UNIVERSIDADE FEDERAL DE MINAS GERAIS
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Fractional p-Laplacian and p-Kirchhoff equations with Sobolev and Choquard critical nonlinearities and weighted singularities

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Orientador: Prof. Ronaldo Brasileiro Assunção.
Coorientador: Prof. Olímpio Hiroshi Miyagaki.

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Fractional \$p\$-Laplacian and \$p\$-Kirchhoff equations with Sobolev and Choquard critical nonlinearities and weighted singularities

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Resumo

Na primeira parte desta tese, estudamos um problema modelo de p-Laplaciano fracionário no espaço $\mathbb{R}^N$ com dupla não linearidades críticas envolvendo um termo crítico local de Sobolev junto com um termo crítico não local de Choquard fracionário; o problema também inclui um termo homogêneo de Hardy; adicionalmente, todas as não linearidades possuem singularidades. Ao estabelecer novos resultados de imersões envolvendo normas de Morrey com peso no espaço homogêneo de Sobolev fracionário, fornecemos condições suficientes sob as quais existe uma solução fraca não trivial para o problema por meio de métodos variacionais. Usando as mesmas técnicas utilizadas para provar este resultado, também podemos tratar problemas envolvendo termos duplamente críticos de Sobolev ou duplamente críticos de Choquard.

Em seguida, estudamos outra variante do problema do p-Laplaciano fracionário com termos de Sobolev-Choquard e um termo de acoplamento crítico. Mais precisamente, consideramos um sistema de equações de p-Laplaciano fracionário no espaço $\mathbb{R}^N$ com dupla não linearidades críticas envolvendo um termo crítico local de Sobolev junto com um termo crítico não local de Choquard; o problema também inclui um termo homogêneo de Hardy; além disso, todas as não linearidades envolvem singularidades; adicionalmente, o termo de acoplamento é crítico no sentido das imersões de Sobolev. Para provar o resultado principal, usamos uma versão da desigualdade de Caffarelli-Kohn-Nirenberg e um refinamento da desigualdade de Sobolev que está relacionada ao espaço de Morrey, pois nosso problema envolve expoentes duplamente críticos. Com a ajuda desses resultados, fornecemos condições suficientes sob as quais existe uma solução fraca não trivial para o problema por meio de métodos variacionais.

Por fim, consideramos uma equação p-Kirchhoff fracionária no espaço $\mathbb{R}^N$ com dupla não linearidades, envolvendo um termo subcrítico não local generalizado de Choquard junto com um termo crítico local de Sobolev; o problema também inclui um termo do tipo Hardy; adicionalmente, todos os termos têm pesos singulares críticos. Focamos nossa atenção na existência de uma solução fraca não trivial para a equação p-Kirchhoff fracionária no espaço $\mathbb{R}^N$. A possibilidade de um crescimento mais lento da não linearidade torna mais difícil estabelecer uma condição de compacidade; para isso, usamos a condição de Cerami. Os pontos cruciais em nosso argumento são a limitação uniforme da parte da convolução e a falta de compacidade das imersões de Sobolev.

Palavras-chave: p-Laplaciano fracionário; potenciais de Hardy; singularidades com peso; equação de Sobolev-Choquard; equação p-Kirchhoff fracionária.

Abstract

In the first part of this dissertation thesis, we study a fractional p-Laplacian model problem in the entire space $\mathbb{R}^N$ featuring doubly critical nonlinearities involving a local critical Sobolev term together with a nonlocal Choquard fractional critical term; the problem also includes a homogeneous Hardy term; additionally, all nonlinearities have singularities. By establishing new embedding results involving weighted Morrey norms in the homogeneous fractional Sobolev space, we provide sufficient conditions under which a weak nontrivial solution to the problem exists via variational methods. By using the same techniques used to prove this result we can also deal with problems involving double critical Sobolev or double critical Choquard terms.

Next, we study another variant of the fractional p-Laplacian problem with Sobolev-Choquard terms and a critical coupling term. More precisely, we consider a fractional p-Laplacian system of equations in the entire space $\mathbb{R}^N$ with doubly critical singular nonlinearities involving a local critical Sobolev term together with a nonlocal Choquard critical term; the problem also includes a homogeneous Hardy term; moreover, all the nonlinearities involve singular critical weights; additionally, the coupling term is critical in the sense of the Sobolev embeddings. To prove the main result we use a version of the Caffarelli-Kohn-Nirenberg inequality and a refinement of Sobolev inequality that is related to Morrey space because our problem involves doubly critical exponents. With the help of these results, we provide sufficient conditions under which a weak nontrivial solution to the problem exists via variational methods.

Finally, we consider a fractional p-Kirchhoff equation in the entire space $\mathbb{R}^N$ featuring double nonlinearities, involving a generalized nonlocal Choquard subcritical term together with a local critical Sobolev term; the problem also includes a Hardy-type term; additionaly, all terms have critical singular weights. We focus our attention on the existence of a nontrivial weak solution for fractional p-Kirchhoff equation in the entire space $\mathbb{R}^N$. The possibility of a slower growth in the nonlinearity makes it more difficult to establish a compactness condition; to do so, we use the Cerami condition. The crucial points in our argument are the uniform boundedness of the convolution part and the lack of compactness of the Sobolev embeddings.

Keywords: fractional p-Laplacian; Hardy potentials; weighted singularities; Sobolev-Choquard equation; fractional p-Kirchhoff equation.

Sumário

Chapter 0

Introduction and main results

In this work we study some elliptic problems involving the fractional p-Laplacian operator in the entire space $\mathbb{R}^N$ with double critical nonlinearities, in the sense of Sobolev and Choquard, and also a Hardy potential; moreover, all nonlinearities have singularities. To address the Sobolev-Choquard and Hardy critical terms that arise in these problems, we must employ Morrey spaces to facilitate the analysis of Palais-Smale sequences. Morrey spaces are particularly useful when the Sobolev embedding is not applicable; moreover, these spaces complement the boundedness properties of the operators that cannot be handled by Lebesgue spaces. Next, we consider a fractional p-Laplacian system where the coupling term is critical in the sense of the Sobolev embeddings. Finally, we consider a fractional p-Kirchhoff equation also featuring doubly critical nonlinearities, namely, a generalized non-local subcritical Choquard term and local critical Sobolev term; the problem also includes a Hardy-type term; additionaly, all terms have critical singular weights. The interest in these problems, which have been extensively studied by several authors, is connected to its applications in modeling steady-state solutions of reaction-diffusion problems arising in biophysics, in plasma physics, in the study of chemical reactions, in elementary particle physics, and also in mathematical finance.

To prove the existence results for these classes of elliptic equations we have to deal with a non-linear and non-local operator for which the method of harmonic extension due to Caffarelli and Silvestre cannot be applied. Since we consider critical behavior of multiple nonlinearities with singularities, we have to make a careful analysis of the energy levels for which we can recover the compactness of the Palais-Smale or Cerami sequences; additionaly, we have to deal with the asymptotic competition between the critical nonlinearities and make sure that one does not dominate the other. So, to deal with the associated difficulties in proving existence results, first we have to prove that the extremals for the Sobolev and Stein-Weiss inequalities are attained; and we also have to prove a refined version of the Caffarelli-Kohn-Nirenberg inequality and some new embeddings involving the weighted Morrey spaces.

0.1 The Sobolev-Choquard problems with Hardy term

In the present section, we consider the following fractional p-Laplacian equation in the entire space $\mathbb{R}^N$ featuring doubly critical nonlinearities, involving a local critical Sobolev term together with a non-local Choquard critical term; the problem also includes a homogeneous Hardy term; additionaly, all terms have critical singular weights. More precisely, we deal with the problem

$$\begin{cases} (-\Delta)^s_{p,\theta} u - \gamma \dfrac{|u|^{p-2}u}{|x|^{sp+\theta}} = \dfrac{|u|^{p^*_s(\beta,\theta)-2}u}{|x|^{\beta}} + \left[I_\mu * F_{\delta,\theta,\mu}(\cdot,u)\right](x) f_{\delta,\theta,\mu}(x,u) \\ u \in \dot{W}^{s,p}_\theta(\mathbb{R}^N) \end{cases} \tag{1}$$

where $0 < s < 1$; $0 < \alpha, \beta < sp + \theta < N$; $0 < \mu < N$; $2\delta + \mu < N$; $\gamma < \gamma_H$ with the best fractional Hardy constant γ_H to be defined below; the Hardy-Sobolev and Stein-Weiss upper critical fractional exponents (this latter also called Hardy-Littlewood-Sobolev upper critical exponent) are respectively defined by

$$p^*_s(\beta,\theta) = \frac{p(N-\beta)}{N-sp-\theta} \qquad \text{and} \qquad p^\sharp_s(\delta,\theta,\mu) = \frac{p(N-\delta-\mu/2)}{N-sp-\theta}.$$

Moreover, $I_\mu(x) = |x|^{-\mu}$ is the Riesz potential of order μ; the functions $f_{\delta,\theta,\mu}, F_{\delta,\theta,\mu} \colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are respectively defined by

$$f_{\delta,\theta,\mu}(x,t) = \frac{|t|^{p^\sharp_s(\delta,\theta,\mu)-2}t}{|x|^{\delta}} \qquad \text{and} \qquad F_{\delta,\theta,\mu}(x,t) = \frac{|t|^{p^\sharp_s(\delta,\theta,\mu)}}{|x|^{\delta}}, \tag{2}$$

that is, $F_{\delta,\theta,\mu}(x,t) = p^\sharp_s(\delta,\theta,\mu)\int_0^{|t|} f_{\delta,\theta,\mu}(x,\tau)\,\mathrm{d}\tau$; and the term with convolution integral,

$$[I_\mu * F_{\delta,\theta,\mu}(\cdot,u)](x) \coloneqq \int_{\mathbb{R}^N} \frac{|u(y)|^{p^\sharp_s(\delta,\theta,\mu)}}{|x-y|^{\mu}|y|^{\delta}}\,\mathrm{d}y\,,$$

is known as Choquard type nonlinearity.

Intuitively, problem (1) is understood as showing the existence of a function $u \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ such that

$$(-\Delta)^s_{p,\theta} u - \gamma \frac{|u|^{p-2}u}{|x|^{sp+\theta}} = \frac{|u|^{p^*_s(\beta,\theta)-2}u}{|x|^{\beta}} + \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{p^\sharp_s(\delta,\theta,\mu)}}{|x-y|^{\mu}|y|^{\delta}}\,\mathrm{d}y\right) \frac{|u(x)|^{p^\sharp_s(\delta,\theta,\mu)-2}u(x)}{|x|^{\delta}}$$

where the fractional p-Laplacian operator is defined for $\theta = \theta_1 + \theta_2$, $x \in \mathbb{R}^N$, and any function $u \in C^\infty_0(\mathbb{R}^N)$, as

$$(-\Delta)^s_{p,\theta} u(x) \coloneqq \text{p.v.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\,\mathrm{d}y\,, \tag{3}$$

and p.v. is the Cauchy's principal value. This operator is the prototype of nonlinear non-local elliptic operator and can also be defined on smooth functions by

$$(-\Delta)^s_{p,\theta} u(x) \coloneqq 2\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\,\mathrm{d}y\,.$$

This definition is consistent, up to a normalization constant $C = C(N,s,\theta)$, with the usual definition of the linear fractional Laplacian operator $(-\Delta)^s$ for $p = 2$ and $\theta = 0$.

Let us now introduce the spaces of functions that are meaningful to our considerations. Throughout this work, we denote the norm of the weighted Lebesgue space $L^p(\mathbb{R}^N, |x|^{-\lambda})$ by

$$\|u\|_{L^p(\mathbb{R}^N;|x|^{-\lambda})} \coloneqq \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{\lambda}}\,\mathrm{d}x\right)^{\frac{1}{p}}$$

for any $0 \leqslant \lambda < N$ and $1 \leqslant p < +\infty$.

We say that a Lebesgue measurable function $u\colon \mathbb{R}^N \to \mathbb{R}$ belongs to the weighted Morrey space $L_M^{p,\gamma+\lambda}(\mathbb{R}^N, |x|^{-\lambda})$ if

$$\|u\|_{L_M^{p,\gamma+\lambda}(\mathbb{R}^n,|x|^{-\lambda})} \coloneqq \sup_{x\in\mathbb{R}^N,\, R\in\mathbb{R}_+} \left\{\left(R^{\gamma+\lambda-N}\int_{B_R(x)} \frac{|u|^p}{|x|^\lambda}\,\mathrm{d}x\right)^{\frac{1}{p}}\right\} < +\infty,$$

where $1 \leqslant p < +\infty$; γ, $\lambda \in \mathbb{R}_+$, and $0 < \gamma + \lambda < N$.

Our concerns involve the homogeneous fractional Sobolev-Slobodeckij space $\dot{W}_\theta^{s,p}(\mathbb{R}^N)$ defined as the completion of the space $C_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo seminorm given by

$$u \mapsto [u]_{\dot{W}_\theta^{s,p}(\mathbb{R}^N)} \coloneqq \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}},$$

i.e., $\dot{W}_\theta^{s,p}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{[\,\cdot\,]}$. We can equip the homogeneous fractional Sobolev space $\dot{W}_\theta^{s,p}(\mathbb{R}^N)$ with the norm

$$\begin{aligned}\|u\|_{\dot{W}_\theta^{s,p}(\mathbb{R}^N)} &\coloneqq \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{sp+\theta}}\mathrm{d}x\right)^{\frac{1}{p}}\\ &\coloneqq \left([u]^p_{\dot{W}_\theta^{s,p}(\mathbb{R}^N)} - \gamma\|u\|^p_{L^p(\mathbb{R}^N;|x|^{-sp-\theta})}\right)^{\frac{1}{p}}.\end{aligned}$$

Here, we assume that $\gamma < \gamma_H$, where the best fractional Hardy constant is defined by

$$\gamma_H \coloneqq \inf_{\substack{u\in\dot{W}_\theta^{s,p}(\mathbb{R}^N)\\ u\neq 0}} \frac{[u]^p_{\dot{W}_\theta^{s,p}(\mathbb{R}^N)}}{\|u\|^p_{L^p(\mathbb{R}^n;|x|^{-sp-\theta})}}.$$

This turns the space $\dot{W}_\theta^{s,p}(\mathbb{R}^N)$ into a Banach space; moreover, this space is uniformly convex; in particular, it is reflexive and separable.

Our main goal in this work is to show that problem (1) admits at least one nontrivial weak solution, by which term we mean a function $u \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)\backslash\{0\}$ such that

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\,\mathrm{d}x\,\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{|u|^{p-2}u\phi}{|x|^{sp+\theta}}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)-2}u\phi}{|x|^\beta}\,\mathrm{d}x\\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}u(x)\phi(x)|u(y)|^{p_s^\sharp(\delta,\theta,\mu)-2}u(y)\phi(y)}{|x|^\delta|x-y|^\mu|y|^\delta}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for any test function $\phi \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$.

Now we define the energy functional $I\colon \dot{W}_\theta^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\begin{aligned}I(u) \coloneqq{}& \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\,\mathrm{d}x\,\mathrm{d}y - \frac{\gamma}{p}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{sp+\theta}}\,\mathrm{d}x\\ &- \frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\,\mathrm{d}x\end{aligned}$$

$$- \frac{1}{2p_s^\sharp(\delta,\theta,\mu)} \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)}|u(y)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta|x-y|^\mu|y|^\delta}\,\mathrm{d}x\,\mathrm{d}y\,.$$

For the parameters in the previously specified intervals, the energy functional I is well defined and is continuously differentiable, i.e., $I \in C^1(\dot{W}^{s,p}_\theta(\mathbb{R}^N);\mathbb{R})$; moreover, a nontrivial critical point of the energy functional I is a nontrivial weak solution to problem (1).

Theorem 0.1. *Problem* (1) *has at least one nontrivial weak solution provided that* $0 < s < 1$*;* $0 < \alpha,\ \beta < sp + \theta < N$*;* $0 < \mu < N$*; and* $\gamma < \gamma_H$.

In this work we also consider the following variants of problem (1), namely one problem with a Hardy potential and double Sobolev type nonlinearities,

$$\begin{cases} (-\Delta)^s_{p,\theta} u - \gamma \dfrac{|u|^{p-2}u}{|x|^{sp+\theta}} = \displaystyle\sum_{k=1}^{2} \frac{|u|^{p_s^*(\beta_k,\theta)-2}u}{|x|^{\beta_k}} \\ u \in \dot{W}^{s,p}_\theta(\mathbb{R}^N); \end{cases} \tag{4}$$

and another one with a Hardy potential and double Choquard type nonlinearities,

$$\begin{cases} (-\Delta)^s_{p,\theta} u - \gamma \dfrac{|u|^{p-2}u}{|x|^{sp+\theta}} = \displaystyle\sum_{k=1}^{2} \left[I_{\mu_k} * F_{\delta_k,\theta,\mu_k}(\cdot,u)\right](x) f_{\delta_k,\theta,\mu_k}(x,u) \\ u \in \dot{W}^{s,p}_\theta(\mathbb{R}^N). \end{cases} \tag{5}$$

The notion of weak solution to problems (4) and (5) can be defined in the same way as that for problem (1), i.e., we multiply the differential equations by test functions and use a kind of integration by parts. Then we recognize these expressions as the derivatives of an energy functionals which, under the appropriate hypotheses on the parameters, are continuously differentiables. This means that weak solutions to these problems are critical points of the appropriate energy functionals. By adaptting the method used in the proof of Theorem 0.1 we deduce the following result.

Theorem 0.2. *Problems* (4) *and* (5) *have at least one nontrivial weak solution under similar assumptions as in Theorem 0.1, i.e.,* $0 < s < 1$*;* $\gamma < \gamma_H$*;* $0 < \alpha_k,\ \beta_k < sp+\theta < N$*; and* $0 < \mu_k < N$ *for* $k \in \{1, 2\}$.

Remark 1. At this point let us mention that, in the past, other authors have attempted to prove existence results for this class of fractional elliptic problems. To be precise, Li & Yang [56] claimed to have established the existence of solution to problem (1) in the case $p = 2$ for the fractional Laplacian, but without singularities in the operator, i.e., $\theta_1 = \theta_2 = 0$, and in the unweighted Choquard term, i.e., $\delta = 0$, or in the unweighted Sobolev term, i.e., $\beta = 0$. Their proofs rely on a related minimization problem. However, we could not check the arguments on which the proof is based; see Yang & Wu [93, inequality (2.8)]; Yang [92, inequality (3.2)]. In this way, we believe that their results in these cases are still open problems; see De Nápoli, Drelichman & Salort [37].

0.2 The Sobolev-Choquard systems with Hardy term

In the present section, we consider the following fractional p-Laplacian system of equations in the entire space $\mathbb{R}^N$ featuring doubly critical nonlinearities, involving a local critical

Sobolev term together with a non-local Choquard critical term; the problem also includes a homogeneous Hardy term; additionally, all terms have critical singular weights. More precisely, we deal with the problem

$$\begin{cases}(-\Delta)^s_{p,\theta}u-\gamma_1\dfrac{|u|^{p-2}u}{|x|^{sp+\theta}}=\left[I_\mu*F(\cdot,u)\right](x)f(x,u)+\dfrac{|u|^{p_s^*(\beta,\theta)-2}u}{|x|^\beta}+\dfrac{\eta a}{a+b}\dfrac{|u|^{a-2}u|v|^b}{|x|^\beta}\\(-\Delta)^s_{p,\theta}v-\gamma_2\dfrac{|v|^{p-2}v}{|x|^{sp+\theta}}=\left[I_\mu*F(\cdot,v)\right](x)f(x,v)+\dfrac{|v|^{p_s^*(\beta,\theta)-2}v}{|x|^\beta}+\dfrac{\eta b}{a+b}\dfrac{|u|^{a}|v|^{b-2}v}{|x|^\beta}\end{cases} \tag{6}$$

where $0<s<1$; $0<\alpha,\ \beta<sp+\theta<N$; $0<\mu<N$; $2\delta+\mu<N$; $\eta\in\mathbb{R}^+$; $\gamma_1,\gamma_2<\gamma_H$ with the best fractional Hardy constant γ_H to be defined below (without lost of generality, to simplify the notation we can consider the only parameter $\gamma=\gamma_1=\gamma_2$).

For simplicity, hereafter we denote the Cartesian product space of two Banach spaces $W=\dot{W}^{s,p}_\theta(\mathbb{R}^N)\times\dot{W}^{s,p}_\theta(\mathbb{R}^N)$, endowed with the norm

$$\|(u,v)\|_W\coloneqq\left(\|u\|^p_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)}+\|v\|^p_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)}\right)^{1/p}.$$

Intuitively, solving problem (6) is understood as showing the existence of a pair $(u,v)\in W\backslash\{0,0\}$ such that

$$\begin{cases}(-\Delta)^s_{p,\theta}u-\gamma\dfrac{|u|^{p-2}u}{|x|^{sp+\theta}}=\left(\displaystyle\int_{\mathbb{R}^N}\dfrac{|u|^{p_s^\sharp}}{|x-y|^\mu|y|^\delta}\mathrm{d}y\right)\dfrac{|u|^{p_s^\sharp-2}u}{|x|^\delta}+\dfrac{|u|^{p_s^*-2}u}{|x|^\beta}+\dfrac{\eta a}{a+b}\dfrac{|u|^{a-2}u|v|^b}{|x|^\beta}\\(-\Delta)^s_{p,\theta}v-\gamma\dfrac{|v|^{p-2}v}{|x|^{sp+\theta}}=\left(\displaystyle\int_{\mathbb{R}^N}\dfrac{|v|^{p_s^\sharp}}{|x-y|^\mu|y|^\delta}\mathrm{d}y\right)\dfrac{|v|^{p_s^\sharp-2}v}{|x|^\delta}+\dfrac{|v|^{p_s^*-2}v}{|x|^\beta}+\dfrac{\eta b}{a+b}\dfrac{|u|^{a}|v|^{b-2}v}{|x|^\beta}\end{cases}$$

where the fractional p-Laplacian operator is defined for $\theta=\theta_1+\theta_2$ and $x\in\mathbb{R}^N$.

Our main goal in this work is to show that problem (6) admits at least one weak solution, by which term we mean a function $(u,v)\in W$ such that

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi_1(x)-\phi_1(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\\&+\iint_{\mathbb{R}^{2N}}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi_2(x)-\phi_2(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\\&-\gamma_1\int_{\mathbb{R}^N}\frac{|u|^{p-2}u\phi_1}{|x|^{sp+\theta}}\mathrm{d}x-\gamma_2\int_{\mathbb{R}^N}\frac{|v|^{p-2}v\phi_2}{|x|^{sp+\theta}}\mathrm{d}x\\&\quad=\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{p_s^\sharp-2}|u(y)|^{p_s^\sharp}u(x)\phi_1(x)}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y\\&\qquad+\iint_{\mathbb{R}^{2N}}\frac{|v(x)|^{p_s^\sharp-2}|v(y)|^{p_s^\sharp}v(x)\phi_2(x)}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y\\&\qquad+\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)-2}u\phi_1(x)}{|x|^\beta}\mathrm{d}x+\int_{\mathbb{R}^N}\frac{|v|^{p_s^*(\beta,\theta)-2}v\phi_2(x)}{|x|^\beta}\mathrm{d}x\\&\qquad+\int_{\mathbb{R}^N}\frac{\eta a|u|^{a-2}u\phi_1|v|^b}{|x|^\beta}\mathrm{d}x+\int_{\mathbb{R}^N}\frac{\eta b|u|^a|v|^{b-2}v\phi_2}{|x|^\beta}\mathrm{d}x\end{aligned}$$

for any pair of test functions $(\phi_1,\phi_2)\in W$. Now we define the energy functional $I\colon W\to\mathbb{R}$ by

$$I(u,v)=\frac{1}{p}\Big[\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y+\iint_{\mathbb{R}^{2N}}\frac{|v(x)-v(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\Big]$$

$$
\begin{aligned}
&-\frac{\gamma_1}{p}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{sp+\theta}}\mathrm{d}x-\frac{\gamma_2}{p}\int_{\mathbb{R}^N}\frac{|v|^p}{|x|^{sp+\theta}}\mathrm{d}x\\
&-\frac{1}{2p_s^\sharp(\delta,\theta,\mu)}\Big[\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{p_s^\sharp}|u(y)|^{p_s^\sharp}}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y+\iint_{\mathbb{R}^{2N}}\frac{|v(x)|^{p_s^\sharp}|v(y)|^{p_s^\sharp}}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y\Big]\\
&-\frac{1}{p_s^*}\Big[\int_{\mathbb{R}^N}\frac{|u|^{p_s^*}}{|x|^\beta}\mathrm{d}x+\int_{\mathbb{R}^N}\frac{|v|^{p_s^*}}{|x|^\beta}\mathrm{d}x\Big]-\frac{1}{p_s^*}\int_{\mathbb{R}^N}\frac{\eta|u|^a|v|^b}{|x|^\beta}\mathrm{d}x. \qquad (7)
\end{aligned}
$$

For the parameters in the previously specified intervals, the energy functional I is well defined and is continuously differentiable, i.e., $I\in C^1(\dot{W}^{s,p}_\theta(\mathbb{R}^N);\mathbb{R})$; moreover, a nontrivial critical point of the energy functional I is a nontrivial weak solution to problem (6).

Theorem 0.3. *Problem* (6) *has at least one nontrivial weak solution provided that* $0<s<1$*;* $0<\alpha,\ \beta<sp+\theta<N$*;* $0<\mu<N$*;* $a+b=p_s^*(\beta,\theta)$*;* $\eta\in\mathbb{R}^+$ *and* $\gamma_1,\gamma_2<\gamma_H$*.*

In this work we also consider the following variants of problem (6), namely one problem with a Hardy potential and double Sobolev type nonlinearities,

$$
\begin{cases}
(-\Delta)^s_{p,\theta}u-\gamma_1\dfrac{|u|^{p-2}u}{|x|^{sp+\theta}}=\displaystyle\sum_{k=1}^{2}\frac{|u|^{p_s^*-2}u}{|x|^{\beta}}+\frac{\eta a}{a+b}\frac{|u|^{a-2}u|v|^b}{|x|^{\beta_k}}\\
(-\Delta)^s_{p,\theta}v-\gamma_2\dfrac{|v|^{p-2}v}{|x|^{sp+\theta}}=\displaystyle\sum_{k=1}^{2}\frac{|v|^{p_s^*-2}v}{|x|^{\beta_k}}+\frac{\eta b}{a+b}\frac{|u|^{a}|v|^{b-2}v}{|x|^{\beta_k}}
\end{cases}
\qquad (8)
$$

and another one with a Hardy potential and double Choquard type nonlinearities,

$$
\begin{cases}
(-\Delta)^s_{p,\theta}u-\gamma_1\dfrac{|u|^{p-2}u}{|x|^{sp+\theta}}=\displaystyle\sum_{k=1}^{2}\left[I_{\mu_k}*F_{\delta,\theta,\mu_k}(\cdot,u)\right](x)f_{\delta,\theta,\mu_k}(x,u)+\frac{\eta a}{a+b}\frac{|u|^{a-2}u|v|^b}{|x|^{\beta_k}}\\
(-\Delta)^s_{p,\theta}v-\gamma_2\dfrac{|v|^{p-2}v}{|x|^{sp+\theta}}=\displaystyle\sum_{k=1}^{2}\left[I_{\mu_k}*F_{\delta,\theta,\mu_k}(\cdot,v)\right](x)f_{\delta,\theta,\mu_k}(x,v)+\frac{\eta b}{a+b}\frac{|u|^{a}|v|^{b-2}v}{|x|^{\beta_k}}
\end{cases}
\qquad (9)
$$

The notion of weak solution to problems (8) and (9) can be defined in the same way as that for problem (6), i.e., we multiply the differential equations by a pair of test functions and use a kind of integration by parts. Then we recognize these expressions as the derivatives of an energy functional which, under the appropriate hypotheses on the parameters, is continuously differentiable. This means that weak solutions to these problems are critical points of the appropriate energy functional. By adaptting the method used in the proof of Theorem 0.3 we deduce the following result.

Theorem 0.4. *Problems* (8) *and* (9) *have at least one nontrivial weak solution under similar assumptions as in Theorem 0.3, i.e.,* $0<s<1$*;* $0<\alpha_k,\ \beta_k<sp+\theta<N$*;* $0<\mu_k<N$*;* $a+b=p_s^*(\beta_k,\theta)$*;* $\eta\in\mathbb{R}^+$ *and* $\gamma_1,\gamma_2<\gamma_H$ *for* $k\in\{1,2\}$*.*

0.3 The Sobolev-Kirchhoff problems with Hardy term

In the present section, we consider the following fractional p-Kirchhoff equation in the entire space $\mathbb{R}^N$ featuring doubly nonlinearities, involving a generalized non-local Choquard subcritical term together with a local critical Sobolev term; the problem also includes a

Hardy-type term; additionaly, all terms have critical singular weights. More precisely, we deal with the problem

$$m(\|u\|^p_{W^{s,p}_\theta(\mathbb{R}^N)})\Big[(-\Delta)^s_{p,\theta}u+V(x)\frac{|u|^{p-2}u}{|x|^\alpha}\Big]=\frac{|u|^{p^*_s(\beta,\theta)-2}u}{|x|^\beta}+\lambda\left[I_\mu*\frac{F_{\delta,\theta,\mu}(\cdot,u)}{|x|^\delta}\right](x)\frac{f_{\delta,\theta,\mu}(x,u)}{|x|^\delta} \tag{10}$$

where $0<s<1$; $0<\alpha<N-\mu$; $0<\beta<sp+\theta<N$; $0<\mu<N$; $2\delta+\mu<N$; $p^*_s(\beta,\theta)=p(N-\beta)/(N-sp-\theta)$. The function $m\colon\mathbb{R}^+_0\to\mathbb{R}^+$ is a Kirchhoff function; the potential function $V\colon\mathbb{R}^N\to\mathbb{R}^+$ is continuous; the nonlinearity $f\colon\mathbb{R}\to\mathbb{R}$ is continuous and define $F(s)=\int_0^s f(t)\mathrm{d}t$; the funtion $I_\mu\colon\mathbb{R}^N\to\mathbb{R}$ is defined by $I_\mu(x)=|x|^{-\mu}$ and is called the Riesz potential. The fractional p-Laplacian operator is defined for $\theta=\theta_1+\theta_2$, $x\in\mathbb{R}^N$, and any function $u\in C_0^\infty(\mathbb{R}^N)$.

Let us now introduce the spaces of functions that are meaningful to our considerations. Throughout this work, we denote the norm of the weighted Lebesgue space $L^p_V(\mathbb{R}^N,|x|^{-\eta})$ by

$$\|u\|_{L^p_V(\mathbb{R}^N;|x|^{-\eta})}\coloneqq\Big(\int_{\mathbb{R}^N}\frac{V(x)|u|^p}{|x|^\eta}\,\mathrm{d}x\Big)^{\frac{1}{p}}$$

for any $0\leqslant\eta<N$ and $1\leqslant p<+\infty$.

We can equip the homogeneous fractional Sobolev space $W^{s,p}_{V,\theta}(\mathbb{R}^N)$ with the norm

$$\begin{aligned}\|u\|_{W^{s,p}_{V,\theta}(\mathbb{R}^N)}=\|u\|_W&\coloneqq\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^N}\frac{V(x)|u|^p}{|x|^\alpha}\mathrm{d}x\Big)^{\frac{1}{p}}\\&\coloneqq\Big([u]^p_{W^{s,p}_\theta(\mathbb{R}^N)}+\|u\|^p_{L^p_V(\mathbb{R}^N;|x|^{-\alpha})}\Big)^{\frac{1}{p}}.\end{aligned}$$

The embedding $W^{s,p}_{V,\theta}(\mathbb{R}^N)\hookrightarrow L^\nu_V(\mathbb{R}^N,|x|^{-\alpha})$ is continuous for any $\nu\in[p,\frac{p(N-\beta)}{N-ps-\theta}]$ and $0<\alpha<N-\mu$, namely there exists a positive constant C_ν such that

$$\|u\|_{L^\nu_V(\mathbb{R}^N,|x|^{-\alpha})}\leqslant C_\nu\|u\|_W\quad\text{for all } u\in W^{s,p}_{V,\theta}(\mathbb{R}^N). \tag{11}$$

The potential function $V\colon\mathbb{R}^N\to\mathbb{R}^+$ verifies the following assumption

(V) V is continuous and there exists $V_0>0$ such that $\inf_{\mathbb{R}^N}V\geqslant V_0$.

Moreover, we assume that the nonlinearities $f,F\colon\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ verify the hypotheses

(F_1) $F\in C^1(\mathbb{R},\mathbb{R})$.

(F_2) There exist constants

$$p^\flat_s(\delta,\mu)\coloneqq\frac{(N-\delta-\mu/2)p}{N}<q_1\leqslant q_2<\frac{(N-\delta-\mu/2)p}{N-sp-\theta}\eqqcolon p^\sharp_s(\delta,\theta,\mu)$$

and $c_0>0$ such that for all $t\in\mathbb{R}$,

$$|f(t)|\leqslant c_0(|t|^{q_1-1}+|t|^{q_2-1}).$$

(F_3) $\displaystyle\lim_{|u(x)|\to\infty}\frac{F(u(x))}{|x|^\delta|u(x)|^{p\xi}}=\infty$ uniformly with respect to $x\in\mathbb{R}^N$ where $\xi\in[1,2p^\flat_s(\delta,\mu)/p)$.

(F_4) There exist constants $r_0 \geqslant 0$, $\kappa > \dfrac{N-\beta}{ps+\theta-\beta}$ and $c_1 \geqslant 0$ such that for $|t| \geqslant r_0$,

$$\frac{|F(t)|^\kappa}{|x|^\delta} \leqslant c_1 |t|^{\kappa p} \mathcal{F}(t), \qquad \mathcal{F}(t) := \frac{1}{p\xi}\frac{f(t)}{|x|^\delta} t - \frac{1}{2}\frac{F(t)}{|x|^\delta} \geqslant 0.$$

With respect to the Kirchhoff function $m\colon \mathbb{R}_0^+ \to \mathbb{R}^+$ we make the following assumptions.

(m_1) m is a continuous function and there exists $m_0 > 0$ such that $\inf_{t\geqslant 0} m(t) = m_0$.

(m_2) There exists $\xi \in [1, 2p_s^\flat(\delta,\mu)/p)$ such that $m(t)t \leqslant \xi M(t)$ for all $t \geqslant 0$, where $M(t) = \int_0^t m(\tau)\mathrm{d}\tau$.

A typical example is $m(t) = a + b\xi t^{\xi-1}$ for $t \geqslant 0$, where $a \geqslant 0$, $b \geqslant 0$, $a+b > 0$, $\xi \in (1, 2p_s^\flat(\delta,\mu)/p)$ if $b > 0$ and $\xi = 1$ if $b = 0$; this is called non-degenerate when $a > 0$ and $b \geqslant 0$ and is called degenerate if $a = 0$ and $b > 0$.

Our main goal in this work is to show that problem (10) admits at least one nontrivial weak solution, by which term we mean a function $u \in W_{V,\theta}^{s,p}(\mathbb{R}^N)$ such that

$$m(\|u\|_W^p)\left[\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} \frac{V(x)|u|^{p-2}u\phi}{|x|^\alpha}\,\mathrm{d}x\right]$$
$$= \int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\beta,\theta)-2}u\phi}{|x|^\beta}\,\mathrm{d}x + \lambda\int_{\mathbb{R}^N}\left(I_\mu * \frac{F_{\delta,\theta,\mu}(u)}{|x|^\delta}\right)\frac{f_{\delta,\theta,\mu}(u)}{|x|^\delta}\phi\mathrm{d}x$$

for any test function $\phi \in W_{V,\theta}^{s,p}(\mathbb{R}^N)$.

Now we define the energy functional $I\colon W_{V,\theta}^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\begin{aligned} I(u) &:= \frac{1}{p}M(\|u\|_W^p) - \frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\,\mathrm{d}x \\ &\quad - \frac{\lambda}{2}\iint_{\mathbb{R}^{2N}} \frac{F_{\delta,\theta,\mu}(u(x))F_{\delta,\theta,\mu}(u(y))}{|x|^\delta|x-y|^\mu|y|^\delta}\,\mathrm{d}x\,\mathrm{d}y \\ &=: \Phi(u) - \Xi(u) - \lambda\Psi(u). \end{aligned} \tag{12}$$

For the parameters in the previously specified intervals, the energy functional I is well defined and is continuously differentiable, i.e., $I \in C^1(W_{V,\theta}^{s,p}(\mathbb{R}^N);\mathbb{R})$; moreover, a nontrivial critical point of the energy functional I is a nontrivial weak solution to problem (10).

The main result is stated as follows.

Theorem 0.5. *Let* $0 < \mu < ps+\theta < N$*; suppose* (V), (m_1)–(m_2) *and* (F_1)–(F_4) *hold. Then problem* (10) *has a nontrivial weak solution for any* $\lambda > 0$.

Overview

The present thesis is organized as follows. In Chapter 1, we prove Theorems 0.1 and 0.2; before that, we present a brief historical background of the problems, mainly over the fractional Laplacian, the Riesz potential, the Choquard equation, the fractional Sobolev spaces, the Stein-Weiss inequality (also known as Hardy-Littlewood-Sobolev inequality), and the Morrey spaces. We also discuss the variational setting for the problems together with some preliminary results such as the Caffarelli-Kohn-Nirenberg, and we mention some related works on fractional elliptic operators. Next, we describe some of the difficulties to prove our first two theorems: we emphasize the nonlocality of the operator, the estimates of the mixed terms, the structure of the Palais-Smale sequences and mainly the auxiliary minimization problems and the asymptotic competition between the two critical nonlinearities; next, comes the outline of the proof, where we define the Brézis-Nirenberg critical level below which we can guarantee that the Palais-Smale sequences have strongly convergent subsequences in the Sobolev spaces. Finally, we present the proofs of the Theorems 0.1 and 0.2.

Chapter 2 is devoted to the proof of Theorems 0.3 and 0.4. We begin this chapter with a brief historical background including, among other things, the fractional Laplacian, the Choquard equation, and some words about systems of fractional elliptic equations. Next, we establish the existence of solutions for some auxiliary problems, namely, that the best Choquard and Sobolev constants are attained, similarly to what have been done in the previous chapter. Then, we study the existence of Palais-Smale sequences and present some analysis with the mountain pass level on the Brézis-Nirenberg critical level below which we can guarantee that the Palais-Smale sequences have strongly convergent subsequences in the Sobolev spaces. Finally, we conclude the proof of Theorems 0.3 and 0.4.

In Chapter 3, we prove Theorem 0.5. We begin the chapter again with a brief historical background including Kirchhoff type problems; the kinds and varieties of potential functions; some comments about Palais-Smale and Cerami conditions; the types of frequently used nonlinearities; the Ambrosetti-Rabinowitz condition; some words about subcritical growth and Cerami conditions; the degraded oscillations and the resonant nonlinearities. Next, we introduce the variational setting of the problem, we present the doubly weighted Stein-Weiss inequality, and we ensure the well-definiteness of the energy functional; the geometry of the mountain pass theorem and the compactness of the Cerami sequences comes next and we conclude the chapter with the proof of Theorem 0.5.

In the final chapter, we briefly summarize this thesis and ennuntiate some open problems.

Notation. For $\rho \in \mathbb{R}_+$, we define $B_\rho(x) := \{y \in \mathbb{R}^N \colon |x-y| < \rho\}$, the open ball centered at x with radius ρ. The constant ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. The arrows $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence, respectively. Given the functions $f, g \colon \mathbb{R}^N \to \mathbb{R}$, we recall that $f = O(g)$ if there is a constant $C \in \mathbb{R}_+$ such that $|f(x)| \leqslant C|g(x)|$ for all $x \in \mathbb{R}^N$; and $f = o(g)$ as $x \to x_0$ if $\lim_{x\to x_0} |f(x)|/|g(x)| = 0$. The pair r and r' denote Hölder conjugate exponents, i.e., $1/r + 1/r' = 1$ or $r + r' = rr'$. The positive and negative parts of a function ϕ are denoted by $\phi_\pm := \max\{\pm\phi, 0\}$. Moreover: $tz := t(u,v) = (tu, tv)$ for all $(u,v) \in W$ and $t \in \mathbb{R}$; (u,v) is said to be nonnegative in $\mathbb{R}^N$ if $u \geqslant 0$ and $v \geqslant 0$ in $\mathbb{R}^N$; (u,v) is said to be positive in $\mathbb{R}^N$ if $u > 0$ and $v > 0$ in $\mathbb{R}^N$. Finally, $C \in \mathbb{R}_+$ denotes a universal constant that may change from line to line; when it is relevant, we will add subscripts to specify the dependence of certain parameters.

Chapter 1

Fractional Sobolev-Choquard critical equation with Hardy term and weighted singularities

1.1 Historical background

Reasons for the recent interest in this class of nonlinear elliptic problems reside in the merits of the subject itself and also in the number and variety of phenomena occurring in real-world applications that can be modeled by these equations. For example, fractional and non-local differential operators arise in a quite natural way in many different problems that involve long-range interactions, such as anomalous diffusion, dislocations in crystals, water waves, phase transitions, stratified materials, semipermeable membranes, flame propagation, non-Newtonian fluid theory in a porous medium, financial mathematics, phase transition phenomena, population dynamics, minimum surfaces, game theory, image processing, etc. In particular, there are some remarkable mathematical models involving the fractional p-Laplacian, such as the fractional Schrödinger equation, the fractional Kirchhoff equation, the fractional porous medium equation, etc. For more information, see the excelent survey papers by Di Nezza, Palatucci & Valdinoci [39], Moroz & Van Schaftingen [68] and Mukherjee & Sreenadh [85] and the references they contain.

We can also mention the diversity of tools used in their study, mainly critical point theory and variational and topological methods.

The fractional Laplacian

There are many equivalent definitions of the fractional Laplacian. In our case, on the Euclidean space $\mathbb{R}^N$ of dimension $N \geqslant 1$, for $\theta = \theta_1 + \theta_2$ and the above specified intervals for the parameters, we define the non-local elliptic p-Laplacian operator with the help of the Cauchy's principal value integral as

$$
\begin{aligned}
(-\Delta)^s_{p,\theta} u(x) &:= \text{p.v.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\,\mathrm{d}y \\
&:= 2\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\,\mathrm{d}y
\end{aligned}
$$

for $x \in \mathbb{R}^N$ and any function $u \in C_0^\infty(\mathbb{R}^N)$. The usual definition of the fractional p-Laplacian carries a normalizing constant dependent on N, s, p, and θ in front of the integral. This constant is irrelevant for our purposes; so, for the sake of clarity, we omit it in the definition and in the formulation of the results. The limit operator (up to a suitable normalizing constant) as $s \to 1^-$ and $\theta_1 = \theta_2 = 0$ is the so called p-Laplacian defined as $\Delta_p u(x) = \nabla \cdot (|\nabla u(x)|^{p-2}\nabla u(x))$. Also, if $p = 2$ and $\theta_1 = \theta_2 = 0$, then the usual notation is $(-\Delta)^s u$ and the definition is consistent, up to a normalizing constant, to the fractional Laplacian defined by the Fourier multiplier $\mathcal{F}[(-\Delta)^s u(x)](\xi) = 2\pi|\xi|^{2s}\mathcal{F}[u(x)](\xi)$ for $x, \xi \in \mathbb{R}^N$. In this formula, $\mathcal{F}[u(x)](\xi) = \int_{\mathbb{R}^N} \exp(-2\pi i x \cdot \xi)u(x)\,\mathrm{d}x$ denotes the Fourier transform of $u \in \mathcal{S}(\mathbb{R}^N)$, Schwartz's space of rapid decaying functions defined by $\mathcal{S}(\mathbb{R}^N) \coloneqq \{g \in C^\infty\colon \sup_{x\in\mathbb{R}^N} |x^\eta \partial_\kappa g(x)| < +\infty\}$ where the supremum is taken over the multi-indices $\kappa, \eta \in \mathbb{N}_0^N$.

Consider the integral functional defined by

$$u \mapsto E(u) \coloneqq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\,\mathrm{d}x\,\mathrm{d}y\,.$$

The Gâteaux derivative of the functional E at u in the direction φ, also called the first variation of the functional, is computed as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}[E(u+\tau\varphi)]\Big|_{\tau=0} &= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\frac{|u(x)-u(y)+\tau(\varphi(x)-\varphi(y))|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\Big]\,\mathrm{d}x\,\mathrm{d}y\,\Big|_{\tau=0}\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

This implies that $(-\Delta)^s_{p,\theta}u$ is the gradient vector field $\nabla E(u)$, i.e., $\nabla E(u) = (-\Delta)^s_{p,\theta}u$; hence, $(-\Delta)^s_{p,\theta}u(x)$ is interpreted as a nonlinear generalization of the usual Laplacian operator. For a variety of interesting problems, their results and the progress of research on the fractional operator, see e.g., Di Nezza, Palatucci & Valdinoci [39]; Molica Bisci, Rădulescu & Servadei [66], Kwaśnicki [54]; Kuusi & Palatucci [53]; del Teso, Castro-Gómez & Vázquez [38], and Lischke et al. [62].

The Riesz potential

On the Euclidean space $\mathbb{R}^N$ of dimension $N \geqslant 1$, for $0 < \mu < N$ and for each point $x \in \mathbb{R}^N\backslash\{0\}$, we set $I_\mu(x) = |x|^{-\mu}$. The Riesz potential of order μ of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is defined as

$$[I_\mu * f](x) \coloneqq \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^\mu}dy,$$

where the convolution integral is understood in the sense of the Lebesgue integral. The usual definition of this potential carries a normalizing constant dependent on N and μ in front of the integral. This constant is chosen to ensure the semigroup property, $I_\mu * I_\nu = I_{\mu+\nu}$ for $\mu, \nu \in \mathbb{R}_+$ such that $\mu + \nu < N$ but it is not considered in this work to simplify the formulation of the results. The Riesz potential $I_\mu\colon L^q(\mathbb{R}^N \to L^r(\mathbb{R}^N)$ is well-defined whenever $1 < q < N/(N-\mu)$ and $1/q - 1/r = (N-\mu)/N$

The Choquard equation

On the Euclidean space $\mathbb{R}^N$ of dimension $N \geqslant 1$ and for $x \in \mathbb{R}^N$, the equation

$$-\Delta u + V(x)u = (I_\mu * |u|^q)|u|^{q-2}u$$

was introduced by Choquard in the case $N = 3$ and $q = 2$ to model one-component plasma. It had appeared earlier in the model of the polaron by Frölich and Pekar, where free electrons interact with the polarisation that they create on the medium. When $V(x) \equiv 1$, the groundstate solutions exist if $2^\flat := 2(N-\mu/2))/N < q < 2(N-\mu/2)/(N-2s) := 2^\sharp$ due to the mountain pass lemma or the method of the Nehari manifold, while there are no nontrivial solution if $q = 2^\flat$ or if $q = 2^\sharp$ as a consequence of the Pohozaev identity. The Choquard equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with Newtonian gravity. This equation is related to several other partial differential equations with non-local interactions. In general, the associated Schrödinger-type evolution equation $i\partial_t\psi = \Delta\psi + \left(I_\mu * |\psi|^2\right)\psi$ is a model for large systems of atoms with an attractive interaction that is weaker and has a longer range than that of the nonlinear Schrödinger equation. Standing wave solutions of this equation are solutions to the Choquard equation. For more information on the various results related to the non-fractional Choquard-type equations and their variants see Moroz & Van Schaftingen [68].

The fractional Sobolev spaces

In the last years, for pure mathematical research and concrete real-world applications, the fractional p-Laplacian operator has been studied on the fractional Sobolev space $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$. It is the natural fractional counterpart of the homogeneous Sobolev space $\mathcal{D}^{1,p}_0(\Omega)$, defined as the completion of the space $C^\infty_0(\mathbb{R}^N)$ with respect to the norm $u \mapsto (\int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x)^{1/p}$. Additionally, in the same way that $\mathcal{D}^{1,p}_0(\mathbb{R}^N)$ is the natural setting for studying variational problems of the type $\inf\{(1/p)\int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x\}$, supplemented with Dirichlet boundary conditions (in the absence of regularity assumptions on the boundary $\partial\Omega$), the space $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$ is the natural framework for studying minimization problems containing functionals of the type

$$u \mapsto \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega f u\,\mathrm{d}x\,,$$

in the presence of non-local Dirichlet boundary conditions, i.e., the values of u prescribed on the whole complement $\mathbb{R}^N \backslash \Omega$, which takes into account long range interactions.

The dual space of $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$ is denoted by $\dot{W}^{s,p}_\theta(\mathbb{R}^N)'$ or by $\dot{W}^{s,-p}_\theta(\mathbb{R}^N)$.

The Stein-Weiss inequality

Here we recall a generalization of the Hardy-Littlewood-Sobolev, also called the doubly weighted inequality or the Stein-Weiss inequality. See, e.g., Stein & Weiss [86]; Lieb & Loss [59, Theorem 4.3]; Yuan, Rădulescu, Chen and Wen [95, Proposition 1], and Han, Lu & Zhu [48].

Proposition 1.1 (Doubly weighted Hardy-Littlewood-Sobolev inequality)**.** *Let* $1 < r,\ t < +\infty$ *and* $0 < \mu < N$ *with* $1/t + \mu/N + 1/r = 2$*; let* $f \in L^t(\mathbb{R}^N)$ *and* $h \in L^r(\mathbb{R}^N)$*. Then there exists a sharp constant* $C(N,\mu,r,t)$*, independent on* f *and* h*, such that*

$$\left|\iint_{\mathbb{R}^{2N}} \frac{\overline{f(x)}h(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right| \leqslant C(N,\mu,r,t)\|f\|_{L^t(\mathbb{R}^N)}\|h\|_{L^r(\mathbb{R}^N)}. \tag{1.1}$$

This inequality was introduced by Hardy and Littlewood in $\mathbb{R}^1$ and generalized by Sobolev to $\mathbb{R}^N$. However, none of them is in its sharp form; namely, neither the sharp constant $C(N,\mu,r,t)$ nor the extremal function such that the inequality (1.1) holds with the sharp constant was known. For a special case when $r=t=2N/(2N-\mu)$, Lieb [58] gave the sharp version of inequality (1.1), i.e., the inequality with the best constant

$$C(t,N,\mu,r)=C(N,\mu)=\pi^{\frac{\mu}{2}}\frac{\Gamma(\frac{n}{2}-\frac{\mu}{2})}{\Gamma(n-\frac{\mu}{2})}\left\{\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right\}^{-1+\frac{\mu}{n}}$$

and showed that its extremals, functions for which the inequality (1.1) is valid with the smallest constant $C(N,\mu,r,t)$, are such that f is a constant multiple of the function h, which must be of the form

$$h(x)=\frac{A}{(\epsilon^2+|x-a|^2)^{N-\mu/2}}$$

for some parameters $A\in\mathbb{C}$, $\epsilon\in\mathbb{R}\setminus\{0\}$, and $a\in\mathbb{R}^N$. For the general case when $r\neq t$, neither the sharp constant $C(N,\mu,r,t)$ nor the extremals are known yet.

Proposition 1.2 (Doubly weighted Stein-Weiss inequality)**.** *Let* $1<r,\ t<+\infty$, $0<\mu<N$, *and* $\eta+\kappa\geqslant 0$ *such that* $\mu+\eta+\kappa\leqslant N$, $\eta<N/r'$, $\kappa<N/t'$ *and* $1/t+(\mu+\eta+\kappa)/N+1/r=2$*; let* $f\in L^t(\mathbb{R}^N)$ *and* $h\in L^r(\mathbb{R}^N)$*. Then there exists a constant* $C(N,\mu,r,t,\eta,\kappa)$*, independent on* f *and* h*, such that*

$$\left|\iint_{\mathbb{R}^{2N}}\frac{\overline{f(x)}h(y)}{|x|^{\eta}|x-y|^{\mu}|y|^{\kappa}}\mathrm{d}x\mathrm{d}y\right|\leqslant C(N,\mu,r,t,\eta,\kappa)\|f\|_{L^t(\mathbb{R}^N)}\|h\|_{L^r(\mathbb{R}^N)}. \tag{1.2}$$

The sharp constant in the Stein-Weiss inequality (1.2) is still unknown as far as we are aware of, even in the special case when $r=t$.

