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Instituto de Ciências Exatas
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Inhomogeneous Processes with Random One-dimensional Reinforcements

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Final Version

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Co-Advisor: Daniel Ungaretti Borges

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"*Que eu havera de viver*
por esse mundo
e morrer ainda em flor"
- Elomar Figueira Mello

Resumo

A teoria da percolação, que surgiu em 1957, modela o comportamento de um fluido fluindo através de um meio poroso. O modelo de percolação de Bernoulli na rede $\mathbb{Z}^d$ trata cada elo de forma independente, designando-o como aberto com probabilidade p e fechado com probabilidade $1-p$. Avanços teóricos significativos na década de 1980 levaram à exploração de vários modelos de percolação não homogênea, onde regiões R do grafo têm elos que são mais propensos a estarem abertos.

No nosso estudo, investigamos percolação de Bernoulli não-homogênea no grafo $G\times\mathbb{Z}$, onde G é um grafo conexo quasi-transitivo. A não homogeneidade é introduzida por meio de uma região aleatória R ao redor do eixo de origem $0\times\mathbb{Z}$, com cada elo em R sendo aberto com probabilidade q e todos os outros elos com probabilidade p. Quando a região R é definida por empilhamento ou sobreposição de caixas com raios aleatórios centrados ao longo do eixo de origem, derivamos condições sobre os momentos dos raios, baseadas nas propriedades de crescimento de G, assegurando que para qualquer p subcrítico e qualquer $q<1$, a fase subcrítica de percolação persiste. Também adaptamos nossas técnicas para o processo de contato em um grafo transitivo G, com não homogeneidades analogamente definidas.

Palavras-chave: Percolação, Processo de Contato, não-homogeneidades, ponto crítico

Abstract

Percolation theory, which emerged in 1957, models the behavior of a fluid flowing through a porous medium. The Bernoulli percolation model on the lattice $\mathbb{Z}^d$ treats each edge independently, designating it as open with probability p and closed with probability $1-p$. Significant theoretical advances in the 1980s led to the exploration of various non-homogeneous percolation models, where regions R of the graph have edges that are more likely to be open.

In our study, we investigate inhomogeneous Bernoulli bond percolation on the graph $G \times \mathbb{Z}$, where G is a connected quasi-transitive graph. The inhomogeneity is introduced via a random region R around the origin axis $0 \times \mathbb{Z}$, with each edge in R being open with probability q and all other edges with probability p. When the region R is defined by stacking or overlapping boxes with random radii centered along the origin axis, we derive conditions on the moments of the radii, based on the growth properties of G, ensuring that for any subcritical p and any $q < 1$, the non-percolative phase persists. We also adapt our techniques to the contact process on a transitive graph G, with analogous inhomogeneities.

Keywords: Percolation; Contact Process; inhomegeneities; critical point.

Contents

Chapter 1

Introduction

Percolation models the spread of a fluid through random medium and it has been object of intensive study since it was introduced in 1957 by Broadbent and Hammersley [7]. Classical models consider a homogeneous medium representing it as a random graph where edges (or sites) are independently present with probability p or absent otherwise.

The fluid is regarded as the connected component, also called the open cluster, of a fixed point that we call the origin of the graph. With the development of many techniques in the 80's, many questions about inhomogeneous percolation were raised. That is, when some region R of privileged flow is considered in the medium. We now formalize these ideas and give concrete examples of such inhomogeneous models and questions.

We say that a graph $G = (V, E)$ is quasi-transitive if there exists a finite set of sites $V_0 \subset V$ such that for every site $w \in V$, there exists $x \in V_0$ and an automorphism τ of G such that $\tau(y) = x$. In this work we consider inhomogeneous Bernoulli bond percolation on the cartesian product $G \times \mathbb{Z}$ where G is an infinite connected quasi-transitive graph and $\mathbb{Z}$ is the set of integers $\mathbb{Z}$. The edges of $G \times \mathbb{Z}$ are pairs of nearest neighbour's sites and this graph is sometimes called the *box product* and denoted $G\Box\mathbb{Z}$. That is, we consider the graph

$$G \times \mathbb{Z} = (V \times \mathbb{Z}, E(G \times \mathbb{Z})), \tag{1.1}$$

where $E(G \times \mathbb{Z})$ is given by the edges which make every layer $V \times \{n\}$ a copy of G and also by the edges connecting the sites corresponding to the same site in G on neighbour layers. More precisely, if $\sim$ denotes the relation "is connected by an edge to", two sites $(v_1, v_2), (w_1, w_2) \in G \times \mathbb{Z}$ are connected by an edge if, and only if,

$$v_1 = w_1 \text{ and } v_2 \sim w_2 \qquad \text{or} \qquad v_1 \sim w_1 \text{ and } v_2 = w_2.$$

We distinguish a vertex in G to be the origin of the graph and we call it 0 as usual. We also denote by 0 the origin $(0, 0) \in G \times \mathbb{Z}$. The set $\{0\} \times \mathbb{Z}$ will be called the vertical line or vertical axis along the origin of $G \times \mathbb{Z}$. We denote by $B^x_G(r) \subset G$ the set of sites that are up to distance r from x, that is

$$B^x_G(r) = \{v \in G; d_G(x, v) \leq r\} \subset G,$$

where d_G denotes the graph distance. For simplicity, we will also write $B_G(r) = B_G^0(r)$. We denote the set of integers $[a, b] \cap \mathbb{Z}$ within some interval $[a, b] \subset \mathbb{R}$ simply as $[a, b] \subset \mathbb{Z}$. We denote the box centered at $(x, a) \in G \times \mathbb{Z}$ with radius r by

$$B^x(a, r) = B_G^x(r) \times [a - r, a + r] \subset G \times \mathbb{Z} \tag{1.2}$$

and we denote for simplicity $B(a, r) = B^0(a, r)$.

We consider inhomogeneous independent percolation on the graph $G \times \mathbb{Z}$. For a subset $R \subset G \times \mathbb{Z}$ let $\mathbb{P}(e \text{ is open}) = q$ for edges e in R and $\mathbb{P}(e \text{ is open}) = p$ otherwise. Let us denote the resulting probability measure by $\mathbb{P}_{p,q}^{(R)}$ and its percolation probability by $\theta^{(R)}(p, q)$, i.e., the probability that there is an infinite open path starting from the origin of $G \times \mathbb{Z}$.

The hyphotesis of quasi-transitivity on G is due to some essential properties for our work that such graphs possess. The first one is that quasi-transitive graphs cannot grow too fast. If Δ_x denotes the degree of $x \in G$ and $\Delta_G = \max_{x \in V_0} \Delta_x$, then, for every $x \in G$,

$$|B_G^x(n)| \leq \Delta_G^{n+1}. \tag{1.3}$$

The second one is that, as it was shown by Antunović, Vaselić (see [3]), subcritical homogeneous percolation on quasi-transitive graphs always have a sharp threshold, see also Beekenkamp, Hulshof [4] for inhomogeneous percolation. Since G is quasi-transitive, this implies that $G \times \mathbb{Z}$ is also quasi-transitive, hence, for any $p < p_c(G \times \mathbb{Z})$, there exists a constant $c = c(p) > 0$ such that for every $(x, a) \in G \times \mathbb{Z}$,

$$\mathbb{P}_p((x, a) \longleftrightarrow \partial B^x(a, r)) \leq e^{-c(p)r}. \tag{1.4}$$

Our results depend solely on these two properties. While we could present our findings in terms of these properties, we believe that the quasi-transitivity setting is sufficiently general for our purposes.

One important class of problems in inhomogeneous percolation is to consider a parameter p such that $\theta(p) = \theta^{(R)}(p, p) = 0$ and a supercritical parameter q. We seek to understand the relation between the size and shape of R and how large q must be to enter the percolating phase of $\mathbb{P}^{(R)}(p, q)$.

Several positive results are present in the literature when one considers random regions R. For instance, Duminil-Copin, Hilário, Kozma, Sidoravicius [9] showed that for Brochette Percolation in the square lattice, that is, when $R \subset \mathbb{Z}^2$ is given by the union of vertical lines chosen independently at random, whenever $q > p_c(\mathbb{Z}^2)$ we can choose $p < p_c(\mathbb{Z}^2)$ such that the origin percolates. See also [6, 14–16] for related problems.

Our work is, though, reminiscent of the following results. In 1994, Madras, Schinazi, and Schonmann [18] showed that for the case where $G = \mathbb{Z}^{d-1}$ and $R =$

$\{0\} \times \mathbb{Z}$ is a vertical line, the critical point of the inhomogeneous model remains the same as in the homogeneous case. Their work is actually for the Contact Process, but it translates naturally for percolation. Later in that year, Zhang [21] showed that for all $q < 1$ we have

$$\mathbb{P}_{1/2,q}^{\{0\}\times\mathbb{Z}}(0 \longleftrightarrow \infty) = 0,$$

recalling the value of the critical point $p_c(\mathbb{Z}^2) = 1/2$. In other words, phase transition also remains continuous in $\mathbb{Z}^2$, which is the two-dimensional case of a problem posed in [18]. In general, we can define

$$q \mapsto p_c(q) = \sup\{p \in [0,1]\,;\, \mathbb{P}_{p,q}^{(R)}(0 \longleftrightarrow \infty) = 0\}. \tag{1.5}$$

Proposition 1.4 of [18] shows, in particular, that $p_c(q)$ is a constant curve for $q \in (0,1)$ in $\mathbb{Z}^d$ for all $d \geq 2$. In 2020, Szabó and Valesin [20] studied this problem for a general graph G and proved that, for any finite subgraph $F \subset G$, $p_c(q)$ is continuous when R is the cylinder $F \times \mathbb{Z}$ in the cartesian product $G \times \mathbb{Z}$, which they call a *ladder graph*, and they conjecture that the curve is constant. Our results, in particular, imply that this is actually the case when G is quasi-transitive. Also in 2020, de Lima and Sanna [8] generalized the result of [20] by replacing $F \times \mathbb{Z}$ by a region R given by the union of an infinite number of *well spaced* cylinders with uniformly bounded radii.

Our objective is to extend beyond the deterministic setting, where the region R is fixed, and investigate models with random thickness in the reinforced one-dimensional region, preventing it from being confined to a deterministic cylinder. We do that in two ways. Firstly, we consider the region R to be the union of boxes centered along the line $\{0\} \times \mathbb{Z}$ having radii given by i.i.d. random variables, this is the Overlap Model defined in Subsection 2.1.1. Then we consider the region R given by stacked boxes with i.i.d. radii also centered along the line $\{0\} \times \mathbb{Z}$, this is the Stack Model defined in Subsection 2.1.2. In both cases we prove that, under mild conditions on the expectation of the radii, for any $p < p_c(G \times \mathbb{Z})$ and for any $q < 1$, the resulting process remains, for almost every environment, in the non-percolating phase.

The text is organized as follows. In Chapter 2 we state and prove the main results of this thesis for the percolation model. In Chapter 3 we discuss our techniques in the scenario of the contact process, which is known to have strong technical relationship with oriented pecolation, via the Harris description. In fact, it is equivalent to the so called oriented continuous-time percolation. See Bezuidenhout, Grimmett [5]. More precisely, as a byproduct of the proof of our main results for percolation, we adapt the results of Chapter 2 to the corresponding inhomogeneous contact process, in which the individuals may present less resistance to the infection and are less likely to heal in some space-time regions. A fundamental difference to Chapter 2 is that we assume G is a transitive graph, since no exponential decay for quasi-transitive graphs such as [3] is known for the Contact Process. Exponential decay in the contact process for transitive graphs G is proved in Aizenman, Jung [2]. In Chapter 4 we describe some open problems that emerged in the analysis of the models. In Appendix (A) we show the ergodicity of the annealed measure for the Overlap Model. Then, in Appendix B, in order to give a self contained exposure we state some general results that are used in the proofs of our main results.

Chapter 2

The Overlap and Stack Models for Percolation

In this chapter we introduce the models we work on this thesis and we prove our main results. We introduce inhomogeneities around the axis $\{0\} \times \mathbb{Z}$ in two different ways. Although both models are analysed by similar techniques, their percolative behavior is different; hence, each model is studied in a separate section.

