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Critical Exponent Inequalities For Bernoulli Percolation

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Critical Exponent Inequalities For Bernoulli Percolation

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Resumo

Nesta dissertação, analisamos técnicas para estabelecer cotas e relações entre expoentes críticos na teoria da percolação. Consideramos o modelo de Percolação de Bernoulli na rede hipercúbica $\mathbb{Z}^d$, explorando três abordagens distintas que fornecem desigualdades diferenciais para funções relevantes do modelo. A primeira baseia-se no estudo do comprimento de correlação fundamental, examinando sua conexão com cruzamentos de caixas. A segunda envolve o estudo da construção do vértice fantasma de Aizenman-Barsky. A última técnica apresentada utiliza a desigualdade OSSS generalizada, com uma aplicação da teoria de algoritmos aleatórios e árvores de decisão.

Palavras-chave: percolação; expoentes críticos; transição de fase.

Abstract

In this dissertation, we analyze techniques for establishing bounds and relations between critical exponents in percolation theory. We consider the Bernoulli Percolation model on the hypercubic lattice $\mathbb{Z}^d$, exploring three distinct approaches that provide differential inequalities for relevant functions of the model. The first is based on the study of the fundamental correlation length, examining its connection to crossings of boxes. The second involves the study of the Aizenman-Barsky ghost vertex construction. The final technique presented is based on the generalized OSSS inequality, with an application of random algorithm theory and decision trees.

Keywords: percolation; critical exponents; phase transition.

Contents

Introduction

In probability theory, the percolation model serves as a rich source of intriguing problems that are simple to state but challenging to solve. Informally, the model involves removing bonds (or vertices) from an infinite graph according to a prescribed probabilistic rule that depends on a parameter. The percolation model originated in the 1950s, with roots in mathematical physics, and it remains an active area of research today. Because of its simplicity, the model can be a natural starting point for studying systems with phase transitions.

Our primary interest lies in describing how the geometry of the random subgraph generated by this process evolves as the parameter changes. A key property of the resulting graph is the presence or absence of an infinite connected component. The transition between these two distinct phases is known as the phase transition of the model. In many percolation models, there is a specific value of the parameter that determines whether an infinite connected component exists, called the critical parameter. However, in some cases, the behavior of the system near or at this critical parameter is still not fully understood.

Certain important functions used to describe the geometry of the system present interesting behavior near this point. Conjectures from the physics literature suggest that each of these functions follows a power-law behavior in terms of the parameter, with the corresponding exponent referred to as the critical exponent of the function. The existence of such powers is a very strong statement; even weaker conjectures, derivable from the existence of these exponents, remain unproven. Furthermore, it is believed that the values of these critical exponents, as well as the nature of the phase transition, depend only on the dimension of the lattice. For instance, in sufficiently high dimensions, the existence of critical exponents has been rigorously established and is well understood. In contrast, for low dimensions, their existence remains unproven.

In this dissertation, we focus on studying certain bounds and relations that critical exponents would satisfy, assuming their existence. These bounds and relations are derived using three distinct techniques, which are applicable to understanding the behavior of the percolation process as it approaches the critical parameter. Although these methods differ fundamentally, they share a common feature: the use of differential inequalities to establish critical bounds. To keep the exposition clear and focused, we restrict our attention to the simplest version of the model, the Bernoulli Percolation on the bonds of the hypercubic lattice $\mathbb{Z}^d$. This choice allows us to present the main results in a more direct and accessible manner, while also providing a clear and concise overview of the theory and its development. Across this work, we summarize these three techniques and present eight key relations concerning critical exponents.

In the first chapter, the model, terminology, and basic results used throughout the text are introduced. The BK and FKG inequalities are stated without proofs, while Russo's formula is both presented and proved. These tools are applied multiple times in the subsequent chapters. Additionally, we introduce the main functions studied in this dissertation and formally define the critical exponents, providing an introductory background to the development of this theory. This chapter is mainly based on two references: *Percolation* by G. Grimmett [8] and *Progress in High-Dimensional Percolation and Random Graphs* by M. Heydenreich and R. van der Hofstad [9].

In the second chapter, we introduce the first technique and the first result. We focus on the study of a function called *the fundamental correlation length*, exploring its equivalent definitions, establishing some basic properties, and investigating its connection to the existence of box-crossings. Specifically, we discuss box-crossings in the supercritical phase for the two-dimensional process and in the subcritical phase for higher dimensions. The primary references for this chapter are Chapters 2 and 3 of *Independent and Dependent Percolation* by Chayes, Puha, and Sweet [3], as well as Chapter 6 of *Percolation* by Grimmett [8].

The third chapter is dedicated to the Aizenman-Barsky ghost vertex construction and its implications. This method introduces a two-parameter double percolation process, adding additional structure to facilitate the analysis of the original model. Within this framework, we establish two key differential inequalities that play a crucial role in deriving bounds for critical exponents. The presentation in this chapter is influenced by the approaches of Grimmett [8] (Chapters 5 and 10), and Heydenreich and van der Hofstad [9] (Chapters 3 and 4), as well as the original work of Aizenman and Barsky [1].

The fourth and final chapter introduces the third method: the generalization of the OSSS inequality. In this chapter, the ghost vertex construction is revisited through the lens of random algorithm theory, leading to the derivation of a differential inequality for the percolation model. This result, originally proved by Hutchcroft in [10] for the Random-Cluster model, is translated here to the Bernoulli Percolation model on the edges of a locally finite transitive graph. The differential inequalities rely on an application of the generalized OSSS inequality established by Duminil-Copin, Raoufi, and Tassion [5].

In this work, we do not present any new theorems or results. All the theorems and propositions discussed here have already been established by their respective authors. The primary objective of this dissertation is to provide a comprehensive and accessible overview of key results in percolation theory concerning critical exponents. What distinguishes this text from existing literature is its combination of conciseness and the integration of both classical and modern approaches. Efforts have been made to unify the notation and terminology from various sources, along with adjustments to some proofs. Additionally, we have reorganized and simplified elements from existing works to create a more cohesive connection between the different references.

To be more precise, we now highlight some of the main contributions of this text. In Chapter 1, while no major differences from the existing literature are introduced, we have unified the notation and provided a concise collection of the tools necessary for the subsequent chapters. In Chapter 2, we formalized the definitions of the Dual Space (Definition 2.9) and Dual Crossing (Definition 2.11), as well as the two rescaling lemmas (Lemma 2.13) and (Lemma 2.14), which are informally presented in the main reference. Additionally, the proof of Theorem 2.18 has been rewritten to emphasize its key steps, which were omitted in the original work.

In Chapter 3, we adjusted some terminology and highlighted specific steps in the proofs from the original works. Moreover, we provided a new and more direct proof for Lemma 3.15, which concerns an asymptotic result for the critical exponent δ, avoiding reliance on advanced theorems from real analysis. Finally, in Chapter 4, the most significant contribution lies in translating the main theorems from the Random-Cluster model to Percolation. Some constants were simplified, and the terminology was adjusted to align with the rest of the text, particularly with Chapter 3, ensuring a seamless connection across the chapters.

Chapter 1

Preliminaries

This chapter introduces the model studied throughout this text, establishing the necessary terminology and tools. The goal of this work is to investigate a specific topic: the relationships between critical exponents and the different techniques used to prove them.

To achieve this, we aim to find the shortest path to these bounds. It is believed that all percolation processes on d-dimensional lattices with finite vertex degrees share a similar nature for their phase transitions, meaning that the critical exponents are "universal", depending only on the number d of dimensions and not on the specific structure of the lattice.

This universality motivates us to focus on a simpler model, Bernoulli Percolation on the hypercubic lattice, which simplifies the process of defining objects while highlighting the differences between the techniques presented.

The first section provides the definition of the model, introduces relevant graph theory terminology, and outlines some results used later in the text. The final section presents the background for the critical bounds proved in this work and discusses how these results relate to other models and contexts.

The primary reference for this introduction to the model is the well-known book *Percolation* by G. Grimmett [8]. We also frequently rely on the book *Progress in High-Dimensional Percolation and Random Graphs* by M. Heydenreich and R. van der Hofstad [9].

1.1 Bernoulli Percolation

First, we define the graph we use throughout the text, the hypercubic lattice with dimension $d \geq 2$. This graph has the set $\mathbb{Z}^d$ as its set of vertices, which has elements x of the form $x = (x_1, ..., x_d)$ where $x_i \in \mathbb{Z}$ for every $i = 1, ..., d$.

On $\mathbb{Z}^d$ we consider a distance D from two vertices x and y as $D(x, y) =$

$\sum_{i=1}^{d}|x_i - y_i|$. Then, we define the set of edges $\mathbb{E}^d$ to be the set with elements $e = xy$, where $x, y \in \mathbb{Z}^d$ have $D(x, y) = 1$. That way, the hypercubic lattice denoted here as $\mathbb{L}^d$ is the graph $(\mathbb{Z}^d, \mathbb{E}^d)$.

The model we construct on the graph $\mathbb{L}^d$ is Bernoulli Percolation on its edges with parameter $p \in [0, 1]$. The configuration space is $\Omega = \{0, 1\}^{\mathbb{E}^d}$ where each element $\omega = (\omega_e)_{e\in\mathbb{E}^d}$ assigns a state (0 or 1) to each edge. This space is equipped with the sigma-algebra $\mathcal{F}$ generated by the cylindrical events. For a given p, the product probability measure P_p independently assigns to each edge a Bernoulli random variable with parameter p, meaning $P_p(\omega_e = 1) = p$ for all $e \in \mathbb{E}^d$. Thus, the probability spaces we work with are $(\Omega, \mathcal{F}, P_p)$ where $p \in [0, 1]$.

For a configuration $\omega \in \Omega$, we say that an edge $e \in \mathbb{E}^d$ is *open* if $\omega_e = 1$ and *closed* if $\omega_e = 0$. We say that two vertices x and y are *connected* in a configuration ω, or simply denote $x \leftrightarrow y$, if there exists a finite *path*, that is, a finite sequence of vertices $(v_0, v_1, ..., v_m)$ where $v_0 = x$ and $v_m = y$, such that each pair v_iv_{i+1} is an edge of $\mathbb{E}^d$ and is open for each $i = 0, ..., m - 1$.

We define the event $\{x \leftrightarrow y\}$ as the set of configurations ω such that x and y are connected. The connection probability is denoted by $\tau_p(x, y) := P_p(x \leftrightarrow y)$. We write e_k for the canonical vertex $(0, ..., 0, 1, 0, ...0)$, where the 1 appears in the k-th coordinate. Then, we simply write $\tau_n(p) := \tau_p(0, ne_1) = P_p(0 \leftrightarrow ne_1)$, that is, the probability that exists a path of open edges joining the origin to the vertex $(n, 0, 0, ..., 0)$ of the hypercubic lattice.

For a subset of vertices V, we write $x \leftrightarrow V$ if $x \leftrightarrow v$ for some $v \in V$. By convention, we assume $x \leftrightarrow x$ for every vertex x of $\mathbb{L}^d$. If S is a subset of vertices (or a subgraph) containing x, y and V, we write $\{x \overset{S}{\leftrightarrow} y\}$ or $\{x \overset{S}{\leftrightarrow} V\}$ for the events where the x is connected to y, or to V respectively, by a path using only edges with both endpoints on S.

We also use the notation $x \leftrightarrow \infty$ if there exist an infinite path, that is, an infinite sequence $\rho = (v_0, v_1, ...)$ starting at $v_0 = x$, such that each pair v_iv_{i+1} is an edge of $\mathbb{E}^d$ and is open for each $i \geq 0$.

Definition 1.1. ***[Cluster of a vertex]*** *For a configuration $\omega \in \Omega$ and a vertex $x \in \mathbb{Z}^d$, the cluster of x, denoted by $C(x) = C_\omega(x)$, is defined as the set of vertices $y \in \mathbb{Z}^d$ that are connected to x in ω, i.e.,*

$$C(x) := \{y \in \mathbb{Z}^d; x \leftrightarrow y\}.$$

Notice that, by our convention, $x \in C(x)$, so $|C(x)| \geq 1$ for all $x \in \mathbb{Z}^d$, where $|A|$ denotes the number of elements in the set A. We now define the *percolation function* as follows.

Definition 1.2. ***[Percolation function]*** *Let $(\Omega, \mathcal{F}, P_p)$ be the percolation* *model defined earlier. The percolation function* $\theta : [0,1] \to [0,1]$ *is defined by*

$$\theta(p) := P_p(|C(0)| = \infty).$$

With this definition, it follows immediately that $\theta(0) = 0$ and $\theta(1) = 1$. However, with this information alone, it is unclear where this function starts taking non-zero values. To address this, we now introduce the concept of the *critical value*.

Definition 1.3. ***[Critical value]*** *We define the critical value, also known as the critical point or critical parameter, and denoted by* $p_c = p_c(d)$, *as*

$$p_c := \inf\{p \in [0,1]; \theta(p) > 0\}.$$

There are fundamental results needed to advance the theory. Some of their proofs are extensive, but they can be found in most books on percolation. Therefore, we omit the proofs of Propositions 1.5, 1.6, 1.9, 1.12 here. For reference, see Chapter 2 of [8].

In some proofs, it will be necessary to consider a truncated version of the percolation function. To define this, for each $n \geq 1$ the box $\Lambda_n = [-n, n]^d$, that is, the subgraph induced by the vertices $z \in \mathbb{Z}^d$ for satisfying $z_i \in [-n, n]$ for each $i = 1, ..., d$. The vertex boundary of Λ_n, denoted by $\partial\Lambda_n$ consists of vertices $z \in \Lambda_n$ that have $\max\{|z_i| \ ; \ i = 1, ..., d\} = n$.

Definition 1.4. ***[Truncated percolation functions]*** *Let* $n \geq 1$. *We define the truncated percolation function* $\theta_n(p)$ *for* $p \in [0,1]$ *by*

$$\theta_n(p) := P_p(0 \leftrightarrow \partial\Lambda_n).$$

Notice that the sequence of functions $\{\theta_n\}_{n\geq 1}$ converges pointwise to the function θ. Also, by coupling all the measures P_p in an increasing way, it is possible to show the following monotonicity property.

Proposition 1.5. *The function* θ *is non-decreasing on* p, *that is, if* $p \leq p'$ *then* $\theta(p) \leq \theta(p')$.

The monotonicity of θ ensures that the probability of the origin being part of an infinite cluster only increases as p grows. This property emphasizes the existence of a phase transition in the model, where the behavior of θ changes significantly at a critical point.

Proposition 1.6. ***[Phase transition for*** $d \geq 2$***]*** *Consider the Bernoulli percolation on the bonds of* $\mathbb{Z}^d$, *where* $d \geq 2$. *Then, we have that* $p_c \in (0,1)$, *that is, the critical parameter is non-trivial.*

This result answers the question of whether the probability of the origin's cluster being infinite is strictly positive or not and establishes that the model exhibits a phase transition. For $d \geq 2$, there exists a non-trivial critical value p_c such that, for $p < p_c$, the probability of the origin's cluster being infinite is zero, while for $p > p_c$, this probability is strictly positive.

The next two propositions provide bounds for the probability of events under specific conditions. To introduce the first inequality, which will be used frequently in this work, we first present a few necessary definitions.

Definition 1.7. ***[Partial order on configurations]*** *The space of configuration* $\Omega = \{0,1\}^{\mathbb{E}^d}$ *allows a natural partial ordering* $\leq$, *where we say that* $\omega \leq \omega'$ *if* $\omega_e \leq \omega'_e$ *for every* $e \in \mathbb{E}^d$.

This leads directly to the following definition.

Definition 1.8. ***[Increasing Events and Random Variables]*** *A random variable* $X : \Omega \to \mathbb{R}$ *defined on* $(\Omega, \mathcal{F})$ *is said to be increasing, if* $X(\omega) \leq X(\omega')$ *whenever* $\omega \leq \omega'$. *We say that an event* $A \in \mathcal{F}$ *is increasing if* 1_A *is an increasing random variable.*

One example of an increasing event is $\{0 \leftrightarrow \infty\}$. If there is an infinite path between 0 and ∞ in ω, this path also connects 0 and ∞ in each ω' where $\omega \leq \omega'$, as additional open edges cannot break existing connections. The following inequality provides a bound for the probability of the intersection of two increasing events when they are not necessarily independent.

Proposition 1.9. ***[FKG inequality (Fortuin, Kasteleyn, Ginibre)]*** *Let* A *and* B *be increasing events, then*

$$P_p(A \cap B) \geq P_p(A)P_p(B),$$

and, more generally, if X *and* Y *are two bounded increasing functions, then*

$$E_p[XY] \geq E_p[X]E_p[Y].$$

We say that an event A is *decreasing* if its complement A^C is increasing. Consequently, the FKG inequality also holds if A and B are both decreasing events.

While the FKG inequality provides a lower bound for the probability of the intersection of increasing or decreasing events, upper bounds are often needed when estimating probabilities. In such cases, we rely on the BK inequality. To formalize this, we introduce the following definitions.

Definition 1.10. ***[Event occurring on a set]*** *Let* A *be an event and* $\Lambda \subset \mathbb{E}^d$. *The event* A *occuring on* Λ *is defined as*

$$A|\Lambda := \{\omega \in A \ ; \ \omega' \in A \text{ for every } \omega' \text{ s.t. } \omega'_e = \omega_e \text{ for every } e \in \Lambda\}.$$

The previous definition says that we are constructing a new event that includes configurations $\omega \in A$ with the property that any other configuration ω' which agrees on Λ with ω, will also belong to the event A. This leads us to the following definition:

Definition 1.11. ***[Disjoint realization]*** *Let A and B be events. We say that A and B occur disjointly on ω, if there exists Λ, Σ, subsets of $\mathbb{E}^d$ depending on ω, where $\Lambda \cap \Sigma = \varnothing$ and $\omega \in A|\Lambda \cap B|\Sigma$. We also write:*

$$A \circ B := \{\omega \in A \cap B \ ; \exists \Lambda, \Sigma \subset \mathbb{E}^d, \Lambda \cap \Sigma = \varnothing, \omega \in A|\Lambda \cap B|\Sigma\}.$$

With these definitions in place, we can state a key inequality that will be used frequently throughout this work.

Proposition 1.12. ***[BK inequality (van den Berg, Kesten)]*** *If A and B are two events depending on finitely many edges, then*

$$P_p(A \circ B) \leq P_p(A) P_p(B).$$

Although the BK inequality is limited to events that depend on a finite number of edges, it remains an essential tool. For events that depend on infinitely many edges, this limitation can often be addressed by truncating the graph, applying the inequality to the truncated case, and then taking appropriate limits.

The next definition is presented to establish the necessary terminology for further results.

Definition 1.13. ***[Covariance]*** *Let X and Y be two random variables on $(\Omega, \mathcal{F})$ with finite second moments. The covariance between X and Y is defined as*

$$\mathrm{Cov}_p[X, Y] := E_p[XY] - E_p[X]E_p[Y].$$

We may use the notation $\mathrm{Cov}_\mu[X, Y]$ when integrating with respect to a different measure μ. If the measure being used is clear from the context, we omit the subscript and write simply $\mathrm{Cov}[X, Y]$.

To estimate the rate of change of probabilities with respect to p, we require additional definitions and results. We begin with the following lemma, which applies to finite graphs.

Lemma 1.14. ***[Derivative of the expected value of a boolean function]*** *Let $G = (V, E)$ be a finite subgraph of $\mathbb{L}^d$, and f be a boolean function, that is, it has the form $f : \{0,1\}^E \to \{0,1\}$. Then, for $p \in [0,1]$*

$$\frac{d}{dp} E_p[f] = \frac{1}{p(1-p)} \sum_{e \in E} \mathrm{Cov}[f, \omega_e].$$

Proof. Set $|\omega| := \sum_{e\in E}\omega_e$ and $\Omega = \{0,1\}^E$. We can then compute the expected value $E_p[f]$ as

$$E_p[f] = \sum_{\omega\in\Omega} f(\omega)p^{|\omega|}(1-p)^{|E|-|\omega|}.$$

Since we are assuming $|E| < \infty$, this expected value is a polynomial in p, and we can compute its derivative with respect to p as

$$\frac{d}{dp}E_p[f] = \sum_{\omega\in\Omega} f(\omega)\frac{d}{dp}\left[p^{|\omega|}(1-p)^{|E|-|\omega|}\right].$$

Developing the right side with basic derivative rules

$$\begin{aligned}\frac{d}{dp}E_p[f] =& \frac{1}{p}\sum_{\omega\in\Omega} f(\omega)|\omega|p^{|\omega|}(1-p)^{|E|-|\omega|} \\ &- \frac{1}{1-p}\sum_{\omega\in\Omega} f(\omega)(|E|-|\omega|)p^{|\omega|}(1-p)^{|E|-|\omega|}.\end{aligned}$$

And rearranging this last expression we get

$$\begin{aligned}\frac{d}{dp}E_p[f] &= \frac{1}{p(1-p)}E_p[f(\omega)(|\omega|-p|E|)] \\ &= \frac{1}{p(1-p)}E_p\left[f(\omega)\left(\sum_{e\in E}\omega_e - \sum_{e\in E}p\right)\right] \\ &= \frac{1}{p(1-p)}\sum_{e\in E}E_p[f(\omega)\cdot(\omega_e-p)] \\ &= \frac{1}{p(1-p)}\sum_{e\in E}\mathrm{Cov}[f,\omega_e],\end{aligned}$$

since

$$\mathrm{Cov}[f(\omega),\omega_e] = E_p[f(\omega)\omega_e] - E[f]\cdot p = E_p[f(\omega)\cdot(\omega_e-p)].$$

□

With a slight variation of the previous proof, we can bound the absolute value of the rate of change of the probability of an increasing event.

Theorem 1.15. *Let A be an event that depends only on bonds on a finite subset Λ of $\mathbb{E}^d$. Let $p\in(0,1)$. Then*

$$\left|\frac{dP_p(A)}{dp}\right| \le \alpha(p)\sqrt{|\Lambda|},$$

where $\alpha(p) := 1/\sqrt{p(1-p)}$.

Proof. Set $f := \mathbb{1}_A$, then as we did in the proof of the previous theorem, we get

$$\begin{aligned}\frac{dP_p(A)}{dp} &= \frac{1}{p(1-p)}\sum_{e\in\Lambda} E_p[\mathbb{1}_A(\omega)(\omega_e - p)] \\ &= \frac{1}{p(1-p)}E_p[\mathbb{1}_A(\omega)(N_\Lambda - p|\Lambda|)] \\ &= \frac{1}{p(1-p)}E_p[\mathbb{1}_A(\omega)(N_\Lambda - E_p[N_\Lambda])],\end{aligned}$$

where, $N_\Lambda := \sum_{e\in\Lambda}\omega_e$, and this random variable has $E[N_\Lambda] = p|\Lambda|$. Thus,

$$\begin{aligned}\left|\frac{dP_p(A)}{dp}\right| &\leq \frac{1}{p(1-p)}E_p[|N_\Lambda - E_p[N_\Lambda]|] \\ &\leq \frac{1}{p(1-p)}\sqrt{E_p[(N_\Lambda - E_p[N_\Lambda])^2]} \\ &= \frac{1}{p(1-p)}\sqrt{\mathrm{Var}_p[N_\Lambda]} \\ &= \frac{1}{p(1-p)}\sqrt{|\Lambda|p(1-p)|} \\ &= \alpha(p)\sqrt{|\Lambda|}.\end{aligned}$$

In the last steps we used Jensen's inequality and the fact that N_Λ is a binomial random variable with parameters $|\Lambda|$ and p, and consequently with variance $|\Lambda|p(1-p)$. □

The main results we aim to study throughout this text involve differential inequalities for functions describing the behavior of the model. To derive these inequalities, we rely on the following proposition, which enables us to compute the derivative of the probability of an increasing event whose occurrence depends only on a finite number of edges in $\mathbb{L}^d$. To formalize this, we first define important concepts.

Definition 1.16. ***[Pivotal edge]*** *Let $e \in \mathbb{E}^d$, $\omega \in \Omega$ and A be an event. We define $\omega^{e,+}$, or $\omega^{e,-}$ to be configurations that agree with ω on $\mathbb{E}^d - \{e\}$, that is $\omega^{e,+}_f = \omega^{e,-}_f = \omega_f$ for all edges $f \neq e$, and also $\omega^{e,+}_e = 1$, and $\omega^{e,-}_e = 0$. We say that the edge e is pivotal for the event A in the configuration ω if*

$$\mathbb{1}_A(\omega^{e,+}) - \mathbb{1}_A(\omega^{e,-}) \neq 0.$$

In some configurations, certain edges can play a decisive role in determining whether a particular event occurs. By “decisive role”, we mean that flipping the state of such an edge necessarily changes the outcome of the event, either causing it to occur or preventing it. For a given event and configuration, it is possible for multiple edges, or none, to exhibit this property simultaneously.

Definition 1.17. ***[Set of pivotal edges]*** *We denote the random subset of edges that are pivotal for an event A in the configuration ω by $\delta A = \delta A(\omega)$.*

The next proposition presents a powerful tool. It relates the rate of change of the probability of an increasing event that depends on a finite number of edges to the expected value of the number of pivotal edges of that event.

