UNIVERSIDADE FEDERAL DE MINAS GERAIS
INSTITUTO DE CIÊNCIAS EXATAS - ICEX
DEPARTAMENTO DE MATEMÁTICA

Jailton Viana da Conceição
Advisor: Mario Jorge Dias Carneiro

Blenders For a Non-Normally Hénon-Like Family

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ABSTRACT The aim of this work is to study the existence of blender structure in a three-dimensional family of diffeomorphisms derived from the canonical Henon family in dimension two by multiply it by a convenient affine function in the z-direction. As in [1] we shall call the topological conjugacy class of this family for Non-normally Henon-like family. A blender is an important concept in Dynamic from the geometric point of view. The first time that the subject appears was in [2] where the authors define and use blender structure to prove the existence of some kind of robust transitive diffeomeorphism far from hyperbolicity. The most deep consequence of the blender structure is that one can have one dimensional submanifold of the ambient space which behaves as two dimensional submanifold. Another fact about blenders is that one can construct affine blender as we will show in this work.

Chapter 1

Introduction

Consider the family $\psi((a,b),(x,y,)) = (1-ax^2+by,x)$. We know by Benedicks- Calerson (see [4]) that this two dimensional family have some strange atractor(see[15] for definition) for some parameters values a and b and we also kow that inside this atractor we have a homoclinic tangence (see [12]).
This work is a detailed study of [1] where is considered the family $\varphi : \mathbb{R}^4 \times \mathbb{R}^3 \to \mathbb{R}^3$ given by $\varphi((a,b,c,d),(x,y,z)) = (1-ax^2+by, x, cz+dx)$.
This is a 3-dimensional version of the Henon family called in [1] Non-Normally Henon Like family. It is proved that if the parameters satisfy

$$\begin{cases} 0 < |b| < \delta; \\ a > \frac{15(1+|b|)^2}{4}; \\ 1+|d| < c < \frac{10}{9}; \\ 0 < |d| < \frac{1}{9}. \end{cases} \qquad (\mathcal{P}.\mathcal{C})$$

then this family have a blender (we are going to define Blender late). This object was used in [10] to obtain robust heterodimensional cycle and in [2] to obtain non-uniformly hyperbolic robustly transitive diffeomorphism.

Chapter 2

Background

Now we are going to present some definitions and results about hyperbolic dynamics. We will not prove no result here but the reader can find a proof in [5, 14].

Definition 2.0.1 *Let $\mathbb{M}$ be a C^k-manifold and let $f \in C^1(\mathbb{M}, \mathbb{M})$. One say that a point $x \in \mathbb{M}$ is a hyperbolic fix point of f when the spectrum of the linear function $df(p) : T_x\mathbb{M} \to T_p\mathbb{M}$ has no intersection with the circle $S^1 \subset \mathbb{C}$. One say that a point $p \in \mathbb{M}$ is periodic hyperbolic point(with period k) of f when it is a hyperbolic fix point of f^k.*

Definition 2.0.2 *Let $\mathbb{M}$ be a metric space and let $f \in C^0(\mathbb{M}, \mathbb{M})$ and $x \in \mathbb{M}$.*

- *One call stable set of a point x the set $W^s := \{y \in \mathbb{M} | \ \ \lim_{n\to\infty} Dist(f^n(y), f^n(x)) = 0\}$*
  If $f \in Homeom(\mathbb{M}, \mathbb{M})$ then
- *One call unstable set of x the set $W^u := \{y \in \mathbb{M} | \ \ \lim_{n\to\infty} Dist(f^{-n}(y), f^{-n}(x)) = 0\}$, that is, the unstable set of x is the stable set of x by f^{-1};*
- *One call local stable set of x (with size $\epsilon > 0$) the set $W^s_\epsilon(x) := \{y \in \mathbb{M} | \ Dist(f^n(y), f^n(x)) < \epsilon, \ \ \forall n \geq 0\}$;*
- *One call local unstable set of x (with size $\epsilon > 0$) the set $W^u_\epsilon(x) := \{y \in \mathbb{M} | \ \ Dist(f^{-n}(y), f^{-n}(x)) < \epsilon, \ \ \forall n \geq 0\}$;*

Theorem 2.0.1 **(The Hartman – Grobman Theorem)**
Let $\mathbb{M}$ be a C^k-manifold and let $f \in Diff^1(\mathbb{M})$ with a hyperbolic fix point $p \in \mathbb{M}$. Then there are neighborhoods $V_p \subset \mathbb{M}$ and $V_0 \subset T_p\mathbb{M}$ and a $h \in Homeom(V_p, V_0)$ such that

$$h \circ f = df(p) \circ h.$$

Corollary 2.0.1 *Let $\mathbb{M}$ be a C^k-manifold, $f \in C^0(\mathbb{M}, \mathbb{M})$ and $p \in \mathbb{M}$ a hyperbolic fix point of f. Then there is $\epsilon_0 > 0$ such that:*

- *$W^s_{\epsilon_0}(p) \subset W^s(p)$ and $W^u_{\epsilon_0}(p) \subset W^u(p)$;*
- *$W^s_{\epsilon_0}(p)$ is a topological sub-manifold with the same dimension of $E^s(p)$;*
- *$W^u_{\epsilon_0}(p)$ is a topological sub-manifold with the same dimension of $E^u(p)$;*

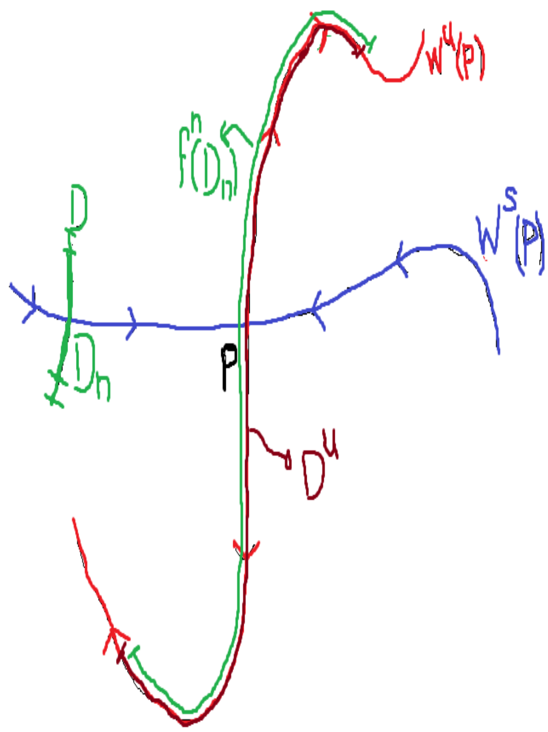

Figure 2.1: λ-Lemma

- $W^s(p) = \bigcup_{n\leq 0} f^{-n}(W^s_{\epsilon_0}(p))$ *and* $W^u(p) = \bigcup_{n\leq 0} f^{n}(W^u_{\epsilon_0}(p))$ *are immersed topological sub-manifolds*

Proof: It follows immediately from the Hartman-Grobman Theorem. ■

Corollary 2.0.2 *Suppose p is a hyperbolic fix point of f thus there is a neighborhood $V_p \subset \mathbb{M}$ such that if $f^n(q) \in V_p$ for all $n \in \mathbb{Z}$ then $q = p$.*

Lemma 2.0.1 (**The $\lambda-$Lemma**)
Suppose that

- *$\mathbb{M}$ is a connected compact boundaryless C^r- Riemannian manifold of dimension n;*
- *$f \in Diff^r(\mathbb{M})$;*
- *$p \in \mathbb{M}$ is a periodic hyperbolic point for f;*
- *D^u is a compact disc in $W^u(p)$;*
- *D is a disc centered in a point $x \in W^s(p)$ such that $\dim(D) = \dim(W^u(p))$ and $x \in D \pitchfork W^s(p)$.*

Then for every $\varepsilon > 0$ there is an integer $n_0 > 0$ such that for all $n \geq n_0$ there is a disc $D_n \subset D$ so that $f^n(D_n)$ is a disc $\varepsilon - C^1$-near of D^u.

Definition 2.0.3 (**Hyperbolic set**)
Let $f \in Diff^r(\mathbb{M})$ and $\Lambda \in \mathbb{M}$. We say that Λ is hyperbolic set for f when:

1. *Λ compact invariant ;*

2. *For every $x \in \Lambda$ there is subspaces $E^s(x)$, $E^u(x) \subset T_x\mathbb{M}$ such that:*

   (a) *$T_x\mathbb{M} = E^s(x) \oplus E^u(x)$;*

   (b) *$df(x)(E^s(x)) = E^s(f(x))$ and $df(x)(E^u(x)) = E^u(f(x))$;*

   (c) *There are constants $c > 0$ and $0 < \lambda < 1$ such that:*

      i. *$\|df^n(x)(v)\| \leq c\lambda^n \|v\|$, $\forall v \in E^s(x)$ and $\forall n \geq 0$;*

      ii. *$\|df^{-n}(x)(v)\| \leq c\lambda^n \|v\|$, $\forall v \in E^u(x)$ and $\forall n \geq 0$.*

The sub-bundles $E^s := \cup_{x\in\Lambda}\{(x,v) | x \in \Lambda \;\; and \;\; v \in E^s(x)\}$ and $E^u := \cup_{x\in\Lambda}\{(x,v) | x \in \Lambda \;\; and \;\; v \in E^u(x)\}$ are called stable sub-bundle and unstable sub-bundle respectively and we have $T_\Lambda\mathbb{M} = E^s \oplus E^u$.
On this way if p is a hyperbolic periodic point of f with period k, then $\Lambda = \{p, f(p), f^2(p), ..., f^{k-1}(p)\}$ is a hyperbolic set for f. Note that hyperbolicity does not depends on the Riemannian Metric.

Proposition 2.0.1 *If $f \in Diff^r(\mathbb{M})$ and $\Lambda \in \mathbb{M}$ is a hyperbolic set for f then the subspaces $E^s(x)$ and $E^u(x)$ depends continuously on the point x.In particular , they have dimensions locally constant.*

Proposition 2.0.2 *(**Adapted Meric**) If $f \in Diff^r(\mathbb{M})$ and $\Lambda \subset \mathbb{M}$ is a hyperbolic set for f, then there is a C^∞- Riemannian Metric on $\mathbb{M}$ and a constant $0 < a < 1$ such that:*

- *$\|df(x)(v^s)\|_* < a\|v^s\|_*$, $\forall v^s \in E^s(x)$ and $\forall x \in \Lambda$;*

- *$\|df^{-1}(x)(v^u)\|_* < a\|v^u\|_*$, $\forall v^u \in E^u(x)$ and $\forall x \in \Lambda$;*

Where $\|.\|_*$ is the norm which come from the metric.

Definition 2.0.4 *Let V a real Hilbert space. We say that a set $C \subset V$ is a cone on V if and only if ,V admits a splitting $V = E \oplus F$ and there is a real number $a > 0$ such that*

$$C = \{(v_E, v_F) \in V | \;\; \|v_E\| \leq a\|v_F\|\}$$

Proposition 2.0.3 *Let V be a real Hilbert space. Then a set $C \subset V$ is a cone on V if, and only if, there is a continuous non-degenerated quadratic form $B : V \to \mathbb{R}$ such that*

$$C = \{v \in V | \;\; B(v) \leq 0\}.$$

Definition 2.0.5 *Let V be a Hilbert space with finite dimension and let C be a cone on V.*

- *We say that the cone C has dimension k when $k = Max\{\dim W | \;\; W \subset C \;\; is \;\; a \;\; subspace \;\; of \;\; V\}$.*

- *We say that a non-degenerated quadratic form $B : V \to \mathbb{R}$ has dimension k if k is the dimension of the cone determined of it.*

Definition 2.0.6 *Let $\mathbb{M}$ be a C^k-manifold of dimension n and let $\Lambda \subset \mathbb{M}$.*

- *One call a continuous field of quadratic form a function B which associates to each point $x \in \Lambda$ a quadratic form $B(x) : T_x\mathbb{M} \to \mathbb{R}$. By continuous one means that the coefficients of the quadratic forms $B(x)$ are continuous real functions on Λ.*

- *One call a continuous cone-field on Λ a function C which to each point $x \in \Lambda$ associates a cone $C(x)$ in $T_x\mathbb{M}$. By continuous cone-field one means that the field of quadratic form determined by C is a continuous field of quadratic form.*

- *If $f \in C^1(\mathbb{M}, \mathbb{M})$ and B is a continuous field of quadratic form on Λ, one call the pull-back of B the field of quadratic form defined by*
$$(f^*B)(x) = B(f(x))(df(x)(v))$$
where we are supposing that $f(\Lambda) \subset \Lambda$.

Theorem 2.0.2 (**Characterization of hyperbolicity via cone – fields**)
Suppose that

- $\mathbb{M}$ *is a connected compact boundaryless C^r- Riemannian manifold of dimension n;*
- $f \in Diff^r(\mathbb{M})$;
- $\Lambda \subset \mathbb{M}$ *is compact f-invariant.*

The following statements are equivalent :

1. *Λ is a hyperbolic set for f;*
2. *There is a continuous field B of non-degenerated quadratic forms on Λ and whose dimension is constant through the orbits of f in points of Λ and such that the quadratic form $f^*B - B$ is positive defined, where $f^*B(v) = B(f(v))$.*
3. *There is on Λ ,two continuous cone-fields C^s and C^u where the dimensions of them are point-wise complements and constant through the orbits of f in points of Λ and such that :*
   (a) *$df(x)(C^u(x)) \subset C^u(f(x))$ and $df^{-1}(x)(C^s(x)) \subset C^s(f^{-1}(x))$;*
   (b) *There is $\sigma > 1$ and $m > 0$ such that*
      i. *$\|df^m(x)(v)\| \leq \sigma\|v\|$, $\forall v \in C^u(x)$ and $\forall x \in \Lambda$;*
      ii. *$\|df^{-m}(x)(v)\| \leq \sigma\|v\|$, $\forall v \in C^s(x)$ and $\forall x \in \Lambda$.*

Theorem 2.0.3 (**The Stable Manifold Theorem**)
Suppose that

- $\mathbb{M}$ *is a connected compact boundaryless C^r- Riemannian manifold of dimension n;*
- $f \in Diff^r(\mathbb{M})$;
- $\Lambda \subset \mathbb{M}$ *is a hyperbolic set for f.*

Then there is $\varepsilon > 0$ such that for every point $x \in \Lambda$ the following holds:

- *$W^s_\varepsilon(x)$ is a C^r-Embedded sub-manifold of $\mathbb{M}$ so that $T_x W^s_\varepsilon(x) = E^s(x)$;*
- *$W^s_\varepsilon(x) \subset W^s(x)$;*
- *$W^s(x) = \bigcup_{n\leq 0} f^{-n}(W^s_\varepsilon(f^n(x)))$ and is a C^r-Immersed sub-manifold of $\mathbb{M}$. Moreover $W^s(x)$ depends continuously on the point x.*

Obviously there is an analogously result for $W^u(x)$ because $W^u(x,f) = W^s(x,f^{-1})$.

Corollary 2.0.3 *In the same hypothesis of the previous theorem one can deduce that there is $\delta > 0$ such that for every $x, y \in \Lambda$ with $d(x,y) < \delta$ one have $W^s_\varepsilon(x) \pitchfork W^u_\varepsilon(y) = \{z\}$.*

Chapter 3

Blender in 3D

In this chapter following [2] we are going to define blender and prove some properties of such object.
Rougly speaking a Blender is a hyperbolic set Λ, locally maximal invariant with a dense orbit such that $W^s(\Lambda)$ is locally homeomorphic to the product of a Cantor set by an interval. But it bahaves as a topological surface in the following sense: There is a conefield $\mathcal{C}^{uu}$ around the strong unstable direction of Λ so that every curve γ tangent to $\mathcal{C}^{uu}$ intersects $W^s(\Lambda)$(see lemma 3.0.6). That is why we need of the hyperbolicity theory in the background.
So here is the definition of Blender.

Definition 3.0.7 *Let $\mathbb{M}^3$ be a boundaryless Riemaniann manifold of dimension 3 and $f \in Diff^1(\mathbb{M}^3)$. Consider the box*
$\mathbb{B} := [-1,1] \times [-1,1] \times [-1,1] \subset \mathbb{R}^3$ *and $\mathbb{D}$ the image of an embedding $\mathcal{E} : \mathcal{O} \supset \mathbb{B} \to \mathbb{M}^3$.*
Decompose the boundary of $\mathbb{D}$ into three parts as follow:

$$\partial^{uu}\mathbb{D} = \mathcal{E}((\{-1\} \cup \{+1\}) \times [-1,1] \times [-1,1])$$
$$\partial^{u}\mathbb{D} = \mathcal{E}([-1,1] \times [-1,1] \times (\{-1\} \cup \{+1\}))$$
$$\partial^{s}\mathbb{D} = \mathcal{E}([-1,1] \times (\{-1\} \cup \{+1\}) \times [-1,1])$$

Suppose that :

- *There is a connected component $\mathbb{A}$ of $\mathbb{D} \cap f(\mathbb{D})$ disjoint from the union $\partial^s\mathbb{D} \cup f(\partial^u\mathbb{D})$;*
- *There are an integer $n_0 > 0$ and a connected component $\mathbb{B}$ of $f^{n_0}(\mathbb{D}) \cap \mathbb{D}$ so that $\mathbb{B}$ is disjoint from $\partial^s\mathbb{D}$, from $f(\partial^{uu}\mathbb{D})$ and from $\mathcal{E}([-1,1] \times [-1,1] \times \{+1\})$. (See Figure)*
- *There is a cone field $C^u(q)$ on $f^{-1}(\mathbb{A}) \cup f^{-n_0}(\mathbb{B})$ such that :*
  - $\forall q \in f^{-1}(\mathbb{A}) \Rightarrow df(q)(C^u(q)) \subset Int(C^u(f(q)))$;
  - $\forall q \in f^{-n_0}(\mathbb{B}) \Rightarrow df^{n_0}(q)(C^u(q)) \subset Int(C^u(f^{n_0}(q)))$.
- *There is a cone field $C^{uu}(q) \subset C^u(q)$ on $f^{-1}(\mathbb{A}) \cup f^{-n_0}(\mathbb{B})$ such that :*
  - $\forall q \in f^{-1}(\mathbb{A}) \Rightarrow df(q)(C^{uu}(q)) \subset Int(C^{uu}(f(q)))$;
  - $\forall q \in f^{-n_0}(\mathbb{B}) \Rightarrow df^{n_0}(q)(C^{uu}(q)) \subset Int(C^{uu}(f^{n_0}(q)))$.

