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Entanglement, decoherence and estimation in neutrino oscillations.

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Abstract

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In this work, we studied entanglement and decoherence in neutrino oscillations from a quantum metrology perspective. A summary of the quantum field theoretic description of neutrinos was done and the main quantum mechanical models were used to analyze the influence of entanglement and decoherence on the quantum Fisher information related to the mixing angle. We have found that, although in a variety of physical settings there is a positive relationship between entanglement and the quantum Fisher information, this is not the case in the neutrino oscillations scenario. This shows that, at least in single-particle settings, entanglement is not directly related to the optimal estimation in metrological tasks.

Keywords: Quantum Fisher information, neutrino oscillations, entanglement.

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"*So the problem is not so much to see what nobody has yet seen, as to think what nobody has yet thought concerning that which everybody sees.* "
[Arthur Schopenhauer]

Contents

Introduction

Neutrinos were first proposed in 1930 by W. Pauli when he tried to solve the discreteness of the spectrum of α and γ rays in radioactive experiments and the spin statistics problem in β–decay [1]. When E. Fermi formulated his theory of β–decay [2] in 1933, no doubts of neutrinos' existence had been left, even though they had not been observed directly in experiments yet, which happened only in 1956 due to F. Reines and C.L. Cowan[1] [3].

The discovery of parity violation by T.D. Lee and C.N. Yang in 1956 [4] led to the formulation of the V-A theory of weak interactions, where neutrinos are left-handed and anti-neutrinos, right-handed. In 1967, the Glashow-Weinberg-Salam Standard Model was formulated [5], which predicted weak neutral currents and the Z boson. Its success was established with the detection of neutral-neutrino interactions in the Gargamelle experiment at CERN, in 1973 [6]. The Standard Model was definitely taken to be the model of leptonic and hadronic weak and electromagnetic interactions in the next year, due to the discovery of the $W^{\pm}$ and Z bosons and the charm quark [7].

In spite of the astonishing success of the Standard Model (SM) to account for the various facets of high-energy Physics, the phenomenon of neutrino oscillations requires neutrinos to be massive, in contrast to their description in the SM where they are massless. This indicates that the SM is an effective theory of another yet unknown theory [1]. Therefore, a clear understanding of the phenomenon of neutrino oscillations is a key step towards the development of new Physics beyond the Standard Model.

[1]This experiment was of the type called reactor neutrino. See fig. 4.

Neutrino oscillations were first proposed by B. Pontecorvo in 1957, in analogy with the strangeness oscillation in the $K^0 \rightleftharpoons \bar{K}^0$ system [8]. The first realistic treatment of the phenomenon was based on the assumption that ν_e and ν_μ are superpositions of the mass eigenstates by Z. Maki, M. Nakagawa and S. Sakata [9], although the first intuitive understanding of neutrino oscillations came up only in 1967 with B. Pontecorvo, completed later by himself and V.N. Gribov [10]. The theory of neutrino oscillations was finally developed in the 70s by S.M. Bilenky and B. Pontecorvo [11] and others.

The most impressive triumph of the theory of neutrino oscillations was the solution of the so-called solar neutrino problem, which began in 1964 with the Homestake experiment [12] and was satisfactorily solved with the results of the SNO experiment[2] in 2002 [13]. It was explained by the oscillations of ν_e into ν_μ and ν_τ inside the Sun amplified by the Mikheev-Smirnov-Wolfenstein resonance effect [14].

Recently, the results of the long-baseline Kam-LAND experiment have confirmed the values of the neutrino oscillations parameters in accordance with the values that have been obtained from all solar neutrino experiments realized so far. These results are all nicely explained in the simplest model of three-neutrino mixing, in which the ν_μ, ν_e and ν_τ are unitary linear combinations of the mass eigenstates [1].

In fig. 1, we show the typical energy and font-detector distance in the different types of neutrino oscillations experiments. The other figures presented in this chapter give an illustration of these neutrino sources.

Nowadays, we have a rather precise knowledge of the values of some neutrino oscillations parameters, such as those of October 2016 provided by the Particle Data Group (PDG) presented in table 1 [15].

As pointed out in the same review [15], an important goal of a research program devoted to the further development of our knowledge about neutrino oscillations should be high precision measurements of the above parameters. That is where the second

[2]See fig. 5.

Source	Type of ν	$\overline{E}$[MeV]	L[km]	min(Δm^2)[eV^2]
Reactor	$\overline{\nu}_e$	~ 1	1	$\sim 10^{-3}$
Reactor	$\overline{\nu}_e$	~ 1	100	$\sim 10^{-5}$
Accelerator	$\nu_\mu, \overline{\nu}_\mu$	$\sim 10^3$	1	~ 1
Accelerator	$\nu_\mu, \overline{\nu}_\mu$	$\sim 10^3$	1000	$\sim 10^{-3}$
Atmospheric ν's	$\nu_{\mu,e}, \overline{\nu}_{\mu,e}$	$\sim 10^3$	10^4	$\sim 10^{-4}$
Sun	ν_e	~ 1	1.5×10^8	$\sim 10^{-11}$

Figure 1: Typical energy and font-detector distance values in the different types of neutrino oscillations experiments. Taken from [15].

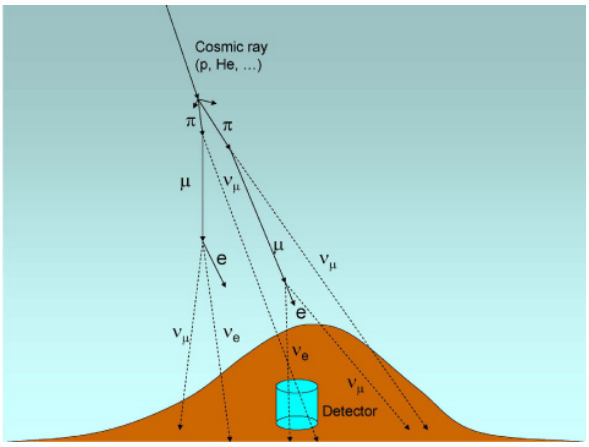


Figure 2: Illustration of neutrinos originated from cosmic rays in the atmosphere. Taken from [16].

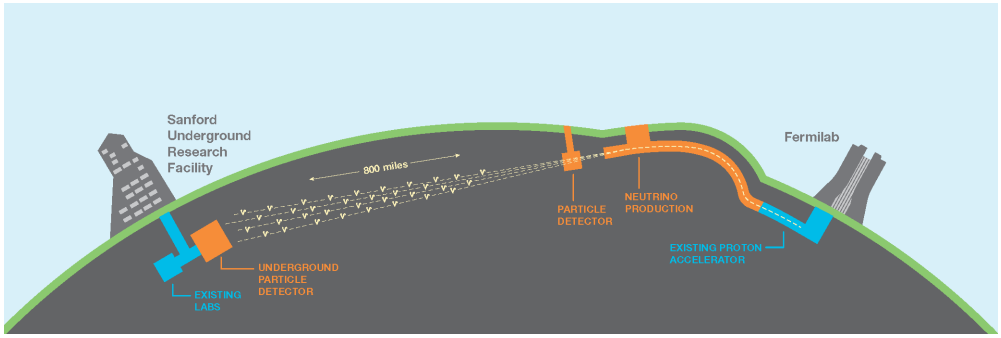


Figure 3: Illustration of neutrinos originated from particle accelerators. Taken from *http://lbnf.fnal.gov/*.

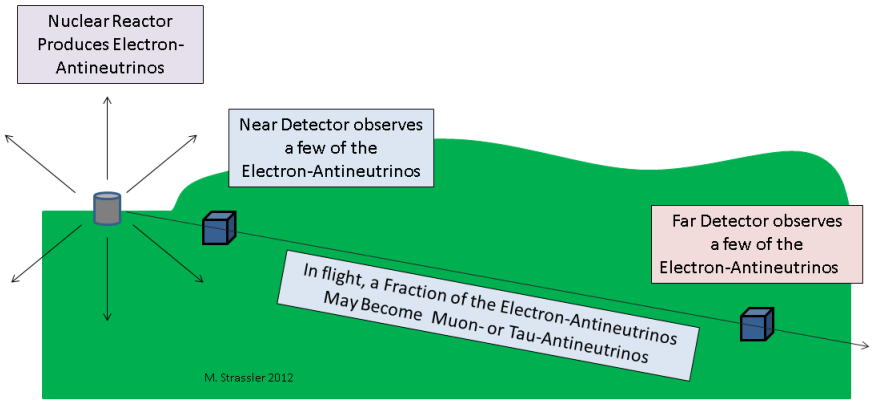


Figure 4: Illustration of neutrinos originated from nuclear reactors. Taken from *https://profmattstrassler.com/2012/04/09/a-neutrino-success-story/*.

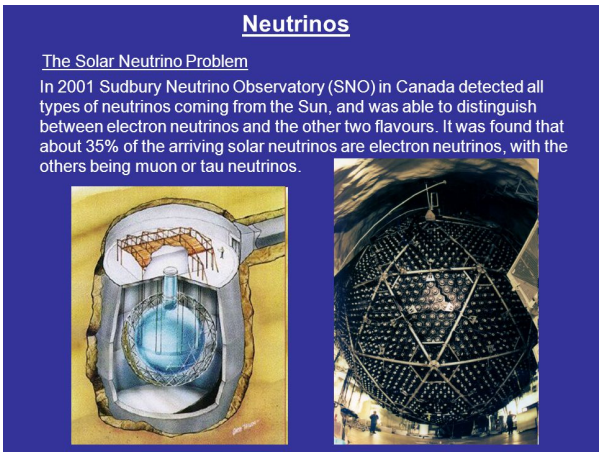


Figure 5: Illustration of Sudbury Neutrino Observatory experiment. Taken from *http://slideplayer.com/slide/4251703/*, published by Erick Arnold.

