\tilde{\mathcal{F}}_2) = 1$ and $\mathcal{D} = 1$. Note that

$$\begin{aligned}\mu_{BR}(\mathcal{F}, X, 0) &= \mu_{BR}(\tilde{\mathcal{F}}, \tilde{X}, q) + \nu^2 - \nu - 1 + \nu m + \mu_0(\tilde{\mathcal{F}}_2) + \frac{m(m-1)}{2} - \mathcal{D} \\ &= 5 + 1 - 1 - 1 + 2 \cdot 1 + 1 + 1 - 1 = 7.\end{aligned}$$

Since π is non-dicritical, item (a) of Proposition 2.3.9 is satisfied.

Example 2.3.11. Consider the foliation $\mathcal{F}$ defined by the 1-form

$$\omega = (2x^7 + 5y^5)dx - xy^2(5y^2 + 3x^5)dy,$$

and let X be the curve not invariant by $\mathcal{F}$ given by $X = \{\phi = y^3 - x^7 = 0\}$. Again, by the local of the coordinates of the blow-up π at $0 \in \mathbb{C}^2$, we have

$$\begin{aligned}\tilde{\mathcal{F}}_1 &= \pi^*\mathcal{F} = x^5(-t^3dx - (2xt^2 + t - 1)dt) \quad \text{and} \\ \tilde{\mathcal{F}}_2 &= \pi^*\mathcal{F} = y^5((y + 1 - u)du - udy),\end{aligned}$$

where $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ are obtained using the local charts $\pi(x, t) = (x, tx)$ and $\pi(u, y) = (uy, y)$, respectively. Set $\tilde{X} = \{t^3 - x^4 = 0\}$, since

$$\phi \circ \pi(x, t) = \phi(x, tx) = x^3(t^3 - x^4).$$

Since $\tilde{X}$ is not invariant by $\tilde{\mathcal{F}}_1$, set $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_1$. In that way, $\tilde{\mathcal{F}}$ is defined by the 1-form

$$\tilde{\omega} = -t^3dx - (2xt^2 + t - 1)dt.$$

Again, by *Singular* [24], we obtain $\mu_{BR}(\mathcal{F}, X, 0) = 56$, $\mu_{BR}(\tilde{\mathcal{F}}, \tilde{X}, q) = 9$, $\nu = 5$, $m = 3$, $\sum_{\substack{p\in\pi^{-1}(0)\\ p\neq q}} \mu_p(\tilde{\mathcal{F}}) = \mu_0(\tilde{\mathcal{F}}_2) = 0$ and $\mathcal{D} = 3$. Then,

$$\begin{aligned}\mu_{BR}(\mathcal{F}, X, 0) &= \mu_{BR}(\tilde{\mathcal{F}}, \tilde{X}, q) + \nu^2 + \nu - 1 + (\nu + 1)m + \mu_0(\tilde{\mathcal{F}}_2) + \frac{m(m-1)}{2} - \mathcal{D} \\ &= 9 + 25 + 5 - 1 + 6 \cdot 3 + 0 + 3 - 3 = 56.\end{aligned}$$

Since π is dicritical, item (b) of Proposition 2.3.9 is satisfied.

Remark 2.3.12. In both examples showed above, the computation of $\mathcal{D} = \dim_{\mathbb{C}} \dfrac{\tilde{\sigma}^*\Omega_{\tilde{X}}}{\sigma^*\Omega_X}$ is not trivial, neither by hand nor by *Singular*. To compute that number, we turned to [72, Section 2], and used some properties. By (1.14), we have that

$$\mathcal{D} = \left(\tau_0(X) - \frac{\mu_0(X)}{2}\right) - \left(\tau_0(\tilde{X}) - \frac{\mu_0(\tilde{X})}{2}\right).$$

Now, we can calculate the value of $\mathcal{D}$ since we know that, when a curve C is defined by a equation written as $y^p - x^q = 0$, we have $\mu_0(C) = \tau_0(C) = (p-1)(q-1)$. Then, for Example 2.3.10, we have

$$\mathcal{D} = \left(\tau_0(X) - \frac{\mu_0(X)}{2}\right) - \left(\tau_0(\tilde{X}) - \frac{\mu_0(\tilde{X})}{2}\right) = (4-2) - (2-1) = 1,$$

and for Example 2.3.11, we have

$$\mathcal{D} = \left(\tau_0(X) - \frac{\mu_0(X)}{2}\right) - \left(\tau_0(\tilde{X}) - \frac{\mu_0(\tilde{X})}{2}\right) = (12-6) - (6-3) = 3.$$

2.3.2 The relative Bruce-Roberts number for foliations on $(\mathbb{C}^2, 0)$

The relative Bruce-Roberts number for foliations can be defined in a similar way as we did in this section, using the same characterization of foliations in dimension two. In that way, the relative Bruce-Roberts number of the foliation $\mathcal{F}$, defined by the 1-form ω with respect to X is given by

$$\mu_{BR}^{-}(\mathcal{F}, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\omega(\Theta_X) + I_X},$$

where ω is a germ of a holomorphic 1-form with isolated singularity at $(\mathbb{C}^2, 0)$ defining $\mathcal{F}$ and X is a complex analytic germ with isolated singularity at $(\mathbb{C}^2, 0)$. Now, for holomorphic foliations on $(\mathbb{C}^2, 0)$, Theorem 2.2.6 can be stated as follows:

Corollary 2.3.13. Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $0 \in \mathbb{C}^2$. Let X be a germ of a reduced curve at $0 \in \mathbb{C}^2$. Assume that X is not invariant by $\mathcal{F}$. Then

$$\mu_{BR}(\mathcal{F}, X) = \mu_0(\mathcal{F}) + \mu_{BR}^{-}(\mathcal{F}, X).$$

The following examples, inspired by our previous examples in dimension two, illustrate Corollary 2.3.13:

Example 2.3.14. We can use *Singular* [24] to compute the relative Bruce-Roberts number for the previous examples:

- In Example 2.3.2, we have $\mathcal{F} : \omega = \lambda x dy + y dx = 0$, $\lambda \neq -p/q$, and $X = \{\phi = y^p - x^q = 0\}$. As we needed numerical examples, for $\lambda = 1$, we have

$$\mu_{BR}^{-}(\mathcal{F}, X) = 6, \text{ when } p = 2 \text{ and } q = 5, \text{ and}$$
$$\mu_{BR}^{-}(\mathcal{F}, X) = 23, \text{ when } p = 11 \text{ and } q = 13.$$

  In both cases, $\mu_{BR}^{-}(\mathcal{F}, X) = p + q - 1 = \mu_{BR}(\mathcal{F}, X) - \mu_0(\mathcal{F})$.

- In Example 2.3.3, we have $\mathcal{F} : \omega = x dy + y dx$, and $X = \{\phi = y^5 - x^6 + x^4 y^3 = 0\}$, with $\mu_{BR}(\omega, X) = 12$ and $\mu_0(\omega) = 1$. Then

$$\mu_{BR}^{-}(\mathcal{F}, X) = \frac{\mathcal{O}_2}{\omega(\Theta_X) + (y^5 - x^6 + x^4 y^3)} = 11.$$

- In Example 2.3.5, we have $\mathcal{F}_\omega : \omega = (y^3 + y^2 - xy)dx - (2xy^2 + xy - x^2)dy = 0$, $\mathcal{F}_\eta : \eta = (2y^2 + x^3)dx - 2xydy = 0$ and $X = \{\phi = y^7 - x^3 = 0\}$, with $\mu_{BR}(\mathcal{F}_\omega, X) = \mu_{BR}(\mathcal{F}_\eta, X) = 17$ and $\mu_0(\mathcal{F}_\omega) = \mu_0(\mathcal{F}_\eta) = 5$. Then

$$\mu^-_{BR}(\mathcal{F}_\omega, X) = \mu^-_{BR}(\mathcal{F}_\eta, X) = 12.$$

In each case, we have $\mu^-_{BR}(\mathcal{F}, X) = \mu_{BR}(\mathcal{F}, X) - \mu_0(\mathcal{F})$, satisfying Corollary 2.3.13.

As seen in (1.10), an interesting property for the Milnor number of a foliation $\mathcal{F}$ is given by

$$\mu_0(\mathcal{F}) \geq \frac{\nu(\nu+1)}{2}.$$

As an application of the above inequality, we have the following corollary:

Corollary 2.3.15. Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $0 \in \mathbb{C}^2$. Let X be a germ of a reduced curve at $0 \in \mathbb{C}^2$. Assume that X is not invariant by $\mathcal{F}$. Then

$$\mu_{BR}(\mathcal{F}, X) - \mu^-_{BR}(\mathcal{F}, X) \geq \frac{\nu(\nu+1)}{2},$$

where ν denotes the algebraic multiplicity of $\mathcal{F}$.

To end this subsection, we present a blow-up formula for the relative Bruce-Roberts number of a foliation $\mathcal{F}$ with respect to a non-invariant curve X. Its proof follows similarly as the proof of Proposition 2.3.9, using the results presented there together with Corollary 2.3.13.

Corollary 2.3.16. Let $\mathcal{F}$ be a germ of a singular holomorphic foliation at $(\mathbb{C}^2, 0)$, let X be a germ of an irreducible reduced curve and let $\pi : \tilde{\mathbb{C}}^2 \to (\mathbb{C}^2, 0)$ be the blow-up at $(\mathbb{C}^2, 0)$. Assume that X is not invariant by $\mathcal{F}$, $\tilde{\mathcal{F}} := \pi^*\mathcal{F}$, and $q \in \pi^{-1}(0) \cap \tilde{X}$, where $\tilde{\mathcal{F}}$ and $\tilde{X}$ are the strict transforms of $\mathcal{F}$ and X respectively. Denote by m the multiplicity of X and by ν the algebraic multiplicity of the foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$. Then, we have the following statements:

(a) If π is non-dicritical, then

$$\mu^-_{BR}(\mathcal{F}, X, 0) = \mu^-_{BR}(\tilde{\mathcal{F}}, \tilde{X}, q) + \nu m + \frac{m(m-1)}{2} - \mathcal{D}.$$

(b) If π is dicritical, then

$$\mu^-_{BR}(\mathcal{F}, X, 0) = \mu^-_{BR}(\tilde{\mathcal{F}}, \tilde{X}, q) + (\nu+1)m + \frac{m(m-1)}{2} - \mathcal{D}.$$

We write $\mu^-_{BR}(\mathcal{F}, X, p)$ to denote the relative Bruce-Roberts number around the point p. Moreover, we have $\mathcal{D} = \dim \dfrac{\tilde{\sigma}^*\Omega_{\tilde{X}}}{\sigma^*\Omega_X}$, where $\tilde{\sigma} : (\overline{X}, 0) \to (\tilde{X}, 0)$ is the normalization, and $\sigma = \pi \circ \tilde{\sigma}$.