Proposition 1.18. ***[Russo's formula]*** *Let A be an increasing event that depends only on a finite set of edges E. Then,*

$$\frac{d}{dp}P_p(A) = E_p[|\delta A|] = \sum_{e \in E} P_p(e \in \delta A).$$

Proof. Letting $f := \mathbb{1}_A$ on Lemma 1.14, we have

$$\frac{d}{dp}P_p(A) = \frac{1}{p(1-p)} \sum_{e \in E} E_p[\mathbb{1}_A(\omega_e - p)].$$

Now, we can make the decomposition

$$\begin{aligned}&E_p[\mathbb{1}_A(\omega_e - p)] \\ &= E_p[\mathbb{1}_A(\omega_e - p)\mathbb{1}_{\{e \in \delta A\}}] + E_p[\mathbb{1}_A(\omega_e - p)\mathbb{1}_{\{e \notin \delta A\}}].\end{aligned}$$

Notice that $\omega_e - p$ depends only on e, also $A \cap \{e \notin \delta A\}$ depends only on edges that are different from e. Therefore,

$$E_p[\mathbb{1}_A(\omega_e - p) \cdot \mathbb{1}_{\{e \notin \delta A\}}] = E_p[\mathbb{1}_A \mathbb{1}_{\{e \notin \delta A\}}]E_p[\omega_e - p] = 0.$$

Notice that since A is increasing we have for $\omega \in \{e \in \delta A\}$, that if $\omega_e = 0$, then $\mathbb{1}_A(\omega) = 0$ and if $\omega_e = 1$, then $\mathbb{1}_A(\omega) = 1$. Thus, since $\{e \in \delta A\}$ does not depend on the value of ω_e, we have

$$\begin{aligned}\frac{d}{dp}P_p(A) &= \frac{1}{p(1-p)} \sum_{e \in E} E_p[\mathbb{1}_A(\omega_e - p)\mathbb{1}_{\{e \in \delta A\}}] \\ &= \frac{1}{p(1-p)} \sum_{e \in E} (1-p)P_p(\{\omega_e = 1\} \cap \{e \in \delta A\}) \\ &= \frac{1}{p(1-p)} \sum_{e \in E} (1-p)pP_p(e \in \delta A) \\ &= \sum_{e \in E} P_p(e \in \delta A) \\ &= E_p[|\delta A|].\end{aligned}$$

□

Another fundamental function in the study of percolation is the expected size of the cluster of the origin.

Definition 1.19. ***[Mean cluster size]*** *For $p \in [0,1]$, we define the function $\chi(p)$ as:*

$$\chi(p) \coloneqq E_p[|C|],$$

where $|C|$ denotes the cardinality of the cluster of the origin.

It is clear that, if $p > p_c$ then $\chi(p) = \infty$, since we can write $\chi(p)$ as:

$$\chi(p) = \sum_{k=1}^{\infty} kP_p(|C| = k) + \infty \cdot P_p(|C| = \infty) = \infty,$$

where we are using that $P_p(|C| = \infty) = \theta(p) > 0$ if $p > p_c$. For this reason, $\chi(p)$ is not particularly interesting in the supercritical phase, as it remains constant at the value ∞. We can then define the truncated version of χ, denoted by χ^f, which describes the mean cluster size on the event where the origin's cluster size is finite. At this point, it is unclear whether this function is finite in the subcritical phase, but we will prove this later in Remark 3.9.

Definition 1.20. ***[Truncated mean cluster size]*** *For $p \in [0,1]$, we define the truncated mean cluster size function χ^f as:*

$$\chi^f(p) \coloneqq E_p[|C|; |C| < \infty] = \sum_{k=1}^{\infty} kP_p(|C| = k).$$

It is clear that, if $p < p_c$ we have $\chi(p) = \chi^f(p)$, since $\theta(p) = 0$.

Finally, we define the function called the fundamental correlation length. [Some intuition].

Definition 1.21. ***[Fundamental correlation length]*** *For $p \in [0,1]$, we define the fundamental correlation length function $\xi(p)$ as the limit:*

$$-\frac{1}{\xi(p)} \coloneqq \lim_{n\to\infty} \frac{\log(\tau_n(p))}{n}.$$

We prove the existence of this limit in Theorem 2.2, here in Chapter 2.

1.2 An Introduction To Critical Exponents

Along with the functions θ and χ^f, there are other functions in percolation theory that play a crucial role in understanding the model. The behavior of some of these functions near the critical parameter is conjectured to be approximately some power of $|p - p_c|$, where the exponent of this power is referred to as the critical exponent of the function. In the following definitions, we formalize what we mean by this "approximately".

Definition 1.22. ***[Little-o notation]*** *We say that* $f(x) = o(g(x))$ *as* $x \to x_0$ *if*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0,$$

where g *is strictly positive in a neighborhood of* x_0*.*

Definition 1.23. ***[Asymptotic equivalence*** $\sim$***]*** *We say that* $f(x)$ *is asymptotically equivalent to* $g(x)$ *as* $x \to x_0$ *if*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1,$$

where g *is strictly positive in a neighborhood of* x_0*. We write this as* $f \sim g$ *as* $x \to x_0$*.*

Definition 1.24. ***[Asymptotic equivalence of logarithms*** $\approx$***]*** *We say that* $f(x)$ *is logarithmically asymptotic to* $g(x)$ *as* $x \to x_0$ *if* $\log(f) \sim \log(g)$ *as* $x \to x_0$*. That is*

$$\lim_{x \to x_0} \frac{\log(f(x))}{\log(g(x))} = 1,$$

where $g > 1$ *in a neighborhood of* x_0*. We write this as* $f \approx g$ *as* $x \to x_0$*.*

Definition 1.25. ***[Comparison up to constants*** $\preceq$***]*** *In this text, we use the notation* $f \preceq g$ *to denote the existence of a positive constant* $C > 0$ *such that*

$$f(x) \leq Cg(x),$$

for every x *where the positive functions* f *and* g *are defined, or a neighborhood around a certain point of interest.*

The previous notation is useful when we are dealing with the asymptotic equivalence $\approx$: since the logarithm transforms products into sums, the constants multiplying the function become irrelevant in some calculations.

We can now state the definition of our first critical exponent, the one concerning the function θ.

Definition 1.26. ***[The critical exponent*** β***]*** *We define the critical exponent* β *to be the number for which the following limit holds:*

$$\lim_{p \downarrow p_c} \frac{\log(\theta(p))}{\log(p - p_c)^{\beta}} = 1.$$

Or equivalently

$$\theta(p) \approx (p - p_c)^{\beta} \text{ as } p \downarrow p_c.$$

In this text, we will prove that if this number exists, then it must be at most 1. While this proof applies to any dimension, it has been established that for sufficiently large dimensions, β indeed exists and is exactly equal to 1.

In the next definition, we introduce the critical exponent γ. This exponent characterizes the rate of divergence of the mean cluster size function χ^f near the critical parameter p_c.

Definition 1.27. ***[The critical exponent** γ**]*** *We define the critical exponent* γ *to be the number for which the following limit holds:*

$$\lim_{p \uparrow p_c} \frac{\log(\chi^f(p))}{\log |p_c - p|^{-\gamma}} = 1,$$

as $p \uparrow p_c$*. Or equivalently*

$$\chi(p) \approx |p - p_c|^{-\gamma} \text{ as } p \uparrow p_c.$$

In this text, we will prove that if this critical exponent exists, then it must be at least 1. While this proof applies to any dimension, it has been established that for sufficiently large dimensions, γ indeed exists and is exactly equal to 1.

Similar definitions can be provided for the remaining critical exponents δ, Δ and ν that we will study in this text. To avoid repetition, we summarize all the definitions in Table 1.1.

Critical Exponent	**Definition**
β	$\theta(p) \approx \|p - p_c\|^{\beta}$ as $p \downarrow p_c$
γ	$\chi(p) \approx \|p_c - p\|^{-\gamma}$ as $p \uparrow p_c$
δ	$P_{p_c}(\|C\| \geq n) = cn^{-1/\delta}(1 + o(1))$ as $n \to \infty$
Δ	$E_p[\|C\|^k] \approx \|p_c - p\|^{-(k-1)\Delta+\gamma}$ as $p \uparrow p_c$
ν	$\xi(p) \approx \|p - p_c\|^{-\nu}$ as $p \uparrow p_c$

Table 1.1: Critical exponents present in this text

The study of critical exponents extends beyond the standard Bernoulli percolation model. As demonstrated in Durrett's paper [6], some results concerning the critical exponents β, γ, and δ also apply to oriented percolation, discrete-time contact processes, and Bernoulli percolation on the binary tree.

For Bernoulli percolation on the edges of a binary tree T, calculations become simpler, as multiple paths connecting pairs of vertices do not need to be considered. In this specific graph, the critical exponents can be computed exactly, and their values are independent of the degree of the regular tree. These values are: $\beta_T = 1$, $\gamma_T = 1$, $\delta_T = 2$, $\Delta_T = 2$, $\nu_T = \frac{1}{2}$. As an illustration, using the theory of Galton-Watson Branching Processes (see Chapter 10 of [8]), the asymptotic behavior of the function θ is given by:

$$\theta(p) \sim 8(p - p_c),$$

as $p \downarrow p_c$, implying that $\beta = 1$. Furthermore, for percolation on T, the expected size of the cluster at the origin is given by a simple series:

$$E_p[|C_0|] = \sum_{n \geq 0} (2p)^n = \frac{1}{2(p_c(T) - p)}.$$

Implying that $\gamma = 1$, since $p_c(T) = 1/2$. In general, the behavior observed in other percolation models can be bounded from above or below by the results for the binary tree, as we will explore further in the mean field bounds discussed in Chapter 3.

The term "mean field" can have different meanings in the mathematical physics literature. In percolation on $\mathbb{L}^d$, it is believed that for sufficiently large dimensions ($d \geq 6$), the critical exponents are the same as those on the binary tree T. Therefore, in this text, by "mean field critical exponents", we refer to, both the critical exponents on the binary tree itself and those on the hypercubic lattice $\mathbb{L}^d$, when d is large.

In Chapter 4 of Heydenreich and Hofstad [9] the results of Aizenman and Newman [2] are discussed. They established that, under a condition known as the *triangle condition*, the mean field critical exponents exist. This condition is stated as $\Delta_{p_c} < \infty$, where:

$$\Delta_{p_c} := \sum_{x,y \in \mathbb{Z}^d} \tau_{p_c}(0, x)\tau_{p_c}(x, y)\tau_{p_c}(y, 0). \tag{1.1}$$

Also, in [9], it is shown that if a transitive graph satisfies the triangle condition, the critical exponents are the same as those on the binary tree. That is stated in Theorem 4.1 presented in [9] as:

Theorem 1.28. *For percolation on a transitive graph, if the triangle condition holds, then $\beta = \gamma = 1$ and $\delta = 2$.*

It can be shown that this condition is very strong and implies that $\theta(p_c) = 0$, meaning that the function θ is continuous. Proving that this condition implies the mean field critical exponents' values from the previous theorem is quite technical and requires substantial effort. Additionally, it is necessary to check for which dimensions d, the lattice $\mathbb{L}^d$ satisfies the triangle condition (1.1).

As presented in Theorem 10.52 in Grimmett [8], we have the following results summarized:

Theorem 1.29. ***[Mean field critical exponents]*** *If $d \geq 19$, then*

$$(p - p_c) \preceq \theta(p) \preceq (p - p_c), \text{ as } p \downarrow p_c;$$

$$\frac{1}{(p_c - p)} \preceq \chi(p) \preceq \frac{1}{(p_c - p)}, \text{ as } p \uparrow p_c;$$

$$\frac{1}{(p_c - p)^{1/2}} \preceq \xi(p) \preceq \frac{1}{(p_c - p)^{1/2}}, \text{ as } p \uparrow p_c;$$

$$\frac{1}{(p_c - p)^2} \preceq \frac{E_p[|C|^{k+1}; |C| < \infty]}{E_p[|C|^k; |C| < \infty]} \preceq \frac{1}{(p_c - p)^2}, \text{ as } p \uparrow p_c.$$

Its proof relies on a technical method to verify the triangle condition for large d called *lace expansion*. This is expected to apply for dimensions $d \geq 6$, though progress has only been made for $d \geq 11$, as shown by Fitzner and Hofstad in [7].

Although the values of the critical exponents stabilize for high dimensions, in lower dimensions, it is expected that they assume different values from the mean field behavior, as discussed in Kesten's paper [13]. For the two-dimensional process, for example, there are predictions for the values of the critical exponents, such as $\beta = 5/36$, $\gamma = 43/18$, $\delta = 91/5$ and $\nu = 4/3$. However, it is only proved rigorously that if these exponents exist, then $\beta < 1$, $\gamma \geq 8/5$, $\delta \geq 5$, and $\nu > 1$. These rigorous bounds were taken from the table in [13].

Apart from the exact values that these critical exponents may assume, we also seek to understand how they relate to one another. It is believed that, in sufficiently low dimensions, if they exist, they must satisfy certain relations. For example, Kesten [13] proved several relations for the two-dimensional process:

Theorem 1.30. *Consider the two-dimensional percolation model. Assume that the critical exponents β, γ, ν, and δ exist. Then, they must satisfy:*

$$\beta = \frac{2\nu}{\delta + 1}, \ \gamma = 2\nu\frac{\delta - 1}{\delta + 1}, \text{ and } \Delta = 2\nu\frac{\delta}{\delta + 1}.$$

This brief discussion provides an insight into the challenges of working with critical exponents. The conjectures regarding their existence form an interesting area of study. While proving the existence of these exponents in low dimensions remains ambitious, we can explore the implications of their existence and the consequences that arise.

The background presented in this section lays the foundation for understanding the critical exponents discussed in the following chapters. The remainder of this text is dedicated to exploring three techniques that enable us to derive relations for these exponents.

Chapter 2

Scaling Theory

In this chapter, we study a technique that rescales certain events, depending on a finite number of edges, to uncover information about the asymptotic behavior of the fundamental correlation length function ξ. The primary goal is to establish the bound $\nu \geq 2/d$ for the critical exponent ν.

In the first section, we analyze the function itself, establishing some of its key properties. Next, we introduce different types of crossings in finite boxes and examine how these crossings relate to the fundamental correlation length. Finally, we derive a differential inequality that leads to the desired bound.

The main references for this chapter are Chapters 2 and 3 of *Independent and Dependent Percolation* by Chayes, Puha and Sweet [3]. We also use Chapter 6 of *Percolation* by Grimmett [8] to give an alternative approach when proving the properties of the function ξ. Alongside the results for ξ, we establish certain properties of the susceptibility function χ, which will be used in the subsequent chapter.

2.1 The Fundamental Correlation Length

We begin by demonstrating the existence of the limit in the definition of the function ξ, as presented in the previous chapter.

The following proof is very straightforward, it combines a natural inclusion of events with a classic real analysis result called Fekete's subadditive lemma. Since we are using this result a few more times in this text, we are going to give its statement:

Lemma 2.1. ***[Fekete's lemma]*** *Let $\{a_n\}_{n\geq 1}$ be a subbaditive sequence, that is $a_{m+n} \leq a_m + a_n$ for all $n, m \geq 1$. Then, the sequence $\{\frac{a_n}{n}\}_{n\geq 1}$ converges to its infimum $a := \inf_n \frac{a_n}{n}$. The limit can be $-\infty$.*

We omit the proof of this lemma since it is a classical real analysis result. Now, we proceed to the existence of the fundamental correlation length.

Theorem 2.2. ***[Existence of the fundamental correlation length]*** *Let $p \in (0,1)$. Then, the following limit exists:*

$$-\frac{1}{\xi(p)} := \lim_{n\to\infty} \frac{\log(\tau_n(p))}{n}.$$

Proof. Fix $n > m \geq 1$. Then, we have the inclusion of events:

$$\{0 \leftrightarrow ne_1\} \supset \{0 \leftrightarrow me_1\} \cap \{me_1 \leftrightarrow ne_1\}.$$

Using the FKG inequality and the translation invariance,

$$\begin{aligned}\tau_n(p) &\geq \tau_m(p) P_p(me_1 \leftrightarrow ne_1)\\ &= \tau_m(p)\tau_{n-m}(p).\end{aligned}$$

Therefore

$$-\log(\tau_n(p)) \leq -\log(\tau_m(p)) - \log(\tau_{n-m}(p)).$$

This tells that the sequence $\{-\log(\tau_n(p))\}_n$ is subadditive. By Lemma 2.1, the sequence $\{-\log(\tau_n(p))/n\}_n$ converges to its infimum, and we denote this limit by $1/\xi(p)$. □

To prove certain properties of the function ξ, it will be helpful to have an equivalent definition. Before proceeding, we establish the following lemma, which states that $\{-\log(\theta_n(p))\}_n$ is almost additive.

Lemma 2.3. *For every $p \in [0,1]$ and every $n, m \in \mathbb{N}$ the following holds:*

$$\frac{1}{2d|\partial\Lambda_m|}\theta_m(p)\theta_n(p) \leq \theta_{n+m}(p) \leq |\partial\Lambda_m|\theta_m(p)\theta_n(p). \tag{2.1}$$

Proof. Recall that $\theta_n(p) = P_p(0 \leftrightarrow \partial\Lambda_n)$. The upper bound is rather simple to prove. By inclusion of events, we can make the following decomposition:

$$\{0 \leftrightarrow \partial\Lambda_{m+n}\} \subset \bigcup_{x\in\partial\Lambda_m} \{0 \leftrightarrow x\} \circ \{x \leftrightarrow x + \partial\Lambda_n\}.$$

Since we are dealing with a finite environment, it is possible to use BK, which along with the union bound leads to

$$\theta_{m+n}(p) \leq \sum_{x\in\partial\Lambda_m} \tau_p(0,x) P_p(x \leftrightarrow x + \partial\Lambda_n).$$

Now, we also have that, if $x \in \partial\Lambda_m$, then $\tau_p(0,x) \leq \theta_m(p)$ by inclusion of events, and by translation invariance that $P_p(x \leftrightarrow x + \partial\Lambda_n) = \theta_n(p)$. Using these two statements we get

$$\theta_{n+m}(p) \leq |\partial\Lambda_m|\theta_m(p)\theta_n(p).$$

To prove the other inequality, we start by setting

$$F_k^+(\Lambda_n) := \{x \in \partial\Lambda_n; x_k = n\}$$

and

$$F_k^-(\Lambda_n) := \{x \in \partial\Lambda_n; x_k = -n\},$$

to represent the front and back faces of Λ_n along the direction of e_k. We also write

$$\gamma_n^k(p) := P_p(0 \xleftrightarrow{\Lambda_n} F_k^+(\Lambda_n)) = P_p(0 \xleftrightarrow{\Lambda_n} F_k^-(\Lambda_n))$$

for $k = 1, ..., d$. It is clear that $\gamma_n^j(p) = \gamma_n^i(p)$ for every $i, j = 1, ..., d$, so we can simply write this probability as $\gamma_n(p)$. By inclusions of events and the union bound, we get the relation:

$$\gamma_n(p) \leq \theta_n(p) \leq 2d\gamma_n(p).$$

Now, let x be any vertex in $\partial\Lambda_m$. Choose an integer k such that $x \in F_k^+(\Lambda_m)$ or $x \in F_k^-(\Lambda_m)$. Assume that the first one occurs, the proof of the other case is just a matter of changing the symbol $+$ to $-$. Let $U_x := \{0 \xleftrightarrow{\Lambda_m} x\}$ and $V_x := \{x \leftrightarrow F_k^+(x + \Lambda_n)\}$. Then, $U_x \cap V_x \subset \{0 \leftrightarrow \partial\Lambda_{m+n}\}$.

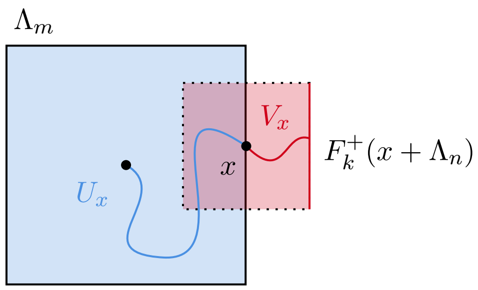


Figure 2.1: The event $U_x \cap V_x$.

Using FKG, the translation invariance, and the previous inclusion of events we get:

$$\begin{aligned} \theta_{m+n}(p) &\geq P_p(U_x)P_p(V_x) \\ &= P_p(U_x)\gamma_n(p) \\ &\geq P_p(U_x)\frac{\theta_n(p)}{2d}. \end{aligned} \tag{2.2}$$

However, we also have that

$$\theta_m(p) = P_p\left(\bigcup_{x\in\partial\Lambda_m} U_x\right) \le \sum_{x\in\partial\Lambda_m} P_p(U_x),$$

and this implies that there is a vertex $\hat{x}$ in $\partial\Lambda_m$ such that $\theta_m(p) \le |\partial\Lambda_m| P_p(U_{\hat{x}})$. Using this $\hat{x}$ in (2.2) we get that

$$\theta_{n+m}(p) \ge \frac{1}{2d|\partial\Lambda_m|}\theta_m(p)\theta_n(p). \tag{2.3}$$

□

We now introduce an alternative definition for the fundamental correlation length. This approach utilizes the functions θ_n instead of τ_n, providing a more convenient method for proving certain properties of ξ, as demonstrated in the following propositions.

Theorem 2.4. *The limit*

$$\phi(p) := \lim_{n\to\infty} -\frac{\log(\theta_n(p))}{n}$$

exists, and $\phi(p) = \frac{1}{\xi(p)}$.

Proof. Let $n \ge m \ge 1$. Notice that $|\partial\Lambda_m| = 2d(2m+1)^{d-1} \le d3^d m^{d-1}$. Thus, if we take the natural logarithm on both inequalities of (2.1), we get

$$\begin{aligned}
&-\log(d^2 3^{d+1} m^{d-1}) + \log(\theta_m(p)) + \log(\theta_n(p)) \\
&\le \log(\theta_{m+n}(p)) \\
&\le \log(\theta_m(p)) + \log(\theta_n(p)) + \log(d3^d m^{d-1}).
\end{aligned}$$

Set $g(m) := \log(d^2 3^{d+1} m^{d-1})$, use the previous inequalities and the fact that $\log(d3^d m^{d-1}) \le g(m)$ to write:

$$\begin{aligned}
&\log(\theta_m(p)) + \log(\theta_n(p)) - g(m) \\
&\le \log(\theta_{m+n}(p)) \\
&\le \log(\theta_m(p)) + \log(\theta_n(p)) + g(m).
\end{aligned} \tag{2.4}$$

Now, consider the sequence $\{a_k\}_{k\ge1}$ given by

$$a_k := g(k) + (d-1)\log(2) + \log(\theta_k(p)).$$

If we prove that this sequence is subadditive, then by Lemma 2.1 we have that $\{a_k/k\}$ converges to a limit we can call $-\phi(p)$. The limit also has the property that $-\phi(p) = \inf_k\{a_k/k\}$. More than that, we will also get that $\lim a_k/k = \lim \log(\theta_k(p))/k$, since $\lim g(k)/k = 0$.

To see that this is the case let $n \geq m \geq 1$. Just by using basic properties of the logarithm function:

$$\begin{aligned} g(m+n) - g(n) &\coloneqq \log(d^2 3^{d+1}(m+n)^{d-1}) - \log(d^2 3^{d+1} n^{d-1}) \\ &= \log\left\{\left(\frac{m+n}{n}\right)^{d-1}\right\} \\ &= (d-1)\log\left(1+\frac{m}{n}\right) \\ &\leq (d-1)\log 2. \end{aligned} \tag{2.5}$$

Therefore, $g(m+n) \leq g(n) + (d-1)\log 2$. Thus, adding $g(n) + 2(d-1)\log 2$ to both sides of the right inequality in (2.4), we get

$$\begin{aligned} a_{m+n} &\coloneqq g(m+n) + (d-1)\log 2 + \log(\theta_{m+n}(p)) \\ &\leq g(n) + 2(d-1)\log 2 + \log(\theta_{m+n}(p)) \\ &\leq \log(\theta_m(p)) + g(m) + \log(\theta_n(p)) + g(n) + 2(d-1)\log 2 \\ &= a_m + a_m. \end{aligned}$$

This is the subadditivity we wanted. It also follows that the sequence $\{b_k\}_{k\geq 1}$ given by

$$b_k \coloneqq g(k) + (d-1)\log(2) - \log(\theta_k(p))$$

is subadditive. Recall that, the left inequality in (2.4) tells that

$$\log(\theta_{m+n}(p)) \geq -g(m) + \log(\theta_m(p)) + \log(\theta_n(p)).$$

Thus, adding $-g(n) - 2(d-1)\log(2)$ to both sides and using the estimate in (2.5) leads us to $b_{m+n} \leq b_m + b_n$. Using Lemma 2.1 one more time, we conclude that $b_k/k \to \inf_k\{b_k/k\} = \phi(p)$.