2.1 Definition of the models and statement of the results

Consider a collection $\{X_n; n \in \mathbb{Z}\}$ of iid random variables supported on $\mathbb{N} = \{1, 2, 3, \dots\}$ in the probability space $(\Xi, \mathcal{G}, \nu)$, where $\Xi = \mathbb{N}^{\mathbb{Z}}$ and $\mathcal{G} = \sigma(X_n, n \in \mathbb{Z})$, with associated expectation operator denoted by E. One might assume that the random variables X_n are supported on $\mathbb{R}_+$, however, this assumption would not introduce any new phenomena to our results. A configuration $\Lambda \in \Xi$ will be called an environment. From now on, we focus on two possible models of random environments describing one-dimensional reinforcements.

2.1.1 The Overlap Model

In this subsection we introduce formally the Overlap Model and we state the main theorem. Let G be a quasi-transitive graph and consider the cartesian product $G \times \mathbb{Z}$ as defined in (1.1). In each site $(0, n)$ of the vertical line $\{0\} \times \mathbb{Z}$, we place a box

$$B_n = B(n, X_n) \tag{2.1}$$

with radius X_n and we consider the improved region R to be the union of these boxes. More precisely, we define

$$R = \bigcup_{n\in\mathbb{Z}} B_n. \tag{2.2}$$

We let every edge $e \in R$ be open with probability q and every edge $e \notin R$ open with probability p. In Figure 2.1, we sketch an environment for the Overlap Model for $G = \mathbb{Z}$.

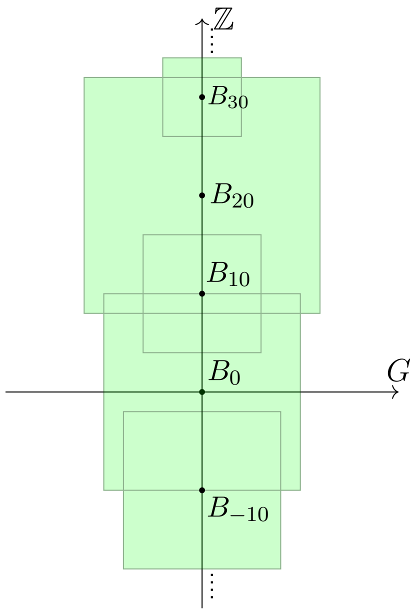

Figure 2.1: An environment for the Overlap Model. To let the drawing clear and to not lose proportionality, we choose to draw only boxes with centers 10 units apart. The reader is invited to imagine the other boxes in between them.

The region $R = R(\Lambda)$ is well defined and we write simply $\mathbb{P}_{p,q}^{(R(\Lambda))} = \mathbb{P}_{p,q}^{\Lambda}$ and $\theta^{(R(\Lambda))}(p,q) = \theta^{\Lambda}(p,q)$. We call $\mathbb{P}_{p,q}^{\Lambda}$ the *quenched* probability measure associated with the environment Λ.

Our main result for this model is the following:

Theorem 1. *Let G be a quasi-transitive graph and consider the Overlap Model defined by i.i.d. random variables $\{X_n; n \in \mathbb{Z}\}$ with common distribution X in $G \times \mathbb{Z}$. For $0 < p < p_c < q < 1$, we have*

$$\theta^{\Lambda}(p,q) = 0 \quad \textit{for } \nu\textit{-a.e.}\Lambda \qquad \Longleftrightarrow \qquad EX < \infty. \tag{2.3}$$

Remark 1. Considering environments with radii given by random variables such that $X \geq r$ a.s., it is clear that any cylinder $F \times \mathbb{Z}$ is contained in the region R if r is chosen properly. In this setting the Overlap Model dominates the model where the improved region R is a cylinder. In the context of quasi-transitive graphs G, this proves the conjecture by Szabó and Valesin [20] that the curve $p_c(q)$ defined in (1.5) is constant if the improved region is a deterministic cylinder.

Remark 2. In the Overlap Model as defined above, each edge in the enhanced region is open with a fixed probabilily $q < 1$ and one could suppose that this probability increases depending on how many boxes the edge belongs to. As it will be clear, our proof could be easily adapted to such case and the result would be the same. This relates to the fact that almost surely, either every edge belongs to a finite number of boxes or the boxes cover the whole space, see Proposition 1.

2.1.2 The Stack Model

In this subsection we define formally the Stack Model and state the main theorem.

To start the construction of the environment, place the box $B(0, X_0)$ at the origin of $G \times \mathbb{Z}$ and then stack successivelly the remaining boxes in both directions along $\{0\} \times \mathbb{Z}$. More precisely. Let $Z(0) = 0$ and, for $n \in \mathbb{Z} \setminus \{0\}$,

$$Z(n) = \begin{cases} X_0 + 2\sum_{i=1}^{n-1} X_i + X_n, & \text{if } n \geq 1 \\ -X_0 - 2\sum_{i=1}^{-n-1} X_{-i} - X_n, & \text{if } n \leq -1 \end{cases} \tag{2.4}$$

be the center of the n-th box and set

$$B_n = B(Z(n), X_n). \tag{2.5}$$

The improved region R is again given by

$$R = \bigcup_{n \in \mathbb{Z}} B_n. \tag{2.6}$$

As in the Overlap Model, the region $R = R(\Lambda)$ is well defined and we let every edge $e \in R$ be open with probability q and every edge $e \notin R$ open with probability p (see Figure 2.2). For the sake of simplicity, we use the same notation to denote the probability measure associated to the Stack Model, that is $\mathbb{P}_{p,q}^{(R(\Lambda))} = \mathbb{P}_{p,q}^{\Lambda}$ and $\theta^{(R(\Lambda))}(p, q) = \theta^{\Lambda}(p, q)$.

In the Overlap Model, the arrangement of the center of the boxes produces more density of boxes along a determined height, causing the region R to be *large* when the random variables X_n are fixed. In the Stack Model, instead, at every fixed

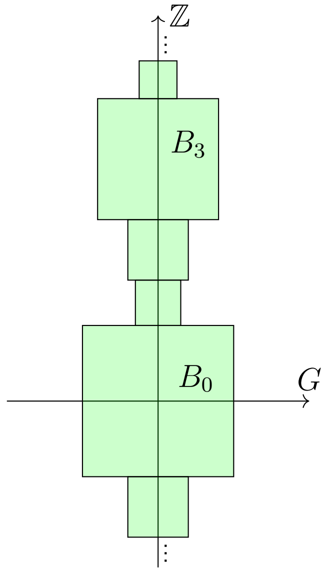

Figure 2.2: An environment for the Stack Model

height the sample of only one box will determine the thickness of the improved region R. Moreover, we emphasize that in the Stack Model R cannot ever be the whole space. Considering connectedness of R preserved, this is the most sparse way that the boxes can be arranged along $\{0\} \times \mathbb{Z}$, being in a sense the extreme opposite of the Overlap Model. For that reason, although we see no straightforward coupling so that Overlap Model dominates Stack Model, the Stack Model produces sparser improved regions, at a heuristic level. As an effect, a weaker hypothesis is needed to prove essentially the same result.

Theorem 2. *Let G be any quasi-transitive graph and consider the Stack Model in $G \times \mathbb{Z}$. If $E \log |B_G(X)| < \infty$, then*

$$\theta^{\Lambda}(p, q) = 0 \quad \textit{for } p \in [0, p_c), \ q \in [0, 1)$$

almost surely on Λ.

The condition $E \log |B_G(X)| < \infty$ in Theorem 2 expresses a relationship between the distribution of X, the growth of the graph G and the exponential decay (1.4) that is sufficient for the theorem to hold. In fact, as we will see in Section 2.3, the log function appears in the expression as a consequence of the exponential decay of $\theta_n(p)$.

By the fact that quasi-transitive graphs grow at most exponentially fast, see (1.3), Theorem 2 guarantees, for any such graph, that there is no percolation in the Stack Model whenever $EX < \infty$. But if the asymptotic behavior of $|B_G(n)|$ is known to be lesser than exponential, we can give an explicit condition on the moment of the radii for the theorem to hold. For example, if $G = \mathbb{Z}^{d-1}$, then $G \times \mathbb{Z} = \mathbb{Z}^d$ and in this case we know that there exists a constant $C_d > 0$ such that $|B_G(n)| \leq C_d n^d$, so it is sufficient to choose i.i.d. radii with $E \log X < \infty$ for the theorem to hold in this case.

2.2 The Overlap Model: Proof of Theorem 1

Consider the Overlap Model with radii given by independent random variables with common distribution X of finite expectation. Take f and g as in Lemma 3 and using the estimate from this lemma, we have

$$\sum_{n\geq 1} \nu(X_n \geq g(n)) = \sum_{n\geq 1} \nu(X_n f(X_n) \geq n) = \sum_{n\geq 1} \nu(X f(X) \geq n) < \infty.$$

By Borel-Cantelli's Lemma,

$$\nu(\liminf_n \{X_n \leq g(n)\}) = 1. \tag{2.7}$$

Hence, although the region R is random, all except finitely many boxes in R are almost surely contained in a deterministic *cone*, whose growth we can estimate. It will be useful to find a deterministic n_0 such that all boxes with center above n_0 are within some deterministic region with positive probability.

To formalize these ideas, we start by defining the deterministic cones properly.

Definition 1. Given $n_0 \in \mathbb{N}$, such that $g(n) < n/2$ for all $n \geq n_0$. The *upwards* and the *downwards cone* are, respectively, the sets

$$W^+ = \bigcup_{n\geq 2n_0} B_G(g(n)) \times \left[\frac{n}{2}, \infty\right) \quad \text{and} \quad W^- = \bigcup_{n\geq 2n_0} B_G(g(n)) \times \left(-\infty, -\frac{n}{2}\right].$$

Recall the notation $B_n = B(n, X_n)$. See (1.2) and (2.1).

Definition 2. Given $n_0 \in \mathbb{N}$ and W^+, W^- as in Definition 1, we say that Λ is a *good* environment if it satisfies

1. The boxes B_n with $n \geq 2n_0$ are contained in W^+: $\bigcup_{n\geq 2n_0} B_n \subseteq W^+$;
2. The boxes B_{-n} with $n \geq 2n_0$ are contained in W^-: $\bigcup_{n\geq 2n_0} B_{-n} \subseteq W^-$;
3. The boxes B_n with $n \in (-2n_0, 2n_0)$ have radius $X_n \leq g(2n_0)$.

We denote by $\mathcal{A} = \mathcal{A}(n_0)$ the set of all good environments.

We notice that in a good environment given by Definition 2 the random region R is covered by the deterministic region $\big(B_G(g(2n_0)) \times [-n_0, n_0]\big) \cup W^+ \cup W^-$. Cones W^+ and W^- are illustrated in Figure 2.3 below, where they are decomposed into layers.

Lemma 1. *There exists n_0 such that the set of good environments $\mathcal{A}$ has $\nu(\mathcal{A}) > 0$.*

Proof. First notice that the finite intersection of independent events

$$\bigcap_{n=-2n_0+1}^{2n_0-1} \{X_n \le g(2n_0)\}$$

has positive probability for all n_0 such that $g(2n_0) \ge m$, where m is some constant depending on the distribution of X. By Lemma 3 we can choose n_0 sufficiently large so that $g(n) \le n/2$ for all $n \ge n_0$. Fix n_0 with these properties. In order to show that

$$\nu\Big(\bigcup_{n \ge 2n_0} B_n \subseteq W^+\Big) > 0$$

it is sufficient to show that with positive probability $B(n, X_n) \subseteq B(n, g(n))$ for all $n \ge 2n_0$. It follows from (2.7) that

$$\nu\Big(\bigcap_{n \ge 2n_0} \{X_n \le g(n)\}\Big) > 0.$$

By the same reasoning,

$$\nu\Big(\bigcap_{n \ge 2n_0} \{X_{-n} \le g(n)\}\Big) > 0.$$

We conclude that $\nu(\mathcal{A}) > 0$, being the intersection of three independent events of positive probability. □

Proceeding to the proof of the theorem, first we show that one of the implications of the statement is trivial.

