# EVOLUTIONARY MODELS OF ROTATING LOW MASS STARS 

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# EVOLUTIONARY MODELS OF ROTATING LOW MASS STARS 

Thesis submitted to the UNIVERSIDADE FEDERAL DE MINAS GERAIS as a partial requirement for obtaining the Ph.D. degree in Physics.

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## Prólogo

O presente trabalho, intitulado "Modelos Evolutivos de Estrelas de Baixa Massa com rotação", tem por objetivo investigar os efeitos da rotação e redistribuição de momento angular na evolução de estrelas de baixa massa.

Graças a uma colaboração científica com os Drs. Francesca D'Antona (Osservatorio Astronomico di Roma, Itália) e Italo Mazzitelli ( $C N R$, Itália) iniciada pelo Dr. Luiz Paulo Ribeiro Vaz (Universidade Federal de Minas Gerais), nos foi possível ter acesso ao código evolutivo ATON 2.0, que foi utilizado como ferramenta para tal investigação. Estes pesquisadores italianos foram, ao longo do trabalho, mais do que simples colaboradores, tendo contribuído também na supervisão do trabalho, em especial durante os períodos que cumprimos na Itália.

Por estas razões, decidimos escrever este trabalho na língua inglesa, de forma a permitir que o mesmo pudesse ser examinado e criticado também pelos pesquisadores italianos. Foi incluído, entretanto, um capítulo em língua portuguesa contendo uma síntese de todo o trabalho executado.

## Prologue

This work, entitled "Evolutionary models of rotating low-mass stars", is aimed at investigating the effects of rotation and angular momentum redistribution in the evolution of low mass stars.

Thanks to a scientific collaboration settled between Drs. Francesca D'Antona (Osservatorio Astronomico di Roma, Italy) and Italo Mazzitelli ( $C N R$, Italy) and Dr. Luiz Paulo Ribeiro Vaz (Universidade Federal de Minas Gerais), the aton 2.0 evolutionary code was made available to us as a scientific tool for that investigation. Those italian researchers have been more than simple collaborators, and have contributed also in supervising the work especially during our stays in Italy.

For these reasons, we choosed to write this thesis in English, in order to allow them to read it and make any suggestions or criticisms. There is, however, a chapter written in Portuguese containing a synopsis of the entire work.

## Agradecimentos

Gostaria de expressar minha sincera gratidão às seguintes pessoas e instituições, cujo apoio e incentivo permitiram-me iniciar e concluir este trabalho:

- Meu orientador, Prof. Dr. Luiz Paulo Ribeiro Vaz, por sua orientação, incentivo e paciência durante todos estes anos, e também por seu particular senso de equilíbrio entre praticidade e perfeição;
- Dra. Francesca D'Antona, do Osservatorio Astronomico di Roma, pela sua ajuda com relação ao tema desta tese, pela sua hospitalidade durante minhas estadias na Itália, e por muitas discussões frutíferas;
- Dr. Italo Mazzitelli, do Istituto di Astrofisica Spaziale, por nos ter permitido acesso ao programa evolutivo estelar ATON, pelo seu apoio constante e por sua amizade;
- Meus amigos italianos - Fabrizio Massi, Michele di Toro, Paolo Ventura e Lorenzo Amati, entre tantos - pela sua amizade, hospitalidade e pelas muitas partidas de futebol;
- Meus colegas do Laboratório de Astrofísica - em especial Wagner Corradi, Sergio Luiz, Silvia Alencar e Nuno Cunha, pelos muitos momentos de diversão e descontração;
- A bibliotecária do Departamento de Física, Maria Cristina Lacerda, por ter me conseguido várias cópias de artigos raros e pelo tempo extra na biblioteca (além do horário normal de funcionamento);
- Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), pelo suporte financeiro durante a maior parte deste trabalho;
- Consiglio Nazionale delle Ricerche (CNR), pelo apoio financeiro parcial durante minhas estadias na Itália, onde parte do trabalho foi realizado;
- Centro Nacional de Computação de Alto Desempenho (CENAPAD-MG), por disponibilizar o acesso aos seus recursos para a execução da maior parte dos cálculos;
- E, finalmente, minha esposa Auxiliadora, pela inesgotável ternura e por ter me trazido nossa adorável filha Marília, cujo nascimento me trouxe uma nova visão de nossas existências.


## Acknowledgements

I would like to express my deep gratitude to the following people and institutions, whose support and encouragement enabled me to start and complete this work:

- My thesis advisor, Prof. Dr. Luiz Paulo Ribeiro Vaz, for his direction, encouragement and patience during all these years, along with his unique sense of balance between practicality and perfection;
- Dr. Francesca D'Antona, from the Osservatorio Astronomico di Roma, not only for her guidance with respect to the subject of this thesis, but also for her warm hospitality during my stays in Italy and for many fruitful discussions;
- Dr. Italo Mazzitelli, from the Istituto di Astrofisca Spaziale, for providing us with the ATON stellar evolutionary code and for his continuous support and friendship;
- My italian friends - Fabrizio Massi, Michele di Toro, Paolo Ventura, and Lorenzo Amati, to name just a few - for their friendship, hospitality and for organizing the soccer matches;
- My colleagues from our Astrophysics Lab, in special Wagner Corradi, Sergio Luiz, Silvia Alencar and Nuno Cunha, for the many moments of fun and relaxation;
- The librarian of the Physics Department, Maria Cristina Lacerda, for providing me with rare paper copies and extra time at the library (beyound office hours);
- The Brazilian Council for Scientific and Technological Development (Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq), for the financial support during most of this work;
- The Consiglio Nazionale delle Ricerche (CNR), for providing some financial support during my stays in Italy, where part of this work was carried out;
- The National Center for High Performance Computing (Centro Nacional de Computação de Alto Desempenho- CENAPAD-MG), for making available its facilities for doing most of our computations;
- And finally, my wife Auxiliadora, for her inexhaustible tenderness and for bringing me our lovely daughter Marília, whose birth gave me new eyes to our existences.


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## Resumo

Neste trabalho nós investigamos os efeitos combinados da rotação e da redistribuição de momento angular em estrelas de baixa massa, desde a etapa pré-sequiência principal até a seqüência principal. Como uma ferramenta para tal estudo, o código evolutivo estelar ATON (Mazzitelli 1989; Ventura et al. 1998b) foi alterado de forma a incluir tais efeitos. A rotação foi implementada segundo a técnica de superfícies equipotenciais desenvolvida por Kippenhahn \& Thomas (1970) e posteriormente aperfeiçoada por Endal \& Sofia (1976). A redistribuição de momento angular foi modelada por intermédio de uma equação diferencial parcial do tipo difusivo-advectivo, com base na teoria originalmente desenvolvida por Chaboyer \& Zahn (1992), a qual considera como único pressuposto que o transporte turbulento é muito mais forte na direção horizontal que na vertical. O coeficiente de difusão desta equação é obtido a partir de comprimentos e velocidades característicos de instabilidades hidrodinâmicas induzidas pela rotação. Este novo código foi empregado para calcular uma série de modelos com rotação de estrelas de baixa massa (na faixa de $0.6 M_{\odot}$ a $\left.1.2 M_{\odot}\right)$. As características gerais destes modelos, com relação aos efeitos estruturais (hidrostáticos) da rotação, mostram que estrelas com rotação comportam-se como se fossem estrelas sem rotação porém de massa ligeiramente menor, estando de acordo com resultados anteriores por outros pesquisadores. Um estudo deste "efeito de abaixamento de massa" para a faixa de massas considerada revela que a rotação diminui a queima de lítio enquanto a estrela é totalmente convectiva, mas aumenta a mesma tão logo a estrela desenvolve um núcleo radiativo. O efeito líquido é um aumento da queima de lítio, em desacordo com dados observacionais que mostram que as estrelas de aglomerados abertos jovens que giram mais rápido são as que apresentam menor queima de lítio. A redistribuição de momento angular nos modelos considerados torna-se muito eficaz na suavização do gradiente interno de velocidade angular tão logo a estrela atinja a idade zero na seqüência principal, mas não é capaz de reproduzir a curva de rotação solar obtida da heliosismologia, indicando que o transporte de momento angular no Sol é mais eficiente do que o predito pelos atuais modelos. O transporte interno de momento angular também contribui para uma queima ainda maior de lítio com relação aos modelos calculados somente com os efeitos estruturais. Isto sugere que outros fenômenos físicos devem ser importantes tanto para a queima de lítio quanto para a evolução da curva de velocidade angular das estrelas.

## Abstract

We have investigated the combined effects of rotation and internal angular momentum redistribution on the structure and evolution of low mass stars, from the pre-main sequence to the main sequence phase. As a tool for that study, the ATON stellar evolutionary code (Mazzitelli 1989; Ventura et al. 1998b) has been modified in order to include those effects. Rotation was implemented according to the equipotential technique developed by Kippenhahn \& Thomas (1970) and later improved by Endal \& Sofia (1976). Angular momentum redistribution in radiative regions was modeled through an advection-diffusion partial differential equation based on the framework originally introduced by Chaboyer \& Zahn (1992), which is based on the sole assumption of stronger turbulent transport in the horizontal direction than in the vertical one. The diffusion coefficient of this equation is obtained from characteristic lengths and velocities of typical rotation-induced hydrodynamical instabilities. This improved code was used to compute a series of rotating low mass stellar models (with masses ranging from $1.2 M_{\odot}$ down to $0.6 M_{\odot}$ ). Regarding the structural (hydrostatic) effects of rotation, the general features of these models show that rotating stars behave as if they were non-rotating stars of slightly lower masses, in accordance with previous results by other researchers. A study of this mass-lowering effect for the considered range of masses shows that rotation decreases lithium depletion while the star is fully convective but increases it as soon as the star develops a radiative core. The net effect is a enhanced lithium depletion, in disagreement with observational data which suggest that faster rotators in young open clusters experience less lithium depletion. Angular momentum redistribution in the considered models is very effective in smoothing their internal angular velocity profile as soon as the star reaches the zero age main sequence, but fails to reproduce the flat solar rotation rate obtained from helioseismology, indicating that, in the Sun, angular momentum transport is more efficient than current models. The internal angular momentum transport also contributes to a still higher lithium depletion than the models computed with only the structural effects of rotation, thus suggesting that other physical phenomena must play a role regarding both lithium depletion and the rotation profile evolution of these stars.

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## Chapter 1

## INTRODUCTION

For almost three decades standard stellar evolutionary models (i.e. models without rotation, magnetic fields, etc.) have been very successful in explaining the general features of stars. Nevertheless, the increasing high-precision observational data available to astronomers and astrophysicists, as well as some significant advances in correlated research areas such as helio- and asteroseismology, are now evidencing the weaknesses and limitations of these standard models. The internal structure of rotating and non-rotating stars are different, and these differences can effectively affect fundamental properties such as the age of star clusters and the luminosity function on the main sequence (Pinsonneault et al. 1988). Besides, the angular momentum excess in protostellar clouds, as compared with the angular momentum of stars, shows that rotation can play a major role in the star formation mechanisms.

One of the major discrepancies between observational data and the results of standard theoretical models of stars refers to the initial abundance of light elements such as lithium and beryllium. This problem is even worse in the Sun, whose abundance of ${ }^{7} \mathrm{Li}$ is lower by a factor of 200 than the ${ }^{7} \mathrm{Li}$ abundance found in meteorites (D'Antona \& Mazzitelli 1984). Furthermore, evolved stars often show anomalous surface abundances, which reveal the presence of mixing at deeper levels. These stars, however, lie in regions of the H-R diagram for which the standard models do not predict such mixing. Rotation, as a physical property shown by all stars, is a good agent for the mixing, as it produces mass motions that in turn generate the mixing of stellar material.

It is also worth mentioning that rotating models have another advantage: the possibility of using the observed surface rotational velocity as a constraint for the theoretical models, a possibility obviously unavailable for the standard non-rotating models.

The main goal of the present work is to study the effects of introducing rotation and internal angular momentum redistribution in stellar evolutionary codes, with particular
emphasis on low-mass stars starting at the pre-main sequence. The evolutionary code used is the ATON 2.0 code, to which we had access through a scientific collaboration with Dr. Francesca D'Antona (Osservatorio Astronomico di Roma, Italy) and Dr. Italo Mazzitelli ( $C N R$, Italy) started by Dr. Luiz Paulo Ribeiro Vaz (incidentally my thesis advisor), from the Physics Department of the Universidade Federal de Minas Gerais (UFMG)

This work is structured in the following way: Chapter 2 gives a brief account of the current research status in stellar modeling with rotation. Chapter 3 describes the main features of the ATON 2.0 stellar evolutionary code and the techniques used for introducing rotation and internal angular momentum redistribution in it, as well as some implementation details. Chapter 4 explores the major results obtained with this new version of the evolutionary code. Chapter 5 presents the general conclusions and some suggestions for improving the work done. And, finally, Chapter 6 presents a synopsis of the entire work in the portuguese language.

First things first, but not necessarily in that order.

[^0]
## Chapter 2

## THE IMPORTANCE OF ROTATING STELLAR MODELS

### 2.1 Background on stellar models

The foundations of stellar structure theory were established in the first half of this century and consolidated in many seminal works such as those of Chandrasekhar (1939) and Schwarzschild (1958) to name just a couple of the most important. Since the only way of getting information about the structure of stars is through their emitted radiation and gravitational interaction with the neighbourhood, theoretical models of stars have been very important tools for understanding the internal constitution of stars. Such theoretical models are constructed by numerically integrating the four stellar structure equations, based on prescribed physical parameters such as mass and chemical composition. A sequence of stellar models progressing in time defines the "evolutionary path" of a given model star.

The technique of constructing theoretical models of stars received a strong impulse in the sixties due to the advent of faster digital computers and efficient numerical methods for solving the stellar structure equations such as the now universally used relaxation method (Henyey et al. 1964). These factors, combined with important theoretical improvements in stellar nucleosynthesis (Clayton 1968) and opacities (Cox \& Stewart 1970), have made the theoretical models an invaluable tool for astrophysicists. The details of computing theoretical stellar models are given in the classical paper of Kippenhahn et al. (1967).

Nevertheless, it is very important to keep in mind that theoretical models are no more than idealizations of the many complex physical processes inside the stars and, though very
encouraging, cannot accurately reproduce the enormous quantity of observational data gathered about stars. This is somewhat obvious as some physical processes, despite having well founded and sound theoretical basis, are usually implemented in stellar models with varying degrees of simplifications in order to overcome typical problems such as numerical instabilities and accuracy, mathematical complexity and (even today) computing time, while others (e.g. turbulence) are still very poorly understood. As a result, there are many important issues to be addressed such as chemical mixing (Goupil \& Zahn 1989), the depletion of light elements (D'Antona e Mazzitelli 1984), the equations of state for stellar interiors (Magni \& Mazzitelli 1979, Mihalas et al. 1988), turbulent convection (Forestini et al. 1991, Canuto \& Mazzitelli 1991), and the effects of rotation (Kippenhahn \& Thomas 1970; Pinsonneault et al. 1989).

Turning our attention to rotation, the first attempts to include its effects in existing stellar evolutionary codes date back to the sixties, when the major methods, still in use today, were developed (Faulkner et al. 1968; Sackmann \& Anand 1969; Kippenhahn \& Thomas 1970; Papaloizou \& Wheelan 1972). From all these approaches, the equipotential surface method of Kippenhahn \& Thomas (1970) seems to be the most used due to its ease of implementation in existing one-dimensional evolutionary codes (see Endal \& Sofia 1976 for a comparison of those methods). Rotation not only causes a number of other physical effects (e.g. meridional circulation) but also triggers some hydrodynamical instabilities in radiative regions of stars, both of which lead to internal momentum redistribution. There are also many other issues directly related to rotation such as the interaction between rotation and convection, magnetic breaking by stellar winds, and the angular momentum transfer from circumstellar disks to the central star, all of which still challenge (from both the physical and mathematical points of view) the present generation of theoretical stellar models.

One of the major applications of stellar evolutionary codes has been, of course, the modeling of the different phases that occur along the stellar evolution. These phases, as it is well known, depend mainly on the star's mass and chemical composition and are traditionally grouped under three main phases: the pre-main sequence phase, where a self-gravitating gas cloud contracts in a quasi-static manner until its central temperature becomes high enough to stop the gravitational contraction and to start the nuclear burning; the main sequence, which corresponds to the long-standing stability phase in which the star core burns hydrogen; and the post main-sequence phase, which starts when the hydrogen in the star core is exhausted and the star (depending on its mass) begins to burn helium in the core and hydrogen in the shell surrounding it. The star's initial mass determines not only the duration of each phase but also the star's fate in the post-main sequence phase (which can range from white dwarfs, for low-mass stars, to neutron stars for the massive ones).

Due to the apparent simplicity of the physical processes that occur in the pre-main sequence phase (detailed for example in Henyey et al. 1955 and Hayashi 1961), which after
the eventual mass accretion and/or loss periods and deuterium burning result in fully convective stars under hydrostatic contraction, most researchers have put their efforts in the main- and post-main sequence phases over the three last decades. However, there is now a growing understanding that several still poorly known physical phenomena, which have little effect in the main sequence phase, do have a great importance in the pre-main sequence phase, casting doubt on its supposed simplicity. As examples we can cite the superadiabatic convection, the equation of state, the radiative opacity, convective overshooting (Mazzitelli 1989, Massaguer et al. 1984), rotation and magnetic fields.

On the other hand, there are not yet as many available observational data of pre-main sequence stars as there are of main sequence ones. There are two main reasons for this scarcity of data: first, pre-main sequence stars are usually located inside huge concentrations of gas and dust, that not only absorb part of the emitted radiation from the pre-main sequence star but also reprocess it, and so superimposing its own spectrum on that of the pre-main sequence star; second, there is just a handful of pre-main sequence eclipsing binaries, which are the only objects which yield direct determinations of physical parameters such as mass, radius, temperature and orbit inclination.

The growing evidence that the pre-main sequence phase is much more complex than previously assumed, and the scarcity of available observational data for it, have lead to an increasing and renewed interest in pre-main sequence stars. Theoretical evolutionary models will be a key tool to understand the properties of these stars.

### 2.2 Rotation and internal angular momentum redistribution

### 2.2.1 General effects of rotation

The physical effects of rotation in stars can be divided in two main categories: the structural (or hydrostatic) effects and the evolutionary effects. The structural effects are the most fundamental ones and come from the simple fact that rotation gives rise to a centrifugal force that counteracts to some extent the gravitational force, thus affecting the condition of hydrostatic equilibrium. The evolutionary effects, on the other hand, are related to the changes in the internal angular velocity profile due to angular momentum redistribution.

### 2.2.1.1 Structural effects

The major techniques for introducing the structural effects of rotation in stellar evolutionary codes were developed mainly during the 60 's and 70 's. Nevertheless, since then, very
few evolutionary codes have considered those effects, probably due to the high success of standard* models in explaining the bulk of available observational data on stars (such as the mass-luminosity relationship, the H-R diagrams of clusters, etc.), and also to the high computational cost of those techniques at that time. For a review of those techniques, as well as a comprehensive discussion of the results brought by their application, we refer the reader to the excellent book by Tassoul (1978). For our purposes, it suffices to say that perhaps the most important result regarding the structural effects is that the evolutionary tracks of rotating stars shift towards lower effective temperatures and luminosities, simulating the evolutionary path of a non-rotating star of lower mass (the so-called mass-lowering effect; see Sackmann 1970). Section 3.3.1 discusses the technique we adopted for introducing the structural effects of rotation in the ATON 2.0 code.

### 2.2.1.2 Evolutionary effects

Since Eddington (1925) and Von Zeipel (1924a,b), it is well known that rotation causes a thermal imbalance in stars, which in turn drives meridional circulation currents. Even if the initial rotation state corresponds to rigid body rotation, those circulation currents alter the internal angular velocity profile leading to differential rotation, which in turn can trigger a number of hydrodynamical instabilities in the almost non-viscous stellar plasma, resulting in turbulent motions. In order to preserve its stability against those turbulent motions, the star must allow for internal redistribution of angular momentum, readjusting its angular velocity profile whenever such instabilities arise.

The internal redistribution of angular momentum in stellar evolutionary models was pioneered by the Yale group (Endal \& Sofia (1978), who used a purely diffusive approach for the angular momentum transport and applied it to follow the main sequence and postmain sequence evolutionary stages of $7 M_{\odot}$ and $10 M_{\odot}$ stellar models. In that approach, diffusion coefficients are estimated from characteristic lenghts and velocities of some considered hydrodynamical instabilities. Since that work, the subject remained unexplored for nearly ten years, when other people at Yale extended the models to lower masses starting at the pre-main sequence phase (Pinsonneault 1988; Pinsonneault et al. 1989, 1990, 1992) and, later on, included other enhancements such as microscopic diffusion (Chaboyer et al. 1995a).

The purely diffusive approach of the Yale group has been somewhat criticized in the literature, mainly regarding the treatment of the meridional circulation. A substantial progress in this field was brought by Chaboyer \& Zahn (1992) and Zahn (1992), who presented a consistent picture of the effects of meridional circulation on the transport of both chemicals and angular momentum. In that picture, the internal transport of chemicals is still described by a purely diffusion equation, but the internal angular momentum obeys an

[^1]advection-diffusion equation. At this moment, we postpone the discussion of these issues to Sect. 3.4, where the details of our implementation will also be presented.

### 2.2.2 Evolutionary changes driven by rotation

Despite the success of standard models in explaining the relevant observational data on stars, there are still many other features that cannot be accounted for by those models (an excellent review on the shortcomings of standard models, with particular emphasis on the Sun, can be found in Dziembowski 1998). For example, as noted by Meynet (1999), in the case of massive stars some observed features such as the number ratio of red to blue supergiants, the chemical enrichment observed at the surface of blue supergiants, or the axisymmetrical features observed in evolved stars such as $\eta$ Carinae, cannot be reproduced by standard models even if changes in metallicity, mass loss rate or the size of the convective cores are considered. These features have deep implications not only for stellar evolution itself but also for our understanding of the evolution of galaxies; as for the low-mass stars, the lithium depletion pattern observed in young open clusters is not consistent with the predictions of standard stellar models (Pinsonneault 1997) either.

These discrepancies between the observational data and predictions of standard models have led to a growing consensus that other non-standard physical effects must be included in evolutionary codes. From these, rotation emerges as a very promising candidate not only due to the enormous amount of observational data that can be used as constraints to stellar models, but also due to its relative theoretical simplicity when compared to other effects such as magnetic fields and turbulence.

Perhaps the most considered structural and evolutionary effects of rotation nowadays are those related to the chemical constituents of stars. This should be natural for, after all, stellar evolution is driven by the continuous changes in the chemical composition due to nuclear reactions. In the remaining of this section, we discuss some of the relevant questions related to this issue for both high- and low-mass stars.

The dynamical effects of rotation in massive stars can also be seen in two-dimensional mag-neto-hydrodynamical simulations of the formation of bipolar nebulae by García-Segura et al. $(1998,1999)$. These authors have demonstrated that both luminous blue variables (LBV) and asymptotic giant branch (AGB) stars reach the so-called $\Omega$-limit $t^{\dagger}$ of critical rotation independently of their rotation rate. The result is an equatorially confined outflow that can reach the shape of a double lobe structure by the action of magnetized winds. These calculations are very promising in explaining the nice axisymmetric shapes of planetary nebulae such as those surrounding the famous $\eta$ Carinae.

[^2]
### 2.2.2.1 Mixing in massive stars

Massive ( $M>8 M_{\odot}$ ) stars are the objects most responsible for the nucleosynthesis and distribution of elements in galaxies. During the last fifteen years, many observational data have been gathered about unusual surface abundances in these stars, such as helium and nitrogen enrichment in main sequence O and B stars (Gies \& Lambert 1992) and boron depletion in main sequence B stars (Venn et al. 1996). These anomalous surface abundances are not reproduced by standard models. Moreover, as pointed out by Langer et al. (1997), there are many other unresolved issues concerning the nucleosynthesis in massive stars such as the production of ${ }^{12} \mathrm{C}$ and ${ }^{14} \mathrm{~N}$, the source of ${ }^{26} \mathrm{Al}$ in the galaxy, and the main site of the r-process, that cannot be explained by standard evolutionary codes either.

These incompatibilities between standard models predictions and observations regarding the element abundances have led to the consideration of some non-standard, alternative mechanisms for mixing such as rotation, internal waves, magnetic fields, gravitational settling, thermal diffusion and others (see Pinsonneault 1997 for a review on this subject). In particular, in the last few years rotational mixing has gained increasing theoretical support as a plausible agent for the observed anomalous abundances in massive stars; this picture is consistent with the observational findings of Herrero et al. (1992) that slowly rotating main sequence OB stars show normal abundances, while the rapid rotating OB stars show surface helium enrichment.

Fliegner et al. (1996) computed the stellar evolution of 10 and $15 M_{\odot}$ including the effects of angular momentum redistribution, and found that the nitrogen to boron ratio obtained for the rotating models is in very good agreement with the observed ratio for a sample of 5 main sequence B stars. Talon et al. (1997) followed the main sequence evolution of a rotating $9 M_{\odot}$ star, finding that fast rotation $\left(v \sim 100 \mathrm{~km} \mathrm{~s}^{-1}\right)$ leads to significant changes in the $\mathrm{C} / \mathrm{N}$ and $\mathrm{O} / \mathrm{N}$ surface ratios. Langer et al. (1997) reported results on rotating models in massive stars, showing that rotation has three major effects in the chemical yields of massive stars, namely: larger convective core sizes, a larger enhancement of H-burning products, and the opening of new nucleosynthesis channels due to the transport of the products of a given burning phase to the site of a neighboring burning stage through the rotational mixing. And very recently, Langer et al. (1999) computed rotating stellar models for stars of intermediate mass ( $3 M_{\odot}$ ) up to the thermally pulsing asymptotic giant branch (TP-AGB), including angular momentum redistribution, and found that rotation can indeed produce the ${ }^{13} \mathrm{C}$-rich layer required for the occurrence of the s-process in TP-AGB stars.

To close this section, we would like to stress the importance of rotation in evolutionary models by examining a recent work by Meynet (1999) who shows that the surface enrichment of blue supergiants is deeply affected by rotation. As can be seen from Fig. 2.1, for non-rotating models the enrichment occurs only after the star has become a red super-


Figure 2.1: Surface values of $\Delta \log (N / H)=\log (N / H)-\log \left(N_{i} / H_{i}\right)$, where the index $i$ refers to initial values, as a function of the effective temperature for a $20 M_{\odot}$ model. (From Meynet 1999.)
giant (at $\log T_{\text {eff }} \sim 3.7$ ), while for rotating models the enrichment begins while the star is still on the main sequence.

### 2.2.2.2 Lithium depletion in low-mass stars

Lithium ${ }^{\ddagger}$ is an element produced by Big Bang nucleosynthesis, and so its abundance in the universe has cosmological implications. The discovery of measurable ${ }^{7} \mathrm{Li}$ in metal-poor, population II stars (Spite \& Spite 1982) has opened the possibility of obtaining the primordial lithium abundance, but it remains controversial whether the Li abundances detected in these stars reflect their initial values or have been uniformly depleted from a higher initial value (Pinsonneault 1997). A good review of the present status of lithium detection in population II stars can be found in Chaboyer (1998).

Since lithium is destroyed at relatively low temperatures ( $T \sim 2.5 \times 10^{6}$ ), it can survive only in the outer regions of low-mass stars, and so in standard models its surface abundance can be affected only by deep convective envelopes, which carry it to deeper regions where the temperature is high enough to burn it. This is not the case for solar type stars, as their lithium destruction region is located below the surface convection zone. So,

[^3]standard models predict that lithium should not be depleted at the surface of solar type stars. However, observations of the lithium surface abundance in the Sun show that it is significantly depleted when compared with the abundance obtained from meteorites. This fact poses a strong challenge for stellar structure theory, which is still unresolved.

Young open clusters have been studied for a long time regarding the lithium content of their members and represent extremely favorable places for testing the pre-main sequence lithium depletion from standard models and also for investigating non-standard stellar physics, since it is usually assumed that they are coeval groups of stars with nearly the same chemical composition. The pattern of lithium depletion in young clusters such as $\alpha$ Persei ( 50 Myr ) and Pleiades ( $70-110 \mathrm{Myr}$ ) show a reasonable agreement with the depletion trend predicted by standard models, as shown in the lower part of Fig. 2.2 for the Pleiades. The situation is very different for intermediate-age clusters such as M34 (200 Myr), Ursa Major (300Myr), Hyades and Praesepe (both 500-700 Myr), whose lithium depletion patterns cannot be reproduced by standard models even if overshooting (which can have a dramatic impact on the pre-main sequence) is accounted for, as depicted in the upper part of Fig. 2.2 for the Hyades.


Figure 2.2: Lithium depletion patterns for the Pleiades and Hyades open clusters, with the predicted values from standard models. The dotted and dashed lines refer to models with overshooting included as shown in the box. The initial ${ }^{7} \mathrm{Li}$ abundance was set to 3.3 , and the metallicities of the models are $[\mathrm{Fe} / \mathrm{H}]=0.0$ for the Pleiades and $[\mathrm{Fe} / \mathrm{H}]=0.1$ for the Hyades. (From Chaboyer et al. 1995b).

Furthermore, these intermediate-age clusters show an impressive "lithium dip" around
the mid- F ( $\mathrm{T}_{\text {eff }}$ between 6400 and 6800 K ) stars, that extends nearly two orders of magnitude for the Hyades and Praesepe, as shown in more detail in Fig. 2.3. This feature demonstrates once more the deficiencies of standard models, since for that mass range the convection zones are too thin to burn lithium.


Figure 2.3: Observations of Li in the Hyades open cluster, showing the so-called lithium dip at temperatures corresponding to mid-F stars. (From Chaboyer 1998.)

Several physical mechanisms that deplete Li have been proposed to explain the observational features seen in open clusters, such as overshooting, mass loss, internal (or gravity) waves, microscopic diffusion and rotation. Overshooting seems to be ruled out because, as most of its effects in low-mass stars occur during the pre-main sequence phase, it would lead to very low Li abundances in young, cool stars, a situation that does not happen to any open cluster so far studied, including the Hyades (Chaboyer et al. 1995b). Swenson \& Faulkner (1992) demonstrated that mass loss alone cannot reproduce the lithium depletion pattern observed in open clusters; additional observational and theoretical arguments against mass loss are also given in Pinsonneault (1997) and Chaboyer (1998). Mixing driven by internal waves, with the suitable addition of pre-main sequence depletion to main-sequence models, can fit well the lithium depletion pattern in the Hyades and Praesepe clusters (Montalban \& Schatzman 1996) including the lithium dip, but is not able to reproduce the observed dispersion in the abundances (Pinsonneault 1997). Microscopic diffusion can also solve the lithium dip in mid-F stars (Michaud 1986), but its time scale is too long for a significant contribution to lithium depletion in cool stars; besides, it predicts the same level of depletion for lithium and beryllium, while the observations show that beryllium is much less depleted (Chaboyer 1998). As pointed out by Pinsonneault (1997), microscopic diffusion should be considered in models but probably is not the only agent for the observed depletion pattern.

Due to the circulation currents that occur in rotating stars, rotation has long been considered a natural agent for mixing in stars (good reviews on the role of rotation in the mixing of light elements can be found in Pinsonneault 1997 and Chaboyer 1998). The Yale models (Pinsonneault et al. 1990, 1992; Chaboyer et al. 1995b) have been successful in explaining a number of features related to the lithium depletion in open clusters, such as the cool side of the lithium dip and the main sequence depletion of lithium in the Sun and similar temperature cluster stars, as can be seen from Fig. 2.4 for both Pleiades and Hyades clusters. These models also reproduce the beryllium observations in the lithium dip stars (Deliyannis \& Pinsonneault 1997) and predict a dispersion of Li abundances for stars of the same age, mass and metallicity as is observed in open clusters, due to a dispersion in the initial rotation velocities. However, the models are not able to explain the presence of rapid late G and K rotators in the Pleiades, as well as the absence of rapid rotating F and early G stars. Another difficulty in these rotating models is that higher rotation rates lead to increased lithium depletion, in contradiction with observational data according to which the faster rotators in the Pleiades show less lithium depletion (García-López et al. 1994; Jones et al. 1996).


Figure 2.4: Same as Fig. 2.2, except that the metallicity of the Pleiades was set to the solar metallicity, and the predicted values are from non-standard models that include the combined effects of microscopic diffusion and rotation. (From Chaboyer et al. 1995b).

Regarding this latter point, it is worth mentioning that Martín \& Claret (1996) obtained rotating models for which increasing rotation rates result in decreasing lithium depletion, a result coherent with the observational pattern but in contrast with the models of the

Yale group. In Chap. 4 of the present work we discuss our own results regarding this point, and we anticipate that they match those of the Yale group.

### 2.2.3 The role of hydrodynamical instabilities in the angular momentum transport

In the previous section we mentioned a number of rotating evolutionary models with internal redistribution of angular momentum from several authors, applied to either massive or low-mass stars. Broadly speaking, all these models fall into two categories: either the pure diffusive approach (Pinsonneault et al. 1990, 1992; Chaboyer et al. 1995a, 1995b; Fliegner et al. 1996; Langer et al. 1997, 1999) or the framework of Zahn (1992) in which the transport of angular momentum is governed by an advection-diffusion equation (Talon et al. 1997; Meynet 1999). On the other hand, the latter works consider only the shear instability resulting from the anisotropic turbulence, while the former ones take in account a number of other hydrodynamical instabilities associated with rotation.

In our approach we adopt the framework of Zahn (1992) for the internal transport of angular momentum, but we also consider the major hydrodynamical instabilities related to rotation. The details of that implementation are discussed in the next chapter; however, before proceeding we think it is worthwhile to discuss, in greater detail, those instabilities, in order to have a clear understanding of their importance.

The physical mechanisms responsible for internal angular momentum redistribution in stellar interiors are associated with hydrodynamical instabilities generated by the existing velocity field. These instabilities are usually classified as either dynamical or secular (Zahn 1974; Endal \& Sofia 1978; Knobloch \& Spruit 1982), depending on the time scales involved. Dynamical instabilities have timescales short enough that the related perturbations can be considered adiabatic; a typical dynamical time scale is the free fall time, which for the Sun is about 26 minutes. Secular instabilities, on the other hand, have time scales long enough for energy exchange to occur; in this case, the characteristic time scale is the Kelvin-Helmholtz (in the Sun, $t_{\mathrm{KH}} \sim 30 \times 10^{6}$ years). Below we review the main instabilities that are currently considered in the literature.

### 2.2.3.1 Dynamical instabilities

## Axisymmetric baroclinic instability

In a compressing fluid with angular velocity $\omega(s, z)$ (where $s=r \sin ^{2} \theta$ is the perpendicular distance to the rotation axis, and $z$ the usual z-coordinate), the stability criteria against axisymmetric, adiabatic perturbations are the Solberg-Høiland criteria, given by
the inequalities (Zahn 1974, Tassoul 1978)

$$
\begin{align*}
& \frac{1}{r^{3}} \frac{\partial j^{2}}{\partial s}+\frac{1}{c_{p}} \frac{\gamma-1}{\Gamma_{3}-1}(-\mathbf{g}) \cdot \nabla S>0  \tag{2.1}\\
& -g_{z}\left(\frac{\partial j^{2}}{\partial s} \frac{\partial S}{\partial z}-\frac{\partial j^{2}}{\partial z} \frac{\partial S}{\partial s}\right)>0 \tag{2.2}
\end{align*}
$$

where $j$ is the specific angular momentum, $r$ the radius, $\Gamma_{3}=(\partial \ln T / \partial \ln \rho)_{\mathrm{ad}}+1$ is one of the adiabatic exponents, $\gamma=c_{p} / c_{v}$, and $S$ the entropy.

In the non-rotating case, the inequality given by Eq. (2.1) reduces to the Schwarzschild criterion for stability against convection: $(-\mathbf{g}) \cdot \nabla S>0$. Therefore, rotation acts as a stabilizing effect against convection as long as $\partial j / \partial r>0$ (Law 1980).

On the other hand, if the star is homentropic (Tassoul 1978), that is, if $\nabla S=0$, that inequality reduces to the Rayleigh criterion for stability in an non-viscous, non-gravitating homogeneous fluid:

$$
\begin{equation*}
\frac{1}{r^{3}} \frac{\partial j^{2}}{\partial s}>0 \tag{2.3}
\end{equation*}
$$

If we take into account that condition 2.1 is strictest (i.e. most difficult to be satisfied) in the equatorial plane, it can be rewritten as (Endal \& Sofia 1978)

$$
\begin{equation*}
\frac{1}{r^{3}} \frac{\partial j^{2}}{\partial r}+\frac{g}{\rho}\left[\left(\frac{d \rho}{d r}\right)_{\mathrm{ad}}-\frac{d \rho}{d r}\right]>0 \tag{2.4}
\end{equation*}
$$

The inequality (2.2), by its turn, indicates that there will be instability if the specific angular momentum $j$ decreases, from the poles to the equator, along a constant entropy surface (Tassoul 1978; Law 1980; Zahn 1983). It also shows that, in a compressible fluid (and so with a density stratification) the buoyancy forces allow a rotating law of the form $\omega(s, z)$ to be sustained, and hence breaking the Taylor-Proudman theorem ${ }^{\S}$ for stability in a compressible fluid: $\partial \omega / \partial z=0$ (Chandrasekhar 1981). (Here we take for the buoyant force its classical meaning in fluid dynamics, i.e. the resulting upward force from the pressure exerted by the neighboring fluid and the gravity force).

