Dissertação de Mestrado:

Análise de Dados de Degradação para Amostras sob Diferentes Condições de Uso: um Estudo de Caso em Rodas de Trens Ferroviários

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ANÁLISE DE DADOS DE DEGRADAÇÃO PARA AMOSTRAS SOB DIFERENTES CONDIÇÕES DE USO: UM ESTUDO DE CASO DE RODAS DE TRENS FERROVIÁRIOS

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Dedico este trabalho
a minha perseverança,
vontade de vencer e dedicação.

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Resumo

Modelos para análise de degradação são aplicados em sistemas com alta confiabilidade, nos quais a ocorrência de falhas deve ser totalmente evitada, ou que a proporção de censuras é alta, mesmo sob testes de vida acelerados. Esta característica compromete a qualidade dos resultados obtidos através dos métodos de análise de confiabilidade tradicionais. Nos sistemas com tais características, a informação acerca da confiabilidade é obtida e monitorada através das chamadas medidas de desempenho, *que são diretamente relacionadas com o modo de falha do sistema. Neste trabalho é apresentado um estudo de caso do desgaste de rodas de trens ferroviários, que é relacionado ao modo de falha do descarrilamento dos mesmos. Para estimar a função de confiabilidade, os métodos aproximado e numérico, descritos em Meeker and Escobar (1998), foram aplicados à base de dados em estudo, que possui unidades amostrais sob diferentes condições de operação. Numa primeira etapa, o método aproximado foi utilizado como ferramenta para análise dos dados, com consequente formalização do modelo não-linear de efeitos mistos com efeitos aleatórios normalmente distribuídos. Em seguida, os parâmetros deste modelo foram estimados a partir do método numérico, utilizando-se o ferramental proposto por Pinheiro and Bates (2000), cujos resultados foram base para a determinação da função de confiabilidade das rodas e de algumas quantidades de interesse, em função da condição operacional das mesmas.*

Palavras-chave: confiabilidade, análise de degradação, modelos de efeitos mistos, desgaste de rodas de trem.

Abstract

Traditionally, reliability assessment of devices has been based on life tests (LT) or accelerated life tests (ALT). However these approaches are not practical for high reliable devices which are not likely to fail in experiments of reasonable length. For those devices, LT or ALT will end up with a high censoring rate compromising the traditional estimation methods. An alternative approach is to monitor the devices for a period of time and assess their reliability from the changes in performance (degradation) observed during the experiment. In this article we present a model to evaluate the problem of train wheels degradation, which is related to failure modes of train derailments. We first identify the most significant working conditions affecting the wheels wear by using a nonlinear mixed effects model (NLME) where the log-rate of wear is a linear function of some working conditions such as side, truck and axle position. Next, we estimate the failure time distribution by working condition analytically. Point and interval estimates of reliability figures by working condition are also obtained. We compare the results of the analysis via NLME to the ones obtained by an approximate degradation analysis.

Key Words: reliability; degradation tests; mixed effects models; restricted maximum likelihood; train wheels wear.

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Degradation data analysis for samples under unequal operating conditions: a case study on train wheels.

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Key Words: reliability; degradation tests; mixed effects models; restricted maximum likelihood; train wheels wear.

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1 Introduction

1.1 Background

In the manufacturing industry there is much interest in providing inference on the lifetime of products. Traditionally, this is addressed by estimating failure time distributions from failure data. However, when products are highly reliable the collection of failure time data can be expensive and impracticable because products take too long to fail. In those cases, a traditional life test will probably result in very few or no failures, so that the traditional failure time analysis - FTA including censoring (Meeker and Escobar (1998)) is no longer effective in assessing product reliability. An alternative approach is that of accelerated life tests – ALT (Nelson, 1990), in which the units (products, components) are subjected to elevated stress levels (*e.g.* use rate, temperature, voltage or pressure). The information gathered under the known stressed environment is then extrapolated, through a physically reasonable statistical model, to obtain estimates of the statistics of the underlying failure time distribution at normal use conditions. However, even this approach may be inadequate for very high-reliability components under severe test time constraints.

Recently, degradation tests have been shown to be a superior alternative to lifetime data in such situations because they are more informative. Most failures arise from degradation mechanisms at work for which there are characteristics that degrade (or grow) over time (*e.g.*, amount of material displaced by electro migration). If there exist product characteristics whose degradation over time can be closely related to failure and can be accurately measured, then collecting "degradation data" can provide useful information about product reliability. Examples are loss of tread on rubber tires and degradation of the active ingredient of a drug because of chemical reactions with oxygen and water or microbial. In order to conduct a degradation test, one has to prespecify a threshold level of degradation, obtain measurements of degradation at different times, and define that *failure occurs when the amount of degradation for a test unit exceeds this level*. For example, a crack grows over time, and failure is defined to occur when the crack reaches a specified length. Another example is the luminosity of fluorescent lights (or luminous flux) usually measured in lumens. Because it degrades over time as the fluorescent material darkens, the fluorescent lamp industry has traditionally defined failure in terms of the amount of degradation in the luminous flux. More specifically, the industry's standard definition for lifetime is the time t when a lamp's luminous flux $\omega(t)$ falls below 60% of its luminous flux after 100 hours of use aging, *i.e.*, $0.6\omega(100)$ (Tseng et al. (1995)). Such failures are referred as "soft" failures because the units are still working, but their performance has become unacceptable. In most situations, degradation data have some important practical advantages (Lu and Meeker (1993); Nelson (1990)), such as: 1) Degradation data can be analyzed earlier, before a failure actually occurs and a degradation test can provide some information about unfailed units; 2) Degradation data may yield more accurate life estimates than the accelerated life tests with few or no failures; 3) Degradation data can provide better information of degradation processes, which helps one to find the appropriate mechanistic model for degradation.

This work was motivated by a real practical situation concerning train wheel degradation. A small part of the database has already been analyzed by Freitas et al. (2009). The authors presented the data analysis of a subset of wheels submitted to the same working conditions. In this paper, a complete analysis of this database is performed, considering the information available on the other wheels and operational issues that might have influence on wheels wear over time. The situation under study is described in the sequel.

1.2 Motivation: train wheel degradation data

Most railways keep in a database descriptions of maintenance actions performed on their trains. Our partner has such a database and they provided us with a subset that contains, among other information, the diameter measurements of the wheels, taken at 13 equally spaced inspection times: $t_0 = 0$ km; $t_1 = 50{,}000$ km; $t_2 = 100{,}000$ km, $\ldots$, $t_{13} = 600{,}000$ km. These measurements were taken for 14 trains. First,

it is important to set out some definitions. In the particular case of the study, a train consists of a locomotive (to provide power) and 3 (three) unpowered vehicles (cars) attached to it. Each car (either a locomotive or an unpowered car) has two trucks; each truck has two axles with two wheels each (Figure 1). The wheels are labeled according to their working positions in a given car using a three-dimension indicator vector, representing in this order: position (side) within an axle (left=0; right=1); truck position (front=0; back=1) and axle position within a truck (outer=0; inner=1). The data used in this paper refer to the diameter measurements of the wheels of the locomotive cars only (8 wheels for each one of the 14 locomotives). Freitas et al. (2009) used the same database but the authors have analyzed only the degradation data of the wheels labeled [000].

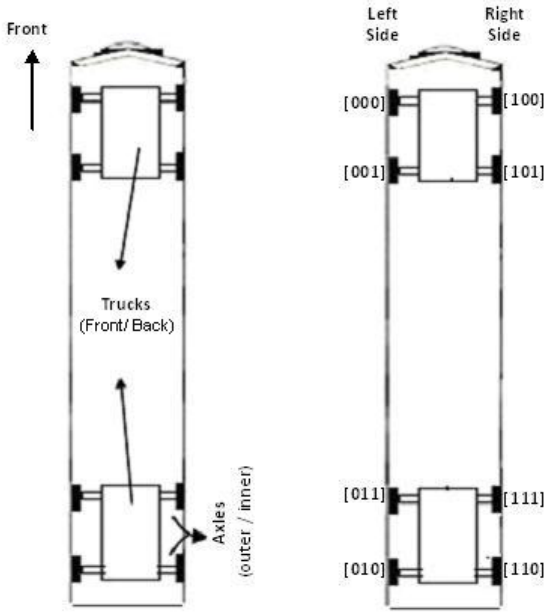

Figure 1: The location of the wheels: side, axles and trucks within a car and their corresponding labels.

The nominal diameter of a new wheel is 966 mm. When the diameter reaches 889 mm, the wheel is replaced by a new one. Figure 2 presents the degradation profiles of the 110 wheels under study. The points on each plot are the amount of wear (in mm) (*i.e.*, 966 mm - [observed diameter measure at time t]), at each inspection time (distance in Km). The event "failure" occurs when the degradation (wear) reaches the threshold level D_f = 77 mm = (966 mm − 889 mm). In this paper, only 110 of the total 112 wheels are analyzed, since the diameter measures for two of them, both labeled [0,0,0] were not considered reliable due to equipment problems.

