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SWITCHING RECONSTRUCTION PROBLEMS FOR SIGNED GRAPHS

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Switching Reconstruction Problems for Signed Graphs

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RESUMO

Um **grafo simples** G é um par ordenado $(V(G), E(G))$, onde $V(G)$ é dito o **conjunto de vértices** do grafo G e $E(G)$ é o conjunto de subconjuntos de dois elementos de $V(G)$, chamado **conjunto de arestas** do grafo G. A análise do multiconjunto obtido ao se apagar um vértice de G, junto com as arestas incidentes a este vértice, de todas as formas possíveis nos leva à conjectura de reconstrução. Esta conjectura nos diz que a partir do multiconjunto supracitado podemos obter um grafo isomorfo ao grafo original, se o grafo original possui pelo menos três vértices. Esta conjectura foi proposta em 1941, por Kelly e por Ulam (veja [3]), e levou a diversos outros problemas de reconstrução, como o problema de reconstrução por arestas apagadas. Neste trabalho abordamos variações deste problema para grafos com sinais.

Seja X um grafo simples e finito. Ao associarmos as arestas deste grafo a uma função $g\colon E(X) \to \{+,-\}$ que associa a cada aresta um sinal positivo ou negativo, obtemos um grafo com sinal (G, X), onde X é o grafo subjacente associado e G é o grafo com o mesmo conjunto de vértices que X que contém todas as arestas positivas de (G, X). Chamamos G^c o grafo com o mesmo conjunto de vértices que X que contém todas as arestas negativas de (G, X).

Nós denotamos por $(G, X)^*$ um grafo não rotulado que é isomorfo a (G, X) e por $(G, X)_e$ o grafo obtido ao se trocar o sinal da aresta e. O multiconjunto $\{(G, X)_e^* \mid e \in E(G)\}$ de grafos não rotulados é chamado **baralho de aresta com sinal** de (G, X). Se um grafo com sinal é determinado, a menos de isomorfismo, a partir do baralho de arestas com sinal, dizemos que o mesmo é reconstrutível. O problema de reconstrução de grafos com sinais consiste em determinar quais grafos com sinais são reconstrutíveis.

Problema. *Sejam* (G, X) *e* (H, X) *grafos com sinais com mais de seis arestas tais que* $\{(G, X)_e^* \mid e \in E(G)\} = \{(H, X)_e^* \mid e \in E(H)\}$. *Então* $(G, X) \cong (H, X)$*?*

Podemos reescrever este problema como

Inspirados pelo problema de reconstrução original e principalmente pelo problema de reconstrução por arestas apagadas apresentamos resultados relacionados ao problema de reconstrução de grafos com sinais.

Dois importantes resultados relacionados ao problema de reconstrução para grafos com sinais que serão apresentados neste trabalho estão listados a seguir. A prova destes teoremas será apresentada no Capítulo 3.

Teorema 1. *Seja (G, X) um grafo com sinal com mais de 2 arestas. Se o número de arestas positivas é diferente do número de arestas negativas, então (G, X) é reconstrutível.*

Teorema 2. *Seja (G, X) um grafo com sinal com mais de 2 arestas não reconstrutível. Seja $(H, X) \ncong (G, X)$ tal que $\{[(G, X)^*_e] \mid e \in E(G)\} = \{[(H, X)^*_e] \mid e \in E(G)\}$. Se o número de arestas positivas de (G, X) é par, então $(G, X) \cong (G^c, X)$. Se o número de arestas positivas de (G, X) é ímpar, então $(H, X) \cong (G^c, X)$.*

Esta tese está dividida da forma a seguir. No Capítulo 1 apresentamos algumas definições e notações em teoria de grafos, em especial é apresentada alguma terminologia sobre reconstrução de grafos por sinais. No Capítulo 2 apresentamos o problema de reconstrução de grafos por sinais. No Capítulo 3 apresentamos alguns problemas enumerativos relacionados ao problema de reconstrução de grafos com sinais. No Capítulo 4 nós resolvemos o problema de reconstrução de grafos com sinais para algumas classes de grafos, em especial para árvores, e no Capítulo 5 nós reconstruímos a sequência de pares de graus de grafos com sinais e apresentamos um novo problema de reconstrução de grafos com sinais.

Palavras chave: reconstrução de grafos; grafos com sinais.

ABSTRACT

Let G be a simple graph. Consider the multiset of all *unlabelled* graphs obtained by deleting a vertex v of G together with all the edges incident with v. This multiset is called the collection of vertex-deleted subgraphs of G. The Reconstruction Conjecture asserts that every finite simple graph, with at least three vertices, is determined, up to isomorphism, by its collection of unlabelled vertex-deleted subgraphs. This conjecture was first formulated in 1941, by Kelly and Ulam (see [3]).

We will consider a variation of the reconstruction problem. Let X be a finite simple graph and let G be a spanning subgraph of X. We call (G, X) a signed graph with underlying unsigned graph X, where we consider the edges of G to be the positive edges of (G, X). We denote by $(G, X)_e$ the graph obtained by switching the sign on edge e. The multiset $\{(G, X)_e^* : e \in E(G)\}$ is called the signed deck of (G, X). In this thesis we study the problem of reconstructing signed graphs from their signed deck. We call the problem the sign switching reconstruction problem. We begin by presenting some enumerative results related to the reconstruction of signed graphs. The main result in this part is an analogue of Lovász's result that says that a signed graph that has different number of positive and negative edges is sign switching reconstructible. Next we reconstruct some special classes of graphs, in particular, we prove that trees are sign switching reconstructible. In Chapter 5 we reconstruct the degree pair sequence of signed graphs and present a new reconstruction problem related to signed graphs.

Keywords: graph reconstruction; signed graphs.

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List of Symbols

Structures and families of graphs

$\Delta(G)$ Maximum degree of G

$\delta(G)$ Minimum degree of G

$\emptyset_n$ Empty graph with n vertices

$\deg(v)$ Degree of v in G

$\overline{G}$ Complement of the graph G

$\{u, v\}$ Edge between the vertices u and v

C_n Cycle on n vertices

$d_G(v)$ Degree of a vertex v in G

$E(G)$ Edge set of G

$e(G)$ Number of edges of G

$F \leq G$ F is isomorphic to a subgraph of G

$F \subseteq G$ F is a subgraph of G

$G + H$ Disjoint union of the graphs G and H

$G \cong H$ Graph G is isomorphic to the graph H

$G \cong_{S_n} H$ Graph G is isomorphic to the graph H with respect to the group S_n

$G \cup H$ Union of the graphs G and H

G^* Representative of the isomorphism class of G

K_n Complete graph on n vertices

$K_{m,n}$ Complete bipartite graph with parts of cardinality m and n

P_n Path on n edges

$s(F, G)$ Number of subgraphs of G that are isomorphic to F

$s_e(F, G)$ Number of subgraphs of G that are isomorphic to F that contain a specified edge e of G

uv Edge between the vertices u and v

$V(G)$ Vertex set of G

$v(G)$ Number of vertices of G

Signed graph reconstruction

(a^+, a^-) Ordered pair, where a^+ and a^- are the number of positive and negative edges of A, respectively

$(C, X)^*$ Representative of the set (C, X) in $\mathscr{R}_\Gamma$

$(d^+(v), d^-(v))$ Degree pair of the vertex v

$|F \to T|$ $= |\mathrm{Aut}(F, X)| s(F, T)_X$

$|S \to T|_F$ $= |\{S \to T\}_F|$

$D(G, X)$ Sign switching deck of (G, X)

$D^-(G, X)$ $= \{(G - e, X) \mid e \in E(G)\}$

$D^+(G,X)$ $=\{(G+e,X) \mid e \in E(G)\}$

$(F,X) \subseteq (G,X)$ (F,X) is a subgraph of (G,X)

$(F,Y) \subseteq (G,X)$ (F,Y) is a subgraph of (G,X)

(G^c,X) Signed graph obtained from (G,X) by change the signs of all its edges

(G,X) Signed graph with underlying subgraph X and positive subgraph G

$(G,X) \cong (H,X)$ Signed graph (G,X) is isomorphic to the signed graph (H,X)

$(G,X) \cong_\Gamma (H,Y)$ Signed graph (G,X) is Γ-isomorphic to the signed graph (H,Y)

$(G,X)^*$ Representative of (G,X) in $\mathscr{R}_\Gamma$

$(G,X)_E$ Graph obtained from (G,X) by change the sign of the edges in E

$(G,X)_e$ Graph obtained from (G,X) by change the sign of e

$\mathscr{R}_\Gamma$ Set of all representatives from all Γ-isomorphism classes, when $\Gamma = \mathrm{Aut}(X)$ this is only the set of all representatives with underlying unsigned graph X

$VD^-(G,X)$ $=\{(G-v,X) \mid v \in V(X)\}$

$VD^+(G,X)$ $=\{(G+v,X) \mid v \in V(X)\}$

$\{S \to T\}_F$ $=\{\sigma \in \mathrm{Aut}(X) \mid \sigma(S) \cap T = \sigma(F)\}$

$d(v)$ Degree pair of the vertex v

$d^+(v)$ Positive degree of the vertex v

$d^-(v)$ Negative degree of the vertex v

$E(G,X)$ Edge set of (G,X)

$G \cong_\Gamma H$ Graph G is isomorphic to the graph H with respect to the group $\mathrm{Aut}(X)$

$G \cong_X H$ Graph G is isomorphic to the graph H with respect to the group $\mathrm{Aut}(X)$

G^c Complement of G in X

$i((F,Y) \xrightarrow{\Gamma} (G,X))$ Number of subgraphs of (G,X) that are isomorphic to (F,Y) with respect to Γ.

M Sign switching matrix

m number of edges of G

$n(k,l)$ Number of vertices of degree pair (k,l) in (G,X)

$s(F,G)_X$ Number of spanning subgraphs F' of G such that $(F',X) \cong (F,X)$

$S^1((A,Y) \xrightarrow{\Gamma} (B,Z))$ Cardinality of the set $\{e \in E(Y) \mid (A,Y)_e \cong_\Gamma (B,Z)\}$

$s_e(F,G)_X$ Number of subgraphs of G that are X-isomorphic to F and contain a specified edge e of G

$t((F,Y) \xrightarrow{\Gamma} (G,X))$ Number of occurrences of (F,Y) as induced subgraph of (G,X) with respect to Γ.

$V(G,X)$ Vertex set of (G,X)

Group notation

$\mathrm{Aut}(F)$ Group of automorphism of F

$\mathrm{Aut}(G,X)$ $= \{\pi \in \mathrm{Aut}(X) : \pi(G) = G\}$

Γ Permutation group of $V(X)$, in this work is $\mathrm{Aut}(X)$

S_n Symmetric group on n elements

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Introduction

In this work all graphs are finite and simple. Let G be a graph. We denote the number of vertices of G by $v(G)$ and the number of edges of G by $e(G)$. Consider the multiset of all *unlabelled* graphs obtained by deleting a vertex v of G together with all the edges incident with v. This multiset is called the **collection of vertex-deleted subgraphs** or the **deck** of G. The (vertex) reconstruction conjecture 0.0.1 was first formulated in 1941 by Kelly and Ulam (see [3]).

Conjecture 0.0.1 (Reconstruction conjecture)**.** *Every finite simple graph, with at least three vertices, is determined, up to isomorphism, by its collection of unlabelled vertex-deleted subgraphs.*

A graph H with the same deck as that of G is called a **reconstruction** of G. If each reconstruction of G is isomorphic to G, we say that G is **reconstructible**. We say that a parameter or an invariant of G is **reconstructible** if the parameter or invariant takes the same value for each reconstruction of G. Some important results in reconstruction are next stated.

Lemma 0.0.2 (Kelly, 1957)**.** *For any two graphs F and G such that $v(F) < v(G)$, the number of subgraphs of G that are isomorphic to F is reconstructible.*

Theorem 0.0.3 (Kelly, 1957)**.** *The degree sequence of a graph with more than 2 vertices is reconstructible.*

Theorem 0.0.4 (Kelly, 1957)**.** *Trees with at least 3 vertices are reconstructible.*

Theorem 0.0.5 (Kelly, 1957)**.** *Disconnected graphs with at least 3 vertices are reconstructible.*

We know that some other classes of graphs are reconstructible, for example, the regular graphs [12], the outerplanar graphs [9], graphs with at most nine vertices and maximal planar graphs (see [5]). Many graph invariants are known to be reconstructible, for example, the dichromatic polynomial, Tutte polynomial, the characteristic polynomial [25], the number of Hamilton cycles, the degree of deleted vertex and the degrees of its neighbours are reconstructible (see Bondy and Hemminger [3]).

In 1964, Harary proposed the edge reconstruction conjecture, analogous to the vertex reconstruction conjecture 0.0.1, which conjecture is weaker than the vertex reconstruction conjecture.

Consider the multiset of all *unlabelled* graphs obtained by deleting an edge e of G. This multiset is called the **collection of edge-deleted subgraphs** or the **edge deck** of G.

Conjecture 0.0.6 (Edge reconstruction conjecture)**.** *A graph with at least four edges is determined, up to isomorphism, by its collection of unlabelled edge-deleted subgraphs.*

A graph H is called an **edge reconstruction** of G if H has the same edge deck as G. We say that a graph G is **edge reconstructible** if all edge reconstructions of G are isomorphic to G. We say that a parameter or an invariant of G is **edge reconstructible** if the parameter or invariant takes the same value for all edge reconstructions of G.