- *There is a cone field $C^s(q)$ on $\mathbb{A} \cup \mathbb{B}$ such that :*
  - $\forall q \in \mathbb{A} \Rightarrow df^{-1}(q)(C^s(q)) \subset Int(C^s(f^{-1}(q)))$;
  - $\forall q \in \mathbb{B} \Rightarrow df^{-n_0}(q)(C^s(q)) \subset Int(C^s(f^{-n_0}(q)))$.

Then we have the followings definitions:

- *We say that a segment L of smooth curve in $\mathbb{D}$ is an unstable segment when L is tangent to $\mathcal{C}^{uu}$ and its boundary ∂L is contained in $\partial^{uu}\mathbb{D}$. We shall denote an unstable segment for L^u.*

- *We say that a segment L of smooth curve in $\mathbb{D}$ is a stable segment when L is tangent to $\mathcal{C}^s$ and its boundary ∂L is contained in $\partial^s\mathbb{D}$ and we shall denote a stable segment for L^s.*

  Note that if L^s is a stable segment in $\mathbb{D}$ such that $L^s \cap \partial^u\mathbb{D} = \emptyset$. Then in $\mathbb{D}/L^s$ there are just two homotopy classes of unstable segments. One is the homotopy class of $L^+ := \mathcal{E}([-1,1] \times \{0\} \times \{+1\})$ and the other one is the homotopy class of $L^- := \mathcal{E}([-1,1] \times \{0\} \times \{-1\})$.

- *We say that an unstable segment L^u is on the upper region of L^s when $L^s \cap L^u = \emptyset$ and L^u is homotopic to L^+ in $\mathbb{D}/L^s$.*

- *We say that a stable segment L^s is on the lower region of L^s when $L^s \cap L^u = \emptyset$ and L^u is homotopic to L^- in $\mathbb{D}/L^s$.*

- *One call an unstable strip trough $\mathbb{D}$ an embedding $\Phi : [-1,1] \times [-1,1] \to \mathbb{D}$ such that for every $t \in [-1,1]$ the image c_t of $[-1,1] \times \{t\}$ is an unstable segment through $\mathbb{D}$ and the image $\mathcal{S}$ of $[-1,1] \times [-1,1]$ is tangent to $\mathcal{C}^u$.*

  For the sake of notational simplicity given an unstable strip $\Phi : [-1,1] \times [-1,1] \to \mathbb{D}$ we also call unstable strip the image $\mathcal{S}$ of $[-1,1] \times [-1,1]$ by Φ.

- *The unstable boundary $\partial\mathcal{S}$ of an unstable strip $\mathcal{S}$ is the union of $\partial^{\pm}\mathcal{S}$, where $\partial^{\pm}\mathcal{S} = \Phi(\{\pm 1\} \times [-1,1])$.*

- *An unstable strip $\mathcal{S}$ is maximal if its unstable boundary $\partial\mathcal{S}$ is contained in $\partial^u\mathbb{D}$.*

- *The width of an unsatable strip $\mathcal{S}$, $\omega d(\mathcal{S})$, is*

$$\inf\{\ell(\alpha)/\alpha \ \ is \ \ an \ \ arc \ \ in \ \ \mathcal{S} \ \ joining \ \ the \ \ two \ \ components \ \ \partial^{\pm}\mathcal{S} \ \ of \ \ \partial\mathcal{S}\}.$$

Lemma 3.0.2 *The following holds.*

$$\sup\{wd(\mathcal{S})/\mathcal{S} \subset \mathbb{D} \ \ is \ \ unstable \ \ strip\} < +\infty.$$

To proof see [2] page 364.

Definition 3.0.8 *Let $\mathbb{M}^3$ be a boundaryless Riemaniann manifold and $f \in Diff^1(\mathbb{M}^3)$. Consider the box $\mathbb{B} := [-1,1] \times [-1,1] \times [-1,1] \subset \mathbb{R}^3$ and $\mathbb{D}$ the image of an embedding $\mathcal{E} : \mathcal{O} \supset \mathbb{B} \to \mathbb{M}^3$.*

We say that the pair $(\mathbb{D}, f)$ is a blender if it satisfies the following hypothesis:

- **(H1)** *There is a connected component $\mathbb{A}$ of $\mathbb{D} \cap f(\mathbb{D})$ disjoint from the union $\partial^s\mathbb{D} \cup f(\partial^u\mathbb{D})$;*

- **(H2)** *There are an integer $n_0 > 0$ and a connected component $\mathbb{B}$ of $f^{n_0}(\mathbb{D}) \cap \mathbb{D}$ so that $\mathbb{B}$ is disjoint from $\partial^s\mathbb{D}$, from $f(\partial^{uu}\mathbb{D})$ and from $\mathcal{E}([-1,1] \times [-1,1] \times \{+1\})$.*

- (**H3**) *(hyperbolicity conditions).*
  - *There is a cone field* $C^u(q)$ *on* $f^{-1}(\mathbb{A}) \cup f^{-n_0}(\mathbb{B})$ *such that :*
    - $\forall q \in f^{-1}(\mathbb{A}) \Rightarrow df(q)(C^u(q)) \subset Int(C^u(f(q)))$;
    - $\forall q \in f^{-n_0}(\mathbb{B}) \Rightarrow df^{n_0}(q)(C^u(q)) \subset Int(C^u(f^{n_0}(q)))$.
  - *There is a cone field* $C^{uu}(q) \subset C^u(q)$ *on* $f^{-1}(\mathbb{A}) \cup f^{-n_0}(\mathbb{B})$ *such that :*
    - $\forall q \in f^{-1}(\mathbb{A}) \Rightarrow df(q)(C^{uu}(q)) \subset Int(C^{uu}(f(q)))$;
    - $\forall q \in f^{-n_0}(\mathbb{B}) \Rightarrow df^{n_0}(q)(C^{uu}(q)) \subset Int(C^{uu}(f^{n_0}(q)))$.
  - *There is a cone field* $C^s(q)$ *on* $\mathbb{A} \cup \mathbb{B}$ *such that :*
    - $\forall q \in \mathbb{A} \Rightarrow df^{-1}(q)(C^s(q)) \subset Int(C^s(f^{-1}(q)))$;
    - $\forall q \in \mathbb{B} \Rightarrow df^{-n_0}(q)(C^s(q)) \subset Int(C^s(f^{-n_0}(q)))$.
  - *There is an expanding constant* $\rho > 1$ *such that the derivatives* df, df^{-1}, df^{n_0} *and* df^{-n_0} *are uniformly* ρ*-expanding through the cones fields above defined.*

  The Lemma below will assures us that as a consequence of (**H1**) *and* (**H3**) *the diffeo* f *has a hyperbolic fix point* q *whose index is 2. We denote by* $W^s_{\mathbb{D}}(q)$ *the connected component of the intersection* $W^s \cap \mathbb{D}$ *containing the point* q*. It is immediate that* $W^s_{\mathbb{D}}(q)$ *is a stable segment through* $\mathbb{D}$*.*

- (**H4**) *There is a neighborhood* U^- *of the lower face* $\mathcal{E}([-1,1] \times [-1,1] \times \{-1\})$ *of* $\mathbb{D}$ *so that every unstable segment* L^u *on the upper region of* $W^s_{\mathbb{D}}(q)$ *has no intersection with* U^-*;*

- (**H5**) *There are a neighborhood* $\mathcal{O}^+$ *of the upper face* $\mathcal{E}([-1,1] \times [-1,1] \times \{+1\})$ *of* $\mathbb{D}$ *and a neighborhood* $\mathcal{V}$ *of* $W^s_{\mathbb{D}}(q)$ *so that for every unstable segment* L^u *on the upper region of* $W^s_{\mathbb{D}}(q)$ *one of the two possibilities holds:*
  - $f(L^u) \cap \mathbb{A}$ *contains an unstable segment on the upper region of* $W^s_{\mathbb{D}}(q)$ *and disjoint of* $\mathcal{O}^+$*;*
  - $f^{n_0}(L^u) \cap \mathbb{B}$ *contains an unstable segment on the upper region of* $W^s_{\mathbb{D}}(q)$ *and disjoint of* $\mathcal{V}$*.*

Remark 3.0.1 *One can note that both definitions above are the same of Bonatti-Dias [2] ,pages 365-369. Here we just consider 3-dimensional manifolds and a slightly different nomination. But essentially are the same things.*

Lemma 3.0.3 *Let* $(\mathbb{D}, f)$ *be pair satisfying* (**H1**) *and* (**H3**) *in the definition of blender. Then*

- *For every stable segment* L^s *the intersection* $\mathbb{A} \cap L^s$ *is a segment of curve whose boundary* ∂L^s *is contained in* $\partial^s \mathbb{A}$*. In particular,* $f^{-1}(\mathbb{A} \cap L^s)$ *is a stable segment on* $\mathbb{D}$*;*

- *For every maximal unstable strip* $\mathcal{S}$ *the intersection* $f(\mathcal{S}) \cap \mathbb{A}$ *is a maximal unstable strip;*

- *The diffeo* f *has an unique fixed point ,say* q*, in* $\mathbb{A}$ *and whose index is 2. Moreover* $W^s_{\mathbb{D}}(q)$ *is a stable segment through* $\mathbb{D}$*;*

- *If* L^u *is an unstable segment, then the intersection* $f(L^u) \cap \mathbb{A}$ *contains at most one unstable segment through* $\mathbb{D}$*.*

To proof see Bonatti-Dias Paper [2] page 366.

Lemma 3.0.4 *Let $(\mathbb{D}, f)$ be a blender as in definition above and $\mathcal{S}$ an unstable strip through $\mathbb{D}$ on the upper region of $W^s_{\mathbb{D}}(q)$.*

If $\widetilde{\mathcal{S}}$ is an unstable strip through $\mathbb{D}$ that is either a connected component of $f(\mathcal{S}) \cap \mathbb{A}$ or a component of $f^{n_0}(\mathcal{S}) \cap \mathbb{B}$. Suppose yet that the unstable boundary $\partial\widetilde{\mathcal{S}}$ is included in the image of the one of $\mathcal{S}$. Then the width $wd(\widetilde{\mathcal{S}})$ of $\widetilde{\mathcal{S}}$ is bigger than $\rho.wd(\mathcal{S})$.

Proof: Suppose that $\widetilde{\mathcal{S}}$ is a connected component of $f(\mathcal{S}) \cap \mathbb{A}$. Then every arc $\widetilde{\alpha}$ in $\widetilde{\mathcal{S}}$ joining $\partial^-\widetilde{\mathcal{S}}$ to $\partial^+\widetilde{\mathcal{S}}$ is the image by f of an arc γ in $\mathcal{S}$ joining $\partial^-\mathcal{S}$ to $\partial^+\mathcal{S}$.Thus , from the expanding conditions for the cone field C^u,

$$\rho.\ell(\gamma) < \ell(\widetilde{\alpha}).$$

■

Lemma 3.0.5 *Let $(\mathbb{D}, f)$ be a blender as in definition above and $\mathcal{S}$ an unstable strip through $\mathbb{D}$ on the upper region of $W^s_{\mathbb{D}}(q)$. Then there are two possibilities:*

- *Either $f(\mathcal{S}) \cup f^{n_0}(\mathcal{S})$ intersects $W^s_{\mathbb{D}}(q)$ or*
- *There is an unstable strip $\widetilde{\mathcal{S}}$ on the upper region of $W^s_{\mathbb{D}}(q)$ and contained in $f(\mathcal{S}) \cup f^{n_0}(\mathcal{S})$ so that its width $wd(\widetilde{\mathcal{S}})$ is bigger than $\rho.wd(\mathcal{S})$.*

To proof see Bonatti-Dias Paper [2] page 368.

Lemma 3.0.6 *For any unstable strip $\mathcal{S}$ on the upper region of $W^s_{\mathbb{D}}(q)$ there is an integer $m > 0$ such that $f^m(\mathcal{S})$ intersects $W^s_{\mathbb{D}}(q)$.*

Proof: Let $\mathcal{S}$ be an unstable strip as in the statement. By previous lemma either $f(\mathcal{S}) \cup f^{n_0}(\mathcal{S})$ intersects $W^s_{\mathbb{D}}(q)$ and in this case we are done or there is an unstable strip $\mathcal{S}_0$ on the upper region of $W^s_{\mathbb{D}}(q)$ and contained in $f(\mathcal{S}) \cup f^{n_0}(\mathcal{S})$ so that its width $wd(\mathcal{S}_0)$ is bigger than $\rho.wd(\mathcal{S})$.

- If $(f(\mathcal{S}_0) \cup f^{n_0}(\mathcal{S}_0)) \cap W^s_{\mathbb{D}}(q) \neq \emptyset$ then $[f^{2n_0}(\mathcal{S}) \cup f^{n_0+1}(\mathcal{S}) \cup f^2(\mathcal{S})] \cap W^s_{\mathbb{D}}(q) \neq \emptyset$ and we are done.
- If not then we take $\mathcal{S}_1 \subset [f^{n_0}(\mathcal{S}_0) \cup f(\mathcal{S}_0)] \cap \mathbb{D}$ with width $wd(\mathcal{S}_1) \geq \rho wd(\mathcal{S}_0) \geq \rho^2 wd(\mathcal{S})$.

- If $[f^{n_0}(\mathcal{S}_1) \cup f(\mathcal{S}_1)] \cap W^s_{\mathbb{D}}(q) \neq \emptyset$ then as $\mathcal{S}_1 \subset [f^{n_0}(\mathcal{S}_0) \cup f(\mathcal{S}_0)] \cap \mathbb{D} \Rightarrow [f^{2n_0}(\mathcal{S}_0) \cup f^{n_0+1}(\mathcal{S}_0) \cup f^2(\mathcal{S}_0)] \cap W^s_{\mathbb{D}}(q) \neq \emptyset$. But $\mathcal{S}_0 \subset (f(\mathcal{S} \cup f^{n_0}(\mathcal{S}))$. Thus $[f^{3n_0}(\mathcal{S}) \cup f^{2n_0+1}(\mathcal{S}) \cup f^{n_0+2}(\mathcal{S}) \cup f^3(\mathcal{S})] \cap W^s_{\mathbb{D}}(q) \neq \emptyset$ and we are done .
- If not ,then we take $\mathcal{S}_2 \subset [f^{n_0}(\mathcal{S}_1) \cup f(\mathcal{S}_1)] \cap \mathbb{D}$ with width $wd(\mathcal{S}_2) \geq \rho.wd(\mathcal{S}_1) \geq \rho^2.wd(\mathcal{S}_0) \geq \rho^3.wd(\mathcal{S})$. We can make this process ever and by induction we are going to obtain a sequence $(\mathcal{S}_n)$ with width $wd(\mathcal{S}_n) \geq \rho^{n+1}.wd(\mathcal{S})$. But by lemma 3.0.2 above that sequence must stop in some moment, that is, we can find an $m > 0$ such that $f^m(\mathcal{S}) \cap W^s_{\mathbb{D}}(q) \neq \emptyset$.

■

Lemma 3.0.7 *The stable manifold $W^s(q)$ of the hyperbolic point q intersects transversally every unstable strip on the upper region of $W^s_{\mathbb{D}}(q)$.*

Proof: Its immediate from the previous lemma. ■

Now we are going to prove the amazing global fact about the stable manifold of a blender which is :

Lemma 3.0.8 *Suppose that there is a hyperbolic fixed point p , of f, with index 1 so that $W^u(p) \cap \mathbb{D}$ contains an unstable segment through $\mathbb{D}$ on the upper region of $W^s_{\mathbb{D}}(q)$. Then $W^s(p) \subset \overline{W^s(q)}$.*

Proof: We fix a point $\widehat{q}$ in $W^s(p)$ and then we take an arbitrary neighborhood $\mathcal{V}_{\widehat{q}}$ of it. The λ-Lemma assures us that for some integer $j > 0$ the set $f^j(\mathcal{V}_{\widehat{q}})$ must contain an unstable strip on the upper region of $W^s_{\mathbb{D}}(q)$. But from the previous Lemma we must have $f^j(\mathcal{V}_{\widehat{q}}) \cap W^s(q) \neq \emptyset$. Since f is a diffeo and $W^s(q)$ is a stable manifold the result is obvious. ■

Chapter 4

Examples of Blenders

4.1 The Affine Blender

In this section we will present an example of blender called affine blender. It is the most simple model of a Blender. The reference is [8]
To begin let $f \in Diff^1(\mathbb{R}^2)$ and $R = [0,1] \times [0,1]$ such that (f, R) is a Smale horseshoe for f, that is :

- $R \cap f(R)$ has has two connected components J_1 and J_2 such that $J_i = I_i \times [0,1]$ for some compact interval $I_i \subset Int[0,1]$ where $i = 1,2$;
- $R \cap f^{-1}(R)$ has has two connected components R_1 and R_2 such that $R_i = I_i \times [0,1]$ for some compact interval $I_i \subset Int[0,1]$ where $i = 1,2$;
- The restriction of f to $R_1 \cup R_2$ is affine with linear parts:

$$\begin{bmatrix} \pm\frac{1}{3} & 0 \\ 0 & \pm 3 \end{bmatrix}$$

  In particular, such restrictions preserves the horizontal and vertical directions (see figures 2.1).

Then one obtain that f has an unique hyperbolic fixed point, say $q = (y_0, z_0)$ in J_1 whose stable manifold is a horizontal segment of line $W^s(q) = [0,1] \times \{z_0\}$ and another fixed point $p = (y_1, z_1)$ in J_2.