[^4]We must keep in mind that this instability applies only to baroclinic stars, that is, those stars for which the constant entropy (or density) surfaces do not coincide with the constant pressure surfaces, which implies a rotation law $\omega=\omega(s, z)$. For barotropic stars (i.e. stars with $P=P(\rho)$ ), one will always have $\partial \omega / \partial z=0$, that is, $\omega=\omega(s)$; in this case $j$ does not vary with $z$ along an isentropic surface (Tassoul 1978). An excellent description of this kind of instability is given in Drazin \& Reid (1981).

The existence of the axisymmetric baroclinic instability is, however, controlled by the angular momentum conservation condition (Zahn 1983, 1993). In the case of axisymmetric and strictly adiabatic perturbations, such instability can only occur if the Railegh criterion is violated:

$$
\begin{equation*}
\frac{1}{s^{3}} \frac{\partial j^{2}}{\partial s}<0 \tag{2.5}
\end{equation*}
$$

(Zahn 1993; Knobloch \& Spruit 1982). If there is thermal diffusion or viscous transport of angular momentum, the system can no longer be considered in the adiabatic state, and this instability loses its dynamical characteristics.

## Non-axisymmetric baroclinic instability

In the case of non-axisymmetric perturbations, the fluid elements exchange angular momentum with each other (Knobloch \& Spruit 1982 and references therein) and so the angular momentum is not conserved, allowing instabilities with large wavelengths along an isobaric surface.

The adiabatic and non-viscous case is the only one studied with some degree of detail in the literature (see Zahn 1983). This case has a dynamical time scale, and its stability criterion is given by (Zahn 1993)

$$
\begin{equation*}
\left|\frac{\partial \ln \Omega}{\partial \ln \rho}\right|>C\left(\frac{N^{2}}{\Omega^{2}}\right) \frac{H_{\Omega}}{r^{2}} \min \left(H_{\rho}, H_{\mathrm{N}}, H_{\Omega}\right) \tag{2.6}
\end{equation*}
$$

where $\Omega$ is the angular velocity, $C$ is a constant of order unity, $N$ is the Brunt-Väisälä frequency

$$
\begin{equation*}
N^{2}=\frac{g \delta}{H_{\mathrm{P}}}\left(\nabla_{\mathrm{ad}}-\nabla\right), \tag{2.7}
\end{equation*}
$$

and $H_{\rho}, H_{\mathrm{N}}, H_{\Omega}$ are the scale heights of the given quantities. This instability condition is usually satisfied only very near the star's core or near a convective region, where $N^{2}$ is
very small (Zahn 1983, 1993; Spruit et al. 1983). If there is a vertical gradient of chemical composition, the Brunt-Väisälä frequency is composed by two terms:

$$
\begin{align*}
N^{2} & =\frac{g \delta}{H_{\mathrm{P}}}\left(\nabla_{\mathrm{ad}}-\nabla+\nabla_{\mu}\right) \\
& =N_{\mathrm{T}}^{2}+N_{\mu}^{2} \tag{2.8}
\end{align*}
$$

with $\nabla_{\mu}=d \ln \mu / d \ln P$.
Knobloch \& Spruit (1982) give the stability criterion against non-axisymmetric perturbations as

$$
\begin{equation*}
\frac{L^{2}}{R_{\star}^{2}}>\chi \frac{H_{\mathrm{P}}}{R_{\star}} \frac{1}{\nabla-\nabla_{\mathrm{ad}}}, \tag{2.9}
\end{equation*}
$$

where $L$ is the height of the given fluid shell, $R_{\star}$ is the stellar radius, $H$ is the pressure scale height and $\chi=\Omega^{2} R_{\star} / \tilde{g}$ is the ratio between the centrifugal and gravitational accelerations in the star. For stars with low rotation, $\chi \ll 1$ and the instability can only occur if $L \ll R_{\star}$. On the other hand, in the case of a star with a smooth rotation profile, one has $L / R_{\star} \approx 1$, and the instability can occur only if the star's rotation is high enough: $\chi>R_{\star}\left(\nabla-\nabla_{\mathrm{ad}}\right) / H_{\mathrm{P}}$. Since the maximum value of $\chi$ for any star is 1 , and $R_{\star}\left(\nabla-\nabla_{\mathrm{ad}}\right) / H_{\mathrm{P}}>1$ except near a convective region, such an instability does not exist for that rotation profile. To conclude, we can say that only in regions near convective regions can the non-axisymmetric baroclinic instability be important. This conclusion becomes even more stringent if we consider that, in the more realistic viscous case, the thermal diffusion due to the dissipation tends to relax this instability (Zahn 1993).

## Dynamical shear instability

According to Zahn (1983), the basic mechanism for this instability is the extraction and clumping of vorticity of a laminar shear flow.

In the case of an unstratified shear flow with no viscosity and no rotation (in other words, a plane-parallel flow without viscosity), a sufficient condition for stability is given by the Rayleigh theorem for the inflexion point (Drazin \& Reid 1981; Knobloch \& Spruit 1982; Zahn 1983), according to which the velocity profile $U(z)$ along the fluid cannot have any inflexion. This means that

$$
\begin{equation*}
U^{\prime \prime}(z) \neq 0 \tag{2.10}
\end{equation*}
$$

The above condition, however, is not a necessary one, as there are fluids that have an inflexion point and are stable (see Drazin \& Reid 1981, Chapter 4, for a discussion of the necessary and sufficient conditions for that).

In the case of a fluid with cylindrical rotation $(\partial \omega / \partial z=0)$, the stability condition becomes (Zahn 1974, 1983)

$$
\begin{equation*}
\frac{d}{d r}\left[\frac{1}{r} \frac{d j}{d r}\right] \neq 0 \tag{2.11}
\end{equation*}
$$

When the stability condition is not satisfied, any infinitesimal perturbation grows exponentially in a dynamical time scale $t_{\mathrm{d}}=D / U$, where $U$ is a typical value of the fluid differential velocity and $D$ its cross-stream size (Zahn 1983).

The instability condition $U^{\prime \prime}(z)=0$ (which is a necessary but not sufficient condition; see Drazin \& Reid 1981 and Zahn 1983) relaxes when the viscosity is taken into account. In this case all fluids become unstable whenever the Reynolds number

$$
\begin{equation*}
\Re=\frac{U D}{\nu}, \tag{2.12}
\end{equation*}
$$

exceeds a given critical value $\Re_{\mathrm{c}}$. If the fluid has an entropy (density) stable stratification, with the gravity force along the $z$ direction, then the buoyant force acts as a restoring force and balances that instability. The importance of this stabilizing effect, when compared to the shear strength, is given by the Richardson number

$$
\begin{equation*}
\mathrm{Ri}=\frac{N^{2}}{(d U / d z)^{2}} \tag{2.13}
\end{equation*}
$$

One can prove (Knobloch \& Spruit 1982 and references therein) that a sufficient condition for adiabatic stability is

$$
\begin{equation*}
\mathrm{Ri}>\mathrm{Ri}_{\mathrm{c}} \equiv \frac{1}{4} \tag{2.14}
\end{equation*}
$$

where $\mathrm{Ri}_{\mathrm{c}}$ is a critical value. Later studies have shown that, for a variety of velocity and density profiles, $\mathrm{Ri}<1 / 4$ is also a sufficient condition for instability (Zahn 1983, 1993).

For a fluid under differential rotation, the Richardson number is given by (Zahn 1974)

$$
\begin{equation*}
\mathrm{Ri}=\cos ^{2} \alpha \frac{g}{c_{\mathrm{p}}} \frac{d S}{d z}\left(\frac{1}{s \nabla \omega}\right)^{2} \tag{2.15}
\end{equation*}
$$

where $\omega=\omega(s, z)$. As before, the stability condition is $\mathrm{Ri}>1 / 4$. In Eq. (2.15), $\alpha$ is the angle between $\nabla S$ and $\nabla \rho$, which characterizes the baroclinic condition. In the equator,
where the fluid's linear velocity is maximum, the Richardson number is lower and then (Endal \& Sofia 1978)

$$
\begin{equation*}
\mathrm{Ri}=\frac{g}{c_{\mathrm{p}}} \frac{d S}{d r}\left[\frac{1}{(d \omega / d \ln r)}\right]^{2} \tag{2.16}
\end{equation*}
$$

which, by using the identity $d S=c_{\mathrm{p}}\left(\nabla-\nabla_{\mathrm{ad}}\right) d \ln P$, becomes

$$
\begin{equation*}
\operatorname{Ri}=g\left(\nabla-\nabla_{\mathrm{ad}}\right) \frac{1}{P} \frac{d P}{d r}\left[\frac{1}{(d \omega / d \ln r)}\right]^{2} \tag{2.17}
\end{equation*}
$$

We notice that, under hydrostatic equilibrium, $d P / d r=-m(r) G \rho / r^{2}$ decreases with $r$ leading to an even smaller Richardson number in the stellar surface, where $d P / d r$ can be approximated by $-\rho g$. Hence

$$
\begin{equation*}
\operatorname{Ri}=\frac{\rho}{P}\left(\nabla-\nabla_{\mathrm{ad}}\right)\left[\frac{g}{(d \omega / d \ln r)}\right]^{2}>\frac{1}{4} \tag{2.18}
\end{equation*}
$$

in agreement with the similar expression presented by Endal \& Sofia (1978).
We must keep in mind that the previous result is valid for a rotation law of the form $\omega=\omega\left(r_{\Psi}\right)^{\boldsymbol{q}}$, which is adopted by Endal \& Sofia (1978) under the assumption that the $\rho=$ const. surfaces do not deviate significantly from the level surfaces, even in the case of non-conservative rotation laws $(\partial \omega / \partial z \neq 0)$.

If the fluid is viscous or subject to thermal diffusion (which means dissipative effects), the restoring buoyant force becomes weaker, its stabilizing effect decreases and hence this instability changes its behaviour from a dynamical to a secular one.

More recently, Talon \& Zahn (1997) investigated the role of a strong anisotropic turbulence in such a criterion, considering a much stronger turbulent transport in the horizontal than in the vertical direction, which is the basic concept established in the Zahn (1992) framework to be discussed in Sect. 3.4.2. Those authors found that the anisotropic turbulence substantially reduces the critical shear which is able to overcome the $\mu$-gradient; in this case the Richardson number Ri is given by the expression

$$
\begin{equation*}
\operatorname{Ri}=\frac{\nu \Re_{\mathrm{c}}}{6}\left(\frac{d U}{d z}\right)^{-2}\left(\frac{N_{T}^{2}}{K+D_{h}}+\frac{N_{\mu}^{2}}{D_{h}}\right) \tag{2.19}
\end{equation*}
$$

and the instability criterion is, as before,

$$
\begin{equation*}
\mathrm{Ri}<\mathrm{Ri}_{\mathrm{c}}=\frac{1}{4} . \tag{2.20}
\end{equation*}
$$

[^5]In Eq. (2.19) $\nu$ is the kinematic viscosity, $\Re_{\mathrm{c}}$ the critical Reynolds number, $d U / d z$ the fluid velocity gradient, $K=4 a c T^{3} /\left(3 \kappa \rho C_{\mathrm{P}}\right)$ is the thermal diffusivity, $N_{T}^{2}$ and $N_{\mu}^{2}$ the components of the Brunt-Väisälä frequency, and $D_{h}$ the turbulent horizontal diffusivity. With this new form for the Richardson number, Talon \& Zahn (1997) showed that the vertical turbulent diffusivity $D_{v}$ due to the dynamical shear is given by the expression

$$
\begin{equation*}
D_{v} \simeq \frac{2 R i_{c}(d U / d z)^{2}}{N_{T}^{2} /\left(K+D_{h}\right)+N_{\mu}^{2} / D_{h}} \tag{2.21}
\end{equation*}
$$

or, as given by Talon et al. (1997),

$$
\begin{equation*}
D_{v}=\frac{8 R i_{c}}{5} \frac{(r d \Omega / d r)^{2}}{N_{T}^{2} /\left(K+D_{h}\right)+N_{\mu}^{2} / D_{h}} . \tag{2.22}
\end{equation*}
$$

To compute $D_{v}$, it remains to calculate the horizontal diffusivity $D_{h}$ which, in Zahn's (1992) framework, is given by

$$
\begin{equation*}
D_{h}=\frac{r U}{C_{h}}\left[\frac{1}{3} \frac{d \ln \left(\rho r^{2} U\right)}{d \ln r}-\frac{1}{2} \frac{d \ln \left(r^{2} \Omega\right)}{d \ln r}\right], \tag{2.23}
\end{equation*}
$$

with $U$ being the meridional circulation velocity and $C_{h}$ a constant of order unity.

### 2.2.3.2 Secular instabilities

## Secular shear

In a fluid with a stable density stratification (such as the one resulting from a gravitational field), the buoyant force acts as a restoring force and then inhibits the shear instability along an horizontal flow $U=U(z)$, as discussed in the previous section. But this situation changes drastically if the perturbations cease to be adiabatic, for in this case the dissipative effects brought by viscosity need to be considered. This can be understood if we keep in mind that the thermal diffusion smooths out the temperature gradient (considering a gas with no chemical composition gradients), and consequently also the density gradient. As a result the buoyant force decreases or even disappears: the thermal diffusion has then a destabilizing effect.

One can show (Zahn 1974, 1983, 1993 and references therein) that when the radiative cooling time $t_{\text {cool }}$ becomes shorter than the characteristic time $|d U / d z|^{-1}$ of the shear, the critical Richardson number is amplified by the factor $\left(t_{\mathrm{cool}} d U / d z\right)^{-1}$, and then the instability criterion becomes

$$
\begin{equation*}
\frac{N^{2} t_{\text {cool }}}{d U / d z}<\mathrm{Ri}_{\mathrm{c}} \tag{2.24}
\end{equation*}
$$

For a medium with thermal diffusivity $\kappa, t_{\text {cool }}=l^{2} / \kappa$ with $l$ being the characteristic length of the instability (Zahn 1993). If we adopt Zahn's (1974) assumption that $l$ is of the order of the smallest length for which the Reynolds number remains supercritical (that is, for which the instability is sustained, $\Re \sim 10^{3}$ ), the instability criterion can be written as

$$
\begin{equation*}
\mathrm{Ri}<\frac{\mathrm{Ri}_{\mathrm{c}}}{\Re_{\mathrm{c}} P_{\mathrm{r}}} \tag{2.25}
\end{equation*}
$$

In the above equation, $\Re_{\mathrm{c}}$ is the critical Reynolds number, $P_{\mathrm{r}}=\nu / \kappa$ is the Prandtl number and $\mathrm{Ri}_{\mathrm{c}}$, as before, is the critical Richardson number $\left(\mathrm{Ri}_{\mathrm{c}}=1 / 4\right)$. That equation shows that, under the effects of the thermal diffusion, the Richardson number is amplified by the factor $1 /\left(\Re_{\mathrm{c}} P_{\mathrm{r}}\right)$, which for stellar interiors is of the order $10^{3}-10^{4}$ (Zahn 1974). This means that the maximum angular velocity gradient tolerated by the secular shear instability is $10^{3}-10^{4}$ lower than the one tolerated by the dynamical instability. However, as pointed out by Endal \& Sofia (1978), both criteria need to be considered, because a secularly unstable gradient can last for an appreciable amount of time (in terms of the evolutionary time scales), while a dynamically unstable one cannot.

If a molecular weight gradient exists (referred to as $\mu$-gradient from now on), it also acts as a stabilizing factor, since in this case the density gradient cannot be totally cancelled by the thermal diffusion. In other words, only the "thermal" part of the buoyant force is supressed by thermal diffusion (Zahn 1993). In this case the stability criterion takes its original form with $N^{2}$ substituted for $N_{\mu}^{2}$ :

$$
\begin{equation*}
\frac{N_{\mu}^{2}}{(d U / d z)^{2}}>\frac{1}{4} \tag{2.26}
\end{equation*}
$$

remembering that $N^{2}=g \delta\left(\nabla_{\mathrm{ad}}-\nabla+\nabla_{\mu}\right) / H_{\mathrm{p}}=N_{\mathrm{T}}^{2}+N_{\mu}^{2}$. It is worth emphasizing that a density (or molecular weight) stratification can only stabilize a vertical shear flow; it has no effect on a horizontal shear flow.

Another way of estimating the stabilizing effect of a $\mu$-gradient is to take the original Richardson criterion for the non-viscous case and to substitute $\nabla_{\text {ad }}$ for $\nabla_{\mu}$ in it, which results in (Endal \& Sofia 1978)

$$
\begin{equation*}
\mathrm{Ri}_{\mathrm{s}}=\frac{\rho}{T}\left[-\left(\frac{\partial \ln T}{\partial \ln \mu}\right)_{\mathrm{P}, \rho}\left(\frac{\partial \ln \mu}{\partial \ln P}\right)\right]\left[\frac{g}{(d \omega / d \ln r)}\right]^{2}>\frac{1}{4} . \tag{2.27}
\end{equation*}
$$

According to Endal \& Sofia (1978), there will be shear instability if both stability conditions (with and without a $\mu$-gradient) are violated. The time scale for the amplification of this instability is $\tau \sim \Re_{\mathrm{c}} /|d \omega / d \ln r|$, which is higher than the Kelvin-Helmholtz time
scale. The propagation velocity of this instability is obtained from the time scale and characteristic length $l^{2} \sim \Re_{\mathrm{c}} /|d \omega / d \ln r|$ of the turbulent elements, that is,

$$
\begin{equation*}
v=\frac{l}{\tau}=\left[\frac{\nu(d \omega / d \ln r)}{\Re_{\mathrm{c}}}\right]^{1 / 2} . \tag{2.28}
\end{equation*}
$$

Maeder (1995) and Maeder \& Meynet (1996) tackled the Richardson criterion for shear instabilities in the radiative and semi-convective regions of stars with rotation. These authors, besides taking into account the effects of both thermal diffusion and the presence of a $\mu$-gradient, examined the contribution of the shear turbulent eddies for the heat transfer. They concluded that there will be shear instability whenever

$$
\begin{equation*}
\frac{g \varphi \nabla_{\mu}}{H_{\mathrm{P}}} \frac{1}{(d U / d z)^{2}}<\frac{1}{4} \tag{2.29}
\end{equation*}
$$

a result very close to that obtained by Zahn (1993) and discussed in the previous section. The instability will happen in radiative regions with a diffusion coefficient given by

$$
\begin{equation*}
D=2 \kappa \frac{\frac{1}{4}\left(\frac{d U}{d z}\right)^{2}-\frac{g \varphi \nabla_{\mu}}{H_{\mathrm{P}}}}{\frac{g \delta}{H_{\mathrm{P}}}\left(\nabla_{\mathrm{ad}}-\nabla\right)} \tag{2.30}
\end{equation*}
$$

## Goldreich-Schubert-Fricke instability

We have seen that, as the Prandtl number $\nu / \kappa$ is very small in stellar interiors, heat diffuses much more quickly than momentum, smoothing the density gradient and weakening the buoyant force. The linear theory of this mechanism was independently worked out by Goldreich \& Schubert (1967) and Fricke (1968), and since then it is usually referred to as Goldreich-Schubert-Fricke instability or GSF instability for short.

In the limit of zero viscosity, and for a rotation law $\omega=\omega(s, z)$, the conditions for secular stability against perturbations with small wavelength are

$$
\begin{align*}
& \frac{\partial}{\partial s}\left(\omega s^{2}\right)>0  \tag{2.31}\\
& \frac{\partial \omega}{\partial z}=0 \tag{2.32}
\end{align*}
$$

Instabilities will arise whenever either of the above conditions is met.

The condition given by Eq. (2.31) reduces to the Rayleigh criterion for stability. That is, the specific angular momentum must increase along the $s$ coordinate in order to maintain stability. As is well known, an angular momentum distribution that violates that condition can be dynamically stable in a thermally stable layer provided that the buoyant force is strong enough to overcome the centrifugal force. However, the thermal diffusion will smooth out the density stratification over time, and the centrifugal force will eventually become dominant within the thermal diffusion time scale, rendering secularly unstable the angular momentum distribution.

In turn, the condition expressed by Eq. (2.32) states that an angular velocity distribution which depends on the $z$ coordinate is intrinsically unstable. In the case of a barotropic star, the hydrostatic equilibrium implies a conservative rotation force (i.e. a rotation force that can be derived from a potential), which only occurs when rotation is constant on cylinders; if not, there will not be hydrostatic equilibrium and the star will be unstable. For baroclinic stars, as their $T=$ const and $\rho=$ const surfaces do not coincide, the buoyant force (which in general is not conservative) can balance the centrifugal force, allowing the star to attain hydrostatic equilibrium even if the condition (2.32) is violated (Kippenhahn 1969). However, once again the thermal diffusion will smooth out the thermal gradient, weakening the buoyant force and continuously approximating the $T=$ const and $\rho=$ const surfaces, so that, in order to have hydrostatic equilibrium, $\partial \omega / \partial z=0$. In other words, the star slowly readjusts its angular momentum redistribution and gradually evolves to the state given by condition (2.32) (Tassoul 1978; Law 1980).

The stability conditions stated by Eqs. (2.31) and (2.32) are the same as those obtained for the case of axisymmetric, adiabatic perturbations with no density gradient: under these conditions Eq. (2.31) reduces to the familiar Rayleigh criterion for stability in non-viscous, incompressible fluids, while Eq. (2.32) is equivalent to the Taylor-Proudmann theorem for non-viscous, incompressible and low-rotation fluids (Tassoul 1978; Chandrasekhar 1981). These equivalences come from the fact that, as the thermal diffusion smooths the temperature gradients and so the density gradients, the final state mimics the case of a non-viscous and incompressible fluid (Law 1980).

As we have seen, a rotation law $\omega=\omega(r)$ breaks down the stability condition, and the star will tend to readjust its angular momentum distribution within the thermal diffusion time scale, in order to restore the stability condition (Law 1980); the differential rotation that will arise along the surfaces $\rho=$ const will induce dynamical instabilities, and the resulting turbulent motions will try to establish $\omega=\omega(r)$. The competition between these processes causes an angular momentum transfer on a time scale given by the GSF instability, with the final state, according to Endal \& Sofia (1978), being rigid body rotation.

Since the secular shear instability is dominant at the base of the convective envelope, the effects of the GSF instability will be greater in the deeper stellar regions (Pinsonneault 1988). For the case of finite, non-zero viscosity, the GSF instability criteria can be written
as (Zahn 1983)

$$
\begin{align*}
& \frac{1}{s^{3}} \frac{\partial}{\partial s}\left(\omega s^{2}\right)^{2}<\frac{\nu}{\kappa} N_{\mathrm{T}}^{2}  \tag{2.33}\\
& \left|s \frac{\partial \omega}{\partial s}\right|>\frac{\nu}{\kappa} N_{\mathrm{T}}^{2} \tag{2.34}
\end{align*}
$$

where $N_{\mathrm{T}}$ is the thermal component of the Brunt-Väisälä frequency,

$$
\begin{equation*}
N_{\mathrm{T}}^{2}=\frac{g}{H_{\mathrm{P}}}\left(\nabla_{\mathrm{ad}}-\nabla\right) \tag{2.35}
\end{equation*}
$$

If there is a $\mu$-gradient given by $\nabla_{\mu}=-g \partial \ln \mu / \partial z$, the expressions (2.33) and (2.34) are still valid provided that we substitute $N_{T}^{2}$ for $N^{2}$ (Zahn 1993), with

$$
\begin{equation*}
N^{2}=N_{\mathrm{T}}^{2}+N_{\mu}^{2}=\frac{g}{H_{\mathrm{P}}}\left(\nabla_{\mathrm{ad}}-\nabla+\nabla_{\mu}\right) . \tag{2.36}
\end{equation*}
$$

However, the molecular weight stratification can stabilize the fluid as long as (Zahn 1993)

$$
\begin{equation*}
N_{\mu}^{2}=\frac{g}{H_{\mathrm{P}}} \nabla_{\mu}>\frac{\nu}{\kappa} N_{\mathrm{T}}^{2} . \tag{2.37}
\end{equation*}
$$

The large scale circulation velocity due to the GSF instability is estimated by Endal \& Sofia (1978) as

$$
\begin{equation*}
v_{\mathrm{GS}}=\frac{2 H_{\mathrm{T}} r}{H_{\mathrm{j}}^{2}}\left[\frac{2 \omega}{\partial \omega / \partial \ln r}+1\right]^{-1} v_{\mathrm{E}} \tag{2.38}
\end{equation*}
$$

where $H_{\mathrm{T}}$ is the temperature scale height, $H_{\mathrm{j}}$ the specific angular momentum scale height and $v_{\mathrm{E}}$ the meridional circulation velocity.

## Meridional circulation

Since the first quarter of this century it has been known that rotation breaks down the radiative equilibrium in stars, as first demonstrated by Von Zeipel (1924a, 1924b) for rigid body rotation (culminating in the famous "Von Zeipel's paradox") and later by Baker \& Kippenhahn (1959) for a generic rotation law. This thermal imbalance arises from the conflict between the trend to spherical symmetry due to the heat transport to the surface and the oblateness imposed by the centrifugal force. Two excellent reviews on this subject can be found in Tassoul (1978) and Zahn (1993).

According to Tassoul (1978), Von Zeipel's paradox can be solved in two ways:
a. by allowing the angular velocity to depend on the $z$ coordinate;
b. by admitting that it is not possible to attain radiative equilibrium in a rotationally distorted star.

In case (a), we have just seen that $\partial \omega / \partial z \neq 0$ triggers the GSF instability which, by its turn, readjusts the rotation rate to $\partial \omega / \partial z=0$ in the thermal diffusion time scale, and so leading to case (b).

In case (b), Eddington $(1925,1929)$ and Vogt (1925) showed that the inexistence of radiative equilibrium will destroy the constancy of $T$ and $P$ over level surfaces, and the resulting pressure gradient between the poles and the equator will cause mass motions along these level surfaces. This flow of matter continues over time, generating a steady meridional circulation. Due to this result, the meridional circulation is also referred to in the literature as Eddington-Vogt circulation.

As shown by Endal \& Sofia (1978), the predominance of the shear instability along the level surfaces allows one to consider only the currents perpendicular to those surfaces. In this case the corresponding velocities are estimated as (Kippenhahn \& Möllenhof 1974)

$$
\begin{equation*}
v_{\mathrm{E}}=-\frac{1}{3} \frac{\nabla_{\mathrm{ad}}}{\delta\left(\nabla_{\mathrm{ad}}-\nabla\right)} \frac{\omega^{2} r^{3} L_{\mathrm{r}}}{\left(G M_{\mathrm{r}}\right)^{2}}\left(\frac{2 \epsilon r^{2}}{L_{\mathrm{r}}}-\frac{2 r^{2}}{M_{\mathrm{r}}}-\frac{3}{4 \pi \rho r}\right) \tag{2.39}
\end{equation*}
$$

where $\delta=-(\partial \ln \rho / \partial \ln T)_{\mathrm{P}, \mu}$.
If there is a $\mu$-gradient, the meridional circulation will induce a non-spherical distribution of chemical composition, which will tend to counteract the meridional circulation (the so-called " $\mu$-barriers"; see e.g. Mestel 1953, Kippenhahn \& Weigert 1994). These effects are usually handled by considering fictitious " $\mu$-currents" which (except for some special situations) always oppose the circulation currents. The velocities of those $\mu$-currents are estimated by Kippenhahn (1974) as

$$
\begin{equation*}
v_{\mu}=\frac{\varphi H_{\mathrm{P}}}{\delta\left(\nabla_{\mathrm{ad}}-\nabla\right) \tau_{\mathrm{KH}}^{*}} \frac{|\Delta \mu|}{\mu} \tag{2.40}
\end{equation*}
$$

where $H_{\mathrm{P}}$ is the pressure scale height, $\nabla_{\mathrm{ad}}$ and $\nabla$ the adiabatic and radiative gradients respectively, $\mu$ the mean molecular weight, $\Delta \mu$ the local variation in $\mu$ over the region affected by the currents, $\tau_{\mathrm{KH}}^{*}$ the thermal adjustment time of a blob of matter with molecular weight different from its surroundings, and $\varphi=(\partial \ln \rho / \partial \ln \mu)_{\mathrm{P}, \mathrm{T}}$ (for an ideal $\operatorname{gas} \varphi=\delta=1$ ).

Endal \& Sofia (1978) further approximated the above expression as

$$
\begin{equation*}
v_{\mu}=\frac{15}{2} \frac{\varphi}{\delta\left(\nabla_{\mathrm{ad}}-\nabla\right)} \frac{c}{c_{\mathrm{P}}} \frac{P}{\rho T} \frac{|\Delta \mu|}{\mu}, \tag{2.41}
\end{equation*}
$$

where $c$ is the light speed and the remaining symbols have their usual meaning.
The meridional circulation, being proportional to $\omega^{2}$, is more important in regions with faster rotation, becoming negligible (if compared to other mechanisms) in lower rotation regions such as the Sun's more external layers. Besides, as it is less sensitive to $\mu$-gradients than the secular shear, the meridional circulation (as also happens with the GSF instability) will dominate the angular momentum transport in the deeper regions of stars.

## Multi-diffusive instabilities

Knobloch \& Spruit (1983) made further progress regarding the GSF instability by introducing the molecular weight as a third component of the problem. The results of their analysis are rather complex, and have a simplified form only in the case of small baroclinity, where the entropy gradient is nearly parallel to the gravity and the $\mu$-gradient is purely vertical. Under these conditions the GSF instability becomes

$$
\begin{equation*}
M N_{\Omega}^{2}>\frac{\nu}{\kappa} N_{\mathrm{T}}^{2}+\frac{\nu}{\kappa_{4}} N_{\mu}^{2}, \tag{2.42}
\end{equation*}
$$

where $\kappa_{4}$ is the helium molecular diffusivity and $M$ a geometrical factor given by

$$
\begin{equation*}
M=\frac{\sin ^{2} \Gamma}{4 \sin \Lambda \sin (\Lambda-\Gamma)} \tag{2.43}
\end{equation*}
$$

$\Gamma$ and $\Lambda$ being the angles between the equatorial plane and both the effective gravity $\mathbf{g}_{\text {eff }}$ and the angular momentum gradient $\nabla J$, respectively.

In addition, those authors found two new hydrodynamical instabilities, namely the triple diffusive instability (TD instability) and the diffusive axisymmetric baroclinic instability (ABCD instability). The first is an instability of the same nature as the GSF instability, while the second corresponds to the diffusive mode of the axisymmetric baroclinic instability previously discussed. The respective instability criteria are

$$
\begin{array}{ll}
\mathrm{TD}: & M N_{\Omega}^{2}>\left(\frac{\nu}{\kappa}+\frac{\kappa_{4}}{\kappa}\right) N_{\mathrm{T}}^{2}+N_{\mu}^{2}, \\
\mathrm{ABCD}: & \frac{1}{2} M N_{\Omega}^{2}>\frac{\nu}{\kappa}\left(N_{\mathrm{T}}^{2}+N_{\mu}^{2}\right) . \tag{2.45}
\end{array}
$$

God used beautiful mathematics in creating the world.

Paul Dirac

## Chapter 3

## METHODS USED FOR ROTATION AND ANGULAR MOMENTUM REDISTRIBUTION

### 3.1 Introductory remarks

In this chapter we present the techniques used for introducing rotation and internal angular momentum transport in the ATON 2.0 stellar evolutionary code, as well as their implementation details.

We first note that rotation and internal angular momentum redistribution were implemented as distinct modules, allowing us to compute rotating models with or without internal transport of angular momentum. One could ask if it makes sense to compute rotating stellar models without internal redistribution of angular momentum, as the latter is certainly a physical mechanism that operates in all stars. The answer is positive: there are some effects of rotation that depend very little on angular momentum redistribution, as we will see later on. For those effects, the computational cost of calculating the internal transport of angular momentum is not worth the slightly better accuracy of the results.

This chapter will present first a brief discussion of the main features of the original ATON 2.0 stellar code, followed by detailed material on introducing both rotation and angular momentum transport in it. From now on, the ATON 2.0 code with these new implementations will be referred to as the ATON 2.1 code.

### 3.2 Overview of the ATON 2.0 stellar evolutionary code

The ATON 2.0 stellar evolution code is a very powerful tool for probing the interior structure and the evolution of stars. In its present version, the code can be used to follow the evolution of stars from early pre-main sequence phases to the final evolutionary stages of brown or white dwarfs cooling.

As usually happens in such evolutionary codes, the code is somewhat complex regarding both its program implementation and the physical mechanisms accounted for, and so in this section we will restrict ourselves to highlighting its main features. A full, detailed account of the ATON 2.0 features can be found throughout a number of papers in the literature, e.g. Mazzitelli (1989), Mazzitelli et al. (1995), D'Antona \& Mazzitelli (1997) and Ventura et al. (1998b).

### 3.2.1 General features

The ATON 2.0 stellar evolution code allows one to follow the evolution of spherically symmetric stellar objects in hydrostatic equilibrium from early pre-main sequence phases prior to deuterium ignition down to final brown or white dwarf cooling or, optionally, to the onset of ${ }^{12} \mathrm{C}+{ }^{12} \mathrm{C}$ reactions. More advanced evolutionary phases, such as the ignition of ${ }^{16} \mathrm{O}+{ }^{16} \mathrm{O}$ and the following (pre-supernova) phases, are not yet accounted for.

The pre-main sequence models start in an arbitrary fully extended convective configuration with a central temperature of usually $\log T_{\mathrm{c}}=5.7$ (and thus prior to the onset of deuterium burning), to which $t=0$ is attributed (see D'Antona \& Mazzitelli 1997 for a discussion of attributing theoretical "ages" to pre-main sequence models).

The internal structure is integrated via the usual Henyey relaxation method, from the center to the base of the optical atmosphere. The independent variable is the mass, and the dependent ones are the radius, total luminosity, total pressure and temperature. Close to the surface, where $\Delta M / M_{\text {tot }}$ can be as small as $\sim 10^{-12}-10^{-18}$, the traditional approach has been to compute grids of subatmospheres (with pressure as the independent variable) down to a given fraction of $M_{\text {tot }}$, in order to avoid numerical precision problems. Instead of doing that, the ATON 2.0 code rezones and performs numerical derivatives directly on the $\Delta M$ vector, where the local value of mass at each grid is obtained as a summation of $\Delta M$ from the center. This approach allows straightforward evaluation of the gravothermal energy generation rate up to the stellar surface, especially when mass loss (e.g. in contact binaries) plays a dominant role in determining the structure.

The calculation of derivatives at the bottom of the optical atmosphere is done by computing three gray atmospheres with very small steps in both luminosity and radius $\left(\delta \ln \left[L, T_{\text {eff }}\right]=0.001\right)$ at each iteration. In this way interpolations over fixed, larger grids
of atmospheres are avoided, as they can lead to numerical disturbances especially when approaching the surface H ionization with a very thin subatmosphere. A simple $\mathrm{T}(\tau)$ relation (Henyey et al. 1968) is adopted all through the gray atmosphere.

The models evolve in time using time steps which are computed according to maximum allowed logarithmic derivatives for the structural quantities and to parameters such as mass loss, chemistry, integrals over the CNO, triple-alpha and ${ }^{12} \mathrm{C}+{ }^{12} \mathrm{C}$ luminosities, and some others. The final time step chosen is the shortest of the $\sim$ fifteen evaluated. A further requirement for computing each time step is that the relaxation (Henyey) procedure must converge within three iterations at most; otherwise the time step must be further decreased. Typically, for a solar-like star, evolution from pre-main sequence to turn-off phases takes $\sim 1000$ physical time steps; He-flash is reached after 15000-20000 time steps (no shell projection); in horizontal branch stars, the central He-exhaustion is reached in about 2000 time steps; and one fully developed thermal pulse cycle, from peak to peak, takes $\sim 5000$ time steps.

After each time step, the internal zoning of the whole structure is re-evaluated according to the principle that the derivatives for all five structural quantities must never exceed given values, fixed by experience and differing from region to region. Special care is taken in sensitive regions such as the center and the surface, the vicinities of convective boundaries and of H or He -burning shells, and regions close to the superadiabaticity peak. Typically, pre-main sequence and main sequence structures are resolved in 800-1200 grid points; red giants in 1200-1500 grid points, $\sim 200$ of which in the thin H -shell; horizontal branch structures contain 1500-2000 grid points and, during thermal pulses, up to $\sim 3000$ grid points are often reached.