There are known operational issues that might have some impact on wheels wear, such as position, inherent part-to-part mechanical properties variability (leading to different wear rates), railway topographic conditions, curves characteristics (side predominance, length and radius), working load and engine power of the wheel, among others. The goal of this work is to try to answer some specific engineering questions such as: 1) do the different working positions have (statistically) significant effect on the wheels wear? 2) if that is the case, what is the time-to-failure distribution of wheels on different working positions? In addition, it is important to get estimates of key reliability summary figures, such as the *MTTF* (mean time to failure, or more specifically, mean distance to failure) and some quantiles of

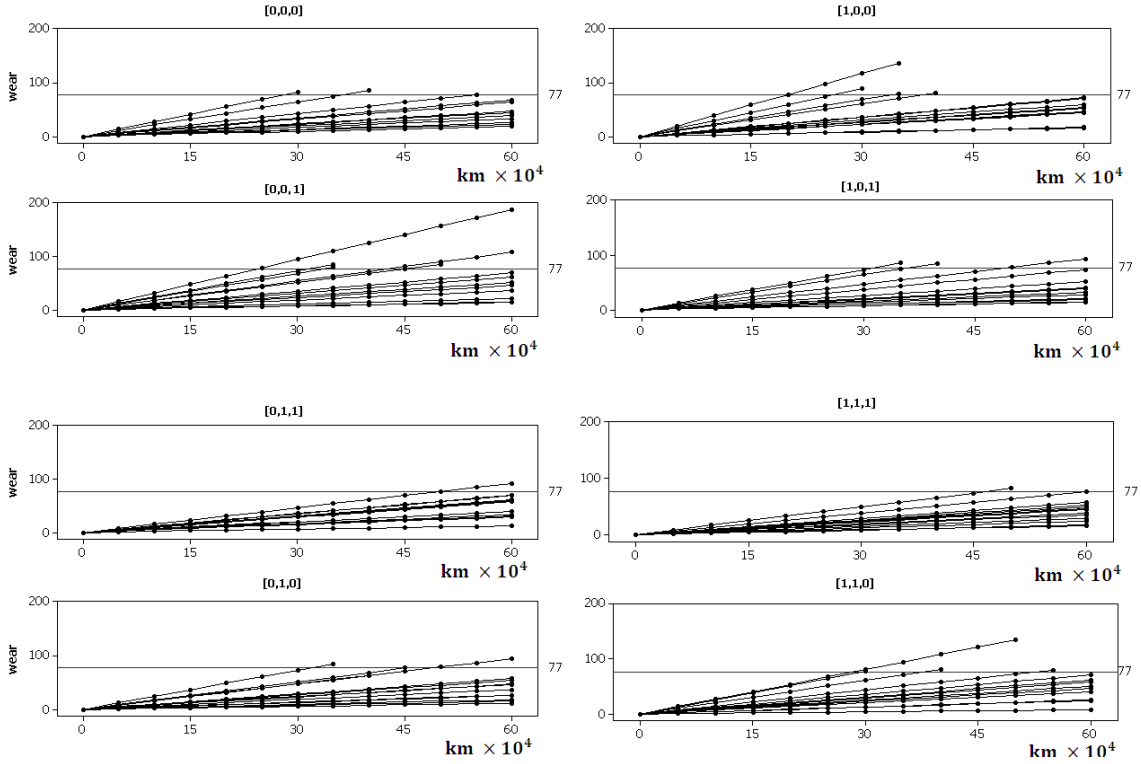

Figure 2: Wheels degradation profiles by working position.

the time-to-failure distribution (*e.g.* 0.01, 0.10 and the median, 0.50). This task will be accomplished by modeling the wheels wear rate as a function of the wheel´s position.

1.3 Literature review on degradation modeling

The amount of literature on degradation data analysis has increased quickly in the past two decades and many of them have targeted specific products and their degradation characteristics (*e.g.* Tseng et al. (1995); Tang and Chang (1995); Yacout et al. (1996); Oliveira and Colosimo (2004); Freitas et al. (2009); Park and Bae (2010)). Basically, both stochastic process models and general statistical path models have been used for modeling degradation.

Stochastic degradation models can be traced back to the material fatigue study of crack developments by Birnbaum and Saunders (1969), where the crack growth was modeled by a sum of independent random crack extensions as a Wiener process. Battacharyya and Fries (1982) pointed out that the product's failure time is the first passage time of the Wiener process to a critical value, so it has an inverse Gaussian distribution. Whitmore and Shenkelberg (1997), and Padgett (2004) used Wiener diffusion processes with a drift to explain the degradation paths, with the advantage being that a time-to-failure distribution is readily available, *i.e.* the inverse Gaussian distribution. In general a Wiener process has found application as a degradation model in many studies due to its good properties (see for example, Doksum (1991); Whitmore (1995) and Doksum and Normand (1995)). But in certain physical situations, it is often the case that a degradation process should be always positive and strictly increasing. Therefore, the gamma process (a stochastic process with independent, non-negative increments having a gamma distribution with an identical scale parameter), has also been used by many authors for the stochastic modeling of monotonic and gradual degradation over time in a sequence of tiny increments.

Examples are the works by Bagdonavicius and Nikulin (2000), Lawless and Crowder (2004) and Pan and Balakrishanan (2011). The latter assumed that a product has two performance characteristics whose degradation can be governed by a gamma process. The authors used a bivariate Birnbaum-Saunders distribution and its marginal distributions to approximate the reliability function of the product and develop inferential methods for the model parameters. All the papers listed so far considered degradation processes consisting of one phase of behavior only. Ng (2008) focused on the estimation of a class of degradation models where the degradation rates could possibly increase or decrease in a non-smooth manner at a particular point in time, for example when the underlying degradation processes changes phase. The degradation path of a given device was modeled using an independent-increments stochastic process with a single unobserved change-point (allowed to vary randomly from device-to-device). The problem of parameter estimation via maximum likelihood in such a situation (*i.e.* when the change-points are unobserved and random) was solved via an application of the EM algorithm.

An alternative approach to degradation data modeling is to consider more general statistical degradation path models. Nelson (1990) (chapter 11) reviewed the degradation literature, surveyed applications, described basic ideas on accelerated-test degradation models, and, using a specific example, showed how to analyze a type of degradation data. Carey and Koenig (1991) studied the performance of devices from an integrated logic family, and used a non-linear degradation path model, motivated by the physics of diffusion of impurities through the devices. Lu and Meeker (1993) developed statistical methods using degradation measures to estimate a time-to-failure distribution for a broad class of degradation models. They considered a nonlinear mixed-effects model (NLME) and used a two-stage method to obtain point estimates and confidence intervals of percentiles of the failure-time distribution. Lu et al. (1997) proposed a model with random regression coefficients and standard-deviation function for analyzing linear degradation data from semiconductors. Su et al. (1999) considered a random coefficient degradation model with random sample size and used maximum likelihood for parameter estimation. A data set from a semiconductor application was used to illustrate their methods. Other applications and model developments for specific degradation tests include Wu and Tsai (2000); Crk (2000); Oliveira and Colosimo (2004); Freitas et al. (2009); Peng and Tseng (2009). A good reference on degradation path models is Meeker and Escobar (1998), chapters 13 and 21. The authors introduced the concepts of degradation analysis, presented useful degradation models and discussed important topics such as the connection between degradation models and failure time models, methods for degradation data analysis and reliability inference based on likelihood methods.

Bayesian approaches have also been used for degradation data analysis in pharmacokinetics and engineering applications (*e.g.*, Wakefield et al. (1994); Gelman et al. (1996); Wakefield (1996); Robinson and Crowder (2000); Hamada (2005); Hamada et al. (2008); Freitas et al. (2010)).

In this paper, we use the general degradation path model to analyze the wheel degradation data. We propose a nonlinear mixed effects model (NLME) where the log-rate is a linear function of some working conditions such as side,truck and axle positions (and interactions among them). With the proposed model it is possible to identify the most significant working conditions affecting the wheels wear and to obtain analytically the failure time distribution by working condition.

1.4 Overview

The rest of the paper is organized as follows. In Section 2, the general degradation path model along with the parameter estimation issues is briefly presented. The basic reference is Meeker and Escobar (1998). In Section 3 a preliminary exploratory analysis of the wheel degradation data is described, including the preliminary empirical approach used for model construction. In section 4, the results of the model fitting are presented. Finally, some concluding remarks end the paper in Section 5.

2 The general degradation path model

For each unit in a random sample of size n, it is assumed that degradation measurements are available at prespecified times $t_{ij}(i = 1,\ldots,n;\ j = 1,2,\ldots,m_i)$, where m_i is the number of measurements taken on the i^{th} unit. Thus the general approach is to model the degradation of the individual units using the same functional form and differences among individual units using random effects. The model is

$$Y_{ij} = D_{ij} + \varepsilon_{ij} = D(t_{ij};\alpha;\beta_i) + \varepsilon_{ij}, \tag{1}$$

where:

- Y_{ij} is the random variable representing the amount of degradation of the i^{th} unit at a prespecified time t_{ij} $i = 1,2,\ldots,n;\ j = 1,2,\ldots,m_i$);
- $D(t_{ij};\alpha;\beta_i)$ as the actual degradation path of unit i at time t_{ij};
- $\alpha = (\alpha_1;\alpha_2;\ldots;\alpha_p)^t$ is a $p \times 1$ vector of fixed effects describing population characteristics (they are modeled as common across all units);
- $\beta_i = (\beta_{i1},\beta_{i2},\ldots,\beta_{ik})^t$ is a $k \times 1$ vector or the i^{th} unit random effects representing the individual unit´s characteristics (variations in the manufacturing of the components, such as properties of the raw material, component dimensions, etc.);
- ε_{ij} is the random error associated to the i^{th} unit at time t_{ij}.

The deterministic form of $D(t_{ij};\alpha;\beta_i)$ might be based on empirical analysis of the degradation process under study, but whenever possible it should be based on the physical-chemical phenomenon associated with it. It is usually a linear or non-linear function in α and β_i. The *time* t_{ij} could be real-time, operating time, or some other appropriate measures of use, like distance (in km or miles) for automobiles, trains or number of cycles in fatigue tests.