Now consider a diffeomorphism $F \in Diff^1(\mathbb{R}^3)$ such that on the box $\mathbb{D} := [-1,1] \times R$ has the following aspect:

$$F(x,y,z) = \begin{cases} \left(\frac{5x}{4}, f(y,z)\right); & if \quad (x,y,z) \in \mathbb{H}_1 := [-1,1] \times R_1; \\ \left(\frac{5x}{4} - \frac{1}{2}, f(y,z)\right); & if \quad (x,y,z) \in \mathbb{H}_2 := [-1,1] \times R_2 \end{cases}$$

Now we denote $\mathbb{V}_1 = [-1,1] \times J_1$ and $\mathbb{V}_2 = \big[-1, \frac{3}{4}\big] \times J_2$. It follows that

$$\mathbb{V}_1 \subset F(\mathbb{H}_1) = [\frac{-5}{4}, \frac{5}{4}] \times J_1 \subset F(\mathbb{D});$$

$$\mathbb{V}_2 \subset F(\mathbb{H}_2) = [\frac{-7}{4}, \frac{3}{4}] \times J_2 \subset F(\mathbb{D}).$$

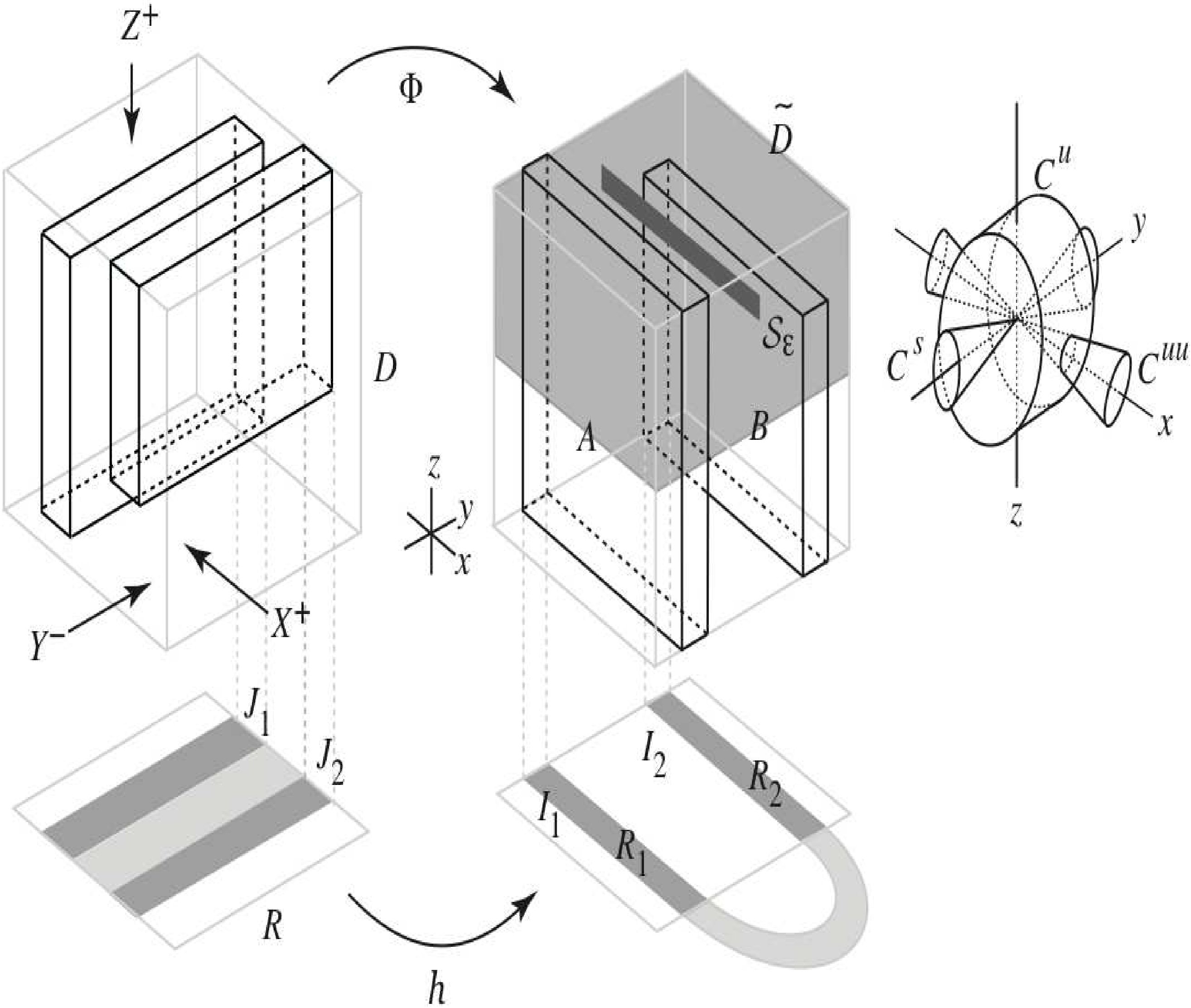

Figure 4.1: The Affine blender map and its invariant cones (figure from [1])

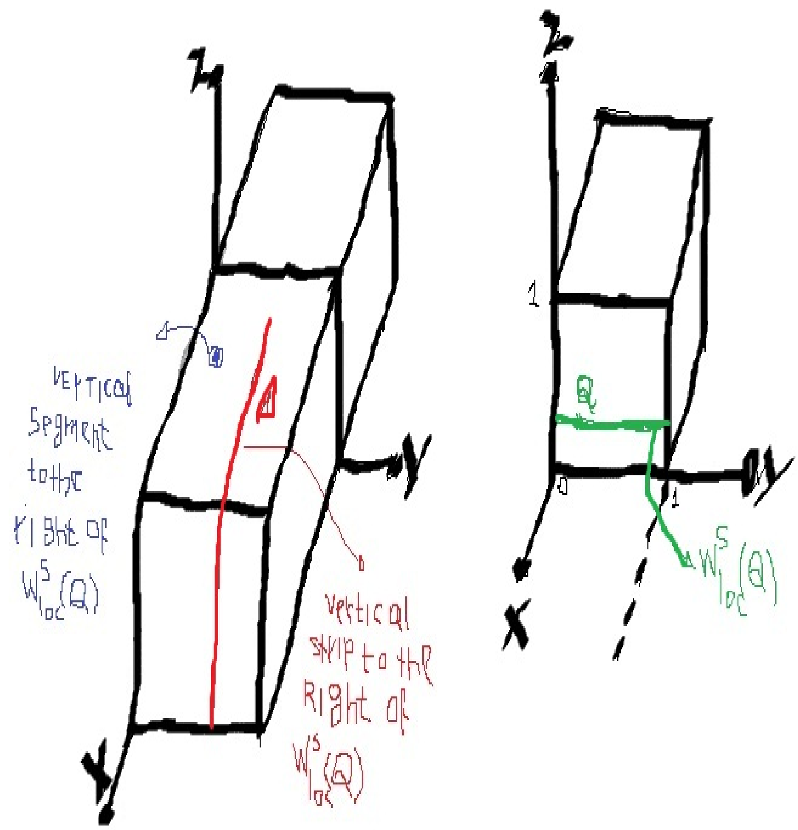

Figure 4.2: On red color we have vertical strip and on blue a vertical segment

Then we get $\mathbb{D} \cap F(\mathbb{D}) = \mathbb{V}_1 \cup \mathbb{V}_2$ and $\mathbb{D} \cap F^{-1}(\mathbb{D}) = \mathbb{H}_1 \cup \mathbb{H}_2$ (disjoint unions !).
Note that

$$F^{-1}(u,v,w) = \begin{cases} \left(\frac{4u}{5}, f^{-1}(v,w)\right); & se \ \ (u,v,w) \in \mathbb{V}_1; \\ \left(\frac{4u}{5} + \frac{2}{5}, f^{-1}(v,w)\right); & se \ \ (u,v,w) \in \mathbb{V}_2. \end{cases}$$

See figure 2.2.
So let us prove the existence of Blender.
Claim1. F has an unique hyperbolic fix point Q in $\mathbb{V}_1$ whose index is equal 2,that is, $\dim W^u(Q) = 2$.
In fact, we take $Q = (0,q) = (0, y_0, z_0)$ where q is the fix point of f in V_1. Then $F(Q) = F(0, y_0, z_0) = (0, f(y_0, z_0)) = (0, y_0, z_0) = Q$ and obviously $Q \in \mathbb{V}_1$.

Let $W^s_{loc}(Q) = \{0\} \times [0,1] \times \{z_0\}$ be the connected component of $W^s(Q) \cap \mathbb{D}$ which contains Q.

Now consider the sets
$X^{\pm} := \{\pm 1\} \times [0,1] \times [0,1]$
$Y^{+} := [-1,1] \times \{0\} \times [0,1]$
$Y^{-} := [-1,1] \times \{1\} \times [0,1]$
$Z^{+} := [-1,1] \times [0,1] \times \{1\}$
$Z^{-} := [-1,1] \times [0,1] \times \{0\}$

$$\partial^{uu}\mathbb{D} = Z^{+} \cup Z^{-};$$

$$\partial^{u}\mathbb{D} = X^{+} \cup X^{-} \cup Z^{+} \cup Z^{-};$$

$$\partial^{s}\mathbb{D} = Y^{+} \cup Y^{-}.$$

Claim 2. $\mathbb{V}_1 \cap \partial^s\mathbb{D} = \emptyset$;
In fact, this follows from the construction of horseshoe (f, B) where we have $I_1 \subset Int([0,1])$.

Claim 3. $\mathbb{V}_1 \cap F(\partial^u\mathbb{D}) = \emptyset$;
Of course, if there is $(u, v, w) \in \mathbb{V}_1 \cap F(\partial^u\mathbb{D}) \Rightarrow (u, v, w) = F(x, y, z)$ where $(x, y, z) \in \partial^u\mathbb{D}$. Then $x = \pm 1$ and $(\frac{4u}{5}, 3v, \frac{w}{3}) = (\pm 1, y, z) \Rightarrow u = \pm\frac{5}{4}$ absurd!

Claim 4. $\mathbb{V}_2 \cap \partial^s\mathbb{D} = \emptyset$;
In fact, otherwise we should have $\{0, 1\} \cap I_2 \neq \emptyset$ an absurd!

Claim 5. $\mathbb{V}_2 \cap F(\partial^{uu}\mathbb{D}) = \emptyset$;
Of course,By construction of the Smale's horseshoe (f, B) we have that $f(\partial^{uu}B) = f([0,1] \times (\{0\} \cup \{+1\}))$ has no intersection with B. Then by construction of F we have the result.

Now note that as F is affine in $\mathbb{D} \cap F(\mathbb{D}) = \mathbb{V}_1 \cup \mathbb{V}_2$ and the tangent space is decomposed in one unstable plane $X \bigoplus Z$ where Z is the strong unstable direction and X is the weak unstable direction, and the stable direction which is Y-direction. This implies immediately that F satisfies the hyperbolicity condition $\mathbf{H}_3$ in the definition of blender .Actually the maximal invariant set of F in $\mathbb{D}$ is hyperbolic set for F.

Claim 6. There is a neighborhood U^- of the face $\{-1\} \times [0,1] \times [0,1]$ of $\mathbb{D}$ such that every vertical segment L to the right of $W^s_{\mathbb{D}}(Q)$ has no intersection with U^-.
Of course,in this case any vertical segment to the right of $W^s_{\mathbb{D}}(Q)$ is exactly a segment of straight line parallel to z-axis and it is far from that face.

Claim 7. For every vertical segment to the right of $W^s_{\mathbb{D}}(Q)$ one of the two things holds:

- $f(L) \cap \mathbb{V}_1$ is a vertical segment to the right of $W^s_{\mathbb{D}}(Q)$;
- $f(L) \cap \mathbb{V}_2$ is a vertical segment to the right of $W^s_{\mathbb{D}}(Q)$.

Of course,If L is a vertical segment to the right of $W^s_{\mathbb{D}}(Q)$ then $L = \{t\} \times \{y\} \times [0,1]$ with $t > 0$. Thus the x-coordinate of $f(L) \cap \mathbb{V}_1$ is $\frac{5t}{4}$ and the x-coordinate of $f(L) \cap \mathbb{V}_2$ is $\frac{5t}{4} - \frac{1}{2}$. Then we have two possibilities:
If $t < \frac{4}{5}$ then we get $f(L) \cap \mathbb{V}_1$ to the right of $W^s_{\mathbb{D}}(Q)$.
If $\frac{4}{5} \leq t \leq 1$ then we get $\frac{5t}{4} - \frac{1}{2} > \frac{1}{2}$ and from that $f(L) \cap \mathbb{V}_2$ is to the right of $W^s_{\mathbb{D}}(Q)$.

Claim 8. There is a neighborhood $\mathcal{O}^+$ of the face $X^+ := \{+1\} \times [0,1] \times [0,1]$ of $\mathbb{D}$ and a neighborhood $\mathcal{V}$ of $W^s_{\mathbb{D}}(Q)$ so that for every vertical segment L^u to the right of $W^s_{\mathbb{D}}(Q)$ one of the two possibilities holds:

- $f(L) \cap \mathbb{V}_1$ is disjoint of $\mathcal{O}^+$;
- $f(L) \cap \mathbb{V}_2)$ is disjoint of $\mathcal{V}$.

In fact, suppose this is not true. Then we can find a sequence L_n of vertical segment to the right of $W^s_{\mathbb{D}}(Q)$ such that as $n \to \infty$ we have $\lim_{n\to\infty} Dist(F(L_n) \cap \mathbb{V}_1, X^+) = 0 = \lim_{n\to\infty} Dist(F(L_n) \cap \mathbb{V}_2, W^s_{\mathbb{D}}(Q))$. So denote by $L^i_n = F(L_n \cap \mathbb{V}_i$, x^i_n the x-coordinate of L^i_n where $i = 1, 2$ and x_n the x-coordinate of L_n. Since F preserves vertical directions then we must have $\lim_{n\to\infty} x^1_n = 1$ and $\lim_{n\to\infty} x^2_n = 0$. On the other hand by using the definition of F^{-1} in $\mathbb{V}_1 \cup \mathbb{V}_2$ we must have $x_n = \frac{4x^1_n}{5} = \frac{4x^2_n}{5} + \frac{2}{5}$. It follows that the sequence x_n has two different limits which impossible.
Note that the segments $F^{-1}(L^1_n) = L_n \cap \mathbb{H}_1$ and $F^{-1}(L^2_n) = L_n \cap \mathbb{H}_2$ must have the same x-coordinate.

Hence the pair $(F, \mathbb{D})$ satisfies the blender conditions.
Now let us prove directly the main properties of blenders to $(F, \mathbb{D})$.

Lemma 4.1.1 *If Δ is a vertical strip to the right of $W^s_{loc}(Q)$ then either $F(\Delta)$ intersects $W^s_{loc}(Q)$ or else $F(\Delta) \cap \mathbb{D}$ must contain another vertical strip $\widehat{\Delta}$ to the right of $W^s_{loc}(Q)$ with $w(\widehat{\Delta}) = \frac{5}{4}w(\Delta)$.*

Proof: Let $\Delta = [x_1, x_2] \times \{y\} \times [0, 1]$ be a vertical strip to the right of $W^s_{loc}(Q)$.
Denote $\Delta_1 = F(\Delta)\cap\mathbb{V}_1 \subset \mathbb{D}\cap F(\mathbb{D})$ and $\Delta_2 = F(\Delta)\cap\mathbb{V}_2 \subset \mathbb{D}\cap F(\mathbb{D})$. Then $F^{-1}(\Delta_1) = \Delta\cap F^{-1}(\mathbb{V}_1) = \Delta\cap\mathbb{H}_1$. It follows that we must have $F^{-1}(\Delta_1) = \Delta \cap \mathbb{H}_1 = [x_1, x_2] \times \{y\} \times [0, z]$. So by definition of F the strip Δ_1 has $wd(\Delta_1) = \frac{5x_2}{4} - \frac{5x_1}{4}$. On the other hand we have $F^{-1}(\Delta_2) = \Delta \cap F^{-1}(\mathbb{V}_2) = \Delta \cap \mathbb{H}_2$ which implies in $F^{-1}(\Delta_2) = \Delta \cap \mathbb{H}_2 = [x_1, x_2] \times \{y\} \times [0, z]$. So again by definition of F the strip Δ_2 has $wd(\Delta_2) = (\frac{5x_2}{4} - \frac{1}{2}) - (\frac{5x_1}{4} - \frac{1}{2})$.

Now We have the following cases:
CASE 1 :If $x_2 \leq \frac{4}{5}$, then $[\frac{5}{4}x_1, \frac{5}{4}x_2] \subset (0, 1]$ and this implies that Δ_1 must be a vertical strip to the right of $W^s_{loc}(Q)$ and more $w(\Delta_1) = \frac{5}{4}w(\Delta)$.

CASE 2: If $\frac{4}{5} < x_2 \leq 1$ then $\frac{5}{4}x_2 - \frac{1}{2} \in (\frac{1}{2}, \frac{3}{4}]$.By definition of F there exists a y' such that $\Delta_2 = F(\Delta)\cap\mathbb{V}_2 = [\frac{5}{4}x_1 - \frac{1}{2}, \frac{5}{4}x_2 - \frac{1}{2}] \times \{y'\} \times [0, 1]$.This gives us more two subcases.

CASE 2.1 :If $\frac{5}{4}x_1 - \frac{1}{2} > 0$ then Δ_2 shall be a vertical strip to the right of $W^s_{loc}(Q)$ with width $w(\Delta_2) = \frac{5}{4}w(\Delta)$.

CASE 2.2 : If $\frac{5}{4}x_1 - \frac{1}{2} \leq 0$ then Δ_2 meets $W^s_{loc}(Q)$ because $W^s_{loc}(Q) = \{0\} \times [0, 1] \times \{z_0\}$ and the proof is concluded.

■

Lemma 4.1.2 *For any vertical strip Δ to the right of $W^s_{loc}(Q)$ there exists an integer $n > 0$ such that $F^n(\Delta)$ intersects $W^s_{loc}(Q)$. In particular every vertical strip Δ to the right of $W^s_{loc}(Q)$ intersects $W^s(Q)$. See figure 4.3.*

Proof: Let Δ be a vertical strip to the right of $W^s_{loc}(Q)$. If $F(\Delta)$ intersects $W^s_{loc}(Q)$ it is finished.Otherwise, as we know there is a vertical strip $\hat{\Delta_1}$ to the right of $W^s_{loc}(Q)$ so that $\hat{\Delta_1} \subset F(\Delta)\cap\mathbb{B}$ and $\omega(\hat{\Delta_1}) = \frac{5}{4}\omega(\Delta)$. If $F(\hat{\Delta_1})$ intersects $W^s_{loc}(Q)$ it is finished because $F(\hat{\Delta_1}) \subset F^2(\Delta) \cap F(\mathbb{B}) \subset F^2(\Delta)$. Otherwise there is a vertical strip $\hat{\Delta_2}$ to the right of $W^s_{loc}(Q)$ such that $\hat{\Delta_2} \subset F(\hat{\Delta_1} \cap \mathbb{B}$ and $\omega(\hat{\Delta_2}) = \frac{5}{4}\omega(\hat{\Delta_1}) = (\frac{5}{4})^2\omega(\Delta)$. If $F(\hat{\Delta_2})$ intersects $W^s_{loc}(Q)$ it finished since $F(\hat{\Delta_2}) \subset F(\hat{\Delta_1}) \cap F(\mathbb{B}) \subset F^3(\Delta) \cap F^2(\mathbb{B}) \subset F^3(\Delta)$. Otherwise we may apply the previous proposition again and since $\frac{5}{4} > 1$ it follows that for some $n > 0$ we shall obtain a vertical strip $\hat{\Delta_n}$ to the right of $W^s_{loc}(Q)$ such that $\omega(\hat{\Delta_n}) = (\frac{5}{4})^n\omega(\Delta) > 1$ and $\hat{\Delta_n} \subset F(\hat{\Delta_{n-1}}) \subset F^n(\Delta)$. Hence for some $n > 0$ $F^n(\Delta)$ must intersects $W^s_{loc}(Q)$. ■

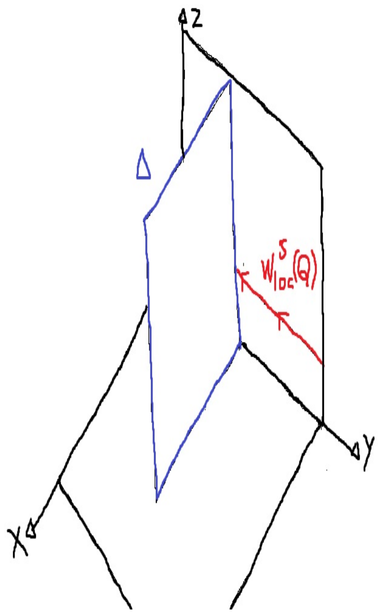

Figure 4.3: vertical strip eventually intersects stable manifold

4.2 Blender in the Henon-Like Family

On this section our main objective is study the blender structure for the non-normally Hénon-like family defined as follows.

