Universidade Federal de Minas Gerais
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Tese de Doutorado

Estimativas a Priori e Soluções Positivas para Sistemas Elípticos de Lane-Emden Fracionários

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Belo Horizonte - 9 de novembro de 2015

À minha família

Agradecimentos

Primeiramente a Deus, meu guia, meu caminho e minha salvação.

Ao meu orientador, Prof. Dr. Marcos da Silva Montenegro, pela escolha do tema, pela paciência, pela disposição, pela orientação durante o desenvolvimento do trabalho e pela compreensão nos momentos difíceis. Graças ao seu profissionalismo e dedicação tem sido possível este trabalho. Findo o qual, creio que posso assim dizer, tornou-se antes de orientador, uma pessoa amiga.

Aos meus familiares, pela compreensão, pelo incentivo nos momento difíceis e por sempre me propiciar um ambiente familiar sadio, dentro dos ensinamentos de Deus.

À comissão julgadora pela disponibilidade e atenção despendida ao trabalho.

Ao meu orientador de mestrado Prof. Dr. Edson Agustini, pelo incentivo e apoio para continuar os estudos, pelas conversas instrutivas e pela sua amizade.

Ao pessoal técnico-administrativo e docentes da PPGMAT-UFMG pela atenção e dedicação aos alunos de pós-graduação.

Aos meus colegas de pós-graduação da UFMG pelas inúmeras e agradáveis conversas.

A CAPES, que financiou minha pesquisa durante o doutorado.

Homem de pouca fé, por que duvidaste?

In this thesis we discuss the existence, nonexistence, uniqueness and a priori bounds of positive viscosity solutions of the following coupled system involving the fractional Laplace operator on a smooth bounded domain Ω in $\mathbb{R}^n$:

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \Omega \\ (-\Delta)^t v = u^q & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

We divided this thesis into two parts:

In the first part we deal with strongly coupled elliptic systems in non-variational form. By mean of Liouville type theorems we establish a priori bounds of positive solutions for subcritical and superlinear nonlinearities in a suitable sense. We then derive the existence of positive solutions through topological methods.

In the second part we deal with strongly coupled elliptic systems in variational form, i.e., $s = t$. By mean of an appropriate variational framework and a Hölder regularity result, we prove that the above system admits at least one positive viscosity solution for any power $0 < s < 1$, provided that $p, q > 0$, $pq \neq 1$ and the couple (p, q) is below the hyperbole

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2s}{n}\,.$$

Moreover, by using maximum principles for the fractional Laplace operator, we show that uniqueness occurs whenever $pq < 1$. Lastly, assuming Ω is star-shaped, by using a Rellich type variational identity, we prove that no such a solution exists if (p, q) is on or above the same hyperbole. As a byproduct, we obtain the critical hyperbole associated to the above system for any $0 < s < 1$.

Keywords: Fractional Laplace operator, a priori bounds, Liouville type theorem, variational methods, critical hyperbole, Lane-Emden system

Sumário

Introduction

The present thesis is divided into two parts: non-variational and variational.

The first part of this thesis deals with a priori bounds and existence of positive solutions for elliptic systems of the form

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \Omega \\ (-\Delta)^t v = u^q & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{1}$$

where Ω is a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 2$, $s, t \in (0,1)$, $p, q > 0$ and the fractional Laplace operator (or fractional Laplacian) of order $2s$, with $0 < s < 1$, denoted by $(-\Delta)^s$, is defined as

$$(-\Delta)^s u(x) = C(n,s)\, P.V. \int\limits_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy\,, \tag{2}$$

or equivalently,

$$(-\Delta)^s u(x) = -\frac{1}{2} C(n,s) \int\limits_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy$$

for all $x \in \mathbb{R}^n$, where P.V. denotes the principal value of the integral and

$$C(n,s) = \left(\int\limits_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}}\, d\zeta \right)^{-1}$$

with $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$.

Remark that $(-\Delta)^s$ is a nonlocal operator on functions compactly supported in $\mathbb{R}^n$. The convergence property

$$\lim_{s \to 1^-} (-\Delta)^s u = -\Delta u$$

pointwise in $\mathbb{R}^n$ holds for every function $u \in C_0^\infty(\mathbb{R}^n)$, so that the operator $(-\Delta)^s$ interpolates the Laplace operator in $\mathbb{R}^n$.

Factional Laplace operators arise naturally in several different areas such as Probability, Finance, Physics, Chemistry and Ecology, see [5]. Moreover, fractional Laplace operator appear naturally also in other contexts such as Image processing, Fluid Mechanics, Geometry, ultra-relativistic limits of quantum mechanic and nonlocal electrostatics, as explained in Chapter 1.

A closely related operator but different from $(-\Delta)^s$, the spectral fractional Laplace operator $\mathcal{A}^s$, is defined in terms of the Dirichlet spectra of the Laplace operator on Ω. Roughly, if (φ_k) denotes a L^2-orthonormal basis of eigenfunctions corresponding to eigenvalues (λ_k) of the Laplace operator with zero Dirichlet boundary values on $\partial\Omega$, then the operator $\mathcal{A}^s$ is defined as $\mathcal{A}^s u = \sum_{k=1}^{\infty} c_k \lambda_k^s \varphi_k$, where c_k, $k \geq 1$, are the coefficients of the expansion $u = \sum_{k=1}^{\infty} c_k \varphi_k$.

After the work [41] on the characterization for any $0 < s < 1$ of the operator $(-\Delta)^s$ in terms of a Dirichlet-to-Neumann map associated to a suitable extension problem, a great deal of attention has been dedicated in the last years to nonlinear nonlocal problems of the kind

$$\left\{ \begin{array}{rcll} (-\Delta)^s u & = & f(x,u) & \text{in } \Omega \\ u & = & 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{array} \right. \tag{3}$$

where Ω is a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$.

Several works have focused on the existence [86, 88, 89, 93, 124, 133, 138, 164, 165, 166, 167], nonexistence [81, 167], symmetry [11, 53] and regularity [8, 32, 154] of viscosity solutions, among other qualitative properties [1, 84]. For developments related to (3) involving the spectral fractional Laplace operator $\mathcal{A}^s$, we refer to [9, 24, 35, 48, 55, 56, 176, 177, 185] and references therein.

A specially important example arises for the power function $f(x,u) = u^p$, with $p > 0$, in which case (3) is called the fractional Lane-Emden problem. Recently, it has been proved in [164] that this problem admits at least one positive viscosity solution for $1 < p < \frac{n+2s}{n-2s}$. The nonexistence has been established in [155] whenever $p \geq \frac{n+2s}{n-2s}$ and Ω is star-shaped. These results were known long before for $s = 1$, see the classical references [3, 99, 141, 148] and the survey [144].

Systems like (1) are strongly coupled vector extensions closely related to (3) with the power function $f(x,u) = u^p$, which have been addressed for $s = t = 1$ by several authors during the two last decades (we refer to the survey [63] and references therein). More specifically, a priori bounds and existence of positive solutions have been considered in these cases. In view of what is known for scalar equations and for systems of the type (1) with $s = t = 1$, one expects that a priori bounds depend on the values of the exponents

p and q. Indeed, the values p and q should be related to Sobolev embedding theorems.

A rather classical fact is that a priori bounds allow to establish existence of positive solutions for systems by mean of topological methods such as degree theory and Krasnoselskii's index theory. For a list of works concerning with non-variational elliptic systems involving Laplace operators we refer to [10, 58, 67, 68, 135, 146, 170, 187], among others.

One goal of this thesis is also establishing existence of positive classical solutions of non-variational strongly coupled systems of the type (1) by mean of a priori bounds for a family of exponents p and q. By a classical solution of the system (1), we mean a couple $(u, v) \in (C^\alpha(\mathbb{R}^n))^2$ for some $0 < \alpha < 1$ satisfying (1) in the usual classical sense.

Our main result of first part is

Theorem 0.0.1. *Let Ω be a bounded open subset of C^2 class of $\mathbb{R}^n$. Assume that $n \geq 2$, $s, t \in (0, 1)$, $n > 2s + 1$, $n > 2t + 1$, $p, q \geq 1$, $pq > 1$ and either*

$$\left(\frac{2s}{p} + 2t\right) \frac{p}{pq - 1} \geq n - 2s \quad \text{or} \quad \left(\frac{2t}{q} + 2s\right) \frac{q}{pq - 1} \geq n - 2t \,. \tag{4}$$

Then, the system (1) admits, at least, one positive classical solution. Moreover, all such solutions are uniformly bounded in the L^∞-norm by a constant that depends only on s, t, p, q and Ω.

Remark 0.0.2. When $0 < s = t < 1$ and $p, q > 1$, a priori bounds and existence of positive classical solutions of (1) for the spectral fractional Laplace operator $\mathcal{A}^s$ have been derived in [55] provided that

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n - 2s}{n} \,. \tag{5}$$

Remark 0.0.3. When $0 < s = t < 1$ and $pq > 1$, the condition (4) implies (5).

The approach used in the proof of Theorem 0.0.1 is based on the blow-up method, firstly introduced by Gidas and Spruck in [100] to treat the scalar case and later extended to strongly coupled systems like (1) with $s = t = 1$ in [135] and then in [67, 68, 170, 187]. This method consists of a contradiction argument, which in turn relies on Liouville type results for equations or systems in the whole space $\mathbb{R}^n$ or in a half-space of it. Proving these last ones is usually the main obstacle in applying the Gidas-Spruck method.

For this purpose, we first shall establish Liouville type theorems for the system

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \; G \\ (-\Delta)^t v = u^q & \text{in } \; G \end{cases} \tag{6}$$

for $G = \mathbb{R}^n$ and $G = \mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \; x_n > 0\}$. In this latter one, we assume the Dirichlet condition $u = 0 = v$ in $\mathbb{R}^n \setminus \mathbb{R}^n_+$.

We recall that viscosity supersolution for the above system is a couple (u, v) of continuous functions in $\mathbb{R}^n$ such that $u, v \geq 0$ in $\mathbb{R}^n \setminus \overline{G}$ and for each point $x_0 \in G$ there exists a neighborhood U of x_0 with $\overline{U} \subset G$ such that for any $\varphi, \psi \in C^2(\overline{U})$ satisfying $u(x_0) = \varphi(x_0)$, $v(x_0) = \psi(x_0)$, $u \geq \varphi$ and $v \geq \psi$ in U, the functions defined by

$$\overline{u} = \begin{cases} \varphi & \text{in } U \\ u & \text{in } \mathbb{R}^n \setminus U \end{cases} \quad \text{and} \quad \overline{v} = \begin{cases} \psi & \text{in } U \\ v & \text{in } \mathbb{R}^n \setminus U \end{cases} \tag{7}$$

satisfy

$$(-\Delta)^s \overline{u}(x_0) \geq v^p(x_0) \quad \text{and} \quad (-\Delta)^t \overline{v}(x_0) \geq u^q(x_0)\,.$$

In a natural way, we have the notions of viscosity subsolution and viscosity solution.

Theorem 0.0.4. *Assume that $n \geq 2$, $s, t \in (0, 1)$, $n > 2s$, $n > 2t$, $p, q > 0$ and $pq > 1$. Then, the only nonnegative viscosity supersolution of the system (6) with $G = \mathbb{R}^n$ is the trivial if and only if (4) holds.*

Theorem 0.0.5. *Assume that $n \geq 2$, $s, t \in (0, 1)$, $n > 2s + 1$, $n > 2t + 1$, $p, q \geq 1$ and $pq > 1$. If the condition (4) holds, then the only nonnegative viscosity bounded solution of the system (6) with $G = \mathbb{R}^n_+$ is the trivial one.*

Remark 0.0.6. Nonexistence results of positive solutions have been established for the scalar problem

$$(-\Delta)^s u = u^p \quad \text{in} \quad G$$

in both cases $G = \mathbb{R}^n$ and $G = \mathbb{R}^n_+$ by assuming that $n > 2s$ and $1 < p < \frac{n+2s}{n-2s}$, see [99, 100] for $s = 1$ and [112, 147] for $0 < s < 1$.

Remark 0.0.7. A number of works have focused attention on nonexistence of positive solutions of (6) for $G = \mathbb{R}^n$ and $G = \mathbb{R}^n_+$ when $s = t = 1$ and $0 < s = t < 1$. We refer for instance to [10, 28, 65, 132, 142, 162, 171, 172] for $s = t = 1$ and [147] for $0 < s = t < 1$ and other references therein.

Several arguments have been employed in the proof of nonexistence results of positive solutions of elliptic systems. Our approach is inspired on a powerful technique, based on maximum principles, developed by Quaas and Sirakov in [146] to treat systems involving different uniformly elliptic linear operators. Particularly, some maximum principles and related results for fractional operators due to Silvestre [169] and Quaas and Xia [147] as well as some auxiliary tools to be proved in the Section 3.1 of Chapter 3 will be used in the proof of Theorems 0.0.1, 0.0.4 and 0.0.5.

The second part of this thesis is devoted to the study of existence, uniqueness and nonexistence of positive viscosity solutions of following vector extension of the fractional Lane-Emden problem on a smooth bounded domain Ω in $\mathbb{R}^n$:

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \Omega \\ (-\Delta)^s v = u^q & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{8}$$

where $p, q > 0$, $n \geq 1$ and $s \in (0, 1)$.

For $s = 1$, the problem (8) and a number of its generalizations have been widely investigated in the literature during the two last decades, see for instance the survey [63] and references therein. Specially, the notions of sublinearity, superlinearity and criticality (subcriticality, supercriticality) have been introduced in [83, 131, 132, 161]. Indeed, the behavior of (8) is sublinear when $pq < 1$, superlinear when $pq > 1$ and critical (subcritical, supercritical) when $n \geq 3$ and (p, q) is on (below, above) the hyperbole, known as critical hyperbole,

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}\,.$$

When $pq = 1$, its behavior is resonant and the corresponding eigenvalue problem has been addressed in [136]. The sublinear case has been studied in [83] where the existence and uniqueness of positive classical solution is proved. The superlinear-subcritical case has been completely covered in the works [58], [64], [66] and [107] where the existence of at least one positive classical solution is derived. Lastly, the nonexistence of positive classical solutions has been established in [131] on star-shaped domains.

In this part we discuss existence and nonexistence of positive viscosity solutions of (8) for $0 < s < 1$. We determine the precise set of exponents p and q for which the problem (8) admits always a positive viscosity solution. In particular, we extend the above-mentioned results corresponding to the fractional Lane-Emden problem for $0 < s \leq 1$ and to the Lane-Emden system involving the Laplace operator. As a byproduct, the notions of sublinearity, superlinearity and criticality (subcriticality, supercriticality) to the problem (8) appear naturally for $0 < s < 1$.

The ideas involved in our proofs base on variational methods, C^β regularity of weak solutions and a variational identity satisfied by positive viscosity solutions of (8). We shall introduce a suitable variational framework in order to establish the existence of nontrivial nonnegative weak solutions of (8). In our variational formulation, the function u arises as a nonzero critical point and then, in a natural way, one defines v so that the couple (u, v) is a weak solution of (8). Using the C^β regularity result (to be proved in

Section 4.2) for weak solutions of (8) and maximum principles for the fractional Laplace operator, we deduce that the constructed couple (u, v) is a positive viscosity solution of (8). Moreover, we prove its uniqueness in some cases. The key tool used in the nonexistence proof is a Rellich type variational identity (to be proved in Section 4.3) to positive viscosity solutions of (8). The proof of the C^β regularity consists in first showing that weak solutions of (8) belong to $L^\delta(\Omega) \times L^\delta(\Omega)$ for every $\delta \geq 1$ and then applying to each equation the C^β regularity result up to the boundary proved recently in [153]. The proof of the variational identity to Lane-Emden systems uses the Pohozaev variational identity to fractional elliptic equations obtained recently in [155].