It is known that if a graph G has at least 4 edges, then if G is vertex reconstructible, then it is also edge reconstructible, and if an invariant of G is vertex reconstructible, then it is also edge reconstructible (see [10]). Additionally, bidegreed graphs [19], claw-free graphs [6], chordal graphs [24], maximal planar graphs [8], and planar graphs with minimum degree 5 are edge reconstructible [13]. The end-vertex degrees of the deleted edge for each graph in the edge deck are edge reconstructible, see the surveys of Bondy [2] and Maccari et al. [15]. There are some important results about edge reconstruction, Nash-Williams [20] give one of these important results.

Let G and H be graphs at the same vertex set and let F be a subgraph of G. We define $\{G \to H\}_F \coloneqq |\{\sigma \in S_n \mid \sigma(G) \cap H = \sigma(F)\}|$.

Lemma 0.0.7 (Nash-Williams, 1978)**.** *Let G be a graph and let F be a spanning subgraph of G. If H is an edge reconstruction of G such that $G \ncong H$, then*

$$\{G \to G\}_F - \{G \to H\}_F = (-1)^{e(G)-e(F)}\, \mathrm{aut}(G).$$

Other important results were presented by Lovász [14] and Müller's [18], both can be seen as consequences of Nash-Williams' Lemma, but was found independently. These results are presented in Theorems 0.0.8 and 0.0.9.

Theorem 0.0.8 (Lovász , 1972)**.** *A graph G is edge reconstructible if* $e(G) > \frac{1}{2}\binom{v(G)}{2}$.

Theorem 0.0.9 (Müller , 1977)**.** *A graph G is edge reconstructible if* $2^{e(G)-1} > v(G)!$.

Another problem about reconstruction was proposed in 1987 by Mnukhin [16] he proved that orbits are reconstructible from its suborbits. Using this idea he also proved that necklaces are reconstructible and presented another point of view of the edge reconstruction problem and a generalisation of Nash-Williams' Lemma.

In 1985, Stanley [22] proposed the vertex-switching reconstruction problem. Let v be a vertex of G, the graph G_v, obtained from G by deleting all edges incidents to v and adding edges joining v to every vertex not adjacent to v in G, is called a **vertex-switching**. The multiset $\{G_v^* : v \in V(G)\}$ of unlabelled graphs is called the **switching deck** of G.

Conjecture 0.0.10 (Vertex-switching reconstruction conjecture)**.** *Every graph, with at least five vertices is determined, up to isomorphism, by its switching deck.*

A graph H with the same switching deck as that of G is called a **switching reconstruction** of G. If each switching reconstruction of G is isomorphic to G, we say that G is **switching reconstructible**. We say that a parameter or an invariant of G is **switching reconstructible** if the parameter or invariant takes the same value for each switching reconstruction of G. In the results about switching reconstruction we will always consider $v(G) > 5$.

It is known that regular graphs [7], triangle-free graphs [7], disconnected graphs with at least two nontrivial components [7] are switching reconstructible. We also know that some parameters are reconstructible, as the degree sequence [22]. In a more general way Stanley [22] has shown that if $v(G) \not\equiv 0 \pmod 4$, then G is vertex-switching reconstructible.

In this work we will consider the following variation of the reconstruction problems.

Let X be a finite set and let G be a group of permutations of X. Consider a k-colouring of X, that is, a function $f\colon X \to \{1,\ldots,k\}$. We say that two colourings f and g of X are equivalent with respect to G if one of them can be obtained from the other by an application a permutation in G, i.e., if there exists $\pi \in G$ such that for all $x \in X$, we have $g(x) = f(\pi(x))$.

If the group is cyclic or dihedral, we obtain a necklace, and when $k = 2$, we think of a necklace with 2 types of beads, red and green. This is the problem proposed by Mnukhin [17].

Let X be the set of edges of a graph Γ, and $k = 2$, and let G be the automorphism group of Γ. In this case, we think of Γ as a signed graph with + or - signs on its edges.

Associated with these structures, we define a reconstruction problem, where a deck is obtained by switching the signs in a signed graph. This reconstruction problem is presented in Chapter 2. The above variation of reconstruction problems have features of both edge reconstruction problem and the switching reconstruction problem.

In Chapter 1 we give some definitions and notation in graph theory, in special about reconstruction of signed graphs. In Chapter 2 we present the definition of the problem of reconstruction of signed graphs. In Chapter 3 we will present some enumerative problems about reconstruction of signed graphs, in Chapter 4 we will solve problem of reconstruction of signed graphs for some classes of graphs and in Chapter 5 we will reconstruct a parameter of signed graphs.

Chapter 1

Basic definitions and notations

1.1 Definitions and notation

We follow Bondy [4] for most of the graph theoretic definitions. Here we give a few selected definitions and fix the notation used in the thesis.

In this work G represent a simple graph, while $V(G)$ denote the vertex set of G and $E(G)$ the edge set of G. We can denote an edge of G by uv or $\{u, v\}$. Also, we consider only finite simple graphs. The number of vertices in G is denoted by $v(G)$, while $e(G)$ represent the number of edges in G. The **degree** of v in G, denoted by $d_G(v)$ or deg(v), is the number of edges incidents to v in G. We denote by $\delta(G) = \delta$ and $\Delta(G) = \Delta$ the minimum and maximum degree of G. A graph whose vertices only have exactly two degrees, δ and Δ, is called a **bidegreed graph**.

We say that a graph F is a **subgraph** of G if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. The expression $F \subseteq G$ means that F is a subgraph of G. The expression $F \leq G$ denote that there is at least one subgraph of G that is isomorphic to F. The number of subgraphs of G that are isomorphic to F is represented by $s(F, G)$. We say that a subgraph F of G is **spanning** if it have the same vertex set as G and $E(F) \subseteq E(G)$. We say that a subgraph F of G is **edge spanning** if F is a spanning subgraph of G and has the same number of isolated vertices as G. The symbol K_n represents a complete graph on n vertices.

A complete graph on three vertices is often called a **triangle**. A graph which contains no triangle is called **triangle-free**. An **empty graph** is one which no two vertices are adjacent, the symbol $\emptyset_n$ represents the empty graph with n vertices, if the number of vertices is clear we can omit the number n and only write $\emptyset$. The **complement** of a graph G, denoted by $\overline{G}$, is the graph with same vertex set of G and whose edges are the pair of nonadjacent vertices of G. A **star** is a graph whose vertex graph can be partitioned into two subsets, one of it with only one vertex. A star with n vertices of degree 1 is denoted by $K_{1,n}$. We say that a graph S is **claw-free** if there is no induced subgraph of S that is isomorphic to $K_{1,3}$. A **cycle** is a graph whose vertices can be arranged in a cyclic sequence, in such manner two vertices are adjacent if they are consecutive in this sequence. The symbol C_n represents a cycle on n vertices. A **chordless cycle** is an induced subgraph of G isomorphic to a cycle of length four or more. A **chordal graph** is a graph that have no chordless cycles. We denote by $G \cup H$ the union o the graphs G and H, while $G + H$ denote the disjoint union. If $V = V(G) = V(H)$, the intersection $G \cap H$ of the graphs G and H is the graph with vertex set V and edge set $E(G) \cap E(H)$. A connected graph without cycles is called a tree. A tree can be central or bicentral, that is can have exactly one or two central vertices. If the tree is central, we define a **branch** of a tree as a maximal subtree that contains a single edge incident with the centre such that this vertex has degree 1 in this subtree. And we define a **branch** of a tree as a maximal subtree that contains a single vertex of the centre such that this vertex has degree 1 in this subtree, if the tree is bicentral.

Two graphs G and H are **isomorphic** if there exists a bijection $\psi\colon V(G) \to V(H)$ such that u and v are adjacent in G if and only if $\psi(u)$ and $\psi(v)$ are adjacent in H. In this case ψ is called an isomorphism; we denote the fact that G is isomorphic to H by $G \cong H$ or by $G \cong_{S_n} H$. An isomorphism of a graph to itself is called an **automorphism**. The relation of isomorphism is an equivalence relation. We will denote a representative of an isomorphism class by G^*. We define an **unlabelled graph** as this representative of the isomorphism class.

1.2 Definitions on signed graphs

Let X be a finite simple graph. Let $g\colon E(X) \to \{+,-\}$. Let G be the spanning subgraph of X consisting of all positive edges. We call (G, X) a **signed graph** with **underlying unsigned graph** X. We will denote by G^c the complement of G in X, that is, the spanning subgraph of X consisting of all negative edges. We denote by (G^c, X) the signed graph obtained from (G, X) by switch the signs of all its edges. We say that two signed graphs (G, X) and (H, Y) are **isomorphic** if there is an isomorphism $f\colon V(X) \to V(Y)$ such that $f(G) = H$ and $f(G^c) = H^c$ we write $(G, X) \cong (H, Y)$. The isomorphism is an equivalence relation and divide the signed graphs into classes. We will denote a representative of an isomorphism class by $(G, X)^*$. We define a **unlabelled signed graph** as this representative of the isomorphism class.

Let $\mathrm{Aut}(G, X)$ represent the subgroup of $\mathrm{Aut}(X)$ which fixes (G, X), i.e., the group with components $\{\pi \in \mathrm{Aut}(X) : \pi(G) = G\}$. We define $|\mathrm{Aut}(G, X)|$ as the cardinality of this subgroup.

We can generalise the definition of signed graphs to a general group of permutations of V. We will assume that all signed graphs under consideration have the same vertex set V, but sometimes we will refer to the vertex set of (G, X) as $V(G, X)$. Let Γ be a group of permutations of V. For signed graphs (G, X) and (H, Y) on the vertex set V, we say that (G, X) is **Γ-isomorphic** to (H, Y) if there is a permutation π in Γ that is an isomorphism from (G, X) to (H, Y). We write $(G, X) \cong_\Gamma (H, Y)$. If (G, X) and (H, X) are Γ-isomorphic we can say that G is **Γ-isomorphic** to H. We write $G \cong_\Gamma H$. The relation of Γ-isomorphism partitions all signed graphs on V into isomorphism classes. For a signed graph (G, X), we denote by $(G, X)^*$ a fixed representative of the Γ-isomorphism class of (G, X), and by $\mathscr{R}_\Gamma$ we denote the set of representative signed graphs from all Γ-isomorphism classes. In the case where $\Gamma = \mathrm{Aut}(X)$ we say that G is **X-isomorphic** to H if there is a permutation in $\mathrm{Aut}(X)$ that is an isomorphism from G to H and write $G \cong_X H$. In this case $\mathscr{R}_X$ denote the set of representative signed graphs from all X-isomorphism classes, note that this is only the set of all representatives that have underlying unsigned graph X.

Let F and G be spanning subgraphs of X. We say that (F, X) is a **subgraph** of

(G, X), and represent this by $(F, X) \subseteq (G, X)$, if there is an automorphism σ of X such that $\sigma(F) \subseteq G$. Let (F, X) and (G, X) be signed graphs. We define $s(F, G)_X$ to be the number of spanning subgraphs F' of G such that $(F', X) \cong (F, X)$. We have

$$s(F, G)_X = |\{\sigma \in \mathrm{Aut}(X) \mid \sigma(F) \subseteq G\}| / \mathrm{Aut}(F, X).$$

We will denote by $s_e(F, G)$ the number of subgraphs of G that are isomorphic to F that contain a specified edge e of G, and by $s_e(F, G)_X$ the number of subgraphs of G that are X-isomorphic to F and contain a specified edge e of G.

We say that (F, Y) is a **subgraph** of (G, X) if Y is a subgraph of X and F is a subgraph of G, and represent this by $(F, Y) \subseteq (G, X)$.

Let $v_1, v_2, \ldots, v_n$ be the vertices of (G, X). Let $d^+(v_i)$ be the degree of the vertex v_i in G and let $d^-(v_i)$ be the degree of the vertex v_i in G^c. We call the pair $d(v_i) = (d^+(v_i), d^-(v_i))$ the **degree pair** of the vertex v_i. We define the **degree pair sequence** as the sequence of pairs

$$((d^+(v_1), d^-(v_1)), (d^+(v_2), d^-(v_2)), \ldots, (d^+(v_n), d^-(v_n))).$$

We define the sequence $(d^+(v_1), d^+(v_2), \ldots, d^+(v_n))$ as the **positive degree sequence** and the sequence $(d^-(v_1), d^-(v_2), \ldots, d^-(v_n))$ as the **negative degree sequence**. When the vertices are not ordered, the terms degree sequence, degree pair sequence, etc. refer to the multiset of degrees or degree pairs.

Chapter 2

Sign switching reconstruction problems

In this chapter we define the problem of reconstruction for signed graph and present some counter examples for this reconstruction problem.

2.1 Problem definition

Denote by $(G,X)_e$ the graph obtained by switching the sign on e. When e is a positive edge, we write $(G-e,X) := (G,X)_e$, and when e is a negative edge, we write $(G+e,X) := (G,X)_e$. Let (G,X) and (H,Y) be signed graphs. Suppose that there is a bijection $f\colon E(X) \to E(Y)$ such that for all $e \in E(X)$, we have $(G,X)_e \cong (H,Y)_{f(e)}$. We call f a **sign switching hypomorphism** from (G,X) to (H,Y), and say that (G,X) and (H,Y) are **sign switching hypomorphic**. Since X and Y are required to be isomorphic in the above definition, we may consider G and H to be spanning subgraphs of the same underlying unsigned graph X; hence all isomorphisms between $(G,X)_e$ and $(H,X)_{f(e)}$ are elements of $\mathrm{Aut}(X)$.

The above notions may be defined equivalently as follows.