2.3.3 Generalized curve foliations and the Bruce-Roberts numbers

To end this section, we give a characterization of the Bruce-Roberts numbers defined above with a particular type of foliation. First, we remember that Theorem 1.2.13 - originally [17, Theorem 4] - enunciated in Section 1.1.2 says that when $\mathcal{F}$ is a germ of non-dicritical foliation at $0 \in \mathbb{C}^2$ and $C = Sep_0(\mathcal{F})$ is the union of the separatrices of $\mathcal{F}$, then

$$\mu_0(\mathcal{F}) \geq \mu_0(C),$$

and the equality holds if and only if $\mathcal{F}$ is a generalized curve foliation. As a consequence of this and our previous results, we can establish new characterizations of non-dicritical generalized curve foliations. Since the two next results are direct consequences of Theorem 1.2.13, we announce them as the following corollaries:

Corollary 2.3.17. Let $\mathcal{F}$ be a germ of a non-dicritical holomorphic foliation at $0 \in \mathbb{C}^2$. Let X be a germ of a reduced curve at $0 \in \mathbb{C}^2$, and $C = Sep_0(\mathcal{F}) = \{f = 0\}$ be a reduced equation of $Sep_0(\mathcal{F})$. Assume that X is not invariant by $\mathcal{F}$. Then

$$\mu_{BR}(\mathcal{F}, X) - \mu_{BR}(f, X) \geq \mu_{BR}^{-}(\mathcal{F}, X) - \mu_{BR}^{-}(f, X),$$

and the equality holds if and only if $\mathcal{F}$ is a generalized curve foliation.

Proof. The proof follows directly from Theorem 1.2.13, applied to $\mathcal{F}$ and f, and Corollary 2.3.13. In fact,

$$\mu_{BR}(\mathcal{F}, X) - \mu_{BR}^{-}(\mathcal{F}, X) = \mu_0(\mathcal{F}) \geq \mu_0(C) = \mu_{BR}(f, X) - \mu_{BR}^{-}(f, X).$$

□

Corollary 2.3.18. Let $\mathcal{F}$ be a germ of a non-dicritical holomorphic foliation at $0 \in \mathbb{C}^2$. Let X be a germ of a reduced curve at $0 \in \mathbb{C}^2$, and $C = Sep_0(\mathcal{F}) = \{f = 0\}$ be a reduced equation of $Sep_0(\mathcal{F})$. Assume that X is irreducible and not invariant by $\mathcal{F}$. Then $\mathcal{F}$ is a generalized curve foliation if, and only if

$$\mu_{BR}(\mathcal{F}, X) - \mu_{BR}(f, X) = \operatorname{ord}_{t=0} \varphi^*\omega - \operatorname{ord}_{t=0} \varphi^*(df),$$

where φ is a Puiseux parametrization of X.

Proof. Since X is an irreducible curve, we can recur to (2.10), and then we have

$$\mu_{BR}(\mathcal{F}, X) = \mu_0(\mathcal{F}) + \operatorname{ord}_{t=0} \varphi^*\omega + \mu_0(X) - \tau_0(X).$$

On the other hand, Remark 2.3.8 can be used to rewrite the result in Corollary 2.1.4 in dimension two as

$$\mu_{BR}(f, X) = \mu_0(C) + \operatorname{ord}_{t=0} \varphi^*(df) + \mu_0(X) - \tau_0(X).$$

Now, subtracting the two equalities above, we have

$$\mu_{BR}(\mathcal{F}, X) - \mu_{BR}(f, X) = \mu_0(\mathcal{F}) - \mu_0(C) + \operatorname{ord}_{t=0} \varphi^*\omega - \operatorname{ord}_{t=0} \varphi^*(df).$$

The proof is concluded using the equivalence shown in Theorem 1.2.13. □

2.3.4 Applications to Global Foliations

To finish this chapter, we present some results involving the Bruce-Roberts number for foliations, in the case that $\mathcal{F}$ is a global foliation on a compact complex surface S, as described in Subsection 1.1.3. Hence, we have our first result:

Theorem 2.3.19. Let $\mathcal{F}$ be a holomorphic foliation on a compact complex surface S, and let $X \subset S$ be a compact curve, none of whose components are invariant by $\mathcal{F}$. Then

$$\sum_{p \in \mathsf{Sing}(\mathcal{F}) \cap X} \mu_{BR}(\mathcal{F}, X, p) = T_{\mathcal{F}} \cdot T_{\mathcal{F}} + T_{\mathcal{F}} \cdot K_S + c_2(S) + N_{\mathcal{F}} \cdot X - \chi(X) - \tau(X),$$

where $\chi(X) = -K_S \cdot X - X \cdot X$ is the virtual Euler characteristic of X, $c_2(S)$ is the second Chern class of S and $\tau(X) = \sum_{p \in X} \tau_p(X)$ is the global Tjurina number of X.

Proof. For each $p \in \mathsf{Sing}(\mathcal{F}) \cap X$, we have

$$\mu_{BR}(\mathcal{F}, X, p) = \mu_p(\mathcal{F}) + \operatorname{tang}(\mathcal{F}, X, p) - \tau_p(X)$$

by Corollary 2.3.1. On the other hand, according to Proposition 1.1.20 and Proposition 1.1.21 we get

$$\sum_{p \in \mathsf{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = T_{\mathcal{F}} \cdot T_{\mathcal{F}} + T_{\mathcal{F}} \cdot K_S + c_2(S)$$

and

$$\sum_{p \in \mathsf{Sing}(\mathcal{F}) \cap X} \operatorname{tang}(\mathcal{F}, X, p) = N_{\mathcal{F}} \cdot X - \chi(X).$$

Hence, we have

$$\begin{aligned}
\sum_{p \in \mathsf{Sing}(\mathcal{F}) \cap X} \mu_{BR}(\mathcal{F}, X, p) &= \sum_{p \in \mathsf{Sing}(\mathcal{F}) \cap X} \Big[\mu_p(\mathcal{F}) + \operatorname{tang}(\mathcal{F}, X, p) - \tau_p(X) \Big] \\
&= \sum_{p \in \mathsf{Sing}(\mathcal{F}) \cap X} \mu_p(\mathcal{F}) + \sum_{p \in \mathsf{Sing}(\mathcal{F}) \cap X} \operatorname{tang}(\mathcal{F}, X, p) - \sum_{p \in \mathsf{Sing}(\mathcal{F}) \cap X} \tau_p(X) \\
&= T_{\mathcal{F}} \cdot T_{\mathcal{F}} + T_{\mathcal{F}} \cdot K_S + c_2(S) + N_{\mathcal{F}} \cdot X - \chi(X) - \tau(X).
\end{aligned}$$

□

With that in mind, and since $\mu_{BR}(\mathcal{F}, X, p) \geq 0$, for all $p \in \mathsf{Sing}(\mathcal{F}) \cap X$, we can obtain an upper bound for the global Tjurina number of X.

Corollary 2.3.20. Let $\mathcal{F}$ be a holomorphic foliation on a compact complex surface S, and let $X \subset S$ be a compact curve, none of whose components are invariant by $\mathcal{F}$. Then

$$\tau(X) \leq T_{\mathcal{F}} \cdot T_{\mathcal{F}} + T_{\mathcal{F}} \cdot K_S + c_2(S) + N_{\mathcal{F}} \cdot X - \chi(X).$$

Example 2.3.21. In the particular case of $S = \mathbb{P}^2_{\mathbb{C}}$, $\mathcal{F}$ a foliation on $\mathbb{P}^2_{\mathbb{C}}$ of degree $\deg(\mathcal{F}) = d$, and $X \subset \mathbb{P}^2_{\mathbb{C}}$ an algebraic curve of $\deg(X) = r$, we get

$$\tau(X) \leq d^2 + d + 1 + r(d + r - 1).$$

In [61, Theorem 3.2], an upper bound for $\tau(X)$ is derived under the assumption that there exists a holomorphic foliation on $\mathbb{P}^2$ that leaves X invariant. In contrast, in Corollary 2.3.20, the bound for the global Tjurina number of X is established using a holomorphic foliation that does not leave X invariant.

On the other hand, since (again by Proposition 1.1.20)

$$\sum_{p\in \mathsf{Sing}(\mathcal{F})} \mu_p(\mathcal{F}) = T_{\mathcal{F}} \cdot T_{\mathcal{F}} + T_{\mathcal{F}} \cdot K_S + c_2(S),$$

we have the following corollary, using the difference of the Bruce-Roberts and the relative Bruce-Roberts numbers, but now globally.

Corollary 2.3.22. Let $\mathcal{F}$ be a holomorphic foliation on a compact complex surface S, and let $X \subset S$ be a compact curve, none of whose components are invariant by $\mathcal{F}$. Then

$$\sum_{p\in \mathsf{Sing}(\mathcal{F})\cap X} \left[\mu_{BR}(\mathcal{F}, X, p) - \mu_{BR}^{-}(\mathcal{F}, X, p)\right] = T_{\mathcal{F}} \cdot T_{\mathcal{F}} + T_{\mathcal{F}} \cdot K_S + c_2(S).$$

Proof. The proof follows immediately by taking sums in Corollary 2.3.13. $\square$

Chapter 3

The Bruce-Roberts Tjurina Number for Holomorphic 1-Forms

In this chapter, our main goal is to define a version of the *Tjurina number* for foliations, first defined in [7]. As in the previous chapter, the idea is to generalize an existing index and check which properties and results can be obtained. The inspiration for this definition is the *Bruce-Roberts Tjurina number* $\tau_{BR}(f, X)$ of a function f along X, defined by Ahmed-Ruas and Bivià-Ausina–Ruas, and defined in Definition 1.2.58. The idea of our index also came after studying the Bruce-Roberts number defined in (2.1), and can be seen in the following definition:

Definition 3.0.1. Let $(X, 0)$ denote the germ of a complex analytic variety at $(\mathbb{C}^n, 0)$, and let ω be the germ of a holomorphic 1-form with isolated singularity at $(\mathbb{C}^n, 0)$. Let V be a germ of a complex analytic hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$ invariant by ω. The *Bruce-Roberts Tjurina number* of ω relative to the pair (X, V) is defined as

$$\tau_{BR}(\omega, X, V) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X) + I_V},$$

where I_V is the ideal of germs of holomorphic functions vanishing on $(V, 0)$.

Observe that if $\omega = df$ is an exact holomorphic 1-form with an isolated singularity at $0 \in \mathbb{C}^n$, then $V = \{f = 0\}$ is a complex analytic hypersurface invariant by ω, and

$$\tau_{BR}(df, X, V) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X) + I_V} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_X) + \langle f \rangle} = \tau_{BR}(f, X). \tag{3.1}$$

In other words, $\tau_{BR}(df, X, V)$ coincides with the Bruce-Roberts Tjurina number of f along X. In this case, when $X = \mathbb{C}^n$, we can also note that

$$\tau_{BR}(df, \mathbb{C}^n, V) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\Theta_{\mathbb{C}^n}) + I_V} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, f \right\rangle} = \tau_0(V)$$

is the classical *Tjurina number* of V.

Before we state our main theorem, we're gonna need some important results. First, we enunciate two lemmas regarding commutative algebra, both of them inspired by [5, Proposition 2.1].

Lemma 3.0.2. Let A be a ring. If $L \supseteq M \supseteq N$ are A-modules, then

$$0 \longrightarrow \frac{M}{N} \longrightarrow \frac{L}{N} \longrightarrow \frac{L}{M} \longrightarrow 0$$

is an exact sequence.