Corollary 1.3. *Let* $0<s<1$*;* $0\leqslant\alpha<sp+\theta<N$*;* $0<\mu<N$*; given a function* $u\in\dot W^{s,p}_{\theta}(\mathbb{R}^N)$ *consider Proposition 1.2 with* $\eta=\kappa=\delta$*;* $2\delta+\mu\leqslant N$*;* $t=r=N/(N-\delta-\mu/2)$*; and* $f(x)=h(x)=|u(x)|^{p^{\sharp}_s(\delta,\theta,\mu)}$*. Then* $f,h\in L^{\frac{N}{N-\delta-\mu/2}}(\mathbb{R}^N)$ *and*

$$\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{p^{\sharp}_s(\delta,\theta,\mu)}|u(y)|^{p^{\sharp}_s(\delta,\theta,\mu)}}{|x|^{\delta}|x-y|^{\mu}|y|^{\delta}}\mathrm{d}x\mathrm{d}y\leqslant C(N,\delta,\theta,\mu)\Big(\int_{\mathbb{R}^N}|u|^{p^*_s(0,\theta)}\mathrm{d}x\Big)^{\frac{2(N-\delta-\mu/2)}{N}} \tag{1.3}$$

In general, for $\eta=\kappa=\delta$ and $t=r$, the map

$$u\mapsto\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^q|u(y)|^q}{|x|^{\delta}|x-y|^{\mu}|y|^{\delta}}\,\mathrm{d}x\,\mathrm{d}y$$

is well-defined if

$$p^{\flat}_s(\delta,\mu):=\frac{p(N-\delta-\mu/2)}{N}<q<\frac{p^*_s(0,\theta)(N-\delta-\mu/2)}{N}=:p^{\sharp}_s(\delta,\theta,\mu).$$

Consider the integral functional defined by

$$u\mapsto J(u):=\frac{1}{qr}\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u(x)|^q|u(y)|^r}{|x|^{\eta}|x-y|^{\mu}|y|^{\kappa}}\,\mathrm{d}y\Big)\,\mathrm{d}x\,.$$

The Gâteaux derivative of the functional J at u in the direction φ, also called the first variation of the functional, is computed as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[J(u+\tau\varphi)]\Big|_{\tau=0}=\frac{1}{qr}\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\frac{|u(x)+\tau\varphi(x)|^q|u(y)+\tau\varphi(y)|^r}{|x|^{\eta}|x-y|^{\mu}|y|^{\kappa}}\Big]\,\mathrm{d}y\Big)\,\mathrm{d}x\Big|_{\tau=0}$$

$$= \frac{1}{q}\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u(x)|^q|u(y)|^{r-2}u(y)\varphi(y)}{|x|^\eta|x-y|^\mu|y|^\kappa}\,\mathrm{d}y\Big)\,\mathrm{d}x$$
$$+\frac{1}{r}\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|u(x)|^{q-2}u(x)\varphi(x)|u(y)|^r}{|x|^\eta|x-y|^\mu|y|^\kappa}\,\mathrm{d}y\Big)\,\mathrm{d}x\,.$$

The Stein-Weiss inequality provides quantitative information to characterize the integrability for the integral operators present in the energy functional. It is intrinsically determined by their weighted scaling invariance; however, the appearance of the Stein-Weiss convolution integral generates the lack of translation invariance. The study of this inequality have aroused an increasing interest by many authors due to its application in partial differential equations; in particular, in the study of the regularity properties of solutions. They are crucial in the analysis developed in this work.

The Morrey spaces

The study of Morrey spaces is motivated by many reasons. Initially, these spaces were introduced by Morrey in order to understand the regularity of solutions to elliptic partial differential equations. Regularity theorems, which allow one to conclude higher regularity of a function that is a solution of a differential equation together with a lower regularity of that function, play a central role in the theory of partial differential equations. One example of this kind of regularity theorem is a version of the Sobolev embedding theorem which states that $W^{j+m,p}(\Omega) \subset C^{j,\lambda}(\overline{\Omega})$ for $0 < \lambda \leqslant m - N/p$, where $j \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain.

Morrey spaces can complement the boundedness properties of operators that Lebesgue spaces can not handle. In line with this, many authors study the boundedness of various integral operators on Morrey spaces. The theory of Morrey spaces may come in useful when the Sobolev embedding theorem is not readily available. The main results about Morrey spaces are summarized as follows.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (i.e., an open and connected set); let $1 \leqslant p \leqslant +\infty$ and $\gamma \geqslant 0$. The Morrey spaces, denoted by $L^{p,\gamma}_M(\Omega)$, are the collection of all functions $u \in L^p(\Omega)$ such that

$$\|u\|_{L^{p,\gamma}_M(\Omega)} := \sup_{x\in\Omega,\, 0<R<\mathrm{diam}(\Omega)}\Big\{\Big(R^{\gamma-N}\int_{\Omega\cap B_R(x)}|u|^p\,\mathrm{d}x\Big)^{1/p}\Big\} < +\infty,$$

where $\mathrm{diam}(\Omega)$ is the diameter of the subset $\Omega \subset \mathbb{R}^N$.

Lemma 1.4. *1. The map $u \mapsto \|u\|_{L^{p,\gamma}_M(\Omega)}$ defines a norm on the Morrey space $L^{p,\gamma}_M(\Omega)$, making it into a normed vector space.*

2. The Morrey space $L^{p,\gamma}_M(\Omega)$ is a Banach space.

Lemma 1.5. *1. For $1 \leqslant p < +\infty$ we have $L^{p,N}_M(\Omega) = L^p(\Omega)$, i.e., $L^{p,N}_M(\Omega)$ and $L^p(\Omega)$ are continuously embedded in each other.*

2. For $1 \leqslant p < +\infty$ we have $L^\infty(\Omega) \hookrightarrow L^{p,0}_M(\Omega)$.

3. For $1 \leqslant p < +\infty$ and $\lambda < 0$ we get $L^{p,\lambda}_M(\Omega) = \{0\}$.

4. For $1 \leqslant p \leqslant q < +\infty$ and λ, $\mu \geqslant 0$ with $\gamma/p \leqslant \mu/q$ it holds $L^{q,\mu}_M(\Omega) \hookrightarrow L^{p,\gamma}_M(\Omega)$.

Remark 2. Lemma 1.5 suggests that for fixed $1 \leqslant p < +\infty$ the Morrey space $L_M^{p,\gamma}(\Omega)$ with $0 \leqslant \gamma \leqslant N$ provides a certain scaling of the spaces between $L^p(\Omega)$ and $L^\infty(\Omega)$. Also, taking $p = q$ in Lemma 1.5–4, we have $L_M^{p,\gamma_2}(\Omega) \hookrightarrow L_M^{p,\gamma_1}(\Omega)$ whenever $\gamma_1 \leqslant \gamma_2$, just like for finite L^p spaces.

In general, the Morrey space $L_M^{p,\gamma+\lambda}(\mathbb{R}^N, |x|^{-\lambda})$ is the collection of all measurable functions $u \in L^p(\mathbb{R}^N, |y|^{-\lambda})$ such that

$$\|u\|_{L_M^{p,\gamma+\lambda}(\mathbb{R}^n,|x|^{-\lambda})} := \sup_{x\in\mathbb{R}^N,\, R\in\mathbb{R}_+} \left\{ \left(R^{\gamma+\lambda-N} \int_{B_R(x)} \frac{|u|^p}{|x|^\lambda}\, \mathrm{d}x \right)^{\frac{1}{p}} \right\} < +\infty,$$

where $1 \leqslant p < +\infty$; $\gamma,\, \lambda \in \mathbb{R}_+$, and $0 < \gamma + \lambda < N$.

Lemma 1.6. *The following fundamental properties are true.*

1. $L^{p\rho}(\mathbb{R}^N, |y|^{-\rho\lambda}) \hookrightarrow L^{p,\gamma+\lambda}(\mathbb{R}^N, |y|^{-\lambda})$ *for* $\rho = \frac{N}{\gamma+\lambda} > 1$.

2. *For any* $p \in (1, +\infty)$, *we have* $L^{p,\gamma+\lambda}(\mathbb{R}^N, |y|^{-\lambda}) \hookrightarrow L^{1,\frac{\gamma}{p}+\frac{\lambda}{p}}(\mathbb{R}^N, |y|^{-\frac{\lambda}{p}})$.

3. *For* $1 \leqslant p < +\infty$ *and* $\gamma + \lambda = N$, *we have*

$$L_M^{p,N}(\mathbb{R}^N, |y|^{-\lambda}) = L^p(\mathbb{R}^N, |y|^{-\lambda}),$$

i.e., $L_M^{p,N}(\mathbb{R}^N, |y|^{-\lambda})$ *and* $L^p(\mathbb{R}^N, |y|^{-\lambda})$ *are continuously embedded in each other.*

Moreover, if we assume that $s \in (0,1)$ *and* $0 < \alpha < sp + \theta < N$, *then we have*

4. *For* $1 \leqslant q < p_s^*(\alpha,\theta)$ *and* $r = \frac{\alpha}{p_s^*(\alpha,\theta)}$, *it holds*

$$\dot{W}_\theta^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N, |y|^{-\alpha}) \hookrightarrow L_M^{q,\frac{(N-sp-\theta)q}{p}+qr}(\mathbb{R}^N, |y|^{-pr}) \tag{1.4}$$

and the norms in these spaces share the same dilation invariance.

5. *For any* $q \in [1, p_s^*(0,\theta))$, $\dot{W}_\theta^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*(0,\theta)}(\mathbb{R}^N) \hookrightarrow L^{q,\frac{(N-sp-\theta)q}{p}}(\mathbb{R}^N)$.

Proof. 1. Since

$$\|u\|_{L^{p,\gamma+\lambda}(\mathbb{R}^N)} = \sup_{R>0, x\in\mathbb{R}^N} \left\{ R^{\gamma+\lambda-N} \int_{B_R(x)} \frac{|u(y)|^p}{|y|^\lambda} \mathrm{d}y \right\}^{\frac{1}{p}} < +\infty,$$

by Hölder's inequality

$$\begin{aligned}
\left[R^{\gamma+\lambda-N} \int_{B_R(x)} \frac{|u(y)|^p}{|y|^\lambda} \mathrm{d}y \right]^{\frac{1}{p}} &\leqslant R^{\frac{\gamma+\lambda-N}{p}} \left[\int_{B_R(x)} \frac{|u(y)|^{p\rho}}{|y|^{\lambda\rho}} \mathrm{d}y \right]^{\frac{1}{p\rho}} \cdot \left[\int_{B_R(x)} 1^{\rho'} \mathrm{d}y \right]^{\frac{1}{p\rho'}} \\
&= R^{\frac{\gamma+\lambda-N}{p}} \left[\int_{B_R(x)} \frac{|u(y)|^{p\rho}}{|y|^{\lambda\rho}} \mathrm{d}y \right]^{\frac{1}{p\rho}} \cdot R^{\frac{N}{p\rho'}} \\
&= R^{\frac{\gamma+\lambda-N}{p} + \frac{N}{p\rho'}} \left[\int_{B_R(x)} \frac{|u(y)|^{p\rho}}{|y|^{\lambda\rho}} \mathrm{d}y \right]^{\frac{1}{p\rho}}.
\end{aligned}$$

Taking $\frac{\gamma+\lambda-N}{p} + \frac{N}{p\rho'} = 0$, then

$$\gamma\rho + \lambda\rho - N\rho + N\rho - N = 0 \Rightarrow \rho(\gamma+\lambda) = N \Rightarrow \rho = \frac{N}{\gamma+\lambda}.$$

Therefore, $L^{p\rho}(\mathbb{R}^N, |y|^{-\rho\lambda}) \hookrightarrow L^{p,\gamma+\lambda}(\mathbb{R}^N, |y|^{-\lambda})$ for $\rho = \frac{N}{\gamma+\lambda} > 1$.

2. We know that

$$\|u\|_{L^{1,\frac{\gamma}{p}+\frac{\lambda}{p}}(\mathbb{R}^N)} = \sup_{R>0, x\in\mathbb{R}^N} \left\{ R^{\frac{\gamma+\lambda}{p}-N} \int_{B_R(x)} \frac{|u(y)|^p}{|y|^{\frac{\lambda}{p}}} \mathrm{d}y \right\}^{\frac{1}{p}},$$

by Hölder's inequality

$$\begin{aligned} R^{\frac{\gamma+\lambda}{p}-N} \int_{B_R(x)} \frac{|u(y)|}{|y|^{\frac{\lambda}{p}}} \mathrm{d}y &\leqslant R^{\frac{\gamma+\lambda}{p}-N} \left[\int_{B_R(x)} \frac{|u(y)|^p}{|y|^{\frac{\lambda}{p}\cdot p}} \mathrm{d}y \right]^{\frac{1}{p}} \cdot \left[\int_{B_R(x)} 1^{p'} \right]^{\frac{1}{p'}} \\ &= R^{\frac{\gamma+\lambda}{p}-N} \left[\int_{B_R(x)} \frac{|u(y)|^p}{|y|^{\frac{\lambda}{p}\cdot p}} \mathrm{d}y \right]^{\frac{1}{p}} \cdot R^{\frac{N}{p'}} \\ &= R^{\frac{\gamma+\lambda}{p}-N+\frac{N(p-1)}{p}} \left[\int_{B_R(x)} \frac{|u(y)|^p}{|y|^{\lambda}} \mathrm{d}y \right]^{\frac{1}{p}} \\ &= R^{\frac{\gamma+\lambda-N}{p}} \left[\int_{B_R(x)} \frac{|u(y)|^p}{|y|^{\lambda}} \mathrm{d}y \right]^{\frac{1}{p}} \\ &= \left[R^{\gamma+\lambda-N} \int_{B_R(x)} \frac{|u(y)|^p}{|y|^{\lambda}} \mathrm{d}y \right]^{\frac{1}{p}} \end{aligned}$$

Therefore, for any $p \in (1, +\infty)$, we have $L^{p,\gamma+\lambda}(\mathbb{R}^N, |y|^{-\lambda}) \hookrightarrow L^{1,\frac{\gamma}{p}+\frac{\lambda}{p}}(\mathbb{R}^N, |y|^{-\frac{\lambda}{p}})$.

3. Take $\gamma + \lambda = N$, then, for $s \in (0,1)$ and $0 < \alpha < sp + \theta < N$, we have

$$\|u\|_{L^{p,\gamma+\lambda}(\mathbb{R}^N)} = \|u\|_{L^{p,N}(\mathbb{R}^N)} = \sup_{R>0, x\in\mathbb{R}^N} \left\{ \int_{B_R(x)} \frac{|u(y)|^p}{|y|^{\lambda}} \mathrm{d}y \right\}^{\frac{1}{p}} \leqslant \left(\int_{\mathbb{R}^N} \frac{|u(y)|^p}{|y|^{\lambda}} \mathrm{d}y \right)^{\frac{1}{p}}.$$

4. For any $q \in [1, p_s^*(\alpha,\theta))$, we have $u \in L^{q,\frac{N-sp-\theta}{p}q+qr}(\mathbb{R}^N, |y|^{-qr})$ if

$$\|u\|_{L^{q,\frac{N-sp-\theta}{p}q+qr}(\mathbb{R}^N,|y|^{-qr})} = \sup_{R>0, x\in\mathbb{R}^N} \left\{ R^{\frac{N-sp-\theta}{p}q+qr-N} \int_{B_R(x)} \frac{|u(y)|^q}{|y|^{qr}} \mathrm{d}y \right\}^{\frac{1}{q}} < +\infty.$$

By Hölder's inequality,

$$\begin{aligned} &\left(R^{\frac{N-sp-\theta}{p}q+qr-N} \int_{B_R(x)} \frac{|u(y)|^q}{|y|^{qr}} \mathrm{d}y \right)^{\frac{1}{q}} \\ &\leqslant R^{\frac{N-sp-\theta}{p}+r-\frac{N}{q}} \left(\int_{B_R(x)} \frac{|u(y)|^{q\frac{p_s^*(\alpha,\theta)}{q}}}{|y|^{qr\frac{p_s^*(\alpha,\theta)}{q}}} \mathrm{d}y \right)^{\frac{q}{p_s^*(\alpha,\theta)}} \cdot \left(\int_{B_R(x)} 1 \mathrm{d}y \right)^{\frac{p_s^*(\alpha,\theta)-q}{p_s^*(\alpha,\theta)}\cdot\frac{1}{q}} \\ &= R^{\frac{N-sp-\theta}{p}+r-\frac{N}{q}} \left(\int_{B_R(x)} \frac{|u(y)|^{p_s^*(\alpha,\theta)}}{|y|^{rp_s^*(\alpha,\theta)}} \mathrm{d}y \right)^{\frac{q}{p_s^*(\alpha,\theta)}} \cdot R^{\frac{N(p_s^*(\alpha,\theta)-q)}{p_s^*(\alpha,\theta)}\cdot\frac{1}{q}} \end{aligned}$$

Taking $r = \frac{\alpha}{p_s^*(\alpha,\theta)}$, then

$$\frac{N-sp-\theta}{p} + r - \frac{N}{q} + \frac{N(p_s^*(\alpha,\theta)-q)}{p_s^*(\alpha,\theta)} \cdot \frac{1}{q}$$

$$
\begin{aligned}
&= \frac{N-sp-\theta}{p} + \frac{\alpha}{p_s^*(\alpha,\theta)} - \frac{N}{q} + \frac{N(p_s^*(\alpha,\theta)-q)}{p_s^*(\alpha,\theta)} \cdot \frac{1}{q} \\
&= \frac{p_s^*(\alpha,\theta)q(N-sp-\theta) + pq\alpha - Npp_s^*(\alpha,\theta) + Np(p_s^*(\alpha,\theta)-q)}{pp_s^*(\alpha,\theta)q} \\
&= \frac{p_s^*(\alpha,\theta)q(N-sp-\theta) - pq(N-\alpha)}{pp_s^*(\alpha,\theta)q} \\
&= \frac{N-sp-\theta}{p} - \frac{(N-\alpha)}{p_s^*(\alpha,\theta)} \\
&= \frac{N-sp-\theta}{p} - \frac{(N-\alpha)(N-sp-\theta)}{p(N-\alpha)} = 0.
\end{aligned}
$$

Therefore,

$$
L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N, |y|^{-\alpha}) \hookrightarrow L^{q,\frac{N-sp-\theta}{p}\cdot q+qr}(\mathbb{R}^N, |y|^{-\alpha}).
$$

5. Consequence of the previous item for $\alpha = 0$.

□

For more properties of Lebesgue spaces, integral inequalities and boundedness properties of the operators in generalized Morrey spaces, see Sawano [78].

The variational setting

The variational structure of problem (1) as well as that of problems (4) and (5) can be established with the help of several inequalities. To ensure the well-definiteness of the energy functional, first we deal with the Hardy potential with a singularity.

Lemma 1.7 (Fractional Hardy inequality)**.** *Let $s \in (0,1)$ and $N > sp+\theta$. Then the best fractional Hardy constant γ_H is attained, where*

$$
\gamma_H := \inf_{\substack{u\in \dot{W}_\theta^{s,p}(\mathbb{R}^N)\\ u\neq 0}} \frac{[u]^p_{\dot{W}_\theta^{s,p}(\mathbb{R}^N)}}{\|u\|^p_{L^p(\mathbb{R}^n;|x|^{-sp-\theta})}}.
$$

Proof. See Abdellaoui & Bentifour [2, Lemma 2.7]; see also Franck & Seiringer [45]. □

Next, we use the following versions of the fractional Hardy-Sobolev and Caffarelli-Kohn-Nirenberg inequalities; see Nguyen & Squassina [72, Theorem 1.1]; see also Abdellaoui & Bentifour [1].

Lemma 1.8. *Let $N \geqslant 1, p \in (1,+\infty), s \in (0,1), 0 \leqslant \alpha \leqslant sp+\theta < N, \theta, \theta_1, \theta_2, \beta \in \mathbb{R}$ be such that $\theta = \theta_1 + \theta_2$. If $1/p_s^*(\alpha,\theta) - \alpha/Np_s^*(\alpha,\theta) > 0$, then there exists a positive constant $C(N,\alpha,\theta)$ such that*

$$
\left(\int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\alpha,\theta)}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{p}{p_s^*(\alpha,\theta)}} \leqslant C(N,\alpha,\theta) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y \tag{1.5}
$$

for all $u \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$.

We note that the norm $\|\cdot\|$ is comparable with the Gagliardo seminorm $[\,\cdot\,]_{\dot W^{s,p}_\theta(\mathbb{R}^N)}$ as stated in the next result.

Corollary 1.9. *Under the hypotheses of Lemma 1.8, if $\gamma < \gamma_H$ then*

$$\left(1-\frac{\gamma_+}{\gamma_H}\right)\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y \leqslant \|u\|^p \leqslant \left(1+\frac{\gamma_-}{\gamma_H}\right)\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y, \tag{1.6}$$

where $\gamma_\pm = \max\{\pm\gamma, 0\}$.

Using inequality (1.3) from Lemma 1.3 together with Hölder's inequality and Lemma 1.8 we can deduce another useful inequality.

Corollary 1.10. *Under the hypotheses of Lemma 1.8 we have*

$$\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{p^\sharp_s(\delta,\theta,\mu)}|u(y)|^{p^\sharp_s(\delta,\theta,\mu)}}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y \leqslant C(N,\delta,\mu)\|u\|^{2p^\sharp_s(\delta,\theta,\mu)}_{\dot W^{s,p}_\theta(\mathbb{R}^N)}. \tag{1.7}$$

Based on the embeddings (1.4) we establish the following improved weighted fractional Caffarelli-Kohn-Nirenberg inequality.

Lemma 1.11 (Fractional Caffarelli-Kohn-Nirenberg inequality)**.** *Let $s \in (0,1)$ and $0 < \beta < sp+\theta < N$. Then there exists $C = C(N,s,\beta) > 0$ such that for any $\zeta \in (\bar\zeta, 1)$ and for any $q \in [1, p^*_s(\beta,\theta))$, it holds*

$$\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{p^*_s(\beta,\theta)}}{|y|^\beta}\mathrm{d}y\right)^{\frac{1}{p^*_s(\beta,\theta)}} \leqslant \|u\|^\zeta_{\dot W^{s,p}_\theta}\|u\|^{1-\zeta}_{L^{q,\frac{N-sp-\theta}{p}q+qr}(\mathbb{R}^N,|y|^{-qr})} \tag{1.8}$$

for all $u \in \dot W^{s,p}_\theta(\mathbb{R}^N)$, where $\bar\zeta = \max\{p/p^*_s(\beta,\theta), (p^*_s(0,\theta)-1)/p^*_s(\beta,\theta)\} > 0$ and $r = \beta/p^*_s(\beta,\theta)$.

Related works on fractional elliptic operators

Problems with one or two nonlinearities involving the p-Laplacian and the fractional p-Laplacian have been studied by many authors. Filippucci, Pucci & Robert [42] proved that there exists a positive solution for a p-Laplacian problem with critical Sobolev and Hardy–Sobolev terms, i.e., problem (4) with $s = 1$, $p = 2$, $\theta_1 = \theta_2 = 0$, $\beta_1 = 0$ and no singularity in the Hardy potential. As is well known, to show existence results it is natural to consider variants of Lions's concentration–compactness principle for critical problems. However, due to the non-local feature of the fractional p-Laplacian, it is difficult to use the concentration–compactness principle directly, since one needs to estimate commutators of the fractional Laplacian and smooth test functions. A possible strategy, which is known as s-harmonic extension, is to transform the non-local problem in $\mathbb{R}^N$ into a local problem in $\mathbb{R}^{N+1}_+$ with Neumann boundary condition, as performed by Caffarelli & Silvestre [23]. Since that, many interesting results in the classical elliptic problems have been extended to the setting of the fractional Laplacian. For example, Ghoussoub & Shakerian [47] considered problem (4) with $p = 2$, $\theta_1 = \theta_2 = 0$, $\beta_1 = \beta_2 = 0$ and no singularity in the Hardy potential; Chen [27] also studied problem (4) and extended this result to the case

$p = 2$, $\theta_1 = \theta_2 = 0$ but with $\beta_1 \neq 0$ and $\beta_2 \neq 0$. In both papers, the authors combined the s-harmonic extension with the concentration-compactness principle to investigate the existence of solutions for a doubly critical problem involving the fractional Laplacian. Assunção, Miyagaki & Silva [14] considered problem (4) with no singularity in the Hardy potential, that is, $\theta_1 = \theta_2 = 0$ and $\beta_1 \neq 0$ and $\beta_2 \neq 0$. Li & Yang [56] studied problem (1) involving the fractional Laplacian with a Hardy potential and two nonlinearities, one of Sobolev type and the other of Choquard type. More precisely, they considered problem (1) with $p = 2$, $\theta_1 = \theta_2 = 0$. The proof of the existence result is achieved in the setting of Morrey spaces to avoid the use of the concentration-compactness principle. They also studied problems (4) and (5) in the case $p = 2$, $\theta_1 = \theta_2 = 0$ and the proof follows basically the same steps. They claim to have considered also the cases $\alpha = 0$ or $\beta = 0$; however, their proof is based on a flawed argument; see Remark 1. Recently, Su [87] considered the general p and $\theta_1 = \theta_2 = 0$ and proved existence, decaying and regularity results for problem problem (1) with a general condition $0 < sp < N$ and $\theta_1 = \theta_2 = 0$.

Our contribution to the problem

Inspired by the previously mentioned papers, we mainly extend the results by Li & Yang [56]. We consider the general fractional p-Laplacian with $p > 1$ and $\theta = \theta_1 + \theta_2$ not necessarily zero. By establishing new embedding results involving weighted Morrey norms in the homogeneous fractional Sobolev space, we provide sufficient conditions under which a weak nontrivial solution to the problem exists via variational methods.

1.2 Some of the difficulties to prove the theorems

In the process of proving Theorem 0.1, there are several technical and substantial difficulties.

The non-locality of the operator

First, we mention that the procedure based upon the Caffarelli & Silvestre approach through s-harmonic extension can overcome the difficulty of the non-locality of the operator only in the case $p = 2$ for the fractional Laplacian operator $(-\Delta)^s$; still, the method is more complicated and less straightforward. See Ghoussoub & Shakerian [47] and Chen [30]. So, we have to study our problem in the non-local context.

Estimatives of mixed terms

Second, the truncation technique adopted by Filipucci, Pucci & Robert [42] is not suitable when we work in the homogeneous Sobolev space $\dot{W}_{\theta}^{s,p}(\mathbb{R}^N)$ to consider the non-local operator $(-\Delta)^s_{p,\theta}$. More precisely, for local problems, e.g., problem (4) in the case $s = 1$, it is useful to prove an inequality of the type $\big||\nabla(\phi u_k)|^p - |\phi\nabla u_k|\big| \leqslant C_p(|u_k\nabla\phi|^p + |\phi\nabla u_k|^{p-1}|u_k\nabla\phi|)$ for a test function ϕ and a bounded sequence $\{u_k\}_{k\in\mathbb{R}}$ in a suitable Sobolev space. This inequality, together with the fact that the gradient of a function with compact support also has compact support, allows one to prove that $\int_{\mathbb{R}^N} |\nabla(\phi u_k)|^p \,\mathrm{d}x = \int_{\mathbb{R}^N} |\phi\nabla u_k|^p \,\mathrm{d}x + o(1)$ as $k \to +\infty$. Based on this estimate, coupled with a careful analysis and some refined estimates, the concentration properties of weakly null Palais-Smales sequences can be obtained, which is crucial to obtain the existence of nontrivial solution to the problem. Recall that the Palais-Smale condition is a substitute for compactness in some calculus of variations problems. Inequalities of this type are important in dealing with problems involving local differential operators; e.g., to investigate the existence of extremals for the related inequalities by the concentration-compactness principle. However, for the problems we consider in this work, there does not seem to exist a similar estimate like this for the fractional p-Laplacian operator $(-\Delta)^s_{p,\theta}$.

The structure of Palais-Smale sequences

Third, since we consider problems with critical nonlinearities in the entire space $\mathbb{R}^N$, the compactness of the corresponding Palais-Smale sequences can not hold for any energy level $c > 0$ since the problem is invariant under the scaling $u(x) \mapsto \lambda^{(N-sp-\theta)/p}u(\lambda x)$. In fact, towards a contradiction, assume that the compactness of the corresponding $(PS)_c$ sequence holds for some $c > 0$, and let $\{u_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ be a $(PS)_c$ sequence, i.e., $I(u_k) \to c$ and $I'(u_k) \to 0$ as $k \to +\infty$. Then, up to a subsequence, we may assume that $u_k \to u$ strongly in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$ as $k \to +\infty$. Define the new sequence $\{v_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ by $v_k(x) = \lambda^{(N-sp-\theta)/p}u_k(\lambda x)$; then it is easy to check that $\{v_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ is also a $(PS)_c$ sequence and $v_k \rightharpoonup 0$ weakly in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$. But this implies that $v_k \to 0$ strongly in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$, which contradicts the hypothesis $c > 0$. This means that, once we have established the mountain pass geometry, it does not yield critical points but only Palais-Smale sequences. Therefore, it very important to understand the behavior of these sequences.

The auxiliary minimization problems

Fourth, we have to show that the best constants in two auxiliary minimization problems are attained; this is a crucial step in our work. More precisely, we consider a minimization problem involving the Choquard convolution integral

$$S_\mu(N,s,p,\theta,\gamma,\alpha) = \inf_{u\in\dot{W}^{s,p}_\theta(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^p}{Q_\sharp(u,u)^{\frac{p}{2p^\sharp_\mu(\alpha,\theta)}}} \tag{1.9}$$

where the quadratic form $Q^\sharp\colon \dot{W}^{s,p}_\theta(\mathbb{R}^N) \times \dot{W}^{s,p}_\theta(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$Q^\sharp(u,v) = \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p^\sharp_s(\delta,\theta,\mu)}|v(y)|^{p^\sharp_s(\delta,\theta,\mu)}}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y.$$

We also consider another minimization problem involving the Sobolev term,

$$\Lambda(N,s,p,\theta,\gamma,\beta) = \inf_{u\in\dot{W}^{s,p}_\theta(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^p}{\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*_s(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)^{\frac{p}{p^*_s(\beta,\theta)}}}. \tag{1.10}$$

To simplify the notation, hereafter we simply denote $S_\mu = S_\mu(N,s,p,\theta,\gamma,\alpha)$ and $\Lambda = \Lambda(N,s,p,\theta,\gamma,\beta)$. To show that S_μ and Λ are attained we have to proof a version of the Caffarelli-Kohn-Nirenberg's inequality, which estimates the norm of a function in the critical weighted Sobolev space and the norm of the same function in the fractional Sobolev and Morrey spaces; see Lemma 1.11. In our setting, that is, $1 < p < +\infty$, we have to use a version of the Caffarelli-Silvestre extension as given by del Teso, Castro-Gómez & Vázquez [38, Theorem 3.1].

Additionaly, we have to consider the fractional Hardy type potential, which is related to the best constant in the fractional Hardy inequality.

The asymptotic competition

Finally, as it was already mentioned by Filippucci, Pucci & Robert [42] and in several other papers, we observe here the main difficulty is that there is an asymptotic competition between the energy carried by two critical nonlinearities. If one dominates the other, then there is vanishing of the weakest one and we obtain solution of an equation with only one critical nonlinearity. Therefore the crucial step in the proof is to avoid the dominance of one term over the other. To overcome this difficulty, we choose the Palais-Smale sequence at suitable energy level and make a careful analysis of the concentration; afterwards, we show that there is a balance between the energies of the two nonlinearities mentioned above, and therefore none can dominate the other. Moreover, we can make the full use of conformal invariance of problem (1) under the above defined scaling and we recover the solution to the problem in the critical case.

1.3 Method of proof and outline of the work

The method adopted in previous works, such as Filippucci, Pucci & Robert [42], Yang & Wu [93], is not applicable to problem (1). For this reason, we develop a new tool which is based on the weighted Morrey space. To be more precise, we discover the embeddings (1.4).

Now we give an outline of the proof of Theorem 0.1. We already know that weak solutions to problem (1) correspond to critical points of the energy functional I defined on the homogeneous Sobolev space $\dot{W}^{s,p}_{\theta}(\mathbb{R}^n)$. Moreover, this functional has the appropriate geometry to use the mountain pass theorem; see Ambrosetti & Rabinowitz [12] and Willem [90]. However, since in our problem we consider doubly critical nonlinearities, this theorem does not yield critical points but only Palais-Smale sequences. Thus, we require the mountain pass level of the Palais-Smale sequences $(PS)_c$ to verify the condition $c < c^*$ for some suitable threshold level c^*. This is crucial in ruling out the vanishing of the sequence.

After showing that the minimizers of S_μ and Λ are attained, we can prove that the mountain pass level verifies the required inequality $c < c^*$, where

$$c^* := \min\left\{\left(\frac{1}{p} - \frac{1}{2p^\sharp_s(\delta,\theta,\mu)}\right) S_\mu^{\frac{2p^\sharp_s(\delta,\theta,\mu)}{2p^\sharp_s(\delta,\theta,\mu)-p}}, \left(\frac{1}{p} - \frac{1}{p^*_s(\beta,\theta)}\right)\Lambda^{\frac{p^*_s(\beta,\theta)}{p^*_s(\beta,\theta)-p}}\right\}. \tag{1.11}$$

Moreover, the $(PS)_c$ sequence $\{u_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_{\theta}(\mathbb{R}^N)$ verifies the conditions

$$\lim_{k\to+\infty} I(u_k) = c < c^* \qquad \text{and} \qquad \lim_{k\to+\infty} I'(u_k) = 0 \text{ strongly in } \dot{W}^{s,p}_{\theta}(\mathbb{R}^N). \tag{1.12}$$

This sequence is bounded; so, up to the passage to a subsequence we may assume that $u_k \rightharpoonup u$ weakly in $\dot{W}^{s,p}_{\theta}(\mathbb{R}^N)$ for some $u \in \dot{W}^{s,p}_{\theta}(\mathbb{R}^N)$. But it may occur that $u \equiv 0$.

To show that this does not occur, denote

$$d_1 := \lim_{k\to+\infty} \int_{\mathbb{R}^N} \frac{|u_k|^{p^*_s(\beta,\theta)}}{|x|^\beta} \mathrm{d}x \qquad \text{and} \qquad d_2 := \lim_{k\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)|^{p^\sharp_s(\delta,\theta,\mu)} |u_k(y)|^{p^\sharp_\mu(\delta,\theta,\mu)}}{|x|^\delta |x-y|^\mu |y|^\delta} \, \mathrm{d}x \, \mathrm{d}y \, .$$

By the definitions of S_μ and Λ together with the properties of the Palais-Smale sequence, we can prove that

$$d_1^{\frac{p}{p^*_s(\beta,\theta)}}\left(\Lambda - d_1^{\frac{p^*_s(\beta,\theta)-p}{p^*_s(\beta,\theta)}}\right) \leqslant d_2 \qquad \text{and} \qquad d_2^{\frac{1}{p^\sharp_s(\delta,\theta,\mu)}}\left(S_\mu - d_2^{\frac{p^\sharp_s(\delta,\theta,\mu)-1}{p^\sharp_s(\delta,\theta,\mu)}}\right) \leqslant d_1.$$

And since $0 < c < c^*$, we can also deduce that

$$\Lambda - d_1^{\frac{p_s^*(\beta,\theta)-p}{p_s^*(\beta,\theta)}} > 0 \qquad \text{and} \qquad S_\mu - d_2^{\frac{p_s^\sharp(\delta,\theta,\mu)-1}{p_s^\sharp(\delta,\theta,\mu)}} > 0.$$

Thus, $d_1 \geqslant \varepsilon_0 > 0$ and $d_2 \geqslant \varepsilon_0 > 0$ for some $\varepsilon_0 \in \mathbb{R}_+$; indeed, if $d_1 = 0$ and $d_2 = 0$, then $c = 0$, which is a contradiction.

Using the embeddings (1.4) and the improved Sobolev inequality (1.8), we deduce that, for $k \in \mathbb{N}$ large enough,

$$0 < C_2 \leqslant \|u_k\|_{L_M^{p,N-sp-\theta+pr}(\mathbb{R}^N,|y|^{-pr})} \leqslant C_1,$$

where $r = \alpha/p_s^*(\alpha,\theta)$. For $k \in \mathbb{N}$ large enough, we may find sequences $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ and $\{x_k\}_{k\in\mathbb{N}} \in \mathbb{R}^N$ such that

$$\lambda_k^{(N-sp-\theta+pr)-N} \int_{B_{\lambda_k}(x_k)} \frac{|u_k(y)|^p}{|y|^{pr}} \mathrm{d}y > \|u_k\|^p_{L_M^{p,N-sp-\theta+pr}(\mathbb{R}^N,|y|^{-pr})} - \frac{C}{2k} \geqslant \tilde{C} > 0.$$

And with the help of these two sequences we can define another sequence $\{v_k\}_{k\in\mathbb{N}} \subset \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ defined by the scaling

$$v_k(x) = \lambda_k^{(N-sp-\theta)/p} u_k(\lambda_k x).$$

This new sequence verifies the condition $\|v_k\| = \|u_k\| \leqslant C$ and up to the passage to a subsequence we may assume that $v_k \rightharpoonup v$ weakly in $\dot{W}_\theta^{s,p}(\mathbb{R}^N)$ for some $v \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ and $v_k \to v$ a.e. in $\mathbb{R}^N$, up to the passage to a subsequence, as $k \to +\infty$. Again, it may occur that $v \equiv 0$; however, the sequence $\{v_k\}_{k\in\mathbb{N}}$ is of a very structured form and we can prove that $v \not\equiv 0$.

To do this, we consider the sequence $\{\tilde{x}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ defined by $\tilde{x}_k = x_k/\lambda_k$ and show that it is bounded; then, we can find $R \in \mathbb{R}_+^N$ such that the ball $B(0,R)$ contains all unitary balls centered in $\tilde{x}_k$ for $k \in \mathbb{N}$; moreover,

$$\int_{B_R(0)} \frac{|v_k(x)|^p}{|x|^{pr+\theta}} \mathrm{d}x \geqslant C_1 > 0.$$

Additionaly, we can show that $|x|^{-r-\frac{\theta r}{sp}} u_k \to |x|^{-r-\frac{\theta r}{sp}} u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$; therefore,

$$\int_{B_R(0)} \frac{|v(x)|^p}{|x|^{pr+\theta}} \mathrm{d}x \geqslant C_1 > 0,$$

and we deduce that $v \not\equiv 0$.

Again, the boundedness of the sequence $\{v_k\}_{k\in\mathbb{N}}$ in $\dot{W}_\theta^{s,p}(\mathbb{R}^N)$ implies the boundedness of the sequence $\{|v_k|^{p_s^*(\beta,\theta)-2} v_k\}_{k\in\mathbb{N}}$ in $L^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-1}}(\mathbb{R}^N,|x|^{-\beta})$, and this implies that $|v_k|^{p_s^*(\beta,\theta)-2} v_k \rightharpoonup |v|^{p_s^*(\beta,\theta)-2} v$ weakly in $L^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-1}}(\mathbb{R}^N,|x|^{-\beta})$. For any $\phi \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$, with the help of the embedding $\dot{W}_\theta^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N,|x|^{-\alpha})$ and a variant of the Brézis-Lieb lemma, we can show that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot,v_k)\right](x) f_\alpha(x,v_k)\phi(x)\mathrm{d}x = \int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot,v)\right](x) f_\alpha(x,v)\phi(x)\mathrm{d}x.$$

We still need to check that the sequence $\{v_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N) \hookrightarrow L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N, |x|^{-\alpha})$ is also a $(PS)_c$ sequence for the energy functional I at the level $c < c^*$. To do this, we notice that the norms in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$ and $L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N, |x|^{-\alpha})$ are invariant under the special scaling used,

$$\|v_k\|^p_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)} = \|u_k\|^p_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)} \qquad \text{and} \qquad \|v_k\|^{p_s^*(\alpha,\theta)}_{L^{p_s^*(\alpha,\theta)}} = \|u_k\|^{p_s^*(\alpha,\theta)}_{L^{p_s^*(\alpha,\theta)}}.$$

Thus, we have $\lim_{k\to+\infty} I(v_k) = c$. Moreover, for all $\phi \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$, we also have $\phi_k(x) = \lambda_k^{(N-sp-\theta)/p}\phi(x/\lambda_k) \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$. And from the strong convergence $I'(u_k) \to 0$ in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)'$, we can deduce that $\langle I'(v), \phi\rangle = \lim_{k\to+\infty}\langle I'(v_k), \phi\rangle = \lim_{k\to+\infty}\langle I'(u_k), \phi\rangle = 0$. Hence, v is a nontrivial weak solution of 1.

To conclude the proof, it remains to show the crucial step that the quantities S_μ and Λ, defined in (1.9) and (1.10), respectively, are attained. To this end, we need some kind of compactness. These problems can be solved in a direct way using the embeddings (1.4) and the improved Sobolev inequality (1.8).

1.4 Proof of Theorems 0.1 and 0.2

Preliminary results

In this section, we give some preliminary results that will be usefull in the proof of Theorem 0.1.

We begin by stating a result about local convergence.

Lemma 1.12. *Let $s \in (0,1)$ and $0 < r < s + \frac{\theta}{p} < \frac{N}{p}$. If $\{u_k\}$ is a bounded sequence in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$ and $u_k \rightharpoonup u$ in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$, then as $k \to +\infty$,*

$$\frac{u_k}{|x|^{r+\frac{\theta r}{sp}}} \to \frac{u}{|x|^{r+\frac{\theta r}{sp}}} \text{ in } L^p_{\text{loc}}(\mathbb{R}^N).$$

Proof. Since $u_k \rightharpoonup u$ in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$, we have

$$u_k \to u \text{ in } L^q_{\text{loc}}(\mathbb{R}^N)\ (1 \leqslant q \leqslant p_s^*(0,\theta)) \quad \text{and} \quad u_k \to u \text{ a.e. on } \mathbb{R}^N.$$

From Lemma 1.7, we have

$$\int_{\mathbb{R}^N} \frac{|u_k|^p}{|x|^{sp+\theta}}\mathrm{d}x \leqslant C_{s,N} \iint_{\mathbb{R}^{2N}} \frac{|v_k(x) - v_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y.$$

For any compact set $\Omega \Subset \mathbb{R}^N$, using Hölder's inequality we have

$$\begin{aligned}
\int_\Omega \frac{|u_k - u|^p}{|x|^{pr+\frac{\theta r}{s}}}\mathrm{d}x &= \int_\Omega \frac{|u_k - u|^{\frac{p(s-r)}{s}+\frac{pr}{s}}}{|x|^{pr+\frac{\theta r}{s}}}\mathrm{d}x \\
&\leqslant \left[\int_\Omega \left(\frac{|u_k - u|^{\frac{pr}{s}}}{|x|^{pr+\frac{\theta r}{s}}}\right)^{\frac{s}{r}}\mathrm{d}x\right]^{\frac{r}{s}} \left[\int_\Omega \left(|u_k - u|^{\frac{p(r-s)}{s}}\right)^{\frac{s}{s-r}}\mathrm{d}x\right]^{\frac{s-r}{s}} \\
&= \left[\int_\Omega \frac{|u_k - u|^p}{|x|^{sp+\theta}}\mathrm{d}x\right]^{\frac{r}{s}} \left[\int_\Omega |u_k - u|^p\mathrm{d}x\right]^{\left(1-\frac{r}{s}\right)}
\end{aligned}$$

$$\leqslant C\left[\int_\Omega |u_k-u|^p \mathrm{d}x\right]^{\left(1-\frac{r}{s}\right)}.$$

This means that

$$\lim_{k\to+\infty}\int_\Omega \frac{|u_k-u|^p}{|x|^{pr+\frac{\theta r}{s}}}\mathrm{d}x=0,$$

that is, as $k\to+\infty$

$$\frac{u_k}{|x|^{r+\frac{\theta r}{sp}}}\to\frac{u}{|x|^{r+\frac{\theta r}{sp}}}\ \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N).$$

This concludes the proof of the lemma. □

Next, we state a variant of the classic Brézis–Lieb lemma that will be useful to prove a similar result for the convolution terms.

Lemma 1.13 (A variant of Brézis–Lieb lemma)**.** *Let $r>1, q\in[1,r]$ and $\delta\in[0,Nq/r)$. Assume that $\{w_k\}$ is a bounded sequence in $L^r(\mathbb{R}^N,|x|^{-\delta r/q})$ and $w_k\to w$ a.e. on $\mathbb{R}^N$. Then,*

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}\left|\frac{|w_k|^q}{|x|^\delta}-\frac{|w_k-w|^q}{|x|^\delta}-\frac{|w|^q}{|x|^\delta}\right|^{\frac{r}{q}}\mathrm{d}x=0 \tag{1.13}$$

and

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}\left|\frac{|w_k|^{q-1}w_k}{|x|^\delta}-\frac{|w_k-w|^{q-1}(w_k-w)}{|x|^\delta}-\frac{|w|^{q-1}w}{|x|^\delta}\right|^{\frac{r}{q}}\mathrm{d}x=0 \tag{1.14}$$

Proof. For the case $\delta=0$, one can refer to [82, Lemma 2.3]; here we focus on the case $\delta>0$. Fixing $\epsilon>0$ small, there exists $C(\epsilon)>0$ such that for all $a,b\in\mathbb{R}$ and $q\geqslant 1$, we have

$$||a+b|^q-|a|^q|\leqslant\epsilon|a|^q+C(\epsilon)|b|^q.$$

Recalling that $(a+b)^p\leqslant 2^{p-1}(a^p+b^p)$ for $a,b\geqslant 0$ and $p\geqslant 1$ and using the previous inequality, we obtain

$$||a+b|^q-|a|^q|^{\frac{r}{q}}\leqslant(\epsilon|a|^q+C(\epsilon)|b|^q)^{\frac{r}{q}}\leqslant\tilde{\epsilon}|a|^r+\tilde{C}(\epsilon)|b|^r, \tag{1.15}$$

Taking $a=(w_k-w)/|x|^{\delta/q}$ and $b=w/|x|^{\delta/q}$ in inequality (1.15), we obtain

$$\begin{aligned}
|f_{N,\epsilon}| &:= \left(\left|\frac{|w_k|^q}{|x|^\delta}-\frac{|w_k-w|^q}{|x|^\delta}-\frac{|w|^q}{|x|^\delta}\right|^{\frac{r}{q}}-\tilde{\epsilon}\left(\frac{|w_k-w|}{|x|^{\frac{\delta}{q}}}\right)^r\right)^+\\
&\leqslant\left|\frac{|w_k|^q}{|x|^\delta}-\frac{|w_k-w|^q}{|x|^\delta}\right|^{\frac{r}{q}}+\left|\frac{w}{|x|^{\delta/q}}\right|^r-\tilde{\epsilon}\left(\frac{|w_k-w|}{|x|^{\frac{\delta}{q}}}\right)^r\\
&\leqslant\tilde{\epsilon}\left(\frac{|w_k-w|}{|x|^{\frac{\delta}{q}}}\right)^r+\tilde{C}(\epsilon)\left|\frac{w}{|x|^{\delta/q}}\right|^r+\left|\frac{w}{|x|^{\delta/q}}\right|^r-\tilde{\epsilon}\left(\frac{|w_k-w|}{|x|^{\frac{\delta}{q}}}\right)^r\\
&\leqslant\left|\frac{w}{|x|^{\delta/q}}\right|^r+\tilde{C}(\epsilon)\left|\frac{w}{|x|^{\delta/q}}\right|^r
\end{aligned}$$

$$= \left(1+\tilde{C}(\epsilon)\right)\left|\frac{w}{|x|^{\delta/q}}\right|^r.$$

Now using Lebesgue Dominated Convergence theorem, we have

$$\int_{\mathbb{R}^N} |f_{N,\epsilon}|\mathrm{d}x \to 0 \quad \text{as } k\to\infty.$$

Therefore, we get

$$\left|\frac{|w_k|^q}{|x|^\delta} - \frac{|w_k-w|^q}{|x|^\delta} - \frac{|w|^q}{|x|^\delta}\right|^{\frac{r}{q}} \leqslant |f_{N,\epsilon}| + \tilde{\epsilon}\left(\frac{|w_k-w|}{|x|^{\frac{\delta}{q}}}\right)^r,$$

which gives

$$\limsup_{k\to\infty}\int_{\mathbb{R}^N}\left|\frac{|w_k|^q}{|x|^\delta} - \frac{|w_k-w|^q}{|x|^\delta} - \frac{|w|^q}{|x|^\delta}\right|^{\frac{r}{q}}\mathrm{d}x \leqslant \tilde{\epsilon}\sup_{k\in\mathbb{N}}\int_{\mathbb{R}^N}\frac{|w_k-w|^r}{|x|^{\frac{\delta r}{q}}}\mathrm{d}x < \infty.$$

Further, letting $\epsilon\to 0$ we conclude (1.13).

The limit (1.14) can be proved in the same way. In fact, fixing $\epsilon > 0$ small, there exists $C(\epsilon) > 0$ such that for all $a, b\in\mathbb{R}$ and $q\geqslant 1$, we have

$$\left||a+b|^{q-1}(a+b) - |a|^{q-1}a\right| \leqslant \epsilon|a|^q + C(\epsilon)|b|^q.$$

Using the previous inequality, we obtain

$$\left||a+b|^{q-1}(a+b) - |a|^{q-1}a\right|^{\frac{r}{q}} \leqslant (\epsilon|a|^q + C(\epsilon)|b|^q)^{\frac{r}{q}} \leqslant \tilde{\epsilon}|a|^r + \tilde{C}(\epsilon)|b|^r, \tag{1.16}$$

where $\tilde{\epsilon} = 2^{\frac{r}{q}-1}\epsilon^{\frac{r}{q}}$ and $\tilde{C}(\epsilon) = 2^{\frac{r}{q}-1}C(\epsilon)^{\frac{r}{q}}$. Now we can adapt the same arguments already used to conclude (1.14) □

Also recall that pointwise convergence of a bounded sequence implies weak converge.

Lemma 1.14. *Let $\Omega\subset\mathbb{R}^N$ be a domain, $q\in(1,\infty)$ and $\{u_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^q(\Omega)$. If $u_k\to u$ almost everywhere on Ω as $k\to\infty$, then $u_k\rightharpoonup u$ weakly in $L^q(\Omega)$.*

Proof. See Willem [90, Proposition 5.4.7]. □

Lemma 1.15 (Weak Young inequality)**.** *Let $n\in\mathbb{N}, \mu\in(0,N), \hat{p},\hat{r} > 1$ and $\frac{1}{\hat{p}}+\frac{\mu}{N} = 1+\frac{1}{\hat{r}}$. If $v\in L^{\hat{p}}(\mathbb{R}^N)$, then $I_\mu * v\in L^{\hat{r}}(\mathbb{R}^N)$ and*

$$\left(\int_{\mathbb{R}^N}|I_\mu * v|^{\hat{r}}\right)^{\frac{1}{\hat{r}}} \leqslant C(N,\mu,\hat{p})\left(\int_{\mathbb{R}^N}|v|^{\hat{p}}\right)^{\frac{1}{\hat{p}}}, \tag{1.17}$$

where $I_\mu(x) = |x|^{-\mu}$. In particular, we can set $\hat{r} = \frac{N\hat{p}}{N-(N-\mu)\hat{p}}$ for $\hat{p}\in\left(1,\frac{N}{N-\mu}\right)$.

Proof. See Lieb & Loss [59, Section 4.3] □

We will use the Lemmas 1.13, 1.14 and 1.15 to prove the next result, which is a generalization of Moroz & Van Schaftingen [67, Lemma 2.4].