Proposition 1. *$EX = \infty$ if, and only if, $R = G \times \mathbb{Z}$ almost surely.*

Proof. We show that any fixed vertex v will be covered by some ball. By translation invariance in the $\mathbb{Z}$ direction, we can assume $v = (w, 0)$ and denote $r = d_G(w, 0)$. Notice that a ball centered at $(0, n)$ with $n > r$ will cover v if its radius satisfies $X_n \ge n$. Since

$$\sum_{n \ge r} \nu(X_n \ge n) = \infty \quad \text{is equivalent to} \quad EX = \infty,$$

and the events $\{X_n \ge n\}$ are independent, the result follows from Borel-Cantelli's lemma. □

This proves the first implication since when $R = G \times \mathbb{Z}$ we are in the supercritical phase of homogeneous percolation whenever $q > p_c(G \times \mathbb{Z})$.

Proof of Theorem 1. We decompose W^+ and W^- into layers $(L_n^+; n \geq n_0)$. Define

$$L_n^+ := B_G(g(2n)) \times \{n\}. \tag{2.8}$$

Notice that $W^+ = \bigcup_{n \geq n_0} L_n^+$. In fact, let $n \geq n_0$, the radius of L_n^+ at $G \times \{n\}$ is $g(2n)$, which is also the radius of W^+ at this height, because the cylinder in W^+ with the largest radius that has non empty intersection with $G \times \{n\}$ is $B_G(g(2n)) \times [n, \infty)$.

It follows by quasi-transitivity of G that $B_G(n)$ grows at most exponentially fast with n. As a consequence, the number of sites in each layer is

$$|L_n^+| = |B_G(g(2n))| \leq e^{c_G g(2n)}, \tag{2.9}$$

where $c_G > 0$ is some constant depending only on G. We define analogously the quantities L_n^- for W^-.

Let us estimate $\mathbb{P}_p(W^+ \leftrightarrow W^-)$. We have

$$\mathbb{P}_p(W^+ \leftrightarrow W^-) \leq \sum_{n,m \geq n_0/2} \mathbb{P}_p(L_n^+ \leftrightarrow L_m^-).$$

Notice that any $w^+ \in L_n^+$ and $w^- \in L_m^-$ have vertical distance $m + n$. Condider $c = c(p)$ as in (1.4).

Summing over all possible pairs of sites, we have

$$\mathbb{P}_p(L_n^+ \leftrightarrow L_m^-) \leq |L_n^+||L_m^-| e^{-c(m+n)} \leq e^{c_G(g(2n)+g(2m))} e^{-c(m+n)}.$$

Summing for $n, m \geq n_0/2$ we obtain

$$\mathbb{P}_p(W^+ \leftrightarrow W^-) \leq \sum_{n,m \geq n_0/2} e^{c_G(g(2n)+g(2m))} e^{-c(m+n)} = \Big(\sum_{n \geq n_0/2} e^{n(-c + c_G 2 \frac{g(2n)}{2n})} \Big)^2. \tag{2.10}$$

Recall that, by Lemma 3 we have $g(n)/n \to 0$. In particular, the series above is convergent and we can actually make it as close to zero as we want by increasing n_0.

Now we proceed to transport this result to the measure $\mathbb{P}_{p,q}^{\Lambda}$. Fix n_0 large enough so that

$$\mathbb{P}_p(W^+ \longleftrightarrow W^-) \leq \frac{1}{2}.$$

Let $B = B_G(g(2n_0)) \times [-n_0, n_0]$. Recall that for good environments $\Lambda \in \mathcal{A}$, the region R is a subset of $B \cup W^+ \cup W^-$. Let $D = \{W^+ \not\longleftrightarrow W^-\}$ and $F = \{\text{every edge inside } B \text{ is closed}\}$. Notice that,

$$\begin{aligned}
\mathbb{P}_{p,q}^{\Lambda}(D^c \cap F) &= \mathbb{P}_{p,q}^{\Lambda}(\{W^+ \overset{B^c}{\longleftrightarrow} W^-\} \cap F) \\
&= \mathbb{P}_{p,q}^{\Lambda}(W^+ \overset{B^c}{\longleftrightarrow} W^-) \mathbb{P}_{p,q}^{\Lambda}(F) \\
&= \mathbb{P}_p(W^+ \overset{B^c}{\longleftrightarrow} W^-) \mathbb{P}_{p,q}^{\Lambda}(F) \\
&\leq \mathbb{P}_p(D^c) \mathbb{P}_{p,q}^{\Lambda}(F).
\end{aligned} \tag{2.11}$$

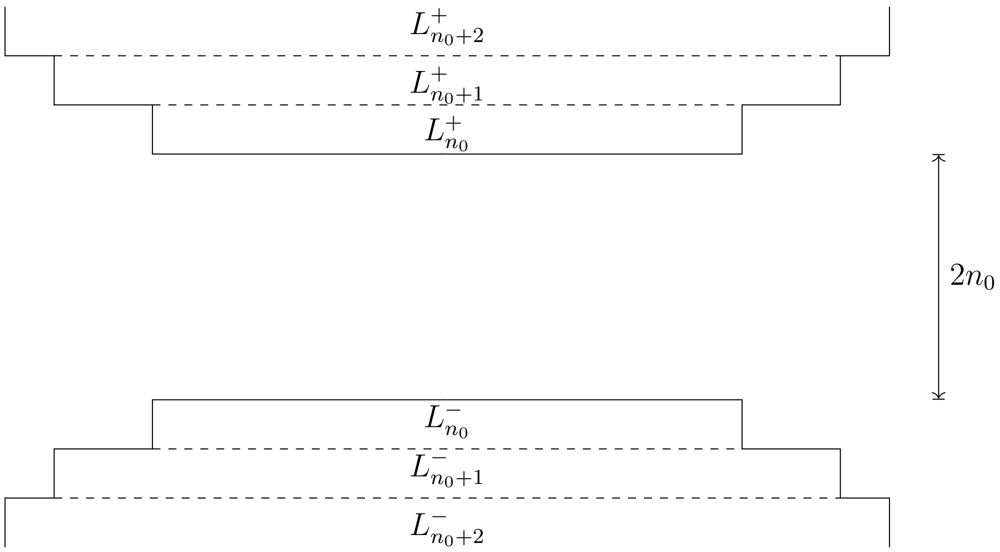

Figure 2.3: Cones and their layers

As a consequence,

$$\mathbb{P}^{\Lambda}_{p,q}(D \cap F) = \mathbb{P}^{\Lambda}_{p,q}(F) - \mathbb{P}^{\Lambda}_{p,q}(D^c \cap F) \geq \mathbb{P}^{\Lambda}_{p,q}(F)\mathbb{P}_p(D). \tag{2.12}$$

Let

$$R^+ = \bigcup_{n \geq n_0} B(n, X_n) \quad \text{and} \quad R^- = \bigcup_{n \geq n_0} B(-n, X_{-n}). \tag{2.13}$$

Since $\Lambda \in \mathcal{A}$, we have that $R^\pm \subset W^\pm$. Using this fact and (2.12), we get

$$\begin{aligned} \mathbb{P}^{\Lambda}_{p,q}(R^+ \nleftrightarrow R^-) &\geq \mathbb{P}^{\Lambda}_{p,q}(D) \geq \mathbb{P}^{\Lambda}_{p,q}(D \cap F) \\ &\geq \mathbb{P}^{\Lambda}_{p,q}(F)\mathbb{P}_p(D) \geq \frac{1}{2}(1-q)^{c(G,n_0)}, \end{aligned} \tag{2.14}$$

where $c(G, n_0)$ is is the number of edges in $B_G(g(2n_0)) \times [-n_0, n_0]$.

Now, in order to use ergodic properties, we define the *annealed* law

$$\mathbb{P}^{\nu}_{p,q}(\cdot) = \int_{\Xi} \mathbb{P}^{\Lambda}_{p,q}(\cdot) d\nu(\Lambda).$$

The main idea is that when we show that an event has probability 1 in the annealed law, then it also has probability 1 for ν-almost all environment Λ. Since $EX < \infty$, the annealed measure is ergodic with respect to the unit vertical translation, τ.

We have,

$$\mathbb{P}^{\nu}_{p,q}(D) = \int_{\Xi} \mathbb{P}^{\Lambda}_{p,q}(D) d\nu(\Lambda) \geq \int_{\mathcal{A}} \mathbb{P}^{\Lambda}_{p,q}(D) d\nu(\Lambda) \geq \frac{1}{2}(1-q)^{c(G,n_0)}\nu(\mathcal{A}) > 0.$$

Now, consider the event where there exists $n \in \mathbb{Z}$ such that $\tau^n(D)$ occurs, which is invariant by τ. By ergodicity, we conclude that there are infinitely many vertical disconnections almost surely in the annealed law, and this implies that $\theta^\Lambda(p,q) = 0$ ν-almost surely. Indeed, whenever $p < p_c(G \times \mathbb{Z})$, in order for the cluster of the origin to be infinite it necessarily has infinite intersections with R. □

2.3 Stack Model: Proof of Theorem 2

In this section we prove Theorem 2. We actually prove it in a slightly more general setting. Instead of $\log |B_G(n)|$, we choose a function $\varphi \geq 0$ such that $\varphi \nearrow \infty$, $E\varphi(X) < \infty$ and we investigate what additional properties it has to satisfy for the result to hold.

Let $\varphi \geq 0$ be any unbounded increasing function and suppose $E\varphi(X) < \infty$. As X is unbounded, the set $\mathcal{L} = \{m \in \mathbb{N} \,|\, \nu(m \leq X \leq 2m) > 0\}$ is infinite. We denote by φ^{-1} the generalized inverse of φ.

Lemma 2. *Let $L = \min\{l \in \mathbb{N} \,|\, \nu(\varphi(X) \leq l) > 0\}$, $l_0 \in \mathcal{L}$ and consider the event*

$$A_k = \{l_0 \leq X_k \leq 2l_0\} \cap \bigcap_{j\geq 1}\{X_{k+j} \leq \varphi^{-1}(j+L)\} \cap \bigcap_{j\geq 1}\{X_{k-j} \leq \varphi^{-1}(j+L)\}.$$

Then, for every $k \in \mathbb{Z}$, $\nu(A_k) > 0$. Moreover, $\nu(A_k)$ is constant as a function of k.

Proof. From the fact that the sequence of random variables is i.i.d., it follows that

$$\nu(A_k) = \nu(l_0 \leq X \leq 2l_0)\left(\prod_{j\geq 1}\nu(\varphi(X) \leq j+L)\right)^2$$

which is positive since $E\varphi(X) < \infty$. Furthermore, The identity above also implies that $\nu(A_k)$ does not depend on k. □

Recall from (2.5) that $B_n = B(Z(n), X_n)$. For $k \in \mathbb{Z}$, define the subregions

$$R_k^+ = \bigcup_{n>k} B_n \qquad \text{and} \qquad R_k^- = \bigcup_{n<k} B_n. \tag{2.15}$$

Notice that for any $k \in \mathbb{Z}$, $R = R_k^+ \cup B_k \cup R_k^-$. We now proceed to define analogous structures as done for the Overlap Model in Definition 1.