Proof. By the proof of [5, Proposition 2.1 (i)], $\theta : L/N \to L/M$, defined by $\theta(x+N) = x+M$, is a well-defined A-module isomorphism, whose kernel is M/N. Since $N \subseteq M$, it's easy to see that θ is surjective. Now, induced by the inclusion $M \subseteq L$, we define $\alpha : M/N \to L/N$ as $\alpha(x+N) = x+L$. We have that α is injective, and $\text{Im}(\alpha) = M/N$. The proof concludes by writing

$$0 \longrightarrow \frac{M}{N} \xrightarrow{\alpha} \frac{L}{N} \xrightarrow{\theta} \frac{L}{M} \longrightarrow 0.$$

□

Lemma 3.0.3. Let A be a ring and M_1, M_2 and M_3 be submodules of an A-module M, with $M_3 \subset M_2$. Then

$$\frac{M_1 + M_2}{M_1 + M_3} \cong \frac{M_2}{M_3 + (M_1 \cap M_2)}.$$

Proof. By [5, Proposition 2.1 (ii)], we have

$$\frac{M_1 + M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}. \tag{3.2}$$

Since $M_3 \subset M_2$, the proof ends by substituting M_1 with $M_1 + M_3$. □

Inspired by the definition given in (1.16), we introduce a new version of the GSV-index for a pair (X, V):

Definition 3.0.4. Let $X = \phi^{-1}(0)$ be an ICIS at $0 \in \mathbb{C}^n$, defined by $\phi = (\phi_1, \ldots, \phi_k)$, and let $V = \{f = 0\}$ be an isolated complex hypersurface invariant by ω. Assume that X is not invariant by ω. The *GSV-index of ω with respect to the pair* (X, V) is defined by:

$$\text{Ind}_{\text{GSV}}(\omega; X, V, 0) := \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I_X + I_{k+1}\begin{pmatrix} \omega \\ d\phi \end{pmatrix} + \langle f \rangle}. \tag{3.3}$$

The next result was given in [49, Corollary 3.7], giving another characterization of the Milnor number when the Bruce-Roberts number defined in Definition 2.0.1 is finite.

Proposition 3.0.5. If ω is a holomorphic 1-form at $0 \in \mathbb{C}^n$ and $(X, 0)$ is an ICIS such that $\mu_{BR}(\omega, X) < \infty$, then

$$\tau_0(X) = \dim_{\mathbb{C}} \frac{\Theta_X}{\Theta_X^T} = \dim_{\mathbb{C}} \frac{\omega(\Theta_X)}{\omega(\Theta_X^T)}. \tag{3.4}$$

3.1 Main Results

Finally, we can enunciate our main theorem.

Theorem 3.1.1. Let ω be a germ of a holomorphic 1-form with isolated singularity at $(\mathbb{C}^n, 0)$. Suppose that the pair (X, V) are isolated complex analytic hypersurfaces at $0 \in \mathbb{C}^n$, V is invariant by ω and X is not invariant by ω. Then

$$\tau_{BR}(\omega, X, V) = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) + \tau_0(\omega, V) - \tau_0(X) + \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V},$$

where Θ_X^T is the submodule of Θ_X of trivial vector fields.

Proof. Consider the following sequence of $\mathbb{C}$-vector spaces:

$$0 \longrightarrow \frac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V} \xrightarrow{\alpha} \frac{\mathcal{O}_n}{\omega(\Theta_X^T) + I_V} \xrightarrow{\beta} \frac{\mathcal{O}_n}{\omega(\Theta_X) + I_V} \longrightarrow 0. \tag{3.5}$$

Since $\omega(\Theta_X^T) + I_V \subseteq \omega(\Theta_X) + I_V \subseteq \mathcal{O}_n$, note that (3.5) is an exact sequence, using Lemma 3.0.2. The exactness of the sequence implies that

$$\tau_{BR}(\omega, X, V) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T) + I_V} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T) + I_V} - \dim_{\mathbb{C}} \frac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V}. \tag{3.6}$$

To compute $\dim_{\mathbb{C}} \frac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V}$, we apply Lemma 3.0.3 with $M_1 = I_V$, $M_2 = \omega(\Theta_X)$, and $M_3 = \omega(\Theta_X^T)$, giving us:

$$\frac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V} \cong \frac{\omega(\Theta_X)}{\omega(\Theta_X^T) + (\omega(\Theta_X) \cap I_V)}.$$

Using again Lemma 3.0.2, with the exact sequence

$$0 \longrightarrow \frac{\omega(\Theta_X^T) + (\omega(\Theta_X) \cap I_V)}{\omega(\Theta_X^T)} \longrightarrow \frac{\omega(\Theta_X)}{\omega(\Theta_X^T)} \longrightarrow \frac{\omega(\Theta_X)}{\omega(\Theta_X^T) + (\omega(\Theta_X) \cap I_V)} \longrightarrow 0,$$

we deduce

$$\dim_{\mathbb{C}} \frac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V} = \dim_{\mathbb{C}} \frac{\omega(\Theta_X)}{\omega(\Theta_X^T)} - \dim_{\mathbb{C}} \frac{\omega(\Theta_X^T) + (\omega(\Theta_X) \cap I_V)}{\omega(\Theta_X^T)}. \tag{3.7}$$

Now, using (3.2) to rewrite $\frac{\omega(\Theta_X^T) + (\omega(\Theta_X) \cap I_V)}{\omega(\Theta_X^T)}$, we have

$$\frac{\omega(\Theta_X^T) + (\omega(\Theta_X) \cap I_V)}{\omega(\Theta_X^T)} \cong \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap (\omega(\Theta_X) \cap I_V)} = \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V}. \tag{3.8}$$

Substituting (3.8) into equation (3.7), we obtain

$$\dim_{\mathbb{C}} \frac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V} = \dim_{\mathbb{C}} \frac{\omega(\Theta_X)}{\omega(\Theta_X^T)} - \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V}. \tag{3.9}$$

Now, suppose $X = \{\phi = 0\}$, $V = \{f = 0\}$, and $\omega = \sum_{j=1}^{n} A_j(x)dx_j$. By Proposition 1.0.3 we have that

$$\omega(\Theta_X^T) = I_{k+1} \begin{pmatrix} \omega \\ d\phi \end{pmatrix} + \langle \phi_i \cdot A_j,\ i = 1, \ldots, k,\ j = 1, \ldots, n \rangle,$$

where $I_{k+1} \begin{pmatrix} \omega \\ d\phi \end{pmatrix}$ is the ideal generated by the $(k+1)$-minors of the matrix

$$\begin{pmatrix} \omega \\ d\phi \end{pmatrix} = \begin{pmatrix} A_1 & \cdots & A_n \\ \dfrac{\partial \phi_1}{\partial x_1} & \cdots & \dfrac{\partial \phi_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial \phi_k}{\partial x_1} & \cdots & \dfrac{\partial \phi_k}{\partial x_n} \end{pmatrix}.$$

Then, we can compute $\dim_{\mathbb{C}} \dfrac{\mathcal{O}_n}{\omega(\Theta_X^T) + I_V}$ as follows:

$$\begin{aligned} \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T) + I_V} &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I_2 \begin{pmatrix} \omega \\ d\phi \end{pmatrix} + \langle \phi A_i \rangle_{1 \le i \le n} + \langle f \rangle} \\ &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi A_1, \ldots, \phi A_n, \frac{\partial \phi}{\partial x_j} A_k - \frac{\partial \phi}{\partial x_k} A_j, f \rangle_{(j,k) \in \Lambda}}, \end{aligned}$$

where $\Lambda = \{(j,k) : j,k = 1, \ldots, n; j \neq k\}$. Since V is invariant by ω and X is not invariant by ω, and using Lemma 2.1.1, we obtain

$$\begin{aligned} \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T) + I_V} &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \phi, \frac{\partial \phi}{\partial x_j} A_k - \frac{\partial \phi}{\partial x_k} A_j, f \rangle_{(j,k) \in \Lambda}} \\ &\quad + \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle A_1, \ldots, A_n, \frac{\partial \phi}{\partial x_j} A_k - \frac{\partial \phi}{\partial x_k} A_j, f \rangle_{(j,k) \in \Lambda}}. \end{aligned}$$

Now, note that $\dfrac{\partial \phi}{\partial x_j} A_k - \dfrac{\partial \phi}{\partial x_k} A_j \in \langle A_1, \ldots, A_n \rangle$. With that, and using (3.3), we deduce

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T) + I_V} = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) + \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle A_1, \ldots, A_n, f \rangle}.$$

Thus, by (1.15), we have

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T) + I_V} = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) + \tau_0(\omega, V). \tag{3.10}$$

Finally, substituting (3.9), (3.10) and (3.4) in (3.6), we complete the proof of Theorem 3.1.1, since

$$\begin{aligned}
\tau_{BR}(\omega, X, V) &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T) + I_V} - \dim_{\mathbb{C}} \frac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V} \\
&= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\omega(\Theta_X^T) + I_V} - \left(\dim_{\mathbb{C}} \frac{\omega(\Theta_X)}{\omega(\Theta_X^T)} - \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V} \right) \\
&= \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) + \tau_0(\omega, V) - \dim_{\mathbb{C}} \frac{\omega(\Theta_X)}{\omega(\Theta_X^T)} + \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V} \\
&= \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) + \tau_0(\omega, V) - \tau_0(X) + \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V}.
\end{aligned}$$

□

As a consequence, we obtain the following corollary:

Corollary 3.1.2. Let $f \in \mathcal{O}_n$ be a function germ with an isolated singularity over an hypersurface $(X, 0)$, and the pair (X, V) are isolated complex analytic hypersurfaces at $0 \in \mathbb{C}^n$. Suppose that V is invariant by ω and X is not invariant by ω. Then

$$\tau_{BR}(f, X) = \mathrm{Ind}_{\mathrm{GSV}}(df; X, V, 0) + \tau_0(V) - \tau_0(X) + \dim_{\mathbb{C}} \frac{df(\Theta_X) \cap \langle f \rangle}{df(\Theta_X^T) \cap \langle f \rangle},$$

where $V = \{f = 0\} \subset (\mathbb{C}^n, 0)$.