It is generally assumed that:

1. The random errors ε_{ij} are independent and identically distributed (i.i.d.) according to a Normal distribution with mean $\mu = 0$ and variance σ_ε^2 (fixed and unknown).
2. The vectors $\beta_i = (\beta_{i1},\beta_{i2},\ldots,\beta_{ik})^t$ $i = 1,\ \ldots,\ n$ are i.i.d. as $\Lambda(\beta|\theta)$, where $\Lambda(\beta|\theta)$ is a multivariate distribution function, which may depend on an unknown (fixed) $q \times 1$ parameter vector $\theta = (\theta_1,\ldots,\theta_q)^t$ that must be estimated from the degradation data.
3. The random vectors β_i are independent of the ε_{ij} deviations.

2.1 Estimation of degradation model parameters: the Maximum Likelihood method

In order to obtain the expression of the likelihood function using the general model (1), let us define:

- $Y_i = (Y_{i1},\ Y_{i2},\ \ldots\ Y_{im_i})^t$ as the $m_i \times 1$ random vector representing the degradation measures of the i^{th} unit and $y_i = (y_{i1},\ y_{i2},\ \ldots\ y_{im_i})^t$ the actual observed values;
- $Y = (Y_1,\ Y_2,\ \ldots\ Y_n)^t$ as the $N \times 1$ random vector representing the full set of degradation measures ($N = \sum_{i=1}^{n} m_i$) and $y = (y_1,\ y_2,\ \ldots\ y_n)^t$ the actual full set of y-observed values.
- $B = (\beta_1,\ \beta_2,\ \ldots\ \beta_n)^t$ as the vector of order $nk \times 1$ combining the n random effects vectors.

In addition, let $f(y_i|\alpha,\ \beta_i,\ \theta,\ \sigma_\varepsilon^2)$ and $f(\beta_i|\theta)$ denote the probability density function of Y_i and β_i respectively. Then, using the general model assumptions 1 to 3, the probability density functions of Y and B are given respectively by

$$f(y|\alpha,\ \beta_i,\ \theta,\ \sigma_\varepsilon^2) = \prod_{i=1}^{n} f(y_i|\alpha,\ \beta_i,\ \theta\ \sigma_\varepsilon^2) \tag{2}$$

and

$$f(B|\theta) = \prod_{i=1}^{n} f(\beta_i|\theta). \tag{3}$$

Although the values of $\beta_1,\ \ldots,\ \beta_k$ for the individual units may be of interest in some applications (*e.g.*), to predict the future degradation of a particular unit, based on a few early readings), subsequent developments in this work concentrate on the use of degradation data to make inferences about the population or process or predictions about future units.

In this case, the underlying model parameters are α, θ and σ_ε^2 and the general expression of the likelihood function is:

$$\begin{aligned}
L(\alpha,\ \theta,\ \sigma_\varepsilon^2) = f(y|\alpha,\ \theta,\ \sigma_\varepsilon^2) &= \int_{\Xi_B} f(y,B|\alpha,\ \theta,\sigma_\varepsilon^2)\ \mathrm{d}B \\
&= \int_{\Xi_{\beta_1}} \cdots \int_{\Xi_{\beta_n}} f(y,B|\alpha,\ \theta,\ \sigma_\varepsilon^2)\ \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n \\
&= \int_{\Xi_{\beta_1}} \cdots \int_{\Xi_{\beta_n}} \left\{ f(y|\alpha,\ B,\ \theta,\ \sigma_\varepsilon^2) f(B|\theta) \right\} \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n \\
&= \int_{\Xi_{\beta_1}} \cdots \int_{\Xi_{\beta_n}} \left\{ \left[\prod_{i=1}^{n} f(y_i|\alpha,\ \beta_i,\ \theta,\ \sigma_\varepsilon^2) \right] \left[\prod_{i=1}^{n} f(\beta_i|\theta) \right] \right\} \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n \\
&= \int_{\Xi_{\beta_1}} \cdots \int_{\Xi_{\beta_n}} \left\{ \prod_{i=1}^{n} f(y_i|\alpha,\ \beta_i,\ \theta,\ \sigma_\varepsilon^2,) f(\beta_i|\theta) \right\} \mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n \\
&= \prod_{i=1}^{n} \left\{ \int_{\Xi_{\beta_i}} f(y_i|\alpha,\ \beta_i,\ \theta,\ \sigma_\varepsilon^2,) f(\beta_i|\theta) \mathrm{d}\beta_i \right\},
\end{aligned} \tag{4}$$

where Ξ_B and Ξ_{β_i} denote the limits of the multiple integrals in B and β_i respectively.

Now, since the random errors ε_{ij} are *i.i.d.* according to a Normal distribution with mean zero and variance σ_ε^2 (*i.e.*, $\varepsilon_{ij} \overset{\text{i.i.d.}}{\sim} N(0, \sigma_\varepsilon^2)$) then conditional on α, β_i, θ and σ_ε^2, the random vectors Y_i ($i = 1,\ \ldots,\ n$) are independent and Multivariate Normal distributed. In other words,

$$Y_i|\alpha,\ \beta_i,\ \theta,\ \sigma_\varepsilon^2 \quad \overset{\text{indep.}}{\sim} \quad N_{m_i}(\mu_{Y_i}, \Sigma_{Y_i})$$

where $N_{m_i}(\mu_{Y_i}, \Sigma_{Y_i})$ is a m_i- Multivariate Normal distribution with mean vector and variance-covariance matrix given respectively by

$$\mu_{Y_i} = E(Y_i) = (D(t_{i1},\ \alpha,\ \beta_i);\ \ldots;\ D(t_{im_i},\ \alpha,\ \beta_i))^t \quad \text{and } \Sigma_{Y_i} = \sigma_\varepsilon^2 I_{m_i};\quad i = 1,\ \ldots,\ n$$

where I_{m_i} is the $m_i \times m_i$ identity matrix.

In addition, conditional on β_i the random variables Y_{ij} ($j = 1, \ldots, m_i$) are independent and Normal distributed. More specifically,

$$Y_{ij}|\alpha, \beta_i, \theta, \sigma_\varepsilon^2 \overset{\text{indep.}}{\sim} N\left[D(t_{ij}, \alpha, \beta_i); \sigma_\varepsilon^2\right]; \quad j = 1, \ldots, m_i.$$

Therefore, the likelihood function (4) takes the form:

$$\begin{aligned} L(\alpha, \theta, \sigma_\varepsilon^2) = f(y|\alpha, \theta, \sigma_\varepsilon^2) &= \prod_{i=1}^{n} \left\{ \int_{\Xi_{\beta_i}} \left[\prod_{j=1}^{m_i} N(\mathrm{D}(t_{ij}, \alpha, \beta_i); \sigma_\varepsilon^2) \right] f(\beta_i|\theta) \mathrm{d}\beta_i \right\} \\ &= \prod_{i=1}^{n} \left\{ \int_{\Xi_{\beta_i}} \left[\prod_{j=1}^{m_i} \frac{1}{\sigma_\varepsilon} \phi_{NOR}(z_{ij}) \right] f(\beta_i|\theta) \mathrm{d}\beta_i \right\}, \end{aligned} \tag{5}$$

where $z_{ij} = \frac{[y_{ij} - D(t_{ij}, \alpha, \beta_i)]}{\sigma_\varepsilon}$ and ϕ_{NOR} is the Standard Normal probability density function.

The maximum likelihood (ML) estimate of the degradation model parameter vector $(\alpha, \theta, \sigma_\varepsilon^2)^t$ is found by maximizing the likelihood function (5). Each evaluation of (5) will, in general, require numerical approximation of n integrals (the number of sample paths) of dimension k (the number of random parameters in each path). Thus maximizing (5) with respect to α, θ, σ_ε^2 directly, even with today's computational capabilities, is extremely difficult.

A common procedure is to suppose that the parameter vector of random effects follows a k-Multivariate Normal distribution. This situation is discussed next.

2.2 Maximum likelihood and the Multivariate Normal model

If the Multivariate Normal assumption is used, in other words, $\beta_i \overset{\text{i.i.d.}}{\sim} N_k(\mu_\beta, \Sigma_\beta)$, $i = 1, \ldots, n$, then $f(\beta_i|\theta)$ in (5)takes the form of the k-Multivariate Normal probability density function, with mean vector (fixed and unknown) μ_β and variance-covariance matrix Σ_β (fixed and unknown). Consequently, $\theta = (\mu_\beta, \Sigma_\beta)$.

Because the deterministic form of the degradation path $D(t_{ij}; \alpha; \beta_i)$ can be nonlinear in the random effects, the integral in (5) generally does not have a closed-form expression even in the Multivariate Normal model. To make the numerical optimization of the likelihood function a tractable problem, different approaches including approximations to the likelihood function (all under the Multivariate Normal assumption) were proposed. Lu and Meeker (1993) proposed a two-stage estimation method for the case where the vector of random effects β or a known reparameterization of β, follows a Multivariate Normal distribution.

In the case of the approximation methods, some consist of taking a first-order Taylor expansion of the degradation path function $D(t_{ij}; \alpha; \beta_i)$ around the expected value of the random effects (Sheiner and Beal (1980), Vonesh and Carter (1992)) or around the conditional modes of the random effects (Lindstrom and Bates (1990)). In particular, Pinheiro and Bates (1995) used Lindstrom and Bates (1990) approximation method to obtain restricted maximum likelihood (REML) estimates (Patterson and Thompson (1971)) to non-linear mixed effects models (NLME). Maximum likelihood estimates of variance components such as Σ_β and σ_ε^2 tend to underestimate these parameters. Therefore many analysts prefer the REML estimates for these quantities. Pinheiro and Bates (2000) implemented the method proposed by Lindstrom and Bates (1990) in the NLME function available in the software S-PLUS (TIBCO Software Inc.). This function is now also available in the software R (GNU general public licence; www.r-project.org). Another option is the procedure PROC NLMIXED, available in the software SAS/STAT(SAS Institute Inc.).