### 3.2.2 Physical inputs

### 3.2.2.1 Thermodynamics

Thermodynamics is taken from Magni \& Mazzitelli (1979, thereinafter referred to as $\mathrm{MM})$, recently updated for full ionization for pure hydrogen, pure helium, carbon, oxygen, and intermediate mixtures. The influence of covolume, van der Waals, Coulomb, and degeneracy effects are taken into account in partial ionization regions, both as a whole (i.e., for their partial pressures) and for their influence on the internal partition functions and the continuum level. Degeneracy (including the relativistic case), Coulomb, and exchange effects are considered in regions of full ionization. Crystallization is included following Hansen (1973) and Pollock and Hansen (1973).

The thermodynamic tables are initially constructed based on the MM tables for five values of H contents $(\mathrm{X}=1.0-\mathrm{Z}, \mathrm{X}=0.8-\mathrm{Z}, \mathrm{X}=0.5-\mathrm{Z}, \mathrm{X}=0.2-\mathrm{Z}, \mathrm{X}=0.0)$ and equivalent $\mathrm{H} / \mathrm{He}$ abundances. These tables are then carefully combined with the Mihalas et al. (1988) EOS
and the OPAL EOS tables (Rogers et al. 1996) in such a way as to give as maximum extent as possible to the $\rho-T$ plane, and also to minimize some biasing that exists in some regions of each individual EOS. In this way, the best features of each EOS are preserved. Full details of this scheme are described in Ventura et al. (1998b).

After constructing the tables, bicubic logarithmic interpolation on both P and T are performed on the tables at fixed H -abundances, followed by linear interpolation on the chemical composition in order to evaluate the various thermodynamical quantities.

### 3.2.2.2 Opacities

A scheme similar to that used for constructing the thermodynamic tables is also employed for the opacity tables: for each given Z, the OPAL opacities (Iglesias \& Rogers 1993) are properly combined with the conductive opacities from Itoh \& Kohyama (1993) to form the ground level, and then completed by the Alexander \& Ferguson's (1994) molecular opacities (plus electron conduction when in full ionization) for temperatures lower than 6000 K. As an option, the Kurucz (1993) low temperature ( $T<6000 \mathrm{~K}$ ) opacities can also be used, but for just one $\mathrm{H} / \mathrm{He}$ ratio.

The opacities for $\mathrm{He} / \mathrm{C} / \mathrm{O}$ mixtures are obtained through interpolation on fifteen out of the 60 OPAL tables (plus conductive opacities), which gives an agreement better than $2 \%$ in the worst case. The final opacity value is then obtained by first doing bicubic logarithmic interpolation on both $T$ and $\rho$ on the tables at a fixed $\mathrm{H} / \mathrm{He}$ ratio, followed by quadratic interpolation on H .

### 3.2.3 Nuclear reactions

The nuclear network includes 14 elements: ${ }^{1} \mathrm{H},{ }^{2} \mathrm{H},{ }^{3} \mathrm{He},{ }^{4} \mathrm{He},{ }^{7} \mathrm{Li},{ }^{7} \mathrm{Be},{ }^{12} \mathrm{C},{ }^{13} \mathrm{C},{ }^{14} \mathrm{~N},{ }^{15} \mathrm{~N}$, ${ }^{16} \mathrm{O},{ }^{17} \mathrm{O},{ }^{18} \mathrm{O}$ and ${ }^{22} \mathrm{Ne}$. The following 22 reactions are considered:

$$
\begin{aligned}
& p+p \rightarrow{ }^{2} \mathrm{H}+e^{+}+\nu \\
& p+p+e^{-} \rightarrow{ }^{2} \mathrm{H}+\nu \\
& { }^{2} \mathrm{H}+p \rightarrow{ }^{3} \mathrm{He}+\gamma \\
& { }^{3} \mathrm{He}+{ }^{4} \mathrm{He} \rightarrow \alpha+2 p \\
& { }^{3} \mathrm{He}+{ }^{4} \mathrm{He} \rightarrow{ }^{7} \mathrm{Be}+\gamma \\
& { }^{7} \mathrm{Li}+p \rightarrow 2 \alpha \\
& { }^{7} \mathrm{Be}+p\left(\rightarrow{ }^{8} \mathrm{Be}+\gamma \rightarrow{ }^{8} \mathrm{Be}\right) \rightarrow 2 \alpha \\
& { }^{7} \mathrm{Be}+e^{-} \rightarrow{ }^{7} \mathrm{Li}+\gamma \\
& { }^{12} \mathrm{C}+p\left(\rightarrow{ }^{13} \mathrm{~N}+\gamma\right) \rightarrow{ }^{13} \mathrm{C}+e^{+}+\nu \\
& { }^{13} \mathrm{C}+p \rightarrow{ }^{14} \mathrm{~N}+\gamma
\end{aligned}
$$

$$
\begin{aligned}
& { }^{14} \mathrm{~N}+p\left(\rightarrow{ }^{15} \mathrm{O}+\gamma\right) \rightarrow{ }^{15} \mathrm{~N}+e^{+}+\nu \\
& { }^{15} \mathrm{~N}+p \rightarrow \alpha+{ }^{12} \mathrm{C} \\
& { }^{16} \mathrm{O}+p\left(\rightarrow{ }^{17} \mathrm{~F}+\gamma\right) \rightarrow{ }^{17} \mathrm{O}+e^{+}+\nu \\
& { }^{17} \mathrm{O}+p \rightarrow \alpha+{ }^{14} \mathrm{~N} \\
& { }^{17} \mathrm{O}+p\left(\rightarrow{ }^{18} \mathrm{~F}+\gamma\right) \rightarrow{ }^{18} \mathrm{O}+e^{+}+\nu \\
& { }^{18} \mathrm{O}+p \rightarrow{ }^{15} \mathrm{~N}+\alpha \\
& 3 \alpha \rightarrow{ }^{12} \mathrm{C}+\gamma \\
& { }^{12} \mathrm{C}+\alpha \rightarrow{ }^{16} \mathrm{O}+\gamma \\
& { }^{14} \mathrm{~N}+\alpha\left(\rightarrow{ }^{18} \mathrm{~F}+\gamma\right) \rightarrow{ }^{18} \mathrm{O}+e^{+}+\nu \\
& { }^{12} \mathrm{C}+12^{\mathrm{C}} \rightarrow \alpha / 2+{ }^{22} \mathrm{Ne} \\
& { }^{18} \mathrm{O}+\alpha \rightarrow{ }^{22} \mathrm{Ne}+\gamma
\end{aligned}
$$

The reactions in parenthesis are not explicitly considered due to their extremely high decay time. All reactions have their cross sections taken from Caughlan \& Fowler (1988); low, intermediate and strong screening coefficients are from Graboske et al. (1973). All nuclear cross sections have their logarithmic values stored in tables with a very small spacing in $\log T$, from which cubic interpolation is done.

Neutrinos from the most relevant physical processes (i.e. pair, photo, brehmsstrahlung and plasma neutrinos) are taken from Itoh et al. (1992), with the exception of recombination neutrinos since they are important only for more advanced (pre-supernova) evolutionary phases, which currently is not included in the aTON 2.0 code. To avoid the computational costs of calculating the various neutrino fluxes with the fitting formulae in the literature, logarithmic tables of neutrino rates for various elements are used. Bicubic logarithmic interpolation on both T and $\rho$ are then performed on the tables at a fixed chemistry, followed by linear interpolation on the chemistry to compute the neutrino emission.

### 3.2.4 Convection Model

The ATON 2.0 code handles two convection models: the classical mixing length formalism (MLT) or the Full Spectrum of Turbulence (FST) developed by Canuto \& Mazzitelli (1991, 1992) and later improved by Canuto et al. (1996). Despite its severe limitations, the MLT has been adopted over the last decades as the standard convection treatment in stellar evolutionary codes. In contrast, the FST, in spite of also relying on free parameters, is a much more modern and sound treatment of convection that takes into account the whole spectrum of turbulence generated in the nearly non-viscous stellar interiors.

It is not our aim to discuss here the full details and consequences of adopting either treatment. Nevertheless, for the sake of completeness we will present here a brief comparison of the two models.

Convection in stellar envelopes is a very complex process, mainly due to its turbulent behaviour, where the largest turbulent eddies have a size comparable to that of the entire convective region ( $\sim 10^{9}$ to $10^{12} \mathrm{~cm}$ ). These eddies break down into successive chains of smaller ones, with kinectic energy being forward- and back-scattered between the eddy scales. The whole process continues until the characteristic length of the eddies is comparable with the one that makes up the local viscosity; at this point the eddies' kinectic energy is thermally dissipated. As there is no available analytical or numerical solution for the non-linear, non-local Navier-Stokes equations that describe this process, we usually resort to local models for computing both the energy fluxes and the convective scale lengths.

In the MLT (which was originally developed for viscous fluids), the spectral distribution of eddies is simulated by one "average" eddy only. The scale length $\Lambda$ was originally chosen to match the distance $z$ from the convective boundary, consistent with the Von Kármán law for incompressible fluids (Prandtl 1925). Since however the one-eddy flux distribution is very crude, no realistic fit in the case of the Sun could be obtained. It is then now customary to apply the MLT with $\Lambda=\alpha H_{\mathrm{P}}$, where $\alpha$ is a free parameter tuned on the solar model; depending on the micro-physical inputs, $1.5 \leq \alpha \leq 2.2$.

On the other hand, Canuto \& Mazzitelli $(1991,1992)$ and Canuto et al. (1996) developed a new treatment of turbulence for low-viscosity fluids, as is the case for stellar interiors, which takes into account the whole eddy spectrum. The stronger physical foundation of that treatment led Stothers \& Chin (1995) to name it as "Full Spectrum of Turbulence" (FST), in contrast with the one-eddy spectrum MLT. In the FST framework, the scale length $\Lambda$ is taken as the harmonic mean between the distance of the layer from the top and the distance from the bottom of the convective region: $z=z^{\text {up }} z_{\text {low }} /\left(z^{\text {up }}+z_{\text {low }}\right)$. This choice matches the requirements that $\Lambda$ must approach $z^{\text {up }}$ or $z_{\text {low }}$ near the upper or lower convective boundaries respectively, as required by the Von Kármán law for incompressible flows, while it must converge to the hydrostatic scale lenght $H_{\mathrm{P}}$ far from the boundaries.

Nevertheless, it should be emphasized that the FST is still far from a perfect theory of convection. Some tuning of the models is still required, for instance, if we want to make a fit of the Sun. In the MLT, this is done by adjusting the completely free parameter $\alpha$, while in the FST framework this is achieved by allowing a small amount of overshooting (OV) from the top (or the bottom) of the surface convective layer: $\Lambda=z+O V$. As the overshooting $O V$ is not yet modelled from first principles, in the FST it is parametrized as $O V=\beta H_{\mathrm{P}}^{\mathrm{top}}$ ( or $O V=\beta H_{\mathrm{P}}^{\text {bot }}$ ), with $\beta$ constrained to $0 \leq \beta \leq 0.25$ (see Ventura et al. 1998b for details).

The FST framework has given good results for stellar structures in many different evolutionary phases, such as the main sequence, red giants, asymptotic giant branch stars, and white dwarfs. See D'Antona et al. (1997) for references on various tests performed by other authors.

### 3.2.5 Chemical evolution and diffusion

The nuclear evolution of the chemical species is treated by the ATON 2.0 code according to the linearized implicit scheme by Arnett \& Truran (1969). This scheme couples the chemical abundances and reaction rates of the species considered, since during a given timestep the chemical abundance of the reacting elements is changing, and so do the reaction rates.

The chemical mixing of species can be handled by the ATON 2.0 code in two different ways: by instantaneous mixing or by using a diffusive approach. In the case of instantaneous mixing, the code treats the entire region to be mixed as a single grid point, with average values of nuclear rates and chemistry. The diffusive approach, on the other hand, is a much more sound treatment and is implemented through the diffusion equation of Cloutman \& Eoll (1976), which is solved by means of a fully implicit scheme. Full details on the difference equations used for chemical diffusion, as well as the matrix algorithm, are given in the Appendix of Ventura et al. (1998b).

For the purpose of computing chemical evolution and diffusion, each physical time step is divided into at least 10 diffusion time steps. This procedure is usually required in order to attain negligible numerical errors in the nuclear evolution integration. In each diffusion time step, the structural quantities such as pressure and temperature are projected to get also projected nuclear reaction rates.

### 3.2.6 Helium sedimentation

Gravitational settling and chemical and thermal diffusion of ${ }^{4} \mathrm{He}$ are optionally included in ATON 2.0. The sedimentation velocities are calculated according the equations given by Proffit \& Michaud (1991) and Paquette et al. (1986), and the resistance coefficients are taken from Iben \& McDonald (1985). The resulting continuity equation which describes these processes is solved by a semi-implicit finite difference method of first-order in time and second-order in radius.

### 3.3 Introduction of rotation

### 3.3.1 Brief description of the chosen strategy

The main strategies for introducing rotation in evolutionary stellar models are reviewed by Tassoul (1978). From all those strategies, we have chosen the Kippenhahn \& Thomas (1970, hereafter KT) method because, although it is not the most accurate one, it presents a very convenient way of introducing rotation in existing evolutionary codes with a mini-
mum of changes. This ease of implementation has made it the preferred method in the available literature (e.g. Endal \& Sofia 1976; Law 1980; Pinsonneault 1988; Martín \& Claret 1996).

Since rotation is a property that renders tri-dimensional any approach to stellar evolution, the main difficulty is how to overcome the prohibitive computational cost of such a triaxial treatment. This is where the KT method enters: under the reasonable assumption of axial symmetry about the rotational axis, it offers a convenient way of averaging over the equipotential (level) surfaces that replace the spherical ones, so that the whole problem is reduced to a one-dimensional approach. In this way, the existing one-dimensional stellar evolutionary codes can be changed, with some minor modifications, in order to account for rotation.

The essence of the KT method is as follows. The spherical surfaces of the non-rotating models are replaced by equipotential, nonspherical surfaces, each of them characterized by the total potential $\Psi$, the mass $M_{\Psi}$ enclosed by the equipotential surface and the radius $r_{\Psi}$ of a topologically equivalent sphere with the same volume $V_{\Psi}$ as the equipotential. For any quantity $f$ that varies over an equipotential surface, we can define its mean value as

$$
\begin{equation*}
\langle f\rangle=\frac{1}{S_{\Psi}} \int_{S_{\Psi}} f d \sigma, \tag{3.1}
\end{equation*}
$$

where $S_{\Psi}$ is the surface area of the equipotential and $d \sigma$ its surface element. Since the local gravity, defined as

$$
\begin{equation*}
g=\frac{d \Psi}{d n} \tag{3.2}
\end{equation*}
$$

where $d n$ is the separation between two successive equipotentials $\Psi$ and $\Psi+d \Psi$, is nonconstant over equipotential surfaces, we have

$$
\begin{align*}
& \langle g\rangle=\frac{1}{S_{\Psi}} \int_{S_{\Psi}} \frac{d \Psi}{d n} d \sigma  \tag{3.3}\\
& \left\langle g^{-1}\right\rangle=\frac{1}{S_{\Psi}} \int_{S_{\Psi}}\left(\frac{d \Psi}{d n}\right)^{-1} d \sigma . \tag{3.4}
\end{align*}
$$

The volume between the surfaces $\Psi$ and $\Psi+d \Psi$ is given by

$$
\begin{align*}
d V_{\Psi} & =\int_{\Psi=\text { const }} d n d \sigma \\
& =d \Psi \int_{\Psi=\text { const }}\left(\frac{d n}{d \Psi}\right) d \sigma \\
& =d \Psi S_{\Psi}\left\langle g^{-1}\right\rangle \tag{3.5}
\end{align*}
$$

from which it follows that

$$
\begin{align*}
d \Psi & =\frac{1}{S_{\Psi}\left\langle g^{-1}\right\rangle} d V_{\Psi} \\
& =\frac{1}{S_{\Psi}\left\langle g^{-1}\right\rangle} \frac{d M_{\Psi}}{\rho(\Psi)} . \tag{3.6}
\end{align*}
$$

On the other hand, the volume of the topologically equivalent sphere is given by

$$
\begin{equation*}
V_{\Psi}=\frac{4 \pi}{3} r_{\Psi}^{3} \tag{3.7}
\end{equation*}
$$

These two expressions allow us to properly modify the usual stellar structure equations. For example, the hydrostatic equilibrium equation is

$$
\begin{equation*}
\frac{d P}{d \Psi}=\rho(\Psi) \tag{3.8}
\end{equation*}
$$

In the spherical case we have $\Psi=G M / r, d \Psi=-G M d r / r^{2}, d r=d M / 4 \pi r^{2} \rho$, and then

$$
\begin{equation*}
\frac{d P}{d M}=-\frac{G M}{4 \pi r^{4}} \tag{3.9}
\end{equation*}
$$

while in the non-spherical case $d \Psi$ is given by Eq. (3.6), leading to

$$
\begin{equation*}
\frac{d P}{d M_{\Psi}}=-\frac{G M_{\Psi}}{4 \pi r_{\Psi}^{4}} f_{\mathrm{p}} \tag{3.10}
\end{equation*}
$$

with $f_{\mathrm{p}}$ given by

$$
\begin{equation*}
f_{\mathrm{p}}=\frac{4 \pi r_{\Psi}^{4}}{G M_{\Psi}} \frac{1}{S_{\Psi}\left\langle g^{-1}\right\rangle} \tag{3.11}
\end{equation*}
$$

With these corrections in mind, the four stellar structure equations become, with $M_{\Psi}$ as the independent variable,

$$
\begin{align*}
\frac{d P}{d M_{\Psi}} & =-\frac{G M_{\Psi}}{4 \pi r_{\Psi}^{4}} f_{\mathrm{p}}  \tag{3.12a}\\
\frac{d r_{\Psi}}{d M_{\Psi}} & =\frac{1}{4 \pi r_{\Psi}^{2} \rho} \tag{3.12b}
\end{align*}
$$

$$
\begin{align*}
& \frac{d L_{\Psi}}{d M_{\Psi}}=\epsilon-T \frac{\partial S}{\partial t}  \tag{3.12c}\\
& \frac{d T_{\Psi}}{d M_{\Psi}}=-\frac{G M_{\Psi} T}{4 \pi r_{\Psi}^{4} P} \nabla, \quad \nabla=\left\{\begin{array}{c}
\nabla_{\text {rad }} \frac{f_{\mathrm{t}}}{f_{\mathrm{P}}} \\
\nabla_{\text {conv }}
\end{array}\right. \tag{3.12d}
\end{align*}
$$

with $f_{\mathrm{p}}$ given by Eq. (3.11) and

$$
\begin{equation*}
f_{\mathrm{t}}=\left(\frac{4 \pi r_{\Psi}^{2}}{S_{\Psi}}\right)^{2} \frac{1}{\langle g\rangle\left\langle g^{-1}\right\rangle} . \tag{3.13}
\end{equation*}
$$

In the case of no rotation $f_{\mathrm{p}}=f_{\mathrm{t}}=1$, and the original stellar structure equations are recovered.

We can readily see that, in this method, the distortion produced by rotation is entirely included in the total potential function $\Psi$. Kippenhahn \& Thomas used a Roche-type potential function of the form

$$
\begin{equation*}
\Psi(r, \cos \theta)=\frac{G M}{r}+\frac{1}{2} \Omega^{2} r^{2} \sin ^{2} \theta \tag{3.14}
\end{equation*}
$$

where $\Omega$ is the angular velocity. In this kind of potential, the first term represents the spherically symmetric part of the gravitational potential, while the second term is the cylindrically symmetric potential directly due to rotation. Instead of using this Roche potential, our choice was to follow the more refined approach presented by Endal \& Sofia (1976; hereafter ES) to the potential function. These authors, by using the ClairautLegendre expansion for the gravitational potential of a self-gravitating body (Kopal 1959; Tassoul 1978), introduced a third term $\Psi_{\mathrm{d}}$ related to the cylindrically symmetric part of the gravitational potential due to the distortion of the figure of the star, given by

$$
\begin{equation*}
\Psi_{\mathrm{d}}=\sum_{j=2}^{\infty} \frac{4 \pi G}{(2 j+1) r^{j+1}} \int_{0}^{r_{0}} \rho \frac{\partial}{\partial r_{0}^{\prime}}\left(r_{0}^{\prime j+3} Y_{j}\right) d r_{0}^{\prime} \tag{3.15}
\end{equation*}
$$

where $r_{0}$ is a characteristic quantity related to the radius of an equipotential surface,

$$
\begin{equation*}
r\left(r_{0}, \theta\right)=r_{0}\left[1+\sum_{j} Y_{j}\left(r_{0}, \theta\right)\right] \tag{3.16}
\end{equation*}
$$

and $Y_{\mathrm{j}}$ is the axisymmetric tesseral harmonic defined by

$$
\begin{equation*}
Y_{j}=-\frac{\Omega^{2} r_{0}^{j+1}}{3 G M_{\Psi}} \frac{5}{2+\eta_{j}\left(r_{0}\right)} P_{j}(\cos \theta) \tag{3.17}
\end{equation*}
$$

$\eta_{j}$ being the logarithmic derivative of $Y_{j}$ with respect to $r_{0}$ :

$$
\begin{equation*}
\eta_{j}=\frac{r_{0}}{Y_{j}} \frac{\partial Y_{j}}{\partial r_{0}} \tag{3.18}
\end{equation*}
$$

The quantity $\eta_{j}$ can be calculated by means of "Radau's equation" (Kopal 1959),

$$
\begin{equation*}
r \frac{d \eta_{j}}{d r}+6 \frac{\rho}{\langle\rho\rangle}\left(\eta_{j}+1\right)+\eta_{j}\left(\eta_{j}-1\right)=j(j+1) \tag{3.19}
\end{equation*}
$$

subject to the initial condition $\eta_{j}(0)=2-j$.
By using a first-order approximation (so that only the first term in the expansion for $\Psi_{\mathrm{d}}$ is retained), ES obtained for the total potential function

$$
\begin{align*}
\Psi & =\Psi_{\mathrm{s}}+\Psi_{\mathrm{d}}+\frac{1}{2} \Omega^{2} r^{2} \sin ^{2} \theta \\
& =\frac{G M_{\Psi}}{r}-\frac{4 \pi}{3 r^{3}} P_{2}(\cos \theta) \int_{0}^{r_{0}} \frac{\rho r_{0}^{\prime 7}}{M_{\Psi}} \Omega^{2} \frac{5+\eta_{2}}{2+\eta_{2}} d r_{0}^{\prime}+\frac{1}{2} \Omega^{2} r^{2} \sin ^{2} \theta \tag{3.20}
\end{align*}
$$

In order to evaluate the above expression, we need a relation between $r_{0}$ and $r_{\Psi}$, which is deduced by ES as

$$
\begin{equation*}
r_{\Psi}=r_{0}\left[1+\frac{3}{5} A^{2}-\frac{2}{35} A^{3}\right]^{\frac{1}{3}} \tag{3.21}
\end{equation*}
$$

where $A$ is the radial part of $Y_{2}$, namely

$$
\begin{equation*}
A=\frac{\Omega^{2} r_{0}^{3}}{3 G M_{\Psi}} \frac{5}{2+\eta_{2}} \tag{3.22}
\end{equation*}
$$

Since one usually has $r_{\Psi}, r_{0}$ can be calculated through Eq. (3.21) by means of an iterative procedure.

### 3.3.2 Remarks on the Kippenhahn-Thomas method

The KT method for introducing rotation, in its original form, applies only for the case of conservative rotation, that is, for the case where the centrifugal acceleration can be derived from a potential,

$$
\begin{equation*}
\Omega^{2} s \mathbf{e}_{s}=-\nabla V_{\mathrm{rot}} \tag{3.23}
\end{equation*}
$$

where $s=r \sin \theta$ is the perpendicular distance to the rotation axis of the star. A necessary and sufficient condition for the conservative case is that $\Omega=\Omega(s)$ (Kippenhahn \& Weigert 1994), which means rotation is constant on cylinders. Rigid body rotation is obviously a special case of conservative rotation.

For non-conservative rotation laws, such as the shellular rotation $\Omega=\Omega(r)$ used in Zahn's (1992) framework (see Sect. 3.4.4), one should use the more rigorous treatment presented by Meynet \& Maeder (1997), which extends the KT method for non-conservative rotation laws. However, for the case of shellular rotation the original KT method can still be regarded as an acceptable approximation: as pointed out by Endal \& Sofia (1978), even for non-conservative rotation laws the baroclinity induced by rotation will be small, and so the angular velocity can be approximated as constant along equipotential (level) surfaces. In the present work, we adopt this latter view, as the Meynet \& Maeder (1997) treatment implies substantial modifications in existing one-dimensional stellar evolutionary codes. Nevertheless, we are planning to include it in a future version of the ATON code.

### 3.3.3 Overview of the code implementation

Given the procedure outlined in the previous subsection, we now present an overview of its implementation in the original ATON 2.0 evolutionary code.

### 3.3.3.1 Interior calculations

The interior calculations were implemented both in the initial stellar model, computed by Runge-Kutta integration to a fitting point, and in the relaxation (Henyey) algorithm. However, we found that the changes brought to the initial stellar model by rotation were completely negligible and, for that reason, we removed the effects of rotation from it. As a matter of fact, it was already expected that the inclusion of rotational effects in the first model should not bring any appreciable change to the results for, as the initial model is only used as a starting point for the calculations, the evolutionary code should quickly put the star in the same evolutionary path.

As the relaxation procedure solves the four stellar structure equations by means of the corresponding difference equations over a mesh of points running from the star's interior to the base of the atmosphere, we must calculate the $f_{\mathrm{p}}$ and $f_{\mathrm{t}}$ correction factors for each mesh point. Labeling a given mesh point by its $r_{\Psi}$ coordinate, our implementation works as follows:

1. We first calculate the characteristic radius $r_{0}$ of the equipotential surface related to $r_{\Psi}$, by solving Eq. (3.21) through a Newton-Raphson scheme. At each iteration of this scheme, Radau's equation (Eq. 3.19) is solved by direct integration through a 4th-order Runge-Kutta method.
2. Once we have $r_{0}$, the integral term in Eq. (3.20) is evaluated by a 32 -point Gaussian quadrature, and then the total potential is obtained.
3. The $f_{\mathrm{p}}$ and $f_{\mathrm{t}}$ correction factors are then evaluated by again integrating Eqs. (3.3) and (3.4) through 32-point Gaussian quadrature. At this point we also evaluate the rotational inertia of the rotationally distorted mass shell related to the given mesh point, through the expression given by Law (1980)

$$
\begin{equation*}
\Delta I=\frac{2}{3} d m_{\psi} r_{\phi}^{2}\left(\frac{r_{0}}{r_{\psi}}\right)^{5} \frac{d \ln r_{0}}{d \ln r_{\psi}}\left[1+\frac{3}{20} \sum_{i=1}^{5} \alpha_{i} A^{i}\left(i \eta_{2}+5\right)\right], \tag{3.24}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha_{i}=\frac{5!}{i!(5-i!)} \int_{-1}^{1} P_{2}^{i}(\mu)\left(1-\mu^{2}\right) d \mu, \quad \mu=\cos \theta \tag{3.25}
\end{equation*}
$$

(the rotational inertia of each mass shell will be needed later on when enforcing angular momentum conservation throughout the star).
4. The difference equations, previously modified in order to incorporate the $f_{\mathrm{p}}$ and $f_{\mathrm{t}}$ correction factors, are subsequently solved by the usual relaxation technique.
5. After the relaxation procedure has converged, the new values of the angular velocity profile throughout the star are calculated based on the new values of the rotational inertia, always enforcing angular momentum conservation according the specified rotation law.

### 3.3.3.2 Atmospheric calculations

The atmospheric calculations are modified simply by changing the local gravity to an "effective gravity" given by (Law 1980)

$$
\begin{equation*}
g_{\mathrm{eff}}=\frac{G M}{r^{2}} f_{\mathrm{p}} \tag{3.26}
\end{equation*}
$$

where $f_{\mathrm{p}}$ is that one calculated for the outermost interior point, under the assumption that it is constant through the thin atmospheric layer.

### 3.3.3.3 Rotation laws

We currently implement three possible rotation laws in the evolutionary code: rigid body rotation, local conservation of angular momentum throughout the whole star, and local
conservation of angular momentum in radiative regions and rigid body rotation in convective ones. Following ES and Pinsonneault et al. (1989), these laws are meant to represent a set of physically plausible laws. The rotation law is specified as an input value to the evolutionary code; a special value signals the non-rotating case, so that the modified evolutionary code can behave exactly as the non-rotating original one.

In the case of solid body rotation, we enforce overall conservation of angular momentum except where the star's angular velocity reaches the equatorial breakup velocity; if this happens, then we arbitrarily remove $10 \%$ of the angular momentum in order to keep the star rotating at a maximum of $90 \%$ of the breakup velocity.

In the case of local conservation of angular momentum, we start with solid body rotation so that each mass shell has the same initial angular velocity; as the star contracts, this generates differential rotation. Along the star's evolution each mass shell is checked against its breakup velocity, and the former criterion is again applied in order to keep each shell rotating at a maximum of $90 \%$ of its breakup velocity.

The last case (differential rotation in radiative layers and rigid body rotation in convective regions) is surely the most complex one. Though the rotation law in convective regions is still very uncertain on both theoretical and observational grounds, we have adopted this model, following Pinsonneault et al. (1989), as a first step towards the introduction of internal angular momentum redistribution mechanisms. However, this case brings an additional difficulty, represented by the steep angular velocity gradient that can develop at the boundary between a convective and a radiative region. This gradient can force the level surfaces to intersect themselves at that boundary, a situation that we regard as unphysical (since the potential would be double-valued at the intersection points). Whenever this condition occurs, we are compelled to remove an extra quantity of angular momentum at that boundary in order to smooth out the angular velocity gradient; in order to do that, we calculate the maximum angular velocity at which there will be no intersection of level surfaces and apply a factor going from 0 to 0.9 (specified as a program input) to it.

The arbitrary removal of angular momentum for the three cases above, however, is applied only for models without angular momentum redistribution.

### 3.3.3.4 Initial conditions

In order to start the calculations of a given rotating stellar model, it is necessary to assign an initial rotation rate to it. This can be done in two different ways, either by supplying the star's initial surface rotation velocity or its initial total angular momentum. In either case we assume that the star's very first initial model (that is, the initial hydrostatic model computed by Runge-Kutta integration) corresponds to rigid body rotation, which
is a plausible assumption since that model will always be fully convective; consequently, each mass shell starts with the same angular velocity.

In our computations we always assign the star an initial angular momentum based on the mass-angular momentum relations investigated by Kawaler (1987) for stars of spectral type later than F2. These relations follow from the well-known empirical mass-angular momentum relation discovered by Kraft (1970) for stars of spectral types O to F5,

$$
\begin{equation*}
J \propto M^{1.57} \tag{3.27}
\end{equation*}
$$

Based on theoretical considerations regarding the internal structure of both massive and low mass stars, Kawaler (1987) not only rederived the Kraft's relation (although with a higher exponent) but also extrapolated it for low mass stars. Specifically for the mass range between 0.6 and 1.25 solar masses, his mass-angular momentum relation can be derived as

$$
\begin{equation*}
J=1.566 \times 10^{50}\left(\frac{M}{M_{\odot}}\right)^{0.985} \quad \mathrm{~g} \mathrm{~cm}^{2} \mathrm{~s}^{-1} \tag{3.28}
\end{equation*}
$$

which is the one adopted in the present work.

### 3.4 Internal angular momentum redistribution

### 3.4.1 Background

Since Eddington (1925) it has been well known that the thermal imbalance that develops inside the stars due to their rotation induces a meridional circulation.

The standard theory of meridional circulation (e.g. Sweet 1950, Mestel 1953) assumes uniform rotation, which is clearly a very simplified treatment as the circulation advects* angular momentum and then continuously changes the internal angular velocity profile. As a result, the star does not rotate uniformly on cylinders, and this gives rise to additional baroclinic forces that are absent in the assumed initial state. Therefore, a consistent description of circulation and rotation must include baroclinic effects and is two-dimensional by its own nature.

As shown in Sect. 2.2.3, differential rotation triggers some dynamical and secular instabilities in the almost non-viscous stellar plasma, which by their turn generate turbulent

[^6]motions. On the other hand, as pointed out by many authors (see e.g. Zahn 1992 and references therein), the star's internal rotation profile depends also on the balance between the meridian advection and those turbulent stresses. Unfortunately, very little is known about the properties of such turbulent motions in the physical conditions of stellar radiative zones. What is usually done is to take some guidance from the results of laboratory experiments or the processes in the Earth's atmosphere and oceans.

Some progress has been made by Chaboyer \& Zahn (1992) and Zahn (1992), who presented a treatment of the meridional circulation and turbulence in rotating stars based on the plausible assumption that turbulence is anisotropic in radiative regions, with horizontal motions stronger than the vertical ones. That treatment allows one to reduce the rotation pattern description from two- to one-dimensional, which is obviously of great interest as the existing one-dimensional stellar evolutionary codes could be adapted to include the combined effects of meridional circulation, rotation and turbulence, even if under such a simplified approach.

These combined effects are naturally very important to the study of stellar evolution. To cite just one example, the meridional circulation may be one of the possible mixing mechanisms in radiative zones of stars, a subject that has been extensively debated over the last years.

### 3.4.2 Modeling of internal angular momentum transport

In their pioneering work, Endal \& Sofia (1978) treated the dynamical instabilities in such a way that whenever a dynamically unstable angular velocity gradient was detected it was instantaneously smoothed to a marginally stable value by means of radial angular momentum exchange with the neighboring shells. For ease of computations we preferred (following Pinsonneault 1988) to treat the dynamical instabilities in the same way as the secular ones (see below), with a large diffusion velocity.

As pointed out by Endal \& Sofia (1978), the secular instabilities occur on time scales of the same order as those in which $\mu$-gradients can be formed and in which the angular velocity profile is influenced by the contraction or expansion of the star. These time scales must be explicitly included in the calculations of the angular momentum and chemical composition distributions. To do so, these authors (later followed by Pinsonneault et al. 1989) used the coupled nonlinear diffusion equations given by

$$
\begin{align*}
& \frac{\partial j}{\partial t}=\frac{1}{\rho r^{2}} \frac{\partial}{\partial r}\left(\rho r^{2} D \frac{\partial j}{\partial r}\right)  \tag{3.29}\\
& \frac{\partial X_{i}}{\partial t}=\frac{1}{\rho r^{2}} \frac{\partial}{\partial r}\left(\rho r^{2} D \frac{\partial X_{i}}{\partial r}\right) \tag{3.30}
\end{align*}
$$

where $j(r, t)=r^{2} \Omega(r, t)$ is the specific angular momentum and $X_{i}(r, t)$ the relative abundances of the chemicals.

Despite the valuable contribution of these authors, the use of a genuine diffusion process has been somewhat criticized in the literature (see e.g. Charbonneau 1992a), on the grounds that modeling the transport of angular momentum and chemicals due to meridional circulation by a purely radial diffusion equation is an extremely poor approximation. However, Chaboyer \& Zahn (1992) showed that the one-dimensional treatment for the redistribution of angular momentum and chemicals is justified if one assumes that the horizontal turbulence is strong enough to enforce $\Omega$ and $X_{i}$ constant along spherical surfaces, a result later confirmed by numerical experiments (Charbonneau 1992b).

In the present work we adopt the treatment of Chaboyer \& Zahn for the internal angular momentum transport, as detailed in Sect. 3.4.4. As for the transport of the chemicals, we do not use the treatment by the above authors since the ATON 2.0 code already treats the diffusion of chemicals (see Sect. 3.2.5); however, since the transport of angular momentum and chemicals are coupled, we do calculate the contribution to the diffusion coefficient of chemicals due to the action of meridional circulation, allowing it to be included in the scheme used for the diffusion of chemicals.

### 3.4.3 Treatment of hydrodynamical instabilities

In the present work the following hydrodynamical instabilities are considered:

- dynamical instabilities: Solberg-Høiland and dynamical shear;
- secular instabilities: Secular shear, Goldreich-Schubert-Fricke instability and meridional circulation.

These instabilities are included according to the prescriptions given in Sect. 2.2.3. Other instabilities, such as the TD and the ABCD instabilities, are not considered, as there are no reliable estimates of their characteristic velocities.

The critical Reynolds number $\Re_{\mathrm{c}}$ that appears in the expression for the secular shear instability is obtained as an input parameter for the aton 2.1 code. For all models discussed in the present work, $\Re_{\mathrm{c}}$ was set to 3000 , according to the suggestions of Zahn (1993).

### 3.4.3.1 Remarks on the dynamical shear instability

Pinsonneault (1988) showed that treating separately the dynamical shear and the other redistribution mechanisms leads to composition transport problems; for this reason he preferred to treat the dynamical shear as a "secular" mechanism with a large diffusion
coefficient. In our work we adopt the same approach; however, for this particular case, the diffusion coefficient is not computed as the product of a characteristic length and a diffusion velocity as in the secular mechanisms, but rather as a direct calculation following the prescription given by Talon \& Zahn (1997; see also Sect. 2.2.3.1 regarding the topic on the dynamical shear).