Now, it only remains to prove the equality $\phi = 1/\xi$. By inclusion of events, we have that $\tau_n(p) \leq \theta_n(p)$, taking logarithms, dividing by n and making $n \to \infty$ gives us the inequality $1/\xi(p) \geq \phi(p)$. For the other direction, recall from the argument that leads to the equation (2.3), that we can find a vertex $\hat{x}$ in $\partial\Lambda_m$ where, for the event $U_x \coloneqq \{0 \xleftrightarrow{\Lambda_m} x\}$, the following inequality holds:

$$P_p(U_{\hat{x}}) \geq \frac{1}{|\partial\Lambda_m|}\theta_m(p).$$

By rotation symmetry, we can assume without loss of generality that $\hat{x} \in F_1^+(\Lambda_m)$. Notice also that the following inclusion of events holds:

$$\{0 \leftrightarrow 2me_1\} \supset \{0 \leftrightarrow \hat{x}\} \cap \{\hat{x} \leftrightarrow 2me_1\}.$$

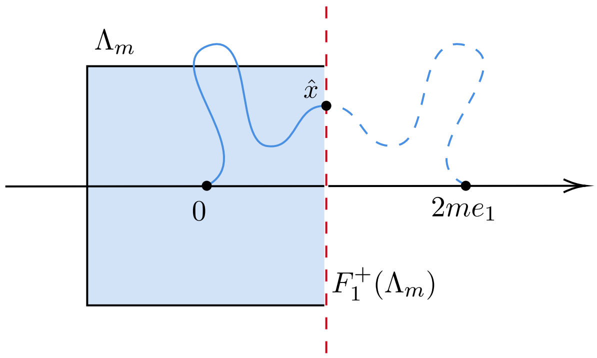


Figure 2.2: Reflection symmetry.

Therefore, using the FKG inequality we get

$$\tau_{2m}(p) \geq P_p(0 \leftrightarrow \hat{x})P_p(\hat{x} \leftrightarrow 2me_1).$$

However, by reflection symmetry $P_p(0 \leftrightarrow \hat{x}) = P_p(\hat{x} \leftrightarrow 2me_1)$. Along with that, $P_p(0 \leftrightarrow \hat{x}) \geq P_p(U_{\hat{x}})$. Hence,

$$\tau_{2m}(p) \geq |\partial\Lambda_m|^{-2}\theta_m(p)^2. \tag{2.6}$$

Since $-\phi(p) = \inf_k\{a_k/k\}$, for every $k \geq 1$ it follows that

$$\log\theta_k(p) \geq -k\phi(p) - (d-1)\log(2) - g(k).$$

Therefore, applying the logarithm function on (2.6) and using the previous inequality for $\log(\theta_m)$ we get

$$\begin{aligned}\log(\tau_{2m}(p)) &\geq -2\log(|\partial\Lambda_m|) + 2\log(\theta_m)\\ &\geq -2(\log(A_1 m^{d-1}))\\ &+ 2\Bigg\{ - m\phi(p) - (d-1)\log(2) - g(m)\Bigg\},\end{aligned}$$

for some positive constant A_1. Dividing by $2m$ and making $m \to \infty$ leads us to the bound $\frac{1}{\xi} \leq \phi$. □

Now, we will demonstrate two key properties of the function ϕ, which by Theorem 2.4 also provide insights into $1/\xi$.

Theorem 2.5. ***[Properties of the function*** ϕ***]*** *The function* ϕ *in Theorem 2.4 satisfies:*

1. ϕ *is continuous on* $(0, 1]$.
2. $\phi(p) = 0$ *for* $p \in (p_c, 1]$.
3. ϕ *is strictly positive on* $(0, p_c)$

As a consequence of items 1 *and* 2*,* $\phi(p_c) = 0$.

Proof. In the proof of Theorem 2.4 we found that both sequences a_k and b_k are subadditive. Consequently, by Lemma 2.1 the sequences a_k/k and b_k/k converge to their respective infima. That is, for each $m \geq 1$, we have

$$\phi(p) \leq \frac{b_m}{m} = \frac{g(m) + (d-1)\log(2) - \log(\theta_m(p))}{m}, \tag{2.7}$$

and

$$-\phi(p) \leq \frac{a_m}{m} = \frac{g(m) + (d-1)\log(2) + \log(\theta_m(p))}{m}. \tag{2.8}$$

By combining inequalities (2.7) and (2.8), we conclude that

$$\left| \frac{\log(\theta_m(p))}{m} + \phi(p) \right| \leq \frac{g(m)}{m} + \frac{(d-1)\log(2)}{m} \to 0$$

as we make $m \to \infty$. This implies that the convergence $\log \theta_m/m \to -\phi$ is uniform on $(0, 1]$. Since $\{0 \leftrightarrow \partial\Lambda_m\}$ depends only on a finite number of edges, the functions θ_m are polynomials in p and, consequently, continuous. Given the uniform convergence of continuous functions, the limit is also a continuous function. This concludes the proof of the first item.

Notice that, by the definition of $g(m)$ and inequality (2.7), we have that

$$\log(\theta_m(p)) \leq \rho + (d-1)\log(m) - m\phi(p),$$

for some positive constant ρ. Therefore, $\theta_m(p) \leq c_1 m^{d-1} e^{-m\phi(p)}$, for another positive constant c_1. Analogously, using inequality (2.8), we find a positive constant c_2 such that $\theta_m(p) \geq c_2 m^{1-d} e^{-m\phi(p)}$. Thus, it follows that, for some constants c_1 and c_2, not depending on $p \in (0, 1]$, and every $m \geq 1$:

$$c_2 m^{1-d} e^{-m\phi(p)} \leq \theta_m(p) \leq c_1 m^{d-1} e^{-m\phi(p)}. \tag{2.9}$$

Now, if $p > p_c$ then $\theta(p) > 0$ by definition of p_c. Additionally, by inclusion of events we have that $\theta_n(p) \geq \theta(p) > 0$. Therefore, for $p > p_c$ and for every $m \geq 1$ we get

$$0 < \theta(p) \leq \theta_m(p) \leq c_1 m^{d-1} e^{-m\phi(p)}.$$

Therefore, if $\phi(p) \neq 0$ we would have a contradiction as we make $m \to \infty$. This concludes the proof for the second item.

For the last item, we rely on the fact that, for $p < p_c$ the sequence $\theta_n(p)$ decays exponentially. For a reference, see Chapter 6 of [8]. Additionally, from (2.9), we deduce that $\phi(p)$ must be strictly positive on $(0, p_c)$. □

The next theorem provides insight into the relationship between the functions χ and ξ, showing that the fundamental correlation length is bounded above by the mean cluster size function in the subcritical phase.

Later in this text, we will prove that $\chi(p) < \infty$ for $p < p_c$. See Remark 3.9 following Theorem 3.8. Combined with the earlier observation from the definition of χ that $\chi(p) = \infty$ for $p > p_c$ his establishes that the finiteness of χ characterizes the subcritical phase.

Theorem 2.6. *Let $p \in (0, p_c)$. Let χ be the mean cluster size function 1.19 and ξ be the fundamental correlation length. Then, for all $u \in \mathbb{Z}^d$*

$$\tau_p(0, u) \leq \left\{1 - \frac{1}{\chi(p)}\right\}^{|u|}.$$

And, as consequence, for all $p \in (0, p_c)$

$$\chi(p) \geq \xi(p).$$

Proof. Let $|x| := D(0, x)$ where $D(x, y) = \sum_{i=1}^{d} |x_i - y_i|$. Denote the ball with radius n and distance D by S_n, that is, $S_n := \{x \in \mathbb{Z}^d; |x| \leq n\}$. Let

$$M_n = \sum_{x \in \partial S_n} \mathbb{1}_{\{x \leftrightarrow 0\}}.$$

That is the random variable that tells the number of vertices on the boundary of S_n that are connected to 0. Then, taking the expected value

$$E_p[M_n] = \sum_{x \in \partial S_n} \tau_p(x, 0).$$

Summing over $n \geq 0$:

$$\sum_{n \geq 0} E_p[M_n] = \sum_{n \geq 0} \sum_{x \in \partial S_n} \tau_p(0, x) = \sum_{x \in \mathbb{Z}^d} \tau_p(0, x) = \chi(p). \tag{2.10}$$

Now, we have $\chi(p)$ written in terms of the connection probabilities $\tau_p(0, x)$. Next, we are going to prove the estimate

$$\tau_p(0, u) \leq E_p[M_n]^{|u|/n}, \tag{2.11}$$

for every vertex $u \in \mathbb{Z}^d$ and every $n \geq 1$.

Before proving this claim, we are showing what follows from it. Let $p < p_c$, so that $\chi(p) < \infty$ (See Remark 3.9). Assume that (2.11) is true for every $u \in \mathbb{Z}^d$ and every $n \geq 1$. In addition, we also claim that we can find an $N \geq 1$ such that

$$E_p[M_N] \leq \left\{1 - \frac{1}{\chi(p)}\right\}^{N}.$$

This follows since otherwise, we would have the contradiction:

$$\chi(p) = \sum_{n\geq 0} E_p[M_n] > \sum_{n\geq 0} \left\{1 - \frac{1}{\chi(p)}\right\}^n = \frac{1}{1-(1-\frac{1}{\chi(p)})} = \chi(p).$$

Now, using the assumption we made, it follows that

$$\tau_p(0,u) \leq (E_p[M_N])^{|u|/N} \leq \left\{1 - \frac{1}{\chi(p)}\right\}^{|u|},$$

for every $u \in \mathbb{Z}^d$. To prove the consequence $\chi(p) \geq \xi(p)$ stated in the theorem, take $u = ne_1$ and use the bound $1 - t \leq e^{-t}$:

$$\tau_n(p) \leq \left\{1 - \frac{1}{\chi(p)}\right\}^n \leq (\exp(-1/\chi(p)))^n.$$

Thus, $\log(\tau_n(p))/n \leq -1/\chi(p)$ implying that $\chi(p) \geq \xi(p)$. Now, it remains only to prove that the estimate (2.11) holds for every $u \in \mathbb{Z}^d$ and every $n \geq 1$. We proceed by spliting the computation into simpler steps. First, notice that if some vertex z has $|z| > m$, then we can make the decomposition

$$\{0 \leftrightarrow z\} \subset \bigcup_{x\in\partial S_m} \{0 \leftrightarrow x\} \circ \{x \leftrightarrow z\}.$$

Therefore, using BK inequality we have that

$$\tau_p(0,z) \leq \sum_{x\in\partial S_m} \tau_p(0,x)\tau_p(x,z). \tag{2.12}$$

Using the trivial bound $\tau_p(x,z) \leq 1$ we can also write

$$\tau_p(0,z) \leq \sum_{x\in\partial S_m} \tau_p(0,x) = E[M_m].$$

Next, for any vertex z with $|z| = n$ and fixing an integer $m \geq 1$ we can write $n = mr + s$ with $0 \leq s < m$. Then, using (2.12) r times and using the translation invariance:

$$\begin{aligned} \tau_p(0,z) &\leq \sum_{x^1\in\partial S_m} \tau_p(0,x^1) \sum_{x^2\in\partial(x^1+S_m)} \tau_p(x^1,x^2) \; ... \\ &... \sum_{x^r\in\partial(x^{r-1}+S_m)} \tau_p(x^{r-1},x^r)\cdot\tau_p(x^r,z) \\ &\leq \left(\sum_{x\in\partial S_m} \tau_p(0,x)\right)^r \\ &= E[M_m]^r. \end{aligned} \tag{2.13}$$

Where x^i are the vertices we find on the boundary of the balls, and r is exactly $\lfloor n/m \rfloor = \lfloor |z|/m \rfloor$. To conclude, take an arbitrary vertex u and an integer $k \geq 1$, just by inclusion of events, translation invariance and FKG inequality we have that $\tau_p(0, ku) \geq \tau_p(0, u)^k$. Then, using (2.13) on $\tau_p(0, ku)$ we find that

$$\tau_p(0, u) \leq \tau_p(0, ku)^{1/k} \leq \left\{ E_p[M_m]^{\lfloor |ku|/m \rfloor} \right\}^{1/k} \to E_p[M_m]^{|u|/m}$$

as we make $k \to \infty$. □

Corollary 2.7. *If $\chi(p)$ is the mean cluster size function, then $\chi(p_c) = \infty$.*

Proof. It follows directly from Theorems 2.5 and 2.6. □

2.2 Box-crossing Probabilities

In this section, we explore the relationship between the function ξ and certain box-crossing probabilities. Specifically, we consider two types of crossings within the box $B_{N,M} := [0, N] \times [0, M]^{d-1}$:

- **Bond crossing ($\mathcal{C}_{N,M}$):** the existence of a path of open edges connecting the two opposite faces of $B_{N,M}$ along the direction of e_1.
- **Dual crossing ($\mathcal{D}_{N,M}$):** the presence of an unbroken hypersurface formed by dual cells that cross the box $B_{N,M}$ perpendicularly to e_1.

These objects will be defined throughout this section, however, we must give first an intuition on how they will be used. These two types of crossings behave differently depending on the dimensionality and the regime. In the supercritical phase of the two-dimensional lattice, bond crossings ($\mathcal{C}_{N,M}$) become more probable as the box size increases. Intuitively, this occurs because larger boxes are more likely to intersect the infinite open cluster characteristic of the supercritical regime. However, in dimensions $d > 2$, the situation changes. Since paths are one-dimensional objects, constructing a larger crossing by combining smaller ones becomes more difficult due to the additional spatial directions available for small crossings to deviate. In this case, dual crossings ($\mathcal{D}_{N,M}$) will be crucial for analyzing the subcritical regime.

Later in this section, we introduce a function $L_\lambda(p)$ that measures the box length L required for the adequate crossing probability $B_{L,2L}$ to exceed a given threshold depending on λ. This function will serve as a tool for rescaling probabilities in a way that will be formalized.

Our goal is to rigorously define these crossing events and establish a relationship between L_λ and ξ. This connection will provide insight into the asymptotic behavior of ξ and the bound for ν that we seek.

Definition 2.8. ***[Bond crossing*** $\mathcal{C}_{N,M}$***]*** *Consider the box* $B_{N,M} := [0, N] \times [0, M]^{d-1}$*. Let* F_0 *and* F_1 *denote the two opposite faces of* $B_{N,M}$ *along the* e_1 *direction. Then, we write* $\mathcal{C}_{N,M} := \{F_0 \xleftrightarrow{B_{N,M}} F_1\}$ *for the event where there exists a path of open edges in* $B_{N,M}$ *connecting* F_0 *and* F_1*. The probability of this event is denoted by* $C_{N,M} := P_p(\mathcal{C}_{N,M})$*.*

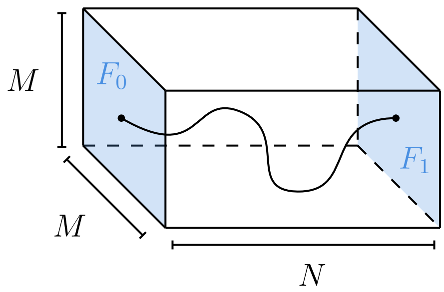


Figure 2.3: The bond crossing $C_{N,M}$.

To define the other type of crossing, we need to construct a dual space associated with the graph $\mathbb{L}^d$. For the two-dimensional hypercubic lattice $\mathbb{Z}^2$ we consider this dual space to be exactly the graph-theoretic dual for planar graphs, which in this case is also $\mathbb{Z}^2$. For dimensions $d > 2$ we will construct a space based on the same idea.

Definition 2.9. ***[The dual space of*** $\mathbb{L}^d$***]*** *Let* $d \geq 2$*. For each edge* $e \in \mathbb{E}^d$*, we associate a dual cell* σ_e*, that is a unitary hypercube of dimension* $d - 1$ *centered at the midpoint of* e*. Let* $\Sigma = \Sigma(d) = \{\sigma_e\}_{e \in \mathbb{E}^d}$ *be the dual space. For a configuration* $\omega \in \Omega$ *resulting from the usual percolation on* $\mathbb{L}^d$*, we declare that* σ_e *is open if and only if* e *is closed. This defines a space of configurations* $\Omega^* = \{0, 1\}^\Sigma = \{\omega^*(\sigma_e)\}_{\sigma_e \in \Sigma}$ *where the relation* $\omega^*(\sigma_e) = 1 - \omega(e)$ *holds.*

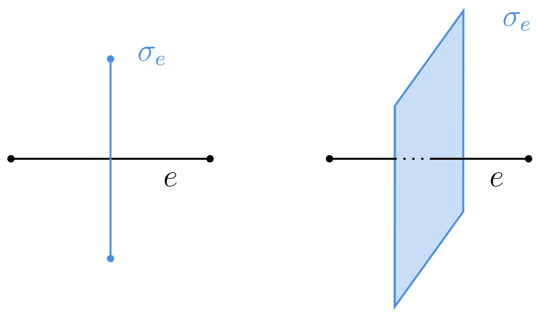


Figure 2.4: The dual cell σ_e of the edge e in $d = 2$ (left) and $d = 3$ (right).

Before proceeding with the next definitions and results, it is worth making a few observations about this construction. Firstly, the d-dimensional dual space Σ is a structure that heavily relies on the specific geometry of the d-dimensional hypercubic lattice, which may make it challenging to generalize to other types of lattices. Secondly, the dual crossing we are about to define represents a scenario where the box of the hypercubic lattice is traversed by a structure from the dual space. The term "dual" reflects the fact that this crossing exists precisely when the usual bond crossing within the same box does not occur. Intuitively, in the subcritical phase of the percolation model, as the size of a box increases, the likelihood of a dual crossing also increases, since the probability of a regular bond crossing becomes lower in this regime.

Definition 2.10. ***[Unbroken hypersurface]*** *Let $S \subset \Sigma$. We say that S is connected, and refer to it by a unbroken hypersurface, if for every pair $\sigma, \sigma' \in S$, there is a sequence $\{\sigma_k\}_{k=1}^n$, where $\sigma_1 = \sigma$ and $\sigma_n = \sigma'$ and for each $k = 1, ..., n-1$ we have that σ_k and σ_{k+1} are adjacent, that is, they share a common hyperface, where, by hyperface we mean a usual (d-2)-dimensional face of the (d-1)-hypercube. We also say that S is open, if σ is open for every $\sigma \in S$.*

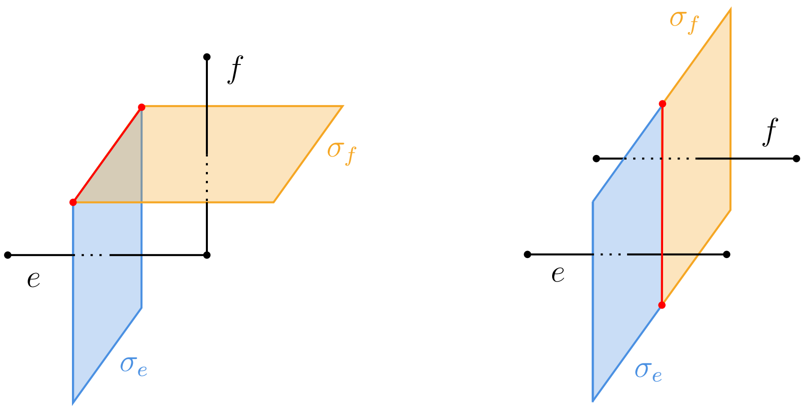


Figure 2.5: Two examples where σ_e and σ_f are adjacent ($d = 3$).

Definition 2.11. ***[Dual crossing $\mathcal{D}_{N,M}$]*** *Consider the box $B_{N,M} \coloneqq [0, N] \times [0, M]^{d-1}$. Let F_0 and F_1 be the opposite faces of $B_{N,M}$ along the e_1 direction. Then, we write $\mathcal{D}_{N,M}$ for the event where there is an open unbroken hypersurface S, where every path of edges of the graph $\mathbb{L}^d$ connecting F_0 and F_1 contains an edge e such that $\sigma_e \in S$. We refer to this unbroken hyperface with this property as dual crossing of the box. The probability of this event is denoted by $D_{N,M} = P_p(\mathcal{D}_{N,M})$.*

Now, consider both the box formed by bonds of the hypercubic lattice and the dual crossing composed of dual cells, embedded in the Euclidean space $\mathbb{R}^d$. Informally, this dual crossing acts as a (d-1)-dimensional surface, similar to the faces F_0 and F_1, slicing the box into two halves. It functions analogously to a bond crossing in a two-dimensional box.

The dual space is introduced because the strategy we aim to employ involves understanding how crossings of larger boxes become more probable by assum-

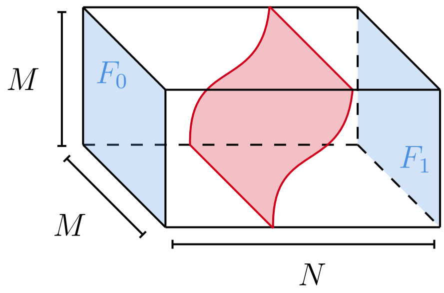


Figure 2.6: The dual crossing $D_{N,M}$.

ing that crossings in smaller boxes are already likely. To achieve this, we rely on patching arguments, that is, constructing larger crossings by combining smaller ones. In dimensions $d > 2$, however, this patching becomes significantly harder with bonds, which are one-dimensional structures.

The following remark is very straightforward.

Remark 2.12. $C_{N,M} = 1 - D_{N,M}$.

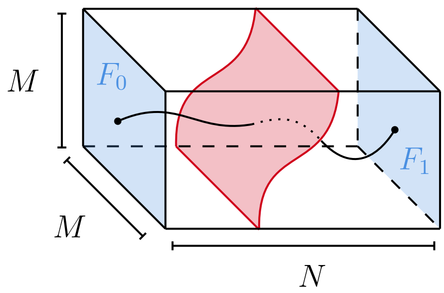


Figure 2.7: Duality of the two crossings.

Now, we provide an illustration of how these "patching arguments" work. First, let $d = 2$. Our goal is to demonstrate how a bond crossing of $B_{4L,L}$ can be constructed by combining multiple translated and rotated copies of any bond crossings of $B_{2L,L}$. This process is better explained in Figure 2.8.

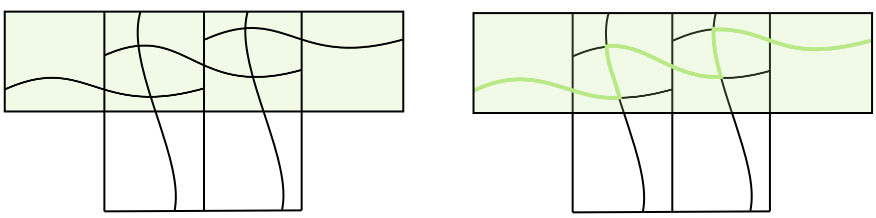


Figure 2.8: Patching argument in $d = 2$.

Notice that we used five paths, say $\rho_1, \ldots, \rho_5$, which are translated and rotated versions of crossings of $B_{2L,L}$. The union $\rho := \rho_1 \cup \ldots \cup \rho_5$ contains a bond crossing of the box $B_{4L,L}$. The specific path used in each rotated and translated smaller box is irrelevant. Once we determine how many such boxes are needed, along with their positions and orientations, simply ensuring that these boxes are crossed guarantees the existence of the larger crossing. This process of taking unions of rotated and translated crossings to ensure the existence of a larger crossing is what we call *patching*.

In dimensions $d > 2$, this approach will be applied to dual crossings in a similar fashion. For dual crossings, the result we will prove later uses crossings of $B_{L,2L}$ to construct a larger dual crossing for the box $B_{L,4L}$. It is evident that a finite number of such smaller dual crossings can be patched together to achieve this goal. However, calculating the exact number of crossings required is challenging. Therefore, we introduce the term *patching constant*, denoted by $v = v(d)$, to represent the minimum number of patchings needed. Naturally, this number depends on the dimension d.

The next two lemmas establish how the probability of a larger crossing becomes large as the probability of a smaller crossing surpasses a threshold. These results are known as the ACCFR (Aizenman, Chayes, Chayes, Fröhlich, Russo) 2-dimensional and d-dimensional Rescaling Lemmas.

Lemma 2.13. ***[ACCFR 2-dimensional Re-scaling Lemma]*** *Suppose $d = 2$. Let $c := 1/25$ and $\lambda \in (0, 1)$. If $C_{2L,L} \geq 1 - c\lambda$, then $C_{4L,2L} \geq 1 - c\lambda^2$.*

Proof. Consider the box $B_{4L,2L}$. It is possible to split this box horizontally into two disjoint $4L$ by L boxes B_1 and B_2. That is, $B_1 := [0, 4L] \times [0, L]$ and $B_2 := [0, 4L] \times [L, 2L]$. By inclusion of events, the existence of a bond crossing of either B_1 or B_2 implies the bond crossing of $B_{4L,2L}$. Since these boxes are disjoint, their crossings occur independently and with the same probability, we obtain

$$1 - C_{4L,2L} \leq (1 - C_{4L,L})^2. \tag{2.14}$$

Now it is just a matter of bounding the right hand side in terms of $C_{2L,L}$. We are going to do this by creating a crossing for $B_{4L,L}$ using five crossed L by $2L$ boxes since we can bound their probabilities from below by the assumption. Recall Figure 2.8. Now, writing $A_1, ..., A_5$ for the events where these translated and rotated $2L$ by L boxes $B_1, ..., B_5$ are crossed, we have:

$$\begin{aligned} C_{4L,L} &\geq P_p(A_1 \cap ... \cap A_5) \\ &\geq (C_{2L,L})^5 \\ &\geq (1 - c\lambda)^5 \\ &\geq 1 - 5c\lambda. \end{aligned} \tag{2.15}$$

Now, just combining (2.14) and (2.15) we get that

$$C_{4L,2L} \geq 1 - (5c\lambda)^2 = 1 - c\lambda^2.$$

□

For dimensions $d > 2$ we prove a similar result.