The **sign switching deck** of (G,X), denoted by $D(G,X)$, is the multiset of unlabelled graphs $\{(G,X)_e^* \mid e \in E(G)\}$. Each unlabelled graph $(G,X)_e^*$ in the sign switching deck of (G,X) is called a **card**. Note that two graphs (G,X) and (H,X)

have the same sign switching deck if and only if they are sign switching hypomorphic. In this case (H, X) is called a **sign switching reconstruction** of (G, X). We say that a given signed graph (G, X) is **sign switching reconstructible** if every sign switching reconstruction of (G, X) is isomorphic to (G, X). Furthermore we say that an invariant or a parameter of (G, X) is sign switching reconstructible if it takes the same value on every sign switching reconstruction of (G, X). In Chapters 3 and 4, since we are working only with sign switching reconstruction problems, we will omit the words sign switching when we use the previous terms in referenced chapters. The following examples show pairs of nonisomorphic signed graphs with the same sign switching deck.

Example 2.1.1. If $X \cong K_{1,2}$ or $X \cong K_2 + K_2$, then $(G, X) := (X, X)$ and $(H, X) := (G^c, X)$ have the same sign switching deck, but are not isomorphic. See Figure 2.1.

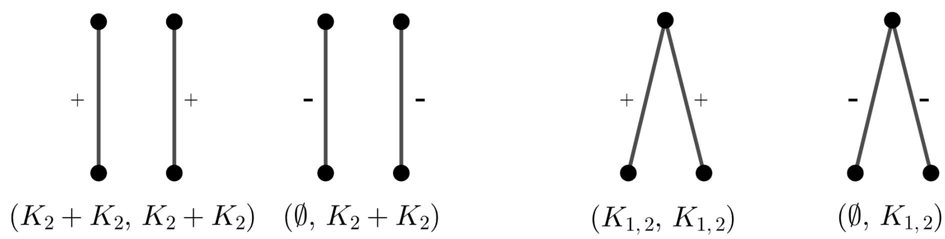


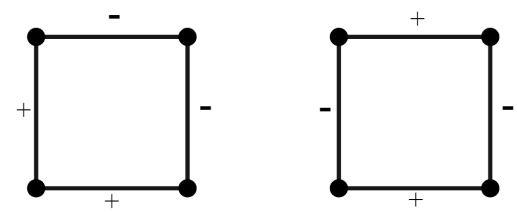
Figure 2.1: Nonreconstructible signed graphs with $e(X) = 2$

Example 2.1.2. Let $X \cong C_4$. Let G be a subgraph of X isomorphic to $K_{1,2} + K_1$ and let H be a subgraph of X isomorphic to $K_2 + K_2$. Then (G, X) and (H, X) have the same sign switching deck, but are not isomorphic. See Figure 2.2.

Example 2.1.3. Let $X \cong K_4$. Let G be a subgraph of X isomorphic to $K_{1,3}$ and let H be a subgraph of X isomorphic to $K_3 + K_1$. Then (G, X) and (H, X) have the same sign switching deck, but are not isomorphic. See Figure 2.3.

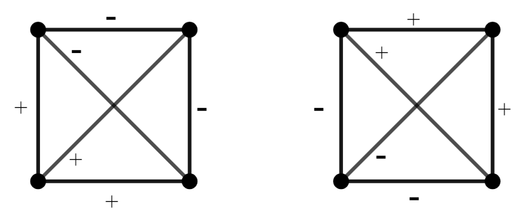
Example 2.1.4. Let $X \cong K_{1,2} + K_{1,2}$. Let G be a subgraph of X isomorphic to $K_{1,2}$ and let H be a subgraph of X isomorphic to $K_2 + K_2$. Then (G, X) and (H, X) have the same sign switching deck, but are not isomorphic. See Figure 2.4.

Example 2.1.5. Let $X \cong P_4$. Let G be a subgraph of X isomorphic to P_2 and let H be a subgraph of X isomorphic to $P_1 + P_1$. Then (G, X) and (H, X) have the same sign switching deck, but are not isomorphic. See Figure 2.5.



$(K_{1,2} + K_1,\ C_4)$ $(K_2 + K_2,\ C_4)$

Figure 2.2: Nonreconstructible signed graphs with $X \cong C_4$



$(K_{1,3},\ K_4)$ $(K_3 + K_1,\ K_4)$

Figure 2.3: Nonreconstructible signed graphs with $X \cong K_4$

$(K_{1,2} + \emptyset_3,\ K_{1,2} + K_{1,2})$ $(K_2 + K_2 + \emptyset_2,\ K_{1,2} + K_{1,2})$

Figure 2.4: Nonreconstructible signed graphs with $X \cong K_{1,2} + K_{1,2}$

$(K_{1,2} + \emptyset_2,\ P_4)$ $(K_2 + K_2 + K_1,\ P_4)$

Figure 2.5: Nonreconstructible signed graphs with $X \cong P_4$

Remark 2.1.6. The above examples are related to the known examples of pairs of nonisomorphic graphs with the same edge deck. The graphs $K_{1,2} + K_1$ and $K_2 + K_2$ have the same edge deck; similarly the graphs $K_{1,3}$ and $K_3 + K_1$ have the same edge deck. Here we have a *packing* of these pairs in X.

These are the only examples of pairs of signed graphs with the same sign switching deck that we know.

Remark 2.1.7. A signed graph (G, X) is sign switching reconstructible if and only if (G^c, X) is also sign switching reconstructible, since (G, X) and (H, X) have the same sign switching deck if and only if (G^c, X) and (H^c, X) have the same sign switching deck, and (G, X) and (H, X) are isomorphic if and only if (G^c, X) and (H^c, X) are isomorphic.

The graphs in Example 2.1.1 are the only examples of nonreconstructible signed graphs for which the number of positive (or negative) edges is not a reconstructible parameter, which is shown in Lemma 2.1.8.

Lemma 2.1.8. *Suppose that $e(X) > 2$ and suppose that (H, X) is a sign switching reconstruction of (G, X).*

1. *If G or G^c is the empty subgraph of X, then $(G, X) \cong (H, X)$.*
2. *We have $e(G) = e(H)$ and $e(G^c) = e(H^c)$.*
3. *If $f\colon E(X) \to E(X)$ is a sign switching hypomorphism from (G, X) to (H, X), then for all $e \in E(X)$, the edges e and $f(e)$ have the same sign.*

Proof.

1. Since $e(X) > 2$, we have $e(G) = 0$ and $e(G^c) = e(X)$ if and only if each graph in the sign switching deck has exactly one positive edge and $e(X) - 1$ negative edges. In this case (G, X) is obtained by switching the sign on the unique positive edge in any graph in the sign switching deck. The same argument works when $e(G) = e(X)$ and $e(G^c) = 0$.
2. Now we assume that both $e(G)$ and $e(G^c)$ are positive. In this case, $D(G, X)$ contains exactly $e(G)$ graphs with $e(G) - 1$ positive edges and $e(G^c) + 1$ negative edges, and $e(G^c)$ graphs with $e(G) + 1$ positive edges and $e(G^c) - 1$ negative edges. Therefore, $e(G)$ and $e(G^c)$ are reconstructible, i.e., $e(G) = e(H)$ and $e(G^c) = e(H^c)$.

3. Suppose, for contradiction, that $e \in E(G)$ and $f(e) \in E(H^c)$. Then we have $(e(G)-1, e(G^c)+1) = (e(H)+1, e(H^c)-1)$, which implies that $e(G) = e(H)+2$ and $e(G^c) = e(H^c) - 2$, which contradicts the second part. $\square$

We will write $m = e(G)$ and $m^c = e(G^c)$. We define the multisets

$$D^+(G, X) \coloneqq \{(G, X)_e \mid e \in E(G)\} = \{(G + e, X) \mid e \in E(G)\}$$

and

$$D^-(G, X) \coloneqq \{(G, X)_e \mid e \in E(G^c)\} = \{(G - e, X) \mid e \in E(G^c)\},$$

hence $D(G, X) = D^+(G, X) \cup D^-(G, X)$, which is a multiset union. We call $D^+(G, X)$ the **positive sign switching deck** of (G, X) and we call $D^-(G, X)$ the **negative sign switching deck** of (G, X).

Lemma 2.1.9. *Suppose that* $e(X) > 2$. *Then* $D^-(G, X)$ *is sign switching reconstructible.*

Proof. We have that m and m^c are sign switching reconstructible, then the number of signed graphs in $D^-(G, X)$ is also sign switching reconstructible. Since $D(G, X) = D^+(G, X) \cup D^-(G, X)$ is a disjoint union, thus $D^-(G, X)$ is sign switching reconstructible. $\square$

We will say that a signed graph is D^--reconstructible, when the signed graph is reconstructible from $D^-(G, X)$, and we will say that a signed graph is D-reconstructible, when the signed graph is reconstructible from $D(G, X)$. For any result that is valid for D^-, there is an analogous result for D^+. We now have the following formulation of the sign switching reconstruction problem.

Problem 2.1.10 (Suggested by Ilia Krasikov)**.** *Suppose that* $e(X) > 6$. *If* (H, X) *is a sign switching reconstruction of* (G, X), *then is* $(G, X) \cong (H, X)$*?*

Adding isolated vertices to X does not make any difference to the sign switching reconstructibility, hence we will consider only signed graphs such that there are no isolated vertices in X and $e(X) > 2$.

Chapter 3

Enumerative methods for sign switching reconstruction

In this chapter, we assume that (G, X) is a graph to be reconstructed from its sign switching deck, and (H, X) is one of its reconstructions. We will present some enumerative methods related to the problem of sign switching reconstruction. These enumerative methods give us analogues of Kelly, Nash-Williams, Lovász and Müller's results. An important result in this chapter is related to the parity of the edges of G. We prove that if $e(G)$ is even, then $(G, X) \cong (G^c, X)$, and if $e(G)$ is odd, then $(H, X) \cong (G^c, X)$.

3.1 Kelly's lemma for sign switching reconstruction

Recall that $s_e(F, G)$ is the number of subgraphs of G that are isomorphic to F that contain a specified edge e of G, and $s_e(F, G)_X$ is the number of subgraphs of G that are X-isomorphic to F that contain a specified edge e of G.

Lemma 3.1.1 (First version of Kelly's Lemma for signed graphs)**.** *Let (F, X) and (G, X) be signed graphs. Suppose that $e(F) < e(G)$. Then*

1. *$s(F, G)_X$ is reconstructible from $D^-(G, X)$, and hence also from $D(G, X)$;*

2. *$s_e(F,G)_X$ is reconstructible from $D^-(G,X)$, and hence also from $D(G,X)$.*

Proof. The proof of this lemma is similar to the proof of Kelly's lemma for edge reconstruction. Let F' be a subgraph of G that is X-isomorphic to F. If e is a positive edge not in F', then F' is a subgraph of $G-e$, and is X-isomorphic to F. Therefore,

$$s(F,G)_X = \frac{\sum_{e\in E(G)} s(F,G-e)_X}{e(G)-e(F)},$$

which is reconstructible from $D^-(G,X)$. Now $s(F,G)_X$ is also reconstructible from $D(G,X)$, since $D^-(G,X)$ is reconstructible from $D(G,X)$ when $e(X) > 2$ from Lemma 2.1.9.

Now $s(F,G)_X - s(F,G-e)_X$ is reconstructible from $D^-(G,X)$, and also from $D(G,X)$ when $e(X) > 2$, which implies the second part. □

Proposition 3.1.2. *The positive degree sequence is D^--reconstructible if $m > 3$. Moreover, for each graph $(G-e,X)$ in $D^-(G,X)$, where $e := v_1v_2$ of G, the unordered pair (multiset) $\{d^+_{v_1}, d^+_{v_2}\}$ is D^--reconstructible.*

Proof. The positive degree sequence and the pair of degrees of the end points of e are determined as in edge reconstruction (see Lemma 1.3 in Greenwell and Hemminger [11]). □

Corollary 3.1.3. *Whether G is an edge spanning subgraph of X is determined by $D^-(G,X)$.*

Proof. From Lemma 1.3 of Greenwell and Hemminger [11] we have that the number of isolated vertices is edge reconstructible. This implies we can determine whether G is an edge spanning subgraph of X from D^-. □

3.2 Lemmas of Nash-Williams, Lovász and Müller for sign switching reconstruction

Now we prove a result analogous to Nash-Williams' lemma (Lemma 0.0.7) with techniques from Alon et al [1]. For that, we use the following notation.