Our three main theorems of the second part are

Theorem 0.0.8. *(sublinear case) Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. Assume that $p, q > 0$ and $pq < 1$. Then the problem (8) admits a unique positive viscosity solution.*

Theorem 0.0.9. *(superlinear-subcritical case) Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n > 2s$ and $0 < s < 1$. Assume that $p, q > 0$, $pq > 1$ and*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2s}{n}\,. \tag{9}$$

Then the problem (8) admits at least one positive viscosity solution.

Theorem 0.0.10. *(critical and supercritical cases) Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n > 2s$ and $0 < s < 1$. Assume that Ω is star-shaped, $p, q > 0$ and*

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2s}{n}\,. \tag{10}$$

Then the problem (8) admits no positive viscosity solution.

For dimension $n > 2s$, these theorems motivate the hyperbole

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2s}{n} \tag{11}$$

to be called critical hyperbole for Lane-Emden systems involving the fractional Laplace operator. Similarly, the curve that splits the system (8) into sublinear and superlinear behaviors is given by the hyperbole $pq = 1$.

Thus, we are led to basically divide this thesis into four parts (chapters) as outlined following.

In Chapter 1, we present a detailed overview of the fractional Laplace operator from its origin. We talked about the various applications of this operator and its main results.

In Chapter 2, we remember some topics essential to our research work, in order to make accessible the reading of this work to those who do not have more consistent knowledge of the subjects dealt with.

The Chapter 3 is organized into four sections. In Section 3.1 we prove some key lemmas required in the proof of Theorem 0.0.4 which will be presented in Section 3.2. Section 3.3 is devoted to the proof of Theorem 0.0.5. In Section 3.4, we use Theorems 0.0.4 and 0.0.5 in order to prove Theorem 0.0.1.

The Chapter 4 is organized into six sections. In Section 4.1 we introduce the variational framework to be used in the existence proofs. In Section 4.2, we prove the C^β regularity result to weak solutions of (8) into the subcritical context according to the hyperbole (11). In Section 4.3, we obtain the Rellich variational identity to positive viscosity solutions of (8). In Section 4.4, we prove Theorem 0.0.8 by using a direct minimization approach, the regularity result provided in the third section and maximum principles. In section 4.5, Theorem 0.0.9 is proved by using the mountain pass theorem and the same regularization. Finally, in Section 4.6, we prove Theorem 0.0.10 by applying the variational identity proved in Section 4.3.

CAPÍTULO 1

Fractional Laplace operator: historical overview

This chapter is inspired in the thesis of Ros-Oton [152] and adapted to fractional Laplace operator.

Partial Differential Equations are relations between the values of an unknown function and its derivatives of different orders. In order to check whether a PDE holds at a particular point, one needs to know only the values of the function in an arbitrarily small neighborhood, so that all derivatives can be computed. A nonlocal equation is a relation for which the opposite happens. In order to check whether a nonlocal equation holds at a point, information about the values of the function far from that point is needed. Most of the times, this is because the equation involves integral operators. The most canonical example of such operator is

$$(-\Delta)^s u(x) = C(n,s)\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \tag{1.1}$$

for all $x \in \mathbb{R}^n$, where P.V. denotes the principal value of the integral and

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}}\, d\zeta \right)^{-1}$$

with $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$. The Fourier symbol of this operator is $|\xi|^{2s}$ and, thus, one has that $(-\Delta)^t \circ (-\Delta)^s = (-\Delta)^{s+t}$, this is why it is called fractional Laplace operator.

1.1 Why studying fractional Laplace operator?

To a great extent, the study of fractional Laplace operator is motivated by real world applications. Indeed, there are many situations in which a nonlocal equation gives a significantly better model than a PDE.

In Mathematical Finance it is particularly important to study models involving jump processes, since the prices of assets are frequently modeled. Note that jump processes are very natural in this situation, since asset prices can have sudden changes. These models have become increasingly popular for modeling market uctuations since the work of Merton [128] in 1976, both for risk management and option pricing purposes. For example, the obstacle problem for the fractional Laplace operator can be used to model the pricing of American options [122, 140]; see also the nice introduction of [37] and also [40, 169]. Good references for financial modeling with jump processes are the books [60] and [158]; see also [139].

Fractional Laplace operators appear also in Ecology. Indeed, optimal search theory predicts that predators should adopt search strategies based on long jumps where prey is sparse and distributed unpredictably, Brownian motion being more efficient only for locating abundant prey; see [108, 149, 182]. Thus, reaction-diffusion problems with nonlocal diffusion such as

$$u_t + (-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n \tag{1.2}$$

arise naturally when studying such population dynamics. Equation (1.2) appear also in physical models of plasmas and flames; see [126], [129], and references therein.

It is worth saying that in these problems the nonlocal diffusion (instead of a classical one) changes completely the behavior of the solutions. For example, consider problem (1.2) with $f(u) = u - u^2$, and with compactly supported initial data. Then, in both cases $s = 1$ and $s \in (0, 1)$, there is an invasion of the unstable state $u = 0$ by the stable one, $u = 1$. However, in the classical case ($s = 1$) the invasion front position is linear in time, while in case $s \in (0, 1)$ the front position will be exponential in time. This was heuristically predicted in [69] and [126], and rigorously proved in [31].

In Fluid Mechanics, many equations are nonlocal in nature. A clear example is the surface quasi-geostrophic equation, which is used in oceanography to model the temperature on the surface [59]. The regularity theory for this equation relies on very delicate regularity results for nonlocal equations in divergence form; see [36, 45, 46].

Another important example is the Benjamin-Ono equation

$$(-\Delta)^{1/2} u = -u + u^2$$

which describes one-dimensional internal waves in deep water [4, 91]. Also, the half Laplace operator $(-\Delta)^{1/2}$ plays a very important role in the understanding of the gravity water waves equations in dimensions 2 and 3; see [94].

In Elasticity, there are also many models that involve nonlocal equations. An important example is the Peierls-Nabarro equation, arising in crystal dislocation models

[74, 125, 181]. Also, other nonlocal models are used to take into account that in many materials the stress at a point depends on the strains in a region near that point [78, 115]. Long range forces have been also observed to propagate along fibers or laminae in composite materials [109], and nonlocal models are important also in composite analysis; see [76] and [134].

Other Physical models arising in macroscopic evolution of particle systems or in phase segregation lead to nonlocal diffusive models such as the fractional porous media equation; see [47, 97, 159]. Related evolution models with nonlocal effects are used in superconductivity [51, 184]. Moreover, other continuum models for interacting particle systems involve nonlocal interaction potentials; see [49].

Other examples in which fractional Laplace operator are used are Image Processing (where nonlocal denoising algorithms are able to detect patterns and contours in a better way than the local PDE based models [27, 102, 114, 186]), Geometry (where the conformally invariant operators, which encode information about the manifold, involve fractional powers of the Laplace operator [50, 103]), ultra-relativistic limits of quantum mechanic [82], nonlocal electrostatics [106, 110, 150] etc.

1.2 Mathematical background

Let us describe briefly the mathematical literature on fractional Laplace operator. As we will see, for many years these equations were studied by people in Probability. More recently, these equations have attracted much interest from people in Analysis and PDEs, with nonlinear equations being the focus of research.

1.3 Probability

The study of fractional Laplace operator started in the fifties with the works of Getoor, Blumenthal, and Kac, among others. In 1959, the continuity up to the boundary of solutions was established, and also some spectral properties of such operators [95]. The asymptotic distribution of eigenvalues was obtained, as well as some comparison results between the Green's function in a domain and the fundamental solution in the entire space [12].

Later, sharp decay estimates for the heat kernel of the fractional Laplace operator in the whole $\mathbb{R}^n$ were proved [13], and an explicit formula for the solution of

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } B_1 \\ u = 0 & \text{in } \mathbb{R}^n \setminus B_1 \end{cases} \tag{1.3}$$

was found [96, 113]. Moreover, Green's function and the Poisson kernel for the fractional Laplace operator in the unit ball B_1 were also explicitly computed by Getoor [14] and Riesz [151], respectively.

Potential theory for the fractional Laplace operator in $\mathbb{R}^n$ enjoys an explicit formulation in terms of the Riesz potential, and thus it is similar to that of the Laplace operator; see for example the classical book of Landkov [118]. However, the boundary potential theory for this operator presents more difficulties mainly due to its nonlocal character.

Fine boundary estimates for the Green's function and the heat kernel near the boundary have been established in the last twenty years. Namely, Green's function estimates were obtained by Kulczycki [116] and Chen-Song [52] in 1997 for $C^{1,1}$ domains, and in 2002 by Jakubowski for Lipschitz domains [111]. Later, Chen-Kim-Song [54] gave sharp explicit estimates for the heat kernel on $C^{1,1}$ domains, recently extended to Lipschitz and more general domains by Bogdan-Grzywny-Ryznar [16].

Related to this, Bogdan [15] in 1997 established the boundary Harnack principle for s-harmonic functions - solutions to $(-\Delta)^s u = 0$ - in Lipschitz domains; see also [17] for an extension of this result to general bounded domains.

For the fractional Laplace operator it is also possible to develop interior regularity results and boundary potential theory by using the associated fundamental solution; see for example [18, 19, 20, 143, 174, 175].

1.4 Analysis and PDEs: nonlinear equations

In the last ten years the study of fractional Laplace operator has attracted much interest from people in Analysis and PDEs. The main motivation for this, as explained above, is that fractional Laplace operator appear in many models in different sciences.

In contrast with the probabilistic works above for linear equations, more recent results using analytical methods often concern nonlinear fractional Laplace operator. In [34], Cabré and Solà-Morales studied layer solutions to a boundary reaction problem in $\mathbb{R}^{n+1}_+$

$$\begin{cases} -\Delta v &= 0 \quad \text{in } \mathbb{R}^{n+1}_+ \\ \frac{\partial v}{\partial \nu} &= f(v) \quad \text{on } \partial\mathbb{R}^{n+1}_+ \end{cases} . \tag{1.4}$$

An important example is the Peierls-Nabarro equation, which corresponds to $f(v) = \sin(\pi v)$. As noticed in previous works of Amick and Toland [4, 181], this boundary reaction problem in all of $\mathbb{R}^{n+1}_+$ is equivalent to the fractional Laplace operator

$$(-\Delta)^{1/2} u = f(u) \text{ in } \mathbb{R}^n.$$

Indeed, given a function u in $\mathbb{R}^n$, one can compute its harmonic extension v in one more dimension, i.e, the solution to $\Delta v = 0$ in $\mathbb{R}^{n+1}_+$, $v = u$ in $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$. Then, it turns out that the normal derivative $\frac{\partial v}{\partial \nu}$ on $\mathbb{R}^n$ is exactly the half Laplace operator $(-\Delta)^{1/2}u$.

On the other hand, motivated by applications to mathematical finance, Silvestre [169] studied the regularity of solutions to the obstacle problem for the fractional Laplace operator $(-\Delta)^s$, $s \in (0,1)$. He obtained an almost-optimal regularity result for its solution, more precisely he proved the solution to be $C^{1+s-\varepsilon}$ for all $\varepsilon > 0$.

In case $s = 1/2$, thanks to the aforementioned extension method, the obstacle problem for the half Laplace operator in $\mathbb{R}^n$ is equivalent to the thin obstacle problem for the Laplace operator in $\mathbb{R}^{n+1}$. For this latter problem, the optimal regularity of solutions and of free boundaries was well known; see [6, 7]. However, for fractional Laplace operator with $s \neq 1/2$ no similar extension problem was available.

This situation changed when Caffarelli and Silvestre [41] introduced the extension problem for the fractional Laplace operator $(-\Delta)^s$, $s \in (0,1)$. Thanks to this extension, in a joint work with Salsa [40] they established the optimal regularity of the solution and of the free boundary for the obstacle problem for the fractional Laplace operator, for all $s \in (0,1)$.

These developments and specially the extension problem for the fractional Laplace operator, have led to a huge amount of new discoveries on nonlinear equations for fractional Laplace operator. Just to mention some of them, we recall the important works on uniqueness of solutions for the equation $(-\Delta)^s u = f(u)$ in $\mathbb{R}^n$ [32, 33, 91, 92]; on the fractional Allen-Cahn equation [29, 30, 57, 156]; on nonlocal minimal surfaces [39, 43, 44, 87, 157]; on free boundary problems involving the fractional Laplace operator [38, 71]; and many others [79, 81, 179, 180].

The nonlinear Dirichlet problems of form

$$\begin{cases} (-\Delta)^s u &= f(x,u) \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{1.5}$$

have attracted much attention in the last years. Many of the mathematical works in the literature deal with existence [86, 88, 89, 93, 124, 133, 138, 164, 165, 166, 167], nonexistence [81, 167], symmetry [11, 53], regularity of solutions [8, 32, 154], and other qualitative properties of solutions [1, 84].

For linear equations, the Lax-Milgram theorem and the Fredholm alternative lead to existence of solutions for fractional Laplace operators [86]. For semilinear equations, other variational methods (like the mountain pass lemma or linking theorems) lead also to existence results for subcritical nonlinearities [164, 165]. In case of critical nonlinearities like $f(u) = u^{\frac{n+2s}{n-2s}} + \lambda u$, a Brezis-Nirenberg type result has been obtained by Servadei and Valdinoci [166, 167].

A very important tool to obtain symmetry results for second order (local) equations $-\Delta u = f(u)$ is the moving planes method [98, 160]. This method was first adapted to nonlocal equations by Birkner, López-Mimbela, and Wakolbinger [11], who proved the radial symmetry of nonnegative solutions to $(-\Delta)^s u = f(u)$ in the unit ball B_1. Later, the moving planes method has been used to solve Serrin's problem for the fractional Laplace operator [62, 80], and also to show nonexistence of nonnegative solutions to supercritical and critical equations $(-\Delta)^s u = u^{\frac{n+2s}{n-2s}}$ in star-shaped domains [81].

This nonexistence result for the fractional Laplace operator by Fall and Weth [81] uses the extension problem of Caffarelli-Silvestre and the fractional Kelvin transform to then apply the moving planes method.

1.5 Spectral fractional Laplace operator

An extension for this operator was devised by Cabré and Tan [35] and Capella, Dávila, Dupaigne, and Sire [48] (see Brändle, Colorado, de Pablo, and Sánchez [24] and Tan [177] also). Thanks to these advances, the boundary fractional problem

$$\begin{cases} \mathcal{A}^s u &= u^p \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.6}$$

has been widely studied on a smooth bounded open subset Ω of $\mathbb{R}^n$, $n \geq 2$, $s \in (0,1)$ and $p > 0$. Particularly, a priori bounds and existence of positive solutions for subcritical exponents ($p < \frac{n+2s}{n-2s}$) has been proved in [24, 35, 55, 177] and nonexistence results has also been proved in [24, 176, 177] for critical and supercritical exponents ($p \geq \frac{n+2s}{n-2s}$). The regularity result has been proved in [48, 177, 185].

When $s = 1/2$, Cabré and Tan [35] established the existence of positive solutions for equations having nonlinearities with the subcritical growth, their regularity, the symmetric property, and a priori estimates of the Gidas-Spruck type by employing a blow-up argument along with a Liouville type result for the square root of the Laplace operator in the half-space. Then [177] has the analogue to $1/2 < s < 1$. Brändle, Colorado, de Pablo, and Sánchez [24] dealt with a subcritical concave-convex problem. For $f(u) = u^q$ with the critical and supercritical exponents $q \geq \frac{n+2s}{n-2s}$, the nonexistence of solutions was proved in [9, 176, 177] in which the authors devised and used the Pohozaev type identities. The Brezis-Nirenberg type problem was studied in [176] for $s = 1/2$ and [9] for $0 < s < 1$. The Lemma's Hopf and Maximum Principe was studied in [177].