Definition 3. For $k \in \mathbb{Z}$, we define the *upwards cone* W_k^+ and the *downwards cone* W_k^- as the sets

$$W_k^+ = \bigcup_{n\geq 0} B_G(\varphi^{-1}(n+L+1)) \times [Z(k)+l_0+n, \infty)$$

and

$$W_k^- = \bigcup_{n\geq 0} B_G(\varphi^{-1}(n+L+1)) \times (-\infty, Z(k)-l_0-n].$$

Proposition 2. *Let $c = c(p)$ be as in (1.4) and suppose that*

$$\sum_{n\geq 1} |B_G(\varphi^{-1}(n+L))|e^{-cn} < \infty. \tag{2.16}$$

There exists $l_0 \in \mathcal{L}$ large enough and a constant $c(G, l_0)$ such that for any environment $\Lambda \in A_k$ we have

$$\mathbb{P}^{\Lambda}_{p,q}(R^+_k \not\leftrightarrow R^-_k) \geq \frac{1}{2}(1-q)^{c(G,l_0)} > 0. \tag{2.17}$$

Proof. For any environment $\Lambda \in A_k$ we have $R^+_k \subset W^+_k$ and $R^-_k \subset W^-_k$ for all $k \geq 1$, so with the same flavor of (2.10), for $\Lambda \in A_k$, we have

$$\begin{aligned}\mathbb{P}_p(R^+_k \longleftrightarrow R^-_k) &\leq \mathbb{P}_p(W^+_k \longleftrightarrow W^-_k)\\ &\leq e^{-2cl_0} \sum_{m,n\geq 1} |B_G(\varphi^{-1}(n+L))||B_G(\varphi^{-1}(m+L))|e^{-c(m+n)}\\ &= e^{-2cl_0}\left(\sum_{n\geq 1} |B_G(\varphi^{-1}(n+L))|e^{-cn}\right)^2. \end{aligned} \tag{2.18}$$

By hypothesis $\sum_{n\geq 1} |B_G(\varphi^{-1}(n+L))|e^{-cn}$ is convergent, hence we can take l_0 large enough so that the probability in (2.18) is at most $1/2$, uniformly on k. Define

$$F_k := \{\text{every edge of } B_k \text{ is closed}\}.$$

Since $\Lambda \in A_k$, we have $l_0 \leq X_k \leq 2l_0$. Hence, the number of edges in $B(0, 2l_0)$, denoted by $c(G, l_0)$, is larger than the number of edges in B_k and we have $\mathbb{P}^{\Lambda}_{p,q}(F_k) \geq (1-q)^{c(G,l_0)}$.

In this stage, we follow the same procedure of (2.11)-(2.14) in Theorem 1 to obtain

$$\mathbb{P}^{\Lambda}_{p,q}(R^+_k \not\longleftrightarrow R^-_k) \geq \frac{1}{2}(1-q)^{c(G,l_0)} > 0. \qquad \square$$

The next proposition shows that the lower bound (2.17) in Proposition 2 holds under the hypothesis of Theorem 2.

Proposition 3. *Let X be a random variable such that $E \log |B_G(X)| < \infty$. Then, for all $p < p_c(G \times \mathbb{Z})$, there exists an increasing function $\varphi = \varphi_p$ such that $E\varphi(X) < \infty$ and*

$$\sum_{n\geq 1} |B_G(\varphi^{-1}(n+L))|e^{-cn} < \infty$$

Proof. Let $\alpha = c(p)/2$ and define $g(n) = |B_G(n)|$, $\varphi(n) = \alpha^{-1}\log(g(n))$. Then $E\varphi(X) < \infty$, $\varphi^{-1}(n) = g^{-1}(e^{\alpha n})$ and

$$\sum_{n\geq 1} |B_G(\varphi^{-1}(n+L))|e^{-cn} = e^{\alpha L} \sum_{n\geq 1} e^{-(c-\alpha)n} < \infty. \qquad \square$$

Remark 3. There is a crucial difference on the dynamics of the models with respect to the shift transformation regarding the existence or not of the first moment of X, which is a key point for both proofs. In Theorem 2 we cannot always rely on ergodicity in the same way as in Theorem 1 because the renewal process with interarrival distribution given by the law of X has no stationary measure if $EX = \infty$.

Proof of Theorem 2. We start by defining a sequence of explorations on the boundaries of the downward cones in order to verify the existence of infinitely many vertical blockades for the origin's cluster almost surely. Loosely speaking, we explore the cluster of all sites in the downwards cone to see if any of them intersects the corresponding upwards cone. That is, given $\omega \in \Omega$, we choose an enumeration $(e_k)_k$ of the edges of $G \times \mathbb{Z}$ and, for every $l \in \mathbb{N}$, define $\mathcal{C}_0 = \mathcal{C}_0(l) = W_l^-$, $\mathcal{D}_0(l) = \emptyset$. Given a set of sites $S \in G \times \mathbb{Z}$, the *edge boundary* of S is the set $\partial S = \{e = vw\,;\, v \in S, w \notin S\}$.

1. Let m_1 be the smallest index k such that $e_k \in \partial\mathcal{C}_0$. We set $\mathcal{D}_1 = \{e_{m_1}\}$ and if the edge $e_{m_1} = v_{m_1}w_{m_1}$ is open, with $w_{m_1} \notin \mathcal{C}_0$, then we set $\mathcal{C}_1 = \mathcal{C}_0 \cup \{w_{m_1}\}$, otherwise we set $\mathcal{C}_1 = \mathcal{C}_0$.

2. For $n \geq 2$, we let m_n be the smallest index k such that $e_k \in \partial\mathcal{C}_{n-1} \setminus \mathcal{D}_{n-1}$. We set $\mathcal{D}_n = \mathcal{D}_{n-1} \cup \{e_{m_n}\}$ and
$$\mathcal{C}_n = \begin{cases} \mathcal{C}_{n-1} \cup \{w_{m_n}\}, \text{if } e_{m_n} = v_{m_n}w_{m_n} \text{ is open, } w_{m_n} \notin \mathcal{C}_{n-1} \\ \mathcal{C}_{n-1}, \text{ otherwise.} \end{cases}$$

3. If $\mathcal{C}_n \cap W_l^+ \neq \emptyset$ the exploration stops and we say that *the exploration failed* and we define $\mathcal{C}_i = \mathcal{C}_n$ for $i \geq n$. If the exploration does not stop, then we say that *the exploration succeeded.*

Notice that by construction the exploration stops only in configurations where there is a crossing $\partial W_l^- \leftrightarrow \partial W_l^+$. The exploration of ∂W_l^- just defined is denoted by $\mathcal{E}_l$.

Recall the definition of A_k from Lemma 2. Since $(X_n)_n$ is an i.i.d. sequence, it is ergodic with respect to the shift transformation, thus
$$\nu(A_k \ i.o.) = 1.$$
In fact, by Birkhoff's Ergodic Theorem,
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{A_k} = \nu(A_0) > 0$$
ν-almost surely.

In other words, the set $\mathcal{N}(\Lambda) = \{l \in \mathbb{N} \,|\, A_l \text{ occurs}\}$ is infinite almost surely. Let $n_1 = n_1(\Lambda) = \min \mathcal{N}$ and consider the exploration $\mathcal{E}_{n_1}$. Suppose the exploration failed and for $m \geq 1$ let $\mathcal{C}(W_m^-) = \bigcup_{i=1}^{\infty} \mathcal{C}_i(m)$. In this case, as the explored region is finite, we can find a translation of W^- that contains it. That is, we can choose the next index $n_2 = n_2(\omega, \Lambda) \in \mathcal{N}$ such that

$$W_{n_1}^- \cup \mathcal{C}(W_{n_1}^-) \subseteq W_{n_2}^-,$$

and perform independently the same exploration on $\partial W_{n_2}^-$.

More generally, the exploration $\mathcal{E}_{n_k}$ of the cluster of $W_{n_k}^-$ is well defined for every k and explored only a finite number of edges, where $n_k = n_k(\omega, \Lambda)$ is the smallest index such that

$$W_{n_{k-1}}^- \cup \mathcal{C}(W_{n_{k-1}}^-) \subseteq W_{n_k}^-,$$

and it is conditionally independent given n_k of the preceeding explorations.

Define $\mathcal{F}_k = \sigma(\mathcal{E}_{n_1}, \ldots, \mathcal{E}_{n_k})$, the smallest σ-algebra that contains all the information of the first k explorations and let T^+ denote the index of the first exploration to succeed. That is

$$T^+(\Lambda, \omega) = \min\{k \geq 1;\ \mathcal{E}_{n_k} \text{ succeeded}\}. \tag{2.19}$$

Notice that, as $\mathcal{E}_{n_m}$ is conditionally independent of $\mathcal{F}_{m-1}$ given n_m, we have

$$\begin{aligned} \mathbb{P}_{p,q}^{\Lambda}\left(T^+ > m\right) &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\bigcap_{k=1}^{m}\{\mathcal{E}_{n_k} \text{ failed}\}} \middle| \mathcal{F}_{m-1}, n_m\right]\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\bigcap_{k=1}^{m-1}\{\mathcal{E}_{n_k} \text{ failed}\}} \mathbb{E}\left[\mathbb{1}_{\{\mathcal{E}_{n_m} \text{ failed}\}} \middle| \mathcal{F}_{m-1}, n_m\right]\right] \\ &= \mathbb{P}_{p,q}^{\Lambda}(\mathcal{E} \text{ failed}) \mathbb{P}_{p,q}^{\Lambda}\left(T^+ > m-1\right). \end{aligned} \tag{2.20}$$

Proceeding by induction, by Proposition 2 and Proposition 3, we have that for every $m \geq 1$,

$$\mathbb{P}_{p,q}^{\Lambda}\left(T^+ > m\right) = \prod_{k=1}^{m} \mathbb{P}_{p,q}^{\Lambda}(\mathcal{E}_{n_k} \text{ failed}) \leq \left(\frac{1}{2}(1-q)^{c(G,l_0)}\right)^m \leq \frac{1}{2^m},$$

hence

$$\mathbb{P}_{p,q}^{\Lambda}\left(T^+ = \infty\right) = \mathbb{P}_{p,q}^{\Lambda}\left(\bigcap_{m=1}^{\infty}\{T^+ > m\}\right) = \lim_{m\to\infty} \mathbb{P}_{p,q}^{\Lambda}\left(T^+ > m\right) \leq \lim_{m\to\infty} 2^{-m} = 0.$$

Until now, we are investigating whether the cluster of the origin is infinite by exploring the boundary of cones such that the initial vertex of exploration is on the positive side of the vertical axis. Analogously, one can define the mirrored explorations $\mathcal{E}_{-k}$ of cones with initial vertex of exploration on the negative side of the vertical axis, we call these *downward explorations.* Notice that the downward explorations are exploring the boundaries of the upward cones. Thus, we can also define the first index T^- such that a downward exploration succeeds. That is,

$$T^-(\Lambda, \omega) = \min\{k \geq 1;\ \mathcal{E}_{n_{-k}} \text{ succeeded}\}. \tag{2.21}$$

By symmetry we also have that $\mathbb{P}^{\Lambda}_{p,q}\left(T^{-}=\infty\right)=0$. Hence, almost surely both upper and lower semispace explorations will succeed, implying that the cluster of the origin is contained in $(W^{+}_{n_{T^+}} \cup W^{-}_{n_{-T^-}})^c$, a region with finitely many improved edges. The proof is, thus, completed.

$\square$

Chapter 3

The Contact Process

The Contact Process is a model introduced in 1974 by Harris [13] used to study the behaviour of dissemination of diseases and population growth.

We consider the Contact Process in the state space $\{0,1\}^G$, where G is a transtive connected graph. The system is described by a family of functions $\xi_t : G \mapsto \{0,1\}$ called states, that is, $\xi_t(x)$ gives the state of the site x at time $t \geq 0$. If $\xi_t(x) = 1$ we say that x is infected or occupied and if $\xi_t(x) = 0$ we say that x is healthy or vacant. Let $d(x,y)$ denote the graph distance between $x, y \in G$. The neighborhood $\mathcal{N}_x$ of a site $x \in G$ is the set

$$\mathcal{N}_x = \{y \in G \; ; \; y \text{ is connected to } x \text{ by an edge}\}.$$

The number of infected neighbors $\eta(x,\xi)$ of x at the state ξ is the quantity

$$\eta(x,\xi) = |\{y \in \mathcal{N}_x \; ; \; \xi(y) = 1\}|.$$

The evolution of the Homogeneous Contact Process is described by the *flip rates*

$$c_0(x,\xi_t) = 1, \;\; c_1(x,\xi_t) = \lambda\eta(x,\xi_t), \;\; \text{for all } t \geq 0$$

satisfying for $i = 0,1$ and $\xi_t(x) \neq i$,

$$\lim_{s\to 0} \frac{P(\xi_{t+s}(x) = i|\xi_t = \xi)}{s} = c_i(x,\xi).$$

That is, all individuals become healthy at rate 1 independently of the others, and get infected at a rate that is proportional to the number of infected neighbors. A result by Harris shows that the flip rates specify a unique Markov process. See Durrett [10] Section 2 for details. Let x^* be the configuration where $x \in G$ is the only infected site and let $\xi_0 = 0^*$. The survival probability is

$$\psi(\lambda) = \psi(\lambda, 0^*) = P(\text{the infection exists at all times } t \geq 0).$$

Harris also proved that there exists $\lambda_c > 0$ such that

$$\psi(\lambda) \begin{cases} = 0, & \text{if } \lambda < \lambda_c \\ > 0, & \text{if } \lambda > \lambda_c. \end{cases}$$

See Liggett [17] Chapters 3 and 6. This process has been largely studied, and many results and techniques are obtained via percolation theory, due to the following geometrical description.