Notation. Let F, S, T be spanning subgraphs of X, where $F \subseteq S$. We define

$$\{S \to T\}_F := \{\sigma \in \mathrm{Aut}(X) \mid \sigma(S) \cap T = \sigma(F)\},$$

$$|S \to T|_F := |\{S \to T\}_F|, \tag{3.1}$$

$$|F \to T| := |\{\sigma \in \mathrm{Aut}(X) \mid \sigma(F) \subseteq T\}| = |\mathrm{Aut}(F, X)| s(F, T)_X. \tag{3.2}$$

From Equations 3.1 and 3.2, we have

$$\begin{aligned}\sum_{F' \mid F \subseteq F' \subseteq S} |S \to T|_{F'} &= \sum_{F' \mid F \subseteq F' \subseteq S} |\{\sigma \in \mathrm{Aut}(X) \mid \sigma(S) \cap T = \sigma(F')\}| \\ &= |\{\sigma \in \mathrm{Aut}(X) \mid \sigma(F) \subseteq T\}| \\ &= |F \to T|. \end{aligned} \tag{3.3}$$

Lemma 3.2.1 (Nash-Williams's lemma for signed graphs)**.** *Let (H, X) be a reconstruction of (G, X) such that $(G, X) \ncong (H, X)$. Let F be a spanning subgraph of G. We have*

$$|G \to G|_F - |G \to H|_F = (-1)^{e(G)-e(F)} |\mathrm{Aut}(G, X)|. \tag{3.4}$$

Proof. From Equations 3.2 and 3.3, we obtain

$$\sum_{F' \mid F \subseteq F' \subseteq G} |G \to H|_{F'} = |\mathrm{Aut}(F, X)| s(F, H)_X.$$

Applying Möbius inversion formula, we obtain

$$|G \to H|_F = \sum_{F' | F \subseteq F' \subseteq G} (-1)^{e(G)-e(F)} |\mathrm{Aut}(F', X)| s(F', H)_X.$$

Hence

$$|G \to G|_F - |G \to H|_F = \sum_{F' | F \subseteq F' \subseteq G} (-1)^{e(G)-e(F)} |\mathrm{Aut}(F', X)| \left(s(F', G)_X - s(F', H)_X \right). \tag{3.5}$$

By Lemma 3.1.1, we have $s(F', G)_X - s(F', H)_X = 0$ for all proper subgraphs F' of G. We have $s(G, G)_X = 1$ and $s(G, H)_X = 0$ (since (H, X) is a nonisomorphic reconstruction of (G, X)). Hence

$$\begin{aligned} |G \to G|_F - |G \to H|_F &= (-1)^{e(G)-e(F)} |\mathrm{Aut}(G, X)| \left(s(G, G)_X - s(G, H)_X \right). \\ &= (-1)^{e(G)-e(F)} |\mathrm{Aut}(G, X)|. \end{aligned}$$

□

Corollary 3.2.2. *If $e(G) - e(F)$ is even, then $|G \to G|_F$ is positive, which implies that there exists $(G', X) \cong (G, X)$ such that $G' \cap G = F$. Similarly, if $e(G) - e(F)$ is odd, then $|G \to H|_F$ is positive, which implies that there exists $(H', X) \cong (H, X)$ such that $H' \cap G = F$.*

Remark 3.2.3. Lemma 3.2.1 and Corollary 3.2.2 are valid for $D^-(G, X)$. But, if we have $D(G, X)$, then we have a similar result for G^c.

Now, as corollaries, we have results similar to the results of Lovász and Müller (Corollaries 3.2.4 and 3.2.10 respectively).

Corollary 3.2.4 (Lovász's lemma for signed graphs)**.** *Let (G, X) be a signed graph.*

1. *If $m > m^c$, then (G, X) is reconstructible from $D^-(G, X)$.*
2. *If $m \neq m^c$, then (G, X) is reconstructible from $D(G, X)$.*

Proof. 1. Let (G, X) be such that $m > m^c$. We take F as an empty subgraph of G. Now for any spanning subgraph T of X, we have $|G \to T|_F = |G \to T^c|$. Since

$m > m^c$, taking T to be G or H, we have $|G \to G^c| = |G \to H^c| = 0$, which contradicts Equation 3.4 since the right side of Equation 3.4 is nonzero. Hence (G, X) is reconstructible from $D^-(G, X)$.

2. If $D(G, X)$ is given and $m \neq m^c$, then if $m > m^c$ then we apply part 1 to (G, X), and if $m < m^c$ then we apply part 1 to (G^c, X). Thus (G, X) is reconstructible from $D(G, X)$. □

The next result is an immediate consequence of Corollary 3.2.4, but it can be proved in a constructive way using a technique of a proof presented by Alon (see Stanley [22]).

Corollary 3.2.5. *If $e(X)$ is odd, then (G, X) is reconstructible from $D(G, X)$.*

Remark 3.2.6. From now on, we will assume that $m = m^c$ when we are considering the problem of reconstruction from $D(G, X)$, and we will assume that $m \leq m^c$ when we are considering the problem of reconstruction from $D^-(G, X)$.

Corollary 3.2.7. *Suppose that (G, X) is not D-reconstructible. If m is odd, then $(H, X) \cong (G^c, X)$, and if m is even, then $G \cong_X G^c$.*

Proof. We take F to be an empty subgraph of G, and apply Equation 3.4. We have

$$|G \to G|_F - |G \to H|_F = |G \to G^c| - |G \to H^c| = (-1)^m |\mathrm{Aut}(G, X)|.$$

If m is odd, then $|G \to H^c|$ is positive, and since we have assumed that $m = m^c$, we have $(H, X) \cong (G^c, X)$. Similarly, if $m = m^c$ is even, then $|G \to G^c|$ is positive, hence $G \cong_X G^c$. □

Remark 3.2.8. If G is edge reconstructible, then $H \cong_{S_n} G$, which implies that $G^c \cong_{S_n} G$, but G^c may not be X-isomorphic to G.

Corollary 3.2.9. *Let (G, X) be a signed graph, with m even. Suppose that (G, X) is not D-reconstructible. If a vertex has degree pair (k, l), then there is another vertex in (G, X) with degree pair (l, k) or $k = l$.*

Proof. Since m is even, we have $G \cong_X G^c$. Let u be a vertex of degree pair (k, l), then there is an isomorphism $\pi \in \mathrm{Aut}(X)$ such that $\pi(G) = G^c$ and $\pi(u) = v$, where the degree pair of v is (l, k) or u remains fixed by π. In the last case $k = l$. $\square$

Corollary 3.2.10 (Müller's result for signed graphs)**.** *Let* (G, X) *be a signed graph. If* $2^{m-1} > |\mathrm{Aut}(X)|$*, then* (G, X) *is reconstructible from* $D^-(G, X)$*.*

Proof. We will apply Corollary 3.2.2 to all proper subgraphs A of G such that $m - e(A)$ is even. There are 2^{m-1} such subgraphs. Thus there are at least 2^{m-1} copies of G, i.e., signed graphs $(G', X) \cong (G, X)$, such that $G' \cap G = A$. But there are precisely $\frac{|\mathrm{Aut}(X)|}{|\mathrm{Aut}(G,X)|}$ signed graphs isomorphic to (G, X). So, if $2^{m-1} > \frac{|\mathrm{Aut}(X)|}{|\mathrm{Aut}(G,X)|}$, then (G, X) is reconstructible from $D^-(G, X)$. In particular, if $2^{m-1} > |\mathrm{Aut}(X)|$, then (G, X) is reconstructible from $D^-(G, X)$. $\square$

3.3 Kelly's Lemma for signed graphs

The main goal of this section is to prove a version of Kelly's Lemma related to the number of subgraphs isomorphic to (F, Y). This version says that the number of subgraphs of (G, X), with less than m edges, X-isomorphic to a given signed graph (F, Y) is reconstructible. With this result we will be able to prove that some special class of signed graphs are reconstructible. This version of Kelly's Lemma is an analogue of a result of Ellingham and Royle [7], we will prove this result by using a similar technique of the paper of Ellingham and Royle [7]. First, we will need some definitions. In this section, we will consider all graphs to be on the same vertex set V, unless stated otherwise.

Recall that $\mathscr{R}_\Gamma$ is the set of representative from all Γ-isomorphism classes of signed graphs. We say that two signed graphs (A, Y) and (B, Z) are **switching equivalent** if there exists $E \subseteq E(Y)$ such that $(A, Y)_E \cong_\Gamma (B, Z)$, where $(A, Y)_E$ is the graph obtained by switching the signs of all edges in E. Switching equivalence is an equivalence relation on the set of signed graphs on V. Note that Γ-isomorphism is a finer relation on the the set of signed graphs on V. We take the **switching equivalence class** of a graph (F,Y) to mean the set of representatives of isomorphism classes of graphs that are switching equivalent to (F,Y).

In this thesis, we assume that $\Gamma = \mathrm{Aut}(X)$.

We define

$$\begin{aligned} i((F,Y) \xrightarrow{\Gamma} (G,X)) &\coloneqq |\{(B,Z) \mid (B,Z) \subseteq (G,X) \text{ and } (B,Z) \cong_\Gamma (F,Y)\}| \\ S^1((A,Y) \xrightarrow{\Gamma} (B,Z)) &\coloneqq |\{e \in E(Y) \mid (A,Y)_e \cong_\Gamma (B,Z)\}| \text{ and} \\ t((F,Y) \xrightarrow{\Gamma} (G,X)) &\coloneqq \sum_{e \in E(X)} i((F,Y) \xrightarrow{\Gamma} (G,X)_e). \end{aligned}$$

Lemma 3.3.1. *For any signed graph (F,Y) such that $e(Y) = k$, we have*

$$\begin{aligned} t((F,Y) \xrightarrow{\Gamma} (G,X)) &= (e(X)-k)i((F,Y) \xrightarrow{\Gamma} (G,X)) \\ &\quad + \sum_{(B,Z)\in\mathscr{R}_\Gamma} S^1((B,Z) \xrightarrow{\Gamma} (F,Y))i((B,Z) \xrightarrow{\Gamma} (G,X)). \end{aligned} \tag{3.6}$$

Proof. We will count the number of ways in which a signed graph that is Γ-isomorphic to (F,Y) can appear as a subgraph of some graph in $D(G,X)$. Suppose that $(B,Z) \subseteq (G,X)$ is Γ-isomorphic to (F,Y). If an edge e not in (B,Z) is switched, then (B,Z) appears as a subgraph in $(G,X)_e$, and continues to be Γ-isomorphic to (F,Y). There are $e(X)-k$ such edges. Thus we have the first term.

A signed graph that is Γ-isomorphic to (F,Y) may also be obtained by switching the sign of an edge of a subgraph (B,Z) of (G,X) in $S^1((B,Z) \xrightarrow{\Gamma} (F,Y))$ ways, thus we obtain the second term. □

Let $\{(F_1,Y)^*, (F_2,Y)^*, \ldots, (F_l,Y)^*\}$ be the switching equivalence class of a graph (F,Y). Without loss of generality, we have assumed that the representative elements in this class have the same underlying unsigned graph Y. We will write the Equation 3.6 for each graph in this class, and obtain a linear system in variables $i((F_j,Y)^* \xrightarrow{\Gamma} (G,X))$, $(1 \leq j \leq l)$. We can turn our attention to a switching equivalence class, because $S^1((B,Z) \xrightarrow{\Gamma} (F,Y)) = 0$ if (B,Z) and (F,Y) are not in the same switching equivalence class. Let $M = (m_{ij})$ be a matrix with entries defined by

$$m_{ij} = S^1((F_j, Y)^* \xrightarrow{\Gamma} (F_i, Y)^*);$$

we call it the **sign switching matrix** of (F, Y). Now the equation in Lemma 3.3.1 can be rewritten as

$$\bar{t} = ((e(X) - k)I + M)\bar{i},$$

where $\bar{t} := (t((F_1, Y)^* \xrightarrow{\Gamma} (G, X)), \ldots, t((F_l, Y)^* \xrightarrow{\Gamma} (G, X)))^t$ and $\bar{i} := (i((F_1, Y)^* \xrightarrow{\Gamma} (G, X)), \ldots, i((F_l, Y)^* \xrightarrow{\Gamma} (G, X)))^t$. If the matrix $(e(X) - k)I + M$ is invertible, then the variables $i((F_j, Y)^* \xrightarrow{\Gamma} (G, X))$, $(1 \le j \le l)$ are uniquely determined.

Proposition 3.3.2. *Let (F, Y) be a graph such that $e(Y) = k$. The system $\bar{t} = ((e(X) - k)I + M)\bar{i}$ has unique solution if and only if $k - e(X)$ is not an eigenvalue of M.*

Proof. We have $(e(X) - k)I + M$ is not invertible if and only if 0 is an eigenvalue of $(e(X) - k)I + M$ if and only if $k - e(X)$ is an eigenvalue of M. □

Let $(A_1, Y), (A_2, Y), \ldots, (A_K, Y)$ be all the labeled signed graphs with underlying unsigned graph Y, where $k := e(Y)$ and $K := 2^k$. We define $A = (a_{ij})$ as

$$a_{ij} = \begin{cases} 1, & \text{if } (A_j, Y)_e = (A_i, Y) \text{ for some } e \text{ edge of } (A_j, Y) \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore A is a $K \times K$ matrix. This matrix has exactly k 1's in each row and in each column.

Definition 3.3.3. A **k-cube** is a graph whose vertices are the partition of a k-set into two subsets. Two partitions are **adjacent** if their common refinement contains a set of size one.

Lemma 3.3.4. *A is the adjacency matrix of a graph isomorphic to the k-cube.*

Proof. Each signed graph with underlying unsigned graph Y is represented by a binary string with 0 entries corresponding to negative edges and 1 entries corresponding to positive edges. Now $a_{ij} = 1$ if and only if the corresponding binary strings are adjacent points of the k-cube. □

Corollary 3.3.5 (See Stanley [23])**.** *The eigenvalues of A are $\theta_j = k - 2j$ with multiplicity $\binom{k}{j}$, for $0 \leq j \leq k$.*

Consider the partition of $\{(A_1, Y), (A_2, Y), \ldots, (A_K, Y)\}$ into isomorphism classes with respect to Γ; we denote the isomorphism classes by $(C_1, Y), (C_2, Y), \ldots, (C_L, Y)$. Let P be the matrix $L \times K$ with rows indexed by the isomorphism classes (C_i, Y) and columns indexed by the signed graphs (A_j, Y) such that

$$p_{ij} = \begin{cases} 1, & \text{if } (A_j, Y) \in (C_i, Y) \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.3.6. *We have $PA = MP$. Therefore any eigenvalue of M is also an eigenvalue of A.*

Proof. We will consider the ij entry of each matrix. First consider $(PA)_{ij}$:

$$(PA)_{ij} = \sum_{l=1}^{K} p_{il}\, a_{lj}.$$

We have $p_{il} = 1$ if and only if $(A_l, Y) \in (C_i, Y)$ and $a_{lj} = 1$ if and only if $(A_j, Y)_e = (A_l, Y)$ for some e edge of (A_j, Y). So $(PA)_{ij}$ counts the number of ways to switch the sign on some edge of (A_j, Y) to obtain a graph in the isomorphism class (C_i, Y).

$$(MP)_{ij} = \sum_{l=1}^{K} m_{il}\, p_{lj}.$$

We have $p_{lj} = 1$ if and only if $(A_j, Y) \in (C_l, Y)$ and $m_{il} = S^1((C_l, Y)^* \xrightarrow{\Gamma} (C_i, Y)^*)$. So $(MP)_{ij}$ counts the number of ways to switch the sign on some edge of (A_j, Y) to obtain a graph in the isomorphism class (C_i, Y). Therefore, $MP = PA$.