Proof. The proof follows from applying Theorem 3.1.1 for the 1-form ω given by $\omega = df$, and using that $\tau_{BR}(df, X, V) = \tau_{BR}(f, X)$ (by (3.1)) and

$$\tau_0(df, V) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, f \right\rangle} = \tau_0(V).$$

□

We use the result of Gómez-Mont, presented in Proposition 1.2.36, to prove the next corollary:

Corollary 3.1.3. Let $\mathcal{F} : \omega = 0$ be a singular holomorphic foliation at $0 \in \mathbb{C}^2$. Suppose that $V = \{f = 0\}$ is a complex analytic curve at $0 \in \mathbb{C}^2$ invariant by ω. Then

$$\begin{aligned}
\tau_{BR}(\mathcal{F}, X, V) - \tau_{BR}(f, X) \;&=\; \mathrm{Ind}_{\mathrm{GSV}}(\mathcal{F}; X, V, 0) - \mathrm{Ind}_{\mathrm{GSV}}(df; X, V, 0) + \mathrm{GSV}_0(\mathcal{F}, V) \\
&\quad + \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap \langle f \rangle}{\omega(\Theta_X^T) \cap \langle f \rangle} - \dim_{\mathbb{C}} \frac{df(\Theta_X) \cap \langle f \rangle}{df(\Theta_X^T) \cap \langle f \rangle}.
\end{aligned}$$

Proof. According to Theorem 3.1.1 and Corollary 3.1.2, we have

$$\tau_{BR}(\mathcal{F}, X, V) = \mathrm{Ind}_{\mathrm{GSV}}(\mathcal{F}; X, V, 0) + \tau_0(\mathcal{F}, V) - \tau_0(X) + \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap \langle f \rangle}{\omega(\Theta_X^T) \cap \langle f \rangle} \quad \text{and}$$

$$\tau_{BR}(f, X) = \mathrm{Ind}_{\mathrm{GSV}}(df; X, V, 0) + \tau_0(V) - \tau_0(X) + \dim_{\mathbb{C}} \frac{df(\Theta_X) \cap \langle f \rangle}{df(\Theta_X^T) \cap \langle f \rangle}.$$

The proof is concluded subtracting the above equations and using (1.21). □

Now, we present some examples of our main result. In the next two cases, we will see that Theorem 3.1.1 is verified in dimension two and three, respectively.

Example 3.1.4. Let $X = \{\phi(x,y) = y^p - x^q = 0\}$ and $V = \{f(x,y) = xy = 0\}$, representing germs of complex analytic curves on $(\mathbb{C}^2, 0)$, and let $\mathcal{F}$ be the foliation defined $\omega = \lambda x dy + y dx$, with $\lambda \neq -\dfrac{p}{q}$, $\lambda \neq 1$. Note that X is not invariant by $\mathcal{F}$, while V is invariant by $\mathcal{F}$, since

$$\begin{aligned}\omega \wedge d\phi &= (\lambda x dy + y dx) \wedge (p y^{p-1} dy - q x^{q-1} dx) = (p y^p + \lambda q x^q)\, dx \wedge dy \quad \text{and}\\ \omega \wedge df &= (\lambda x dy + y dx) \wedge (y dx + x dy) = ((1-\lambda) xy)\, dx \wedge dy.\end{aligned}$$

Since $\lambda \neq -\dfrac{p}{q}$, $\lambda \neq 1$, we find that $\tau_0(X) = (p-1)(q-1)$ (as seen in Example 2.3.2) and

$$\tau_0(\omega, V) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle \lambda x, y, xy \rangle} = 1.$$

Next, we compute the remaining indices. First, observe that

$$\frac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V} = \frac{\langle\ (p+\lambda q)xy,\ p y^p + \lambda q x^q, xy\ \rangle}{\langle\ y(y^p - x^q),\ \lambda x(y^p - x^q),\ p y^p + \lambda q x^q, xy\ \rangle} = \frac{I}{\langle y^{p+1}, x^{q+1}\rangle + I},$$

where $I = \langle xy, p y^p + \lambda q x^q \rangle$. Since, $y^p = \alpha x^q$ on I, with $\alpha \in \mathbb{C}$, we have:

$$y^{p+1} = y \cdot y^p = y \cdot \alpha x^q \in I,$$

and similarly, $x^{q+1} \in I$. Thus, $\dim_{\mathbb{C}} \dfrac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V} = 0$. Then, from (3.9), we have

$$\dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V} = \dim_{\mathbb{C}} \frac{\omega(\Theta_X)}{\omega(\Theta_X^T)} = \tau_0(X) = (p-1)(q-1).$$

Now, for $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0)$, note that

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle y^p - x^q, p y^p + \lambda q x^q, xy \rangle}.$$

Rewriting the given ideal, we get

$$\begin{aligned}\langle y^p - x^q, p y^p + \lambda q x^q, xy \rangle &= \langle y^p - x^q, p(y^p - x^q) + p x^q + \lambda q x^q, xy \rangle\\ &= \langle y^p - x^q, (p + \lambda q) x^q, xy \rangle\\ &= \langle y^p, x^q, xy \rangle,\end{aligned}$$

and then,

$$\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle y^p - x^q, p y^p + \lambda q x^q, xy \rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle y^p, x^q, xy \rangle} = p + q - 1.$$

Finally, let's compute the Bruce-Roberts Tjurina number. From [35, Example 1], we know that

$$\Theta_X = \left\langle qy\frac{\partial}{\partial y} + px\frac{\partial}{\partial x}, py^{p-1}\frac{\partial}{\partial x} + qx^{q-1}\frac{\partial}{\partial y} \right\rangle.$$

Hence, we have

$$\tau_{BR}(\omega, X, V) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle(\lambda q + p)xy, py^p + \lambda qx^q, xy\rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle py^p + \lambda qx^q, xy\rangle} = p + q.$$

Therefore, since

$$\begin{aligned} p + q = \tau_{BR}(\omega, X, V) &= \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) + \tau_0(\omega, V) - \tau_0(X) + \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V} \\ &= (p + q - 1) + 1 - (p - 1)(q - 1) + (p - 1)(q - 1), \end{aligned}$$

we conclude that Theorem 3.1.1 is verified.

Example 3.1.5. Let $X = \{\phi(x, y, z) = x^3 + yz = 0\}$ and $V = \{f(x, y, z) = x^2 + y^2 + z^2 = 0\}$ be germs of isolated complex hypersurfaces on $(\mathbb{C}^3, 0)$. Consider

$$\omega = df + f(zdx + xdy + ydz) = (2x + zf)dx + (2y + xf)dy + (2z + yf)dz.$$

By the definition of ω, is easy to see that V is invariant by ω, and since

$$\begin{aligned} \omega \wedge d\phi &= [((2x + zf)dx + (2y + xf)dy + (2z + yf)dz)] \wedge (3x^2dx + zdy + ydz) \\ &= (2xz - 6x^2y + (z^2 - 3x^2)f)dx \wedge dy + (2y^2 - 2z^2 + (xy - yz)f)dy \wedge dz \\ &+ (2xy - 6x^2 + (yz - 3x^2y)f)dx \wedge dz, \end{aligned}$$

we have that X is not invariant by ω. Additionally, by Theorem 1.2.47, we can compute Θ_X as

$$\Theta_X = \left\langle z\frac{\partial}{\partial x} - 3x^2\frac{\partial}{\partial y},\ y\frac{\partial}{\partial x} - 3x^2\frac{\partial}{\partial z},\ y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z},\ x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right\rangle.$$

Using *Singular* [24] to compute the indices, we get $\tau_{BR}(\omega, X, V) = 5$, $\tau_0(\omega, V) = 1$, $\mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) = 5$, $\tau_0(X) = 2$ and $\dim_{\mathbb{C}} \dfrac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V} = 1$. With that and (3.9), we have

$$\dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V} = \dim_{\mathbb{C}} \frac{\omega(\Theta_X)}{\omega(\Theta_X^T)} - \dim_{\mathbb{C}} \frac{\omega(\Theta_X) + I_V}{\omega(\Theta_X^T) + I_V} = 2 - 1 = 1.$$

Hence, Theorem 3.1.1 is satisfied, since

$$\begin{aligned} 5 = \tau_{BR}(\omega, X, V) &= \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) + \tau_0(\omega, V) - \tau_0(X) + \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V} \\ &= 5 + 1 - 2 + 1. \end{aligned}$$

Remark 3.1.6. In a similar way as made in Remark 2.3.6, we show below the codes used on the software *Singular* [24] to calculate the indices in Example 3.1.5. The same codes were used along this chapter.

```
> ring r=0,(x,y,z),ds; // local ring
> poly f=x2+y2+z2; // polynomial that defines V
> poly g=x3+yz; // polynomial that defines X
> poly f1=diff(f,x); // parcial derivatives of f
> poly f2=diff(f,y); //
> poly f3=diff(f,z); //
> poly g1=diff(g,x); // parcial derivatives of g
> poly g2=diff(g,y); //
> poly g3=diff(g,z); //
> ideal I1=(2x+z*f,2y+x*f,2z+y*f,f); // defining the ideals that appear
in each index
> ideal I2=(f,g,g1*f2-g2*f1,g2*f3-g3*f2,g3*f1-g1*f3);
> ideal I3=(g,g1,g2,g3);
> ideal I4=(3x2*(2y+x*f)-z*(2x+z*f),-y*(2x+z*f)+3x2*(2z+y*f),
-y*(2y+x*f)+z*(2z+y*f),x*(2x+z*f)+2y*(2y+x*f)+z*(2z+y*f),f);

> size(kbase(groebner(I1)));
1 // Tjurina number of the foliation

> size(kbase(groebner(I2)));
5 // GSV-Index of the foliation

> size(kbase(groebner(I3)));
2 // Tjurina number of X

> size(kbase(groebner(I4)));
5 // Bruce-Roberts Tjurina number of the foliation
```

3.2 The relation between $\mu_{BR}(\omega, X)$ and $\tau_{BR}(\omega, X, V)$

The natural next step of this work is to study what results we can obtain from the definitions presented in the last two chapters. More specifically, the definitions of the Bruce-Roberts number of an 1-form and the Bruce-Roberts Tjurina number of an 1-form, defined in Definition 2.0.1 and Definition 3.0.1, respectively. It follows directly from those definitions that

$$\tau_{BR}(\omega, X, V) \leq \mu_{BR}(\omega, X).$$

Therefore, if $\mu_{BR}(\omega, X) < \infty$ then $\tau_{BR}(\omega, X, V) < \infty$.

In order to find a relation between those indices, we are going to define two new numbers, and make a construction inspired by the work of Bivià-Ausina–Kourliouros–Ruas in [8, Section 2]. For that, we need to establish some notations: consider $V = \{f = 0\}$ a germ of complex hypersurface invariant by ω. With that, we write

$$\Theta_V^\omega = \{\delta \in \Theta_n : \omega(\delta) \in \langle f \rangle\},$$

and we can also define by

$$H_\omega = \{\zeta \in \Theta_n : \omega(\zeta) = 0\}$$

the submodule of vector fields tangent to ω. Observe that $H_\omega \subset \Theta_V^\omega$.

Now, we can present the following definition, inspired by [8, Section 2, p. 5]:

Definition 3.2.1. Let $(X, 0)$ be a complex analytic subvariety, ω be a germ of a holomorphic 1-form with isolated singularity at $0 \in \mathbb{C}^n$ and V a germ of a complex hypersurface. Suppose that $\mu_{BR}(\omega, X) < \infty$ and V is invariant by ω. Then, we define

$$\overline{\mu}_X(\omega) := \dim_{\mathbb{C}} \frac{\Theta_n}{\Theta_X + H_\omega}$$

and

$$\overline{\tau}_X(\omega, V) := \dim_{\mathbb{C}} \frac{\Theta_n}{\Theta_X + \Theta_V^\omega},$$

in the case where these two numbers are finite.