2.3 Evaluation of F(t)

A specified model for D(t) and D_f defines a failure time distribution. In general this distribution can be written as a function of the degradation model parameters. Suppose that a unit fails at time t if the degradation level reaches D_f at time t. Then

$$F(t) = P(T \leq t) = P[D(t,\alpha,\beta) \geq D_f], \tag{6}$$

when the degradation measurements are increasing with time, or

$$F(t) = P(T \leq t) = P[D(t,\alpha,\beta) \leq D_f], \tag{7}$$

when the degradation measurements are decreasing with time.

For a fixed D_f, the distribution of T depends on the distribution of β, which in turn, depends on the basic path parameters. There are basically three procedures that might be used to evaluate F(t):

1. Analytical solution: in some simple cases it is possible to write down a closed-form expression for F(t). To illustrate the procedure suppose the actual degradation path of a particular unit is given by

   $$D(t) = \alpha + \beta t,$$

   where α is fixed and β varies according to lognormal distribution with scale parameter $exp(\mu_\beta)$ and shape σ_β ($\beta \sim logn(\mu_\beta, \sigma_\varepsilon)$). The failure time T occurs when $D(t) = D_f$. Then,

   $$D_f = \alpha + \beta T \quad \text{e} \quad T = g(\beta; \alpha; D_f) = \frac{D_f - \alpha}{\beta}.$$

   Therefore,

   $$\begin{aligned} F(t) = P(T \leq t) &= P\left(\frac{D_f - \alpha}{\beta} \leq t\right) \\ &= P\left(\beta \geq \frac{D_f - \alpha}{t}\right) = 1 - \Phi_{\text{NOR}}\left(\frac{log(D_f - \alpha) - log(t) - \mu_\beta}{\sigma_\beta}\right) \\ &= \Phi_{\text{NOR}}\left(\frac{log(t) - [log(D_f - \alpha) - \mu_\beta]}{\sigma_\beta}\right), \qquad t > 0, \end{aligned}$$

   where $\Phi_{\text{NOR}}(\cdot)$ is the Standard Normal cumulative distribution function. Consequently, if $\beta \sim logn(\mu_\beta, \sigma_\beta)$ then, T has a lognormal distribution with scale parameter $exp(\mu_T) = exp(log(D_f - \alpha) - \mu_\beta)$ and shape parameter $\sigma_T = \sigma_\beta$.

   In such case, one can estimate F(t) by substituting the estimates $\hat{\alpha}$, $\hat{\mu}$ and $\hat{\sigma}_\beta$ and D_f into the expression of the lognormal cumulative distribution function. Other examples can be found in Lu and Meeker (1993) and Freitas *et al.*(2009).

2. Numerical evaluation of F(t) by direct integration: for most practical path models, especially when D(t) is nonlinear and β has dimension $k > 1$, it is necessary to evaluate the integral (6) (or (7)). Usually it will be necessary to evaluate F(t) with numerical methods. The amount of computational time needed to evaluate the multidimensional integral will, however, increase exponentially with the dimension of the integral.

3. Evaluation of F(t) using Monte Carlo simulation: Monte Carlo simulation is a particularly useful method for evaluating F(t). The evaluation is done by using the estimates $\hat{\alpha}$, $\hat{\mu}_\beta$, $\hat{\Sigma}_\beta$ and $\hat{\sigma}_\varepsilon^2$ obtained by maximization of the likelihood (5). A large number M of random sample degradation paths is generated from the assumed path model. For each one of them, the "failure time" t_j^* (the first crossing time) is calculated ($j = 1, \ldots, M$). For any desired values of t, use the proportion of paths crossing D_f by time t as an evaluation of F(t). In other words, an estimate $\hat{F}(t)$ of the failure time at any $t > 0$ is given by

$$\hat{F}(t) = \frac{\sum_{j=1}^{M} I_{(t_j^* \leq t)}}{M}, \qquad t > 0, \tag{8}$$

where I is an indicator function ($I_A(x) = 1$ if $x \in A$ and zero otherwise) and M should be large enough to reduce the standard deviation of the Monte Carlo error (usually, $M \geq 10^5$).

In order to generate the M random sample degradation paths, it is necessary to generate M simulated realizations $\Theta_{[l]}^* = (\alpha_1^*, \ldots, \alpha_p^*; \beta_1^*, \ldots, \beta_k^*)_{[l]}^t$ ($l = 1, \ldots, M$),. For instance, if the Multivariate Normal distribution assumption for the random effects is being used then those simulated realizations come from a Multivariate Normal distribution with mean $\hat{\mu}_\Theta$ and variance-covariance matrix $\hat{\Sigma}_\Theta$. Note that α is a vector of fixed effects. Consequently, $\mu_\Theta = (\alpha_1, \ldots, \alpha_p\ ;\ \mu_{\beta_1}, \ldots, \mu_{\beta_k})^t$ and $\Sigma_\Theta = \begin{bmatrix} A & B \\ B^t & \Sigma_\beta \end{bmatrix}$; where $\underset{p\times p}{A} = Cov(\alpha)$, $\underset{p\times k}{B} = Cov(\alpha, \beta)$, and $\underset{k\times k}{\Sigma_\beta} = Cov(\beta)$. A and B are matrices with null elements.

A similar procedure can be used with other distributions.

Confidence intervals can be obtained using the bias-corrected Bootstrap method (Efron and Tibshirani (1993)).

3 Wheel degradation data revisited: exploratory data analysis and model building

In this section we return to the practical situation presented in Section 1.2. First we describe the steps followed for model construction, including the choice of the functional form of the degradation path and an empirical search of the time-to-failure distribution of the wheels. Then, in section 4 we give the results of the model fitting: the estimate of F(t) and of some other important reliability summary figures such as the MTTF (mean time to failure or, more specifically, the mean distance to failure) and quantiles of the time-to-failure distribution.

3.1 Empirical search with an approximate degradation analysis

Meeker and Escobar (1998) described an alternative (but only approximately correct) method of analyzing degradation data. It will be referred to along this text as "the approximate method". Despite the well known potential problems and limitations of the approximate method (see Meeker and Escobar (1998)), it is used here only as an exploratory data analysis. It should shed some light on modeling issues such as (1) the wheel´s failure time distribution and possibly the suitable distribution for the random effects included in the model, and (2) the significance of main effects of the working positions (side, axle and truck) and interactions among them. There are two steps in the approximate method. The first step consists of a separate analysis for each unit to predict the time at which the unit will reach the critical degradation level (D_f) corresponding to failure. These times are called "pseudo failure times". In the second step, the n pseudo-failure times (one for each sample unit) are analyzed as a complete sample of failure times to estimate F(t). For the wheel degradation data (profiles shown in Figure 2) it is reasonable

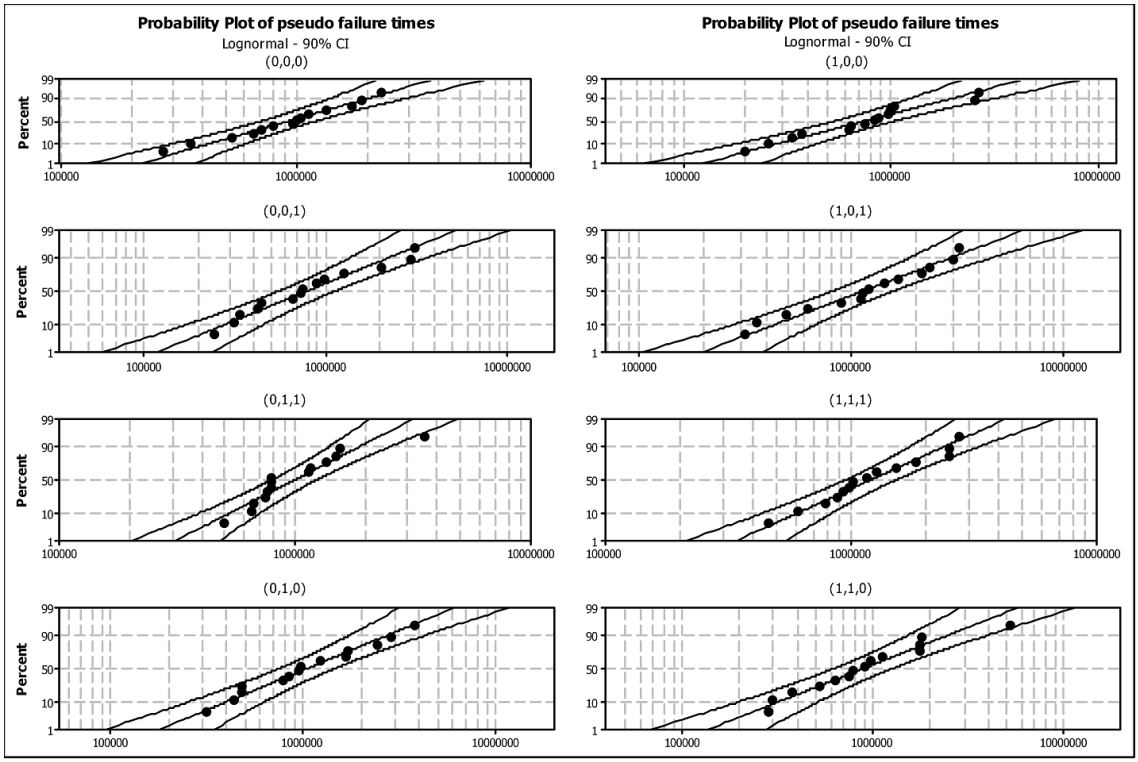

Figure 3: Lognormal probability plots of the pseudo failure times by working positions (labels in parentheses).