CAPÍTULO 2

Preliminaries

This chapter aims to present the main definitions and results of fractional Laplace operator, especially those relating to the following chapters. The theorems are presented without proofs. These can be found in the references cited in the sequencing of the results presented here.

2.1 The fractional Sobolev space $W^{s,p}$: embedding theorems

This section is devoted to the definition and preliminary notions of the fractional Sobolev spaces. We will see that under certain regularity assumptions on the domain Ω, any function in $W^{s,p}(\Omega)$ may be extended to a function in $W^{s,p}(\mathbb{R}^n)$. This section is also devoted to the Sobolev embedding involving $W^{s,p}$ in particular the compact embedding theorem.

No prerequisite is needed. We just recall the definition of the Fourier transform of a distribution. First, consider the Schwartz space $\mathcal{S}$ of rapidly decaying C^∞ functions in $\mathbb{R}^n$. The topology of this space is generated by the seminorms

$$p_n(\varphi) = \sup_{x\in\mathbb{R}^n} (1+|x|)^n \sum_{|\alpha|\leq n} |D^\alpha \varphi(x)|,\ n = 0, 1, 2, \dots,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Thus

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) \text{ such that } p_n(\varphi) < \infty\}.$$

Let $\mathcal{S}'(\mathbb{R}^n)$ be the set of all tempered distributions, that is the topological dual of $\mathcal{S}(\mathbb{R}^n)$. An important property is that for $1 \leq p \leq \infty$, $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. As usual, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we denote by

$$\mathcal{F}\varphi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \varphi(x) dx$$

the Fourier transform of φ and we recall that one can extend $\mathcal{F}$ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 2.1.1. *The space $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.*

Proof. See [70], page 180. ■

We start by fixing a parameter $0 < s < 1$. Let Ω be an open subset of $\mathbb{R}^n$, with $n \geq 1$. For any $p \in (1, +\infty)$, one defines the fractional Sobolev space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \mid \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}, \tag{2.1}$$

that is, an intermediary Banach space between $L^r(\Omega)$ and $W^{1,r}(\Omega)$ induced with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}, \tag{2.2}$$

where the term

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}$$

is the so-called Gagliardo semi-norm of u.

Proposition 2.1.2. *The space $W^{s,p}(\Omega)$ is of local type, that is, for every u in $W^{s,p}(\Omega)$ and for every $\phi \in C_0^\infty(\Omega)$, the product ϕu belongs to $W^{s,p}(\Omega)$.*

Proof. See [70], page 194. ■

Proposition 2.1.3. *The space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$.*

Proof. See [70], pages 195 to 197. ■

The following result concerning the existence of embeddings in $\mathbb{R}^n$.

Proposition 2.1.4. *The spaces $W^{s,p}(\mathbb{R}^n)$ satisfy the following embedding properties:*

(i) If $0 < s \leq s' < 1$, then $W^{s',p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n)$.

(ii) If $s \in (0,1)$, then $W^{1,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n)$. Moreover,

$$[u]_{W^{s,p}(\mathbb{R}^n)} \leq \left(\frac{2\omega_n}{ps(1-s)} \right)^{1/p} \|\nabla u\|^s_{L^p(\mathbb{R}^n)} \|u\|^{1-s}_{L^p(\mathbb{R}^n)},$$

where ω_n is the volume of unit ball in $\mathbb{R}^n$.

Proof. See [70], pages 202 and 203 and see also [72]. ■

It is worth noticing that, as in the classical case with s being an integer, the space $W^{s',p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$ when $s \leq s'$ and Ω be an open set in $\mathbb{R}^n$, as next result points out.

Proposition 2.1.5. *Let $p \in [1, +\infty)$ and $0 < s \leq s' < 1$. Let Ω be an open set in $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s',p}(\Omega)}$$

for some suitable positive constant $C = C(n, s, p) \geq 1$. In particular,

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Proof. See [73], pages 524 and 525. ■

Proposition 2.1.6. *Let $p \in [1, +\infty)$ and $s \in (0, 1)$. Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary and $u : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for some suitable positive constant $C = C(n, s, p) \geq 1$. In particular,

$$W^{1,p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Proof. See [73], page 526. ■

We say that Ω admits an (s, p)-extension if there exists a continuous linear operator T that sends $u \in W^{s,p}(\Omega)$ to $T(u) = \overline{u} \in W^{s,p}(\mathbb{R}^n)$, such that

$$\forall\ x \in \Omega,\ Tu(x) = u(x).$$

In the case of a class C^1 or Lipschitz open set, we have the following result.

Proposition 2.1.7. *Any Lipschitz open set Ω admits an (s, p)-extension.*

Proof. See [70], pages 206 to 209. ■

For an open subset Ω of $\mathbb{R}^n$ and for any nonnegative integer m, let $C_b^m(\Omega)$ be the subset of $C^m(\Omega)$ consisting of the functions whose partial derivatives of order $\leq m$ are bounded and uniformly continuous on Ω. By endowing this subspace with the norm

$$\|\varphi\|_{C_b^m(\Omega)} = \sup_{|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha \varphi(x)|,$$

we obtain a Banach space. Note that when Ω is a bounded open subset, any function on this space, as well as all its partial derivatives, admits a continuous extension to Ω. The space $C_b^m(\Omega)$ is therefore identical to $C^m(\overline{\Omega})$. Consider the following important subspace of $C_b^m(\Omega)$. For $0 < \lambda \leq 1$, $C_b^{0,\lambda}(\Omega)$ denotes the space of Hölder continuous functions of order λ on Ω, defined as follows:

$$C_b^{0,\lambda}(\Omega) = \{\varphi \in C_b(\Omega) \mid \exists\, C > 0, \forall\, (x,y) \in \Omega^2, |\varphi(x) - \varphi(y)| \leq C|x-y|^\lambda\}.$$

When $\lambda = 1$, these are called the Lipschitz continuous functions. More generally, we define $C_b^{m,\lambda}(\Omega)$ to be the subset of $C_b^m(\Omega)$ of functions φ such that

$$\exists\, C > 0, \forall\, \alpha, |\alpha| = m, \forall\, (x,y) \in \Omega^2, |D^\alpha\varphi(x) - D^\alpha\varphi(y)| \leq C|x-y|^\lambda.$$

Endowed with the norms

$$\|\varphi\|_{m,\lambda} = \|\varphi\|_{C_b^m(\Omega)} + \sup_{|\alpha|=m} \sup_{\{(x,y)\in\Omega^2 | x\neq y\}} \frac{|D^\alpha\varphi(x) - D^\alpha\varphi(y)|}{|x-y|^\lambda},$$

these are Banach spaces. Moreover, we have

$$\forall\, (\nu,\lambda),\ 0 < \nu < \lambda < 1 \Longrightarrow C_b^{m,\lambda}(\Omega) \hookrightarrow C_b^{m,\nu}(\Omega) \hookrightarrow C_b^m(\Omega),$$

where the inclusions are strict.

Theorem 2.1.1. *Let $s \in (0,1)$ and let $p \in (1,\infty)$. We have:*

(i) If $sp < n$, then $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for every $q \leq np/(n-sp)$.

(ii) If $n = sp$, then $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for every $q < \infty$.

(iii) If $sp > n$, then $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ and, more precisely,

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow C_b^{0,s-n/p}(\mathbb{R}^n).$$

Proof. See [70], pages 210 to 215. ∎

Proposition 2.1.8. *Let $s \in (0,1)$ and let $p > 1$. Let Ω be an open set that admits an (s,p)-extension; then $C_0^\infty(\overline{\Omega})$, the space of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^n)$, is dense in $W^{s,p}(\Omega)$.*

Proof. See [70], page 216. ■

The following is a corollary to Proposition 2.1.8 and Theorem 2.1.1.

Corollary 2.1.1. *Let $s \in (0,1)$ and let $p \in (1,\infty)$. Let Ω be a Lipschitz open set. We then have:*

(i) If $sp < n$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \leq np/(n-sp)$.

(ii) If $n = sp$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < \infty$.

(iii) If $sp > n$, then $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and, more precisely,

$$W^{s,p}(\Omega) \hookrightarrow C_b^{0,s-n/p}(\Omega).$$

Theorem 2.1.2. *Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$. Let $s \in (0,1)$, let $p \in (1,\infty)$ and let $n \geq 1$. We then have:*

(i) If $sp < n$, then the embedding of $W^{s,p}(\Omega)$ into $L^q(\Omega)$ is compact for every $q < np/(n-sp)$.

(ii) If $n = sp$, then the embedding of $W^{s,p}(\Omega)$ into $L^q(\Omega)$ is compact for every $q < \infty$.

(iii) If $sp > n$, then the embedding $W^{s,p}(\Omega)$ into $C_b^{0,\lambda}(\Omega)$ is compact for $\lambda < s - n/p$.

Proof. See [70], pages 217 and 218. ■

Before going ahead, it is worth explaining why the definition in (2.1) cannot be plainly extended to the case $s \geq 1$. Suppose that Ω is a connected open set in $\mathbb{R}^n$, then any measurable function $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy < \infty$$

is actually constant (see [[25], Proposition 2]). This fact is a matter of scaling and it is strictly related to the following result that holds for any $u \in W^{1,p}(\Omega)$:

$$\lim_{s \to 1^-} (1-s) \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy = C_1 \int_\Omega |\nabla u|^p dx$$

for a suitable positive constant C_1 depending only on n and p (see [23]).

In the same spirit, in [127], Maz'ya and Shaposhnikova proved that, for a function $u \in \bigcup_{0<s<1} W^{s,p}(\mathbb{R}^n)$, it yields

$$\lim_{s\to 0^+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy = C_2 \int_{\mathbb{R}^n} |u|^p dx$$

for a suitable positive constant C_2 depending only on n and p.

Let $s \in \mathbb{R} \setminus \mathbb{N}$ with $s \geq 1$. The space $W^{s,p}(\Omega)$ is defined to be

$$W^{s,p}(\Omega) = \{u \in W^{[s],p}(\Omega) \mid D^j u \in W^{s-[s],p}(\Omega), \forall j, |j| = [s]\},$$

where $[s]$ is the largest integer smaller than s, j denotes the n-uple $(j_1, \dots, j_n) \in \mathbb{N}^n$ and $|j|$ denotes the sum $j_1 + \dots + j_n$.

It is clear that $W^{s,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{W^{[s],p}(\Omega)} + [u]^p_{W^{s-[s],p}(\Omega)}\right)^{\frac{1}{p}} \tag{2.3}$$

is a reflexive Banach space, see [130].

Clearly, if $s = m$ is an integer, the space $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$.

Lemma 2.1.3. *For any $s > 0$ and $1 \leq p < \infty$, then $\mathcal{S}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n)$.*

Proof. See [130], pages 164 to 166. ■

Corollary 2.1.2. *Let $p \in [1, \infty)$ and $s, s' > 1$. Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$. Then, if $s \leq s'$, we have*

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Proof. See [73], page 527. ■

As in the classic case with s being an integer, any function in the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ can be approximated by a sequence of smooth functions with compact support.

Theorem 2.1.4. *For any $s > 0$, the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$.*

A proof can be found in [2], Theorem 7.38.

Let $W_0^{s,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,p}(\Omega)}$ defined in (2.3). Note that, in view of Theorem 2.1.4, we have

$$W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$$

but in general, for $\Omega \subset \mathbb{R}^n, W^{s,p}(\Omega) \neq W_0^{s,p}(\Omega)$, i.e, $C_0^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$. Furthermore, it is clear that the same inclusions stated above for $W^{s,p}(\Omega)$ hold for the spaces $W_0^{s,p}(\Omega)$.

Remark 2.1.5. For $s < 0$ and $p \in (1,\infty)$, we can define $W^{s,p}(\Omega)$ as the dual space of $W_0^{-s,p'}(\Omega)$ where $p' = p/(p-1)$. The norm $\|\cdot\|_{s,p,\Omega}^*$ in the dual space $W^{s,p}(\Omega)$ of $W_0^{-s,p'}(\Omega)$ is defined in the usual way by duality:

$$\|g\|_{s,p,\Omega}^* = \sup_{v \in W_0^{-s,p'}(\Omega)} \frac{|\langle g, v\rangle_{s,p,\Omega}|}{\|v\|_{W^{-s,p'}(\Omega)}},\ (v \neq 0).$$

Here $\langle \cdot,\cdot\rangle_{s,p,\Omega}$ denotes the canonical duality pairing on $W^{s,p}(\Omega) \times W_0^{-s,p'}(\Omega)$; i.e., if $g \in W^{s,p}(\Omega)$, then $g(v) = \langle g, v\rangle_{s,p,\Omega}$ for every $v \in W_0^{-s,p'}(\Omega)$. Notice that, in this case, the space $W^{s,p}(\Omega)$ is actually a space of distributions on Ω, since it is the dual of a space having $C_0^\infty(\Omega)$ as density subset.

Proposition 2.1.9. *Let Ω be a Lipschitz open set of $\mathbb{R}^n$. For any $s > 0$ and any $p \in [1,\infty)$, the following occurs:*

(i) If $p < \infty$, then $W^{s,p}(\Omega)$ is separable.

(ii) If $1 < p < \infty$, then $W^{s,p}(\Omega)$ is uniformly convex (hence reflexive).

The following embedding theorem is similar to the previous ones.

Theorem 2.1.6. *Let Ω be a Lipschitz open set of $\mathbb{R}^n$. We then have:*

(i) If $sp < n$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \leq np/(n-sp)$.

(ii) If $n = sp$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < \infty$.

(iii) If $sp > n$, then we have:

- *If $s - n/p \notin \mathbb{N}$, then $W^{s,p}(\Omega) \hookrightarrow C_b^{[s-n/p],s-n/p-[s-n/p]}(\Omega)$.*
- *If $s - n/p \in \mathbb{N}$, then $W^{s,p}(\Omega) \hookrightarrow C_b^{s-n/p-1,\lambda}(\Omega)$ for every $\lambda < 1$.*

Proof. See [70], pages 219 and 220. ∎

Theorem 2.1.7. *Let $\Omega \subseteq \mathbb{R}^n$. Then we have*

(i) $\quad W^{s,p}(\Omega) \hookrightarrow C_b^\mu(\Omega)$, *if $s - n/p > \mu$.*

$\quad W^{s,p}(\Omega) \hookrightarrow C_b^\mu(\Omega)$, *if $s - n/p = \mu \neq$ nonnegative integer.*

(ii) if $q \neq \infty$, then

$$W^{s_1,p}(\Omega) \hookrightarrow W^{s_2,q}(\Omega) \Longleftrightarrow q \geq p, s_1 - \frac{n}{p} \geq s_2 - \frac{n}{q}.$$

In particular case, $p = 1, s_1 = n, s_2 = 0, q = \infty$, we have

$$W^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega).$$

Proof. See [145]. ■

For a bounded open set we also have results concerning compact injections.

Theorem 2.1.8. *Let Ω be a bounded Lipschitz open set of $\mathbb{R}^n$. We then have:*

(i) If $sp < n$, then the embedding $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for all exponents q satisfying $q < np/(n - sp)$.

(ii) If $n = sp$, then the embedding $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $q < \infty$.

(iii) If $sp > n$, then we have:

- *If $s - n/p \notin \mathbb{N}$, then the embedding $W^{s,p}(\Omega) \hookrightarrow C_b^{[s-n/p],\lambda}(\Omega)$ is compact for every $\lambda < s - n/p - [s - n/p]$.*
- *If $s - n/p \in \mathbb{N}$, then the embedding $W^{s,p}(\Omega) \hookrightarrow C_b^{s-n/p-1,\lambda}(\Omega)$ is compact for every $\lambda < 1$.*

Proof. See [70]. ■

Theorem 2.1.9. *(Rellich-Kondrachov Compactness Embedding Theorem) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Then the following compactness embeddings hold:*

(i) $W^{s,p}(\Omega) \hookrightarrow\hookrightarrow C_b^\mu(\Omega) \Longleftrightarrow s - n/p > \mu$.