### 3.4.4 Implementation details

The internal angular momentum redistribution was implemented according to the framework established by Chaboyer \& Zahn (1992, later on improved by other authors such as Zahn 1992, Urpin et al. 1996, Talon et al. 1997, and Maeder \& Zahn 1998), which we now briefly describe. The major advantage of that approach is that it allows one to follow the internal angular velocity evolution under the combined action of the meridional circulation and shear turbulence.

The main hypothesis brought by Chaboyer \& Zahn (1992) and Zahn (1992) is that turbulence is highly anisotropic inside stars, with horizontal transport (i.e., along surfaces of contant pressure) stronger than vertical transport (that is, perpendicular to those surfaces). As noted by Urpin et al. (1996), this seems plausible because, in the Earth's atmosphere, turbulent motions in regions where the stratification is stable are highly anisotropic, though it is not certain at the moment if such highly anisotropic turbulence really exists in stellar radiative zones.

The stronger horizontal transport, among other effects, tends to establish a rotation rate that depends very little on the latitude. This near latitude-independent rotation rate, dubbed as "shellular" by Zahn (1992), allows one to substantially simplify the whole problem by treating it as one-dimensional.

### 3.4.4.1 Internal transport of angular momentum

Assuming that the effect of the turbulent stresses on the large scale flow can be described by an anisotropic eddy viscosity with vertical and horizontal components $\nu_{v}$ and $\nu_{h}$, respectively, the transport of angular momentum follows an advection-diffusion equation given by

$$
\begin{align*}
& \frac{\partial}{\partial t}\left[\rho r^{2} \sin ^{2} \theta\right]+\nabla \cdot\left[\rho r^{2} \sin ^{2} \theta \Omega \mathbf{u}\right]  \tag{3.31}\\
& \quad=\frac{\sin ^{2} \theta}{r^{2}} \frac{\partial}{\partial r}\left[\rho \nu_{v} r^{4} \frac{\partial \Omega}{\partial r}\right]+\frac{1}{\sin \theta} \frac{\partial}{\partial r}\left[\rho \nu_{h} \sin ^{3} \theta \frac{\partial \Omega}{\partial \theta}\right]
\end{align*}
$$

where $\Omega(r, \theta)$ is the angular velocity and $\mathbf{u}$ is the meridional circulation velocity.

By virtue of the main initial assumption, the horizontal component of the turbulent viscosity is considered to be much larger than the vertical one. In this case the departure from sphericity of the equipotential surfaces are very small and can be neglected to lowest order. We can then split the angular velocity in its mean and latitudinal parts

$$
\begin{equation*}
\Omega(r, \theta)=\bar{\Omega}(r)+\widehat{\Omega}(r, \theta) \quad \text { with } \quad \bar{\Omega}(r)=\frac{\int \Omega \sin ^{3} \theta d \theta}{\int \sin ^{3} \theta d \theta} \tag{3.32}
\end{equation*}
$$

and take $|\widehat{\Omega}| \ll \bar{\Omega}$. For this case of shellular rotation, the meridional circulation is simply given by

$$
\begin{equation*}
u(r, \theta)=U(r) P_{2}(\cos \theta) \tag{3.33}
\end{equation*}
$$

It remains to average Eq. (3.31) horizontally. This is done by multiplying it by $\sin \theta$ and integrating it over $d \theta$, resulting in

$$
\begin{equation*}
\frac{\partial}{\partial t}\left[\rho r^{2} \bar{\Omega}\right]=\frac{1}{5 r^{2}} \frac{\partial}{\partial r}\left(\rho r^{4} \bar{\Omega} U\right)+\frac{1}{r^{2}} \frac{\partial}{\partial r}\left[\rho \nu_{v} r^{4} \frac{\partial \bar{\Omega}}{\partial r}\right] \tag{3.34}
\end{equation*}
$$

Equation (3.34) is the main advection-diffusion equation governing the internal transport of angular momentum. However, it was derived by using eulerian coordinates and generally applies to stationary stars, such as those on the main sequence. In the case of evolving stars, the effects due to the star's contraction or expansion must be taken into account. This task was undertaken by Maeder \& Zahn (1998), who rederived that equation for the case of evolving stars so that, for the case of lagrangian coordinates (those used by most of stellar evolution codes including ATON 2.0), the advection-diffusion equation reads

$$
\begin{equation*}
\rho \frac{d}{d t}\left[r^{2} \bar{\Omega}\right]_{M_{r}}=\frac{1}{5 r^{2}} \frac{\partial}{\partial r}\left(\rho r^{4} \bar{\Omega} U\right)+\frac{1}{r^{2}} \frac{\partial}{\partial r}\left[\rho \nu_{v} r^{4} \frac{\partial \bar{\Omega}}{\partial r}\right] \tag{3.35}
\end{equation*}
$$

where the subscrit $M_{r}$ on the left-hand side denotes the corresponding mass level in the star. This lagrangian form has already been adopted by other authors, e.g. Talon et al. (1997). Equation (3.35) is the basic angular momentum transport equation implemented in the aton 2.1 code.

### 3.4.4.2 The meridional circulation velocity

The meridional circulation velocity for the case of shellular rotation is computed by Zahn (1992) as follows. One starts from the well-known heat equation

$$
\begin{equation*}
\rho T \frac{\partial S}{\partial t}+\rho T \mathbf{u} \cdot \nabla S=\nabla \cdot(\chi \nabla T)+\rho \varepsilon \tag{3.36}
\end{equation*}
$$

where $\mathbf{u}$ is the fluid velocity, $\chi=16 \sigma T^{3} / 3 \kappa \rho$ is the thermal conductivity and the other symbols have their usual meaning. The time derivative is usually neglected since the timescales involved are of the order of the Kelvin-Helmholtz timescale. The rotation rate is taken as constant on equipotential surfaces and varying only with the depth, as done before when obtaining Eq. (3.34). But since all other physical variables take part in the thermal imbalance that drives the meridional circulation, their horizontal variation must be considered. These horizontal variations are assumed to be small enough so that they can be treated as first order perturbations around their mean values. Depending on the nature of the problem, the surface of reference for the expansions is taken either as a spherical layer or an equipotential surface:

$$
\begin{align*}
\rho & =\rho_{0}(r)+\widehat{\rho}(r) P_{2}(\cos \theta)  \tag{3.37}\\
\rho & =\bar{\rho}(P)+\widetilde{\rho}(P) P_{2}(\cos \theta) \tag{3.38}
\end{align*}
$$

Besides, as the centrifugal force is not conservative for the case of shellular rotation, all physical variables (except the pressure) vary in latitude on an equipotential surface. These variations are linked to the rotation rate through the expression

$$
\begin{equation*}
\frac{\widetilde{\mu}}{\bar{\mu}}-\frac{\widetilde{T}}{\bar{T}}=\frac{\widetilde{\rho}}{\bar{\rho}}=\frac{1}{3} \frac{r^{2}}{\bar{g}} \frac{d \Omega^{2}}{d r} \tag{3.39}
\end{equation*}
$$

By using the above expansions, the heat expression can be rewritten as a sum of two components, the first one related to the deviations from spherical symmetry due to the shellular rotation, and the second one associated with the horizontal variations of the molecular weight:

$$
\begin{equation*}
\nabla \cdot(\chi \nabla T)+\rho \varepsilon=\bar{\rho}+\frac{L}{M}\left(E_{\Omega}+E_{\mu}\right) P_{2}(\cos \theta) \tag{3.40}
\end{equation*}
$$

where

$$
\begin{align*}
E_{\Omega}= & 2\left[1-\frac{\Omega^{2}}{2 \pi G \rho}-\frac{\varepsilon}{\varepsilon_{\mathrm{m}}}\right] \frac{\widetilde{g}}{g}-  \tag{3.41}\\
& \frac{\rho_{\mathrm{m}}}{\rho}\left[\frac{r}{3} \frac{d}{d r}\left(H_{\mathrm{T}} \frac{d \Theta}{d r}-\chi_{\mathrm{T}} \Theta\right)-2 \frac{H_{\mathrm{T}}}{r} \Theta+\frac{2}{3} \Theta\right]- \\
& \frac{\varepsilon}{\varepsilon_{\mathrm{m}}}\left[H_{\mathrm{T}} \frac{d \Theta}{d r}+\left(\varepsilon_{\mathrm{T}}+\chi_{\mathrm{T}}-1\right) \Theta\right]-\Theta, \\
E_{\mu}= & \frac{\rho_{\mathrm{m}}}{\rho}\left[\frac{r}{3} \frac{d}{d r}\left(H_{\mathrm{T}} \frac{d \Lambda}{d r}-\left(\chi_{\mu}+\chi_{\mathrm{T}}+1\right) \Lambda\right)-2 \frac{H_{\mathrm{T}}}{r} \Lambda\right]-  \tag{3.42}\\
& \frac{\varepsilon}{\varepsilon_{\mathrm{m}}}\left[H_{\mathrm{T}} \frac{d \Lambda}{d r}+\left(\varepsilon_{\mu}+\varepsilon_{\mathrm{T}}-\chi_{\mu}-\chi_{\mathrm{T}}-1\right) \Lambda\right] .
\end{align*}
$$

In these expressions, $H_{\mathrm{T}}=|d r / d l n \bar{T}|$ is the temperature scale height, $\varepsilon$ the nuclear energy generation rate, $\rho_{\mathrm{m}}$ the mean density inside of the considered level surface, and the quantities $\Theta, \Lambda, \varepsilon_{\mathrm{m}}$ are respectively given by

$$
\begin{align*}
& \Theta(r)=\frac{\widetilde{\rho}}{\bar{\rho}}=\frac{1}{3} \frac{r^{2}}{\bar{g}} \frac{d \Omega^{2}}{d r}  \tag{3.43}\\
& \Lambda(r)=\frac{\widetilde{\mu}}{\bar{\mu}}=\frac{\widetilde{T}}{\bar{T}}+\Theta(r),  \tag{3.44}\\
& \varepsilon_{\mathrm{m}}(r)=\frac{L(r)}{M(r)} \tag{3.45}
\end{align*}
$$

while the logarithmic derivatives $\chi_{\mathrm{T}}, \chi_{\mu}, \varepsilon_{\mathrm{T}}$ and $\varepsilon_{\mu}$ are defined as

$$
\begin{array}{ll}
\chi_{\mathrm{T}}=\left(\frac{\partial \ln \chi}{\partial \ln T}\right)_{P, \mu}, & \chi_{\mu}=\left(\frac{\partial \ln \chi}{\partial \ln \mu}\right)_{P, T} \\
\varepsilon_{\mathrm{T}}=\left(\frac{\partial \ln \varepsilon}{\partial \ln T}\right)_{P, \mu}, & \varepsilon_{\mu}=\left(\frac{\partial \ln \varepsilon}{\partial \ln \mu}\right)_{P, T} \tag{3.47}
\end{array}
$$

The quantities $\Theta$ and $\Lambda$ are baroclinic functions that give a measure of the variations of respectively the density and the mean molecular weight over an equipotential surface. While $\Theta$ is directly computed from the angular velocity gradient, $\Lambda$ is obtained through the horizontal variation of the concentration $\widetilde{c_{i}}$ of each chemical constituent,

$$
\begin{equation*}
\widetilde{c_{i}}=-\frac{r^{2} U}{6 D_{h}} \frac{\partial c_{i}}{\partial r} \tag{3.48}
\end{equation*}
$$

(Zahn 1992; Talon et al. 1997). In our case, we have considered the horizontal variations of just hydrogen and helium, as we are dealing mainly with low-mass stars. Then, by using the classical approximation for the mean molecular weight of ionized matter (e.g. Cox \& Giuli 1968),

$$
\begin{equation*}
\mu \simeq \frac{2}{3 X+Y / 2+1} \tag{3.49}
\end{equation*}
$$

it is a simple matter to obtain

$$
\begin{equation*}
\frac{\widetilde{\mu}}{\bar{\mu}}=-\bar{\mu}\left(\frac{3 \widetilde{X}}{2}+\frac{\widetilde{Y}}{4}\right) . \tag{3.50}
\end{equation*}
$$

Furthermore, in the calculation of the quantity

$$
\begin{equation*}
\frac{\widetilde{g}}{\bar{g}}=\frac{1}{3} \Omega^{2} \frac{d}{d r}\left(\frac{r^{2}}{g_{0}}\right)-\frac{d}{d r}\left(\frac{\widehat{\Phi}}{g_{0}}\right) \tag{3.51}
\end{equation*}
$$

where $\widehat{\Phi}$ is the fluctuating part of the gravitational potential and $g_{0}=G M / r^{2}$, we take advantage of the fact that $\widehat{\Phi}$ is already obtained as part of the Clairaut-Legendre expansion of $\Phi$ used in calculating the Kippenhahn-Thomas $f_{p}$ and $f_{t}$ factors (see Eq. 3.15). In this way we do not need to calculate it by means of the second-order ordinary differential equation presented by Zahn (1992).

The amplitude of the radial component of the meridional circulation velocity is then finally given by the expression

$$
\begin{equation*}
U(r)=\frac{L}{M g}\left(\frac{P}{C_{P} \rho T}\right) \frac{1}{\nabla_{\mathrm{ad}}-\nabla}\left(E_{\Omega}+E_{\mu}\right) . \tag{3.52}
\end{equation*}
$$

Further improvements in the above framework have been pursued by some authors. Urpin et al. (1996) made a minor correction in the expressions for $E_{\Omega}$ and $E_{\mu}$, but their work is not well suited for our purposes since they neglect all terms related to the energy generation rate, which are important in the case of low mass stars as shown below. Talon et al. (1997) also presented modified expressions for $E_{\Omega}$ and $E_{\mu}$ in which the role of the horizontal variations of the chemical species $\widetilde{c_{i}}$ was made clearer. And very recently Maeder \& Zahn (1998) also brought some enhancements to Zahn's (1992) original framework by considering a general equation of state and the effect of a $\mu$-gradient in the resulting expression for the meridional circulation.

All these improvements, however, do not represent any substantial modification in Zahn's original expressions. So, as we had already been working with those expressions when the works of Talon et al. and Maeder \& Zahn came to our attention, we preferred to continue with it, deferring these improvements to a later work.

Returning back to Zahn's original expressions for $E_{\Omega}$ and $E_{\mu}$, we notice that they were again derived by considering a stationary situation such as the main sequence, where the star does not undergo any significant expansion or contraction. Once more, as noted before for the case of the advection-diffusion PDE equation, these expressions must be corrected for the case of non-stationarity such as the pre-main sequence phase. The corrections we consider in this work are the following:

1. to explicitly consider the time derivative of the entropy in Eq. (3.36), which cannot be neglected in contracting or expanding evolutionary phases. In lagragian coordinates we have

$$
\begin{equation*}
T \frac{d S}{d t}=-\varepsilon_{\mathrm{G}} \tag{3.53}
\end{equation*}
$$

(Kippenhahn \& Weigert 1994) where $\varepsilon_{\mathrm{G}}$ is the gravitational energy release rate. For practical purposes, the effect of including this time derivative amounts to substituting all terms in $\varepsilon / \varepsilon_{m}$ by $\left(\varepsilon+\varepsilon_{\mathrm{G}}\right) / \varepsilon_{m}$;
2. to abandon the simplification made by Zahn of the term $L / M$, which originally should be read $L / M_{\star}$ where $M_{\star}$ is the "reduced mass" given by $M\left(1-\Omega^{2} / 2 \pi G \rho_{m}\right)$. The reason is that the original work of Zahn was intended for high mass stars, which have extended radiative envelopes where the term $\Omega^{2} / 2 \pi G \rho_{m}$ is always very small, and so justifying that simplification. However, low-mass stars develop a radiative core, in which the values of $\bar{\rho}$ and $\rho_{\mathrm{m}}$ are nearly equal especially at the center. So, for low-mass stars, in the first term of the expression for $E_{\Omega}$ we cannot use the same simplification as done by Zahn.

### 3.4.4.3 Approximations used for computing $\mathrm{U}(\mathbf{r})$

In order to properly calculate the terms $E_{\Omega}$ and $E_{\mu}$ by the ATON 2.1 code, some approximations have also been made, and are worth discussing. First, the ideal gas approximation is used. Second, and most important, it would be very difficult for the aton code to directly compute the logarithmic derivatives $\chi_{\mu}$ and $\varepsilon_{\mu}$, as the code works with fixed values of the chemical species previously input by the user as the $X$ and $Z$ relative abundances. To do that would require substantial modifications in the code. For this reason, those derivatives are analytically obtained by considering the relationship between the thermodynamical quantities $P, T, \mu$ and $\rho$ through the equation of state for an ideal gas,

$$
\begin{equation*}
P=\frac{\mathcal{R}}{\mu} \rho T . \tag{3.54}
\end{equation*}
$$

In this way, $\partial \chi / \partial \mu$ is computed as follows:

$$
\begin{equation*}
\frac{\partial \chi}{\partial \mu}=\frac{\partial}{\partial \mu}\left(\frac{16 \sigma}{3} \frac{T^{3}}{\kappa \rho}\right)=\frac{16 \sigma}{3}\left(\frac{3 T^{2}}{\kappa \rho} \frac{\partial T}{\partial \mu}-\frac{T^{3}}{\kappa^{2} \rho} \frac{\partial \kappa}{\partial \mu}-\frac{T^{3}}{\kappa \rho^{2}} \frac{\partial \rho}{\partial \mu}\right) \tag{3.55}
\end{equation*}
$$

In the above expression $\partial T / \partial \mu$ and $\partial \rho / \partial \mu$ are obtained through the ideal gas approximation, while $\partial \kappa / \partial \mu$ is again expanded by considering $\kappa=\kappa(T, P, \rho)$ as indeed it is numerically computed in the code (for a previously chosen chemical composition). We then get

$$
\begin{equation*}
\frac{\partial \kappa}{\partial \mu}=\frac{\partial \kappa}{\partial T} \frac{\partial T}{\partial \mu}+\frac{\partial \kappa}{\partial P} \frac{\partial P}{\partial \mu}+\frac{\partial \kappa}{\partial \rho} \frac{\partial \rho}{\partial \mu} \tag{3.56}
\end{equation*}
$$

The same reasoning applies in computing $\partial \varepsilon / \partial \mu$ : here it suffices to consider $\mu=\mu(T, P, \rho)$ through the ideal gas approximation, which leads to

$$
\begin{equation*}
\frac{\partial \varepsilon}{\partial \mu}=\frac{\partial \varepsilon}{\partial T} \frac{\partial T}{\partial \mu}+\frac{\partial \varepsilon}{\partial P} \frac{\partial P}{\partial \mu}+\frac{\partial \varepsilon}{\partial \rho} \frac{\partial \rho}{\partial \mu} . \tag{3.57}
\end{equation*}
$$

In computing Eqs. (3.56) and (3.57) above, we notice that all intermediate derivatives were already available in the ATON 2.0 code, with exception of $\partial \kappa / \partial \rho$ and $\partial \varepsilon / \partial \rho$ which were easily introduced in the 2.1 version.

To close this section, we discuss now an important numerical issue related to the calculation of $U(r)$. Equation (3.42) requires the calculation of the third derivative of the angular velocity (through the quantity $\Theta$ ). As it is well known, numerical differentiation almost always gives poor results regardless of the method used unless one has data that can be represented by well behaved functions. In our case this situation is even worse because all physical quantities of interest have grids irregularly spaced.

In this sense the quantity $\Theta$ poses a problem, since it is obtained as the product of the radius squared and the derivative $d \Omega / d r$. Moreover, the term in $\rho_{\mathrm{m}} / \rho$ in Eq. (3.42) involves the product of the radius and the 2nd-derivative of $\Theta$. Since the typical scale of stellar radii (for solar-type stars) is $10^{10} \mathrm{~cm}$, any small numerical fluctuation in the computed 1st-, 2nd- and 3rd-order derivatives will be greatly amplified, thus compromising the numerical accuracy of the quantity $E_{\Omega}$.

The available literature on numerical methods for computing derivatives focuses on regularly spaced grids. (We could of course transform the irregulary spaced grid in an equally spaced one by means of interpolation, do the calculations and then convert back to the irregularly spaced one, but this procedure would also introduce numerical errors). To minimize the errors due to the inacurracy of numerical derivatives computed on irregulary spaced grids, we adopted the standard approach of smoothing the original data before computing any derivatives (see e.g. Hildebrand 1956). For practical purposes, we first use a moving median average to smooth the angular velocity, in order to remove major irregularities; the smoothed angular velocity profile is then used to compute the quantity $\Theta$, which by its turn is smoothed by a weighted least-squares smoothing spline. The same procedure is applied when calculating the quantity $E_{\mu}$.

### 3.4.4.4 Results for $\mathrm{U}(\mathrm{r})$

We have found that the meridional circulation velocity $\mathrm{U}(\mathrm{r})$ obtained with the method and approximations discussed in the previous subsection exhibits a very complex behaviour, that most of the time (along the star's evolution from the pre-main sequence to the main sequence) is not compatible with what one would expect from both theory (e.g. Tassoul
1978) and numerical results obtained by other authors (Urpin et al. 1996; Talon et al. 1997; Denissenkov et al. 1999).

Figure 3.1 shows $\mathrm{U}(\mathrm{r})$ along the radiative interior of a $1 M_{\odot}$ model at the end of the premain sequence phase (age $=10^{7} \mathrm{yr}$ ). We see that it has a sharp drop at the origin, and remains negative during most of the radiative interior. Such a negative velocity would imply that the circulation currents would rise at the equator and sink at the poles, which is clearly unphysical since it would enhance the thermal imbalance in radiative regions of rotating stars instead of suppressing it. In the same figure, the change from negative to positive values in the upper region of the radiative zone would suggest the existence of a double-cell circulation, a result that as a matter of fact is also obtained by many authors; but, again, the velocity direction of the upper cell also gets inverted in our calculations.


Figure 3.1: Meridional circulation velocity (computed through Eq. [3.52]) as a function of the radiative core radius, for a $1 M_{\odot}$ model at age $10^{7} \mathrm{yr}$.

For this particular case, the reasons of this direction inversion can be easily traced back, as can the sharp drop at the center. The former is due to the fact that the ratio $\varepsilon / \varepsilon_{\mathrm{m}}$ is almost unity at the center of the nuclear burning core of stars in both pre-main sequence and main sequence stages. To better visualize this, let us rewrite the contribution $\left(E_{\Omega}+E_{\mu}\right)$ to the meridional circulation velocity as the sum

$$
\begin{equation*}
E_{\Omega}+E_{\mu}=E_{\mathrm{rb}}+E^{\prime} \tag{3.58}
\end{equation*}
$$

with $E_{\mathrm{rb}}$ given by

$$
\begin{equation*}
E_{\mathrm{rb}}=2\left[1-\frac{\Omega^{2}}{2 \pi G \rho}-\frac{\varepsilon}{\varepsilon_{\mathrm{m}}}\right] \frac{\tilde{g}}{g} . \tag{3.59}
\end{equation*}
$$

$E_{\mathrm{rb}}$ is the term that corresponds to rigid body rotation, while $E^{\prime}$ represents the contribution to $\mathrm{U}(\mathrm{r})$ by the baroclinic effects induced by the assumed shellular rotation law $\omega=\omega(r)$. As can be seen from Fig. 3.2, the closeness to unity of $\varepsilon / \varepsilon_{\mathrm{m}}$ makes $E_{\mathrm{rb}}$ smaller than the absolute value of $E^{\prime}$ for most of the radiative core, resulting in the observed velocity inversion. The sharp drop of $U(r)$ near the center is due to the presence of the temperature scale height $H_{\mathrm{T}}$ in both expressions for $E_{\Omega}$ and $E_{\mu}$, as it formally diverges at the star's center. As soon as the star begins its journey onto the main sequence,


Figure 3.2: Quantities $E^{\prime}$ and $E_{\mathrm{rb}}$ (see text) for the same model as in Fig. (3.1), as a function of the radiative core radius.
the situation gets much more complicated, as depicted in Fig. 3.3 for the same model at the age of $3.16 \times 10^{7}$ yr. Now the quantity $E_{\mathrm{rb}}$ becomes larger than $E^{\prime}$ in the deep core, resulting in a intermediate region with negative $U(r)$ surrounded by positive values of $U(r)$ in the deep core and at the outer radiative layers. In this case, the reasons of these inversions cannot be traced back as easily as in the preceding case.

As a matter of fact, none of these shortcomings are encountered in the case of higher


Figure 3.3: Meridional circulation velocity (computed through Eq. [3.52]) as a function of the radiative core radius, for a $1 M_{\odot}$ model at age of $3.16 \times 10^{7} \mathrm{yr}$.
mass stars, as they usually have a convective core surrounded by a radiative envelope, so that $\varepsilon / \varepsilon_{\mathrm{m}}$ starts at a value less than 1 even in the deeper radiative layers; for the same reason, the radiative envelopes of these stars will always be characterized by the smooth part of the temperature scale height. In addition, in spite of being quite general, the framework developed by Zahn (1992) regarding the meridional circulation was never tested with low-mass stars until now. So, the problems just described suggest that Zahn's framework should be re-examined in order to check its adequacy for low-mass stars. It is worth mentioning that although the ATON 2.1 code was carefully checked for coding errors, they cannot be discarded as a possible reason for the observed discrepancies.

We feel it is appropriate to stress that one of the main goals of the present work is to study the effects of internal angular momentum transport in the evolution of low-mass stars and, as so, we are mainly interested in obtaining estimates of the characteristic velocities of the various instabilities considered, including meridional circulation. So, in view of the obtained results, we decided to approximate the meridional velocity circulation by the local value it would have for the case of rigid body rotation, which is given by the quantity $E_{\mathrm{rb}}$. Similar estimates for $\mathrm{U}(\mathrm{r})$ have already been used in other works when computing internal angular momentum redistribution, such as those by the Yale group (Endal \& Sofia 1978; Pinsonneault et al. 1989) and others (Fliegner et al. 1996; Langer et al. 1997, 1999). Regarding that approximation, it remains to compute the effects of the nonspherical distribution of molecular weight generated by the circulation (as discussed in Sect. 2.2.3.2), which in Zahn's framework are represented by the quantity $E_{\mu}$. We adopt
the estimate of Kippenhahn (1974) for the " $\mu$-currents" (see Eq. 2.40), substituting $\tau_{\text {KH }}^{*}$ for the local Kelvin-Helmholtz timescale $\tau_{\mathrm{KH}}=G M / r^{2} L$ as in Pinsonneault et al. (1989) so that, in the ideal gas approximation,

$$
\begin{equation*}
v_{\mu}=\frac{H_{\mathrm{P}}}{\left(\nabla_{\mathrm{ad}}-\nabla\right) \tau_{\mathrm{KH}}} \frac{|\Delta \mu|}{\mu} . \tag{3.60}
\end{equation*}
$$

The computed approximation for $\mathrm{U}(\mathrm{r})$ is shown in Fig. 3.4, for the same ages previously discussed. We note that, except for the change in sign, the approximation used and that obtained with the inclusion of the baroclinic terms $E_{\Omega}$ and $E_{\mu}$ are of the same magnitude.


Figure 3.4: Meridional circulation velocity (computed by using the local rigid body approximation) as a function of the radiative core radius, for a $1 M_{\odot}$ model at ages $10^{7} \mathrm{yr}$ and $3.16 \times 10^{7} \mathrm{yr}$.

The peaks in $U(r)$ at both the beginning and end of the radiative region are due to the superadiabatic gradient $\left(\nabla_{\mathrm{ad}}-\nabla\right)$, which is formally zero at convective-radiative interfaces (note that the ZAMS model also has a small convective core already developed), as depicted in Fig. 3.5.

### 3.4.4.5 Calculation of diffusion coefficients

The diffusion coefficients are computed as the product of the velocity and the characteristic path length of the redistribution currents associated with each instability (Endal \& Sofia


Figure 3.5: Superadiabatic gradient $\nabla_{\mathrm{ad}}-\nabla$ as a function of the radiative core radius, for the same models as in Fig. 3.4.
1978). The characteristic length is taken as the minimum value between the local value of the radius and the velocity scale height,

$$
\begin{equation*}
l=\min \left(r,\left|\frac{\partial \ln v}{\partial r}\right|^{-1}\right) \tag{3.61}
\end{equation*}
$$

According to Endal \& Sofia (1978), the velocity scale height $|\partial \ln v / \partial r|^{-1}$ is a more accurate representation of the characteristic length $l$ when the extent of the unstable region is significantly smaller than $r$.

Equation (3.61) is numerically computed as

$$
\begin{equation*}
l_{i}=\min \left(r_{i},\left|\frac{d \ln v}{d r}\right|_{i}^{-1}\right) \tag{3.62}
\end{equation*}
$$

with $(d \ln v / d r)_{i}$ computed from $\ln v_{i+1}$ and $\ln v_{i-1}$.
The final diffusion coefficient to be used in Eq. (3.35) is taken as the sum of the individual diffusion coefficients computed so far.

### 3.4.4.6 Method used for solving the advection-diffusion equation

To compute the internal transport of angular momentum at each physical time step we must solve the advection-diffusion partial differential given by Eq. (3.35). To do so, that equation is first converted to a finite difference equation and then solved by a fully implicit scheme. These steps are explained with details in the appendix A. The equation is solved from the center to the surface of the star. Each convective region is treated as if all its mass was concentrated in a single mass shell with a zero total diffusion coefficient. Also, as a first approximation, the thin radiative region that usually appears at the surface of model stars with masses around $1 M_{\odot}$ was neglected for purposes of calculating the meridional circulation velocity and its derived diffusion coefficients; we expect to address this issue in the near future.

The boundaries between radiative and convective regions must be modelled with great care, as the overadiabacity $\left(\nabla_{\mathrm{ad}}-\nabla\right)$ formally approaches zero there. To avoid numerical problems at those boundaries, the ATON 2.0 code automatically puts a larger number of mesh points through it. Some attempts at modeling these boundaries for purposes of computing diffusion coefficients have already been made by some authors. For example, Pinsonneault (1988) modeled those boundaries by a transition region with the fixed extent of 0.05 pressure scale heights.

In our case, for the sake of simplicity we prefer to use another approach. We first look for mesh points at those boundaries for which the superadiabatic gradient $\left(\nabla_{a d}-\nabla\right)$ is lower than a pre-specified value ( $1 \times 10^{-2}$ in our computations). For those points (if found) we assign the same value of the meridional circulation velocity that was calculated for the previous point (or next one, depending on the case). In this way we avoid the formal discontinuities that may occur at those boundaries.

It must be kept in mind that a transition region actually develops at the radiativeconvective interface, due to still poorly understood phenomena such as overshooting and semi-convection. Besides, as first pointed out by Pinsonneault (1988), there is no compelling physical argument for any assumed behaviour for the velocities within such transition layers.

### 3.4.4.7 Angular momentum loss at the surface

It is well known from observations of young clusters that low mass stars lose angular momentum as they evolve (Kraft 1970; Soderblom 1983). The loss of angular momentum is generally believed to occur at the star's surface layers through the action of a magnetized wind (Schatzman 1962), and since it affects the star's rotational evolution, it must be included in rotating models. We adopted the prescription used in Chaboyer et al. (1995b),
which reads

$$
\begin{array}{ll}
\frac{\partial J}{\partial t}=K_{\mathrm{w}}\left(\frac{R}{R_{\odot}}\right)^{2-n}\left(\frac{M}{M_{\odot}}\right)^{-n / 3}\left(\frac{\dot{M}}{10^{-14}}\right)^{1-2 n / 3} \Omega^{1+4 n / 3}, & \Omega<\Omega_{\text {crit }}, \\
\frac{\partial J}{\partial t}=K_{\mathrm{w}}\left(\frac{R}{R_{\odot}}\right)^{2-n}\left(\frac{M}{M_{\odot}}\right)^{-n / 3}\left(\frac{\dot{M}}{10^{-14}}\right)^{1-2 n / 3} \Omega \Omega_{\text {crit }}^{4 n / 3}, & \Omega \geq \Omega_{\text {crit }}, \tag{3.64}
\end{array}
$$

where $\Omega_{\text {crit }}$ introduces a critical rotation level at which the angular momentum loss saturates, and $\dot{M}$ is the mass loss rate in units of $10^{-14} M_{\odot} \mathrm{yr}^{-1}$. For a "wind index" $n=1.5$, these relations reproduce well the Skumanich (1972) law $v \propto t^{-1 / 2}$.

The constant $K_{w}$ in Eqs. (3.63) and (3.64) is usually calibrated by requiring that the model surface velocity matches the current solar rotation rate at the equator. Since this calibration is rather uncertain as it depends on the assumed initial rotational velocity for the Sun, we postpone it for a later work and, as a first approximation, adopt the value of $K_{w}$ given by Krishnamurthi et al. (1997) for their models with differential rotation, namely $K_{w}=1.019 \times 10^{48}$.

The dependence on mass loss is very weak (Chaboyer et al. 1995a); in addition, for T Tauri stars the mass loss and mass accretion rates are correlated, with the ratio $\dot{M}_{\text {loss }} / \dot{M}_{\text {acc }}$ being of the order $10^{-2}$ (Hartigan et al. 1995). Since these are competing mechanisms for angular momentum gain and loss, and since the duration of mass accretion phase is highly uncertain, we prefer to use a conservative value and set $\dot{M}_{\text {loss }}$ fixed at $2 \times 10^{-14}$ $M_{\odot} \mathrm{yr}^{-1}$ for all calculations, the same value used by Chaboyer et al. (1995a).

The angular momentum loss computed at each diffusion timestep enters the advectiondiffusion equation as a boundary condition at the star's surface.

### 3.4.5 Overview of the code implementation

Figure 3.6 shows a flowchart with the major steps used for computing rotating stellar models with internal redistribution of angular momentum. Since all steps were already described in Sects. 3.2.1 and 3.2.5 except the internal transport of angular momentum, we will discuss here only the details of the latter.

The internal transport of angular momentum is computed through the following steps:

1. On entering the calculations, we have the new values of the structural quantities (radius, pressure, temperature, etc.) and the old values of the angular velocity.
2. The previous physical time step is divided in the same number of diffusion time steps used to compute the chemical evolution and microscopic diffusion (usually 10 diffusion time steps).
3. At each diffusion time step, the values of the structural quantities are obtained by linear interpolation between their respective values prior to and after the relaxation procedure.
4. The entire radiative region is then checked for the presence of any of the dynamical and secular instabilities considered; whenever such instabilities are detected, their respective diffusion coefficients are calculated. The meridional circulation velocity is also calculated at each point of that region, since it drives the advective term of the advection-diffusion partial differential equation (PDE) used to compute the internal transport of angular momentum (see Sect. 3.4.4.1).
5. At each mesh point, the computed diffusion coefficients (if any) corresponding to each instability are summed up to form the total diffusion coefficient of that mesh point, and the advection-diffusion PDE is then solved for the entire radiative region, thus obtaining the new angular velocity profile at the end of the current diffusion time step.


Figure 3.6: Simplified flowchart showing the main steps performed by the ATON 2.1 stellar evolution code.

The most exciting phrase to hear in science, the one that heralds new discoveries, is not "Eureka!" (I found it!) but "Hmm... That's funny ..."

Isaac Asimov

## Chapter 4

## APPLICATIONS OF ROTATING STELLAR MODELS

### 4.1 Evolutionary differences between rotating and non-rotating stellar models

Since the end of the 60 's it has been known that rotation lowers the stellar luminosity, and Sackmann (1970) demonstrated that a uniformly rotating star with mass $M$ has central properties similar to those of a non-rotating star with mass $M-\Delta M$. This effect, generally referred to in the literature as the mass-lowering effect, can be easily understood if we consider that the basic structural effect of rotation is to generate a centrifugal force that counteracts the gravitational force. The net effect is that a rotating star with a given mass will experience a lower gravitational pulldown, exactly as if it had a lower mass.

The results of our rotating models confirm this general property, as can be seen from Fig. 4.1 which shows the evolutionary paths of a $1 M_{\odot}$ model star with and without uniform rotation in the H-R diagram. The rotating model has a lower effective temperature than its non-rotating counterpart, mimicking a lower mass, non-rotating model. From that figure we also note that the effects due to rotation are very small, being appreciable only for high values of the initial angular momentum. It is worth noting that the qualitative results obtained are the same for all three rotation laws used, as depicted in Fig. 4.2 which shows the individual effects of rotation for the adopted rotation laws. These results are in full agreement with previous work from other authors, since, as stated by Tassoul (1978), the structural effects of rotation do not depend on the chosen rotation law when the initial angular momentum is kept fixed.


Figure 4.1: Evolutionary paths for a $1 M_{\odot}$ model star without rotation (solid line) and with uniform rotation (dashed line) generated by the ATON 2.1 code. The rotating model was assigned an initial angular momentum of $1.566 \times 10^{50} \mathrm{~g} \mathrm{~cm}^{2} \mathrm{~s}^{-1}$.