Let λ be an eigenvalue of M associated to the left eigenvector v^t. We have

$$\begin{aligned} v^tPA &= v^tMP \\ &= v^t\lambda P \\ &= v^tP\lambda. \end{aligned}$$

Thus we found that v^tP is a left eigenvector of A corresponding to λ. □

In Theorem 3.3.7 we present an other proof of Corollary 3.2.5, that say that if $e(X)$ is odd, then (G, X) is reconstructible.

Theorem 3.3.7. *If* $e(X) \not\equiv 0 \pmod 2$, *then* (G, X) *is reconstructible.*

Proof. Let (H, X) be a sign switching reconstruction of (G, X) not isomorphic to (G, X). So $(G, X) \in (C_i, Y)$ and $(H, X) \in (C_j, Y)$ for some $i \neq j$. Hence two columns of M are identical and 0 is an eigenvalue of M. Since the eigenvalues of M are of the form $e(X) - 2j$ we conclude that $e(X) \equiv 0 \pmod 2$. □

Theorem 3.3.8 (Kelly's Lemma for signed graphs)**.** *Given signed graphs* (G, X) *and* (F, Y) *such that* $e(Y) < e(X)/2$, *then the number of subgraphs of* (G, X) *that are* Γ*-isomorphic to* (F, Y) *is reconstructible.*

Proof. We have $e(X)/2 > k \implies e(X) > 2k$, then

$$k - e(X) < k - 2k = -k. \tag{3.7}$$

Let λ be an eigenvalue of M. From Lemma 3.3.6 we obtain λ is also an eigenvalue of the matrix A, and from Corollary 3.3.5 we have $\lambda = k - 2j$, for some $0 \leq j \leq k$. So, we have

$$-k \leq \lambda \leq k. \tag{3.8}$$

From Expressions 3.7 and 3.8 $\lambda \neq k - e(X)$, and so $k - e(X)$ is not an eigenvalue of M. So, the number of subgraphs of (G, X) isomorphic to (F, Y) is reconstructible, from Proposition 3.3.2. □

In Corollary 3.3.9 we present other proof for Corollary 3.2.4.

Corollary 3.3.9. *If $m \neq m^c$, then (G, X) is reconstructible.*

Proof. Suppose without lose of generality that $m < m^c$. By Lemma 3.3.8, subgraphs (F, Y) such that $e(Y) = m$ are reconstructible from $D(G, X)$; in particular, (G, X) is reconstructible. □

Theorem 3.3.10. *If X has n vertices and more than $2(n-1)$ edges, then the degree pair sequence is reconstructible.*

Proof. Let (G, X) be a signed graph such that $e(X) > 2(n-1)$. We will look at the degree pair sequence of (G, X) as a multiset of degree pairs. We can see each degree pair as a star $(F, K_{1,l})$, with $l \leq n-1$. We have $e(K_{1,l}) \leq n-1$ and $e(X)/2 > n-1$. We have all subgraphs $(F, K_{1,l})$ such that $e(K_{1,l}) \leq n-1$ are reconstructible by Lemma 3.3.8, thus we can count the number of all stars of the form $(F, K_{1,l})$, and then we can reconstruct the degree pair sequence of (G, X). □

Chapter 4

Reconstructing special classes of signed graphs

4.1 Reconstructing (G, X) when X is disconnected

The main result of this section is that if X is disconnected and has more than 4 edges, then (G, X) is sign switching reconstructible. We divide the proof in three parts: when $e(X) = 2m > 2$ and each component of X has fewer than m edges (Proposition 4.1.1); when $e(X) = 2m > 4$ and a unique component of X has at least m edges (Proposition 4.1.2); when $e(X) = 2m > 4$ and X has two components each with m edges (Proposition 4.1.3). These propositions, together with all small counter examples to sign switching reconstruction, which we have listed in Examples 2.1.1, 2.1.2, 2.1.3, 2.1.4,2.1.5 imply the main result.

Proposition 4.1.1. *If a signed graph (G, X) is such that X is disconnected and all components of X have fewer than $m > 1$ edges, then it is sign switching reconstructible.*

Proof. Let $A_1, A_2, \ldots, A_k$ be the components of (G, X). Suppose, without lose of generality, that $e(A_1) \leq e(A_2) \leq \ldots \leq e(A_k)$. From Lemma 3.3.8 we can reconstruct the number of components isomorphic to A_k. Let

- $c(A_i)$ be the number of components of (G, X) that are isomorphic to A_i;
- $s(A_i)$ be the number of subgraphs of (G, X) that are isomorphic to A_i;
- $s(A_i, A_j)_\Gamma$ be the number of subgraphs of A_j that are Γ-isomorphic to A_i.

We have that

$$c(A_{k-1}) = s(A_{k-1}) - c(A_k)s(A_{k-1}, A_k)_\Gamma.$$

From Lemma 3.3.8 this number is reconstructible. In a general way, we can calculate the number of components isomorphic to each A_i, $i = 1, 2, \ldots, k-1$, using a recursive equation

$$c(A_i) = s(A_i) - \sum_{l=i}^{l=k-1} c(A_{l+1})s(A_i, A_{l+1})_\Gamma,$$

and Lemma 3.3.8. We can count the quantity $s(A_i, A_{l+1})_\Gamma$ from the deck. Therefore the number $c(A_i)$ is reconstructible and (G, X) is reconstructible. □

Proposition 4.1.2. *If a signed graph (G, X) is such that X is disconnected and has a unique component with at least $m > 2$ edges, then it is sign switching reconstructible.*

Proof. Let A be the unique largest component with a positive edges and b negative edges, where $a + b \geq m$.

First we claim that if (G, X) is not sign switching reconstructible, then $a > 0$ and $b > 0$. Suppose to the contrary that (G, X) is not sign switching reconstructible and A has all positive or all negative edges. Since $m > 2$, by Lemma 3.2.1, we can replace 2 edges of A to obtain a graph (G', X) isomorphic to (G, X); moreover the replacing edges are not in A. This implies that in (G', X) all edges of the largest component do not have the same sign, which is a contradiction.

Now suppose that $a > 0$ and $b > 0$. Then there are precisely a cards in which the largest component has $a - 1$ positive edges and $b + 1$ negative edges, precisely b cards in which the largest component has $a + 1$ positive edges and $b - 1$ negative edges, and precisely $2m - (a + b)$ cards in which the largest component has a positive edges and b negative edges. Thus the component A is uniquely determined.

Other components of (G, X) are now uniquely determined in a similar way of the proof of Proposition 4.1.1, where the recursive equation is

$$c(A_i) = s(A_i) - \sum_{l=i}^{l=k-2} c(A_{l+1})s(A_i, A_{l+1})_\Gamma - s(A_i, A)_\Gamma.$$

□

Proposition 4.1.3. *If a signed graph (G, X) is such that X is disconnected with exactly two m-edge components, with $m > 2$, then it is sign switching reconstructible.*

Proof. First we claim that if (G, X) is not sign switching reconstructible, then we do not have a component with all positive or all negative edges. Suppose to the contrary that (G, X) is not sign switching reconstructible and a component A has all positive edges. Since $m > 2$, by Lemma 3.2.1, we can replace two edges of A to obtain a graph (G', X) isomorphic to (G, X); moreover the replacing edges are not in A. This implies that in (G', X) we do not have a component with all positive or all negative edges, which is a contradiction.

Let A and B be components of (G, X). We associate to A an ordered pair (a, b), where the first term is the number of positive edges of A, while the second term is the number of negative edges of A. We have that the ordered pair associated to B is (b, a). Thus the multiset of ordered pairs associated to the graph (G, X) is $\{(a, b), (b, a)\}$. The multiset of sets associated with the deck have the following sets

$$\{(a{-}1, b{+}1), (b, a)\};\ \{(a{+}1, b{-}1), (b, a)\};\ \{(a, b), (b{-}1, a{+}1)\};\ \{(a, b), (b{+}1, a{-}1)\}.$$

Without loss of generality, suppose that $0 < a \leq b$. Thus $a-1$ and $b+1$ are, respectively, the smaller and the larger number in this multiset. So we can identify a and b. We have some cases

Case (1). $a = b$. The multiset of sets associated with the deck will have the following sets

$$\{(a-1,a+1),(a,a)\};\ \{(a+1,a-1),(a,a)\};\ \{(a,a),(a-1,a+1)\};\ \{(a,a),(a+1,a-1)\}.$$

And the graphs with associated ordered pair (a,a) are isomorphic to the two components of (G,X), then we can identify (G,X). So, (G,X) is sign switching reconstructible.

Case (2). $a+1=b$. The multiset of sets associated with the deck will have the following sets

$$\{(a-1,a+2),(a+1,a)\};\ \{(a+1,a),(a+1,a)\};\ \{(a,a+1),(a,a+1)\};\ \{(a,a+1),(a+2,a-1)\}.$$

In the card that have an associated set $\{(a-1,a+2),(a+1,a)\}$, we have that the graph with associated ordered pair $(a+1,a)$ is isomorphic to a component of (G,X). In the card that have an associated set $\{(a,a+1),(a+2,a-1)\}$, we have that the graph with associated ordered pair $(a,a+1)$ is isomorphic to a component of (G,X). Thus we can identify (G,X). So (G,X) is sign switching reconstructible.

Case (3). $a\neq b$, $a+1\neq b$. The multiset of sets associated with the deck will have the following sets

$$\{(a-1,b+1),(b,a)\};\ \{(a+1,b-1),(b,a)\};\ \{(a,b),(b-1,a+1)\};\ \{(a,b),(b+1,a-1)\}.$$

The graphs with associated ordered pairs (a,b) and (b,a) are the components of (G,X). Thus we can identify (G,X). So (G,X) is sign switching reconstructible. □

Remark 4.1.4. If $m\leq 2$, the only possible disconnected graph are Example 2.1.4 or Example 2.1.1.

Theorem 4.1.5. *A signed graph* (G,X) *such that* X *is disconnected and* $m>4$ *is sign switching reconstructible.*

Proof. For $1<m\leq 2$, looking at the Examples 2.1.1, 2.1.2, 2.1.4 and 2.1.5 we can see the unique disconnected graphs with $1<m\leq 2$ are that in Example 2.1.1 and 2.1.4. Now the result follows from Propositions 4.1.1, 4.1.2, 4.1.3 for all $m>4$. □

4.2 Reconstructing (G, X) when X is a tree

Now we will prove that if X is a tree, then (G, X) is reconstructible, see Theorem 4.2.2. We will divide the proof in two cases: the case when X is bicentral (i.e., the centre consists of two adjacent vertices), and the case when X is central (i.e., the centre consists of a single vertex). First we will prove that if X is a star, then (G, X) is reconstructible.

Proposition 4.2.1. *Let (G, X) be a signed graph. If X is a star, then (G, X) is reconstructible.*

Proof. Suppose that $X \cong K_{1,2m}$, then $G \cong K_{1,m}$. We have m cards in the deck such that $G \cong K_{1,m-1}$. If we change a negative edge to positive in one of this cards we obtain (G, X). Thus (G, X) is reconstructible. □

Theorem 4.2.2. *Let (G, X) be a signed graph. If X is a tree, then (G, X) is reconstructible.*

Proof. Suppose that X is a bicentral tree. Let e be the central edge. Without lose of generality, suppose that e is a positive edge. We have $2m - 1$ cards in which e is positive and $2m - 1 > 1$, and a unique card $(G, X)_e$ in which e is negative. The original graph is obtained by switching edge e in $(G, X)_e$.

Now suppose that X is central. Since we have proved that (G, X) is reconstructible if X is a star, in Proposition 4.2.1, we assume that X is central with central vertex v, and has at least one edge not adjacent to the centre. Suppose that the degree pair of the central vertex is (a, b). Let $A_1, \ldots, A_a$ be the branches of (G, X) that have a positive edge incident to v. Let $B_1, \ldots, B_b$ be the branches of (G, X) that have a negative edge incident to v.

We reconstruct the degree pair (a, b) of v as follows. We have $a > 0$ and $b > 0$ if and only if there are three types of cards present in the deck - cards in which v has degree pair $(a - 1, b + 1)$, cards in which v has degree pair (a, b), and cards in which v has degree pair $(a + 1, b - 1)$, and in this case (a, b) is uniquely determined. The other case is when the degree pair of v is either $(a, 0)$ or $(0, b)$. Since we have assumed that

X is not a star, the degree pair of v is $(a, 0)$ if and only if there are exactly two types of cards present in the deck - cards in which v has degree pair $(a, 0)$ and cards in which v has degree pair $(a-1, 1)$. In this case the degree pair $(a, 0)$ is uniquely determined. Similarly, if the degree pair of v is $(0, b)$, it is uniquely determined.