Lemma 1.16 (Another variant of Brézis–Lieb lemma)**.** *Let $N \in \mathbb{N}, \mu \in (0, N)$, $\frac{N-\delta-\mu/2}{N-\beta} \leqslant q < \infty$ and let $\{u_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^{\frac{(N-\beta)q}{N-\delta-\mu/2}}(\mathbb{R}^N)$. If $u_k \to u$ a.e. on $\mathbb{R}^N$ as $k \to \infty$, then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \left[\left(I_\mu * \frac{|u_k|^q}{|x|^\delta} \right) \frac{|u_k|^q}{|x|^\delta} - \left(I_\mu * \frac{|u_k - u|^q}{|x|^\delta} \right) \frac{|u_k - u|^q}{|x|^\delta} \right] = \int_{\mathbb{R}^N} \left(I_\mu * \frac{|u|^q}{|x|^\delta} \right) \frac{|u|^q}{|x|^\delta}. \tag{1.18}$$

Proof. For every $k \in \mathbb{N}$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[\left(I_\mu * \frac{|u_k|^q}{|x|^\delta} \right) \frac{|u_k|^q}{|x|^\delta} - \left(I_\mu * \frac{|u_k - u|^q}{|x|^\delta} \right) \frac{|u_k - u|^q}{|x|^\delta} \right] \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left(I_\mu * \frac{|u_k|^q}{|x|^\delta} \right) \frac{|u_k|^q}{|x|^\delta} \mathrm{d}x - \int_{\mathbb{R}^N} \left(I_\mu * \frac{|u_k - u|^q}{|x|^\delta} \right) \frac{|u_k|^q}{|x|^\delta} \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \left(I_\mu * \frac{|u_k|^q}{|x|^\delta} \right) \frac{|u_k - u|^q}{|x|^\delta} \mathrm{d}x + \int_{\mathbb{R}^N} \left(I_\mu * \frac{|u_k - u|^q}{|x|^\delta} \right) \frac{|u_k - u|^q}{|x|^\delta} \mathrm{d}x \\
&\quad + 2 \int_{\mathbb{R}^N} \left(I_\mu * \frac{|u_k|^q}{|x|^\delta} \right) \frac{|u_k - u|^q}{|x|^\delta} \mathrm{d}x - 2 \int_{\mathbb{R}^N} \left(I_\mu * \frac{|u_k - u|^q}{|x|^\delta} \right) \frac{|u_k - u|^q}{|x|^\delta} \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left(I_\mu * \left(\frac{|u_k|^q}{|x|^\delta} - \frac{|u_k - u|^q}{|x|^\delta} \right) \right) \frac{|u_k|^q}{|x|^\delta} \mathrm{d}x - \int_{\mathbb{R}^N} \left(I_\mu * \left(\frac{|u_k|^q}{|x|^\delta} - \frac{|u_k - u|^q}{|x|^\delta} \right) \right) \frac{|u_k - u|^q}{|x|^\delta} \mathrm{d}x \\
&\quad + 2 \int_{\mathbb{R}^N} \left(I_\mu * \left(\frac{|u_k|^q}{|x|^\delta} - \frac{|u_k - u|^q}{|x|^\delta} \right) \right) \frac{|u_k - u|^q}{|x|^\delta} \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left[I_\mu * \left(\frac{|u_k|^q}{|x|^\delta} - \frac{|u_k - u|^q}{|x|^\delta} \right) \right] \left(\frac{|u_k|^q}{|x|^\delta} - \frac{|u_k - u|^q}{|x|^\delta} \right) \mathrm{d}x \\
&\quad + 2 \int_{\mathbb{R}^N} \left[I_\mu * \left(\frac{|u_k|^q}{|x|^\delta} - \frac{|u_k - u|^q}{|x|^\delta} \right) \right] \left(\frac{|u_k - u|^q}{|x|^\delta} \right) \mathrm{d}x.
\end{aligned}$$

By Lemma 1.13 with $r = \frac{2(N-\beta)q}{2N-2\delta-\mu}$, one has

$$\frac{|u_k - u|^q}{|x|^\delta} - \frac{|u_k|^q}{|x|^\delta} \to \frac{|u|^q}{|x|^\delta} \qquad \text{in } L^{\frac{2(N-\beta)}{2N-2\delta-\mu}}(\mathbb{R}^N),$$

as $k \to +\infty$. Using this convergence and Lemma 1.15 with $\hat{p} = \frac{2(N-\beta)}{2N-2\delta-\mu}$ and $\hat{r} = \frac{2(N-\beta)}{\mu-2\delta-2\beta}$, we have

$$I_\mu * \left(\frac{|u_k - u|^q}{|x|^\delta} - \frac{|u_k|^q}{|x|^\delta} \right) \to I_\mu * \frac{|u|^q}{|x|^\delta} \qquad \text{in } L^{\frac{2(N-\beta)}{2N-2\delta-\mu}}(\mathbb{R}^N).$$

Finally, by Lemma 1.14 we deduce that

$$\left| \frac{|u_k - u|^q}{|x|^\delta} \right| \rightharpoonup 0 \qquad \text{in } L^{\frac{2(N-\beta)}{2N-2\delta-\mu}}(\mathbb{R}^N),$$

as $k \to \infty$, and we reach the conclusion. □

Lemma 1.17. *Let $s \in (0, 1), 0 \leqslant \alpha < sp + \theta < N, \mu \in (0, N)$ and $2\delta + \mu < N$. If $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$ and $u_k \rightharpoonup u$ in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$, then we have*

$$\lim_{k\to\infty} Q^\sharp(u_k, u_k) = \lim_{k\to\infty} Q^\sharp(u_k - u, u_k - u) + Q^\sharp(u, u).$$

Proof. Consider $s \in (0,1), 0 < sp + \theta < N$ and $2\delta + \mu < N$. For $p \geqslant 2$, we have

$$\begin{aligned} p_s^\sharp(\delta,\theta,\mu) &:= \frac{p(N-\delta-\mu/2)}{N-sp-\theta} > \frac{p(N-\delta-\mu/2)}{N} \\ &= \frac{p-(2\delta+\mu)/2}{N} > \frac{p(N-N/2)}{N} \geqslant 1. \end{aligned}$$

For $1 < p < 2$, we use the above specified intervals and we also impose the additional condition $\delta + \mu/2 < sp + \theta < N$; therefore, in this case we also have $p_s^\sharp(\delta,\theta,\mu) > 1$.

Taking $q = p_s^\sharp(\delta,\theta,\mu)$ in Lemma 1.16, we obtain

$$\frac{(N-\beta)q}{N-\delta-\mu/2} = \frac{N-\beta}{N-\delta-\mu/2} p_s^\sharp(\delta,\theta,\mu) = \frac{N-\beta}{N-\delta-\mu/2}\frac{p(N-\delta-\mu/2)}{N-sp-\theta} = p_s^*(\beta,\theta).$$

Since $\{u_k\}_{k\in\mathbb{N}} \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ and $u_k \rightharpoonup u$ in $\dot{W}_\theta^{s,p}(\mathbb{R}^N)$, the embedding $\dot{W}_\theta^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*(\beta,\theta)}(\mathbb{R}^N, |x|^{-\beta})$ in the Lemma 1.8 implies that

$$\left(\int_{\mathbb{R}^N} \frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)^{\frac{p}{p_s^*(\beta,\theta)}} \leqslant C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k(x)-u_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}y\mathrm{d}x \leqslant C.$$

Therefore, $u_k, u \in L^{p_s^*(\beta,\theta)}(\mathbb{R}^N, |x|^{-\beta})$ and as $k \to +\infty$,

$$\frac{u_k}{|x|^{\frac{\beta}{p_s^*(\beta,\theta)}}} \to \frac{u}{|x|^{\frac{\beta}{p_s^*(\beta,\theta)}}} \quad \text{a.e. on } \mathbb{R}^N.$$

Consequently, Lemma 1.16 gives the desired equality. □

Lemma 1.18. *Let $s \in (0,1), 0 \leqslant \alpha, \beta < sp + \theta < N$ and $\mu \in (0,N)$ and let $\{u_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N, |x|^{-\alpha})$. If $u_k \to u$ a.e. on $\mathbb{R}^N$ as $k \to +\infty$, then for any $\phi \in L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N, |x|^{-\alpha})$, we have*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot,u_k)\right](x) f_\alpha(x,u_k)\phi(x)\mathrm{d}x = \int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot,u)\right](x) f_\alpha(x,u)\phi(x)\mathrm{d}x, \tag{1.19}$$

where F_α and f_α were introduced in (2).

Proof. Using $\phi = \phi_+ - \phi_-$, it is enough to prove our lemma for $\phi \geqslant 0$. Denote $\tilde{u}_k = u_k - u$ and observe that

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot,u)\right](x) f_\alpha(x,u)\phi(x)\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^N}\left[I_\mu * \frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right]\frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot u(x)}{|x|^\delta}\phi(x)\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^N}\left[I_\mu * \frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right]\frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot u(x)}{|x|^\delta}\phi(x)\mathrm{d}x \\ &+ \int_{\mathbb{R}^N}\left[I_\mu * \frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right]\frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot \tilde{u}(x)}{|x|^\delta}\phi(x)\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^N}\left[I_\mu * \frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right]\frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot\tilde{u}(x)}{|x|^\delta}\phi(x)\mathrm{d}x\\
&+\int_{\mathbb{R}^N}\left[I_\mu * \frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot u(x)}{|x|^\delta}\right]\frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\phi(x)\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\left[I_\mu * \frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right]\frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot u(x)}{|x|^\delta}\phi(x)\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^N}\left[I_\mu * \left(\frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}-\frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right)\right]\frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot u(x)}{|x|^\delta}\phi(x)\mathrm{d}x\\
&+\int_{\mathbb{R}^N}\left[I_\mu * \left(\frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot u(x)}{|x|^\delta}-\frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot\tilde{u}(x)}{|x|^\delta}\right)\right]\frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\phi(x)\mathrm{d}x\\
&+\int_{\mathbb{R}^N}\left[I_\mu * \frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right]\frac{|\tilde{u}(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot\tilde{u}(x)}{|x|^\delta}\phi(x)\mathrm{d}x \qquad (1.20)
\end{aligned}
$$

Now we apply Lemma 1.13 with $q = p_s^\sharp(\delta,\theta,\mu)$ and $r = \alpha/p^*(\beta,\theta)$, by taking $(w_k, w) = (u_k, u)$. We find that, as $k \to +\infty$,

$$
\left|\frac{|u_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}-\frac{|u_k-u|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}-\frac{|u|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right|\to 0 \quad \text{in } L^{\frac{N-\beta}{N-\delta-\mu/2}}(\mathbb{R}^N),
$$

i.e., as $k \to +\infty$,

$$
\left|\frac{|u_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}-\frac{|u_k-u|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right|\to\frac{|u|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta} \quad \text{strongly in } L^{\frac{N-\beta}{N-\delta-\mu/2}}(\mathbb{R}^N). \qquad (1.21)
$$

Analogously applying the same reasoning to $(w_k, w) = (u_k\phi^{1/p_s^\sharp(\delta,\theta,\mu)}, u\phi^{1/p_s^\sharp(\delta,\theta,\mu)})$, we obtain

$$
\begin{aligned}
&\left|\frac{|u_k\phi^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot u_k\phi}{|x|^\delta}-\frac{|u_k\phi^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}-u\phi^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot\left(u_k\phi^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}-u\phi^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}\right)}{|x|^\delta}\right.\\
&\left.-\frac{|u\phi^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}|^{p_s^\sharp(\delta,\theta,\mu)-2}u\phi^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}}{|x|^\delta}\right|\to 0,
\end{aligned}
$$

in $L^{\frac{r}{p_s^\sharp(\delta,\theta,\mu)}}(\mathbb{R}^N) = L^{\frac{N-\beta}{N-\delta-\mu/2}}(\mathbb{R}^N)$, i.e.,

$$
\left|\frac{|u_k|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot u_k\phi}{|x|^\delta}-\frac{|u_k-u|^{p_s^\sharp(\delta,\theta,\mu)-2}(u_k-u)\phi}{|x|^\delta}\right|\to\frac{|u|^{p_s^\sharp(\delta,\theta,\mu)-2}u\phi}{|x|^\delta}
$$

strongly in $L^{\frac{N-\beta}{N-\delta-\mu/2}}(\mathbb{R}^N)$.

Now, we apply Lemma 1.15 with the choices $\hat{p} = \frac{p_s^*(\beta,\theta)}{p_s^\sharp(\delta,\theta,\mu)} = \frac{N-\beta}{N-\delta-\mu/2}$ and

$$
\frac{1}{\hat{r}} = \frac{1}{\hat{p}} + \frac{\mu}{N} - 1 = \frac{N-\delta-\mu/2}{N-\beta} + \frac{\mu}{N} - 1 = \frac{\mu/2-\delta-\mu\beta N+\beta}{N-\beta},
$$

i.e., $\hat{r} = \frac{p_s^\sharp(\delta,\theta,\mu)N - p_s^*(\beta,\theta)(N-\mu)}{p_s^*(\beta,\theta)N} = \frac{N-\beta}{\mu/2-\delta-\mu\beta N+\beta}$, together with limits (1.21). We obtain

$$\left(\int_{\mathbb{R}^N}\left|I_\mu * \left(\frac{|u_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta} - \frac{|u_k-u|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta} - \frac{|u|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right)\right|^{\frac{N-\beta}{\mu/2-\delta-\mu\beta N+\beta}} \mathrm{d}x\right)^{\frac{\mu/2-\delta-\mu\beta N+\beta}{N-\beta}}$$
$$\leqslant C(N,\mu,\hat{p})\left(\int_{\mathbb{R}^N}\left|\frac{|u_k|^{p_\mu^\sharp(\alpha,\theta)}}{|x|^\delta} - \frac{|u_k-u|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta} - \frac{|u|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right|^{\frac{N-\beta}{N-\delta-\mu/2}} \mathrm{d}x\right)^{\frac{N-\delta-\mu/2}{N-\beta}}$$
$$\to 0 \text{ strongly in } L^{\frac{N-\beta}{\mu/2-\delta-\mu\beta N+\beta}}(\mathbb{R}^N)$$

as $k \to +\infty$. Therefore,

$$I_\mu * \left(\frac{|u_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta} - \frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right) \to I_\mu * \frac{|u|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta} \text{ strongly in } L^{\frac{N-\beta}{\mu/2-\delta-\mu\beta N+\beta}}(\mathbb{R}^N) \quad (1.22)$$

as $k \to +\infty$.

In the same way we can obtain

$$I_\mu * \left(\frac{|u_k|^{p_\mu^\sharp(\delta,\theta,\mu)-2}u_k\phi}{|x|^\delta} - \frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-2}\tilde{u}_k\phi}{|x|^\delta}\right) \to I_\mu * \frac{|u|^{p_s^\sharp(\delta,\theta,\mu)-2}u\phi}{|x|^\delta} \text{ strongly in } L^{\frac{N-\beta}{\mu/2-\delta-\mu\beta N+\beta}}(\mathbb{R}^N) \quad (1.23)$$

as $k \to +\infty$.

Since $u_k \rightharpoonup u$ weakly in $L^{p_s^*(\beta,\theta)}(\mathbb{R}^N,|x|^{-\delta})$ as $k \to +\infty$, we also have

$$\begin{cases} |u_k|^{p_s^\sharp(\delta,\theta,\mu)-2}u_k\phi \rightharpoonup |u|^{p_s^\sharp(\delta,\theta,\mu)-2}u\phi \\ |u_k-u|^{p_s^\sharp(\delta,\theta,\mu)} \rightharpoonup 0 \Rightarrow |\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)} \rightharpoonup 0 \qquad \text{in } L^{\frac{N-\beta}{\mu/2-\delta-\mu\beta N+\beta}}(\mathbb{R}^N,|x|^{-\delta}) \\ |\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-2}\tilde{u}_k\phi \rightharpoonup 0 \end{cases} \quad (1.24)$$

Combining (1.22), (1.23) and (1.24) we have

$$\begin{cases} \lim_{k\to\infty}\int_{\mathbb{R}^N}\left[I_\mu * \left(\frac{|u_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta} - \frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right)\right]\left(|\frac{|u_k|^{p_s^\sharp(\delta,\theta,\mu)-2}u_k\phi}{|x|^\delta} - \frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-2}\tilde{u}_k\phi}{|x|^\delta}\right)\mathrm{d}x \\ \qquad = \int_{\mathbb{R}^N}\left(I_\mu * \frac{|u|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right)\frac{|u|^{p_s^\sharp(\delta,\theta,\mu)-2}u\phi}{|x|^\delta}\,\mathrm{d}x\,, \\ \lim_{k\to\infty}\int_{\mathbb{R}^N}\left[I_\mu * \left(\frac{|u_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta} - \frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right)\right]\frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-2}\tilde{u}_k\phi}{|x|^\delta}\,\mathrm{d}x = 0, \\ \lim_{k\to\infty}\int_{\mathbb{R}^N}\left[I_\mu * \left(\frac{|u_k|^{p_s^\sharp(\delta,\theta,\mu)-2}u_k\phi}{|x|^\delta} - \frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-2}\tilde{u}_k\phi}{|x|^\delta}\right)\right]\frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\,\mathrm{d}x = 0. \end{cases} \quad (1.25)$$

By Hölder's inequality together with Lemma 1.15 we obtain

$$\left|\int_{\mathbb{R}^N}\left(I_\mu * \frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right)\frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-2}\tilde{u}_k\phi}{|x|^\delta}\right|$$
$$\leqslant \left|\left[\int_{\mathbb{R}^N}\left(I_\mu * \frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right)^{\frac{N-\beta}{\mu/2-\delta-\mu/\beta N+\beta}}\right]^{\frac{\mu/2-\delta-\mu\beta N+\beta}{N-\beta}}\right| \cdot \left|\left[\int_{\mathbb{R}^N}\left(\frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-1}\phi}{|x|^\delta}\right)^{\frac{p_s^*(\beta,\theta)}{p_s^\sharp(\delta,\theta,\mu)}}\right]^{\frac{p_s^\sharp(\delta,\theta,\mu)}{p_s^*(\beta,\theta)}}\right|$$

$$
\begin{aligned}
&\leqslant C\left[\int_{\mathbb{R}^N}\left(\frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right)^{\frac{N-\beta}{\mu/2-\delta-\mu\beta N+\beta}}\right]^{\frac{\mu/2-\delta-\mu\beta N+\beta}{N-\beta}}\cdot\left|\left[\int_{\mathbb{R}^N}\left(\frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-1}\phi}{|x|^\delta}\right)^{\frac{p_s^*(\beta,\theta)}{p_s^\sharp(\delta,\theta,\mu)}}\right]^{\frac{p_s^\sharp(\delta,\theta,\mu)}{p_s^*(\beta,\theta)}}\right| \\
&= C\left[\int_{\mathbb{R}^N}\frac{|\tilde{u}_k|^{p_s^*(\alpha,\theta)}}{|x|^\delta}\right]^{\frac{\mu/2-\delta-\mu\beta N+\beta}{N-\beta}}\cdot\left|\left[\int_{\mathbb{R}^N}\left(\frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-1}\phi}{|x|^\delta}\right)^{\frac{p_s^*(\beta,\theta)}{p_s^\sharp(\delta,\theta,\mu)}}\right]^{\frac{p_s^\sharp(\delta,\theta,\mu)}{p_s^*(\beta,\theta)}}\right| \\
&= \||\tilde{u}_k|\|_{L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N,|x|^{-\delta})}^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}\cdot\left\||\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-1}\phi\right\|_{L^{\frac{p_s^*(\beta,\theta)}{p_s^\sharp(\delta,\theta,\mu)}}(\mathbb{R}^N,|x|^{-\delta})} \\
&\leqslant C\left\||\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-1}\phi\right\|_{L^{\frac{p_s^*(\beta,\theta)}{p_s^\sharp(\delta,\theta,\mu)}}(\mathbb{R}^N,|x|^{-\delta})}. \qquad (1.26)
\end{aligned}
$$

In the last inequality, we used the assumption that $\{\tilde{u}_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^{p_s^*}(\mathbb{R}^N,|x|^{-\alpha})$ and the fact that the parameters in Lemma 1.13, the variant of Brézis-Lieb lemma, are in the admissible range.

On the other hand, using $q=p_s^\sharp(\delta,\theta,\mu)$ in Lemma 1.14, we have

$$
|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)} \rightharpoonup 0 \text{ weakly in } L^{\frac{p_s^*(\beta,\theta)}{p_s^\sharp(\delta,\theta,\mu)}}(\mathbb{R}^N,|x|^{-\delta})
$$

as $k\to+\infty$; but this is equivalent to

$$
|\tilde{u}_k| \rightharpoonup 0 \text{ weakly in } L^{p_s^*(\beta,\theta)}(\mathbb{R}^N,|x|^{-\delta})
$$

as $k\to+\infty$ which, in turn, is equivalent to

$$
|\tilde{u}_k|^{\frac{p_s^*(\beta,\theta)(p_s^\sharp(\delta,\theta,\mu)-1)}{p_s^\sharp(\delta,\theta,\mu)}} \rightharpoonup 0 \text{ weakly in } L^{\frac{p_s^\sharp(\delta,\theta,\mu)}{p_s^\sharp(\delta,\theta,\mu)-1}}(\mathbb{R}^N,|x|^{-\delta}),
$$

as $k\to+\infty$. Consequently,

$$
\left\||\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-1}\phi\right\|_{L^{\frac{p_s^*(\beta,\theta)}{p_s^\sharp(\delta,\theta,\mu)}}(\mathbb{R}^N,|x|^{-\delta})} = \left(\int_{\mathbb{R}^N}\frac{|\tilde{u}_k|^{\frac{p_s^*(\beta,\theta)(p_s^\sharp(\delta,\theta,\mu)-1)}{p_s^\sharp(\delta,\theta,\mu)}}\phi^{\frac{p_s^*(\beta,\theta)}{p_s^\sharp(\delta,\theta,\mu)}}}{|x|^\delta}\right)^{\frac{p_s^\sharp(\delta,\theta,\mu)}{p_s^*(\beta,\theta)}} \to 0.
$$

Thus, from (1.26), we obtain

$$
\lim_{k\to\infty}\int_{\mathbb{R}^N}\left(I_\mu*\frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right)\frac{|\tilde{u}_k|^{p_s^\sharp(\delta,\theta,\mu)-2}\tilde{u}_k\phi}{|x|^\delta}=0. \qquad (1.27)
$$

Passing to the limit in (1.20), from (1.25) and (1.27) we reach

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{\mathbb{R}^N}\left[I_\mu*F_\alpha(\cdot,u_k)\right](x)f_\alpha(x,u_k)\phi(x)\mathrm{d}x \\
&=\lim_{k\to\infty}\int_{\mathbb{R}^N}\left[I_\mu*\left(\frac{|u_k(x)|^{p_s^\sharp}}{|x|^\delta}-\frac{|\tilde{u}_k(x)|^{p_s^\sharp}}{|x|^\delta}\right)\right]\frac{|u_k(x)|^{p_s^\sharp-2}\cdot u_k(x)}{|x|^\delta}\phi(x)\mathrm{d}x
\end{aligned}
$$

$$+\lim_{k\to\infty}\int_{\mathbb{R}^N}\left[I_\mu*\left(\frac{|u_k(x)|^{p_s^\sharp-2}\cdot u_k(x)}{|x|^\delta}-\frac{|\tilde{u}_k(x)|^{p_s^\sharp-2}\cdot\tilde{u}_k(x)}{|x|^\delta}\right)\right]\frac{|\tilde{u}_k(x)|^{p_s^\sharp}}{|x|^\delta}\phi(x)\mathrm{d}x$$
$$+\lim_{k\to\infty}\int_{\mathbb{R}^N}\left[I_\mu*\frac{|\tilde{u}_k(x)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta}\right]\frac{|\tilde{u}_k(x)|^{p_s^\sharp(\delta,\theta,\mu)-2}\cdot\tilde{u}_k(x)}{|x|^\delta}\phi(x)\mathrm{d}x$$
$$=\int_{\mathbb{R}^N}\left(I_\mu*\frac{|u|^{p_s^\sharp}}{|x|^\delta}\right)\frac{|u|^{p_s^\sharp-2}u}{|x|^\delta}\phi$$
$$=\int_{\mathbb{R}^N}\left[I_\mu*F_\alpha(\cdot,u)\right](x)f_\alpha(x,u)\phi(x)\mathrm{d}x.$$

The lemma is proved. □

Proof of Lemma 1.11

To prove the Caffarelli-Kohn-Nirenberg's inequality stated in Lemma 1.11, first we have to deal with the generalization of the extension problem. The main goal here is to write a formula that extends, to the nonlinear setting, the extension obtained by Caffarelli & Silvestre in the linear setting. More precisely, for $0<s<1$ and $1<p<+\infty$, and $0<sp+\theta<N$, consider the extension problem

$$\begin{cases}[(-(\Delta_x)^s_{p,\theta}]u(x,z)+\dfrac{1-sp-\theta}{z}u_z(x,z)+u_{zz}(x,z)=0 & x\in\mathbb{R}^N,\ z\in\mathbb{R}_+\\ u(x,0)=g(x) & x\in\mathbb{R}^N.\end{cases}\tag{1.28}$$

The solution of this problem can be obtained by the convolution

$$u(x,z)=\int_{\mathbb{R}^N}P(x-\xi,z)g(\xi)\,\mathrm{d}\xi$$

where the Poisson kernel P is given, up to a multiplicative constant, by

$$P(x,z):=\frac{y^{sp+\theta}}{(|x|^2+y^2)^{\frac{N+sp+\theta}{2}}}.$$

By means of these formulas, we define $E_{s,p,\theta}[g](x,z):=u(x,z)$, called the extension operator. This operator allows one to give a representation formula for the fractional p-Laplacian; see del Teso, Castro-Gómez & Vázquez [38, Theorem 3.1].

Proposition 1.19. *Let $0<s<1$, $1<p<+\infty$, $0<sp+\theta<N$, $x_0\in\mathbb{R}^N$. Suppose that $u\in C^2(\mathbb{R}^N)$ is a continuously bounded function. If $1<p<2/(2-s-\theta/p)$, assume additionally that $\nabla u(x_0)\neq 0$. Then the fractional p-Laplacian operator $(-\Delta)^s_{p,\theta}$ can be represented by*

$$(-\Delta)^s_{p,\theta}u(x_0)=\lim_{z\to0}\frac{E_{s,p,\theta}[|u(x_0)-u(\cdot)|^{p-2}(u(x_0)-u(\cdot))](x_0,z)}{z^{sp+\theta}}.\tag{1.29}$$

Now we state an estimate by Sawyer & Wheeden [79, Theorem 1]; see also Muckenhoupt & Wheeden [70, Theorem D].

Lemma 1.20. *Suppose that $0<\tilde{s}<N, 1<\tilde{p}\leqslant\tilde{q}<+\infty, 0<\tilde{s}\tilde{p}+\tilde{\theta}<N, \tilde{p}'=\frac{\tilde{p}}{\tilde{p}-1}$ and that V and W are nonnegative measurable functions on $\mathbb{R}^N$, $N\geqslant1$. If, for some $\sigma>1$,*

$$|Q|^{\frac{\tilde{s}}{N}+\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}}\left(\frac{1}{|Q|}\int_Q V^\sigma\mathrm{d}y\right)^{\frac{1}{\tilde{q}\sigma}}\left(\frac{1}{|Q|}\int_Q W^{(1-\tilde{p}')\sigma}\mathrm{d}y\right)^{\frac{1}{\tilde{p}'\sigma}}\leqslant C_\sigma\tag{1.30}$$

for all cubes $Q \subset \mathbb{R}^N$*, then for any functions* $f \in L^{\tilde{p}}(\mathbb{R}^N, W(y))$*, we have*

$$\left(\int_{\mathbb{R}^N} |E_{\tilde{s},\tilde{p},\tilde{\theta}}[f](y)|^{\tilde{q}} V(y)\mathrm{d}y\right)^{\frac{1}{\tilde{q}}} \leqslant CC_\sigma \left(\int_{\mathbb{R}^N} |f(y)|^{\tilde{p}} W(y)\mathrm{d}y\right)^{\frac{1}{\tilde{p}}}, \tag{1.31}$$

where $C = C(\tilde{p}, \tilde{q}, N)$ *and* $E_{s,p,\theta}$ *is the extension operator denotes the Riesz potential of order* $\tilde{s}$*, namely*

$$E_{\tilde{s},\tilde{p},\tilde{\theta}} f(x) = \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\tilde{s}}}\mathrm{d}y. \tag{1.32}$$

Proof of Lemma 1.11. For $g \in L^p(\mathbb{R}, |x|^{-\alpha})$, we define the operator

$$E_{s,p,\theta}[g](x) := \int_{\mathbb{R}^N} \frac{|g(y)|^{p-2} g(y)}{|x|^{\theta_1}|x-y|^{N-sp}|y|^{\theta_2}}\, \mathrm{d}y\,. \tag{1.33}$$

If

$$\begin{aligned} g(x) &:= (-\Delta)^s_{p,\theta} u(x) \\ &:= 2\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\, \mathrm{d}y\, \mathrm{d}x \end{aligned}$$

then

$$u(x) = E_{s,p,\theta}[g](x) = E_{s,p,\theta}[(-\Delta)^s_{p,\theta} u(x)]. \tag{1.34}$$

First, we take $\tilde{s} = s$, $\tilde{p} = p$, $\max\{p, p^*_s(0,\theta) - 1\} < \tilde{q} < p^*_s(\alpha,\theta)$, $\sigma = \frac{1}{p^*_s(0,\theta)-\tilde{q}} > 1$ and

$$W(y) \equiv 1, \qquad V(y) := \frac{|u(y)|^{p^*_s(\alpha,\theta)-\tilde{q}}}{|y|^\alpha}$$

in Lemma 1.20. Then, the left side of inequality (1.30) becomes

$$|Q|^{\frac{s}{N}+\frac{1}{\tilde{q}}-\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q V^\sigma \mathrm{d}y\right)^{\frac{1}{\tilde{q}\sigma}} = |Q|^{\frac{s}{N}+\frac{1}{\tilde{q}}-\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q \left|\frac{|u(y)|^{p^*_s(\alpha,\theta)-\tilde{q}}}{|y|^\alpha}\right|^{\frac{1}{p^*_s(0,\theta)-\tilde{q}}} \mathrm{d}y\right)^{\frac{p^*_s(0,\theta)-\tilde{q}}{\tilde{q}}}. \tag{1.35}$$

Secondly, we verify that this expression is bounded by a constant dependent only on the parameter σ. To do this, we consider an arbitrary fixed $x \in \mathbb{R}^N$ and, without loss of generality, we can substitute the cube $Q \subset \mathbb{R}^N$ with an open ball $B_R(x)$. For the chosen parameters, we define $t := \frac{\tilde{q}}{p^*_s(\alpha,\theta)}$; then, $0 < [p^*_s(\alpha,\theta) - \tilde{q}]\sigma < 1$ and $\frac{t\sigma\alpha}{1-[p^*_s(\alpha,\theta)-\tilde{q}]\sigma} < N$. Using Hölder's inequality, we deduce that

$$\begin{aligned} R^{-N}\int_{B_R(x)} V^\sigma \mathrm{d}y &= R^{-N}\int_{B_R(x)} \frac{|u(y)|^{(p^*_s(\alpha,\theta)-\tilde{q})\sigma}}{|y|^{\alpha\sigma}}\mathrm{d}y \\ &= R^{-N}\int_{B_R(x)} \frac{1}{|y|^{t\sigma\alpha}} \cdot \frac{|u(y)|^{(p^*_s(\alpha,\theta)-\tilde{q})\sigma}}{|y|^{(1-t)\alpha\sigma}}\mathrm{d}y \\ &\leqslant R^{-N}\left(\int_{B_R(x)} \frac{1}{|y|^{\frac{t\alpha\sigma}{1-[p^*_s(\alpha,\theta)-\tilde{q}]\sigma}}}\mathrm{d}y\right)^{1-[p^*_s(\alpha,\theta)-\tilde{q}]\sigma} \cdot \left(\int_{B_R(x)} \frac{|u|}{|y|^r}\mathrm{d}y\right)^{[p^*_s(\alpha,\theta)-\tilde{q}]\sigma}, \end{aligned}$$

where $r := \frac{(1-t)\alpha}{p_s^*(\alpha,\theta)-\tilde q} = \frac{\alpha}{p_s^*(\alpha,\theta)}$. To evaluate the first integral on the right-hand side of the previous inequality we use the integration in polar coordinates formula and we obtain

$$\begin{aligned}
&R^{-N}\int_{B_R(x)} V^\sigma \mathrm{d}y \\
&\leqslant R^{-N}\left(\int_0^R \tilde w^{N-1-\frac{t\alpha\sigma}{1-[p_s^*(\alpha,\theta)-\tilde q]\sigma}}\mathrm{d}\tilde w\right)^{1-[p_s^*(\alpha,\theta)-\tilde q]\sigma}\cdot\left(\int_{B_R(x)}\frac{|u|}{|y|^r}\mathrm{d}y\right)^{[p_s^*(\alpha,\theta)-\tilde q]\sigma}\\
&= R^{-N}\left(CR^{N-\frac{t\alpha\sigma}{1-[p_s^*(\alpha,\theta)-\tilde q]\sigma}}\right)^{1-[p_s^*(\alpha,\theta)-\tilde q]\sigma}\cdot\left(\int_{B_R(x)}\frac{|u|}{|y|^r}\mathrm{d}y\right)^{[p_s^*(\alpha,\theta)-\tilde q]\sigma}\\
&= CR^{-t\alpha\sigma-N[p_s^*(\alpha,\theta)-\tilde q]\sigma}\left(\int_{B_R(x)}\frac{|u|}{|y|^r}\mathrm{d}y\right)^{[p_s^*(\alpha,\theta)-\tilde q]\sigma}.
\end{aligned}$$

This implies that

$$\left\{R^{-N}\int_{B_R(x)} V^\sigma \mathrm{d}y\right\}^{\frac{1}{\tilde q\sigma}} \leqslant \left\{CR^{-t\alpha\sigma-N[p_s^*(\alpha,\theta)-\tilde q]\sigma}\left(\int_{B_R(x)}\frac{|u|}{|y|^r}\mathrm{d}y\right)^{[p_s^*(\alpha,\theta)-\tilde q]\sigma}\right\}^{\frac{1}{\tilde q\sigma}}.$$

Now we multiply both sides of this inequality by $R^{s+\frac{N}{\tilde q}-\frac{N}{p}}$ to get

$$\begin{aligned}
&R^{s+\frac{N}{\tilde q}-\frac{N}{p}}\left\{R^{-N}\int_{B_R(x)} V^\sigma \mathrm{d}y\right\}^{\frac{1}{\tilde q\sigma}}\\
&\leqslant R^{s+\frac{N}{\tilde q}-\frac{N}{p}}\left\{CR^{-t\alpha\sigma-N[p_s^*(\alpha,\theta)-\tilde q]\sigma}\left(\int_{B_R(x)}\frac{|u|}{|y|^r}\mathrm{d}y\right)^{[p_s^*(\alpha,\theta)-\tilde q]\sigma}\right\}^{\frac{1}{\tilde q\sigma}}\\
&\leqslant C\left\{R^{\left(s+\frac{N}{\tilde q}-\frac{N}{p}\right)\tilde q\sigma}\cdot R^{-t\alpha\sigma-N[p_s^*(\alpha,\theta)-\tilde q]\sigma}\left(\int_{B_R(x)}\frac{|u|}{|y|^r}\mathrm{d}y\right)^{[p_s^*(\alpha,\theta)-\tilde q]\sigma}\right\}^{\frac{1}{\tilde q\sigma}}\\
&= C\left\{R^{\left(s+\frac{N}{\tilde q}-\frac{N}{p}\right)\tilde q\sigma\frac{1}{[p_s^*(\alpha,\theta)-\tilde q]\sigma}}\cdot R^{\frac{-t\alpha\sigma-N[p_s^*(\alpha,\theta)-\tilde q]\sigma}{[p_s^*(\alpha,\theta)-\tilde q]\sigma}}\int_{B_R(x)}\frac{|u|}{|y|^r}\mathrm{d}y\right\}^{\frac{[p_s^*(\alpha,\theta)-\tilde q]\sigma}{\tilde q\sigma}}\\
&= C\left\{R^{\left(s+\frac{N}{\tilde q}-\frac{N}{p}\right)\frac{\tilde q}{p_s^*(\alpha,\theta)-\tilde q}}\cdot R^{\frac{-t\alpha-N[p_s^*(\alpha,\theta)-\tilde q]}{p_s^*(\alpha,\theta)-\tilde q}}\int_{B_R(x)}\frac{|u|}{|y|^r}\mathrm{d}y\right\}^{\frac{p_s^*(\alpha,\theta)-\tilde q}{\tilde q}}\\
&= C\left\{R^{\left(s+\frac{N-t\alpha}{\tilde q}-\frac{N}{p}\right)\frac{\tilde q}{p_s^*(\alpha,\theta)-\tilde q}}\cdot R^{-N}\int_{B_R(x)}\frac{|u|}{|y|^r}\mathrm{d}y\right\}^{\frac{p_s^*(\alpha,\theta)-\tilde q}{\tilde q}}.
\end{aligned}$$

A simple computation shows that

$$\left(s+\frac{N-t\alpha}{\tilde q}-\frac{N}{p}\right)\frac{\tilde q}{p_s^*(\alpha,\theta)-\tilde q} = \frac{N-sp}{p}+r;$$

hence,

$$\begin{aligned}
R^{s+\frac{N}{\tilde q}-\frac{N}{p}}\left\{R^{-N}\int_{B_R(x)} V^\sigma \mathrm{d}y\right\}^{\frac{1}{\tilde q\sigma}} &\leqslant C\left\{R^{\frac{N-sp}{p}+r-N}\int_{B_R(x)}\frac{|u|}{|y|^r}\mathrm{d}y\right\}^{\frac{p_s^*(\alpha,\theta)-\tilde q}{\tilde q}}\\
&= C\|u\|_{L^{1,\frac{N-sp}{p}+r}(\mathbb{R}^N,|y|^{-r})}^{\frac{p_s^*(\alpha,\theta)-\tilde q}{\tilde q}} := C_\sigma
\end{aligned}$$

Using (1.33) and (1.34) and inequality (1.31) in Lemma 1.20, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|u(y)|^{p_s^*(\alpha,\theta)}}{|y|^\alpha}\mathrm{d}y &= \int_{\mathbb{R}^N}|u(y)|^{\tilde q}\frac{|u(y)|^{p_s^*(\alpha,\theta)-\tilde q}}{|y|^\alpha}\mathrm{d}y\\ &= \int_{\mathbb{R}^N}|E_{s,p,\theta}[g](y)|^{\tilde q}V(y)\mathrm{d}y\\ &\leqslant (CC_\sigma)^{\tilde q}\|g\|^{\tilde q}_{L^p(\mathbb{R}^N)}\\ &= C\|u\|^{p_s^*(\alpha,\theta)-\tilde q}_{L_M^{1,\frac{N-sp}{p}+r}(\mathbb{R}^N,|y|^{-r})}\|g\|^{\tilde q}_{L^p(\mathbb{R}^N)}\\ &\leqslant C\|u\|^{p_s^*(\alpha,\theta)-\tilde q}_{L_M^{1,\frac{N-sp}{p}+r}(\mathbb{R}^N,|y|^{-r})}\|u\|^{\tilde q}_{\dot W_\theta^{s,p}(\mathbb{R})}.\end{aligned}$$

Finally, we choose $p\in[1,p_s^*(\alpha,\theta))$ and define $\zeta:=\frac{\tilde q}{p_s^*(\alpha,\theta)}$; for the parameters in the specified intervals we deduce that $\max\left\{\frac{p}{p_s^*(\alpha,\theta)},\frac{p_s^*(\alpha,\theta)-1}{p_s^*(\alpha,\theta)}\right\}<\zeta<1$. Hence, for every function $u\in\dot W_\theta^{s,p}(\mathbb{R}^N)$, it holds the inequality

$$\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{p_s^*(\alpha,\theta)}}{|y|^\alpha}\mathrm{d}y\right)^{\frac{1}{p_s^*(\alpha,\theta)}}\leqslant\|u\|^{1-\zeta}_{L^{\tilde q,\frac{N-sp}{p}\tilde q+\tilde q r}(\mathbb{R}^N,|y|^{-\tilde q r})}\|u\|^{\zeta}_{\dot W_\theta^{s,p}}.$$

This concludes the proof of Lemma 1.11. □

Lemma 1.21. *(Theorem 1 in [73]) Let $s\in(0,1)$, $N>sp+\theta$ and $p_s^*(0,\theta)=\frac{pN}{N-sp-\theta}$. Then there exists a constant $C=C(N,s)$ such that for any $\max\left\{\frac{p}{p_s^*(0,\theta)},\frac{p_s^*(0,\theta)-1}{p_s^*(0,\theta)}\right\}<\zeta<1$ and for any $1\leqslant q<p_s^*(0,\theta)$,*

$$\|u\|_{L^{p_s^*(0,\theta)}(\mathbb{R}^N)}\leqslant C\|u\|^{\zeta}_{\dot W_\theta^{s,p}(\mathbb{R}^N)}\|u\|^{1-\zeta}_{L^{q,\frac{N-sp}{p}q}(\mathbb{R}^N)} \tag{1.36}$$

for all $u\in\dot W_\theta^{s,p}(\mathbb{R}^N)$.

Proof. The proof of this result follows the same steps of the previous proof and is omitted. □

Solving the minimization problems (1.9) and (1.10)

In this section, we deal with a crucial step in the proof of the Theorem 0.1. More precisely, we solve the minimization problems (1.9) and (1.10). Using the embeddings of the fractional Sobolev space into the weighted Lebesgue space and the Morrey space in Lemma 1.6 together with the Caffarelli-Kohn-Nirenberg's inequality in Lemma 1.11, we can prove the existence of minimizers for

$$S_\mu(N,s,\gamma,\alpha)=\inf_{u\in\dot W_\theta^{s,p}(\mathbb{R}^N)\setminus\{0\}}\frac{\|u\|^p}{Q^\sharp(u,u)^{\frac{p}{2p_\mu^\sharp(\delta,\theta,\mu)}}} \tag{1.37}$$

where the quadratic form $Q^\sharp\colon\dot W_\theta^{s,p}(\mathbb{R}^N)\times\dot W_\theta^{s,p}(\mathbb{R}^N)\to\mathbb{R}$ is defined by

$$Q^\sharp(u,v):=\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)}|v(y)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y \tag{1.38}$$

and

$$\Lambda(N,s,\gamma,\beta)=\inf_{u\in\dot W^{s,p}_\theta(\mathbb{R}^N)\setminus\{0\}}\frac{\|u\|^p}{\left(\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)^{\frac{p}{p_s^*(\beta,\theta)}}}. \tag{1.39}$$

We can derive the following results:

Proposition 1.22. *For $s\in(0,1)$ the best constants $S_\mu(N,s,\gamma,\alpha)$ and $\Lambda(N,s,\gamma,\beta)$ verify the following items.*

1. *If $0<\alpha<sp+\theta<N,\mu\in(0,N)$ and $\gamma<\gamma_H$, then $S_\mu(N,s,\gamma,\alpha)$ is attained in $\dot W^{s,p}_\theta(\mathbb{R}^N)$;*
2. *If $0<\beta<sp+\theta<N$ and $\gamma<\gamma_H$, then $\Lambda(N,s,\gamma,\beta)$ is attained in $\dot W^{s,p}_\theta(\mathbb{R}^N)$;*
3. *If $N>sp+\theta,\mu\in(0,N)$ and $0\leqslant\gamma<\gamma_H$, then $S_\mu(N,s,\gamma,0)$ is attained in $\dot W^{s,p}_\theta(\mathbb{R}^N)$;*
4. *If $N>sp+\theta$ and $0\leqslant\gamma<\gamma_H$, then $\Lambda(N,s,\gamma,0)$ is attained in $\dot W^{s,p}_\theta(\mathbb{R}^N)$.*

Proof. 1. If $0<\alpha<sp+\theta<N$ and $\gamma<\gamma_H$, let $\{u_k\}_{k\in\mathbb{N}}\subset\dot W^{s,p}_\theta(\mathbb{R}^N)$ be a minimizing sequence of $S_\mu(N,s,\gamma,\alpha)$ such that

$$Q^\sharp(u_k,u_k)=1,\qquad \|u_k\|^p\to S_\mu(N,s,\gamma,\alpha) \tag{1.40}$$

as $k\to+\infty$. Recall that $r=\frac{\alpha}{p_s^*(\alpha,\theta)}$. The embeddings (1.4) and the Caffarelli-Kohn-Nirenberg's inequality (1.8) imply that

$$\begin{aligned}\|u_k\|_{L_M^{q,\frac{N-sp-\theta}{p}q+qr}(\mathbb{R}^N,|y|^{-pr})}&\leqslant C\|u_k\|_{L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N,|y|^{-\alpha})}\\&\leqslant C_1\|u_k\|^{\zeta}_{\dot W^{s,p}_\theta(\mathbb{R}^N)}\|u_k\|^{1-\zeta}_{L_M^{q,\frac{N-sp-\theta}{p}q+qr}(\mathbb{R}^N,|y|^{-pr})}.\end{aligned}$$

Therefore,

$$\|u_k\|_{L_M^{q,\frac{N-sp-\theta}{p}q+qr}(\mathbb{R}^N,|y|^{-pr})}\leqslant C_1\|u_k\|_{\dot W^{s,p}_\theta(\mathbb{R}^N)}.$$

On the other hand, using Caffarelli-Kohn-Nirenberg's inequality once again, together with the inequality (1.3) and the properties (1.40), we get

$$\begin{aligned}\|u_k\|^{1-\zeta}_{L_M^{q,\frac{N-sp-\theta}{p}q+qr}(\mathbb{R}^N,|y|^{-pr})}&\geqslant C\frac{\|u_k\|_{L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N,|y|^{-\alpha})}}{\|u_k\|^{\zeta}_{\dot W^{s,p}_\theta(\mathbb{R}^N)}}\\&\geqslant C\frac{(Q^\sharp(u_k,u_k))^{\frac{N}{(2N-\mu)p_s^*(\alpha,\theta)}}}{\|u_k\|^{\zeta}_{\dot W^{s,p}_\theta(\mathbb{R}^N)}}\\&=C\frac{1}{\|u_k\|^{\zeta}_{\dot W^{s,p}_\theta(\mathbb{R}^N)}}.\end{aligned}$$

By the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in $\dot W^{s,p}_\theta(\mathbb{R}^N)$, we deduce that

$$\|u_k\|_{L_M^{q,\frac{N-sp-\theta}{p}q+qr}(\mathbb{R}^N,|y|^{-pr})}\geqslant C_2.$$

Putting these results together, we have

$$C_2 \leqslant \|u_k\|_{L_M^{q,\frac{N-sp-\theta}{p}q+qr}(\mathbb{R}^N,|y|^{-pr})} \leqslant C_1. \tag{1.41}$$

For any $k \in \mathbb{N}$ large enough, we may find $\lambda_k > 0$ and $x_k \in \mathbb{R}^N$ such that

$$\lambda_k^{-sp-\theta+pr} \int_{B_{\lambda_k(x_k)}} \frac{|u_k(y)|^p}{|y|^{pr}} \mathrm{d}y > \|u_k\|^p_{L_M^{p,N-sp-\theta+pr}(\mathbb{R}^N,|y|^{-pr})} - \frac{C_3}{2k} \geqslant C_4 > 0$$

for constants $C_3, C_4 \in \mathbb{R}_+$.

Our goal is to pass to the limit as $k \to +\infty$ in the minimizing sequence. To do this, we create another sequence that will help us to control the radius and the centers of these balls. Let

$$v_k(x) = \lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k x)$$

be the appropriate scaling for the class of problems that we consider and define $\tilde{x}_k := \frac{x_k}{\lambda_k}$. Then, using the change of variables $y = \lambda_k x$ with $\mathrm{d}y = \lambda_k^N \,\mathrm{d}x$, we have

$$\begin{aligned}
\int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^p}{|x|^{pr}} \mathrm{d}x &= \int_{B_1\left(\frac{x_k}{\lambda_k}\right)} \frac{\left(\lambda_k^{\frac{N-sp-\theta}{p}} |u_k(\lambda_k x)|\right)^p}{|x|^{pr}} \mathrm{d}x \\
&= \int_{B_1\left(\frac{x_k}{\lambda_k}\right)} \frac{\lambda_k^{N-sp-\theta} |u_k(\lambda_k x)|^p}{|x|^{pr}} \mathrm{d}x \\
&= \int_{B_1\left(\frac{x_k}{\lambda_k}\right)} \lambda_k^{N-sp-\theta} \frac{|u_k(y)|^p}{\left|\frac{y}{\lambda_k}\right|^{pr}} \frac{\mathrm{d}y}{\lambda_k^N} \\
&= \int_{B_1\left(\frac{x_k}{\lambda_k}\right)} \lambda_k^{-sp-\theta+pr} \frac{|u_k(y)|^p}{|y|^{pr}} \mathrm{d}y \\
&= \int_{B_{\lambda_k}(x_k)} \lambda_k^{-sp-\theta+pr} \frac{|u_k(y)|^p}{|y|^{pr}} \mathrm{d}y \geqslant C > 0.
\end{aligned} \tag{1.42}$$

Now we claim that $S_\mu(N, s, \gamma, \alpha)$ is invariant under the previously defined dilation.

In fact, $Q^\sharp(v_k, v_k) = 1$. To show this property, we use the change of variables $\bar{x} = \lambda_k x$ and $\bar{y} = \lambda_k y$, we have

$$\begin{aligned}
Q^\sharp(v_k, v_k) &= \iint_{\mathbb{R}^{2N}} \frac{|v_k(x)|^{p_s^\sharp(\delta,\theta,\mu)} |v_k(y)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta |x-y|^\mu |y|^\delta} \mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathbb{R}^{2N}} \frac{|\lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k x)|^{p_s^\sharp(\delta,\theta,\mu)} |\lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k y)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta |x-y|^\mu |y|^\delta} \mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathbb{R}^{2N}} \lambda_k^{\frac{N-sp-\theta}{p} \cdot 2p_s^\sharp(\delta,\theta,\mu)} \frac{|u_k(\bar{x})|^{p_s^\sharp(\delta,\theta,\mu)} |u_k(\bar{y})|^{p_s^\sharp(\delta,\theta,\mu)}}{\left|\frac{\bar{x}}{\lambda_k}\right|^\delta \left|\frac{\bar{x}}{\lambda_k} - \frac{\bar{y}}{\lambda_k}\right|^\mu \left|\frac{\bar{y}}{\lambda_k}\right|^\delta} \frac{\mathrm{d}\bar{x}}{\lambda_k^N} \frac{\mathrm{d}\bar{y}}{\lambda_k^N} \\
&= \iint_{\mathbb{R}^{2N}} \lambda_k^{\frac{N-sp-\theta}{p} \cdot 2p_s^\sharp(\delta,\theta,\mu) + 2\delta + \mu - 2N} \frac{|u_k(\bar{x})|^{p_s^\sharp(\delta,\theta,\mu)} |u_k(\bar{y})|^{p_s^\sharp(\delta,\theta,\mu)}}{|\bar{x}|^\delta |\bar{x}-\bar{y}|^\mu |\bar{y}|^\delta} \mathrm{d}\bar{x}\mathrm{d}\bar{y}
\end{aligned}$$

$$
\begin{aligned}
&= \iint_{\mathbb{R}^{2N}} \frac{|u_k(\bar{x})|^{p_s^\sharp(\delta,\theta,\mu)}|u_k(\bar{y})|^{p_s^\sharp(\delta,\theta,\mu)}}{|\bar{x}|^{\delta_\mu(\alpha)}|\bar{x}-\bar{y}|^{\mu}|\bar{y}|^{\delta_\mu(\alpha)}}\mathrm{d}\bar{x}\mathrm{d}\bar{y} \\
&= Q^\sharp(u_k,u_k) = 1,
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
&\frac{N-sp-\theta}{p}\cdot 2p_s^\sharp(\delta,\theta,\mu)+2\delta+\mu-2N \\
&\qquad = \frac{N-sp-\theta}{p}\cdot 2\frac{p(N-\delta-\mu/2)}{N-sp-\theta}+2\delta+\mu-2N \\
&\qquad = 2N-2\delta-\mu+2\delta+\mu-2N = 0.
\end{aligned}
$$

Furthermore, $\|v_k\|^p \to S_\mu(N,s,\gamma,\alpha)$. In fact, we know that $\{u_k\}_{k\in\mathbb{N}}$ is a minimizing sequence for $S_\mu(N,s,\gamma,\alpha)$. Using the same change of variables $\bar{x}=\lambda_k x$ and $\bar{y}=\lambda_k y$, we obtain

$$
\begin{aligned}
\|v_k\|^p &= \iint_{\mathbb{R}^{2N}} \frac{|v_k(x)-v_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{|v_k(x)|^p}{|x|^{sp+\theta}}\mathrm{d}x \\
&= \iint_{\mathbb{R}^{2N}} \frac{\left(\lambda_k^{\frac{N-sp-\theta}{p}}|u_k(\lambda_k x)-u_k(\lambda_k y)|\right)^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{\lambda_k^{N-sp-\theta}|u_k(\lambda_k x)|^p}{|x|^{sp+\theta}}\mathrm{d}x \\
&= \iint_{\mathbb{R}^{2N}} \lambda_k^{N-sp-\theta}\frac{|u_k(\bar{x})-u_k(\bar{y})|^p}{\lambda^{-\theta_1}|\bar{x}|^{\theta_1}\lambda_k^{-N-sp}|\bar{x}-\bar{y}|^{N+sp}\lambda^{-\theta_2}|\bar{y}|^{\theta_2}}\frac{\mathrm{d}\bar{x}}{\lambda_k^N}\frac{\mathrm{d}\bar{y}}{\lambda_k^N} \\
&\quad - \gamma\int_{\mathbb{R}^N}\frac{\lambda_k^{N-sp-\theta}|u_k(\bar{x})|^p}{\lambda_k^{-sp-\theta}|\bar{x}|^{sp+\theta}}\frac{\mathrm{d}\bar{x}}{\lambda_k^N} \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_k(\bar{x})-u_k(\bar{y})|^p}{|\bar{x}|^{\theta_1}|\bar{x}-\bar{y}|^{N+sp}|\bar{y}|^{\theta_2}}\mathrm{d}\bar{x}\mathrm{d}\bar{y} - \gamma\int_{\mathbb{R}^N}\frac{|u_k(\bar{x})|^p}{|\bar{x}|^{sp+\theta}}\mathrm{d}\bar{x} \\
&= \|u_k\|^p.
\end{aligned}
$$

And since $\|u_k\|^p \to S_\mu(N,s,\gamma,\alpha)$ as $k\to+\infty$, we deduce that $\|v_k\|^p \to S_\mu(N,s,\gamma,\alpha)$ as $k\to+\infty$.