to consider a linear functional form (a straight line) for the degradation path. Therefore, for a given experimental unit i, the degradation path is given by

$$y_{ij} = D_i\left(t_{ij}; \beta_{0i}; \beta_{1i}\right) + \epsilon_{ij} = \beta_{0i} + \beta_{1i} t_{ij} + \varepsilon_{ij} \qquad (j = 1, \ldots, 13) \tag{9}$$

and, by using ordinary least squares estimation, the pseudo failure time for the i^{th} unit (wheel) is given by

$$\hat{t}_i = \frac{D_f - \hat{\beta}_{0i}}{\hat{\beta}_{1i}} \tag{10}$$

where D_f is the specified threshold and $\hat{\beta}_{0i}$ and $\hat{\beta}_{1i}$ are the individual fitted intercept and slope respectively (recall that n= 110 wheels, 12 for the position [0,0,0] and 14 in each one of the other seven positions). Next the pseudo failure times $\hat{t}_i$ are analyzed as a complete sample of observed failure times. Two families of distributions are considered, the Weibull and the lognormal (as in the work by Freitas et al. (2009) with the labeled [000] wheels only). The Weibull distribution turned out to be inadequate for some of the positions (probability plots are not shown here).On the other hand, the probability plots of the lognormal distribution by working position(Figure 3), suggest that this family of distributions is a reasonable choice since it describes well the pseudo failure times independent of the working position. In addition, as it will be shown along this work, its close relation to the Normal distribution turned out to be useful at the modeling step, resulting in a normally distributed random effect parameterization. This in turn made it possible to use the NLME (nonlinear mixed effects) function developed by Pinheiro and Bates (2000) and implemented in the software R, for the model fitting and parameter estimation.

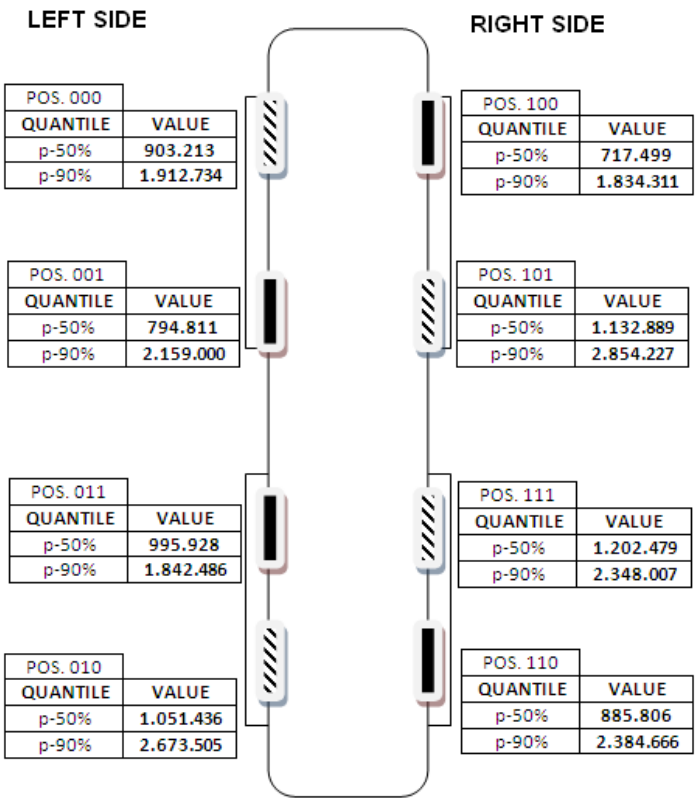

Figure 4: Point estimates of quantiles (0.50 and 0.90) of the failure time distributions based on pseudo failure times data (by working positions).

Figure 4 shows the point estimates of the 0.50 (median) and 0.90 quantiles by working position. These two quantiles provide an idea of center and tail of the fitted lifetime distributions. For a given axle, the side with the highest point estimates is indicated by a hachured area. There is an indication of possible interaction between the axle position (outer or inner) and the side (right or left). For the wheels located on the inner axles, the estimated quantiles values *decrease* when we move *from the right to the left side* of these axles. The opposite pattern can be observed for the outer axles.

Figure 5 presents the two-factor interaction plots (in a 3×3 matrix layout) based on the mean pseudo failure times. The plots in positions (3,1) and (1,3) both represent the interaction between the axle position (outer or inner) and the side (right or left). In particular the plot (3,1) indicates that for the wheels located on the inner axles (dashed line), the mean pseudo failure times *decrease* when we move *from the right (1) to the left (0)* side of these axles. The opposite pattern can be observed for the outer axles. This pattern was already observed in Figure (4) with some percentiles of the estimated failure time distributions based on the pseudo failure time data.

The plots (3,2) and (2,3) also indicate a possible interaction between truck position (front or back) and axle position (outer or inner). In particular the plot (3,2) shows, for the outer axle (black continue line), an increase in the mean pseudo failure time when we move from the front (0) to the back truck (1). On the other hand, in the case of the inner axles (dashed line) the truck position does not seem to have a significant effect on the mean pseudo failure times.

Using the information provided by this exploratory data analysis it is possible to move forward and postulate the model for the data under study.

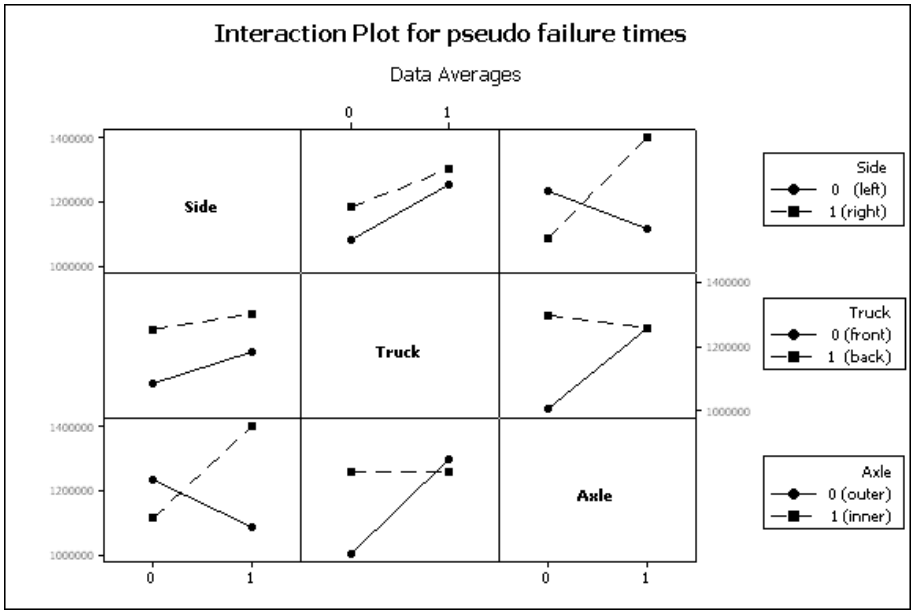

Figure 5: Interaction plots for the pseudo failure times data.

3.2 Model specification

From the results of the exploratory data analysis it is possible to say that:

1. The patterns of the degradation profiles (Figure 2) suggest a linear (straight line) functional form for the degradation path for all the working positions, with positive degradation rate (slope).
2. The lognormal distribution is a good candidate for the distribution of time-to-failure distribution of the wheels.
3. As a consequence of 1 and 2, the degradation rate (the slope) should also have a lognormal distribution (see example in Section 2.3).
4. There is an indication of interactions between the working condition factors (in particular, side and axle) that might be affecting the degradation rate and should be included in the model, possibly by writing the degradation rate as a function of these factors and interactions.

Therefore, we use the following nonlinear mixed effects model for the i^{th} sample unit (wheel):

$$Y_{ij} = \alpha_0 + e^{\eta_i}\, t_j + \varepsilon_{ij} \quad (i = 1, \ldots, n;\ j = 1, \ldots, m_i) \tag{11}$$

$$\eta_i = \eta\left(X_{ij}^t\,,\, \alpha\,,\, \beta_i\right) = \beta_i + X_{ij}^t\, \alpha$$

where:

- $n = 110$ wheels, m_i is the number of measurements per wheel $(i = 1, \ldots, n)$ and $m_i \leq 13$;
- η_i is the log-wear rate of the i^{th} sample unit (wheel); it is a function of the working positions and individual unit characteristics;
- β_i is the random effect associated to the i^{th} sample unit; it represents individual unit characteristics;

- $X_{ij} = \left[X_{ij1}, X_{ij2}, X_{ij3}, \left(X_{ij1} \times X_{ij2}\right), \left(X_{ij1} \times X_{ij3}\right), \left(X_{ij2} \times X_{ij3}\right)\right]^t$ is a 6×1 vector of covariates associated with Y_{ij}. The X_{ijl} (l = 1,2,3) are dummy variables indicating the working positions (side, truck, and axle). Specifically $X_{ij1} = 1$ if the i^{th} wheel unit is positioned on the *right side* and zero otherwise; $X_{ij2} = 1$ if the i^{th} wheel unit is positioned on the *back truck* and zero otherwise, and $X_{ij3} = 1$ if the i^{th} wheel unit is positioned on the *inner axle* within the truck and zero otherwise. In addition, $X_{ijl} \times X_{ijl^*}(l, l^* = 1,2,3, l \neq l^*)$ are dummy variables indicating the three second order interactions (*side* $\times$ *truck, side* $\times$ *axle, truck* $\times$ *axle*);
- $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_{12}, \alpha_{13}, \alpha_{23})^t$ is a 6×1 vector of fixed effects. The first three components, namely α_1, α_2 and α_3 represent the (population) main effects of side, truck and axle, respectively. The other three ones are associated to the second order interactions *side* $\times$ *truck, side* $\times$ *axle* and *truck* $\times$ *axle*, respectively;
- α_0 is the intercept, corresponding to the mean initial degradation level of the wheel;
- ε_{ij} is the associated random error for unit i at time (distance) t_j.