Lemma 2.14. ***[ACCFR d-dimensional Re-scaling Lemma]*** *Suppose $d > 2$. Let $c(d) := v(d)^{-2}$ where $v(d)$ is the patching constant and $\lambda \in (0, 1)$. If $D_{L,2L} \geq 1 - c(d)\lambda$, then $D_{2L,4L} \geq 1 - c(d)\lambda^2$.*

Proof. Consider the box $B_{2L,4L}$. It is possible to split this box vertically into two disjoint $L \times 4L \times ... \times 4L$ sided boxes B_1 and B_2. That is, $B_1 := [0, L] \times [0, 4L]^{d-1}$ and $B_2 := [L, 2L] \times [0, 4L]^{d-1}$. By inclusion of events, the existence of a dual crossing of either B_1 or B_2 implies the dual crossing of $B_{2L,4L}$.

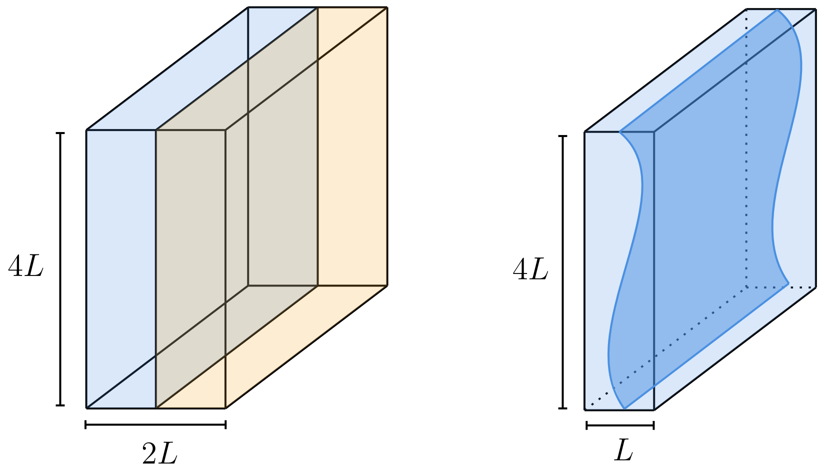


Figure 2.9: Box B_1 is crossed.

Since these boxes are disjoint, their crossings occur independently and with the same probability, we obtain

$$1 - D_{2L,4L} \leq (1 - D_{L,4L})^2. \tag{2.16}$$

Now it is just a matter of bounding the right hand side in terms of $D_{L,2L}$. We are going to do this by *patching* rotation and translations of dual crossings of $B_{L,2L}$. The goal is to obtain a crossing $\mathcal{D}_{L,4L}$ using a certain amount $v = v(d)$ of crossed $L \times 2L \times ... \times 2L$ boxes, since we can bound their probabilities from below by the assumption. For $i = 1, ..., v$, we denote by A_i the event where the i-th such box presents a dual crossing.

$$\begin{aligned} D_{L,4L} &\geq P_p(A_1 \cap ... \cap A_v) \\ &\geq (D_{L,2L})^v \\ &\geq (1 - c\lambda)^v \\ &\geq 1 - cv\lambda. \end{aligned} \tag{2.17}$$

Now, just combining (2.16) and (2.17) we get that

$$D_{2L,4L} \geq 1 - (cv\lambda)^2 = 1 - c\lambda^2.$$

□

At this point, it is natural to define, for each p, the smallest length L of the box $B_{L,2L}$ such that the probability of the crossings exceeds the threshold required to satisfy the rescaling lemmas.

Definition 2.15. ***[Supercitical Correlation Length** ($d = 2$)**]*** *For $p \in (p_c, 1]$, and $\lambda \in (0,1)$ we define*

$$L^*_\lambda(p) := \min\{L \geq 1 \ ; \ C_{2L,L} \geq 1 - c(2)\lambda\}.$$

Definition 2.16. ***[Subcritical Correlation Length** ($d \geq 3$)**]*** *Let $d > 2$. For $p \in [0, p_c)$ and $\lambda \in (0,1)$ we define*

$$L_\lambda(p) := \min\{L \geq 1 \ ; \ D_{L,2L} \geq 1 - c(d)\lambda\}.$$

These functions are in fact well defined, that is, they assume finite values for their respective restrictions on values of p. The reason is that, in two dimensions, if $p > p_c$ then $C_{2L,L}(p) \to 1$ as $L \to \infty$, and in dimensions $d > 2$ if $p < p_c$ then $D_{L,2L}(p) \to 1$ as $L \to \infty$. The proof for the two-dimensional case can be found in Chayes and Chayes [4], for dimensions $d > 2$ we can adapt that proof by using unbroken hypersurfaces in the dual space. However, we do not present these proofs here, since they are rather technical and long.

Next, we have an important theorem that says that ξ and L_λ are equivalent in some sense.

Theorem 2.17. *Let $p < p_c$, $\lambda \in (0,1)$, and $d > 2$. There exist nonzero, finite, constants c_1, c_2 and c_3, depending on d and λ, such that*

$$\frac{c_1 \log(L_\lambda(p)) + c_2}{L_\lambda(p) - 1} \geq \frac{1}{\xi(p)} \geq \frac{c_3}{L_\lambda(p)}.$$

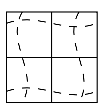
Proof. Let $p < p_c$. and $L = 2^k L_\lambda$. Since $L_\lambda = \min\{L \geq 1 \ ; \ D_{L,2L} \geq 1 - c(d)\lambda\}$ we have that $D_{L_\lambda, 2L_\lambda} \geq 1 - c(d)\lambda$. Using Lemma 2.14 k times, it follows that

$$D_{L,2L} = D_{2^k L_\lambda, 2^{k+1} L_\lambda} \geq 1 - c(d)\lambda^{2^k}. \tag{2.18}$$

Also, by the inclusion of events and FKG inequality, it follows that

$$1 - \tau_L(p) = P_p(0 \nleftrightarrow Le_1)$$

$$\geq P_p\left\{ \right\} \tag{2.19}$$



$$\geq (D_{L,2L})^{2d}.$$

Since the origin can be enclosed by $2d$ open dual crossings $\mathcal{D}_{L,2L}$, this ensures that no open path of bonds can connect the origin to the vertex Le_1. Combining equations (2.19) and (2.18) we get

$$\begin{aligned}\tau_L(p) &\leq 1 - \{1 - c(d)\lambda^{2^k}\}^{2d} \\ &\leq 2dc(d)\lambda^{2^k}.\end{aligned}$$

Taking logarithms, dividing by L and recalling that $2^k = L/L_\lambda$ we obtain

$$\begin{aligned}\frac{\log \tau_L(p)}{L} &\leq \frac{\log(2dc(d))}{L} + \frac{2^k \log(\lambda)}{L} \\ &= \frac{\log(2dc(d))}{L} + \frac{\log(\lambda)}{L_\lambda}.\end{aligned}$$

Making $L \to \infty$ we find that $c_3/L_\lambda \leq 1/\xi$, where $c_3 = -\log(\lambda)$.

By the definition of L_λ, it follows that $D_{(L_\lambda-1),2(L_\lambda-1)} \leq 1 - c(d)\lambda$. Notice that, the dual-crossing of $B_{(L_\lambda-1),2(L_\lambda-1)}$ does not occur, if, and only if there is a path connecting some vertex x of F_0 to another vertex y of F_1, using the notation from Definition 2.8 for the box $B_{(L_\lambda-1),2(L_\lambda-1)}$:

$$\begin{aligned}c(d)\lambda &\leq 1 - D_{(L_\lambda-1),2(L_\lambda-1)} \\ &= P_p\left(\bigcup_{x\in F_0}\bigcup_{y\in F_1}\{x \leftrightarrow y\}\right) \\ &\leq \sum_{x\in F_0}\sum_{y\in F_1}\tau_p(x,y) \\ &\leq \sum_{x\in F_0}\sum_{y\in F_1}\exp\left(\frac{-(L_\lambda-1)}{\xi}\right) \\ &\leq (2L_\lambda)^{2(d-1)}\exp\left(\frac{-(L_\lambda-1)}{\xi}\right).\end{aligned}$$

Summarizing,

$$c(d)\lambda \leq (2L_\lambda)^{2(d-1)}\exp\left(\frac{-(L_\lambda-1)}{\xi}\right).$$

Rearranging and taking logarithms we get:

$$\log(c(d)\lambda 2^{-2(d-1)}) \leq 2(d-1)\log L_\lambda - \frac{(L_\lambda-1)}{\xi}.$$

Therefore,

$$\frac{1}{\xi} \leq \frac{c_1 \log L_\lambda + c_2}{L_\lambda - 1},$$

where $c_1 = 2(d-1)$ and $c_2 = -\log(c(d)\lambda 2^{-2(d-1)})$. □

2.3 A Lower Bound For ν

At this point, we have collected all the necessary tools and results to prove the bound $\nu \geq 2/d$. This bound will follow directly from the asymptotic behavior described in the following result:

Theorem 2.18. *Let $d > 2$ and ξ be the fundamental correlation length. Then*

$$\liminf_{p \uparrow p_c} \frac{\log(\xi(p))}{|\log(p_c - p)|} \geq \frac{2}{d}.$$

Proof. We start by creating an auxiliary function. Set

$$\psi = \psi(p) \coloneqq \max\{L \geq 1; C_{L,2L} > M\},$$

where $M = (c(d)e^{-1})/2$. Notice that, for L_λ as in Definition 2.16, the relation $L_\lambda - 1 \leq \psi$ holds, where $\lambda = e^{-1}/2$.

Now, assume that $C_{L,2L} \leq C_{L_\lambda,2L_\lambda}$ holds for every $L \geq 2L_\lambda$, (we are going to prove this later in Lemma 2.20). The definition of ψ implies that $\psi \leq 2L_\lambda$. Therefore, we have bounded ψ from above and below in terms of L_λ, for which we already know, by Theorem 2.17, how to compare with ξ.

Now, we study the behaviour of the function ψ, by proving that there is a function $f(p)$, bounded away from 0 near p_c from below, for which

$$\psi(p_S) \geq \{f(p_S)(p_c - p_S)\}^{-2/d},$$

for some sequence $\{p_S\}_{S\geq 1}$ converging to p_c.

Set $g_L(p) \coloneqq C_{L,2L}(p)$. Since $B_{L,2L}$ is a finite box, for every $L \geq 1$, we have that $g_L(p)$ is a polynomial on p, and therefore it is differentiable. Write $|B_{L,2L}| \coloneqq A(L)$. By Theorem 1.15, we have the square root estimate

$$\left|\frac{dg_L(p)}{dp}\right| \leq \alpha(p)\sqrt{A(L)},$$

where $\alpha(p) \coloneqq 1/\sqrt{p(1-p)}$. This is used to obtain the following estimate for $p < p_c$

$$\begin{aligned} |g_L(p) - g_L(p_c)| &= \left|\int_p^{p_c} \frac{dg_L(p)}{dp} dp\right| \\ &\leq \int_p^{p_c} \left|\frac{dg_L(p)}{dp}\right| dp \\ &\leq H_L(p)(p_c - p), \end{aligned}$$

where

$$\begin{aligned} H_L(p) &= \frac{KL^{d/2}}{p_c - p} \int_p^{p_c} |\alpha(p)| dp \\ &= \frac{KL^{d/2}}{p_c - p} \int_p^{p_c} \sqrt{\frac{1}{u(1-u)}} du, \end{aligned}$$

since $\sqrt{A(L)} \leq KL^{d/2}$ for some constant $K > 0$ and every L big enough. Therefore

$$g_L(p) \geq g_L(p_c) - H_L(p)(p_c - p).$$

Since $\xi(p_c) = \infty$ by Theorem 2.5, the estimate in Theorem 2.17 tells that $L_\lambda(p) = \min\{L \geq 1 \,;\, g_L(p) \leq c\lambda\}$ also diverges at p_c, for every $\lambda \in (0,1)$. Taking $\lambda = 2M$ we have that $g_L(p_c) \geq 2M = c(d)e^{-1}$ for every $L \geq 1$. Hence,

$$g_L(p) \geq 2M - H_L(p)(p_c - p). \tag{2.20}$$

Notice that, as we increase S, we can find a sequence of parameters p_S converging to p_c such that the following equality holds:

$$\int_{p_S}^{p_c} \frac{1}{\sqrt{s(1-s)}} ds = \frac{M}{S^{d/2}}.$$

By multiplying both sides of the last equality with $(p_c - p_S)$ it is possible to see that this sequence is such that $p_S = p_c - M/H_S(p_S)$. Therefore, using (2.20), $g_S(p_S) \geq M$ holds for $S \geq 1$. By the definition of ψ:

$$\begin{aligned} \psi(p_S) &= \max\{L \geq 1; g_L(p_S) > M\} \\ &\geq S \\ &= f(p_S)^{-2/d}(p_c - p_S)^{-2/d}. \end{aligned}$$

Where

$$f(p) := \frac{K}{M(p_c - p)} \int_p^{p_c} \sqrt{\frac{1}{u(1-u)}} du \geq \frac{2K}{M} > 0.$$

Since $f(p)$ is bounded away from 0 near p_c from below, since $\sqrt{1/(x(1-x))} \geq 2$ if $x \in (0,1)$. Therefore,

$$\begin{aligned} \limsup_{p \uparrow p_c} \frac{\log \psi(p)}{|\log(p_c - p)|} &\geq \lim_{p_S \to p_c} \frac{\log \psi(p_S)}{|\log(p_c - p_S)|} \\ &\geq \lim_{p_S \to p_c} \frac{\log(f(p_S)^{-2/d}(p_c - p_S)^{-2/d})}{|\log(p_c - p_S)|} \\ &\geq -\frac{2}{d} \lim_{p_S \to p_c} \left(\frac{\log(2K/M) + \log(p_c - p_S)}{|\log(p_c - p_S)|} \right) \\ &= \frac{2}{d}. \end{aligned}$$

From Theorem 2.17 we have that

$$\xi(p) \geq \frac{L_\lambda(p) - 1}{c_1 \log L_\lambda(p) + c_2}.$$

Using the fact that $L_\lambda(p) \to \infty$ as $p \uparrow p_c$ and that, for every $\varepsilon > 0$, and real numbers A and B follows that

$$x^{1-\varepsilon} = o\left(\frac{x-1}{A\log(x) + B}\right) \text{ as } x \to \infty,$$

we can find a neighborhood around p_c for which $\xi(p) \geq (L_\lambda(p))^{1-\varepsilon}$ for every p. Therefore, for every $\varepsilon > 0$ we found region around p_c where

$$\xi(p) \geq (L_\lambda(p))^{1-\varepsilon} \geq \left(\frac{1}{2}\psi(p)\right)^{1-\varepsilon}.$$

Applying the natural logarithm to both sides:

$$\begin{aligned}\liminf_{p\uparrow p_c} \frac{\log \xi(p)}{|\log(p_c - p)|} &\geq (1-\varepsilon) \liminf_{p\uparrow p_c} \frac{\log(1/2) + \log(\psi(p))}{|\log(p_c - p)|} \\ &= (1-\varepsilon)\frac{2}{d}.\end{aligned}$$

We get the result making $\varepsilon \to 0$. □

Corollary 2.19. *If the critical exponent ν exists, then $\nu \geq 2/d$.*

Proof. If the critical exponent ν exists, then, using Theorem 2.18 and the definition of ν we get

$$\nu = -(-\nu) = \lim_{p\uparrow p_c} \frac{\log(\xi(p))}{|\log((p_c - p)|} \geq \frac{2}{d}.$$

□

Lemma 2.20. *$C_{L,2L} \leq C_{L_\lambda,2L_\lambda}$ holds for every $L \geq 2L_\lambda$.*

Proof. Notice first that $C_{N.M}$ is increasing in M and decreasing in N. For this reason, if $k \leq 2L$, then

$$C_{2L+k,2(2L+k)} \leq C_{2L,8L}.$$

And, since the existence of a bond crossing $C_{2L,8L}$ implies the existence of two disjoint bond crossings of L by $8L$ boxes we have

$$C_{2L+k,2(2L+k)} \leq C_{2L,8L} \leq C^2_{L,8L}.$$

If the bond crossing of $B_{L,8L}$ does not occur, we have a dual crossing of this box in the sense we defined previously. We can obtain such dual crossing of $B_{L,8L}$ by patching dual crossings of boxes $B_{L,2L}$. Let $A(d)$ be the patching constant for $B_{L,8L}$ with boxes $B_{L,2L}$. Then, by FKG:

$$(1 - C_{L,2L})^{A(d)} \leq 1 - C_{L,8L}.$$

Implying that $1 - A(d)C_{L,2L} \leq 1 - C_{L,8L}$. Recall that $g_L(p) = C_{L,2L}$. Combining what we have obtained so far

$$\begin{aligned} g_p(2L+k) &= C_{2L+k,2(2L+k)} \\ &\leq C_{2L,8L} \\ &\leq C_{L,8L}^2 \\ &\leq A(d)^2 C_{L,2L}^2 \\ &= A(d)^2 g_p(L)^2. \end{aligned}$$

Iterating the result m times, we get for $k \leq 2^m L$, that

$$g_p(2^m L + k) \leq \left(A(d)^2 g_p(L) \right)^{2^m - 1} g_p(L).$$

By the definition of L_λ, it is possible to choose λ small enough such that $A(d)^2 g_p(L_\lambda) \leq 1$. Thus, for every $m \geq 1$ and $k \leq 2^m L_\lambda$

$$g_p(2^m L_\lambda + k) \leq g_p(L_\lambda).$$

This means that, for every $L \geq 2L_\lambda$ it follows that $C_{L,2L} = g_p(L) \leq g_p(L_\lambda) = C_{L_\lambda,2L_\lambda}$. □

Chapter 3

Aizenman-Barsky Mean Field Bounds

In this chapter, we present the Aizenman-Barsky construction and some results about the critical exponents that can be proven with it. The first two sections describe the construction and prove two differential inequalities.

To prove these inequalities, we truncate the graph and use the FKG and BK inequalities along with Russo's Formula. Then, we take the appropriate limits to extend these inequalities to functions defined on the infinite graph. This construction will be revisited in the last chapter, where we use a different method to prove other differential inequalities.

The next sections of this chapter focus on results for the critical exponents β, γ, and δ, respectively. For these results, we refer to Chapters 5 and 10 of *Percolation* by Grimmett [8], Chapter 3 of *Progress in High-Dimensional Percolation and Random Graphs* by Heydenreich nad van der Hofstad [9] and Chapter 3 of *Independent and Dependent Percolation* by Chayes, Puha and Sweet [3].

3.1 The Ghost Vertex Construction

Let $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ represent the usual hypercubic lattice. First, we add a new vertex to this graph, which we call g. Sometimes it will be referred to as the *ghost vertex* and we can think of it as a point "at infinity". With this new vertex, we also create new edges that connect the original vertices to g. In this way, we construct a new graph $\mathbb{G}^d$, with the following sets of vertices and edges:

$$\mathbb{G}^d = (\mathcal{Z}^d, \mathcal{E}^d) \coloneqq (\mathbb{Z}^d \cup \{g\}, \mathbb{E}^d \cup \mathcal{G}^d).$$

Where $\mathcal{G}^d = \{xg\}_{x \in \mathbb{Z}^d}$. On this new graph, we perform the usual Bernoulli percolation with parameter $p \in (0,1)$ on the edges of $\mathbb{E}^d$. Additionally, we perform a Bernoulli percolation with parameter $s \in (0,1)$ on the edges xg, ensuring that these two processes are independent. For this two-parameter percolation, we also consider a new configuration space Ω:

$$\Omega \coloneqq \{0,1\}^{\mathbb{E}^d} \times \{0,1\}^{\mathcal{G}^d}.$$

We denote the new percolation measure by $P_{p,s}$, which is the natural product measure defined for each pair $(p,s) \in (0,1)^2$. The corresponding expected value is denoted by $E_{p,s}$.

This percolation process on $\mathbb{G}^d$ can also be seen as a simultaneous bond and vertex percolation on the graph $\mathbb{Z}^d$. Here, each bond is kept with probability p, and each vertex is "colored green" with probability s. Using this interpretation, we can often omit the vertex g from expressions, focusing instead on certain subsets of vertices and edges of $\mathbb{Z}^d$ to express the quantities of interest. Let $G = G(\omega)$ be the random subset of green vertices of $\mathbb{Z}^d$, that is, the vertices for which the edge xg is open. As before, let $C = C(\omega)$ denote the open cluster at the origin.

The main function studied in this chapter is the two-parameter percolation function, defined as:

$$\Theta = \Theta(p,s) \coloneqq P_{p,s}(C \cap G \neq \varnothing).$$

In words, this function gives the probability, for given p and s, that the cluster of the origin contains at least one green vertex. The next result shows that the standard percolation function can be recovered by taking the limit $s \downarrow 0$ in Θ.

Lemma 3.1. *If $s \downarrow 0$ then $\Theta(p,s) \downarrow \theta(p)$.*

Proof. We can write $\Theta(p,s) = 1 - P_{p,s}(C \cap G = \varnothing)$. Then, if we make a decomposition of the space with respect to the cluster size $|C|$ we get:

$$\begin{aligned} 1 - \Theta(p,s) = \sum_{n \geq 1} & P_{p,s}(C \cap G = \varnothing \,|\, |C| = n) P_p(|C| = n) + \\ & + P_{p,s}(C \cap G = \varnothing \,|\, |C| = \infty) P_p(|C| = \infty). \end{aligned}$$

Notice that if $s > 0$ then, $P_{p,s}(C \cap G = \varnothing \,|\, |C| = \infty) = \prod_{i=1}^{\infty}(1-s) = 0$ meaning that the last term of the right side vanishes. For the other terms of the sum, if $C \cap G = \varnothing$ when $|C| = n$, then none of the n vertices in C are green, and that occurs with probability exactly $(1-s)^n$. Rewriting the expression above with these observations we have:

$$\Theta(p,s) = 1 - \sum_{n \geq 1} (1-s)^n P_p(|C| = n). \tag{3.1}$$

Since $\sum_{n\geq 1} P_p(|C|=n) < \infty$, using the Abel's Theorem for power series

$$F(1-s) := \sum_{n\geq 1}(1-s)^n P_p(|C|=n) \uparrow \sum_{n\geq 1} P_p(|C|=n),$$

as we make $s \downarrow 0$. Therefore:

$$\Theta(p,s) \downarrow 1 - \sum_{n\geq 1} P_p(|C|=n) = \theta(p).$$

□

Another important function that will be used in this chapter is the expected value of the cluster size $|C|$ on the event where the origin is not connected to any green vertex by a path of open edges, that is the function $\mathcal{X}$ given as:

$$\mathcal{X} = \mathcal{X}(p,s) := E_{p,s}(|C|; C \cap G = \varnothing).$$

Lemma 3.2. *If $s \downarrow 0$ then $\mathcal{X}(p,s) \uparrow \chi^f(p)$.*

Proof. As before, we can write $\mathcal{X}(p,s)$ as

$$\mathcal{X}(p,s) = \sum_{n\geq 1} nP_{p,s}(C \cap G = \varnothing \,|\, |C|=n) P_p(|C|=n).$$

Since we can calculate $P_{p,s}(C \cap G = \varnothing \,|\, |C|=n) = (1-s)^n$:

$$\mathcal{X}(p,s) = \sum_{n\geq 1} n(1-s)^n P_p(|C|=n).$$

Therefore, if we make $s \downarrow 0$ we get that $\mathcal{X}(p,s) \uparrow \chi^f(p)$, recalling that $\chi^f(p) = \sum_{n\geq 1} nP_p(|C|=n)$. □

Notice that $\Theta(p,s)$ as given by (3.1) is a power series with the radius of convergence at least 1. For this reason we can differentiate Θ with respect to s to get to the relation:

$$\frac{\partial}{\partial s}\Theta(p,s) = \sum_{n\geq 1} n(1-s)^{n-1} P_p(|C|=n) = \frac{1}{1-s}\mathcal{X}(p,s). \tag{3.2}$$

Another property worth mentioning is a linear lower bound for Θ in terms of the parameter s.