(ii) $W^{s_1,p}(\Omega) \hookrightarrow\hookrightarrow W^{s_2,q}(\Omega) \Longleftrightarrow q \geq p, s_1 - \frac{n}{p} > s_2 - \frac{n}{q}$.

Proof. See [145]. ■

Corollary 2.1.3. *(Gagliardo-Nirenberg type inequalities). We have*

(i) for $0 \leq s_1 < s_2 < \infty, 1 < p_1 < \infty, 1 < p_2 < \infty$,

$$s = \theta s_1 + (1-\theta)s_2, \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2},$$

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|^{\theta}_{W^{s_1,p_1}(\mathbb{R}^n)} + \|u\|^{1-\theta}_{W^{s_2,p_2}(\mathbb{R}^n)}.$$

(ii) for $0 < s < \infty, 1 < p < \infty, 0 < \theta < 1$,

$$\|u\|_{W^{s\theta,p/\theta}(\mathbb{R}^n)} \leq C\|u\|^{\theta}_{W^{s,p}(\mathbb{R}^n)} + \|u\|^{1-\theta}_{L^\infty(\mathbb{R}^n)}.$$

Proof. See [26]. ■

For any $s \in (0,1)$ and any $p \in [1,\infty)$, we say that an open set $\Omega \subseteq \mathbb{R}^n$ is an extension domain for $W^{s,p}$ if there exists a positive constant $C = C(n,p,s,\Omega)$ such that: for every function $u \in W^{s,p}(\Omega)$ there exists $\overline{u} \in W^{s,p}(\mathbb{R}^n)$ with $\overline{u}(x) = u(x)$ for all $x \in \Omega$ and $\|\overline{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)}$.

Now, we present a important theorem, that states that every open Lipschitz set Ω with bounded boundary is an extension domain for $W^{s,p}$.

Theorem 2.1.10. *Let* $p \in [1,\infty), s \in (0,1)$ *and* $\Omega \subseteq \mathbb{R}^n$ *be an open set of class* $C^{0,1}$ *with bounded boundary. Then* $W^{s,p}(\Omega)$ *is continuously embedded in* $W^{s,p}(\mathbb{R}^n)$*, namely for any* $u \in W^{s,p}(\Omega)$ *there exists* $\overline{u} \in W^{s,p}(\mathbb{R}^n)$ *such that* $\overline{u}_{|\Omega} = u$ *and*

$$\|\overline{u}\|_{W^{s,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\Omega)},$$

where $C = C(n,p,s,\Omega)$.

Proof. See [73], pages 548 and 549. ■

For every $s > 0$, we denote by $W^{s,p}(\overline{\Omega})$ the space of all distributions in Ω which are restrictions of elements of $W^{s,p}(\mathbb{R}^n)$ and by $\widehat{W}^{s,p}(\Omega)$ the space of functions $u \in W^{s,p}(\overline{\Omega})$ such that the extension $\overline{u}$ by zero outside of belongs to $W^{s,p}(\mathbb{R}^n)$. Recall now some density results ([2, 104]):

(i) The space $C_0^\infty(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$ for any real s.

(ii) The space $C_0^\infty(\Omega)$ is dense in $\widehat{W}^{s,p}(\Omega)$ for all $s > 0$.

(iii) The space $C_0^\infty(\Omega)$ is dense in $W^{s,p}(\Omega)$ for all $0 < s < \frac{1}{p}$, that means that $W^{s,p}(\Omega) = W_0^{s,p}(\Omega)$.

Theorem 2.1.11. *(Traces of functions living in* $W^{s,p}(\Omega)$*) ([2, 104]) Let* Ω *be a bounded open set of class* $C^{k,1}$*, for some integer* $k \geq 0$*. Let* s *be real number such that* $s \leq k+1, s - \frac{1}{p} = m + \sigma$*, where* $m \geq 0$ *is an integer and* $0 < \sigma < 1$.

(i) The following mapping $\gamma_0 : W^{s,p}(\Omega) \to W^{s-\frac{1}{p},p}(\partial\Omega)$ *given by* $\gamma_0(u) = u_{|\partial\Omega}$ *is continuous and surjective. When* $\frac{1}{p} < s < 1 + \frac{1}{p}$*, we have* $Ker(\gamma_0) = W_0^{s,p}(\Omega)$.

(ii) For $m \geq 1$, the following mapping

$$(\gamma_0, \gamma_1) : W^{s,p}(\Omega) \to (W^{s-\frac{1}{p},p}(\partial\Omega) \times W^{s-1-\frac{1}{p},p}(\partial\Omega))$$

given by $(\gamma_0, \gamma_1)(u) = (u_{|\partial\Omega}, \frac{\partial u}{\partial \nu}_{|\partial\Omega})$ is continuous and surjective, where ν is the outward-oriented unit normal vector field on $\partial\Omega$. When $1 + \frac{1}{p} < s < 2 + \frac{1}{p}$, we have $Ker(\gamma_0, \gamma_1) = W_0^{s,p}(\Omega)$.

Theorem 2.1.12. *(Traces) Let $1 \leq p \leq \infty, s > \frac{1}{p}$, and Ω be a bounded domain of $\mathbb{R}^n$ of class $C^{0,1}$. There exists a (unique) linear and continuous trace operator $\gamma_0 : W^{s,p}(\Omega) \to L^p(\partial\Omega)$ such that $\gamma_0(v) = v$ on $\partial\Omega$ for any $v \in C_0^\infty(\overline{\Omega})$.*

Theorem 2.1.13. *(Normal Traces) Let $1 \leq p \leq \infty, s > 1+\frac{1}{p}$, and Ω be a bounded domain of $\mathbb{R}^n$ of class $C^{1,1}$, and ν be the outward-oriented unit normal vector field on $\partial\Omega$. There exists a (unique) linear and continuous normal trace operator $\gamma_1 : W^{s,p}(\Omega) \to L^p(\partial\Omega)$ such that $\gamma_1(v) = \frac{\partial v}{\partial \nu}$ on $\partial\Omega$ for any $v \in C_0^\infty(\overline{\Omega})$.*

Notice that the Theorem 2.1.10 implies that the quotient norm:

$$\|u\|_{W^{s,p}(\Omega),e} = \inf_{U_{|\Omega}=u} \|U\|_{W^{s,p}(\mathbb{R}^n)} \tag{2.4}$$

is equivalent to $\|u\|_{W^{s,p}(\Omega)}$, since it is always true that $\|u\|_{W^{s,p}(\Omega)} \leq \|\overline{u}\|_{W^{s,p}(\mathbb{R}^n)}$. Therefore we one can now deduce a characterization of $W^{s,p}(\Omega)$ in the case of a bounded open set of $\mathbb{R}^n$ with a Lipschitz boundary, with $0 < s < 1$ and $1 \leq p < \infty$.

Lemma 2.1.14. *If Ω is a bounded open set of $\mathbb{R}^n$ with a Lipschitz boundary, then for $0 < s < 1$ and $1 \leq p < \infty$ we have*

$$W^{s,p}(\Omega) = \{f_{|\Omega} \mid f \in W^{s,p}(\mathbb{R}^n)\}$$

with the quotient norm $\|u\|_{W^{s,p}(\Omega),e}$ defined in (2.4).

Proof. See [178], pages 169 and 170. ■

Let us call $r_\Omega : W^{s,p}(\mathbb{R}^n) \to W^{s,p}(\Omega)$ the restriction. By the above discussion, it is clear that r_Ω is surjective. Therefore,

$$Ker(r_\Omega) = \{f \in W^{s,p}(\mathbb{R}^n) \mid r_\Omega(f) = f_{|\Omega} = 0\}$$

thus r_Ω gives the isomorphism

$$W^{s,p}(\Omega) = \frac{W^{s,p}(\mathbb{R}^n)}{Ker(r_\Omega)}.$$

The characterizations of $W^{s,p}(\Omega)$ and $W^{s,p}(\mathbb{R}^n)$ provide a characterization of which functions $u \in W^{s,p}(\Omega)$ are such that the extension of u by 0 outside Ω, denoted by $\overline{u}$, belong to $W^{s,p}(\mathbb{R}^n)$.

Lemma 2.1.15. *Let Ω be a bounded open set of $\mathbb{R}^n$ with a Lipschitz boundary. Then, for $0 < s < 1$ and $1 \leq p < \infty$ one has $\overline{u} \in W^{s,p}(\mathbb{R}^n)$ if and only if $u \in W^{s,p}(\Omega)$ and $d^{-s}u \in L^p(\Omega)$, where $d(x)$ denotes the distance from x to the boundary $\partial\Omega$.*

Proof. See [178], page 173. ■

For Ω is a bounded Lipschitz open set and $0 < s < 1$, we have

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^n) \mid u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

For more details on the above claims, see [178].

The Fractional Hardy inequalities is given by

Theorem 2.1.16. *Let $n \geq 1, s > 1$ and $1 \leq p < \infty$. If Ω is a bounded Lipschitz domain, then there exists $C(n,p,s,\Omega) > 0$ so that*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+s}} dxdy \geq C(n,p,s,\Omega) \int_\Omega \frac{|f(x)|^p}{d(x)^s} dx$$

for all $f \in W_0^{s/p,p}(\Omega)$.

Proof. See [77]. ■

2.2 The fractional Laplace operator

In this section, we focus on the case $p = 2$. This is quite an important case since the fractional Sobolev spaces $W^{s,2}(\mathbb{R}^n)$ and $W_0^{s,2}(\mathbb{R}^n)$ turn out to be Hilbert spaces. They are usually denoted by $H^s(\mathbb{R}^n)$ and $H_0^s(\mathbb{R}^n)$, respectively. In fact, we can define an inner product

$$\langle u, v\rangle = \langle u, v\rangle_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dxdy,$$

which the reader can easily check satisfies all of the properties of an inner product. Moreover, they are strictly related to the fractional Laplace operator $(-\Delta)^s$ (see Proposition 2.2.8), where, for any $u \in \mathcal{S}$ and $s \in (0,1)$, $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = C(n,s)\, P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy \tag{2.5}$$

for all $x \in \mathbb{R}^n$, where P.V. denotes the principal value of the integral and

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}}\, d\zeta \right)^{-1} \tag{2.6}$$

with $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$. The choice of this constant is motived by Proposition 2.2.6 (see [73]).

Remark 2.2.1. Due to the singularity of the Kernel, the right hand-side of (2.5) is not well defined in general. In the case $s \in (0, 1/2)$ the integral in (2.5) is not really singular near x. Let $u \in \mathcal{S}$ then every $D^\alpha u$ is bounded so u is Lipschitz, we have

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}} dy &= \int\limits_{B_R} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}} dy + \int\limits_{\mathbb{R}^n \setminus B_R} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}} dy \\
&\leq C \int\limits_{B_R} \frac{|x-y|}{|x-y|^{n+2s}} dy + \|u\|_{L^\infty(\mathbb{R}^n)} \int\limits_{\mathbb{R}^n \setminus B_R} \frac{1}{|x-y|^{n+2s}} dy \\
&\leq C \left(\int\limits_{B_R} \frac{1}{|x-y|^{n+2s-1}} dy + \int\limits_{\mathbb{R}^n \setminus B_R} \frac{1}{|x-y|^{n+2s}} dy \right),
\end{aligned}$$

using polar coordinates $\rho z = y - x$ with $|y - x| = \rho$,

$$\int\limits_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}} dy \leq C \left(\int\limits_{B_R} \frac{1}{\rho^{2s}} dy + \int\limits_{\mathbb{R}^n \setminus B_R} \frac{1}{|x-y|^{n+2s}} dy \right)$$

up to relabeling the positive constant C that depends only on n and the norm $L^\infty(\mathbb{R}^n)$ of u at some steps.

We have

$$\frac{C(n,s)}{2} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dx dy = \int\limits_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} v dx$$

forall $u, v \in H^s(\mathbb{R}^n)$. We also have $(-\Delta)^s$ is a bounded linear operator from $W^{2s,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. The operators $(-\Delta)^{-s}$, are inverse the former ones and are now given by standard convolution expressions

$$(-\Delta)^{-s} f(x) = \frac{1}{\gamma(s)} \int\limits_{\mathbb{R}^n} \frac{f(z)}{|x-z|^{n-2s}} dz, \; 0 < 2s < n \tag{2.7}$$

in terms of Riesz potentials, where

$$\gamma(s) = \frac{\pi^{n/2} 2^{2s} \Gamma(s)}{\Gamma\left(\frac{n-2s}{2}\right)}.$$

We have $\frac{1}{\gamma(s)} = C(n, -s)$. The basic reference for these operators are the books by Landkof [118] and Stein [173]. From (2.7), we see that

$$F(x) = \frac{C(n,-s)}{|x|^{n-2s}}$$

is the fundamental solution of $(-\Delta)^s$, i.e., $(-\Delta)^s F = \delta_0$ when $n > 2s$ (see [41]). This function is generally known as the Riesz kernel.

Proposition 2.2.1. *The space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.*

Proof. See [70], page 184. ■

Hölder's inequality can be used to prove the following

Theorem 2.2.2. *(Interpolation inequality). Let $0 < r < t < 1$, and $s = \alpha r + (1-\alpha)t$ for some $\alpha \in (0,1)$. Then*

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C\|u\|^{\alpha}_{H^r(\mathbb{R}^n)}\|u\|^{1-\alpha}_{H^t(\mathbb{R}^n)}.$$

The following Lemma show that we may write the singular integral in (2.5) as a weighted second order differential quotient without the P.V.

Lemma 2.2.3. *Let $s \in (0,1)$ and let $(-\Delta)^s$ be the fractional Laplace operator defined by (2.5). Then, for any $u \in \mathcal{S}$,*

$$(-\Delta)^s u(x) = -\frac{1}{2}C(n,s)\int\limits_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy$$

for all $x \in \mathbb{R}^n$.

Proof. See [73], pages 529 and 530. ■

The following result shows that $(-\Delta)^s$ interpolates the Laplace operator in $\mathbb{R}^n$.

Proposition 2.2.2. *Let $n > 1$. For any $u \in C_0^\infty(\mathbb{R}^n)$ the following statements hold pointwise in $\mathbb{R}^n$:*

(i) $\lim\limits_{s\to 0^+} (-\Delta)^s u = u.$

(ii) $\lim\limits_{s\to 1^-} (-\Delta)^s u = -\Delta u.$

Proof. See [73], pages 543 to 545. ■

2.2.1 Relation and difference between $(-\Delta)^s$ and $\mathcal{A}^s$

• The spectral fractional Laplace operator: For Ω be a smooth bounded open subset of $\mathbb{R}^n$. The spectral fractional Laplace operator $\mathcal{A}^s$ is defined as follows. Let φ_k be an eigenfunction of $-\Delta$ given by

$$\begin{cases} -\Delta\varphi_k &= \lambda_k\varphi_k \quad \text{in } \Omega \\ \quad\varphi_k &= 0 \qquad\; \text{on } \partial\Omega \end{cases}, \tag{2.8}$$

where λ_k is the corresponding eigenvalue of $\varphi_k, 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \to +\infty$. Then, $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basic of $L^2(\Omega)$ satisfying

$$\int_{\Omega} \varphi_j \varphi_k dx = \delta_{j,k}.$$

We define the operator $\mathcal{A}^s$ for any $u \in C_0^{\infty}(\Omega)$ by

$$\mathcal{A}^s u = \sum_{k=1}^{\infty} \lambda_k^s \xi_k \varphi_k, \tag{2.9}$$

where

$$u = \sum_{k=1}^{\infty} \xi_k \varphi_k \text{ and } \xi_k = \int_{\Omega} u \varphi_k dx.$$

• The restricted fractional Laplace operator: In this case we materialize the zero Dirichlet condition by restricting the operator to act only on functions that are zero outside Ω. We will call the operator defined in such a way the restricted fractional Laplace operator. So defined, $(-\Delta)^s$ is a self-adjoint operator on $L^2(\Omega)$, with a discrete spectrum: we will denote by $\mu_k > 0,\ k = 1, 2, \ldots$ its eigenvalues written in increasing order and repeated according to their multiplicity and we will denote by $\{\psi_k\}_k$ the corresponding set of eigenfunctions, normalized in $L^2(\Omega)$, where $\psi_k \in H_0^{2s}(\Omega)$. Eigenvalues μ_k (including multiplicities) satisfy

$$0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \to +\infty.$$

The spectral fractional Laplace operator $\mathcal{A}^s$ is related to (but different from) the restricted fractional Laplace operator $(-\Delta)^s$.