Definition 4.2.1 ***(Non-Normally Henon-Like Family)****Consider the function* $\varphi : \mathbb{R}^4 \times \mathbb{R}^3 \to \mathbb{R}^3$ *given by* $\varphi((a, b, c, d), (x, y, z)) = (1 - ax^2 + by, x, cz + dx)$. *Thus for each point* $(a, b, c, d) \in \mathbb{R}^4$ *one have a smooth function,which we shall call* φ *too, given by* $\varphi(x, y, z)) = (1 - ax^2 + by, x, cz + dx)$ *and this family depends smoothly on the parameters* (a, b, c, d)

Let us now state the main result of the reference [1] which is the following theorem:

Theorem 4.2.1 *There exists a constant* $0 < \delta < \frac{1}{4}$ *such that if the parameters* a, b, c *and* d *satisfies:*

$$\begin{cases} 0 < |b| < \delta; \\ a > \frac{15(1+|b|)^2}{4}; \\ 1 + |d| < c < \frac{10}{9}; \\ 0 < |d| < \frac{1}{9}. \end{cases} \qquad (\mathcal{P}.\mathcal{C})$$

Then each diffeomorphisim φ *has a blender* $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\mathbb{D})$, *for some cube* $\mathbb{D} \subset \mathbb{R}^3$, *containing a saddle fix point* $p = (x_p, y_p, z_p)$ *with index 2 and satisfying*

$$\dim \overline{\left(\Pi_{yz}\big(W^s(p)\big) \cap \mathbb{D}\right)} = 2.$$

Remark 4.2.1 *Numerical simulations in figure 4.4 also support this main theorem. In fact,although uniformly hyperbolicity of* φ *does not break down under* $(\mathcal{P}.\mathcal{C})$, *geometrical dispersions of stable segment abruptly occurs if* d *cross* 0, *which corresponds to the phase transition from the non-normally hyperbolic horseshoe to the blenders.*

To prove the main theorem we are going to make various lemma as support for the proof. First of all let us obtain some properties of φ around fix point and then classify it under hyperbolicity point of view.

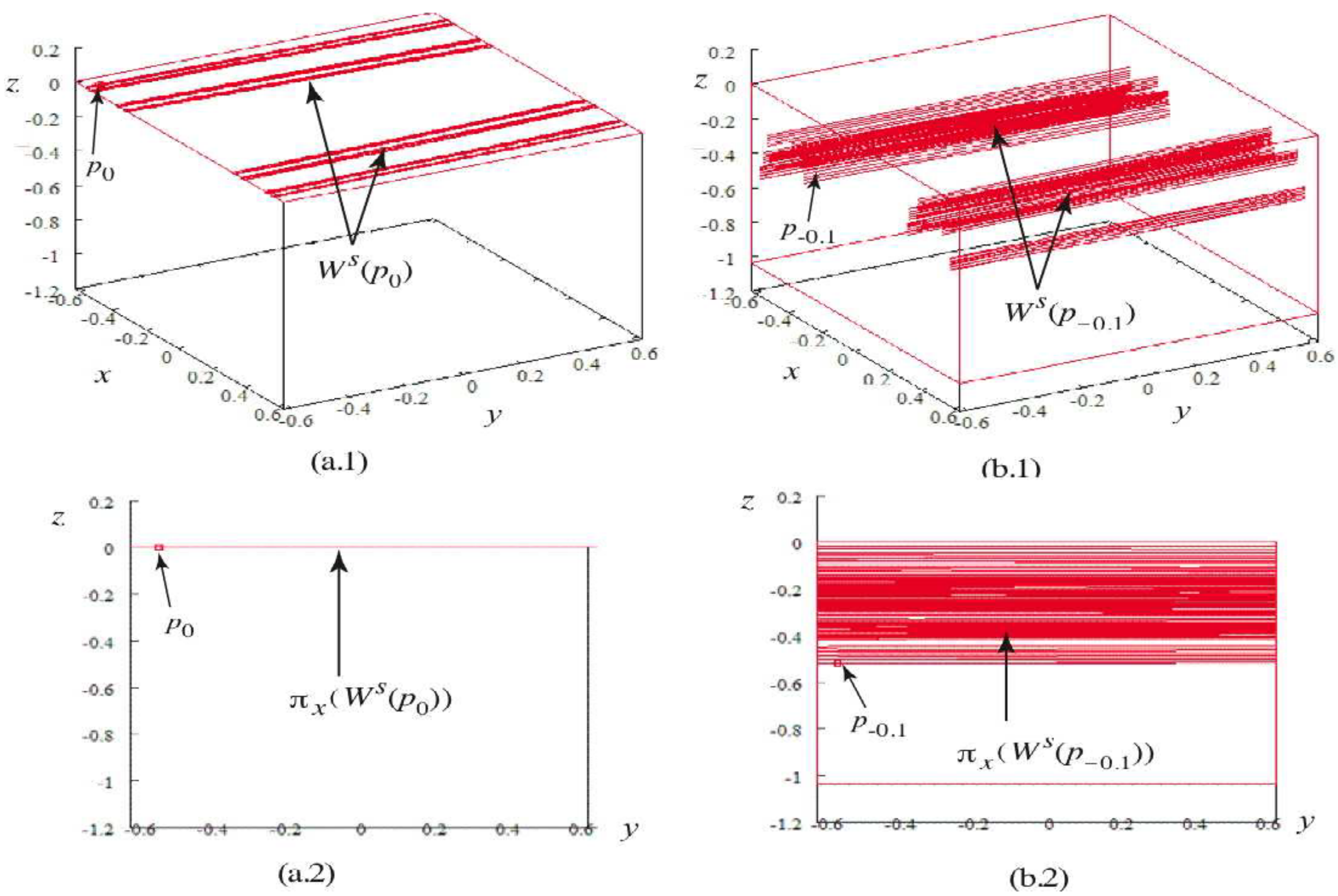

Figure 4.4: (a.1) Stable segments for $\varphi_{a,b,c,d}$ for $d = 0$ and (a.2) their projective images on the yz-plane , (b.1) Stable segments for $\varphi_{a,b,c,d}$ when $d = -0.1$ and (b.2) their projective images on the yz-plane ,where (a, b, c) is fixed near $(5.0, -0.1, 1.11)$.(Figure from [1])

When in the family above we have $b \neq 0$ and $c \neq 0$,then each function $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism. In fact, it is easy verify that φ is a bijective function. Moreover in any point (x, y, z) the jacobian matrix is

$$J\varphi(x,y,z) = \begin{bmatrix} -2ax & b & 0 \\ 1 & 0 & 0 \\ d & 0 & c \end{bmatrix}$$

whose determinant is $det\big[J\varphi(x,y,z)\big] = -bc \neq 0$. So from now on we shall consider only the case $b \neq 0$ and $c \neq 0$ and with that we have a family of diffeomorphism .

Does that family $\varphi(x,y,z) = (1 - ax^2 + by, x, cz + dx)$ have some fix point ?
Well

$$\begin{cases} 1 - ax^2 + by = x \\ x = y \\ cz + dx = z \end{cases} \Rightarrow \begin{cases} 1 - ax^2 + by - x = 0 \\ x = y \\ (1-c)z = dx \end{cases} \Rightarrow \begin{cases} x = \frac{(b-1) \pm \sqrt{4a + (b-1)^2}}{2a} \\ y = x \\ (1-c)z = dx \end{cases}$$

Thus if we have $d \neq 0$ then $z \neq 0$ and $c \neq 1$ because $x \neq 0$ since (x, y, z) is a fix point.So in the case $d \neq 0$ the point (x, y, z) is a fix point of φ if and only if

$$\begin{cases} x = \frac{(b-1) \pm \sqrt{4a + (b-1)^2}}{2a} \\ y = x \\ z = \frac{dx}{1-c} \end{cases}$$

In the case $d = 0$ the fix point must be only

$$\begin{cases} x = \frac{(b-1) \pm \sqrt{4a + (b-1)^2}}{2a} \\ y = x \\ z = 0 \end{cases}$$

For now suppose $d \neq 0$ and let (x_0, y_0, z_0) be the fixed point of φ. Then

$$J\varphi(x,y,z) = \begin{bmatrix} (1-b) \pm \sqrt{4a + (1-b)^2} & b & 0 \\ 1 & 0 & 0 \\ d & 0 & c \end{bmatrix}$$

which gives us the characteristic polynomial

$$P_+(\lambda) = \det \begin{bmatrix} (1-b) + \sqrt{4a + (1-b)^2} - \lambda & b & 0 \\ 1 & -\lambda & 0 \\ d & 0 & c - \lambda \end{bmatrix}$$

$\Rightarrow P_+(\lambda) = -\lambda^3 + \big[\sqrt{4a + (1-b)^2} + c + 1 - b\big]\lambda^2 + \big[b(1+c) - c\big[1 + \sqrt{4a + (1-b)^2}\big]\big]\lambda - bc.$
For the other case we obtain

$$P_-(\lambda) = \det \begin{bmatrix} (1-b) - \sqrt{4a + (1-b)^2} - \lambda & b & 0 \\ 1 & -\lambda & 0 \\ d & 0 & c - \lambda \end{bmatrix}$$

$\Rightarrow P_-(\lambda) = -\lambda^3 + \big[-\sqrt{4a + (1-b)^2} + c + 1 - b\big]\lambda^2 + \big[b(1+c) - c(1 - \sqrt{4a + (1-b)^2})\big]\lambda - bc.$
Now we have the following

Lemma 4.2.1 *There exists* $0 < \delta < \frac{1}{4}$ *such that if the family* $\varphi(x, y, z)$ *satisfies* $(\mathcal{P}.\mathcal{C})$ *then* $\varphi(x, y, z) = (1 - ax^2 + by, x, cz + dx)$ *has a fixed point* $p = (x_0, y_0, z_0)$ *of index* 2.

Proof: We already saw that if $d \neq 0$ then the points of coordinates

$$x = y = \frac{(b-1) \pm \sqrt{4a + (b-1)^2}}{2a}, z = \frac{d}{1-c}x$$

are fixed points of φ.
Consider the fix point of coordinates

$$x = y = \frac{(b-1) - \sqrt{4a + (b-1)^2}}{2a}, z = \frac{d}{1-c}x$$

As we know the characteristic polynomial of the jacobian matrix of φ at this fixed point is

$$P_{-}(\lambda) = -\lambda^3 + [(1 - b + c) - \sqrt{4a + (1-b)^2}]\lambda^2 + [b(1 + c) - c(1 - \sqrt{4a + (1-b)^2})]\lambda - bc$$

- Claim 1: $P_{-}(\lambda)$ has at least one real root in the interval $(-1, 1)$
  In fact, we have that:

$$P_{-}(1) = -1 + (1 - b + c) - \sqrt{4a + (1-b)^2} + b + bc - c + c\sqrt{4a + (1-b)^2} - bc =$$

$$= (c - 1)\sqrt{4a + (1-b)^2}$$

  Since $c > 1$ it follows that $P_{-}(1) > 0$. On the other side we have that:

$$P_{-}(-1) = 1 + 1 - b + c - \sqrt{4a + (1-b)^2} - b - bc + c - c\sqrt{4a + (1-b)^2} - bc =$$

$$= 2(1 - bc - b + c) - (c + 1)\sqrt{4a + (1-b)^2} =$$

$$= 2(1 - b)(1 + c) - (c + 1)\sqrt{4a + (1-b)^2}$$

  Therefore

$$P_{-}(-1) < 0 \Leftrightarrow$$

$$2(1 - b) < \sqrt{4a + (1-b)^2} \Leftrightarrow$$

$$4(1 - 2b + b^2) < 4a + 1 - 2b + b^2 \Leftrightarrow$$

$$\frac{3(1-b)^2}{4} < a$$

  But by $(\mathcal{P}.\mathcal{C})$ we have $a > \frac{15(1+|b|)^2}{4} > \frac{3(1-b)^2}{4}$ which implies $P_{-}(-1) < 0$.
  Hence by Intermediate Value Theorem for Continuous Functions follows that the claim is true.

- Claim 2: $P_{-}(\lambda)$ has at least one real root in the interval $(1, +\infty)$.
  In fact, since $P_{-}(1) > 0$ and $\lim_{\lambda \to +\infty} P(\lambda) = -\infty$ this claim is true by Intermediate Value Theorem.

- Claim 3: $P_{-}(\lambda)$ has at least one real root in the interval $(-\infty, -1)$.
  In fact, since $P_{-}(-1) < 0$ and $\lim_{\lambda \to -\infty} P(\lambda) = +\infty$ this claim is also true by Intermediate Value Theorem.

Hence since we have a polynomial of degree 3 then by the claims above the fix point of coordinates

$$x = y = \frac{(b-1) - \sqrt{4a + (b-1)^2}}{2a}, z = \frac{d}{1-c}x$$

must be hyperbolic and its index is equal 2. ∎

Now let us make some conventions.

1. $r_1 := \frac{3(1+|b|)+2\sqrt{4a+(1+|b|)^2}}{4a}$;
2. $r_2 := \frac{|d||(b-1)-\sqrt{4a+(b-1)^2}|}{2a(c-1)}$
3. $\mathbb{D} = [-r_1, r_1] \times [-r_1, r_1] \times [z_p - r_2, z_p + r_2]$, where (x_0, y_0, z_0) is the fixed point on of the lemma 4.2.1. As we can see $\mathbb{D}$ depends on the parameters (a, b, c, d);
4. $X^+ := \mathbb{D} \cap (\{r_1\} \times \mathbb{R}^2)$ and $X^- := \mathbb{D} \cap (\{-r_1\} \times \mathbb{R}^2)$;
5. $Y^+ := \mathbb{D} \cap (\mathbb{R} \times \{r_1\} \times \mathbb{R})$ and $Y^- := \mathbb{D} \cap (\mathbb{R} \times \{-r_1\} \times \mathbb{R})$;
6. $Z^+ := \mathbb{D} \cap (\mathbb{R}^2 \times \{z_0 + r_2\})$ and $Z^- := \mathbb{D} \cap (\mathbb{R}^2 \times \{z_0 - r_2\})$.

Lemma 4.2.2 *The followings holds(see figure 4.5):*
(I) $\frac{|b|+1+\sqrt{4a+(|b|+1)^2}}{2a} < r_1 < \frac{4}{5}$;

(II) $d < 0 \Rightarrow Z^- \subset \mathbb{R}^2 \times \{-2r_2\}$ *and* $Z^+ \subset \mathbb{R}^2 \times \{0\}$;

(III) $d > 0 \Rightarrow Z^- \subset (\mathbb{R}^2 \times \{0\}) \cap \mathbb{D}$ *and* $Z^+ \subset (\mathbb{R}^2 \times \{2r_2\}) \cap \mathbb{D}$.

Proof: (I) The inequality $\frac{|b|+1+\sqrt{4a+(|b|+1)^2}}{2a} < r_1$ is immediate as a consequence of the definition of r_1. So let us prove that $r_1 < \frac{4}{5}$.

$$\left\{ \begin{array}{l} 0 < |b| < \delta < \frac{1}{4} \\ a > \frac{15(1+|b|)^2}{4} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} 1 + |b| < \frac{5}{4} \\ \frac{1}{4a} < \frac{1}{15(1+|b|)^2} < \frac{1}{15} \end{array} \right.$$

So $\frac{1}{16a^2} < \frac{1}{225}$. Thus :

$$r_1 = \frac{3(1+|b|)}{4a} + 2\sqrt{\frac{(1+|b|)^2}{16a^2} + \frac{1}{4a}} < 3\frac{1}{15}\frac{5}{4} + 2\sqrt{\frac{1}{225}(\frac{5}{4})^2 + \frac{1}{15}} = \frac{1}{4} + 2\sqrt{\frac{53}{720}} < \frac{4}{5}$$

. (II) Suppose we have $(x, y, z) \in Z^-$. Then $z = z_0 - r_2$ where
$z_0 = \frac{d\left[(b-1)-\sqrt{4a+(b-1)^2}\right]}{2a(1-c)}$ and $r_2 = \frac{-d\left|b-1-\sqrt{4a+(b-1)^2}\right|}{2a(c-1)}$. It follows that:

$$z = \frac{d}{2a(1-c)}\left[(b-1) - \sqrt{4a + (b-1)^2} - \left|(b-1) - \sqrt{4a + (b-1)^2}\right|\right] =$$

$$\frac{d}{2a(1-c)}\left[2\left|(b-1) - \sqrt{4a + (b-1)^2}\right|\right] = -2r_2 \Rightarrow$$

$$\Rightarrow (x, y, z) \in (\mathbb{R}^2 \times \{-2r_2\}) \cap \mathbb{D}.$$

Analogously one can show that $z^+ \subset (\mathbb{R}^2 \times \{0\}) \cap \mathbb{D}$ and the same for the case $d > 0$.