Note that we use t_j instead of t_{ij} in (11) since for the data set under study, $t_{ij} = t_j$ for all $i = (1, \ldots, n)$.

In addition, we assume that:

- The random errors ε_{ij} are independent and identically distributed (*i.i.d.*), according to a Normal distribution with mean zero and variance σ_ε^2 fixed and unknown.
- β_i ($i = 1, \ldots, n$) are independent and identically distributed according to a Normal distribution with mean μ_β and variance σ_β^2, (both fixed and unknown), *i.e.*, $\beta_i \overset{\text{iid}}{\sim} N\left(\mu_\beta, \sigma_\beta^2\right)$.
- The random effects β_i are independent of the random errors ε_{ij}.

This model parameterization has some advantages. First, the wear rate is always positive as suggested by the profiles (Figure 2). Second, by using the same argument presented in Section 2.3, it is possible to obtain the distribution of the time to failure F(t) analytically. In fact suppose that the actual degradation path of a particular wheel is given by

$$D(t) = \alpha_0 + e^\eta \times t$$

where $\eta = \eta(X^t\alpha; \beta)$, X is a $nc \times 1$ vector of covariates (dummy variables associated to the working positions and interactions, and nc is the number of covariates), α is a fixed parameter vector of order $nc \times 1$ and β varies from unit to unit according to a $N\left(\mu_\beta, \sigma_\beta^2\right)$. The failure time T is achieved when $D(t) = D_f$, the threshold level, or equivalently when $D_f = \alpha_0 + e^\eta T$. Therefore, as in Section 2.3, $T = \frac{D_f - \alpha_0}{e^\eta}$.

Now since $\eta = X^t\alpha + \beta$, then η is also normally distributed with mean $\mu_\eta = X^t\alpha + \mu_\beta$ and variance $\sigma_\eta^2 = \sigma_\beta^2$, *i.e.*, $\eta \sim N\left(X^t\alpha + \mu_\beta; \sigma_\beta^2\right)$. Consequently, the degradation rate e^η has a lognormal distribution, *i.e.*, $e^\eta \sim \text{logn}\left(X^t\alpha + \mu_\beta; \sigma_\beta^2\right)$ and the time to failure T of the wheels has a lognormal distribution. More specifically,

$$T \sim \text{logn}\left(\mu_T; \sigma_T^2\right)$$

with

$$\mu_T = log(D_f - \alpha_0) - \mu_\eta = log(D_f - \alpha_0) - (X^t\alpha + \mu_\beta) \text{ and } \sigma_T = \sigma_\eta = \sigma_\beta \tag{12}$$

Once the restricted maximum likelihood (REML) point estimates $\hat{\alpha_0}$, $\hat{\alpha}$ and $\hat{\sigma}_\beta$ are obtained, it is possible to estimate the failure time distribution F(t) by substituting those REML estimates into (12)(invariance property of the REML estimators). Similarly, point estimates of the reliability figures (*e.g.*, MTTF, quantiles) can also be obtained by the invariance property of the REML estimators.

Finally, this parameterization allows one to use the functions already implemented in softwares such as SAS, S-Plus and R, which assume a Normal distribution for the random effects. We use for the model fitting and parameter estimation, the function NLME developed by Pinheiro and Bates (2000) and implemented in the software R.

4 Model fitting and results

The results of the model fitting, after a complete evaluation from the full model up to the elimination of the non-significant interactions and factors are presented in Table 1. The residual analysis indicates no violation of the normality assumption for the random errors (Figure 6).

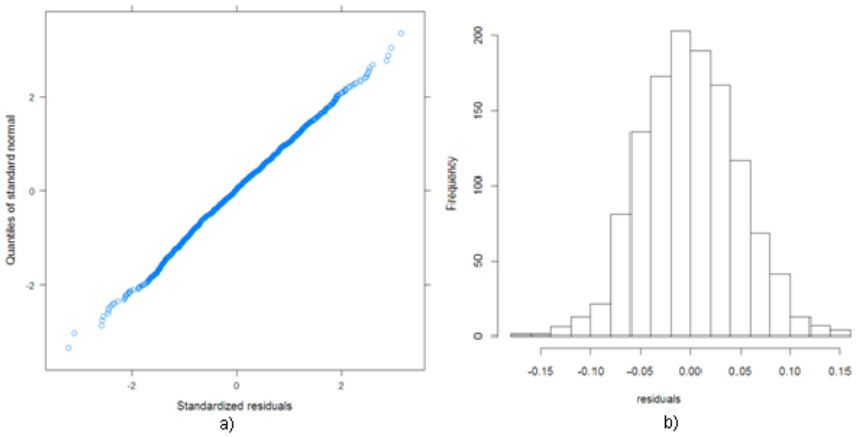

Figure 6: (a)Normal probability plot; (b) histogram of the model residuals.

Table 1: Estimated coefficients (fixed effects and standard errors) for the final model.

Parameter	Point Estimates	Std.Error	p-value
α_0	0.993688	0.00303564	0.000
μ_β	-9.491146	0.13979668	0.000
α_1 (side)	0.209818	0.19413978	0.280
α_3 (axle position)	0.100194	0.19418977	0.6059
α_{13} (side x axle)	-0.490497	0.27200106	0.0716(*)
Components of Variance			
σ^2_ε	$(0.05000057)^2$		
σ^2_β	$(0.71282520)^2$		

(*) Significant at 10% level

The initial degradation α_0 is significantly different from zero ($p < 10^{-4}$) and the estimated value has a small standard deviation, indicating one of the two following possibilities: 1) the actual initial diameter is not 996 mm but 1 mm smaller or, most likely, 2) there is a burn-in process occurring during the first covered distances (between zero and 50,000 km) due to the forging effect under the operational conditions (a very common phenomenon in high load mechanical systems). In addition, the *side* × *axle* interaction is significant at 10% level ($p = 0.0716$) while the *truck* × *axle* is not significant (p=0.573). The main effects of side and axle remain in the model even though they turned out to be non significant ($p = 0.28$ and 0.6059 respectively).

Table 2 shows the specific final degradation path models and the time to failure distributions for each one of the four working conditions.

Finally, we estimate the time to failure distribution and the reliability figures of interest for each one of the working conditions by substituting the estimated parameter values (Table 1) in the lognormal parameter expressions shown in Table 2. These results are presented in Table 3.

Table 2: Degradation path models and time to failure distributions for each one of the working conditions.

Working condition [side,axle]	**Degradation path model**	**Time to failure distribution lognormal(μ_T,σ_T)**
[0,0]=[left,outer]	$D_{ij} = \alpha_0 + \exp(\beta_i)t_j$	$\mu_T = log\left(D_f - \alpha_0\right) - \left(\mu_\beta\right)$; $\sigma_T = \sigma_\beta$
[0,1]=[left,inner]	$D_{ij} = \alpha_0 + \exp(\beta_i + \alpha_3)t_j$	$\mu_T = log\left(D_f - \alpha_0\right) - \left(\alpha_3 + \mu_\beta\right)$; $\sigma_T = \sigma_\beta$
[1,0]=[right,outer]	$D_{ij} = \alpha_0 + \exp(\beta_i + \alpha_1)t_j$	$\mu_T = log\left(D_f - \alpha_0\right) - \left(\alpha_1 + \mu_\beta\right)$; $\sigma_T = \sigma_\beta$
[1,1]=[right,inner]	$D_{ij} = \alpha_0 + \exp(\beta_i + \alpha_1 + \alpha_3 + \alpha_{13})t_j$	$\mu_T = log\left(D_f - \alpha_0\right) - \left(\alpha_1 + \alpha_3 + \alpha_{13} + \mu_\beta\right)$; $\sigma_T = \sigma_\beta$

4.1 Estimated reliability figures and comparison to the approximate degradation analysis.

In this section we compare the estimates of the reliability figures (quantiles and MTTF) obtained with the nonlinear model to the ones obtained with the approximate degradation analysis (Meeker and Escobar (1998)). In the approximate degradation analysis the pseudo failure times are analyzed as a complete sample of failure times. Table 3 shows the point estimates and two sided 90% confidence intervals for those figures. For the approximate method, we use the *delta method* (Meeker and Escobar (1998)), while for the nonlinear model approach we use a nonparametric bootstrap and bias-corrected percentile confidence interval (Efron and Tibshirani (1993)). The bootstrap confidence intervals were obtained by using the bootstrap simulation with B = 10,000. Figure 7 presents the estimated failure time distribution (F(t)) for each one of the working positions.

Some observations from Table 3 and Figure 7 are:

1. Although the point estimates are very similar, the confidence intervals based on the nonlinear model (and restricted maximum likelihood estimation) are wider than the ones obtained with the approximate analysis. This result should not come as a surprise. The approximate analysis on the pseudo failure times disregards the estimation errors by considering those data as actual realizations.

2. The wheels assigned to the working position [side,axle]=[1,0]=[right, outer] have the worst performance (independent of the truck position) while the ones assigned to the position [side, axle]=[1,1]=[right, inner], the best. This result can be seen by comparing the point estimates of the reliability figures presented in Table 3. The working position [1,0] has the smallest values, no matter which figure we choose. Another way to confirm this statement is by observing the F(t) curves shown in Figure 7. The curve located above all the others (i.e., indicating less reliability) is the one corresponding to the position [1,0]. On the other hand, the curve corresponding to position [1,1] is below all the others, indicating higher reliability levels.