Note that $\overline{\mu}_X(\omega) \geq \overline{\tau}_X(\omega, V)$, and since the sequence

$$0 \longrightarrow \frac{\Theta_X + \Theta_V^\omega}{\Theta_X + H_\omega} \longrightarrow \frac{\Theta_n}{\Theta_X + H_\omega} \longrightarrow \frac{\Theta_n}{\Theta_X + \Theta_V^\omega} \longrightarrow 0$$

is exact, by Lemma 3.0.2, we have

$$\overline{\mu}_X(\omega) - \overline{\tau}_X(\omega, V) = \dim_{\mathbb{C}} \frac{\Theta_X + \Theta_V^\omega}{\Theta_X + H_\omega} = \dim_{\mathbb{C}} \frac{\Theta_V^\omega}{H_\omega + (\Theta_X \cap \Theta_V^\omega)},$$

where the last step follows from the equivalence of Lemma 3.0.3.

The next proposition uses $\overline{\mu}_X$ and $\overline{\tau}_X$ to show new characterizations of the Bruce-Roberts number and the Bruce-Roberts Tjurina number, and then, to compute the difference between these indices.

Proposition 3.2.2. Let (ω, X) be a pair (of a germ of a holomorphic 1-form and a complex analytic subvariety) in $(\mathbb{C}^n, 0)$ with $\mu_{BR}(\omega, X) < \infty$. Suppose that V is a complex hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$ invariant by ω. Then

$$\mu_{BR}(\omega, X) = \mu_0(\omega) + \overline{\mu}_X(\omega),$$
$$\tau_{BR}(\omega, X, V) = \tau_0(\omega, V) + \overline{\tau}_X(\omega, V).$$

In particular,

$$\mu_{BR}(\omega, X) - \tau_{BR}(\omega, X, V) = \mu_0(\omega) - \tau_0(\omega, V) + \overline{\mu}_X(\omega) - \overline{\tau}_X(\omega, V).$$

Proof. Consider ω defined by $\omega = \sum_{j=1}^{n} A_j dx_j$, and then, set $\langle\omega\rangle = \langle A_1, \ldots, A_n\rangle$. Consider the following sequences of $\mathcal{O}_n$-modules:

$$0 \longrightarrow \frac{\Theta_n}{\Theta_X + H_\omega} \xrightarrow{\cdot\omega} \frac{\mathcal{O}_n}{\omega(\Theta_X)} \xrightarrow{\pi} \frac{\mathcal{O}_n}{\langle\omega\rangle} \longrightarrow 0, \tag{3.11}$$

$$0 \longrightarrow \frac{\Theta_n}{\Theta_X + \Theta_V^\omega} \xrightarrow{\cdot\omega} \frac{\mathcal{O}_n}{\omega(\Theta_X) + \langle f\rangle} \xrightarrow{\pi} \frac{\mathcal{O}_n}{\langle\omega\rangle + \langle f\rangle} \longrightarrow 0, \tag{3.12}$$

where $\cdot\omega$ is the evaluation map and π is induced by the inclusion $\omega(\Theta_X) \subseteq \langle\omega\rangle$. We claim that both of the sequences are exact.

For (3.11), suppose that $g \in \operatorname{Im}(\omega)$. Then, $g = \omega(h + \Theta_X + H_\omega)$, where $h \in \Theta_n$. With that,

$$\pi(g) = \pi(\omega(h + \Theta_X + H_\omega)) = \pi(\omega(h) + \omega(\Theta_X)) = \omega(h) + \langle\omega\rangle \in \langle\omega\rangle,$$

which gives us $g \in \ker\pi$. On the other hand, consider $j \in \ker\pi$. Then, there exists $k \in \mathcal{O}_n$, such that $j = k + \omega(\Theta_X)$. Since $\pi(j) = \pi(k + \omega(\Theta_X)) = k + \langle\omega\rangle$, then $k \in \langle\omega\rangle$. With that, we can write $k = \omega(\eta)$, for some $\eta \in \Theta_n$. Therefore,

$$\omega(\eta + \Theta_X + H_\omega) = \omega(\eta) + \omega(\Theta_X) = k + \omega(\Theta_X) = j,$$

which gives us $j \in \operatorname{Im}(\omega)$.

This shows us that $\operatorname{Im}(\omega) = \ker\pi$, and the sequence (3.11) is exact. With a similar proof, (3.12) is also an exact sequence. The exactness of (3.11) and (3.12) concludes the proof. ☐

As a consequence, we obtain an algebraic characterization of the equality of the Bruce-Roberts Milnor and Tjurina numbers.

Corollary 3.2.3. Let (ω, X) be a pair in $(\mathbb{C}^n, 0)$ with $\mu_{BR}(\omega, X) < \infty$. Suppose that V is a complex hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$ invariant by ω. Then the following conditions are equivalent:

1. $\mu_{BR}(\omega, X) = \tau_{BR}(\omega, X, V)$;
2. $\mu_0(\omega) = \tau_0(\omega, V)$ and $\overline{\mu}_X(\omega) = \overline{\tau}_X(\omega, V)$, and the last equality is equivalent to

$$\Theta_V^\omega = H_\omega + \Theta_X \cap \Theta_V^\omega.$$

In that result, when $\omega = df$, for some $f \in \mathcal{O}_n$, and $\mu_{BR}(f, X) < \infty$, we recover Proposition 1.2.63. Now, in the case that X is an isolated complex hypersurface at $0 \in \mathbb{C}^n$, we obtain the following corollary:

Corollary 3.2.4. Let (ω, X) be a pair in $(\mathbb{C}^n, 0)$ with $\mu_{BR}(\omega, X) < \infty$. Suppose that V is a complex hypersurface with an isolated singularity at $0 \in \mathbb{C}^n$ invariant by ω. Then

$$\overline{\mu}_X(\omega) = \operatorname{Ind}_{\mathrm{GSV}}(\omega; X, 0) - \tau_0(X)$$

and

$$\overline{\tau}_X(\omega, V) = \operatorname{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) - \tau_0(X) + \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap \langle f\rangle}{\omega(\Theta_X^T) \cap \langle f\rangle}.$$

Proof. From Theorem 2.1.3, we have

$$\mu_{BR}(\omega, X) = \mu_0(\omega) + \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, 0) - \tau_0(X),$$

and from Theorem 3.1.1, we have

$$\tau_{BR}(\omega, X, V) = \mathrm{Ind}_{\mathrm{GSV}}(\omega; X, V, 0) + \tau_0(\omega, V) - \tau_0(X) + \dim_{\mathbb{C}} \frac{\omega(\Theta_X) \cap I_V}{\omega(\Theta_X^T) \cap I_V}.$$

The proof is concluded by comparing both equalities above with Proposition 3.2.2. □

For our next result, we remind that, from Definition 1.2.59, the number $r_f(I)$ is the minimum of $r \in \mathbb{Z}_{\geq 1}$ such that $f^r \in I$, with R being a ring and I an ideal of R such that $f \in R$. If such r does not exist, then $r_f(I) = \infty$.

With that, it is possible to enunciate the following corollary:

Corollary 3.2.5. Let X be a complex analytic subvariety of $(\mathbb{C}^n, 0)$. Let ω be a germ of a holomorphic 1-form such that $\mu_{BR}(\omega, X) < \infty$. Suppose that $(V, 0)$ is determined by $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Then

$$\frac{\mu_{BR}(\omega, X)}{\tau_{BR}(\omega, X, V)} \leq r_f(\omega(\Theta_X)).$$

Proof. It follows directly by taking $R = \mathcal{O}_n$ and $I = \omega(\Theta_X)$ in Theorem 1.2.60:

$$\frac{\ell\left(\dfrac{R}{I}\right)}{\ell\left(\dfrac{R}{\langle f \rangle + I}\right)} \leq r_f(I),$$

with $\ell(M)$ being the length of the set M. □

We present an example that illustrates Corollary 3.2.5.

Example 3.2.6. Let $V = \{f(x, y) = x^{2m+1} + x^m y^{m+1} + y^{2m} = 0\}$ and $X = \{\phi(x, y) = xy = 0\}$ be germs of complex analytic curves at $0 \in \mathbb{C}^2$. Consider the foliation $\mathcal{F}$ at $(\mathbb{C}^2, 0)$ defined by

$$\omega = (f_x + yf)dx + (f_y + xf)dy = df + f(xdy + ydx).$$

Note that X is not invariant by $\mathcal{F}$, since

$$\omega \wedge d\phi = ((f_x + yf)dx + (f_y + xf)dy) \wedge (ydx + xdy) = (xf_x - yf_y)dx \wedge dy,$$

while V is invariant by $\mathcal{F}$, which is evident by the definition of V. Thus, since $\Theta_X = \left\langle x\dfrac{\partial}{\partial x}, y\dfrac{\partial}{\partial y}\right\rangle$, we have

$$\omega(\Theta_X) = \langle xf_x + xyf, yf_y + xyf \rangle.$$

In [7, Example 4.1], we used *Singular* [24] to compute the indices $\mu_{BR}(\omega, X)$ and $\tau_{BR}(\omega, X, V)$. For each value of m, we found the following results:

m	$\mu_{BR}(\omega, X)$	$\tau_{BR}(\omega, X, V)$	$\dfrac{\mu_{BR}(\omega, X)}{\tau_{BR}(\omega, X)}$
1	6	6	1
2	20	17	1.17647...
3	42	34	1.23529...
4	72	57	1.26315...
10	420	321	1.30841...
20	1640	1241	1.32151...
1000	4002000	3002001	1.33311...

Additionally, in any of the cases above, we observe that $f \notin \omega(\Theta_X)$, but $f^2 \in \omega(\Theta_X)$. Therefore, Corollary 3.2.5 is satisfied, as

$$\frac{\mu_{BR}(\omega, X)}{\tau_{BR}(\omega, X, V)} < \frac{4}{3} \leq 2 = r_f(\omega(\Theta_X)). \tag{3.13}$$

Furthermore, we managed to show that (3.13) is verified for each value of $m \in \mathbb{N}, m \geq 2$. In fact, considering the ideal

$$J = \omega(\Theta_X) = \langle (2m+1)x^{2m+1} + mx^m y^{m+1} + fxy, (m+1)y^{m+1}x^m + (2m)y^{2m} + fxy \rangle,$$

we have

$$\mu_{BR}(\omega, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\omega(\Theta_X)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{J} = 4m^2 + 2m$$

and

$$\tau_{BR}(\omega, X, V) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\omega(\Theta_X) + I_V} = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{J + \langle f \rangle} = 3m^2 + 3m + 1.$$

Although $\lim_{m \to \infty} \dfrac{\mu_{BR}(\omega, X)}{\tau_{BR}(\omega, X, V)} = \dfrac{4}{3}$, we still have $\dfrac{\mu_{BR}(\omega, X)}{\tau_{BR}(\omega, X, V)} < \dfrac{4}{3} \leq r_f(\omega(\Theta_X))$, and the result is satisfied.