∎

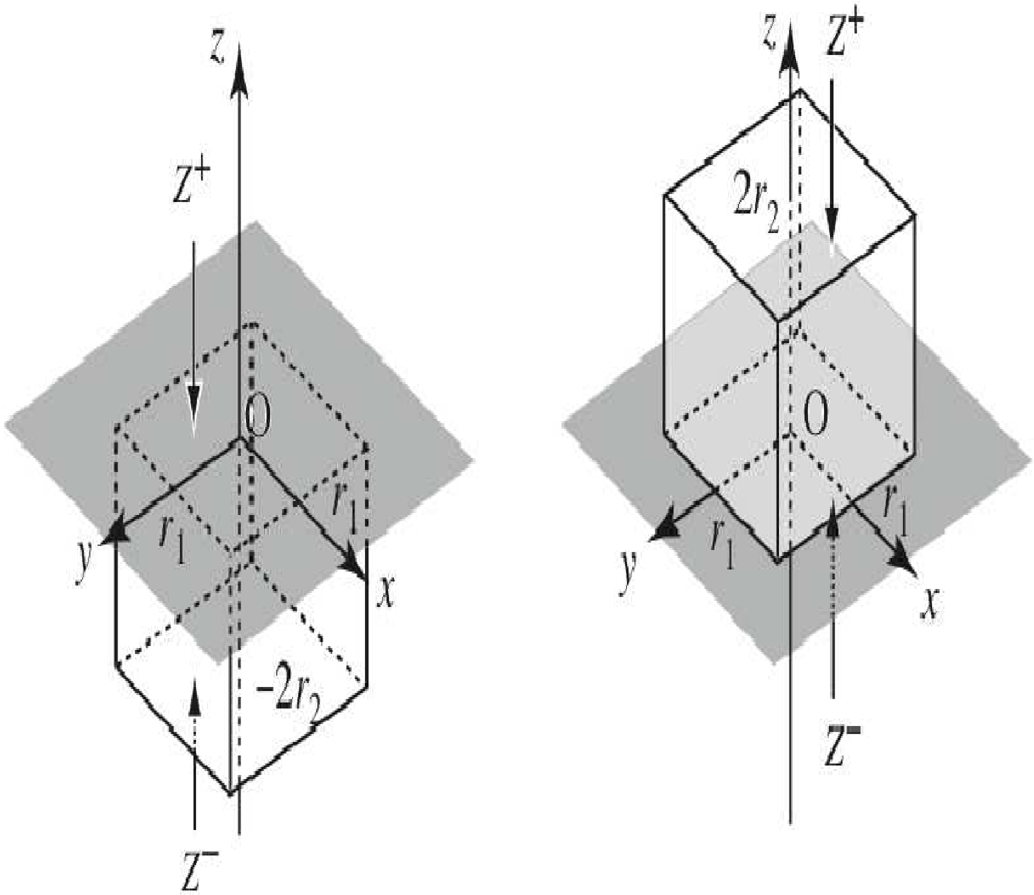

Figure 4.5: Position of $\mathbb{D}$ with respect to the xy-plane in the cases $d < 0$ and $d > 0$. (Figure from [1])

Lemma 4.2.3 *Suppose that the family $\varphi(x, y, z)$ satisfies the $\mathcal{P}.\mathcal{C}$ with $d \neq 0$*

1. *If $d < 0$ then the set $\varphi(\mathbb{D}) \cap \mathbb{D}$ possesses two connected components $\mathbb{A}$ and $\mathbb{B}$ on $\mathbb{D}$ such that:*
   - $\mathbb{A} \cap \left[Y^+ \cup \varphi(X^+ \cup Z^+)\right] = \emptyset$;
   - $\mathbb{A} \cap \left[Y^- \cup \varphi(X^- \cup Z^-)\right] = \emptyset$;
   - $\mathbb{B} \cap \left[(Y^+ \cup Z^+) \cup \varphi(X^+ \cup Z^-)\right] = \emptyset$;
   - $\mathbb{B} \cap \left[(Y^- \cup Z^+) \cup \varphi(X^- \cup Z^-)\right] = \emptyset$
2. *If $d > 0$ then the set $\varphi(\mathbb{D}) \cap \mathbb{D}$ possesses two connected components $\mathbb{A}$ and $\mathbb{B}$ on $\mathbb{D}$ such that:*
   - $\mathbb{A} \cap \left[Y^+ \cup \varphi(X^+ \cup Z^+)\right] = \emptyset$;
   - $\mathbb{A} \cap \left[Y^- \cup \varphi(X^- \cup Z^-)\right] = \emptyset$;
   - $\mathbb{B} \cap \left[(Y^+ \cup Z^-) \cup \varphi(X^+ \cup Z^-)\right] = \emptyset$;
   - $\mathbb{B} \cap \left[(Y^- \cup Z^-) \cup \varphi(X^- \cup Z^-)\right] = \emptyset$.

Proof: If $d < 0$ we fixed some special points in the edge of $\mathbb{D}$.

- $P_1 := (0, -r_1, 0)$ $\quad P_2 := (0, -r_1, -2r_2)$;
- $P_1^{\pm} := (\pm r_1, -r_1, 0)$ $\quad P_2^{\pm} := (\pm r_1, -r_1, -2r_2)$;
- $Q_1 := (0, r_1, 0)$ $\quad Q_2 := (0, r_1, -2r_2)$;
- $Q_1^{\pm} := (\pm r_1, r_1, 0)$ $\quad Q_2^{\pm} := (\pm r_1, r_1, -2r_2)$;

In the figure 4.6 we can see that points in $\mathbb{D}$.
Now consider $\Pi_{xy} : \mathbb{R}^3 \to \mathbb{R}^2$ the projection in the xy-plane. Then $\Pi_{xy} \circ \varphi(x, y, z) = (1 - ax^2 + by, x,)$ that is , the projection on the xy-plane gives us the Henon-Family.Let us analyze the points :

- $\Pi_{xy} \circ \varphi(P_1) = (1 - br_1, 0)$;
- $\Pi_{xy} \circ \varphi(P_1^-) = (1 - ar_1^2 - br_1, -r_1) = \Pi_{xy} \circ \varphi(P_2^-)$;
- $\Pi_{xy} \circ \varphi(P_1^+) = (1 - ar_1^2 - br_1, r_1) = \Pi_{xy} \circ \varphi(P_2^+)$;
- $\Pi_{xy} \circ \varphi(Q_1^-) = (1 - ar_1^2 + br_1, r_1) = \Pi_{xy} \circ \varphi(Q_2^-)$;
- $\Pi_{xy} \circ \varphi(Q_1^+) = (1 - ar_1^2 + br_1, r_1) = \Pi_{xy} \circ \varphi(Q_2^+)$;
- $\Pi_{xy} \circ \varphi(Q_1) = (1 + br_1, 0)$;

Let us now From the parameter condition and by definition of r_1 we have that $1 \pm br_1 \geq 1 - |b|r_1 > \frac{4}{5} > r_1$. This shows that the points $\Pi_{xy} \circ \varphi(P_1)$ and $\Pi_{xy} \circ \varphi(Q_1)$ are points of the semi-plane $\{(x, y) \in \mathbb{R}^2 / x > r_1\}$.

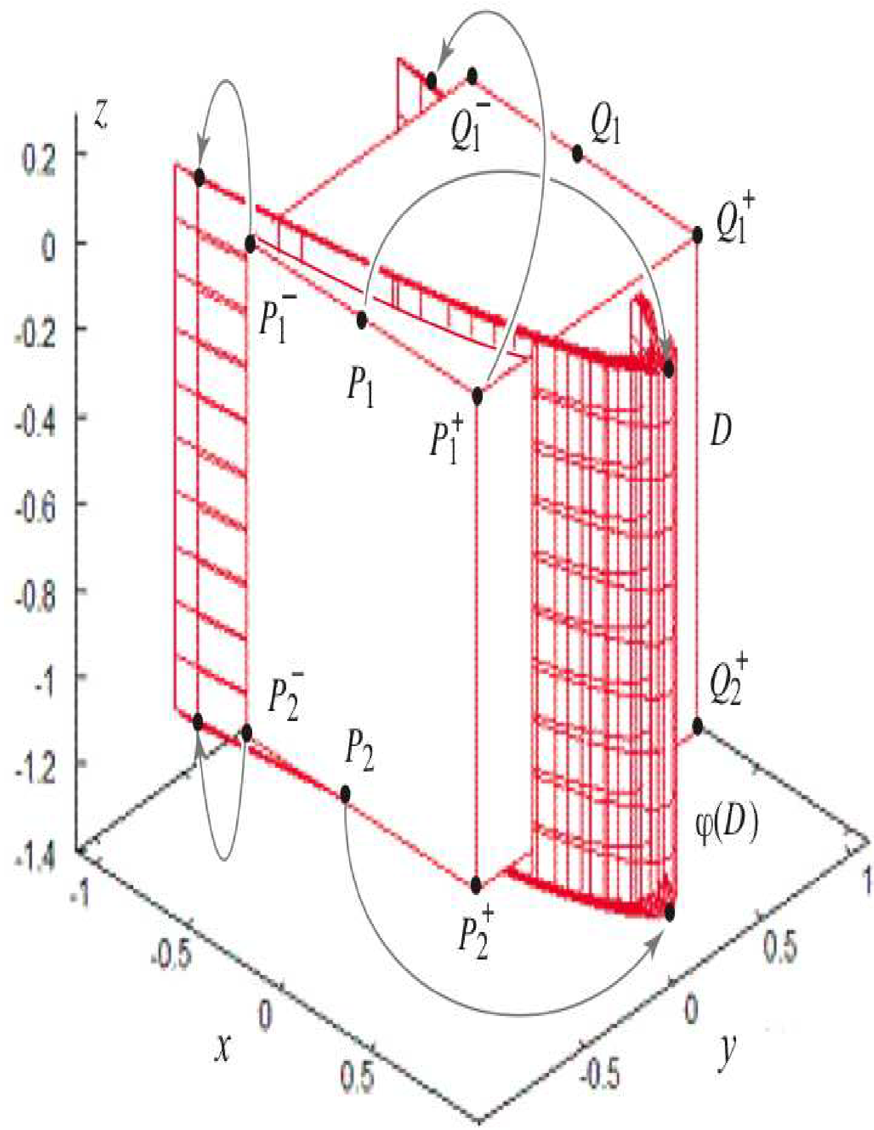

Figure 4.6: $\mathbb{D}$ and $\varphi(\mathbb{D})$ when $b, d < 0$.(Figure from [1])

- Claim 1: $1 - ar_1^2 + |b|r_1 < -r_1$.
  In fact ,we have that $r_1 = \frac{3(1+|b|)^2+2\sqrt{4a+(1+|b|)^2}}{4a}$. Thus $1 - ar_1^2 + |b|r_1 < -r_1 \Leftrightarrow 1 - ar_1^2 + (1+|b|)r_1 < 0 \Leftrightarrow$
  $\Leftrightarrow 1 - \left[\frac{9(1+|b|)^2+12(1+|b|)\sqrt{4a+(1+|b|)^2}+16a}{16a}\right] + \frac{3(1+|b|)^2+2(1+|b|)\sqrt{4a+(1+|b|)^2}}{4a} < 0 \Leftrightarrow$
  $\Leftrightarrow 1 - \left[\frac{9(1+|b|)^2+12(1+|b|)\sqrt{4a+(1+|b|)^2}}{16a}\right] - \frac{16a}{16a} + \frac{3(1+|b|)^2+2(1+|b|)\sqrt{4a+(1+|b|)^2}}{4a} < 0 \Leftrightarrow$
  $\Leftrightarrow \frac{3(1+|b|)^2-4(1+|b|)\sqrt{4a+(1+|b|)^2}}{16a} < 0 \Leftrightarrow$
  $\leftrightarrow 3(1+|b|) < 4\sqrt{4a+(1+|b|)^2}$. But this last inequality is true by the parameters conditions.
  As $1 - ar_1^2 \pm br_1 \leq 1 - ar_1^2 + |b|r_1$. It follows from the previous claim that $1 - ar_1^2 \pm br_1 < -r_1$.
  This shows that all of the points $\Pi_{xy} \circ \varphi(P_1^{\pm}), \Pi_{xy} \circ \varphi(P_2^{\pm}), \Pi_{xy} \circ \varphi(Q_1^{\pm})$ and $\Pi_{xy} \circ \varphi(Q_2^{\pm})$ lies in the semi-plane $\{(x,y) \in \mathbb{R}^2/x < -r_1\}$.

- Claim 2: $\Pi_{xy} \circ \varphi(Y^+)$ is a quadratic curve between the points $\Pi_{xy} \circ \varphi(Q_1^-) = \Pi_{xy} \circ \varphi(Q_2^-)$, $\Pi_{xy} \circ \varphi(Q_1^+) = \Pi_{xy} \circ \varphi(Q_2^+)$ and whose critical point is $\Pi_{xy} \circ \varphi(Q_1) = \Pi_{xy} \circ \varphi(Q_2)$.
  Of course ,as $Y^+ = \{(x,y,z) \in \mathbb{D}/y = r_1\}$ then given $(x,y,z) \in Y^+ \Rightarrow$
  $\Rightarrow \varphi(x,y,z) = (1 - ax^2 + br_1, x, cz + dx) \Rightarrow \Pi \circ \varphi(x,y,z) = (1 - ax^2 + br_1, x)$. But $(x,y,z) \in \mathbb{D} \Rightarrow |x| \leq r_1$.
  On this way we have:

  - when $x = -r_1 \Rightarrow \Pi \circ \varphi(x,y,z) = (1 - ar_1^2 + br_1, -r_1)$;
  - when $x = 0 \Rightarrow \Pi \circ \varphi(x,y,z) = (1 + br_1, 0)$;
  - when $x = r_1 \Rightarrow \Pi \circ \varphi(x,y,z) = (1 - ar_1^2 + br_1, r_1)$;

  which means that the quadratic curve $(1 - ax^2 + br_1, x)$ pass by the points
  $\Pi_{xy} \circ \varphi(Q_1^-) = \Pi_{xy} \circ \varphi(Q_2^-) = (1 - ar_1^2 + br_1, -r_1)$,
  $\Pi_{xy} \circ \varphi(Q_1^+) = \Pi_{xy} \circ \varphi(Q_2^+) = (1 - ar_1^2 + br_1, r_1)$ and
  $\Pi_{xy} \circ \varphi(Q_1) = \Pi_{xy} \circ \varphi(Q_2)$.
  See the figure 2.7

- Claim 3:$\Pi_{xy} \circ \varphi(Y^-)$ is a quadratic curve between the points $\Pi_{xy} \circ \varphi(P_1^-) = \Pi_{xy} \circ \varphi(P_2^-)$, $\Pi_{xy} \circ \varphi(P_1^+) = \Pi_{xy} \circ \varphi(P_2^+)$ and $\Pi_{xy} \circ \varphi(P_1) = \Pi_{xy} \circ \varphi(P_2.)$
  Of course, as $Y^+ = \{(x,y,z) \in \mathbb{D}/y = -r_1\}$ then given $(x,y,z) \in Y^- \Rightarrow$
  $\Rightarrow \varphi(x,y,z) = (1 - ax^2 + br_1, x, cz + dx) \Rightarrow \Pi \circ \varphi(x,y,z) = (1 - ax^2 - br_1, x)$. This sows that $\Pi_{xy} \circ \varphi(Y^-)$ is a quadratic curve.In another side $(x,y,z) \in Y^- \Rightarrow |x| \leq r_1$. Thus

  - when $x = -r_1 \Rightarrow \Pi \circ \varphi(x,y,z) = (1 - ar_1^2 - br_1, -r_1)$;
  - when $x = 0 \Rightarrow \Pi \circ \varphi(x,y,z) = (1 - br_1, 0)$ and
  - when $x = r_1 \Rightarrow \Pi \circ \varphi(x,y,z) = (1 - ar_1^2 - br_1, r_1)$.

  But
  $\Pi_{xy} \circ \varphi(P_1^-) = \Pi_{xy} \circ \varphi(P_2^-) = (1 - ar_1^2 - br_1, -r_1)$,
  $\Pi_{xy} \circ \varphi(P_1^+) = \Pi_{xy} \circ \varphi(P_2^+) = (1 - ar_1^2 - br_1, r_1)$,
  $\Pi_{xy} \circ \varphi(P_1) = \Pi_{xy} \circ \varphi(P_2) = (1 - br_1, 0)$.
  See the figure 2.7

Now fix a number $\delta, -r_1 < \delta < r_1$ and denote $\Omega := \{(x,y,z) \in \mathbb{D}/y = \delta\}$.

- Claim 4: $\Pi_{xy} \circ \varphi(\Omega)$ is a quadratic curve between the curves $\Pi_{xy} \circ \varphi(Y^+)$ and $\Pi_{xy} \circ \varphi(Y^-)$.
In fact, if $(x, y, z) \in \Omega$ then $\varphi(x, y, z) = (1 - ax^2 + br_1, x, cz + dx) \Rightarrow \Pi \circ \varphi(x, y, z) = (1 - ax^2 + br\delta, x)$.
This sows that $\Pi_{xy} \circ \varphi(\Omega)$ is a quadratic curve.
Now as
$$\begin{cases} \Pi_{xy} \circ \varphi(Y^+) = (1 - ax^2 + br_1, x), |x| \leq r_1 \\ \Pi_{xy} \circ \varphi(Y^-) = (1 - ax^2 - br_1, x), |x| \leq r_1. \end{cases}$$
and we have $-r_1 < \delta < r_1$ then

  - If $b > 0 \Rightarrow -br_1 < b\delta < br_1 \Leftrightarrow 1 - ax^2 - br_1 < 1 - ax^2 + b\delta < 1 - ax^2 + br_1$.
  - If $b < 0$ and $-r_1 < 0 < \delta < r_1$ then $br_1 < b\delta < 0 < -br_1 \Leftrightarrow$
  $\Leftrightarrow 1 - ax^2 + br_1 < 1 - ax^2 + b\delta < 1 - ax^2 - br_1$.