As before, if x is vacant, then it becomes occupied at a rate $\lambda\eta(x,\xi_t)$ and if x is occupied, it becomes vacant at rate 1. Now, for each $x,y \in G$ such that $d(x,y)=1$, let $\{T_n^{(x,y)}\ ;\ n\geq 1\}$ be a Poisson process with rate λ and $\{U_n^x\ ;\ n\geq 1\}$ be a Poisson process with rate 1. When the clock $T_n^{(x,y)}$ ticks, let us say, when $T_n^{(x,y)}=t$, draw an arrow from (x,t) to (y,t) to indicate the potential infection. That is, if x is occupied, y will become occupied as well. If y is already occupied the arrow has no effect on the state of y. At times U_n^x we say that there is a *death mark*, or *d.m.* for short, at x and the site becomes vacant if it is occcupied. If x is already vacant, a death mark will have no effect.

Given $t_0<t_1$ and $x,y\in G$, we say that there is a path from (x,t_0) to (y,t_1), denoted $\{(x,t_0)\longrightarrow(y,t_1)\}$, if there exists a sequence of times $t_0=s_0<s_1<\cdots<s_n=t_1$ and a sequence of sites $x=x_0,x_1,\ldots,x_n=y$ such that

1. For $i=1,\ldots,n$, there exists an arrow from x_{i-1} to x_i at the time s_{i-1}
2. For $i=1,\ldots,n$, the segment $\{x_{i-1}\}\times[s_{i-1},s_i]$ has no death marks.

An event $\mathcal{I}$ is said to be increasing if, for any realization of the graphical construction in $\mathcal{I}$, the realizations which consists of the addition of arrows or removal of death marks are also in $\mathcal{I}$. We refer to Section 2.1 of [5] for a formal definition. When two events $\mathcal{I}_1$, $\mathcal{I}_2$ are both increasing, FKG inequality states that

$$\mathbb{P}(\mathcal{I}_1\cap\mathcal{I}_2)\geq\mathbb{P}(\mathcal{I}_1)\mathbb{P}(\mathcal{I}_2). \tag{3.1}$$

See Statement 2.11 on Bezuidenhout, Grimmett [5].

3.1 Results for the Inhomogeneous Contact Process

As in the Percolation Model, we consider a collection $\{X_n;n\in\mathbb{N}_0\}$ of iid unbounded random variables in the probability space $(\Xi,\mathcal{G},\nu)$, where $\mathcal{G}=\sigma(X_n,n\in\mathbb{N}_0)$, supported on $\mathbb{N}$, with associated expectation operator denoted by E. Notice that here the random variables are indexed by $\mathbb{N}_0$ instead of $\mathbb{Z}$, the point being that we now consider only the half-space $G\times\mathbb{R}^+$. We call a configuration $\Lambda\in\Xi$ an *environment*. Fix a distinguished site $0\in G$ and call it the *origin* of the graph. The box $B_G(r)\subset G$ with radius r centered at $0\in G$ is the set $B_G(r)=\{y\in G\ ;\ d(0,y)\leq r\}$.

Definition 4. For $(x,s),(y,t) \in G \times \mathbb{R}^+$, we define the distance

$$\Delta((x,s),(y,t)) = \max\{d(x,y), |t-s|\}.$$

The box $B(n,r)$ of radius r centered at $(0,n) \in G \times \mathbb{R}^+$ is the set

$$B(n,r) = B_G(r) \times [n-r, n+r] = \{y \in G \times \mathbb{R}^+ \ ; \ \Delta((0,n),y) \leq r\}.$$

The boundary $\partial B(n,r)$ of $B(n,r)$ is the set

$$\partial B(n,r) = \{y \in G \times \mathbb{R}^+ \ ; \ \Delta((0,n),y) = r\}.$$

Definition 5. The *site boundary* and the *interior* of a set $S \subset G$ are, respectively, the sets

$$\partial_v S = \{x \in S \ ; \ \exists y \notin S, \, d(x,y) = 1\},$$
$$\text{int}_v S = S \setminus \partial_v S.$$

For a random region $R = R(\Lambda) \subset G \times \mathbb{R}^+$ we consider the inhomogeneous contact process with inhomogeneities in R by setting the flip rates in the following way. We define $c_1(x,t) = \lambda \eta(x, \xi_t)$ as in the homogeneous model, and

$$c_0^{\Lambda}(x,t) = \begin{cases} \delta, \ \text{if } (x,t) \in R(\Lambda) \\ 1, \ \text{otherwise.} \end{cases} \tag{3.2}$$

That is, the parameter δ modifies the infection behavior when it is within the region R. In other words, each individual has the probability of exhibiting higher or lower resistance to infection at certain time intervals. If δ is close to zero within the region R, we observe a low healing rate, meaning that the infection is more resistant, while larger values of δ mimic weaker infections. More specifically, fixing $\lambda < \lambda_c$, we are interested in choosing a parameter δ such that $\lambda/\delta > \lambda_c$, which is equivalent to choosing a supercritical parameter λ within the region R and keeping the healing rate unchanged. When the region $R = R(\Lambda)$ is well defined, we write the associated probability measure simply as $\mathbb{P}^{\Lambda}_{\lambda,\delta}$, which is called the *quenched* probabilty associated with the environment Λ.

We define

$$\psi^{\Lambda}(\lambda,\delta) = \mathbb{P}^{\Lambda}_{\lambda,\delta}(\text{the infection exists at all times } t \geq 0).$$

We note that if G is transitive, then $G \times \mathbb{R}^+$ has exponential decay for the homogeneous Contact Process in the subcritical phase. That is, for all $\lambda < \lambda_c$, there exists $\gamma = \gamma(\lambda) > 0$ such that,

$$\mathbb{P}((0,0) \longrightarrow \partial_v B(0,r)) \leq e^{-\gamma r}. \tag{3.3}$$

In fact, Bezuidenhout, Grimmett [5] successfully adapted the results of Aizenmann, Barski [1] to the Contact Process in $\mathbb{Z}^d$ and claim that the result is still valid for every transitive lattice, even pointing out that transitivity might not be essential, as known for Bernoulli Percolation. See [5] page 986. However, there is no analogous version of the result for quasi-transitive graphs in the literature of the Contact Process, as shown in [3] for Bernoulli Percolation. In 2007, Aizenman, Jung [2] gave a new proof of the result which works for general transitive lattices. See also Swart [19]. We focus on two possible models of random improved regions, with essentially the same structure defined in the Overlap and Stack models for percolation. See (2.2) and (2.6).

3.1.1 The Overlap Model

Let G be a transitive graph and consider the Cartesian product $G \times \mathbb{R}^+$. In each site $(0, n)$ of the vertical line $\{0\} \times \mathbb{Z}^+$, we place a box

$$B_n = B(n, X_n) \tag{3.4}$$

with radius X_n and we consider the improved region R to be the union of these boxes. More precisely, we define

$$R = \bigcup_{n \in \mathbb{Z}^+} B_n. \tag{3.5}$$

The Overlap Model for the Contact Process is the model that considers the region $R = R(\Lambda)$ as defined in (3.5) and the flip rates c_1 and c_0^Λ as defined in (3.2).

The analogous version of Theorem 1 for the Contact Process is the following.

Theorem 3. *Consider the Overlap Model for the Contact Process in a transitive graph G. For $0 < \lambda < \lambda_c < \lambda/\delta < \infty$, we have*

$$\psi^\Lambda(\lambda, \delta) = 0 \quad \Lambda - a.s \qquad \Longleftrightarrow \qquad EX < \infty \tag{3.6}$$

3.1.2 The Stack Model

To start the construction of the environment, place the box $B(0, X_0)$ at the origin of $G \times \mathbb{Z}$ and then stack succssivelly the remaining boxes along $\{0\} \times \mathbb{Z}^+$. More precisely, Let $Z(0) = 0$ and, for $n \in \mathbb{Z}^+$,

$$Z(n) = X_0 + 2\sum_{i=1}^{n-1} X_i + X_n, \tag{3.7}$$

be the center of the n-th box. For $n \in \mathbb{Z}^+$ we use the notation

$$B_n = B(Z(n), X_n). \tag{3.8}$$

The improved region R is given by

$$R = \bigcup_{n \in \mathbb{Z}^+} B_n. \tag{3.9}$$

The Stack Model for the Contact Process is the model that considers the region $R = R(\Lambda)$ as defined in (3.9) and the flip rates c_1 and c_0^Λ as defined in (3.2). Theorem 2 translates to the Contact Process in the following way.

Theorem 4. *Consider the Stack Model for the Contact Process in a transitive graph G. If $E \log |B_G(X)| < \infty$, then, for $0 < \lambda < \lambda_c$, $\delta > 0$,*

$$\psi^\Lambda(\lambda, \delta) = 0 \quad \textit{almost surely on } \Lambda.$$

It should be noted that the discussion after Theorem 2 applies for Theorem 4.

3.2 The Overlap Model: Proof of Theorem 3

The environment setting here is fundamentally the same as in the Percolation case, the difference being that here we consider the truncated versions of the objects in $\mathbb{R}^+$. More precisely, given any set $A \in G \times \mathbb{Z}$ in Chapter 2, here we may regard it as a subset of $G \times \mathbb{R}$ and consider only the intersection $A \cap (G \times \mathbb{R}^+)$. With that in mind, we refer appropriately to the previous chapter for some definitions and results.

Let $\mathcal{A}$ be as in Definition 2 and fix $\Lambda \in \mathcal{A}$. By Lemma 1 we have $\nu(\mathcal{A}) > 0$. Consider W^+ and W^- as defined in Definition 1, $(0, k) \in G \times \mathbb{R}^+$, and define $W_k^+ = (0, k) + W^+$, $W_k^- = (0, k) + W^-$. Now we wish to bound $\mathbb{P}(W_k^+ \longleftrightarrow W_k^-)$ uniformly for every $k \in \mathbb{N}$. In the Bernoulli Percolation case, it is sufficient to use union bound and control the probability of connection between the layers (defined in (2.8)), as by the construction of the cones, there are no sites in between subsequent layers. Here, instead, connection can happen at any time in the size 1 interval in between subsequent layers. For that reason, we give a slightly different definition of layer.

The m-th layer of the upwards cone is the set

$$L_m^+ = (B_G(g(2m)) \times \{m\}) \cup (\partial_v B_G(g(2m)) \times [m, m+1)).$$

We call $\partial_T L_m^+ = (\partial_v B_G(g(2m)) \times [m, m+1))$ the time boundary of L_m^+. We define L_m^- and $\partial_T L_m^-$ analogously.