Now we consider the following cases.

Case (1). $a > 1$ and $b > 1$. Each card in which v has degree pair $(a-1, b+1)$ is obtained by switching a positive edge incident with v. Each such card contains $a-1$ branches that have a positive edge incident with v; all these branches are the original branches of (G, X). We construct a multiset S of all such branches from all cards in which v has degree pair $(a-1, b+1)$. Each original branch A_i appears exactly $a-1$ times in S (and when considered up to isomorphism, a multiple of $a-1$ times). Hence the multiset of branches $A_i, i = 1, \ldots, a$ is uniquely determined from S. By a similar argument we obtain all B_i looking at the cards such that the central vertex has degree pair $(a+1, b-1)$.

Case (2). $a = 1$ and $b > 1$. We can obtain all branches $B_1, \ldots, B_b$ as in Case (1). Now we consider the unique card $(G, X)_e$ in which all edges incident with v are negative. This card must be obtained by switching e which is in A_1. From the multiset branches of $(G, X)_e$, we remove the branches $B_1, \ldots, B_b$. Now A_1 is obtained by switching the sign on the edge incident with v in the remaining branch.

Case (3). $a > 1$ and $b = 1$. The proof is analogous to the Case (2).

Case (4). $a = 0$ and $b > 1$. The branches $B_1, \ldots, B_b$ are obtained as in Case (1) by considering all cards in which v has degree pair $(1, b-1)$.

Case (5). $a > 1$ and $b = 0$. The proof is analogous to the Case (4).

Case (6). $a = 1$ and $b = 1$. When one of the branches has more than m edges, the proof proceeds similar to Proposition 4.1.2, and when both branches have m edges, the proof is similar to the proof of Proposition 4.1.3.

With these cases, the proof when X is a central tree is completed. □

4.3 Reconstructing (T, X) when T is a spanning tree of X

In this section, we assume that (T, X) is a signed graph, where T is a spanning tree of X. The main result is that except when T is $K_{1,3}$, the signed graph (T, X) is reconstructible. Note that in the Examples 2.1.2 and 2.1.5 the tree $K_{1,2}$ is not spanning.

We will first present some definitions given by Sheehan and Clapham [21] for edge reconstruction.

Definition 4.3.1. Let G be a graph. Let $1 \leq k \leq e(G)$. We say that G is k**-free** if, for every subset A of $E(G)$ such that $|A| = e(G) - k$, there exists an automorphism ϕ of K_n such that $E(G) \cap E(\phi(G)) = A$. Let $E \subseteq E(G)$ such that $|E| = k$. If there exists $F \subseteq E(\overline{G})$ such that $G - E + F \cong_{S_n} G$, then we say that the set E is **replaceable**, and that F is a replacing set of E. Thus a graph G is k-free if and only if every subset of $E(G)$ of size k is replaceable. A graph is **even-free** if it is k-free, for all k even.

Nash-Williams' Lemma 0.0.7 implies that if G is not edge reconstructible, then G is even-free.

The proof that (T, X) is reconstructible, if T is a spanning tree, depends on the following result of Sheehan and Clapham [21].

Theorem 4.3.2 (Sheehan and Clapham, 1992)**.** *Apart from paths, the only 2-free trees are those in Figure 4.1.*

Trees were proved to be reconstructible (hence also edge reconstructible) by Kelly [12] (see 0.0.4). The result of Sheehan and Clapham 4.3.2 says, as an application of Nash-Williams' Lemma 0.0.7, that all trees except paths and those in the Figure 4.1 are edge reconstructible. We use the same approach to show that if T is a spanning subtree of X, then (T, X) is sign switch reconstructible.

We now give analogous definitions for signed graphs.

Definition 4.3.3. Let (G, X) be a signed graph. Let $1 \leq k \leq e(G)$. We say that (G, X) is k**-free** if, for every $A \subseteq E(G)$ such that $|A| = e(G) - k$, there exists an automorphism

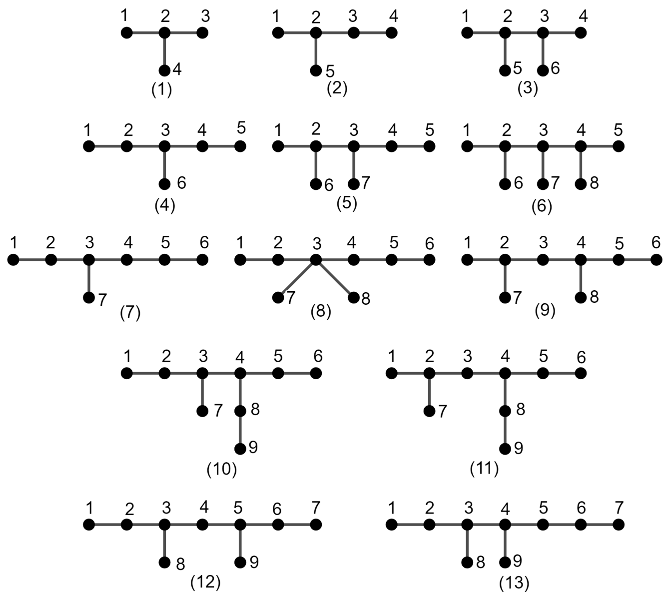


Figure 4.1: Two-free trees

ϕ of X such that $E(G) \cap E(\phi(G)) = A$. Equivalently, (G, X) is k-free if for every $E \subseteq E(G)$ such that $|E| = k$, there exists $F \subseteq E(G^c)$, called the **replacing set** of E, such that $(G - E + F, X) \cong (G, X)$. A signed graph is **even-free** if is k-free for all k even.

From Nash-Williams' Lemma 3.2.1, if (G, X) is not reconstructible, then for every $E \subseteq E(G)$ of even cardinality, there exists $F \subseteq E(G^c)$ such that $(G, X) \cong (G - E + F, X)$ and for every $E \subseteq E(G)$ of odd cardinality, there exists $F \subseteq E(G^c)$ such that $(G-E+F, X) \cong (H, X)$. But if G is edge reconstructible, then we know that $G \cong_{S_n} H$, hence for every $E \subseteq E(G)$ there exists $F \subseteq E(G^c)$ such that $G - E + F \cong_{S_n} G$. Then we have the following corollary.

Corollary 4.3.4. *If (G, X) is not reconstructible then for every $E \subseteq E(G)$ of even cardinality, there exists $F \subseteq E(G^c)$ such that $(G, X) \cong (G - E + F, X)$ and for every $E \subseteq E(G)$ of odd cardinality, there exists $F \subseteq E(G^c)$ such that $(G - E + F, X) \cong (H, X)$. If G is edge reconstructible, then $G - E + F \cong_{S_n} G$, in both cases.*

Proposition 4.3.5. *Let T be a spanning tree of X. If (T, X) is 2-free, then T is 2-free.*

Proof. This follows from the fact that $\mathrm{Aut}(X)$ is a subgroup of S_n. □

We observe that if T is not spanning we cannot guarantee that $\mathrm{Aut}(X)$ is a subgroup of S_n. We also have the following observations.

Remark 4.3.6. It follows from Proposition 4.3.5 that if T is a spanning tree of X and (T, X) is 2-free, then T must be a path or one of the trees presented in Figure 4.1. We know that $(K_{1,3}, K_4)$ is not sign switching reconstructible, therefore, it is 2-free. Therefore, to prove that (T, X) is reconstructible whenever T is a spanning subtree of X, we only need look at paths and the other 12 trees in Figure 4.1.

Remark 4.3.7. Let (T, X) be a signed graph such that T has more than 3 edges. Suppose that (T, X) is not reconstructible. From Corollary 3.2.7 it follows that if T has even number of edges, then $T^c \cong_X T$. If T has odd number of edges, then we claim that $T \cong_{S_n} T^c$. In fact, by Lemma 3.2.1, (T^c, X) is the nonisomorphic reconstruction of (T, X), and from Theorem 0.0.4, trees are vertex reconstructible, hence they are also edge reconstructible, therefore, $T \cong_{S_n} T^c$.

Lemma 4.3.8. *If T is isomorphic to a path, then (T, X) is reconstructible.*

Proof. Suppose that vertices of X are labelled $1, 2, \ldots, n$ so that 1 to n appear consecutively on T. Suppose that (T, X) is not reconstructible.

If $n = 3$, then there is no signed graph (T, X) such that T is a spanning path. If $n = 4$, then we have a unique possible signed graph (T, X) such that T is a path. So (T, X) is reconstructible. Now we assume that $n > 4$.

Hence every edge set of T has a replacing set. The only replacing edge of $\{1, 2\}$ is $\{1, n\}$, hence $\{1, n\}$ is an edge of T^c. The only replacing edge set of $\{\{1, 2\}, \{2, 3\}\}$ is $\{\{1, 3\}, \{2, n\}\}$; similarly, the only replacing edge set of $\{\{n-2, n-1\}, \{n-1, n\}\}$ is $\{\{n-2, n\}, \{1, n-1\}\}$. Hence $\{1, 3\}$ and $\{1, n-1\}$ are also edges of T^c. This shows that the degree of 1 in T^c is 3, which is a contradiction since we know that T^c must be a path. □

We now prove that if T is one of the trees in Figure 4.1, other than $K_{1,3}$, then (T, X) is reconstructible. For most of the trees in Figure 4.1, the reconstructibility of

(T, X) follows from the existence of an edge that cannot be replaced, as shown in the following proposition.

Proposition 4.3.9. *If T is isomorphic to one of the trees (3)–(6), (8)–(13) of Figure 4.1, then (T, X) is reconstructible.*

Proof. By Corollary 4.3.4 and Remark 4.3.7, if (H, X) is a reconstruction of (T, X), then $H \cong_{S_n} T$. Hence, if (T, X) is not reconstructible, then for every $E \subseteq E(T)$, there is a replacing set $F \subseteq E(T^c)$ such that $T - E + F \cong_{S_n} T$. We show that each of the trees (3)–(6) and (8)–(13) of Figure 4.1 contains an edge e that does not have a replacing edge (even with respect to S_n), i.e., there is no edge f such that $T - e + f \cong_{S_n} T$. To verify this, consider edge $e = \{2, 3\}$ in (3), edge $e = \{3, 6\}$ in (4), edge $e = \{3, 7\}$ in (5), edge $e = \{3, 7\}$ in (6), edge $e = \{3, 7\}$ in (8), edge $e = \{1, 2\}$ in (9), edge $e = \{3, 7\}$ in (10), edge $e = \{2, 7\}$ in (11), edge $e = \{3, 8\}$ in (12) and edge $e = \{4, 9\}$ in (13). □

Proposition 4.3.10. *If T is isomorphic to (2) in Figure 4.1, then (T, X) is reconstructible.*

Proof. To simplify our counts we will suppose T is equal (2). We will first look at the edges of T that have unique replacement. The only replacing edge of $\{1, 2\}$, $\{2, 3\}$ and $\{2, 5\}$ are respectively $\{1, 3\}$, $\{2, 4\}$ and $\{3, 5\}$. The edge $\{3, 4\}$ have two possible replacing edge: $\{1, 4\}$ and $\{4, 5\}$, but in both cases we have that T' is isomorphic to P_4. A contradiction with Remark 4.3.7. □

Lemma 4.3.11. *Let (T, X) be a signed graph that is not sign switching reconstructible, and such that T is a tree with even number of edges. If there is only one vertex u of degree k in T, then there is only one vertex v of degree k in T^c. Moreover, if u and v are distinct, then they cannot be adjacent.*

Proof. From Corollary 3.2.7 there is a $\pi \in \mathrm{Aut}(X)$ such that $\pi(G) = G^c$. Since u and v are the unique vertices of degree k in G and G^c respectively, we have that $\pi(u) = v$ and $\pi(v) = u$. Suppose that u and v are distinct vertices. If there is an edge uv in (T, X), then $\pi(uv) = vu$, a contradiction with the fact that $\pi(G) = G^c$. □

Proposition 4.3.12. *If T is isomorphic to (7), in Figure 4.1, then (T, X) is reconstructible.*

Proof. To simplify our counts we will suppose T is equal (7). We will first look at the edges of T that have unique replacement. The only replacing edge of $\{1,2\}$, $\{3,7\}$ and $\{5,6\}$ are respectively $\{1,7\}$, $\{4,7\}$ and $\{1,6\}$. Suppose that (T,X) is not reconstructible, this implies that (T,X) is 2-free. We will try to replace the edge set $\{\{4,5\},\{5,6\}\}$. We have three possible replacing sets $\{\{1,5\},\{4,6\}\}$, $\{\{1,6\},\{5,7\}\}$ and $\{\{1,5\},\{6,7\}\}$. If $\{4,6\}$ is an edge in T', then we have a cycle in T', a contradiction with Remark 4.3.7. If $\{5,7\}$ or $\{6,7\}$ is an edge in T', then we have a vertex of degree 3 in T that is adjacent in X to a vertex of degree 3 in T', a contradiction with Lemma 4.3.11. Thus (T,X) is reconstructible. □

Lemma 4.3.8 and Propositions 4.3.9, 4.3.10 and 4.3.12 implies Theorem 4.3.13.

Theorem 4.3.13. *If T is a spanning tree not isomorphic to $K_{1,3}$, then (T,X) is reconstructible.*

Chapter 5

Sign switching vertex reconstruction

In this chapter we will present a new reconstruction problem. Results related to this problem will be useful to prove that the degree sequence of a signed graph is reconstructible, in the sense of reconstructibity presented in Chapter 2.