The following figure shows $\Pi_{xy} \circ \varphi(\mathbb{D})$ in the case $b < 0$ and $d < 0$.
Since the connectedness is invariant by continuous functions we obtain that $\mathbb{D} \cap \varphi(\mathbb{D})$ must possesses two connected components in $\mathbb{D}$.
Now we are going to determine the projection of $\mathbb{D}$ and $\varphi(\mathbb{D})$ in the xz-plane,that is ,the sets $\Pi_{xz}(\mathbb{D})$ and $\Pi_{xz} \circ \varphi(\mathbb{D})$. As we have $d < 0$ it is immediate verify that $\Pi_{xz}(\mathbb{D})$ is the rectangle $[-r_1, r_1] \times [z_p - r_2, z_p + r_2] = [-r_1, r_1] \times [-2r_2, 0]$. Also is immediate verify that

  - $\Pi_{xz} \circ \varphi(P_1) = (1 - br_1, 0)$, $\Pi_{xz} \circ \varphi(P_2) = (1 - br_1, -2cr_2)$;
  - $\Pi_{xz} \circ \varphi(Q_1) = (1 + br_1, 0)$, $\Pi_{xz} \circ \varphi(Q_2) = (1 + br_1, -2cr_2)$;
  - $\Pi_{xz} \circ \varphi(P_1^-) = (1 - ar_1^2 - br_1, -dr_1)$;
  - $\Pi_{xz} \circ \varphi(P_2^-) = (1 - ar_1^2 - br_1, -2cr_1 - dr_1)$;
  - $\Pi_{xz} \circ \varphi(P_1^+) = (1 - ar_1^2 - br_1, dr_1)$;
  - $\Pi_{xz} \circ \varphi(P_2^+) = (1 - ar_1^2 - br_1, -2cr_1 + dr_1)$;
  - $\Pi_{xz} \circ \varphi(Q_1^-) = (1 - ar_1^2 + br_1, -dr_1)$;
  - $\Pi_{xz} \circ \varphi(Q_2^-) = (1 - ar_1^2 + br_1, -2cr_1 - dr_1)$;
  - $\Pi_{xz} \circ \varphi(Q_1^+) = (1 - ar_1^2 + br_1, dr_1)$;
  - $\Pi_{xz} \circ \varphi(Q_2^+) = (1 - ar_1^2 + br_1, -2cr_1 + dr_1)$;

- Claim 5: The segment of line between $\Pi_{xz} \circ \varphi(P_1)$ and $\Pi_{xz} \circ \varphi(P_1^-)$ has no intersection with the segment $\Pi_{xz}(Z^+)$.
Of course,as $Z^+ = \{(x, y, z) \in \mathbb{D}/z = z_p + r_2\}$ then $\Pi_{xz}(Z^+) = \{(x, z) \in \mathbb{R}^2/|x| \leq r_1, z = z_p + r_2\} = \{(x, 0) \in \mathbb{R}^2/|x| \leq r_1\}$ since by Claim 1 which precedes the proposition 1 we have $Z^+ \subset \mathbb{R}^2 \times \{0\}$.
The segment of line between $\Pi_{xz} \circ \varphi(P_1)$ and $\Pi_{xz} \circ \varphi(P_1^-)$ is given by :
$\{t\Pi_{xz} \circ \varphi(P_1) + (1 - t)\Pi_{xz} \circ \varphi(P_1^-)\} =$
$= \{((1 - br_1)t, 0) + ((1 - t)(1 - ar_1^2 - br_1), -(1 - t)dr_1)/t \in [0.1]\} =$
$\{([1 - br_1]t + [1 - t][1 - ar_1^2 - br_1], [t - 1]dr_1)/t \in [0, 1]\}$.
Thus if the segment intersects $\Pi_{xz}(Z^+)$ there would be a parameter $t_0 \in [0, 1]$ such that $(t_0 - 1)dr_1 = 0$.
As $d \neq 0$ and $r_1 > 0 \Rightarrow t_0 - 1 = 0 \Rightarrow t_0 = 1$. But this would means that the point $\Pi_{xz} \circ \varphi(P_1) = (1 - br_1, 0)$ should be in the $\Pi_{xz}(Z^+) = \{(x, 0)/|x| \leq r_1\}$. However we have $1 \pm br_1 \geq 1 - |b|r_1 > \frac{4}{5} > r_1$ what shows that the point $\Pi_{xz} \circ \varphi(P_1)$ is not on the segment $\Pi_{xz}(Z^+)$.

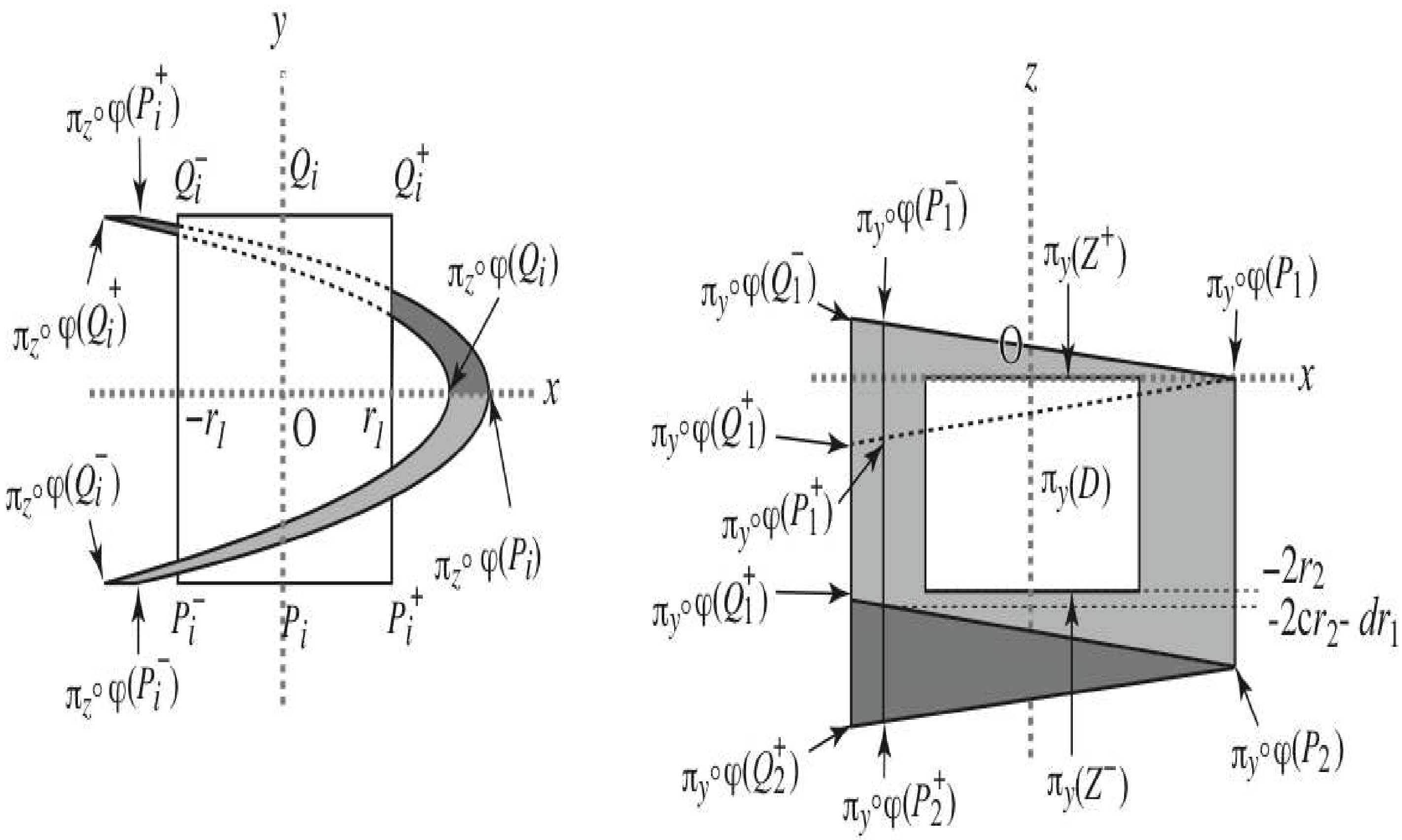

Figure 4.7: projections of $\mathbb{D}$ and $\varphi(\mathbb{D})$ in xy-plane and xz-plane when $b, d < 0$. (Figure from [1])

- Claim 6 : $dr_1 > -2r_2$.
  In fact, as $dr_1 > -2r_2 \Leftrightarrow -dr_1 < 2r_2 \Leftrightarrow |d|r_1 < 2r_2 = \frac{2|d||x_p|}{c-1} \Leftrightarrow r_1 < \frac{2|x_p|}{c-1} \Leftrightarrow$
  $\Leftrightarrow 3(1+|b|) + 2\sqrt{4a+(1+|b|)^2} < \frac{2[(1-b)+\sqrt{4a+(1-b)^2}]}{2a(c-1)} \Leftrightarrow$
  $\Leftrightarrow 3(1+|b|) + 2\sqrt{4a+(1+|b|)^2} < \frac{4[(1-b)+\sqrt{4a+(1-b)2}]}{c-1}$.
  Thus it is enough show the following inequality $\Leftrightarrow 3(1+|b|) + 2\sqrt{4a+(1+|b|)^2} < \frac{4[(1-b)+\sqrt{4a+(1-b)2}]}{c-1} \Leftrightarrow$
  $\Leftrightarrow (c-1)[3(1+|b|) + 2\sqrt{4a+(1+|b|)^2}] < 4[(1-|b|) + \sqrt{4a+(1-|b|)^2}]$.
  By parameters conditions $c - 1 > \frac{1}{9}$ and thus the last inequality is true since the inequality
  $\frac{3(1+|b|)+2\sqrt{4a+(1+|b|)^2}}{9} < 4[(1-|b|) + \sqrt{4a+(1-|b|)^2}] \Leftrightarrow$
  $3(1+|b|) + 2\sqrt{4a+(1+|b|)^2} < 36(1-|b|) + 36\sqrt{4a+(1-|b|)^2}$ $(**)$ is true.
  Finally $(**)$ will be true since the followings inequalities hold.
  $$\begin{cases} (\alpha) & 36\sqrt{4a+(1-|b|)^2} > 2\sqrt{4a+(1+|b|)^2} \\ (\beta) & 36(1-|b|) > 3(1+|b|). \end{cases}$$
  But (α) is true iff $18^2[(1-|b|)^2 + 4a] > (1+|b|)^2 + 4a \Leftrightarrow 4a(18^2-1) > (1+|b|)^2 - 18^2(1-|b|)^2 \Leftrightarrow$
  $\Leftrightarrow 4a > \frac{(1+|b|)^2-18^2(1-|b|)^2}{18^2-1} \Leftrightarrow 4a > \frac{1+2|b|+|b|^2-324(1-2|b|+|b|^2)}{323} \Leftrightarrow 4a > \frac{-323|b|^2+650|b|-323}{323} \Leftrightarrow$
  $\Leftrightarrow 4a > -|b|^2 + \frac{650|b|}{323} - 1$ which is true by parameter condition.
  On the other side ,the inequality on (β) is true iff $33 \geq 39|b|$ and this last is true again by parameters conditions.

- Claim 7: The segment of line between $\Pi_{xz}\circ\varphi(P_1)$ and $\Pi_{xz}\circ\varphi(P_1^+)$ not intersects the segment $\Pi_{xz}(Z^+)$.But it intersects the segments $\Pi_{xz}(X^+)$ and $\Pi_{xz}(X^-)$ passing trough the interior of $\Pi_{xz}(\mathbb{D})$.
  In fact ,as we have $d < 0$ then by claim 1 before the proposition 1, we obtain
  $\Pi_{xz}(\mathbb{D}) = \{(x,z) \in \mathbb{R}^2/|x| \leq r_1 \ \ and \ \ -2r_2 \leq z \leq 0\}$
  Since $\Pi_{xz}\circ\varphi(P_1) = (1-br_1, 0)$ and $\Pi_{xz}\circ\varphi(P_1^+) = (1-ar_1^2-br_1, dr_1)$ then from $1\pm br_1 \geq 1-|b|r_1 > \frac{4}{5} > r_1$ we see that $\Pi_{xz}\circ\varphi(P_1)$ is not on $Pi_{xz}(\mathbb{D})$. On the other hand from claim 1 above $1-ar_1^2\pm br_1 \leq 1-ar_1^2+|b|r_1 < r_1$ implies that $\Pi_{xz}\circ\varphi(P_1^+)$ is not on $\Pi_{xz}(\mathbb{D})$. So from that and from claim 6 above the result follows.See figure 4.7

- Claim 8 : $-2cr_2 - dr_1 < -2r_2$.
  In fact, this is immediate from the claim 6 above .

- Claim 9 : The segment of line between $\Pi_{xz}\circ\varphi(P_2)$ and $\Pi_{xz}\circ\varphi(P_2^-)$ has no intersection with the segment $\Pi_{xz}(Z^-)$.
  Of course, we have $\Pi_{xz}(Z^-) = \{(x, -2r_2)/|x| \leq r_1\}$, $\Pi_{xz}\circ\varphi(P_2) = (1-br_1, -2cr_2)$ and
  $\Pi_{xz}\circ\varphi(P_2^-) = (1-ar_1^2-br-1, -2cr_2-dr_1)$. Thus the segment of line between $\Pi_{xz}\circ\varphi(P_2)$ and $\Pi_{xz}\circ\varphi(P_2^-)$ is the set
  $\hat{S} := \{(t[1-br_1] + [1-t][1-ar_1^2-br_1], -2tcr_2 + [1-t][-2cr_2-dr_1])/t \in [0,1]\}$ so one point of $\hat{S}$ also belongs to $\Pi_{xz}(Z^-)$ iff there exists $t_0, 0 \leq t_0 \leq 1$ such that
  $-2t_0cr_2 + [1-t_0][-2cr_2-dr_1] = -2r_2 \Leftrightarrow t_0 = \frac{2cr_2+dr_1-2r_2}{dr_1}$. But $d < 0$ and $r_1 > 0$ which means that $t_0 \in [0,1]$ implies in $2cr_2 + dr_1 \leq 2r_2$ what is a contradiction by claim 8 above.

- Claim 10 : The segment of line between $\Pi_{xz}\circ\varphi(P_2)$ and $\Pi_{xz}\circ\varphi(P_2^+)$ has no intersection with the segment $\Pi_{xz}(Z^-)$.

Of course, since $\Pi_{xz}(Z^-) = \{(x, -2r_2)/|x| \leq r_1\}$, $\Pi_{xz} \circ \varphi(P_2) = (1 - br_1, -2cr_2)$ and $\Pi_{xz} \circ \varphi(P_2^-) = (1 - ar_1^2 - br_1, -2cr_2 + dr_1)$. So the segment of line between $\Pi_{xz} \circ \varphi(P_2)$ and $\Pi_{xz} \circ \varphi(P_2^+)$ is the set $S = \{([1-t][1 - ar_1^2 - br_1] + t[1 - br - 1], [1-t][-2cr_1 + dr_1] + -2cr_2t)/ \quad 0 \leq t \leq 1\}$. Then one point of S belongs to $\Pi_{xz}(Z^-)$ iff there is $0 \leq t_0 \leq 1$ such that $-2cr_2t - 0 + [1 - t_0][-2cr_2 + dr_1] = -2r_2$. This implies in $t_0 = \frac{2r_2 - 2cr_2 + dr_1}{dr_1}$. But $d < 0$, $c > 1$ and $r_1 > 0 \Rightarrow t_0 = 1 + \frac{2r_2(1-c)}{dr_1} > 1$ which a contradiction .

Finally let us determine the sets $\mathbb{A}$ and $\mathbb{B}$.
Now fixing $d < 0$ we have two cases to treat which are $b < 0$ and $b > 0$.Here we are going to treat the case $b < 0$ the other one is similar.
We start by fixing one point x_0 , $-r_1 \leq x_0 \leq r_1$. Now fix a point $(\hat{x}, \hat{y}, \hat{z}) \in \mathbb{D} \cap \varphi(\mathbb{D})$ such that $\hat{x} = x_0$. Then there exist an unique point $(x, y, z) \in \mathbb{D}$ such that $(\hat{x}, \hat{y}, \hat{z}) = \varphi(x, y, z) = \big(1 - ax^2 + by, x, cz + dx\big)$. Thus

$$\begin{cases} \hat{x} = 1 - ax^2 + by = x_0 \\ \hat{y} = x \qquad\qquad \textit{where} \ \ |y| \leq r_1 \ \ \textit{and} \ \ |x| \leq r_1 \\ \hat{z} = cz + dx. \end{cases}$$

From $1 - ax^2 + by = x_0 \Rightarrow ax^2 = 1 + by - x_0$. But $b < 0$ and $-r_1 \leq y \leq r_1 \Rightarrow r_1 < 1 + br_1 \leq 1 - br_1$. Since $-r_1 \leq x_0 \leq r_1$ we obtain $1 + br_1 - r_1 \leq 1 + by - x - 0 \leq 1 - br_1 + r_1$. But $1 + br_1 > r_1 \Rightarrow$ $0 < 1 + br_1 - r_1 \leq 1 + by - x_0 \leq 1 + br_1 + r_1$. So we have $x = \pm\sqrt{\frac{1+by-x_0}{a}}$. Hence

$$\hat{y} = \pm\sqrt{\frac{1 + by - x_0}{a}}, \quad -r_1 \leq y \leq r_1$$

As $b < 0$ the function $y \mapsto \sqrt{\frac{1+by-x_0}{a}}$ is decreasing and $y \mapsto -\sqrt{\frac{1+by-x_0}{a}}$ is increasing.

- If $x_0 = -r_1$ then $(-r_1, \hat{y}, \hat{z}) \in \mathbb{D} \cap \varphi(\mathbb{D}) \Rightarrow \sqrt{\frac{1+br_1+r_1}{a}} \leq \hat{y} = \sqrt{\frac{1+by+r_1}{a}} \leq \sqrt{\frac{1-br_1+r_1}{a}}$ or $-\sqrt{\frac{1-br_1+r_1}{a}} \leq \hat{y} = -\sqrt{\frac{1+by+r_1}{a}} \leq -\sqrt{\frac{1+br_1+r_1}{a}}$ where $-r_1 \leq y \leq r_1$.
  From the Claim 1 above it follows that $1 - ar_1^2 - br_1 < r_1 \Leftrightarrow ar_1^2 > 1 - br_1 + r_1 \Leftrightarrow r_1 > \sqrt{\frac{1-br_1+r_1}{a}} \Leftrightarrow$ $-\sqrt{\frac{1-br_1+r_1}{a}} > -r_1$. Thus $\hat{y} > -r_1$ ever.
  Also by Claim 1 above we obtain $\sqrt{\frac{1-br_1+r_1}{a}} < r_1$, that is, $\hat{y} < r_1$ ever.
- If $x_0 = r_1$ then $(r_1, \hat{y}, \hat{z}) \in \mathbb{D} \cap \varphi(\mathbb{D}) \Rightarrow \sqrt{\frac{1+br_1-r_1}{a}} \leq \hat{y} = \sqrt{\frac{1+by-r_1}{a}} \leq \sqrt{\frac{1-br_1-r_1}{a}}$ or $-\sqrt{\frac{1-br_1-r_1}{a}} \leq \hat{y} = -\sqrt{\frac{1+by-r_1}{a}} \leq -\sqrt{\frac{1+br_1-r_1}{a}}$ where $-r_1 \leq y \leq r_1$.

Hence ,fixed $-r_1 \leq x_0 \leq r_1$ and given a point $(\hat{x}, \hat{y}, \hat{z}) \in \mathbb{D} \cap \varphi(\mathbb{D})$ such that $\hat{x} = x_0$ then $\hat{y} = \pm\sqrt{\frac{1+bt-x_0}{a}}$ where $-r_1 \leq t \leq r_1$.