Theorem 2.2.4. *The operators $(-\Delta)^s$ and $\mathcal{A}^s$ are not the same, since they have different eigenvalues and eigenfunctions. More precisely:*

(i) the first eigenvalues of $(-\Delta)^s$ is strictly less than the one of $\mathcal{A}^s$.

(ii) the eigenfunctions of $(-\Delta)^s$ are only Hölder continuous up to the boundary, differently from the ones of $\mathcal{A}^s$ that are as smooth up the boundary as the boundary allows.

Proof. See [163]. ■

• Common notation. In the sequel we use $\mathcal{L}$ to refer to any of the two types of operators $\mathcal{A}^s$ or $(-\Delta)^s$, $0 < s < 1$. Each one is defined on a Hilbert space

$$\Theta^s(\Omega) = \{u = \sum_{k=1}^{\infty} u_k \psi_k \in L^2(\Omega) \mid \sum_{k=1}^{\infty} \mu_k |u_k|^2 < +\infty\}$$

with values in its dual $\Theta^s(\Omega)'$. The Spectral Theorem allows to write $\mathcal{L}$ as

$$\mathcal{L}u = \sum_{k=1}^{\infty} \mu_k u_k \psi_k$$

for any $u \in \Theta^s(\Omega)$. Thus the inner product of $\Theta^s(\Omega)$ is given by

$$\langle u, v\rangle_{\Theta^s(\Omega)} = \int_\Omega \mathcal{L}^{1/2}u\mathcal{L}^{1/2}v dx = \int_\Omega u\mathcal{L}v dx = \int_\Omega v\mathcal{L}u dx.$$

We denote by $\|\cdot\|_{\Theta^s(\Omega)}$ the norm derived from this inner product. The notation in the formula copies the one just used for the second operator. When applied to the first one we put here $\psi_k = \varphi_k$, and $\mu_k = \lambda_k^s$. Note that $\Theta^s(\Omega)$ depends in principle on the type of operator and on the exponent s. It turns out that $\Theta^s(\Omega)$ independent of operator for each s, see [22]. We remark that $\Theta^s(\Omega)'$ can be described as the completion of the finite sums of the form

$$f = \sum_{k=1}^{\infty} c_k \psi_k$$

with respect to the dual norm

$$\|f\|_{\Theta^s(\Omega)'} = \sum_{k=1}^{\infty} \mu_k^{-1}|c_k|^2 = \|\mathcal{L}^{-1/2}f\|^2_{L^2(\Omega)} = \int_\Omega f\mathcal{L}^{-1}f dx$$

and it is a space of distributions. Moreover, the operator $\mathcal{L}$ is an isomorphism between $\Theta^s(\Omega)$ and $\Theta^s(\Omega)' \simeq \Theta^s(\Omega)$, given by its action on the eigenfunctions. If $u, v \in \Theta^s(\Omega)$ and $f = \mathcal{L}u$ we have, after this isomorphism,

$$\langle f, v\rangle_{\Theta^s(\Omega)'\times\Theta^s(\Omega)} = \langle u, v\rangle_{\Theta^s(\Omega)\times\Theta^s(\Omega)} = \sum_{k=1}^{\infty} \mu_k u_k v_k.$$

If it also happens that $f \in L^2(\Omega)$, then clearly we get

$$\langle f, v\rangle_{\Theta^s(\Omega)'\times\Theta^s(\Omega)} = \int_\Omega f v dx.$$

We have $\mathcal{L}^{-1} : \Theta^s(\Omega)' \to \Theta^s(\Omega)$ can be written as

$$\mathcal{L}^{-1}f(x) = \int_\Omega G_\Omega(x, y)f(y)dy,$$

where G_Ω is the Green function of operator $\mathcal{L}$ (see [21, 111]). It is known that

$$\Theta^s(\Omega) = \begin{cases} L^2(\Omega) & \text{if } \ s = 0 \\ H^s(\Omega) = H_0^s(\Omega) & \text{if } \ s \in (0, \frac{1}{2}) \\ H_{00}^{\frac{1}{2}}(\Omega) & \text{if } \ s = \frac{1}{2} \\ H_0^s(\Omega) & \text{if } \ s \in (\frac{1}{2}, 1] \\ H^s(\Omega) \cap H_0^1(\Omega) & \text{if } \ s \in (1, 2] \end{cases}, \tag{2.10}$$

where $H_{00}^{\frac{1}{2}}(\Omega) := \{u \in H^{1/2}(\Omega) \mid \int_\Omega \frac{u^2(x)}{d(x)}dx < +\infty\}$.

The next theorem gives a relation between the spectral fractional Laplace operator $\mathcal{A}^s$ and the restricted fractional Laplace operator $(-\Delta)^s$.

Theorem 2.2.5. *For $u \in H^s(\mathbb{R}^n)$, $u \geq 0$ and supp$(u) \subset \overline{\Omega}$, the following relation holds in the sense of distributions:*

$$\mathcal{A}^s u \geq (-\Delta)^s u.$$

If $u \neq 0$ then this inequality holds with strict sign.

Proof. See [137]. ■

By weak solutions, we mean the following: Let $f \in L^{\frac{2n}{n+2s}}(\Omega)$. Given the problem

$$\begin{cases} \mathcal{A}^s u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.11}$$

we say that a function $u \in \Theta^s(\Omega)$ is a weak solution of (2.11) provided

$$\int_\Omega \mathcal{A}^{s/2}u\mathcal{A}^{s/2}\phi dx = \int_\Omega f(x)\phi(x)dx$$

for all $\phi \in \Theta^s(\Omega)$. Given the problem

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{2.12}$$

we say that a function u is a weak solution of (2.12) if $u \in H_0^s(\Omega)$, and

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2}u(-\Delta)^{s/2}\varphi dx = \int_\Omega f\varphi dx \tag{2.13}$$

for all $\varphi \in H_0^s(\Omega)$.

The most notable difference these operators is the constraints on the surface $\partial\Omega$ for the spectral fractional Laplace operator against the volume-constraint for the restricted fractional Laplace operator one in $\mathbb{R}^n \setminus \Omega$. In fact, the restriction from the space $H^s(\Omega)$ onto $\partial\Omega$ is not defined in the fractional Sobolev space $H^s(\Omega)$ when $s \in [0, 1/2]$. Since the restriction to $\partial\Omega$ is not defined for functions that are not smooth enough. The importance such case $s \in [0, 1/2]$ is that if we want to have solution with jump discontinuity, we should deal with spaces such as H^s for $s \in [0, 1/2]$ (see [183] and [[119], Volume Constraint 1.1]). In the stochastic sense, a path of a sample for a symmetric process with jump is not continuous, it can jump at an exterior point of the bounded domain. The set of those points is the volume-constraint, it is the exterior domain (see [183] and [[119], Volume Constraint 3.4]).

Recently, it was shown in [41] that the fractional Laplace operator in the whole space (see (2.5)) can be realized in a local way by using one more variable and the so-called s-harmonic extension.

More precisely, if u is a regular function in $\mathbb{R}^n$, we say that $w = E_s(u)$ is its s-harmonic extension to the upper half-space, $\mathbb{R}^{n+1}_+$, if w is a solution to the problem

$$\begin{cases} -div(y^{1-2s}\nabla w) &= 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+ \\ \quad w &= u \quad \text{on} \quad \mathbb{R}^n \times \{y = 0\} \end{cases}. \tag{2.14}$$

In [41] it is proved that

$$\frac{1}{k_s} \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y) = -(-\Delta)^s u(x),$$

where $k_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$.

The appropriate functional spaces to work with are

$$H^s_{0,L}(\mathbb{R}^{n+1}_+) \text{ and } H^s(\mathbb{R}^n),$$

where $H^s_{0,L}(\mathbb{R}^{n+1}_+)$ defined as the completion of $C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$, under the norm

$$\|w\|^2_{H^s_{0,L}(\mathbb{R}^{n+1}_+)} = \int_{\mathbb{R}^{n+1}_+} y^{1-s} |\nabla w|^2.$$

The s-harmonic extension for restricted fractional Laplace operator is the following: given $u \in \Theta^s(\Omega)$, we solve

$$\begin{cases} -div(y^{1-2s}\nabla w) &= 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+ \\ \quad w &= 0 \quad \text{on} \quad \mathbb{R}^n \setminus \Omega \times (0, \infty) \subset \mathbb{R}^{n+1}_+ \\ \quad w &= u \quad \text{on} \quad \Omega \end{cases} \tag{2.15}$$

for $w \in \mathcal{H}^s(\mathbb{R}^{n+1}_+)$ defined as the completion of $C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$, under the norm

$$\|w\|^2_{H^s_{0,L}(\mathbb{R}^{n+1}_+)} = \int_{\mathbb{R}^{n+1}_+} y^{1-s} |\nabla w|^2,$$

where w vanishes outside of $\Omega \times (0, \infty)$. Then,

$$\frac{1}{k_s} \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y) = -(-\Delta)^s u(x).$$

The s-harmonic extension for spectral fractional Laplace operator is the following: we consider the cylinder

$$C := \{(x, y) \mid x \in \Omega, y \in \mathbb{R}_+\} \subset \mathbb{R}^{n+1}_+$$

and denote by $\partial_L C$ its lateral boundary, i.e, $\partial_L C := \partial\Omega \times (0,\infty)$.

We first define the extension operator for smooth functions. Given a smooth function u, we define its s-harmonic extension $w = E_s(u)$ to the cylinder C as the solution to the problem

$$\left\{\begin{array}{rcl} -div(y^{1-2s}\nabla w) &=& 0 \ \text{ in } \ C \\ w &=& 0 \ \text{ on } \ \partial_L C \\ w &=& u \ \text{ on } \ \Omega \end{array}\right. . \tag{2.16}$$

We define function space

$$H^s_{0,L}(C) = \left\{ v \mid v = 0 \text{ on } \partial_L C, \|v\|_{H^s_{0,L}(C)} = \left(\int_C y^{1-s} |\nabla v|^2 \right)^{1/2} < \infty \right\}.$$

Then

$$\Theta^s(\Omega) = \{ u = tr_\Omega v \mid v \in H^{2s}_{0,L}(C) \},$$

see [48].

2.2.2 Estimates for the fractional Laplace operator

An important tool in PDEs is the classical L^p to $W^{2,p}$ estimate for the Laplace equation. Namely, if u is the solution of

$$\left\{\begin{array}{rcl} -\Delta u &=& f \ \text{ in } \ \Omega \\ u &=& 0 \ \text{ on } \ \partial\Omega \end{array}\right. \tag{2.17}$$

with $f \in L^p(\Omega)$, $1 < p < \infty$, then

$$\|u\|_{W^{2,p}(\Omega)} \le C \|f\|_{L^p(\Omega)}.$$

This estimate and the Sobolev embeddings lead to $L^q(\Omega)$ or $C^\alpha(\overline{\Omega})$ estimates for the solution u, depending on whether $1 < p < \frac{n}{2}$ or $p > \frac{n}{2}$, respectively.

A very important tool used in this thesis is the following:

Proposition 2.2.3. *Let Ω be a bounded $C^{1,1}$ open subset of $\mathbb{R}^n$, $s \in (0,1), n > 2s$ and u be the solution of*

$$\left\{\begin{array}{rcl} (-\Delta)^s u &=& f \ \text{ in } \ \Omega \\ u &=& 0 \ \text{ in } \ \mathbb{R}^n \setminus \Omega \end{array}\right. . \tag{2.18}$$

(i) If $f \in L^1(\Omega)$, then for each $1 \le r < \frac{n}{n-2s}$ there exists a constant C, depending only on n, s, r and $|\Omega|$, such that

$$\|u\|_{L^r(\Omega)} \le C \|f\|_{L^1(\Omega)}.$$

(ii) *Let* $1 < p < \frac{n}{2s}$. *If* $f \in L^p(\Omega)$, *then there exists a constant* C, *depending only on* n, s *and* p, *such that*

$$\|u\|_{L^q(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \ \textit{where } q = \frac{np}{n-2ps}.$$

(iii) *Let* $\frac{n}{2s} < p < +\infty$. *If* $f \in L^p(\Omega)$, *then there exists a constant* C, *depending only on* n, s, p *and* Ω, *such that*

$$\|u\|_{C^\beta(\mathbb{R}^n)} \leq C\|f\|_{L^p(\Omega)}, \ \textit{where } \beta = \min\left\{s, 2s - \frac{n}{p}\right\}.$$

Proof. See [154]. ■

Proposition 2.2.3 follows from Theorem 2.2.6 and Proposition 2.2.4 below. The first one contains some classical results concerning embeddings for the Riesz potential, and reads as follows.

Theorem 2.2.6. *(see [173]) Let* $s \in (0,1), n > 2s$ *and* u *and* f *be such that*

$$u = (-\Delta)^{-s} f \ \textit{ in } \mathbb{R}^n$$

in the sense that u *is the Riesz potential of order* $2s$ *of* f. *Assume that* u *and* f *belong to* $L^p(\mathbb{R}^n)$, *with* $1 \leq p < \infty$.

(i) *If* $p = 1$, *then there exists a constant* C, *depending only on* n *and* s, *such that*

$$\|u\|_{L^q_{weak}(\mathbb{R}^n)} \leq C\|f\|_{L^1(\mathbb{R}^n)}, \ \textit{where } q = \frac{n}{n-2s}.$$

(ii) *If* $1 < p < \frac{n}{2s}$, *then there exists a constant* C, *depending only on* n, *and* s, *such that*

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \ \textit{where } q = \frac{np}{n-2ps}. \tag{2.19}$$

(iii) *If* $\frac{n}{2s} < p < +\infty$, *then there exists a constant* C, *depending only on* n, s *and* p *such that*

$$[u]_{C^\beta(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \ \textit{where } \beta = 2s - \frac{n}{p},$$

where $[\cdot]_{C^\beta(\mathbb{R}^n)}$ *denotes the* C^β *seminorm.*

Parts (i) and (ii) of Theorem 2.2.6 are proved in the book of Stein [[173], Chapter V]. Part (iii) is also a classical result, but it seems to be more difficult to find an exact reference for it.

Proposition 2.2.4. *Let Ω be a bounded $C^{1,1}$ open subset of $\mathbb{R}^n$, $s \in (0,1), g \in C^\alpha(\mathbb{R}^n \setminus \Omega)$ for some $\alpha > 0$ and u be the solution of*

$$\begin{cases} (-\Delta)^s u &= 0 \quad \text{in} \quad \Omega \\ \quad u &= g \quad \text{in} \quad \mathbb{R}^n \setminus \Omega \end{cases}. \tag{2.20}$$

Then, $u \in C^\beta(\mathbb{R}^n)$, with $\beta = \min\{s, \alpha\}$, and

$$\|u\|_{C^\beta(\mathbb{R}^n)} \leq C\|g\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)},$$

where C is a constant depending only on Ω, α and s.