Parameter	best-fit	3σ
$\Delta m^2_{21}\,[10^{-5}eV^2]$	7.37	6.93 – 7.97
$\Delta m^2\,[10^{-3}eV^2]$	2.50 (2.46)	2.37 – 2.63 (2.33 – 2.60)
$\sin^2(\theta_{12})$	0.297	0.250–0.354
$\sin^2(\theta_{23})$, $\Delta m^2 > 0$	0.437	0.379–0.616
$\sin^2(\theta_{23})$, $\Delta m^2 < 0$	0.569	0.383–0.637
$\sin^2(\theta_{13})$, $\Delta m^2 > 0$	0.0214	0.0185–0.0246
$\sin^2(\theta_{13})$, $\Delta m^2 < 0$	0.0218	0.0186–0.0248
$^{\delta}/_{\pi}$	1.35 (1.32)	0.92 – 1.99 (0.83 – 1.99)

Table 1: The best-fit values and 3σ allowed ranges of the 3-neutrino oscillation parameters, derived from a global fit of the current neutrino oscillation data (from [15]) . For the Dirac phase δ it is given the best fit value and the 2σ allowed ranges; at 3σ no physical values of δ are disfavored. The values (values in brackets) correspond to $m_1 < m_2 < m_3$ ($m_3 < m_1 < m_2$). The definition of Δm^2 used is: $\Delta m^2 = m_3^2 - \left(m_2^2 + m_1^2\right)/2$. Thus, $\Delta m^2 > 0$ if $m_1 < m_2 < m_3$, and $\Delta m^2 < 0$ if $m_3 < m_1 < m_2$.

component of this work comes into action: quantum estimation theory (QET).

QET is the appropriate framework whenever it is of interest to determine the value of some quantity that does not correspond directly to an ordinary quantum observable of the system [17, 18]. The idea is to measure some other observables and, based on the results of such measurements, infer the value of the quantity we are interested in. More specifically, this is the so-called *local* QET [3].

Given that we already know the mean value of the parameter we want to measure, we intend to find the measurement scheme that allows us to achieve the least uncertainty, i.e. variance. This is quantified by the Quantum Fisher Information (QFI) [17, 19], which gives the value of the least variance possible to reach with any measurement protocol. Moreover, as one might expect, it is extremely relevant to review some aspects of measurement theory [19, 20] in order to understand the subtletites involved in the determination of which measurement one should implement on the system.

This formalism has been applied to a variety of quantum systems and for different purposes such as in the estimation of a quantum phase [21] and estimation problems with

[3]Another paradigm in QET is known as *global* QET [17], but we will not be concerned about it in this work at all.

open quantum systems and non-unitary processes [22]. Besides this, the geometric structure of the QFI has been exploited to give an operational characterization of multipartite entanglement [17, 23].

Still related to entanglement, there has been a large piece of evidence that in multiparticle settings entanglement can be used as a resource for quantum estimation protocols [24]. Therefore, it would be of interest to investigate such a connection in the context of single-particle multipartite entanglement, which is the case in the neutrino mixing scenario [25].

That being said, this work is organized as follows. In chapter 1, we briefly review the main features of the quantum mechanical models of neutrino oscillations and point out their strong and weak points. Although it will not be used throughout the work, we discuss quantum field theory and the Dirac equation in the appendix A for the sake of completeness, since a proper treatment of spin-$1/2$ particles, such as the neutrinos, requires this mathematical framework.

In chapter 2, we cover entanglement and decoherence, two key ingredients for the purposes of this work. We discuss the main features of entanglement measures and stress the role of single-particle multimode entanglement in the context of neutrino mixing. Regarding decoherence, we recall the basic features of the formalism of open quantum systems and discuss some models of decoherence, their importance and adequacy to neutrino oscillations.

In chapter 3, we talk about the statistical and measure theoretical aspects of quantum mechanics, since these concepts will be of absolute relevance to the development of the ideas we will come up with later, and the concepts of QET that we will need.

After that, in chapter 4, we apply the formalism we have just explained to the case of two-neutrino mixing, within an open quantum system model to implement the decoherence due to different velocities of propagation of the neutrino mass eigenstates, in the estimation of the mixing angle. We show that the maximum precision is achieved with the

mass measurement and compare such precision with that of flavor measurement, which is the measurement scheme implemented in real neutrino oscillation experiments. The main result of this work is then presented: in contrast to multiparticle settings, where entanglement is directly related to the precision in metrological tasks, that is not the case in the neutrino oscillations, which shows that regarding quantum metrology single-particle multimode entanglement is different from multiparticle entanglement.

Finally, in chapter 5, we show our conclusions and works in progress regarding this theme.

Chapter 1

Quantum mechanics of neutrino oscillations

Although a proper treatment of neutrino oscillations ultimately requires a QFT approach, its main ingredient, the mismatch between mass and flavor, can be accounted for in simple quantum mechanical models [26]. We will focus our discussion in a two-flavor neutrino oscillation scenario, since this is relevant in the context of atmospheric neutrinos and allows for the analytical computation of all formulas of interest to us.

1.1 Plane wave model

We can understand the mechanism of neutrino oscillations as follows [27]: There is an hermitian flavor operator $\hat{F}$ which does not commute with the Hamiltonian $\hat{H}$ that governs the propagation of the particles, of which the corresponding eigenstates have well defined masses m_i and momenta $\vec{p_i}$:

$$\hat{H} \left|\nu_i\right\rangle = E_i \left|\nu_i\right\rangle = \sqrt{\vec{p_i}^2 + m_i^2} \left|\nu_i\right\rangle . \tag{1.1}$$

Therefore, although we don't have a common basis for both operators, we can write each set of eigenstates in terms of the other, since they are both bases for the Hilbert

space[1]:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle \,, \tag{1.2}$$

where $U_{\alpha i}$ is the unitary Pontecorvo-Maki-Nakagawa-Sakata matrix [27]. Accordingly to quantum mechanics [28], the probability amplitude for a flavor neutrino ν_α to propagate from the space-time origin to a position $\vec{x}$ in a time t and be detected as a flavor neutrino ν_β is[2]

$$\mathcal{A}_{\alpha\to\beta}(t,\vec{x}) = \langle\nu_\beta| \exp(-i\hat{H}t + i\hat{\vec{P}}\cdot\vec{x}) |\nu_\alpha\rangle \,. \tag{1.3}$$

Using eqs. (1.1) and (1.2), we can rewrite this amplitude as

$$\mathcal{A}_{\alpha\to\beta}(t,\vec{x}) = \sum_j U_{\alpha j} U^\dagger_{j\beta} e^{-i\phi_j}, \tag{1.4}$$

which renders the oscillation probability

$$P_{\alpha\to\beta}(t,\vec{x}) = \sum_{j,k} U_{\alpha j} U_{\alpha k} U^\dagger_{j\beta} U^\dagger_{k\beta} e^{-i(\phi_j-\phi_k)}, \tag{1.5}$$

where $\phi_j = E_j t - \vec{p}_j\cdot\vec{x}$.

Now, we assume that the differences in masses and momenta between the propagation eigenstates is very small, so that we can expand the phases ϕ_j around the average mass m and average momentum $\vec{p}$ [26]. Exploiting the invariance of eq. (1.5) under an arbitrary rephasing $U_{\alpha i} \longrightarrow e^{i\psi_a} U_{\alpha i} e^{i\psi_b}$ [29], we can take the PMNS-matrix as a real unitary matrix with only one parameter, the *mixing angle* θ:

$$U = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{1.6}$$

Since neutrino masses are extremely small, we may assume that neutrinos are usually ultra-relativistic (propagate with speed close to the speed of light). Under these conditions, the probability (1.5) reads, for the two-flavor scenario:

$$P_{\alpha\to\beta}(\vec{x}) = \frac{1}{2}\sin^2(2\theta)\left(1 - \cos\left(\frac{2\pi|\vec{x}|}{L^{osc}_{12}}\right)\right), \tag{1.7}$$

[1]We will use the convention that greek indices refer to flavor eigenstates and latin indices to mass eigenstates.

[2]For the sake of convenience, we shall adopt the so-called natural units, where $\hbar = c = 1$.

with

$$L_{12}^{osc} \equiv \frac{4\pi E}{m_1^2 - m_2^2} \tag{1.8}$$

being the *oscillation length* and $E = \sqrt{\vec{p}^2 + m^2}$.

Hence, we see that, in this model, the probability of flavor oscillations oscillates in space between 0 and the maximum value $\sin^2(2\theta)$, and it also elucidates how this effect takes place due to the different masses of the different propagating neutrino eigenstates.

However, in spite of its simplicity, this model suffers from some pathological inconsistencies as it has been pointed out by many authors [26, 27, 30] such as: plane waves are completely delocalized in space, so that propagating particles must be described by wave packets; a correct oscillation formula must take into account that if the energy-momentum uncertainty in the production process is much less than the mass differences of the neutrinos, then one can determine which mass eigenstate is propagating and no oscillations will be observed, since they are a consequence of the superposition of mass eigenstates. Therefore, in the next subsection, we will refine our understanding of the quantum mechanics of neutrino oscillations by looking at a wave packet treatment.

1.2 Wave packet approach and decoherence

In order to remedy the issues remarked at the last paragraph of the preceding section we will now regard each mass eigenstate $|\nu_i\rangle$ as a gaussian state with definite mean momentum p_i and a momentum spread σ_p in one spatial dimension, which for simplicity we will assume to be the same for each mass eigenstate, i.e. [26]:

$$|\nu_i\rangle = \int \frac{dp}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{2\pi}\sigma_p}\right)^{\frac{1}{2}} e^{-\frac{(p-p_i)^2}{4\sigma_p^2}} |p, m_i\rangle \,. \tag{1.9}$$

Substitution of the above expression and eq. (1.2) into eq. (1.3) and again expanding the energy and momentum around the mean values yields, instead of eq. (1.5), the

probability

$$P_{\alpha\to\beta}(t,x) = \sum_{j,k} U_{\alpha j}U_{\alpha k}U^{\dagger}_{j\beta}U^{\dagger}_{k\beta}\frac{1}{\sqrt{2\pi}\sigma_x} \times \exp\left[-i(E_j-E_k)t + i(p_j-p_k)x - \frac{1}{4\sigma_x^2}\left((v_jt-x)^2+(v_kt-x)^2\right)\right], \tag{1.10}$$

where $E_i = \sqrt{p_i^2+m_i^2}$ and $v_i = \dfrac{p_i}{E_i}$.