In this way, the sequence $\{v_k\}_{k\in\mathbb{N}} \subset \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ is also a minimizing sequence for $S_\mu(N,s,\gamma,\alpha)$ such that we have

$$
Q^\sharp(v_k,v_k)=1, \qquad \|v_k\|^p \to S_\mu(N,s,\gamma,\alpha). \tag{1.43}
$$

From inequality (1.42) together with Hölder's inequality,

$$
\begin{aligned}
0 < C &\leqslant \int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^p}{|x|^{pr}}\mathrm{d}x \\
&\leqslant \left(\int_{B_1(\tilde{x}_k)} 1\mathrm{d}x\right)^{1-\frac{p}{p_s^*(\alpha,\theta)}}\left(\int_{B_1(\tilde{x}_k)}\left(\frac{|v_k(x)|^p}{|x|^{pr}}\right)^{\frac{p_s^*(\alpha,\theta)}{p}}\mathrm{d}x\right)^{\frac{p}{p_s^*(\alpha,\theta)}} \\
&\leqslant C\left(\int_{B_1(\tilde{x}_k)}\frac{|v_k(x)|^{p_s^*(\alpha,\theta)}}{|x|^{\alpha}}\mathrm{d}x\right)^{\frac{p}{p_s^*(\alpha,\theta)}}.
\end{aligned}
$$

Therefore,

$$\left(\int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^{p_s^*(\alpha,\theta)}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{p}{p_s^*(\alpha,\theta)}} \geqslant C > 0. \tag{1.44}$$

We claim that the sequence $\{\tilde{x}_k\} \subset \mathbb{R}^N$ of the centers of the balls is bounded. We argue by contradiction and suppose that $|\tilde{x}_k| \to +\infty$ as $k \to +\infty$; then for any $x \in B_1(\tilde{x}_k)$, we have $|x| \geqslant |\tilde{x}_k| - 1$ for $k \in \mathbb{N}$ large enough. By Hölder's inequality, we obtain

$$\begin{aligned}
\int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^{p_s^*(\alpha,\theta)}}{|x|^\alpha}\mathrm{d}x &\leqslant \int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^{p_s^*(\alpha,\theta)}}{(|\tilde{x}_k|-1)^\alpha}\mathrm{d}x \\
&= \frac{1}{(|\tilde{x}_k|-1)^\alpha}\int_{B_1(\tilde{x}_k)} |v_k(x)|^{p_s^*(\alpha,\theta)}\mathrm{d}x \\
&\leqslant \frac{1}{(|\tilde{x}_k|-1)^\alpha}\left(\int_{B_1(\tilde{x}_k)} 1\mathrm{d}x\right)^{\frac{p_s^*(0,\theta)-p_s^*(\alpha,\theta)}{p_s^*(0,\theta)}} \left(\int_{B_1(\tilde{x}_k)} \left(|v_k(x)|^{p_s^*(\alpha,\theta)}\right)^{\frac{p_s^*(0,\theta)}{p_s^*(\alpha,\theta)}}\mathrm{d}x\right)^{\frac{p_s^*(\alpha,\theta)}{p_s^*(0,\theta)}} \\
&\leqslant \frac{C}{(|\tilde{x}_k|-1)^\alpha}\left(\int_{B_1(\tilde{x}_k)} |v_k(x)|^{p_s^*(0,\theta)}\mathrm{d}x\right)^{\frac{p_s^*(\alpha,\theta)}{p_s^*(0,\theta)}} \\
&\leqslant \frac{C}{(|\tilde{x}_k|-1)^\alpha}\left(\int_{\mathbb{R}^N} |v_k(x)|^{p_s^*(0,\theta)}\mathrm{d}x\right)^{\frac{p_s^*(\alpha,\theta)}{p_s^*(0,\theta)}} \\
&= \frac{C}{(|\tilde{x}_k|-1)^\alpha}\|v_k(x)\|_{L^{p_s^*(0,\theta)}}^{p_s^*(\alpha,\theta)}.
\end{aligned}$$

From this inequality, together with the embeddings in Lemma 1.6–(4), we deduce that

$$\begin{aligned}
\int_{B_1(\tilde{x}_k)} \frac{|v_k(x)|^{p_s^*(\alpha,\theta)}}{|x|^\alpha}\mathrm{d}x &\leqslant \frac{C}{(|\tilde{x}_k|-1)^\alpha}\|v_k(x)\|_{L^{p_s^*(\alpha,\theta)}}^{p_s^*(0,\theta)} \\
&\leqslant \frac{C}{(|\tilde{x}_k|-1)^\alpha}\|v_k(x)\|_{\dot{W}_\theta^{s,p}(\mathbb{R}^N)}^{p_s^*(\alpha,\theta)} \\
&\leqslant \frac{C}{(|\tilde{x}_k|-1)^\alpha} \to 0 \qquad (k \to +\infty)
\end{aligned}$$

where we used the boundedness of the minimizing sequence $\{v_k\}_{k\in\mathbb{N}} \subset \dot{W}_\theta^{s,p}(\mathbb{R}^N)$. This is a contradiction with inequality (1.44) and this implies that the sequence $\{\tilde{x}_k\} \subset \mathbb{R}^N$ is bounded.

From inequality (1.42) and the boundedness of the sequence $\{\tilde{x}_k\} \subset \mathbb{R}^N$ of the centers of the balls, we may find $R > 0$ such that $B_R(0)$ contains all balls of center $\tilde{x}_k$ and radius 1; moreover, with

$$\int_{B_R(0)} \frac{|v_k(x)|^p}{|x|^{pr}}\mathrm{d}x \geqslant C_1 > 0. \tag{1.45}$$

Since $\|v_k\| = \|u_k\| \leqslant C$ for $k \in \mathbb{N}$ large enough, there exists a function $v \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ such that

$$v_k \rightharpoonup v \quad \text{in } \dot{W}_\theta^{s,p}(\mathbb{R}^N), \qquad v_k \to v \text{ a.e.} \quad \text{on } \mathbb{R}^N, \tag{1.46}$$

as $k \to +\infty$, up to subsequences. According to Lemma 1.12, we have

$$\frac{u_k}{|x|^r} \to \frac{u}{|x|^r} \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N);$$

hence,

$$\int_{B_R(0)} \frac{|v(x)|^p}{|x|^{pr}} \mathrm{d}x \geqslant C_1 > 0,$$

and we deduce that $v \not\equiv 0$.

We may verify by Lemma 1.17 that

$$1 = Q^\sharp(v_k, v_k) = Q^\sharp(v_k - v, v_k - v) + Q^\sharp(v, v) + o(1). \tag{1.47}$$

By definition (1.37), by weak convergence $v_k \rightharpoonup v$ in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$ together with Brézis-Lieb's lemma and by the estimate (1.47), we have

$$\begin{aligned}
S_\mu(N,s,\gamma,\alpha) &= \lim_{k\to\infty} \|v_k\|^p \\
&= \|v\|^p + \lim_{k\to\infty} \|v_k - v\|^p \\
&\geqslant S_\mu(N,s,\gamma,\alpha)(Q^\sharp(v,v))^{\frac{p}{p^\sharp_s(\delta,\theta,\mu)}} \\
&\quad + S_\mu(N,s,\gamma,\alpha)\left(\lim_{k\to\infty} Q^\sharp(v_k - v, v_k - v)\right)^{\frac{p}{p^\sharp_s(\delta,\theta,\mu)}} \\
&\geqslant S_\mu(N,s,\gamma,\alpha)\Big(Q^\sharp(v,v) + \lim_{k\to\infty} Q^\sharp(v_k - v, v_k - v)\Big)^{\frac{p}{p^\sharp(\delta,\theta,\mu)}} \\
&= S_\mu(N,s,\gamma,\alpha),
\end{aligned}$$

where in the last but one passage above we used the inequality

$$(a+b)^q \leqslant a^q + b^q, \tag{1.48}$$

valid for all $a, b \in \mathbb{R}^*_+$ and $q > 1$. So we have equality in all passages, that is,

$$Q^\sharp(v,v) = 1, \qquad \lim_{k\to\infty} Q^\sharp(v_k - v, v_k - v) = 0, \tag{1.49}$$

since $v \not\equiv 0$. It turns out that, since

$$S_\mu(N,s,\gamma,\alpha) = \|v\|^p + \lim_{k\to\infty} \|v_k - v\|^p,$$

then

$$S_\mu(N,s,\gamma,\alpha) = \|v\|^p \qquad \text{and} \qquad \lim_{k\to\infty} \|v_k - v\|^p = 0.$$

Finally, by inequality

$$\iint_{\mathbb{R}^{2N}} \frac{||u(x)| - |u(y)||^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \leqslant \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y$$

we deduce that $|v| \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ is also a minimizer for $S_\mu(N,s,\gamma,\alpha)$; so we can assume that $v \geqslant 0$. Thus, $S_\mu(N,s,\gamma,\alpha)$ is achieved by a non-negative function in the case $0 < \alpha < sp + \theta$ and $\gamma < \gamma_H$.

2. For $0 < \beta < sp + \theta < N$ and $\gamma < \gamma_H$, let $\{u_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ be a minimizing sequence for $\Lambda(N,s,\gamma,\beta)$ such that

$$\int_{\mathbb{R}^N} \frac{|u_k|^{p_s^*(\alpha,\theta)}}{|x|^\alpha} \mathrm{d}x = 1, \qquad \|u_k\|^p \to \Lambda(N,s,\gamma,\beta)$$

as $k \to +\infty$.

Now we claim that $\Lambda(N,s,\gamma,\beta)$ is invariant under the previously defined dilation.

Let $v_k(x) = \lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k x)$ and $\tilde{x}_k = \frac{x_k}{\lambda_k}$ as in the previous case. In this way, the sequence $\{v_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ is also a minimizing sequence for $\Lambda(N,s,\gamma,\beta)$ such that we have

$$\int_{\mathbb{R}^N} \frac{|v_k|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x = 1, \qquad \|v_k\|^p \to \Lambda(N,s,\gamma,\beta). \tag{1.50}$$

In fact, using variable change $\bar{x} = \lambda_k x$, for every $k \in \mathbb{N}$ we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|v_k|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x &= \int_{\mathbb{R}^N} \frac{|\lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k x)|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \frac{\lambda_k^{N-\beta} |u_k(\lambda_k x)|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \frac{\lambda_k^{N-\beta} u_k(\bar{x})^{p_s^*(\beta,\theta)}}{\lambda_k^{-\beta} |\bar{x}|^\beta} \frac{\mathrm{d}\bar{x}}{\lambda_k^N} \\
&= \int_{\mathbb{R}^N} \frac{|u_k|^{p_s^*(\beta,\theta)}}{|\bar{x}|^\beta} \mathrm{d}\bar{x} = 1.
\end{aligned}$$

We have already shown that $\|v_k\| = \|u_k\|$ for every $k \in \mathbb{N}$. Hence, $\|v_k\|^p \to \Lambda(N,s,\gamma,\beta)$.

We claim that the sequence $\{\tilde{x}_k\} \subset \mathbb{R}^N$ is bounded and the proof follows the same steps already presented. From this boundedness and inequality (1.42), we may find $R > 0$ such that $B_R(0)$ contains all the unitary balls $B_1(\tilde{x}_k)$ centered in $\tilde{x}_k$ and

$$\int_{B_R(0)} \frac{|v_k(x)|^p}{|x|^{pr}} \mathrm{d}x \geqslant C_1 > 0. \tag{1.51}$$

Since $\|v_k\| = \|u_k\| \leqslant C$, there exists a $v \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ such that

$$v_k \rightharpoonup v \quad \text{in } \dot{W}^{s,p}_\theta(\mathbb{R}^N), \qquad v_k \to v \text{ a.e.} \quad \text{on } \mathbb{R}^N, \tag{1.52}$$

as $k \to +\infty$, up to subsequences. According to Lemma 1.12, we have

$$\frac{v_k}{|x|^r} \to \frac{v}{|x|^r} \qquad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N),$$

as $k \to +\infty$, where $r = \frac{\beta}{p_s^*(\beta,\theta)}$. Therefore,

$$\int_{B_R(0)} \frac{|v(x)|^p}{|x|^{pr}} \mathrm{d}x \geqslant C_1 > 0,$$

and we deduce that $v \not\equiv 0$.

We may verify by Lemma 1.13 that, if $q = p_s^*(\beta,\theta)$ and $\delta = \beta$, then

$$1 = \int_{\mathbb{R}^N} \frac{|v_k|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x = \int_{\mathbb{R}^N} \frac{|v_k - v|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x + \int_{\mathbb{R}^N} \frac{|v|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x + o(1).$$

By definition (1.39) and by weak convergence $v_k \rightharpoonup v$ in $\dot{W}_\theta^{s,p}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned}
\Lambda(N,s,\gamma,\beta) &= \lim_{k\to\infty} \|v_k\|^p \\
&= \|v\|^p + \lim_{k\to\infty} \|v_k - v\|^p \\
&\geqslant \Lambda(N,s,\gamma,\beta) \left(\int_{\mathbb{R}^N} \frac{|v|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x \right)^{\frac{p}{p_s^*(\beta,\theta)}} \\
&+ \Lambda(N,s,\gamma,\beta) \Big(\lim_{k\to\infty} \int_{\mathbb{R}^N} \frac{|v_k - v|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x \Big)^{\frac{p}{p_s^*(\beta,\theta)}} \\
&\geqslant \Lambda(N,s,\gamma,\beta) \Big(\int_{\mathbb{R}^N} \frac{|v|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x + \lim_{k\to\infty} \int_{\mathbb{R}^N} \frac{|v_k - v|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x \Big)^{\frac{p}{p_s^*(\beta,\theta)}} \\
&= \Lambda(N,s,\gamma,\beta)
\end{aligned}$$

where we used the inequality (1.48). So we have equality in all passages, that is,

$$\int_{\mathbb{R}^N} \frac{|v|^{p_s^*(\alpha,\theta)}}{|x|^\beta} \mathrm{d}x = 1, \qquad \lim_{k\to\infty} \int_{\mathbb{R}^N} \frac{|v_k - v|^{p_s^*(\alpha,\theta)}}{|x|^\beta} \mathrm{d}x = 0, \tag{1.53}$$

since $v \not\equiv 0$. It turns out that, since

$$\Lambda(N,s,\gamma,\beta) = \|v\|^p + \lim_{k\to\infty} \|v_k - v\|^p,$$

then

$$\Lambda(N,s,\gamma,\beta) = \|v\|^p \qquad \text{and} \qquad \lim_{k\to\infty} \|v_k - v\|^p = 0.$$

As in the previous case, we deduce that $|v| \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ is also a minimizer for $\Lambda(N,s,\gamma,\beta)$ is achieved by a non-negative function in the case $0 < \beta < sp + \theta$ and $\gamma < \gamma_H$.

3. In the case $\alpha = 0$ and $0 \leqslant \gamma < \gamma_H$, we were inspired by the method introduced by Filippucci et. al [42] and Dipierro et. al. [40]. Let $\{u_k\}_{k\in\mathbb{N}} \subset \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ be a minimizing sequence for $S_\mu(N,s,\gamma,0)$. Without loss of generality, we can choose this sequence such that

$$Q^\sharp(u_k,u_k) = 1, \qquad S_\mu(N,s,\gamma,0) \leqslant \|u_k\|^p < S_\mu(N,s,\gamma,0) + \frac{1}{k}. \tag{1.54}$$

Indeed, by definition (1.37), if we normalize $Q^\sharp(u_k,u_k) = 1$, then

$$S_\mu(N,s,\gamma,0) \leqslant \frac{\|u_k\|^p}{(Q^\sharp(u_k,u_k))^{\frac{p}{2p_\mu^\sharp(0,\theta)}}} \leqslant \|u_k\|^p < S_\mu(N,s,\gamma,0) + \frac{1}{k}$$

for $k \in \mathbb{N}$ large enough. By inequality

$$\iint_{\mathbb{R}^{2N}} \frac{||u_k(x)|^* - |u_k(y)|^*|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \leqslant \iint_{\mathbb{R}^{2N}} \frac{|u_k(x) - u_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y, \tag{1.55}$$

where $|u_k|^*$ is the symmetric decreasing rearrangement of $|u_k|$, we deduce that $|u_k|^* \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ is also a minimizer for $S_\mu(N,s,\gamma,\alpha)$; so we can assume that $u_k \geqslant 0$.

Furthermore,

$$1 = Q^\sharp(|u_k|,|u_k|) \leqslant Q^\sharp(|u_k|^*,|u_k|^*) \tag{1.56}$$

and

$$\int_{\mathbb{R}^N} \frac{|u_k|^p}{|x|^{sp+\theta}}\mathrm{d}x \leqslant \int_{\mathbb{R}^N} \frac{||u_k|^*|^p}{|x|^{sp+\theta}}\mathrm{d}x \tag{1.57}$$

Denoting $w_k \coloneqq |u_k|^*$, we have that v_k is radially symmetric and decreasing. Since $0 \leqslant \gamma < \gamma_H$, by the definition of S_μ and by inequalities (1.55), (1.56) and (1.57), we deduce that

$$\begin{aligned}
S_\mu \leqslant S_{\mu,\mathrm{rad}} \coloneqq & \inf_{w_k\in \dot{W}^{s,p}_\theta(\mathbb{R}^N)\setminus\{0\}} \frac{\iint_{\mathbb{R}^{2N}} \frac{|w_k(x)-w_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{|w_k|^p}{|x|^{sp+\theta}}\mathrm{d}x}{Q^\sharp(w_k,w_k)^{\frac{p}{2p^\sharp_\mu(0,\theta)}}} \\
&\leqslant \frac{\iint_{\mathbb{R}^{2N}} \frac{|w_k(x)-w_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{|w_k|^p}{|x|^{sp+\theta}}\mathrm{d}x}{Q^\sharp(w_k,w_k)^{\frac{p}{2p^\sharp_\mu(0,\theta)}}} \\
&\leqslant \frac{\iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{|u_k|^p}{|x|^{sp+\theta}}\mathrm{d}x}{Q^\sharp(u_k,u_k)^{\frac{p}{2p^\sharp_\mu(0,\theta)}}} \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)-u_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{|u_k|^p}{|x|^{sp+\theta}}\mathrm{d}x \\
&= \|u_k\|^p < S_\mu(N,s,\gamma,0) + \frac{1}{k},
\end{aligned}$$

for $k\in\mathbb{N}$ large enough, where in the last passage we used inequality (1.54). Therefore, $\{w_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ is a minimizing sequence of $S_\mu(N,s,\gamma,0)$ and $\{\|w_k\|\}_{k\in\mathbb{N}} \subset \mathbb{R}$ is a uniformly bounded sequence. Noticing that $Q^\sharp(w_k,w_k) \geqslant 1$, the embeddings

$$\dot{W}^{s,p}_\theta(\mathbb{R}^N) \hookrightarrow L^{p^*_s(0,\theta)}(\mathbb{R}^N) \hookrightarrow L^{p,N-sp}_M(\mathbb{R}^N),$$

together Lemma 1.21 and the Caffarelli-Kohn-Nirenberg's inequality (1.8) imply that

$$\begin{aligned}
\|w_k\|_{L_M^{q,\frac{N-sp-\theta}{p}q}(\mathbb{R}^N)} &\leqslant C\|w_k\|_{L^{p^*_s(0,\theta)}(\mathbb{R}^N,|y|^{-\alpha})} \\
&\leqslant C\|w_k\|^\zeta_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)}\|w_k\|^{1-\zeta}_{L_M^{q,\frac{N-sp-\theta}{p}q}(\mathbb{R}^N)}.
\end{aligned}$$

Therefore,

$$\|w_k\|_{L_M^{q,\frac{N-sp-\theta}{p}q}(\mathbb{R}^N)} \leqslant C_1\|w_k\|_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)}.$$

On the other hand, using Caffarelli-Kohn-Nirenberg's inequality once again, together with the inequality (1.3) and the properties (1.40), we get

$$\|w_k\|^{1-\zeta}_{L_M^{q,\frac{N-sp-\theta}{p}q}(\mathbb{R}^N)} \geqslant C\frac{\|w_k\|_{L^{p^*_s(0,\theta)}(\mathbb{R}^N)}}{\|w_k\|^\zeta_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)}}$$

$$\geqslant C\frac{(Q^\sharp(w_k,w_k))^{\frac{N}{(2N-\mu)p_s^*(0,\theta)}}}{\|w_k\|^\zeta_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)}}$$
$$= C\frac{1}{\|w_k\|^\zeta_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)}}.$$

By the boundedness of the sequence $\{w_k\}_{k\in\mathbb{N}}$ in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$ and the previous inequality, we deduce that there exists a positive constant $C_2>0$ such that

$$\|w_k\|_{L_M^{q,\frac{N-sp-\theta}{p}q}(\mathbb{R}^N)}\geqslant C_2.$$

Putting these results together, we have

$$C_2\leqslant\|u_k\|_{L_M^{q,\frac{N-sp-\theta}{p}q}(\mathbb{R}^N)}\leqslant C_1.$$

Using this inequality, we may find $\lambda_k>0$ and $x_k\in\mathbb{R}^N$ such that

$$\lambda_k^{-sp-\theta}\int_{B_{\lambda_k}(x_k)}|w_k(y)|^p\mathrm{d}y>\|w_k\|^p_{L_M^{p,N-sp-\theta}(\mathbb{R}^N)}-\frac{C}{2k}\geqslant C>0.$$

Letting $v_k(x)=\lambda_k^{\frac{N-sp-\theta}{p}}w_k(\lambda_k x)$ and $\tilde{x}_k=\frac{x_k}{\lambda_k}$, we see that $\{v_k\}_{k\in\mathbb{N}}\subset\dot{W}^{s,p}_\theta(\mathbb{R}^N)$ is also a minimizing sequence of $S_\mu(N,s,\gamma,0)$ and satisfies

$$\int_{B_1(\tilde{x}_k)}|v_k(x)|^p\mathrm{d}x\geqslant C>0. \tag{1.58}$$

In fact, using the change of variables $y=\lambda_k x$, we obtain

$$\begin{aligned}\int_{B_1(\tilde{x}_k)}|v_k(x)|^p\mathrm{d}x&=\int_{B_1(\tilde{x}_k)}\left|\lambda_k^{\frac{N-sp-\theta}{p}}w_k(\lambda_k x)\right|^p\mathrm{d}x\\&=\int_{B_1(\tilde{x}_k)}\lambda_k^{N-sp-\theta}|w_k(\lambda_k x)|^p\mathrm{d}x\\&=\int_{B_1\left(\frac{y_k}{\lambda_k}\right)}\lambda_k^{N-sp-\theta}|w_k(y)|^p\frac{\mathrm{d}y}{\lambda_k^N}\\&=\int_{B_1\left(\frac{y_k}{\lambda_k}\right)}\lambda_k^{-sp-\theta}|w_k(y)|^p\mathrm{d}y\\&=\int_{B_1(\tilde{y}_k)}\lambda_k^{-sp-\theta}|w_k(y)|^p\mathrm{d}y\geqslant C>0.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|v_k\|^p&=\iint_{\mathbb{R}^{2N}}\frac{|v_k(x)-v_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y-\gamma\int_{\mathbb{R}^N}\frac{|v_k(x)|^p}{|x|^{sp+\theta}}\mathrm{d}x\\&=\iint_{\mathbb{R}^{2N}}\frac{\left(\lambda_k^{\frac{N-sp-\theta}{p}}|w_k(\lambda_k x)-w_k(\lambda_k y)|\right)^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y-\gamma\int_{\mathbb{R}^N}\frac{\lambda_k^{N-sp-\theta}|w_k(\lambda_k x)|^p}{|x|^{sp+\theta}}\mathrm{d}x.\end{aligned}$$

Using the change of variables $\bar{x}=\lambda_k x$ and $\bar{y}=\lambda_k y$, we obtain

$$\|v_k\|^p=\iint_{\mathbb{R}^{2N}}\lambda_k^{N-sp-\theta}\frac{|w_k(\bar{x})-w_k(\bar{y})|^p}{\lambda^{-\theta_1}|\bar{x}|^{\theta_1}\lambda_k^{-N-sp}|\bar{x}-\bar{y}|^{N+sp}\lambda^{-\theta_2}|\bar{y}|^{\theta_2}}\frac{\mathrm{d}\bar{x}}{\lambda_k^N}\frac{\mathrm{d}\bar{y}}{\lambda_k^N}$$

$$
\begin{aligned}
&- \gamma \int_{\mathbb{R}^N} \frac{\lambda_k^{N-sp-\theta} |w_k(\bar{x})|^p}{\lambda_k^{-sp-\theta} |\bar{x}|^{sp+\theta}} \frac{\mathrm{d}\bar{x}}{\lambda_k^N} \\
&= \iint_{\mathbb{R}^{2N}} \frac{|w_k(\bar{x}) - w_k(\bar{y})|^p}{|\bar{x}|^{\theta_1} |\bar{x} - \bar{y}|^{N+sp} |\bar{y}|^{\theta_2}} \mathrm{d}\bar{x}\mathrm{d}\bar{y} - \gamma \int_{\mathbb{R}^N} \frac{|w_k(\bar{x})|^p}{|\bar{x}|^{sp+\theta}} \mathrm{d}\bar{x} \\
&= \|w_k\|^p.
\end{aligned}
$$

Since $\|v_k\| = \|w_k\| \leqslant C$, there exists $v \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ such that $v_k \rightharpoonup v$ in $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$ weakly as $k \to +\infty$ up to subsequences.

Now, we need to prove $v \not\equiv 0$. For this purpose, we will consider separatelly the cases where $\{\tilde{x}_k\}_{k\in\mathbb{N}}$ unbounded and $\{\tilde{x}_k\}_{k\in\mathbb{N}}$ bounded.

Case (1). If $\{\tilde{x}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ is an unbounded sequence, we assume that $|\tilde{x}_k| \to +\infty$ up to a subsequence. Since the sequence $\{v_k(x)\}_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)}$ is radially symmetric and decreasing, from inequality (1.58), for all $k \in \mathbb{N}$ we have that

$$
\int_{B_2(0)} |v_k(x)|^p \mathrm{d}x \geqslant \int_{B_1(0)} |v_k(x + \tilde{x}_k)|^p \mathrm{d}x = \int_{B_1(\tilde{x}_k)} |v_k(x)|^p \mathrm{d}x \geqslant C > 0.
$$

Since the embedding $\dot{W}^{s,p}_\theta(\mathbb{R}^N) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^N)$ is compact, we have

$$
\int_{B_2(0)} |v(x)|^p \mathrm{d}x \geqslant C > 0.
$$

So, in the unbounded case we have $v \not\equiv 0$.

Case (2). If $\{\tilde{x}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ is a bounded sequence, from (1.58) we may find $R > 0$ such that

$$
\int_{B_R(0)} |v_k(x)|^p \mathrm{d}x \geqslant C > 0,
$$

and from this inequality we deduce that

$$
\int_{B_R(0)} |v(x)|^p \mathrm{d}x \geqslant C > 0.
$$

Thus, in the bounded case we also have $v \not\equiv 0$.

4. The proof of this item is similar to the last item and is omitted. □

Existence of Palais-Smale sequence

We shall now use the minimizers of $S_\mu = S_\mu(N, s, p, \theta, \alpha)$ and $\Lambda = \Lambda(N, s, p, \theta, \gamma, \beta)$ obtained in Proposition 1.22 to prove the existence of a nontrivial weak solution for equation (1). Recall that the energy functional associated to (1) is

$$
\begin{aligned}
I(u) = \frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x|^{\theta_1} |x - y|^{N+sp} |y|^{\theta_2}} \mathrm{d}x\mathrm{d}y - \frac{\gamma}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{sp+\theta}} \mathrm{d}x \\
- \frac{1}{p_s^*(\beta, \theta)} \int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x - \frac{1}{2p_s^\sharp(\delta, \theta, \mu)} \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)} |u(y)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta |x - y|^\mu |y|^\delta} \mathrm{d}x\mathrm{d}y,
\end{aligned}
\tag{1.59}
$$

for all $u \in \dot{W}^{s,p}_{\theta}(\mathbb{R}^N)$. The fractional Sobolev and fractional Hardy-Sobolev inequalities imply that $I \in C^1(\dot{W}^{s,p}_{\theta}(\mathbb{R}^N), \mathbb{R})$ and that

$$\langle I'(u), \phi\rangle = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y - \gamma\int_{\mathbb{R}^N}\frac{|u|^{p-2}u\phi}{|x|^{sp+\theta}}\mathrm{d}x - \int_{\mathbb{R}^N}\frac{|u|^{p^*_s(\beta,\theta)-2}u(x)\phi}{|x|^{\beta}}\mathrm{d}x + \iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{p^\sharp_s(\delta,\theta,\mu)}|u(y)|^{p^\sharp_s(\delta,\theta,\mu)}}{|x|^{\delta}|x-y|^{\mu}|y|^{\delta}}\mathrm{d}x\mathrm{d}y.$$

Note that a nontrivial critical point of I is a nontrivial weak solution to equation (1).

Recall that a Palais-Smale sequence for the energy functional I at the level $c \in \mathbb{R}$ is a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_{\theta}(\mathbb{R}^N)$ such that

$$\lim_{k\to+\infty} I(u_k) = c \quad \text{and} \quad \lim_{k\to+\infty} I'(u_k) = 0 \quad \text{strongly in } \dot{W}^{s,p}_{\theta}(\mathbb{R}^N)'. \tag{1.60}$$

This sequence is refered to as a $(PS)_c$ sequence.

Now we state a result that ensures the existence of a Palais-Smale sequence for the energy functional.

Proposition 1.23. *Let $s \in (0,1), 0 < \alpha, \beta < sp+\theta < N, \mu \in (0,N)$ and $\gamma < \gamma_H$. Consider the functional $I\colon \dot{W}^{s,p}_{\theta}(\mathbb{R}^N) \to \mathbb{R}$ defined in (1.59) on the Banach space $\dot{W}^{s,p}_{\theta}(\mathbb{R}^N)$. Then there exists a $(PS)_c$ sequence $\{u_k\} \subset \dot{W}^{s,p}_{\theta}(\mathbb{R}^N)$ for I at some level $c \in (0, c^*)$, where*

$$c^* := \min\left\{\left(\frac{1}{p} - \frac{1}{2p^\sharp_s(\delta,\theta,\mu)}\right) S_\mu^{\frac{2p^\sharp_s(\delta,\theta,\mu)}{2p^\sharp_s(\delta,\theta,\mu)-p}}, \left(\frac{1}{p} - \frac{1}{p^*_s(\beta,\theta)}\right)\Lambda^{\frac{p^*_s(\beta,\theta)}{p^*_s(\beta,\theta)-p}}\right\}. \tag{1.61}$$

To prove Proposition 1.23 we need the following version of the mountain pass theorem by Ambrosetti and Rabinowitz [12].

Lemma 1.24. *(Montain Pass Lemma) Let $(E, \|\cdot\|)$ be a Banach space and let $I \in C^1(E,\mathbb{R})$ a functional such that the following conditions are satisfied:*

(1) *$I(0) = 0$;*

(2) *There exist $\rho > 0$ and $r > 0$ such that $I(u) \geqslant \rho$ for all $u \in E$ with $\|u\| = r$;*

(3) *There exist $v_0 \in E$ such that $\lim_{t\to+\infty} \sup I(tv_0) < 0$. Let $t_0 > 0$ be such that $\|t_0 v_0\| > r$ and $I(t_0 v_0) < 0$; define*

$$c := \inf_{g\in\Gamma} \sup_{t\in[0,1]} I(g(t)),$$

where

$$\Gamma := \left\{g \in C^0([0,1], E)\colon g(0) = 0, g(1) = t_0 v_0\right\}.$$

Then $c \geqslant \rho > 0$, and there exists a $(PS)_c$ sequence $\{u_k\} \subset E$ for I at level c, i.e.,

$$\lim_{k\to+\infty} I(u_k) = c \quad \text{and} \quad \lim_{k\to+\infty} I'(u_k) = 0 \quad \text{strongly in } E'.$$

The proof of Proposition 1.23 follows from the next two lemmas.

Lemma 1.25. *The functional I verifies the assumptions of Lemma 1.24.*

Proof. Clearly, we have $I(0) = 0$. We now verify the second assumption of Lemma 1.24. Recalling the definition (1.38) of the quadratic form $Q^\sharp$ and using inequality (1.3), for any $u \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ we obtain

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|^p - \frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x - \frac{1}{2p_\mu^\sharp(\alpha,\theta)}Q^\sharp(u,u) \\ &\geqslant \frac{1}{p}\|u\|^p - C_1\|u\|^{p_s^*(\beta,\theta)} - \frac{1}{2p_\mu^\sharp(\alpha,\theta)}Q^\sharp(u,u) \\ &\geqslant \frac{1}{p}\|u\|^p - C_1\|u\|^{p_s^*(\beta,\theta)} - C_2\|u\|^{2p_s^\sharp(\delta,\theta,\mu)}. \end{aligned}$$

Since $s \in (0,1), 0 < \alpha, \beta < sp + \theta < N$ and $\mu \in (0,N)$, we have that $p_s^*(\beta,\theta) > p$ and $2p_s^\sharp(\delta,\theta,\mu) > p_s^*(\alpha,\theta) > p$. Therefore, there exists $r > 0$ small enough such that

$$\inf_{\|u\|=r} I(u) > 0 = I(0),$$

so item (2) of Lemma 1.24 are satisfied.

For $u \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ and $t \in \mathbb{R}_+$, we have

$$I(tu) = \frac{t^p}{p}\|u\|^p - \frac{t^{p_s^*(\beta,\theta)}}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x - \frac{t^{2p_\mu^\sharp(\alpha,\theta)}}{2p_s^\sharp(\delta,\theta,\mu)}Q^\sharp(u,u);$$

since $2p_s^\sharp(\delta,\theta,\mu) > p_s^*(\alpha,\theta) > p$, we deduce that

$$\lim_{t\to+\infty} I(tu) = -\infty \quad \text{for any } u \in \dot{W}_\theta^{s,p}(\mathbb{R}^N).$$

Consequently, for any fixed $v_0 \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$, there exists $t_{v_0} > 0$ such that $\|t_{v_0}v_0\| > r$ and $I(t_{v_0}v_0) < 0$. Thus, item (3) of Lemma 1.24 is satisfied. □

From Lemma 1.25 above, we guarante by Lemma 1.24 the existence of a Palais-Smale sequence $\{u_k\} \subset \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ such that

$$\lim_{k\to+\infty} I(u_k) = c \quad \text{and} \quad \lim_{k\to+\infty} I'(u_k) = 0 \quad \text{strongly in } \dot{W}_\theta^{s,p}(\mathbb{R}^N)'.$$

Moreover, by the definition of c we deduce that $c \geqslant \rho > 0$. Therefore $c > 0$ for all function $u \in \dot{W}_\theta^{s,p}(\mathbb{R}^N) \setminus \{0\}$.

Lemma 1.26. *Let $\mu \in (0,N)$ and $0 < \alpha < sp + \theta$. Then, there exists a function $u \in \dot{W}_\theta^{s,p}(\mathbb{R}^N) \setminus \{0\}$ such that the level c of the functional I satisfies $0 < c < c^*$, where c^* is defined as in (1.61).*

Proof. Using (1) and (2) in Proposition 1.22, we obtain the minimizers $U_{\gamma,\alpha} \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ for $S_\mu(N,s,\gamma,\alpha)$ and $V_{\gamma,\beta} \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ for $\Lambda(N,s,\gamma,\beta)$, respectively. Thus, there exist a function $v_0 \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ defined by

$$v_0(x) = \begin{cases} U_{\gamma,\alpha}(x), & \text{if } \dfrac{2p_s^\sharp(\delta,\theta,\mu) - p}{2pp_s^\sharp(\delta,\theta,\mu)}S_\mu(N,s,\gamma,\alpha)^{\frac{2p_\mu^\sharp(\alpha,\theta)}{2p_\mu^\sharp(\alpha,\theta)-p}} \leqslant \frac{sp-\beta}{p(N-\beta)}\Lambda(N,s,\gamma,\beta)^{\frac{N-\beta}{sp-\beta}} \\ V_{\gamma,\beta}(x), & \text{if } \dfrac{2p_s^\sharp(\delta,\theta,\mu) - p}{2pp_s^\sharp(\delta,\theta,\mu)}S_\mu(N,s,\gamma,\alpha)^{\frac{2p_\mu^\sharp(\alpha,\theta)}{2p_\mu^\sharp(\alpha,\theta)-p}} > \frac{sp-\beta}{p(N-\beta)}\Lambda(N,s,\gamma,\beta)^{\frac{N-\beta}{sp-\beta}} \end{cases}$$

and a positive number $t_0 > 0$ such that $\|t_0 v_0\| > r$ and $I(t_0 v_0) < 0$. We can define

$$c \coloneqq \inf_{g\in\Gamma} \sup_{t\in[0,1]} I(g(t)),$$

where

$$\Gamma \coloneqq \left\{ g \in C^0([0,1], \dot{W}^{s,p}_\theta(\mathbb{R}^N)) \colon g(0) = 0, g(1) = t_0 v_0 \right\}.$$

Clearly, we have that $c > 0$. For the case where $v_0(x) = U_{\gamma,\alpha}(x)$, we can deduce that

$$0 < c < \frac{2p^\sharp_s(\delta,\theta,\mu) - p}{2pp^\sharp_s(\delta,\theta,\mu)} S_\mu(N,s,\gamma,\alpha)^{\frac{p^\sharp_s(\delta,\theta,\mu)}{p^*_\mu(\alpha,\theta)-1}}.$$

In fact, for all $t \geqslant 0$, by the definition of the functional I, we have that

$$I(tv_0) = I(tU_{\gamma,\alpha}) \leqslant \frac{t^p}{p}\|U_{\gamma,\alpha}\|^p - \frac{t^{2p^\sharp_s(\delta,\theta,\mu)}}{2p^\sharp_s(\delta,\theta,\mu)} Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha}) \eqqcolon f_1(t).$$

It is easy to see that

$$\begin{aligned} f_1'(t) &= t^{p-1}\|U_{\gamma,\alpha}\|^p - t^{2p^\sharp_s(\delta,\theta,\mu)-1} Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha}) \\ &= t^{p-1}\Big[\|U_{\gamma,\alpha}\|^p - t^{2p^\sharp_s(\delta,\theta,\mu)-p} Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})\Big]. \end{aligned}$$

So, $f_1'(\tilde{t}) = 0$ for

$$\tilde{t} = \left(\frac{\|U_{\gamma,\alpha}\|^p}{Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})}\right)^{\frac{1}{2p^\sharp_s(\delta,\theta,\mu)-p}}, \tag{1.62}$$

and this is a point of maximum for f_1. Additionally, this maximum value is

$$\begin{aligned} f_1(\tilde{t}) &= \frac{\left(\frac{\|U_{\gamma,\alpha}\|^p}{Q^\sharp(U_{\gamma,\alpha},U_{\gamma,\alpha})}\right)^{\frac{p}{2p^\sharp_s(\delta,\theta,\mu)-p}}}{p}\|U_{\gamma,\alpha}\|^p - \frac{\left(\frac{\|U_{\gamma,\alpha}\|^p}{Q^\sharp(U_{\gamma,\alpha},U_{\gamma,\alpha})}\right)^{\frac{2p^\sharp_s(\delta,\theta,\mu)}{2p^\sharp_s(\delta,\theta,\mu)-p}}}{2p^\sharp_s(\delta,\theta,\mu)} Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha}) \\ &= \left(\frac{\|U_{\gamma,\alpha}\|^p}{Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})}\right)^{\frac{p}{2p^\sharp_s(\delta,\theta,\mu)-p}} \frac{\|U_{\gamma,\alpha}\|^p}{p} - \left(\frac{\|U_{\gamma,\alpha}\|^p}{Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})}\right)^{\frac{2p^\sharp_s(\delta,\theta,\mu)}{2p^\sharp_s(\delta,\theta,\mu)-p}} \frac{Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})}{2p^\sharp_s(\delta,\theta,\mu)} \\ &= \frac{\|U_{\gamma,\alpha}\|^{\frac{p^2}{2p^\sharp_s(\delta,\theta,\mu)-p}+p}}{pQ^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})^{\frac{p}{2p^\sharp_s(\delta,\theta,\mu)-p}}} - \frac{\|U_{\gamma,\alpha}\|^{\frac{2pp^\sharp_s(\delta,\theta,\mu)}{2p^\sharp_s(\delta,\theta,\mu)-p}}}{2p^\sharp_s(\delta,\theta,\mu) Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})^{\frac{2p^\sharp_s(\delta,\theta,\mu)}{2p^\sharp_s(\delta,\theta,\mu)-p}-1}} \\ &= \frac{\|U_{\gamma,\alpha}\|^{\frac{2pp^\sharp_s(\delta,\theta,\mu)}{2p^\sharp_s(\delta,\theta,\mu)-p}}}{pQ^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})^{\frac{p}{2p^\sharp_s(\delta,\theta,\mu)-p}}} - \frac{\|U_{\gamma,\alpha}\|^{\frac{2pp^\sharp_s(\delta,\theta,\mu)}{2p^\sharp_s(\delta,\theta,\mu)-p}}}{2p^\sharp_s(\delta,\theta,\mu) Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})^{\frac{p}{2p^\sharp_s(\delta,\theta,\mu)-p}}} \\ &= \left[\frac{1}{p} - \frac{1}{2p^\sharp_s(\delta,\theta,\mu)}\right] \frac{\|U_{\gamma,\alpha}\|^{\frac{2pp^\sharp_s(\delta,\theta,\mu)}{2p^\sharp_s(\delta,\theta,\mu)-p}}}{Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})^{\frac{p}{2p^\sharp_s(\delta,\theta,\mu)-p}}} \\ &= \left[\frac{2p^\sharp_s(\delta,\theta,\mu) - p}{2pp^\sharp_s(\delta,\theta,\mu)}\right] \frac{\|U_{\gamma,\alpha}\|^{\frac{2pp^\sharp_s(\delta,\theta,\mu)}{2p^\sharp_s(\delta,\theta,\mu)-p}}}{Q^\sharp(U_{\gamma,\alpha}, U_{\gamma,\alpha})^{\frac{p}{2p^\sharp_s(\delta,\theta,\mu)-p}}} \end{aligned}$$

$$= \left[\frac{2p_s^\sharp(\delta,\theta,\mu)-p}{2pp_s^\sharp(\delta,\theta,\mu)}\right]\left(\frac{\|U_{\gamma,\alpha}\|^p}{Q^\sharp(U_{\gamma,\alpha},U_{\gamma,\alpha})^{\frac{p}{2p_s^\sharp(\delta,\theta,\mu)}}}\right)^{\frac{2p_s^\sharp(\delta,\theta,\mu)}{2p_s^\sharp(\delta,\theta,\mu)-p}}$$
$$= \left[\frac{2p_s^\sharp(\delta,\theta,\mu)-p}{2pp_s^\sharp(\delta,\theta,\mu)}\right] S_\mu(N,s,\gamma,\alpha)^{\frac{2p_s^\sharp(\delta,\theta,\mu)}{2p_s^\sharp(\delta,\theta,\mu)-p}}.$$

Therefore,

$$\sup_{t\geqslant 0} I(tU_{\gamma,\alpha}) \leqslant \sup_{t\geqslant 0} f_1(t) = \frac{2p_s^\sharp(\delta,\theta,\mu)-p}{2pp_s^\sharp(\delta,\theta,\mu)} S_\mu(N,s,\gamma,\alpha)^{\frac{2p_s^\sharp(\delta,\theta,\mu)}{2p_s^\sharp(\delta,\theta,\mu)-p}} \tag{1.63}$$

The equality does not hold in (1.63); otherwise, we would have that $\sup_{t\geqslant 0} I(tU_{\gamma,\alpha}) = \sup_{t\geqslant 0} f_1(t)$. Let $t_1>0$ be the point where $\sup_{t\geqslant 0} I(tU_{\gamma,\alpha})$ is attained. We have

$$f_1(t_1) - \frac{t_1^{p_s^*(\beta,\theta)}}{p_s^*(\beta,\theta)} \int_{\mathbb{R}^N} \frac{|U_{\gamma,\alpha}|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x = f_1(\tilde{t})$$

which means that $f_1(t_1) > f_1(\tilde{t})$, since $t_1>0$. This contradicts the fact that $\tilde{t}$ is the unique maximum point for f_1. Thus, we have strict inequality in (1.63), that is,

$$\sup_{t\geqslant 0} I(tU_{\gamma,\alpha}) < \sup_{t\geqslant 0} f_1(t) = \frac{2p_\mu^\sharp(\alpha,\theta)-p}{2pp_\mu^\sharp(\alpha,\theta)} S_\mu(N,s,\gamma,\alpha)^{\frac{2p_\mu^\sharp(\alpha,\theta)}{2p_\mu^\sharp(\alpha,\theta)-p}}. \tag{1.64}$$

Therefore, $0 < c < \frac{2p_\mu^\sharp(\alpha,\theta)-p}{2pp_\mu^\sharp(\alpha,\theta)} S_\mu(N,s,\gamma,\alpha)^{\frac{2p_\mu^\sharp(\alpha,\theta)}{2p_\mu^\sharp(\alpha,\theta)-p}}$.

Similarly, for the case of $v_0(x) = V_{\gamma,\beta}(x)$, we can verify that

$$\sup_{t\geqslant 0} I(tV_{\gamma,\beta}) < \frac{sp-\beta}{p(N-\beta)} \Lambda(N,s,\gamma,\beta)^{\frac{N-\beta}{sp-\beta}}. \tag{1.65}$$

In fact, for all $t\geqslant 0$, by functional I definition we have that

$$I(tv_0) = I(tV_{\gamma,\beta}) \leqslant \frac{t^p}{p}\|V_{\gamma,\beta}\|^p - \frac{t^{p_s^*(\beta,\theta)}}{p_s^*(\beta,\theta)} \int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x := g_1(t).$$

It is easy to see that

$$g_1'(t) = t^{p-1}\|V_{\gamma,\beta}\|^p - t^{p_s^*(\beta,\theta)-1} \int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x$$
$$= t^{p-1}\left[\|V_{\gamma,\beta}\|^p - t^{p_s^*(\beta,\theta)-p} \int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right].$$

So, $g_1(\tilde{t}) = 0$ for

$$\tilde{t} = \left(\frac{\|V_{\gamma,\beta}\|^p}{\int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x}\right)^{\frac{1}{p_s^*(\beta,\theta)-p}},$$

and this is a point of maximum for g_1. Additionally, this maximum value is

$$\begin{aligned}
g_1(\tilde{t}) &= \frac{\left(\frac{\|V_{\gamma,\beta}\|^p}{\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x}\right)^{\frac{p}{p_s^*(\beta,\theta)-p}}}{p}\|V_{\gamma,\beta}\|^p - \frac{\left(\frac{\|V_{\gamma,\beta}\|^p}{\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x}\right)^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-p}}}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x \\
&= \left(\frac{\|V_{\gamma,\beta}\|^p}{\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x}\right)^{\frac{p}{p_s^*(\beta,\theta)-p}}\frac{\|V_{\gamma,\beta}\|^p}{p} - \left(\frac{\|V_{\gamma,\beta}\|^p}{\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x}\right)^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-p}}\frac{\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x}{p_s^*(\beta,\theta)} \\
&= \frac{\|V_{\gamma,\beta}\|^{\frac{p^2}{p_s^*(\beta,\theta)-p}+p}}{p\cdot\left(\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)^{\frac{p}{p_s^*(\beta,\theta)-p}}} - \frac{\|V_{\gamma,\beta}\|^{\frac{pp_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-p}}}{p_s^*(\beta,\theta)\cdot\left(\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-p}-1}} \\
&= \frac{\|V_{\gamma,\beta}\|^{\frac{pp_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-p}}}{p\cdot\left(\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)^{\frac{p}{p_s^*(\beta,\theta)-p}}} - \frac{\|V_{\gamma,\beta}\|^{\frac{pp_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-p}}}{p_s^*(\beta,\theta)\cdot\left(\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)^{\frac{p}{p_s^*(\beta,\theta)-p}}} \\
&= \left[\frac{1}{p}-\frac{1}{p_s^*(\beta,\theta)}\right]\left(\frac{\|V_{\gamma,\beta}\|^p}{\left(\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)^{\frac{p}{p_s^*(\beta,\theta)}}}\right)^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-p}} \\
&= \left[\frac{p_s^*(\beta,\theta)-p}{p\cdot p_s^*(\beta,\theta)}\right]\left(\frac{\|V_{\gamma,\beta}\|^p}{\left(\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)^{\frac{p}{p_s^*(\beta,\theta)}}}\right)^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-p}} \\
&= \frac{sp+\theta-\beta}{p(N-\beta)}\left(\frac{\|V_{\gamma,\beta}\|^p}{\left(\int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)^{\frac{p}{p_s^*(\beta,\theta)}}}\right)^{\frac{N-\beta}{sp+\theta-\beta}} \\
&= \frac{sp+\theta-\beta}{p(N-\beta)}\Lambda(N,s,\gamma,\beta)^{\frac{N-\beta}{sp+\theta-\beta}}.
\end{aligned}$$

Therefore,

$$\sup_{t\geqslant 0} I(tV_{\gamma,\beta}) \leqslant \sup_{t\geqslant 0} g_1(t) = \frac{sp+\theta-\beta}{p(N-\beta)}\Lambda(N,s,\gamma,\beta)^{\frac{N-\beta}{sp+\theta-\beta}}. \tag{1.66}$$

The equality does not hold in (1.66), otherwise, we would have that $\sup_{t\geqslant 0} I(tV_{\gamma,\beta}) = \sup_{t\geqslant 0} g_1(t)$. Let $t_1 > 0$, where $\sup_{t\geqslant 0} I(tV_{\gamma,\beta})$ is attained. We have

$$g_1(t_1) - \frac{t_1^{2p_s^\sharp(\delta,\theta,\mu)}}{2p_s^\sharp(\delta,\theta,\mu)}Q^\sharp(V_{\gamma,\alpha},V_{\gamma,\alpha}) = g_1(\tilde{t})$$

which means that $g_1(t_1) > g_1(\tilde{t})$, since $t_1 > 0$. This contradicts the fact that $\tilde{t}$ is the unique maximum point for $g_1(t)$. Thus

$$\sup_{t\geqslant 0} I(tV_{\gamma,\beta}) < \sup_{t\geqslant 0} g_1(t) = \frac{sp+\theta-\beta}{p(N-\beta)}\Lambda(N,s,\gamma,\beta)^{\frac{N-\beta}{sp+\theta-\beta}}. \tag{1.67}$$

Therefore, $0 < c < \frac{sp+\theta-\beta}{p(N-\beta)}\Lambda(N,s,\gamma,\beta)^{\frac{N-\beta}{sp+\theta-\beta}}$.

From the definition (1.61) of c^* and from inequalities (1.64) and (1.67), we have

$$0 < c < c^* := \min\left\{\frac{2p_s^\sharp(\delta,\theta,\mu)-p}{2pp_s^\sharp(\delta,\theta,\mu)}S_\mu(N,s,\gamma,\alpha)^{\frac{2p_s^\sharp(\delta,\theta,\mu)}{2p_s^\sharp(\delta,\theta,\mu)-p}}, \frac{sp+\theta-\beta}{p(N-\beta)}\Lambda(N,s,\gamma,\beta)^{\frac{N-\beta}{sp+\theta-\beta}}\right\}.$$

The lemma is proved. □

Proof of Proposition 1.23. Follows immediately from Lemmas 1.25 and 1.26. □

Proposition 1.27. *Let $s \in (0,1), N > sp+\theta, \alpha = 0 < \beta < sp+\theta$ or $\beta = 0 < \alpha < sp+\theta, \mu \in (0,N)$ and $0 \leqslant \gamma < \gamma_H$. Consider the functional I defined in (1.59) on the Banach space $\dot{W}^{s,p}_\theta(\mathbb{R}^N)$. Then there exists a (PS) sequence $\{u_k\} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ for I at some $c \in (0,c^*)$, i.e.,*

$$\lim_{k\to+\infty} I(u_k) = c \quad \text{and} \quad \lim_{k\to+\infty} I'(u_k) = 0 \quad \text{strongly in } \dot{W}^{s,p}_\theta(\mathbb{R}^N)',$$

where c^* is defined in (1.61).

Proof. The proof is similar to that of Proposition 1.23. Since $0 \leqslant \gamma < \gamma_H$, using items (3) and (4) in Proposition 1.22, we obtain a minimizer $U_{\gamma,0} \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ for $S_\mu(N,s,\gamma,0)$ and $V_{\gamma,0} \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ for $\Lambda(N,s,\gamma,0)$. The rest is standard. □

Conclusion of the proof of Theorem 0.1

The existence of a solution will follow from the proof of the Theorem 0.1.

Proof of Theorem 0.1. Suppose that $s \in (0,1), 0 < \alpha, \beta < sp+\theta, \mu \in (0,N)$ and $\gamma < \gamma_H$.