Remark 3.2.7. In the context of singular curves in dimension two, the search for an upper bound for $\dfrac{\mu}{\tau}$ has been the study of recent works. In [51, Theorem 1.1], the author shows that $\dfrac{\mu_0(X)}{\tau_0(X)} \leq n$, when $X = f^{-1}(0)$ and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ at the origin with only isolated singularity. In particular, in the case where $V = \{f = 0\}$ is a reduced complex analytic curve, $\omega = df$, and $X = \mathbb{C}^2$, the inequality (3.13) reminds us of

$$\frac{\mu_0(V)}{\tau_0(V)} < \frac{4}{3},$$

which is exactly the *Dimca-Greuel inequality* proposed in [25, Question 4.2] and recently proved in [4, Theorem 6]. However, in the case of holomorphic foliations on $(\mathbb{C}^2, 0)$, the Dimca-Greuel inequality does not generally hold, as shown in [31, Example 4.2].

3.3 The Bruce-Roberts Tjurina Number for Foliations on $(\mathbb{C}^2, 0)$

In a similar way to Section 2.3.1, our main goal in this section is to define the Bruce-Roberts Tjurina number for foliations in dimension two, since we can define them with the notion of a germ of a holomorphic 1-form. In that case, the *Bruce-Roberts Tjurina number* of a foliation $\mathcal{F}$ relative to the pair (X, V) can be written as

$$\tau_{BR}(\mathcal{F}, X, V) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\omega(\Theta_X) + I_V},$$

where ω is a germ of a holomorphic 1-form that defines $\mathcal{F}$. This definition follows directly from Definition 3.0.1. Besides that, we are going to work with the notion of *relatively quasihomogeneous* of a germ $f \in \mathcal{O}_n$ along a variety X. This concept was defined by Bivià-Ausina–Kourliouros–Ruas in [8, Definition 1.1]:

Definition 3.3.1. A pair (f, X) in $(\mathbb{C}^n, 0)$ is called *relatively quasihomogeneous* if there exists a vector of positive rational numbers $w = (w_1, \dots, w_n) \in \mathbb{Q}^n_+$, a system of coordinates $x = (x_1, \dots, x_n)$ and a system of generators $\langle h_1, \dots, h_m \rangle = I_X$ of the ideal of functions vanishing on X, such that

$$f(x) = \sum_{\langle w,m \rangle = 1} a_m x^m, \ a_m \in \mathbb{C},$$
$$h_i(x) = \sum_{\langle w,m \rangle = d_i} b_{m,i} x^m, \ b_{m,i} \in \mathbb{C}, \ i = 1, \dots, m,$$

where each $d_i \in \mathbb{Q}_+$ is the quasihomogeneous degree of h_i.

To prove the main theorem of this section, the following proposition is essential. To that, we refer to [8, Corollary 3.4].

Proposition 3.3.2. A pair (f, X) is relatively quasihomogeneous if, and only if there exists a logarithmic vector field $\delta \in \Theta_X$, $\delta(0) = 0$, with positive rational eigenvalues, which admits f as an eigenfunction (we can always choose the eigenvalue equal to 1):

$$\delta \in \Theta_X, \delta(f) = f \text{ and } \operatorname{sp}(\delta) = (w_1, \dots, w_n) \in \mathbb{Q}^n_+,$$

where $\operatorname{sp}(\delta)$ is the set of eigenvalues of the linear part of δ.

Now, and motivated by [8, Theorem 4.1], we can present the following theorem, as an application of Theorem 3.1.1:

Theorem 3.3.3. Let $\mathcal{F}$ be a germ of a non-dicritical generalized curve foliation at $(\mathbb{C}^2, 0)$, and let X be a germ of a reduced curve at $(\mathbb{C}^2, 0)$ not invariant by $\mathcal{F}$. Let $V = \{f = 0\}$ be the reduced equation of the total set of separatrices of $\mathcal{F}$. If $\mu_{BR}(\mathcal{F}, X) = \tau_{BR}(\mathcal{F}, X, V)$, then there exist coordinates $(u, v) \in \mathbb{C}^2$, $g, h \in \mathcal{O}_2$, with $u(0) = 0$, $v(0) = 0$, $g(0) \neq 0$ and integers $\alpha, \beta, \zeta \in \mathbb{N}$ such that

$$f(u, v) = \sum_{\alpha i+\beta j=\zeta} P_{i,j} u^i v^j, \quad P_{i,j} \in \mathbb{C},$$

$$g\omega = df + h(\beta v du - \alpha u dv),$$

and the pair (f, X) is relatively quasihomogeneous in these coordinates.

Proof. Consider ω the holomorphic 1-form that defines $\mathcal{F}$, given by $\omega = Adx + Bdy$. Since $\mu_{BR}(\mathcal{F}, X) = \tau_{BR}(\mathcal{F}, X, V)$, we have $\mu_0(\omega) = \tau_0(\omega, V)$ and

$$\Theta_V^\omega = H_\omega + \Theta_X \cap \Theta_V^\omega \tag{3.14}$$

by Corollary 3.2.3. Thus,

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle A, B \rangle} = \mu_0(\omega) = \tau_0(\omega, V) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{\langle A, B, f \rangle}$$

and that implies that $f \in \langle A, B \rangle$. According to [55, Théorème A], that means that there exist coordinates $(u, v) \in (\mathbb{C}^2, 0)$, $g, h \in \mathcal{O}_2$, with $u(0) = 0$, $v(0) = 0$, $g(0) \neq 0$ and integers $\alpha, \beta, \zeta \in \mathbb{N}$ such that

$$f(u, v) = \sum_{\alpha i+\beta j=\zeta} P_{i,j} u^i v^j \tag{3.15}$$

and

$$g\omega = df + h(\beta v du - \alpha u dv). \tag{3.16}$$

Now, consider the vector field given by $X_{\alpha,\beta} = \alpha u \frac{\partial}{\partial u} + \beta v \frac{\partial}{\partial v} \in \Theta_n$. We claim that $X_{\alpha,\beta} \in \Theta_V^\omega$. In fact, (3.15) gives us that

$$df = \left(\sum_{\alpha i+\beta j=\zeta} i\ P_{i,j} u^{i-1} v^j \right) du + \left(\sum_{\alpha i+\beta j=\zeta} j\ P_{i,j} u^i v^{j-1} \right) dv,$$

and, from (3.16), we got

$$\begin{aligned} g\omega(X_{\alpha,\beta}) &= df(X_{\alpha,\beta}) + h(\alpha\beta uv - \alpha\beta uv) \\ &= \left(\sum_{\alpha i+\beta j=\zeta} i\ P_{i,j} u^{i-1} v^j \right) \alpha u + \left(\sum_{\alpha i+\beta j=\zeta} j\ P_{i,j} u^i v^{j-1} \right) \beta v \\ &= \alpha \sum_{\alpha i+\beta j=\zeta} i\ P_{i,j} u^i v^j + \beta \sum_{\alpha i+\beta j=\zeta} j\ P_{i,j} u^i v^j \\ &= \sum_{\alpha i+\beta j=\zeta} (\alpha i + \beta j)\ P_{i,j} u^i v^j = f\zeta. \end{aligned}$$

Therefore, $\omega(X_{\alpha,\beta}) \in \langle f \rangle$.

Suppose now that $X_{\alpha,\beta} \notin \Theta_X$ in these coordinates (otherwise, there is nothing to prove by Proposition 3.3.2). By (3.14), we get that there exists $\gamma \in H_\omega$ such that $\delta = X_{\alpha,\beta} - \gamma \in \Theta_X \cap \Theta_V^\omega$. Since $\omega(\gamma) = 0$, γ is a vector field defining $\mathcal{F}$, and then f is an eigenfunction for γ. Moreover, (3.15) gives us that f is also an eigenfunction for $X_{\alpha,\beta}$. Thus, f is an eigenfunction for $\delta \in \Theta_X$, and from Proposition 3.3.2, we obtain that the pair (f, X) is relatively quasihomogeneous in these coordinates. $\square$

Chapter 4

Conclusion and Further Research

In this last chapter, we will talk about the main results of this work and present some questions that we encountered in the construction of this thesis.

In Chapter 2, we define the Bruce-Roberts number of a holomorphic 1-form with an isolated singularity at $0 \in \mathbb{C}^n$, with respect to a germ of a complex analytic variety X, and also in the case where $n = 2$ and ω defines a holomorphic foliation. Although we have plenty of examples in the case of foliations, we would like to know what happens to the Bruce-Roberts number of some other families of foliations and if (or how) these numbers change.

Inspired by the work made in [45], we wonder if it is possible to have a result that relates the Bruce-Roberts number $\mu_{BR}(\omega, X)$ (or the relative Bruce-Roberts number) when we perform some deformations on the 1-form ω. For example, looking at the 1-form given in (1.20), and also motivated by Corollary 2.2.9, if we consider the foliation $\omega = d(fg) + fg\eta$ in $(\mathbb{C}^n, 0)$, with $f, g \in \mathcal{O}_n$, what results can be obtained from the difference $\mu_{BR}(\omega, X) - \mu_{BR}(fg, X)$? We know so far that in this case, the pullback of omega $\pi^*\omega$ induces a complex saddle $df = 0$, with $f = xy$, similar to the ones studied in [45], and we believe that this may possibly lead to new results.

In Chapter 3, our main work is to define the Bruce-Roberts Tjurina number of a 1-form relative to a pair (X, V), with $(X, 0)$ a germ of a complex analytic variety at $(\mathbb{C}^n, 0)$ and V a germ of a complex analytic hypersurface. Naturally, the next step is to wonder how this number can be defined in the case where X or V are an isolated complete intersection singularity - or simply an ICIS. With that kind of definition (in particular, in the case where X is an ICIS) we can proceed with a work similar to what we made in Section 3.2, comparing the numbers μ_{BR} and τ_{BR}. In the case of the Bruce-Roberts number of ω with respect to an ICIS X, some results were shown in [49], in which they used our definition given in Definition 2.0.1 (and first presented in [6]).

Right after Theorem 2.1.3, we state that $\mu_{BR}(\omega, X)$ is a topological invariant under homeomorphisms of $(\mathbb{C}^n, 0)$ that fix $(X, 0)$, which follows from the topological invariance of both $\text{Ind}_{\text{GSV}}(\omega; X, 0)$ and $\mu_0(\omega)$. That result was also illustrated in Examples 2.3.4 and 2.3.5. Naturally, we wonder if the Bruce-Roberts Tjurina number $\tau_{BR}(\omega, X, V)$ is also an invariant. We guess that it is better to approach the analytical invariance instead of the topological invariance, since the number in question is a generalization of the Tjurina number, but more investigations in that line are still required.

In Corollary 3.2.5 , we present a relation between the Bruce-Roberts and the Bruce-Roberts Tjurina numbers of a 1-form with respect to a curve X (and V, in the case of Tjurina), which says that their quotient is limited by the number $r_f(\omega(\Theta_X)$. Naturally, a question that arises is whether it is possible to have an upper bound that does not depend of X, or V, as mentioned in Remark 3.2.7, since

$$\frac{\mu_0(V)}{\tau_0(V)} < \frac{4}{3}.$$

That result, named the *Dimca-Greuel inequality*, answers this question when it comes to a reduced complex analytic curve. The attempt to reach a similar result gave us Example 3.2.6 that, coincidentally or not, limited $\dfrac{\mu_{BR}(\omega, X)}{\tau_{BR}(\omega, X, V)}$ by $\dfrac{4}{3}$ too. The idea is to investigate in what other cases we can have the same (or lower) upper bound, making changes not only on X, or V, but maybe doing some deformations in ω as well.