A last consideration regarding the above expression is that it involves time explicitly but, in actual experiments in neutrino oscillations, one has no information about the time interval between the production and detection process, and the detections at specific font-detector distances account for all possible times. Therefore, what is actually measured is the time integral of eq. (1.10), which may be conveniently done by completing squares in the exponential argument. The result is:

$$\begin{aligned} P_{\alpha\to\beta}(x) = &\sum_{j,k} U_{\alpha j}U_{\alpha k}U^{\dagger}_{j\beta}U^{\dagger}_{k\beta}\frac{1}{\sqrt{2\pi}\sigma_x}\sqrt{\frac{2}{v_j^2+v_k^2}} \\ &\times \exp\left[i\left(\frac{v_j+v_k}{v_j^2+v_k^2}(E_j-E_k) - (p_j-p_k)\right)x\right] \quad \text{(a)} \\ &\times \exp\left[-\left(\frac{x}{L^{coh}_{jk}}\right)^2\right] \quad \text{(b)} \\ &\times \exp\left[-\frac{(E_j-E_k)^2}{4\sigma_p^2(v_j^2+v_k^2)}\right] \quad \text{(c)} \\ &. \end{aligned} \tag{1.11}$$

with

$$L^{coh}_{jk} \equiv \frac{2\sigma_x\sqrt{v_j^2+v_k^2}}{|v_j-v_k|}, \tag{1.12}$$

being the *jk coherence length*.

In order to understand what the wave packet approach brings to us, we examine each of the three factors in evidence in eq. (1.11):

- factor (a) is the oscillating factor that appears in the plane wave approach. In fact,

(a) $\longrightarrow \exp[-i2\pi\frac{x}{L_{jk}^{osc}}]$, with $L_{jk}^{osc} = \dfrac{4\pi E}{m_j^2 - m_k^2}$ and E defined as in the plane wave approach in the ultra-relativistic limit [26, 31];

- term (b) is exclusively due to the wave packet approach and admits the following physical interpretation [26]: since the different massive wave packets have different group velocities, there must be a distance such that after distinct wave packets have traveled it they will be no longer in a superposition, causing the effects of neutrino oscillations to be suppressed. As one might expect, such distance is proportional to the uncertainty in the localization of the wave packets σ_x and inversely proportional to the velocity difference between the packets.

- finally, the term (c) gives us one more desired property of any satisfying model for neutrino oscillations, namely the fact that if the momentum uncertainty is much less that the energy difference of the mass eigenstates, one is able to determine which mass eigenstate is propagating and the effects of neutrino oscillations are washed out.

Now we emphasize two important aspects of the preceding analysis. To make the discussion still more clear, we specialize the expression (1.11) to the case of two-flavor neutrino oscillations again, as it was done in eq. (1.7). This time, the result is:

$$P_{\alpha\to\beta}(x) = \frac{\sin^2(2\theta)}{2}\left[1 - \exp\left(-\left(\frac{x}{L_{12}^{coh}}\right)^2\right)\exp\left(-\mu\left(\frac{\Delta p_{12}}{\sigma_p}\right)^2\right)\cos\left(2\pi\frac{x}{L_{12}^{osc}}\right)\right], \tag{1.13}$$

with μ a positive constant. Note that even when decoherence takes place, i.e. after the distance traveled is greater than the coherence length, or when one can resolve which mass eigenstate is propagating, we still have a "classical" probability that a υ_e neutrino propagates as a m_i and then it is detected as a ν_μ, plus the probability with the m_2 as the intermediate state [26]. So that when we say "neutrino oscillation effects are suppressed" we actually mean the effects caused by interference due to the superposition of wave

packets of distinct massive wave packets.

The last remark, and the more important one for the rest of this work, is that the essential feature introduced with the wave packet approach is the decoherence due to different group velocities of the associated wave packets or due to the uncertainty in momentum being much smaller than the energy difference between the massive neutrinos. However, the plane wave model allows us to understand the main features of the neutrino oscillation physics and to compute explicitly all quantities of interest to us. So, in our analyses from now on, we will implement the effects of decoherence in a plane wave model by means of an open systems formalism [32, 33], since this will make it feasible to study a physically more consistent model and at the same time to compute and study all the relevant physical quantities.

Chapter 2

Entanglement and decoherence

Entanglement might be considered as the fundamental feature of quantum mechanics. It is manifested, for instance, when the state of a compound system of two subsystems cannot be factorized as a product of pure states of the corresponding subsystems [34] which causes the two subsystems to share quantum correlations that are stronger than any classical ones, as it was shown by Bell in 1964 [28, 35]. Although, it was brought up to Physics only as an auxiliary concept of almost philosophical nature, from the end of the last century on it has become clear that it is a physical resource on the same footing as energy, and in fact it is the heart of a whole new branch of Physics called quantum information, which exploits [35] the unusual correlations between entangled subsystems to perform information protocols such as processing and distribution in ways that were impossible, or at least very inefficent, with only classical resources [34].

Nevertheless, entanglement theory is far from being fully developed. Some fundamental issues such as the quantification of bipartite entanglement in mixed states and the characterisation and interpretation of multipartite entanglement are waiting for a complete understanding [34]. Therefore, we will make a summary of the main aspects of entanglement theory that will be needed for our purposes here. For more detailed discussions and a broader exposition of entanglement theory, the reader might consult, for example, the review article [35].

2.1 Bipartite pure state entanglement

We will begin by discussing entanglement in bipartite pure states, since this is the case that is almost completely understood. Given a compound system formed by two subsystems with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we say that a state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ of the compound system is separable if there exist states $|\psi_1\rangle \in \mathcal{H}_1$, $|\psi_2\rangle \in \mathcal{H}_2$ such that $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. When that is not the case, we say that the state $|\psi\rangle$ is entangled [34].

To determine if a state is entangled or not, we can proceed as follows: let the dimensions of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ be d_1 and d_2, respectively. Then, a generic state $|\psi\rangle$ of the compound system may be written as

$$|\psi\rangle = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} \alpha_{ij} |u_i\rangle |w_j\rangle, \tag{2.1}$$

where $\{u_i\}$ and $\{w_j\}$ are bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We recall the definition of the so-called partial-trace operator ρ_1 (ρ_2), which encodes all the information regarding measurements performed only on the subsystem 1 (2)[36]:

$$\begin{aligned} \rho_1 &= \sum_{i,i'=1}^{d_1}\sum_{j=1}^{d_2} \alpha_{ij}\alpha^*_{i'j} |i\rangle \langle i'| ; \\ \rho_2 &= \sum_{i=1}^{d_1}\sum_{j,j'=1}^{d_2} \alpha_{ij}\alpha^*_{ij'} |j\rangle \langle j'| . \end{aligned} \tag{2.2}$$

From now on we assume $d_1 \leq d_2$. If we take in $\mathcal{H}_1$ the basis $|u_i\rangle$ that diagonalizes ρ_1, we can write eq. (2.1) as $|\psi\rangle = \sum_{i=1}^{d_1} |u_i\rangle |v_i\rangle$, with $|v_i\rangle = \sum_{j=1}^{d_2} \alpha_{ij} |w_j\rangle$. We say that the states $\{v_i\}$ are the mirror states in $\mathcal{H}_2$ of the states $\{u_i\}$ that diagonalize ρ_1 [36]. As a consequence, these mirror states are also orthogonal, since $\langle u_i| \rho_1 |u_j\rangle = \lambda_j \delta_{ij} = \langle v_i| v_j\rangle$. Therefore, introducing the states $|\hat{u}_i\rangle = \dfrac{|v_i\rangle}{\sqrt{\lambda_i}}$, we conclude that every pure state of a bipartite system can be written in the so-called *Schmidt expansion* [36]:

$$|\psi\rangle = \sum_{i=1}^{d_1} \sqrt{\lambda_i} |u_i\rangle |\hat{u}_i\rangle . \tag{2.3}$$

We emphasize the simplification that eq. (2.3) brings to us, both computationally and conceptually. Regarding the former aspect, we notice that whereas expression (2.1) might have $d_1 d_2$ terms, the expansion (2.3) has at most d_1 terms, independently of the dimension of the larger Hilbert space. On the conceptual aspect, it is clear that the state $|\psi\rangle$ will be entangled if, and only if, there is only one term in the Schmidt expansion.

Moreover, it is also evident from the Schmidt form that the partial trace operators from eq. (2.2) have the same eigenvalues. These eigenvalues still make it possible to define the canonical measure of entanglement for this particular class of states, through the entropy of entanglement (Von Neumann entropy) [34, 36]:

$$S_1 = S_2 = -\sum_{i=1}^{d_1} \lambda_i \log(\lambda_i). \tag{2.4}$$

When there is no non-zero degenerated eigenvalue, the Schmidt expansion (2.3) is unique. Otherwise, the same entangled state can be represented by different, but equivalent (from the point of view of entanglement), Schmidt forms [36].

2.2 Bipartite mixed state entanglement

Now we consider bipartite entanglement in mixed states. A generic bipartite state can always be written as a convex combination of pure states: $\rho = \sum_i \lambda_i |\psi_i\rangle \langle\psi_i|$. However, this decomposition is not unique [28]. Therefore, if one performs local measurements on the subsystems and detect correlations, the ambiguity in the description makes it impossible to determine if the correlations are due to a genuine quantum interaction (entanglement) or have been induced by means of Local Operations and Classical Communication (LOCC) [34]. So this is the essential idea in entanglement theory: we say that a state is separable when there is at least one way of engineering the state relying only on LOCC. Otherwise, we say that the state is entangled, which means that there are correlations between the subsystems that can be generated only through a genuinely quantum interaction [37]. Mathematically, this is expressed as follows: a bipartite state

ρ is called separable if there exist coefficients $\{p_k \geq 0 : \sum_k p_k = 1\}$ and states $\{\sigma_k\}$ acting on $\mathcal{H}_1$ and $\{\tau_k\}$ acting on $\mathcal{H}_2$ such that [34]

$$\rho = \sum_k p_k \, \sigma_k \otimes \tau_k. \tag{2.5}$$

Otherwise, the state ρ is said to be entangled.

Although the above definition is well motivated and physically reasonable, it is highly unfeasible computationally. How would one examine all possible decompositions of the form (2.5) to check that a state is entangled? So it is necessary to have practical criteria to decide about entanglement in the mixed state scenario. For the class of 2×2 systems, that we will consider throughout this dissertation, there is the celebrated Peres-Horodecki criterion, also called Positive Partial Transpose (PPT) criterion, that is simple and equivalent to the definition of entanglement [38, 39]: a state ρ is separable if and only if the partial transpose state is positive semidefinite, i.e. it is also a valid quantum state. The partial transposition is defined as follows: given bases $\{u_i\}$ of $\mathcal{H}_1$ and $\{v_i\}$ of $\mathcal{H}_2$, the elements of the partial transpose (with respect to subsystem 2) ρ^{T_2} are defined as:

$$\langle u_i| \langle v_j| \rho^{T_2} |u_k\rangle |v_l\rangle \equiv \langle u_i| \langle v_l| \rho |u_k\rangle |v_j\rangle . \tag{2.6}$$

This criterion is always necessary for separability, but we emphasize here that it is also sufficient *only* for 2×2 and 2×3 bipartite systems. An example of a class of PPT states (i.e. entangled states that do *not* violate the PPT criterion) in a 3×3 system is:

$$\rho_a = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}, \tag{2.7}$$

with $0 < a < 1$.