Proof. See [154]. ■

We recall that a domain satisfies the exterior ball condition if there exists a positive radius ρ_0 such that all the points on $\partial\Omega$ can be touched by some exterior ball of radius ρ_0.

Proposition 2.2.5. *Let Ω be a bounded Lipschitz domain satisfying the exterior ball condition, $f \in L^\infty(\Omega)$, and u be a weak solution of (2.18). Then, $u \in C^s(\mathbb{R}^n)$ and*

$$\|u\|_{C^s(\mathbb{R}^n)} \leq C\|f\|_{L^\infty(\Omega)},$$

where C is a constant depending only on Ω and s.

Proof. See [153]. ■

Theorem 2.2.7. *Let Ω be a bounded $C^{1,1}$ domain, $f \in L^\infty(\Omega)$, u be a weak solution of (2.18) and $d(x) = d(x, \partial\Omega)$. Then, $u/d^s \mid_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, we have $u/d^s \in C^\alpha(\overline{\Omega})$ and*

$$\|u/d^s\|_{C^\alpha(\overline{\Omega})} \leq C\|f\|_{L^\infty(\Omega)}$$

for some $\alpha > 0$ satisfying $\alpha < \min\{s, 1-s\}$. The constants α and C depend only on Ω and s.

Proof. See [153]. ■

Note that the boundary term $u/d^s \mid_{\partial\Omega}$ has to be understood in the limit sense - note that one of the statements of the theorem is that u/d^s is continuous up to the boundary.

From inequality (2.19) we have the following fractional Gagliardo-Nirenberg inequalities.

Corollary 2.2.1. *Let* $1 \leq p, p_2 < \infty, 0 < \alpha < p < \infty, 0 < 2s < n$ *and* $1 < p_1 < n/2s$. *We have*

$$\|u\|_{L^p(\mathbb{R}^n)} \leq B^{\alpha/p} \|(-\Delta)^s u\|^{\alpha/p}_{L^{p_1}(\mathbb{R}^n)} \|u\|^{\frac{p-\alpha}{p}}_{L^{p_2}(\mathbb{R}^n)}$$

with

$$\alpha \left(\frac{1}{p_1} - \frac{2s}{n} \right) + \frac{p-\alpha}{p_2} = 1$$

and

$$B = 2^{-2s} \pi^{-s} \frac{\Gamma((n-2s)/2)}{\Gamma((n+2s)/2)} \left(\frac{\Gamma(n)}{\Gamma(n/2)} \right)^{2s/n}. \tag{2.21}$$

Proof. See [105]. ■

The best constant for the inequality (2.19) is B given by (2.21), see [123].

We introduce fractional logarithmic Gagliardo-Nirenberg inequalities which imply the L^p-logarithmic Sobolev inequalities for fractional Laplace operator.

Theorem 2.2.8. *Let* $1 < q < \infty, 0 < 2s < n$ *and* $1 < p_1 < n/2s$. *Then the inequality*

$$\exp\left(\left(\frac{1}{q} + \frac{2s}{n} - \frac{1}{p_1} \right) \int_{\mathbb{R}^n} \frac{|u(x)|^q}{\|u\|^q_{L^q(\mathbb{R}^n)}} \ln \left(\frac{|u(x)|^q}{\|u\|^q_{L^q(\mathbb{R}^n)}} \right) dx \right) \leq B \frac{\|(-\Delta)^s u\|_{L^{p_1}(\mathbb{R}^n)}}{\|u\|_{L^q(\mathbb{R}^n)}}$$

holds for

$$\frac{1}{q} + \frac{2s}{n} - \frac{1}{p_1} > 0$$

and B *given by (2.21).*

Proof. See [105]. ■

2.2.3 An approach via the Fourier transform

Now, we take into account an alternative definition of the space $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ via the Fourier transform. Precisely, we may define

$$\widehat{H}^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < +\infty\} \tag{2.22}$$

and we observe that the above definition, unlike the ones via the Gagliardo norm in (2.2), is valid also for any real $s \geq 1$. We may also use an analogous definition for the case $s < 0$ by setting

$$\widehat{H}^s(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 d\xi < +\infty\}$$

although in this case the space $\widehat{H}^s(\mathbb{R}^n)$ is not a subset of $L^2(\mathbb{R}^n)$ and, in order to use the Fourier transform, one has to start from an element of $\mathcal{S}'(\mathbb{R}^n)$, (see also Remark 2.1.5).

The equivalence of the space $\widehat{H}^s(\mathbb{R}^n)$ defined in (2.22) with the one defined in the section 2.1 via the Gagliardo norm (see (2.1)) is stated and proven in the forthcoming Proposition 2.2.7.

First we give a new equivalence to the fractional Laplace operator $(-\Delta)^s$ as a pseudo-differential operator of symbol $|\xi|^{2s}$.

Proposition 2.2.6. *Let $s \in (0,1)$ and let $(-\Delta)^s : \mathcal{S} \to L^2(\mathbb{R}^n)$ be the fractional Laplace operator defined by (2.5). Then, for any $u \in S$,*

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)) \text{ for all } \xi \in \mathbb{R}^n.$$

Proof. See [73], pages 530 to 532. ■

Notice that $(-\Delta)^s u \notin \mathcal{S}$ since $|\xi|^{2s}$ introduces a singularity at the origin in its Fourier transform. That singularity is going to translate in a lack of rapid decay for $(-\Delta)^s u$. However, $(-\Delta)^s u$ is still C^∞.

Proposition 2.2.7. *Let $s \in (0,1)$. Then the fractional Sobolev space $H^s(\mathbb{R}^n)$ defined in (2.1) coincides with $\widehat{H}^s(\mathbb{R}^n)$ defined in (2.22). In particular, for any $u \in H^s(\mathbb{R}^n)$*

$$[u]^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi,$$

where $C(n,s)$ is defined by (2.6).

Proof. See [73], pages 532 and 533. ■

Finally, the relation between the fractional Laplace operator $(-\Delta)^s$ and the fractional Sobolev space H^s.

Proposition 2.2.8. *Let $s \in (0,1)$ and let $u \in H^s(\mathbb{R}^n)$. Then,*

$$[u]^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1} \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^n)},$$

where $C(n,s)$ is defined by (2.6).

Proof. See [73], page 533. ■

For $p \in (1,\infty)$ and $s > 0$, we use $\widehat{H}^{s,p}(\mathbb{R}^n)$ to denote the Bessel potential space

$$\widehat{H}^{s,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) \mid (1-\Delta)^{s/2} u \in L^p(\mathbb{R}^n)\}$$

which is equipped with the norm

$$\|u\|_{\widehat{H}^{s,p}(\mathbb{R}^n)} = \|(1-\Delta)^{s/2}u\|_{L^p(\mathbb{R}^n)} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}(u)(\xi)]\|_{L^p(\mathbb{R}^n)},$$

where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^n$. The homogeneous space is denoted by

$$H^{s,p}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid (-\Delta)^{s/2}u \in L^p(\mathbb{R}^n)\}.$$

We use the semi-norm

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} = \|(-\Delta)^{s/2}u\|_{L^p(\mathbb{R}^n)}.$$

Note that by the inequalities

$$N_1(1+|\xi|^s) \le (1+|\xi|^2)^{s/2} \le N_2(1+|\xi|^s)$$

we have

$$\|u\|_{\widehat{H}^{s,p}(\mathbb{R}^n)} \sim \|u\|_{L^p(\mathbb{R}^n)} + \|u\|_{H^{s,p}(\mathbb{R}^n)},$$

see [75, 117].

The space $C_0^\infty(\mathbb{R}^n)$ is dense in $\widehat{H}^{s,p}(\mathbb{R}^n)$ for any $s > 0$. It is known also that $\widehat{H}^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ if s is an integer or if $p = 2$. Furthermore, for $s \in \mathbb{R}$, we have that $W^{s,p}(\mathbb{R}^n) \hookrightarrow \widehat{H}^{s,p}(\mathbb{R}^n)$ if $p \le 2$ and $\widehat{H}^{s,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(\mathbb{R}^n)$ if $p \ge 2$.

When $q = p_1$, the previous logarithmic Gagliardo-Nirenberg inequalities imply the following logarithmic Sobolev inequalities for fractional Laplace operator.

Corollary 2.2.2. *For any $0 < 2s < n, 1 < p < n/2s, u \in H^{2s,p}(\mathbb{R}^n)$ such that $\|u\|_{L^p(\mathbb{R}^n)} = 1$, we have*

$$\exp\left(\frac{2s}{n}\int\limits_{\mathbb{R}^n} |u(x)|^p \ln |u(x)|^p dx\right) \le B\|(-\Delta)^s u\|_{L^p(\mathbb{R}^n)}$$

with B given by (2.21).

2.3 Preliminary results: viscosity and weak solutions

For a given Ω subset of $\mathbb{R}^n$ and functions f and g, we consider the equation of the form:

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega \\ \quad u &= g \quad \text{in } \mathbb{R}^n \setminus \Omega \end{cases}. \tag{2.23}$$

We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ continuous in Ω is a viscosity supersolution (subsolution) of (2.23) if $u \geq g$ (resp. $u \leq g$) in $\mathbb{R}^n \setminus \overline{\Omega}$ and for every point $x_0 \in \Omega$ and some neighborhood V of x_0 with $\overline{V} \subset \Omega$ and for any $\phi \in C^2(\overline{V})$ such that $u(x_0) = \phi(x_0)$ and

$$u(x) \geq \phi(x) \text{ (resp. } u(x) \leq \phi(x)) \text{ for all } x \in V$$

defining

$$\overline{u} = \begin{cases} \phi \text{ in } V \\ u \text{ in } \mathbb{R}^n \setminus V \end{cases} \tag{2.24}$$

we have

$$(-\Delta)^s \overline{u}(x_0) \geq f(x_0) \text{ (resp. } (-\Delta)^s \overline{u}(x_0) \leq f(x_0)).$$

In a natural way, we have the notions of viscosity subsolution and viscosity solution.

The quantity $R(\Omega)$ is defined to be the smallest positive constant R such that

$$|B_R(x) \setminus \Omega| \geq \frac{1}{2}|B_R(x)|$$

for all $x \in \Omega$. If no such radius R exists, we define $R(\Omega) = +\infty$. It is easy to say that whenever the subset Ω contained between two parallel hyperplanes at a distance d, we have

$$R(\Omega) \leq \frac{2^n d}{\omega_n},$$

where ω_n is the volume of unit ball in $\mathbb{R}^n$.

Theorem 2.3.1. *(weak Harnack inequality) Let $u \in C(\overline{B}_{2R})$ satisfies*

$$\begin{cases} (-\Delta)^s u & \geq & f & \text{in} & B_{2R} \\ u & \geq & 0 & \text{in} & \mathbb{R}^n \end{cases}, \tag{2.25}$$

where $f \in C(\overline{B}_{2R})$. Then

$$\left(\frac{1}{|B_R|} \int_{B_R} u^{p_0} \right)^{1/p_0} \leq C \left(\inf_{B_R} u + R^{2s} \|f\|_{L^\infty(B_{2R})} \right),$$

where p_0 and C are positive universal constants.

Proof. See [147], pages 5 and 6. ■

The follow an ABP estimate. It applies in any domain satisfying $R(\Omega) < \infty$. Notice that here we do not need the subset is bounded.

Theorem 2.3.2. *Let Ω be an open domain with $R(\Omega) < +\infty$. Suppose $u \in C(\overline{\Omega})$ and $f \in C(\overline{\Omega})$ satisfy $\sup_{\Omega} u < \infty$ and*

$$\begin{cases} (-\Delta)^s u & \geq & f & \text{in} & \Omega \\ u & = & 0 & \text{in} & \mathbb{R}^n \setminus \Omega \end{cases}. \tag{2.26}$$

Then

$$\sup_{\Omega} u \leq CR(\Omega)^{2s} \|f\|_{L^\infty(\Omega)},$$

where C is a positive constant.

Proof. See [147], page 7. ■

Now we give maximum principle in domains (not necessarily bounded) for which $R(\Omega)$ is sufficiently small.

Theorem 2.3.3. *Let Ω be an open domain. Suppose that $\phi : \Omega \to \mathbb{R}$ is in $L^\infty(\Omega)$ and $u \in C(\overline{\Omega})$ is solution of*

$$\begin{cases} (-\Delta)^s u & \geq & \phi(x)u(x) & \text{in} & \Omega \\ u & \geq & 0 & \text{in} & \mathbb{R}^n \setminus \Omega \end{cases} \tag{2.27}$$

with $\phi u \in C(\overline{\Omega})$. Then there exist a number $\overline{R}$ such that $R\left(\{x \in \Omega \mid u(x) < 0\}\right) \leq \overline{R}$ implies that each solution satisfies $u \geq 0$ in Ω.

Proof. See [147], page 8. ■

Next we give a regularity theorem.

Theorem 2.3.4. *Let g bounded in $\mathbb{R}^n \setminus \Omega$ and $f \in C^{\beta}_{loc}(\Omega)$ for some $0 < \beta < 1$ and u be a viscosity solution of*

$$\begin{cases} (-\Delta)^s u & = & f & \text{in} & \Omega \\ u & = & g & \text{in} & \mathbb{R}^n \setminus \Omega \end{cases}, \tag{2.28}$$

then there exists $\gamma > 0$ such that $u \in C^{2s+\gamma}_{loc}(\Omega)$.

Proof. See [147], page 8. ■

Remark 2.3.5. We say that a function u continuous in Ω and bounded in $\mathbb{R}^n$ is a classical solution of (2.23) if $(-\Delta)^s u(x)$ is well defined for all $x \in \Omega$,

$$(-\Delta)^s u(x) = f, \text{ for all } x \in \Omega$$

and $u(x) = g$ a.e. in $\mathbb{R}^n \setminus \Omega$. Classical super and subsolutions are defined similarly.

The Maximum Principle is key tool in the analysis, one can see that a nonnegative solution u is either strictly positive or identically zero in $\mathbb{R}^n$. The strong maximum principle involving $(-\Delta)^s$ due to Silvestre [169] is given by:

Lemma 2.3.6. *Let Ω be an open set of $\mathbb{R}^n$ and let u be a lower semicontinuous function in $\overline{\Omega}$ such that*

$$\begin{cases} (-\Delta)^s u & \geq & 0 & \text{in} & \Omega \\ u & \geq & 0 & \text{in} & \mathbb{R}^n \setminus \Omega \end{cases}. \tag{2.29}$$

Then $u \geq 0$ in $\mathbb{R}^n$. Moreover, if $u(x) = 0$ for some point inside Ω, then $u \equiv 0$ in all $\mathbb{R}^n$.

Proof. See [169], pages 81 and 82. ■

Remark 2.3.7. When $s = 1$, Ω must be connected so that the strong maximum principle holds, but here it is not necessary. Look at the two definitions below to get the matter; with the classical Laplace operator, the integration is over the ball while in the fractional Laplace operator the integration is in the whole space except in the ball. Thus, it is not worth to suppose connectivity.

$$-\Delta u(x) = \lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(x) - u(y) dy; \tag{2.30}$$

$$(-\Delta)^s u(x) = \lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{\mathbb{R}^n \setminus B_r(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy. \tag{2.31}$$

We see that apart from the nonlocal property that distinguishes them, they are very similar.

As a consequence of the strong maximum principle we prove the comparison principle Theorem.

Theorem 2.3.8. *Assume u and v are supersolution and subsolution of the equation*

$$(-\Delta)^s u = f$$

in Ω, where Ω is a bounded open subset of $\mathbb{R}^n$ and f is a continuous function in $\overline{\Omega}$. Moreover, assume that $u \geq v$ in $\mathbb{R}^n \setminus \Omega$. Then $u \geq v$ in Ω. In addiction if $u(x) - v(x) = 0$, for some point inside Ω, then $u = v$ in $\mathbb{R}^n$.