It should be emphasized that during the stage of model fitting, the adequacy of the independence assumption among the unit-specific random effects $\beta_i (i = 1, \ldots, n)$ (and, consequently, among the unit-specific log-rates $\eta_i (i = 1, \ldots, n)$) was verified and turned out to be satisfied. Recall that each car is a *cluster* of eight wheels and such layout can indeed induce a correlation structure among the random effects of

wheels within the same car. To verify the existence of this cluster effect, a random effect associated to the cluster *train* was included in the model but it was not kept in the final model due to the small value of the standard deviation (1.4397×10^{-8}).

Table 3: Interval and point estimates of the reliability figures by working position

Working condition [side,axle]	Method	Estimates ($\times 10^3$ *Km*)				
		MTTF	$t_{0.01}$	$t_{0.05}$	$t_{0.10}$	$t_{0.50}$
[0,0]=[left,outer]	NLME	1,306.6 [979.1;1,713.6]*	196.9 [142.2;281.1]	314.6 [233.3;438.0]	408.6 [307.1;558.5]	1,005.5 [769.2;1,347.2]
	Approximate	1,238.8 [1,019.3;1,458.2]*	198.6 [149.6;263.5]	316,9 [245.8;408.6]	406,6 [319.3;517.8]	979.2 [748.8;1,221.7]
[0,1]=[left,inner]	NLME	1,186.8 [893.2;1,519.1]	171.2 [131.6;242.2]	258,5.6 [217.6;382.7]	365.8 [284.0;489.1]	922.1 [706.8;1,178.9]
	Approximate	1,125.3 [933.1;1,318.0]	180.5 [136.8;238.1]	288,0 [224.9;368.7]	369,4 [292.2;467.0]	889.7 [718.9;1,101.1]
[1,0]=[right,outer]	NLME	1,056.5 [773.1;1,421.8]	157.3 [116.9;222.7]	254.6 [192.1;349.9]	329.4 [249.7;449.6]	819.5 [612.2;1,087.3]
	Approximate	1,008.6 [836.1;1,181.0]	161.7 [122.6;213.3]	258,0 [201.5;330.4]	331,0 [261.1;418.5]	797.2 [644.2;1,986.7]
[1,1]=[right,inner]	NLME	1,573.5 [1,195,4;1,960.9]	225.1 [169.4;329.7]	367.3 [279.0;508.9]	477.4 [367.2;664.3]	1,207.4 [935.9;1,524.2]
	Approximate	1,476.6 [1,224.1;1,729.1]	236.7 [179.4;312.3]	377,7 [295.0;483.7]	484,6 [383.4;612.7]	1,167.2 [943.1;1,445.5]

90% confidence interval.

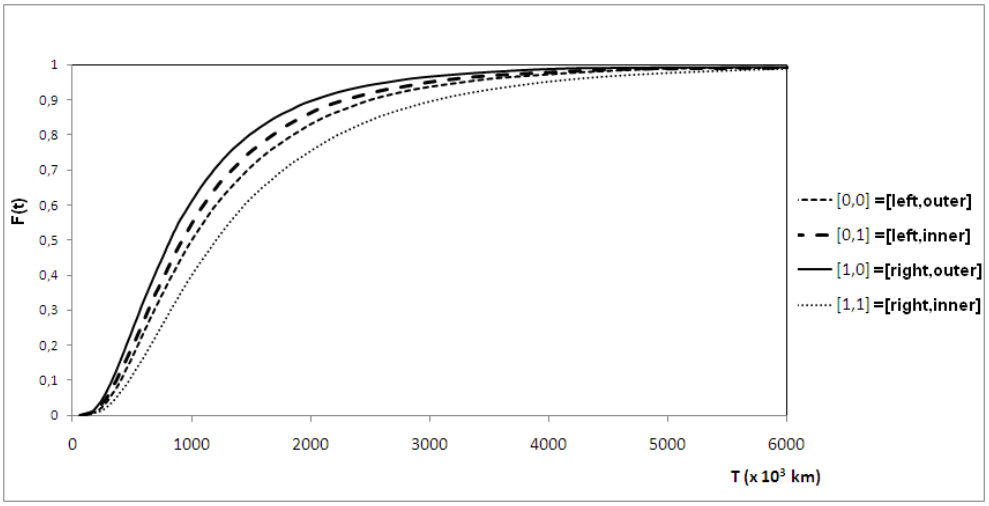

Figure 7: Fitted time to failure distributions $F(T)$ by working condition

5 Conclusion and final comments

In this paper we introduce a model to analyze the degradation data of train wheels with linear trend (over time) degradation profiles. Before considering the practical results of the data analysis it is worth stepping back and appreciate some of the main features of the proposed model. Some include the following:

1. the model is linear in time with positive degradation rate (in agreement with the observed wheels degradation profiles);
2. by modeling the unit-specific log-rates ($\eta_i, i = 1, \ldots, n$) as linear functions of covariates (working positions) and unit-specific normally distributed random effect ($\beta_i;\ i = 1, \ldots, n$), it is possible to investigate possible effects of working conditions (side,axle and truck position) on the failure time distribution;
3. in addition, since the unit-specific log-rates are linear functions of normally distributed random effects, it is possible to use functions developed for Normal random effects which are already implemented in a number of softwares such as S-Plus, SAS and R. Here we used the NLME function available in R.
4. finally but not less important, even with the inclusion of the covariates in the log-rate equation, it is still possible to obtain the failure time distributions by working positions **analytically** which in turn reduces dramatically the computational effort.

Now, as far as the practical results are concerned, we can say that:

1. the approximate method and the nonlinear mixed effects model (with restricted maximum likelihood estimation) provided quite similar point estimates for the key figures although the latter presented wider and more reliable confidence intervals;
2. we were able to identify working positions where the wheels are subject to higher levels of stress and whose difference in wear rate affected substantially the time to failure distributions;
3. the results presented in this paper can be used to support decisions in main areas, for instance:
    - Maintenance: the determination of different maintenance follow up procedures (such as frequency of inspections) for different wheel positions and
    - Research and Development: the design of reliability tests with new materials assigned to the different working conditions (from the mildest condition [1,1] up to the stressful one [1,0]).