Nevertheless, as it was already mentioned, our discussions will deal only with 2×2 systems, so we can rely upon the PPT criterion and, also in this context, use an appropriate measure of entanglement based on the violation of this criterion. We first define the *negativity* $\mathcal{N}$ by [34]:

$$\mathcal{N}(\rho) = \max\left\{0, -\sum_k \lambda_k^-\right\}, \tag{2.8}$$

with $\{\lambda_k\}$ being the negative eigenvalues of the partial transpose.

Finally, we define the logarithmic negativity [34] $E_{\mathcal{N}}$ by

$$E_{\mathcal{N}}(\rho) = \log\left(1 + 2\mathcal{N}(\rho)\right). \tag{2.9}$$

We will use the logarithmic negativity because it satisfies the physical requirement of being a full monotone under LOCC, i.e. $E_{\mathcal{N}}\left(\hat{O}_{LOCC}(\rho)\right) \leq E_{\mathcal{N}}(\rho)$, expressing the fact that entanglement cannot be created under the action of LOCC only. Moreover, it is related to other entanglement measures that are important in other contexts [34], so that any possible result obtained with it can be investigated in other scenarios as well.

2.3 Open quantum systems formalism

Actual quantum systems always interact with their environment. As a consequence, a proper description of these systems always need to take into account the environment, whose correlations with the system of interest washes out, in general, the quantum coherences of the state of the system in much the same way that the information about the path in a Young slit experiment destroys the interference pattern [36]. Hence, in general, we will need to describe the system by a mixed state.

We could describe completely the evolution of the quantum system by using the postulates of quantum mechanics: we should write down the state ρ of the system A plus environment E, the total Hamiltonian H of the system plus environment, evolve the state ρ and then trace over the degrees of freedom of the environment to obtain, at any instant

we want, the state ρ_A of the system. However, as in the definition of entanglement (2.5), this turns out to be out completely unfeasible: a generic environment has a huge amount of degrees of freedom that one would not be able to compute the state ρ_A of the system in the aforementioned manner.

A generic evolution of a quantum system A with Hilbert space $\mathcal{H}_A$ of dimension N_A, being it caused by the usual Hamiltonian, by a generalized measurement or interaction with an environment, can always be described by a *quantum dynamical map* $\Gamma(t)$ which maps an initial state ρ at time $t_0 = 0$ of the system into a state $\rho(t) = \Gamma(t)\rho$ at a time $t > 0$. By this, we mean a collection $\Gamma(t),\ t \in \mathbb{R}^+$ of linear maps satisfying [36, 40]:

$$\text{Trace preserving: } \operatorname{Tr}(\Gamma(t)\rho) = 1; \tag{2.10}$$

$$\text{Positivity preserving: } \begin{cases} [\Gamma(t)\rho]^\dagger = \Gamma(t)\rho; \\ \langle\psi|\,\Gamma(t)\rho\,|\psi\rangle \geq 0 \quad |\psi\rangle \in \mathcal{H}_A. \end{cases} \tag{2.11}$$

$$\text{Complete positivity: } \Gamma(t) \otimes \mathbb{I}_n \geq 0, \quad \forall n \in \mathbb{N}; \tag{2.12}$$

$$\text{Strong continuity: } \lim_{t\to 0} \Gamma(t)\rho = \rho, \quad \forall \rho \text{ acting on } \mathcal{H}_A. \tag{2.13}$$

Conditions (2.10) and (2.11) are needed to keep, at any instant, the probabilistic interpretation of ρ. The necessity of condition (2.12) can be seen as follows: imagine that the system was entangled with another system B, described by a Hilbert space of dimension n, by an unknown interaction and that they have been separated so that they no longer interact. Then, we must consider the whole state ρ_{AB}. Since the systems no longer interact, the operator acting on ρ_{AB} is $\Gamma(t) \otimes \mathbb{I}_n$. So the maps $\Gamma(t)$ must preserve the positivity of ρ_{AB} in order to represent a physically sound process. Since this must be valid for whichever system B, n can be an arbitrary natural number and this property is called complete positivity.

A remarkable result in this context is that any trace preserving completely positive linear map can be cast into a sum [40]:

$$\Gamma(t)\rho = \sum_{i=1}^{N_K} M_i(t)\rho M_i^\dagger(t), \quad N_K \leq N_A^2, \quad \sum_{i=1}^{N_K} M_i(t)M_i^\dagger(t) = I \tag{2.14}$$

called the *Kraus sum representation* [41]. It means that no matter how complicated the environment might be, or whatever process the system might be undergoing such as decoherence, the evolution of the quantum system can be cast into a relatively simple form as a sum of at most N_A^2 operators.

Moreover, under certain conditions found in a variety of optical system [36] and, more important for us, in the context of neutrino oscillations as well, such representation can be cast into an even simpler representation through a differential equation for the density operator of the system, called the *Lindblad master equation* [33, 40, 42]:

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_{i=1}^{N_K-1} L_i \rho(t) L_i^\dagger - \{J, \rho(t)\}, \tag{2.15}$$

with H being the renormalized Hamiltonian[1] of the system under study alone, $J = \frac{1}{2}\sum_{i=1}^{N_K-1} L_i^\dagger L_i$ and $\{L_i\}$ are called the Lindbladian operators. In this context, we can interpret eq. (2.15) as a quantum master equation, with the operators L_i being associated to the different quantum jumps that the system might undergo [36].

The validity of such approximation is that the interaction with the environment and the system is weak, in the sense that any "memory effects" on the evolution of the system can be neglected. By this, we mean that any entanglement between the environment and the system has a typical time scale τ_c that is much less than the typical time scale T_e of the evolution of the system itself. Then, we can always assume that, at any fixed instant, the total state of the system plus environment is of the form $\rho_{AB}(t) = \rho_A \otimes \rho_B$. In this way, the state of the system $\rho(t+\tau)$ depends only on the state $\rho(t)$ and not on the state at any prior times, with $\tau_c \ll \tau \ll T_e$, which permits one to pass from (2.14) to (2.15).

[1]The nomenclature "renormalized" here means that H will have the same form of the free Hamiltonian of the system, but the energy levels will be shifted due to the interaction of A with the environment. For example, in the case of spontaneous emission of a two-level atom, the energy levels contain the Lamb shift due to the interaction of the atom and the vacuum energy [36].

Chapter 3

Quantum estimation and quantum measurements

According to the theory of quantum mechanics, a physical observable, such as energy or angular momentum, is represented by an Hermitian operator acting on a Hilbert space that represents the physical system under study. The value we attribute to such physical observable is then the expectation value of the corresponding operator on the state the physical system is found in [28]. However, a quantum state might carry information encoded in some parameter that does not correspond to a physical observable. An example of such situation is when one is interested in determining the value of the noise parameter of depolarizing or amplitude-damping [43]. Then, to determine the value of the parameter one has to perform some measurements on the system and, based on the results of these measurements, infer the value of the quantity of interest. This situation is properly addressed in the framework of Quantum Estimation Theory (QET)[44, 45], which we will summarize in this chapter.

3.1 Generalized measurements

To begin with, the concept of measurement is central to estimation theory, so we briefly digress about the theory of quantum measurements before to proceed. The fundamental measurement postulate of quantum mechanics, as given by Von Neumann [46], says that

a measurement is described by a set of projectors $\{\Pi_j\}$ such that, for a system in a state ρ, the outcome j occurs with probability $\mathrm{Tr}(\Pi_j\rho)$ and, when it indeed occurs, the state of the system changes to $\dfrac{\Pi_j\rho}{\mathrm{Tr}(\Pi_j\rho)}$. However, this is not the most general way of measuring, i.e. acquiring information about, a quantum system [47].

Suppose we have two systems A, the "target" that we want to acquire information about, and B, the "probe" that is initially prepared in a known state and then it is allowed to interact with system A. After the interaction, one performs a projective measurement on the probe. Since the interaction induces correlations between the target and the probe, such a projective measurement on the probe gives us information about the target [47]. Such a procedure is called a *generalized measurement*. Now that we have discussed the physics of this process, let us go through the maths.

Let $\{|n\rangle, n = 0, \ldots, N-1\}$ be the orthonormal basis corresponding to the projective measurement to be performed on the probe and assume the probe to be in the state $|0\rangle$ before the interaction. Then, if ρ is the initial state of the target, the initial state ρ_{tot} of the total system target+probe is

$$\rho_{tot} = |0\rangle\langle 0| \otimes \rho. \tag{3.1}$$

Now, the interaction between the target and the probe should be described by a unitary matrix U acting on the total Hilbert space $\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_A$. We expand it in terms of a basis $\{|i\rangle, i = 0 \ldots, M-1\}$ of the target and the basis of the projective measurement:

$$\begin{aligned} U &= \sum_{i',i=0}^{M-1} \sum_{n,n'=0}^{N-1} \alpha_{n,n',i,i'} |n\rangle |i\rangle \langle n'| \langle i'| \\ &= \sum_{n,n'=0}^{N-1} |n\rangle\langle n'| \otimes A_{n,n'}, \end{aligned} \tag{3.2}$$

where $A_{n,n'} = \sum_{i,i'=1}^{M-1} \alpha_{n,n',i,i'} |i\rangle\langle i'|$ is the sub-block of U acting on the target corresponding to a pair of probe states n, n' [47]. Now, since U is unitary and denoting simply by A_n

the sub-blocks A_{n0}, we have

$$U^\dagger U = \sum_{n,n',p=0}^{N-1} |n\rangle \langle n'| \otimes A_{pn}^\dagger A_{pn'} = I_{NM}. \tag{3.3}$$

In particular, the term $n = n' = 0$ gives

$$\sum_{n=0}^{N-1} A_n^\dagger A_n = I_N. \tag{3.4}$$

Conversely, given N matrices $\{A_n\}$ satisfying eq. (3.4) and $M \in \mathbb{N}$, the matrix

$$U = \begin{pmatrix} & & \overbrace{\quad}^{M-3 \text{ zeros}} & & \\ A_0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{N-1} & 0 & \cdots & 0 & 0 \\ 0_{1\times N} & 1 & 0 & \cdots & 0 \\ 0_{1\times N} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0_{1\times N} & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is unitary. Hence, we conclude that any set $\{A_n\}$ of matrices satysfing eq. (3.4) can represent an interaction between the target and a probe.