Bibliography

[1] Ahmed, Imran. "Weighted homogeneous polynomials with isomorphic Milnor algebras". In: *J. Prime Res. Math.* 8 (2012), pp. 106–114. ISSN: 1817-2725.

[2] Ahmed, Imran and Ruas, Maria Aparecida Soares. "Invariants of relative right and contact equivalences". In: *Houston J. Math.* 37.3 (2011), pp. 773–786. ISSN: 0362-1588.

[3] Ahmed, Imran, Ruas, Maria Aparecida Soares, and Tomazella, João Nivaldo. "Invariants of topological relative right equivalences". In: *Math. Proc. Cambridge Philos. Soc.* 155.2 (2013), pp. 307–315. ISSN: 0305-0041,1469-8064. DOI: 10.1017/S0305004113000297. URL: https://doi.org/10.1017/S0305004113000297.

[4] Almirón, Patricio. "On the quotient of Milnor and Tjurina numbers for two-dimensional isolated hypersurface singularities". In: *Math. Nachr.* 295.7 (2022), pp. 1254–1263. ISSN: 0025-584X,1522-2616.

[5] Atiyah, M. F. and Macdonald, I. G. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969, pp. ix+128.

[6] Barbosa, Pedro, Fernández-Pérez, Arturo, and León, Víctor. *The Bruce-Roberts number of holomorphic 1-forms along complex analytic varieties*. 2024. arXiv: 2409.01237 [math.CV]. URL: https://arxiv.org/abs/2409.01237.

[7] Barbosa, Pedro, Fernández-Pérez, Arturo, and León, Víctor. *The Bruce-Roberts Tjurina number of holomorphic 1-forms along complex analytic varieties*. 2024. arXiv: 2409.19814 [math.CV]. URL: https://arxiv.org/abs/2409.19814.

[8] Bivià-Ausina, C., Kourliouros, K., and Ruas, M. A. S. "Bruce-Roberts numbers and quasihomogeneous functions on analytic varieties". In: *Res. Math. Sci.* 11.3 (2024), Paper No. 46, 23. ISSN: 2522-0144,2197-9847. DOI: 10.1007/s40687-024-00458-7. URL: https://doi.org/10.1007/s40687-024-00458-7.

[9] Bivià-Ausina, Carles, Kourliouros, Konstantinos, and Ruas, Maria Aparecida Soares. *Modules of derivations, logarithmic ideals and singularities of maps on analytic varieties*. 2024. arXiv: 2407.02947 [math.AG]. URL: https://arxiv.org/abs/2407.02947.

[10] Bivià-Ausina, Carles and Ruas, Maria Aparecida Soares. “Mixed Bruce-Roberts numbers”. In: *Proc. Edinb. Math. Soc. (2)* 63.2 (2020), pp. 456–474. ISSN: 0013-0915,1464-3839. DOI: 10.1017/s0013091519000543. URL: https://doi.org/10.1017/s0013091519000543.

[11] Brasselet, Jean-Paul, Seade, José, and Suwa, Tatsuo. *Vector fields on singular varieties*. Vol. 1987. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2009, pp. xx+225. ISBN: 978-3-642-05204-0. DOI: 10.1007/978-3-642-05205-7. URL: https://doi.org/10.1007/978-3-642-05205-7.

[12] Brieskorn, E. and Greuel, G.-M. “Singularities of complete intersections”. In: *Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)*. Math. Soc. Japan, Tokyo, 1975, pp. 123–129.

[13] Bruce, J. W. and Roberts, R. M. “Critical points of functions on analytic varieties”. In: *Topology* 27.1 (1988), pp. 57–90. ISSN: 0040-9383. DOI: 10.1016/0040-9383(88)90007-9. URL: https://doi.org/10.1016/0040-9383(88)90007-9.

[14] Brunella, Marco. *Birational geometry of foliations*. Vol. 1. IMPA Monographs. Springer, Cham, 2015, pp. xiv+130. ISBN: 978-3-319-14309-5; 978-3-319-14310-1. DOI: 10.1007/978-3-319-14310-1. URL: https://doi.org/10.1007/978-3-319-14310-1.

[15] Brunella, Marco. “Some remarks on indices of holomorphic vector fields”. In: *Publ. Mat.* 41.2 (1997), pp. 527–544. ISSN: 0214-1493,2014-4350. DOI: 10.5565/PUBLMAT_41297_17. URL: https://doi.org/10.5565/PUBLMAT_41297_17.

[16] Callejas-Bedregal, Roberto, Morgado, Michelle F. Z., and Seade, José. “Milnor number and Chern classes for singular varieties: an introduction”. In: *Handbook of geometry and topology of singularities III*. Springer, Cham, [2022] ©2022, pp. 493–564. ISBN: 978-3-030-95759-9; 978-3-030-95760-5. DOI: 10.1007/978-3-030-95760-5_7. URL: https://doi.org/10.1007/978-3-030-95760-5_7.

[17] Camacho, César, Lins Neto, Alcides, and Sad, Paulo. “Topological invariants and equidesingularization for holomorphic vector fields”. In: *J. Differential Geom.* 20.1 (1984), pp. 143–174. ISSN: 0022-040X,1945-743X. URL: http://projecteuclid.org/euclid.jdg/1214438995.

[18] Camacho, César and Sad, Paulo. “Invariant varieties through singularities of holomorphic vector fields”. In: *Ann. of Math. (2)* 115.3 (1982), pp. 579–595. ISSN: 0003-486X. DOI: 10.2307/2007013. URL: https://doi.org/10.2307/2007013.

[19] Cano, Felipe, Cerveau, Dominique, and Déserti, Julie. *Théorie élémentaire des feuilletages holomorphes singuliers*. Echelles. Belin, 2013, 208 p. URL: https://hal.science/hal-00805694.

[20] Cano, Felipe, Corral, Nuria, and Mol, Rogério. "Local polar invariants for plane singular foliations". In: *Expo. Math.* 37.2 (2019), pp. 145–164. ISSN: 0723-0869,1878-0792. DOI: 10.1016/j.exmath.2018.01.003. URL: https://doi.org/10.1016/j.exmath.2018.01.003.

[21] Cerveau, D. and Mattei, J.-F. *Formes intégrables holomorphes singulières*. Vol. 97. Astérisque. With an English summary. Société Mathématique de France, Paris, 1982, p. 193.

[22] Cisneros Molina, José Luis, Lê, Dũng Tráng, and Seade, José, eds. *Handbook of geometry and topology of singularities. I.* Springer, Cham, [2020] ©2020, pp. xviii+601. ISBN: 978-3-030-53060-0; 978-3-030-53061-7. DOI: 10.1007/978-3-030-53061-7. URL: https://doi.org/10.1007/978-3-030-53061-7.

[23] De Góes Grulha Jr., Nivaldo. "The Euler obstruction and Bruce-Roberts' Milnor number". In: *Q. J. Math.* 60.3 (2009), pp. 291–302. ISSN: 0033-5606,1464-3847. DOI: 10.1093/qmath/han011. URL: https://doi.org/10.1093/qmath/han011.

[24] Decker, Wolfram et al. SINGULAR *4-4-0 — A computer algebra system for polynomial computations*. http://www.singular.uni-kl.de. 2024.

[25] Dimca, Alexandru and Greuel, Gert-Martin. "On 1-forms on isolated complete intersection curve singularities". In: *J. Singul.* 18 (2018), pp. 114–118. ISSN: 1949-2006. DOI: DOI:10.5427/jsing.2018.18h. URL: http://dx.doi.org/10.5427/jsing.2018.18h.

[26] Ebeling, W. and Gusein-Zade, S. M. "Indices of 1-forms on an isolated complete intersection singularity". In: vol. 3. 2. Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday. 2003, pp. 439–455, 742–743. DOI: 10.17323/1609-4514-2003-3-2-439-455. URL: https://doi.org/10.17323/1609-4514-2003-3-2-439-455.

[27] Ebeling, W., Gusein-Zade, S. M., and Seade, J. "Homological index for 1-forms and a Milnor number for isolated singularities". In: *Internat. J. Math.* 15.9 (2004), pp. 895–905. ISSN: 0129-167X,1793-6519. DOI: 10.1142/S0129167X04002624. URL: https://doi.org/10.1142/S0129167X04002624.

[28] Ebeling, Wolfgang and Gusein-Zade, Sabir M. "Indices of vector fields and 1-forms". In: *Handbook of geometry and topology of singularities IV*. Springer, Cham, [2023] ©2023, pp. 251–305. ISBN: 978-3-031-31924-2; 978-3-031-31925-9. DOI: 10.1007/978-3-031-31925-9_5. URL: https://doi.org/10.1007/978-3-031-31925-9_5.

[29] Essen, Arno van den. "Reduction of singularities of the differential equation $Ady = Bdx$". In: *Équations différentielles et systèmes de Pfaff dans le champ complexe (Sem., Inst. Rech. Math. Avancée, Strasbourg, 1975)*. Vol. 712. Lecture Notes in Math. Springer, Berlin, 1979, pp. 44–59. ISBN: 3-540-09250-1.

[30] Fernández-Pérez, Arturo, Costa, Gilcione Nonato, and Rosas Bazán, Rudy. "On the Milnor number of non-isolated singularities of holomorphic foliations and its topological invariance". In: *J. Topol.* 16.1 (2023), pp. 176–191. ISSN: 1753-8416,1753-8424. DOI: 10.1112/topo.12281. URL: https://doi.org/10.1112/topo.12281.

[31] Fernández-Pérez, Arturo, García Barroso, Evelia R., and Saravia-Molina, Nancy. "An upper bound for the GSV-index of a foliation". In: *Rend. Circ. Mat. Palermo (2)* 74.3 (2025), Paper No. 95, 8. ISSN: 0009-725X,1973-4409. DOI: 10.1007/s12215-025-01215-7. URL: https://doi.org/10.1007/s12215-025-01215-7.

[32] Fernández-Pérez, Arturo, García Barroso, Evelia R., and Saravia-Molina, Nancy. "On Milnor and Tjurina Numbers of Foliations". In: *Bull. Braz. Math. Soc. (N.S.)* 56.2 (2025), Paper No. 23. ISSN: 1678-7544,1678-7714. DOI: 10.1007/s00574-025-00447-6. URL: https://doi.org/10.1007/s00574-025-00447-6.

[33] Fulton, William. *Algebraic curves*. Advanced Book Classics. An introduction to algebraic geometry, Notes written with the collaboration of Richard Weiss, Reprint of 1969 original. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989, pp. xxii+226. ISBN: 0-201-51010-3.

[34] García Barroso, Evelia R. and Płoski, Arkadiusz. "On the Milnor formula in arbitrary characteristic". In: *Singularities, algebraic geometry, commutative algebra, and related topics*. Springer, Cham, 2018, pp. 119–133. ISBN: 978-3-319-96826-1; 978-3-319-96827-8.