Proposition 4. *Let $S, T \subseteq G \times \mathbb{Z}$ and $\alpha(S)$ be the number of sets of the form $\{x\} \times [i, i+1)$, with $x \in G$, $i \in \mathbb{N}_0$, which have nonempty intersection with S. Then, there exists a constant $C = C(\gamma) > 0$ such that*

$$\mathbb{P}(S \longrightarrow T) \leq C\alpha(S)e^{-\gamma\Delta(S,T)}.$$

Proof. First, Let $D = \{\text{There is no death mark in } \{0\} \times [0, 1]\}$. Then, we have

$$\mathbb{P}(D \cap \{\{0\} \times [0, 1] \longrightarrow \partial B(0, r)\}) = \mathbb{P}((0, 0) \longrightarrow \partial B(0, r)) \leq e^{-\gamma r}. \tag{3.10}$$

As both events are increasing, FKG inequality (3.1) yields,

$$\mathbb{P}(D \cap \{\{0\} \times [0, 1] \longrightarrow \partial B(0, r)\}) \geq \mathbb{P}(D)\mathbb{P}(\{0\} \times [0, 1] \longrightarrow \partial B(0, r)). \tag{3.11}$$

By (3.10) and (3.11) we conclude that, for $C = \mathbb{P}(D)^{-1} > 0$,

$$\mathbb{P}(\{0\} \times [0, 1] \longrightarrow \partial B(0, r)) \leq Ce^{-\gamma r}.$$

Notice that $S \subset \bigcup \{x\} \times [i, i+1)$, where the union is taken over the sets that have nonempty intersection with S. By union bound, translation invariance and an inclusion of events,

$$\begin{aligned}
\mathbb{P}(S \longrightarrow T) &\leq \sum \mathbb{P}(\{x\} \times [i, i+1) \longrightarrow T) \\
&\leq \sum \mathbb{P}(\{x\} \times [i, i+1) \longrightarrow \partial B(x, \Delta(S, T))) \\
&\leq \alpha(S)\mathbb{P}(\{0\} \times [0, 1) \longrightarrow \partial B(0, \Delta(S, T))) \\
&\leq C\alpha(S)e^{-\gamma\Delta(S,T)}.
\end{aligned}$$

□

Notice that by union bound and the exponential decay (3.3),

$$\mathbb{P}(W_k^- \longrightarrow W_k^+) \leq \sum_{n,m \geq n_0/2} \mathbb{P}_p(L_m^- \longrightarrow L_n^+).$$

From definition implies $\Delta(L_n^+, L_m^-) \geq m + n$. By Proposition 4, for every m, n we have

$$\mathbb{P}(L_m^- \longrightarrow L_n^+) \leq C\alpha(L_m^-)e^{-\gamma(m+n)} \leq Ce^{c_G g(2m)}e^{-\gamma(m+n)}.$$

Summing for $n, m \geq n_0/2$ we obtain

$$\begin{aligned}\mathbb{P}(W_k^- \longrightarrow W_k^+) &\leq \sum_{n,m \geq n_0/2} \mathbb{P}(L_m^- \longrightarrow L_n^+) \\ &\leq C \sum_{n,m \geq n_0/2} e^{c_G g(2m)} e^{-\gamma(m+n)} \\ &\leq C \Big(\sum_{n \geq n_0/2} e^{n(-\gamma + c_G 2 \frac{g(2n)}{2n})} \Big)^2 .\end{aligned} \tag{3.12}$$

Recalling that by Lemma 4 we have $g(n)/n \to 0$, we conclude that the sum of the series above is convergent and we can actually make it as close to zero as we want by increasing n_0 if needed.

Fix n_0 large enough so that

$$\mathbb{P}(W_k^- \longrightarrow W_k^+) \leq \frac{1}{2} \tag{3.13}$$

and consider the following events.

- $F_1 = \{$There is a death mark in $\{x\} \times [k - n_0, k - n_0 + 1)$, $\forall x \in B_G(g(2n_0))\}$,
- $F_2 = \{$There is no arrow in $\{x\} \times [k - n_0, k - n_0 + 1)$, $\forall x \in B_G(g(2n_0))\}$,
- $F_3 = \{$There is no arrow pointing towards $\partial B_G(g(2n_0)) \times [k - n_0, k + n_0)\}$.

Observe that, by independence of the Poisson processes,

$$\mathbb{P}^{\Lambda}_{\lambda,\delta}(F_1) = c = c(n_0, G, \delta) \geq (1 - e^{-\delta})^{|B_G(g(2n_0))|} > 0.$$

Similarly, $\mathbb{P}^{\Lambda}_{\lambda,\delta}(F_2)$ and $\mathbb{P}^{\Lambda}_{\lambda,\delta}(F_3)$ are positive constants. So, by independence, the event $F = F_1 \cap F_2 \cap F_3$ has positive probability

$$\mathbb{P}^{\Lambda}_{\lambda,\delta}(F) > 0. \tag{3.14}$$

Combining (3.13) and (3.14), it follows that, for $\Lambda \in \mathcal{A}$,

$$\begin{aligned}\mathbb{P}^{\Lambda}_{\lambda,\delta}(W_m^- \not\longrightarrow W_m^+) &\geq \mathbb{P}^{\Lambda}_{\lambda,\delta}(\{W_m^- \not\longrightarrow W_m^+\} \cap F) \\ &\geq \mathbb{P}(W_m^- \not\longrightarrow W_m^+) \mathbb{P}^{\Lambda}_{\lambda,\delta}(F) \\ &\geq \frac{1}{2} \mathbb{P}^{\Lambda}_{\lambda,\delta}(F) > 0.\end{aligned} \tag{3.15}$$

Now, we could finish the proof by extending the process to the real line, that is, to consider the contact process in $G \times \mathbb{R}$ to rely on ergodicity, as done by Madras, Schinazi, and Schonmann (See page 1157 of [18]). Instead, we rely on the exploration of the downwards cone W_m^-, as we have already done for the Stack Model for percolation in the previous chapter. Although some technicalities do arise from the continuity aspects of the model, we will see that the fact that in the contact process we work on the half-space $\mathbb{R}^+$ actually makes the exploration easier than in the Percolation Model, because here the exploration of W_m^- will be always finite.

We shall explore the boundary of W_m^- in order to see if there is a connection with W_m^+ in a way that any path starting in a site in the interior of W_m^- is analysed if it leaves W_m^-. Notice that, if only a finite number of sites $x \in G$ are occupied, then the first time t_0 such that an infection or a death mark occurs is well defined and satisfies $t_0 > 0$ almost surely. Suppose W_m^- intersects $G \times \{0\}$ on L_{i+1}^- for some $i > 0$, that is,

$$W_m^- \cap (G \times \{0\}) = B(g(2(i+1)) \times \{0\}.$$

We perform a discretization of the time boundary $\partial_T L_i^-$ in the following way. Consider $x \in B(g(2(i+1)))$ and let N be such that no two consecutive ocurrences (of death mark or infection) happen in an interval smaller than $1/N$. Notice that $N < \infty$ almost surely. Divide all the lenght one intervals in the time boundary of L_i^- in N equal parts $0 = l_0, l_1 = 1/N, \dots, l_N = 1$. Let

$$\mathcal{C}_0^1 = B(g(2(i+1))) \setminus \text{int}_v B(g(2i)).$$

For every $x \in \mathcal{C}_0^1$, let s_0 be the smallest time such that there is any occurrence. If $s_0 < l_1$ and the occurrence is a death mark, let $\mathcal{C}_0^2 = \mathcal{C}_0^1 \setminus \{x\}$. We write $x \mapsto y$ to indicate that x has infected y. If $s_0 < l_1$ and the occurrence is an infection $x \mapsto y$, then we set $\mathcal{C}_0^2 = \mathcal{C}_0^1 \cup \{y\}$ and we may redefine N to take into account consecutive occurrences in y. As the contact process is well defined, no path visits an infinite amount of sites in finite time, so the exploration of a site cluster always gets to any time t in a finite number of steps. As a consequence, redefining N does not lead to the accumulation of the process at any given time t. If $l_{k-1} < s_0 < l_k < l_N$, we set $\mathcal{C}_0^k = \mathcal{C}_0^{k-1} = \dots = \mathcal{C}_0^1$ and

$$\mathcal{C}_0^{k+1} = \begin{cases} \mathcal{C}_0^k \cup \{y\}, & \text{if } x \mapsto y \\ \mathcal{C}_0^k \setminus \{x\}, & \text{if there is a death mark in } x. \end{cases}$$

Now, we proceed inductively. Suppose the first j occurrences $s_0 < s_1 < \dots < s_{j-1}$ have been explored and $\mathcal{C}_0^1, \dots, \mathcal{C}_0^{k_j}$ have been defined. If $l_{k-1} < s_j < l_k < l_N$, we define $\mathcal{C}_0^{k_{j+1}-1} = \dots = \mathcal{C}_0^{k_j+1} = \mathcal{C}_0^{k_j}$ and

$$\mathcal{C}_0^{k_{j+1}} = \begin{cases} \mathcal{C}_0^{k_j} \cup \{y\}, & \text{if } x \mapsto y \\ \mathcal{C}_0^{k_j} \setminus \{x\}, & \text{if there is a death mark in } x. \end{cases}$$

If $l_{N-1} < s_j < l_N$, define $\mathcal{C}_0^N = \dots = \mathcal{C}_0^{k_j+1} = \mathcal{C}_0^{k_j}$ and

$$\mathcal{C}_1^1 = \begin{cases} \mathcal{C}_0^N \cup [B(g(2i)) \setminus \text{int}_v B(g(2(i-1)))] \cup \{y\}, & \text{if } x \mapsto y \\ \mathcal{C}_0^N \cup [B(g(2i)) \setminus \text{int}_v B(g(2(i-1)))] \setminus \{x\}, & \text{if there is a death mark in } x. \end{cases}$$

That is, we proceed the same way, but we add the sites corresponding to the (just reached) top of the layer L_i^- . Now repeat the process above for $\mathcal{C}_1^1$ to construct $\mathcal{C}_1^k$, $k = 1, \dots, N$. Again by induction, suppose defined $\mathcal{C}_m^k$, $k = 1, \dots, N$, $m < j < i$. We define

$$\mathcal{C}_j^1 = \begin{cases} \mathcal{C}_{j-1}^L \cup [B(g(2(i-j+1))) \setminus \mathrm{int}_v B(g(2(i-j)))] \cup \{y\}, & \text{if } x \mapsto y \\ \mathcal{C}_{j-1}^L \cup [B(g(2(i-j+1))) \setminus \mathrm{int}_v B(g(2(i-j)))] \setminus \{x\}, & \text{if there is a d.m. in } x \end{cases}$$

and again we define $\mathcal{C}_j^2, \dots, \mathcal{C}_j^N$ inductively. Finally, we let

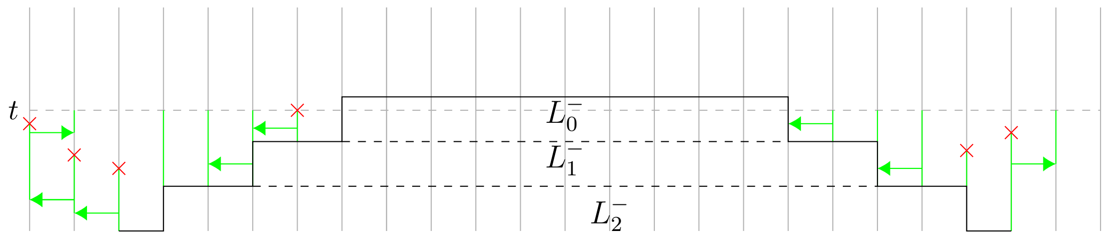

Figure 3.1: Exploration until time $t \in (i-1, i)$

$$\mathcal{C}_1 = \begin{cases} \mathcal{C}_{i-1}^N \cup B(g(2)) \cup \{y\}, & \text{if } x \mapsto y \\ \mathcal{C}_{i-1}^N \cup B(g(2)) \setminus \{x\}, & \text{if there is a death mark in } x. \end{cases}$$

Notice that $\mathcal{C}_1$ contains the set of individuals x such that $W_m^- \longrightarrow (x, i)$. For $x \in \mathcal{C}_1$, at each occurrence $i < t_1 < t_2 < \dots$ we exclude x from the set in the case of death mark and include any individuals x may infect. This way we have a well defined sequence of random sets $\mathcal{C}_1(t_n)$. We say that the the exploration failed if there exists $n \in \mathbb{N}$ and $x \in \mathcal{C}_1(t_n)$ such that $(x, t_n) \in W_m^+$ and in this case the exploration stops at t_n. Otherwise we say that the exploration succeeded.