After the interaction, the state of the total system is

$$\overline{\rho_{tot}} = U \rho_{tot} U^\dagger = \sum_{p,m=0}^{N-1} |p\rangle \langle m| \otimes A_p \rho A_m^\dagger. \tag{3.5}$$

Now we can ask: what is the probability of getting the value n in the projective measurement on the probe? According to the Von Neumann's postulate, it is:

$$p_n = \mathrm{Tr}(|n\rangle \langle n| \otimes I_M \overline{\rho_{tot}}) = \mathrm{Tr}(A_n^\dagger A_n \rho). \tag{3.6}$$

Once more in accordance to the Von Neumann's postulate, if such a result occurs, the total system will be found in the state

$$\widetilde{\rho} = \frac{|n\rangle \langle n| \otimes I_M \overline{\rho_{tot}} |n\rangle \langle n| \otimes I_M}{p_n} = |n\rangle \langle n| \otimes \frac{A_n \rho A_n^\dagger}{p_n}. \tag{3.7}$$

Therefore, after result n is obtained by the projective measurement on the probe, the target is found in the state given by eq. (3.7). So, we see that, mathematically, a

generalized measurement on a quantum system, described by a Hilbert space $\mathcal{H}$ and a state ρ, is a set $\{A_n, n = 0, \ldots, N-1\}$ of operators satisfying $\sum_{n=0}^{N-1} A^\dagger A = I_{\mathcal{H}}$ corresponding to N possible outcomes, such that the outcome n occurs with probability $p_n = \mathrm{Tr}(A^\dagger A \rho)$ and the state of the system changes to $\frac{A\rho A^\dagger}{p_n}$ when result n happens [36, 47].

A final remark on quantum measurements is that the terminology Positive Operator Valued Measure (POVM) is also widely used to refer to them. The motivation for this terminology is that a measure is a map that ascribes a number to every subset of a given set, so a POVM ascribes a positive operator to every subset of a given set. In the case of a generalized measurement, given any subset $\mathcal{M}$ of the possible outcomes, the probability for the measurement to be in $\mathcal{M}$ is $\sum_{n\in\mathcal{M}} \mathrm{Tr}(A_n^\dagger A_n \rho) = \mathrm{Tr}\left(\sum_{n\in\mathcal{M}} A_n^\dagger A_n \rho\right)$, which associates with $\mathcal{M}$ the positive operator $\sum_{n\in\mathcal{M}} A_n^\dagger A_n$ [47].

3.2 Quantum estimation theory and the Quantum Fisher information

Now, we move on to the quantum estimation theory. As mentioned before, this is the framework we should use when we want to know the value of a parameter that does not correspond directly to the eigenvalue of some Hermitian operator. Then, we measure the system to infer the value of this parameter. There are two paradigms regarding QET: global QET and local QET [45, 44]. In the global aspect, one is looking for a POVM that extremize some cost functional averaged over all values of the parameter. In the local aspect, one looks, at a *fixed* value of the parameter, for a POVM maximizing the Fisher information [48], and thus minimizing the variance of the parameter. We will treat only the local aspect in this work. For a general discussion and examples of the global paradigm, the interested reader might consult ref. [44].

In the local paradigm, we will restrict ourselves to the case of *unbiased* estimators [19, 49] and treat the case of only one parameter θ to be estimated by measuring some

observable M. By an *estimator*, we mean a function $\lambda_\theta : Spec(M) \to \mathbb{R}$ from the set of eigenvalues (spectrum) of M to the set of real numbers that ascribes, to each result x of the measurement, a value $\lambda_\theta(x)$ to be assigned to the parameter θ [45, 49]. The relationship between parameter and measurement outcomes is given by a probability distribution $f(x;\theta)$, which gives the probability of the measurement outcome to take a value x given that the value of the parameter is θ. We say that the estimator is unbiased when

$$\langle \lambda_\theta \rangle = \sum_x \lambda_\theta(x) f(x;\theta) = \theta. \tag{3.8}$$

Now, a central result of estimation theory is that the precision that can be achieved by any classical data processing procedure and any unbiased estimator λ_θ is given by the Cramer-Rao bound [44, 48]

$$Var(\lambda_\theta) \geq \frac{1}{NF(\theta)}, \tag{3.9}$$

with N being the number of measurements performed and

$$F(\theta) = \sum_x f(x;\theta) \left(\frac{\partial \log f(x;\theta)}{\partial \theta} \right)^2 = \sum_x \frac{1}{f(x;\theta)} \left(\frac{\partial f(x;\theta)}{\partial \theta} \right)^2 \tag{3.10}$$

is the *Fisher information* [49].

In the quantum scenario, an estimation problem is stated in terms of states ρ_θ on some Hilbert space $\mathcal{H}$, with θ ranging over some open interval in $\mathbb{R}$. A quantum estimator $\hat{\lambda_\theta}$ is then a self-adjoint operator that corresponds to a quantum measurement followed by a classical data processing protocol. This indirect way of estimation brings an additional uncertainty to the measured value and the goal of local QET is to optimize the inference procedure by minimizing this additional uncertainty [44, 45].

According to the postulates of quantum mechanics and the formalism of generalized measurements we have seen in the previous section, we have

$$f(x;\theta) = \mathrm{Tr}(\Pi_x \rho_\theta), \tag{3.11}$$

with $\{\Pi_x\}$ being elements of a POVM representing the generalized measurement procedure. Now we introduce the self-adjoint operator called *Symmetric Logarithmic Derivative* (SLD) L_θ [45], defined to be the solution of

$$\frac{\partial \rho_\theta}{\partial \theta} = \frac{\rho_\theta L_\theta + L_\theta \rho_\theta}{2}. \tag{3.12}$$

With the aid of this operator, we can write the derivative of the distribution f given by eq. (3.11) as $\partial_\theta f(x;\theta) = \mathrm{Tr}(\partial_\theta \rho_\theta \Pi_x) = \mathrm{Re}(\mathrm{Tr}(\rho_\theta \Pi_x L_\theta))$[1]. Then, the Fisher information (3.10) associated with this measurement scheme becomes

$$F(\theta) = \sum_x \frac{\mathrm{Re}(\mathrm{Tr}(\rho_\theta \Pi_x L_\theta))^2}{\mathrm{Tr}(\rho_\theta L_\theta)}. \tag{3.13}$$

The above expression gives the optimal precision in estimating the parameter θ by any classical data processing scheme performed on the measurement outcomes of the POVM $\{\Pi_x\}$. So, it is reasonable to ask whether or not there exist an optimal POVM, i.e. a POVM such that eq. (3.13) is greater than with any other POVM. Remarkably, the answer is positive as the following reasoning shows [45]:

$$\begin{aligned}
F(\theta) &= \sum_x \frac{\mathrm{Re}(\mathrm{Tr}(\rho_\theta \Pi_x L_\theta))^2}{\mathrm{Tr}(\rho_\theta L_\theta)} \\
&\leq \sum_x \left| \frac{\mathrm{Tr}(\rho_\theta \Pi_x L_\theta)}{\sqrt{\mathrm{Tr}(\rho_\theta L_\theta)}} \right|^2 \\
&= \sum_x \left| Tr\left(\frac{\sqrt{\rho_\theta}\sqrt{\Pi_x}}{\sqrt{\mathrm{Tr}(\rho_\theta L_\theta)}} \right) \sqrt{\Pi_x} L_\theta \sqrt{\rho_\theta} \right|^2 \\
&\leq \sum_x \mathrm{Tr}(\Pi_x L_\theta \rho_\theta L_\theta) \\
\implies F(\theta) &\leq H(\theta) \equiv \mathrm{Tr}(\rho_\theta L_\theta^2).
\end{aligned} \tag{3.14}$$

The quantity $H(\theta)$ is called the *Quantum Fisher Information* (QFI) and determines the ultimate precision limit in the estimation of the parameter θ allowed by the laws of quantum mechanics, since it is independent of the measurement scheme. The explicit

[1] From now on, we adopt the shorter notation ∂_θ for $\frac{\partial}{\partial \theta}$.

form of this assertion is the so-called *quantum Cramer-Rao* bound [45, 49]

$$Var(\hat{\lambda_\theta}) \geq \frac{1}{NH(\theta)}. \tag{3.15}$$

In this context, an optimal estimator is one that achieves the equality in eq. (3.15). The inequality that leads to the bound (3.14) is saturated when the POVM $\{\Pi_x\}$ correspond to the projectors over the eigenspaces of the operator L_θ [45]. However, this does not mean that L_θ is the optimal observable to be measured, since the POVM gives complete information only about the probabilities of the measurement outcomes and not on the measurement procedure itself. Nevertheless, one explicit form for the optimal quantum estimator is

$$\hat{\lambda_\theta} = \theta\mathbb{I} + \frac{L_\theta}{H(\theta)}. \tag{3.16}$$

Indeed, using the fact that $\mathrm{Tr}(\rho_\theta L_\theta) = 0$, it is straightforward to show that $\langle\hat{\lambda_\theta}\rangle = \theta$ and $var(\hat{\lambda_\theta}) = \dfrac{1}{H(\theta)}$.

The solution of eq. (3.12) is well known from linear system's theory [50] and is:

$$L_\theta = 2\int_0^\infty dt \exp(\rho_\theta t)\partial_\theta\rho_\theta \exp(\rho_\theta t). \tag{3.17}$$

By writing $\rho_\theta = \sum_n \varrho_n |\psi_n\rangle\langle\psi_n|$, the SLD can also be represented as

$$L_\theta = 2\sum_{\{nm\}} \frac{\langle\psi_m|\,\partial_\theta\rho_\theta|\psi_n\rangle}{\varrho_n + \varrho_m} |\psi_m\rangle\langle\psi_n|, \tag{3.18}$$

where $\{nm\}$ means that the sum is carried only over the terms such that $\varrho_n + \varrho_m \neq 0$. Consequently, the QFI (3.14) can be computed as

$$H(\theta) = 2\sum_{\{nm\}} \frac{|\langle\psi_m|\,\partial_\theta\rho_\theta|\psi_n\rangle|^2}{\varrho_n + \varrho_m}. \tag{3.19}$$

Chapter 4

Quantum metrology in neutrino oscillations

We are now in a position to study neutrino oscillations from a quantum metrology perspective. We apply the ideas introduced in the previous chapters to the case of two-flavor oscillations. We will follow closely ref. [51], which is the article that resulted from the work presented in this dissertation.