5.1 Problem definition

Denote by $(G-v, X)$ the signed graph obtained by changing the sign of all positive edges incident to vertex v to negative. Let (G, X) and (H, Y) be signed graphs. Suppose that there is a bijection $f\colon V(X) \to V(Y)$ such that for all $v \in V(X)$, we have $(G - v, X) \cong (H - f(v), Y)$. We call f a **negative vertex hypomorphism** from (G, X) to (H, Y). We say that (G, X) and (H, Y) are **negative vertex hypomorphic**. Since X and Y are required to be isomorphic in the above definition, we may consider G and H to be spanning subgraphs of the same underlying graph X; hence all isomorphisms between $(G - v, X)$ and $(H - f(v), X)$, are elements of $\mathrm{Aut}(X)$. The above notions may be defined equivalently as follows. We define the multiset

$$VD^{-}(G, X) := \{(G - v, X)^{*} \mid v \in V(X)\}.$$

We call $VD^{-}(G,X)$ the **negative vertex deck** of (G,X). Note that two graphs (G,X) and (H,X) have the same negative vertex deck if and only if they are negative vertex hypomorphic. In this case (H,X) is called a **negative vertex reconstruction** of (G,X). We say that a given signed graph (G,X) is **negative vertex reconstructible** if every negative vertex reconstruction of (G,X) is isomorphic to (G,X). Furthermore we say that an invariant or a parameter of (G,X) is **negative vertex reconstructible** if it takes the same value on every negative vertex reconstruction of (G,X).

The above notions are similar to the operation of vertex deletion, vertex hypomorphism, vertex deck, and so on. But note that we do not have any deletions of vertices. In $(G-v,X)$ we have that $v \in V(G-v) = V(X)$ and v is a isolated vertex in $G-v$. The notions of positive vertex hypomorphism, positive vertex deck $VD^{+}(G,X)$, and so on are similarly defined, where we switch the sign of the negative edges to positive at each vertex.

We define $VD(G,X) \coloneqq VD^{-}(G,X) \uplus VD^{+}(G,X)$ and call $VD(G,X)$ the **sign vertex deck** of (G,X). If two signed graphs (G,X) and (H,X) have the same sign vertex deck, then we say that (H,X) is a **sign vertex reconstruction** of (G,X). We say that a given signed graph (G,X) is **sign vertex reconstructible** if every sign vertex reconstruction of (G,X) is isomorphic to (G,X).

Remark 5.1.1. The vertex deck of G is obtained from $VD^{-}(G,X)$, so every property of G that is vertex reconstructible is also negative vertex reconstructible. In particular, the positive degree sequence is reconstructible from $VD^{-}(G,X)$ if (G,X) has more than 2 vertices. Similarly, the negative degree sequence is reconstructible from $VD^{+}(G,X)$ if (G,X) has more than 2 vertices. Also, the neighbourhood positive degree sequence. (See Theorem 0.0.3.)

Problem 5.1.2 (sign vertex reconstruction problem)**.** *Is* (G,X) *reconstructible from the sign vertex deck, up to isomorphism?*

Example 5.1.3. Let $X \cong K_{1,1}$. The two signed graphs (G,X) and (H,X), with $G \cong K_{1,1}$ and $H \cong \emptyset$, have the same sign vertex deck. We have that $VD^{+}(G,X) = \{(H,X)^{*}, (H,X)^{*}\}$, since switching positive edges to negative will change the sign of the unique edge in (G,X); $VD^{+}(H,X) = \{(H,X)^{*}, (H,X)^{*}\}$, since switching positive edges to negative leaves the graph unchanged; $VD^-(G,X) = \{(G,X)^*, (G,X)^*\}$, since switching negative edges to positive leaves the graph unchanged; and $VD^-(H,X) = \{(G,X)^*, (G,X)^*\}$, since switching negative edges to positive change the sign of the unique edge in (H,X). But $(G,X) \not\cong (H,X)$. We do not know any other pair of nonisomorphic signed graphs that have the same signed vertex deck.

Proposition 5.1.4. *If X has isolated vertices then (G,X) is both positive and negative vertex reconstructible.*

Proof. If X has isolated vertices then (G,X) is the graph in $VD^+(G,X)$ that has the least number of positive edges, which is the same as the graph in $VD^-(G,X)$ that has the most number of positive edges. □

Proposition 5.1.5. *Given the sign vertex deck of (G,X), we can obtain the negative and the positive vertex deck.*

Proof. Consider a vertex v of X. If v is isolated in G, then $(G-v,X) = (G,X)$; otherwise, $(G-v,X)$ has fewer positive edges than (G,X). Similarly, if v is isolated in G^c, then $(G+v,X) = (G,X)$; otherwise, $(G+v,X)$ has more positive edges than (G,X). We order the graphs in $VD(G,X)$ in the increasing order of the number of positive edges. In this order, the first $v(X)$ graphs constitute the negative vertex deck $VD^-(G,X)$ and the last $v(X)$ graphs constitute the positive vertex deck $VD^+(G,X)$. □

Corollary 5.1.6. *If X has isolated vertices, then (G,X) is reconstructible.*

Proof. Suppose that X has isolated vertices. From Proposition 5.1.4 we can reconstruct (G,X) from $VD^-(G,X)$ and from $VD^+(G,X)$. Since Proposition 5.1.5 says that we can recognise both positive and negative vertex deck, then (G,X) is reconstructible. □

Proposition 5.1.7. *If G has isolated vertices and $v(X) > 2$, then (G,X) is reconstructible from $VD^-(G,X)$.*

Proof. Since the positive degree sequence of (G,X) is reconstructible from $VD^-(G,X)$ (see Remark 5.1.1), we can determine if G has isolated vertices or not. If G has isolated vertices, then the graph in VD^- that has the same number of positive edges as G is (G, X). □

From this point, we will only consider signed graphs with more than 2 vertices, and assume that neither X nor G has isolated vertices.

Let (F, Y) be a signed graph. We denote by $v^\bullet(F)$ the number of isolated vertices in F.

Lemma 5.1.8. *Let (F, X) and (G, X) be signed graphs. If $v(F) - v^\bullet(F) < v(G)$, then we can construct the quantity*

$$s(F, G)_X \coloneqq |\{\sigma \in \mathrm{Aut}(X) \mid \sigma(F) \subseteq G\}|.$$

Proof. The proof is similar to the proof of Lemma 0.0.2. Let (H, X) be a reconstruction of (G, X). Then

$$\begin{aligned} s(F, G)_X &= \frac{\sum_v s(F, G - v)_X + \sum_v s(F, G + v)_X}{v(X) - (v(F) - v^\bullet(F))} \\ &= \frac{\sum_v s(F, H - f(v))_X + \sum_v s(F, H + f(v))_X}{v^\bullet(F)} \\ &= s(F, H)_X. \end{aligned}$$

□

5.2 Reconstructing the degree pair sequence from the negative vertex deck

The positive and negative degree sequences are individually reconstructible from $VD^-(G, X)$ and $VD^+(G, X)$, respectively (see Remark 5.1.1); moreover for every card $(G - v, X)$ in $VD^-(G, X)$, we can compute $d^+(v)$, and similarly, for every card $(G + v, X)$ in $VD^+(G, X)$, we can compute $d^-(v)$ (see Remark 5.1.1). But computing $d^-(v)$ for every card $(G - v, X)$ in $VD^-(G, X)$ does not seem obvious. We can easily reconstruct $d(v)$ if for every card $(G - v, X)$ we have a unique isolated vertex in G, in this case v is this isolated vertex of G. But, if G has two or more isolated vertices we cannot immediate recognize v. Our goal in this section is to compute the degree pair sequence of (G, X), i.e., the multiset $\{d(v) = (d^+(v), d^-(v)) \mid v \in V(G, X)\}$ given $VD^-(G, X)$.

Theorem 5.2.1. *The degree pair sequence of (G, X) is reconstructible from its negative vertex deck. More than this, for each $v \in V(G, X)$ we can compute $d(v)$.*

Proof. Let $n(k, l)$ denote the number of vertices in $V(G, X)$ such that $d(v) = (k, l)$.

1. We have $n(0, l) = 0$ for all l since G is spanning.

2. Now we compute $n(1, k)$ for all k. For each vertex v in $V(G, X)$ that has degree pair $(1, k)$ for some k, there is a card $(G - v, X)$ containing a vertex with degree pair $(0, k + 1)$. Moreover, for each card $(G - v, X)$, we know $d^+(v)$ in (G, X). Now consider a card $(G - v, X)$ such that $d^+(v) = 1$ in (G, X). If there is a unique vertex in the card with positive degree 0, say with degree pair $(0, l)$, then that vertex must be v, in which the degree pair of v is $(1,\ l - 1)$. If there are two vertices a and b in the card, both with positive degree 0, then, since G has no isolated vertices, there must be a negative edge ab in $(G - v, X)$, and (G, X) is uniquely constructed by switching the sign on ab from negative to positive. It is not possible for there to be more than 2 vertices in $(G - v, X)$ of positive degree 0. Thus by looking at all cards in $VD^-(G, X)$, we can either construct (G, X) uniquely or know all vertices of degree $(1, k)$ for all k.

3. Now we compute $n(k, l)$ for all $k > 1$ and l. For each vertex v in (G, X) that has degree pair (k, l) for some $k > 1$ and some l, there is a card $(G - v, X)$ containing a vertex with degree pair $(0, k + l)$. Moreover, for each card $(G - v, X)$, we know $d^+(v)$ in (G, X). Now consider a card $(G - v, X)$ such that $d^+(v) = k$ in (G, X). If there is a unique vertex in the card with positive degree 0, say with degree pair $(0, q)$, then that vertex must be v, in which the degree pair of v is $(k,\ q - k)$. Suppose that there are two or more vertices in the card with positive degree 0. Thus the positive degree of these vertices only can be k or 1 in (G, X). Let a and

b be two of these vertices, such that $d(a) = (0, k + l)$ and $d(b) = (0, i)$. Suppose that (H, X) is a reconstruction of (G, X) that is not isomorphic to (G, X). Since the positive degree sequence is reconstructible, we have that $d(a) = (1, k + l - 1)$ and $d(b) = (k, i - k)$. Then

$$(1, k + l - 1) = (1, i - 1)$$
$$k + l = i.$$

We know k and i from the positive degree sequence, and thus l is known. So we know $d(v)$. □

5.3 Constructing the negative vertex deck from the negative sign switching deck

In this section, we assume that (G, X) is a graph to be D^--reconstructed, and (H, X) is one of its reconstructions.

Theorem 5.3.1. *If (G, X) is reconstructible from $VD^-(G, X)$ and $e(X) > 6$, then (G, X) is reconstructible from $D^-(G, X)$.*

Proof. Note that $D^-(G, X)$ has m signed graphs. We will look at the set $\{(G{-}e{-}v, X) \mid e \in E(G),\ v \in V(X)\}$. This set has mn graphs, saying $H_1, H_2, \ldots, H_{mn}$, consider that this graphs are ordering by a decreasing order of edges. Furthermore, $VD^-(G, X) \subset \{(G - e - v, X) \mid e \in E(G),\ v \in V(X)\}$ Note that $((G - e) - v, X) = (G - u, X)$ if and only if e is incident to the vertex u, so H_1 is in $VD^-(G, X)$. If $H_i \not\subset H_j$ for any H_j then H_i is also in $VD^-(G, X)$. □

Corollary 5.3.2. *Let (G, X) be a signed graph such that G and G^c are edge spanning and all reconstructions (H, X) are such that H and H^c are edge spanning. Then the degree pair sequence is reconstructible from $D^-(G, X)$. More than this, for each $v \in V(G, X)$ we can compute $d(v)$.*

Lemma 5.3.3. *The only pair of graphs (H, X) and (G, X) such that H and H^c are edge spanning and G is not edge spanning are the graphs in the Example 2.1.2.*

Proof. There is a vertex v in (H, X) such that the degree pair of v is $(0, k)$, for some k. Thus v has degree $(0, k)$ in all graphs in $D^-(G, X)$. But this vertex has degree (l, k), with $l \neq 0$ in (G, X), then it only can occur in Example 2.1.2. □

Lemma 5.3.4. *The only pair of graphs (H, X) and (G, X) such that H and H^c are edge spanning and G^c is not edge spanning are the graphs in the Example 2.1.2.*

Proof. Since H^c is not edge spanning there is a vertex v of degree pair $(k, 0)$ for some k. Let e_i, $i = 1, \ldots j$ be all the positive edges incident to v. The vertex v has degree pair $(k-1, 0)$ in $(H, X)_{e_i}$, for any $i = 1, \ldots, j$. Let f_i, $i = 1, \ldots l$ be all the others positive edges in (H, X). We have that v has degree pair $(k, 0)$. Since (G, X) and (H, X) have the same deck the edges f_i, $i = 1, \ldots l$ can not occurs. Then $H \cong K_{1,m}$ for some m. Furthermore G and G^c are edge spanning subgraphs such that v has degree $(k-1, 0)$ in all graphs in the deck, then there is only two positive edges and the only possible pair of graphs are in the Example 2.1.2. □

Theorem 5.3.5. *If (G, X) is such that G is k-regular and G is edge spanning, then (G, X) is reconstructible from $VD^-(G, X)$.*

Proof. Since G is k-regular and from Remark 5.1.1 the positive degree sequence is reconstructible, then any reconstruction (H, X) is such that H is also k-regular. We will look at (G_v^-, X), it is a signed graph with $n-k$ vertices of positive degree k and a vertex with positive degree zero. We can identify the vertex of degree zero, that is, the vertex v, and so reconstruct the signed graph. For this we only need join the vertex v to each vertex of positive degree $n-k$ with a positive edge. The proof when G^c is k-regular and spanning is analogous. □

5.4 Results for positive deck

The results of Sections 5.2 and 5.3 can be obtained for $VD^+(G,X)$ in an analogous way.