Let $\{u_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ be a Palais-Smale sequence $(PS)_c$ as in Proposition 1.23, i.e.,

$$I(u_k) \to c,\ I'(u_k) \to 0 \qquad \text{strongly in } \dot{W}^{s,p}_\theta(\mathbb{R}^N)' \text{ as } k \to +\infty.$$

Then

$$I(u_k) = \frac{1}{p}\|u_k\|^p - \frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x - \frac{1}{2p_s^\sharp(\delta,\theta,\mu)}Q^\sharp(u_k,u_k) = c + o(1) \tag{1.68}$$

and

$$\langle I'(u_k), u_k\rangle = \|u_k\|^p - \int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x - Q^\sharp(u_k,u_k) = o(1). \tag{1.69}$$

From (1.68) and (1.69), if $2p_s^\sharp(\delta,\theta,\mu) \geqslant p_s^*(\beta,\theta) > p$, we have

$$\begin{aligned} c + o(1)\|u_k\| &= I(u_k) - \frac{1}{p_s^*(\beta,\theta)}\langle I'(u_k), u_k\rangle \\ &= \frac{1}{p}\|u_k\|^p - \frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x - \frac{1}{2p_s^\sharp(\delta,\theta,\mu)}Q^\sharp(u_k,u_k) \\ &\quad - \frac{1}{p_s^*(\beta,\theta)}\|u_k\|^p + \frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x + \frac{1}{p_s^*(\beta,\theta)}Q^\sharp(u_k,u_k) \end{aligned}$$

$$= \frac{p_s^*(\beta,\theta)-p}{p\cdot p_s^*(\beta,\theta)}\|u_k\|^p + \left(\frac{1}{p_s^*(\beta,\theta)} - \frac{1}{2p_s^\sharp(\delta,\theta,\mu)}\right) Q^\sharp(u_k,u_k)$$
$$\geqslant \frac{p_s^*(\beta,\theta)-p}{p\cdot p_s^*(\beta,\theta)}\|u_k\|^p.$$

From (1.68) and (1.69), if $p_s^*(\beta,\theta) > 2p_s^\sharp(\delta,\theta,\mu) > p$, we have

$$\begin{aligned}
c+o(1)\|u_k\| &= I(u_k) - \frac{1}{2p_s^\sharp(\delta,\theta,\mu)}\langle I'(u_k),u_k\rangle \\
&= \frac{1}{p}\|u_k\|^p - \frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x - \frac{1}{2p_s^\sharp(\delta,\theta,\mu)}Q^\sharp(u_k,u_k) \\
&\quad - \frac{1}{2p_s^\sharp(\delta,\theta,\mu)}\|u_k\|^p + \frac{1}{2p_s^\sharp(\delta,\theta,\mu)}\int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x + \frac{1}{2p_s^\sharp(\delta,\theta,\mu)}Q^\sharp(u_k,u_k) \\
&= \frac{2p_s^\sharp(\delta,\theta,\mu)-p}{p\cdot 2p_s^\sharp(\delta,\theta,\mu)}\|u_k\|^p + \left(\frac{1}{2p_s^\sharp(\delta,\theta,\mu)} - \frac{1}{p_s^*(\beta,\theta)}\right)\int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x \\
&\geqslant \frac{2p_s^\sharp(\delta,\theta,\mu)-p}{p\cdot 2p_s^\sharp(\delta,\theta,\mu)}\|u_k\|^p.
\end{aligned}$$

Thus, $\{u_k\}_{k\in\mathbb{N}}$ is bounded $\dot{W}_\theta^{s,p}(\mathbb{R}^N)$, so from the estimate (1.69) there exists a subsequence, still denoted by $\{u_k\}_{k\in\mathbb{N}} \subset \dot{W}_\theta^{s,p}(\mathbb{R}^N)$, such that

$$\|u_k\|^p \to b, \qquad \int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x \to d_1, \qquad Q^\sharp(u_k,u_k)\to d_2,$$

as $k\to+\infty$; additionally,

$$b = d_1 + d_2.$$

By the definitions of $\Lambda(N,s,\gamma,\beta)$ and $S_\mu(N,s,\gamma,\alpha)$, we get

$$d_1^{\frac{p}{p_s^*(\beta,\theta)}}\Lambda(N,s,\gamma,\beta) \leqslant b = d_1+d_2, \qquad d_2^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}S_\mu(N,s,\gamma,\alpha) \leqslant b = d_1+d_2.$$

From the first inequality we have $d_1^{\frac{p}{p_s^*(\beta,\theta)}}\Lambda(N,s,\gamma,\beta) - d_1 \leqslant d_2$, that is

$$d_1^{\frac{p}{p_s^*(\beta,\theta)}}\left(\Lambda(N,s,\gamma,\beta) - d_1^{\frac{p_s^*(\beta,\theta)-p}{p_s^*(\beta,\theta)}}\right) \leqslant d_2. \tag{1.70}$$

And from the second inequality we have $d_2^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}S_\mu(N,s,\gamma,\alpha) - d_2 \leqslant d_1$, that is,

$$d_2^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}\left(S_\mu(N,s,\gamma,\alpha) - d_2^{\frac{p_s^\sharp(\delta,\theta,\mu)-1}{p_s^\sharp(\delta,\theta,\mu)}}\right) \leqslant d_1. \tag{1.71}$$

Claim 1. *We have*

$$\Lambda(N,s,\gamma,\beta) - d_1^{\frac{p_s^*(\beta,\theta)-p}{p_s^*(\beta,\theta)}} > 0, \qquad S_\mu(N,s,\gamma,\alpha) - d_2^{\frac{p_s^\sharp(\delta,\theta,\mu)-1}{p_s^\sharp(\delta,\theta,\mu)}} > 0.$$

Proof. In fact, since $c+o(1)\|u_k\| = I(u_k) - \frac{1}{p}\langle I'(u_k),u_k\rangle$, we have

$$\begin{aligned} I(u_k) - \frac{1}{p}\langle I'(u_k),u_k\rangle &= \frac{1}{p}\|u_k\|^p - \frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x - \frac{1}{2p_\mu^\sharp(\alpha,\theta)}Q^\sharp(u_k,u_k) \\ &\quad - \frac{1}{p}\|u_k\|^p + \frac{1}{p}\int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x + \frac{1}{p}Q^\sharp(u_k,u_k) \\ &= \left(\frac{1}{p} - \frac{1}{p_s^*(\beta,\theta)}\right)\int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x + \left(\frac{1}{p} - \frac{1}{2p_\mu^\sharp(\alpha,\theta)}\right)Q^\sharp(u_k,u_k) \\ &= c + o(1)\|u_k\|. \end{aligned}$$

Passing to the limit as $k\to+\infty$, we get

$$\left(\frac{1}{p} - \frac{1}{p_s^*(\beta,\theta)}\right)d_1 + \left(\frac{1}{p} - \frac{1}{2p_\mu^\sharp(\alpha,\theta)}\right)d_2 = c; \tag{1.72}$$

so,

$$d_1 \leqslant \left(\frac{1}{p} - \frac{1}{p_s^*(\beta,\theta)}\right)^{-1} c = \frac{p(N-\beta)}{sp+\theta-\beta}c, \qquad d_2 \leqslant \left(\frac{1}{p} - \frac{1}{2p_\mu^\sharp(\alpha,\theta)}\right)^{-1} c = \frac{2pp_s^\sharp(\delta,\theta,\mu)}{2p_s^\sharp(\delta,\theta,\mu)-p}c.$$

Using these upper bounds for d_1, d_2 and the fact $0<c<c^*$, we have

$$\begin{aligned} &\Lambda(N,s,\gamma,\beta) - d_1^{\frac{p_s^*(\beta,\theta)-p}{p_s^*(\beta,\theta)}} \\ &\geqslant \Lambda(N,s,\gamma,\beta) - \left[\frac{p(N-\beta)}{sp+\theta-\beta}c\right]^{\frac{p_s^*(\beta,\theta)-p}{p_s^*(\beta,\theta)}} \\ &> \Lambda(N,s,\gamma,\beta) - \left[\frac{p(N-\beta)}{sp+\theta-\beta}c^*\right]^{\frac{p_s^*(\beta,\theta)-p}{p_s^*(\beta,\theta)}} \\ &\geqslant \Lambda(N,s,\gamma,\beta) - \left[\frac{p(N-\beta)}{sp+\theta-\beta}\cdot\frac{(sp+\theta-\beta)}{p(N-\beta)}\Lambda(N,s,\gamma,\beta)^{\frac{N-\beta}{(sp+\theta-\beta)}}\right]^{\frac{p_s^*(\beta,\theta)-p}{p_s^*(\beta,\theta)}} \\ &\quad = \Lambda(N,s,\gamma,\beta) - \Lambda(N,s,\gamma,\beta) = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} &S_\mu(N,s,\gamma,\alpha) - d_2^{\frac{p_s^\sharp(\delta,\theta,\mu)-1}{p_s^\sharp(\delta,\theta,\mu)}} \\ &\geqslant S_\mu(N,s,\gamma,\alpha) - \left[\frac{2pp_s^\sharp(\delta,\theta,\mu)}{2p_s^\sharp(\delta,\theta,\mu)-p}c\right]^{\frac{p_s^\sharp(\delta,\theta,\mu)-1}{p_s^\sharp(\delta,\theta,\mu)}} \\ &> S_\mu(N,s,\gamma,\alpha) - \left[\frac{2pp_s^\sharp(\delta,\theta,\mu)}{2p_s^\sharp(\delta,\theta,\mu)-p}c^*\right]^{\frac{p_s^\sharp(\delta,\theta,\mu)-1}{p_s^\sharp(\delta,\theta,\mu)}} \\ &\geqslant S_\mu(N,s,\gamma,\alpha)\left[\frac{2pp_s^\sharp(\delta,\theta,\mu)}{(2p_s^\sharp(\delta,\theta,\mu)-p)}\cdot\frac{(2p_s^\sharp(\delta,\theta,\mu)-p)}{2pp_s^\sharp(\delta,\theta,\mu)}S_\mu(N,s,\gamma,\alpha)^{\frac{p_s^\sharp(\delta,\theta,\mu)}{p_s^\sharp(\delta,\theta,\mu)-1}}\right]^{\frac{p_s^\sharp(\delta,\theta,\mu)-1}{p_s^\sharp(\delta,\theta,\mu)}} \\ &\quad = S_\mu(N,s,\gamma,\alpha) - S_\mu(N,s,\gamma,\alpha) = 0. \end{aligned}$$

This concludes the proof of the claim. □

Following up, inequalities (1.70) and (1.71) imply, respectively, that

$$\left[\Lambda(N,s,\gamma,\beta)-\left(\frac{p(N-\beta)}{sp+\theta-\beta}c\right)^{\frac{p_s^*(\beta,\theta)-p}{p_s^*(\beta,\theta)}}\right]d_1^{\frac{p}{p_s^*(\beta,\theta)}}$$
$$\leqslant\left[\Lambda(N,s,\gamma,\beta)-d_1^{\frac{p_s^*(\beta,\theta)-p}{p_s^*(\beta,\theta)}}\right]d_1^{\frac{p}{p_s^*(\beta,\theta)}}\leqslant d_2$$

and

$$\left[S_\mu(N,s,\gamma,\alpha)-\left(\frac{2pp_s^\sharp(\delta,\theta,\mu)}{2p_s^\sharp(\delta,\theta,\mu)-p}c\right)^{\frac{p_s^\sharp(\delta,\theta,\mu)-1}{p_s^\sharp(\delta,\theta,\mu)}}\right]d_2^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}$$
$$\leqslant\left[S_\mu(N,s,\gamma,\alpha)-d_2^{\frac{p_s^\sharp(\delta,\theta,\mu)-1}{p_s^\sharp(\delta,\theta,\mu)}}\right]d_2^{\frac{1}{p_s^\sharp(\delta,\theta,\mu)}}\leqslant d_1.$$

If $d_1=0$ and $d_2=0$, then (1.72) implies that $c=0$, which is in contradiction with $c>0$. Therefore, $d_1>0$ and $d_2>0$ and we can choose $\epsilon_0>0$ such that $d_1\geqslant\epsilon_0>0$ and $d_2\geqslant\epsilon_0>0$; moreover, there exists a $K>0$ such that

$$\int_{\mathbb{R}^N}\frac{|u_k|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x>\frac{\epsilon_0}{2},\qquad Q^\sharp(u_k,u_k)>\frac{\epsilon_0}{2}$$

for every $k>K$. Then the inequality (1.3), the embeddings (1.4), and the improved Sobolev inequality (1.8) imply that there exists $C_1,C_2>0$ such that

$$0<C_2\leqslant\|u_k\|_{L_M^{p,N-sp-\theta+pr}(\mathbb{R}^N,|y|^{-pr})}\leqslant C_1,$$

where $r=\frac{\alpha}{p_s^*(\alpha,\theta)}$. For any $k>K$, we may find $\lambda_k>0$ and $x_k\in\mathbb{R}^N$ such that

$$\lambda_k^{(N-sp-\theta+pr)-N}\int_{B_{\lambda_k}(x_k)}\frac{|u_k(y)|^p}{|y|^{pr}}\mathrm{d}y>\|u_k\|^p_{L_M^{p,N-sp-\theta+pr}(\mathbb{R}^N,|y|^{-pr})}-\frac{C}{2k}\geqslant\tilde{C}>0.$$

Now we define the sequence $\{v_k\}_{k\in\mathbb{N}}\subset\dot{W}_\theta^{s,p}(\mathbb{R}^N)$ by $v_k(x)=\lambda_k^{\frac{N-sp-\theta}{p}}u_k(\lambda_kx)$. As we have already shown, $\|v_k\|=\|u_k\|\leqslant C$ for every $k\in\mathbb{N}$; so, there exists a $v\in\dot{W}_\theta^{s,p}(\mathbb{R}^N)$ such that, after passage to subsequence, still denoted in the same way,

$$v_k\rightharpoonup v\quad\text{in }\dot{W}_\theta^{s,p}(\mathbb{R}^N)$$

as $k\to+\infty$. In a fashion similar to the proof of Proposition 1.22-(1), we can prove that $v\not\equiv0$.

In addition, the boundedness of the sequence $\{v_k\}_{k\in\mathbb{N}}\subset\dot{W}_\theta^{s,p}(\mathbb{R}^N)$ implies that the sequence $\{|v_k|^{p_s^*(\beta,\theta)-2}v_k\}_{k\in\mathbb{N}}\subset L^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-1}}(\mathbb{R}^N,|x|^{-\beta})$ is bounded also. In fact, by embeddings (1.4), we obtain

$$\int_{\mathbb{R}^N}\frac{\left||v_k|^{p_s^*(\beta,\theta)-2}\cdot v_k\right|^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-1}}}{|x|^\beta}dx=\int_{\mathbb{R}^N}\frac{\left||v_k|^{p_s^*(\beta,\theta)-1}\right|^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-1}}}{|x|^\beta}dx$$

$$= \int_{\mathbb{R}^N} \frac{|v_k|^{p_s^*(\beta,\theta)}}{|x|^\beta} \mathrm{d}x < C.$$

Then, after passage to subsequence, still denoted in the same way, we deduce that

$$|v_k|^{p_s^*(\beta,\theta)-2} v_k \rightharpoonup |v|^{p_s^*(\beta,\theta)-2} v \quad \text{in } L^{\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-1}}(\mathbb{R}^N, |x|^{-\beta}) \tag{1.73}$$

as $k \to +\infty$.

For any $\phi \in L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N, |x|^{-\alpha})$, Lemma 1.18 implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot, v_k)\right](x) f_\alpha(x, v_k)\phi(x)\mathrm{d}x = \int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot, v)\right](x) f_\alpha(x, v)\phi(x).\mathrm{d}x. \tag{1.74}$$

Since $\dot{W}_\theta^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N, |x|^{-\alpha})$, (1.74) holds for any $\phi \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$.

Finally, we need to check that $\{v_k\}_{k\in\mathbb{N}} \subset \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ is also a $(PS)_c$ sequence for the functional I at energy level c. For this, the norms in $\dot{W}_\theta^{s,p}(\mathbb{R}^N)$ and $L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N, |x|^{-\alpha})$ are invariant under the special dilatation $v_k = \lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k x)$. In fact

$$\begin{aligned}
\|v_k\|^p_{\dot{W}_\theta^{s,p}(\mathbb{R}^N)} &= \iint_{\mathbb{R}^{2N}} \frac{|v_k(x) - v_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathbb{R}^{2N}} \frac{\left|\lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k x) - \lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k y)\right|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathbb{R}^{2N}} \frac{\lambda_k^{N-sp-\theta}|u_k(\lambda_k x) - u_k(\lambda_k y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathbb{R}^{2N}} \frac{\lambda_k^{N-sp-\theta}|u_k(\bar{x}) - u_k(\bar{y})|^p}{\lambda_k^{-\theta_1}|\bar{x}|^{\theta_1}|\bar{x}-\bar{y}|^{N+sp}\lambda_k^{-\theta_2}|\bar{y}|^{\theta_2}\lambda_k^{-N-sp}} \frac{\mathrm{d}\bar{x}}{\lambda_k^N}\frac{\mathrm{d}\bar{y}}{\lambda_k^N} \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_k(\bar{x}) - u_k(\bar{y})|^p}{|\bar{x}|^{\theta_1}|x-y|^{N+sp}|\bar{y}|^{\theta_2}} \mathrm{d}\bar{x}\mathrm{d}\bar{y} \\
&= \|u_k\|^p_{\dot{W}_\theta^{s,p}(\mathbb{R}^N)}
\end{aligned}$$

and

$$\begin{aligned}
\|v_k\|^{p_s^*(\alpha,\theta)}_{L^{p_s^*(\alpha,\theta)}} &= \int_{\mathbb{R}^N} \frac{|v_k(x)|^{p_s^*(\alpha,\theta)}}{|x|^\alpha} \\
&= \int_{\mathbb{R}^N} \frac{\lambda_k^{\frac{N-sp-\theta}{p}p_s^*(\alpha,\theta)}|u_k(\lambda_k x)|^{p_s^*(\alpha,\theta)})}{|x|^\alpha} \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \frac{\lambda_k^{N-\alpha}|u_k(\lambda_k x)|^{p_s^*(\alpha,\theta)})}{|x|^\alpha} \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \frac{\lambda_k^{N-\alpha}|u_k(\bar{x})|^{p_s^*(\alpha,\theta)})}{\lambda_k^{-\alpha}|\bar{x}|^\alpha} \frac{\mathrm{d}\bar{x}}{\lambda^N} \\
&= \int_{\mathbb{R}^N} \frac{|u_k(\bar{x})|^{p_s^*(\alpha,\theta)}}{|\bar{x}|^\alpha} \mathrm{d}\bar{x} \\
&= \|u_k\|^{p_s^*(\alpha,\theta)}_{L^{p_s^*(\alpha,\theta)}}.
\end{aligned}$$

Thus, we have

$$\lim_{k\to+\infty} I(v_k) = c.$$

Moreover, for all $\phi \in \dot{W}_{\theta}^{s,p}(\mathbb{R}^N)$, we have $\phi_k(x) = \lambda_k^{\frac{N-sp-\theta}{p}} \phi\left(x/\lambda_k\right) \in \dot{W}_{\theta}^{s,p}(\mathbb{R}^N)$. From $I'(u_k) \to 0$ in $\dot{W}_{\theta}^{s,p}(\mathbb{R}^N)'$ as $k \to +\infty$, we can deduce that

$$\lim_{k\to+\infty} \langle I'(v_k), \phi\rangle = \lim_{k\to+\infty} \langle I'(u_k), \phi\rangle = 0.$$

Thus (1.73) and (1.74) lead to

$$\langle I'(v), \phi\rangle = \lim_{k\to+\infty} \langle I'(v_k), \phi\rangle = 0.$$

Hence v is a nontrivial weak solution to problem 1. □

Proof of Theorem 0.2. The proof follows the same steps of the proof of Theorem 0.1. Here we only remark that for problem (4) with a Hardy potential and double Sobolev type nonlinearities we have to define the value below which we can recover the compactness of the Palais-Smale sequences by

$$c^* := \min_{k\in\{1,2\}} \left\{ \frac{sp+\theta-\beta_k}{p(N-\beta_k)} \Lambda(N,s,\gamma,\beta_k)^{\frac{N-\beta_k}{sp+\theta-\beta_k}} \right\}.$$

Similarly, for problem (5) with a Hardy potential and double Choquard type nonlinearities we have to define the corresponding number by

$$c^* := \min_{k\in\{1,2\}} \left\{ \frac{2p_s^{\sharp}(\delta_k,\theta,\mu_k)-p}{2pp_s^{\sharp}(\delta_k,\theta,\mu_k)} S_{\mu_k}(N,s,\gamma,\alpha)^{\frac{2p_s^{\sharp}(\delta_k,\theta,\mu_k)}{2p_s^{\sharp}(\delta_k,\theta,\mu_k)-p}} \right\}.$$

The details are omitted. □

Chapter 2

Fractional Sobolev-Choquard critical systems with Hardy term and weighted singularities

2.1 Historical background

The fractional Laplacian

There are many equivalent definitions of the fractional Laplacian. In our case, on the Euclidean space $\mathbb{R}^N$ of dimension $N \geqslant 1$, for $\theta = \theta_1 + \theta_2$ and the above specified intervals for the parameters, we define the non-local elliptic p-Laplacian operator with the help of the Cauchy's principal value integral as in (3).

For problems with two nonlinearities involving the Laplacian operator, see Filipucci, Pucci & Robert [42]. For similar problems involving the fractional Laplacian, see Servadei & Valdinoci [81], Ghoussoub & Shakerian [47], Chen & Squassina [30] Chen [27, 28], Assunção, Silva & Miyagaki [14]. For a survey paper on the subject of fractional Sobolev spaces, see Di Nezza, Palatucci & Valdinoci [39]; see also Molica Bisci, Rădulescu & Servadei [66].

The Choquard equation

On the Euclidean space $\mathbb{R}^N$ of dimension $N \geqslant 1$ and for $x \in \mathbb{R}^N$, the equation $-\Delta u + V(x)u = (I_\mu * |u|^q)|u|^{q-2}u$ was introduced by Choquard in the case $N = 3$ and $q = 2$ to model a certain approximation to Hartree-Fock theory of one-component plasma and to describe a electron trapped in its own hole. Also in this situation it finds physical significance in the work by Frölich and Pekar on the description of the quantum mechanics of a polaron at rest. When $V(x) \equiv 1$, the groundstate solutions exist if $2^\flat := 2(N - \mu/2))/N < q < 2(N - \mu/2)/(N - 2s) := 2^\sharp$ due to the mountain pass lemma or the method of the Nehari manifold, while there are no nontrivial solution if $q = 2^\flat$ or if $q = 2^\sharp$ as a consequence of the Pohozaev identity.

In general, the associated Schrödinger-type evolution equation $i\partial_t\psi = \Delta\psi + \Big(I_\mu * |\psi|^2\Big)\psi$ is a model for large systems of atoms with an attractive interaction that is weaker and has a longer range than that of the nonlinear Schrödinger equation. Standing wave solutions of this equation are solutions to the Choquard equation. For more information on the various results related to the non-fractional Choquard-type equations and their variants see Moroz & Van Schaftingen [68] and Mukherjee & Sreenadh [71].

The Morrey spaces

The study of Morrey spaces is motivated by many reasons. Initially, these spaces were introduced by Morrey in order to understand the regularity of solutions to elliptic partial differential equations. Regularity theorems, which allow one to conclude higher regularity of a function that is a solution of a differential equation together with a lower regularity of that function, play a central role in the theory of partial differential equations. One example of this kind of regularity theorem is a version of the Sobolev embedding theorem which states that $W^{j+m,p}(\Omega) \subset C^{j,\lambda}(\overline{\Omega})$ for $0 < \lambda \leqslant m - N/p$, where $j \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain.

Morrey spaces can complement the boundedness properties of operators that Lebesgue spaces can not handle. In line with this, many authors study the boundedness of various integral operators on Morrey spaces. The theory of Morrey spaces may come in useful when the Sobolev embedding theorem is not readily available. For more information on Morrey spaces, see Gantumur [46] and Sawano [78].

Systems of fractional elliptic equations

The subject of two or more fractional elliptic equations have been widely studied in recent years. We devote this section on briefly glimpsing the results that have already been proved in the context of existence, non-existence, uniqueness and multiplicity of solutions to systems of fractional elliptic equations.

Liu & Wang [63] gave a sufficient condition on large coupling coefficients for the existence of a nontrivial ground state solution in a system of nonlinear Schrödinger equation; they also considered bound state solutions. Chen & Deng [29] investigated the existence of two nontrivial solutions to the fractional p-Laplacian system involving concave-convex nonlinearities via the Nehari method. Chen [25] obtained the existence of infinitely many nonnegative solutions for a class of the quasilinear Schrödinger system in $\mathbb{R}^N$ in the Laplacian setting and investigate the multiplicity of solutions for a p-Kirchhoff system driven by a non-local integro-differential operator with zero Dirichlet boundary data. Xiang, Zhang & Rădulescu [91] studied the multiplicity of solutions for a p–Kirchhoff system driven by a non-local integro-differential operator with zero Dirichlet boundary data. Chen & Squassina [30] used Nehari manifold techniques to obtain the existence of multiple solutions to a fractional p–Laplacian system involving critical concave-convex nonlinearities. Fiscella, Pucci & Saldi [43], using several variational methods, dealt with the existence of nontrivial nonnegative solutions of Schrödinger–Hardy systems driven by two possibly different fractional p–Laplacian operators. The main features of this paper is the presence of the Hardy term and the fact that the nonlinearities do not necessarily satisfy the Ambrosetti–Rabinowitz condition. Wang, Zhang & Zhang [89] are interested in a fractional Laplacian system in the whole space $\mathbb{R}^N$, which involves critical Sobolev-type nonlinearities and critical Hardy-Sobolev-type nonlinearities. Yang [94] considered the existence of nontrivial weak solutions to a doubly critical system involving fractional Laplacian in $\mathbb{R}^N$ with subcritical weight. More recently, Lu & Shen [64] studied a critical fractional p-Laplacian system with homogeneous nonlinearity; they used a concentration compactness principle associated with fractional p-Laplacian system for the fractional order Sobolev spaces in bounded domains, which is significantly more difficult to prove than in the case of a single fractional p-Laplacian equation and is of independent interest.

Most of the existing results have been developed for systems with two equations. For a general system, Lin & Wei [60, 61] studied a system with several coupled nonlinear Schrödinger equations in the whole space up to three dimensions which has some applications in nonlinear optics. The existence of ground state solutions may depend on the coupling constants that model the interaction between the components of the system. If a constant is positive, the interaction is attractive; otherwise, it is repulsive. When all the constants are positive and some associated matrix is positively definite, they proved the existence of a radially symmetric ground state solution; however, if all the constants are negative, our if one of them is negative and the matrix is positively definite, there is no ground state solution. They also obtained the existence of a bound state solution which is non-radially symmetric in the three dimensional case.

Our contribution to the problem and some of its difficulties

The present work is motivated by Assunção, Miyagaki & Siqueira [13], Wang, Zhang & Zhang [89] and Yang [94]. Our existence result can be regarded as an extension and improvement of the corresponding existence results in these works. More precisely, we will extend the result in [13] to a system of coupled equations in $\mathbb{R}^N$ with the general fractional p-Laplacian with $p > 1$ and $\theta = \theta_1 + \theta_2$ not necessarily zero. Moreover, we use a refinement of Sobolev inequality that is related to Morrey space because our problem involves doubly critical exponents. As one can expect, the non-locality of the fractional p-Laplacian makes it more difficult to study. In our case, one of the main difficulties when dealing with this problem is the lack of compactness of Sobolev embedding theorem for the critical exponent. Therefore, we have to develop a precise analysis of the level of the Palais-Smale sequences obtained with the application of the mountain pass theorem and study their behavior concerning strong convergence of one of its scaled subsequences.

2.2 Existence of solutions for auxiliary minimization problems

We begin this section by introducing two important and sharp Rayleigh-Ritz constants. The first one is related to the Gagliardo seminorm, the Hardy term, and a convolution involving the upper Stein-Weiss exponent,

$$S^\sharp := \inf_{(u,v)\in W\setminus\{0\}} \frac{\|(u,v)\|_W^p}{\left(Q^\sharp(u,v)\right)^{\frac{p}{2p_s^\sharp(\delta,\theta,\mu)}}}, \tag{2.1}$$

where the quadratic form $Q^\sharp \colon W \to \mathbb{R}$ is given by

$$Q^\sharp(u,v) := \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_s^\sharp(\delta,\theta,\mu)}|u(y)|^{p_s^\sharp(\delta,\theta,\mu)} + |v(x)|^{p_s^\sharp(\delta,\theta,\mu)}|v(y)|^{p_s^\sharp(\delta,\theta,\mu)}}{|x|^\delta|x-y|^\mu|y|^\delta}\,\mathrm{d}x\,\mathrm{d}y\,. \tag{2.2}$$

The second one is related to the Gagliardo seminorm, ther Hardy term, and the norm in the critical weighted Lebesgue space, that is, the Sobolev constant,

$$S^* := \inf_{(u,v)\in W\setminus\{0\}} \frac{\|(u,v)\|_W^p}{\left(Q^*(u,v)\right)^{\frac{p}{p_s^*(\beta,\theta)}}}, \tag{2.3}$$

where the quadratic form $Q^*: W \to \mathbb{R}$ is given by

$$Q^*(u,v) := \int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*(\beta,\theta)} + |v(x)|^{p_s^*(\beta,\theta)} + \eta |u(x)|^a |v(x)|^b}{|x|^\beta} \, \mathrm{d}x\,. \tag{2.4}$$

For general $p \neq 2$, the explicit formula for the extremal functions for the p-fractional Sobolev inequality is not known yet, though it is conjectured that it is of the form

$$U(x) = \frac{C}{(1+|x|^{\frac{p}{p-1}})^{\frac{N-sp}{p}}}$$

up to translation and scaling. However, there is a result about the asymptotic behavior of U, as seen in Brasco, Mosconi & Squassina [20] and Mosconi, Perera, Squassina & Yang [69].

One of the first major difficulties that we encounter is the lack of an explicit formula for a minimizer of the quantity S^*. There is a conjecture about the minimizers which states that they have the form

$$U(x) = \frac{c}{[1+(|x-x_0|/\varepsilon))^{\frac{p}{p-1}}]^{\frac{N-sp}{p}}}$$

where $C \neq \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$, and $\varepsilon \in \mathbb{R}_+$. This conjecture was proved by Lieb [58] in the case $p = 2$; however, for $p \neq 2$ it is not even known if these functions are minimizers. To overcome this difficulty, we will work with some asymptotic estimates for minimizers recently obtained by Brasco, Mosconi & Squassina [21]; see also Mosconi, Perera, Squassina & Yang [69].

Proposition 2.1. *For $s \in (0,1)$ the best constants $S^\sharp$ and S^* verify the following items.*

1. *If $0 < \alpha < sp + \theta < N, \mu \in (0,N)$ and $\gamma < \gamma_H$, then $S^\sharp$ is attained in W;*
2. *If $0 < \beta < sp + \theta < N$ and $\gamma < \gamma_H$, then S^* is attained in W.*

Proof. 1. If $0 < \alpha < sp + \theta < N$ and $\gamma < \gamma_H$, let $\{(u_k, v_k)\}_{k\in\mathbb{N}} \subset W$ be a minimizing sequence of $S^\sharp$ such that

$$Q^\sharp(u_k, v_k) = 1, \qquad \|(u_k, v_k)\|^p \to S^\sharp \tag{2.5}$$

as $k \to +\infty$. Recall that $r = \frac{\alpha}{p_s^*(\alpha,\theta)}$. In the same way that we argued in (1.41) on the page 45, the embeddings (1.4) and the Caffarelli-Kohn-Nirenberg's inequality (1.8) enables us to find $C_1, C_2 > 0$ such that

$$C_1 \leqslant \|u_k\|_{L_M^{q,\frac{N-sp-\theta}{p}q+qr}(\mathbb{R}^N,|y|^{-pr})} \leqslant C_2. \tag{2.6}$$

For any $k \in \mathbb{N}$ large enough, we may find $\lambda_k > 0$ and $x_k \in \mathbb{R}^N$ such that

$$\lambda_k^{-sp-\theta+pr} \int_{B_{\lambda_k(x_k)}} \frac{|u_k(y)|^p}{|y|^{pr}} \mathrm{d}y > \|u_k\|^p_{L_M^{p,N-sp-\theta+pr}(\mathbb{R}^N,|y|^{-pr})} - \frac{C}{2k} \geqslant C > 0$$

for constants $C \in \mathbb{R}_+$.

Our goal is to pass to the limit as $k \to +\infty$ in the minimizing sequence. To do this, we adapt the Levy's concentration principle; more precisely, we create another sequence that will help us to control the radius and the centers of these balls. Let

$$\tilde{u}_k(x) = \lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k x) \quad \text{and} \quad \tilde{v}_k(x) = \lambda_k^{\frac{N-sp-\theta}{p}} v_k(\lambda_k x)$$

be the appropriate scaling for the class of problems that we consider. Then, in the same way as we do in the equation (1.42) on the page 45, we have

$$\int_{B_1\left(\frac{x_k}{\lambda_k}\right)} \frac{|\tilde{u}_k(x)|^p}{|x|^{pr}} \mathrm{d}x \geqslant C > 0 \quad \text{and} \quad \int_{B_1\left(\frac{x_k}{\lambda_k}\right)} \frac{|\tilde{v}_k(x)|^p}{|x|^{pr}} \mathrm{d}x \qquad \geqslant C > 0. \tag{2.7}$$

Now we claim that $S^\sharp$ is invariant under the previously defined dilation.

In fact, $Q^\sharp(\tilde{u}_k, \tilde{v}_k) = 1$. To show this property, we use the change of variables $\bar{x} = \lambda_k x$ and $\bar{y} = \lambda_k y$, we have

$$\begin{aligned}
Q^\sharp(\tilde{u}_k, \tilde{v}_k) &= \iint_{\mathbb{R}^{2N}} \frac{|u_k(x)|^{p_s^\sharp}|u_k(y)|^{p_s^\sharp}}{|x|^\delta|x-y|^\mu|y|^\delta} \mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{|v_k(y)|^{p_s^\sharp}|v_k(y)|^{p_s^\sharp}}{|x|^\delta|x-y|^\mu|y|^\delta} \mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathbb{R}^{2N}} \lambda_{k,1}^{\frac{N-sp-\theta}{p}2p_s^\sharp} \frac{|u_k(\lambda_k x)|^{p_s^\sharp}|u_k(\lambda_{k,1} y)|^{p_s^\sharp}}{|x|^\delta|x-y|^\mu|y|^\delta} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint_{\mathbb{R}^{2N}} \lambda_k^{\frac{N-sp-\theta}{p}2p_s^\sharp} \frac{|v_k(\lambda_k x)|^{p_s^\sharp}|v_k(\lambda_k y)|^{p_s^\sharp}}{|x|^\delta|x-y|^\mu|y|^\delta} \,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_k(\bar{x})|^{p_s^\sharp}|u_k(\bar{y})|^{p_s^\sharp}}{|\bar{x}|^\delta|\bar{x}-\bar{y}|^\mu|\bar{y}|^\delta} \mathrm{d}\bar{x}\mathrm{d}\bar{y} + \iint_{\mathbb{R}^{2N}} \frac{|v_k(\bar{x})|^{p_s^\sharp}|v_k(\bar{y})|^{p_s^\sharp}}{|\bar{x}|^\delta|\bar{x}-\bar{y}|^\mu|\bar{y}|^\delta} \mathrm{d}\bar{x}\mathrm{d}\bar{y} \\
&= Q^\sharp(u_k, v_k) = 1.
\end{aligned}$$

Furthermore, $\|(\tilde{u}_k, \tilde{v}_k)\|^p \to S^\sharp$. In fact, we know that $\{(\tilde{u}_k, \tilde{v}_k)\}_{k\in\mathbb{N}}$ is a minimizing sequence for $S^\sharp$. Using the same change of variables $\bar{x} = \lambda_k x$ and $\bar{y} = \lambda_k y$, we obtain

$$\begin{aligned}
\|(\tilde{u}_k, \tilde{v}_k)\|^p &= \iint_{\mathbb{R}^{2N}} \frac{|u_k(x) - u_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{|v_k(x) - v_k(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \\
&\quad - \gamma_1 \int_{\mathbb{R}^N} \frac{|u_k|^p}{|\bar{x}|^{sp+\theta}} \mathrm{d}x - \gamma_2 \int_{\mathbb{R}^N} \frac{|v_k|^p}{|\bar{x}|^{sp+\theta}} \mathrm{d}\bar{x} \\
&= \iint_{\mathbb{R}^{2N}} \lambda_{k,1}^{N-sp-\theta} \frac{|u_k(\lambda_k x) - u_k(\lambda_k y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \\
&\quad + \iint_{\mathbb{R}^{2N}} \lambda_k^{N-sp-\theta} \frac{|v_k(\lambda_k x) - v_k(\lambda_k y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \\
&\quad - \gamma_1 \int_{\mathbb{R}^N} \lambda_k^{N-sp-\theta} \frac{|u_k|^p}{|x|^{sp+\theta}} \mathrm{d}x - \gamma_2 \int_{\mathbb{R}^N} \lambda_{k,2}^{N-sp-\theta} \frac{|v_k|^p}{|x|^{sp+\theta}} \mathrm{d}x \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_k(\bar{x}) - u_k(\bar{y})|^p}{|\bar{x}|^{\theta_1}|\bar{x}-\bar{y}|^{N+sp}|\bar{y}|^{\theta_2}} \mathrm{d}\bar{x}\mathrm{d}\bar{y} + \iint_{\mathbb{R}^{2N}} \frac{|v_k(\bar{x}) - v_k(\bar{y})|^p}{|\bar{x}|^{\theta_1}|\bar{x}-\bar{y}|^{N+sp}|\bar{y}|^{\theta_2}} \mathrm{d}\bar{x}\mathrm{d}\bar{y} \\
&\quad - \gamma_1 \int_{\mathbb{R}^N} \frac{|u_k|^p}{|\bar{x}|^{sp+\theta}} \mathrm{d}x - \gamma_2 \int_{\mathbb{R}^N} \frac{|v_k|^p}{|\bar{x}|^{sp+\theta}} \mathrm{d}\bar{x} \\
&= \|(u_k, v_k)\|^p.
\end{aligned}$$

And since $\|(u_k, v_k)\|^p \to S^\sharp$ as $k \to +\infty$, we deduce that $\|(\tilde{u}_k, \tilde{v}_k)\|^p \to S^\sharp$ as $k \to +\infty$.

In this way, the sequence $\{(\tilde{u}_k, \tilde{v}_k)\}_{k\in\mathbb{N}} \subset W$ is also a minimizing sequence for $S^\sharp$ such that we have

$$Q^\sharp(\tilde{u}_k, \tilde{v}_k) = 1, \qquad \|(\tilde{u}_k, \tilde{v}_k)\|^p \to S^\sharp. \tag{2.8}$$

Consider $\{\tilde{x}_k\} = \{\frac{x_k}{\lambda_k}\}$, from inequality (2.7) together with Hölder's inequality

$$\begin{aligned}
0 < C &\leqslant \int_{B_1(\tilde{x}_k)} \frac{|\tilde{u}_k(x)|^p}{|x|^{pr}} \mathrm{d}x \\
&\leqslant \left(\int_{B_1(\tilde{x}_k)} 1 \mathrm{d}x \right)^{1-\frac{p}{p_s^*(\alpha,\theta)}} \left(\int_{B_1(\tilde{x}_k)} \left(\frac{|\tilde{u}_k(x)|^p}{|x|^{pr}} \right)^{\frac{p_s^*(\alpha,\theta)}{p}} \mathrm{d}x \right)^{\frac{p}{p_s^*(\alpha,\theta)}} \\
&\leqslant C \left(\int_{B_1(\tilde{x}_k)} \frac{|\tilde{u}_k(x)|^{p_s^*(\alpha,\theta)}}{|x|^\alpha} \mathrm{d}x \right)^{\frac{p}{p_s^*(\alpha,\theta)}}.
\end{aligned}$$

Therefore,

$$\left(\int_{B_1(\tilde{x}_k)} \frac{|\tilde{u}_k(x)|^{p_s^*(\alpha,\theta)}}{|x|^\alpha} \mathrm{d}x \right)^{\frac{p}{p_s^*(\alpha,\theta)}} \geqslant C > 0. \tag{2.9}$$

We claim that the sequence $\{\tilde{x}_k\} \subset \mathbb{R}^N$ of the centers of the balls is bounded. We argue by contradiction and suppose that $|\tilde{x}_k| \to +\infty$ as $k \to +\infty$; then for any $x \in B_1(\tilde{x}_k)$, we have $|x| \geqslant |\tilde{x}_k| - 1$ for $k \in \mathbb{N}$ large enough. By Hölder's inequality, we obtain

$$\begin{aligned}
\int_{B_1(\tilde{x}_k)} \frac{|\tilde{u}_k(x)|^{p_s^*(\alpha,\theta)}}{|x|^\alpha} \mathrm{d}x &\leqslant \frac{1}{(|\tilde{x}_k| - 1)^\alpha} \int_{B_1(\tilde{x}_k)} |\tilde{u}_k(x)|^{p_s^*(\alpha,\theta)} \mathrm{d}x \\
&\leqslant \frac{C}{(|\tilde{x}_k| - 1)^\alpha} \left(\int_{B_1(\tilde{x}_k)} |\tilde{u}_k(x)|^{p_s^*(0,\theta)} \mathrm{d}x \right)^{\frac{p_s^*(\alpha,\theta)}{p_s^*(0,\theta)}} \\
&\leqslant \frac{C}{(|\tilde{x}_k| - 1)^\alpha} \|\tilde{u}_k(x)\|_{L^{p_s^*(0,\theta)}}^{p_s^*(\alpha,\theta)} \\
&\leqslant \frac{C}{(|\tilde{x}_k| - 1)^\alpha} \|u_k(x)\|_{\dot{W}_\theta^{s,p}(\mathbb{R}^N)}^{p_s^*(\alpha,\theta)} \\
&\leqslant \frac{C}{(|\tilde{x}_k| - 1)^\alpha} \to 0 \qquad (k \to +\infty)
\end{aligned}$$

where we used the boundedness of the minimizing sequence $\{\tilde{u}_k\}_{k\in\mathbb{N}} \subset \dot{W}_\theta^{s,p}(\mathbb{R}^N)$. This is a contradiction with inequality (2.9) and this implies that the sequence $\{\tilde{x}_k\} \subset \mathbb{R}^N$ is bounded.

From inequality (2.7) and the boundedness of the sequence $\{\tilde{x}_k\} \subset \mathbb{R}^N$ of the centers of the balls, we may find $R > 0$ such that $B_R(0)$ contains all balls of center $\tilde{x}_k$ and radius 1; moreover, with

$$\int_{B_R(0)} \frac{|\tilde{u}_k(x)|^p}{|x|^{pr}} \mathrm{d}x \geqslant C_1 > 0. \tag{2.10}$$

Since $\|(\tilde{u}_k, \tilde{v}_k)\| = \|(u_k, v_k)\| \leqslant C$ for $k \in \mathbb{N}$ large enough, there exists a function pair $(\tilde{u}, \tilde{v}) \in W$ such that

$$(\tilde{u}_k, \tilde{v}_k) \rightharpoonup (\tilde{u}, \tilde{v}) \quad \text{weakly in } \dot{W}_\theta^{s,p}(\mathbb{R}^N) \times \dot{W}_\theta^{s,p}(\mathbb{R}^N) \tag{2.11}$$

$$(\tilde{u}_k, \tilde{v}_k) \to (\tilde{u}, \tilde{v}) \quad \text{a.e. on } \mathbb{R}^N \times \mathbb{R}^N, \tag{2.12}$$

as $k \to +\infty$, up to subsequences. According to Lemma 1.12, we have

$$\left(\frac{\tilde{u}_k}{|x|^r}, \frac{\tilde{v}_k}{|x|^r}\right) \to \left(\frac{\tilde{u}}{|x|^r}, \frac{\tilde{v}}{|x|^r}\right) \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N) \times L^p_{\mathrm{loc}}(\mathbb{R}^N);$$

hence,

$$\int_{B_R(0)} \frac{|\tilde{u}(x)|^p}{|x|^{pr}} \mathrm{d}x \geqslant C_1 > 0,$$

and we deduce that $\tilde{u} \not\equiv 0$. Similarly we can get $\tilde{v} \not\equiv 0$.

We can verify in the same way as we did in (1.17) that

$$1 = Q^\sharp(\tilde{u}_k, \tilde{v}_k) = Q^\sharp(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v}) + Q^\sharp(\tilde{u}, \tilde{v}) + o(1). \tag{2.13}$$

By a Brézis-Lieb's lemma, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(I_\mu * \frac{|\tilde{u}_k - \tilde{u}|^{p_s^\sharp}}{|y|^\delta}\right) \frac{|\tilde{u}_k - \tilde{u}|^{p_s^\sharp}}{|x|^\delta} \mathrm{d}x + \int_{\mathbb{R}^N} \left(I_\mu * \frac{|\tilde{u}|^{p_s^\sharp}}{|y|^\delta}\right) \frac{|\tilde{u}|^{p_s^\sharp}}{|x|^\delta} \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left(I_\mu * \frac{|\tilde{u}_k|^{p_s^\sharp}}{|y|^\delta}\right) \frac{|\tilde{u}_k|^{p_s^\sharp}}{|x|^\delta} \mathrm{d}x + o(1), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(I_\mu * \frac{|\tilde{v}_k - \tilde{v}|^{p_s^\sharp}}{|y|^\delta}\right) \frac{|\tilde{v}_k - \tilde{v}|^{p_s^\sharp}}{|x|^\delta} \mathrm{d}x + \int_{\mathbb{R}^N} \left(I_\mu * \frac{|\tilde{v}|^{p_s^\sharp}}{|y|^\delta}\right) \frac{|\tilde{v}|^{p_s^\sharp}}{|x|^\delta} \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left(I_\mu * \frac{|\tilde{v}_k|^{p_s^\sharp}}{|y|^\delta}\right) \frac{|\tilde{v}_k|^{p_s^\sharp}}{|x|^\delta} \mathrm{d}x + o(1), \end{aligned} \tag{2.15}$$

Therefore, by definition (2.1), by weak convergence $(\tilde{u}_k, \tilde{v}_k) \rightharpoonup (\tilde{u}, \tilde{v})$ in $\dot{W}^{s,p}_\theta(\mathbb{R}^N) \times \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ together with the Brézis–Lieb lemma and by the estimate (2.13), we have

$$\begin{aligned} S^\sharp &= \lim_{k\to\infty} \|(\tilde{u}_k, \tilde{v}_k)\|^p \\ &= \|(\tilde{u}, \tilde{v})\|^p + \lim_{k\to\infty} \|(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v})\|^p \\ &\geqslant S^\sharp (Q^\sharp(\tilde{u}, \tilde{v}))^{\frac{p}{p_s^\sharp}} + S^\sharp \left(\lim_{k\to\infty} Q^\sharp(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v})\right)^{\frac{p}{p_s^\sharp}} \\ &\geqslant S^\sharp \Big(Q^\sharp(\tilde{u}, \tilde{v}) + \lim_{k\to\infty} Q^\sharp(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v})\Big)^{\frac{p}{p^\sharp}} \\ &= S^\sharp, \end{aligned}$$

where in the last but one passage above we used the inequality

$$(a+b)^q \leqslant a^q + b^q, \tag{2.16}$$

valid for all $a, b \in \mathbb{R}^*_+$ and $0 < q < 1$. So we have equality in all passages, that is,

$$Q^\sharp(\tilde{u}, \tilde{v}) = 1, \qquad \lim_{k\to\infty} Q^\sharp(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v}) = 0, \tag{2.17}$$

since $\tilde{u},\tilde{v}\not\equiv 0$. It turns out that, since

$$S^{\sharp}=\|(\tilde{u},\tilde{v})\|^{p}+\lim_{k\to\infty}\|(\tilde{u}_k-\tilde{u},\tilde{v}_k-\tilde{v})\|^{p},$$

then

$$S^{\sharp}=\|(\tilde{u},\tilde{v})\|^{p}\qquad\text{and}\qquad\lim_{k\to\infty}\|(\tilde{u}_k-\tilde{u},\tilde{v}_k-\tilde{v})\|^{p}=0.$$

Finally, by inequality

$$\begin{aligned}\iint_{\mathbb{R}^{2N}}&\frac{||\tilde{u}(x)|-|\tilde{u}(y)||^{p}}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y+\iint_{\mathbb{R}^{2N}}\frac{||\tilde{v}(x)|-|\tilde{v}(y)||^{p}}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\\&\leqslant\iint_{\mathbb{R}^{2N}}\frac{|\tilde{u}(x)-\tilde{u}(y)|^{p}}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y+\iint_{\mathbb{R}^{2N}}\frac{|\tilde{v}(x)-\tilde{v}(y)|^{p}}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\end{aligned}$$

we deduce that $(|\tilde{u}|,|\tilde{v}|)\in W$ is also a minimizer for $S^{\sharp}$; so we can assume that $\tilde{u}\geqslant 0,\tilde{v}\geqslant 0$. Thus, $S^{\sharp}$ is achieved by a non-negative function in the case $0<\alpha<sp+\theta$ and $\gamma<\gamma_H$.

2. For $0<\beta<sp+\theta<N$ and $\gamma<\gamma_H$, let $\{(u_k,v_k)\}_{k\in\mathbb{N}}\subset W$ be a minimizing sequence for S^{*} such that

$$Q^{*}(u_k,v_k)=1,\qquad\|(u_k,v_k)\|^{p}\to S^{*}\tag{2.18}$$

as $k\to+\infty$.

Now we claim that S^{*} is invariant under the previously defined dilation. Let

$$\tilde{u}_k(x)=\lambda_k^{\frac{N-sp-\theta}{p}}u_k(\lambda_k x),\qquad\qquad\tilde{v}_k(x)=\lambda_k^{\frac{N-sp-\theta}{p}}v_k(\lambda_k x)$$

and $\tilde{x}_k=\frac{x_k}{\lambda_k}$ as in the previous case. In this way, the sequence $\{(\tilde{u}_k,\tilde{v}_k)\}_{k\in\mathbb{N}}\subset W$ is also a minimizing sequence for S^{*} such that we have

$$Q^{*}(u_k,v_k)=1,\qquad\|(u_k,v_k)\|^{p}\to S^{*}$$

We have already shown that $\|(\tilde{u}_k,\tilde{v}_k)\|=\|(u_k,v_k)\|$ for every $k\in\mathbb{N}$. Hence, $\|(\tilde{u}_k,\tilde{v}_k)\|^{p}\to S^{*}$.

We claim that the sequence $\{\tilde{x}_k\}\subset\mathbb{R}^N$ is bounded and the proof follows the same steps already presented. From this boundedness and inequality (1.42), we may find $R>0$ such that $B_R(0)$ contains all the unitary balls $B_1(\tilde{x}_k)$ centered in $\tilde{x}_k$ and

$$\int_{B_R(0)}\frac{|v_k(x)|^{p}}{|x|^{pr}}\mathrm{d}x\geqslant C_1>0.\tag{2.19}$$

Since $\|v_k\|=\|u_k\|\leqslant C$, there exists a $v\in\dot{W}^{s,p}_{\theta}(\mathbb{R}^N)$ such that

$$v_k\rightharpoonup v\quad\text{in }\dot{W}^{s,p}_{\theta}(\mathbb{R}^N),\qquad v_k\to v\text{ a.e.}\quad\text{on }\mathbb{R}^N,\tag{2.20}$$

as $k\to+\infty$, up to subsequences. According to Lemma 1.12, we have

$$\frac{v_k}{|x|^{r}}\to\frac{v}{|x|^{r}}\qquad\text{in }L^{p}_{\mathrm{loc}}(\mathbb{R}^N),$$

as $k \to +\infty$, where $r = \frac{\beta}{p_s^*}$. Therefore,

$$\int_{B_R(0)} \frac{|v(x)|^p}{|x|^{pr}} \mathrm{d}x \geqslant C_1 > 0,$$

and we deduce that $v \not\equiv 0$.

We may verify by Lemma 1.13 that, if $q = p_s^*(\beta, \theta)$ and $\delta = \beta$, then

$$1 = \int_{\mathbb{R}^N} \frac{|v_k|^{p_s^*}}{|x|^\beta} \mathrm{d}x = \int_{\mathbb{R}^N} \frac{|v_k - v|^{p_s^*}}{|x|^\beta} \mathrm{d}x + \int_{\mathbb{R}^N} \frac{|v|^{p_s^*}}{|x|^\beta} \mathrm{d}x + o(1).$$

By definition (1.39) and by weak convergence $(\tilde{u}_k, \tilde{v}_k) \rightharpoonup (\tilde{u}, \tilde{v})$ in $\dot{W}_\theta^{s,p}(\mathbb{R}^N) \times \dot{W}_\theta^{s,p}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned} S^* &= \lim_{k\to\infty} \|(\tilde{u}_k, \tilde{v}_k)\|^p \\ &= \|(\tilde{u}, \tilde{v})\|^p + \lim_{k\to\infty} \|(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v})\|^p \\ &\geqslant S^*(Q^*(\tilde{u}, \tilde{v}))^{\frac{p}{p_s^*}} + S^* \left(\lim_{k\to\infty} Q^*(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v}) \right)^{\frac{p}{p_s^*}} \\ &\geqslant S^* \Big(Q^*(\tilde{u}, \tilde{v}) + \lim_{k\to\infty} Q^*(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v}) \Big)^{\frac{p}{p_s^*}} \\ &= S^* \end{aligned}$$

where we used the inequality (2.16). So we have equality in all passages, that is,

$$Q^*(\tilde{u}, \tilde{v}) = 1, \qquad \lim_{k\to\infty} Q^*(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v}) = 0, \tag{2.21}$$

since $\tilde{u}, \tilde{v} \not\equiv 0$. It turns out that, since

$$S^* = \|(\tilde{u}, \tilde{v})\|^p + \lim_{k\to\infty} \|(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v})\|^p,$$

then

$$S^* = \|(\tilde{u}, \tilde{v})\|^p \qquad \text{and} \qquad \lim_{k\to\infty} \|(\tilde{u}_k - \tilde{u}, \tilde{v}_k - \tilde{v})\|^p = 0.$$

As in the previous case, we deduce that $|(\tilde{u}, \tilde{v})| \in W$ is also a minimizer for S^* is achieved by a non-negative function in the case $0 < \beta < sp + \theta$ and $\gamma < \gamma_H$.