Figure 4.2: Detail of the evolutionary tracks for rotating models of $1 M_{\odot}$ under different initial angular momentum and different rotation laws, for the MLT convection model. $R B$ stands for rigid body rotation; $D R$, local conservation of angular momentum throughout the whole star; and $D R+R B$, local conservation of angular momentum in radiative regions and rigid body rotation in convective ones. (From Mendes et al. 1999.)

### 4.2 Lithium depletion in rotating low-mass stars

In recent years a great effort has been conducted by many researchers toward a better understanding of the depletion of light elements such as beryllium and lithium in young, low mass stars, as the formers can be used as tracers of the stellar evolution; however, the current picture is still subject to many uncertainties. Lithium depletion is related to a number of physical parameters such as stellar mass (Cayrel et al. 1984), metallicity (Hobbs \& Duncan 1987) and rotation (Marcy et al. 1985; Soderblom 1983; Martín et al. 1994). García López et al. (1994) have found that the observed spread in lithium abundances of the Pleiades low-mass stars is related to rotation in such a way that rapid rotators have less lithium depletion than slow rotators. Evidence for such a rotationlithium depletion relation has also been found by Cunha et al. (1995), who have studied the lithium abundance over a sample of late F to early $G$ young stars in the Orion association (however, recent results by King et al. [1999] show that slow rotators in the Pleiades have abundances similar to the fast rotators, casting doubt on such a correlation at least in the Pleiades). Binaries in the Hyades present a lower scatter in abundances than single stars in the same cluster, and close binaries show conspicuous overabundances (Barrado y Navascués \& Stauffer 1996).

On the theoretical side, the standard stellar models used so far are unable to explain all observations (see Pinsonneault 1994 for a review on this subject). Of all non-standard physical processes that could significantly influence the lithium depletion, rotation is undoubtedly the most quoted, since it is a natural agent for chemical mixing inside the stars (e.g. Charbonnel \& Vauclair 1991; Pinsonneault 1991; Zahn 1992, 1993, 1994; Charbonnel et al. 1994; Strom 1994).

Nevertheless, so far very few attempts have been made to include rotation in evolutionary codes, and even less to check the effects of rotation on lithium depletion. The Yale group (Pinsonneault et al. 1990) has found that rotation increases the lithium depletion, for both models with only rigid body rotation and those with differential rotation and internal angular momentum redistribution, as shown in Fig. 4.3. On the other hand, Martín \& Claret (1996) found that their rigid-body rotating models deplete less lithium than the non-rotating ones, in accordance with the observational picture. Their results can be seen in Fig. 4.4. Martín \& Claret attributed the differences between their results and those of Pinsonneault et al. (1990) mainly to the adopted opacities, as the latter authors used the older Cox \& Stewart (1970) libraries. Motivated by these conflicting results, we ran a series of stellar models of 0.6 to 1.2 solar masses in order to check the effects of rotation on lithium depletion. Our results show that rotation indeed increases the lithium depletion in low mass, pre-main sequence stars, as shown in Fig. 4.5 for the particular case of a $0.8 M_{\odot}$ model star with the same initial angular momentum and chemical composition as in Martín \& Claret's paper. Our results can be easily understood in the light of the early mentioned mass-lowering effect. As shown by Bodenheimer (1965) and D'Antona \& Mazzitelli (1984), the temperature at the base of the convective region gets higher as we


Figure 4.3: Surface lithium abundance according to the rotating models of Pinsonneault et al. (1990). The solid lines refer to the non-rotating models, and the dashed ones to rotating models with no internal angular momentum redistribution. (From Pinsonneault et al. 1990.)
go toward lower masses. So, as a rotating star mimics a lower mass, non-rotating one, it burns the light elements such as lithium and beryllium faster than the non-rotating case.

In this way, the qualitative results from the ATON 2.1 code regarding the lithium depletion in low mass, pre-main sequence stars are very similar to those of the Yale group. These results were obtained with very updated opacities (as described in Subsect. 3.2.2.2), and so the discrepancies between Martín \& Claret's models and both ours and the Yale group's models cannot be attributed to the adopted opacities. At the moment we cannot explain what might have produced Martín \& Claret's results.

A full account of the results obtained with the ATON 2.1 code concerning lithium depletion in low mass, pre-main sequence stars can be found in the published papers included in Appendices B and C.


Figure 4.4: Surface lithium abundances as a function of age for a $0.8 M_{\odot}$, model star, according to the evolutionary models of Martín \& Claret (1994). The solid line stands for a non-rotating model, the dotted one a rotating model with $\omega_{i}=2 \times 10^{-6}$, and the dashed one a rotating model with $\omega_{i}=2.5 \times 10^{-6}$. All rotating models were obtained for uniform rotation and no internal angular momentum redistribution. (From Martín \& Claret 1994).


Figure 4.5: Surface lithium abundance (in the scale $N[\mathrm{Li}]=12+\log \frac{N[L i]}{N[H]}$ ), as a function of age, for a $0.8 M_{\odot}$ star, obtained with the ATON 2.1 code. The solid line represents the non-rotating case, while the dashed line indicates the case of uniform rotation for the same initial angular momentum and chemical composition as the equivalent model of Martín \& Claret (1996). In spite of using a different (but equivalent) scale for the lithium abundance, our results show that lithium depletion is increased in rotating models.

### 4.3 Effects of the internal redistribution of angular momentum

In order to check the effects of angular momentum redistribution on the evolution of low-mass stars through our particular implementation of the techniques described in the previous chapter, we have run several low-mass stellar models of $0.6,0.8$ and $1 M_{\odot}$, exploring different values of certain input data (listed in table 4.1) in order to compare the obtained results and to study their sensivity to those input values.

As explained in Sect. 2.2.2.2, the main interest in computing models of rotating low-mass stars is related to the rotational mixing due to the internal angular momentum transport and its influence on the lithium contents of these stars. For the particular case of the Sun, accurate data obtained from helioseismology give us detailed information of its present angular velocity profile, up to approximately $r / R_{\odot}=0.4$ in depth. So, for the $1 M_{\odot}$ model we also have the unique opportunity of evolving it up to the present solar age and to verify to what extent the theoretical results match the solar observations. This, of course, can only be done with the Sun since no other star can have its oscillations resolved as accurately with currently available technology, despite the progresses in asteroseismology (see e.g. Goupil et al. 1996).

For this reason, most of our analysis will be centered on the $1 M_{\odot}$ model in the present work. We first begin discussing some aspects related to the techniques used, and then proceed to the comparison with observational data.

Table 4.1 summarizes the input data used for each computed model. In that table column 1 indicates the model identification for further references; column 2 gives the model mass; column 3 gives the initial angular momentum (recalling that these values are taken from Kawaler's [1987] mass-momentum angular relations, as explained in Sect. 3.3.3.4); column 4 specifies the convection model (Mixing Length Theory [MLT] or the Full Spectrum of Turbulence [FST]); columns 5 and 6 indicate the helium and metals concentrations, respectively; column 7 gives the multiplicative factor of the total diffusion coefficient obtained from the hydrodynamical instabilities (to be explained in Sect. 4.3.5 below); column 8 specifies the "wind index" adopted for the surface angular momentum loss; and column 9 indicates if the advection term was included or not in the PDE that controls the angular momentum redistribution (see Sect. 4.3.4 below). Besides the input values listed in Table 4.1, other relevant input data adopted for all computed models were the following:

1. OPAL (Iglesias \& Rogers 1993) opacities supplemented by those of Alexander \& Ferguson (1994) for lower ( $T<6000 \mathrm{~K}$ ) temperatures;
2. $\alpha=1.5$ for the models using the MLT convection model, and $\beta=0.15 H_{\mathrm{P}}$ (see Sect. 3.2.4) for those using the FST treatment.

Table 4.1: Input parameters for the computed models

| Model <br> ID | Mass $\left(M_{\odot}\right)$ | Initial J <br> $\left(\mathrm{g} \mathrm{cm}^{2} \mathrm{~s}^{-1}\right)$ | Convection <br> Model | Y | Z | $K_{\mathrm{d}}$ | Wind <br> index | J advection |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| OA | 1.0 | $J_{0}=1.566 \times 10^{50}$ | MLT | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
| OB | 1.0 | $J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
| OC | 1.0 | $J_{0}$ | MLT | 0.271 | 0.0175 | 10.0 | 1.5 | Yes |
| OD | 1.0 | $J_{0}$ | FST | 0.271 | 0.0175 | 10.0 | 1.5 | Yes |
| OE | 1.0 | $J_{0}$ | MLT | 0.271 | 0.0175 | 1.0 | 1.5 | No |
| OF | 1.0 | $J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 1.5 | No |
| OG | 1.0 | $J_{0}$ | MLT | 0.271 | 0.0175 | 1.0 | 2.0 | Yes |
| OH | 1.0 | $J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 2.0 | Yes |
| OI | 1.0 | $2 J_{0}$ | MLT | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
| OJ | 1.0 | $2 J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
|  |  |  |  |  |  |  |  |  |
| EA | 0.8 | $J_{0}=1.257 \times 10^{50}$ | MLT | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
| EB | 0.8 | $J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
| EC | 0.8 | $J_{0}$ | MLT | 0.271 | 0.0175 | 10.0 | 1.5 | Yes |
| ED | 0.8 | $J_{0}$ | FST | 0.271 | 0.0175 | 10.0 | 1.5 | Yes |
| EE | 0.8 | $J_{0}$ | MLT | 0.271 | 0.0175 | 1.0 | 1.5 | No |
| EF | 0.8 | $J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 1.5 | No |
| EG | 0.8 | $J_{0}$ | MLT | 0.271 | 0.0175 | 1.0 | 2.0 | Yes |
| EH | 0.8 | $J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 2.0 | Yes |
| EI | 0.8 | $2 J_{0}$ | MLT | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
| EJ | 0.8 | $2 J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
|  |  |  |  |  |  |  |  |  |
| SA | 0.6 | $J_{0}=9.470 \times 10^{49}$ | MLT | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
| SB | 0.6 | $J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
| SC | 0.6 | $J_{0}$ | $J_{0}$ | $J_{0}$ | FST | 0.271 | 0.0175 | 10.0 |
| SD | 0.6 | $J_{0}$ | MLT | 0.271 | 0.0175 | 10.0 | 1.5 | Yes |
| SE | 0.6 | $J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
| SF | 0.6 | $J_{0}$ | MLT | 0.271 | 0.0175 | 1.0 | 1.5 | No |
| SG | 0.6 | $2 J_{0}$ | FST | 0.271 | 0.0175 | 1.0 | 2.0 | Yo |
| SH | 0.6 | $2 J_{0}$ | MLT | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |
| SI | 0.6 | FST | 0.271 | 0.0175 | 1.0 | 1.5 | Yes |  |
| SJ | 0.6 |  |  |  |  |  |  |  |

The rotation laws used are the same adopted for studying the structural effects of rotation, namely local conservation of angular momentum in radiative zones and rigid body rotation in convective ones.

### 4.3.1 Evolution of the angular velocity profile

We first start discussing model OA, which from now on will be the "reference" model for most of the following discussion. The effects of the internal angular momentum distribution can be readily seen in Fig. 4.6, which shows the angular velocity profile plotted against the total stellar radius at four representative ages, two of them in the pre-main sequence phase (curves labeled A and B), one very close to the ZAMS (curve C) and the last one corresponding to the current solar age (curve D). These ages are also indicated in the corresponding evolutionary track shown in Fig. 4.7. As expected, during most of the pre-main sequence phase (indicated by curves A and B) the star's contraction is much more effective than the redistribution mechanisms, since the contraction along the Hayashi track occurs on a very short timescale. This leads to a fast rotating radiative core, which will reach its peak velocity near the ZAMS (curve C). As the star enters the main sequence, it remains stable and the effects of the redistribution mechanisms become


Figure 4.6: Angular velocity profile as a function of radius for the reference model OA, at four representative stellar ages.


Figure 4.7: Evolutionary track for the $1 M_{\odot}$ reference model. The points labeled A through D correspond to the same ages as in Fig. 4.6.


Figure 4.8: Angular velocity profile as a function of radius for model OB, which uses the FST convection model. The indicated ages are the same as in Fig. 4.6.
apparent, transferring angular momentum from the core to the external layers (curve D). The constant regions at the beginning and end of the plotted curves correspond to the convective core (if present) and the convective envelope respectively. We note that the initial angular momentum of $J_{0}=1.566 \times 10^{50} \mathrm{~g} \mathrm{~cm}^{2} \mathrm{~s}^{-1}$ adopted for the reference model corresponds to an initial surface velocity of about $3 \mathrm{~km} \mathrm{~s}^{-1}$ at the beginning of the Hayashi track, which seems to be very modest but which will reach $\sim 52 \mathrm{~km} \mathrm{~s}^{-1}$ by the (pre-main sequence) age of $10^{7} \mathrm{yr}-\mathrm{a}$ value typical of the rapid rotators found in young open clusters such as the Pleiades.

The general behaviour exhibited in Fig. 4.6 remains the same regardless of the chosen convection model, as shown in Fig. 4.8 for model OB. This should present no surprise, since we adopt rigid body rotation in convective regions for both MLT and FST convection models. The FST model, however, shows moderately higher rotation rates at the first three indicated ages, due to the fact that, in general, the FST treatment results in radii slightly lower than the corresponding MLT ones, implying lower rotational inertia and consequently higher rotation rates.

Increasing the initial angular momentum of the models has the trivial consequence of producing higher internal rotation rates, as exemplified in Fig. 4.9 for models OB and EB. However, we also note that the model with higher initial angular momentum suffers a stronger smoothing in its angular velocity profile with time; this in fact merely proves the consistency of our implementation, because the hydrodynamical instabilities show up


Figure 4.9: Angular velocity profile as a function of radius for models OB and OJ. The plotted curves indicates that the higher initial angular momentum, the higher the transport of angular momentum.
more frequently with higher rotation rates, leading to an enhanced transfer of angular momentum from the core to the surface.

All features exhibited by the $1 M_{\odot}$ models are also present in those corresponding to 0.8 and $0.6 M_{\odot}$. The angular velocity curves of the latter models also show their steepest profile near the ZAMS, due to the contraction along the Hayashi track. Then, as the models cease their contraction, the redistribution mechanisms attain their maximum efficiency so that at typical main sequence ages the rotation profile gets smoother. Figures 4.10 to 4.13 shows examples of the rotation profiles for those masses.

Angular momentum losses at the star's surface can be very important for the rotational evolution of late type stars. In the pre-main sequence phase these losses have little impact, as the contraction along the Hayashi track has a much shorter timescale than the angular momentum loss by winds (Bouvier et al. 1997). During the main sequence, however, it is believed that these losses are the major mechanism for their observed rotational spindown (see Barnes 1999 for a recent review on this subject). In our models, we find that the angular momentum losses computed through the mechanism described in Sect. 3.4.4.7 are barely noticeable, even for a model with a higher "wind index" of $n=2.0$ (model OG) that, in principle, would experience enhanced angular momentum loss. A possible reason for this inefficiency is the fact that we have used a constant $K_{\mathrm{w}}$ whose calibration was not based upon our own models; a proper calibration of $K_{\mathrm{w}}$ is highly desirable, and should be made as soon as possible.


Figure 4.10: Angular velocity profile as a function of radius for the model EA ( $0.8 M_{\odot}$ ), at four representative stellar ages.


Figure 4.11: Evolutionary track for the model EA $\left(0.8 M_{\odot}\right)$. The points labeled A through D correspond to the same ages as in Fig. 4.10.


Figure 4.12: Angular velocity profile as a function of radius for the model SA ( $0.6 M_{\odot}$ ), at four representative stellar ages.


Figure 4.13: Evolutionary track for the model SA $\left(0.6 M_{\odot}\right)$. The points labeled A through D correspond to the same ages as in Fig. 4.12.

### 4.3.2 Internal diffusion coefficients

The efficiency of the internal angular momentum redistribution is obviously directly related to the computed diffusion coefficients for each hydrodynamical instability considered. The individual diffusion coefficients for those instabilities are plotted in Fig. 4.14 for the $1 M_{\odot}$ reference model, along with the total diffusion coefficient.


Figure 4.14: Diffusion coefficients for the $1 M_{\odot}$ reference model, at the indicated ages in the lower right box. $D_{\mathrm{DYN}}, D_{\mathrm{SEC}}$ and $D_{\mathrm{GSF}}$ stand respectively for the diffusion coefficients obtained for the dynamical shear, secular shear, and Goldreich-Schubert-Fricke (GSF) instabilities, while $D_{\text {TOT }}$ is the total diffusion coefficient.

Some interesting results regarding the relative influence of these instabilities (as indicated by the magnitudes of the corresponding diffusion coefficients) can be drawn from that
figure. First, we note that the dynamical instability is only moderately effective, being more active just in the radiative regions closer to the convective boundaries. This is because the controlling factor in that instability is the superadiabatic gradient $\left(\nabla_{\mathrm{ad}}-\nabla\right)$, which formally diverges at the convective boundaries. As the Richardson number Ri is directly proportional to $\left(\nabla_{\mathrm{ad}}-\nabla\right)$ (see e.g. Eq. 2.18), this means that the dynamical shear tolerates no differential rotation at those boundaries.

Second, the secular shear largely dominates over all other instabilities. The reason is that it depends on the Prandtl number (the ratio of the viscosity to the thermal diffusivity), which is very low in stellar conditions. Similar results had already been found by Pinsonneault (1988). We can also see that the secular shear is ineffective in the star's center during the pre-main sequence phase; this is explained as the result of its sensivity to molecular weight gradients, which of course are very small in the pre-main sequence.

### 4.3.3 Rotational mixing due to angular momentum transport

As explained earlier in Sect. 2.2.2.2, rotational mixing has been very successful in explaining some lithium depletion features observed in low mass stars of young open clusters. We have also seen (Sect. 4.2) that the structural effects of rotation alone result in increased lithium depletion with regard to non-rotating models. Since the angular momentum redistribution in low mass stars reduces the angular velocity gradients, effectively extracting angular momentum from the core to the surface (as shown in Fig. 4.6), one would expect that it would also contribute to enhanced lithium depletion. The reason is that the convective envelope rotates faster, attaining higher temperatures at the radiative-convective interface (Mendes et al. 1999). Figures 4.15 and 4.16 show that this really happens; our results, then, confirm previous findings by other authors (Pinsonneault et al. 1990) that internal angular momentum contributes to a still higher lithium depletion with respect to rotating models without it. We can also see that, though this increase in lithium depletion is barely noticeable for $1 M_{\odot}$ models, it is much higher for lower mass stars. In fact, comparing Fig. 4.16 with Fig. 3 from Mendes et al. (1999; also in appendix C), we see that the $0.8 M_{\odot}$ model with $J=J_{0}$ and angular momentum redistribution suffers stronger lithium depletion than the same model with no redistribution but with $J=2 J_{0}$ ! Angular momentum redistribution, then, can have a dramatic impact on the lithium depletion of stars with masses in the range $0.6-0.8 M_{\odot}$.

### 4.3.4 Consequences of including Zahn's advection term

In Sect. 3.4.2 we mentioned that the pure diffusive approach taken by the Yale group has been regarded by some authors as a poor approximation for the transport of both chemicals and angular momentum by meridional circulation. As we have seen, in Zahn's (1992)


Figure 4.15: Lithium depletion as a function of age for model OA ( $1 M_{\odot}$ ). The term "AMR" in the legend stands for Angular Momentum Redistribution.


Figure 4.16: Lithium depletion as a function of age for model EA ( $0.8 M_{\odot}$ ). As before, the term "AMR" means Angular Momentum Redistribution.
framework the meridional circulation not only diffuses but also advects both quantities. A simple test of the importance of the advective transport can easily be done with the ATON 2.1 code by just dropping the advective term in the right hand side of Eq. (3.35), reducing it to a pure diffusive partial differential equation, and then adding the contribution of the meridional circulation to the total diffusion coefficient, exactly as done by the Yale group. Model OE was computed in this way, but the resulting rotation profile showed a completely negligible difference with respect to the reference model. Thus, at least for low-mass stars of the considered range, the diffusion part largely dominates over the advective one, implying that the purely diffusive approach is indeed a valid approximation for these stars.

Figure 4.17 shows the profile of the radial component of the meridional circulation velocity $U(r)$ for the reference model. As explained in Sect. 3.4.4.4, the peaks in $U(r)$ at the beginning and end of the radiative regions are due to the formal discontinuities in $\left(\nabla_{\mathrm{ad}}-\nabla\right)$ at the boundary with convective regions. From that figure, we can see that $U(r)$ attains a typical value of $10^{-6} \mathrm{~cm} \mathrm{~s}^{-1}$ in most of the radiative region. In view of the negligible


Figure 4.17: Meridional circulation along the radiative interior for the $1 M_{\odot}$ reference model.
importance of the advective contribution for the low-mass models, the approximation made in Sect. 3.4.4.4 for the computation of $U(r)$ is fairly acceptable, at least for the mass range considered in this work.

For intermediate and higher mass stars, however, one would indeed expect that the effects of meridional circulation should be much more expressive than for low-mass stars. As they are much hotter than lower mass stars, their thermal imbalance due to rotation (Von Zeipel's theorem) is stronger, and the resulting circulation currents are expected to have greater velocities, contributing to higher advection of angular momentum. These higher values of $U(r)$ for massive stars have, in fact, been obtained in numerical calculations that also adopt Zahn's framework; for example, Urpin et al. (1996) found values of $U(r)=10^{-4}-10^{-5} \mathrm{~cm} \mathrm{~s}^{-1}$ for their $20 M_{\odot}$ models, while Denissenkov et al. (1999) have found typical values of $10^{-3}$ and $10^{-2} \mathrm{~cm} \mathrm{~s}^{-1}$ for their 10 and $30 M_{\odot}$ models respectively. So, massive stars can attain circulation velocities up to four orders of magnitude larger than those of our solar models, and the advection of angular momentum is significantly enhanced in those stars.

### 4.3.5 Comparison with observations

Inversion techniques of the solar oscillation frequencies obtained by helioseismology give us a very accurate description of many solar properties such as the location of the radiativeconvective interface, fair estimates of both envelope helium abundance and the hydrogen profile in the radiative interior (Christensen-Dalsgaard 1998), and its internal rotation profile (Kosovichev et al. 1997).

What matters to our study is, of course, the solar rotation rate which shows a latitudinal variation in the convective zone and then suffers a transition to a near rigid-body profile throughout the radiative interior up to the fractional radius of $r / R_{\odot}=0.4$, when the inversion results begin to lose accuracy (Fig. 4.18).

This observational result places very stringent constraints on the current non-standard stellar models wich include rotation. Comparing the obtained rotation curve at solar age from our reference model (curve labeled D in Fig. 4.6) with the solar rotation profile (Fig. 4.18), we clearly see that internal angular momentum redistribution cannot reproduce the helioseismic observations, since it predicts differential rotation all along the radiative interior*. As a matter of fact, other rotating models that include transport of angular momentum also fail to reproduce the flat rotation profile in the Sun from $r=0.4 R_{\odot}$ up to the base of the convection zone (Pinsonneault et al. 1989; Chaboyer et al. 1995a).

[^7]

Figure 4.18: Solar rotation curve inferred from $p$-mode frequency splittings. The vertical line at the center of the figure indicates the bottom of the convective zone. The angles shown correspond to the respective solar latitudes. The shaded region on each rotation curve indicate the error estimates. (From Kosovichev et al. 1997).

Since both the pure diffusive and the diffusive-advective approaches rest on diffusion coefficients to compute the transport of angular momentum, and since these coefficients are calculated on the basis of very crude approximations to the hydrodynamical instabilities considered, it seems natural to check if higher values of the computed diffusion coefficients could reproduce a flat rotation profile in the radiative interior as found in the Sun. To allow for this kind of test, we introduced in the ATON 2.1 code a new input parameter $K_{\mathrm{d}}$ which acts as an arbitrary multiplicative factor of the total diffusion coefficient used in the PDE diffusion-advection equation; we then ran some models (OC, OI and OJ) with $K_{\mathrm{d}}$ set to 10,100 and $10^{5}$ respectively. The results obtained with these models are quite disappointing: moderate (but physically acceptable, in view of the uncertainties in the estimates of the diffusion coefficients) values of $K_{\mathrm{d}}$ (10 and 100) result in negligible differences in the angular velocity profile with respect to the original diffusion coefficients; only if we use a very high multiplicative factor $\left(K_{\mathrm{d}}=10^{5}\right)$ can we obtain a flat profile for $\Omega(r)$ (except for a discontinuity at the interface of the inner convective core), as indicated in Fig. 4.19. But we believe that such a high value of the diffusion coefficients is physically implausible, as it would require that either the instability velocities or their characteristic lengths should be magnified by a factor of at least $10^{3}$.

Our results, then, basically confirm that angular momentum redistribution in the Sun is much more efficient than it can be modeled with the current methods and techniques for angular momentum transport. However, it is important to stress that, as pointed out by Chaboyer et al. (1995a), a fast rotating solar core is not ruled out by helioseismology: inversion of frequency splitting of low- $l p$-mode by Toutain \& Fröhlich (1992) and Loudagh


Figure 4.19: Angular velocity profile as a function of radius for $1 M_{\odot}$ models with $K_{\mathrm{d}}=1$ and $K_{\mathrm{d}}=10^{5}$, at the present solar age.
et al. (1993) suggest that at $r \approx 0.2 R_{\odot}$ the rotation rate is $\sim 4$ times faster than the surface. Furthermore, data on subgiant and horizontal branch stars indicates that stars must have large reservoirs of angular momentum that is transported to the surface when they leave the main sequence (Chaboyer et al. 1995a and references therein).

Regarding the lithium depletion of low mass stars, the results presented in Sect. 4.3.3 confirm that models with angular momentum redistribution deplete more lithium than rotating models without it, in accordance with previous work by other researchers (e.g. Pinsonneault et al. 1990). We note that our current implementation of the ATON 2.1 code does not consider the direct effect of meridional circulation on the diffusion of chemicals - an effect that in principle could lead to an even higher lithium depletion than in the considered models. As discussed in Mendes et al. (1999, see appendix C), the obtained results do not match the observations of low mass stars in young open clusters such as the Pleiades and the Orion association, whose spread in lithium depletion is correlated with rotational velocities in such a way that the faster rotators show less lithium depletion. This discrepancy also suggests that other physical mechanisms should play an important role in the lithium depletion of low-mass stars.

The problem with computers is that they only give answers.
P. Picasso (attributed)

## Chapter 5

## SUMMARY

The ATON 2.0 evolutionary code is a very powerful tool for studying stellar astrophysics. The improvements brought to its version 2.1 - which were the main subject of the present work - represent just a small enhancement of its features, yet they place it among the very few non-standard evolutionary codes available today. In spite of this effort, we can foresee a long road of still more improvements to be made. Questions such as turbulence in convective envelopes, magnetic fields, equation of state, opacities and many others must be addressed in order to obtain better agreement with observations and to solve the major shortcomings of stellar modeling.

In what follows, we will present a short summary of results obtained so far, as well as a discussion on the limitations of the current implementation and some directions for future work still in the field of rotation.

### 5.1 Structural effects of rotation and their influence on lithium depletion

The structural effects of rotation were introduced in the ATON 2.1 code by means of the technique developed by Kippenhahn \& Thomas (1970), which basically computes correction factors to be applied in the standard stellar structure equations. This technique, though not the most accurate, is the most used since it allows existing one-dimensional evolutionary codes to be adapted with few modifications.

The results obtained with the ATON 2.1 code regarding the structural effects of rotation are fully compatible with the results from other authors (e.g. Sackmann 1970; Pinsonneault
et al. 1989, 1990): with respect to the dynamical quantities such as the pressure, temperature, density and so on, a rotating star behaves as if it was a non-rotating one of slightly lower mass. As a consequence, its evolutionary path along the H-R diagram is oriented toward lower effective temperatures and luminosity than those showed by a non-rotating star of the same mass. However, we find that this effect is very small, being more noticeable only for very high initial rotation rates.

This change in the location of the evolutionary tracks of rotating stars in the H-R diagram with regard to non-rotating ones cannot be ignored, since these tracks are often the only available tool for estimating some fundamental properties of observed stars such as their ages and masses.

In the case of low-mass stars, an imediate application of rotating models refers to the long standing question of lithium evolution in young open clusters. The late type stars belonging to those clusters show a large spread in their lithium abundances, most of it created during the the main sequence by the mixing due to angular momentum redistribution. Furthermore, intermediate-age clusters such as Hyades and Praesepe show a "lithium dip" around the mid-F stars that extends up to two orders of magnitude. Existing evolutionary models with rotation are able to explain some features of these clusters, such as the cool side of the lithium dip and the main sequence lithium depletion of their late-type stars. The spread in the lithium abundances of these clusters seems to be related to rotation, in such a way that the faster rotators show less lithium depletion. Regarding this point, current results of evolutionary models with rotation are contradictory: Pinsonneault et al. (1992) found that rotation increases lithium depletion, while Martín \& Claret (1996) found the opposite.

In our work, we were able to confirm that rotation indeed increases lithium depletion in low mass stars, contrary to the observational evidence. The higher depletion in these low mass stars has a very simple explanation: since rotating stars behave as if they were non-rotating ones with slightly lower masses, the temperatures at the bottom of their convective envelopes will be higher enough to burn the surface lithium convected to their interiors. This result implies that other physical mechanisms must be sought out in order to reproduce the reduction in lithium depletion required by the observations.

### 5.2 Properties of rotating models with angular momentum redistribution

Angular momentum is a natural property of stars and must be internally redistributed over time in order to account for the observed surface rotation rates of stars and, at least in the case of the Sun, also their internal rotation curves. The most studied mechanism of angular momentum redistribution is related to the meridional circulation currents that
arise in rotating stars as a consequence of the thermal imbalance caused by rotation (von Zeipel's theorem). These currents transport hot material from the poles to the colder equatorial regions, allowing the stars to regain their thermal equilibrium. In that process, those currents not only mix the star's material but also transport angular momentum. A number of other hydrodynamical instabilities related to rotation also cause internal angular momentum transport.

Though some other physical processes such as internal waves and magnetic fields have also been proposed regarding the mixing of the chemical constituents of stars, rotation and internal angular momentum transport have gained considerable theoretical support in the last few years, due to the ever increasing failures of the so-called standard models in explaining many observed anomalous surface abundances in stars, such as helium and nitrogen enrichment in O and B stars and boron depletion in B stars. These discrepancies have made it clear that existing evolutionary codes should be updated in order to seek a better agreement with observational data.

Until a decade ago, the only technique available for inclusion of angular momentum redistribution in evolutionary codes was based on a pure diffusive scheme, in which the angular momentum transport is solved by a diffusion equation, with the diffusion coefficient obtained as the sum of individual coefficients calculated for each hydrodynamical instability considered. Although this technique has been adopted by some researchers with considerable success, it has also been criticized mainly due to the treatment of meridional circulation as a diffusive process. In recent years, Zahn (1992) and collaborators have proposed an improved technique, based on the sole assumption that the horizontal component of the turbulent transport is much stronger than the vertical one. Under this assumption, the angular velocity can be approximated by a shellular rotation law of the form $\omega=\omega(r)$. The angular momentum transport is modeled as an advective-diffusion process, in which the meridional circulation drives the advective part.

In our work we adopted Zahn's technique and implemented it in the ATON code, in addition to the structural effects of rotation. Stellar models from 1.0 down to $0.6 M_{\odot}$ were run with this new ATON 2.1 version, showing that angular momentum redistribution can be very effective in smoothing the internal angular velocity profile of these models. For the particular models of $1 M_{\odot}$, however, the degree of smoothing obtained was not enough to reproduce the flat internal rotation curve inferred for the Sun through helioseismology. This indicates that angular momentum transport in the Sun is more efficient than that predicted by current theoretical models. Other mechanisms (to be discussed in the next section) should be investigated and included in evolutionary codes for the purposes of stellar modeling.

Regarding the effects due to the rotational mixing induced by the angular momentum redistribution, we have found that lithium depletion in low mass stars becomes still higher than in our previous rotating models that considered only the structural effects of rotation. This result confirms the findings of other researchers, such as those from the Yale group,
and bring more support to the conviction that other physical phenomena must play a role in the depletion of light elements in low mass stars.

### 5.3 Directions for future work

This section gives a brief account of some improvements that should be done in the ATON 2.1 code regarding the effects of rotation. The first three (rotation-induced chemical diffusion, inclusion of the effects due to a $\mu$-gradient, and extension of the code for handling intermediate and massive stellar models) are of straightforward implementation, while the remaining ones (disk-locking, the interaction between rotation and magnetic fields, and gravity waves) are more complex and will demand a substantial effort.

### 5.3.1 Rotation-induced diffusion coefficient for the chemicals

The general framework of Chaboyer \& Zahn (1992) and Zahn (1992), discussed in Sect. 3.4 of the present work, also allows for the effects on the transport of the chemicals due to the interaction between the shear turbulence and meridional circulation in radiative regions of stars. Considering once again as the main hypothesis that turbulent motions are stronger in the horizontal direction than in the vertical one, these authors find that the transport of chemicals can indeed be modeled by a pure diffusion equation:

$$
\begin{equation*}
\rho \frac{\partial \bar{c}}{\partial t}=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left[\rho r^{2} D_{\mathrm{eff}} \frac{\partial \bar{c}}{\partial r}\right] \tag{5.1}
\end{equation*}
$$

where $\bar{c}$ is the mean value of the concentration of a given chemical constituent and $D_{\text {eff }}$ the effective diffusion coefficient. The latter, by its turn, is estimated as

$$
\begin{equation*}
D_{\mathrm{eff}}=D_{v}+\frac{|r U(r)|^{2}}{30 D_{h}} \tag{5.2}
\end{equation*}
$$

with $D_{v}$ and $D_{h}$ standing respectively for the vertical and horizontal diffusivities.
The rotation-induced diffusion of chemicals is very important for the stellar evolution of both low- and high-mass stars, for the reasons described in Sect. 2.2.2. So, it is highly desirable to include this effect in the ATON 2.1 code. This should take only two steps: to compute the diffusion coefficient $D_{\text {eff }}$ and to solve the diffusion Eq. (5.1) at each evolutionary time step.

As a matter of fact, the first step is already done in the code since we have all the necessary ingredients to compute $D_{\text {eff }}$. This is done in the following way: for $D_{v}$ we
simply take the diffusion coefficient already calculated for the dynamical shear according to the prescription of Talon \& Zahn (1997), as described in Sect. 2.2.3.1 under the subtitle "dynamical shear". The horizontal diffusivity $D_{h}$ is obtained according to the expression (2.23), which we reproduce here:

$$
D_{h}=\frac{r U}{C_{h}}\left[\frac{1}{3} \frac{d \ln \left(\rho r^{2} U\right)}{d \ln r}-\frac{1}{2} \frac{d \ln \left(r^{2} \Omega\right)}{d \ln r}\right] .
$$

As for the second step, fortunately the ATON 2.1 code already implements the microscopic diffusion of chemicals by means of a diffusion equation (see Sect. 3.2.5). So, in principle it would be a simple matter to use this microscopic diffusion process to include the rotationinduced diffusion, by merely adding the value of $D_{\text {eff }}$ computed at each diffusion time step to the microscopic diffusion coefficient. The differences in the respective timescales of these processes would be automatically accounted for through the relative values of the competing diffusion coefficients.

This last step, despite its relative simplicity, is not yet implemented in the ATON 2.1 code, and we are planning to do it in the near future.

### 5.3.2 Inclusion of the effects due to $\mu$-gradients

Mestel (1953) investigated the effects of a gradient in the chemical composition (the socalled " $\mu$-gradient") in the mixing caused by meridional circulation in rotating stars, and found that whenever such a gradient exists, the meridional circulation creates a nonspherical distribution of chemical composition which tends to prevent the circulation itself. In this way, the presence of $\mu$-gradients has a inhibiting effect on the rotation-induced mixing. The Yale models take care of this effect by considering that the meridional circulation is opposed by fictitious " $\mu$-currents", whose velocity is estimated and then added to the meridional circulation velocity. In Zahn's (1992) framework this effect manifests itself through the factor $E_{\mu}$ that appears in the expression for the meridional circulation velocity (see Eq. 3.52).

However, the extent of such inhibition is not yet well understood, and the available results from non-standard stellar models that include rotation are not enough to bring a clear picture of this subject. For example, Chaboyer et al. (1995a) found that for their $1 M_{\odot}$ models the inhibition of rotational mixing due to $\mu$-gradients was significantly smaller than expected from theoretical considerations. On the other hand, Meynet \& Maeder (1997) computed rotating models of $9,20,40$ and $60 M_{\odot}$ and found that $\mu$-gradients are always strong enough to inhibit mixing. So, the sensitivity of rotation-induced mixing to $\mu$-gradients is at the moment still uncertain, and the available theoretical models do not allow any firm conclusion to be drawn.