Let $\mathcal{A}$ be as in Definition 2. By ergodicity of $(X_n)_n$ with respect to the shift transformation $\tau(X_0, X_1, \dots) = (X_1, X_2, \dots)$, for $\mathcal{A}_n = \tau^n(\mathcal{A})$, we have

$$\nu(\mathcal{A}_k) = 1.$$

If the first exploration, which we call $\mathcal{E}_1$, failed, then we consider $n_1 = n_1(\Lambda, (\xi_t)_t)$ the first index j such that all the explored area in $\mathcal{E}_1$ is contained in W_{m+j}^- and $\mathcal{A}_j$ is observed. We run a new exploration $\mathcal{E}_2$ of the boundary of $W_{m+n_1}^-$. Proceeding inductively, if the first $k-1$ explorations $\mathcal{E}_1, \dots, \mathcal{E}_{k-1}$ have failed, then we define an exploration $\mathcal{E}_k$ of the boundary of $W_{m+n_{k-1}}$, where $n_{k-1} = n_{k-1}(\Lambda, (\xi_t)_t)$ is the first index j such that all the explored area in $\mathcal{E}_{k-1}$ is contained in W_{m+j}^- and $\mathcal{A}_j$ is observed.

Let T^+ be as in (2.19). To prove the theorem it is sufficient to show that

$$\mathbb{P}_{\lambda,\delta}^{\Lambda}(T^+ = \infty) = 0.$$

By (2.20) and (3.15), we have

$$\mathbb{P}^{\Lambda}_{\lambda,\delta}\left(T^{+} > m\right) = \prod_{k=1}^{m} \mathbb{P}^{\Lambda}_{\lambda,\delta}(\mathcal{E}_k \text{ failed}) \leq \left(1 - \frac{1}{2}\mathbb{P}^{\Lambda}_{\lambda,\delta}(F)\right)^{m}.$$

So,

$$\begin{aligned}\mathbb{P}^{\Lambda}_{\lambda,\delta}\left(T^{+} = \infty\right) &= \mathbb{P}^{\Lambda}_{\lambda,\delta}\left(\bigcap_{m=1}^{\infty}\{T^{+} > m\}\right) \\ &= \lim_{m\to\infty} \mathbb{P}^{\Lambda}_{\lambda,\delta}\left(T^{+} > m\right) \\ &\leq \lim_{m\to\infty}\left(1 - \frac{1}{2}\mathbb{P}^{\Lambda}_{\lambda,\delta}(F)\right)^{m} \\ &= 0.\end{aligned} \tag{3.16}$$

This concludes the proof. □

3.3 The Stack Model: Proof of Theorem 4

In this section we prove Theorem 4. Likewise Section 2.3, we prove it in a slightly more general setting. We refer to several definitions and results in Chapter 2, specially in Section 2.3, and in this case it should be clear that we are actually considering the analogous notions and results in the contact process in $G\times\mathbb{R}^{+}$ instead of percolation in $G \times \mathbb{Z}$.

Let $\varphi \geq 0$ be any increasing function and suppose $E\varphi(X) < \infty$. Let the center $Z(n)$ of the boxes be as in (2.4). For $n \in \mathbb{N}_0$, let B_n be as in (2.5). As in the previous chapter, the improved region R for the Stack Model is given by $R = \bigcup_{n\in\mathbb{Z}} B_n$. For $k \in \mathbb{Z}$, we also let R_k^{+} and R_k^{-} as in (2.15), respectively. Let A_k be as in Lemma 2. We have that $\nu(A_j) > 0$. Let W_k^{+} and W_k^{-} be as in Definition 3 and recall the notation $L = \min\{l \in \mathbb{N} \,|\, \nu(\varphi(X) \leq l) > 0\}$.

Proposition 5. *Let $\gamma = \gamma(\lambda)$ be as in (3.3). There exists l_0 large enough such that for an environment $\Lambda \in A_k$, if*

$$\sum_{n\geq 1} |B_G(\varphi^{-1}(n+L))| e^{-\gamma n} < \infty \tag{3.17}$$

then, there exists a constant $c(G, l_0)$ such that

$$\mathbb{P}^{\Lambda}_{\lambda,\delta}(R_k^{-} \not\to R_k^{+}) \geq c(G, l_0) > 0.$$

Proof. For any environment $\Lambda \in A_k$ we have $R_k^+ \subset W_k^+$ and $R_k^- \subset W_k^-$ for all $k \geq 1$. As in (3.12), we use Proposition 4 to calculate,

$$\begin{aligned} \mathbb{P}(R_k^- \longrightarrow R_k^+) &\leq \mathbb{P}(W_k^- \longrightarrow W_k^+) \\ &\leq Ce^{-\gamma l_0}\left(\sum_{n\geq 0}|B_G(\varphi^{-1}(n+L))|e^{-\gamma n}\right)^2. \end{aligned} \tag{3.18}$$

If $\sum_{n\geq 0}|B_G(\varphi^{-1}(n+L))|e^{-\gamma n} < \infty$, we can choose l_0 large enough so that the probability is uniformly bounded in k by $1/2$. Now, consider the event F where

- There is a death mark in $\{x\} \times [Z(k) - 2l_0, Z(k) - 2l_0 + 1)\ \forall x \in B_G(2l_0)$,
- There is no arrow in $\{x\} \times [Z(k) - 2l_0, Z(k) - 2l_0 + 1)\ \forall x \in B_G(2l_0)$,
- There is no arrow pointing towards $\partial B_G(2l_0) \times [Z(k) - 2l_0, Z(k) + 2l_0)$.

Then, as in (3.15), we have

$$\mathbb{P}^{\Lambda}_{\lambda,\alpha}(R_k^+ \not\longleftrightarrow R_k^-) \geq \frac{1}{2}\mathbb{P}^{\Lambda}_{\lambda,\alpha}(F) > 0. \tag{3.19}$$

□

The next proposition shows that Proposition 5 holds under the hypothesis of Theorem 4.

Proposition 6. *Let X be a random variable such that $E\log|B_G(X)| < \infty$. Then, for all $\lambda < \lambda_c(G\times\mathbb{Z})$, there exists an increasing function $\varphi = \varphi_\lambda$ such that $E\varphi(X) < \infty$ and*

$$\sum_{n\geq 1}|B_G(\varphi^{-1}(n+L))|e^{-\gamma n} < \infty.$$

Proof. Let $\alpha = \gamma(\lambda)/2$ and define $g(n) = |B_G(n)|$, $\varphi(n) = \alpha^{-1}\log(g(n))$. Then $E\varphi(X) < \infty$, $\varphi^{-1}(n) = g^{-1}(e^{\alpha n})$ and

$$\sum_{n\geq 1}|B_G(\varphi^{-1}(n+L))|e^{-\gamma n} = e^{\alpha L}\sum_{n\geq 1}e^{-(\gamma-\alpha)n} < \infty.$$ □

Proof of Theorem 4. Notice that since $(X_n)_n$ is an iid sequence, it is ergodic with respect to the shift transformation, thus

$$\nu(A_k\ i.o.) = 1.$$

In fact, by Birkhoff's Ergodic Theorem (See Theorem 5),

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}_{A_k} = \nu(A_0) > 0$$

ν-almost surely.

Now, the proof is analogous to the proof of Theorem 4. We define a sequence of explorations $\mathcal{E}_i$ of the boundary of $W_{n_i}^-$ to see if there is a connection $W_{n_i}^- \longrightarrow W_{n_i}^+$, conditioning on failure of the previous exploration, that at n_i A_{n_i} is observed in the configuration and that the previous finite explored area is contained in $W_{n_i}^-$. Then, for T^+ as in (2.19) we show that $\mathbb{P}_{\lambda,\delta}^{\Lambda}(T^+ = \infty) = 0$ as in (3.16). We also refer to the proof of Theorem 3 for details of the explorations. $\square$

Chapter 4

Future Works

4.1 Percolation for heavy-tailed distributions

Many questions regarding Overlap and Stack Models remain open. If the distribution of the radii X_n has a very heavy tail, in the sense that $E\log(X) = \infty$, does the Stack Model percolate in $\mathbb{Z}^d$ for some $q > p_c(\mathbb{Z}^d)$? In this case, we cannot guarantee that the series in the discussion after Theorem 2 is convergent, and therefore, we cannot use the theorem. To demonstrate percolation in this case, i.e., to show that the condition $E\log(X) < \infty$ in Theorem 2 is necessary, one idea is to find a sequence of *enormous* boxes, with radii growing superexponentially in relation to the distance between them, and show, using the *multiscale method*, that at each observed scale, the probability of connection between the boxes remains uniformly away from zero. The multiscale method has been successfully used in many recent models, such as in [9] and more recently in [14]. On the other hand, there may be finer estimates for $\mathbb{P}_p(R^+ \longleftrightarrow R^-)$ than those in (2.18) obtained using deterministic cones, so that Theorem 2 is generalized. There is also the possibility that models like these have a trivial critical point $p_c = 0$, since the typical argument showing that the model has a nontrivial critical point does not work due to the super-exponential growth of the radii.

4.2 Critical percolation in the square lattice

Zhang's technique in [21] to demonstrate that $\theta^R(p_c, q) = 0$ in $\mathbb{Z}^2$ when R is a line uses intrinsic properties of percolation in planar graphs. More specifically, it uses an adaptation of the celebrated Russo-Seymour-Welsh Theorem for this non-homogeneous percolation model. Zhang's proof can be generalized when R is any fixed deterministic cylinder, but it faces difficulties in estimating the probability of a subcritical path in the dual lattice when crossing the region R in Stack or Overlap models, precisely because there is no uniform estimate for the width of the enhanced region to be crossed by the cluster. The technique used, for example, in Theorem 1, also does not work since at the critical point, we do not have the exponential decay estimate. The known decay for the cluster diameter at the critical point of $\mathbb{Z}^2$ is polynomial and insufficient for the convergence of the series obtained in the estimate equivalent to (2.18). See, for example, Grimmett [12] pages 236 and 279.

4.3 Percolation with a disconnected improved region

The Stack Model results in a connected region. Alternatively, one could consider boxes with spacing between them and set the parameter $q = 1$. Our methods might be applicable in this scenario, and it would be interesting to establish conditions on the radii and spacings necessary to achieve the percolating phase.

4.4 Exponential Decay

The theorems we have proven about the Overlap and Stack Models state that, under certain hypotheses, the critical point of the homogeneous model remains unchanged when we open edges in the region R with high probability. A natural question is whether in the subcritical region of the models, we still have exponential decay of $\theta_n^\Lambda(p,q) := \mathbb{P}_{p,q}^\Lambda(0 \longleftrightarrow \partial B(0,n))$. In other words, is it true that for the Stack or Overlap model, there exists $c > 0$ such that $\theta_n^\Lambda(p,q) \leq e^{-cn}$? The constant c, if it exists, besides depending on the parameter p, as in the homogeneous case, certainly also depends on q and EX, since the cluster will behave as in the supercritical homogeneous case inside R, where the *mean-field lower bound* $\theta_n(q) \geq \theta(q) \geq \alpha(q - p_c)$ holds for some α. It is not clear, though, if our proof can be adapted to show that there is exponential decay, since the conclusion of both proofs rely on indirect arguments such as ergodicity.