Proposition 3.3. *If $s, p \in [0,1]$, then $\Theta(p,s) \geq s$.*

Proof. If $0 \leq s \leq 1$, then $(1-s)^k \leq 1-s$ for every $k \geq 1$. Therefore, multiplying both sides by a positive term gives us that for every $k \geq 1$

$$(1-s)^k P_p(|C|=k) \leq (1-s)P_p(|C|=k).$$

Finally, summing for k:

$$1-\Theta \leq 1-s.$$

□

3.2 Two Differential Inequalities

The goal of this section is to discuss the proof of the two important differential inequalities:

Lemma 3.4. *[Differential Inequality A] If $s \in (0,1)$ and $p \in (0,1)$, then:*

$$(1-p)\frac{\partial \Theta}{\partial p} \leq 2d(1-s)\Theta\frac{\partial \Theta}{\partial s}.$$

Lemma 3.5. *[Differential Inequality B] If $s \in (0,1)$ and $p \in (0,1)$, then:*

$$\Theta \leq s\frac{\partial \Theta}{\partial s} + \Theta^2 + p\Theta\frac{\partial \Theta}{\partial p}.$$

These differential inequalities were first introduced by Aizenman and Barsky (1987) [1]. Since the tools available to us are restricted to finite environments, we must construct a finite-volume approximation. To preserve translation invariance in a finite setting, we impose periodic boundary conditions. Below, we provide an informal outline of this construction to avoid excessive technical detail.

For each $N \geq 1$, we define a truncated version of the graph $\mathbb{G}^d$ as follows. Consider the box $\Lambda_N = [-N,N]^d$, and let I be any subset of $[d] \coloneqq \{1,...,d\}$. We then identify opposite faces of the cube by merging into a single vertex the set:

$$X(I) \coloneqq \{x \in \Lambda_N; |x_i| = N \text{ for every } i \in I \text{ , and } x_i = z_i \text{ for } j \notin I\},$$

where z ranges over $\{-N+1,...,N-1\}^d$. We also identify parallel edges. This construction can be vizualized by imagining that Λ_N is embedded in the torus.

This process generates a new set of vertices $Z(N)$, and a corresponding set of edges $E(N)$, forming the graph $L(N) = (Z(N), E(N))$ where every vertex $x \in Z(N)$ have $2d$ neighboors. Adding the ghost vertex g and reproducing the construction we did before, we define the graph:

$$\mathbb{G}^d_N = (\mathcal{Z}(N), \mathcal{E}(N)) = (Z(N) \cup \{g\}, E(N) \cup \cup_{x \in Z(N)}\{xg\}).$$

We also denote by G_N the random subset of green vertices in $Z(N)$ and by $C_N(x)$ the open cluster containing the vertex x, and we write $C_N = C_N(0)$. As in the previous construction, we define:

$$\Theta_N(p,s) := P_{p,s}(C_N \cap G_N \neq \varnothing),$$

and

$$\chi_N(p,s) := E_{p,s}[|C_N|; C_N \cap G_N = \varnothing].$$

For a discussion on the convergence of the truncated functions, see Section A.

In the next two subsections, we establish the differential inequalities stated in Lemmas 3.4 and 3.5 for the truncated functions. The desired result then follows by taking the limit as $N \to \infty$.

Proof of Lemma 3.4. Let both p and s take values in $(0,1)$. We apply Russo's formula to the event $\{C_N \cap G_N \neq \varnothing\}$. To do this, we must first condition on the event $\{G_N = \Gamma\}$, where Γ is a subset of vertices. Under this conditioning, the probability of the event $\{C_N \cap \Gamma \neq \varnothing\}$ depends only on the parameter p.

Let $A = A_N(\Gamma) = \{C_N \cap \Gamma \neq \varnothing\}$. Notice that this event is increasing and depends only on a finite number of edges. Thus, by Russo's formula, the derivative of its probability with respect to p is given by:

$$\frac{d}{dp}P_p(A) = \sum_{e \in E(N)} P_p(e \text{ is pivotal for } A).$$

Let us carefully analyze what it means for an edge $e = xy$ to be pivotal for the event A. On one hand, if the edge $e = xy$ is pivotal for A , then in the graph $\mathbb{G}_N^d - \{e\}$ the following must happen:

1. $0 \not\leftrightarrow \Gamma$.

2. Exactly one of the two vertices x and y belongs to C_N.

3. The other vertex is connected to Γ by a path of open edges.

These conditions follow directly from the definition of an edge being pivotal. On the other hand, if conditions $1, 2$ and 3 hold, the edge e is necessarily pivotal for the event A. Since the property of being pivotal for an event does not depend on the state of the edge itself, the events A and $\{e \text{ is closed}\}$ are independent:

$$\begin{aligned}(1-p)\frac{d}{dp}P_p(A) &= \sum_{e \in E(N)} (1-p)P_p(e \text{ is pivotal for } A)\\ &= \sum_{e \in E(N)} P_p(e \text{ is closed}, e \text{ is pivotal for } A).\end{aligned}$$

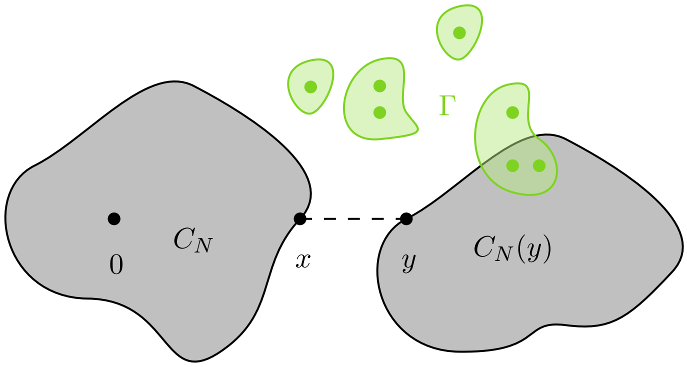


Figure 3.1: Edge $e = xy$ being pivotal for the event A.

Since the pivotal edge e being closed implies that the edge e is removed from the graph $\mathbb{G}_N^d$, we can rewrite the last equation based on our observations about the occurrence of the event A:

$$(1-p)\frac{d}{dp}P_p(A) = \sum_{x\sim y} P_p(0 \nleftrightarrow \Gamma, x \in C_N, C_N(y) \cap \Gamma \neq \varnothing). \tag{3.3}$$

Considering the finite collection $\mathcal{G}$ of all such subsets Γ, we establish the following relation:

$$\begin{aligned}
&\sum_{\Gamma\in\mathcal{G}} P_s(G_N = \Gamma)\frac{d}{dp}P_p(A) \\
&= \frac{\partial}{\partial p}\sum_{\Gamma\in\mathcal{G}} P_s(G_N = \Gamma)P_p(A) \\
&= \frac{\partial}{\partial p}\sum_{\Gamma\in\mathcal{G}} P_s(G_N = \Gamma)P_p(C_N \cap \Gamma \neq \varnothing) \\
&= \frac{\partial}{\partial p}\sum_{\Gamma\in\mathcal{G}} P_{p,s}(G_N = \Gamma, C_N \cap \Gamma \neq \varnothing) \\
&= \frac{\partial}{\partial p}\sum_{\Gamma\in\mathcal{G}} P_{p,s}(G_N = \Gamma, C_N \cap G_N \neq \varnothing) \\
&= \frac{\partial}{\partial p}P_{p,s}(C_N \cap G_N \neq \varnothing) \\
&= \frac{\partial}{\partial p}\Theta_N(p,s).
\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we get a useful expression for the partial derivative

$\partial_p\Theta$:

$$\begin{aligned}
&(1-p)\frac{\partial}{\partial p}\Theta_N(p,s)\\
&=\sum_{\Gamma\in\mathcal{G}} P_s(G_N=\Gamma)(1-p)\frac{d}{dp}P_p(A)\\
&=\sum_{\Gamma\in\mathcal{G}} P_s(G_N=\Gamma)\sum_{x\sim y} P_p(0\not\leftrightarrow\Gamma, x\in C_N, C_N(y)\cap\Gamma\neq\varnothing)\\
&=\sum_{x\sim y}\sum_{\Gamma\in\mathcal{G}} P_{p,s}(G_N=\Gamma, 0\not\leftrightarrow\Gamma, x\in C_N, C_N(y)\cap\Gamma\neq\varnothing)\\
&=\sum_{x\sim y}\sum_{\Gamma\in\mathcal{G}} P_{p,s}(G_N=\Gamma, 0\not\leftrightarrow G_N, x\in C_N, C_N(y)\cap G_N\neq\varnothing)\\
&=\sum_{x\sim y} P_{p,s}(0\not\leftrightarrow G_N, x\in C_N, C_N(y)\cap G_N\neq\varnothing).
\end{aligned}$$

Summarizing:

$$(1-p)\frac{\partial}{\partial p}\Theta_N(p,s)=\sum_{x\sim y} P_{p,s}(0\not\leftrightarrow G_N, x\in C_N, C_N(y)\cap G_N\neq\varnothing). \tag{3.5}$$

The previous equation is important and will also be used in the final step of the proof of Lemma 3.5. Now, we aim to find a way to bound the right-hand side of Equation (3.5). To achieve this, we condition the event inside the summation on $\{C_N=\Sigma\}$ where Σ is a finite, connected subset of vertices of $Z(N)$ that contains the origin and the vertex x, but does not include y.

Since also $\{x\in C_N, C_N=\Sigma\}=\{C_N=\Sigma\}$, we write the right-hand side of (3.5) as:

$$=\sum_{x\sim y}\sum_{\Sigma} P_{p,s}(0\not\leftrightarrow G_N, C_N(y)\cap G_N\neq\varnothing|C_N=\Sigma)P_p(C_N=\Sigma).$$

It follows that, conditioned on $\{C_N=\Sigma\}$, the events $\{0\not\leftrightarrow G_N\}$ and $\{C_N(y)\cap G_N\neq\varnothing\}$ are independent. The former depends only on the edges incident to vertices of Σ, while the latter depends only on the edges incident to vertices of Σ^C. Consequently, the probability inside the summation is:

$$P_{p,s}(0\not\leftrightarrow G_N|C_N=\Sigma)P_{p,s}(C_N(y)\cap G_N\neq\varnothing|C_N=\Sigma)P_p(C_N=\Sigma).$$

Informally speaking, since $\{C_N(y)\cap G_N\neq\varnothing\}$ is an increasing event, its probability under the conditional measure $P_{p,s}(\ \cdot\ |C_N=\Sigma)$ will be no greater than its probability under $P_{p,s}$, as the first measure restricts the possible configurations by fixing the realization of a set of edges. Therefore, the second factor in the expression above can be bounded by $P_{p,s}(C_N(y)\cap G_N\neq\varnothing)$ that corresponds exactly to $\Theta_N(p,s)$ due to translation invariance. Thus,

$$\begin{aligned}(1-p)\frac{\partial}{\partial p}\Theta_N(p,s) &\le \sum_{x\sim y}\sum_{\Sigma} P_{p,s}(0 \not\leftrightarrow G_N | C_N = \Sigma) P_p(C_N = \Sigma)\Theta_N \\ &= \Theta_N \sum_{x\sim y} P_{p,s}(x \in C_N, y \notin C_N, 0 \not\leftrightarrow G_N) \\ &\le \Theta_N \sum_{x\sim y} P_{p,s}(x \in C_N, 0 \not\leftrightarrow G_N) \\ &= \Theta_N 2d \sum_{x\in V(N)} P_{p,s}(x \in C_N, 0 \not\leftrightarrow G_N) \\ &= \Theta_N 2d \mathcal{X}_N(p,s) \\ &= 2d\Theta_N(1-s)\frac{\partial}{\partial s}\Theta_N.\end{aligned}$$

In the last step, we used the truncated version of Equation (3.2), which completes the proof. □

Proof of Lemma 3.5. We can rewrite Θ_N as:

$$\Theta_N = P_{p,s}(|C_N \cap G_N| = 1) + P_{p,s}(|C_N \cap G_N| \ge 2).$$

Now, we condition the first term of the right-hand side on the event $\{|C_N| = n\}$ to get:

$$\begin{aligned}&P_{p,s}(|C_N \cap G_N| = 1) \\ &= \sum_{n\ge 1} P_{p,s}(|C_N \cap G_N| = 1 | |C_N| = n) P_{p,s}(|C_N| = n) \\ &= \sum_{n\ge 1} ns(1-s)^{n-1} P_{p,s}(|C_N| = n) \\ &= s\sum_{n\ge 1} n(1-s)^{n-1} P_{p,s}(|C_N| = n) \\ &= s\frac{\partial}{\partial s}\Theta_N.\end{aligned}$$

This corresponds exactly to the first term on the right-hand side of the differential inequality we aim to prove. The two remaining terms, involving the partial derivative with respect to the parameter p, will be derived from an upper bound for the probability $P_{p,s}(|C_N \cap G_N| \ge 2)$.

Let x be a vertex, and consider the event $A_x \coloneqq \{x \in G_N\} \cup \{x \leftrightarrow G_N\}$, which means that either x is in G_N itself or x is connected to another vertex in G_N. Then, $A_x \circ A_x$ corresponds to the event where there are two edge-disjoint paths connecting x to two distinct vertices in G_N. Using this, we decompose the probability as follows:

$$\begin{aligned}P_{p,s}(|C_N \cap G_N| \geq 2) &= P_{p,s}(|C_N \cap G_N| \geq 2, A_0 \circ A_0) \\ &\quad + P_{p,s}(|C_N \cap G_N| \geq 2, (A_0 \circ A_0)^C).\end{aligned}$$

The expression above can be simplified using the inclusion $\{|C_N \cap G_N| \geq 2\} \supset A_0 \circ A_0$. Thus, the first term becomes $P_{p,s}(A_0 \circ A_0)$, which can be bounded from above by $P_{p,s}(A_0)^2 = P_{p,s}(C_N \cap G_N \neq \varnothing)^2 = \Theta_N^2$ using the BK inequality. To apply the BK inequality, we need to work with the truncated functions Θ_N instead of Θ. Additionally, it should be verified that the BK inequality is applicable to this graph, but we will not address this verification in this text. Hence, the expression above simplifies to:

$$P_{p,s}(|C_N \cap G_N| \geq 2) \leq \Theta_N^2 + P_{p,s}(|C_N \cap G_N| \geq 2, (A_0 \circ A_0)^C).$$

Now, let us analyze what the occurrence of the event $B \coloneqq \{|C_N \cap G_N| \geq 2\} \cap (A_0 \circ A_0)^C$ implies. First, if B occurs, there are at least two distinct vertices in the cluster of the origin C_N that are also green.

However, since $A_0 \circ A_0$ does not occur, there are no pairwise edge-disjoint paths from 0 to each of these green vertices in the cluster of the origin. It follows that this event occurs if, and only if, there exists an edge $e = xy$ where $x, y \in C_N$ such that e is open, and removing e from the graph $\mathbb{G}_N^d$, while keeping its vertices, results in the following events holding:

- $U \coloneqq \{C_N \cap G_N = \varnothing\}$.
- $V \coloneqq \{x \in C_N\}$.
- $W \coloneqq \{A_y \circ A_y\}$.

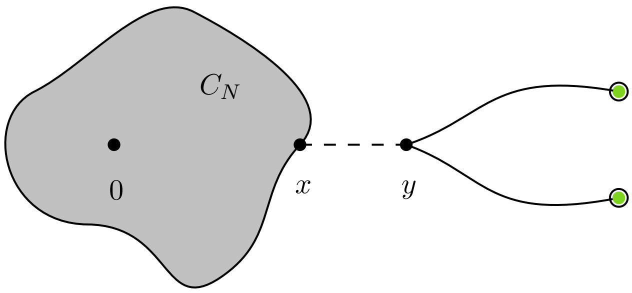


Figure 3.2: The event $U \cap V \cap W$.

The events $H \coloneqq U \cap V \cap W$ and $\{e \text{ is open}\}$ are independent because the former does not depend on the edge e. Therefore,

$$\begin{aligned}P_{p,s}(e \text{ is open}, H) &= P_p(e \text{ is open}) P_{p,s}(H) \\ &= \frac{p}{1-p} P_{p,s}(e \text{ is closed}, H).\end{aligned} \tag{3.6}$$

Let Σ be some connected subset of vertices containing 0 and x but not containing y. The events $\{C_N \cap G_N = \varnothing\}$ and $A_y \circ A_y$ are independent if conditioned on $\{C_N = \Sigma\}$ since the former depends only on the inside of Σ, and the latter depends on the outside. That yields:

$$\begin{aligned} &P_{p,s}(C_N \cap G_N = \varnothing, A_y \circ A_y | C_N = \Sigma) \\ &= P_{p,s}(C_N \cap G_N = \varnothing | C_N = \Sigma) P_{p,s}(A_y \circ A_y | C_N = \Sigma). \end{aligned}$$

Using the BK inequality, the second factor can be estimated by $P_{p,s}(A_y|C_N = \Sigma)^2$ which can be bounded from above by $P_{p,s}(A_y)P_{p,s}(A_y|C_N = \Sigma)$. Once again, we omit the discussion on why the BK inequality holds for the conditional measure in this text. For the same reason discussed previously A_y and $\{C_N \cap G_N = \varnothing\}$ are also independent when conditioned on $\{C_N = \Sigma\}$. Consequently, the previous equation is no greater than:

$$P_{p,s}(C_N \cap G_N = \varnothing, A_y | C_N = \Sigma) P_{p,s}(A_y).$$

Since, $P_{p,s}(A_y) = \Theta_N$, by the translation invariance, the estimate we get is summarized in

$$\begin{aligned} &P_{p,s}(C_N \cap G_N = \varnothing, A_y \circ A_y | C_N = \Sigma) \\ &\leq P_{p,s}(C_N \cap G_N = \varnothing, A_y | C_N = \Sigma)\Theta_N. \end{aligned} \tag{3.7}$$

Now, returning to the probability we seek to bound:

$$\begin{aligned} &P_{p,s}(|C_N \cap G_N| \geq 2, (A_0 \circ A_0)^C) \leq \\ &\leq P_{p,s}\left(\bigcup_{e \in E(N)} \{e \text{ is open}, H\} \right) \\ &\leq \sum_{xy \in E(N)} P_{p,s}(e \text{ is open}, H). \end{aligned}$$

Using (3.6) and the inclusion of events, the last sum is no greater than

$$\begin{aligned} &\frac{p}{1-p} \sum_{xy \in E(N)} P_{p,s}(H) \\ &= \frac{p}{1-p} \sum_{xy \in E(N)} P_{p,s}(C_N \cap G_N = \varnothing, x \in C_N, A_y \circ A_y) \\ &= \frac{p}{1-p} \sum_{xy \in E(N)} \sum_{\Sigma} P_{p,s}(C_N \cap G_N = \varnothing, C_N = \Sigma, A_y \circ A_y). \end{aligned}$$

The right side of the last expression is equal to

$$\frac{p}{1-p} \sum_{xy \in E(N)} \sum_{\Sigma} P_{p,s}(C_N \cap G_N = \varnothing, A_y \circ A_y | C_N = \Sigma) P_{p,s}(C_N = \Sigma).$$

Which, by Equation (3.7) is bounded from above by

$$\begin{aligned}
&\frac{p\Theta_N}{1-p} \sum_{xy \in E(N)} \sum_{\Sigma} P_{p,s}(C_N \cap G_N = \varnothing, A_y | C_N = \Sigma) P_{p,s}(C_N = \Sigma) \\
&= \frac{p\Theta_N}{1-p} \sum_{xy \in E(N)} \sum_{\Sigma} P_{p,s}(C_N \cap G_N = \varnothing, A_y, C_N = \Sigma) \\
&= \frac{p\Theta_N}{1-p} \sum_{xy \in E(N)} \sum_{\Sigma} P_{p,s}(C_N \cap G_N = \varnothing, C_N(y) \cap G_N \neq \varnothing, x \in C_N, C_N = \Sigma) \\
&= \frac{p\Theta_N}{1-p} \sum_{xy \in E(N)} P_{p,s}(C_N \cap G_N = \varnothing, C_N(y) \cap G_N \neq \varnothing).
\end{aligned} \tag{3.8}$$

Recalling Equation (3.5):

$$p\Theta_N \left(\frac{1}{1-p} \sum_{xy \in E(N)} P_{p,s}(C_N \cap G_N = \varnothing, C_N(y) \cap G_N \neq \varnothing) \right) = p\Theta_N \left(\frac{\partial}{\partial p} \Theta \right).$$

This is exactly the last term of the right side of the differential inequality. □

3.3 Upper Bound for β

In this section, we present a proof for the mean-field bound $\beta \leq 1$, derived from the two differential inequalities discussed in the previous sections.

To establish the result, we first examine how a lower bound for the function Θ can be derived from these differential inequalities. The following result will also be used in Section 3.5, and its proof relies primarily on the analytical properties of the functions Θ and $\mathcal{X}$ previously discussed.

Before proceeding with the mathematical details, it is worth mentioning some historical context. This upper mean-field bound for β appears in the literature with various proofs. One notable proof, by Chayes and Chayes [4], uses the BK inequality and Russo's formula to establish a more direct differential inequality:

$$\theta \leq \theta^2 + p\theta\theta',$$

for $p > p_c$.

This inequality is derived similarly to the methods in the previous section, but with a more detailed analysis of the infinite cluster in the supercritical phase. The cluster is decomposed into the *backbone* and its *dangling ends*, allowing a better understanding of its geometry. Along the way, it is shown that the event $\{0 \longleftrightarrow \infty\}$ almost surely has a finite number of pivotal edges in the supercritical phase. This differential inequality implies the existece of a constant $a > 0$ such that $\theta(p) \geq a(p-p_c)$ for $p > p_c$, which implies that $\beta \leq 1$, given its existence.

Another interesting approach derives the lower linear bound $\theta(p) \geq c(p-p_c)$ for $p > p_c$ using differential inequalities for the functions θ_n obtained from the generalized OSSS. This is done in Kern's paper [12] which is inspired by Duminil-Copin, Raoufi and Tassion [5].

We are not going to discuss more details of the first approach, as the differential inequality by Chayes and Chayes does not directly relate Θ and θ as we do here. Similarly, we avoid further discussion of the application of the OSSS inequality to obtain the mean-field bound for β as a similar approach is presented in Hutchcroft's work [10] and is explored in Chapter 4 of this text.

Now, we introduce an inequality for the function Θ, derived from the two previous differential inequalities. This result will be a key tool in proving the main theorem of this section.

Proposition 3.6. *If $p \in (0,1)$ and $s \in (0,1)$, then*

$$\Theta(p,s) \geq -\frac{1}{2A\chi^f(p)} + \frac{1}{2}\sqrt{\left(\frac{1}{A\chi^f(p)}\right)^2 + \frac{4s}{A}},$$

where $A = A(p) \coloneqq 1 + 2dp/(1-p)$.

Proof. Let $s \in (0,1)$ and $p \in (0,1)$. Fix p and write $f(s) = \Theta(p,s)$. Notice first that f is continuously differentiable and strictly increasing, due to the fact that it can be represented as a power series whose derivative, given by Equation (3.1), is strictly positive.

The relation $f'(s) = (1-s)^{-1}\mathcal{X}(p,s)$ implies that $f'(0) = \chi^f(p)$. Notice that $f(0) = 0$, from Lemma 3.1 and $f(1) = 1$. Therefore, it is possible to consider the inverse g for the function f, which also has $g(0) = 0$ and $g(1) = 1$. Using the fact that $g'(f(s)) = 1/f'(s)$ we obtain

$$g'(0) = \frac{1}{\chi^f(p)}.$$

Now, we combine the differential inequalities obtained in the last sections:

$$(1-p)\partial_p\Theta \leq 2d(1-s)\Theta\partial_s\Theta,$$

and

$$\Theta \leq s\partial_s\Theta + \Theta^2 + p\Theta\partial_p\Theta.$$

In the second equation above, $\partial_p\Theta$ can be bounded by the first term in the first equation divided by $1-p$. We get a differential inequality where the only partial derivative showing up is the one with respect to s. Recalling that our parameter p is fixed, we can write in terms of the function f producing the following expression:

$$f \leq sf' + f^2 + \frac{2dp(1-s)}{1-p} f^2 f'.$$

Writing $f(s) = \phi > 0$ and $g(\phi) = s > 0$ and using the relation between the derivatives of f and g, the differential inequality is translated in terms of the function g as:

$$\phi \leq g(\phi)\frac{1}{g'(\phi)} + \phi^2 + \frac{2dp(1-g(\phi))}{1-p}\phi^2 \frac{1}{g'(\phi)}.$$

After rearranging a little we get an expression that is easier to integrate:

$$\frac{1}{\phi}g' - \frac{1}{\phi^2}g \leq \frac{2dp}{1-p} + g' = (A-1) + g',$$

where we also used the trivial bound $1 - g \leq 1$. Now it is possible to integrate the expression above, from 0 to some value $\phi < 1$:

$$\begin{aligned} \frac{g(\phi)}{\phi} - g'(0) &= \int_0^\phi \frac{g'(u)}{u} - \frac{g(u)}{u^2} du \\ &\leq \int_0^\phi \Big((A-1) + g'(u) \Big) du \\ &= (A-1)\phi + g(\phi) - g(0). \end{aligned}$$

With $g(0) = 0$, $g'(0) = 1/\chi^f(p)$, and substituting back $g(\phi) = s$ and $\phi = f(s) = \Theta(p,s) = \Theta$ we get

$$\frac{s}{\Theta} - \frac{1}{\chi^f(p)} \leq (A-1)\Theta + s.$$

Now, we recall from Proposition 3.3 that $s \leq \Theta$ to bound the right side as

$$\frac{s}{\Theta} - \frac{1}{\chi^f(p)} \leq A\Theta.$$

Which implies that

$$\Theta^2 + \frac{\Theta}{A\chi^f(p)} - \frac{s}{A} \geq 0.$$

Finally, solving for Θ.

$$\Theta(p,s) \geq -\frac{1}{2A\chi^f(p)} + \frac{1}{2}\sqrt{\left(\frac{1}{A\chi^f(p)}\right)^2 + \frac{4s}{A}}.$$

□

The last proposition has the following result as an immediate corollary.