Acknowledgements

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A Código Comentado do Modelo

```
####Procedimento para ajuste do modelo de degradação de
# todas as rodas de trem
####  Elaborado por Julio Cesar Ferreira
# Nao esquecer de acertar o diretorio de execucao e o nome do
# arquivo de dados
### Verificar tambem a forma funcional do modelo - se com ou
# sem intercepto


setwd("D:/Julio/Meus Documentos/Mestrado/Dissertação/Testes computacionais")

#Lendo as medidas de Desgaste do arquivo .txt, organizando e
# exibindo os dados...

dados<-read.table('dados_ma_empilhados_mod_2_semrodas_7_13_MA11.txt',
head=T,sep=";",dec=",")

attach(dados)
LADO <- as.factor(LADO)
POS_TRUCK <- as.factor(POS_TRUCK)
POS_EIXO <- as.factor(POS_EIXO)
ID_RODA <-as.factor(ID_RODA)
TREM <-as.factor(TREM)
dados

#### Abrindo as bibliotecas de interesse
library(nlme)
library(lattice)


####DETERMINACAO DA FORMA FUNCIONAL E AJUSTE DO MODELO #####

equacaodesgaste <- function(beta1,beta2,tempo) beta1+exp(beta2)*tempo

## Ajustando o modelo...

modelo<-nlme(DESGASTE ~ equacaodesgaste(beta1,beta2,DISTANCIA),
data=dados,fixed = list(beta1~1,beta2~LADO*POS_EIXO),random =
beta2~1|ID_RODA,start=c(1,-2,1,1,1),method="REML")

### Exibindo os resultados do modelo

summary(modelo)

### Agora, serão estimadas as distribuições do tempo de falha
#para as diversas posições de roda

# 1a rotina: estimacao pontual de F(t)
# Deve-se ajustar o parametro "N.bt" (numero de amostras bootstrap)
# df é a degradação limitrofe = 77mm
```

```
######## Declaração de funções úteis:

# calcular.percentis: retorna um vetor com os tempos quantis,
#dados os percentis de interesse
## Será usada no bootstrap

calcular.percentis <- function(vetor.tempos,vetor.percentis){
vetor.distribuicao<-c(rep(0,length(vetor.percentis)))
for (z in 1:length(vetor.distribuicao))
 vetor.distribuicao[z]<-quantile(vetor.tempos,vetor.percentis[z])
return(vetor.distribuicao)
}


## calcular.quantis: calcula percentis de interesse,
# dados os quantis de interesse

calcular.quantis<-function(vetor.tempos,vetor.quantis){
      resultado<-c(rep(0,length(vetor.quantis)))
      for(z in 1:length(vetor.quantis))
resultado[z]<-length(vetor.tempos[vetor.tempos<=vetor.quantis[z]])/
length(vetor.tempos)
      return(resultado)
}

####  Parametros da Simulação

df=77   #Limiar de desgaste
N.bt=10000
# Selecionar um numero PAR bootstrap para
# obter percentis corretos

vetor.posicoes = matrix(c(0,0,1,1,0,1,0,1),nrow=4)
rownames(vetor.posicoes) = c("esq/ext","esq/int","dir/ext","dir/int")
# é o vetor que armazena as posicoes de uso (lado, eixo)

media.mc = matrix(0,length(vetor.posicoes[,1]),3)
rownames(media.mc) = rownames(media.mc,do.NULL=FALSE,prefix="Posicao.")
colnames(media.mc)<-c("\%IC_inf","Muˆ","\%IC_sup")
#vetor que irá armazenar a média e seus limites de 90\% de confiança

percentis.ft<-c(seq(0.01,0.99,0.01),.999)
#Vetor com os percentis de F(t) para geração do grafico

ft<-c(rep(0,length(vetor.posicoes[,1])*length(percentis.ft)*3))
dim(ft)<-c(length(percentis.ft),3,length(vetor.posicoes[,1]))
dimnames(ft)<-list(NULL,c("\%IC_inf","F(t)","\%IC_sup"),
c("pos1","pos2","pos3","pos4"))
# Matriz que armazena a estimação de F(t) pontual e intervalar

ft.bt<-c(rep(0,length(percentis.ft)*N.bt*length(vetor.posicoes[,1])))
```

```
dim(ft.bt)<-c(length(percentis.ft),N.bt,length(vetor.posicoes[,1]))
#matriz que armazena as distribuicoes empiricas dos
#tempos de falha no bootstrap

medias.bt<-matrix(0,length(vetor.posicoes[,1]),N.bt)
rownames(medias.bt) = rownames(medias.bt,do.NULL=FALSE,prefix="Posicao.")
#Vetor que irá armazenar as medias do tempo de falha
#por posicao de cada iteracao bootstrap

percentis.interesse <-c(0.01,0.05,0.1,0.5)
#vetor com os percentis de interesse - para construção
#da tabela de resultados

percentis.interesse.bt<-c(rep(0,N.bt*4*length(vetor.posicoes[,1])))
dim(percentis.interesse.bt)<-c(N.bt,4,length(vetor.posicoes[,1]))
dimnames(percentis.interesse.bt)<-list(NULL,c("1\%","5\%","10\%","50\%"),
c("pos1","pos2","pos3","pos4"))
### Matriz que armazena os percentis de interesse a
# cada iteração bootstrap

# Gerando um vetor com os valores de f(t) e a média,
# para gerar gráfico e tabela de resultado

for (k in 1:length(vetor.posicoes[,1]))
{
  ft[,2,k]<-qlnorm(percentis.ft,(log(df-modelo$coefficients$fixed[1])-
(modelo$coefficients$fixed[2]+modelo$coefficients$fixed[3]*
vetor.posicoes[k,1]+modelo$coefficients$fixed[4]*
vetor.posicoes[k,2]+modelo$coefficients$fixed[5]*
vetor.posicoes[k,1]*vetor.posicoes[k,2])),sd(ranef(modelo)))

  media.mc[k,2]<-exp(log(df-modelo$coefficients$fixed[1])-
(modelo$coefficients$fixed[2]+modelo$coefficients$fixed[3]*
vetor.posicoes[k,1]+modelo$coefficients$fixed[4]*vetor.posicoes[k,2]+
modelo$coefficients$fixed[5]*vetor.posicoes[k,1]*vetor.posicoes[k,2])
+0.5*(sd(ranef(modelo))^2))
}
# Fim da geraçao da estimação de F(t) baseada no Modelo
# com os dados originais

# Agora faremos a geraçao das distribuicoes F*(t)
# Bootstrap, para construcao do intervalo de confianca das
# quantidades de interese
# O modelo2 será baseado na reamostragem dos perfis

#######Algoritmo Bootstrap Não-Parametrico ############

for(j in 1:N.bt)
 {

   dados.bt<-dados[dados$ID_RODA==0]
##"dados.bt" inicialmente recebe "dados" LIMPO
```

```
##Amostrando dos perfis originais com reposição

   rodas.sorteio<-sample(unique(dados$ID_RODA),replace=TRUE)
   for(i in 1:length(rodas.sorteio))
    {
      dados.bt.aux<-dados[dados$ID_RODA==rodas.sorteio[i],]
      dados.bt.aux$ID_RODA<-
        rep(i,length(dados[dados$ID_RODA==rodas.sorteio[i],1]))
      dados.bt<-rbind(dados.bt,dados.bt.aux)
    }

### Fim da geração das amostras bootstrap, agora
# faremos o ajuste do modelo NLME bootstrap

modelo2<-nlme(DESGASTE ~equacaodesgaste(beta1,beta2,DISTANCIA),
data=dados.bt,fixed = list(beta1~1,beta2~LADO*POS_EIXO),random =
beta2~1|ID_RODA,start=c(1,-2,1,1,1),method="REML")

#Agora vamos estimar e armazenar a F*(t)
# da j-esima iteracao bootstrap

     for(k in 1:length(vetor.posicoes[,1]))
{

ft.bt[,j,k]<-plnorm(ft[,2,k],(log(df-modelo2$coefficients$fixed[1])-
(modelo2$coefficients$fixed[2]+modelo2$coefficients$fixed[3]*
vetor.posicoes[k,1]+modelo2$coefficients$fixed[4]*vetor.posicoes[k,2]+
modelo2$coefficients$fixed[5]*vetor.posicoes[k,1]*vetor.posicoes[k,2]))
,sd(ranef(modelo2)))

medias.bt[k,j]<-exp(log(df-modelo2$coefficients$fixed[1])-
(modelo2$coefficients$fixed[2]+modelo2$coefficients$fixed[3]*
vetor.posicoes[k,1]+modelo2$coefficients$fixed[4]*vetor.posicoes[k,2]+
modelo2$coefficients$fixed[5]*vetor.posicoes[k,1]*vetor.posicoes[k,2])
+0.5*(sd(ranef(modelo2))^2))

percentis.interesse.bt[j,,k]<-qlnorm(percentis.interesse,
(log(df-modelo2$coefficients$fixed[1])-(modelo2$coefficients$fixed[2]+
modelo2$coefficients$fixed[3]*vetor.posicoes[k,1]+
modelo2$coefficients$fixed[4]*vetor.posicoes[k,2]+
modelo2$coefficients$fixed[5]*vetor.posicoes[k,1]*
vetor.posicoes[k,2])),sd(ranef(modelo2)))
}
}
#### Fim da Simulação Bootstrap

## Agora, iremos calcular os intervalos de confianca
# bootstrap para as quantidades de interesse

##Cálculo dos percentis de interesse
# (quantis 1\%, 5\%, 10\%, 50\%)
```

```
ic.percentis.interesse<-c(rep(0,2*length(percentis.interesse)*
length(vetor.posicoes[,1])))
dim(ic.percentis.interesse)<-c(2,length(percentis.interesse),
length(vetor.posicoes[,1]))
dimnames(ic.percentis.interesse)<-list(c("2,5\%","97,5\%"),
c("1\%","5\%","10\%","50\%"),c("pos1","pos2","pos3","pos4"))

for(k in 1:length(vetor.posicoes[,1]))
{

  for(j in 1:length(percentis.interesse))
  {
  l<-ceiling(N.bt*0.05)
  u<-floor(N.bt*.95)
  ic.percentis.interesse[1,j,k]<-sort(percentis.interesse.bt[,j,k])[l]
  ic.percentis.interesse[2,j,k]<-sort(percentis.interesse.bt[,j,k])[u]
  }

}

### Calculando o intervalo de confiança para a media

for(k in 1:length(vetor.posicoes[,1]))
{
   l<-ceiling(N.bt*0.05)
   u<-floor(N.bt*.95)
   media.mc[k,1]<-sort(medias.bt[k,])[l]
   media.mc[k,3]<-sort(medias.bt[k,])[u]
}

### Calculando os intervalos de confianca bootstrap para F(t)

for(k in 1:length(vetor.posicoes[,1]))

  for(j in 1:length(percentis.ft))
{
l<-ceiling(N.bt*pnorm(2*qnorm((length(ft.bt[j,,k][ft.bt[j,,k]<=
    percentis.ft[j]])/N.bt),0,1)+qnorm(.05,0,1),0,1))
   u<-N.bt*pnorm(2*qnorm(1-(length(ft.bt[j,,k][ft.bt[j,,k]<=
percentis.ft[j]])/N.bt),0,1)+qnorm(.95,0,1),0,1)
   ft[j,1,k]<-sort(ft.bt[j,,k])[l]
   ft[j,3,k]<-sort(ft.bt[j,,k])[u]
}

###Plotando as estimativas das distribuições das quatro posições

windows()
plot(ft[,2,1],percentis.ft,type="n",xlim=c(0,6000000),ylim=c(0,1),
xlab="Distancia",ylab="P(T<t)",main="Distribuições do tempo de
falha por posição")
for (j in 1:length(vetor.posicoes[,1]))
```

```
{
lines(ft[,2,j],percentis.ft,type="l",lwd=(j-2),lty=j+6)
legend(legend=paste("Lado",vetor.posicoes[j,1],"Eixo",
    vetor.posicoes[j,2]),x=3000000,y=j/20,bty="n",lwd=j-2,
    lty=j+6,cex=0.8)
}

##Plotando as distribuições do tempo de falha por posição
#IC de 90\% para F(t)

for(k in 1:length(vetor.posicoes[,1]))
{
 windows()
 plot(ft[,2,k],percentis.ft,type="n",xlim=c(0,6000000),
ylim=c(0,1),xlab="Distancia",ylab="P(T<t/X)",main=paste
("Distribuicao F(t/X)-IC 90\%",vetor.posicoes[k,1],
vetor.posicoes[k,2]))lines(ft[,2,k],percentis.ft,type="l",lty=1)
      lines(ft[,2,k],ft[,1,k],type="l",lty=2)
      lines(ft[,2,k],ft[,3,k],type="l",lty=2)
}

# Gravando os resultados em arquivo
write.csv(ft,"ft.csv")
write.csv(ic.percentis.interesse,"ic.percentis.interesse.csv")
write.csv(media.mc,"medias.csv")
```