□

2.3 Existence of Palais-Smale sequences

We shall now use the minimizers of $S^\sharp$ and S^* obtained in Proposition 2.1 to prove the existence of a nontrivial weak solution for equation (6). Recall the definition (7) of the energy functional associated to problem (6). The fractional Sobolev and fractional Hardy-Sobolev inequalities imply that $I \in C^1(W, \mathbb{R})$ and that

$$\begin{aligned} &\langle I'(u, v), (\phi_1, \phi_2) \rangle \\ &= \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi_1(x) - \phi_1(y))}{|x|^{\theta_1}|x - y|^{N+sp}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&+\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi_2(x)-\phi_2(y))}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\\
&-\gamma_1\int_{\mathbb{R}^N}\frac{|u|^{p-2}u\phi_1}{|x|^{sp+\theta}}\mathrm{d}x-\gamma_2\int_{\mathbb{R}^N}\frac{|v|^{p-2}v\phi_2}{|x|^{sp+\theta}}\mathrm{d}x\\
&-\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{p_s^\sharp-2}|u(y)|^{p_s^\sharp}u(x)\phi_1(x)}{|x|^{\delta}|x-y|^{\mu}|y|^{\delta}}\mathrm{d}x\mathrm{d}y-\iint_{\mathbb{R}^{2N}}\frac{|v(x)|^{p_s^\sharp-2}|v(y)|^{p_s^\sharp}v(x)\phi_2(x)}{|x|^{\delta}|x-y|^{\mu}|y|^{\delta}}\mathrm{d}x\mathrm{d}y\\
&-\int_{\mathbb{R}^N}\frac{|u|^{p_s^*-2}u(x)\phi_1}{|x|^{\beta}}\mathrm{d}x-\int_{\mathbb{R}^N}\frac{|v|^{p_s^*-2}v(x)\phi_2}{|x|^{\beta}}\mathrm{d}x\\
&+\int_{\mathbb{R}^N}\frac{\eta a|u|^{a-2}u\phi_1|v|^{b}}{|x|^{\beta}}\mathrm{d}x+\int_{\mathbb{R}^N}\frac{\eta b|u|^{a}|v|^{b-2}v\phi_2}{|x|^{\beta}}\mathrm{d}x.
\end{aligned}
$$

Note that a nontrivial critical point of I is a nontrivial weak solution to equation (6).

Recall that a Palais-Smale sequence for the energy functional I at the level $c \in \mathbb{R}$ is a sequence $\{(u_k, v_k)\}_{k\in\mathbb{N}} \subset W$ such that

$$
\lim_{k\to+\infty} I(u_k, v_k) = c \quad \text{and} \quad \lim_{k\to+\infty} I'(u_k, v_k) = 0 \quad \text{strongly in } W'. \tag{2.22}
$$

This sequence is referred to as a $(PS)_c$ sequence.

Now we state a result that ensures the existence of a Palais-Smale sequence for the energy functional.

Proposition 2.2. *Let $s \in (0,1), 0 < \alpha, \beta < sp + \theta < N, \mu \in (0, N)$ and $\gamma < \gamma_H$. Consider the functional $I\colon W \to \mathbb{R}$ defined in (7) on the Banach space W. Then there exists a $(PS)_c$ sequence $\{(u_k, v_k)\} \subset W$ for I at some level $c \in (0, c^*)$, where*

$$
c^* := \min\left\{\left(\frac{1}{p}-\frac{1}{2p_s^\sharp(\delta,\theta,\mu)}\right)S^{\sharp\,\frac{2p_s^\sharp}{2p_s^\sharp(\delta,\theta,\mu)-p}},\left(\frac{1}{p}-\frac{1}{p_s^*(\beta,\theta)}\right)S^{*\,\frac{p_s^*(\beta,\theta)}{p_s^*(\beta,\theta)-p}}\right\}. \tag{2.23}
$$

To prove Proposition 1.23 we need the following version of the mountainpass theorem by Ambrosetti and Rabinowitz [12].

Lemma 2.3. *(Montain Pass Lemma) Let $(W, \|\cdot\|)$ be a Banach space and let $I \in C^1(W, \mathbb{R})$ a functional such that the following conditions are satisfied:*

(1) *$I(0, 0) = 0$;*

(2) *There exist $\rho > 0$ and $r > 0$ such that $I(u, v) \geqslant \rho$ for all $u, v \in \dot{W}_\theta^{s,p}(\mathbb{R}^N)$ with $\|(u, v)\| = r$;*

(3) *There exist $e \in W$ with $\|e\| > r$ such that $I(e) < 0$; define*

$$
c := \inf_{g\in\Gamma}\sup_{t\in[0,1]} I(g(t)),
$$

where

$$
\Gamma := \left\{g \in C^0([0,1], \dot{W}_\theta^{s,p}(\mathbb{R}^N))\colon g(0) = 0, g(e) < 0\right\}.
$$

Then $c \geqslant \rho > 0$, and there exists a $(PS)_c$ sequence $\{(u_k, v_k)\} \subset W$ for I at level c, i.e.,

$$
\lim_{k\to+\infty} I(u_k, v_k) = c \quad \text{and} \quad \lim_{k\to+\infty} I'(u_k, v_k) = 0 \quad \text{strongly in } W'.
$$

The proof of Proposition 1.23 follows from the next two lemmas.

Lemma 2.4. *The functional I verifies the assumptions of Lemma 2.3.*

Proof. Clearly, we have $I(0,0)=0$. We now verify the second assumption of Lemma 2.3. Recalling the definition (2.4) of the quadratic form Q^* and using inequality (1.7), for any $(u,v)\in W$ we obtain

$$\begin{aligned}
I(u,v) &= \frac{1}{p}\Big[\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\Big] \\
&\quad - \frac{\gamma_1}{p}\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{sp+\theta}}\mathrm{d}x - \frac{\gamma_2}{p}\int_{\mathbb{R}^N}\frac{|v|^p}{|x|^{sp+\theta}}\mathrm{d}x \\
&\quad - \frac{1}{2p_s^\sharp}\Big[\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{p_s^\sharp}|u(y)|^{p_s^\sharp}}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}}\frac{|v(x)|^{p_s^\sharp}|v(y)|^{p_s^\sharp}}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y\Big] \\
&\quad - \frac{1}{p_s^*}\Big[\int_{\mathbb{R}^N}\frac{|u|^{p_s^*}}{|x|^\beta}\mathrm{d}x + \int_{\mathbb{R}^N}\frac{|v|^{p_s^*}}{|x|^\beta}\mathrm{d}x + \int_{\mathbb{R}^N}\frac{\eta|u|^a|v|^b}{|x|^\beta}\mathrm{d}x\Big] \\
&\geqslant \frac{1}{p}\|(u,v)\|_W^p - \frac{C}{2p_s^\sharp}\Big[\|u\|^{2p_s^\sharp}_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)} + \|v\|^{2p_s^\sharp}_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)}\Big] - \frac{1}{p_s^*}Q^*(u,v) \\
&\geqslant \frac{1}{p}\|(u,v)\|_W^p - \frac{C}{2p_s^\sharp}\Big[\|u\|^{p}_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)} + \|v\|^{p}_{\dot{W}^{s,p}_\theta(\mathbb{R}^N)}\Big]^{\frac{2p_s^\sharp}{p}} - \frac{1}{p_s^*}Q^*(u,v) \\
&\geqslant \frac{1}{p}\|(u,v)\|_W^p - C_1\|(u,v)\|^{2p_s^\sharp} - C_2\|(u,v)\|^{p_s^*}.
\end{aligned}$$

Since $s\in(0,1), 0<\alpha,\beta<sp+\theta<N$ and $\mu\in(0,N)$, we have that $p_s^*(\beta,\theta)>p$ and $2p_s^\sharp>p_s^*(\alpha,\theta)>p$. Therefore, there exists $r>0$ small enough such that

$$\inf_{\|(u,v)\|=r} I(u,v)>\rho,$$

so item (2) of Lemma 2.3 are satisfied.

For $(u,v)\in W$ and $t\in\mathbb{R}_+$, we have

$$I(tu,tv)=\frac{t^p}{p}\|(u,v)\|^p-\frac{t^{2p_s^\sharp}}{2p_s^\sharp}Q^\sharp(u,v)-\frac{t^{p_s^*}}{p_s^*}Q^*(u,v);$$

since $2p_s^\sharp>p_s^*(\alpha,\theta)>p$, we deduce that

$$\lim_{t\to+\infty} I(tu,tv)=-\infty \quad \text{for any } (u,v)\in W.$$

Consequently, for any fixed $e\in W$, there exists $t_e>0$ such that $\|t_e e\|>r$ and $I(t_e e)<0$. Thus, item (3) of Lemma 1.24 is satisfied. ☐

From Lemma 2.4 above, we guarante by Lemma 2.3 the existence of a Palais-Smale sequence $\{(u_k,v_k)\}\subset W$ such that

$$\lim_{k\to+\infty} I(u_k,v_k)=c \quad\text{and}\quad \lim_{k\to+\infty} I'(u_k,v_k)=0 \quad\text{strongly in } W'.$$

Moreover, by the definition of c we deduce that $c\geqslant\rho>0$. Therefore $c>0$ for all function $(u,v)\in W\setminus\{(0,0)\}$.

Lemma 2.5. *Suppose that $\mu \in (0, N)$ and that $0 < \alpha < sp + \theta$. Then there exists $(u, v) \in W \setminus \{(0, 0)\}$ such that $c \in (0, c^*)$, where c^* is defined in (2.23).*

Proof. Using Proposition 2.1, we obtain the minimizers $(u_1, v_1) \in W$ for $S^\sharp$ and $(u_2, v_2) \in W$ for S^*, respectively. Thus, there exist a function $(e_1, e_2) \in W$ defined by

$$(e_1, e_2) = \begin{cases} (u_1, v_1), & \text{if } \frac{2p_s^\sharp - p}{2pp_s^\sharp} S_\mu(N, s, \gamma, \alpha)^{\frac{2p_\mu^\sharp(\alpha,\theta)}{2p_\mu^\sharp(\alpha,\theta)-p}} \leqslant \frac{sp-\beta}{p(N-\beta)} \Lambda(N, s, \gamma, \beta)^{\frac{N-\beta}{sp-\beta}} \\ (u_2, v_2), & \text{if } \dfrac{2p_s^\sharp(\delta, \theta, \mu) - p}{2pp_s^\sharp} S_\mu(N, s, \gamma, \alpha)^{\frac{2p_\mu^\sharp(\alpha,\theta)}{2p_\mu^\sharp(\alpha,\theta)-p}} > \frac{sp-\beta}{p(N-\beta)} \Lambda(N, s, \gamma, \beta)^{\frac{N-\beta}{sp-\beta}} \end{cases}$$

such that $\|(e_1, e_2)\| > r$ and $I(e_1, e_2) < 0$. We can define

$$c := \inf_{g \in \Gamma} \sup_{t \in [0,1]} I(g(t)),$$

where

$$\Gamma := \left\{ g \in C^0([0, 1], \dot{W}_\theta^{s,p}(\mathbb{R}^N)) \colon g(0) = 0, g(e_1, e_2) < 0 \right\}.$$

Clearly, we have that $c > 0$. For the case where $e = (u_1, v_1)$, we can deduce that

$$0 < c < \frac{2p_s^\sharp - p}{2pp_s^\sharp} S_\mu(N, s, \gamma, \alpha)^{\frac{p_s^\sharp}{p_\mu^*(\alpha,\theta)-1}}.$$

In fact, for all $t \geqslant 0$, by the definition of the functional I, we have that

$$I(tu_1, tv_1) \leqslant \frac{t^p}{p} \|(u_1, v_1)\|^p - \frac{t^{2p_s^\sharp}}{2p_s^\sharp} Q^\sharp(u_1, v_1) =: f_1(t).$$

It is easy to see that

$$\begin{aligned} f_1'(t) &= t^{p-1} \|(u_1, v_1)\|^p - t^{2p_s^\sharp - 1} Q^\sharp(u_1, v_1) \\ &= t^{p-1}[\|(u_1, v_1)\|^p - t^{2p_s^\sharp - p} Q^\sharp(u_1, v_1)]. \end{aligned}$$

So, $f_1'(\tilde{t}) = 0$ for

$$\tilde{t} = \left(\frac{\|(u_1, v_1)\|^p}{Q^\sharp(u_1, v_1)} \right)^{\frac{1}{2p_s^\sharp - p}}, \tag{2.24}$$

and this is a point of maximum for f_1. Besides of that, this maximum value is

$$f_1(\tilde{t}) = \left[\frac{1}{p} - \frac{1}{2p_s^\sharp} \right] \frac{\|(u_1, v_1)\|^{\frac{2pp_s^\sharp}{2p_s^\sharp - p}}}{Q^\sharp(u_1, v_1)^{\frac{p}{2p_s^\sharp - p}}} = \left[\frac{2p_s^\sharp - p}{2pp_s^\sharp} \right] S^{\sharp \frac{2p_s^\sharp}{2p_s^\sharp - p}}.$$

Therefore,

$$\sup_{t \geqslant 0} I(tu_1, tv_1) \leqslant \sup_{t \geqslant 0} f_1(t) = \frac{2p_s^\sharp - p}{2pp_s^\sharp(\delta, \theta, \mu)} S_\sharp^{\frac{2p_s^\sharp}{2p_s^\sharp - p}} \tag{2.25}$$

The equality does not hold in (2.25); otherwise, we would have that $\sup_{t\geqslant 0} I(tu_1,tv_1) = \sup_{t\geqslant 0} f_1(t)$. Let $t_1>0$ be the point where $\sup_{t\geqslant 0} I(tu_1,tv_1)$ is attained. We have

$$f_1(t_1) - \frac{t_1^{p_s^*(\beta,\theta)}}{p_s^*(\beta,\theta)} Q^*(u_1,v_1) = f_1(\tilde{t})$$

which means that $f_1(t_1) > f_1(\tilde{t})$, since $t_1>0$. This contradicts the fact that $\tilde{t}$ is the unique maximum point for f_1. Thus, we have strict inequality in (2.25), that is,

$$\sup_{t\geqslant 0} I(tu_1,tv_1) < \sup_{t\geqslant 0} f_1(t) = \frac{2p_\mu^\sharp(\alpha,\theta)-p}{2pp_\mu^\sharp(\alpha,\theta)} S^{\sharp\frac{2p_\mu^\sharp(\alpha,\theta)}{2p_\mu^\sharp(\alpha,\theta)-p}}. \tag{2.26}$$

Therefore, $0<c<\frac{2p_\mu^\sharp(\alpha,\theta)-p}{2pp_\mu^\sharp(\alpha,\theta)} S^{\sharp\frac{2p_\mu^\sharp(\alpha,\theta)}{2p_\mu^\sharp(\alpha,\theta)-p}}$.

Similarly, for the case of $e=(u_2,v_2)$, we can verify that

$$\sup_{t\geqslant 0} I(tu_2,tv_2) < \frac{sp-\beta}{p(N-\beta)} \Lambda(N,s,\gamma,\beta)^{\frac{N-\beta}{sp-\beta}}. \tag{2.27}$$

In fact, for all $t\geqslant 0$, by functional I definition we have that

$$I(tu_2,tv_2) \leqslant \frac{t^p}{p}\|(u_2,v_2)\|^p - \frac{t^{p_s^*}}{p_s^*}Q^*(u,v) := g_1(t).$$

It is easy to see that

$$\begin{aligned} g_1'(t) &= t^{p-1}\|(u_2,v_2)\|^p - t^{p_s^*-1}Q^*(u,v) \\ &= t^{p-1}\left[\|(u_2,v_2)\|^p - t^{p_s^*-p}Q^*(u,v)\right]. \end{aligned}$$

So, $g_1(\tilde{t})=0$ for

$$\tilde{t} = \left(\frac{\|(u_2,v_2)\|^p}{Q^*(u,v)}\right)^{\frac{1}{p_s^*-p}},$$

and this is a point of maximum for g_1. Besides of that, this maximum value is

$$g_1(\tilde{t}) = \left[\frac{1}{p}-\frac{1}{p_s^*}\right]\left(\frac{\|(u_2,v_2)\|^p}{Q^{*\frac{p}{p_s^*}}}\right)^{\frac{p_s^*}{p_s^*-p}} = \frac{sp+\theta-\beta}{p(N-\beta)} S^{*\frac{N-\beta}{sp+\theta-\beta}}.$$

Therefore,

$$\sup_{t\geqslant 0} I(tu_2,tv_2) \leqslant \sup_{t\geqslant 0} g_1(t) = \frac{sp+\theta-\beta}{p(N-\beta)} S^{*\frac{N-\beta}{sp+\theta-\beta}}. \tag{2.28}$$

The equality does not hold in (2.28), otherwise, we would have that $\sup_{t\geqslant 0} I(tu_2,tv_2) = \sup_{t\geqslant 0} g_1(t)$. Let $t_1>0$, where $\sup_{t\geqslant 0} I(tu_2,tv_2)$ is attained. We have

$$g_1(t_1) - \frac{t_1^{2p_s^\sharp}}{2p_s^\sharp} Q^\sharp(u_2,v_2) = g_1(\tilde{t})$$

which means that $g_1(t_1) > g_1(\tilde{t})$, since $t_1 > 0$. This contradicts the fact that $\tilde{t}$ is the unique maximum point for $g_1(t)$. Thus

$$\sup_{t\geqslant 0} I(tu_2, tv_2) < \sup_{t\geqslant 0} g_1(t) = \frac{sp+\theta-\beta}{p(N-\beta)} S^{*\frac{N-\beta}{sp+\theta-\beta}}. \tag{2.29}$$

Therefore, $0 < c < \frac{sp+\theta-\beta}{p(N-\beta)} S^{*\frac{N-\beta}{sp+\theta-\beta}}$.

From the definition (2.23) of c^* and from inequalities (2.26) and (1.67), we have

$$0 < c < c^* := \min\left\{\left(\frac{1}{p} - \frac{1}{2p_s^\sharp}\right) S^{\sharp\frac{2p_s^\sharp}{2p_s^\sharp - p}}, \left(\frac{1}{p} - \frac{1}{p_s^*}\right) S^{*\frac{p_s^*}{p_s^* - p}}\right\}.$$

The lemma is proved. □

Proof of Proposition 2.2. Follows immediately from Lemmas 2.4 and 2.5. □

2.4 Proof of Theorems 0.3 and 0.4

The existence of a solution will follow from the proof of the Theorem 0.3.

Proof of Theorem 0.3. Suppose that $s \in (0,1), 0 < \alpha, \beta < sp + \theta, \mu \in (0, N)$ and $\gamma < \gamma_H$.

Let $\{(u_k, v_k)\}_{k\in\mathbb{N}} \subset W$ be a Palais-Smale sequence $(PS)_c$ as in Proposition 2.2, i.e.,

$$I(u_k, v_k) \to c,\ I'(u_k, v_k) \to 0 \qquad \text{strongly in } W' \text{ as } k \to +\infty.$$

Then

$$I(u_k, v_k) = \frac{1}{p}\|(u_k, v_k)\|^p - \frac{1}{2p_s^\sharp} Q^\sharp(u_k, u_k) - \frac{1}{p_s^*} Q^*(u_k, v_k) = c + o(1) \tag{2.30}$$

and

$$\langle I'(u_k, v_k), (u_k, v_k)\rangle = \|(u_k, v_k)\|^p - Q^\sharp(u_k, v_k) - Q^*(u_k, v_k) = o(1). \tag{2.31}$$

From (2.30) and (2.31), if $2p_s^\sharp \geqslant p_s^* > p$, we have

$$\begin{aligned} c + o(1)\|(u_k, v_k)\| &= I(u_k, v_k) - \frac{1}{p_s^*}\langle I'(u_k, v_k), (u_k, v_k)\rangle \\ &= \frac{1}{p}\|(u_k, v_k)\|^p - \frac{1}{2p_s^\sharp} Q^\sharp(u_k, v_k) - \frac{1}{p_s^*} Q^*(u_k, v_k) \\ &\quad - \frac{1}{p_s^*}\|(u_k, v_k)\|^p + \frac{1}{p_s^*} Q^\sharp(u_k, v_k) + \frac{1}{p_s^*} Q^*(u_k, v_k) \\ &= \frac{p_s^* - p}{p \cdot p_s^*}\|(u_k, v_k)\|^p + \left(\frac{1}{p_s^*} - \frac{1}{2p_s^\sharp}\right) Q^\sharp(u_k, v_k) \\ &\geqslant \frac{p_s^* - p}{p \cdot p_s^*}\|(u_k, v_k)\|^p. \end{aligned}$$

Again from (2.30) and (2.31), if $p_s^* > 2p_s^\sharp > p$, we have

$$c + o(1)\|(u_k, v_k)\| = I(u_k, v_k) - \frac{1}{2p_s^\sharp}\langle I'(u_k, v_k), (u_k, v_k)\rangle$$

$$
\begin{aligned}
&= \frac{1}{p}\|(u_k,v_k)\|^p - \frac{1}{2p_s^\sharp}Q^\sharp(u_k,u_k) - \frac{1}{p_s^*}Q^*(u_k,v_k)\\
&\quad - \frac{1}{2p_s^\sharp}\|(u_k,v_k)\|^p + \frac{1}{2p_s^\sharp}Q^\sharp(u_k,u_k) + \frac{1}{2p_s^\sharp}Q^*(u_k,v_k)\\
&= \frac{2p_s^\sharp - p}{p\cdot 2p_s^\sharp}\|(u_k,v_k)\|^p + \left(\frac{1}{2p_s^\sharp} - \frac{1}{p_s^*}\right)Q^*(u_k,v_k)\\
&\geqslant \frac{2p_s^\sharp - p}{p\cdot 2p_s^\sharp}\|(u_k,v_k)\|^p.
\end{aligned}
$$

Thus, $\{(u_k,v_k)\}_{k\in\mathbb{N}} \subset W$ is a bounded sequence; so from the estimate (2.31) there exists a subsequence, still denoted by $\{(u_k,v_k)\}_{k\in\mathbb{N}} \subset W$, such that

$$
\|(u_k,v_k)\|^p \to b, \qquad Q^*(u_k,v_k) \to d_1, \qquad Q^\sharp(u_k,u_k) \to d_2,
$$

as $k \to +\infty$; additionally,

$$
b = d_1 + d_2.
$$

By the definitions of $S^\sharp$ and S^*, we get

$$
d_1^{\frac{p}{p_s^*}} S^* \leqslant b = d_1 + d_2, \qquad d_2^{\frac{1}{p_s^\sharp}} S^\sharp \leqslant b = d_1 + d_2.
$$

From the first inequality we have $d_1^{\frac{p}{p_s^*}} S^* - d_1 \leqslant d_2$, that is

$$
d_1^{\frac{p}{p_s^*}}\left(S^* - d_1^{\frac{p_s^*-p}{p_s^*}}\right) \leqslant d_2. \tag{2.32}
$$

And from the second inequality we have $d_2^{\frac{1}{p_s^\sharp}} S^\sharp - d_2 \leqslant d_1$, that is,

$$
d_2^{\frac{1}{p_s^\sharp}}\left(S^\sharp - d_2^{\frac{p_s^\sharp-1}{p_s^\sharp}}\right) \leqslant d_1. \tag{2.33}
$$

Claim 2. *We have*

$$
S^* - d_1^{\frac{p_s^*-p}{p_s^*}} > 0, \qquad S^\sharp - d_2^{\frac{p_s^\sharp-1}{p_s^\sharp}} > 0.
$$

Proof. In fact, since $c + o(1)\|(u_k,v_k)\| = I(u_k,v_k) - \frac{1}{p}\langle I'(u_k,v_k),(u_k,v_k)\rangle$, we have

$$
\begin{aligned}
I(u_k,v_k) - \frac{1}{p}\langle I'(u_k,v_k),(u_k,v_k)\rangle &= \frac{1}{p}\|(u_k,v_k)\|^p - \frac{1}{2p_\mu^\sharp(\alpha,\theta)}Q^\sharp(u_k,u_k) - \frac{1}{p_s^*}Q^*(u_k,v_k)\\
&\quad - \frac{1}{p}\|(u_k,v_k)\|^p + \frac{1}{p}Q^\sharp(u_k,u_k) + \frac{1}{p}Q^*(u_k,v_k)\\
&= \left(\frac{1}{p} - \frac{1}{2p_\mu^\sharp(\alpha,\theta)}\right)Q^\sharp(u_k,u_k) + \left(\frac{1}{p} - \frac{1}{p_s^*}\right)Q^*(u_k,v_k)\\
&= c + o(1)\|(u_k,v_k)\|.
\end{aligned}
$$

Passing to the limit as $k \to +\infty$, we get

$$
\left(\frac{1}{p} - \frac{1}{p_s^*}\right)d_1 + \left(\frac{1}{p} - \frac{1}{2p_\mu^\sharp(\alpha,\theta)}\right)d_2 = c; \tag{2.34}
$$

so,

$$d_1 \leqslant \left(\frac{1}{p} - \frac{1}{p_s^*}\right)^{-1} c = \frac{p(N-\beta)}{sp+\theta-\beta}\, c, \qquad d_2 \leqslant \left(\frac{1}{p} - \frac{1}{2p_\mu^\sharp(\alpha,\theta)}\right)^{-1} c = \frac{2pp_s^\sharp}{2p_s^\sharp - p}\, c.$$

Using these upper bounds for d_1, d_2 and the fact $0 < c < c^*$, we have

$$\begin{aligned}
S^* - d_1^{\frac{p_s^*-p}{p_s^*}} &\geqslant S^* - \left[\frac{p(N-\beta)}{sp+\theta-\beta}\, c\right]^{\frac{p_s^*-p}{p_s^*}} \\
&> S^* - \left[\frac{p(N-\beta)}{sp+\theta-\beta}\, c^*\right]^{\frac{p_s^*-p}{p_s^*}} \\
&\geqslant S^* - \left[\frac{p(N-\beta)}{sp+\theta-\beta} \cdot \frac{(sp+\theta-\beta)}{p(N-\beta)}\, S^{*\frac{N-\beta}{(sp+\theta-\beta)}}\right]^{\frac{p_s^*-p}{p_s^*}} \\
&= S^* - S^* = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
S^\sharp - d_2^{\frac{p_s^\sharp-1}{p_s^\sharp}} &\geqslant S^\sharp - \left[\frac{2pp_s^\sharp}{2p_s^\sharp - p}\, c\right]^{\frac{p_s^\sharp-1}{p_s^\sharp}} \\
&> S^\sharp - \left[\frac{2pp_s^\sharp}{2p_s^\sharp - p}\, c^*\right]^{\frac{p_s^\sharp-1}{p_s^\sharp}} \\
&\geqslant S^\sharp \left[\frac{2pp_s^\sharp}{(2p_s^\sharp - p)} \cdot \frac{(2p_s^\sharp - p)}{2pp_s^\sharp}\, S^{\sharp\frac{p_s^\sharp}{p_s^\sharp-1}}\right]^{\frac{p_s^\sharp-1}{p_s^\sharp}} \\
&= S^\sharp - S^\sharp = 0.
\end{aligned}$$

This concludes the proof of the claim. □

Following up, inequalities (2.32) and (2.33) imply, respectively, that

$$\left[S^* - \left(\frac{p(N-\beta)}{sp+\theta-\beta} c\right)^{\frac{p_s^*-p}{p_s^*}}\right] d_1^{\frac{p}{p_s^*}} \leqslant \left[S^* - d_1^{\frac{p_s^*-p}{p_s^*}}\right] d_1^{\frac{p}{p_s^*}} \leqslant d_2$$

and

$$\left[S^\sharp - \left(\frac{2pp_s^\sharp}{2p_s^\sharp - p} c\right)^{\frac{p_s^\sharp-1}{p_s^\sharp}}\right] d_2^{\frac{1}{p_s^\sharp}} \leqslant \left[S^\sharp - d_2^{\frac{p_s^\sharp-1}{p_s^\sharp}}\right] d_2^{\frac{1}{p_s^\sharp}} \leqslant d_1.$$

If $d_1 = 0$ and $d_2 = 0$, then (2.34) implies that $c = 0$, which is in contradiction with $c > 0$. Therefore, $d_1 > 0$ and $d_2 > 0$ and we can choose $\epsilon_0 > 0$ such that $d_1 \geqslant \epsilon_0 > 0$ and $d_2 \geqslant \epsilon_0 > 0$; moreover, there exists a $K \in \mathbb{N}$ such that

$$Q^*(u_k, v_k) > \frac{\epsilon_0}{2}, \qquad Q^\sharp(u_k, u_k) > \frac{\epsilon_0}{2}$$

for every $k > K$. The embeddings (1.4), and the improved Sobolev inequality (1.8) imply that there exist $C_1, C_2 > 0$ such that

$$0 < C_2 \leqslant \|u_k\|_{L_M^{p,N-sp-\theta+pr}(\mathbb{R}^N,|y|^{-pr})} \leqslant C_1,$$

where $r = \frac{\alpha}{p_s^*(\alpha,\theta)}$. For any $k > K$, we may find $\lambda_k > 0$ and $x_k \in \mathbb{R}^N$ such that

$$\lambda_k^{(N-sp-\theta+pr)-N} \int_{B_{\lambda_k}(x_k)} \frac{|u_k(y)|^p}{|y|^{pr}} \mathrm{d}y > \|u_k\|^p_{L_M^{p,N-sp-\theta+pr}(\mathbb{R}^N,|y|^{-pr})} - \frac{C}{2k} \geqslant \tilde{C} > 0.$$

Now we define the sequence $\{\tilde{u}_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ by $\tilde{u}_k(x) = \lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k x)$ and the sequence $\{\tilde{v}_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ by $\tilde{v}_k(x) = \lambda_k^{\frac{N-sp-\theta}{p}} v_k(\lambda_k x)$. As we have already shown, $\|\tilde{u}_k\| = \|u_k\| \leqslant C$ and $\|\tilde{v}_k\| = \|v_k\| \leqslant C$ for every $k \in \mathbb{N}$; so, there exist $u \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ and $v \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ such that, after passage to subsequence, still denoted in the same way,

$$\tilde{u}_k \rightharpoonup u \text{ in } \dot{W}^{s,p}_\theta(\mathbb{R}^N) \qquad \text{and} \qquad \tilde{v}_k \rightharpoonup v \text{ in } \dot{W}^{s,p}_\theta(\mathbb{R}^N)$$

as $k \to +\infty$. In a fashion similar to the proof of Proposition 1.22-(1), we can prove that $u \not\equiv 0$ and $v \not\equiv 0$.

In addition, the boundedness of the sequences $\{\tilde{u}_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ and $\{\tilde{v}_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ implies that the sequences $\{|\tilde{u}_k|^{p_s^*-2}\tilde{u}_k\}_{k\in\mathbb{N}} \subset L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N, |x|^{-\beta})$ and $\{|\tilde{v}_k|^{p_s^*-2}\tilde{v}_k\}_{k\in\mathbb{N}} \subset L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N, |x|^{-\beta})$ are bounded too. In fact, by embeddings (1.4), we obtain

$$\int_{\mathbb{R}^N} \frac{\left||\tilde{u}_k|^{p_s^*-2} \cdot \tilde{u}_k\right|^{\frac{p_s^*}{p_s^*-1}}}{|x|^\beta} \mathrm{d}x = \int_{\mathbb{R}^N} \frac{|\tilde{u}_k|^{p_s^*}}{|x|^\beta} \mathrm{d}x < C.$$

and

$$\int_{\mathbb{R}^N} \frac{\left||\tilde{v}_k|^{p_s^*-2} \cdot \tilde{v}_k\right|^{\frac{p_s^*}{p_s^*-1}}}{|x|^\beta} \mathrm{d}x = \int_{\mathbb{R}^N} \frac{|\tilde{v}_k|^{p_s^*}}{|x|^\beta} \mathrm{d}x < C.$$

Then, after passage to subsequence, still denoted in the same way, we deduce that

$$|\tilde{u}_k|^{p_s^*-2}\tilde{u}_k \rightharpoonup |u|^{p_s^*-2}u \quad \text{in } L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N, |x|^{-\beta}) \tag{2.35}$$

and

$$|\tilde{v}_k|^{p_s^*-2}\tilde{v}_k \rightharpoonup |v|^{p_s^*-2}v \quad \text{in } L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N, |x|^{-\beta}) \tag{2.36}$$

as $k \to +\infty$.

For any $\phi_1, \phi_2 \in L^{p_s^*(\alpha,\theta)}(\mathbb{R}^N, |x|^{-\alpha})$, Lemma 1.18 implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot, \tilde{u}_k)\right](x) f_\alpha(x, \tilde{u}_k)\phi_1(x)\mathrm{d}x = \int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot, u)\right](x) f_\alpha(x, u)\phi_1(x)\mathrm{d}x \tag{2.37}$$

and

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot, \tilde{v}_k)\right](x) f_\alpha(x, \tilde{v}_k)\phi_2(x)\mathrm{d}x = \int_{\mathbb{R}^N} \left[I_\mu * F_\alpha(\cdot, v)\right](x) f_\alpha(x, v)\phi_2(x)\mathrm{d}x. \tag{2.38}$$

Since $\dot{W}^{s,p}_\theta(\mathbb{R}^N) \hookrightarrow L^{p^*_s(\alpha,\theta)}(\mathbb{R}^N, |x|^{-\alpha})$, (2.37) and (2.38) hold for any $\phi_1, \phi_2 \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$.

Finally, we need to check that $\{\tilde{u}_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ and $\{\tilde{v}_k\}_{k\in\mathbb{N}} \subset \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ are also a $(PS)_c$ sequence for the functional I at energy level c. Do to this, we note that the norms in $L^{p^*_s(\alpha,\theta)}(\mathbb{R}^N, |x|^{-\alpha})$ are invariant under the special dilatation $\tilde{u}_k = \lambda_k^{\frac{N-sp-\theta}{p}} u_k(\lambda_k x)$ and $\tilde{v}_k = \lambda_k^{\frac{N-sp-\theta}{p}} v_k(\lambda_k x)$. In fact

$$\|\tilde{u}_k\|^{p^*_s(\alpha,\theta)}_{L^{p^*_s(\alpha,\theta)}} = \int_{\mathbb{R}^N} \frac{\lambda_k^{\frac{N-sp-\theta}{p}p^*_s(\alpha,\theta)} |u_k(\lambda_k x)|^{p^*_s(\alpha,\theta)})}{|x|^\alpha} dx = \int_{\mathbb{R}^N} \frac{|u_k(\bar{x})|^{p^*_s(\alpha,\theta)}}{|\bar{x}|^\alpha} d\bar{x} = \|u_k\|^{p^*_s(\alpha,\theta)}_{L^{p^*_s(\alpha,\theta)}}$$

and

$$\|\tilde{v}_k\|^{p^*_s(\beta,\theta)}_{L^{p^*_s(\beta,\theta)}} = \int_{\mathbb{R}^N} \frac{\lambda_k^{\frac{N-sp-\theta}{p}p^*_s(\beta,\theta)} |v_k(\lambda_k x)|^{p^*_s(\beta,\theta)})}{|x|^\beta} dx = \int_{\mathbb{R}^N} \frac{|v_k(\bar{x})|^{p^*_s(\beta,\theta)}}{|\bar{x}|^\beta} d\bar{x} = \|v_k\|^{p^*_s(\beta,\theta)}_{L^{p^*_s(\beta,\theta)}}.$$

Besides of that, we have

$$\int_{\mathbb{R}^N} \frac{\eta |\tilde{u}_k(x)|^a |\tilde{v}_k(x)|^b}{|x|^\beta} dx = \int_{\mathbb{R}^N} \frac{\eta \lambda_k^{\frac{N-sp-\theta}{p}(a+b)} |u_k(\lambda_k x)|^a |v_k(\lambda_k x)|^b}{|x|^\beta} dx = \int_{\mathbb{R}^N} \frac{\eta |u_k(\bar{x})|^a |v_k(\bar{x})|^b}{|\bar{x}|^\beta} d\bar{x}.$$

Thus, we have

$$\lim_{k\to+\infty} I(\tilde{u}_k, \tilde{v}_k) = c.$$

Moreover, for all $\phi_1, \phi_2 \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$, we have $\phi_{1,k}(x) = \lambda_k^{\frac{N-sp-\theta}{p}} \phi_1(x/\lambda_k) \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ and $\phi_{2,k}(x) = \lambda_k^{\frac{N-sp-\theta}{p}} \phi_2(x/\lambda_k) \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$. From $I'(u_k, v_k) \to 0$ in W' as $k \to +\infty$, we can deduce that

$$\lim_{k\to+\infty} \langle I'(\tilde{u}_k, \tilde{v}_k), (\phi_1, \phi_2)\rangle = \lim_{k\to+\infty} \langle I'(u_k, v_k), (\phi_1, \phi_2)\rangle = 0.$$

Thus (2.35), (2.36), (2.37) and (2.38) lead to

$$\langle I'(u, v), (\phi_1, \phi_2)\rangle = \lim_{k\to+\infty} \langle I'(\tilde{u}_k, \tilde{v}_k), (\phi_1, \phi_2)\rangle = 0.$$

Hence (u, v) is a nontrivial weak solution to problem 6. □

Proof of Theorem 0.4. The proof follows the same steps of the proof of Theorem 0.3. Here we only remark that for problem (8) with a Hardy potential and double Sobolev type nonlinearities we have to define the value below which we can recover the compactness of the Palais-Smale sequences by

$$c^* := \min_{k\in\{1,2\}} \left\{ \left(\frac{1}{p} - \frac{1}{p^*_s}\right) S^*(N, s, \gamma, \beta_k)^{\frac{p^*_s(\beta_k,\theta)}{p^*_s(\beta_k,\theta)-p}} \right\}.$$

Similarly, for problem (9) with a Hardy potential and double Choquard type nonlinearities we have to define the corresponding number by

$$c^* := \min_{k\in\{1,2\}} \left\{ \left(\frac{1}{p} - \frac{1}{2p^\sharp_s(\delta, \theta, \mu_k)}\right) S^\sharp(N, s, \gamma, \beta_k)^{\frac{2p^\sharp_s(\delta,\theta,\mu_k)}{2p^\sharp_s(\delta,\theta,\mu_k)-p}} \right\}.$$

The details are omitted. □

Chapter 3

Fractional Kirchhoff equation with Sobolev-Choquard singular nonlinearities

3.1 Historical background

The study of non-local problems driven by the fractional and non-local operators has received a tremendous popularity because of their intriguing structure and the great application in the number and variety of phenomena occurring in real-world applications that can be modeled by these equations such as optimization, finance, phase transition phenomena, anomalous diffusion, dislocations in crystals, quantum mechanics, game theory, water waves, phase transitions, stratified materials, semipermeable membranes, population dynamics. For more information on non-local and fractional problems, see the excelent survey papers by Di Nezza, Palatucci & Valdinoci [39] and Moroz & Van Schaftingen [68]; see also the book by Molica Bisci, Rădulescu & Servadei [66].

The Choquard equation

On the Euclidean space $\mathbb{R}^N$, the equation

$$-\Delta u + V(x)u = (I_\mu * |u|^q)|u|^{q-2}u \qquad (x \in \mathbb{R}^N)$$

was introduced by Choquard in the case $N = 3$ and $q = 2$ to model one-component plasma. It had appeared earlier in the model of the polaron by Frölich and Pekar, where free electrons interact with the polarisation that they create on the medium. A remarkable feature in the Choquard nonlinearity is the appearance of a lower nonlinear restriction, usually called the lower critical exponent $2^\flat > 1$, that is, the nonlinearity is superlinear. When $V(x) \equiv 1$, the groundstate solutions exist if $2^\flat := 2(N - \mu/2))/N < q < 2(N - \mu/2)/(N - 2s) := 2^\sharp$ due to the mountain pass lemma or the method of the Nehari manifold, while there are no nontrivial solution if $q = 2^\flat$ or if $q = 2^\sharp$ as a consequence of the Pohozaev identity.

In general, the associated Schrödinger-type evolution equation $i\partial_t\psi = \Delta\psi + \Big(I_\mu * |\psi|^2\Big)\psi$ is a model for large systems of atoms with an attractive interaction that is weaker and has a longer range than that of the nonlinear Schrödinger equation. Standing wave solutions of this equation are solutions to the Choquard equation. For more information on the various results related to the non-fractional Choquard-type equations and their variants see Moroz & Van Schaftingen [68].

Kirchhoff type problems

Kirchhoff type problems have been widely studied in recent years. After Lions has presented an abstract functional framework to use for Kirchhoff type equations, this problem has been widely studied in extensive literature. Alves, Corrêa & Figueiredo [7] investigated the existence of positive solutions to the class of non-local boundary value problems of the Kirchhoff type with the classical Laplace operator. For other papers involving the Kirchhoff type problems with the classical Laplace operator see Chen & Li [26] and Figueiredo [41]. For the p-Laplacian case, Colasuanno & Pucci [32] established the existence of infinitely many solutions for Dirichlet problems involving the p-polyharmonic operators on bounded domains. For the fractional Kirchhoff problem, Fiscella & Valdinoci [44] proposed a stationary fractional Kirchhoff variational problem which takes into account the non-local aspect of the tension arising from non-local measurements of the fractional length of the string. It was pointed out in Pucci, Xiang & Zhang [75] that equations like $-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\Big)\Delta u+u=kf(u)+|u|^{2^*-2}u$ in the whole space $\mathbb{R}^N$ can be applied to describe the growth and movement of a specific species. Song & Shi [84] considered a class of degenerate fractional p-Laplacian equation of Schrödinger–Kirchhoff with critical Hardy–Sobolev nonlinearities. The main feature and difficulty of this article is the fact that the Kirchhoff term could be zero at zero, that is, the problem equation is degenerate. The degenerate case was studied by Autuori, Fiscella & Pucci in [16], by introducing a new technical approach based on the asymptotic property of the critical mountain pass level; they established the existence and the asymptotic behavior of non-negative solutions to the problem. Furthermore, the existence of a solution for different critical fractional Kirchhoff problems set on the whole space $\mathbb{R}^N$ is given by Liang & Shi [57]. More recently, Chen [28] establish the existence of solutions to the fractional p-Kirchhoff type equations with a generalized Choquard nonlinearities without assuming the Ambrosetti Rabinowitz condition.

The kinds and varieties of potential functions

Several hypotheses have been used on the potential function included in the class of elliptical problems. For example, Berestycki & Lions [19] considered the case in which the potential function $V\colon\mathbb{R}^N\to\mathbb{R}$ is constant. Afterwards, Pankov [74] studied a problem with $V\in L^\infty$ being a periodic function with unit period in each variable, that is, $V(x+z)=V(x)$ for all $x\in\mathbb{R}^N$ and with $z\in\mathbb{Z}^N$; for other articles about problems with periodic potentials, see Coti-Zelati & Rabinowitz [83] and Kryszewski & Szulkin [52]. Zhu & Yang [51] studied problems with an asymptotic potential V at a positive constant, i.e., there is $V_\infty\in\mathbb{R}_+$ such that $|V(x)-V_\infty|\to 0$ when $|x|\to+\infty$ and $V(x)\leqslant V_\infty$ for all $x\in\mathbb{R}^N$. For the case where the potential V is strictly positive and the Lebesgue measure of the set $\{x\in\mathbb{R}^N\colon V(x)\leqslant M\}$ is finite for all $M\in\mathbb{R}_+$, see Bartsch & Wang [18]. Costa [33] and Miyagaki [65] studied the case of a coercive potential $V\colon\mathbb{R}^N\to\mathbb{R}$, that is, $\lim_{|x|\to+\infty}V(x)=+\infty$. For the case of a radial potential $V\colon\mathbb{R}^N\to\mathbb{R}$, i.e., $V(x)=W(r)$ such that $W\colon\mathbb{R}_+^*\to\mathbb{R}$ and $r=|x|$ for all $x\in\mathbb{R}^N$, see Alves, de Morais Filho & Souto [8]. Alves & Souto [5, 6] considered a continuous, non-negative potential function V which can vanish at infinity, that is, $V(x)\to 0$ when $|x|\to\infty$; for other problems involving this kind of potential, see Alves, Assunção & Miyagaki [9] and Alves & Assunção [10].

Palais-Smale and Cerami conditions

Let B be a Banach space such that $J\colon B \to \mathbb{R}$ is a C^1 functional defined on B and (u_n) is a sequence in B. The Palais-Smale condition $(PS)_c$ at level c means that if the sequence $\{u_n\}_{n\in\mathbb{N}}$ is such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to +\infty$, then $\{u_n\}$ has a convergent subsequence. The Cerami condition $(C)_c$ at level c means that if the sequence$\{u_n\}_{n\in\mathbb{N}}$ is such that $J(u_n) \to c$ and $(1 + \|u_n\|)J'(u_n) \to 0$ as $n \to +\infty$, then $c \in \mathbb{R}$ is a critical value of J. With the above conditions it is possible to show that if a sequence $\{u_n\}_{n\in\mathbb{N}}$ verifies the Palais-Smale condition $(PS)_c$, then it also verifies the Cerami condition $(C)_c$; for more details see Costa [34]. That the Cerami condition $(C)_c$ does not imply the Palais-Smale condition $(PS)_c$ can be seen by the function $z\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $z(x, y) = \ln(1 + x^2) - \ln(1 + y^2)$; this function verifies the Cerami condition $(C)_0$ but does not verify the Palais-Smale condition $(PS)_0$, for the level set $z^{-1}(0)$ is $|x| = |y|$; see Robinson [77]. A Cerami sequence can produce a critical point even when a (PS) sequence does not. A condition similar to $(C)_c$ was introduced by Cerami and was applied to the search for critic points of a functional on an unbounded Riemannian manifold. It should be mentioned that this weakening of the Palais-Smale condition seems essential in the study of variational problems in the strong resonance case because in general the Palais-Smale condition is not satisfied. For more information on these kinds of compactness conditions see the following comments about the nonlinearities.

Some types of frequently used nonlinearities

Several interesting questions arise when we consider the nonlinearities that appear in the study of partial differential equations. For example, inspired by Harrabi [49], consider the general prototype equation $-\Delta u = f(x, u)$ where $x \in \Omega$ with $\Omega \subset \mathbb{R}^N$ a bounded, open subset. We look for weak solutions in the Sobolev space $H_0^1(\Omega) := \{w \in H^1(\Omega)\colon w = 0 \text{ on } \Omega\}$. As usual, a weak solution to this problem is any function $u \in H_0^1(\Omega)$ such that $\langle u, v\rangle_{H_0^1(\Omega)} = \int_\Omega f(x, u)v \, \mathrm{d}x$ for every function $v \in H_0^1(\Omega)$; here, the inner product is defined by $\langle u, v\rangle_{H_0^1(\Omega)} := \int_\Omega uv \, \mathrm{d}x$. It is well known that a function $u \in H_0^1(\Omega)$ is a weak solution to this problem if, and only if, it is a critical point of the Euler-Lagrange energy functional defined by $J(u) = (1/2)\|u\|^2_{H_0^1(\Omega)} - \int_\Omega F(x, u) \, \mathrm{d}x$ where $F(x, s) = \int_0^s f(x, t) \, \mathrm{d}t$.

One can ask whether the differential equations have any nontrivial solutions; one can also ask whether it is possible to give a lower bound to the number of solutions by using some topological facts related to the nonlinearity $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ (e.g., if it is odd in the first variable). For example, see Amann & Zehnder [11] or Castro & Lazer [24].

For general operators, the blow up argument can be used to get existence of positive solution when the nonlinearity f has an assymptotical behavior like $f(s) = |s|^{q-2}s$ at infinity with $1 < q < 2^* = 2N/(N - 2)$.

Most results use some hypotheses on the nonlinearity f to make variational methods work. For example, $f \in C(\Omega \times \mathbb{R}; \mathbb{R})$ satisfying the large subcritical growth condition,

- (h_1) there exist $C_0 \in \mathbb{R}_+$ and $s_0 \in \mathbb{R}_+$ such that $|f(x, s)| \leqslant C|s|^{2^*-1}$ for every $|s| \geqslant s_0$ and for every $x \in \Omega$.

Under hypothesis (h_1) the energy functional is well defined in the Sobolev space $H_0^1(\Omega)$ and belongs to $C^1(H_0^1(\Omega); \mathbb{R})$. If we impose some more additional conditions on the nonlinearity f, for example if $f(x, s) = a(x)g(s)$ where $a \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R})$, then it is possible to prove that any weak solution to the problem belongs to the space $C^2(\overline{\Omega})$.

The use of critical point theory needs a compactness condition, usually the Palais-Smale (PS) condition or the Cerami (C) condition. Our main goal in this brief survey is to revise the required standard assumptions used to get one these conditions. See Clément, Figueiredo & Mitidieri [31] or Ramos & Rodrigues [76].

In great part of the literature the (PS) condition is proved by using standard assumptions. Mainly, the Ambrosetti-Rabinowitz condition, the (AR) condition in short, which supposes the existence of $\xi \in \mathbb{R}$ with $\xi > 2$ and $s_0 \in \mathbb{R}_+$ such that $sf(x,s) \geqslant \xi F(x,s)$ for $|s| > s_0$ and $x \in \Omega$.

Another typical hypothesis is the subcritical polynomial growth condition,

(h_2) there exist $C \in \mathbb{R}_+$ and $q \in \mathbb{R}$ such that $1 \leqslant q < 2^*$ such that $|f(x,s)| \leqslant C(|s|^{q-1}+1)$ for every $x \in \Omega$ and for every $s \in \mathbb{R}$.

Under the (AR) condition and hypothesis (h_2), the Euler-Lagrange energy functional associated to the differential equation verifies the (PS) condition.

The Ambrosetti-Rabinowitz condition revisited

The major difficulty in the use of the (PS) condition consists often in proving the boundedness of any Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$. In contrast, for the (AR) condition one has $(\xi/2-1)\|u\|^2 \leqslant C_0(\|u\|+1)$. However, the (AR) condition is too restrictive and one requires instead the strong superlinear condition,

(h_3) there exists $C \in \mathbb{R}_+$ and $q \in \mathbb{R}$ with $q = \xi - 1 > 0$ such that $|f(x,s)| \geqslant C(|s|^q - 1)$ for every $x \in \Omega$ and for every $s \in \mathbb{R}$.

In the particular case of the Sobolev space $H_0^1(\Omega)$, many new existence results have been obtained when (AR) is relaxed by condition (h_3). Therefore, some mild oscillations of the nonlinearity f can be allowed. See de Figueiredo & Yang [36] and Jeanjean [50].

However, condition (h_3) is also violated by many nonlinearities, as for example, $f(s) = as$ or $f(s) = as\ln(s)$ at infinity with $a \in \mathbb{R}_+$. Some special attention has been given to the value $\xi = 2$ to introduce weaker condition than (AR) and no longer require the strong superlinear condition. For example,

(h_4) there exist $c \in \mathbb{R}_+$ and $s_0 \in \mathbb{R}_+$ such that $c|f(x,s)|^{2N/(N+2m)} \leqslant sf(x,s) - 2F(x,s)$ for every $|s| > s_0$ and for every $x \in \Omega$.

The key ingredient in this approach is the Riesz-Fréchet representation theorem, which permits one to write $J'(u_n)$ as a variational equation by supposing the existence of $v_n \in H_0^1(\Omega)$ such that $J'(u_n)\phi = \langle v_n, \phi\rangle_{H_0^1(\Omega)}$ for every $\phi \in H_0^1(\Omega)$ and $|J'(u_n)|_{(H_0^1(\Omega))'} = |v_n|_{H_0^1(\Omega)}$. Thus, $u_n - v_n$ could be seen as a weak solution in $H_0^1(\Omega)$ of the equation $\langle u_n - v_n, \phi\rangle_{H_0^1(\Omega)} = \int_\Omega f(x, u_n(x))\phi \,\mathrm{d}x$ for every $\phi \in H_0^1(\Omega)$. Another new aspect in this argument is the use of the Lesbesgue space theory to show the boundedness of the sequence $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$. To accomplish this, a regularity result due to Agmon, Douglis & Nirenberg [3] can be useful.

Notice that from condition (h_1), $C_0|f(x,s)|^{2^*} \leqslant |sf(x,s)|$; and from the (AR) condition, $0 < (1-2/\xi)sf(x,s) \leqslant sf(x,s) - 2F(x,s)$ for every $|s| > s_0$ and for every

$x \in \Omega$. Hence, condition (h_4) is weaker than the (AR) condition. On the other hand, if $F(x, \pm s_0) > 0$, then condition (h_4) implies condition (h_1).

From condition (h_4) it follows that $\int_\Omega |f(x, u_n)|^{2^*} = O(\|u\| + 1)$ for Palais-Smale sequences. So, it seems that condition (h_4) is an optimal condition ensuring the boundedness of the sequence.