Proof. We have that $u - v \geq 0$ in $\mathbb{R}^n \setminus \Omega$ and

$$(-\Delta)^s u \leq f \text{ and } (-\Delta)^s v \geq f \text{ in } \Omega.$$

Thus $(-\Delta)^s(u-v) \geq 0$ in Ω. Therefore from Lemma 2.3.6, we have $u \geq v$ in Ω. In addiction if $u(x) - v(x) = 0$, for some point inside Ω, then $u = v$ in $\mathbb{R}^n$. ■

We also need the following C^β estimate for fractional Laplace operator $(-\Delta)^s$, which is a direct conclusion of Theorem 2.6 in [42].

Theorem 2.3.9. *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$. If $u \in C(\overline{\Omega})$ satisfies the inequalities*

$$(-\Delta)^s u \geq -C_0 \text{ and } (-\Delta)^s u \leq C_0 \text{ in } \Omega,$$

then for $\Omega' \Subset \Omega$ there exist constant $\beta > 0$ such that $u \in C^\beta(\Omega')$ and

$$\|u\|_{C^\beta(\Omega')} \leq C\left\{\|u\|_{L^\infty(\Omega)} + C_0\right\}$$

for some constant $C > 0$ which depents on n.

Remark 2.3.10. Theorems 2.3.9 and 2.3.4 imply that if u is a viscosity and bounded solution of $(-\Delta)^s u = u^p$ in Ω with $p > 0$, then u is classical. In fact, if u is bounded, we have u^p is bounded. So Theorem 2.3.9 implies there exist constants $\beta, \gamma > 0$ such that $u \in C^\beta$ and then $u^p \in C^\gamma$. Finally by Theorem 2.3.4 u is a classical solution.

We are to use the following convergence result for fractional Laplace operator $(-\Delta)^s$ (see Corollary 4.6 in [42]).

Theorem 2.3.11. *Let $\{u_k\}, k \in \mathbb{N}$ be a sequence of functions that are bounded in $\mathbb{R}^n$ and continuous in Ω, f_k and f are continuous in Ω such that*

(a) $(-\Delta)^s u_k = f_k$ in Ω in viscosity sense.

(b) $u_k \to u$ locally uniformly in Ω.

(c) $u_k \to u$ a.e. in $\mathbb{R}^n$.

(d) $f_k \to f$ locally uniformly in Ω.

Then $(-\Delta)^s u = f$ in Ω in viscosity sense.

Perron method involving $(-\Delta)^s$, see [168].

Theorem 2.3.12. *Let Ω be open, bounded and with C^2-boundary and let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, there exists a viscosity solution $u \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ of the problem*

$$\begin{cases} (-\Delta)^s u &= 0 \text{ in } \Omega \\ u &= g \text{ in } \mathbb{R}^n \setminus \Omega \end{cases}. \tag{2.32}$$

Theorem 2.3.13. *Let Ω be open, bounded and with C^2-boundary and let $f \in C(\mathbb{R}^n)$. Let $u \in H_0^s(\Omega) \cap L^\infty(\mathbb{R}^n)$ be a weak solution of*

$$\begin{cases} (-\Delta)^s u & = & f & \text{in} & \Omega \\ u & = & 0 & \text{in} & \mathbb{R}^n \setminus \Omega \end{cases} . \tag{2.33}$$

Then, u is a viscosity solution of this problem.

Proof. See [168]. ■

Proposition 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $f \in H_0^s(\Omega)'$. Then there is a unique weak solution $u \in H_0^s(\Omega)$ of the problem 2.33.*

Proof. See [86]. ■

We establish here the Pohozaev identity for the fractional Laplace operator, which reads as follows.

Theorem 2.3.14. *Let $\Omega \subset \mathbb{R}^n$ be any bounded $C^{1,1}$ domain, and let $d(x) = dist(x, \partial\Omega)$. Let u be any bounded weak solution of*

$$\begin{cases} (-\Delta)^s u & = & f(x,u) & \text{in} & \Omega \\ u & = & 0 & \text{in} & \mathbb{R}^n \setminus \Omega \end{cases} . \tag{2.34}$$

Then u/d^s is Hölder continuous in $\overline{\Omega}$, and it holds the identity

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u dx = \frac{2s-n}{2} \int_\Omega u(-\Delta)^s u dx - \frac{1}{2}\Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{d^s}\right)^2 (x \cdot \nu) d\sigma, \tag{2.35}$$

where ν is the unit outward normal to $\partial\Omega$ at x and Γ is the Gamma function.

Proof. See [155]. ■

Let us mention some consequences of Theorem 2.3.14. First, when $f(x,u)$ does not depend on x, our identity can be written as

$$(2s-n)\int_\Omega uf(u)dx + 2n\int_\Omega F(u)dx = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{d^s}\right)^2 (x \cdot \nu) d\sigma, \tag{2.36}$$

where $F' = f$. Thus, when Ω is star-shaped, it immediately leads to the nonexistence of nontrivial solutions for supercritical nonlinearities, and also of nonnegative solutions for the critical power $f(u) = u^{\frac{n+2s}{n-2s}}$, this was previously showed in [81] for nonnegative solutions.

Lemma 2.3.15. *(weak maximum principles) Let $w \in H_0^s(\Omega)$, consider the following problem*

$$\begin{cases} (-\Delta)^s w & \geq & 0 & \text{in} & \Omega \\ w & \geq & 0 & \text{in} & \mathbb{R}^n \setminus \Omega \end{cases}. \tag{2.37}$$

Then $w \geq 0$ in Ω.

Proof. See [138]. ■

Proposition 2.3.2. *(Generalised Hopf Lemma) If a smooth function $v(x)$ satisfies $(-\Delta)^s v = 0$ in some smooth domain Ω of $\mathbb{R}^n$, if v is nonnegative and nonzero in $\mathbb{R}^n$, and if there is a point $x_0 \in \partial\Omega$ for which $v(x_0) = 0$, then there exists $\lambda > 0$ such that $v(x) \geq \lambda((x - x_0) \cdot \nu(x_0))^s$, where $\nu(x_0)$ is the inner normal to $\partial\Omega$ at x_0.*

Proof. See [38]. ■

Remark 2.3.16. ([153]) Assume that u and v satisfy

$$\begin{cases} (-\Delta)^s u = g & \text{in} \;\; \Omega \\ (-\Delta)^t v = h & \text{in} \;\; \Omega \\ u = v = 0 & \text{in} \;\; \mathbb{R}^n \setminus \Omega \end{cases}, \tag{2.38}$$

where $g, h \in L^\infty(\Omega)$ and h, g are positive in Ω. Then, by Theorem 2.2.7 we have that u/d^s and v/d^t are $C^\alpha(\overline{\Omega})$ functions. In addition, by the Hopf lemma for the fractional Laplace operator we find that $u/d^s, v/d^t \geq c > 0$ in $\overline{\Omega}$.

CAPÍTULO 3

Proof of non-variational contributions

3.1 Preliminary lemmas

We next present three lemmas which will be used in the proof of Theorem 0.0.4. Their proofs are inspired in the work of Felmer and Quaas [85] and adapted to fractional operators.

Throughout the paper, it is assumed that $p, q > 0$ and $pq > 1$. So, thanks to a suitable rescaling of u and v, we can assume that $C(n, s) = 1$ and $C(n, t) = 1$.

Given a nonnegative continuous function $u : \mathbb{R}^n \to \mathbb{R}$, define

$$m_u(r) = \min_{|x| \leq r} u(x)$$

for $r > 0$.

Lemma 3.1.1. *Let $s \in (0, 1)$, $n > 2s$ and $u \neq 0$ be a nonnegative viscosity supersolution of*

$$(-\Delta)^s u = 0 \text{ in } \mathbb{R}^n . \tag{3.1}$$

Then, for each $R_0 > 1$ and $\sigma \in (-n, -n+2s)$, there exists a constant $C > 0$, independent of u, such that

$$m_u(r) \geq C m_u(R_0) r^\sigma \tag{3.2}$$

for all $r \geq R_0$.

Proof of Lemma 3.1.1. Let R_0, σ and u be as in the above statement. Given $R > R_0$ and $\varepsilon > 0$, we consider the function

$$w(r) = \begin{cases} \varepsilon^\sigma \text{ if } 0 < r \leq \varepsilon \\ r^\sigma \text{ if } \varepsilon \leq r \end{cases} \tag{3.3}$$

We first assert that $(-\Delta)^s w(r) < 0$ for all $R_0 < r < R$ and $\varepsilon > 0$ small enough. In fact, for $|x| = r$, we have

$$\begin{aligned}
2(-\Delta)^s w(r) &= -\int\limits_{B_\varepsilon(-x)} \frac{\varepsilon^\sigma}{|y|^{n+2s}}\, dy - \int\limits_{B_\varepsilon(x)} \frac{\varepsilon^\sigma}{|y|^{n+2s}}\, dy - \int\limits_{B_\varepsilon^c(-x)} \frac{|x+y|^\sigma}{|y|^{n+2s}}\, dy \\
&\quad - \int\limits_{B_\varepsilon^c(x)} \frac{|x-y|^\sigma}{|y|^{n+2s}}\, dy + 2\int\limits_{\mathbb{R}^n} \frac{|x|^\sigma}{|y|^{n+2s}}\, dy \\
&= -\int\limits_{\mathbb{R}^n} \frac{|x+y|^\sigma + |x-y|^\sigma - 2|x|^\sigma}{|y|^{n+2s}}\, dy \\
&\quad + \left(\int\limits_{B_\varepsilon(-x)} \frac{|x+y|^\sigma - \varepsilon^\sigma}{|y|^{n+2s}}\, dy + \int\limits_{B_\varepsilon(x)} \frac{|x-y|^\sigma - \varepsilon^\sigma}{|y|^{n+2s}}\, dy \right) \\
&= 2(-\Delta)^s |x|^\sigma + \left(\int\limits_{B_\varepsilon(-x)} \frac{|x+y|^\sigma - \varepsilon^\sigma}{|y|^{n+2s}}\, dy + \int\limits_{B_\varepsilon(x)} \frac{|x-y|^\sigma - \varepsilon^\sigma}{|y|^{n+2s}}\, dy \right).
\end{aligned}$$

Since $R_0 > 1$, the two last above integral converge uniformly to 0 for $|x| > R_0$ as $\varepsilon \to 0$.

On the other hand, using that $R_0 > 1$ and $\sigma \in (-n, -n+2s)$ and the fact that $|x|^{-n+2s}$ is the fundamental solution of the fractional Laplace operator $(-\Delta)^s$ (see [41]), one easily checks that $(-\Delta)^s |x|^\sigma < 0$ for all $|x| > R_0$, see [85]. Thus, the above claim follows for $\varepsilon > 0$ small enough.

For such a parameter ε and $|x| = r$, we set

$$\varphi(x) = m_u(R_0) \frac{w(r) - w(R)}{w(\varepsilon) - w(R)}$$

for all $|x| < R$ and $\varphi(x) = 0$ for $|x| \geq R$. As can easily be checked, $(-\Delta)^s \varphi \leq 0$ for all $R_0 < |x| < R$. Moreover, we have $u(x) \geq \varphi(x)$ for $|x| \leq R_0$ or $|x| \geq R$, so that the Silvestre's strong maximum principle readily yields $u(x) \geq \varphi(x)$ for all $R_0 \leq |x| \leq R$. Finally, letting $R \to \infty$ in this last inequality, we achieve the expected conclusion with $C = \varepsilon^{-\sigma}$. ■

Our second auxiliary lemma is

Lemma 3.1.2. *Let $s \in (0,1)$, $n > 2s$ and $u \neq 0$ be a nonnegative viscosity supersolution of (3.1). Then, there exist constants $C > 0$ and $R_0 > 0$, independent of u, such that*

$$m_u(r/2) \leq C m_u(r) \tag{3.4}$$

for all $r \geq R_0$.

Proof of Lemma 3.1.2. Given $r > 0$ and $\varepsilon > 0$, set

$$R = r\left[\frac{\varepsilon}{1+\varepsilon 2^{-n+2s}}\right]^{1/(n-2s)},$$

where ε is chosen such that $R < r/2$.

Consider the functions

$$w_r(\overline{r}) = \begin{cases} (R)^{-n+2s} & \text{if } 0 < \overline{r} \le R \\ \overline{r}^{-n+2s} & \text{if } R \le \overline{r} \le 2r \\ (2r)^{-n+2s} & \text{if } \overline{r} \ge 2r \end{cases}$$

and

$$w(\overline{r}) = \begin{cases} (R)^{-n+2s} & \text{if } 0 < \overline{r} \le R \\ \overline{r}^{-n+2s} & \text{if } R \le \overline{r} \end{cases}$$

Given a fixed function u as in the above statement, we define

$$\varphi(x) = m_u(r/2)\frac{w_r(\overline{r}) - w(2r)}{w(R) - w(2r)}$$

for x with $|x| = \overline{r}$. As a direct consequence, one has $u(x) \ge \varphi(x)$ for all x with $|x| \le r/2$ and $|x| \ge 2r$. Moreover, decreasing ε, if necessary, one gets

$$\begin{aligned} 2(-\Delta)^s w_r(\overline{r}) = &-\int\limits_{B_R(x)} \frac{r^{-n+2s}}{\varepsilon|y|^{n+2s}} + \frac{(2r)^{-n+2s}}{|y|^{n+2s}}\,dy - \int\limits_{B^c_{2r}(x)} \frac{(2r)^{-n+2s}}{|y|^{n+2s}}\,dy \\ &- \int\limits_{B_{2r}(x)\setminus B_R(x)} \frac{|x-y|^{-n+2s}}{|y|^{n+2s}}\,dy + \int\limits_{\mathbb{R}^n} \frac{|x|^{-n+2s}}{|y|^{n+2s}}\,dy \le 0 \end{aligned}$$

for all $r/2 < \overline{r} < 2r$. Thus, $(-\Delta)^s\varphi(x) \le 0$ for all x with $r/2 < |x| < 2r$.

Evoking the Silvestre's maximum principle, we then deduce that $u(x) \ge \varphi(x)$ for all x with $r/2 < |x| < 2r$. Lastly, we assert that this conclusion leads to

$$m_u(r) \ge \varepsilon m_u(r/2)(1 - 2^{-n+2s}).$$

In fact, we have

$$\varphi(x) = m_u(r/2) \ge \varepsilon m_u(r/2)(1 - 2^{-n+2s})$$

if $0 < |x| \le R$, and

$$\varphi(x) = \varepsilon m_u(r/2)\frac{\overline{r}^{-n+2s} - (2r)^{-n+2s}}{r^{-n+2s}} \ge \varepsilon m_u(r/2)(1 - 2^{-n+2s})$$

if $R < |x| \le r$. So, the result follows with $C = (\varepsilon(1-2^{-n+2s}))^{-1}$ by minimizing u on the closed ball $|x| \le r$. ■

Our third lemma concerns with the behavior of fractional Laplace operators applied to the function $\Theta(x) = \log(1+|x|)|x|^{-n+2s}$.

Lemma 3.1.3. *Let $s \in (0,1)$ and $n > 2s$. Then, there exists a constant $C_0 > 0$ such that*

$$(-\Delta)^s \Theta(x) \le C_0 |x|^{-n}$$

for all $x \neq 0$.