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Appendix A

Ergodicity in the Overlap Model

In this appendix we prove the ergodicity of the Annealed measure in the Overlap Model. Let $\Omega = \{0,1\}$ and consider the measurable space $(\Omega^{G\times\mathbb{Z}}, \mathcal{F})$, where $\mathcal{F}$ is the product σ-algebra. We wish to define the shift $\tau : \Omega^{G\times\mathbb{Z}} \longrightarrow \Omega^{G\times\mathbb{Z}}$ in the $\mathbb{Z}$ *direction* in the space of configurations $\Omega^{G\times\mathbb{Z}}$. Let $e = \langle (x_1, y_1), (x_2, y_2)\rangle$, where $x_1, x_2 \in G$ and $y_1, y_2 \in \mathbb{Z}$. We abuse notation to also white

$$\tau(e) = \langle (x_1, y_1 - 1), (x_2, y_2 - 1)\rangle.$$

For $\omega = (\omega_e)_{e\in G\times\mathbb{Z}} \in \Omega$, define $\tau(\omega) = (\omega_{\tau(e)})_{e\in G\times\mathbb{Z}}$. For any event $A \in \mathcal{F}$,

$$\tau^{-1}(A) = \{\omega \in \Omega^{G\times\mathbb{Z}} : \tau(\omega) \in A\}$$

If in the environment Λ we have $R(\Lambda) = \bigcup_{n\in\mathbb{Z}} B(n, X_n(\Lambda))$ as in (2.2). We define $\tau(\Lambda)$ to be the environment where

$$R(\tau(\Lambda)) = \bigcup_{n\in\mathbb{Z}} B(n, X_{n+1}(\Lambda)).$$

Denoting for simplicity $\mathbb{P} = \mathbb{P}^{\nu}_{p,q}$, we have

$$\begin{aligned}
\mathbb{P}(\tau^{-1}(A)) &= \int_{\Xi} \mathbb{P}^{\Lambda}_{p,q}(\tau^{-1}(A))d\nu(\Lambda) \\
&= \int_{\Xi} \mathbb{P}^{\Lambda}_{p,q}(\tau^{-1}(A))d\nu(\Lambda) \\
&= \int_{\Xi} \mathbb{P}^{\tau(\Lambda)}_{p,q}(A)d\nu(\Lambda) \\
&= \mathbb{P}(A).
\end{aligned}$$

So the annealed measure is invariant by τ.

Now,

$$\mathcal{A} = \{(A_\alpha)_{\alpha\in G\times\mathbb{Z}};\ A_\alpha = \Omega \text{ for all but finitely many indexes } \alpha\}$$

is an algebra which generates the product σ-algebra $\mathcal{F}$.

Now we show that the annealed measure is in fact mixing, therefore is ergodic. First we check the mixing condition for elements of the algebra. Let $A_1, A_2 \in \mathcal{A}$ and recall from (2.1) that $B_k = B(k, X_k)$. Notice that considering n large enough we can find disjoint boxes $B(\tau^{-n}(A_1))$ and $B(A_2)$ centered along $\{0\} \times \mathbb{Z}$ that contains all the edges on which $\tau^{-n}(A_1)$ and A_2 depends, respectively. We can suppose without loss of generality that $B(A_2)$ is centered at the origin and that $B(\tau^{-n}(A_1))$ is above it.

Consider the event

$$D_n = \{\text{There exists } k \text{ such that } B_k \text{ intersects } B(\tau^{-n}(A_1)) \text{ and } B(A_2)\}.$$

Let us estimate $\nu(D_n)$. Let R_{A_2} denote the radius of $B(A_2)$ and $m_0 = m_0(n)$ denote the distance between $B(\tau^{-n}(A_1))$ and $B(A_2)$ and notice that $m_0 \to \infty$ as $n \to \infty$. If $k \geq 0$, then, for B_k to have non empty intersection with $B(\tau^{-n}(A_1))$ and with $B(A_2)$, it is sufficient that

$$X_k \geq M(k, n) = m_0 + R_{A_2} + |k|.$$

By hypothesis $EX < \infty$, thus, given $\varepsilon > 0$, we can find $K = K(k)$ large enough so that for every $n \in \mathbb{N}$,

$$\sum_{k \geq K} \nu(X \geq M(k, n)) < \frac{\varepsilon}{4},$$

as well as $N = N(k, n)$ such that for $n \geq N$,

$$\nu(X \geq M(k, n)) < \frac{\varepsilon}{4K}.$$

So, for all $n \geq N$,

$$\begin{aligned}
\nu(D_n) &\leq \nu\left(\bigcup_{k \in \mathbb{Z}} \{X_k \geq M(k, n)\}\right) \\
&\leq \sum_{k \in \mathbb{Z}} \nu(X \geq M(k, n)) \\
&\leq 2 \sum_{k \geq 0} \nu(X \geq M(k, n)) \\
&= 2 \sum_{k=0}^{K-1} \nu(X \geq M(k, n)) + 2 \sum_{k \geq K} \nu(X \geq M(k, n)) \\
&< \varepsilon.
\end{aligned}$$

Notice that

$$\lim_n \int_{D_n} \mathbb{P}^{\Lambda}_{p,q}(\tau^{-n}(A_1) \cap A_2) d\nu(\Lambda) \leq \lim_n \nu(D_n) = 0.$$

As $\mathbb{P}^{\Lambda}_{p,q}(\tau^{-n}(A_1))$ and $\mathbb{P}^{\Lambda}_{p,q}(A_2)$ are independent random variables under the event D_n^c, by the Dominated Convergence Theorem,

$$\begin{aligned}\lim_n \mathbb{P}(\tau^{-n}(A_1) \cap A_2) &= \lim_n \int_{D_n} \mathbb{P}^{\Lambda}_{p,q}(\tau^{-n}(A_1) \cap A_2) d\nu(\Lambda) \\ &\quad + \lim_n \int_{D_n^c} \mathbb{P}^{\Lambda}_{p,q}(\tau^{-n}(A_1) \cap A_2) d\nu(\Lambda) \\ &= \lim_n \int_{D_n^c} \mathbb{P}^{\Lambda}_{p,q}(\tau^{-n}(A_1))\mathbb{P}^{\Lambda}_{p,q}(A_2) d\nu(\Lambda) \\ &= \mathbb{P}(A_1)\mathbb{P}(A_2).\end{aligned}$$

Now, let $A \in \mathcal{F}$. For every $\varepsilon > 0$, there exists $A_1 \in \mathcal{A}$ such that

$$\mathbb{P}(A \Delta A_0) < \varepsilon \tag{A.1}$$

Given $A, B \in \mathcal{F}$, fix $\varepsilon > 0$ and choose $A_0, B_0 \in \mathcal{A}$ such that $\mathbb{P}(A\Delta A_0) < \varepsilon$, $\mathbb{P}(B\Delta B_0) < \varepsilon$. Notice that for any events C, D

$$\begin{aligned}&\mathbb{P}(C \cup D) = \mathbb{P}(C\Delta D) + \mathbb{P}(C \cap D) \\ \Rightarrow &\mathbb{P}(C \cup D) - \mathbb{P}(C \cap D) = \mathbb{P}(C\Delta D) \\ \Rightarrow &|P(C) - P(D)| \le \mathbb{P}(C\Delta D).\end{aligned}$$

Choose n large enough such that

$$|\mathbb{P}(\tau^{-n}(A_0) \cap B_0) - \mathbb{P}(A_0)\mathbb{P}(B_0)| < \varepsilon \tag{A.2}$$

and notice that

$$|\mathbb{P}(A_0)\mathbb{P}(B_0) - \mathbb{P}(A)\mathbb{P}(B)| < 2\varepsilon. \tag{A.3}$$

Moreover, a general property of the symmetric difference allows us to write

$$(\tau^{-n}(A) \cap B)\Delta(\tau^{-n}(A_0) \cap B_0) \subseteq (\tau^{-n}(A)\Delta\tau^{-n}(A_0)) \cup (B\Delta B_0).$$

As τ is measure preserving,

$$\begin{aligned}|\mathbb{P}(\tau^{-n}(A) \cap B) - \mathbb{P}(\tau^{-n}(A_0) \cap B_0)| &\le \mathbb{P}[(\tau^{-n}(A) \cap B)\Delta(\tau^{-n}(A_0) \cap B_0)] \\ &\le \mathbb{P}[(\tau^{-n}(A)\Delta\tau^{-n}(A_0)) \cup (B\Delta B_0)] \\ &\le \mathbb{P}(\tau^{-n}(A)\Delta\tau^{-n}(A_0)) + \mathbb{P}(B\Delta B_0) \\ &\le 2\varepsilon.\end{aligned} \tag{A.4}$$

Collecting (A.1), (A.2), (A.3) and (A.4) we have

$$\begin{aligned}|\mathbb{P}(\tau^{-n}(A) \cap B) - \mathbb{P}(A)\mathbb{P}(B)| &\le |\mathbb{P}(\tau^{-n}(A) \cap B) - \mathbb{P}(\tau^{-n}(A_0) \cap B_0)| \\ &\quad + |\mathbb{P}(\tau^{-n}(A_0) \cap B_0) - \mathbb{P}(A_0)\mathbb{P}(B_0)| \\ &\quad + |\mathbb{P}(A_0)\mathbb{P}(B_0) - \mathbb{P}(A)\mathbb{P}(B)| \\ &\le 5\varepsilon\end{aligned}$$

and this concludes the proof.

Appendix B

Technical Results

In this Appendix, we state some general results which are used along the text.

Theorem 5 (Birkhoff's Ergodic Theorem)**.** *Let τ be a measure-preserving transformation in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{I}$ be the $\sigma-$field of invariant events. For any $X \in L^1$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} X(\tau^n(\omega)) = E(X|\mathcal{I})$$

almost surely.

Moreover, if $(\tau, \mathbb{P})$ is ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} X(\tau^n(\omega)) = E(X).$$

A proof of this theorem can be found at Durrett [11], Theorem 7.2.1.

The following is a classical result of de la Valée Poussin on uniform integrability applied to a single function.

Lemma 3. *Let X be an unbounded r.v. supported on $\mathbb{N}$. If $\mathbb{E}X < \infty$, then there exists a non-decreasing function $f : \mathbb{R} \to [0, \infty)$ with $f(x) \nearrow \infty$ as $x \to \infty$, such that*

$$\mathbb{E}[Xf(X)] < \infty.$$

Proof. Clearly, the construction of such f must depend on the random variable considered. The claim is related to the fact that there is no 'fastest converging series'. A possible construction for f is as follows. We know that $\mathbb{E}X < \infty$ is equivalent to $\sum_{n\geq 1} \mathbb{P}(X \geq n) < \infty$.

Defining $r_n := \sum_{m\geq n} \mathbb{P}(X \geq m)$, we know that $r_n \downarrow 0$ as $n \to \infty$. For $k \geq 1$, define $n_k := \min\{j \geq 1;\ r_j \leq 2^{-k}\}$ and notice that, because X is unbounded, n_k is a non-decreasing sequence that tends to infinity. Define $h(n) = \max\{k;\ n_k \leq n\}$ and notice that

$$\sum_{n\geq 1} h(n)\mathbb{P}(X \geq n) = \sum_{k\geq 1} k \sum_{n;h(n)=k} \mathbb{P}(X \geq n) \leq \sum_{k\geq 1} k r_{n_k} \leq \sum_{k\geq 1} k 2^{-k} < \infty.$$

Now, define $H(x) = \int_0^x h(s)\mathrm{d}s$, considering the extension $h(x) = h(\lfloor x \rfloor)$ for non-integer $x \in \mathbb{R}$. Since $H'(x) = h(x)$ a.s., we conclude that

$$\begin{aligned}\sum_{n\geq 1} h(n)\mathbb{P}(X \geq n) &= \mathbb{E}\Big[\sum_{n\geq 1} h(n)\mathbb{1}_{\{X\geq 1\}}\Big] \\ &= \mathbb{E}\Big[\int_1^\infty h(x)\mathbb{1}_{\{X\geq 1\}}\mathrm{d}x\Big] \\ &= \int_1^\infty h(x)\mathbb{P}(X \geq x)\mathrm{d}x \\ &= \mathbb{E}[H(X)\mathbb{1}_{\{X\geq 1\}}]\end{aligned}$$

is finite, and taking $f(x) = H(x)/x$ satisfies all the properties we wanted. □

Lemma 4. *Let f be as in Lemma 3 and denote by g the inverse function of $xf(x)$. Then,*

$$\lim_{n\to\infty} \frac{g(n)}{n} = 0.$$

Proof. By contradiction, suppose that there exists some $\varepsilon > 0$ and an increasing sequence (n_k) with $g(n_k) \geq \varepsilon n_k$. Then, applying $x \mapsto xf(x)$ we get

$$n_k \geq \varepsilon n_k f(\varepsilon n_k), \text{ for all } k \quad \Longrightarrow \quad f(\varepsilon n_k) \leq \frac{1}{\varepsilon}, \text{ for all } k$$

contradicting that $f(n) \nearrow \infty$. □