The reconstruction of the degree pair sequence from the positive deck is analogous to the degree pair sequence from the negative deck. We also have that if (G,X) is such that G is k-regular, then (G,X) is reconstructible from $VD^+(G,X)$.

Corollary 5.4.1. *The degree pair sequence is reconstructible from $D(G,X)$.*

Theorem 5.4.2. *Let (G,X) be a signed graph such that m is even, G is edge spanning, G^c is edge spanning and $(G,X) \ncong (K_2+K_2,C_4)$. Let (H,X) be such that (G,X) and (H,X) are sign switching hypomorphic. Suppose $(G-e+f,X) \cong (H,X)$. Let v_1 and v_2 be vertices incident to the edge e and u_1 and u_2 be vertices incident to the edge f. If the degree pair of v_1 and v_2 are (k,l) and (i,j) respectively, then the degree pair of u_1 and u_2 are $(k-1,l+1)$ and $(i-1,j+1)$ respectively.*

Proof. Suppose that there is a reconstruction (H,X) such that $(G-e+f,X) \cong (H,X)$, $e \neq f$. Let v_1 and v_2 be vertices incident to the edge e and u_1 and u_2 be vertices incident to the edge f. Suppose that the degree pair of v_1 and v_2 are (k,l) and (i,j) respectively. Thus the degree of v_1 and v_2, in $(G-e+f,X) \cong (H,X)$, are $(k-1,l+1)$ and $(i-1,j+1)$ respectively. From Corollary 3.2.9 the degree pair of u_1 and u_2, in $(G-e+f,X) \cong (H,X)$, are (k,l) and (i,j). And thus the degree pair u_1 and u_2, in (G,X), are $(k-1,l+1)$ and $(i-1,j+1)$. ☐

Theorem 5.4.3. *Let (G,X) be a non reconstructible signed graph such that G and G^c are edge spanning and $(G,X) \ncong (K_2+K_2,C_4)$. Let (H,X) be such that (G,X) and (H,X) are sign switching hypomorphic, that is a bijection $f: E(X) \to E(X)$ such that $f(E(G)) = E(H)$ and for all $e \in E(G)$ we have $(G-e,X) \cong (H-f(e),X)$ and for all $e \in E(X) \setminus E(G)$ we have $(G+e,X) \cong (H+f(e),X)$ and $(G,X) \ncong (H,X)$. Let e be an edge of (G,X). Let v_1 and v_2 be the end vertices of e and u_1 and u_2 be the end vertices incident to the edge $f(e)$. We have that $\{d(v_1),d(v_2)\} = \{d(u_1),d(u_2)\}$, where $d(v_i)$ is the degree pair of the vertex v_i in (G,X) and $d(u_i)$ is the degree pair of the vertex u_i in (H,X), $i=1,2$.*

Proof. Let (p, q) and (r, s) be the degree pair of the endvertices of e. The pair $\{p, r\}$ is known, since the positive degree sequence is reconstructible (see Proposition 3.1.2). Also, the degree pair sequence of (G, X) is known (see Corollary 5.4.1). We will look at the degree pair sequence as a multiset. Suppose without lose of generality that $e \in E(G)$. We have two cases.

Case (1). $p = r$. There is i and j such that (p, i) and (p, j) are in the degree pair sequence of (G, X) but are not in the degree pair sequence of $(G - e, X)$. Thus the degree pair of the endvertices of e are (p, i) and (p, j). Since $(G - e, X) \cong (H - f(e), X)$, we have that $\{d(v_1), d(v_2)\} = \{d(u_1), d(u_2)\} = \{(p, i), (p, j)\}$.

Case (2). $p \neq r$. Suppose, without lose of generality that $p < r$. There is i such that exactly one degree pair $(p - 1, i)$ is new in $(G - e, X)$. Thus one of the degrees of the endvertices of e is $(p, i - 1)$. There is j such that exactly one (r, j) disappear from the degree pair sequence of (G, X). Thus the other degree of the endvertices of e is (r, j). Since $(G - e, X) \cong (H - f(e), X)$, we have that $\{d(v_1), d(v_2)\} = \{d(u_1), d(u_2)\} = \{(p, i - 1), (r, j)\}$. □

Theorem 5.4.4. *Let (G, X) be a signed graph such that G and G^c are edge spanning and $(G, X) \ncong (K_2 + K_2, C_4)$. Let (H, X) be such that (G, X) and (H, X) are sign switching hypomorphic. Let $f : E(X) \to E(X)$ be like in Theorem 5.4.3. Then $(G - e + f(e), X) \cong (H, X)$ for all $e \in E(G)$. Let v_1 and v_2 be vertices incident to the edge e and u_1 and u_2 be vertices incident to the edge $f(e)$. If the degree pair of v_1 and v_2 in (G, X) are (k, l) and (i, j) respectively, then the degree pair of u_1 and u_2 in (G, X) are $(k - 1, l + 1)$ and $(i - 1, j + 1)$ respectively.*

Proof. We have that $(G - e, X) \cong (H - f(e), X)$, for all $e \in E(G)$, then $(G - e + f(e), X) \cong (H - f(e) + f(e), X)$, for all $e \in E(G)$. Since v_1 and v_2 are (k, l) and (i, j) in (G, X), looking at $(H, X) \cong (H - f(e) + f(e), X)$ we obtain that u_1 and u_2 has degree $(k - 1, l + 1)$ and $(i - 1, j + 1)$ respectively in (G, X). □

5.5 Reconstructing (G, X) when G is bidegreed

In this section, we assume that (G, X) is a signed graph to be reconstructed from the sign switching deck and such that G is bidegreed. We set $\Delta = \Delta(G)$ and $\delta = \delta(G)$. Our goal in this section is prove that (G, X) is reconstructible, if G and G^c are edge spanning. In order to prove this we will prove that if (G, X) is not reconstructible the only possible positive edges have endvertices with positive degree Δ.

Lemma 5.5.1. *Let (G, X) be a signed graph such that G is bidegreed. Let (H, X) be its sign switching reconstruction. Suppose that G and G^c are edge spanning, then $\Delta = \delta+1$.*

Proof. Let (G, X) be a signed graph such that G is bidegreed; G and G^c are edge spanning. From Remark 3.2.8 G^c is also bidegreed. Let (H, X) be a sign switching reconstruction of (G, X) such that $(G, X) \ncong (H, X)$. Let $(G - e + f, X) \cong (H, X)$. From Theorem 5.4.4 if the end vertices of e has degree pairs (k, l) and (i, j) in (G, X), then the end vertices of f has degree pairs $(k - 1, l + 1)$ and $(i - 1, j + 1)$ in (G, X), since G and G^c are bidegreed we have $\Delta = \delta + 1$. $\square$

From this point we will assume that G and G^c are edge spanning subgraphs of (G, X).

Proposition 5.5.2. *If there is a positive edge uv in (G, X) such that $d^+(u) = d^+(v) = \delta$, then (G, X) is sign switching reconstructible.*

Proof. Let $e = uv$ be a positive edge in (G, X). Consider $(G - e, X)$. Since G is bidegreed, there are only two vertices u and v with positive degree $\delta - 1$. To obtain (G, X) we only need switch the sign of uv. $\square$

Proposition 5.5.3. *If there are positive edges uv and vw such that $d^+(u) = d^+(w) = \delta$ and $d^+(v) = \Delta$, then (G, X) is sign switching reconstructible.*

Proof. Suppose that (G, X) is not sign switching reconstructible. From Corollary 4.3.4 (G, X) is 2-free and thus $\{uv, vw\}$ has a replacing set. We have that $d^+(u) = d^+(w) = \delta - 1$ and $d^+(v) = \Delta - 2$ in $(G - uv - vw, X)$. There is no e and f such that $(G - uv -$

$vw + e + f, X) \cong (G, X)$, since there is no e and f such that $(G - uv - vw + e + f, X)$ and (G, X) have the same degree pair sequence. A contradiction. Thus (G, X) is sign switching reconstructible. □

Proposition 5.5.4. *If there are positive edges uv and vw such that $d^+(u) = \delta$ and $d^+(v) = d^+(w) = \Delta$, then (G, X) is sign switching reconstructible.*

Proof. If there is a vertex $u' \neq u$ such that $d^+(u') = \delta$ and $u'v$ is a positive edge in (G, X), then (G, X) is reconstructible from Proposition 5.5.3. Thus suppose that there is no such u'. Then each vertex $z \neq u$ that is a neighbour of v in G is such that $d^+(z) = \Delta$. Suppose that (G, X) is not sign switching reconstructible and let (H, X) be its sign switching reconstruction. Let A be the set of all edges incident to v. From Corollary 4.3.4 we have a set of edges B such that $(G - A + B, X) \cong (G, X)$ or $(G - A + B, X) \cong (H, X)$. From Corollary 5.4.1 we have that $(G - A + B, X)$ have the same degree pair sequence as (G, X). Since $d^+(u) = \delta$, looking at $(G - A + B, X)$ we obtain that $d(v) = (\Delta, \delta)$. But $d(v) = (\delta, \Delta)$ in $(G - A + B, X)$ and all z that is a neighbour of v in G is such that $d^+(z) = \delta$ in $(G - A + B, X)$. A contradiction with Corollary 5.4.1 that say the degree pair sequence is sign switching reconstructible. □

From Propositions 5.5.3 and 5.5.4 we can see that if there is a positive edge uv in (G, X) such that $d^+(u) = \delta$ and $d^+(v) = \Delta$, then (G, X) is reconstructible.

Theorem 5.5.5. *If G is bidegreed, then (G, X) is sign switching reconstructible.*

Proof. Suppose that (G, X) is not sign switching reconstructible. From Propositions 5.5.2, 5.5.3 and 5.5.4, there is no edge e such that at least one of its endvertices have positive degree δ. Thus the unique degree in G is Δ. It is a contradiction with the fact that G is bidegreed and edge spanning. □

Conclusion

In this chapter we will summarise the main results obtained in this thesis and make some comments about some natural questions that arise from these results.

In Chapter 3 we proved that a signed graph that has a different number of positive and negative edges is reconstructible. In the same chapter we proved that if (G, X) is not sign switching reconstructible, and (H, X) is a reconstruction not isomorphic to (G, X), then:

1. if the number of edges of G is even then $(G, X) \cong (G^c, X)$;
2. if the number of edges of G is odd then $(H, X) \cong (G^c, X)$, which implies that when the number of edges of G is odd, (G, X) has at most one sign switching reconstruction that is not isomorphic to (G, X).

Is it true that if (G, X) is not sign switching reconstructible, then it has a unique sign switching reconstruction not isomorphic to (G, X)?

We considered the sign switching reconstruction problem when either G or X belongs to some special class of graphs. In Chapter 4 we proved that (G, X) is sign switching reconstructible if G is a tree, and in Chapter 5 we proved that (G, X) is sign switching reconstructible if G is regular or bidegreed. Moreover, we showed that (G, X) is sign switching reconstructible when X is a tree or a disconnected graph. But the question remains open when X is regular or bidegreed, or when G is disconnected.

There are other questions related to special classes of graphs. Since outerplanar graphs, chordal graphs and claw-free graphs are known to be edge reconstructible, we would like to solve the sign switching reconstruction problem when either G or X is outerplanar or chordal or claw-free. Similarly, since triangle-free graphs are vertex-switching reconstructible, it is natural to ask what happens when G or X is triangle-free.

It seems that neither the edge reconstructibility proofs nor the vertex-switching reconstruction proofs work in a straight-forward way for sign switching reconstruction. This may be because the sign switching reconstruction problem resembles both the edge reconstruction problem and the vertex-switching reconstruction problem. Our proofs of Kelly's lemma is based on the technique used by Ellingham and Royle for a result on vertex-switching reconstruction, while our version of Nash-Williams's lemma is similar to the corresponding result for edge reconstruction. On the other hand the known results for edge reconstruction and for vertex-switching reconstruction are quite different.

We have another problem related to the reconstruction of (G, X) when G is disconnected. This problem is related to the reconstructibility of the degree pair sequence, proved in Chapter 5. In Theorem 3.3.10 we proved that the degree pair sequence is reconstructible, if $e(X) > 2(v(X) - 1)$.

Remark 5.5.6. If (G, X) has n vertices and X has at most $2(n-1)$ edges, then G or G^c is disconnected or a tree.

We have proved that if G is a spanning tree, then (G, X) is sign switching reconstructible. Thus if we proved that G disconnected implies (G, X) reconstructible, we would have a simpler proof of the reconstructibility of the degree pair sequence.

In Chapter 5 we presented problems of reconstruction of signed graphs related to the vertex reconstruction problem, in particular we formulated the sign vertex reconstruction problem. Such a problem may be solved for some special class of graphs.

The following variation of Stanley's vertex switching reconstruction problem is also of interest. Let $(G, X)_v$ be the signed graph obtained by switching the sign of all edges incident to the vertex v. Consider the multiset $\{(G, X)^*_v \mid v \in V(G, X)\}$.

Problem 5.5.7. *Is (G, X) reconstructible from $\{(G, X)^*_v \mid v \in V(G, X)\}$, up to isomorphism?*

We can ask about the reconstructibility of (G, X) from $\{(G, X)^*_v \mid v \in V(G, X)\}$

when G or X are contained in some special class of graphs. We can also ask about the reconstructibility of the degree pair sequence in this case.

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