The ATON 2.1 code is already prepared to take into account the effects of $\mu$-gradients according to either Yale group's or Zahn's (1992) approaches. Nevertheless, very recently Maeder \& Zahn (1998) suggested that the effects of a $\mu$-gradient should be considered in the calculations of the resulting meridional circulation velocity not only through the contribution $E_{\mu}$ due to the horizontal variations of the chemical composition, but also by explicitely including it in the expression for the superadiabatic gradient, in the same way it is used e.g. in the Ledoux criterion for convection. Apart from other improvements also developed by those authors, the expression for the radial part of the meridional circulation velocity should then read

$$
\begin{equation*}
U(r)=\frac{L}{M g}\left(\frac{P}{C_{P} \rho T}\right) \frac{1}{\nabla_{\mathrm{ad}}-\nabla+\nabla_{\mu}}\left(E_{\Omega}+E_{\mu}\right), \tag{5.3}
\end{equation*}
$$

instead of expression (3.52). Since $\nabla_{\mu}$ is usually larger than $\left(\nabla_{\mathrm{ad}}-\nabla\right)$ by an order of magnitude or more in stellar interiors, the resulting meridional circulation velocity could be significantly reduced. We defer the study of the effects of a $\mu$-gradient in the calculation of $U(r)$ to a future work, where the above suggestion should be properly considered.

### 5.3.3 Extension of the code to intermediate and high-mass stars

In the present work, rotation and internal redistribution of angular momentum has been applied only to low-mass stars. However, by the reasons cited in Subsect. 2.2.2.1, their influence on the structure and evolution of intermediate- and high-mass stars should also be studied. A clear indication of the importance of higher mass stars is the contribution of the advection of angular momentum, which we found to be negligible for low-mass models but is expected to be much more significant for higher mass stars (cf. Sect. 4.3.4).

In principle, we expect that the ATON 2.1 code should work fairly well for this mass range, but it is likely that some minor adjustments will be required. For example, massive stars have huge radiative envelopes, and the meridional circulation velocity has a formal discontinuity at the star's surface that should be properly worked out in these models.

### 5.3.4 Disk-locking

It is well known that the rotational periods of T Tauri stars show a very characteristic distribution (Edwards 1994, 1995) whose main features are:

- T Tauri stars that show signs of an accretion disk (usually referred to as "classical" T Tauri stars, or CTTS) have a very narrow period distribution, peaked at about 8 days;
- T Tauri stars that do not show such signs ("weak line" T Tauri stars, or WTTS) have a much broader period distribution, with some stars showing lower rotation periods by a factor of 4 relative to those presented by CTTS.

This behaviour is clearly seen in Fig. 5.1. Furthermore, spectroscopic measurements of CTTS rotational velocities reveal that these velocities are systematically an order of magnitude lower than the breakup velocity (that is, $\omega \leq 0.1 \omega_{\text {crit }}$ ).


Figure 5.1: Color excess diagram for the near infrared $H(1.65 \mu \mathrm{~m})$ and $K(2.2 \mu \mathrm{~m})$ bands, corrected for the interstellar reddening, as a function of the rotation periods for a sample of 52 T Tauri stars. An infrared color excess $\geq 0.1$ signals the presence of an accreting disk. (From Edwards 1994)

From a theoretical point of view, CTTS are stellar objects which are still contracting and gaining angular momentum from the inner part of the keplerian disk, at the rate specified by the mass accretion rate (Edwards 1994):

$$
\begin{equation*}
\dot{J}_{\mathrm{acc}} \sim \dot{M}_{\mathrm{acc}} \Omega_{\mathrm{Kep}} R_{\star}^{2} . \tag{5.4}
\end{equation*}
$$

Typical mass accretion rates in T Tauri stars range from $\sim 10^{-5.5} M_{\odot} \mathrm{yr}^{-1}$ for FU Orionis stars, $\sim 10^{-7} M_{\odot} \mathrm{yr}^{-1}$ for the $10 \%$ of the most active CTTS, down to $\sim 10^{-8}$ for the majority of CTTS (Cameron \& Campbell 1993 and references therein). According to Hartmann \& Stauffer (1989), an accretion rate of $10^{-7} M_{\odot} \mathrm{yr}^{-1}$ would lead a fully convective star with rigid body rotation to a rotation rate $\sim \omega_{\text {crit }} / 2$ in just $10^{6}$ years, if one ignores the angular momentum loss. In view of these facts, the observational picture of CTTS
poses an enigma, as they should have a rotation rate near the breakup velocity. The traditional magnetic-braking models based on angular momentum loss by magnetized winds cannot account for these contradictions, primarily because they cannot explain the large dispersion in rotation rates observed at the ZAMS (Krishnamurthi et al. 1997).

The whole picture suggests that, during the phase of active disk accretion, there must be some physical mechanism that regulates the central star's rotation rate. Current studies of these mechanisms utilize either closed or open magnetic field geometry, and are reviewed by Edwards (1995). As the latter approach removes angular momentum from the central star but does not offer a natural explanation for the regulation of its angular velocity, we will concentrate on the former approach (now referred to as disk-locking mechanism in the literature) in this section.

Edwards (1994) proposed the existence of a mechanism that should keep constant the stellar angular velocity, in such a way that the contraction of the star towards the main sequence would lead to a decrease in its angular momentum. In this way, the initial angular momentum of the star would be the one it has when the regulating accretion disk is dissipated. The disk-regulating mechanism should also be uniform and efficient in all young stars with accretion disks, in order to explain the short dispersion in rotation rates observed in CTTS. Additional observational support for a disk-regulated mechanism has been given by Choi \& Herbst (1996) through a study of 75 T Tauri stars in the Orion Nebula Cluster.

Since then, a significative theoretical effort has put in evidence the disk-locking mechanism as a plausible explanation for the rotational evolution of pre-main sequence stars. Most researchers adopt a picture in which the closed lines of the stellar magnetic field couple to the disk at a given radius (the co-rotation radius) and force the star in co-rotation with the accretion disk. The extent of this radius depends on an intricate balance between mass accretion rate $\dot{M}$ and the magnetic field strength $B$. In this picture the magnetic pressure also clears out the inner part of the disk, truncating it at a radius which is nearly coincident with the co-rotation radius, as shown in Fig. 5.2. Needless to say, the hydromagnetic prescriptions for obtaining this picture greatly differ among the theoreticians (e.g. Cameron \& Campbell 1993; Cameron et al. 1995; Clarke et al. 1995; Ghosh 1995; Armitage \& Clarke 1996).

Theoretical stellar models with disk-locking have been used to account for the angular momentum evolution of pre-main sequence stars (Bouvier et al. 1997; Krishnamurthi et al. 1997). A range of disk lifetimes is usually employed to reproduce the transition of low velocity dispersion in the PMS to high dispersion at the ZAMS.

The disk-locking paradigm has recently been challenged by the observational results of Stassun et al. (1999a, 1999b), who determined rotational periods for a large sample of T Tauri stars in the Orion Nebula Cloud. These authors found no connection between rotation periods and near-IR signatures of circumstellar disks. Moreover, they showed


Figure 5.2: Schematic diagram of the disk-locking mechanism for the case of closed magnetic field geometry. (From Stassun et al. 1999b).
that the dispersion in the rotation rates observed in their sample (whose age is estimated as 1 Myr ) resembles that one found in the Pleiades. These findings, though not ruling out the disk-locking mechanism, show that the rotational evolution of pre-main sequence stars is still far from elucidated.

### 5.3.5 Interaction between rotation and magnetic fields

Magnetic fields have been defined as the "swiss-army knife of astrophysics", yet there is no doubt about its importance for stellar structure and evolution. Topics such as stellar dynamos, magnetized winds, bipolar outflows, chromospheric activity and others show that magnetic fields are inextricably linked to stellar formation and evolution.

A large wealth of scientific literature exists on the role of magnetic fields in stellar formation and evolution, and any discussion or review on that subject is obviously beyond the scope of this work. In what follows we limit ourselves to a brief discussion on the interaction between rotation and magnetic fields in late-type stars and the possibilities of introducing it in the ATON 2.1 code.

The stellar chromospheric and coronal activity seen in the Sun and many other late-type stars is related to a stellar magnetic field, which is believed to be generated at the base of the convective envelope as the result of the interaction between convection and rotation. This dynamo effect is by far the most studied aspect of such an interaction (a good review on the present status of dynamo mechanisms can be found in Brandenburg \& Saar 1999).

Although the present work is focused on angular momentum redistribution in radiative regions, the rotational behaviour of convective regions is also relevant for the evolution of low mass stars. The interesting point is that rotation first generates the dynamo, and then the magnetic field reacts back on rotation, forming a highly intricate phenomena. For example, active stars such as T Tauri stars and RS CVn systems show polar activity, which has been demonstrated to reduce angular momentum loss efficiency at the star's surface (Buzasi 1997; Solanki et al. 1997). The effects of meridional circulation on the internal rotation of solar-type star's convective zones was studied by Rüdiger et al. (1998), who showed that the resulting differential rotation is then significantly reduced. Finally, regarding the long-standing question of lithium depletion in low mass stars, Ventura et al. (1998a) demonstrated that the presence of even a low-intensity magnetic field in convective regions can greatly inhibit lithium depletion during the pre-main sequence. All these results clearly show that stellar dynamos are a very important mechanism that sooner or later must be included in stellar models.

Turning our attention to radiative zones, the magnetic fields seen at the surface of latetype stars can barely penetrate the radiative interior, meaning that any poloidal field in it must be the relic of some earlier evolutionary phase (Mestel \& Weiss 1987). In general, the radiative interior is usually regarded as perfectly conducting as a first approximation, as the ohmic decay of a large-scale field is of order $10^{10} \mathrm{yr}$ for a main-sequence star of solar mass.

Independent of the origin of magnetic fields in radiative regions of stars, Mestel (1965) was the first one to point out that stars with large-scale magnetic fields will not be in general in radiative equilibrium, giving rise to circulatory motions. The situation becomes more complicated if the star rotates, for then there will be a competition between the Eddington-Vogt circulation and the magnetic-induced circulation (Moss 1974). Many studies have been conducted on the interaction between rotation and magnetic fields in radiative interiors, but the current theoretical picture is plagued by many uncertainties due not only to the complexity of the subject but also to the conflicting results obtained so far by different researchers. This can be illustrated, for example, by the long standing question as to whether or not magnetic fields can enforce uniform rotation in stellar radiative interiors, a question of great interest in view of the current helioseismological picture of the Sun. Moss $(1987,1992)$ and Mestel et al. (1988) have found that the Lorentz force arising from the toroidal magnetic field that results from the interaction of rotation and a poloidal field will be strong enough to allow just a very slight differential rotation. These results are, however, disputed by Tassoul \& Tassoul (1986, 1989), whose numerical calculations showed that the existence of a large-scale magnetic field in a low-viscosity medium such as the stellar radiative zones enhances the gradients in the angular velocity, thus preventing uniform rotation in the whole of the radiative zone.

Whatever would be the outcome of such calculations, what matters is that in either case magnetic fields provide a mechanism for angular momentum redistribution in stars,
a result which is now not only strongly supported by a number of theoretical works (e.g. Moss et al. 1990, Kitayama et al. 1996) but also used to model some observational features of the Sun such as the magnetic field intensity in its radiative zone (Mestel \& Weiss 1987) and its spin-down along the main sequence (Charbonneau \& MacGregor 1993; Rüdiger \& Kitchatinov 1996). Besides, the influence of angular momentum transport by hydromagnetic means in the depletion of light elements such as beryllium and lithium was very recently investigated by Barnes et al. (1999).

To conclude, we believe that in spite of its relative complexity, the interaction between rotation and magnetic fields in stellar radiative zones could be introduced in the ATON 2.1 code, in a first approach, through some magnetohydrodynamic process for angular momentum transport in stars, such as those recently reviewed by Spruit (1999).

### 5.3.6 Internal waves

The failure of the mechanisms of rotation-induced turbulent diffusion (Endal \& Sofia 1978, Pinsonneault et al. 1989) and wind-driven meridian circulation (Zahn 1992) in extracting enough angular momentum from the solar radiative interior, in order to explain the flat rotation profile revealed by helioseismology, have led some researchers to turn their attention to other physical mechanisms capable of reconciling theory and observations.

It is believed that internal (or gravity) waves are produced by turbulent stresses in the convective zone of the Sun. The theoretical foundations of their properties in stellar interiors were established by Press (1981) and Goldreich \& Nicholson (1989a, 1989b). However, it seems that Schatzman (1993) was the first to propose that gravity waves could be very effective in redistributing angular momentum in stars, to the extent of reproducing the flat solar rotation curve. Schatzman (1993) demonstrated that gravity waves also generate a macroscopic diffusive process that transports chemical elements. Later on, Zahn et al. (1997) and Kumar \& Quataert (1997), using different but complementary approaches to the problem, not only confirmed the results of Schatzman but also found that gravity waves induces angular momentum transport in a very short timescale ( $\sim 10^{7} \mathrm{yr}$ ), enforcing rigid body rotation in the solar interior. More recent results by Talon \& Zahn (1998) and Kumar et al. (1999) support the effectiveness of gravity waves in establishing a rigidly rotating solar radiative interior. Gravity waves were also used by Montalban \& Schatzman (1996) to successfully reproduce the lithium dip observed in mid-F stars of open clusters.

In spite of their success to model the solar rotation curve in the radiative interior, gravity waves also have some shortcomings. Pinsonneault (1997) noted that the dispersion in abundances observed in the lithium depletion pattern of open clusters would not be expected for mixing driven by waves. Ringot (1998) showed that the treatment given so far to gravity waves in the context of angular momentum redistribution is oversimplified,
and that the evolution of the flow profile might be highly complex, making it difficult to obtain the near rigid body rotation in the Sun. Deliyannis et al. (1998) studied the correlation between lithium and beryllium depletion in F stars, and demonstrated that the observed Li-Be trend is better supported by the action of a "slow-mixing" mechanism, such as rotation, than by models with wave-driven mixing.

In principle it would be relatively straightforward to implement the angular momentum transport by gravity waves in the ATON 2.1 code. However, as shown by Schatzman (1993), the effects of gravity waves are not restricted to angular momentum transport: they also have important consequences on the diffusion of chemicals and also on the heat transport. So, the introduction of gravity waves in evolutionary codes should allow for all these combined effects, in order to obtain a coherent and self-consistent picture of their importance in evolutionary models. These difficulties make the introduction of gravity waves in the ATON 2.1 code a feasible but not immediate solution to be worked out.

### 5.4 Final remarks

In the present work we introduced rotation and angular momentum redistribution in the ATON 2.0 stellar evolution code. We think this is an important accomplishment for a number of reasons. First, rotation is a measurable quantity that can act as an additional constraint for stellar modeling, a possibility not open for standard models. In this way, the code can now deal with many important issues in stellar evolution such as the rotational evolution on the giant phase and its associated extra mixing, the anomalous abundances found in massive stars, and so on. Second, the fact that we obtained a worse agreement between the lithium depletion of rotating models and the observations strongly suggests that other physical echanisms must play a role in stellar evolution. At least for the case of low mass stars, the solution for this disagreement can rest in magnetic fields. As demonstrated by Ventura et al. (1998a), parametric computations of low mass stellar models with a small magnetic field pervarding the convective envelope can significantly inhibit lithium depletion in those stars. Since rotation is at the root of stellar magnetic fields through the dynamo mechanism, only evolutionary codes with rotation can provide the foundation for a more physically sound treatment of magnetic fields and its effects.

To conclude, we believe that the new ATON 2.1 code opens up a large field of research possibilities on rotation and its associated effects (including magnetic fields) in stellar structure and evolution. At the dawn of a new millenium, we dare to think that it will allow us to face new challenges in astrophysics.

```
43rd Law of Computing:
    Anything that can go wr
fortune: Segmentation violation -- Core dumped
```

Unix Fortune

## Chapter 6

## SÍNTESE DO TRABALHO EM LÍNGUA PORTUGUESA

Este capítulo contém uma versão resumida contendo a essência do trabalho realizado ao longo da presente tese de doutorado. Detalhes completos podem ser obtidos nos demais capítulos ou nos artigos contidos nos apêndices B a D.

### 6.1 Motivação

Há quase três décadas os modelos evolutivos estelares denominados "padrões" (standard), ou seja, sem a inclusão de efeitos como a rotação, campos magnéticos, etc., têm sido muito bem sucedidos em reproduzir as características básicas de estrelas. Contudo, a crescente precisão dos dados observacionais disponíveis, bem como os avanços em áreas correlatas como a heliosismologia, tem evidenciado muitas deficiências e limitações dos mesmos. Por exemplo, uma das maiores discrepâncias entre observações e os resultados dos modelos evolutivos clássicos refere-se à abundância de elementos leves como o lítio e berílio em estrelas de baixa massa: as observações mostram em geral uma abundância destes elementos significativamente menor do que o predito por tais modelos. No caso do Sol, este problema é ainda mais grave, dado que a abundância de ${ }^{7} \mathrm{Li}$ é menor por um fator 200 em relação à encontrada em meteoros (D'Antona \& Mazzitelli 1984). Além disto, estrelas evoluídas frequentemente apresentam abundâncias anômalas em suas superfícies, o que revela a existência de mistura ( mixing) em camadas mais profundas. A rotação, em particular, por ser uma propriedade exibida em todas as estrelas, tem sido considerada um importante agente para esta mistura, uma vez que provoca movimentos de massa que por sua vez resultam na mistura da matéria estelar.

Estas razões nos motivaram a estudar os efeitos da rotação e da redistribuição interna de momento angular em modelos evolutivos de estrelas de baixa massa, usando como ferramenta para tal o código evolutivo ATON 2.0, ao qual tivemos acesso através de uma colaboração científica com os Drs. Francesca D'Antona (Osservatorio Astronomico di Roma, Itália) e Italo Mazzitelli ( $C N R$, Itália) iniciada pelo Dr. Luiz Paulo Ribeiro Vaz, do Departamento de Física da Universidade Federal de Minas Gerais (UFMG).

A Seção 6.2 seguinte apresenta uma rápida descrição do atual estado de conhecimento referente ao modelamento da evolução estelar com rotação. A Seção 6.3 descreve as principais características do código evolutivo ATON 2.0 e as técnicas que empregamos para introduzir os efeitos da rotação e da redistribuição de momento angular no mesmo. A Seção 6.4 discute os principais resultados obtidos. As conclusões finais e algumas sugestões para prosseguimento do trabalho são apresentadas na Seção 6.5. Observamos que estas seções estão em exata concordância com o conteúdo dos Capítulos de 2 a 5 do presente trabalho.

### 6.2 Importância dos Modelos Estelares com Rotação

### 6.2.1 Breve introdução aos modelos evolutivos estelares

As bases da teoria de interiores estelares foram construídas ao longo da primeira metade deste século, e consolidadas em importantes obras das quais as de Chandrasekhar (1939) e Schwarzschild (1958) são apenas alguns exemplos. Uma vez que os únicos meios de se obter informações sobre a estrutura estelar são através da análise da radiação emitida pelas mesmas e da interação gravitacional das mesmas com o meio vizinho, desde então os modelos teóricos de interiores estelares tornaram-se uma importante ferramenta para nossa compreensão sobre a constituição interna das estrelas; tais modelos teóricos são obtidos pela integração numérica das equações da estrutura estelar. Uma sequiência de modelos estelares ao longo do tempo, obtida desta maneira, define a chamada "trajetória evolutiva" da estrela.

Apesar do vigoroso impulso na construção de modelos evolutivos ocorrido desde a década de 60 com o advento de novas técnicas numéricas (Henyey, Forbes \& Gould 1964) e substanciais progressos na física de interiores estelares tais como a nucleosíntese (Clayton 1968) e a opacidade (Cox \& Stewart 1970), ainda hoje tais modelos não conseguem reproduzir a enorme massa de dados observacionais disponíveis. Isto se deve principalmente às simplificações utilizadas nestes modelos e à pouca compreensão de alguns processos físicos tais como a queima de elementos leves como lítio e berílio (D'Antona e Mazzitelli 1984), a equação de estado para interiores estelares (Magni \& Mazzitelli 1979, Mihalas et al. 1988), a turbulência no processo de convecção (Forestini et al. 1991; Canuto \& Mazzitelli 1991) e os efeitos da rotação sobre a evolução estelar (Kippenhahn \& Thomas 1970; Pinsonneault et al. 1989).

Especificamente quanto à rotação, as principais técnicas utilizadas para sua incorporação em modelos evolutivos foram desenvolvidas ao longo das décadas de 60 e 70 (Faulkner et al. 1968; Sackmann \& Anand 1969; Kippenhahn \& Thomas 1970; Papaloizou \& Wheelan 1972). Contudo, a rotação causa ainda efeitos físicos adicionais como a circulação meridional e instabilidades hidrodinâmicas que, a seu turno, provocam redistribuição interna de momento angular nas estrelas. Outros aspectos diretamente relacionados com a rotação tais como sua interação com a convecção, a frenagem magnética por ventos estelares, e a transferência de momento angular de discos circunstelares para a estrela central, são ainda pouco estudados e portanto ausentes dos atuais modelos.

Uma das principais aplicações dos modelos evolutivos estelares tem sido a investigação das diversas etapas que ocorrem ao longo da evolução estelar. Entre estas, talvez a menos estudada seja a etapa pré-sequiência principal, devido à aparente simplicidade dos processos físicos que ocorrem na mesma (Henyey et al. 1955; Hayashi 1961). Contudo, sabe-se hoje que vários mecanismos físicos ainda não muito bem compreendidos, e que possuem pouco ou nenhum efeito na estrutura das estrelas na etapa seqüência principal, são contudo relevantes na fase pré-seqüência principal, colocando assim em xeque sua suposta simplicidade. Como exemplos podemos citar os processos de convecção superadiabática, a equação de estado, a opacidade radiativa, o overshooting, a rotação e o efeito dos campos magnéticos.

Por outro lado, os dados observacionais disponíveis sobre estrelas pré-seqüência principal são muito mais escassos do que aqueles referentes a estágios evolutivos posteriores, devido basicamente a dois fatores: em primeiro lugar, estrelas da fase pré-sequiência principal são geralmente encontradas em meio a concentrações de gás e de poeira, as quais não só obscurecem a luminosidade intrínseca daquelas estrelas como também reprocessam a radiação absorvida, superpondo seu espectro ao das estrelas pré-seqüência principal; e, em segundo lugar, existem ainda muito poucas detecções de binárias eclipsantes compostas por estrelas pré-seqüência principal, as quais consistem no único meio confiável para a determinação de parâmetros estelares importantes, como por exemplo a massa, raio, temperatura e inclinação da órbita. Tais fatores citados têm levado a um crescente interesse teórico e observacional sobre esta fase, no qual os modelos evolutivos, pelas razões acima expostas, desempenham importante papel. Por estas razões, limitaremos nossa investigação às estrelas de baixa massa no período evolutivo compreendido pelas etapas pré-seqüência principal e seqüência principal.

### 6.2.2 Rotação e Redistribuição interna de momento angular

Os efeitos físicos da rotação são por nós divididos em duas principais categorias: os efeitos estruturais (ou hidrostáticos) e os efeitos evolutivos. Os efeitos estruturais são os mais fundamentais e provêm do simples fato que a rotação origina uma força centrífuga que se contrapõe à força gravitacional, afetando a condição de equilíbrio hidrostático. Os
efeitos evolutivos, por sua vez, referem-se às mudanças no perfil interno da velocidade angular devido à redistribuição de momento angular.

Como dito anteriormente, desde as décadas de 60 e 70 as principais técnicas para incorporação dos efeitos estruturais da rotação já eram conhecidas (uma boa revisão das mesmas pode ser encontrada em Tassoul 1978); contudo, atualmente muito poucos modelos evolutivos incluem tais efeitos. Os resultados fornecidos por tais modelos indicam que o principal efeito estrutural da rotação é o deslocamento da trilha evolutiva de uma estrela com rotação em direção a menores luminosidade e temperaturas efetivas, simulando a trilha de uma estrela sem rotação e com massa ligeiramente menor.

Quanto aos efeitos evolutivos, desde Eddington (1925) e Von Zeipel (1924a,b) já se sabe que a rotação causa um desequilíbrio térmico em estrelas, que por sua vez acarreta correntes circulatórias. Mesmo se o estado inicial corresponder à rotação de corpo rígido, tais correntes alteram o perfil interno da velocidade angular, resultando em rotação diferencial que, a seu turno, induz uma série de instabilidades hidrodinâmicas, originando assim movimentos turbulentos. A fim de preservar sua estabilidade contra estes movimentos turbulentos, a estrela deve sofrer redistribuição interna de momento angular, reajustando seu gradiente de velocidade angular sempre que tais instabilidades ocorrerem.

Pesquisadores da Univ. de Yale (Endal \& Sofia 1978; Pinsonneault 1988; Pinsonneault et al. 1989, 1990, 1992; Chaboyer et al. 1995a) foram os pioneiros no modelamento da redistribuição de momento angular, considerando um processo puramente difusivo no qual o respectivo coeficiente de difusão é estimado a partir de distâncias e velocidades características de certas instabilidades hidrodinâmicas. Esta técnica tem sido criticada na literatura, principalmente quanto ao tratamento da circulação meridional como um processo difusivo. Um progresso importante neste aspecto foi obtido por Chaboyer \& Zahn (1992) e Zahn (1992), que investigaram os efeitos da circulação meridional no transporte tanto dos componentes químicos quanto do momento angular em regiões radiativas. De acordo com o novo quadro apresentado por tais pesquisadores, o transporte dos componentes químicos pode ser realmente modelado por uma equação de difusão, mas o transporte de momento angular deve ser descrito por uma equação de advecção*-difusão.

Apesar do sucesso obtido pelos modelos clássicos, muitas características de estrelas não são ainda reproduzidas pelos mesmos (vide p. ex. Dziembowski 1998). Por exemplo, propriedades observadas em estrelas massivas como a razão entre o número de supergigantes vermelhas e azuis ou as abundâncias anômalas na superfície de supergigantes azuis não podem ser explicadas por tais modelos mesmo se mudanças na metalicidade, na taxa de perda de massa ou no tamanho dos núcleos convectivos são consideradas (Meynet 1999). Estas discrepâncias indicam que outros mecanismos físicos devem ser incorporados nos modelos evolutivos; entre tais mecanismos, a rotação surge como um candidato promissor

[^8]devido não só ao volume de dados disponíveis mas também por sua relativa simplicidade, quando comparada com outros fenômenos tais como campos magnéticos ou turbulência.

Atualmente, os efeitos estruturais e evolutivos da rotação mais investigados são aqueles referentes à composição química das estrelas. Modelos com rotação calculados para estrelas massivas $\left(M>8 M_{\odot}\right)$ têm sido capazes de reproduzir várias propriedades observadas em tais estrelas, tais como a proporção entre as abundâncias superficiais de nitrogênio e boro (Fliegner et al. 1996) e carbono/nitrogênio e oxigênio/nitrogênio (Talon et al. 1997); a camada rica em ${ }^{13} \mathrm{C}$ necessária à ocorrência de reações nucleares lentas (s-process) em estrelas do tipo TP-AGB (Langer et al. 1999); e o estágio evolutivo no qual inicia-se o enriquecimento superficial de estrelas evoluídas (Meynet 1999; vide Fig. 2.1).

No caso de estrelas de baixa massa, um dos aspectos mais investigados têm sido a abundância de lítio, uma vez que este último é produzido na nucleosíntese do "Big Bang" e portanto tem relevância cosmológica. O padrão de esgotamento de lítio observado em aglomerados abertos jovens como $\alpha$ Persei ( 50 Myr ) e Plêiades ( $70-110 \mathrm{Myr}$ ) é razoavelmente reproduzido pelos modelos estelares clássicos, como mostrado na parte inferior da Fig. 2.2 para o caso das Plêiades. A situação, contudo, é muito diferente no caso de aglomerados de idade intermediária como M34 (200 Myr), Ursa Maior (300Myr), Híadas e Praesepe (ambos 500-700 Myr), cuja abundância de lítio não não é reproduzida por tais modelos mesmo se o overshooting (que pode ter uma dramática influência na fase préseqüência principal) é levado em conta, como mostra a parte superior da Fig. 2.2 para o caso das Híadas. Para explicar este comportamento da abundância de lítio em tais casos, diversos mecanismos têm sido propostos na literatura, tais como overshooting, ondas de gravidade, difusão microscópica e rotação; todos, porém, apresentam deficiências (vide Chaboyer 1998). Modelos com rotação (Pinsonneault et al. 1990, 1992; Chaboyer et al. 1995b) têm sido bem sucedidos em reproduzir algumas das caraterísticas da abundância de lítio em aglomerados abertos, como se pode ver da Fig. 2.4 para o caso das Plêiades e das Híadas; contudo, tais modelos não são capazes de explicar a presença de estrelas G e K com alta rotação nestes aglomerados.

Como mencionado anteriormente, a redistribuição de momento angular na técnica desenvolvida pelo grupo de Yale é obtida através de uma equação de difusão na qual o coeficiente de difusão resulta de algumas instabilidades hidrodinâmicas disparadas pela rotação. Estas instabilidades são usualmente classificadas como dinâmicas ou seculares, como explicado a seguir. Instabilidades dinâmicas são aquelas que ocorrem num intervalo de tempo muito curto comparado com a escala de tempo da evolução estelar, sendo portanto consideradas adiabáticas. As instabilidades seculares, por outro lado, ocorrem em escalas de tempo suficientemente grandes para que haja troca de energia. Existe uma extensa literatura acerca do papel destas instabilidades hidrodinâmicas na evolução estelar, entre as quais citamos Tassoul (1978), Endal \& Sofia (1978), Knobloch \& Spruit (1982) e Zahn (1993). No presente trabalho, iremos considerar como instabilidades dinâmicas o dynamical shear e a chamada instabilidade de Solberg-Høiland, e como instabilidades
seculares o secular shear, a instabilidade de Goldreich-Schubert-Fricke e a circulação meridional. Tais instabilidades são as mesmas adotadas pelo grupo de Yale, e uma descrição detalhada das mesmas pode ser obtida nas referências acima.

### 6.3 Técnicas empregadas

### 6.3.1 Principais características do código ATON 2.0

O código evolutivo ATON 2.0 é uma poderosa ferramenta para investigação da estrutura e evolução estelares. Na versão atual, este código pode ser utilizado para seguir a evolução estelar desde a etapa pré-sequiência principal até o estágio de resfriamento de anãs brancas ou anãs marrons. Estágios mais avançados, como a queima de ${ }^{16} \mathrm{O}+{ }^{16} \mathrm{O}$ e etapas subseqüentes, ainda não são contemplados neste código.

Os modelos da etapa pré-seqüência principal iniciam-se em uma configuração arbitrária totalmente convectiva com uma temperatura central de $\log T_{\mathrm{c}}=5.7 \mathrm{e}$, portanto, antes da queima de deutério. A estrutura interna é integrada através do método usual de relaxação (método de Henyey), do centro à base da atmosfera ótica. Próximo da superfície, onde $\Delta M / M_{\text {tot }}$ pode ser tão pequeno quanto $\sim 10^{-12}-10^{-18}$, a prática usual consiste em calcular grades (grids) de sub-atmosferas até uma dada fração de $M_{\text {tot }}$, a fim de evitar problemas numéricos. Ao invés disto, o código ATON 2.0 modifica a densidade local de pontos (rezoning) e calcula as derivadas numéricas diretamente sobre o vetor $\Delta M$, onde o valor local da massa em cada ponto da grade é obtida através da soma de $\Delta M$ a partir do centro. Esta técnica permite a obtenção direta da taxa de geração de energia gravitacional desde o centro até a superfície, especialmente quando a perda de massa (p. ex. em binárias de contato) tem um papel importante na estrutura estelar.

As derivadas na base da atmosfera ótica são obtidas através do cálculo de três atmosferas cinzas com pequenos intervalos tanto no raio quanto na luminosidade ( $\delta \ln \left[L, T_{\text {eff }}\right]=0.001$ ) em cada iteração. Uma relação simples do tipo $\mathrm{T}(\tau)$ (Henyey et al. 1968) é empregada ao longo das atmosferas cinzas.

Os modelos evoluem no tempo através de passos temporais que são calculados de acordo com valores máximos das derivadas logarítmicas correspondentes às variáveis estruturais e também com parâmetros tais como a perda de massa, composição química, integrais sobre as luminosidades CNO, triple-alpha e ${ }^{12} \mathrm{C}+{ }^{12} \mathrm{C}$, e outros. Um requisito adicional para o cálculo do passo temporal é que a etapa de relaxação deve convergir em no máximo três iterações; do contrário, o passo temporal deve ser diminuído ainda mais. Tipicamente, para uma estrela do tipo solar, são empregados $\sim 1000$ passos temporais desde a etapa pré-seqüência até a saída da seqüência principal (turn-off), ao passo que o flash do hélio é alcançado em 15000-20000 passos temporais.

Após cada passo temporal, a densidade de pontos de toda a estrutura é recalculada com base no princípio que as derivadas das cinco quantidades estruturais não podem exceder certos valores, fixados com base na experiência e diferentes de região para região. Estruturas correspondentes às etapas pré-seqüência principal e seqüência principal empregam tipicamente 800-1200 pontos; gigantes vermelhas, 1200-1500 pontos (cerca de 200 dos quais na fina camada de queima de H ); e estruturas no ramo horizontal, 1500-2000 pontos, sendo que durante os pulsos térmicos uma grade de até 3000 pontos pode ser alcançada.

Os valores termodinâmicos empregados são obtidos de Magni \& Mazzitelli (1979), tendo sido recentemente atualizados para o regime de ionização total nos casos de hidrogênio e hélio puros, carbono, oxigênio e misturas intermediárias. A influência de vários processos físicos tais como covolume, forças de van der Waals e Coulomb, degenerescência e outros são adequadamente considerados. As opacidades OPAL (Iglesias \& Rogers 1993) são adotadas, combinadas com as opacidade condutivas de Itoh \& Kohyama (1993) para o estado fundamental e completadas com as opacidades moleculares de Alexander \& Ferguson (1994) ou, opcionalmente, com as de Kurucz (1993) para temperaturas inferiores a 6000 K .

A rede de reações nucleares é composta por 22 reações (vide Subsec. 3.2.3), cujas seções de choque foram obtidas de Caughlan \& Fowler (1988), e os coeficientes de screening de Graboske et al. (1973). Neutrinos gerados pelos processos físicos mais relevantes são obtidos de Itoh et al. (1992). Com relação ao tratamento da convecção, o código ATON 2.0 pode lidar tanto com o formalismo tradicional do "comprimento de mistura" (mixing length) quanto com o tratamento mais moderno de convecção turbulenta (Full Spectrum of Turbulence) desenvolvido por Canuto \& Mazzitelli $(1991,1992)$ e Canuto et al. (1996).

A evolução nuclear dos componentes químicos é tratada segundo o esquema linearizado e implícito de Arnett \& Truran (1969). A mistura dos componentes químicos pode ser feita, no ATON 2.0, como uma mistura instantânea ou através de um processo difusivo (usualmente denominado difusão microscópica), sendo este último implementado segundo o método totalmente implícito sugerido por Cloutman \& Eoll (1976).

A partir deste ponto, o código ATON 2.0 acrescido dos aperfeiçoamentos referentes à introdução de rotação e redistribuição interna de momento angular, a serem discutidos a seguir, passará a ser referido como ATON 2.1.

### 6.3.2 Introdução da Rotação

Os principais métodos para a inclusão de rotação em modelos evolutivos estelares são discutidos em Tassoul (1978). Destes, escolhemos o método de Kippenhahn \& Thomas (1970, doravante referido como KT) porque, mesmo não sendo o mais preciso, é o mais facilmente adaptável aos códigos computacionais uni-dimensionais já existentes. De fato,
esta facilidade o tornou o método mais empregado na literatura (vide p. ex. Endal \& Sofia 1976; Law 1980; Pinsonneault 1988; Martín \& Claret 1996).

O método KT baseia-se na substituição das superfícies esféricas dos modelos sem rotação por superfícies equipotenciais não-esféricas, caracterizadas pelo seu potencial total $\Psi$, a massa $M_{\Psi}$ delimitada por tais superfícies, e o raio $r_{\Psi}$ de uma esfera topologicamente equivalente com o mesmo volume $V_{\Psi}$ delimitado pelas superfícies equipotenciais. Valores médios da gravidade superficial, $\langle g\rangle$, e de seu inverso, $\left\langle g^{-1}\right\rangle$, nestas superfícies são então calculados, levando, por sua vez, a fatores de correção denominados $f_{\mathrm{P}}$ e $f_{\mathrm{T}}$ a serem introduzidos nas equações da estrutura estelar (cf. Eqs. 3.11 a 3.13 ).

O potencial total $\Psi$ que caracteriza cada superfície equipotencial foi originalmente escolhido por KT como o potencial de Roche, que como se sabe é composto por um termo que representa o potencial gravitacional simetricamente esférico e outro que corresponde ao potencial cilindricamente esférico devido à rotação. Posteriormente, Endal \& Sofia (1976) sugeriram o uso de um potencial mais preciso baseado na expansão de Clairaut-Legendre de um corpo auto-gravitatório (Kopal 1959; Tassoul 1978), que adiciona ao potencial de Roche um terceiro termo referente à parte simetricamente cilíndrica do potencial gravitacional devida à distorção causada pela rotação (vide Eqs. 3.14 e 3.20). Este potencial mais acurado é o que adotamos no presente trabalho.