- If $\hat{y} = -\sqrt{\frac{1+bt-x_0}{a}}$ then $-r_1 < -\sqrt{\frac{1-br_1+r_1}{a}} \leq \hat{y} \leq -\sqrt{\frac{1+br_1-r_1}{a}} < 0$.
- If $\hat{y} = \sqrt{\frac{1+bt-x_0}{a}}$ then $0 < \sqrt{\frac{1+br_1-r_1}{a}} \leq \hat{y} \leq \sqrt{\frac{1-br_1+r_1}{a}} < r_1$.

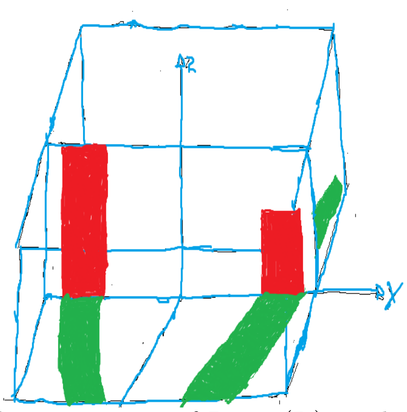

Figure 4.8: In red color we see the projection of $\mathbb{D}\cap\varphi(\mathbb{D})$ in the yz-plane and in green color we see the projection of $\mathbb{D}\cap\varphi(\mathbb{D})$ in the xy-plane with $d,b<0$

Now we are going to project $(\hat{x},\hat{y},\hat{z}) = \big(1-ax^2+by, x, cz+dx\big)$ on the yz-plane.
$\Pi_{yz}(\hat{x},\hat{y},\hat{z}) = (\hat{y},\hat{z}) = \big(x, cz+dx\big)$ where $-2r_2 \leq z \leq 0$ and $x = \pm\sqrt{\frac{1+bt-x_0}{a}}$, $-r_1 \leq t \leq r-1$, $-r_1 \leq x_0 \leq r-1$. So we have to consider $x<0$ and $x>0$.

- If $x<0$ then
  $-r_1 < -\sqrt{\frac{1-br_1+r_1}{a}} \leq x \leq -\sqrt{\frac{1+br_1-r_1}{a}} \Rightarrow 0 < -d\sqrt{\frac{1+br_1-r_1}{a}} \leq dx \leq -d\sqrt{\frac{1-br_1+r_1}{a}} < -dr_1.$
  So from that $-2cr_2 - d\sqrt{\frac{1+br_1-r_1}{a}} \leq cz+dx \leq -d\sqrt{\frac{1-br_1+r_1}{a}} < -dr_1.$
- If $x>0$ then
  $0 < \sqrt{\frac{1+br_1-r_1}{a}} \leq x \leq \sqrt{\frac{1-br_1+r_1}{a}} < r_1 \Rightarrow dr_1 < d\sqrt{\frac{1-br_1+r_1}{a}} \leq dx \leq d\sqrt{\frac{1+br_1-r_1}{a}} < 0.$
  So from that $-2cr_2 + dr_1 < -2cr_2 + d\sqrt{\frac{1-br_1+r_1}{a}} \leq cz+dx \leq d\sqrt{\frac{1+br_1-r_1}{a}} < 0.$

Thus we take $\mathbb{A} := \{(x,y,z)\in\mathbb{D}\cap\varphi(\mathbb{D})/y<0\}$ and $\mathbb{B} := \{(x,y,z)\in\mathbb{D}\cap\varphi(\mathbb{D})/y>0\}$ on the statement. From all facts proved above it is immediate verify that this choice of $\mathbb{A}$ and $\mathbb{B}$ satisfies the requirement in the statement. One can check similarly the case $d>0$.

$\blacksquare$

The next lemma will be usefull for the construction of the cone-fields.

Lemma 4.2.4 *Suppose that*

- *The family $\varphi(x,y,z)$ satisfies the parameters conditions $\mathcal{P}.\mathcal{C}$;*
- *$\mathbb{A}$ and $\mathbb{B}$ are the connected components of $\varphi(\mathbb{D})\cap\mathbb{D}$.*

Then

- $\frac{1+|b|}{a} < |x|$ *for* $(x,y,z)\in\varphi^{-1}(\mathbb{A}\cup\mathbb{B})$*;*
- $\frac{1+|b|}{a} < |y|$ *for* $(x,y,z)\in\mathbb{A}\cup\mathbb{B}$.

Proof: Suppose $(x,y,z)\in\varphi^{-1}(\mathbb{A}\cup\mathbb{B})$. It follows that $|1-ax^2+by|\leq r_1$. Thus $r_1\geq|1-ax^2+by|\geq$ $\geq|1+by|-ax^2\Rightarrow ax^2\geq|1+by|-r_1\geq 1-|b||y|-r_1\geq 1-|b|r_1-r_1=1-r_1(1+|b|)\Rightarrow$ $\Rightarrow|x|\geq\frac{1}{\sqrt{a}}\sqrt{1-r_1(1+|b|)}(*)$. Now we replace $r_1=\frac{3(1+|b|)+2\sqrt{4a+(1+|b|)^2}}{4a}$ in $(*)$ to obtain $|x|\geq\frac{1}{2a}\sqrt{4a-[1+|b|]\big[3(1+|b|)+2\sqrt{4a+(1+|b|)^2}\big]}$ $(**)$. Now we put $\rho:=\frac{15(1+|b|)^2}{4}$ and we consider the function $f:[\rho,+\infty)\to\mathbb{R}$ given by $f(a)=\sqrt{4a-[1+|b|]\big[3(1+|b|)+2\sqrt{4a+(1+|b|)^2}\big]}$.

Since $r_1<\frac{4}{5}$ and $|b|<\frac{1}{4}$ we have f is a smooth function well defined and $f'(a)=\frac{4}{2f(a)}\left[1-\frac{1+|b|}{\sqrt{4a+(1+|b|)^2}}\right]>0$.

So f is increasing and this implies in

$\frac{f(a)}{2a}>\frac{f(\rho)}{2a}=\frac{1}{2a}\sqrt{15(1+|b|)^2-(1+|b|)\left[3(1+|b|)+2\sqrt{15(1+|b|)^2+(1+|b|)^2}\right]}=\frac{1+|b|}{a}$. Thus $\frac{f(a)}{2a}>\frac{1+|b|}{a}$ for any $a>\rho=\frac{15(1+|b|)^2}{4}$. By inequality $(**)$ we obtain $|x|>\frac{1+|b|}{a}$.

The other item can be proved in a similar way. ■

Now let us see the construction of the cone fields.

Lemma 4.2.5 *Suppose that*

- *The family $\varphi(x,y,z)$ satisfies $\mathcal{P}.\mathcal{C}$;*
- *$\mathbb{A}$ and $\mathbb{B}$ are the connected components of $\varphi(\mathbb{D})\cap\mathbb{D}$;*
- *$C^s_b(q):=\{(\alpha,\beta,\gamma)\in T_q\mathbb{D}/\sqrt{|b|}|\beta|\geq\sqrt{\alpha^2+\gamma^2}\}$ for each $q\in\mathbb{A}\cup\mathbb{B}$.*

Then

- *For each $v\in C^s_b(q)\Rightarrow d\varphi^{-1}(q)(v)\in C^s_b(\varphi^{-1}(q)$;*
- *There exists a constant $\rho>1$ such that for every $q\in\mathbb{A}\cup\mathbb{B}$ and every $v\in C^s_b(q)$ one have $\|d\varphi^{-1}(q)(v)\|>\rho\|v\|$.*

Proof: We shall write $q=(x,y,z)\in\mathbb{A}\cup\mathbb{B}$ and fix $v=(\alpha,\beta,\gamma)\in C^s_b(q)$.Thus we have $\varphi^{-1}(x,y,z)=\left(y,\frac{ay^2+x-1}{b},\frac{z-dy}{c}\right)$ and

$$d\varphi^{-1}(q)=\begin{bmatrix}0 & 1 & 0\\ \frac{1}{b} & \frac{2ay}{b} & 0\\ 0 & 0 & \frac{-d}{c}\end{bmatrix}$$

Thus $d\varphi^{-1}(q)(v)=\left(\beta,\frac{\alpha+2ay\beta}{b},\frac{\gamma-d\beta}{c}\right)$. We shall denote $d\varphi^{-1}(q)(v)=v_{-1}=(\alpha_{-1},\beta_{-1},\gamma_{-1})$. So as we have $v\in C^s_b(q)\Rightarrow|b|>\sqrt{|b||\beta|}\geq\sqrt{\alpha^2+\gamma^2}\Rightarrow|\beta|>\sqrt{\alpha^2+\gamma^2}\geq Max\{|\alpha|,|\gamma|\}$. By $\mathcal{P}.\mathcal{C}$, $1+|d|<c<\frac{10}{9}$ and $|d|<\frac{1}{9}$. Thus $\alpha^2_{-1}+\gamma^2_{-1}=\beta^2+\frac{(\gamma-d\beta)^2}{c^2}\leq\beta^2+\frac{(|\gamma|+|d||\beta|)^2}{c^2}<\beta^2+\frac{(|\beta|+|d||\beta|)^2}{c^2}=\beta^2\left[1+\frac{(1+|d|)^2}{c^2}\right]<\beta^2\left[1+\left[1+\frac{1}{9}\right]^2\right]=\frac{181\beta^2}{81}$.

On the other side ,by previous lemma and $0<|b|<\frac{1}{4}$ we obtain

$\beta^2_{-1}=\left(\frac{\alpha+2ay\beta}{b}\right)^2\geq\frac{(2a|y||\beta|-|\alpha|)^2}{|b|^2}\geq\frac{(2a|y||\beta|-|\beta|)^2}{|b|^2}=\frac{\beta^2(2a|y|-1)^2}{|b|^2}\geq\frac{\beta^2(2a|y|-1)^2}{|b|}\geq\frac{4\beta^2[(1+|b|)-1]2}{|b|}>\frac{4\beta^2}{|b|}.\beta^2_{-1}|b|>4\beta^2>$

$\frac{181\beta^2}{81} > \alpha_{-1}^2 + \gamma_{-1}^2 \Rightarrow \sqrt{|b||\beta_1|} \geq \sqrt{\alpha_{-1}^2 + \gamma_{-1}^2}\}$, that is, $v_1 = d\varphi^{-1}(q)(v) = (\alpha_{-1}, \beta_{-1}, \gamma_{-1}) \in C_b^s(\varphi^{-1}(q)$. Moreover , $|\beta_1| > \frac{2|\beta|}{\sqrt{|b|}} > 4|\beta|$. So now we consider in $\mathbb{R}^3$ the maximum norm $\|.\|_M$. Then we have $\|d\varphi^{-1}(q)(v)\|_M = |\beta_{-1}| > 4|\beta| = 4\|v\|_M$. It follows that if $\|.\|$ is the euclidean Norm then $\|d\varphi^{-1}(q)(v)\| \geq \|d\varphi^{-1}(q)(v)\|_M > 4\|v\|_M \geq 4.\frac{1}{3}\|v\| \Rightarrow \|d\varphi^{-1}(q)(v)\| > \frac{4}{3}\|v\|$. So we can take constant $\rho = \frac{4}{3}$. ∎

Lemma 4.2.6 *Suppose that*

- *The family $\varphi(x, y, z)$ satisfies $\mathcal{P}.\mathcal{C}$;*
- *$\mathbb{A}$ and $\mathbb{B}$ are the connected components of $\varphi(\mathbb{D}) \cap \mathbb{D}$;*
- *$C^u(q) := \{(\alpha, \beta, \gamma) \in T_q\mathbb{D}/\sqrt{\alpha^2 + \gamma^2} \geq \sqrt{2}|\beta|\}$ for each $q \in \varphi^{-1}(\mathbb{A} \cup \mathbb{B})$.*

Then

- *For each $v \in C^u(q) \Rightarrow d\varphi(q)(v) \in C^u(\varphi(q)$;*
- *There exists a constant $\rho > 1$ such that for every $q \in \varphi^{-1}(\mathbb{A} \cup \mathbb{B})$ and every $v \in C^u(q)$ one have $\|d\varphi(q)(v)\| > \rho\|v\|$.*

Proof: For any $q = (x, y, z) \in \varphi^{-1}(\mathbb{A} \cup \mathbb{B})$ and $v = (\alpha, \beta, \gamma) \in C^u(q)$ we have

$$d\varphi^{-1}(q) = \begin{bmatrix} -2ax & b & 0 \\ 1 & 0 & 0 \\ d & 0 & c \end{bmatrix}$$

So $d\varphi(q)(v) = \big(-2ax\alpha + b\beta, \alpha, c\gamma + d\alpha\big)$. We shall write $w = (\alpha_1, \beta_1, \gamma_1) := d\varphi(q)(v)$. Now we start by using the maximum norm $\|.\|_M$. Thus by $\mathcal{P}.\mathcal{C}$ $\quad 1 + |d| < c \Rightarrow$
$\Rightarrow \|w\|_M = Max\{|\alpha_1|, |\beta_1|, |\gamma_1|\} = Max\{|-2ax\alpha + b\beta|, |\alpha|, |c\gamma + d\alpha|\} \geq |c\gamma + d\alpha| \geq c|\gamma| - |d||\alpha| \quad (*)$.
Case1 : $|\alpha| \geq |\beta|$.
Then $\alpha_1^2 + \gamma_1^2 = \big(-2ax\alpha + b\beta\big)^2 + \big(c\gamma + d\alpha\big)^2 \geq \big(2a|x||\alpha| - |b||\beta|\big)^2 + \big(c|\gamma| - |d||\alpha|\big)^2 \geq \big[2a|x||\alpha| - |b||\beta|\big]^2 \geq \big[2|\alpha|(1 + |b|) - |b||\alpha|\big]^2$. Note that here we used the lemma above . So $\alpha_1^2 + \gamma_1^2 \geq \big[2|\alpha|(1 + |b|) - |b||\alpha|\big]^2 \geq 4|\alpha|^2 = 4\alpha^2 = 4\beta_1^2 > 2\beta_1^2 \Rightarrow \alpha_1^2 + \gamma_1^2 > \sqrt{2}|\beta_1|$. Thus $w \in C^u(\varphi(q))$. Now we have the followings subcases:
Case1.1 : $|\alpha| \geq |\gamma|$
Then as $\sqrt{\alpha^2 + \gamma^2} \geq \sqrt{2}|\beta| > |\beta|$ it follows that $\|v\|_M = |\alpha|$. By lemma above one have
$\|v\|_M = Max\{|\alpha_1|, |\beta_1|, |\gamma_1|\} \geq |\alpha_1| = |-2ax\alpha + b\beta| \geq 2a|x||\alpha| - |b||\beta| > 2|\alpha|$,that is ,$\|v\|_M > 2\|v\|_M$.
Case1.2 : $|\alpha| < |\gamma|$
That implies in $\|v\|_M = Max\{|\alpha|, |\beta|, |\gamma|\} = |\gamma|$ since $v \in C^u(q)$. By $(*)$ above we obtain $\|w\|_M \geq c|\gamma| - |d||\alpha| > \big(c - |d|\big)|\gamma| > |\gamma|$. So $\|w\|_M \geq \rho_1\|v\|_M$ for some constant $\rho_1 > 1$ and as we can see such constant does not depends on the point q and vector $v \in T_q\mathbb{D}$ as well .
Case2 : $|\beta| > |\alpha|$ and $|\gamma| > \sqrt{2}|\alpha|$
Since $1 + |d| < c$ we have $\alpha_1^2 + \gamma_1^2 \geq \big(2a|x||\alpha| - |b||\beta|\big)^2 + \big(c|\gamma| - |d||\alpha|\big)^2 \geq \big(c|\gamma| - |d||\alpha|\big)^2 > \big(c|\gamma| - |d||\gamma|\big)^2 = |\gamma|^2\big[c - |d|\big]^2 \geq |\gamma|^2 > 2|\alpha|^2 = 2|\beta_1|^2$ that is, $\sqrt{\alpha_1^2 + \gamma_1^2} > \sqrt{2}|\beta_1|$. Hence , $w \in C^u(\varphi(q))$.
Since $\sqrt{2}|\beta| \geq \sqrt{\alpha^2 + \gamma^2} < \sqrt{\beta^2 + \gamma^2}$ one can obtain $2\beta^2 < \beta^2 + \gamma^2 \Rightarrow |\beta| < |\gamma|$ which implies in $\|v\|_M = Max\{|\alpha|, |\beta|, |\gamma|\} = |\gamma| \quad (**)$. So from $(*)$ above $\|w\|_M \geq c|\gamma| - |d||\alpha| \Rightarrow \|w\|_M > |\gamma|\big(|c - |d|\big) > |\gamma|$. Thus $\|w\|_M > \rho_2\|v\|_M$ for some constant $\rho_2 > 1$.
Case3 : $|\beta| > |\alpha|$ and $|\gamma| \leq \sqrt{2}|\alpha|$.

From $\sqrt{2}|\beta| \leq \sqrt{\alpha^2+\gamma^2}$ and $|\beta| > |\alpha|$ we obtain $\sqrt{2}|\alpha| \geq |\gamma| > |\beta| > |\alpha|$. Then from lemma above it follows that
$\alpha_1^2+\gamma_1^2 \geq \big[2a|x||\alpha| - |b||\beta|\big]^2 + \big[||\gamma| - |d||\alpha|\big]^2 \geq \big[2|\alpha|(1+|b|) - |b||\beta|\big]^2 \geq \big[2|\alpha|(1+|b|) - \sqrt{2}|\alpha||b|\big]^2 > 4|\alpha|^2 = 4\beta_1^2 > 2\beta_1^2$, that is, $\sqrt{\alpha_1^2+\gamma_1^2} \geq \sqrt{2}|\beta_1|$. Hence $w \in C^u(\varphi(q))$.
Also the lemma above assures us that $\|w\|_M \geq |\alpha_1| = |-2ax\alpha + b\beta| \geq 2a|x||\alpha| - |b||\beta| \geq 2|\alpha|(1+|b|) - |b||\gamma| > 2(1+|b|)\frac{\sqrt{2}}{2}|\gamma| - |b||\gamma| = \sqrt{2}|\gamma| + \sqrt{2}|b||\gamma| - |b||\gamma| > \sqrt{2}|\gamma| = \sqrt{2}\|v\|_M$. That is, $\|w\|_M \geq \sqrt{2}\|v\|_M$.
Hence we take $\rho := Max\{\rho_1, \rho_2, \sqrt{2}, 2\}$ on the announcement . ■

Lemma 4.2.7 *Suppose that*

- *The family $\varphi(x,y,z)$ satisfies $\mathcal{P}.\mathcal{C}$;*
- *$\mathbb{A}$ and $\mathbb{B}$ are the connected components of $\varphi(\mathbb{D}) \cap \mathbb{D}$;*
- *$C^{uu}(q) := \{(\alpha,\beta,\gamma) \in T_q\mathbb{D} / |\alpha| \geq \sqrt{2}\sqrt{\beta^2+\gamma^2}\}$ for each $q \in \varphi^{-1}(\mathbb{A} \cup \mathbb{B})$.*

Then for vector $v \in C^{uu}(q) \Rightarrow d\varphi(q)(v) \in C^{uu}(\varphi(q))$.