4.1 Mixing angle estimation: plane-wave model

For the reader's convenience, we recall here the main ideas from chapter 2 about neutrino mixing. In the case of only two neutrino flavors (e.g. ν_e and ν_μ), this mixing is effectively described by a mixing angle θ through the relations

$$\begin{pmatrix} |\nu_e\rangle \\ \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ \\ |\nu_2\rangle \end{pmatrix}, \tag{4.1}$$

where $H|\nu_i\rangle = E_i|\nu_i\rangle, i = 1, 2$ and we will assume $m_2 > m_1$.

Neutrinos are always produced in a flavor eigenstate [26], which we will take from now on as an electron neutrino. Moreover, we will always work on the mass basis. Therefore, the density matrix for the neutrino is:

$$\rho = \begin{pmatrix} \cos^2(\theta) & \frac{1}{2}\sin(2\theta)e^{i\phi} \\ \\ \frac{1}{2}\sin(2\theta)e^{-i\phi} & \sin^2(\theta) \end{pmatrix}, \tag{4.2}$$

where $\phi \equiv \delta t \equiv \dfrac{(m_2^2 - m_1^2)\, t}{2E}$ and E is defined as in chapter 1.

Since it is a projector, it will have eigenvalues 0 and 1. The respective eigenvectors are:

$$\left\{ |0\rangle = \begin{pmatrix} -\sin(\theta)e^{i\phi} \\ \\ \cos(\theta) \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} \cos(\theta) \\ \\ \sin(\theta)e^{-i\phi} \end{pmatrix} \right\}. \tag{4.3}$$

Now, we turn to the issue of entanglement in neutrino oscillations. As mentioned before, the natural form of entanglement to consider in this context is single-particle, mode entanglement [25]. First, notice that we can look at the neutrino state space as the two-qubit Hilbert space $\mathcal{H}_\nu \subset \mathcal{H}_1 \otimes H_2$ spanned by $\{|1\rangle_1 \otimes |0\rangle_2, |0\rangle_1 \otimes |1\rangle_2\}$, by means of the equivalence defined on the mass basis by $|\nu_1\rangle \to |1\rangle_1 \otimes |0\rangle_2$ and $|\nu_2\rangle \to |0\rangle_1 \otimes |1\rangle_2$. Then, observe that in this two-qubit representation of occupation number there is a bipartition of the space of quantum states available, relative to which entanglement can be considered. Correspondingly, a neutrino state which is entangled as a two-qubit state is said to be mode entangled.

The evolution of an initial electron neutrino is then[1]

$$|\nu_e(t)\rangle = \tilde{U}_{ee}(t)\,|10\rangle + \tilde{U}_{e\mu}(t)\,|01\rangle\,, \tag{4.4}$$

the matrix $\tilde{U}(t)$ defined as

$$\tilde{U}(t) \equiv \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} e^{-iE_1 t} & 0 \\ \\ 0 & e^{-iE_2 t} \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{4.5}$$

The eq. (4.4) explicitly shows the entanglement between distinct fermionic modes in a single particle scenario, since it the Schmidt decomposition of the pure state (4.2).

Using the explicit form of ρ from eq. (4.2) and the results of the previous chapter, the SLD reads

$$L_\theta = 2 \begin{pmatrix} -\sin(2\theta) & \cos(2\theta)e^{i\phi} \\ \\ \cos(2\theta)e^{-i\phi} & \sin(2\theta) \end{pmatrix}, \tag{4.6}$$

and the QFI is

[1] Using the shorter notation $|1\rangle_1 \otimes |0\rangle_2 \leftrightarrow |01\rangle$ and so on.

$$H(\theta) = 4. \tag{4.7}$$

In this case, the operator L_θ has the eingenvalues ± 2 with corresponding eigenvectors

$$|\pm 2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} (1 \mp \sin(2\theta))^{1/2} \\ \\ \pm e^{-i\phi} (1 \pm \sin(2\theta))^{1/2} \end{pmatrix}. \tag{4.8}$$

Hence, the POVM that should be implemented to achieve the best precision is

$$|\pm 2\rangle \langle \pm 2| = \frac{1}{2} \begin{pmatrix} 1 \mp \sin(2\theta) & \pm \cos(2\theta) e^{i\phi} \\ \\ \pm \cos(2\theta) e^{-i\phi} & 1 \pm \sin(2\theta) \end{pmatrix}. \tag{4.9}$$

For the sake of comparison, we compute the FI associated with the flavor measurement, which is the actual experiment in neutrino oscillations. The result is

$$\left\{ \begin{pmatrix} \cos^2(\theta) & \frac{1}{2}\sin(2\theta) \\ \\ \frac{1}{2}\sin(2\theta) & \sin^2(\theta) \end{pmatrix}, \begin{pmatrix} \sin^2(\theta) & -\frac{1}{2}\sin(2\theta) \\ \\ -\frac{1}{2}\sin(2\theta) & \cos^2(\theta) \end{pmatrix} \right\}, \tag{4.10}$$

which gives the flavor FI

$$F_{flavor}(\theta) = \frac{4\cos^2(2\theta)\sin^2\left(\frac{\phi}{2}\right)}{1 - \sin^2(2\theta)\sin^2\left(\frac{\phi}{2}\right)}. \tag{4.11}$$

In fig. 4.1, we contrast the behaviors of the FI of the flavor measurement and of the QFI for some values of the mixing angle.

To finish this section, we also compare the behaviors of the FI and of the entanglement in the state (4.4), recalling that entanglement in pure bipartite systems is completely characterized by the Von Neumann entropy or by any of its monotones [35, 52]. In this work, we will simply rescale the Von Neumann entropy so that it has the same maximum value as the QFI. As we can see in fig. 4.2, in this model, entanglement is not associated with the best precision dictated by the FI, since the local minima of entanglement match the local maxima of the FI.

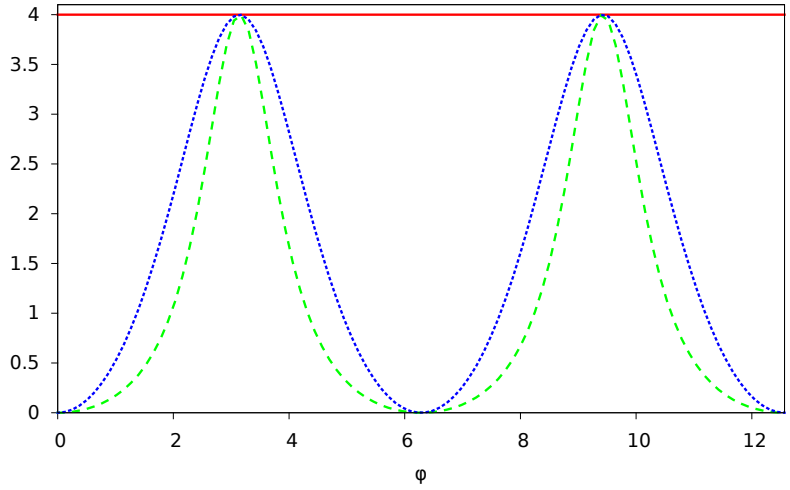


Figure 4.1: Comparison between the FI associated with the flavor measurement for the experimental value of the mixing angle θ_{12}(dashed), for $\theta = \frac{\pi}{8}$ (dotted) and the QFI (solid).

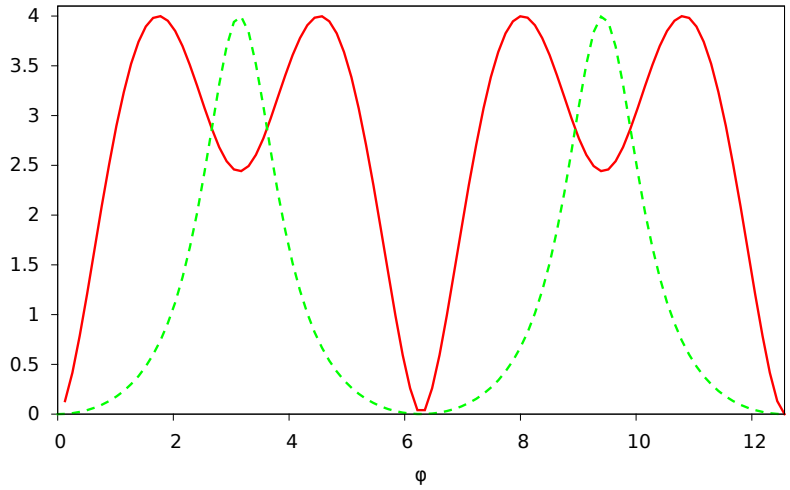


Figure 4.2: Contrast between the flavor FI (dashed) and the scaled Von Neumann entropy (solid) with the mixing angle set at the experimental value [15].

4.2 Decoherence model and the QFI

Now, we investigate effects of decoherence on the QFI of neutrino oscillations. To this aim, we introduce a Markovian term in the dynamics of the neutrinos. The reason for this is twofold. Firstly, given that neutrinos interact so rarely, it is reasonable to assume that the characteristic time of correlations between the neutrinos and any environment is sufficiently short in comparison with the typical time of their evolution, which is the starting point for a Markovian quantum dynamics [33, 36]. Moreover, in more appropriate models for neutrino oscillations as the wave packet model [26, 31], the cause of decoherence in neutrino oscillations is the separation of wave packets of different mass neutrinos due to their different group velocities. This is equivalent to be able to identify which mass neutrino is arriving at the detector. Therefore, although not derived from an underlying microscopic theory of the interaction of the neutrinos, we can incorporate these considerations in the plane wave model by inserting a Lindbladian

$$A = \sqrt{\lambda}\,|\nu_1\rangle\,\langle\nu_1|\,, \tag{4.12}$$

where λ is a decoherence parameter, and solving the resulting Markovian equation in Lindblad form [36, 42]

$$\frac{d}{dt}\rho(t) = -i\,[H,\rho(t)] + A\rho(t)A^\dagger - \frac{1}{2}\left\{\rho(t), A^\dagger A\right\}. \tag{4.13}$$

The solution is

$$\rho(t) = \begin{pmatrix} \cos^2(\theta) & \frac{1}{2}\sin(2\theta)e^{\left(i\delta-\frac{\lambda}{2}\right)t} \\ \frac{1}{2}\sin(2\theta)e^{-\left(i\delta+\frac{\lambda}{2}\right)t} & \sin^2(\theta) \end{pmatrix}. \tag{4.14}$$

This time, the eingenvalues of ρ are

$$\beta_\pm = \frac{1 \pm \sqrt{\cos^2(2\theta) + e^{-\lambda t}\sin^2(2\theta)}}{2}, \tag{4.15}$$

the corresponding eigenvectors being

$$|\beta_\pm\rangle = \frac{1}{\sqrt{2}(1+\alpha^2)^{1/4}} \begin{pmatrix} \frac{\alpha}{\left[(1+\alpha^2)^{1/2}\mp 1\right]^{1/2}} \\ \pm e^{-i\delta t}\left[(1+\alpha^2)^{1/2}\mp 1\right]^{1/2} \end{pmatrix}, \tag{4.16}$$

with $\alpha \equiv \tan(2\theta)e^{-\frac{\lambda}{2}t}$.