Corollary 3.7. *If $s \in (0,1)$ and $p \in (0,1)$ is such that $\chi^f(p) = \infty$:*

$$\Theta(p,s) \geq \sqrt{\frac{s}{A}},$$

where $A(p) = A := 1 + 2dp/(1-p)$.

In order to prove the result from the title of this section we are going to prove the following theorem:

Theorem 3.8. *If p is such that $\chi^f(p) = \infty$, then either*

1. *$\theta(p) > 0$, or*
2. *$\theta(p) = 0$ and $\theta(p') \geq \frac{1}{2p'}(p' - p)$ for $p' \geq p$.*

Remark 3.9. *The fact that $\chi^f(p) < \infty$ for every $p < p_c$ is a consequence of the theorem above. If we assume, by contradiction, that $\chi^f(p) = \infty$ for some $p < p_c$, then, since $\theta(p) = 0$ by definition, we would have that if p' is any value in (p, p_c) then $\theta(p') > 0$, which implies $p \geq p_c$ that is a contradiction.*

Proof of Theorem 3.8. Let $a \in (0,1)$ be such that $\chi^f(a) = \infty$ and let $s \in (0,1)$. We can assume that $\theta(a) = 0$, otherwise we are done. Since $\Theta(p,s)$ is strictly increasing for s we have

$$\Theta(a,s) > \Theta(a,0) = \theta(a) = 0.$$

Since $\Theta(a,s)$ and s are both positive, we can divide the differential inequality $\Theta \leq s\partial_s\Theta + \Theta^2 + p\Theta\partial_p\Theta$ by $s\Theta$ producing the inequality:

$$\frac{1}{s} \leq \frac{1}{\Theta}\partial_s\Theta + \frac{\Theta}{s} + \frac{p}{s}\partial_p\Theta.$$

Rearranging, we get

$$0 \leq \frac{1}{\Theta}\partial_s\Theta + \frac{1}{s}\Bigg(\Theta + p\partial_p\Theta - 1\Bigg).$$

Producing the differential inequality

$$0 \leq \frac{\partial}{\partial s}\log(\Theta) + \frac{1}{s}\frac{\partial}{\partial p}\Big(p\Theta - p\Big) \tag{3.9}$$

Consider b, ε, δ real numbers such that $0 < a < b < 1$ and $0 < \delta < \varepsilon < 1$. We integrate the differential inequality (3.9) on the rectangle $[a,b] \times [\delta, \varepsilon]$:

$$0 \leq \int_a^b \Bigg[\log(\Theta(p,s))\Bigg]_{s=\delta}^{s=\varepsilon} dp + \int_\delta^\varepsilon \frac{1}{s}\Bigg[p\Theta - p\Bigg]_{p=a}^{p=b} ds.$$

Developing further the integrals, the right-hand side is bounded by

$$(b-a)\log\left(\frac{\Theta(b,\varepsilon)}{\Theta(a,\delta)}\right)+b\int_\delta^\varepsilon \frac{\Theta(b,s)}{s}ds$$
$$-a\int_\delta^\varepsilon \frac{\Theta(a,s)}{s}ds+(a-b)\log\left(\frac{\varepsilon}{\delta}\right).$$

On the first term, we used the fact that the integrand can be bounded by $\log(\Theta(b,\varepsilon)/\Theta(a,\delta))$ since both Θ and log are non-decreasing functions. We notice that since $\Theta(a,s)\geq 0$, the third term is negative and we can bound the previous expression from above by

$$(b-a)\log\left(\frac{\Theta(b,\varepsilon)}{\Theta(a,\delta)}\right)+b\Theta(b,\varepsilon)\log\left(\frac{\varepsilon}{\delta}\right)+(a-b)\log\left(\frac{\varepsilon}{\delta}\right)$$
$$=(b-a)\log\left(\frac{\Theta(b,\varepsilon)}{\Theta(a,\delta)}\right)+\Big[a-b+b\Theta(b,\varepsilon)\Big]\log\left(\frac{\varepsilon}{\delta}\right).$$

Dividing by $\log(\varepsilon/\delta)>0$ yields

$$0\leq(b-a)\left[\frac{\log(\Theta(b,\varepsilon))-\log(\Theta(a,\delta))}{\log(\varepsilon)-\log(\delta)}\right]+[a-b+b\Theta(b,\varepsilon)]. \tag{3.10}$$

Now, we use the fact that such a was chosen to satisfy $\chi^f(a)=\infty$, therefore, by Corollary 3.7 we get

$$\log(\Theta(a,\delta))\geq-\frac{1}{2}\log(A)+\frac{1}{2}\log(\delta).$$

Therefore, since $\log(\delta)<0$

$$\frac{\log(\Theta(a,\delta))}{\log(\delta)}\leq\frac{1}{2}-\frac{\log(A(a))}{2\log(\delta)}.$$

From this, when we make $\delta\downarrow 0$ in Equation (3.10) we get

$$\limsup_{\delta\downarrow 0}\frac{\log(\Theta(b,\varepsilon))-\log(\Theta(a,\delta))}{\log(\varepsilon)-\log(\delta)}\leq 1/2.$$

Therefore, as we make $\delta\downarrow 0$ we get that

$$0\leq\frac{1}{2}(b-a)+a-b+b\Theta(b,\varepsilon).$$

Rearranging the expression above we get that $\Theta(b,\varepsilon)\geq(b-a)/2b$. By making $\varepsilon\downarrow 0$ we get our result, that for $b>a$:

$$\theta(b)\geq(b-a)/2b.$$

□

Using this theorem and the fact that $\chi^f(p_c) = \infty$ proved in Corollary 2.7 from the previous chapter we get:

Corollary 3.10. *If $p > p_c$ then*

$$\theta(p) \geq \frac{1}{2p}(p - p_c).$$

Now we are ready to prove the main result of this section, the mean-field upper bound for the critical exponent β.

Theorem 3.11. ***[Mean-field bound for** β**]*** *If the critical exponent β exists, then $\beta \leq 1 = \beta_T$ (critical exponent on the binary tree).*

Proof. Remember that the critical exponent β is defined the number for which the following limit holds:

$$\lim_{p \downarrow p_c} \frac{\log(\theta(p))}{\log((p - p_c))} = \beta.$$

We apply the logarithm function to both sides of the expression obtained in Corollary 3.10 to get

$$\log(\theta(p)) \geq \log\left(\frac{1}{2}\right) + \log(p - p_c).$$

Dividing by $\log(p - p_c) < 0$, and making $p \downarrow p_c$ yields

$$\beta := \lim_{p - p_c} \frac{\log(\theta(p))}{\log(p - p_c)} \leq \lim_{p \downarrow p_c} \frac{\log(1/2)}{\log(p - p_c)} + 1 = 1.$$

□

3.4 Lower Bound for γ

In this section, we establish the mean-field lower bound for the critical exponent γ, which is associated with the expected size of the cluster of the origin. This bound is derived using a straightforward differential inequality, avoiding the need for intricate techniques. The proof relies on Russo's formula, the BK inequality, and the following technical lemma from real analysis:

Lemma 3.12. ***[Derivative of a finite maximum of fuctions]*** *Let I be a finite set of indices, and for $i \in I$, let $p \mapsto f_i(p)$ be a differentiable function. Then, at the points where $F(p) := \max_{i \in I} f_i(p)$ is differentiable, we have:*

$$\frac{d}{dp}F(p) \leq \max_{i \in I}\left\{\frac{d}{dp}f_i(p)\right\}.$$

We omit the proof of this lemma for brevity. Using this result, we derive a lower bound for the function χ:

Theorem 3.13. *If $p < p_c$, then*

$$\chi(p) \geq \frac{1}{2d(p_c - p)}.$$

Proof. We begin by writing $\chi(p) = E_p[|C|]$ as the following sum

$$\chi(p) = E_p\left[\sum_{x\in\mathbb{Z}^d} \mathbb{1}_{\{0\leftrightarrow x\}}\right] = \sum_{x\in\mathbb{Z}^d} \tau_p(0, x).$$

Now, for each $n \geq 1$ and each $v \in \Lambda_n$, we consider an analogous function, this time restricted to paths lying entirely inside the box Λ_n around the origin.

$$\chi_n(p; v) := \sum_{x\in\Lambda_n} P_p(v \overset{\Lambda_n}{\longleftrightarrow} x).$$

Additionally, for each $n \geq 1$ we define the function that takes the maximum at Λ_n of such functions $\chi_n(p; v)$, that is

$$\hat{\chi}_n(p) := \max_{v\in\Lambda_n} \chi_n(p; v).$$

We immediately get that, in particular $\hat{\chi}_n(p) \geq \chi_n(p; 0)$, since $0 \in \Lambda_n$. Also, by the translation invariance $\chi_n(p; v) \leq \chi(p)$. Therefore, if we take the maximum at the left-hand side what we obtain is $\hat{\chi}_n(p) \leq \chi(p)$ for every $n \geq 1$. Write

$$\tau_n(p; x) := P_p(0 \overset{\Lambda_n}{\longleftrightarrow} x).$$

On one hand, since $\{0 \overset{\Lambda_n}{\longleftrightarrow} x\} \nearrow \{0 \leftrightarrow x\}$ as $n \to \infty$, then $\tau_n(p; x) \uparrow \tau_p(x)$. Hence,

$$\chi_n(p, 0) = \sum_{x\in\Lambda_n} \tau_n(p; x) \nearrow \sum_{x\in\mathbb{Z}^d} \tau_p(x) = \chi(p)$$

by the Monotone Convergence Theorem. On the other hand, as we have observed before $\chi(p) \geq \hat{\chi}_n(p) \geq \chi_n(p; 0)$. Therefore, using the previous equation we can conclude that

$$\hat{\chi}_n(p) \to \chi(p).$$

Now, we seek a differential inequality for the derivative of $\chi_n(p, v)$ with respect to p since in this truncated case Russo's Formula is allowed. Then, we hope that it is possible to take limits to translate this differential inequality in terms of the $\chi(p)$, which is the function we are interested in.

Let $A_n(v, x) = \{x \overset{\Lambda_n}{\longleftrightarrow} v\}$, that is, the event where exists a path ρ of open edges lying inside Λ_n connecting v to x. This is an increasing event that depends only on a finite number of edges, which is precisely $|E(\Lambda_n)|$. Applying Russo's Formula:

$$\begin{aligned}\frac{d}{dp}\chi_n(p;v) &= \frac{d}{dp}\left[\sum_{x\in\Lambda_n} P_p(A_n(v,x))\right]\\ &= \sum_{x\in\Lambda_n}\frac{d}{dp}P_p(A_n(v,x))\\ &= \sum_{x\in\Lambda_n}\sum_{e\in E(\Lambda_n)} P_p(e \text{ is pivotal for } A_n(v,x)).\end{aligned}$$

The next step is to understand what it means for an edge e to be pivotal for the event $A_n(v,x)$, or at least what this implies, in order to get an upper bound for this derivative.

Notice that, if e is pivotal for $A_n(x,v)$, then if the state of $e = wz$ is switched, so it is the connection between x and v, this implies that there must be two disjoint paths, one connecting $x \leftrightarrow z$ and $v \leftrightarrow w$. But it is possible that the roles of z and w are interchanged and there are instead two disjoint paths, one connecting $x \leftrightarrow w$ and $v \leftrightarrow z$. We write this as

$$\{e \text{ is pivotal for } A_n(v,x)\} \subset \{x \leftrightarrow z \circ v \leftrightarrow w\} \cup \{x \leftrightarrow w \circ v \leftrightarrow z\}.$$

For each of these sets on the inclusion we can use the BK inequality, hence the derivative $\chi_n(p;v)'$ can be bounded from above by

$$\sum_{x\in\Lambda_n}\sum_{e\in E(\Lambda_n)}\big[P_p(A_n(x,z))P_p(A_n(v,w)) + P_p(A_n(x,w))P_p(A_n(v,z))\big].$$

Summing for $x \in \Lambda_n$ first, we have that this last summation can be bounded from above by

$$\hat{\chi}_n(p)\sum_{e\in E(\Lambda_n)}\big[P_p(A_n(v,w)) + P_p(A_n(v,z))\big].$$

Therefore, using the fact that each vertex has $2d$ neighbors, we find the differential inequality

$$\frac{d}{dp}\chi_n(p;v) \le \hat{\chi}_n(p)2d\hat{\chi}_n(p) = 2d\hat{\chi}_n^2(p).$$

Now, we can apply Lemma 3.12 to obtain

$$\begin{aligned}\frac{d}{dp}\hat{\chi}_n(p) &\le \max_{v\in\Lambda_n}\left\{\frac{d}{dp}\chi_n(p,v)\right\}\\ &\le 2d\hat{\chi}_n^2(p).\end{aligned}$$

Integrating from p to p_c:

$$\frac{1}{\hat{\chi}_n(p)} - \frac{1}{\hat{\chi}_n(p_c)} = \left[-\frac{1}{\hat{\chi}_n(p)} \right]_p^{p_c} \leq 2d(p_c - p).$$

Since $\hat{\chi}_n(p) \to \chi(p)$, as anticipated at the beginning of the proof and we also have $\chi(p_c) = \infty$, we conclude:

$$\frac{1}{\chi(p)} \leq 2d(p_c - p).$$

□

As a consequence of the last result, we get the lower bound for γ.

Theorem 3.14. ***[Mean-field bound for*** γ***]*** *If the critical exponent* γ *exists, then* $\gamma \geq 1 = \gamma_T$ *(critical exponent on the binary tree).*

Proof. Assume that the critical exponent γ exists, since for values of $p < p_c$ we have $\chi^f(p) = \chi(p)$, by Theorem 3.13

$$\chi^f(p) \geq \frac{1}{2d|p_c - p|}.$$

Taking logarithms on both sides, and then dividing by $\log|p_c - p| < 0$

$$-\gamma := \lim_{p \uparrow p_c} \frac{\log \chi^f(p)}{\log|p_c - p|} \leq \lim_{p \uparrow p_c} \frac{\log(1/2d)}{\log|p_c - p|} - 1 = -1.$$

□

3.5 Two Results For δ

In this section, we establish two key relations for the critical exponent δ. The proofs rely on Corollary 3.7, which is derived from differential inequality A and differential inequality B. First, we show that $\delta \geq 2$ by analyzing the asymptotic behavior of Θ from the asymptotic behavior of $P_{p_c}(|C| \geq n)$ from the definition of the critical exponent. Then, we compare the functions θ and Θ near the critical point to establish the relation $\beta \geq 2/\delta$.

Lemma 3.15. *If* $P_{p_c}(|C| \geq n) = c_1 n^{-1/\delta}(1+o(1))$ *as* $n \to \infty$ *for some positive constant* c_1*, then* $\Theta(p_c, s) = c_2 s^{1/\delta}(1+o(1))$ *as* $s \downarrow 0$ *for some positive constant* c_2.

Proof. The hypothesis provides information about the behavior of

$$b_n := P_{p_c}(|C| \geq n)$$

rather than $P_{p_c}(|C| = n)$. To connect this to $\Theta(p, s)$, we manipulate the series accordingly. Recall the original summation:

$$\Theta(p, s) = 1 - \sum_{n \geq 1} (1 - s)^n P_p(|C| = n).$$

Changing the order of the summation leads us to:

$$\Theta(p, s) = \frac{s}{1 - s} \sum_{n \geq 1} (1 - s)^n P_p(|C| \geq n).$$

Now, write $a = 1 - s$, to simplify the expression above, then $s \downarrow 0$ if, and only if $a \uparrow 1$. The asymptotic behavior we want to describe is the one obtained from the following power series

$$\Theta_{p_c}(a) = \frac{1 - a}{a} \sum_{n \geq 1} a^n b_n.$$

For any $\varepsilon > 0$, we can find a large enough N such that:

$$(c_1 - \varepsilon) n^{-1/\delta} \leq b_n \leq (c_1 + \varepsilon) n^{-1/\delta}$$

holds for every $n \geq N$. Then, since the sequence $a^n n^{-1/\delta}$ is monotone decreasing, we can write

$$\Theta_{p_c}(a) \geq \frac{(1 - a)}{a} \left\{ \sum_{1 \leq n < N} a^n b_n + (c_1 - \varepsilon) I(a) \right\}, \tag{3.11}$$

and

$$\Theta_{p_c}(a) \leq \frac{(1 - a)}{a} \left\{ \sum_{1 \leq n < N} a^n b_n + (c_1 + \varepsilon) \left(a^N N^{-1/\delta} + I(a) \right) \right\}, \tag{3.12}$$

where $I(a) = \int_N^\infty a^x x^{-1/\delta} dx$. Making the change of variables $y = -x \log(a)$ we obtain that

$$I(a) = \frac{1}{(-\log(a))^{1 - 1/\delta}} \int_{-N \log(a)}^\infty e^{-y} y^{-1/\delta} dy.$$

Therefore, we have that

$$(c_1 - \varepsilon)\Gamma(1 - 1/\delta) \leq \lim_{a \uparrow 1} \frac{\Theta_{p_c}(a)}{(1 - a)^{1/\delta}} \leq (c_1 + \varepsilon)\Gamma(1 - 1/\delta),$$

where $\varepsilon > 0$ was arbitrarily set. Thus, this implies that $\Theta(p_c, s) = c_2 s^{1/\delta}(1 + o(1))$ as $s \downarrow 0$. $\square$

Now, we combine the previous Lemma with the Corollary 3.7 to get the following theorem:

Theorem 3.16. ***[Mean-field bound for*** δ***]*** *If the critical exponent δ exists, then $\delta \geq 2 = \delta_T$ (critical exponent on the binary tree).*

Proof. Assume that the critical exponent δ exists. Then, by the preceding theorem $\Theta(p_c, s) = c_2 s^{1/\delta}(1 + o(1))$ as we make $s \downarrow 0$. Therefore, for small values of s:

$$\Theta(p_c, s) \preceq s^{1/\delta}.$$

On the other hand, by Corollary 3.7 we can also bound $\Theta(p_c, s)$ from below since $\chi^f(p_c) = \infty$, yielding:

$$s^{1/2} \preceq \Theta(p_c, s) \preceq s^{1/\delta},$$

for s sufficiently small. Now, taking the natural logarithm, dividing by $\log(s) < 0$ and then making $s \downarrow 0$ we get $\delta \geq 2$. □

The next result establishes a relation between the functions Θ and θ near the critical parameter.

Lemma 3.17. *$\theta(p_c+\varepsilon) \leq 1 - \{1 - \Theta(p_c, \psi(\varepsilon))\}^2$ for values of ε small enough, where*

$$\psi(\varepsilon) = 1 - \left\{ \left(1 - \frac{\varepsilon^2}{p_c^2}\right)\left(1 - \frac{\varepsilon^2}{(1-p_c)^2}\right) \right\}^d.$$

Proof. Let A be a connected and finite subgraph of $\mathbb{L}^d$ containing the origin. We write $V(A)$ for its set of vertices, $E(A)$ for its set of edges, and $\triangle A$ for its edge-boundary, that is $\triangle A := \{e = xy \in \mathbb{E}^d; x \in A, y \in A^C\}$. For positive integers n, m, b, consider the collection $\mathcal{A}[n, m, b]$ of such connected and finite subgraphs containing the origin with $V(A) = n$, $E(A) = m$ and $\triangle A = b$.

When the origin's cluster is finite, it assumes the form of one of the elements of the collection $\mathcal{A}[n, m, b]$, for integers n, m, b. Therefore, we can write

$$\theta(p) = 1 - \sum_{n,m,b} \pi_{n,m,b}(p),$$

where $\pi_{n,m,b}(p) := P_p(C \in \mathcal{A}[n, m, b])$. If we denote the size of a collection $\mathcal{A}[n, m, b]$ by $a[n, m, b]$ then we can write

$$\pi_{n,m,b}(p) = a[n, m, b]p^m(1 - p)^b.$$

Notice that, for values of ε, not necessarily positive, the probabilities $\pi_{n,m,b}(p)$ and $\pi_{n,m,b}(p - \varepsilon)$ also obey the following multiplicative relation

$$\pi_{n,m,b}(p - \varepsilon) = \pi_{n,m,b}(p)S_{p,\varepsilon}(m, b),$$

where

$$S_{p,\varepsilon}(m,b) = \left(1 - \frac{\varepsilon}{p}\right)^m \left(1 + \frac{\varepsilon}{1-p}\right)^b.$$

We can then look at the random variable $S_{p,\varepsilon}(C) = S_{p,\varepsilon}(|E(C)|, |\triangle C|)$. Then, suming the previous relation we get

$$\begin{aligned}\sum_{n,m,b} \pi_{n,m,b}(p_c - \varepsilon) &= \sum_{n,m,b} \pi_{n,m,b}(p_c) S_{p_c,\varepsilon}(m,b) \\ &= E_{p_c}[S_{p_c,\varepsilon}(C); |C| < \infty].\end{aligned}$$

In addition, since the cluster of the origin is almost surely finite in the subcritical phase, for $\varepsilon > 0$ we have:

$$1 = P_{p_c - \varepsilon}(|C| < \infty) = \sum_{n,m,b} \pi_{n,m,b}(p_c - \varepsilon). \tag{3.13}$$

In order to prove this lemma we want to find a way to bound the probability $\theta(p_c + \varepsilon)$ from above with some function of $\Theta(p_c, \varepsilon)$. Notice that, by Equation (3.13):

$$\begin{aligned}\theta(p_c + \varepsilon) &= 1 - \sum_{n,m,b} \pi_{n,m,b}(p_c + \varepsilon) \\ &= 1 - \left(\sum_{n,m,b} \pi_{n,m,b}(p_c - \varepsilon)\right)\left(\sum_{n,m,b} \pi_{n,m,b}(p_c + \varepsilon)\right) \\ &= 1 - \Big(E_{p_c}[S_{p_c,\varepsilon}(C); |C| < \infty]\Big)\Big(E_{p_c}[S_{p_c,-\varepsilon}(C); |C| < \infty]\Big).\end{aligned}$$

Using Cauchy-Schwarz inequality,

$$\theta(p_c + \varepsilon) \le 1 - \left(E_{p_c}[(S_{p_c,\varepsilon}(C) S_{p_c,-\varepsilon}(C))^{1/2}; |C| < \infty]\right)^2.$$

And just by substituting back the definition of $S_{p,\varepsilon}(C)$ on the right side of the inequality above, what we get is

$$\theta(p_c + \varepsilon) \le 1 - \Big(E_{p_c}[\Psi(\varepsilon, C); |C| < \infty]\Big)^2,$$

where

$$\Psi(\varepsilon, C) = \left(1 - \frac{\varepsilon^2}{p_c^2}\right)^{|E(C)|/2} \left(1 - \frac{\varepsilon^2}{(1-p_c)^2}\right)^{|\triangle C|/2}.$$

Using the relations $m \le 2dn$ and $b \le 2dn$ from basic graph theory, it follows that $|E(C)|/2 \le d|C|$ and $|\triangle C|/2 \le d|C|$, hence:

$$\Psi(\varepsilon, C) \ge \left(1 - \frac{\varepsilon^2}{p_c^2}\right)^{d|C|} \left(1 - \frac{\varepsilon^2}{(1-p_c)^2}\right)^{d|C|}.$$

However,

$$\begin{aligned}
&E_{p_c}\left[\left(1-\frac{\varepsilon^2}{p_c^2}\right)^{d|C|}\left(1-\frac{\varepsilon^2}{(1-p_c)^2}\right)^{d|C|};|C|<\infty\right]\\
&=\sum_{n\geq 1}(1-\psi(\varepsilon))^n P_{p_c}(|C|=n)\\
&=1-\Theta(p_c,\psi(\varepsilon)).
\end{aligned}$$

With that, we can finally conclude that

$$\theta(p_c+\varepsilon)\leq 1-(1-\Theta(p_c,\psi(\varepsilon)))^2.$$

□

Now that we have established this relation between Θ and θ, the critical relation follows directly by combining the results derived so far.