Proof: For any $q = (x,y,z) \in \varphi^{-1}(\mathbb{A} \cup \mathbb{B})$ and $v = (\alpha,\beta,\gamma) \in C^{uu}(q)$. We shall write $w = (\alpha_1,\beta_1,\gamma_1) := d\varphi(q)(v) = \big(-2ax\alpha + b\beta, \alpha, c\gamma + d\alpha\big)$. Since $v \in C^{uu}(q) \Rightarrow \frac{|\alpha|}{\sqrt{2}} \geq Max\{|\beta|, |\gamma|\}$. So from lemma above one can obtain $\alpha_1^2 \geq [2a|x||\alpha| - |b||\beta|]^2 \geq \left[2(1+|b|)|\alpha| - \frac{|b||\alpha|}{\sqrt{2}}\right]^2 \geq 4\alpha^2$.
From $\mathcal{P}.\mathcal{C}$ we have $|d| + \frac{c}{\sqrt{2}} < 1$. Therefore one can obtain $\beta_1^2+\gamma_1^2 \geq \alpha^2 + \Big[|d||\alpha| + c|\gamma|\Big]^2 \geq |\alpha|^2\Big[1 + \Big[|d| + \frac{c}{\sqrt{2}}\Big]^2\Big] \geq 2\alpha^2 < 4\alpha^2$.
Thus $\alpha_1^2 \geq 4\alpha^2 > 2\big(\beta_1^2+\gamma_1^2\big) \Rightarrow |\alpha_1| > \sqrt{2}\sqrt{\beta_1^2+\gamma_1^2}$, that is, $w \in C^{uu}(\varphi(q))$. ■

The next lemma and its proof will be usefull in the proof of the lemma 4.2.10

Lemma 4.2.8 *Let be the family $\varphi(x,y,z)$ satisfies $\mathcal{P}.\mathcal{C}$, $\partial^+\mathbb{B}$ and $\partial^+\mathbb{B}$ be respectively the upper and the lower boundary of $\mathbb{B}$ and $p \in \mathbb{A}$ be the hyperbolic fixed point of the lemma 4.2.1.*

- *If $d < 0$ there exists a small neighborhood V^+ of $\partial^+\mathbb{B}$ such that $V^+ \cap W^s_{\mathbb{D}}(p) = \emptyset$;*
- *If $d > 0$ there exists a small neighborhood V^- of $\partial^-\mathbb{B}$ such that $V^- \cap W^s_{\mathbb{D}}(p) = \emptyset$.*

Proof: The upper bound δ for $|b|$ in $\mathcal{P}.\mathcal{C}$ will be decided definitely on this proof.
Suppose $d < 0$. As we know $\mathbb{B} := \{(x,y,z) \in \mathbb{D} \cap \varphi\mathbb{D} / y > 0\}$. If $(\hat{x},\hat{y},\hat{z}) \in \partial^-\mathbb{B}$ then there exists an unique point $(x_0,y_0,z_0) \in \mathbb{D}$ such that $(\hat{x},\hat{y},\hat{z}) = \varphi(x_0,y_0,z_0) = \big(1 - ax_0^2 + by_0, x_0, cz_0 + dx_0\big)$. As $(\hat{x},\hat{y},\hat{z}) \in \mathbb{B} \Rightarrow \hat{y} > 0 \Rightarrow x_0 > 0$. But $\hat{z} = cz_0 + dx_0$, $d < 0$ and $-2r_2 \leq z_0 \leq 0$. Then $\hat{z}$ will be maximum if and only if $z_0 = 0$. Thus $\hat{z} = dx_0$. From the calculus on Lemma 2.0.8 we have $dx_0 > dr_1$. So $\hat{z} > dr_1$.
Now by knowing that $W^s_{\mathbb{D}}(p)$ is a segment through $P = (x_p, y_p, z_p)$ for which the tangent space is contained in the stable cone $C^s_b(P)$ with opening slope is smaller than $\sqrt{|b|}$ as given in Lemma 2.0.5. Hence, for any point $(\widetilde{x},\widetilde{y},\widetilde{z}) \in W^s_{\mathbb{D}}(p)$,

$$\frac{|\widetilde{z} - z_p|}{2r_1} < \sqrt{|b|}$$

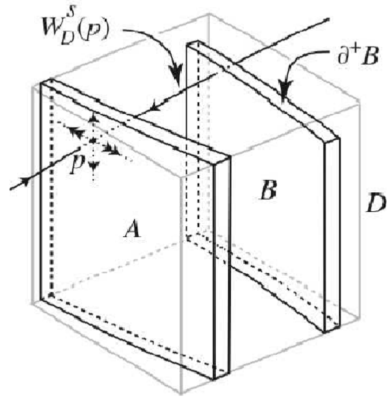

Figure 4.9: Positional relation between $W^s_{\mathbb{D}}(p)$ and $\mathbb{B}$(Figure from [1])

$$\widetilde{z} - z_p \leq |\widetilde{z} - z_p| < 2r_1\sqrt{|b|}$$
$$\widetilde{z} < z_p + 2r1\sqrt{|b|}.$$

Now note that for $|b| \longrightarrow 0$ one get

$$dr_1 = \frac{d[3(1+|b|) + 2\sqrt{4a + (b-1)^2}]}{4a} \longrightarrow \frac{d[3 + 2\sqrt{4a+1}]}{4a}$$

and

$$z_p + 2r_1\sqrt{|b|} \longrightarrow \frac{d(-1-\sqrt{1+4a})}{2a(1-c)}.$$

By $\mathcal{P}.\mathcal{C}$ we obtain $\frac{d(-1-\sqrt{1+4a})}{2a(1-c)} < \frac{d[3+2\sqrt{4a+1}]}{4a}$. Therefore, by taking $\delta > 0$ small enough ,for any $0 < |b| < \delta$, one obtain

$$z_p + 2r_1\sqrt{|b|} < dr_1.$$

This implies that, for $d < 0$, $W^s_{\mathbb{D}}(p)$ is located below $\partial^+\mathbb{B}$. Hence, one can take a neighborhood V^+ of $\partial^+\mathbb{B}$ such that $V^+ \cap W^s_{\mathbb{D}}(p) = \emptyset$.

For $d > 0$,it is clear that the claim can be shown similarly.This ends the proof. ∎

Lemma 4.2.9 *Suppose the family $\varphi(x, y, z)$ satisfies $\mathcal{P}.\mathcal{C}$.*

- *If $d < 0$ there exists a neighborhood U^- of the lower face Z^- of $\mathbb{D}$ such that every unstable curve L^u, trough $\mathbb{D}$, in the upper region of $W^s_{\mathbb{D}}(p)$ has no intersection with U^-;*
- *If $d > 0$ there exists a neighborhood U^+ of the upper face Z^+ of $\mathbb{D}$ such that every unstable curve L^u, trough $\mathbb{D}$, in the lower region of $W^s_{\mathbb{D}}(p)$ has no intersection with U^+;*

Proof: Suppose $d < 0$.

- Claim 1 : $\left[W^s_{\mathbb{D}}(p) \cap \mathbb{A}\right] \cap \left[Z^{\pm} \cup X^{\pm}\right] = \emptyset.$
  Of course, Since $p \in Int(\mathbb{A})$ and $W^s(p)$ is the stable manifold of $p$then, we get $\varphi(W^s(p) \cap \mathbb{A}) \subset Int(\mathbb{A})$. By Lemma 4.2.3 above this claim follows immediately.

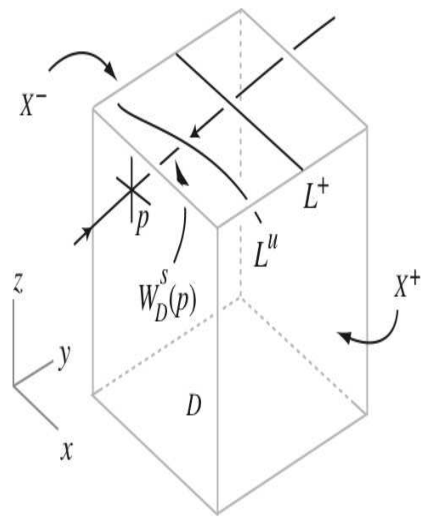

Figure 4.10: L^u is located in the upper region of $W^s_{\mathbb{D}}(p)$ (Figure from [1])

- Claim 2 : $W^s_{\mathbb{D}}(p) \subset Int\big(\varphi^{-1}(\mathbb{A})\big)$.
  In fact, we have $W^s_{\mathbb{D}}(p) := W^s(p) \cap \mathbb{D}$. Thus $\varphi\big(W^s_{\mathbb{D}}(p)\big) = \varphi\big(W^s(p)\big) \cap \varphi\big(\mathbb{D}\big) = W^s(p) \cap \varphi\big(\mathbb{D}\big)$. But the Claim 1 above assures us that $W^s(p) \cap \varphi\big(\mathbb{D}\big) \subset \big[W^s(p) \cap \mathbb{A}\big] \cup \big[W^s(p) \cap \mathbb{B}\big]$. However, $W^s_{\mathbb{D}}(p)$ is a connected set. Thus $\varphi\big(W^s_{\mathbb{D}}(p)\big)$ must be connected too. So $\varphi\big(W^s_{\mathbb{D}}(p)\big) = W^s(p) \cap \mathbb{A} \subset Int\big(\mathbb{A}\big)$ since $p \in W^s_{\mathbb{D}} \cap \mathbb{A}$. Hence $W^s_{\mathbb{D}}(p) \subset Int\big(\varphi^{-1}(\mathbb{A})\big)$.

From Lemma 4.2.3 above we can get immediately

$$\varphi^{-1}\big(\mathbb{A}\big) \cap \big[Z^{\pm} \cup X^{\pm}\big] = \emptyset.$$

That is, it implies the existence of a sufficiently small neighborhood U^- of the lower face Z^- satisfying $U^- \cap \varphi^{-1}\big(\mathbb{A}\big) = \emptyset$. Hence, one can get not only $W^s_{\mathbb{D}}(p) \cap U^- = \emptyset$ but $L^u \cap U^- = \emptyset$ for any unstable segment L^u on the upper region of $W^s_{\mathbb{D}}(p)$.
The case $d > 0$ can be proved in similar way. ■

Lemma 4.2.10 *Suppose that*

- *The family $\varphi(x, y, z)$ satisfies $\mathcal{P}.\mathcal{C}$;*
- *$\mathbb{A}$ and $\mathbb{B}$ are the connected components of $\varphi(\mathbb{D}) \cap \mathbb{D}$.*

Then

- *If $d < 0$ then there exists a neighborhood O^+ of Z^+ and a neighborhood V of $W^s_{\mathbb{D}}(p)$ such that for every unstable segment L^u on the upper region of $W^s_{\mathbb{D}}(p)$, $\varphi(L^u) \cap \mathbb{B}$ contains an unstable segment on the upper region of $W^s_{\mathbb{D}}(p)$ and disjoint of V.*
- *If $d > 0$ then there exists a neighborhood O^- of Z^- and a neighborhood V of $W^s_{\mathbb{D}}(p)$ such that for every unstable segment L^u on the lower region of $W^s_{\mathbb{D}}(p)$, $\varphi(L^u) \cap \mathbb{B}$ contains an unstable segment on the lower region of $W^s_{\mathbb{D}}(p)$ and disjoint of V.*

Proof: We are going to prove the case $d < 0$. The other one can be proved in similar way.

Claim 1. The segments $\varphi(L^u) \cap \mathbb{A}$ and $\varphi(L^u) \cap \mathbb{B}$ has no intersection with $W^s_{\mathbb{D}}(p)$.

Of course, as L^u is on the upper region of $W^s_{\mathbb{D}}(p)$ we have $W^s_{\mathbb{D}}(p) \cap L^u = \emptyset \Leftrightarrow \big[W^s(p) \cap \mathbb{D}\big] \cap L^u = \emptyset$. So $\emptyset = \varphi\big(\big[W^s(p) \cap \mathbb{D}\big] \cap L^u\big) \Rightarrow \emptyset = \mathbb{D} \cap \varphi\big(\big[W^s(p) \cap \mathbb{D}\big] \cap L^u\big) = \mathbb{D} \cap \varphi(W^s(p)) \cap \varphi(\mathbb{D}) \cap \varphi(L^u) = \big[\mathbb{D} \cap \varphi(\mathbb{D}\big] \cap W^s(p) \cap \varphi(L^u) = \big[\mathbb{A} \cup \mathbb{B}\big] \cap W^s(p) \cap \varphi(L^u) = \big[\varphi\big(L^u\big) \cap \mathbb{A}\big] \cap W^s(p) \cup \big[\varphi\big(L^u\big) \cap \mathbb{B}\big] \cap W^s(p)$. Hence $\big[\varphi\big(L^u\big) \cap \mathbb{A}\big] \cap W^s(p) = \emptyset$ and $\big[\varphi\big(L^u\big) \cap \mathbb{B}\big] \cap W^s(p) = \emptyset$.

Claim 2. There is a neighborhood V of $W^s_{\mathbb{D}}(p)$ such that for every unstable segment $L^u \subset \mathbb{D}$ on the upper region of $W^s_{\mathbb{D}}(p)$ one have $\varphi(L^u) \cap \mathbb{B}$ is disjoint of V.

In fact, suppose that is not true. Then we can find a sequence $(L^u_n) \subset \mathbb{D}$ of unstable segment in the upper region of $W^s_{\mathbb{D}}(p)$ so that

$$\lim_{n \to \infty} Dist(\varphi(L^u_n) \cap \mathbb{B}, W^s_{\mathbb{D}}(p)) = 0.$$

Thus there is a sequence of points $q_{n\mathbb{B}} \in \varphi(L^u_n) \cap \mathbb{B}$ and a point $q_{\mathbb{B}} \in W^s_{\mathbb{D}}(p)$ such that $\lim_{n \to \infty} q_{n\mathbb{B}} = q_{\mathbb{B}}$. It follows that $\varphi^{-1}(q_{n\mathbb{B}}) \in L^u_n \cap \varphi^{-1}(\mathbb{B})$ and $\lim_{n \to \infty} \varphi^{-1}(q_{n\mathbb{B}}) = \varphi^{-1}(q_{\mathbb{B}}) \in W^s(p)$ (invariant manifold). But that is a contradiction because as we know $L^u_n \subset \mathbb{D}$, $W^s_{\mathbb{D}}(p) \subset Int(\varphi^{-1}(\mathbb{A}))$ and $\varphi^{-1}(\mathbb{A})$ is disjoint of $\varphi^{-1}(\mathbb{B})$.

Claim 3 For every unstable segment $L^u \subset \mathbb{D}$ on the upper region of $W^s_{\mathbb{D}}(p)$ one have $\varphi(L^u) \cap \mathbb{B}$ is an unstable segment on the upper region of $W^s_{\mathbb{D}}(p)$.

In fact, it is clear that $\varphi(L^u) \cap \mathbb{B}$ is an unstable segment. Now suppose z_0 is the maximum hight of $W^s_{\mathbb{D}}(p)$. Then as we saw in the Lemma 4.2.8 we must have $z_0 < z_p + 2r_1\sqrt{|b|} < dr_1$. Thus $-dr_1 < -z_0 = |z_0| \Leftrightarrow \frac{-dr_1+(c-1)|z_0|}{c} < |z_0|$. So now we take $\delta > 0$ such that $\frac{-dr_1+(c-1)|z_0|}{c} < \delta < |z_0|$ and consider the unstable segment $L^u := [-r_1, r_1] \times \{y_0\} \times \{z_0 + \delta\} \subset \mathbb{D}$. It follows that $\varphi(L^u) \cap \mathbb{B} = \{(1 - ax^2 + by_0, x, cz_0 + c\delta + dx)/\ 0 \leq x \leq r_1\}$. By the choice of δ we have $\frac{(c-1)z_0+c\delta}{|d|} \geq r_1$ which implies $x < \frac{(c-1)z_0+c\delta}{|d|} \Leftrightarrow c(z_0 + \delta) + dx > z_0$ which means that $\varphi(L^u) \cap \mathbb{B}$ is on the upper region of $W^s_{\mathbb{D}}(p)$. Now we take $0 < \delta_1 < \frac{-dr_1+(c-1)|z_0|}{c}$. Let us show that if $L^u_1 := [-r_1, r_1] \times \{y_0\} \times \{z_0 + \delta_1\}$ then $\varphi(L^u_1) \cap \mathbb{B}$ is on the upper region of $W^s_{\mathbb{D}}(p)$. Then suppose that is not true, that is ,suppose $\varphi(L^u_1) \cap \mathbb{B}$ is on the lower region of $W^s_{\mathbb{D}}(p)$ and consider $L^u_2 := [-r_1, r_1] \times \{y_0\} \times \{z_0 + \delta_2\}$ where $\frac{-dr_1+(c-1)|z_0|}{c} < \delta_2 < |z_0|$ and $\mathcal{S} \subset \mathbb{D}$ to be the unstable strip whose edges are L^u_1 and L^u_2. It follows that $\varphi(\mathcal{S}) \cap \mathbb{B}$ is an unstable strip which intersects $W^s_{\mathbb{D}}(p)$. Then $\mathcal{S}$ contains unstable segments L^u on the upper region of $W^s_{\mathbb{D}}(p)$ such that $\varphi(L^u) \cap \mathbb{B}$ intersects the neighborhood V(see claim 2 !) of $W^s_{\mathbb{D}}(p)$ what can not occur. This shows what we desire.

From all arguments above we can conclude that for every unstable segment of straight line ,L^u, on the upper region of $W^s_{\mathbb{D}}(p)$ one have $\varphi(L^u) \cap \mathbb{B}$ on the upper region of $W^s_{\mathbb{D}}(p)$. In the general case where L^u is a segment of curve then we consider a neighborhood $\mathcal{U}$ of L^u constituted by segments unstable of straight line and by continuity of φ we get $\varphi(L^u) \cap \mathbb{B}$ on the upper region of $W^s_{\mathbb{D}}(p)$.

$\blacksquare$

Hence as a consequence of the lemmas 4.2.1, 4.2.3, 4.2.5, 4.2.6, 4.2.7, 4.2.9, 4.2.10 above we see that the pair $(\varphi, \mathbb{D})$ satisfies the blender's conditions.

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