The matrix elements of $\partial_\theta \rho$ in that basis are:

$$\begin{cases} \langle \beta_\pm | \left(\partial_\theta \rho\right) | \beta_\pm \rangle & = \mp \frac{\sin(2\theta)\left(1-e^{-\lambda t}\right)}{\sqrt{1+\tan^2(2\theta)e^{-\lambda t}}} \\ \\ \langle \beta_- | \left(\partial_\theta \rho\right) | \beta_+ \rangle & = -\frac{\mathrm{cosec}(2\theta)|\tan(2\theta)e^{-\frac{\lambda}{2}t}|}{\sqrt{1+\tan^2(2\theta)e^{-\lambda t}}} \end{cases} . \tag{4.17}$$

Inserting them into eq. (3.19), we obtain

$$H_\lambda(\theta) = 4. \tag{4.18}$$

The subscript λ is to remind us that, in this case, we are dealing with a decoherence term in the dynamics. Despite that, the QFI remains the same as in the free case. We interpret this result in the following manner: even though the FI associated with a specific measure decreases in time (see below), one can always find another measurement scheme that allows the optimal estimation with the same precision of when there is no environment.

In the same spirit of the free case, we calculate the FI associated with the flavor measurement. We first note that, in this case:

$$L_\theta = 2 \begin{pmatrix} -\tan(\theta) & 0 \\ \\ 0 & \cot(\theta) \end{pmatrix}. \tag{4.19}$$

This shows that the optimal measurement for the estimation of the mixing angle θ is the mass measurement. On the other hand, we have

$$F^{(\lambda)}_{flavor}(\theta) = \frac{4\cos^2(2\theta)\left[1 - e^{-\frac{\lambda}{2}t}\cos(\delta t)\right]}{2 - \sin^2(2\theta)\left[1 - e^{-\frac{\lambda}{2}t}\cos(\delta t)\right]}. \tag{4.20}$$

We note that in the $\lambda \longrightarrow 0$ limit of the above expression, one reobtains the relation (4.11), as it should be. Another interesting feature is that the FI tends to the *residual FI*

$$F_{res} = \frac{4\cos^2(2\theta)}{1 + \cos^2(2\theta)}. \tag{4.21}$$

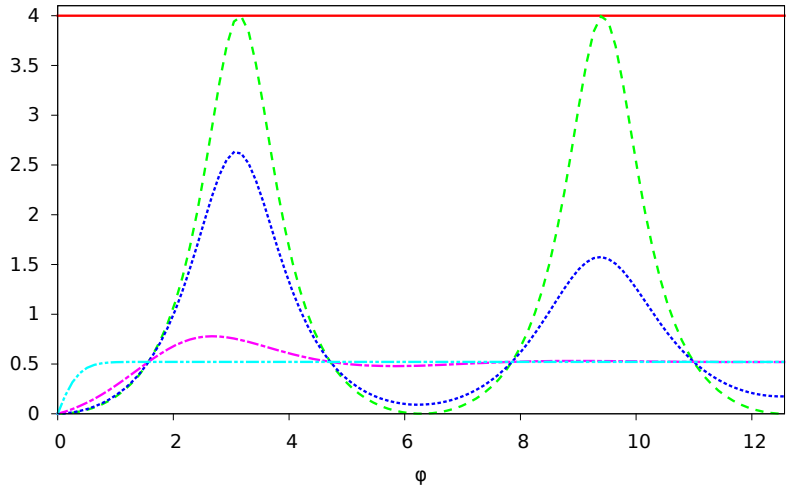
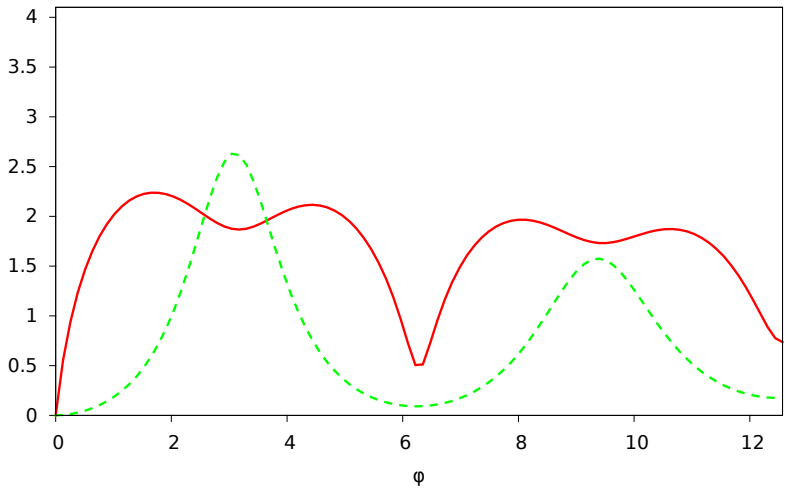


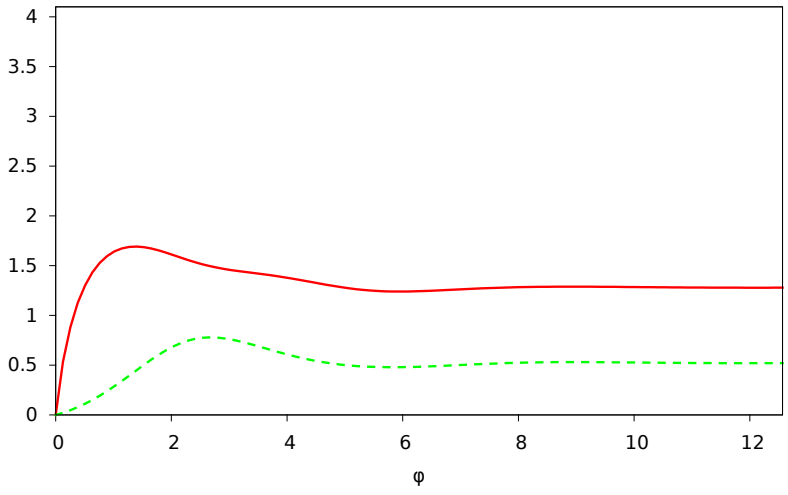
Figure 4.3: Plots of the QFI (solid) and the FI associated with the flavor measurement at the experimental value [15] of the mixing angle. Decoherence parameter λ taken as 0 (dashed), $\frac{\delta}{10}$ (dotted), δ (dot-dashed) and 10δ (double-dot-dashed).

This is the FI after decoherence had taken place, which is well below the upper limit given by the QFI at the experimetal value of the mixing angle (see fig. 4.3).

To analyse entanglement in this model, we can no longer rely on the Von Neumann entropy, since the system will be in a mixed state for every $t > 0$. However, given that the system is formally equivalent to a pair of qubits, we can use the logarithmic negativity [52, 53] as a proper quantifier of entanglement. In fig. 4.4, we do the same comparison between entanglement and the FI as we did in the previous section. We note the local maxima of the FI match the local minima of the entanglement one more time.



(a) Parameter λ set equal to $\frac{\delta}{10}$.



(b) Parameter λ set equal to δ.

Figure 4.4: Comparison between the flavor FI (dashed) and the logarithmic negativity (solid).

Chapter 5

Conclusions and outlooks

We investigated estimation of the mixing parameter θ characterizing two-flavor neutrino oscillations from a Quantum Estimation Theory perspective. For neutrinos initially in a definite flavor state, we considered two types of models to describe the ensuing flavor oscillations: (1) the standard plane-wave approach; (2) a model which takes into account the damping of quantum coherences between the different mass states that compose the flavor state. Since this type of "decoherence effect" always occurs to some extent in experiments, the results obtained in the second case are the ones to relate to the physical applications. In this case, we demonstrated that the measurement scheme which realizes the optimal precision limit established by the Quantum Fisher Information for unbiased estimation of θ can be determined to be the direct mass measurement. Interestingly, direct mass measurements are optimal in the plane-wave model as well. Since the decoherence parameter does not appear in eq. (4.19), one could expect these to remain optimal in the limit of zero decoherence. This is in fact the case, as can be seen by calculating the FI in the plane-wave model for the corresponding POVM and verifying that it also yields $H(\theta) = 4$. Thus, there are at least two distinct ways of saturating the QFI in the model without decoherence.

However, direct detection of neutrino masses in oscillation experiments is not within the reach of current technology. With this in mind, we also analyzed here the Fisher Information associated to the population measurement protocol, which is the one employed

to estimate θ. For both types of models, we saw that the Fisher Information is optimized periodically. Equivalently, this means that its local maxima occur at specific neutrino times-of-flight. Therefore, although the usual flavor measurement is not the one which realizes the QFI, it can in principle be implemented with the best possible sensitivity to the desired parameter θ. We also investigated how the single-particle, mode entanglement in the oscillating neutrino system relates to the Quantum Fisher Information and the Fisher Information for direct flavor detection. We showed that this entanglement does not enhance neither of these quantifiers. In fact, quite to the contrary, we found in all the cases considered that while the Quantum Fisher Information does not change regardless of the entanglement variation in time, the Fisher Information for population measurement has its local maxima simultaneously with the entanglement's local minima. In particular, this demonstrates that in single-particle settings the presence of entanglement correlations in probe states does not imply on precision enhancement in estimations tasks.