Theorem 3.18. *If the critical exponents δ and β exist, then $\beta \geq 2/\delta$.*

Proof. Let $t > 0$. From assuming the existence of β and δ, we have that $\varepsilon^{\beta+t} \preceq \theta(p_c+\varepsilon)$ for small values of ε. In addition, from Lemma 3.15 we have that $\Theta(p_c,\psi(\varepsilon)) \preceq \psi(\varepsilon)^{1/\delta}$ for small values of $\psi(\varepsilon)$. Since also $\psi(\varepsilon) = C\varepsilon^2 + o(\varepsilon^2)$ as $\varepsilon \downarrow 0$, we have that $\psi(\varepsilon) \preceq \varepsilon^2$ for values of ε small enough. As consequence $\Theta(p_c,\psi(\varepsilon)) \preceq \varepsilon^{2/\delta}$. Now, the only part missing is to prove $\theta(p_c+\varepsilon) \preceq \Theta(p_c,\psi(\varepsilon))$, however, from the last result

$$\begin{aligned}
\theta(p_c+\varepsilon) &\leq 1-(1-\Theta(p_c,\psi(\varepsilon)))^2\\
&= 2\Theta(p_c,\psi(\varepsilon))(1-\Theta(p_c,\psi(\varepsilon)/2)\\
&= 2\Theta(p_c,\psi(\varepsilon))(1+o(1)) \text{ as } \varepsilon\downarrow 0\\
&\preceq \Theta(p_c,\psi(\varepsilon))
\end{aligned}$$

Where the last inequality above holds for values of $\varepsilon > 0$ sufficiently small. It follows that $\varepsilon^{\beta+t} \preceq \varepsilon^{2/\delta}$ for small values of $\varepsilon > 0$. Applying the natural logarithm to both sides, dividing by $\log(\varepsilon) < 0$ and then making $\varepsilon \downarrow 0$ we get that $\beta + t \geq 2/\delta$. We get to the result by making $t \downarrow 0$. □

Chapter 4

OSSS Inequalities

In this chapter, we present an alternative method to obtain useful differential inequalities and critical bounds without relying on the BK, FKG inequalities, or Russo's formula. The main result of this chapter is a differential inequality proved by Hutchcroft in [10], which was obtained from a generalization of the OSSS inequality by Duminil-Copin, Raoufi, and Tassion [5].

Although the results in his work [10] extends to a more general setting, the Random-Cluster Model, our focus is on Bernoulli percolation on the edges of $\mathbb{Z}^d$. This choice avoids the need to define the Random-Cluster Model or address the existence of its measures on infinite graphs. By concentrating on the Bernoulli percolation model, we can highlight the parallels between this new method and the approaches discussed in earlier chapters.

In this chapter, we apply the theory of random algorithms and decision trees to the same ghost vertex construction studied in the previous chapter. This new approach highlights both the differences and similarities between some classical methods from the 1980s and modern techniques. Notably, in contrast to earlier methods, our approach eliminates the need for finite-volume approximations in the differential inequalities, resulting in a cleaner technique. Moreover, the limitations imposed by Russo's formula for differentiation and the BK inequality for handling geometric structures do not arise in our application.

The chapter is organized as follows: the first section introduces fundamental concepts from the study of random algorithms, including the OSSS inequality and its generalization, without going into their proofs. We then derive a percolation-specific version of the differential inequality from [10] and explore its implications. In the final section, we apply this inequality to establish relationships among the critical exponents δ, Δ, and γ.

4.1 Background For The Differential Inequality

In this section, we provide key definitions from the theory of random algorithms, including decision trees, revealment probability, and the OSSS inequalities. The terminology follows the original OSSS inequality article [15], its generalization to monotonic measures by Duminil-Copin, Raoufi, and Tassion [5], and Hutchcroft's work [10].

4.1.1 Decision Trees And The OSSS Inequality

We start this section recalling the definition of a boolean function, that is a function $f : \{0,1\}^k \to \{0,1\}$, with $k \geq 1$, that assumes only values 0 and 1. In this chapter, we are interested in an extension of this definition, where the domain is a space of configurations $\{0,1\}^E$ or $\{0,1\}^{E\cup V}$, for the set of edges and vertices of a countable graph $G = (V, E)$.

Now, for a boolean function f, we associate a deterministic algorithm T that represents this function in some sense. This algorithm is a decision tree that reads the values of the coordinates of a configuration and determines the value of the function.

Definition 4.1. ***[Decision tree]*** *Let $E = \{e_1, e_2, ...\}$ be a countable set. A decision tree T is a function $T : \{0,1\}^E \to E^{\mathbb{N}}$, where $\omega \mapsto T(\omega) = \{T_i(\omega)\}_{i\in\mathbb{N}}$, with the following properties:*

- *$T_1(\omega) = e_1$ for every $\omega \in \{0,1\}^E$.*
- *For each $n \geq 2$ there is a function $S_n : (E \times \{0,1\})^{n-1} \to E$ such that $T_n(\omega) = S_n[(T_i, \omega(T_i))_{i=1}^{n-1}]$.*

Definition 4.2. ***[Decision forest]*** *A decision forest F is a collection of decision trees $F = \{T^\alpha\}_{\alpha\in I}$ where I is a countable set.*

It is possible to view a decision forest $F = \{T^\alpha\}_{\alpha\in\mathbb{N}}$ as a decision tree, making an arrangement to run all decision trees T^α at once. One way to achieve this is to consider $F(\omega) = \{F_k(\omega)\}_{k\in\mathbb{N}}$, where $F_1(\omega) = T_1^1(\omega)$ and for $k \geq 2$ we set $F_k(\omega) = T_1^1(\omega)$ if k is not a power of prime, and $F_{p_i^j}(\omega) = T_j^i(\omega)$ where p_i is the i-th prime. In words, a decision tree presents a list of elements of E, that we can interpret as the history of coordinates investigated by the algorithm on a given configuration.

Now, we want to describe how a decision tree T can be related to a function f that assigns values 0 or 1 to configurations on $\Omega = \{0,1\}^E$. That leads us to the definitions:

Definition 4.3. ***[Decision tree computing a function]*** *Let ω be a random* *variable with law* μ *on* $\Omega = \{0,1\}^E$ *and* T *be a decision tree on* Ω*. Let* $\mathcal{F}_n(T) = \sigma\{T_1(\omega), ..., T_n(\omega)\}$*, that is, the sigma algebra generated by these* $n \geq 1$ *random variables. We say that the decision tree* T *computes a function* $f : \{0,1\}^E \to \{0,1\}$ *if* f *is measurable with respect to* $\overline{\mathcal{F}}(T)$*, the* μ *completion of* $\mathcal{F}(T) = \vee_n \mathcal{F}_n$*.*

The following definition is a natural extension of the previous one.

Definition 4.4. ***[Decision forest computing a function]*** *Let* ω *be a random variable with law* μ *on* $\Omega = \{0,1\}^E$ *and* $F = \{T^\alpha\}_{\alpha \in I}$ *be a decision forest on* Ω*. Let* $\mathcal{F}_n(T^\alpha) = \sigma\{T_1^\alpha(\omega), ..., T_n^\alpha(\omega)\}$*, that is, the sigma algebra generated by these* $n \geq 1$ *random variables. We say that the decision forest* F *computes a function* $f : \{0,1\}^E \to \{0,1\}$ *if* f *is measurable with respect to* $\overline{\mathcal{G}}$*, the* μ *completion of the smallest sigma algebra* $\mathcal{G}$ *that contains all the sigma algebras* $\mathcal{F}(T^\alpha) = \vee_n \mathcal{F}_n(T^\alpha)$*.*

The previous definitions are essentially telling that a decision tree T can compute a function f if the information needed to assign values for a configuration can be found by the algorithm behind T.

For each $e \in E$, a decision tree T, or decision forest F, and a probability measure μ such as in the definitions before, we consider the following useful quantity:

Definition 4.5. ***[Revealment probability]*** *For each* $e \in E$*,* T *and* μ *such as in the definitions before, the revealment probability is defined as*

$$\delta_e(T, \mu) = \mu\Bigg(\bigcup_{n \geq 1} \{T_n(\omega) = e\}\Bigg).$$

That is, the probability of such $e \in E$ *being present in some coordinate of* $T(\omega)$*. For decision forests* $F = \{T^\alpha\}_{\alpha \in I}$ *we define the revealment probability by*

$$\delta_e(F, \mu) = \mu\Bigg(\bigcup_{n \geq 1} \bigcup_{\alpha \in I} \{T_n^\alpha(\omega) = e\}\Bigg).$$

The next definition presents the main quantity estimated by the OSSS inequality.

Definition 4.6. *For each probability measure* μ *on* $\{0,1\}^E$ *and for function* $f, g : \{0,1\}^E \to \mathbb{R}$ *we define* $\mathrm{CoVr}_\mu[f,g]$ *by:*

$$\mathrm{CoVr}_\mu[f,g] = \mu \otimes \mu\Big[|f(\omega_1) - g(\omega_2)|\Big] - \mu\Big[|f(\omega_1) - g(\omega_1)|\Big],$$

where ω_1 *and* ω_2 *are drawn independently from the measure* μ*.*

The OSSS inequality, introduced by O'Donnell, Saks, Schramm, and Servedio in [15], gives a useful estimate, in terms of the revealment probability, to the quantity $\mathrm{CoVr}_\mu[f,g]$ for a pair functions f,g when μ is a product measure and f,g are boolean functions. It is originally stated in the following way:

Theorem 4.7. ***[OSSS inequality]*** *Let $E = \{e_1, ..., e_n\}$ be a finite set and μ be a product measure on $\Omega = \{0,1\}^E$. Then for every pair of measurable functions $f,g : \Omega \to \mathbb{R}$ with f increasing and every decision tree T computing g we have that*

$$\frac{1}{2}\left|\mathrm{CoVr}_\mu[f,g]\right| \leq \sum_{i=1}^{n} \delta_{e_i}(T,\mu)\mathrm{Cov}_\mu[f,\omega(e_i)].$$

This theorem was generalized later by D.Copin, Tassion, Raoufi in [5] for monotonic measures that are not necessarily product measures.

Definition 4.8. ***[Monotonic measure]*** *A measure μ on $\Omega = \{0,1\}^E$ is monotonic if, for any $e \in E$, $F \subset E$ and $\xi, \zeta \in \Omega$ with the properties*

- $\xi \leq \zeta$,
- $\mu(\cap_{e\in F}\{\omega_e = \xi_e\}) > 0$,
- $\mu(\cap_{e\in F}\{\omega_e = \zeta_e\}) > 0$,

we have that

$$\mu[\omega_e = 1 | \cap_{e\in F} \{\omega_e = \xi_e\}] \leq \mu[\omega_e = 1 | \cap_{e\in F} \{\omega_e = \zeta_e\}].$$

Theorem 4.9. ***[OSSS inequality for monotonic measures]*** *Let E be a finite or countably infinite set and let μ be a monotonic measure on $\Omega = \{0,1\}^E$. Then, for every pair of measurable, μ-integrable functions $f,g : \Omega \to \mathbb{R}$ with f increasing, and every decision tree T computing g we have that*

$$\frac{1}{2}\left|\mathrm{CoVr}_\mu[f,g]\right| \leq \sum_{e\in E} \delta_e(T,\mu)\mathrm{Cov}_\mu[f,\omega(e)].$$

It is also possible to rephrase the previous theorem in terms of decision forests.

Corollary 4.10. ***[OSSS inequality for decision forests]*** *Let E be a finite or countably infinite set and let μ be a monotonic measure on $\{0,1\}^E$. Then, for every pair of measurable, μ-integrable functions $f,g : \{0,1\}^E \to \mathbb{R}$ with f increasing and every decision forest F computing g we have that*

$$\frac{1}{2}\left|\mathrm{CoVr}_\mu[f,g]\right| \leq \sum_{e\in E} \delta_e(F,\mu)\mathrm{Cov}_\mu[f,\omega(e)].$$

We are not going to present here the proofs of Theorems 4.7 and 4.9, since they are rather technical and are out of the scope of this work. However, other details about the definitions, other applications of these inequalities and their proofs can be found, respectively, in [15], [5]. It is also worth mentioning that some of the considerations and terminology here follow [10].

4.1.2 Dini Derivatives

For some applications, it suffices to consider a weaker version of a derivative of a function. In this subsection, we present a Dini derivative of a function and establish a few of its useful properties. A reference for more background and proofs for the following facts is Kannan and Krueger [11], Chapter 3.

Definition 4.11. ***[Lower-Right Dini Derivative]*** *For a real function $f : [a,b] \to \mathbb{R}$ we define the Lower-Right Dini Derivative of f at a point $x \in [a,b)$ to be*

$$\left(\frac{d}{dx}\right)_+ f(x) = \liminf_{\varepsilon \downarrow 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}.$$

The next proposition tells that a weaker version of the Fundamental Theorem of Calculus holds for the Dini Derivative of an increasing function.

Proposition 4.12. *If $f : [a,b] \to \mathbb{R}$ is increasing, then we have that*

$$f(b) - f(a) \geq \int_a^b \left(\frac{d}{dx}\right)_+ f(x) dx.$$

Finally, the other property we are going to use later is a relation between the Dini Derivative of the logarithm of a function and the function itself.

Proposition 4.13. *For a function $f : [a,b] \to \mathbb{R}$ we have that*

$$\left(\frac{d}{dx}\right)_+ \log f(x) = \frac{1}{f(x)} \left(\frac{d}{dx}\right)_+ f(x).$$

4.1.3 The Differential Inequality

The following result extends Lemma 1.14 to graphs with a countable number of edges. It is important to note that, in this setting, we are no longer limited to functions f that depend on only a finite number of edges.

Proposition 4.14. *If $G = (V,E)$ is a countable graph, $f : \{0,1\}^E \to \{0,1\}$ is an increasing function and $p \in (0,1)$, then*

$$\left(\frac{d}{dp}\right)_+ E_p[f] \geq \frac{1}{p(1-p)} \sum_{e \in E} \mathrm{Cov}[f, \omega_e].$$

Proof. Fix a finite subset of edges $A \subset E$. For a pair of parameters $p, q \in [0,1]$ we construct a percolation measure $P_{p,q}$ on $\Omega = \{0,1\}^E$ that has the property

$$P_{p,q}(\omega_e = 1) = \begin{cases} p & e \in A \\ q & e \notin A \end{cases}.$$

Write $\Omega = \Omega_A \times \Omega_{A^C}$ where $\Omega_A = \{0,1\}^A$ and $\Omega_{A^C} = \{0,1\}^{A^C}$. Therefore, a configuration ω can be written as a pair (ω_A, ω_{A^C}) and the measure $P_{p,q}$ is a product measure $Q_p \otimes Q_q$ where Q_p is the product of Bernoulli measures with parameter p defined on Ω_A and Q_q is the product of Bernoulli measures with parameter q defined on Ω_{A^C}.

Let f be any function defined on $\Omega_A \times \Omega_{A^C}$. The expected value $E_{p,q}$ with respect to the measure $P_{p,q}$ can be computed as

$$E_{p,q}[f] = \sum_{\omega_A} p^{|\omega_A|}(1-p)^{|A|-|\omega_A|} \int f(\omega_A, \omega_{A^C}) dQ_q. \tag{4.1}$$

To simplify the notation we write $F(p,q)$ for $E_{p,q}[f]$ and $I_q(\omega_A)$ for the integral $\int f(\omega_A, \omega_{A^C}) dQ_q$. Thus, Equation (4.1) turns into

$$F(p,q) = \sum_{\omega_A} I_q(\omega_A) p^{|\omega_A|}(1-p)^{|A|-|\omega_A|}.$$

Now, we compute the derivative of $F(p,q)$ with respect to the first variable following the lines of the proof of Lemma 1.14

$$\begin{aligned}
&\frac{\partial}{\partial p} F(p,q) \\
&= \frac{1}{p(1-p)} \sum_{\omega_A} I_q(\omega_A) p^{|\omega_A|}(1-p)^{|A|-|\omega_A|}(|\omega_A| - p|A|) \\
&= \frac{1}{p(1-p)} \sum_{e \in A} \sum_{\omega_A} p^{|\omega_A|}(1-p)^{|A|-|\omega_A|} \int f(\omega_A, \omega_{A^C})(\omega_e - p) dQ_q \\
&= \frac{1}{p(1-p)} \sum_{e \in A} E_{p,q}[f(\omega)(\omega_e - p)].
\end{aligned}$$

And we conclude that

$$\frac{\partial}{\partial p} E_{p,q}[f] = \frac{1}{p(1-p)} \sum_{e \in A} \mathrm{Cov}_{p,q}[f, \omega_e].$$

Now, for parameters $p_0 \leq p$ if the function f is also increasing, we have

$$E_p[f] \geq E_{p,p_0}[f] \geq E_{p_0}[f].$$

And therefore

$$\begin{aligned}\left(\frac{d}{dp}\right)_+ E_p[f]\bigg|_{p=p_0} &= \liminf_{p\downarrow p_0} \frac{E_p[f]-E_{p_0}[f]}{p-p_0} \\ &\geq \liminf_{p\downarrow p_0} \frac{E_{p,p_0}[f]-E_{p_0}[f]}{p-p_0} \\ &= \frac{\partial}{\partial p}F(p,q)\bigg|_{p=q=p_0} \\ &= \frac{1}{p_0(1-p_0)}\sum_{e\in A}\mathrm{Cov}_{p_0}[f,\omega_e].\end{aligned}$$

Since p_0 is arbitrary we conclude for every finite subset of edges $A \subset E$ that

$$\left(\frac{d}{dp}\right)_+ E_p[f] \geq \frac{1}{p(1-p)}\sum_{e\in A}\mathrm{Cov}_p[f,\omega_e].$$

Since also $\mathrm{Cov}_p[f,\omega_e] \geq 0$, taking the supremum over the finite subsets A of edges of E we get the result

$$\begin{aligned}\left(\frac{d}{dp}\right)_+ E_p[f] &\geq \sup_{A\subset E}\frac{1}{p(1-p)}\sum_{e\in A}\mathrm{Cov}_p[f,\omega_e] \\ &= \frac{1}{p(1-p)}\sum_{e\in E}\mathrm{Cov}_p[f,\omega_e].\end{aligned}$$

□

The next proposition is the key result to prove the differential inequality we want. Its proof is a nice illustration of the use of the OSSS inequality.

Proposition 4.15. *Let $\mathbb{G} = (V, E)$ be a countable graph and let μ be a monotonic measure on $\{0,1\}^E$. Let $v \in V$, $n \geq 1$ and $\lambda \in (0,\infty)$. Then,*

$$\sum_{e\in E}\mathrm{Cov}_\mu[\mathbb{1}\{|C(v)|\geq n\},\omega(e)] \geq \left[\frac{(1-e^{-\lambda})-\mu[1-e^{-\lambda|C(v)|/n}]}{2\cdot\sup_{u\in V}\mu[1-e^{-\lambda|C(u)|/n}]}\right]\cdot\mu(|C(v)|\geq n).$$

Proof. On the graph $\mathbb{G} = (V, E)$, we construct two independent processes in an analogous fashion to the Ghost Vertex Construction from Section 3.1. Instead of a usual product measure, we consider on the space $\{0,1\}^E$ a random configuration ω with law μ. Now, we fix a positive constant λ and an integer $n \geq 1$. On the space $\{0,1\}^V$ we consider a Bernoulli percolation, independent from the first process, where we declare a vertex to be green with parameter $\gamma = 1 - e^{-\lambda/n}$.

Consider the product space $\Omega = \{0,1\}^E \times \{0,1\}^V$ equipped with the product measure $P := \mu \otimes Q_\gamma$, where Q_γ denotes the product measure from the Bernoulli percolation on the vertices of $\mathbb{G}$. In this proof, we use the same terminology as before. The random subset of green vertices is denoted by $G = G(\omega)$ and the connected component of the subgraph of open edges containing a vertex u is denoted by $C(u) = C(u, \omega)$.

Now, fixing a vertex v we define the increasing functions $f, g : \{0,1\}^{E \cup V} \to \{0,1\}$ by

$$f := \mathbb{1}\{|C(v)| \geq n\},$$

and

$$g := \mathbb{1}\{C(v) \cap G \neq \varnothing\}.$$

For each $u \in V$, we create an algorithm described by a decision tree $T = T^u$ that starts at the vertex u, and explores the cluster $C(u)$, one edge at a time, for vertices u that are green, and halts if the vertex is not green.

To explore the cluster of open edges of a green vertex, the algorithm looks first at the vertex u, then its adjacent edges, and obeying an ordering previously established, reveals the state of edges that are in the boundary of the set of open edges revealed until the previous step. The algorithm continues until the whole cluster $C(u)$ is discovered. In order to optimize the process, this algorithm needs to avoid going beyond the boundary of the open cluster $C(u)$, which consists of closed edges. In the subsequent paragraphs, we define this algorithm by a decision tree $T = T^u$ formally.

Fix an enumeration for the set of edges $E = \{e_1, e_2, e_3, ...\}$ and let $u \in V$ be any vertex. Set $T_1(\omega) = u$ for the first value for the decision tree $T(\omega) = (T_k(\omega))_{k \in \mathbb{N}}$. If $u \notin G$ we halt the algorithm, setting $T_n(\omega) = u$ for every $n \geq 2$. Otherwise, we continue with the algorithm as follows.

We create three auxiliary sequences of sets to save the data the algorithm gathered until each step $k \geq 1$. They can be described in words in the following manner:

For each integer k, we consider:

- U_k for the set that saves the endpoints of the open edges revealed until step k.
- O_k for the set that saves the open edges revealed until step k.
- C_k for the set that saves the closed edges revealed until step k.

That way, at each step $k \geq 1$ we have the information we need in the vector (U_k, O_k, C_k, T_k). Now, we define formally how they are created in an inductive way:

For the first step, that is $k = 1$, we initialize the algorithm by setting

$$(U_1, O_1, C_1, T_1) \coloneqq (\{u\}, \varnothing, \varnothing, u).$$

Now, assume we have already set (U_k, O_k, C_k, T_k) for every $1 \leq k \leq n$, for some $n > 1$. We define the next vector $(U_{n+1}, O_{n+1}, C_{n+1}, T_{n+1})$ depending on two cases.

Case 1: If there is at least one edge in E with one of its endpoints in U_n that is not in $O_n \cup C_n$, we set $T_{n+1}(\omega)$ to be the edge $e = xy$ that is minimal with respect to the ordering fixed previously.

Subcase 1-a: If the edge $e = T_{n+1}(\omega)$ is open we set

$$(U_{n+1}, O_{n+1}, C_{n+1}, T_{n+1}) \coloneqq (U_n \cup \{x, y\}, O_n \cup \{e\}, C_n, e).$$

Subcase 1-b: If the edge $T_{n+1}(\omega)$ is closed we set

$$(U_{n+1}, O_{n+1}, C_{n+1}, T_{n+1}) \coloneqq (U_n, O_n, C_n \cup \{e\}, e).$$

Case 2: If every edge $e \in E$ that has at least one endpoint in U_n is already in $O_n \cup C_n$, we halt the algorithm by setting

$$(U_{n+1}, O_{n+1}, C_{n+1}, T_{n+1}) \coloneqq (U_n, O_n, C_n, T_n).$$

that means we already have discovered the edges of the cluster $C(u)$.

Notice that, the set of elements that appears in the coordinates of T is, the singleton $\{u\}$ when the vertex is not green, and when u is green, it is the set of edges with at least one endpoint in $C(u)$ along with u. That is

$$\bigcup_{n\geq 1} \{z \in V \cup E \ ; \ T_n(\omega) = z\} = \begin{cases} \{u\} & \text{if } u \notin G \\ \{u\} \cup \overline{\triangle} C(u) & \text{if } u \in G \end{cases}.$$

Where $\overline{\triangle} C(u)$ is the set of edges that are incident to vertices of $C(u)$. Write T^u for the decision tree previously constructed where we fixed an arbitrary vertex u. The collection of such decision trees $F = \{T^u\}_{u\in V}$ is a decision forest that clearly computes g, and therefore we can use Corollary 4.10 to estimate the covariance of f and g under the probability measure P. Since both f and g assume values on $\{0, 1\}$ it is easy to see that $\mathrm{CoVr}_P[f, g] = 2\mathrm{Cov}_P[f, g]$, therefore, we get:

$$\begin{aligned} &\mathrm{Cov}_P[f, g] \\ &\leq \sum_{z\in E\cup V} \delta_z(F, P)\mathrm{Cov}_P[f, \omega_z] \\ &= \sum_{e\in E} \delta_e(F, P)\mathrm{Cov}_P[f, \omega_e] + \sum_{u\in V} \delta_u(F, P)\mathrm{Cov}_P[f, \omega_u] \\ &= \sum_{e\in E} \delta_e(F, P)\mathrm{Cov}_P[f, \omega_e] \\ &= \sum_{e\in E} \delta_e(F, P)\mathrm{Cov}_\mu[f, \omega_e]. \end{aligned} \tag{4.2}$$

Since f and ω_u are clearly independent under P, therefore $\mathrm{Cov}_P[f, \omega_u] = 0$. It is possible to bound the probability of an edge e being revealed by F from a simple observation. From the way we constructed the algorithm, an edge $e = xy$ is revealed if and only if at least one of its endpoints x, y is contained on the cluster $C(u)$ of a green vertex u. This comes from the fact that the algorithm does not explore further the cluster of vertices that are not green. Thus,

$$\bigcup_{n\geq 1}\{F_n(\omega) = e = xy\} = \{C(x)\cap G \neq \varnothing\} \cup \{C(y)\cap G\neq\varnothing\}.$$

Taking the probability of the set, by the definition of $\delta_e(F, P)$ and the union bound we get

$$\begin{aligned}\delta_e(F,P) &\leq 2\sup_{u\in V} P(C(u)\cap G\neq\varnothing)\\ &= 2\sup_{u\in V}\mu\left(1-e^{-\lambda|C(u)|/n}\right),\end{aligned} \tag{4.3}$$

using the notation $\mu(\phi)$ for the expected value of a function ϕ with respect to μ. Equations (4.2) and (4.3) yields

$$\mathrm{Cov}_P[f,g] \leq 2\sup_{u\in V}\mu\left(1-e^{-\lambda|C(u)|/n}\right)\sum_{e\in E}\mathrm{Cov}_\mu[f,\omega_e]. \tag{4.4}$$

Now, developing $\mathrm{Cov}_P[f, g]$ we get

$$\begin{aligned}&\mathrm{Cov}_P[f,g]\\ &= E_P[fg]-E_P[f]E_P[g]\\ &= P(|C(v)|\geq n, C(v)\cap G\neq\varnothing) - \mu(|C(v)|\geq n)\cdot P(C(v)\cap G\neq\varnothing).\end{aligned}$$

We also have that:

$$\begin{aligned}P(|C(v)|\geq n, C(v)\cap G\neq\varnothing) &= \mu\left(\mathbb{1}\{|C(v)|\geq n\}\cdot(1-e^{-\lambda|C(v)|/n})\right)\\ &\geq (1-e^{-\lambda})\mu(|C(v)|\geq n).\end{aligned}$$

In conclusion,

$$\mathrm{Cov}_P[f,g]\geq \mu(|C(v)|\geq n)\cdot\left[(1-e^{-\lambda})-\mu(1-e^{-\lambda|C(v)|/n})\right]. \tag{4.5}$$

Substituting (4.5) into Equation (4.4) we get that

$$\begin{aligned}&\sum_{e\in E}\mathrm{Cov}_\mu[f,\omega(e)]\geq\\ &\left[\frac{(1-e^{-\lambda})-\mu[1-e^{-\lambda|C(v)|/n}]}{2\cdot\sup_{u\in V}\mu[1-e^{-\lambda|C(u)|/n}]}\right]\cdot\mu(|C(v)|\geq n).\end{aligned}$$

□

Now, we can proceed with the main theorem of this chapter, the differential inequality proved by [10] for the random-cluster model, but presented here directly for the percolation model.