Em sua formulação original, o método KT aplica-se somente ao caso de rotação conservativa, ou seja, quando a força centrífuga pode ser derivada de um potencial (vide eq. 3.23). No nosso caso, porém, será utilizada uma rotação da forma $\Omega=\Omega(r)$, que não é conservativa; contudo, como observado por Endal \& Sofia (1978), para moderadas taxas de rotação a velocidade angular pode ser aproximada por um valor constante sobre superfícies equipotenciais, mantendo assim a aplicabilidade do método KT. Um tratamento mais rigoroso do método KT aplicado a leis de rotação não-conservativas pode ser encontrado em Meynet \& Maeder (1997).

Três "leis de rotação" possíveis foram implementadas no código ATON 2.0: (a) rotação de corpo rígido em toda a estrela, (b) conservação local de momento angular em toda a estrela, e (c) uma combinação das anteriores (conservação local de momento angular em regiões radiativas e rotação de corpo rígido em regiões convectivas). Estas leis foram escolhidas de forma a representar um conjunto de leis fisicamente plausíveis. Em todos os três casos, uma perda artificial de momento angular é imposta aos modelos sempre que a velocidade angular exceder à que equivale à taxa de ruptura equatorial e, no caso (c), se o gradiente de velocidade angular na interface radiativa-convectiva for acentuado o suficiente para provocar uma interseção das superfícies equipotenciais em tal interface. Esta remoção artificial não é necessária se o mecanismo de redistribuição de momento angular estiver operante. O momento angular inicial adotado para modelos de quaisquer massas é obtido das relações de Kawaler (1987) para estrelas de tipo espectral mais frio que F2 (Eq. 3.28).

### 6.3.3 Redistribuição interna de momento angular

As correntes circulatórias presentes nas estrelas devido ao desequilíbrio térmico causado pela rotação acarretam rotação diferencial, e o gradiente de velocidade angular resultante, por sua vez, pode causar movimentos turbulentos em função de instabilidades hidrodinâmicas como as citadas na Sec. 6.2.2. Assim, o perfil interno de velocidade angular das estrelas depende de um complexo balanço entre a circulação meridional e estes movimentos turbulentos, dos quais se sabe muito pouco nas condições físicas dos interiores estelares. Chaboyer \& Zahn (1992) e Zahn (1992) trouxeram uma importante contribuição neste aspecto ao sugerir um tratamento integrado entre a circulação meridional e a turbulência em regiões radiativas de estrelas, baseado unicamente no pressuposto de que a turbulência em tais regiões é altamente anisotrópica, com movimentos horizontais muito mais fortes que os verticais. Como notado por Urpin et al. (1996), este pressuposto parece obter algum respaldo de estudos da atmosfera da Terra. Sob tais condições, os movimentos turbulentos horizontais tendem a estabelecer uma taxa de rotação que depende muito pouco da latitude, ou seja, $\Omega=\Omega(r)$, sendo a mesma denominada shellular por Zahn (1992). Desta forma, o tratamento de Chaboyer \& Zahn (1992) e Zahn (1992) permite reduzir o tratamento da rotação de duas para apenas uma dimensão, permitindo sua inclusão nos códigos computacionais já existentes.

Em adição, diferentemente do grupo de Yale que, como já mencionado, adotaram um tratamento puramente difusivo para o transporte interno de momento angular, a teoria de Chaboyer \& Zahn (1992) e Zahn (1992) parece ser mais consistente na medida em que trata os efeitos da circulação meridional não como um processo difusivo mas como um genuíno processo de advecção. Em nosso trabalho adotamos tal teoria para o transporte interno de momento angular implementado no código ATON2.1. A equação diferencial parcial (3.35) é a equação central desta teoria, sendo o seu lado direito composto por um termo advectivo correspondente ao transporte de momento angular pela circulação meridional, e um segundo termo que representa a parte difusiva propriamente dita.

A parte radial da velocidade de circulação meridional $U(r)$ que aparece na Eq. (3.35) é obtida a partir da bem-conhecida equação do calor (Eq. 3.36). Utilizando um processo de linearização dos valores das quantidades físicas envolvidas ao longo de superfícies equipotenciais (com exceção, naturalmente, da pressão $P$ que é constante nestas superfícies), a equação do calor pode ser escrita como a soma de dois componentes denominados $E_{\Omega}$ e $E_{\mu}$, sendo o primeiro referente ao desvio da simetria esférica devido à rotação shellular, e o segundo associado à variação horizontal do peso molecular médio (Eqs. 3.40 a 3.43). A amplitude da componente radial da velocidade de circulação meridional é, enfim, diretamente obtida a partir desta forma linearizada da equação do calor (vide Eq. 3.52).

Uma vez que as expressões originais de Zahn (1992) para o cálculo de $U(r)$ são voltadas para estrelas massivas da sequiência principal, com envelopes radiativos, algumas aproximações usadas pelo mesmo necessitam ser revistas para o caso de estrelas de baixa
massa na etapa pré-seqüência principal. Especificamente, a taxa de geração de energia gravitacional $\varepsilon_{\mathrm{G}}$ necessita ser somada àquela nuclear $(\varepsilon)$ na expressão para $E_{\Omega}$ (Eq. 3.42), e o termo $L / M$ na expressão final para $U(r)$ (Eq. 3.52) deve ser substituído por $L / M_{\star}$ onde $M_{\star}$ é a "massa reduzida" dada por $M\left(1-\Omega^{2} / 2 \pi G \rho_{m}\right)$. Além disto, é importante frisar que outras aproximações foram por nós introduzidas na obtenção de $E_{\Omega}$ e $E_{\mu}$, como o emprego de um gás ideal no cálculo de várias derivadas intermediárias e a utilização de um processo numérico de suavização (smoothing) em algumas destas derivadas.

Os resultados obtidos para $U(r)$ pela expressão (3.52), contudo, quando aplicados a estrelas de baixa massa nas etapas pré-seqüência principal e seqüência principal, não são compatíveis com os resultados obtidos por outros autores no caso de estrelas massivas (Urpin et al. 1996; Talon et al. 1997; Denissenkov et al. 1999). Como se pode ver das Figs. 3.1 e 3.2 , que representam modelos de $1 M_{\odot}$ em idades típicas da pré-seqüência principal ( $10^{7}$ anos) e seqüência principal ( $3.16 \times 10^{7}$ anos), a velocidade de circulação meridional é negativa por uma região apreciável do interior radiativo. Particularmente na etapa pré-seqüência principal (Fig. 3.1), esta velocidade negativa significaria que as correntes circulatórias desceriam nos pólos e subiriam no equador, agravando o desequilíbrio térmico provocado pela rotação ao invés de compensar o mesmo. Nós creditamos estes resultados ao fato que a teoria de Zahn (1992), apesar de ser genérica, utiliza uma série de aproximações voltadas para o caso de estrelas massivas na seqüência principal. Em adição, esta teoria até o momento ainda não tinha sido testada para estrelas de baixa massa. Por estas razões, uma vez que estamos interessados apenas numa estimativa para $U(r)$, decidimos aproximá-la utilizando o valor local que a mesma teria (ou seja, em cada ponto da grade numérica) considerando rotação de corpo rígido (Eq. 3.59). Esta mesma aproximação é a utilizada pelo grupo de Yale, apesar do emprego de uma expressão diferente para $U(r)$. Neste valor aproximado de $U(r)$ devem ser levados em consideração, ainda, os efeitos da "barreira de peso molecular" (Mestel 1953, Kippenhahn \& Weigert 1994), que se traduzem numa velocidade fictícia dada pela Eq. (3.60), a ser subtraída do valor obtido de $U(r)$.

Os coeficientes de difusão referentes a cada instabilidade hidrodinâmica considerada são obtidos como o produto entre as respectivas velocidades e comprimentos de escala característicos (Eq. 3.61). Tais coeficientes são então somados de forma a compor o coeficiente final de difusão contido na equação diferencial parcial de adveç̧ão-difusão. Esta equação, por sua vez, foi convertida para uma equação de diferenças finitas e resolvida por um método totalmente implícito (vide detalhes no apêndice A). Cada região convectiva é tratada como se toda a sua massa estivesse concentrada em um único ponto na grade numérica. A região radiativa estreita que usualmente aparece na camada mais externa dos modelos de $1 M_{\odot}$ foi desprezada, numa primeira aproximação. Os limites entre regiões radiativas e convectivas são tratados como uma zona de transição sempre convectiva até que o gradiente superadiabático $\left(\nabla_{\mathrm{ad}}-\nabla\right)$ supere o valor de 0.01 .

A perda de momento angular na superfície estelar devida à ação de ventos magnetizados
foi também considerada no código ATON 2.1 , sendo modelada através das expressões (3.63) e (3.64), obtidas de Chaboyer et al. (1995b). Estas perdas são consideradas uma condição de contorno na solução da equação de adveç̧ão-difusão.

### 6.4 Aplicações de modelos estelares com rotação

### 6.4.1 Diferenças evolutivas entre modelos com e sem rotação

Os resultados obtidos através do código ATON 2.1 referentes à evolução de estrelas com rotação confirmam as características já observadas por outros pesquisadores desde o final da década de 60 (p. ex. Sackmann 1970): uma estrela de massa $M$ com rotação uniforme tem as mesmas propriedades de uma estrela equivalente sem rotação mas com massa $M-\Delta M$ (o chamado mass-lowering effect), como exemplificado na Fig. 4.1. Este resultado se deve basicamente ao fato de que a força centrífuga que resulta da rotação compensa parcialmente a força de contração gravitacional, e ocorre para qualquer uma das leis de rotação implementadas (Fig. 4.2). Obtivemos ainda que tais efeitos da rotação são muito modestos, tornando-se apreciáveis somente para taxas de rotação muito elevadas.

### 6.4.2 Queima de lítio em estrelas de baixa massa com rotação

Como afirmado anteriormente na Sec. 6.2.2, a questão da abundância de lítio nas estrelas jovens de baixa massa sempre foi objeto de muito interesse. A abundância de lítio nestas estrelas é correlacionada com diversos parâmetros físicos tais como a massa (Cayrel et al. 1984), a metalicidade (Hobbs \& Duncan 1987) e a rotação (Marcy et al. 1985; Soderblom 1983; Martín et al. 1994). No tocante à esta última, os dados observacionais disponíveis sugerem que as estrelas que giram mais rápido apresentam maior conteúdo de lítio que as com menores taxas de rotação (García López et al. 1994, Cunha et al. 1995). No campo teórico, os resultados disponíveis até então são contraditórios, com o grupo de Yale (Pinsonneault et al. 1990) encontrando que a rotação diminui o conteúdo de lítio (Fig. 4.3) ao passo que Martín \& Claret (1996) obtiveram o resultado oposto (Fig. 4.4), em concordância com o quadro sugerido pelas observações.

Motivados por estas discrepâncias, executamos uma série de modelos de massas na faixa $0.6 M_{\odot}$ até $1.2 M_{\odot}$, partindo da etapa pré-seqüência principal, com o objetivo de verificar os efeitos da rotação na queima de lítio. Nossos resultados indicam que a rotação efetivamente contribui para uma maior queima de lítio como indicado na Fig. 4.5, em acordo com os resultados do grupo de Yale; estes resultados podem ser facilmente entendidos com base no efeito de mass lowering citado anteriormente. Uma vez que o código ATON 2.1 utiliza opacidades bem atualizadas, a discrepância entre os resultados do grupo de Yale e os de Martín \& Claret (1996) não podem ser creditadas ao uso, pelo grupo de

Yale, de opacidades mais antigas, como sugerido pelos primeiros. Uma discussão completa sobre esta questão é apresentada nos artigos de Mendes et al. $(1997,1999)$ incluídos nos Apêndices B e C.

### 6.4.3 Efeitos da redistribuição interna de momento angular

Com o objetivo de testar os efeitos da redistribuição interna de momento angular em estrelas de baixa massa desde a etapa pré-seqüência principal, foram calculados diversos modelos de $0.6,0.8$ e $1 M_{\odot}$, explorando diferentes valores de determinados parâmetros de entrada a fim de se comparar os resultados obtidos e analisar sua sensibilidade a estes parâmetros. Para tais estrelas, o principal aspecto dos efeitos da redistribuição de momento angular refere-se à mistura dos componentes químicos mas, no caso particular dos modelos solares, também no tocante ao perfil interno da velocidade angular, uma vez que resultados da heliosismologia fornecem tal perfil com significativa precisão até a profundidade de de $r / R_{\odot}=0.4$ (Kosovichev et al. 1997).

Os parâmetros referentes aos modelos computados estão listados na Tab. 4.1, na qual a coluna 1 indica a identificação do modelo; a coluna 2; a massa; a coluna 3, o momento angular inicial; a coluna 4, o modelo de convecção (Mixing Length Theory [MLT] ou o Full Spectrum of Turbulence [FST]); as colunas 5 e 6, as concentrações de hélio e metais respectivamente; a coluna 7 , o fator multiplicativo do coeficiente total de difusão; a coluna 8 , o wind index utilizado no cálculo da perda de momento angular por ventos magnetizados; e a coluna 9, se o termo advectivo foi incluído ou não na equação de transporte de momento angular. Além disto, para todos os modelos foram utilizadas as opacidades OPAL (Iglesias \& Rogers 1993) complementadas pelas de Alexander \& Ferguson (1994) para temperaturas inferiores a 6000 K , empregando-se ainda $\alpha=1.5$ para os modelos com convecção MLT ou $\beta=0.15 H_{\mathrm{P}}$ para os modelos FST.

De maneira geral, obtivemos que o transporte interno de momento angular é bastante eficaz na suavização do perfil interno de velocidade angular, como demonstrado na Fig. 4.6 para o modelo denominado OA. Esta figura mostra a evolução deste perfil ao longo de quatro idades características, indicadas também na trilha evolutiva correspondente da Fig. 4.7. Como esperado, durante a maior parte da etapa pré-sequiência principal (curvas A e B) os efeitos da contração ao longo da trilha de Hayashi são predominantes, levando a um núcleo com alta rotação cujo máximo é atingido próximo da Idade Zero da Seqüência Principal (ZAMS, curva C). Deste ponto em diante a estrela torna-se estável e os efeitos da redistribuição interna de momento angular tornam-se aparentes (curva D). Este mesmo comportamento é também obtido por modelos de massas inferiores ( 0.6 e $0.8 M_{\odot}$ ), e ocorre qualquer que seja o modelo de convecção escolhido. Aumentando-se o momento angular inicial, obtém-se um efeito de suavização ainda maior, como se pode ver na Fig. 4.9; isto pode ser explicado considerando-se que as instabilidades hidrodinâmicas, e consequentemente o transporte interno de momento angular, dependem diretamente do
gradiente de velocidade angular, e portanto um gradiente mais agudo provocará maior redistribuição. A redistribuição de momento angular contribui também para um aumento da queima de lítio, como exemplificado nas Figs. 4.15 e 4.16, e portanto amplificando a queima que já ocorre devido aos efeitos puramente estruturais da rotação.

Outro aspecto investigado é a real importância, na equação diferencial parcial que governa o transporte de momento angular, do termo advectivo que compõe a mesma, tendo em vista as críticas que existem quando ao modelamento puramente difusivo adotado pelo grupo de Yale. Nossos resultados mostram que os efeitos deste termo são totalmente desprezíveis para a faixa de massas consideradas, devido aos valores muito baixos da velocidade de circulação meridional, justificando portanto a aproximação do grupo de Yale, pelo menos para estrelas de baixa massa.

Finalmente, um teste importante da redistribuição de momento angular refere-se ao perfil interno de rotação do Sol. A Fig. (4.18) mostra este perfil, obtido com base em observações das oscilações solares, no qual verifica-se que a taxa de rotação no interior radiativo é aproximadamente constante até a profundidade de $r / R_{\odot}=0.4$, quando então tais observações perdem precisão. Nossos resultados para os modelos de $1 M_{\odot}$, contudo, somente conseguem reproduzir este perfil multiplicando-se artificialmente o coeficiente total de difusão por valores tão altos quanto $10^{5}$ (vide Fig. 4.19), que consideramos fisicamente implausíveis. Desta forma, conclui-se que no Sol os mecanismos de redistribuição de momento angular são mais eficientes do que as técnicas adotadas são capazes de modelar.

### 6.5 Conclusões

### 6.5.1 Efeitos estruturais da rotação e sua efeito na queima de lítio

A investigação realizada demonstrou que estrelas com rotação comportam-se como uma estrela equivalente sem rotação porém com menor massa, devido basicamente à ação da força centrífuga. As trilhas evolutivas no diagrama H-R das estrelas com rotação, portanto, apresentam menores luminosidades e temperaturas efetivas em relação às estrelas sem rotação. Estes resultados encontram-se de acordo com aqueles obtidos por outros códigos evolutivos (p.ex. Sackmann 1970; Pinsonneault et al. 1989, 1990). No caso de estrelas de baixa massa, este comportamento acarreta, contudo, uma queima de lítio maior que nas estrelas em rotação (tendo em vista que a temperatura na base do envelope convectivo aumenta à medida em que se diminui a massa estelar), em discordância com o quadro sugerido pelas observações de estrelas jovens em aglomerados abertos. Isto sugere que outros mecanismos físicos devem influir na abundância de lítio das estrelas de baixa massa.

### 6.5.2 Propriedades de estrelas de baixa massa com transporte interno de momento angular

O momento angular é uma propriedade intrínseca das estrelas, e deve ser internamente redistribuído de forma a explicar a taxa de rotação superficial das estrelas e também suas curvas internas de rotação, como observado no caso do Sol através da heliosismologia. No presente trabalho, empregou-se a teoria de Chaboyer \& Zahn (1992) e Zahn (1992), que resulta numa equação diferencial parcial de advecção-difusão, para este transporte de momento angular. Constatou-se que o transporte interno de momento angular pode ser bastante eficaz na suavização do perfil interno de velocidade angular em estrelas. No caso solar, contudo, esta suavização não é suficiente para reproduzir a curva de rotação observada no interior do Sol. Com relação aos efeitos de mistura por rotação (rotational mixing), o transporte de momento angular contribui para um acréscimo na queima de lítio ainda maior que o obtido pelos efeitos estruturais da rotação, agravando as discrepâncias com o padrão de esgotamento de lítio sugerido pelas observações. Estes fatores sugerem que outros mecanismos físicos devem operar nos interiores estelares, de forma a permitir uma redistribuição de momento angular mais eficiente e, ao mesmo tempo, compatibilizando as abundâncias de lítio com os dados observacionais.

### 6.5.3 Diretrizes para a continuidade do trabalho

Ao longo do desenvolvimento deste trabalho, constatou-se a importância da inclusão de diversos aperfeiçoamentos adicionais no código ATON 2.1 especificamente relacionados com a rotação. Alguns destes aperfeiçoamentos não apresentam maiores dificuldades e podem ser executados num intervalo de tempo relativamente curto, ao passo que os demais, devido à sua inerente complexidade, demandarão esforços de trabalho e de tempo significativamente maiores.

No primeiro grupo, podemos citar a inclusão do coeficiente de difusão dos componentes químicos, devido à rotação, no processo de difusão microscópica já considerado pelo código ATON 2.1; o tratamento dos efeitos devidos a um gradiente do peso molecular que, como demonstrado por Mestel (1953), pode reduzir significativamente as correntes circulatórias internas (apesar de trabalhos recentes fornecerem resultados contraditórios a respeito, como os de Chaboyer et al. [1995a] e Meynet \& Maeder [1997]); e a extensão do código para lidar com a redistribuição de momento angular em estrelas de massas intermediárias ou maiores. No segundo grupo, incluem-se o mecanismo de regulação do momento angular de estrelas jovens através da interação das mesmas com seus discos circumstelares (disklocking), o qual, contudo, foi posto em xeque pelos resultados recentes de Stassun et al. (1999a, 1999b); a interação entre rotação e campos magnéticos (Mestel et al. 1988, Tassoul \& Tassoul 1986, 1989); e, finalmente, a inclusão dos efeitos devidos às chamadas ondas de gravidade (Schatzman 1993; Talon \& Zahn 1998).

## Appendix A

## Fully implicit algorithm for solving the diffusion-advection equation

This appendix presents the discretization and the fully implicit scheme used for solving the partial differential equation (PDE) that governs the advection and diffusion of angular momentum.

We start from the analytical expression for the PDE, given by Eq. (3.35):

$$
\begin{equation*}
\rho \frac{d}{d t}\left[r^{2} \bar{\Omega}\right]=\frac{1}{5 r^{2}} \frac{\partial}{\partial r}\left(\rho r^{4} \bar{\Omega} U\right)+\frac{1}{r^{2}} \frac{\partial}{\partial r}\left[\rho \nu_{v} r^{\frac{\partial}{\Omega}} \frac{\partial \bar{\Omega}}{\partial r}\right] . \tag{A.1}
\end{equation*}
$$

The first step is to cast the right-hand side of that equation in the more convenient lagrangian formulation. This is simply done by using the well known relation between eulerian and lagrangian coordinates,

$$
\frac{\partial}{\partial r}=4 \pi r^{2} \rho \frac{\partial}{\partial m}
$$

After substituting for the above relation and dividing everything by $\rho$, Eq. (3.35) takes the form (with the overbars dropped)

$$
\begin{equation*}
\frac{d}{d t}\left[r^{2} \Omega\right]-\frac{4 \pi}{5} \frac{\partial}{\partial m}\left(\rho r^{4} \Omega U\right)=16 \pi^{2} \frac{\partial}{\partial m}\left[\rho^{2} \nu_{v} r^{6} \frac{\partial \Omega}{\partial m}\right] . \tag{A.2}
\end{equation*}
$$

Expanding the time derivative on the left-hand side and again dividing the whole equation
by $r^{2}$, we get

$$
\begin{equation*}
\frac{d \Omega}{d t}+\Omega \frac{1}{r^{2}} \frac{d r^{2}}{d t}-\frac{4 \pi}{5 r^{2}} \frac{\partial}{\partial m}\left(\rho r^{4} \Omega U\right)=\frac{16 \pi^{2}}{r^{2}} \frac{\partial}{\partial m}\left[\rho^{2} \nu_{v} r^{6} \frac{\partial \Omega}{\partial m}\right] . \tag{A.3}
\end{equation*}
$$

Now we rewrite Eq. (A.3) as a difference equation based on the lagrangian implicit scheme proposed by Cloutman \& Eoll (1976). By using the notation $\tau_{i}^{n}$ for a given quantity $\tau$ at the mesh point $i$ and time $n$, that equation can be written as

$$
\begin{align*}
& \frac{\Omega_{i}^{n+1}-\Omega_{i}^{n}}{\Delta t}+\Omega_{i}^{n+1} \frac{1}{r_{i}^{2}} \frac{\left(r^{2}\right)_{i}^{n+1}-\left(r^{2}\right)_{i}^{n}}{\Delta t}-  \tag{A.4}\\
& \frac{4 \pi}{5 r_{i}^{2}} \frac{\rho_{i+1} r_{i+1}^{4} U_{i+1} \Omega_{i+1}^{n+1}-r_{i}^{4} \rho_{i} U_{i} \Omega_{i}^{n+1}}{m_{i+1}-m_{i}}= \\
& \frac{16 \pi^{2}}{r_{i}^{2}} \frac{2}{m_{i+1}-m_{i-1}}\left[\frac{G_{i+1 / 2}\left(\Omega_{i+1}^{n+1}-\Omega_{i}^{n+1}\right)}{m_{i+1}-m_{i}}-\frac{G_{i-1 / 2}\left(\Omega_{i}^{n+1}-\Omega_{i-1}^{n+1}\right)}{m_{i}-m_{i-1}}\right]
\end{align*}
$$

where

$$
\begin{aligned}
& G_{i}=\rho_{i}^{2} \nu_{i} r_{i}^{6} \\
& G_{i+1 / 2}=\frac{1}{2}\left(G_{i}+G_{i+1}\right), \\
& G_{i-1 / 2}=\frac{1}{2}\left(G_{i}+G_{i-1}\right) .
\end{aligned}
$$

Now, by virtue of the scheme adopted for advancing the angular momentum distribution in time (cf. Sect. 3.4.5), at each timestep both values $\left(r^{2}\right)_{i}^{n}$ and $\left(r^{2}\right)_{i}^{n+1}$ are already computed, so that the discretized term $d r^{2} / d t$ can be treated as a known quantity. Taking

$$
\begin{equation*}
R_{i}=\frac{\left(r^{2}\right)_{i}^{n+1}-\left(r^{2}\right)_{i}^{n}}{\Delta t} \tag{A.5}
\end{equation*}
$$

and multiplying Eq. (A.5) by $\Delta t$ we get, after some rearrangements,

$$
\begin{align*}
& \Omega_{i}^{n+1}\left\{1+\frac{R_{i} \Delta t}{r_{i}^{2}}+\frac{4 \pi \Delta t \rho_{i} r_{i}^{4} U_{i}}{5 r_{i}^{2}\left(m_{i+1}-m_{i}\right)}+\right. \\
& \left.\quad \frac{32 \pi^{2} \Delta t}{r_{i}^{2}\left(m_{i+1}-m_{i-1}\right)}\left[\frac{G_{i-1 / 2}}{m_{i}-m_{i-1}}+\frac{G_{i+1 / 2}}{m_{i+1}-m_{i}}\right]\right\}= \\
& \quad \Omega_{i+1}^{n+1}\left\{\frac{4 \pi \Delta t \rho_{i+1} r_{i+1}^{4} U_{i+1}}{5 r_{i}^{2}\left(m_{i+1}-m_{i}\right)}+\frac{32 \pi^{2} \Delta t G_{i+1 / 2}}{r_{i}^{2}\left(m_{i+1}-m_{i-1}\right)\left(m_{i+1}-m_{i}\right)}\right\}+ \\
& \quad \Omega_{i-1}^{n+1}\left\{\frac{32 \pi^{2} \Delta t G_{i-1 / 2}}{r_{i}^{2}\left(m_{i+1}-m_{i-1}\right)\left(m_{i}-m_{i-1}\right)}\right\}+\Omega_{i}^{n} . \tag{A.6}
\end{align*}
$$

## Taking

$$
\begin{align*}
\alpha_{i}= & \left\{1+\frac{R_{i} \Delta t}{r_{i}^{2}}+\frac{4 \pi \Delta t \rho_{i} r_{i}^{4} U_{i}}{5 r_{i}^{2}\left(m_{i+1}-m_{i}\right)}+\right. \\
& \left.\frac{32 \pi^{2} \Delta t}{r_{i}^{2}\left(m_{i+1}-m_{i-1}\right)}\left[\frac{G_{i-1 / 2}}{m_{i}-m_{i-1}}+\frac{G_{i+1 / 2}}{m_{i+1}-m_{i}}\right]\right\},  \tag{A.7}\\
\beta_{i}= & \left\{\frac{4 \pi \Delta t \rho_{i+1} r_{i+1}^{4} U_{i+1}}{5 r_{i}^{2}\left(m_{i+1}-m_{i}\right)}+\frac{32 \pi^{2} \Delta t G_{i+1 / 2}}{r_{i}^{2}\left(m_{i+1}-m_{i-1}\right)\left(m_{i+1}-m_{i}\right)}\right\},  \tag{A.8}\\
\gamma_{i}= & \left\{\frac{32 \pi^{2} \Delta t G_{i-1 / 2}}{r_{i}^{2}\left(m_{i+1}-m_{i-1}\right)\left(m_{i}-m_{i-1}\right)}\right\},  \tag{A.9}\\
\delta_{i}= & \Omega_{i}^{n}, \tag{A.10}
\end{align*}
$$

Eq. (A.6) can be cast in the form

$$
\begin{equation*}
\alpha_{i} \Omega_{i}^{n+1}=\beta_{i} \Omega_{i+1}^{n+1}+\gamma_{i} \Omega_{i-1}^{n+1}+\delta_{i} . \tag{A.11}
\end{equation*}
$$

The set of Eqs. (A.11) for each mesh point can be written in matrix form (with $M$ standing for the number of mesh points) as

$$
\left[\begin{array}{ccccccccc}
\alpha_{1} & -\beta_{1} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-\gamma_{2} & \alpha_{2} & -\beta_{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -\gamma_{3} & \alpha_{3} & -\beta_{3} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & -\gamma_{M-1} & \alpha_{M-1} & -\beta_{M-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\gamma_{M} & \alpha_{M}
\end{array}\right]\left[\begin{array}{c}
\Omega_{1}^{n+1} \\
\Omega_{2}^{n+1} \\
\Omega_{3}^{n+1} \\
\vdots \\
\Omega_{M+1}^{n+1} \\
\Omega_{M}^{n+1}
\end{array}\right]=\left[\begin{array}{c}
\Omega_{1}^{n} \\
\Omega_{n}^{n} \\
\Omega_{3}^{n} \\
\vdots \\
\Omega_{M n-1}^{n} \\
\Omega_{M}^{n}
\end{array}\right]
$$

which, due to its tridiagonal nature, can be easily solved by standard techniques (see e.g. Press et al. 1992).

We now proceed to the iterative algorithm we use to solve the above matrix equation. For all mesh points running from 2 to $M-1$, we want to write an expression of the form

$$
\begin{equation*}
\Omega_{i}^{n+1}=\tau_{i}+\mu_{i} \Omega_{i+1}^{n+1} . \tag{A.12}
\end{equation*}
$$

In order to do that, we substitute the equivalent value for $\Omega_{i-1}^{n+1}$ given by the above relation in Eq. (A.11), obtaining

$$
\begin{align*}
\alpha_{i} \Omega_{i}^{n+1} & =\beta_{i} \Omega_{i+1}^{n+1}+\gamma_{i}\left(\tau_{i-1}+\mu_{i-1} \Omega_{i}^{n+1}\right)+\delta_{i} \\
& =\beta_{i} \Omega_{i+1}^{n+1}+\gamma_{i} \tau_{i-1}+\gamma_{i} \mu_{i-1} \Omega_{i}^{n+1}+\delta_{i} \tag{A.13}
\end{align*}
$$

Solving for $\Omega_{i}^{n+1}$, we get

$$
\begin{equation*}
\Omega_{i}^{n+1}=\left(\frac{\gamma_{i} \tau_{i-1}+\delta_{i}}{\alpha_{i}-\gamma_{i} \mu_{i-1}}\right)+\Omega_{i+1}^{n+1}\left(\frac{\beta_{i}}{\alpha_{i}-\gamma_{i} \mu_{i-1}}\right) . \tag{A.14}
\end{equation*}
$$

Comparing Eqs. (A.12) and (A.14), we obtain the recursive formulae for computing $\tau_{i}$ and $\mu_{i}$ at each mesh point,

$$
\begin{align*}
& \tau_{i}=\frac{\gamma_{i} \tau_{i-1}+\delta_{i}}{\alpha_{i}-\gamma_{i} \mu_{i-1}},  \tag{A.15}\\
& \mu_{i}=\frac{\beta_{i}}{\alpha_{i}-\gamma_{i} \mu_{i-1}} . \tag{A.16}
\end{align*}
$$

These equations must of course be supplemented by the equivalent ones for the first valid mesh point, which in the ATON 2.0 evolutionary code is labelled as $i=2$ (with $i=1$ corresponding to $r=0$ ). Applying Eq. (A.11) for $i=2$, we obtain

$$
\begin{equation*}
\alpha_{2} \Omega_{2}^{n+1}=\beta_{2} \Omega_{3}^{n+1}+\gamma_{2} \Omega_{1}^{n+1}+\delta_{2} . \tag{A.17}
\end{equation*}
$$

The boundary condition at the center is

$$
\begin{equation*}
\Omega_{2}^{n+1}=\Omega_{1}^{n+1} \tag{A.18}
\end{equation*}
$$

Substituting it in Eq. (A.17), it follows that

$$
\Omega_{2}^{n+1}=\frac{\beta_{2}}{\alpha_{2}-\gamma_{2}} \Omega_{3}^{n+1}+\frac{\delta_{2}}{\alpha_{2}-\gamma_{2}}
$$

which is the same as

$$
\Omega_{2}^{n+1}=\tau_{2}+\mu_{2} \Omega_{3}^{n+1}
$$

so that

$$
\begin{align*}
\tau_{2} & =\frac{\delta_{2}}{\alpha_{2}-\gamma_{2}}  \tag{A.19}\\
\mu_{2} & =\frac{\beta_{2}}{\alpha_{2}-\gamma_{2}} . \tag{A.20}
\end{align*}
$$

It remains to derive the value of $\Omega_{M}^{n+1}$. This is easily done by applying the same procedure outlined for deriving the quantities $\tau_{2}$ and $\mu_{2}$ : we apply Eq. (A.11) for $i=M$ and substitute in it the boundary condition at $i=M$,

$$
\begin{equation*}
\Omega_{M-1}^{n+1}=\Omega_{M}^{n+1} \tag{A.21}
\end{equation*}
$$

obtaining the final result

$$
\begin{equation*}
\Omega_{m}^{n+1}=\frac{\tau_{M-1}}{1-\mu_{M-1}} . \tag{A.22}
\end{equation*}
$$

The algorithm for solving Eq. (A.2) can now be summarized as follows:

1. Compute quantities $\alpha_{2}, \beta_{2}, \lambda_{2}$ and $\delta_{2}$ by means of Eqs. (A.7) to (A.10).
2. Compute quantities $\tau_{2}$ and $\mu_{2}$ through Eqs. (A.19) and (A.20).
3. For all mesh points from $i=3$ to $i=M-1$ :

- compute $\alpha_{i}, \beta_{i}, \lambda_{i}$ and $\delta_{i}$ again through Eqs. (A.7) to (A.10);
- compute $\tau_{i}$ and $\mu_{i}$ by using Eqs. (A.19) and (A.20).

4. Compute $\Omega_{M}^{n+1}$ (Eq. A.21).
5. For all mesh points from $i=M-1$ down to $i=2$ compute $\Omega_{i}^{n+1}$ through Eq. (A.12).
6. Compute $\Omega_{1}^{n+1}=\Omega_{2}^{n+1}$.

Notice that the above algorithm does not require keeping variables $\alpha_{i}, \beta_{i}, \lambda_{i}$ and $\delta_{i}$ at each iteration.

## Appendix B

Paper published in the Memorie della Società Astronomica Italiana journal

# ROTATING STELLAR MODELS AND LITHIUM DEPLETION 

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#### Abstract

The ATON stellar evolution code has been modified to include rotation, in the approximation by Endal \& Sofia (1976). As a preliminary application of this improved code, we have run some rotating stellar models to investigate the effects of rotation on the Lithium depletion of low mass, pre-main sequence stars. The models were run with rigid body rotation, no angular momentum loss and redistribution, and under two prescriptions for convection, namely the mixing length theory [MLT] and the turbulent convection model by Canuto \& Mazzitelli (1991) [CM]. The calculations show that rotation increases the Lithium depletion for both MLT and CM, a result which is expected from the theory since rotating stars behave as non-rotating stars of lower mass and so must experience greater lithium depletion. This result shows that some other parameter controls lithium depletion in pre-main sequence stars, as it is necessary to explain the observational data on low mass, pre-main sequence stars such as those from the Pleiades (García López et al. 1994) and the $\alpha$-Persei clusters (Balachandran et al. 1988, 1996).


## 1. Introduction

The ATON stellar structure code (Mazzitelli 1989) has been modified in order to introduce rotation. We chose to follow the approach taken by Endal \& Sofia (1976), which in turn adopts the Kippenhahn \& Thomas (1970) method with an improved potential function that includes a third term related to the distortion of the figure of the star. Three rotation laws have been implemented, namely: rigid body rotation, local conservation of angular momentum throughout the whole star, and local conservation of angular momentum in radiative regions and rigid body rotation in convective ones. For all three rotation laws, our results confirm earlier findings from other researchers (e.g. Sackmann 1970; Pinsonneault et al. 1989; Martín \& Claret 1996): the evolutionary tracks of rotating stars shift towards lower effective temperatures and luminosities, simulating the evolutionary path of a non-rotating star of lower mass (the so-called mass-lowering effect; see Sackmann 1970). However, we have also found that these differences are very small, being considerable only for high initial angular momentum. Figure 1 shows the evolutionary tracks of a $1.0 M_{\odot}$ with solar chemical composition, for both rotating (with rigid body rotation) and non-rotating cases.


Fig. 1. Evolutionary tracks for rotating and non-rotating models of $1 M_{\odot}$. The convection model is that from Canuto \& Mazzitelli (1990, 1992), and the initial angular momentum is $3.93 \times 10^{50} \mathrm{~g} \mathrm{~cm}^{2} \mathrm{~s}^{-1}$.