We have studied the two-flavor scenario because we often find simple analytic expressions for all quantities of interest, because it enables us to understand many aspects of neutrino oscillations that certainly will be present in more sophisticated models, and also because of its applicability in several experimental oscillations settings. Nevertheless, it should be interesting to make an analysis similar to the one presented here for three-flavor oscillations. This would allow one to examine the richer case of single-particle multipartite entanglement, and obtain the limits of precision in quantities that are the focus of future neutrino oscillations experiments, like the CP phase δ_{CP} and the mixing angle θ_{13} [54].

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Appendices

Appendix A

Quantum field theory of fermions and the Dirac equation

A.1 Representations of the Lorentz group

The principle of relativity states that any two inertial observers are equivalent, in the sense that both describe the same physical phenomena according to the same physical laws written in their respective coordinates [55]. In mathematical terms, this means that if an inertial observer describe events using coordinates $\{x^\mu\}$ and another one uses coordinates $\{x'^\mu\}$, then the relationship between the two sets of coordinates is a Lorentz transformation [1]:

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \tag{A.1}$$

such that the quantity

$$x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu \tag{A.2}$$

is left invariant [55]. This implies that $\eta_{\mu\nu} = \Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho\sigma}$ or, in matrix form:

$$\eta = \Lambda \eta \Lambda^T. \tag{A.3}$$

Equation (A.3) implies that $det(\Lambda) = \pm 1$ and $\eta_{00} = 1 = (\Lambda^0_0)^2 - (\Lambda^1_0)^2 - (\Lambda^2_0)^2 - (\Lambda^3_0)^2$, i.e., $(\Lambda^0_0)^2 \geq 1$. Therefore, every matrix has an inverse and given that the composition

[1] The notation used in this appendix is explained in the appendix B.

of two transformations is again a valid transformation, as it can be seen from equation (A.3), we conclude that the transformations form a (Lie) group called the Lorentz group.

It has four disconnected components which are classified according to the sign of the determinant and of the 00 entry [56]. From now on, we will restrict our attention to the component with positive determinant and positive 00 entry, since this is the component of the group that contains the identity element, which is where we can define the concept of generators of the group [56]. Considering transformations infinitesimally close to the identity $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$, equation (A.3) implies that $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Therefore, the Lorentz group has six free parameters and, correspondingly, six generators [57].

Labeling the corresponding generators as $J^{\mu\nu}$, noting that the variation of a four-vector under an infinitesimal Lorentz transformation $\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)$ is $\delta V^\rho = \omega^\rho_\nu V^\nu = -\frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho_\sigma V^\sigma$, one finds the explicit form of the generators in the four-vector representation [57]:

$$(J^{\mu\nu})^\rho_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma). \tag{A.4}$$

From the above equation, we can immediately obtain the Lie algebra of the Lorentz group (Lorentz algebra):

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\mu\sigma}J^{\nu\rho} + \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho}). \tag{A.5}$$

Introducing $J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$, $K^i = J^{0i}$ and $J^{\pm,i} = \frac{1}{2}(J^i \pm IK^i)$, we can rewrite the Lorentz algebra as

$$[J^{\pm,i}, J^{\pm,j}] = i\epsilon^{ijk}J^{\pm,k}; \tag{A.6}$$

$$[J^{+,i}, J^{-,j}] = 0. \tag{A.7}$$

Therefore, the Lorentz algebra factories into two commuting copies of the $SU(2)$ algebra, i.e. it is labeled by two half-integers (j_-, j_+) and the dimension of the corresponding

representation is $(2j_- + 1)(2j_+ + 1)$ [57].

Now we examine more closely the representation $(1/2, 0)$. In this representation, we can take the solutions of eq. (A.6) to be $J^{-,i} = \frac{\sigma^i}{2}$, $J^{+,i} = 0$, with σ^i being the Pauli matrices. Introducing the parameters $\theta^i = \frac{1}{2}\epsilon^{ijk}\omega^{jk}$ and $\eta^i = \omega^{0,i}$ too, we can write a generic Lorentz group element in this representation as

$$\Lambda_L = \exp\left((-i\theta^i - \eta^i)\frac{\sigma^i}{2}\right). \tag{A.8}$$

We call an element of this representation a left-handed Weyl spinor ψ_L [57]. If we investigate the representation $(0, 1/2)$, we simply exchange the roles of J^+ and J^-, and following essentially the same steps as before we arrive at the conclusion that the Lorentz group elements in this representation are:

$$\Lambda_R = \exp\left((-i\theta^i + \eta^i)\frac{\sigma^i}{2}\right). \tag{A.9}$$

We call the elements of this representation right-handed Weyl spinors ψ_R.

Finally, we introduce the so called Dirac spinors, which are formally just a combination of left-handed and right-handed Weyl spinors:

$$\Psi_D = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}. \tag{A.10}$$

These are objects of great interest when parity is a symmetry of the theory in question, as it is the case of QED. By what we have seen from the Weyl spinor, a Dirac spinor transform under a Lorentz transformation with a Lorentz element given by

$$\Lambda_D = \begin{pmatrix}\Lambda_L & 0 \\ 0 & \Lambda_R\end{pmatrix}. \tag{A.11}$$

A.2 Spinor field lagrangian and the Dirac equation

A remarkable theorem in representation theory states that there is no non-trivial unitary finite dimensional representation of a noncompact group [56]. This is relevant because, in unitary representations, the generators are Hermitian operators, and therefore

can represent physical observables of the theory. Therefore, we need to consider infinite dimensional representations of the Lorentz group, which are provided by considering fields [57].

A left-handed Weyl field is a function ψ_L of the spacetime such that, under a Lorentz transformation (A.1), it changes as

$$\psi_L(x) \longrightarrow \psi'_L(x') = \Lambda_L \psi_L(x). \tag{A.12}$$

An analogous definition holds for a right-handed Weyl spinor.

Using the results presented in the previous section, it is possible to show [57] that a possible action describing fermions, written in terms of Dirac spinors, is

$$S = \int d^4x \bar{\Psi}(x) \left(i\not{\partial} - m\right) \Psi(x), \tag{A.13}$$

where, for a four-vector A^μ, $\not{A} \equiv \gamma^\mu A_\mu$, with $\{\gamma^\mu\}$ being a representation of the so-called *Dirac gamma matrices* that satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{A.14}$$

and $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$ is the *Dirac adjoint.*

The Euler-Lagrange equation that results from the action (A.13) is

$$\left(i\not{\partial} - m\right) \Psi = 0, \tag{A.15}$$

which is the *Dirac equation.*

The generic solution of the above equation can be written as a superposition of plane-waves [57]

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_s a_{\vec{p},s} u^s(p) e^{-ipx} + b^*_{\vec{p},s} v^s(p) e^{ipx}, \tag{A.16}$$

where $s = 1, 2$ represents the helicity, $p = (E_{\vec{p}}, \vec{p})$, $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$, $px \equiv p^\mu x_\mu$ and the Dirac spinors $u^s(p), v^s(p)$ satisfy

$$\begin{cases} (\not{p} - m) u^s(p) = 0; \\ (\not{p} + m) v^s(p) = 0. \end{cases} \tag{A.17}$$

Taking $\vec{p}$ to be in the z direction, the explicit solution is [57]

$$u^s(p) = \begin{pmatrix} \left[\sqrt{E+p^3}\frac{1-\sigma^3}{2} + \sqrt{E-p^3}\frac{1+\sigma^3}{2}\right]\chi^s \\ \left[\sqrt{E+p^3}\frac{1+\sigma^3}{2} + \sqrt{E-p^3}\frac{1-\sigma^3}{2}\right]\chi^s \end{pmatrix}, \tag{A.18}$$

where $\chi^s,\ s = 1, 2$ are two linearly independent Weyl spinors. The solution $v^s(p)$ differ from $u^s(p)$ only in the sign of the two lower entries.

Having at our disposal the plane-wave expansion, we can canonically quantize the field by promoting the coefficients $a_{\vec{p},s}, b_{\vec{p},s}$ to annihilation operators acting on a Fock space and imposing the canonical anticommutation relations [57]:

$$\{a_{\vec{p},s}, a^\dagger_{\vec{q},r}\} = \{b_{\vec{p},s}, b^\dagger_{\vec{q},r}\} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta_{rs} \tag{A.19}$$

and all other anticommutators equal to zero.

A fundamental quantity for the computation of amplitudes of fermionic process in the QFT framework is the *fermion propagator* D, which is defined as the vacuum expectation value of the time-ordered product of two fields:

$$D(x-y) \equiv \langle 0|\, T\{\Psi(x), \Psi(y)\}\, |0\rangle\,, \tag{A.20}$$

with

$$T\{\Psi(x), \Psi(y)\} = \begin{cases} \Psi(x)\Psi(y), & x^0 > y^0; \\ -\Psi(y)\Psi(x), & x^0 < y^0. \end{cases} \quad . \tag{A.21}$$

With the aid of eqs. (A.16) and (A.19), we obtain the expression [57]

$$D(x-y) = \int \frac{d^4p}{(2\pi)^4}\tilde{D}(p)e^{-ipx}, \tag{A.22}$$

with [2]

$$\tilde{D}(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}. \tag{A.23}$$

[2]The $i\epsilon$ term means that one should take the $\epsilon \longrightarrow 0$ limit after using the Feynman prescription for goind around the poles in the residue integral [57].

Appendix B

Quantum field theory notation

This appendix is devoted to explain the usual notation and conventions of quantum field theory, as can be found in [57].

Regarding units, we shall adopt the so-called natural units where $c = \hbar = 1$.

Greek indices run from 0 to 3 and latin indices from 1 to 3. Repeated indices, one upper and another one lower are summed over (Einstein's convention). In addition, we will make the convention that repeated latin indices are always summed over, even if both appearances are as sub or superscripts.

We will represent a four-vector or tensor by the symbol x^μ alone, although a more correct way would be $\{x^\mu\}$, since it is a set of the four coordinates. However, we believe that there is no risk of confusion since the context makes clear whether we are referencing to the entire set or to a specific element of that set.

The metric is $\eta_{\mu\nu} = diag(1, -1, -1, -1)$.

We also recall the operations of raising and lowering indices given by $x_\mu = \eta_{\mu\nu} x^\nu$ and $x^\mu = \eta^{\mu\nu} x_\nu$, with $\eta_{\mu\nu}\eta^{\nu\rho} = \delta^\rho_\mu$.