Theorem 4.16. *Let $n \geq 1$, $\lambda > 0$, and $p \in (0,1)$. If P_p is the Bernoulli percolation measure on the edges of a transitive graph, then*

$$\left(\frac{d}{dp}\right)_+ \log P_p(|C| \geq n) \geq \frac{1}{2p(1-p)}\left[\frac{(1-e^{-\lambda})n}{\lambda \sum_{m=1}^{\lceil n/\lambda \rceil} P_p(|C| \geq m)} - 1\right], \tag{4.6}$$

where $\lceil x \rceil$ denotes the ceiling function, and C is the origin's cluster.

Proof. Fix an integer $n \geq 1$ and let $f : \{0,1\}^E \to \{0,1\}$ be defined by $f(\omega) = \mathbb{1}\{|C(\omega)| \geq n\}$. By Propositions 4.13 and 4.14 we have that

$$\begin{aligned}\left(\frac{d}{dp}\right)_+ \log(E_p[f]) &= \frac{1}{E_p[f]}\left(\frac{d}{dp}\right)_+ E_p[f] \\ &\geq \frac{1}{E_p[f]p(1-p)} \sum_{e \in E} \mathrm{Cov}[f, \omega_e].\end{aligned}$$

We can now use Proposition 4.15 to get

$$\begin{aligned}&\left(\frac{d}{dp}\right)_+ \log(E_p[f]) \\ &\geq \frac{1}{2E_p[f]p(1-p)}\left(\frac{(1-e^{-\lambda})}{E_p[1-e^{-\lambda|C|/n}]} - 1\right)E_p[f].\end{aligned}$$

Notice that $1 - e^{-\lambda|C|/n} \leq 1 \wedge \lambda|C|/n$, therefore

$$\begin{aligned}E_p[1-e^{-\lambda|C|/n}] &\leq E_p\left[1 \wedge \frac{\lambda|C|}{n}\right] \\ &= \frac{\lambda}{n}E_p\left[\frac{n}{\lambda} \wedge |C|\right] \\ &\leq \frac{\lambda}{n}\sum_{m=1}^{\lceil n/\lambda \rceil} P_p(|C| \geq m).\end{aligned}$$

Which finally implies that

$$\left(\frac{d}{dp}\right)_+ \log(E_p[f]) \geq \frac{1}{2p(1-p)}\left[\frac{(1-e^{-\lambda})n}{\lambda \sum_{m=1}^{\lceil n/\lambda \rceil} P_p(|C| \geq m)} - 1\right].$$

□

We now state some direct consequences of this inequality, which we use in the next section.

Corollary 4.17. *If* $p \in (0, 1)$, *then*

$$\left(\frac{d}{dp}\right)_+ \log P_p(|C| \geq n) \geq \frac{1}{2p(1-p)}\left[\frac{n}{E_p[|C|]} - 1\right].$$

Proof. If we make $\lambda \downarrow 0$, then $(1 - e^{-\lambda})/\lambda \to 1$ and $n/\lambda \to \infty$. Therefore, we get the result just by substituting these limits on Theorem 4.16. □

Corollary 4.18. *If* $0 < p < p_0 < 1$ *and* $\lambda > 0$, *then*

$$P_p(|C| \geq n) \leq P_{p_0}(|C| \geq n) \exp(-A_{n,p_0}(p_0 - p)),$$

where

$$A_{n,p_0} = 2\left[\frac{(1 - e^{-\lambda})n}{\lambda \sum_{m=1}^{\lceil n/\lambda \rceil} P_{p_0}(|C| \geq m)} - 1\right].$$

Proof. Since $\log P_p(|C| \geq n)$ is increasing over p, we can integrate the inequality from Theorem 4.16 using Proposition 4.12 from p to p_0 to get

$$\begin{aligned}
&\log P_{p_0}(|C| \geq n) - \log P_p(|C| \geq n) \\
&\geq \int_p^{p_0} \left(\frac{d}{dt}\right)_+ \log P_t(|C| \geq n) dt \\
&\geq \int_p^{p_0} \frac{1}{2t(1-t)}\left[\frac{(1 - e^{-\lambda})n}{\lambda \sum_{m=1}^{\lceil n/\lambda \rceil} P_{p_0}(|C| \geq m)} - 1\right] dt \\
&\geq \frac{A_{n,p_0}}{4} \int_p^{p_0} \frac{1}{t(1-t)} dt \\
&\geq A_{n,p_0}(p_0 - p).
\end{aligned}$$

Thus, applying the exponential function and rearranging, we conclude that if $p < p_0$, then

$$P_p(|C| \geq n) \leq P_{p_0}(|C| \geq n) \exp(-A_{n,p_0}(p_0 - p)).$$

□

Corollary 4.19. *If* $0 < p < p_0$, *then*

$$P_p(|C| \geq n) \leq P_{p_0}(|C| \geq n) \exp(-B_{n,p_0}(p_0 - p)),$$

where

$$B_{n,p_0} = 2\left[\frac{n}{E_{p_0}[|C|]} - 1\right].$$

Proof. Taking the limit $\lambda \to 0$ in Corollary 4.18 yields the result. □

4.2 Three Bounds With γ, δ, Δ

With the differential inequality (4.6) established and its consequences analyzed, we now turn to the study of critical exponents. The following result shows that, if the critical exponent δ exists, we can derive a bound on the decay of the radius of the open cluster in the subcritical phase. Additionally, it reveals how the k-th moment of the cluster volume diverges at the critical parameter. As a direct consequence, we obtain the critical relations $\gamma \leq \delta - 1$ and $\Delta \leq \delta$.

Theorem 4.20. *Let G be an infinite, connected, locally finite transitive graph, and suppose that there exist constants $M > 0$ and $\delta > 1$ such that*

$$P_{p_c}(|C| \geq n) \leq Mn^{-1/\delta}$$

for every $n \geq 1$. Then the following holds:

1. There exist positive constants B_1 and B_2 such that

$$P_p(|C| \geq n) \leq B_2 n^{-1/\delta} \exp\left[-B_1 n (p_c - p)^\delta\right]$$

for every $p \in [0, p_c)$ and $n \geq 1$.

2. There exists a constant D such that

$$E_p[|C|^k] \leq k! \left[\frac{D}{p_c - p}\right]^{\delta k - 1}$$

for every $p \in [0, p_c)$ and $k \geq 1$.

Proof. Assume the existence of the constants $M > 0$ and $\delta > 1$ such that

$$P_{p_c}(|C| \geq n) \leq Mn^{-1/\delta}$$

holds for any $n \geq 1$. Then, summing the first n terms yields the estimate

$$\begin{aligned} \sum_{m=1}^{n} P_{p_c}(|C| \geq m) &\preceq \sum_{m=1}^{n} m^{-1/\delta} \\ &\preceq \int_0^n x^{-1/\delta} dx \\ &\preceq n^{1-1/\delta}. \end{aligned}$$

We also have, by Corollary 4.18 that, for any $p_1 \in (0, p_c)$

$$\begin{aligned} P_{p_1}(|C| \geq n) &\leq P_{p_c}(|C| \geq n) \exp(-A_{n,p_c}(p_c - p_1)) \\ &\preceq n^{-1/\delta} \exp(-An^{1/\delta}(p_c - p_1)). \end{aligned} \tag{4.7}$$

For a positive constant A, where we considered $\lambda = 1$ in the following way

$$A_{n,p_c} = 2\left[\frac{(1-e^{-1})n}{\sum_{m=1}^{n} P_{p_c}(|C| \geq m)} - 1\right] \geq A\frac{n}{n^{1-1/\delta}} - 2 = An^{1/\delta} - 2.$$

Thus, we get a bound for the expected value of the size of the cluster by summing (4.7) over $n \geq 1$

$$E_{p_1}[|C|] \preceq \sum_{n=1}^{\infty} n^{-1/\delta} \exp(-An^{1/\delta}(p_c - p_1)).$$

It is possible to estimate the sum on the right side by the integral

$$I = \int_1^{\infty} x^{-1/\delta} \exp(-A(p_c - p_1)x^{1/\delta})dx.$$

We can simplify the expressions by writing $\alpha := 1/\delta$ and $\sigma := A^{\delta}(p_c - p_1)^{\delta}$:

$$I = \int_1^{\infty} x^{-\alpha} \exp(-(\sigma x)^{\alpha})dx.$$

Now we can make the change of variables $y = (\sigma x)^{\alpha}$, which implies that $xdy = \alpha y dx$. The integral now becomes

$$\begin{aligned} I &= \int_{\sigma^{\alpha}}^{\infty} \left(\frac{y}{\sigma^{\alpha}}\right)^{-1} e^{-y} \left(\frac{y^{1/\alpha - 1}}{\alpha\sigma}\right) dy \\ &= \sigma^{\alpha-1}\left(\frac{1}{\alpha}\int_{\sigma^{\alpha}}^{\infty} e^{-y} y^{(1/\alpha-1)-1} dy\right) \\ &\leq \frac{\Gamma(1/\alpha - 1)}{\alpha}\sigma^{\alpha-1} \\ &\preceq (p_c - p_1)^{1-\delta}. \end{aligned}$$

Therefore, we conclude that, if $p_1 \in (0, p_c)$ then

$$E_{p_1}[|C|] \preceq (p_c - p_1)^{1-\delta}.$$

Now, consider a $p \in (0, p_1)$. Applying Corollary 4.19, we get that, for any $n \geq 1$ we have

$$\begin{aligned} P_p(|C| \geq n) &\leq P_{p_1}(|C| \geq n)\exp(-B_{n,p_1} n(p_1 - p)) \\ &\preceq n^{-1/\delta} \exp\left(-B_0 n \frac{p_1 - p}{(p_c - p_1)^{1-\delta}}\right). \end{aligned}$$

Since $P_{p_1}(|C| \geq n) \leq P_{p_c}(|C| \geq n) \leq Mn^{-1/\delta}$ and

$$B_{n,p_1} = 2\left[\frac{n}{E_{p_1}[|C|]} - 1\right] \geq B_0 n \frac{1}{(p_c - p_1)^{1-\delta}} - 2,$$

where B_0 is some positive constant. In conclusion, we proved that, if $p < p_1 < p_c$, then, there exist positive constants B_0 and B_2, that do not depend on p and p_1, such that for every $n \geq 1$

$$P_p(|C| \geq n) \leq B_2 n^{-1/\delta} \exp\left(- B_0 n \frac{p_1 - p}{(p_c - p_1)^{1-\delta}} \right).$$

Setting $p_1 := (p + p_c)/2$ we find B_1 such that

$$P_p(|C| \geq n) \leq B_2 n^{-1/\delta} \exp(-B_1 n (p_c - p)^{\delta}).$$

To prove the second part we use the formula for the k-th moment of the positive random variable $|C|$ as

$$E_p[|C|^k] = \int_0^\infty k x^{k-1} P_p(|C| \geq x) dx.$$

In addition, using the previous inequality, since $|C|$ assumes only integer values, if $x > 0$

$$\begin{aligned} P_p(|C| \geq x) &= P_p(|C| \geq \lceil x \rceil) \\ &\leq B_2 \lceil x \rceil^{-1/\delta} \exp(-B_1 \lceil x \rceil (p_c - p)^{\delta}) \\ &\preceq x^{-1/\delta} \exp(-B_1 x (p_c - p)^{\delta}). \end{aligned}$$

Therefore

$$E_p[|C|^k] \preceq k \int_0^\infty x^{k-1} x^{-1/\delta} \exp(-B_1 x (p_c - p)^{\delta}) dx.$$

Now, as done before we simplify the expression writing $\alpha = k - 1 - 1/\delta$ and $\sigma = \sigma(p) = B_1 (p_c - p)^{\delta}$, so the integral can be written as

$$\begin{aligned} E_p[|C|^k] &\preceq k \int_0^\infty x^{\alpha} \exp(-\sigma x) dx \\ &= \frac{k}{\sigma^{\alpha+1}} \Gamma(\alpha + 1) \\ &= \frac{k \Gamma(k - 1/\delta)}{B_1^{\alpha+1} (p_c - p)^{\delta(k-1/\delta)}} \\ &\leq k! \left[\frac{D}{p_c - p} \right]^{\delta k - 1}. \end{aligned}$$

Where on the second line we are using the change of variables $y = \sigma x$. □

As a consequence of the previous theorem, we get the following critical relations.

Corollary 4.21. *If the critical exponents γ, δ and Δ exist, then*

$$\gamma \leq \delta - 1$$

and

$$\Delta \leq \delta.$$

Proof. Assume the existence of these three critical exponents, γ, δ, and Δ. For the first relation, recall the definition of γ, that is

$$E_p[|C|] \approx (p_c - p)^{-\gamma}$$

as $p \uparrow p_c$. This, along with the second part of the previous theorem with $k = 1$, yields that, for a given $\varepsilon > 0$

$$(p_c - p)^{-\gamma+\varepsilon} \preceq (p_c - p)^{1-\delta}$$

for values of p close enough to p_c. Applying the natural logarithm to both sides, then dividing by $\log(p_c - p) < 0$ and finally making $p \uparrow p_c$ and $\varepsilon \downarrow 0$ we get that

$$-\gamma \geq 1 - \delta.$$

For the second relation, for $k \geq 2$, from the definition of Δ and γ

$$E_p[|C|^k] \approx (p_c - p)^{-(k-1)\Delta+\gamma}.$$

Then, using the previous theorem with $k \geq 2$ we get that, for a given $\varepsilon > 0$:

$$(p_c - p)^{-(k-1)\Delta+\gamma+\varepsilon} \preceq (p_c - p)^{1-\delta k}$$

for values of p sufficiently close to p_c. Applying the natural logarithm to both sides, then dividing by $\log(p_c - p) < 0$ and finally making $p \uparrow p_c$ and then $\varepsilon \downarrow 0$ we get that

$$-(k-1)\Delta + \gamma \geq 1 - \delta k.$$

Dividing by $k - 1$ and making $k \to \infty$ implies that $\Delta \leq \delta$. □

Theorem 4.22. *Let G be an infinite, connected, locally finite transitive graph, and suppose that there exist constants $M > 0$ and $\gamma \geq 1$ such that*

$$E_p[|C|] \leq M(p_c - p)^{-\gamma}$$

for every $p \in [0, p_c)$. Then there exists a constant D such that

$$E_p[|C|^k] \leq k! \left[\frac{D}{p_c - p}\right]^{\gamma+(k-1)(\gamma+1)}$$

for every $p \in [0, p_c)$ and $k \geq 1$.

This theorem implies the critical relation $\Delta \leq \gamma + 1$.

Proof. Assume the existence of $M > 0$ and $\gamma \geq 1$ such that

$$E_p[|C|] \leq M(p_c - p)^{-\gamma}$$

holds for every $n \geq 1$ and $p \in (0, p_c)$. If we consider a $p_1 \in (0, p_c)$, and a $p \in [0, p_1)$ Corollary 4.19 tells us that, for every $n \geq 1$:

$$P_p(|C| \geq n) \leq P_{p_1}(|C| \geq n) \exp(-B_{n,p_1}(p_1 - p)).$$

Applying Markov's Inequality for $P_{p_1}(|C| \geq n)$ and rewriting the definition of B_{n,p_1} we get

$$P_p(|C| \geq n) \leq \frac{E_{p_1}[|C|]}{n} \exp\left\{ -2(p_1 - p)\left(\frac{n}{E_{p_1}[|C|]} - 1\right)\right\}.$$

Now, we use the hypothesis to bound $E_{p_1}[|C|]$ twice on the equation above to get

$$P_p(|C| \geq n) \preceq \frac{1}{n}(p_c - p_1)^{-\gamma} \exp\left[- \frac{d_0(p_1 - p)n}{(p_c - p_1)^{-\gamma}}\right],$$

where d_0 is a positive constant. Finally, we set $p_1 := (p + p_c)/2$, the expression then becomes

$$P_p(|C| \geq n) \preceq \frac{1}{n}(p_c - p)^{-\gamma} \exp(-d_1(p_c - p)^{\gamma+1}n),$$

where d_1 is also a positive constant. From here, to bound the growth of the k-th moment for the positive random variable $|C|$ we make a similar calculation to what we did in the previous theorem:

$$\begin{aligned} E_p[|C|^k] &= \int_0^\infty kx^{k-1} P_p(|C| \geq x)dx \\ &\preceq \int_0^\infty kx^{k-1}\frac{1}{x}(p_c - p)^{-\gamma} \exp\Big(- d_1(p_c - p)^{\gamma+1}x\Big)dx \\ &= k(p_c - p)^{-\gamma} \int_0^\infty x^{k-2}e^{-\alpha x}dx, \end{aligned}$$

where $\alpha = d_1(p_c - p)^{\gamma+1}$. Making the change of variables $y = \alpha x$ we get

$$\begin{aligned} E_p[|C|^k] &\preceq \frac{k(p_c - p)^{-\gamma}}{a^{k-1}}\Gamma(k - 1) \\ &= k(k-2)!\left[\frac{D}{(p_c - p)}\right]^{\gamma+(\gamma+1)(k-1)} \\ &\leq k!\left[\frac{D}{(p_c - p)}\right]^{\gamma+(\gamma+1)(k-1)}. \end{aligned}$$

□

Corollary 4.23. *If the critical exponents Δ and γ exist, then*

$$\Delta \leq \gamma + 1.$$

Proof. Assume the existence of both critical exponents. By the existence of γ, the previous theorem tells us that

$$E_p[|C|^k] \preceq (p_c - p)^{-\gamma-(\gamma+1)(k-1)}.$$

Also, by the definition of Δ we have that, for a given $\varepsilon > 0$:

$$(p_c - p)^{-(k-1)\Delta+\gamma+\varepsilon} \preceq E_p[|C|^k],$$

for values of p near p_c. Therefore, we have for values of p sufficiently close to p_c that

$$(p_c - p)^{-(k-1)\Delta+\gamma+\varepsilon} \preceq (p_c - p)^{-\gamma-(\gamma+1)(k-1)}.$$

Taking the natural logarithm, dividing by $\log(p_c - p) < 0$ and then making $p \uparrow p_c$ and $\varepsilon \downarrow 0$ we get that, for every $k \geq 2$ the following relation holds

$$-(k-1)\Delta + \gamma \geq -\gamma - (\gamma+1)(k-1).$$

Dividing by $k - 1$, then making $k \to \infty$ we conclude that

$$\Delta \leq \gamma + 1.$$

□

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Appendices

Appendix A

Finite Volume Approximations

In order to complete the proofs of differential inequality A and differential inequality B we have to prove that Θ is C^1 on p for $s > 0$ and also the convergence of the functions $\Theta_N \to \Theta$ and their partial derivatives $\partial_p \Theta_N \to \partial_p \Theta$, and $\partial_s \Theta_N \to \partial_s \Theta$ as $N \to \infty$.

Let A be a connected and finite subgraph of $\mathbb{L}^d$ containing the origin. We write $V(A)$ for its set of vertices, $E(A)$ for its set of edges, and $\triangle A$ for its boundary of edges. Now, for positive integers n, m, b consider the collection $\mathcal{A}[n, m, b]$ of such connected and finite subgraphs containing the origin with $V(A) = n$, $E(A) = m$ and $\triangle A = b$.

When the cluster of the origin is finite, it assumes the form of one of the elements of the collection $\mathcal{A}[n, m, b]$, for some integers n, m, b. Therefore, we can write

$$P_p(|C| = n) = \sum_{m,b} a_{n,m,b} p^m (1-p)^b, \tag{A.1}$$

and

$$P_p(|C_N| = n) = \sum_{m,b} a_{n,m,b}(N) p^m (1-p)^b, \tag{A.2}$$

where $a_{n,m,b}(N)$ refers to the cardinality of the collection $\mathcal{A}[n, m, b](N)$ of such connected and finite subgraphs of Λ_N containing the origin. We can also write

$$\Theta(p, s) = 1 - \sum_{n \geq 1} (1-s)^n \left\{ \sum_{m,b} a_{n,m,b} p^m (1-p)^b) \right\}, \tag{A.3}$$

and

$$\Theta_N(p, s) = 1 - \sum_{n \geq 1} (1-s)^n \left\{ \sum_{m,b} a_{n,m,b}(N) p^m (1-p)^b) \right\}. \tag{A.4}$$

Proposition A.1. *The function Θ is continuously differentiable in p for $s > 0$.*

Proof. Let $p \in (0,1)$, $s > 0$. It will be sufficient if we show that the series with the derivative of the terms of Θ converges uniformly at some neighborhood of p. Let

$$\psi(p) := \sum_{n\geq 1} \partial_p \Big\{ (1-s)^n P_p(|C| = n) \Big\}.$$

Then, using equation (A.1) we get

$$\psi(p) = \sum_{n\geq 1} (1-s)^n \sum_{m,b} a_{n,m,b} p^m (1-p)^b \left(\frac{m}{p} - \frac{b}{1-p} \right).$$

Using that $b, m \leq 2dn$ we can bound the absolute value of the tail of $\psi(p)$ by

$$\sum_{n\geq M} (1-s)^n P_p(|C| = n) \frac{2dn}{p(1-p)}.$$

This goes uniformly to 0 in every strictly closed subinterval $[a, b]$ of $(0, 1)$ as $M \to \infty$. Therefore, we conclude that $\psi(p)$ is exactly the derivative of $\Theta(p, s)$ with respect to p. □

Proposition A.2. *If p, s are in $(0,1)$ then $\Theta_N(p,s) \to \Theta(p,s)$.*

Proof.

$$|\Theta(p,s) - \Theta_N(p,s)| \leq \sum_{n\geq 1} (1-s)^n |P_p(|C| = n) - P_p(|C_N| = n)|.$$

However, $P_p(|C| = n) = P_p(|C_N| = n)$ if $n < N$ by the construction of $\mathbb{G}_N^d$. Therefore,

$$\begin{aligned} |\Theta(p,s) - \Theta_N(p,s)| &\leq \sum_{n\geq N} (1-s)^n |P_p(|C| = n) - P_p(|C_N| = n)| \\ &\leq 2 \sum_{n\geq N} (1-s)^n \\ &= 2\frac{(1-s)^N}{s}. \end{aligned}$$

That goes to zero as $N \to \infty$. □

Proposition A.3. *If p, s are in $(0,1)$ then $\partial_p \Theta_N(p,s) \to \partial_p \Theta(p,s)$.*

Proof. Let $p, s \in (0,1)$. Using the same estimates presented in the last results we get

$$\begin{aligned} |\partial_p \Theta(p,s) - \partial_p \Theta_N(p,s)| &\leq 2 \sum_{n\geq N} (1-s)^n \left(\frac{m}{p} - \frac{b}{1-p} \right) \\ &\leq \frac{4d}{p(1-p)} \sum_{n\geq N} n(1-s)^n. \end{aligned}$$

That goes to zero as $N \to \infty$. □

Proposition A.4. *If p, s are in $(0,1)$ then $\partial_s \Theta_N(p,s) \to \partial_s \Theta(p,s)$.*

Proof. Let $p, s \in (0,1)$. Using the same estimates presented in the last results we get

$$\begin{aligned} |\partial_s \Theta(p,s) - \partial_s \Theta_N(p,s)| &= \sum_{n \geq 1} ns(1-s)^{n-1} |P_p(|C| = n) - P_p(|C_N| = n)| \\ &\leq 2 \sum_{n \geq N} ns(1-s)^{n-1}. \end{aligned}$$

That goes to zero as $N \to \infty$. □

♣