## 2. Lithium depletion and rotation

The nowadays observational picture indicates that lithium depletion on low mass, premain sequence stars is related to rotation in such a way that rapid rotators have smaller lithium depletion than slow rotators (Balachandran et al. 1988, 1996; García López et al. 1994; Cunha et al. 1995). On the theoretical side, so far very few attempts have been made to include rotation in evolutionary codes, and even less to check the effects of rotation on lithium depletion. Pinsonneault et al. (1990) have found that rotation increases the lithium depletion, while Martín \& Claret (1996) claimed that their rotating models deplete less lithium than the non-rotating ones, in accordance to the observational picture. As these theoretical models are conflicting, it is important indeed to sort out whether the structural differences between the rotating and non-rotating models work towards confirming the observational constraints or not. In the latter case, the differences in lithium depletion must be entirely attributed to the phases of angular momentum transport and associated chemical mixing.

As a preliminary application of the ATON code with its new feature, we have run a series of rotating models of $1.2,1.1,1.0,0.9$ and 0.8 solar masses to check the structural effects of rotation on the lithium depletion of low mass, pre-main sequence stars. The input physics used for the models is as follows. The opacities were taken from Iglesias \& Rogers (1993), supplemented by those at low and intermediate temperatures from Alexander \& Ferguson (1994) or Kurucz (1993). The chemical composition was set to Y $=0.271, \mathrm{Z}=0.0175$. For the convection treatment we have used both the mixing length theory (MLT), with $\alpha=1.5$, and the turbulent convection model of Canuto \& Mazzitelli (1991, 1992) (CM). For comparison purposes, we have chosen rigid body rotation with initial angular momentum estimated according to the relations given by Kawaler (1987) for stars of spectral type later than F2.


Fig. 2. Lithium depletion as function of age for masses of $1.0 M_{\odot}$ (left) and $0.80 M_{\odot}$ (right) with a metallicity of $\mathrm{Z}=0.0175$.

## 3. Main results and discussion

Figure 2 shows the run of lithium concentration (in the scale $N[\mathrm{Li}]=12+\log \frac{N[\operatorname{Li}]}{N[H]}$ ) as a function of age, for 1.0 and $0.8 M_{\odot}$. The initial lithium abundance was set to $\log N[\mathrm{Li}]=$ 3.31. We see that rotation indeed increases the lithium depletion. Since lithium burning is a function of the temperature at the bottom of the convective envelope, greater depletion means higher temperatures, and we can see from figure 3 that actually this is the case.

Martin \& Claret (1996) attributed the differences of their results, with respect to those of Pinsonneault et al. (1990), mainly to the adopted opacities, as the latter authors used the older Cox \& Stewart (1970) libraries. However our results - obtained with updated opacities - are qualitatively similar to those of Pinsonneault et al. . We cannot trace back why the results of Martin \& Claret differ from ours and from those of Pinsonneault et al. but we stress that (1) rotation cannot, by itself, reduce lithium depletion, and (2) recent opacities increase lithium depletion in both standard (non-rotating) and non-standard (rotating) models. These facts, and the need to fit the observational constraints, lead us to conclude that other physical mechanisms must play a role.

If structural effects are to be looked for, it seems worthy recalling that we do not consider the effect of the dynamo-induced magnetic field on the convection. It is possible that the larger field induced in rapid rotators inhibit turbulent mixing (Spruit 1987).

## Acknowledgements

L. T. S. Mendes acknowledges the support from the Conselho Nacional de Pesquisa e Desenvolvimento (CNPq), Brazil, and from the Osservatorio Astronomico di Roma, Italy, through the CNR grant 97.00017.CT02.


Fig. 3. Temperature at the bottom of the convective region for rotating and non-rotating models of $0.8 M_{\odot}$.

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## Appendix C

## Paper published in the Astronomy and Astrophysics main journal

## ASTRONOMY AND ASTROPHYSICS

# Theoretical models of low-mass, pre-main sequence rotating stars 

## I. The effects on lithium depletion

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Received 9 January 1998 / Accepted 29 July 1998


#### Abstract

Rotating stellar models of $1.2 M_{\odot}$ down to $0.6 M_{\odot}$ have been computed to investigate the effects of rotation on the lithium depletion of low-mass, pre-main sequence stars. The models were generated under three different rotation laws (rigid body rotation, local conservation of angular momentum over the whole star, and local conservation of angular momentum in radiative zones and rigid body rotation in convective ones), no angular momentum loss and redistribution, and under two prescriptions for convection, namely the mixing length theory [MLT] and the turbulent convection introduced by Canuto \& Mazzitelli (1991) [CM]. The general features of the rotating models are compatible with previous results by other authors. As for the lithium depletion, our results show that rotation decreases lithium depletion while the star is fully convective but increases it as soon as the star develops a radiative core, a result which is expected from the theory since rotating stars behave as non-rotating stars of lower mass and so must experience greater lithium depletion. The results hold for all three rotation laws assumed, but are specifically presented here for the case of rigid body rotation. This result shows that other physical mechanisms must play a role on the lithium depletion in the pre-main sequence, in order to explain the observational data on low-mass, pre-main sequence stars such as those from the Pleiades (García López et al. 1994) and the $\alpha$-Persei clusters (Balachandran et al. 1988, 1996).


Key words: stars: rotation - stars: pre-main sequence - stars: interiors - stars: evolution

## 1. Introduction

It has long been known that rotation is an important physical parameter in the theory of stellar structure and evolution, due to the changes it brings to the internal structure of stars. Besides, rotation is linked to a number of important physical processes in stars such as magnetic fields and the burning of light elements

[^9]to name just a few. The subject of stellar rotation is discussed in greater detail, e.g., in Tassoul (1978).

Of all non-standard physical processes that could significantly influence the stellar depletion of lithium, rotation is undoubtedly the most quoted, since it is a natural agent for chemical mixing inside the stars (e.g. Charbonnel \& Vauclair 1991; Pinsonneault 1991; Zahn 1992, 1993, 1994; Charbonnel et al. 1994; Strom 1994). Nevertheless, so far very few attempts have been made to include rotation in evolutionary codes, and even less to check the effects of rotation on lithium depletion.

As part of an ongoing project to improve the macro- and micro-physics of the ATON stellar structure code (Mazzitelli 1989 and references therein), we have then modified it to incorporate the hydrostatic structural effects (i.e. the changes in the internal structure of the star) due to presence of rotation.

Despite the great effort that has been conducted towards a better understanding of the depletion of light elements such as beryllium and lithium in young, low mass stars (as these elements can be used as tracers of the stellar evolution), the current picture is still subject to many uncertainties. Lithium depletion is related to a number of physical parameters such as stellar mass (Cayrel et al. 1984), metallicity (Hobbs \& Duncan 1987) and rotation (Marcy et al. 1985; Soderblom 1993; Martín et al. 1994; Randich et al. 1998). Rotating models have been built up by the Yale group (Pinsonneault et al. 1990, and recently Chaboyer et al. 1995). The resulting pre-main sequence lithium depletion is modestly dependent on the stellar rotation rate. In fact during the pre-main sequence phase, rotating models deplete just a bit more lithium than the non-rotating models due to the hydrostatic properties of the rotating models. The main importance of rotation resides in the subsequent redistribution of angular momentum - and the chemical mixing associated to this process - which occurs during the main sequence evolution: this leads to a large spread in the main sequence lithium abundances of stars.

Comparing these models with the observations, however, we have to notice the following:

1. a spread in the lithium abundances is created early in the life of stars, as it is present in young clusters such as $\alpha$ Persei and
the Pleiades ( $\sim 100 \mathrm{Myr}$ ). On the contrary, the mixing due to the redistribution of angular momentum, which causes the lithium spread in rotating models, occurs mostly later during the stellar life.
2. the spread is in the sense that, statistically at least, the stars having the largest rotational velocities seem to deplete less lithium. In fact García López et al. (1994), and later Jones et al. (1996), found that the observed spread in lithium abundances of the Pleiades low-mass stars is related to rotation in such a way that fast rotators have less lithium depletion than slow rotators. Evidence for such a rotation-lithium depletion relation has also been found by Cunha et al. (1995), who have studied the lithium abundance over a sample of late F to early G young stars in the Orion association. Binaries in the Hyades present a lower scatter in abundances than single stars in the same cluster, and close binaries show conspicuous overabundances (Barrado y Navascués \& Stauffer 1996).

Consequently, the observational indications are opposite to the predictions of rotating models. The quest for a mechanism inhibiting lithium depletion in pre-main sequence motivated Martín \& Claret (1996) to a new exploration of the effect of rotation. At variance with Pinsonneault et al. (1990), they found indeed that rigid body rotating models depleted less lithium than the non-rotating models. They attributed the differences of their results, with respect to those of Pinsonneault et al. (1990), mainly to the adopted opacities, as the latter authors used the older Cox \& Stewart (1970) libraries. Therefore, the theoretical models by the Yale group are in conflict with the Martín \& Claret results. On the other hand, the result by Martín \& Claret (1996) would help in solving the problem of lithium, also in view of the fact that modern computations of pre-main sequence lithium depletion in non-rotating models provide too much depletion with respect to what the observations in open clusters demand (D’Antona and Mazzitelli 1997, Martín 1997, Ventura et al. 1998), and a reduction of this depletion is needed.

It is important to sort out whether the structural (hydrostatic) differences between the rotating and non-rotating models work towards confirming the observational constraints or not. If there is no substantial difference between the rotating and non rotating models, as predicted by Pinsonneault et al. (1990), the differences in lithium depletion found among open clusters stars must be entirely attributed to the phases of angular momentum transport and associated chemical mixing, which is less important in fast rotating stars due to the lower degree of differential rotation between the radiative interior and the external convective envelope, but the early spread in abundances remains not understood.

In order to have an independent check of this problem, in this paper we present the results of a series of rotating stellar models computed with the ATON code, with particular emphasis on the structural effects (always meant, throughout this text, as only hydrostatic effects) of rotation on the lithium abundances of low-mass, pre-main sequence stars. Some preliminary results already have been presented (Mendes et al. 1997).

## 2. Model features

The general features of the ATON stellar structure code are described in Mazzitelli (1989), Mazzitelli et al. (1995) and references therein. The pre-main sequence models start in an arbitrary fully extended convective configuration with a central temperature of $\log T_{\mathrm{c}}=5.7$ (and thus prior to the onset of deuterium burning), to which $t=0$ is attributed (see D'Antona \& Mazzitelli 1997 for a discussion of attributing theoretical "ages" to pre-main sequence models).

### 2.1. Rotation

Rotation was implemented according to the approach taken by Endal \& Sofia (1976), which uses the Kippenhahn \& Thomas (1970) method but with an improved potential function due to the inclusion of a third term related to the distortion of the figure of the star. As it is well known, this approach incorporates only the hydrostatic effects of rotation and is far from a full treatment of rotational effects, which should include internal angular momentum redistribution and surface angular momentum loss. Three rotation laws have been implemented, namely rigid body rotation, local conservation of angular momentum throughout the whole star, and local conservation of angular momentum in radiative regions and rigid body rotation in convective ones. Following Endal \& Sofia (1976) and Pinsonneault et al. (1989), these rotation laws were chosen to represent a set of physically plausible rotation laws. Although recent data from both helioseismology (Thompson et al. 1996) and numerical models (e.g. Brummel et al. 1995) give a quite different picture for the present sun, it should be kept in mind that in the sun mechanisms of redistribution of angular momentum have been active for almost 5 billion years, altering its initial distribution.

It should be noted that, in general, local conservation of angular momentum in pre-main sequence stars implies radial differential rotation (as the star contracts along the Hayashi track), which (except for rotation constant on cylinders) is clearly a non-conservative rotation law. Though the Kippenhahn \& Thomas method was originally developed for conservative rotating forces, even for non-conservative ones the baroclinity caused by rotation will be small (Endal \& Sofia 1978), and following these last authors we also assumed that the angular velocity is constant along a level surface. A more rigorous treatment of a non-conservative rotation law of the form $\omega=\omega(r)$ can be found in Meynet \& Maeder (1997).

### 2.2. Input physics

The opacities adopted in the current version of the ATON code are taken from Iglesias \& Rogers (1993), supplemented by those at low and intermediate temperatures from Alexander \& Ferguson (1994) or Kurucz (1993). The initial angular momentum for all models was estimated according to the relations given by Kawaler (1987) for stars of spectral type later than F2 (see Table 1). The chemical composition was set with $Y=0.28$ and $Z=0.0175$. As for the convection model, the calcu-


Fig. 1. Detail of the evolutionary tracks for rotating models of $1 M_{\odot}$ under different initial angular momentum and different rotation laws, for the MLT convection model. $R B$ stands for rigid body rotation; $D R$, local conservation of angular momentum throughout the whole star; and $D R+S B$, local conservation of angular momentum in radiative regions and rigid body rotation in convective ones.

Table 1. Initial angular momentum adopted for the models

| Mass $\left(M_{\odot}\right)$ | $J_{0}\left(\mathrm{~g} \mathrm{~cm}^{2} \mathrm{~s}^{-1}\right)$ |
| :--- | :---: |
| 1.2 | $1.874 \times 10^{50}$ |
| 1.1 | $1.720 \times 10^{50}$ |
| 1.0 | $1.566 \times 10^{50}$ |
| 0.9 | $1.412 \times 10^{50}$ |
| 0.8 | $1.257 \times 10^{50}$ |
| 0.6 | $9.470 \times 10^{49}$ |

lations were done under both mixing length formalism (MLT), with $\alpha$ (ratio of the mixing length $\lambda$ to the pressure scale height) set to 1.5 , and the Full Spectrum of Turbulence (FST) model employing the fluxes by Canuto \& Mazzitelli (1991, 1992, hereinafter referred to as CM). A full description of the characteristics of this model can be found in D'Antona et al. (1997). Here we simply remember that the convective scale is taken to be the distance from the boundary of the convective region, increased by a small "overshooting" length $\beta$, which allows to precisely fit the solar model. Here the $\beta$ is set to $0.175 H_{p}$. Remember that this "overshooting" parameter regards only the mixing scale, and is not a parameter of chemical mixing. In any case, no real overshooting is allowed at the base of the convective layer, which would alter lithium depletion (e.g. Ventura et al. 1998).


Fig. 2. Surface angular velocity, as a function of age, for the rotating models of $1 M_{\odot}$. The convection model is the CM.

## 3. Main results

### 3.1. Structural effects of rotation

We have calculated rotating stellar models of $0.6,0.8,0.9,1.0$, 1.1 and 1.2 solar masses. The results confirm what other researches have already found (e.g. Sackmann 1970; Pinsonneault et al. 1989; Martín \& Claret 1996): the evolutionary tracks of rotating stars shift towards lower effective temperatures and luminosities, simulating the evolutionary path of a non-rotating star of lower mass (the so-called mass-lowering effect; see Sackmann 1970). However, we have also found that these differences are very small, being considerable only for high initial angular momentum, as can be seen from Fig. 1, where we also show the individual effects of rotation for each of the adopted rotation laws. Fig. 2 shows the time evolution of the surface angular velocity for the $1.0 M_{\odot}$ models.

### 3.2. Rotation-induced lithium depletion

For the purpose of comparison with other works, we discuss here only the results for the rigid body rotation case, but we anticipate that, in accordance with previous work (e.g. Tassoul 1978 and references therein), our results are qualitatively the same for all rotation laws used, as the structural effects of rotation do not depend on the chosen rotation law when the initial angular momentum $J_{0}$ is kept fixed. Our main results can be summarized as follows:
a. while the star is fully convective, rotation causes an initial decrease in the lithium depletion rate.
b. soon after a radiative core is developed, this situation is inverted and rotation increases the lithium depletion.
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Fig. 3. Lithium depletion as function of age for $1.0,0.8$ and $0.6 M_{\odot}$ for two different initial angular momentum. The metallicity is $\mathrm{Z}=0.0175$ and the convection model is the MLT. The detail at the lower left corner shows the transition from lower to greater lithium depletion for the 0.6 $M_{\odot}$ model.
c. the higher the initial angular momentum, the higher the lithium depletion after the development of the radiative core.
d. the qualitative effects of the rotation-induced lithium depletion are equal for both mixing length theory and the Canuto \& Mazzitelli turbulent convection model.

Figs. 3 and 4 exemplify these results, and show the variation of the lithium concentration (in the scale $N[\mathrm{Li}]=12+$ $\log \frac{N[\mathrm{Li}]}{N[\mathrm{H}]}$ ) as a function of age, for masses of $1.0,0.8$ and 0.6 solar masses with two different initial angular momentum values (corresponding to $\mathrm{J}_{0}$ and $2 \mathrm{~J}_{0}$ ) and for both MLT and CM convection models. Items (a), (b) and (c) are fully compatible with the previous results obtained by Pinsonneault et al. (1990). This can be easily understood by recalling that rotating stars mimic lower-mass, non-rotating ones. So, as long as the rotating star is fully convective, its central temperature is smaller than that of the non-rotating case; this effect becomes more pronounced as we go toward smaller masses, as they have smaller moments of inertia. On the other hand, this situation changes when the radiative core develops, for then the temperature at the base of the convective envelope results higher in the rotating stars than for the non-rotating ones, and so contributes to greater lithium depletion specially for low-mass stars in their pre-main sequence stages (Bodenheimer 1965; D'Antona \& Mazzitelli 1984). This can be seen in Fig. 5, where we plot the run of the temperatures as a function of age for both rotating and non-rotating $0.8 M_{\odot}$ and $0.6 M_{\odot}$ solar models. Item (d) is a trivial result, and it simply tells us that the structural effects of rotation are not affected by the chosen convection model.


Fig. 4. Same as Fig. 3, but for the CM (1992) turbulent convection model. Note that, for this convection model, the transition from decreased lithium depletion to increased lithium depletion occurs at an earliest stage than for the MLT, as can be seen for the $0.6 M_{\odot}$ model.


Fig. 5. Temperatures at the base of the convection zone for rotating and non-rotating stellar models of $0.6 M_{\odot}$ and $0.8 M_{\odot}$. For the linear part (fully convective phase) this refers to the central temperature. Results shown for the MLT model.

It is also worth to compare the run of lithium concentration against the effective temperature with those of D'Antona \& Mazzitelli (1994). Fig. 6 exemplifies our results, and shows that using the more recent opacities of Iglesias \& Rogers (1993)


Fig. 6. Lithium abundance as a function of the effective temperature at the age of $5 \times 10^{7}$ years; the squares and filled circles mark masses ranging from 1.2 to 0.8 solar masses with steps of 0.1 . The dotted line is the respective one from the "ku cm" model of D'Antona \& Mazzitelli (1994, their Fig. 19). The crosses are the $\alpha$ Persei data from Balachandran (1996). For ease of comparison with the former authors, the initial lithium concentration was normalized at $\log N[L i]=3.5$, though it is probably lower $(\sim 3.3)$.
has a stronger effect on lithium depletion than rotation itself. It also shows that the non-rotating models have a larger $T_{\text {eff }}$ than the rotating models of the same mass, so that, for a given $T_{\text {eff }}$, the rotating sequence actually shows a lower depletion than the non-rotating sequence. Notice however that:

1. the effect is very small, especially compared to what is required to explain the observed spread;
2. when the age increases, the total angular momentum will not be conserved, and the main sequence location of the initially fast rotating models will become even closer to the non-rotating main sequence. (This caveat is even more important when discussing the very old population II stars, whose rotation rate today is certainly much reduced with respect to the rate on arrival to the main sequence, 10 billion years ago).

### 3.3. Low metallicity stars

As we have seen, the lithium depletion of both rotating and nonrotating models of solar metallicity, employing the most updated opacities, equation of state and convective fluxes, is incompatible with the lithium observations of young open clusters, while the D'Antona \& Mazzitelli (1994) results were at least partially compatible with them (see Fig. 6). Previous generation models gave much too low lithium depletion to explain the open


Fig. 7. Lithium depletion as function of effective temperatures for nonrotating and rotating models, at $\mathrm{Z}=0.001$. The open circles refer to the lithium abundances of halo subdwarfs from Bonifacio \& Molaro (1997), while the filled ones have the same meaning as in Fig. 6. The dotted line corresponds to the rotating models but at the same effective temperatures of the non-rotating models.
clusters patterns, so that the necessary additional depletion was generally attributed to secondary mixing mechanisms acting on a timescale much longer than pre-main sequence burning. In the presence of a larger than necessary pre-main sequence lithium burning, a possible escape from this problem would have been that rotating models structurally deplete less lithium (Martín \& Claret 1996), but this is not confirmed by our computations.

We must then answer the following immediate question: does this problem occur also for the population II stars, that is, is the "Spite plateau" of low metallicity subdwarfs preserved in computations adopting the same updates in the physics? To answer this problem, we have computed models of non rotating and rotating stars of $0.7,0.65,0.60,0.55$ and $0.50 M_{\odot}$, with metallicity $\mathrm{Z}=0.002$. The results are presented in Fig. 7, where our model stars, at an arbitrary large age of $10^{10}$ years, are superimposed on the observational data of Bonifacio \& Molaro (1997). Here we plot the rotating models both at their own $T_{\text {eff }}$ (dashed line) and at the $T_{\text {eff }}$ of the non-rotating models (dotted line) considering that, after 10 billion years, rotation will have been drastically slowed down due to mechanisms of angular momentum loss. We see that the plateau is maintained in both rotating and non rotating models. This is reasonable, as the Livermore new computations have acted mainly to increase the population I values of opacities, and the global effect of improvements is less critical when the metallicity is drastically reduced. This result is particularly welcome, as the "Spite plateau" is certainly one of the most firm constraints for our knowledge of population II lithium abundance. The debate rests,
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Table 2. Lithium depletion values for the low-metallicity ( $\mathrm{Z}=0.001$ ) rotating and non-rotating stellar models. The labels NR and $R$ at each stellar mass stand for non-rotating and rotating models, respectively.

| Mass $\left(M_{\odot}\right)$ | Age $=2 \times 10^{7} \mathrm{yr}$ |  | Age $=1 \times 10^{10} \mathrm{yr}$ |  |
| :--- | :---: | :---: | :---: | :---: |
|  | $\frac{N(L i)}{N_{0}(L i)}$ | $\log [\mathrm{Li} / \mathrm{H}]$ | $\frac{N(L i)}{N_{0}(L i)}$ | $\log [\mathrm{Li} / \mathrm{H}]$ |
| $0.70(\mathrm{NR})$ | 0.963 | 2.212 | 0.963 | 2.212 |
| $0.70(\mathrm{R})$ | 0.961 | 2.211 | 0.961 | 2.211 |
| $0.65(\mathrm{NR})$ | 0.921 | 2.192 | 0.919 | 2.191 |
| $0.65(\mathrm{R})$ | 0.917 | 2.190 | 0.911 | 2.187 |
| 0.60 (NR) | 0.816 | 2.140 | 0.786 | 2.123 |
| 0.60 (R) | 0.801 | 2.132 | 0.748 | 2.102 |
| 0.55 (NR) | 0.555 | 1.972 | 0.431 | 1.862 |
| $0.55(\mathrm{R})$ | 0.511 | 1.937 | 0.313 | 1.724 |
| 0.50 (NR) | 0.120 | 1.308 | 0.003 | -0.336 |
| $0.50(\mathrm{R})$ | 0.009 | 1.172 | $6.3 \times 10^{-7}$ | -3.970 |

in this field, on the reality of the dispersion or even trends in abundance among the plateau stars, appreciated by some researchers (e.g. Deliyannis et al. 1993; Thornburn 1994; Ryan et al. 1996) but disputed by others (Molaro et al. 1995; Spite et al. 1996; Bonifacio \& Molaro 1997). Apart from the cosmological implications it raises, the possible existence of a dispersion on the plateau (if any) would indicate the action of some stellar and/or Galactic processing. Candidates for the stellar processing have been considered in the literature, such as rotational mixing (Pinsonneault et al. 1992), microscopic diffusion (Chaboyer \& Demarque 1994) or stellar winds (Vauclair \& Charbonnel 1995). Though we have not yet included the effects of rotational mixing in our models, Fig. 7 shows that additional mixing mechanisms are needed to explain the pattern of lithium abundances below $T_{\text {eff }} \sim 5800 \mathrm{~K}$, which can not be attributed to standard mixing.

As for the structural effects of rotation, our models show that they become significant only for stars with $T_{\text {eff }}<5400 \mathrm{~K}$, that is, away from the plateau. Also burning during the main sequence lifetime is important for these stars (although not as significant as the pre-main sequence burning) as it can be seen from the data in Table 2.

## 4. Discussion and final remarks

It is important to stress that our results are qualitatively similar to those from Pinsonneault et al. (1990), but were obtained with updated opacities; so, the differences between these results and those by Martin \& Claret (1996) cannot be attributed to the adopted opacities, as suggested by the latter authors. Besides, preliminary tests using Martin \& Claret's less accurate gravitational potential did not change our overall results.

We have seen that the structural effects of rotation on the lithium depletion of low-mass, pre-main sequence stars can not reduce the difference between theoretical and observed lithium depletion patterns: in fact these effects increase lithium depletion, while a reduction would be needed, to be consistent with the smaller depletion in faster rotating stars in young open clusters. This situation can become even worse if we take into account the
cumulative effects due to the rotation-induced mixing caused by the internal redistribution of angular momentum, which is highly dependent of the hydrodynamical instabilities triggered by rotation (see e.g. Zahn 1993). In fact, the results of Pinsonneault et al. $(1990,1992)$ suggest that this rotational mixing increases even more the lithium depletion of such stars. In order to check for this effect, current work is in progress to introduce angular momentum loss and redistribution in the ATON code.

On the other side, as pointed out by Strom (1994), the older pre-main sequence stars (those ones with ages close to the Hyades, and whose surface velocities are decreasing due to surface angular momentum loss) exhibit a correlation between rotation and lithium depletion opposed to that seen in pre-main sequence stars, that is, the faster rotators show a greater degree of lithium depletion. This clearly shows that other physical mechanisms must play a role on the lithium depletion pattern of low-mass stars, both pre-main sequence and more evolved ones. For example, Spruit (1987) argued that the larger dynamoinduced magnetic fields in rapid rotators could inhibit turbulent mixing. A preliminary analysis of this possibility is successfully explored by Ventura et al. 1998.

For the low metallicity subdwarfs, we can say that the structural effects of rotation are negligible for those ones which lie on the Spite plateau. However, as the effective temperatures at which significative depletion begins are lower, for the theoretical models, than those observed in such stars, once again we are lead to the conclusion that other physical mechanisms must be taken into account if we want a better match with the observations.

Acknowledgements. The authors wish to thank Drs. L. P. R. Vaz and E. L. Martín for their helpful comments and suggestions, and are also indebted to the referee for pointing out many improvements to the manuscript. Luiz T. S. Mendes thanks the hospitality of the Osservatorio Astronomico di Roma, where part of this work was carried out. Partial financial support from the Conselho Nacional de Pesquisa e Desenvolvimento (CNPq), Brazil, and from the Consiglio Nazionale delle Ricerche (CNR), Italy are acknowledged.

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## Appendix D

Paper to be published in the $A S P$ Conference Series, Vol. 198

# Low-mass Rotating Stellar Models with Angular Momentum Redistribution 

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#### Abstract

We report here preliminary results concerning the introduction of internal angular momentum redistribution in the ATON 2.0 stellar evolutionary code. Angular momentum transport in radiative zones was implemented according to the general framework established by Zahn (1992) based on the assumption of stronger turbulent transport in the horizontal than in the vertical direction. Particular results for the case of a $1 \mathrm{M}_{\odot}$ model star show that angular momentum transport can be very effective in smoothing the internal angular velocity profile and also increases the lithium depletion, but does not suffice to reproduce the flat solar rotation rate obtained from helioseismology.


## 1. Introduction

For more than a decade there has been a growing concern that internal angular momentum transport plays an important role in the mixing of the chemicals of the stars, affecting their surface abundances. In the ongoing effort to improve the macro- and micro-physics of the ATON 2.0 stellar evolutionary code (Ventura et al. 1998), we present here first results concerning the introduction of internal redistribution of angular momentum and the associated rotational mixing.

As it is well known since Eddington (1925) and Von Zeipel (1924a,b), rotation causes a thermal imbalance in stars which in turn drives meridional circulation currents. Even if the initial rotation state corresponds to rigid body rotation, those circulation currents alter the internal angular velocity profile leading to differential rotation, which in turn can trigger a number of hydrodynamical instabilities in the almost non-viscous stellar plasma, resulting in turbulent motions. Both dynamical (Solberg-Hoiland and dynamical shear) and secular (meridional circulation, Goldreich-Schubert-Fricke instability, and secular shear) instabilities were considered in our calculations. Internal angular momentum transport was modeled following the general framework established
by Zahn (1992) and later pursued by other authors (Urpin et al. 1996, Talon et al. 1997, Maeder \& Zahn 1998), based on the assumption of stronger turbulent transport in the horizontal than in the vertical direction. The internal angular momentum transport in radiative zones is then given by the advection-diffusion partial differential equation

$$
\begin{equation*}
\frac{\partial}{\partial t}\left[\rho r^{2} \bar{\Omega}\right]=\frac{1}{5 r^{2}} \frac{\partial}{\partial r}\left(\rho r^{4} \bar{\Omega} U\right)+\frac{1}{r^{2}} \frac{\partial}{\partial r}\left[\rho \nu_{v} r^{4} \frac{\partial \bar{\Omega}}{\partial r}\right] \tag{1}
\end{equation*}
$$

where $U(r)$ is the meridional circulation velocity and the other symbols retain their usual meanings. In Eq. (1), the total diffusion coefficient $\nu_{v}$ is the sum of the individual diffusion coefficients for each hydrodynamical instability considered.

## 2. Input physics

The ATON 2.0 code has many updated and modern features regarding the physics of stellar interiors, including most up to date OPAL (Iglesias \& Rogers 1993) opacities, supplemented by those of Alexander \& Ferguson (1994) for lower ( $T<6000 \mathrm{~K}$ ) temperatures; diffusive mixing and overshooting; convection treated under either mixing length theory or the Full Scale of Turbulence (FST, Canuto et al. 1996); and other (for a full account of the code features see Ventura et al. 1998). The structural effects of rotation were included in the ATON code (Mendes et al. 1999) by using the Kippenhahn \& Thomas (1970) method with the improvements brought by Endal \& Sofia (1976) regarding the calculation of the potential function. Angular momentum losses in the star's external layers due to magnetized stellar winds are also considered, and enter the PDE advection-diffusion equation as a boundary condition at the surface. We adopted the prescription from Chaboyer et al. (1995) with a "wind index" $n=1.5$, which reproduces well the Skumanich (1972) law $v \propto t^{-1 / 2}$ :

$$
\begin{array}{ll}
\frac{\partial J}{\partial t}=K\left(\frac{R}{R_{\odot}}\right)^{2-n}\left(\frac{M}{M_{\odot}}\right)^{-n / 3} \omega^{1+4 n / 3}, & \omega<\omega_{\text {crit }} \\
\frac{\partial J}{\partial t}=K\left(\frac{R}{R_{\odot}}\right)^{2-n}\left(\frac{M}{M_{\odot}}\right)^{-n / 3} \omega \omega_{\text {crit }}^{4 n / 3}, & \omega \geq \omega_{\text {crit }} \tag{3}
\end{array}
$$

where $\omega_{\text {crit }}$ introduces a critical rotation level at which the angular momentum loss saturates. The constant $K$ is usually calibrated by requiring that the model surface velocity match the current solar rotation rate at the equator.

## 3. Results for the $1 \mathrm{M}_{\odot}$ model

The $1 M_{\odot}$ model was run with a chemical composition of $Y=0.271, Z=0.0175$ and under the FST framework for convection. The initial angular momentum was taken from the Kawaler (1987) relations for low-mass stars, and corresponds to $J_{0}=1.566 \times 10^{50}$ (cgs units) for the case of $1 M_{\odot}$. The chosen rotation law corresponds to local angular momentum conservation in radiative regions and rigid body rotation in convective ones. Figure 1 shows the angular velocity


Figure 1. Angular velocity as a function of radius for the $1 M_{\odot}$ model with $J=J_{0}$, at four different ages. Solid line: $\log t=6.5$; short-dashed line: $\log t=7.0 ;$ dotted line: $\log t=7.5 ;$ long-dashed line $: \log t=8.0$. The figure at the right shows the indicated ages in the respective evolutionary track; point E refers to the position corresponding to the ZAMS.
profile along the star's radius for four typical evolutionary ages, three of them in the pre-main sequence phase and the other in the main sequence, along with their respective locations (labeled A through D) at the evolutionary path in the $\mathrm{H}-\mathrm{R}$ diagram. We see that during most of the pre-main sequence phase (points A and B) the star's contraction is much more effective than the redistribution mechanisms, giving rise to a rapidly rotating radiative core. Near the zero-age main sequence (point C), the star is now slowly contracting and the core reaches its peak velocity. From this point on the transfer of angular momentum from the core to the external layers becomes very efficient, as it can be seen at the beginning of the main sequence (point D). However, these effects are not able to reproduce the near rigid body rotation rate found in the radiative interior of the present Sun obtained from helioseismic data, meaning that other mechanisms must play a role in determining the current solar rotation profile.

As for the rotational mixing, we found that internal transport of angular momentum contributes for a higher depletion of the light elements such as lithium, in accordance with earlier results from other researchers (Pinsonneault et al. 1990), though this effect is barely noticeable for the $1 M_{\odot}$ model as it can be seen from Fig. 2. However, we note that in our models the direct effect of the meridional circulation velocity on the diffusion of chemicals is not yet considered, which could in principle contribute to a still higher depletion of the light elements. Work is in progress to include that contribution in the ATON 2.0 code.

## 4. Conclusions

We have confirmed that internal angular momentum transport is not significant in the pre-main sequence phase of a $1 M_{\odot}$ model star, when compared to the


Figure 2. Lithium depletion (in the scale $\left.N[\mathrm{Li}]=12+\log \frac{N(\mathrm{Li}]}{N[\mathrm{H}]}\right)$ as a function of age for the $1 M_{\odot}$ model. Dotted line: non-rotating model. Solid line: rotating model with no angular momentum transport. Dashed line: rotating model with angular momentum transport.
increasing spin-up of the star which results from their contraction along the Hayashi track. As the star approaches the ZAMS, however, the angular momentum redistribution due to the hydrodynamical instabilities triggered by rotation can be very effective in smoothing the internal angular velocity profile. The results also show that the rotational mixing induced by the internal angular momentum transport contributes to higher lithium depletion rates, meaning that other physical effects must be present in order to explain the observed pattern of rotation-lithium depletion in low-mass, pre-main sequence stars of young open clusters (García Lopez et al. 1994).

Acknowledgments. Partial finantial support of the Consiglio Nazionale delle Ricerche (Italy) is gratefully acknowledged.

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[^0]:    The Doctor
    (In John Flanagan and Andrew McCulloch's Meglos)

[^1]:    *Here we consider standard models as the classical ones that do not consider the effects of rotation, microscopic diffusion, magnetic fields, and other "non-standard" physical effects.

[^2]:    ${ }^{\dagger}$ This limit refers to the critical rotational velocity that exceeds the surface gravity when it occurs before the Eddington limit is reached; see Langer 1997.

[^3]:    ${ }^{\ddagger}$ In this work we will always refer to lithium as its ${ }^{7} \mathrm{Li}$ isotope, which is the one relevant to stellar structure and evolution. The ${ }^{6} \mathrm{Li}$ isotope accounts for only $7 \%$ of the observed abundances and burns at a much lower temperature, making its contribution to the stellar Li contents usually negligible.

[^4]:    ${ }^{\S}$ The Taylor-Proudman theorem states that $\int_{\mathrm{S}}(!+2 \boldsymbol{\Omega}) \cdot d \mathbf{S}=$ constant (where $!=\nabla \times \mathbf{u}$ is the fluid vorticity, $\mathbf{u}$ is the fluid velocity, $\boldsymbol{\Omega}$ is the angular velocity, and $d \mathbf{S}$ is the normal vector to a surface element). For the simpler case of steady, slow motions in a rotating, non-viscous fluid, this theorem simply means that all motions are necessarily two-dimensional (Chandrasekhar 1981). In other words, there is no variation of the velocity field in the direction parallel to the rotation axis.

[^5]:    ${ }^{\top}$ Here $r_{\Psi}$ means the radius of equipotential surfaces characterized by their total potential $\Psi$.

[^6]:    *Here the term advection has its classical hydrodynamical meaning, that is, the transport of material by a moving fluid.

[^7]:    *Here we are interested in comparing only the shapes of both figures, as our main goal is to check if angular momentum redistribution is effective enough to flatten out the internal angular velocity profile. Though the surface rotation velocity in Fig. 4.6 reaches 31273 nHz for curve D - near two orders of magnitude higher than the solar surface velocity -, such a direct comparison of absolute values in these figures can be misleading, as the surface angular velocity in our models depend not only on the assumed initial angular momentum but also on the calibration of the constant $K_{\mathrm{w}}$ used to compute angular momentum losses at the surface (see Sect. 3.4.4.7)

[^8]:    *Aqui o termo adveç̧ão tem seu significado hidrodinâmico, ou seja, o transporte de matéria por um fluido em movimento.

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