[35] Genzmer, Y. and Hernandes, M. E. "On the Saito basis and the Tjurina number for plane branches". In: *Trans. Amer. Math. Soc.* 373.5 (2020), pp. 3693–3707. ISSN: 0002-9947,1088-6850. DOI: 10.1090/tran/8019. URL: https://doi.org/10.1090/tran/8019.

[36] Genzmer, Yohann and Mol, Rogério. "Local polar invariants and the Poincaré problem in the dicritical case". In: *J. Math. Soc. Japan* 70.4 (2018), pp. 1419–1451. ISSN: 0025-5645,1881-1167. DOI: 10.2969/jmsj/76227622. URL: https://doi.org/10.2969/jmsj/76227622.

[37] Gómez-Mont, X., Seade, J., and Verjovsky, A. "The index of a holomorphic flow with an isolated singularity". In: *Math. Ann.* 291.4 (1991), pp. 737–751. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01445237. URL: https://doi.org/10.1007/BF01445237.

[38] Gómez-Mont, Xavier. "An algebraic formula for the index of a vector field on a hypersurface with an isolated singularity". In: *J. Algebraic Geom.* 7.4 (1998), pp. 731–752. ISSN: 1056-3911,1534-7486.

[39] Greuel, G.-M., Lossen, C., and Shustin, E. *Introduction to singularities and deformations*. Springer Monographs in Mathematics. Springer, Berlin, 2007, pp. xii+471. ISBN: 978-3-540-28380-5; 3-540-28380-3.

[40] Greuel, Gert-Martin. "Deformation and smoothing of singularities". In: *Handbook of geometry and topology of singularities. I.* Springer, Cham, [2020] ©2020, pp. 389–448. ISBN: 978-3-030-53060-0; 978-3-030-53061-7. DOI: 10.1007/978-3-030-53061-7_7. URL: https://doi.org/10.1007/978-3-030-53061-7_7.

[41] Griffiths, Phillip and Harris, Joseph. *Principles of algebraic geometry.* Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978, pp. xii+813. ISBN: 0-471-32792-1.

[42] Guseĭn-Zade, S. M. and Èbeling, V. "On the index of a holomorphic 1-form on an isolated complete intersection singularity". In: *Dokl. Akad. Nauk* 380.4 (2001), pp. 458–461. ISSN: 0869-5652.

[43] Hamm, H. A. "Topology of isolated singularities of complex spaces". In: *Proceedings of Liverpool Singularities Symposium, II (1969/1970).* Vol. Vol. 209. Lecture Notes in Math. Springer, Berlin-New York, 1971, pp. 213–217.

[44] Kourliouros, Konstantinos. "The Milnor-Palamodov theorem for functions on isolated hypersurface singularities". In: *Bull. Braz. Math. Soc. (N.S.)* 52.2 (2021), pp. 405–413. ISSN: 1678-7544,1678-7714. DOI: 10.1007/s00574-020-00209-6. URL: https://doi.org/10.1007/s00574-020-00209-6.

[45] León, V. and Scárdua, B. "On integral conditions for the existence of first integrals for analytic deformations of complex saddle singularities". In: *J. Dyn. Control Syst.* 29.4 (2023), pp. 1187–1201. ISSN: 1079-2724,1573-8698. DOI: 10.1007/s10883-022-09612-2. URL: https://doi.org/10.1007/s10883-022-09612-2.

[46] Licanic, Sergio. "An upper bound for the total sum of the Baum-Bott indexes of a holomorphic foliation and the Poincaré problem". In: *Hokkaido Math. J.* 33.3 (2004), pp. 525–538. ISSN: 0385-4035. DOI: 10.14492/hokmj/1285851908. URL: https://doi.org/10.14492/hokmj/1285851908.

[47] Lima-Pereira, B. K. et al. "The Bruce-Roberts numbers of a function on an ICIS". In: *Q. J. Math.* 75.1 (2024), pp. 31–50. ISSN: 0033-5606,1464-3847. DOI: 10.1093/qmath/haad038. URL: https://doi.org/10.1093/qmath/haad038.

[48] Lima-Pereira, B. K. et al. "The relative Bruce-Roberts number of a function on a hypersurface". In: *Proc. Edinb. Math. Soc. (2)* 64.3 (2021), pp. 662–674. ISSN: 0013-0915,1464-3839. DOI: 10.1017/S0013091521000432. URL: https://doi.org/10.1017/S0013091521000432.

[49] Lima-Pereira, Bárbara K. et al. *The Bruce-Roberts Numbers of 1-Forms on an ICIS.* 2024. arXiv: 2409.08380 [math.AG]. URL: https://arxiv.org/abs/2409.08380.

[50] Lins Neto, Alcides and Scárdua, Bruno. *Complex algebraic foliations.* Vol. 67. De Gruyter Expositions in Mathematics. De Gruyter, Berlin, [2020] ©2020, pp. viii+241. ISBN: 978-3-11-060205-0; 978-3-11-060107-7; 978-3-11-059451-5. URL: https://doi.org/10.1515/9783110602050-201.

[51] Liu, Yongqiang. "Milnor and Tjurina numbers for a hypersurface germ with isolated singularity". In: *C. R. Math. Acad. Sci. Paris* 356.9 (2018), pp. 963–966. ISSN: 1631-073X,1778-3569. DOI: 10.1016/j.crma.2018.07.004. URL: https://doi.org/10.1016/j.crma.2018.07.004.

[52] MacPherson, R. D. "Chern classes for singular algebraic varieties". In: *Ann. of Math. (2)* 100 (1974), pp. 423–432. ISSN: 0003-486X. DOI: 10.2307/1971080. URL: https://doi.org/10.2307/1971080.

[53] Martinet, Jean and Ramis, Jean-Pierre. "Problèmes de modules pour des équations différentielles non linéaires du premier ordre". In: *Inst. Hautes Études Sci. Publ. Math.* 55 (1982), pp. 63–164. ISSN: 0073-8301,1618-1913. URL: http://www.numdam.org/item?id=PMIHES_1982__55__63_0.

[54] Mattei, J.-F. and Moussu, R. "Holonomie et intégrales premières". In: *Ann. Sci. École Norm. Sup. (4)* 13.4 (1980), pp. 469–523. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1980_4_13_4_469_0.

[55] Mattei, Jean-François. "Quasi-homogénéité et équiréductibilité de feuilletages holomorphes en dimension deux". In: 261. Géométrie complexe et systèmes dynamiques (Orsay, 1995). 2000, pp. xix, 253–276.

[56] Milnor, John. *Singular points of complex hypersurfaces*. Vol. No. 61. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1968, pp. iii+122.

[57] Nabeshima, Katsusuke and Tajima, Shinichi. "A new algorithm for computing logarithmic vector fields along an isolated singularity and Bruce-Roberts Milnor ideals". In: *J. Symbolic Comput.* 107 (2021), pp. 190–208. ISSN: 0747-7171,1095-855X. DOI: 10.1016/j.jsc.2021.03.003. URL: https://doi.org/10.1016/j.jsc.2021.03.003.

[58] Nuño-Ballesteros, J. J., Oréfice, B., and Tomazella, J. N. "The Bruce-Roberts number of a function on a weighted homogeneous hypersurface". In: *Q. J. Math.* 64.1 (2013), pp. 269–280. ISSN: 0033-5606,1464-3847. DOI: 10.1093/qmath/har032. URL: https://doi.org/10.1093/qmath/har032.

[59] Nuño-Ballesteros, J. J. et al. "The Bruce-Roberts number of a function on a hypersurface with isolated singularity". In: *Q. J. Math.* 71.3 (2020), pp. 1049–1063. ISSN: 0033-5606,1464-3847. DOI: 10.1093/qmathj/haaa015. URL: https://doi.org/10.1093/qmathj/haaa015.

[60] Omena, Raphael de. *Bruce-Roberts numbers and indices of vector fields on an ICIS*. 2024. arXiv: 2403.20226 [math.AG]. URL: https://arxiv.org/abs/2403.20226.

[61] Plessis, A. A. du and Wall, C. T. C. "Application of the theory of the discriminant to highly singular plane curves". In: *Math. Proc. Cambridge Philos. Soc.* 126.2 (1999), pp. 259–266. ISSN: 0305-0041,1469-8064. DOI: 10.1017/S0305004198003302. URL: https://doi.org/10.1017/S0305004198003302.

[62] Saito, Kyoji. "Theory of logarithmic differential forms and logarithmic vector fields". In: *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 27.2 (1980), pp. 265–291. ISSN: 0040-8980.

[63] Scárdua, Bruno. *Holomorphic foliations with singularities—key concepts and modern results*. Latin American Mathematics Series. Springer, Cham, [2021] ©2021, pp. xi+167. ISBN: 978-3-030-76704-4; 978-3-030-76705-1. DOI: 10.1007/978-3-030-76705-1. URL: https://doi.org/10.1007/978-3-030-76705-1.

[64] Seidenberg, A. "Reduction of singularities of the differential equation $A\,dy = B\,dx$". In: *Amer. J. Math.* 90 (1968), pp. 248–269. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2373435. URL: https://doi.org/10.2307/2373435.

[65] Tajima, Shinichi. "On polar varieties, logarithmic vector fields and holonomic D-modules". In: *Recent development of micro-local analysis for the theory of asymptotic analysis*. Vol. B40. RIMS Kôkyûroku Bessatsu. Res. Inst. Math. Sci. (RIMS), Kyoto, 2013, pp. 41–51.

[66] Teissier, Bernard. "Cycles évanescents, sections planes et conditions de Whitney". In: *Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972)*. Vol. Nos. 7 et 8. Astérisque. Soc. Math. France, Paris, 1973, pp. 285–362.

[67] Tjurina, G. N. "Locally semi-universal flat deformations of isolated singularities of complex spaces". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 33 (1969), pp. 1026–1058. ISSN: 0373-2436.

[68] Tráng, Lê Dũng. "Computation of the Milnor number of an isolated singularity of a complete intersection". In: *Funkcional. Anal. i Priložen.* 8.2 (1974), pp. 45–49. ISSN: 0374-1990.

[69] Tráng, Lê Dũng and Ramanujam, C. P. "The invariance of Milnor's number implies the invariance of the topological type". In: *Amer. J. Math.* 98.1 (1976), pp. 67–78. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2373614. URL: https://doi.org/10.2307/2373614.

[70] Trotman, David. "Stratification theory". In: *Handbook of geometry and topology of singularities. I.* Springer, Cham, [2020] ©2020, pp. 243–273. ISBN: 978-3-030-53060-0; 978-3-030-53061-7. DOI: 10.1007/978-3-030-53061-7_4. URL: https://doi.org/10.1007/978-3-030-53061-7_4.

[71] Wall, C. T. C. *Singular points of plane curves*. Vol. 63. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2004, pp. xii+370. ISBN: 0-521-83904-1; 0-521-54774-1. DOI: 10.1017/CBO9780511617560. URL: https://doi.org/10.1017/CBO9780511617560.

[72] Wang, Zhenjian. "Monotonic invariants under blowups". In: *Internat. J. Math.* 31.12 (2020), pp. 2050093, 14. ISSN: 0129-167X,1793-6519. URL: https://doi.org/10.1142/S0129167X20500937.