The function $f_\alpha(s) = s[g(|s|)]^\alpha$ where $g(s) = \ln(\ln(\cdots \ln|s|))$ verifies condition (h_4) for every $\alpha \in \mathbb{R}_+$; however, it does not verify the strong superlinear condition (h_3). Moreover, $f(s) = as$ does not verify condition (h_4); however, $f(s) = as - |s|^{\alpha-1}s$ with $(2^* - 1)^{-1} \leqslant \alpha < 1$ and $f(s) = as + s[\ln(|s| + 2)]^{-\alpha'}$ with $\alpha' \in \mathbb{R}_+$ verify condition (h_4) but not condition (h_3).

Subcritical polynomial growth condition revisited

Upon verifying the boundedness of the (PS) sequences, the use of the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ together with condition (h_2) allows one to prove that if the sequence $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ is bounded, then $f(x, u_n)$ has a convergent subsequence in $L^{2N/(N+2)}(\Omega)$. This means that the operator $K(u)v = \int_\Omega f(x, u)v \,\mathrm{d}x$ is compact. But this condition is not satisfied when the nonlinearity is very close to critical growth, as in the example $f_\alpha(s) = |s|^{4/(N-2)}s/\ln^\alpha(|s|+2)$ for $\alpha \in \mathbb{R}_+$. However, the operator K is compact for f_α, which means that condition (h_2) is only a sufficient condition. One way to weaken this condition is to substitute it with the condition

(h_5) $\lim_{s\to+\infty} f(x, s)/|s|^{2^*-1} = 0$ uniformly with respect to $x \in \Omega$.

The operator K is still compact under this assumption. Moreover, the condition (h_5) seems to be nearly optimal because if $f(s) = |s|^{4/(N-2)}s$ at infinity, then K is no longer compact.

Since many existence results are based on the fact that the (PS) condition is satisfied, most cases require the (AR) condition as well as the subcritical polynomial growth. Thus, after verifying (PS) condition under hypotheses (h_4) and (h_5), it is possible to improve some classical existence results having the minimax structure. For example, let λ_1 be the lowest eigenvalue of the self-adjoint $(-\Delta)u = f(x, u)$ problem with Dirichlet condition. Then the energy functional has a nontrivial critical point by the mountain pass theorem if $\lim_{s\to+\infty} f(x, s) > \lambda_1$ uniformly in $\overline{\Omega}$ and $\lim_{s\to 0} f(x, s)/s < \lambda_1$ uniformly in $\overline{\Omega}$.

Consider now the function $f(s) = |s|^{4/(N-2)}s/\ln^{q+q'}(|s|) + [|s|^{4/(N-2)}s/\ln^q(|s|)][\gamma + \sin(\ln(|s|))]$, defined for $|s| > 1$, where $q \in \mathbb{R}_+$ and $0 < q' < 1$. It can be shown that there exists a constant $\underline{\gamma} \in \mathbb{R}$ such that for $\gamma > \underline{\gamma}$, then f verifies the (AR) condition; and if $\gamma < \underline{\gamma}$, then f does not verify neither the (AR) nor the (h_4) conditions. However, if $\gamma = \underline{\gamma}$ and if $q' < \min\{1, q(N-2)/(N+2)\}$, then f does not verify neither the (AR) nor the (h_2) conditions but verifies both conditions (h_4) and (h_5). This shows some improvements brought by these conditions.

Subcritical polynomial growth and Cerami conditions

Consider again our prototype equation $-\Delta u = f(x, u)$ in the bounded, open domain $\Omega \subset \mathbb{R}^N$ with a variational structure; the energy functional $J\colon H_0^1(\Omega) \to \mathbb{R}$ associated to this problem can be defined by $J(u) = (1/2)\|u\|^2 - \int f(x, u(x)) \,\mathrm{d}x$.

If the Cerami condition is verified and since $\|J'(u_n)\|_{(H_0^1(\Omega))'}\|u_n\| \to 0$ as $n \to +\infty$, then every Cerami sequence satisfies $2J(u_n) - J'(u_n)u_n = O(1)$, contrarily to the Palais-Smale sequences, where one only has $2J(u_n) - J'(u_n)u_n = O(\|u_n\| + 1)$. For instance, in case $f(s) = a(x)s + b|s|^{\alpha-1}s$ or $f(s) = a(x)s + b\log(|s|+1)$ where $a \in C(\overline{\Omega;\mathbb{R}})$ is a continuous, positive function with b, $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$, the energy functional verifies the Cerami condition.

The use of the Cerami condition with the sequence $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ usually goes as follows. A common assumption used by some authors is that $H(x,s) = 2F(x,s) - sf(x,s) \geqslant -w_1(x)$ for $x \in \Omega$ and $t \in \mathbb{R}$, where $w_1 \in L^1(\Omega)$ and that $H(x,s) \to +\infty$ a.e. as $|s| \to +\infty$; it is possible to prove that $\|u_n\|^2 - \langle f(\cdot,u_n), u_n\rangle \to 0$ as $n \to +\infty$ and this implies that $\int_\Omega H(x,u_n)\,\mathrm{d}x \leqslant K$ for some constant $K \in \mathbb{R}$. Then, towards a contradiction, it is assumed that the sequence $\{\|u_n\|\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is unbounded, i.e., $\|u_n\| \to +\infty$ as $n \to +\infty$. Another sequence is now created by defining $\tilde{u}_n(x) \coloneqq u_n(x)/\|u_n\|$; therefore, $\|\tilde{u}_n\| = 1$ and, up to passage to a subsequence, we have $\tilde{u}_n \rightharpoonup \tilde{u}$ weakly in $H_0^1(\Omega)$, $\tilde{u}_n \to \tilde{u}$ strongly in $L^2(\Omega)$, and $\tilde{u}_n \to \tilde{u}$ a.e. in Ω; moreover, it can be showed that $\tilde{u} \not\equiv 0$. If we denote $\Omega_0 \coloneqq \{x \in \Omega\colon \tilde{u}(x) \neq 0\}$ and $\Omega_1 \coloneqq \Omega\backslash\Omega_0$, then $|u_n(x)| = \|u_n\|\tilde{u_n}(x) \to +\infty$ as $n \to +\infty$ for every $x \in \Omega_0$ and

$$\int_{\Omega_0\cup\Omega_1} H(x,u_n(x))\,\mathrm{d}x \geqslant \int_{\Omega_0} H(x,u_n(x))\,\mathrm{d}x - \int_{\Omega_1} w_1(x)\,\mathrm{d}x \to +\infty.$$

But this contradicts the boundedness of the lefthand side integral previously mentioned. Therefore, the Cerami sequence $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(\Omega)$ must be bounded and we obtain some compactness to work with in the proof of the existence result.

The crucial element in this argument is the estimate $\|u_n\|^2 - \langle f(\cdot,u_n), u_n\rangle \to 0$ as $n \to +\infty$. If we had been dealing with a Palais-Smale sequence all the time, we could only conclude that $\|u_n\|^2 - \langle f(\cdot,u_n), u_n\rangle = o(\|u_n\|)$ which would only imply that $\int_\Omega H(x,u_n)\,\mathrm{d}x = o(\|u_n\|)$. This would not contradict the estimate $\int_\Omega H(x,u_n(x))\,\mathrm{d}x \to +\infty$ as $n \to +\infty$ and the argument would not go through. For more details, see Schechter [80].

Degraded oscillation case

Consider the nonlinearity with a very sparsed oscilation, for example, $f(s) = \gamma s^p + s^p(1 + \sin(\log(s+2)))$ if $s \geqslant 0$ and $f(s) = 0$ if $s < 0$. Then f satisfies the subcritical polynomial growth condition at infinity for every $\gamma \geqslant 0$. Moreover, there exists $\underline{\gamma} \in \mathbb{R}_+$ such that if $\gamma > \underline{\gamma}$ then f verifies the (AR) condition and if $0 < \gamma \leqslant \underline{\gamma}$ then f verifies only the strong superlinear condition. However, if $\gamma = 0$ then f does not even verify the condition $\lim_{s\to+\infty} f(s)/s = +\infty$ uniformly with respect to $x \in \Omega$ since $f(\exp(\exp((2n - 1/2)\pi - 2))) = 0$. This case is referred to as the degraded oscillation case. Under the strong superlinear condition together using the assumption $\lim_{s\to+\infty}[sf'(s) - pf(s)]/s^p = 0$ and some additional conditions it is possible to prove the existence of at least one positive solution. For example, $f(s) = s^p(1+\sin(\log(\log(s+2))))$ is an instance for this situation. However the last condition is too restrictive; for example, the function $f(s) = \gamma s^p + s^p\sin(\log(s+2))$ for $s \geqslant 0$ with $\gamma > 1$ does not verify it.

The case of resonant nonlinearities.

Other kind of question is related to resonant nonlinearities. More precisely, consider an asymptotically linear function, that is, $\lim_{|t|\to+\infty} t^{-1}f(t) = \alpha$ where $\alpha \in \mathbb{R}$ is finite;

then we can write $f(t) = \alpha t - g(t)$ with $\lim_{|t|\to\infty} t^{-1}g(t) = 0$ where $g\colon \mathbb{R} \to \mathbb{R}$. As usual, we denote by $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ the sequence of eigenvalues of the self-adjoint realization in $L^2(\Omega)$ of the Laplacian operator with Dirichlet boundary condition. We say that the problem is resonant at infinity if $\alpha = \lambda_k$ for some positive integer $k \in \mathbb{N}$. Depending on the growth of the function g at infinity we have different degrees of resonance; that is, the smaller the g, the stronger the resonance. To quantify these degrees of resonance, we can consider some situations:

(l_1) $\lim_{t\to+\infty} g(t) = \ell_+ \neq 0$ and $\lim_{t\to-\infty} g(t) = \ell_- \neq 0$: this weak resonance was first considered by Landesman & Lazer [55];

(l_2) $\lim_{|t|\to+\infty} g(t) = 0$ and $\lim_{t\to+\infty} \int_0^t g(s)\,\mathrm{d}s = \pm\infty$: this mild resonance was first considered by Ahmad, Lazer & Paul [4];

(l_3) $\lim_{|t|\to+\infty} g(t) = 0$ and $\lim_{|t|\to+\infty} \int_0^t g(s)\,\mathrm{d}s = \beta$ where $\beta \in \mathbb{R}$ is finite: this strong resonance at infinity was considered by Thews [88];

(l_4) $\lim_{|t|\to+\infty} tg(t) = 0$, $\lim_{t\to+\infty} \int_{-\infty}^t g(s)\,\mathrm{d}s = 0$, this integral being well-defined and non-negative for every $t \in \mathbb{R}$: this strong resonance at infinity was considered by Bartolo, Benci & Fortunato [17].

In general terms, the existence results mentioned are proved through the application of deformation lemmas whose proofs, in turn, rely on a weakened version of the well-known Palais-Smale condition introduced by Cerami.

To conclude, we mention that when the variational approach can not be employed, the question of existence of solutions may be dealt with via topological methods. In this case, the proof of existence of solutions is essentially reduced to deriving *a priori* estimates for all possible solutions and in general needs that the domain $\Omega \subset \mathbb{R}^N$ be convex or a ball. However, certain behavior of the nonlinearity at infinity is still necessary.

For these and several other existence results above mentioned, see the interesting paper by Harrabi [49].

Our contribution to the problem

Motivated by the above papers, our results improve upon previous work in the following ways: we focus our attention on the existence of a nontrivial weak solution for fractional p-Kirchhoff equation in the entire space $\mathbb{R}^N$, which causes a difficulty due to lack of compactness for Sobolev theorem; moreover, the problem also includes a non-local Choquard subcritical term and a critical Hardy-type term; additionally, we consider singularities in the fractional p-Laplacian with $\theta = \theta_1 + \theta_2$ not necessarily zero and we also add a critical Sobolev nonlinearity. The possibility of a slower growth in the nonlinearity makes it more difficult to establish a compactness condition. In fact, we will not prove the usual Palais-Smale condition, but rather a less restrictive version often credited to Cerami. Our argument has two crucial points: the first one is to prove a uniform boundedness of the convolution part $|I_\mu \star F| < +\infty$, which gives a lot of help when we choose Cerami sequences; the second one to treat the lack of compactness of the Sobolev embeddings.

3.2 The variational setting

Here we recall a generalization of the Hardy-Littlewood-Sobolev, also called the doubly weighted inequality or the Stein-Weiss inequality. It provides quantitative information to characterize the integrability for the integral operators present in the energy functional and is crucial in the analysis developed in this work. See Stein & Weiss [86]; Lieb [58] and Lieb & Loss [59, Theorem 4.3].

Proposition 3.1 (Doubly weighted Stein-Weiss inequality)**.** *Let* $1 < r,\ t < +\infty$, $0 < \mu < N$, *and* $\eta + \kappa \geqslant 0$ *such that* $\mu + \eta + \kappa \leqslant N$, $\eta < N/r'$, $\kappa < N/t'$ *and* $1/t + (\mu + \eta + \kappa)/N + 1/r = 2$; *let* $f \in L^t(\mathbb{R}^N)$ *and* $h \in L^r(\mathbb{R}^N)$. *Then there exists a constant* $C(N, \mu, r, t, \eta, \kappa)$, *independent on* f *and* h, *such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x|^\eta |x-y|^\mu |y|^\kappa} \mathrm{d}x\mathrm{d}y \right| \leqslant C(N, \mu, r, t, \eta, \kappa) \|f\|_{L^t(\mathbb{R}^N)} \|h\|_{L^r(\mathbb{R}^N)}. \tag{3.1}$$

Corollary 3.2. *Let* $0 < s < 1$; $0 \leqslant \alpha < sp + \theta < N$; $0 < \mu < N$; *given a function* $u \in \dot{W}^{s,p}_\theta(\mathbb{R}^N)$ *consider Proposition 3.1 with* $\eta = \kappa = \delta$; $2\delta + \mu \leqslant N$ *and* $t = r = N/(N - \delta - \mu/2)$. *Then* $f, h \in L^{\frac{N}{N-\delta-\mu/2}}(\mathbb{R}^N)$ *and*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)||h(y)|}{|x|^\delta |x-y|^\mu |y|^\delta} \mathrm{d}x\mathrm{d}y \leqslant C(N, \delta, \theta, \mu) \|f\|_{L^t(\mathbb{R}^N)} \|h\|_{L^t(\mathbb{R}^N)}. \tag{3.2}$$

In general, for $\eta = \kappa = \delta$ and $t = r$, the map

$$u \mapsto \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x|^\delta |x-y|^\mu |y|^\delta} \,\mathrm{d}x\,\mathrm{d}y$$

is well defined if

$$p^\flat_s(\delta, \mu) \coloneqq \frac{p(N - \delta - \mu/2)}{N} < q < \frac{p^*_s(0,\theta)(N - \delta - \mu/2)}{N} =: p^\sharp_s(\delta, \theta, \mu).$$

The constant $p^\flat_s(\delta, \mu)$ is termed as the lower critical exponent and $p^\sharp_s(\delta, \theta, \mu)$ is termed as the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality.

The variational structure of problem (10) can be established with the help of several results. To ensure the well-definiteness of the energy functional, we use the following result.

Lemma 3.3. *Let* (V) *and* (m_1) *hold. Then the functional* Φ *defined in* (12) *is of class* $C^1(W^{s,p}_{V,\theta}(\mathbb{R}^N), \mathbb{R})$ *and*

$$\begin{aligned} \langle \Phi'(u), \varphi \rangle = & m(\|u\|^p_W) \Big[\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x|^{\theta_1}|x-y|^{N+ps}|y|^{\theta_2}} \mathrm{d}x dy \\ & + \int_{\mathbb{R}^N} V(x) \frac{|u(x)|^{p-2}u(x)\varphi(x)}{|x|^\alpha} \mathrm{d}x \Big], \end{aligned}$$

for all $u, \varphi \in W^{s,p}_{V,\theta}(\mathbb{R}^N)$. *Moreover,* Φ *is weakly lower semi-continuous in* $W^{s,p}_{V,\theta}(\mathbb{R}^N)$.

Proof. Let $\{u_n\}_n \subset W$ and $u \in W$ satisfy $u_n \to u$ strongly in W as $n \to \infty$. Without loss of generality, we assume that $u_n \to u$ a.e. in $\mathbb{R}^N$. Then the sequence

$$\left\{ \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x|^{\theta_1/p'}|x-y|^{(N+sp)/p'}|y|^{\theta_2/p'}} \right\}_n \quad \text{is bounded in } L^{p'}(\mathbb{R}^{2N}), \tag{3.3}$$

as well as in $\mathbb{R}^{2N}$

$$
\begin{aligned}
U_n(x,y) &:= \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x|^{\theta_1/p'}|x-y|^{(N+sp)/p'}|y|^{\theta_2/p'}}\\
&\to \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x|^{\theta_1/p'}|x-y|^{(N+sp)/p'}|y|^{\theta_2/p'}} := U(x,y).
\end{aligned}
$$

Thus, the Brezis–Lieb lemma implies

$$
\begin{aligned}
&\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} |U_n(x,y)-U(x,y)|^{p'}\mathrm{d}x\mathrm{d}y\\
&\quad=\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} (|U_n(x,y)|^{p'}-|U(x,y)|^{p'})\mathrm{d}x\mathrm{d}y\\
&\quad=\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\Big(\frac{|u_n(x)-u_n(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}-\frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\Big)\mathrm{d}x\mathrm{d}y. \qquad (3.4)
\end{aligned}
$$

The fact that $u_n \to u$ strongly in W yields that

$$
\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\Big(\frac{|u_n(x)-u_n(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}-\frac{|u(x)-u(y)|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\Big)\mathrm{d}x\mathrm{d}y=0. \qquad (3.5)
$$

Moreover, the continuity of m implies that

$$
\lim_{n\to\infty} m([u_n]^p_{W^{s,p}_\theta(\mathbb{R}^N)}) = m([u]^p_{W^{s,p}_\theta(\mathbb{R}^N)}). \qquad (3.6)
$$

From (3.4) it follows that

$$
\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} |U_n(x,y)-U(x,y)|^{p'}\mathrm{d}x\mathrm{d}y = 0. \qquad (3.7)
$$

Similarly, the sequence

$$
\Big\{\frac{V^{1/p'}(x)|u_n(x)|^{p-2}u_n(x)}{|x|^{\alpha/p'}}\Big\}_n \quad \text{is bounded in } L^{p'}(\mathbb{R}^{2N}), \qquad (3.8)
$$

as well as in $\mathbb{R}^{2N}$

$$
K_n(x,y) := \frac{V^{1/p'}(x)|u_n(x)|^{p-2}u_n(x)}{|x|^{\alpha/p'}} \qquad \to \frac{V^{1/p'}(x)|u(x)|^{p-2}u(x)}{|x|^{\alpha/p'}} := K(x,y).
$$

Thus, the Brezis–Lieb lemma implies

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N} |K_n(x,y)-K(x,y)|^{p'}\mathrm{d}x\\
&\quad=\lim_{n\to\infty}\int_{\mathbb{R}^N}(|K_n(x,y)|^{p'}-|K(x,y)|^{p'})\mathrm{d}x\\
&\quad=\lim_{n\to\infty}\int_{\mathbb{R}^N}\Big(\frac{V(x)|u_n(x)|^p}{|x|^\alpha}-\frac{V(x)|u(x)|^p}{|x|^\alpha}\Big)\mathrm{d}x. \qquad (3.9)
\end{aligned}
$$

The fact that $u_n \to u$ strongly in W yields that

$$
\int_{\mathbb{R}^N}\Big(\frac{V(x)|u_n(x)|^p}{|x|^\alpha}-\frac{V(x)|u(x)|^p}{|x|^\alpha}\Big)\mathrm{d}x = 0. \qquad (3.10)
$$

From (3.9) it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|K_n(x,y)-K(x,y)|^{p'}\mathrm{d}x=0. \tag{3.11}$$

From Hölder inequality, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x|^{\theta_1}|x+y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^N}\frac{V(x)|u_n(x)|^{p-2}u_n(x)\phi(x)}{|x|^{\alpha}}\mathrm{d}x\\
&=\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x|^{\frac{\theta_1}{p'}}|x+y|^{\frac{N+sp}{p'}}|y|^{\frac{\theta_2}{p'}}}\cdot\frac{\phi(x)-\phi(y)}{|x|^{\frac{\theta_1}{p}}|x+y|^{\frac{N+sp}{p}}|y|^{\frac{\theta_2}{p}}}\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^N}\frac{(V(x))^{\frac{1}{p'}}|u_n(x)|^{p-2}u_n(x)}{|x|^{\frac{\alpha}{p'}}}\cdot\frac{(V(x))^{\frac{1}{p}}\phi(x)}{|x|^{\frac{\alpha}{p}}}\mathrm{d}x\\
&\leqslant\left(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p}}{|x|^{\theta_1}|x+y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p'}}\cdot\left(\iint_{\mathbb{R}^{2N}}\frac{|\phi(x)-\phi(y)|^{p}}{|x|^{\theta_1}|x+y|^{N+sp}|y|^{\theta_2}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\\
&\quad+\left(\int_{\mathbb{R}^N}\frac{V(x)|u_n|^p}{|x|^{\alpha}}\mathrm{d}x\right)^{\frac{1}{p'}}\cdot\left(\int_{\mathbb{R}^N}\frac{V(x)|\phi|^p}{|x|^{\alpha}}\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

Similarly, we can obtain the previous result for u.

Combining (3.6), (3.7) and (3.11) with the Hölder inequality, we have

$$\|\Phi'(u_n)-\Phi'(u)\|_{W'}=\sup_{\phi\in W,\ \|\phi\|_W=1}|\langle\Phi'(u_n)-\Phi'(u),\phi\rangle|\to 0$$

as $n\to+\infty$. Hence $\Phi\in C^1(W,\mathbb{R})$. Finally, that the map $v\mapsto[v]^p_{W^{s,p}_\theta(\mathbb{R}^N)}$ is lower semi-continuous in the weak topology of $W^{s,p}_{V,\theta}(\mathbb{R}^N)$ and M is nondecreasing and continuous on $\mathbb{R}^+_0$, so the $u\mapsto M([u]^p_{W^{s,p}_\theta(\mathbb{R}^N)})$ is lower semi-continuous in the weak topology of $W^{s,p}_{V,\theta}(\mathbb{R}^N)$. □

One of the main difficulties of this work is to prove the weak strong continuity of the term involving the weighted Sobolev critical exponent. To accomplish this goal, we use the following result.

Lemma 3.4. *The functional Ξ defined in (12) as well as Ξ' are weakly strongly continuous on $W^{s,p}_{V,\theta}(\mathbb{R}^N)$.*

Proof. See Lemma 1.11; see also Assunção, Miyagaki & Siqueira [15, Lemma 1.7]. □

Lemma 3.5. *Assume $((F_2))$ holds, then there exists $K>0$ such that*

$$\left|\mathcal{I}_\mu*\frac{F(v)}{|x|^\delta}\right|\leqslant K\quad\text{for } v\in W^{s,p}_{V,\theta}(\mathbb{R}^N). \tag{3.12}$$

Proof. By the assumptions (F_2) and (V) and using (11), we have

$$\left|\mathcal{I}_\mu*\frac{F(v)}{|x|^\delta}\right|=\left|\int_{\mathbb{R}^N}\frac{F(v)}{|x|^\delta|x-y|^\mu}\mathrm{d}y\right|$$

$$
\begin{aligned}
&\leqslant\left|\int_{|x-y|\leqslant 1}\frac{F(v)}{|x|^{\delta}|x-y|^{\mu}}\mathrm{d}y\right|+\left|\int_{|x-y|\geqslant 1}\frac{F(v)}{|x|^{\delta}|x-y|^{\mu}}\mathrm{d}y\right|\\
&\leqslant c_0\int_{|x-y|\leqslant 1}\frac{|v|^{q_1}+|v|^{q_2}}{|x|^{\delta}|x-y|^{\mu}}\mathrm{d}y+c_0\int_{|x-y|\geqslant 1}\frac{|v|^{q_1}+|v|^{q_2}}{|x|^{\delta}}\mathrm{d}y\\
&=c_0\int_{|x-y|\leqslant 1}\frac{|v|^{q_1}+|v|^{q_2}}{|x|^{\delta}|x-y|^{\mu}}\mathrm{d}y+c_0\int_{|x-y|\geqslant 1}\frac{V_0|v|^{q_1}+V_0|v|^{q_2}}{V_0|x|^{\delta}}\mathrm{d}y\\
&\leqslant c_0\int_{|x-y|\leqslant 1}\frac{|v|^{q_1}+|v|^{q_2}}{|x|^{\delta}|x-y|^{\mu}}\mathrm{d}y+\frac{c_0}{V_0}\int_{|x-y|\geqslant 1}\frac{V(y)|v|^{q_1}+V(y)|v|^{q_2}}{|x|^{\delta}}\mathrm{d}y\\
&\leqslant c_0\int_{|x-y|\leqslant 1}\frac{|v|^{q_1}+|v|^{q_2}}{|x|^{\delta}|x-y|^{\mu}}dy+C(\|v\|_W^{q_1}+\|v\|_W^{q_2})\\
&\leqslant c_0\int_{|x-y|\leqslant 1}\frac{|v|^{q_1}+|v|^{q_2}}{|x|^{\delta}|x-y|^{\mu}}\mathrm{d}y+C.
\end{aligned}
$$

For the first term in the last line above, choosing $t_1\in\left(\frac{N-ps-\theta}{N-\delta-\mu/2},\frac{p(N-\beta)}{(N-ps-\theta)q_1}\right)$ and $t_2\in\left(\frac{N-ps-\theta}{N-\delta-\mu/2},\frac{p(N-\beta)}{(N-ps-\theta)q_2}\right)$, using Hölder inequality and the assumption (V), we obtain

$$
\begin{aligned}
&\int_{|x-y|\leqslant 1}\frac{|v|^{q_1}+|v|^{q_2}}{|x|^{\delta}|x-y|^{\mu}}\mathrm{d}y\\
&\leqslant\left(\int_{|x-y|\leqslant 1}\frac{|v|^{q_1t_1}}{|x|^{\delta t_1}}\mathrm{d}y\right)^{\frac{1}{t_1}}\left(\int_{|x-y|\leqslant 1}|x-y|^{-\frac{\mu t_1}{t_1-1}}\mathrm{d}y\right)^{\frac{t_1-1}{t_1}}\\
&\quad+\left(\int_{|x-y|\leqslant 1}\frac{|v|^{q_2t_2}}{|x|^{\delta t_2}}\mathrm{d}y\right)^{\frac{1}{t_2}}\left(\int_{|x-y|\leqslant 1}|x-y|^{-\frac{\mu t_2}{t_2-1}}\mathrm{d}y\right)^{\frac{t_2-1}{t_2}}\\
&=\left(\int_{|x-y|\leqslant 1}\frac{V_0|v|^{q_1t_1}}{V_0|x|^{\delta t_1}}\mathrm{d}y\right)^{\frac{1}{t_1}}\left(\int_{|x-y|\leqslant 1}|x-y|^{-\frac{\mu t_1}{t_1-1}}\mathrm{d}y\right)^{\frac{t_1-1}{t_1}}\\
&\quad+\left(\int_{|x-y|\leqslant 1}\frac{V_0|v|^{q_2t_2}}{V_0|x|^{\delta t_2}}\mathrm{d}y\right)^{\frac{1}{t_2}}\left(\int_{|x-y|\leqslant 1}|x-y|^{-\frac{\mu t_2}{t_2-1}}\mathrm{d}y\right)^{\frac{t_2-1}{t_2}}\\
&=\frac{1}{V_0^{1/t_1}}\left(\int_{|x-y|\leqslant 1}\frac{V(x)|v|^{q_1t_1}}{|x|^{\delta t_1}}\mathrm{d}y\right)^{\frac{1}{t_1}}\left(\int_{|x-y|\leqslant 1}|x-y|^{-\frac{\mu t_1}{t_1-1}}\mathrm{d}y\right)^{\frac{t_1-1}{t_1}}\\
&\quad+\frac{1}{V_0^{1/t_2}}\left(\int_{|x-y|\leqslant 1}\frac{V(x)|v|^{q_2t_2}}{|x|^{\delta t_2}}\mathrm{d}y\right)^{\frac{1}{t_2}}\left(\int_{|x-y|\leqslant 1}|x-y|^{-\frac{\mu t_2}{t_2-1}}\mathrm{d}y\right)^{\frac{t_2-1}{t_2}}
\end{aligned}
$$

Let $0<\alpha<N-\mu$, we choose $q_1t_1=p, q_2t_2=p$ and $\delta t_1=\alpha, \delta t_2=\alpha$, so using (11), we have

$$
\begin{aligned}
&\int_{|x-y|\leqslant 1}\frac{|v|^{q_1}+|v|^{q_2}}{|x|^{\delta}|x-y|^{\mu}}\mathrm{d}y\\
&\leqslant C\left(\|v\|_W^{q_1}+\|v\|_W^{q_2}\right)\left[\left(\int_{r\leqslant 1}r^{N-1-\frac{\mu t_1}{t_1-1}}\mathrm{d}y\right)^{\frac{t_1-1}{t_1}}+\left(\int_{r\leqslant 1}r^{N-1-\frac{\mu t_2}{t_2-1}}\mathrm{d}y\right)^{\frac{t_2-1}{t_2}}\right]\\
&\leqslant C.
\end{aligned}
$$

□

Lemma 3.6. *Let* (V) *and* (F_1)–(F_2) *hold. Then the functional* Ψ *defined in* (12) *as well as* Ψ' *are weakly strongly continuous on* $W^{s,p}_{V,\theta}(\mathbb{R}^N)$.

Proof. Let $\{u_n\}$ be a sequence in $W^{s,p}_{V,\theta}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W^{s,p}_{V,\theta}(\mathbb{R}^N)$ as $n \to \infty$. Then $\{u_n\}$ is bounded in $W^{s,p}_{V,\theta}(\mathbb{R}^N)$, and then there exists a subsequence denoted by itself, such that

$$u_n \to u \quad \text{in } L^{q_1}(\mathbb{R}^N, |x|^{-\delta}) \cap L^{q_2}(\mathbb{R}^N, |x|^{-\delta}), \qquad \text{and} \quad u_n \to u \quad \text{a.e. in } \mathbb{R}^N \quad \text{as } n \to \infty,$$

and by [22, Theorem 4.9] there exists $\ell \in L^{q_1}(\mathbb{R}^N, |x|^{-\delta}) \cap L^{q_2}(\mathbb{R}^N, |x|^{-\delta})$ such that

$$\frac{|u_n(x)|}{|x|^\delta} \leqslant \ell(x) \text{ a.e. in } \mathbb{R}^N.$$

First, we show that Ψ is weakly strongly continuous on $W^{s,p}_{V,\theta}(\mathbb{R}^N)$. Since $F \in C^1(\mathbb{R}, \mathbb{R})$, we see that $\frac{F(u_n)}{|x|^\delta} \to \frac{F(u)}{|x|^\delta}$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$, and so $\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{F(u_n)}{|x|^\delta} \to \Big(\mathcal{I}_\mu * \frac{F(u)}{|x|^\delta}\Big)\frac{F(u)}{|x|^\delta}$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$. From Lemma 3.5 and (F_2), we have

$$\left|\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{F(u_n)}{|x|^\delta}\right| \leqslant Kc_0\Big(\frac{|u_n(x)|^{q_1}}{q_1|x|^\delta} + \frac{|u_n(x)|^{q_2}}{q_2|x|^\delta}\Big) \in L^1(\mathbb{R}^N).$$

By Lebesgue dominated convergence theorem, we get

$$\int_{\mathbb{R}^N}\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{F(u_n)}{|x|^\delta}\mathrm{d}x \to \int_{\mathbb{R}^N}\Big(\mathcal{I}_\mu * \frac{F(u)}{|x|^\delta}\Big)\frac{F(u)}{|x|^\delta}\mathrm{d}x \quad \text{as } n \to \infty,$$

which implies that $\Psi(u_n) \to \Psi(u)$ as $n \to \infty$. Thus Ψ is weakly strongly continuous on $W^{s,p}_{V,\theta}(\mathbb{R}^N)$.

Next, we prove that Ψ' is weakly strongly continuous on $W^{s,p}_{V,\theta}(\mathbb{R}^N)$. Since $u_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$, $\frac{f(u_n)}{|x|^\delta} \to \frac{f(u)}{|x|^\delta}$ for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. Then

$$\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{f(u_n)}{|x|^\delta} \to \Big(\mathcal{I}_\mu * \frac{F(u)}{|x|^\delta}\Big)\frac{f(u)}{|x|^\delta} \quad \text{a.e. in } \mathbb{R}^N, \quad \text{as } n \to \infty.$$

By (F_2) and Hölder inequality, we have that for any $\varphi \in W^{s,p}_{V,\theta}(\mathbb{R}^N)$,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left|\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{f(u_n)}{|x|^\delta}\varphi(x)\right|\mathrm{d}x \\
&\leqslant c_0K\int_{\mathbb{R}^N}|\Big(\frac{|u_n|^{q_1-1}}{|x|^\delta} + \frac{|u_n|^{q_2-1}}{|x|^\delta}\Big)\varphi(x)|\mathrm{d}x \\
&= c_0K\Big[\int_{\mathbb{R}^N}\frac{|u_n|^{q_1-1}}{|x|^{\delta(q_1-1)/q_1}}\frac{|\varphi(x)|}{|x|^{\delta/q_1}}\mathrm{d}x + \int_{\mathbb{R}^N}\frac{|u_n|^{q_2-1}}{|x|^{\delta(q_2-1)/q_2}}\frac{|\varphi(x)|}{|x|^{\delta/q_2}}\mathrm{d}x\Big] \\
&\leqslant c_0K\Big[\Big(\int_{\mathbb{R}^N}\Big(\frac{|u_n|^{q_1-1}}{|x|^{\delta(q_1-1)/q_1}}\Big)^{\frac{q_1}{q_1-1}}\mathrm{d}x\Big)^{\frac{q_1-1}{q_1}}\Big(\int_{\mathbb{R}^N}\Big(\frac{|\varphi(x)|}{|x|^{\delta/q_1}}\Big)^{q_1}\mathrm{d}x\Big)^{\frac{1}{q_1}} \\
&\quad + \Big(\int_{\mathbb{R}^N}\Big(\frac{|u_n|^{q_2-1}}{|x|^{\delta(q_2-1)/q_2}}\Big)^{\frac{q_2}{q_2-1}}\mathrm{d}x\Big)^{\frac{q_2-1}{q_2}}\Big(\int_{\mathbb{R}^N}\Big(\frac{|\varphi(x)|}{|x|^{\delta/q_2}}\Big)^{q_2}\mathrm{d}x\Big)^{\frac{1}{q_2}}\Big] \\
&= c_0K\Big(\|u_n\|^{q_1-1}_{L^{q_1}(\mathbb{R}^N,|x|^{-\delta})}\|\varphi\|_{L^{q_1}(\mathbb{R}^N,|x|^{-\delta})} + \|u_n\|^{q_2-1}_{L^{q_2}(\mathbb{R}^N,|x|^{-\delta})}\|\varphi\|_{L^{q_2}(\mathbb{R}^N,|x|^{-\delta})}\Big)
\end{aligned}$$

$$\leqslant c_0 K\Big(C_{q_1}\|\ell(x)\|^{q_1-1}_{L^{q_1}(\mathbb{R}^N,|x|^{-\delta})} + C_{q_2}\|\ell(x)\|^{q_2-1}_{L^{q_2}(\mathbb{R}^N,|x|^{-\delta})}\Big)\|\varphi\|_W.$$

Then by Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\|\Psi'(u_n)-\Psi'(u)\|_{\left(W^{s,p,\theta}_V(\mathbb{R}^N)\right)'}\\
&= \sup_{\|\varphi\|_{W^{s,p,\theta}_V(\mathbb{R}^N)}=1} |\langle \Psi'(u_n)-\Psi'(u),\varphi\rangle|\\
&= \sup_{\|\varphi\|_{W^{s,p,\theta}_V(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N}\Big|\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{f(u_n)}{|x|^\delta}\varphi(x) - \Big(\mathcal{I}_\mu * \frac{F(u)}{|x|^\delta}\Big)\frac{f(u)}{|x|^\delta}\varphi(x)\Big|\mathrm{d}x\\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Therefore, we get that $\Psi'(u_n)\to\Psi'(u)$ in $\left(W^{s,p}_{V,\theta}(\mathbb{R}^N)\right)'$ as $n\to\infty$. This completes the proof. □

3.3 The geometry of the mountain pass theorem

In this section, we will prove our main result. First, we introduce the following definition.

Definition 3.7. For $c\in\mathbb{R}$, we say that I satisfies the $(C)_c$ condition if for any sequence $\{u_n\}\subset W^{s,p}_{V,\theta}(\mathbb{R}^N)$ with

$$I(u_n)\to c, \quad \|I'(u_n)\|(1+\|u_n\|_W)\to 0,$$

there is a subsequence $\{u_n\}$ such that $\{u_n\}$ converges strongly in $W^{s,p}_{V,\theta}(\mathbb{R}^N)$.

We will use the following mountain pass theorem to prove our result.

Lemma 3.8 (Theorem 1 in [35])**.** *Let E be a real Banach space, $I\in C^1(E,\mathbb{R})$ satisfies the $(C)_c$ condition for any $c\in\mathbb{R}$, and*

(i) There are constants $\rho,\alpha>0$ such that $I|_{\partial B_\rho}\geqslant\alpha$.

(ii) There is an $e\in E\backslash B_\rho$ such that $I(e)\leqslant 0$.

Then,

$$c=\inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} I(\gamma(t))\geqslant\alpha$$

is a critical value of I, where

$$\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=0,\gamma(1)=e\}.$$

We first show that the energy functional I satisfies the geometric structure.

Lemma 3.9. *Assume that (V), (m_1)–(m_2) and (F_1)–(F_3) hold. Then*

(i) There exists $\alpha,\rho>0$ such that $I(u)\geqslant\alpha$ for all $u\in W^{s,p}_{V,\theta}(\mathbb{R}^N)$ with $\|u\|_W=\rho$.

(ii) $I(u)$ is unbounded from below on $W^{s,p}_{V,\theta}(\mathbb{R}^N)$.

Proof. (i) From Lemma 3.5 and (m_1)–(m_2), (F_2), we have

$$I(u)=\frac{1}{p}M(\|u\|^p_W)-\frac{1}{p^*_s(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u|^{p^*_s(\beta,\theta)}}{|x|^\beta}\mathrm{d}x$$

$$
\begin{aligned}
&-\frac{\lambda}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_{\delta,\theta,\mu}(u(x))F_{\delta,\theta,\mu}(u(y))}{|x|^\delta|x-y|^\mu|y|^\delta}\,\mathrm{d}x\,\mathrm{d}y\\
\geqslant&\frac{1}{p\xi}m(\|u\|_W^p)\|u\|_W^p-\|u\|_W^{p_s^*(\beta,\theta)}-\frac{\lambda c_0K}{2}\int_{\mathbb{R}^N}\left(\frac{|u|^{q_1}}{q_1}+\frac{|u|^{q_2}}{q_2}\right)\mathrm{d}x\\
\geqslant&\left[\frac{m_0}{p\xi}-\|u\|_W^{p_s^*(\beta,\theta)-p}-\frac{\lambda c_0K}{2}\left(C_{q_1}^{q_1}\|u\|_W^{q_1-p}+C_{q_2}^{q_2}\|u\|_W^{q_2-p}\right)\right]\|u\|_W^p.
\end{aligned}
$$

Since $q_2\geqslant q_1>p$ and $p_s^*(\beta,\theta)>p$, the claim follows if we choose ρ small enough.

(ii) Rewriting the inequality of (m_2) in the form of $m(t)/M(t)\leqslant\xi/t$, after integration, we deduce that there is a constant $C\in\mathbb{R}_+$ such that

$$
M(t)\leqslant Ct^\xi\quad\text{for all } t\geqslant 1. \tag{3.13}
$$

By the assumption (F_3), we can take that t_0 such that $F(t_0)\neq 0$, we find

$$
\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu*\frac{F(t_0\chi_{B_1})}{|x|^\delta}\right)\frac{F(t_0\chi_{B_1})}{|x|^\delta}\mathrm{d}x=F(t_0)^2\int_{B_1}\int_{B_1}\frac{1}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y>0,
$$

where B_r denotes the open ball centered at the origin with radius r and χ_{B_1} denotes the standard indicator function of set B_1. By the density theorem, there will be $v_0\in W_{V,\theta}^{s,p}(\mathbb{R}^N)$ with

$$
\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu*\frac{F(v_0)}{|x|^\delta}\right)\frac{F(v_0)}{|x|^\delta}\mathrm{d}x>0.
$$

Define the function $v_t(x)=v_0(\frac{x}{t})$, then, using the change of variables $x/t=\bar{x}$ and $y/t=\bar{y}$, we have

$$
\begin{aligned}
I(v_t)=&\frac{1}{p}M(\|v_t\|_W^p)-\frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|v_t|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x-\frac{\lambda}{2}\iint_{\mathbb{R}^{2N}}\frac{F(v_t(x))F(v_t(y))}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}x\mathrm{d}y\\
\leqslant&\frac{1}{p}C\|v_t\|_W^{p\xi}-\frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|v_t|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x-\frac{\lambda}{2}\iint_{\mathbb{R}^{2N}}\frac{F(v_t(x))F(v_t(y))}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}\bar{x}\mathrm{d}\bar{y}\\
=&\frac{1}{p}C\left[t^{N-ps-\theta}\iint_{\mathbb{R}^{2N}}\frac{|v_0(\bar{x})-v_0(\bar{y})|^p}{|x|^{\theta_1}|x-y|^{N+sp}|y|^{\theta_2}}\mathrm{d}\bar{x}\mathrm{d}\bar{y}+t^{N-sp-\theta}\int_{\mathbb{R}^N}\frac{V(t\bar{x})|v_0|^p}{|\bar{x}|^{sp+\theta}}\mathrm{d}\bar{x}\right]^\xi\\
&-\frac{t^{N-\beta}}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|v_0|^{p_s^*(\beta,\theta)}}{|\bar{x}|^\beta}\mathrm{d}\bar{x}-t^{2(N-\delta-\mu/2)}\frac{\lambda}{2}\iint_{\mathbb{R}^{2N}}\frac{F(v_0(x))F(v_0(y))}{|x|^\delta|x-y|^\mu|y|^\delta}\mathrm{d}\bar{x}\mathrm{d}\bar{y},
\end{aligned}
$$

for sufficiently large t. Therefore, we have that $I(v_t)\to-\infty$ as $t\to\infty$ since $1\leqslant\xi<\frac{2(N-\delta-\mu/2)}{N}$ gives that $2(N-\delta-\mu/2)>N\xi>(N-ps-\theta)\xi$. Furthermore, since $\beta<sp+\theta$, then $N-\beta>N-sp-\theta$. Hence we obtain that the functional I is unbounded from below. □

3.4 The compactness of the Cerami sequences

Next, we prove the important result that the Cerami sequences for the energy functional are bounded.

Lemma 3.10. *Assume that (V), (m_1)–(m_2) and (F_1)–(F_4) hold. Then $(C)_c-$sequence of I is bounded for any $\lambda>0$.*

Proof. Suppose that $\{u_n\} \subset W^{s,p}_{V,\theta}(\mathbb{R}^N)$ is a $(C)_c$−sequence for $I(u)$, that is,

$$I(u_n) \to c, \qquad \|I'(u_n)\|_W(1 + \|u_n\|_W) \to 0,$$

which shows that

$$c = I(u_n) + o(1), \qquad \langle I'(u_n), u_n\rangle = o(1) \tag{3.14}$$

where $o(1) \to 0$ as $n \to +\infty$. We now prove that $\{u_n\}$ is bounded in $W^{s,p}_{V,\theta}(\mathbb{R}^N)$. We argue by contradiction. Suppose that the sequence $\{u_n\}$ is unbounded in $W^{s,p}_{V,\theta}(\mathbb{R}^N)$, then we may assume that

$$\|u_n\|_W \to \infty, \quad \text{as } n \to +\infty. \tag{3.15}$$

Let $\omega_n(x) = \frac{u_n}{\|u_n\|_W}$, then $\omega_n \in W^{s,p}_{V,\theta}(\mathbb{R}^N)$ with $\|\omega_n\|_W = 1$. Hence, up to a subsequence, still denoted by itself, there exists a function $\omega \in W^{s,p,\theta}_V(\mathbb{R}^N)$ such that

$$\omega_n(x) \to \omega(x) \quad \text{a.e. in } \mathbb{R}^N, \qquad \text{and} \quad \omega_n(x) \to \omega(x) \quad \text{a.e. in } L^r(\mathbb{R}^N) \tag{3.16}$$

as $n \to \infty$, for $p \leqslant r < \frac{Np}{N-ps-\theta}$.

Let $\Omega_1 = \{x \in \mathbb{R}^N : \omega(x) \neq 0\}$, then

$$\lim_{n\to\infty} \omega_n(x) = \lim_{n\to\infty} \frac{u_n(x)}{\|u_n\|_W} = \omega(x) \neq 0 \ \text{ in } \Omega_1,$$

and (3.15) implies that

$$|u_n| \to \infty \ \text{ a.e. in } \Omega_1. \tag{3.17}$$

So from the assumption (F_3) and Lemma 3.5, we have

$$\lim_{n\to\infty} \frac{\left(\mathcal{I}_\mu * \frac{F(u_n(x))}{|x|^\delta}\right)\frac{F(u_n(x))}{|x|^\delta}}{|u_n(x)|^{p\xi}} |\omega_n(x)|^{p\xi} = \infty, \quad \text{for a.e. } x \in \Omega_1. \tag{3.18}$$

Hence, there is a constant C such that

$$\frac{\left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right)\frac{F(u_n(x))}{|x|^\delta}}{|u_n(x)|^{p\xi}} |\omega_n(x)|^{p\xi} - \frac{C}{\|u_n\|_W^{p\xi}} \geqslant 0. \tag{3.19}$$

By (3.14) we have that

$$\begin{aligned} c &= I(u_n) + o(1) \\ &= \frac{1}{p} M(\|u_n\|_W^p) - \frac{1}{p_s^*(\beta,\theta)} \int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}\, \mathrm{d}x \\ &\quad - \frac{\lambda}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_{\delta,\theta,\mu}(u_n(x)) F_{\delta,\theta,\mu}(u_n(y))}{|x|^\delta |x-y|^\mu |y|^\delta}\, \mathrm{d}x\, \mathrm{d}y + o(1). \end{aligned} \tag{3.20}$$

Using this estimate, together with (m_1)–(m_2) and the embedding $W^{s,p}_{V,\theta}(\mathbb{R}^N) \hookrightarrow L^{p_s^*(\beta,\theta)}(\mathbb{R}^N, |y|^{-\beta})$, we find

$$\max\left\{\frac{1}{2}, \frac{1}{\lambda p_s^*(\beta,\theta)}\right\}\left(\int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right)\frac{F(u_n)}{|x|^\delta}\mathrm{d}x + \int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}\, \mathrm{d}x\right)$$

$$
\begin{aligned}
&\geqslant \frac{1}{2}\int_{\mathbb{R}^N}\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{F(u_n)}{|x|^\delta}\mathrm{d}x+\frac{1}{\lambda p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\\
&=\frac{1}{p\lambda}M(\|u_n\|_W^p)-\frac{c}{\lambda}+\frac{o(1)}{\lambda}\\
&\geqslant \frac{1}{\xi p\lambda}m(\|u_n\|_W^p)\|u_n\|_W^p-\frac{c}{\lambda}+\frac{o(1)}{\lambda}\\
&\geqslant \frac{m_0}{\xi p\lambda}\|u_n\|_W^p-\frac{c}{\lambda}+\frac{o(1)}{\lambda}\\
&\to\infty, \quad \text{as} \quad n\to\infty.
\end{aligned} \tag{3.21}
$$

We claim that meas$(\Omega_1)=0$. Indeed, if meas$(\Omega_1)\neq 0$. From (3.18) and (3.19), we have

$$
\begin{aligned}
+\infty =&\int_{\Omega_1}\liminf_{n\to\infty}\frac{\Big(\mathcal{I}_\mu * \frac{F(u_n(x))}{|x|^\delta}\Big)\frac{F(u_n(x))}{|x|^\delta}}{|u_n(x)|^{p\xi}}|\omega_n(x)|^{p\xi}\mathrm{d}x\\
&+\int_{\Omega_1}\liminf_{n\to\infty}\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta|u_n(x)|^{p\xi}}|\omega_n(x)|^{p\xi}\mathrm{d}x-\int_{\Omega_1}\limsup_{n\to\infty}\frac{C}{\|u_n\|_W^{p\xi}}\mathrm{d}x\\
\leqslant&\int_{\Omega_1}\liminf_{n\to\infty}\left(\frac{\Big(\mathcal{I}_\mu * \frac{F(u_n(x))}{|x|^\delta}\Big)\frac{F(u_n(x))}{|x|^\delta}}{|u_n(x)|^{p\xi}}|\omega_n(x)|^{p\xi}+\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta|u_n(x)|^{p\xi}}|\omega_n(x)|^{p\xi}-\frac{C}{\|u_n\|_W^{p\xi}}\right)\mathrm{d}x
\end{aligned}
$$

and by Fatou's lemma,

$$
\begin{aligned}
&\leqslant\liminf_{n\to\infty}\int_{\Omega_1}\left(\frac{\Big(\mathcal{I}_\mu * \frac{F(u_n(x))}{|x|^\delta}\Big)\frac{F(u_n(x))}{|x|^\delta}}{|u_n(x)|^{p\xi}}|\omega_n(x)|^{p\xi}+\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta|u_n(x)|^{p\xi}}|\omega_n(x)|^{p\xi}-\frac{C}{\|u_n\|_W^{p\xi}}\right)\mathrm{d}x\\
&=\liminf_{n\to\infty}\int_{\Omega_1}\left(\frac{\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{F(u_n)}{|x|^\delta}}{\|u_n\|_W^{p\xi}}+\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta\|u_n\|_W^{p\xi}}-\frac{C}{\|u_n\|_W^{p\xi}}\right)\mathrm{d}x
\end{aligned}
$$

and by (3.13),

$$
\begin{aligned}
&\leqslant\liminf_{n\to\infty}\int_{\Omega_1}\Big(\frac{C\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{F(u_n)}{|x|^\delta}}{M(\|u_n\|_W^p)}+\frac{C|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta M(\|u_n\|_W^p)}\Big)\mathrm{d}x-\liminf_{n\to\infty}\int_{\Omega_1}\frac{C}{\|u_n\|_W^{p\theta}}\mathrm{d}x\\
&\leqslant\liminf_{n\to\infty}\int_{\mathbb{R}^N}\Big(\frac{C\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{F(u_n)}{|x|^\delta}}{M(\|u_n\|_W^p)}+\frac{C\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}}{M(\|u_n\|_W^p)}\Big)\mathrm{d}x\\
&=\frac{C}{p}\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{F(u_n)}{|x|^\delta}+\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}}{\frac{1}{p}M(\|u_n\|_W^p)}\mathrm{d}x
\end{aligned}
$$

and by (3.20),

$$
=\frac{C}{p}\liminf_{n\to\infty}\frac{\int_{\mathbb{R}^N}\Big(\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{F(u_n)}{|x|^\delta}+\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}\Big)\mathrm{d}x}{\frac{\lambda}{2}\int_{\mathbb{R}^N}\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{F(u_n)}{|x|^\delta}\mathrm{d}x+\frac{1}{p_s^*(\beta,\theta)}\int_{\mathbb{R}^N}\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x+c-o(1)}
$$

$$\leqslant \frac{C}{p}\liminf_{n\to\infty}\frac{\int_{\mathbb{R}^N}\left(\left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right)\frac{F(u_n)}{|x|^\delta}+\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}\right)\mathrm{d}x}{\max\left\{\frac{\lambda}{2},\frac{1}{p_s^*(\beta,\theta)}\right\}\left(\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right)\frac{F(u_n)}{|x|^\delta}\mathrm{d}x+\int_{\mathbb{R}^N}\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\right)+c-o(1)}. \tag{3.22}$$

So by (3.21) and (3.22), we get the contradiction

$$+\infty \leqslant \frac{C}{p\max\left\{\frac{\lambda}{2},\frac{1}{p_s^*(\beta,\theta)}\right\}}.$$

This shows that $\mathrm{meas}(\Omega_1)=0$. Hence $\omega(x)=0$ for almost all $x\in\mathbb{R}^N$. The convergence in (3.16) means that

$$\omega_n(x)\to 0 \quad \text{a.e. in } \mathbb{R}^N, \quad \text{and} \quad \omega_n(x)\to 0 \quad \text{a.e. in } L^r(\mathbb{R}^N) \quad \text{as } n\to\infty, \tag{3.23}$$

for $p\leqslant r<\frac{Np}{N-ps-\theta}$.

Using (3.14), (m_2), $p_s^*(\beta,\theta)>p$ and $\xi\geqslant 1$, we get

$$\begin{aligned}
c+1 \geqslant& I(u_n)-\frac{1}{p\xi}\langle I'(u_n),u_n\rangle\\
=&\frac{1}{p}M(\|u_n\|_W^p)-\frac{1}{p\xi}m(\|u_n\|_W^p)\|u_n\|_W^p\\
&+\left(\frac{1}{p\xi}-\frac{1}{p_s^*(\beta,\theta)}\right)\int_{\mathbb{R}^N}\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\\
&+\lambda\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right)\left(\frac{1}{p\xi}\frac{f(u_n)}{|x|^\delta}u_n-\frac{1}{2}\frac{F(u_n)}{|x|^\delta}\right)\mathrm{d}x\\
\geqslant&\left(\frac{1}{p\xi}-\frac{1}{p_s^*(\beta,\theta)}\right)\int_{\mathbb{R}^N}\frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta}\mathrm{d}x\\
&+\lambda\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right)\left(\frac{1}{p\xi}\frac{f(u_n)}{|x|^\delta}u_n-\frac{1}{2}\frac{F(u_n)}{|x|^\delta}\right)\mathrm{d}x\\
\geqslant&\lambda\int_{\mathbb{R}^N}\left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right)\left(\frac{1}{p\xi}\frac{f(u_n)}{|x|^\delta}u_n-\frac{1}{2}\frac{F(u_n)}{|x|^\delta}\right)\mathrm{d}x\\
=&\lambda\int_{\mathbb{R}^N}(\mathcal{I}_\mu * F(u_n))\mathfrak{F}(u_n)\mathrm{d}x,
\end{aligned} \tag{3.24}$$

for n large enough.

For $a,b\geqslant 0$, let us define

$$\Omega_n^*(a,b):=\left\{x\in\mathbb{R}^N: a\leqslant\frac{|u_n(x)|}{|x|^{\beta/(p_s^*(\beta,\theta)-p)}}\leqslant b\right\}$$
$$\Omega_n^i(a,b):=\left\{x\in\mathbb{R}^N: a\leqslant\frac{|u_n(x)|}{|x|^{\delta/(q_i-p)}}\leqslant b\right\} \quad (i\in\{1,2\}).$$

From (m_1) and (m_2), we have that

$$M(\|u_n\|_W^p)\geqslant\frac{1}{\xi}m(\|u_n\|_W^p)\|u_n\|_W^p\geqslant\frac{m_0}{\xi}\|u_n\|_W^p. \tag{3.25}$$

This inequality, together with (3.15) and (3.20) yields that

$$
\begin{aligned}
0 < \frac{\lambda}{2p} &\leqslant \limsup_{n\to\infty} \frac{\int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right) \frac{F(u_n)}{|x|^\delta} \mathrm{d}x}{M(\|u_n\|_W^p)} \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{\left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right) \frac{F(u_n)}{|x|^\delta}}{M(\|u_n\|_W^p)} \mathrm{d}x \\
&= \limsup_{n\to\infty} \left(\int_{\Omega_n^1 \cap \Omega_n^2 \cap \Omega_n^*(0,r_0)} + \int_{\mathbb{R}^N \setminus \Omega_n^1 \cap \Omega_n^2 \cap \Omega_n^*(0,r_0)} \right) \frac{\left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right) \frac{F(u_n)}{|x|^\delta}}{M(\|u_n\|_W^p)} \mathrm{d}x. \qquad (3.26)
\end{aligned}
$$

To simplify the notation, we denote $\Omega = \Omega_n^1 \cap \Omega_n^2 \cap \Omega_n^*(0, r_0)$. On the one hand, by Lemma 3.5, (3.25), (F_2), and (3.23), we obtain

$$
\begin{aligned}
\int_\Omega \frac{\left(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right) \frac{F(u_n)}{|x|^\delta}}{M(\|u_n\|_W^p)} \mathrm{d}x &\leqslant \int_\Omega \frac{K \frac{|F(u_n)|}{|x|^\delta}}{M(\|u_n\|_W^p)} \mathrm{d}x \\
&\leqslant \frac{K\xi}{m_0} \int_\Omega \frac{\frac{|F(u_n)|}{|x|^\delta}}{\|u_n\|_W^p} \mathrm{d}x \\
&\leqslant \frac{c_0 K\xi}{m_0} \int_\Omega \frac{1}{|x|^\delta} \left(\frac{|u_n|^{q_1}}{q_1 \|u_n\|_W^p} + \frac{|u_n|^{q_2}}{q_2 \|u_n\|_W^p} \right) \mathrm{d}x \\
&\quad + \frac{\xi}{m_0} \int_\Omega \frac{|u_n|^{p_s^*(\beta,\theta)}}{|x|^\beta \|u_n\|_W^p} \mathrm{d}x \\
&= \frac{c_0 K\xi}{m_0} \int_\Omega \frac{1}{|x|^\delta} \left(\frac{|u_n|^{q_1-p}|u_n|^p}{q_1 \|u_n\|_W^p} + \frac{|u_n|^{q_2-p}|u_n|^p}{q_2 \|u_n\|_W^p} \right) \mathrm{d}x \\
&\quad + \frac{\xi}{m_0} \int_\Omega \frac{|u_n|^{p_s^*(\beta,\theta)-p}|u_n|^p}{|x|^\beta \|u_n\|_W^p} \mathrm{d}x \\
&= \frac{c_0 K\xi}{m_0} \int_\Omega \frac{1}{|x|^\delta} \left(\frac{|u_n|^{q_1-p}}{q_1} |\omega_n|^p + \frac{|u_n|^{q_2-p}}{q_2} |\omega_n|^p \right) \mathrm{d}x \\
&\quad + \frac{\xi}{m_0} \int_\Omega \frac{|u_n|^{p_s^*(\beta,\theta)-p}}{|x|^\beta} |\omega_n|^p \mathrm{d}x \\
&\leqslant \frac{c_0 K\xi}{m_0} \left(\frac{r_0^{q_1-p}}{q_1} + \frac{r_0^{q_2-p}}{q_2} \right) \int_\Omega |\omega_n|^p \mathrm{d}x \\
&\quad + \frac{\xi}{m_0} r_0^{p_s^*(\beta,\theta)-p} \int_\Omega |\omega_n|^p \mathrm{d}x \\
&\to 0, \quad \text{as } n \to \infty. \qquad (3.27)
\end{aligned}
$$

On the other hand, using Hölder inequality, (3.23), (3.24) and (F_4), we find

$$
\begin{aligned}
&\int_{\mathbb{R}^N \setminus \Omega} \frac{\left|\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right| \frac{F(u_n)}{|x|^\delta}}{M(\|u_n\|_W^p)} \mathrm{d}x \\
&\leqslant \frac{\xi}{m_0} \int_{\mathbb{R}^N \setminus \Omega} \frac{\left|\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right| \frac{F(u_n)}{|x|^\delta}}{\|u_n\|_W^p} \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\xi}{m_0}\int_{\mathbb{R}^N\setminus\Omega}\frac{\left|\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right|\frac{F(u_n)}{|x|^\delta}}{|u_n|^p}|\omega_n(x)|^p\mathrm{d}x\\
&\leqslant \frac{\xi}{m_0}\left(\int_{\mathbb{R}^N\setminus\Omega}\left(\frac{\left|\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right|\frac{F(u_n)}{|x|^\delta}}{|u_n|^p}\right)^\kappa \mathrm{d}x\right)^{\frac{1}{\kappa}}\left(\int_{\mathbb{R}^N\setminus\Omega}|\omega_n(x)|^{\frac{\kappa p}{\kappa-1}}\mathrm{d}x\right)^{\frac{\kappa-1}{\kappa}}\\
&\leqslant \frac{\xi}{m_0}c_1^{\frac{1}{\kappa}}\left(\int_{\mathbb{R}^N\setminus\Omega}\left|\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right|^\kappa \mathcal{F}(u_n)\mathrm{d}x\right)^{\frac{1}{\kappa}}\left(\int_{\Omega_n(r_0,\infty)}|\omega_n(x)|^{\frac{\kappa p}{\kappa-1}}\mathrm{d}x\right)^{\frac{\kappa-1}{\kappa}}\\
&\leqslant \frac{\xi}{m_0}c_1^{\frac{1}{\kappa}}K^{\frac{\kappa-1}{\kappa}}\left(\int_{\mathbb{R}^N\setminus\Omega}\left|\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\right|\mathcal{F}(u_n)\mathrm{d}x\right)^{\frac{1}{\kappa}}\left(\int_{\mathbb{R}^N\setminus\Omega}|\omega_n(x)|^{\frac{\kappa p}{\kappa-1}}\mathrm{d}x\right)^{\frac{\kappa-1}{\kappa}}\\
&\leqslant \frac{\xi}{m_0}c_1^{\frac{1}{\kappa}}K^{\frac{\kappa-1}{\kappa}}\left(\frac{c+1}{\lambda}\right)^{\frac{1}{\kappa}}\left(\int_{\mathbb{R}^N\setminus\Omega}|\omega_n(x)|^{\frac{\kappa p}{\kappa-1}}\mathrm{d}x\right)^{\frac{\kappa-1}{\kappa}} \to 0, \quad \text{as } n\to\infty. \qquad (3.28)
\end{aligned}
$$

Here we used the fact that $\frac{\kappa p}{\kappa-1} \in (p, \frac{p(N-\beta)}{N-ps-\theta})$ if $\kappa > \frac{N-\beta}{ps+\theta-\beta}$. Thus, we get a contradiction from (3.26)-(3.28). The proof is complete. □

Lemma 3.11. *Assume that* (V)*,* (m_1)*–*(m_2) *and* (F_1)*–*(F_4) *hold. Then the functional* I *satisfies* $(C)_c$*−condition for any* $\lambda > 0$*.*

Proof. Suppose that $\{u_n\} \subset W^{s,p}_{V,\theta}(\mathbb{R}^N)$ is a $(C)_c$−sequence for I, from Lemma 3.10, we have that $\{u_n\}$ is bounded in $W^{s,p}_{V,\theta}(\mathbb{R}^N)$, then if necessary to a subsequence, we have

$$
\begin{aligned}
&u_n \rightharpoonup u \ \text{ in } W^{s,p}_{V,\theta}(\mathbb{R}^N), \quad u_n \to u \ \text{ a.e. in } \mathbb{R}^N,\\
&u_n \to u \ \text{ in } L^{q_1}(\mathbb{R}^N, |x|^{-\delta}) \cap L^{q_2}(\mathbb{R}^N, |x|^{-\delta}), \qquad (3.29)\\
&\frac{|u_n|}{|x|^\delta} \leqslant \ell(x) \ \text{ a.e. in } \mathbb{R}^N, \quad \text{for some } \ell(x) \in L^{q_1}(\mathbb{R}^N, |x|^{-\delta}) \cap L^{q_2}(\mathbb{R}^N, |x|^{-\delta}).
\end{aligned}
$$

For simplicity, let $\varphi \in W^{s,p}_{V,\theta}(\mathbb{R}^N)$ be fixed and denote by B_φ the linear functional on $W^{s,p}_{V,\theta}(\mathbb{R}^N)$ defined by

$$
B_\varphi(v) = \iint_{\mathbb{R}^{2N}}\frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x|^{\theta_1}|x-y|^{N+ps}|y|^{\theta_2}}(v(x)-v(y))\mathrm{d}x\mathrm{d}y
$$

for all $v \in W^{s,p}_{V,\theta}(\mathbb{R}^N)$. By Hölder inequality, we have

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}}\frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x|^{\theta_1}|x-y|^{N+ps}|y|^{\theta_2}}(v(x)-v(y))\mathrm{d}x\mathrm{d}y\\
&= \iint_{\mathbb{R}^{2N}}\frac{|\varphi(x)-\varphi(y)|^{p-1}}{|x|^{\theta_1-\theta_1/p}|x-y|^{(N+ps)-(N+ps)/p}|y|^{\theta_2-\theta_2/p}}\frac{(v(x)-v(y))}{|x|^{\theta_1/p}|x-y|^{(N+ps)/p}|y|^{\theta_2/p}}\mathrm{d}x\mathrm{d}y\\
&\leqslant \left(\iint_{\mathbb{R}^N}\left(\frac{|\varphi(x)-\varphi(y)|^{p-1}}{|x|^{\theta_1-\theta_1/p}|x-y|^{(N+ps)-(N+ps)/p}|y|^{\theta_2-\theta_2/p}}\right)^{\frac{p}{p-1}}\mathrm{d}x\mathrm{d}y\right)^{\frac{p-1}{p}}\\
&\qquad \left(\iint_{\mathbb{R}^N}\left(\frac{(v(x)-v(y))}{|x|^{\theta_1/p}|x-y|^{(N+ps)/p}|y|^{\theta_2/p}}\right)^p\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\\
&\leqslant [\varphi]^{p-1}_{s,p,\theta}[v]_{s,p,\theta} \leqslant \|\varphi\|^{p-1}_W\|v\|_W,
\end{aligned}
$$

for all $v \in W^{s,p}_{V,\theta}(\mathbb{R}^N)$. Hence, (3.29) gives that

$$\lim_{n\to\infty}\Big(m(\|u_n\|_W^p) - m(\|u\|_W^p)\Big)B_u(u_n-u) = 0, \tag{3.30}$$

since $\Big\{m(\|u_n\|_W^p) - m(\|u\|_W^p)\Big\}_n$ is bounded in $\mathbb{R}$.

Since $I'(u_n) \to 0$ in $(W^{s,p}_{V,\theta}(\mathbb{R}^N))'$ and $u_n \rightharpoonup u$ in $W^{s,p}_{V,\theta}(\mathbb{R}^N)$, we have

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

That is,

$$\begin{aligned}
o(1) =&\langle I'(u_n) - I'(u), u_n - u\rangle\\
=&m(\|u_n\|_W^p)\|u_n-u\|_W^p - m(\|u\|_W^p)\|u_n-u\|_W^p\\
&-\int_{\mathbb{R}^N}\frac{|u_n|^{p_s^*(\beta,\theta)}u_n(u_n-u)}{|x|^\beta}\mathrm{d}x + \int_{\mathbb{R}^N}\frac{|u|^{p_s^*(\beta,\theta)}u(u_n-u)}{|x|^\beta}\mathrm{d}x\\
&-\lambda\int_{\mathbb{R}^N}\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{f(u_n)}{|x|^\delta}(u_n-u)\mathrm{d}x + \lambda\int_{\mathbb{R}^N}\Big(\mathcal{I}_\mu * \frac{F(u)}{|x|^\delta}\Big)\frac{f(u)}{|x|^\delta}(u_n-u)\mathrm{d}x\\
=&m(\|u_n\|_W^p)\Big(B_{u_n}(u_n-u) + \int_{\mathbb{R}^N}\frac{V(x)|u_n|^{p-2}u_n(u_n-u)}{|x|^\alpha}\mathrm{d}x\Big)\\
&-m(\|u\|_W^p)\Big(B_u(u_n-u) + \int_{\mathbb{R}^N}\frac{V(x)|u|^{p-2}u(u_n-u)}{|x|^\alpha}\mathrm{d}x\Big)\\
&-\int_{\mathbb{R}^N}\frac{(|u_n|^{p_s^*(\beta,\theta)-2}u_n - |u|^{p_s^*(\beta,\theta)-2}u)(u_n-u)}{|x|^\alpha}\mathrm{d}x\\
&-\lambda\int_{\mathbb{R}^N}\Big[\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{f(u_n)}{|x|^\delta} - \Big(\mathcal{I}_\mu * \frac{F(u)}{|x|^\delta}\Big)\frac{f(u)}{|x|^\delta}\Big](u_n-u)\mathrm{d}x\\
=&m(\|u_n\|_W^p)\Big(B_{u_n}(u_n-u) + \int_{\mathbb{R}^N}\frac{V(x)|u_n|^{p-2}u_n(u_n-u)}{|x|^\alpha}\mathrm{d}x\Big)\\
&-m(\|u\|_W^p)\Big(B_u(u_n-u) + \int_{\mathbb{R}^N}\frac{V(x)|u|^{p-2}u(u_n-u)}{|x|^\alpha}\mathrm{d}x\Big)\\
&+m(\|u_n\|_W^p)B_u(u_n-u) - m(\|u_n\|_W^p)B_u(u_n-u)\\
&+m(\|u_n\|_W^p)\frac{V(x)|u|^{p-2}u(u_n-u)}{|x|^\alpha}\mathrm{d}x - m(\|u_n\|_W^p)\frac{V(x)|u|^{p-2}u(u_n-u)}{|x|^\alpha}\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\frac{(|u_n|^{p_s^*(\beta,\theta)-2}u_n - |u|^{p_s^*(\beta,\theta)-2}u)(u_n-u)}{|x|^\alpha}\mathrm{d}x\\
&-\lambda\int_{\mathbb{R}^N}\Big[\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{f(u_n)}{|x|^\delta} - \Big(\mathcal{I}_\mu * \frac{F(u)}{|x|^\delta}\Big)\frac{f(u)}{|x|^\delta}\Big](u_n-u)\mathrm{d}x\\
=&m(\|u_n\|_W^p)\Big[B_{u_n}(u_n-u) - B_u(u_n-u)\Big]\\
&+\Big(m(\|u_n\|_W^p) - m(\|u\|_W^p)\Big)B_u(u_n-u)\\
&+m(\|u_n\|_W^p)\int_{\mathbb{R}^N}\frac{V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)}{|x|^\alpha}\mathrm{d}x\\
&+[m(\|u_n\|_W^p) - m(\|u\|_W^p)]\int_{\mathbb{R}^N}\frac{V(x)|u|^{p-2}u(u_n-u)}{|x|^\alpha}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^N} \frac{(|u_n|^{p_s^*(\beta,\theta)-2}u_n - |u|^{p_s^*(\beta,\theta)-2}u)(u_n-u)}{|x|^\alpha}\mathrm{d}x \\
&- \lambda \int_{\mathbb{R}^N} \Big[\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{f(u_n)}{|x|^\delta} - \Big(\mathcal{I}_\mu * \frac{F(u)}{|x|^\delta}\Big)\frac{f(u)}{|x|^\delta}\Big](u_n-u)\mathrm{d}x. \qquad (3.31)
\end{aligned}
$$

From Lemma 3.2, we have

$$
\int_{\mathbb{R}^N} \Big[\Big(\mathcal{I}_\mu * \frac{F(u_n)}{|x|^\delta}\Big)\frac{f(u_n)}{|x|^\delta} - \Big(\mathcal{I}_\mu * \frac{F(u)}{|x|^\delta}\Big)\frac{f(u)}{|x|^\delta}\Big](u_n-u)\mathrm{d}x \to 0, \quad \text{as } n\to\infty. \qquad (3.32)
$$

Moreover, using Hölder inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \frac{V(x)|u|^{p-2}u(u_n-u)}{|x|^\alpha}\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \frac{(V(x))^{(p-1)/p}|u|^{p-1}}{|x|^{\alpha(p-1)/p}}\frac{(V(x))^{1/p}|u_n-u|^p}{|x|^{\alpha/p}}\mathrm{d}x \\
&\leqslant \Big(\int_{\mathbb{R}^N}\Big(\frac{(V(x))^{(p-1)/p}|u|^{p-1}}{|x|^{\alpha(p-1)/p}}\Big)^{\frac{p}{p-1}}\mathrm{d}x\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^N}\Big(\frac{(V(x))^{(1)/p}|u_n-u|}{|x|^{\alpha/p}}\Big)^{p}\mathrm{d}x\Big)^{\frac{1}{p}} \\
&= \Big(\int_{\mathbb{R}^N}\frac{V(x)|u|^p}{|x|^\alpha}\mathrm{d}x\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^N}\frac{V(x)|u_n-u|^p}{|x|^\alpha}\mathrm{d}x\Big)^{\frac{1}{p}}. \qquad (3.33)
\end{aligned}
$$

From inequality above and (3.29), we obtain

$$
[m(\|u_n\|_W^p) - m(\|u\|_W^p)]\int_{\mathbb{R}^N}\frac{V(x)|u|^{p-2}u(u_n-u)}{|x|^\alpha}\mathrm{d}x \to 0, \quad \text{as } n\to\infty. \qquad (3.34)
$$

From (3.30)-(3.34) and (m_1), we obtain

$$
\lim_{n\to\infty} m(\|u_n\|_W^p)\left(\Big[B_{u_n}(u_n-u) - B_u(u_n-u)\Big] + \int_{\mathbb{R}^N}\frac{V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)}{|x|^\alpha}\mathrm{d}x\right) = 0.
$$

Since $m(\|u_n\|_W^p)[B_{u_n}(u_n-u) - B_u(u_n-u)] \geqslant 0$ and $\frac{V(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)}{|x|^\alpha} \geqslant 0$ for all n by convexity, (m_1) and (V_1), we have

$$
\begin{aligned}
\lim_{n\to\infty}\Big[B_{u_n}(u_n-u) - B_u(u_n-u)\Big] &= 0, \\
\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)}{|x|^\alpha}\mathrm{d}x &= 0. \qquad (3.35)
\end{aligned}
$$

Let us now recall the well-known Simon inequalities. There exist positive numbers c_p and C_p, depending only on p, such that

$$
|\xi-\eta|^p \leqslant \begin{cases} c_p(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) & \text{for } p\geqslant 2, \\ C_p\Big[(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta)\Big]^{p/2}(|\xi|^p+|\eta|^p)^{(2-p)/2} & \text{for } 1<p<2, \end{cases} \qquad (3.36)
$$

for all $\xi,\eta\in\mathbb{R}^N$. According to the Simon inequality, we divide the discussion into two cases.

Case $p \geqslant 2$: From (3.35) and (3.36), as $n \to \infty$, we have

$$\begin{aligned}[u_n - u]^p_{W^{s,p}_\theta(\mathbb{R}^N)} =& \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u(x) - u_n(y) + u(y)|^p}{|x|^{\theta_1}|x-y|^{N+ps}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \\ =& \iint_{\mathbb{R}^{2N}} \frac{|(u_n(x) - u_n(y)) - (u(x) - u(y))|^p}{|x|^{\theta_1}|x-y|^{N+ps}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \\ \leqslant& c_p \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x|^{\theta_1}|x-y|^{N+ps}|y|^{\theta_2}} \\ & \times \Big(u_n(x) - u(x) - u_n(y) + u(y)\Big) \mathrm{d}x\mathrm{d}y \\ =& c_p \Big[B_{u_n}(u_n - u) - B_u(u_n - u)\Big] = o(1),\end{aligned}$$

and

$$\begin{aligned}\|u_n - u\|^p_{L^p_V(\mathbb{R}^N, |x|^{-\alpha})} =& \int_{\mathbb{R}^N} \frac{V(x)|u_n - u|^p}{|x|^\alpha} \mathrm{d}x \\ \leqslant& c_p \int_{\mathbb{R}^N} \frac{V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)}{|x|^\alpha} \mathrm{d}x = o(1).\end{aligned}$$

Consequently, $\|u_n - u\|_W = ([u_n - u]^p_{W^{s,p}_\theta(\mathbb{R}^N)} + \|u_n - u\|^p_{L^p_V(\mathbb{R}^N, |x|^{-\alpha})})^{\frac{1}{p}} \to 0$ as $n \to \infty$.

Case $1 < p < 2$: taking $\xi = u_n(x) - u_n(y)$ and $\eta = u(x) - u(y)$ in (3.36), as $n \to \infty$, we have

$$\begin{aligned}[u_n - u]^p_{W^{s,p}_\theta(\mathbb{R}^N)} =& \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u(x) - u_n(y) + u(y)|^p}{|x|^{\theta_1}|x-y|^{N+ps}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \\ =& \iint_{\mathbb{R}^{2N}} \frac{|(u_n(x) - u_n(y)) - (u(x) - u(y))|^p}{|x|^{\theta_1}|x-y|^{N+ps}|y|^{\theta_2}} \mathrm{d}x\mathrm{d}y \\ \leqslant& C_p \Big[\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x|^{\theta_1}|x-y|^{N+ps}|y|^{\theta_2}} \\ & \times \Big(u_n(x) - u(x) - u_n(y) + u(y)\Big)\Big]^{\frac{p}{2}} \Big(\frac{|u_n(x) - u_n(y)|^p + |u(x) - u(y)|^p)}{|x|^{\theta_1}|x-y|^{N+ps}|y|^{\theta_2}}\Big)^{\frac{2-p}{2}} \mathrm{d}x\mathrm{d}y \\ =& C_p \Big[B_{u_n}(u_n - u) - B_u(u_n - u)\Big]^{p/2} ([u_n]^p_{W^{s,p}_\theta(\mathbb{R}^N)} + [u]^p_{W^{s,p}_\theta(\mathbb{R}^N)})^{(2-p)/2} \\ \leqslant& C_p \Big[B_{u_n}(u_n - u) - B_u(u_n - u)\Big]^{p/2} ([u_n]^{p(2-p)/2}_{W^{s,p}_\theta(\mathbb{R}^N)} + [u]^{p(2-p)/2}_{W^{s,p}_\theta(\mathbb{R}^N)}) \\ \leqslant& C \Big[B_{u_n}(u_n - u) - B_u(u_n - u)\Big]^{p/2} = o(1).\end{aligned}$$

Here we used the fact that $[u_n]_{W^{s,p}_\theta(\mathbb{R}^N)}$ and $[u]_{W^{s,p}_\theta(\mathbb{R}^N)}$ are bounded, and the elementary inequality

$$(a+b)^{(2-p)/2} \leqslant a^{(2-p)/2} + b^{(2-p)/2} \quad \text{for all } a, b \geqslant 0 \text{ and } 1 < p < 2.$$

Moreover, by Hölder inequality and (3.35), as $n \to \infty$,

$$\begin{aligned}\|u_n - u\|^p_{L^p_V(\mathbb{R}^N, |x|^{-\alpha})} =& \int_{\mathbb{R}^N} \frac{V(x)|u_n - u|^p}{|x|^\alpha} \mathrm{d}x \\ \leqslant& C_p \int_{\mathbb{R}^N} V(x) \Big[\frac{(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)}{|x|^\alpha}\Big]^{p/2} \Big(\frac{|u_n|^p + |u|^p}{|x|^\alpha}\Big)^{(2-p)/2} \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\leqslant C_p\left(\int_{\mathbb{R}^N}\frac{V(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)}{|x|^\alpha}\mathrm{d}x\right)^{p/2}\\
&\quad\times\left(\int_{\mathbb{R}^N}\frac{V(x)(|u_n|^p+|u|^p)}{|x|^\alpha}\mathrm{d}x\right)^{(2-p)/2}\\
&\leqslant C_p\left(\|u_n\|^{p(2-p)/2}_{L^p_V(\mathbb{R}^N,|x|^{-\alpha})}+\|u\|^{p(2-p)/2}_{L^p_V(\mathbb{R}^N,|x|^{-\alpha})}\right)\\
&\quad\times\left(\int_{\mathbb{R}^N}\frac{V(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)}{|x|^\alpha}\mathrm{d}x\right)^{p/2}\\
&\leqslant C\left(\int_{\mathbb{R}^N}\frac{V(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)}{|x|^\alpha}\mathrm{d}x\right)^{p/2}\to 0.
\end{aligned}
$$

Thus $\|u_n-u\|_W=([u_n-u]^p_{W^{s,p}_\theta(\mathbb{R}^N)}+\|u_n-u\|^p_{L^p_V(\mathbb{R}^N,|x|^{-\alpha})})^{\frac{1}{p}}\to 0$ as $n\to\infty$. The proof is complete. □

Proof of Theorem 0.5. By Lemma 3.9 we show the geometry of the functional I associated with the problem (10). Furthermore, by Lemmas 3.10 and 3.11 we show that the Cerami sequences for the functional are limited and that the functional I verifies the Cerami condition, respectively. Therefore, we obtain that there exists a critical point of functional I, so problem (10) has a nontrivial weak solution for any $\lambda>0$. □

Conclusion

Summary of this work

In this work we studied some class of partial differential elliptic equations involving the fractional p-Laplacian operator.

We studied a fractional p-Laplacian model problem in the entire space $\mathbb{R}^N$ featuring doubly critical nonlinearities involving a local critical Sobolev term, a nonlocal Choquard fractional critical term, and a homogeneous Hardy term; all nonlinearities have singular critical weights. We established new embedding results involving weighted Morrey norms in the homogeneous fractional Sobolev space; and we provided sufficient conditions under which a weak nontrivial solution to the problem exists via variational methods. With this knowledge, we formulated similar problems with double critical Sobolev and double critical Choquard terms and discovered that the same technique could be applied to prove existence results for these classes of problems.

Next, we considered a fractional p-Laplacian system of equations in the entire space $\mathbb{R}^N$ with doubly critical singular nonlinearities, a local critical Sobolev term together, a nonlocal Choquard critical term, and a homogeneous Hardy term; all nonlinearities have singular critical weights; additionally, the coupling term is critical in the sense of the Sobolev embeddings. Since the problem involves doubly critical exponents, our proof made use of a version of the Caffarelli-Kohn-Nirenberg inequality and a refinement of Sobolev inequality that is related to Morrey space. We proved our existence theorem using these results and variational methods. Again, we notice that the same technique could be used to prove some variants of this problem.

Finally, we considered a fractional p-Kirchhoff equation in the entire space $\mathbb{R}^N$ featuring double nonlinearities, a generalized nonlocal subcritical Choquard term limited by both the lower and the upper critical Stein-Weiss exponents, a local critical Sobolev term, and a Hardy-type term; additionaly, all terms have critical singular weights. With respect to the compactness condition, we had to use Cerami sequences because in our problem there is a possibly slower growth in the nonlinearities. In this way, we could deal with the lack of compactness of the Sobolev embeddings through a uniform boundedness of the convolution part.

Goals for the near future

In this section we list some open problems related to the fractional p-Laplacian operator. We begin by mentioning that there is an increasing and ever growing literature devoted to the study of improved versions of the Sobolev-Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg type inequalities as a subject with their own interest.

1. We can consider the existence of ground state solutions to fractional p-Laplacian

problems with doubly critical nonlinearity in the sense of Stein-Weiss and involving both the lower and the upper critical exponents, defined respectively by $p_s^\flat(\delta,\theta,\mu) = p(N-\delta-\mu/2)/N$ and $p_s^\sharp(\delta,\theta,\mu) = p(N-\delta-\mu/2)/(N-sp-\theta)$.

2. We can consider the existence of solutions to doubly critical coupled systems involving fractional p-Laplacian in $\mathbb{R}^N$ with Hardy-Sobolev terms and Choquard term, all of them with singular weights.

3. This work deals only with the case $sp+\theta < N$, called subconformal case. Recently, there appeared some papers dealing with the case $sp = N$ (and $\theta_1 = \theta_2 = 0$), called conformal case. Therefore, it seems possible to consider versions of problem (1) also in the conformal case $sp+\theta = N$.

4. Li & Yang [56] considered, among other things, an existence result to problem (1) in the case $p = 2$, $\theta_1 = \theta_2 = 0$ and $\alpha = 0$. Their result rely on the proof of a related inequality in the paper by Yang [92]; see also Yang & Wu [93]. However, we could not check the arguments given by these authors; particularly [92, inequality (3.2)] and [93, inequality (2.8)]. So, it does not seem possible to perform the argument using the refined Sobolev inequality with the Morrey norm in the presence of weights in the case $\alpha = 0$. In fact, in this case the Morrey space coincides with the weighted Lebesgue space and we cannot argue as in the case $\alpha \neq 0$. For this reason, we believe that the analysis of problem (1) in the case $\alpha = 0$ still must be done. Perhaps the proof of the existence result can be achieved in the context of Besov norm as pointed out by De Nápoli, Drelichman & Salort [37]. They proved the existence of minimizers of the Stein-Weiss inequality only for $p = 2$ but, in turn, they managed to deal with any α in an open interval containing the origin. Hence, the analysis of the problem for $\alpha = 0$ still needs to be done even in the case $p = 2$ and $\theta = \theta_1 + \theta_2 \neq 0$.

Bibliography

[1] B. Abdellaoui and R. Bentifour, “Caffarelli-Kohn-Nirenberg type inequalities of fractional order with applications,” *J. Funct. Anal.*, vol. 272, no. 10, pp. 3998–4029, 2017. [Online]. Available: https://doi.org/10.1016/j.jfa.2017.02.007 26

[2] ——, “Caffarelli–Kohn–Nirenberg type inequalities of fractional order with applications,” *Journal of Functional Analysis*, vol. 272, no. 10, pp. 3998–4029, 2017. 26

[3] S. Agmon, A. Douglis, and L. Nirenberg, “Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. i,” *Communications on pure and applied mathematics*, vol. 12, no. 4, pp. 623–727, 1959. 86

[4] S. Ahmad, A. Lazer, and J. Paul, “Elementary critical point theory and perturbations of elliptic boundary value problems at resonance,” *Indiana University Mathematics Journal*, vol. 25, no. 10, pp. 933–944, 1976. 89

[5] C. O. Alves and M. A. Souto, “Existence of solutions for a class of elliptic equations in $\mathbb{R}^N$ with vanishing potentials,” *Journal of Differential Equations*, vol. 252, no. 10, pp. 5555–5568, 2012. 84

[6] ——, “Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity,” *Journal of Differential Equations*, vol. 254, no. 4, pp. 1977–1991, 2013. 84

[7] C. O. Alves, F. Corrêa, and G. M. Figueiredo, “On a class of nonlocal elliptic problems with critical growth,” *Differ. Equ. Appl*, vol. 2, no. 3, pp. 409–417, 2010. 84

[8] C. Alves, D. de Morais Filho, and M. Souto, “Radially symmetric solutions for a class of critical exponent elliptic problems in $\mathbb{R}^n$,” *Electronic Journal of Differential Equation*, 1996. 84

[9] M. J. Alves, R. B. Assunçao, and O. H. Miyagaki, “Existence result for a class of quasilinear elliptic equations with (p-q)-laplacian and vanishing potentials,” *Illinois Journal of Mathematics*, vol. 59, no. 3, pp. 545–575, 2015. 84

[10] M. J. Alves and R. B. Assuncao, “Existence of solutions for a problem with multiple singular weighted p-laplacians and vanishing potentials,” *Electronic Journal of Differential Equations*, 2022. 84

[11] H. Amann and E. Zehnder, “Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations,” *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, vol. 7, no. 4, pp. 539–603, 1980. 85

[12] A. Ambrosetti and P. H. Rabinowitz, "Dual variational methods in critical point theory and applications," *Journal of functional Analysis*, vol. 14, no. 4, pp. 349–381, 1973. 30, 54, 74

[13] R. B. Assunção, O. H. Miyagaki, and R. F. Siqueira, "Fractional Sobolev-Chocard critical equation with hardy term and weighted singularities," *arXiv preprint arXiv:2311.00852*, 2023. 67

[14] R. B. Assunção, J. C. Silva, and O. H. Miyagaki, "A fractional p-Laplacian problem with multiple critical Hardy–Sobolev nonlinearities," *Milan Journal of Mathematics*, vol. 88, no. 1, pp. 65–97, 2020. 28, 65

[15] R. B. Assunção, O. H. Miyagaki, and R. F. S. Siqueira, "Fractional Sobolev-Chocard critical equation with hardy term and weighted singularities," 2023. 92

[16] G. Autuori, A. Fiscella, and P. Pucci, "Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity," *Nonlinear Analysis*, vol. 125, pp. 699–714, 2015. 84

[17] P. Bartolo, V. Benci, and D. Fortunato, "Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity," *Nonlinear analysis: Theory, methods & applications*, vol. 7, no. 9, pp. 981–1012, 1983. 89

[18] T. Bartsch and Z. Qiang Wang, "Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^n$: existence and multiplicity results," *Communications in Partial Differential Equations*, vol. 20, no. 9-10, pp. 1725–1741, 1995. 84

[19] H. Berestycki and P.-L. Lions, "Nonlinear scalar field equations. pt. 1," *Archive for Rational Mechanics and Analysis*, vol. 82, no. 4, pp. 313–346, 1983. 84

[20] L. Brasco, S. Mosconi, and M. Squassina, "Optimal decay of extremals for the fractional Sobolev inequality," *Calculus of Variations and Partial Differential Equations*, vol. 55, pp. 1–32, 2016. 68

[21] ——, "Optimal decay of extremals for the fractional Sobolev inequality," *Calc. Var. Partial Differential Equations*, vol. 55, no. 2, pp. Art. 23, 32, 2016. [Online]. Available: https://doi.org/10.1007/s00526-016-0958-y 68

[22] H. Brezis and H. Brézis, *Functional analysis, Sobolev spaces and partial differential equations.* Springer, 2011, vol. 2. 94

[23] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian," *Comm. Partial Differential Equations*, vol. 32, no. 7-9, pp. 1245–1260, 2007. [Online]. Available: https://doi.org/10.1080/03605300600987306 27

[24] A. Castro and A. C. Lazer, "Critical point theory and the number of solutions of a nonlinear Dirichlet problem," *Ann. Mat. Pura Appl. (4)*, vol. 120, pp. 113–137, 1979. [Online]. Available: https://doi.org/10.1007/BF02411940 85

[25] C. Chen, "Infinitely many solutions to a class of quasilinear Schrödinger system in r^n," *Applied Mathematics Letters*, vol. 52, pp. 176–182, 2016. 66

[26] S.-J. Chen and L. Li, "Multiple solutions for the nonhomogeneous Kirchhoff equation on $\mathbb{R}^N$," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 3, pp. 1477–1486, 2013. 84

[27] W. Chen, "Fractional elliptic problems with two critical Sobolev-Hardy exponents," *Electron. J. Differential Equations*, pp. Paper No. 22, 12, 2018. 27, 65

[28] ——, "Existence of solutions for fractional p-Kirchhoff type equations with a generalized Choquard nonlinearity," *Topol. Methods Nonlinear Anal.*, vol. 54, no. 2, pp. 773–791, 2019. [Online]. Available: https://doi.org/10.12775/tmna.2019.069 65, 84

[29] W. Chen and S. Deng, "The nehari manifold for a fractional p-laplacian system involving concave–convex nonlinearities," *Nonlinear Analysis: Real World Applications*, vol. 27, pp. 80–92, 2016. 66

[30] W. Chen and M. Squassina, "Critical nonlocal systems with concave-convex powers," *Advanced Nonlinear Studies*, vol. 16, no. 4, pp. 821–842, 2016. 28, 65, 66

[31] P. Clément, D. Figueiredo, and E. Mitidieri, "Positive solutions of semilinear elliptic systems," *Communications in Partial Differential Equations*, vol. 17, pp. 923–940, 01 1992. 86

[32] F. Colasuonno and P. Pucci, "Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 17, pp. 5962–5974, 2011. 84

[33] D. G. Costa, "On a class of elliptic systems in $\mathbb{R}^N$," *Electron. J. Differential Equations*, pp. No. 07, approx. 14 pp., 1994. 84

[34] ——, "Remarks on compactness conditions and their application," *Electronic Journal of Differential Equations*, no. Special Issue 02, pp. 101–107, 2023. 85

[35] D. G. Costa and O. H. Miyagaki, "Nontrivial solutions for perturbations of the p-laplacian on unbounded domains," *Journal of Mathematical Analysis and Applications*, vol. 193, no. 3, pp. 737–755, 1995. 95

[36] D. G. de Figueiredo and J. Yang, "On a semilinear elliptic problem without (PS) condition," *Journal of Differential Equations*, vol. 187, no. 2, pp. 412–428, 2003. 86

[37] P. De Nápoli, I. Drelichman, and A. Salort, "Weighted inequalities for the fractional Laplacian and the existence of extremals," *Communications in Contemporary Mathematics*, vol. 21, no. 03, p. 1850034, 2019. 13, 107

[38] F. del Teso, D. Gómez-Castro, and J. L. Vázquez, "Three representations of the fractional p-Laplacian: semigroup, extension and Balakrishnan formulas," *Fract. Calc. Appl. Anal.*, vol. 24, no. 4, pp. 966–1002, 2021. [Online]. Available: https://doi.org/10.1515/fca-2021-0042 20, 29, 40

[39] E. Di Nezza, G. Palatucci, and E. Valdinoci, "Hitchhiker's guide to the fractional Sobolev spaces," *Bull. Sci. Math.*, vol. 136, no. 5, pp. 521–573, 2012. [Online]. Available: https://doi.org/10.1016/j.bulsci.2011.12.004 19, 20, 65, 83

[40] S. Dipierro, L. Montoro, I. Peral, and B. Sciunzi, "Qualitative properties of positive solutions to nonlocal critical problems involving the Hardy-Leray potential," *Calculus of Variations and Partial Differential Equations*, vol. 55, pp. 1–29, 2016. 50

[41] G. M. Figueiredo, "Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument," *Journal of Mathematical Analysis and Applications*, vol. 401, no. 2, pp. 706–713, 2013. 84

[42] R. Filippucci, P. Pucci, and F. Robert, "On a p-Laplace equation with multiple critical nonlinearities," *J. Math. Pures Appl. (9)*, vol. 91, no. 2, pp. 156–177, 2009. [Online]. Available: https://doi.org/10.1016/j.matpur.2008.09.008 27, 28, 30, 50, 65

[43] A. Fiscella, P. Pucci, and S. Saldi, "Existence of entire solutions for Schrödinger–hardy systems involving two fractional operators," *Nonlinear Analysis*, vol. 158, pp. 109–131, 2017. 66

[44] A. Fiscella and E. Valdinoci, "A critical Kirchhoff type problem involving a nonlocal operator," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 94, pp. 156–170, 2014. 84

[45] R. L. Frank and R. Seiringer, "Non-linear ground state representations and sharp Hardy inequalities," *J. Funct. Anal.*, vol. 255, no. 12, pp. 3407–3430, 2008. [Online]. Available: https://doi.org/10.1016/j.jfa.2008.05.015 26

[46] T. Gantumur, "An introduction to the Morrey and Campanato spaces," https://www.math.mcgill.ca/gantumur/math580f12/MorreyCampanato.pdf, January 2012. 66

[47] N. Ghoussoub and S. Shakerian, "Borderline variational problems involving fractional Laplacians and critical singularities," *Adv. Nonlinear Stud.*, vol. 15, no. 3, pp. 527–555, 2015. [Online]. Available: https://doi.org/10.1515/ans-2015-0302 27, 28, 65

[48] X. Han, G. Lu, and J. Zhu, "Hardy-Littlewood-Sobolev and Stein-Weiss inequalities and integral systems on the Heisenberg group," *Nonlinear Anal.*, vol. 75, no. 11, pp. 4296–4314, 2012. [Online]. Available: https://doi.org/10.1016/j.na.2012.03.017 21

[49] A. Harrabi, "On the Palais-Smale condition," *Journal of Functional Analysis*, vol. 267, no. 8, pp. 2995–3015, 2014. 85, 89

[50] L. Jeanjean, "On the existence of bounded Palais-Smale sequences and application to a landesman–lazer-type problem set on $\mathbb{R}^n$," *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 129, no. 4, pp. 787–809, 1999. 86

[51] Y. Jianfu and Z. Xiping, "On the existence of nontrivial solution of a quasilinear elliptic boundary value problem for unbounded domains:(i) positive mass case," *Acta Mathematica Scientia*, vol. 7, no. 3, pp. 341–359, 1987. 84

[52] W. Kryszewski and A. Szulkin, "Generalized linking theorem with an application to a semilinear Schrödinger equation," *Adv. Differential Equations*, vol. 3, no. 3, pp. 441–472, 1998. 84

[53] T. Kuusi, G. Mingione, and Y. Sire, “Regularity issues involving the fractional p-Laplacian,” in *Recent developments in nonlocal theory.* De Gruyter, Berlin, 2018, pp. 303–334. [Online]. Available: https://doi.org/10.1515/9783110571561-010 20

[54] M. Kwaśnicki, “Ten equivalent definitions of the fractional Laplace operator,” *Fract. Calc. Appl. Anal.*, vol. 20, no. 1, pp. 7–51, 2017. [Online]. Available: https://doi.org/10.1515/fca-2017-0002 20

[55] E. M. Landesman and A. Lazer, “Nonlinear perturbations of linear elliptic boundary value problems at resonance,” *Journal of Mathematics and Mechanics*, vol. 19, no. 7, pp. 609–623, 1970. 89

[56] G. Li and T. Yang, “The existence of a nontrivial weak solution to a double critical problem involving a fractional Laplacian in $\mathbb{R}^n$ with a Hardy term,” *Acta Mathematica Scientia*, vol. 40, no. 6, pp. 1808–1830, oct 2020. [Online]. Available: https://doi.org/10.1007 13, 28, 107

[57] S. Liang and S. Shi, “Soliton solutions to Kirchhoff type problems involving the critical growth in $\mathbb{R}^N$,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 81, pp. 31–41, 2013. 84

[58] E. H. Lieb, “Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities,” *Ann. of Math. (2)*, vol. 118, no. 2, pp. 349–374, 1983. [Online]. Available: https://doi.org/10.2307/2007032 22, 68, 90

[59] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., ser. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001, vol. 14. [Online]. Available: https://doi.org/10.1090/gsm/014 21, 34, 90

[60] T.-C. Lin and J. Wei, “Ground state of n coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leqslant 3$,” *Communications in mathematical physics*, vol. 255, pp. 629–653, 2005. 67

[61] ——, “Erratum: “Ground state of N coupled nonlinear Schrödinger equations in R^n, $n \leq 3$” [Comm. Math. Phys. 255 (2005), no. 3, 629–653; mr2135447],” *Comm. Math. Phys.*, vol. 277, no. 2, pp. 573–576, 2008. [Online]. Available: https://doi.org/10.1007/s00220-007-0365-5 67

[62] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, M. M. Meerschaert, M. Ainsworth *et al.*, “What is the fractional Laplacian? a comparative review with new results,” *Journal of Computational Physics*, vol. 404, p. 109009, 2020. 20

[63] Z. Liu and Z.-Q. Wang, “Ground states and bound states of a nonlinear Schrödinger system,” *Advanced Nonlinear Studies*, vol. 10, no. 1, pp. 175–193, 2010. 66

[64] G. Lu and Y. Shen, “Existence of solutions to fractional p-Laplacian systems with homogeneous nonlinearities of critical Sobolev growth,” *Adv. Nonlinear Stud.*, vol. 20, no. 3, pp. 579–597, 2020. [Online]. Available: https://doi.org/10.1515/ans-2020-2098 66

[65] O. H. Miyagaki, "On a class of semilinear elliptic problems in $\mathbb{R}^n$ with critical growth," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 29, no. 7, pp. 773–781, 1997. 84

[66] G. Molica Bisci, V. Rădulescu, and R. Servadei, *Variational methods for nonlocal fractional problems*, ser. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2016, vol. 162, with a foreword by Jean Mawhin. [Online]. Available: https://doi.org/10.1017/CBO9781316282397 20, 65, 83

[67] V. Moroz and J. Van Schaftingen, "Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics," *J. Funct. Anal.*, vol. 265, no. 2, pp. 153–184, 2013. [Online]. Available: https://doi.org/10.1016/j.jfa.2013.04.007 34

[68] ——, "A guide to the Choquard equation," *J. Fixed Point Theory Appl.*, vol. 19, no. 1, pp. 773–813, 2017. [Online]. Available: https://doi.org/10.1007/s11784-016-0373-1 19, 21, 66, 83, 84

[69] S. Mosconi, K. Perera, M. Squassina, and Y. Yang, "The Brezis-Nirenberg problem for the fractional p-Laplacian," *Calc. Var. Partial Differential Equations*, vol. 55, no. 4, pp. Art. 105, 25, 2016. [Online]. Available: https://doi.org/10.1007/s00526-016-1035-2 68

[70] B. Muckenhoupt and R. Wheeden, "Weighted norm inequalities for fractional integrals," *Trans. Amer. Math. Soc.*, vol. 192, pp. 261–274, 1974. [Online]. Available: https://doi.org/10.2307/1996833 40

[71] T. Mukherjee and K. Sreenadh, "Fractional Choquard equation with critical nonlinearities," *NoDEA Nonlinear Differential Equations Appl.*, vol. 24, no. 6, pp. Paper No. 63, 34, 2017. [Online]. Available: https://doi.org/10.1007/s00030-017-0487-1 66

[72] H.-M. Nguyen and M. Squassina, "Fractional Caffarelli-Kohn-Nirenberg inequalities," *J. Funct. Anal.*, vol. 274, no. 9, pp. 2661–2672, 2018. [Online]. Available: https://doi.org/10.1016/j.jfa.2017.07.007 26

[73] G. Palatucci and A. Pisante, "Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces," *Calculus of Variations and Partial Differential Equations*, vol. 50, no. 3-4, pp. 799–829, 2014. 43

[74] A. Pankov, "Periodic nonlinear Schrödinger equation with application to photonic crystals," 2004. 84

[75] P. Pucci, M. Xiang, and B. Zhang, "Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian," *Adv. Calc. Var.*, vol. 12, no. 3, pp. 253–275, 2019. [Online]. Available: https://doi.org/10.1515/acv-2016-0049 84

[76] M. Ramos and P. Rodrigues, "On a fourth order superlinear elliptic problem," *Electronic Journal of Differential Equations*, pp. 243–255, 2001. 86

[77] S. B. Robinson, "Multiple solutions for semilinear elliptic boundary value problems at resonance," *Electron. J. Differential Equations*, pp. Paper No. 1, 14, 1995. 85

[78] Y. Sawano, "A thought on generalized Morrey spaces," *J. Indones. Math. Soc.*, vol. 25, no. 3, pp. 210–281, 2019. 26, 66

[79] E. Sawyer and R. L. Wheeden, "Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces," *Amer. J. Math.*, vol. 114, no. 4, pp. 813–874, 1992. [Online]. Available: https://doi.org/10.2307/2374799 40

[80] M. Schechter, "The use of Cerami sequences in critical point theory," *Abstr. Appl. Anal.*, pp. Art. ID 58 948, 28, 2007. [Online]. Available: https://doi.org/10.1155/2007/58948 88

[81] R. Servadei and E. Valdinoci, "Mountain pass solutions for non-local elliptic operators," *Journal of Mathematical Analysis and Applications*, vol. 389, no. 2, pp. 887–898, 2012. 65

[82] G. Singh, "Nonlocal perturbations of the fractional Choquard equation," *Advances in Nonlinear Analysis*, vol. 8, no. 1, pp. 694–706, 2017. 33

[83] V. C. Z. Sissa and P. H. Rabinowitz, "Homoclinic type solutions for a semilinear elliptic pde on $\mathbb{R}^N$," *Communications on pure and applied mathematics*, vol. 45, no. 10, pp. 1217–1269, 1992. 84

[84] Y. Song and S. Shi, "Existence of infinitely many solutions for degenerate p-fractional Kirchhoff equations with critical Sobolev–Hardy nonlinearities," *Zeitschrift für angewandte Mathematik und Physik*, vol. 68, pp. 1–13, 2017. 84

[85] K. Sreenadh and T. Mukherjee, "Critical growth elliptic problems with Choquard type nonlinearity: a survey," in *Mathematical modelling, optimization, analytic and numerical solutions*, ser. Ind. Appl. Math. Springer, Singapore, [2020] ©2020, pp. 197–229. [Online]. Available: https://doi.org/10.1007/978-981-15-0928-5_10 19

[86] E. M. Stein and G. Weiss, "Fractional integrals on n-dimensional Euclidean space," *J. Math. Mech.*, vol. 7, pp. 503–514, 1958. [Online]. Available: https://doi.org/10.1512/iumj.1958.7.57030 21, 90

[87] Y. Su, "Fractional p-Laplacian problem with critical Stein-Weiss type term," *J. Geom. Anal.*, vol. 33, no. 5, pp. Paper No. 160, 22, 2023. [Online]. Available: https://doi.org/10.1007/s12220-023-01209-w 28

[88] K. Thews, "Non-trivial solutions of elliptic equations at resonance," *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 85, no. 1-2, pp. 119–129, 1980. 89

[89] L. Wang, B. Zhang, and H. Zhang, "Fractional laplacian system involving doubly critical nonlinearities in r^ n," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2017, no. 57, pp. 1–17, 2017. 66, 67

[90] M. Willem, *Minimax Theorems*, ser. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, Inc., Boston, MA, 1996, vol. 24. [Online]. Available: https://doi.org/10.1007/978-1-4612-4146-1 30, 34

[91] M. Xiang, B. Zhang, and V. D. Rădulescu, "Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-laplacian," *Nonlinearity*, vol. 29, no. 10, p. 3186, 2016. 66

[92] J. Yang, "Fractional Sobolev-Hardy inequality in $\mathbb{R}^N$," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 119, pp. 179–185, 2015. 13, 107

[93] J. Yang and F. Wu, "Doubly critical problems involving fractional Laplacians in $\mathbb{R}^n$," *Advanced Nonlinear Studies*, vol. 17, no. 4, pp. 677–690, 2017. 13, 30, 107

[94] T. Yang, "On doubly critical coupled systems involving fractional Laplacian with partial singular weight," *Mathematical Methods in the Applied Sciences*, vol. 44, no. 17, pp. 13 448–13 467, 2021. 66, 67

[95] S. Yuan, V. D. Rădulescu, S. Chen, and L. Wen, "Fractional Choquard logarithmic equations with Stein-Weiss potential," *J. Math. Anal. Appl.*, vol. 526, no. 1, pp. Paper No. 127 214, 45, 2023. [Online]. Available: https://doi.org/10.1016/j.jmaa.2023.127214 21