Proof of Lemma 3.1.3. Using that $|x|^{-n+2s}$ is the fundamental solution of $(-\Delta)^s$ (see [41]), one first has

$$\begin{aligned}
-2(-\Delta)^s\Theta(x) &= \int\limits_{\mathbb{R}^n} \frac{\log(1+|x-y|)|x-y|^{-n+2s}}{|y|^{n+2s}}\,dy \\
&\quad + \int\limits_{\mathbb{R}^n} \frac{\log(1+|x+y|)|x+y|^{-n+2s}}{|y|^{n+2s}}\,dy - 2\int\limits_{\mathbb{R}^n} \frac{\log(1+|x|)|x|^{-n+2s}}{|y|^{n+2s}}\,dy \\
&= \int\limits_{\mathbb{R}^n} \frac{\left(\log(1+|x-y|) - \log(1+|x|)\right)|x-y|^{-n+2s}}{|y|^{n+2s}}\,dy \\
&\quad + \int\limits_{\mathbb{R}^n} \frac{\left(\log(1+|x+y|) - \log(1+|x|)\right)|x+y|^{-n+2s}}{|y|^{n+2s}}\,dy \\
&= \int\limits_{\mathbb{R}^n} \left(\log\left(\frac{1+|x-y|}{1+|x|}\right)|x-y|^{-n+2s}\right)\frac{1}{|y|^{n+2s}}\,dy \\
&\quad + \int\limits_{\mathbb{R}^n} \left(\log\left(\frac{1+|x+y|}{1+|x|}\right)|x+y|^{-n+2s}\right)\frac{1}{|y|^{n+2s}}\,dy \\
&= \int\limits_{\mathbb{R}^n} r^{-n}\left(\log\left(\frac{1+r|e_1-z|}{1+r}\right)|e_1-z|^{-n+2s}\right)\frac{1}{|z|^{n+2s}}\,dz \\
&\quad + \int\limits_{\mathbb{R}^n} r^{-n}\left(\log\left(\frac{1+r|e_1+z|}{1+r}\right)|e_1+z|^{-n+2s}\right)\frac{1}{|z|^{n+2s}}\,dz\,,
\end{aligned}$$

where $x = re_1$ and $z = y/r$. Note that there is no loss of generality in considering $x = re_1$, since $\log(1+|x|)$ and $|x|^{-n+2s}$ are radially symmetric.

In order to complete the proof we just need to find a constant $C_0 > 0$ such that

$$\int\limits_{\mathbb{R}^n} \frac{\left(\log\left(\frac{1+r|e_1-z|}{1+r}\right)|e_1-z|^{-n+2s} + \log\left(\frac{1+r|e_1+z|}{1+r}\right)|e_1+z|^{-n+2s}\right)}{|z|^{n+2s}}\,dz \ge -C_0\,. \tag{3.5}$$

For this purpose, we write for $\rho > 0$, $\gamma \in [0,1)$ and $r \ge 0$,

$$\log\left(\frac{1+r|e_1-z|}{1+r}\right)|e_1-z|^{-n+2s} = g(|e_1-z|,\gamma) \tag{3.6}$$

and

$$\log\left(\frac{1+r|e_1+z|}{1+r}\right)|e_1+z|^{-n+2s} = g(|e_1+z|,\gamma)\,, \tag{3.7}$$

where

$$g(\rho,\gamma) = \rho^{-n+2s}\log(1+\gamma(\rho-1))$$

and

$$\gamma = \frac{r}{1+r}\,.$$

Consider first $B_1 = \{z \mid |z+e_1| \leq 1/2\}$ and note that $g(|e_1-z|,\gamma)$ is bounded in B_1, while $g(|e_1+z|,\gamma)$ has a singularity at $-e_1 \in B_1$. Then, for some constants $C > 0$, independent of γ, we have

$$\begin{aligned}\int_{B_1}\frac{|g(|e_1+z|,\gamma)|}{|z|^{n+2s}}\,dz &= \int_{B_{1/2}(0)}\frac{|g(|z|,\gamma)|}{|z-e_1|^{n+2s}}\,dz \leq -C\int_0^{1/2} g(\rho,\gamma)\rho^{n-1}\,d\rho \\ &\leq -C\int_0^{1/2}\rho^{2s-1}\log(\rho)\,d\rho \leq C\,.\end{aligned}$$

Since $1+\gamma(\rho-1) \geq \rho$ as $\gamma \in [0,1)$, the integral in (3.5), when considered over B_1, is bounded below by a constant independent of r. In a similar way, the conclusion follows for the set $B_2 = \{z \mid |z-e_1| \leq 1/2\}$.

On the set $B_3 = \{z \mid |z| \geq 2\}$, for some constant $C > 0$, independent of γ, we have

$$|g(|e_1-z|,\gamma) + g(|e_1+z|,\gamma)| \leq C|z|^{-2n}\log(|z|)\,.$$

Thus, the integral in (3.5), when considered over B_3, is also bounded below by a constant independent of r.

It then remains to analyze the behavior of the integral over $B_4 = \{z \mid |z| \leq 1/2\}$. For each fixed $r \geq 0$ and $\gamma \in [0,1)$, define $f_r : \mathbb{R}^n \to \mathbb{R}$ given by $f_r(z) = g(|e_1+z|,\gamma) + g(|e_1-z|,\gamma)$. Using that $f_r(0) = 0$ and $D(f_r(0)) = 0$, the Taylor formula provides

$$f_r(z) = z^t \cdot \int_0^1 (1-\rho)D^2(f_r(\rho z))\,d\rho \cdot z\,, \tag{3.8}$$

where all derivatives are taken only with respect to the variable z. Thus, the estimate of the integral (3.5) over B_4 follows if we can show that

$$\left|\frac{\partial^2 f_r(z)}{\partial z_i \partial z_j}\right| \leq C \tag{3.9}$$

for all $|z| \leq 1/2$, where $C > 0$ is a constant independent of r.

On the other hand, a straightforward computation gives

$$\frac{d}{d\rho} g(\rho,\gamma) = (-n+2s)\rho^{-n+2s-1}\log(1+\gamma(\rho-1)) + \frac{\gamma\rho^{-n+2s}}{1+\gamma(\rho-1)}$$

and

$$\begin{aligned}\frac{d^2}{d\rho^2} g(\rho,\gamma) &= (-n+2s)(-n+2s-1)\rho^{-n+2s-2}\log(1+\gamma(\rho-1)) \\ &\quad + \frac{2\gamma(-n+2s)\rho^{-n+2s-1}}{1+\gamma(\rho-1)} - \frac{\gamma^2\rho^{-n+2s}}{(1+\gamma(\rho-1))^2}.\end{aligned}$$

Then, one easily checks that

$$|\frac{d}{d\rho} g(\rho,\gamma)|,\ |\frac{d^2}{d\rho^2} g(\rho,\gamma)| \leq C$$

for all $1/2 \leq \rho \leq 3/2$ and $\gamma \in [0,1)$, where C is a constant independent of ρ and γ. So, for certain bounded functions D_{ij} and d_{ij} in B_4, we have

$$\frac{\partial^2 f_r(z)}{\partial z_i \partial z_j} = \frac{d^2}{d\rho^2} g(|e_1+z|,\gamma)D_{ij} + \frac{d}{d\rho} g(|e_1+z|,\gamma)d_{ij}$$

and (3.9) follows.

Finally, joining the above estimates on the four sets B_i, one gets (3.5) as desired. ■

3.2 Proof of Theorem 0.0.4

We organize the proof of Theorem 0.0.4 into two stages, according to the sufficiency and necessity of the assumption (4).

Proof of the sufficiency of (4). We analyze separately two different cases:

(I) $\left(\frac{2s}{p}+2t\right)\frac{p}{pq-1} > n-2s$ or $\left(\frac{2t}{q}+2s\right)\frac{q}{pq-1} > n-2t$;

(II) $\left(\frac{2s}{p}+2t\right)\frac{p}{pq-1} = n-2s$ or $\left(\frac{2t}{q}+2s\right)\frac{q}{pq-1} = n-2t$.

We first assume the situation (I). Let (u,v) be a nonnegative viscosity supersolution of the system (6) with $G=\mathbb{R}^n$ and $\eta:[0,+\infty)\to\mathbb{R}$ be a C^∞ cutoff function satisfying $0\le\eta\le 1$, η is nonincreasing, $\eta(r)=1$ if $0\le r\le 1/2$ and $\eta(r)=0$ if $r\ge 1$. Clearly, there exists a constant $C>0$ such that $(-\Delta)^s\eta(|x|)\le C$ and $(-\Delta)^t\eta(|x|)\le C$.

Choose $R_0>0$ as in Lemma 3.1.2 for s and t, simultaneously, and consider the functions

$$\xi_u(x)=m_u(R_0/2)\eta(|x|/R_0) \text{ and } \xi_v(x)=m_v(R_0/2)\eta(|x|/R_0)\,.$$

For some constant $C_0>0$, independent of R_0, u and v, we have

$$(-\Delta)^s(\xi_u(x))\le C_0\frac{m_u(R_0/2)}{{R_0}^{2s}} \text{ and } (-\Delta)^t(\xi_v(x))\le C_0\frac{m_v(R_0/2)}{{R_0}^{2t}}\,.$$

Moreover, $\xi_u(x)=0\le u(x)$ if $|x|>R_0$ and $\xi_u(x)=m_u(R_0/2)\le u(x)$ if $|x|\le R_0/2$. Similarly, $\xi_v(x)=0\le v(x)$ if $|x|>R_0$ and $\xi_v(x)=m_v(R_0/2)\le v(x)$ if $|x|\le R_0/2$. Thus, the functions $u-\xi_u$ and $v-\xi_v$ attain their global minimum values at points x_u and x_v with $|x_u|<R_0$ and $|x_v|<R_0$, respectively.

Now let $\varphi(x):=\xi_u(x)-\xi_u(x_u)+u(x_u)$ and $\psi(x):=\xi_v(x)-\xi_v(x_v)+v(x_v)$. Note that $\varphi(x_u)=u(x_u)$, $\psi(x_v)=v(x_v), u(x)\ge\varphi(x)$ and $v(x)\ge\psi(x)$ for all $x\in B(0,R_0)$. Let $\overline{u}$ and $\overline{v}$ be defined as in (7) with $U=B(0,R_0)$. Since (u,v) is a viscosity supersolution of (6), one has

$$(-\Delta)^s(\overline{u})(x_u)\ge v^p(x_u) \text{ and } (-\Delta)^t(\overline{v})(x_v)\ge u^q(x_v)\,. \tag{3.10}$$

We now assert that

$$(-\Delta)^s(\overline{u})(x_u)\le(-\Delta)^s(\xi_u)(x_u) \text{ and } (-\Delta)^t(\overline{v})(x_v)\le(-\Delta)^t(\xi_v)(x_v)\,.$$

In fact, note that $w_u(x):=\overline{u}(x)-\xi_u(x)\ge 0$ for all $x\in\mathbb{R}^n$ and x_u is a global minimum point of w_u. Thus, we have $(-\Delta)^s(w_u)(x_u)\le 0$ and thus the first inequality follows. The other inequality also follows in an analogous way. Therefore, from (3.10), one gets

$$m_u^q(R_0)\le u^q(x_v)\le C_0\frac{m_v(R_0/2)}{{R_0}^{2t}} \text{ and } m_v^p(R_0)\le v^p(x_u)\le C_0\frac{m_u(R_0/2)}{{R_0}^{2s}}\,. \tag{3.11}$$

Applying Lemma 3.1.2 in the above inequalities, one then derives

$$m_u(R_0)\le\frac{C_1}{{R_0}^{\left(\frac{2s}{p}+2t\right)\frac{p}{pq-1}}} \text{ and } m_v(R_0)\le\frac{C_2}{{R_0}^{\left(\frac{2t}{q}+2s\right)\frac{q}{pq-1}}}\,. \tag{3.12}$$

We now consider the case (I). It suffices to assume that $(\frac{2s}{p}+2t)\frac{p}{pq-1} > n-2s$, since the argument is analogous for the second inequality in (I). Choose $-n < \sigma_1 < -n+2s$ such that

$$\left(\frac{2s}{p}+2t\right)\frac{p}{pq-1}+\sigma_1 > 0\,.$$

By Lemma 3.1.1, we have

$$m_u(r) \le m_u(R_0) \le \frac{C}{r^{\left(\frac{2s}{p}+2t\right)\frac{p}{pq-1}+\sigma_1}}$$

for all $r \ge R_0 \ge 1$. Therefore, $m_u(r)$ goes to 0 as $r \to +\infty$, providing the contradiction $(u,v)=(0,0)$.

Finally, assume the situation (II). In a similar way, we analyze only the equality $(\frac{2s}{p}+2t)\frac{p}{pq-1} = n-2s$. Let (u,v) be nonnegative viscosity supersolution of (6) with $G=\mathbb{R}^n$. We begin by proving that for certain $C>0$ and $R_0>0$, we have

$$m_u(r) \ge C m_u(R_0) r^{-n+2s} \tag{3.13}$$

for all $r \ge R_0$. Indeed, by Lemma 3.1.1 and (3.11), for any $-n<\sigma<-n+2s$, we have

$$(-\Delta)^s u(x) \ge v^p(x) \ge m_v(r)^p \ge C(m_u(2r))^{pq} r^{2tp} \ge C(m_u(R_0))^{pq} r^{\sigma pq+2tp} \tag{3.14}$$

for all x with $|x|=r\ge R_0$.

Now consider the function

$$w(r) = \begin{cases} \varepsilon^{-n+2s} \text{ if } 0<r\le\varepsilon \\ r^{-n+2s} \text{ if } \varepsilon \le r \end{cases} \tag{3.15}$$

where $0<\varepsilon<R_0/2$. Since $|x|^{-n+2s}$ is the fundamental solution of the fractional Laplace operator $(-\Delta)^s$ (see [41]), we have

$$2(-\Delta)^s w(r) = \left(\int\limits_{B_\varepsilon(-x)} \frac{|x+y|^{-n+2s}-\varepsilon^{-n+2s}}{|y|^{n+2s}}\,dy + \int\limits_{B_\varepsilon(x)} \frac{|x-y|^{-n+2s}-\varepsilon^{-n+2s}}{|y|^{n+2s}}\,dy \right),$$

where $|x|=r$. It is clear that $|y|\ge |x|/2$ whenever $|x|\ge R_0$ and $y\in B_\varepsilon(x)$. Thus,

$$\int\limits_{B_\varepsilon(x)} \frac{|x-y|^{-n+2s}-\varepsilon^{-n+2s}}{|y|^{n+2s}}\,dy \le \frac{C}{r^{n+2s}}$$

for some constant $C>0$ and then, by symmetry of the integrals, one obtains

$$2(-\Delta)^s w(r) \le \frac{C}{r^{n+2s}}\,.$$

For fixed $R_1 > R_0$, we define the comparison function

$$\varphi(x) = m_u(R_0)\frac{w(r) - w(R_1)}{w(\varepsilon) - w(R_1)}$$

for all x with $|x| < R_1$ and $\varphi(x) = 0$ for $|x| \geq R_1$. As can easily be checked,

$$(-\Delta)^s \varphi(x) \leq \frac{C_1}{|x|^{n+2s}} \tag{3.16}$$

for all x with $R_0 < |x| < R_1$. On the other hand, since $n = pq(n-2s) - 2tp$, we can choose $\sigma \in (-n, -n+2s)$ such that $-\sigma pq - 2tp < n + 2s$. Then, using (3.14) and (3.16), one gets

$$(-\Delta)^s \varphi(x) \leq \frac{C_1}{|x|^{n+2s}} \leq \frac{C_1}{|x|^{-\sigma pq - 2tp}} \leq (-\Delta)^s u(x)$$

for all x with $R_0 < |x| < R_1$ and $u(x) \geq \varphi(x)$ for $|x| \leq R_0$ or $|x| \geq R_1$, so that the Silvestre's maximum principle readily yields $u(x) \geq \varphi(x)$ for all $R_0 \leq |x| \leq R_1$. Finally, letting $R_1 \to +\infty$ in this last inequality, the claim (3.13) follows.

In the sequel, we split the proof into two cases according to the value of $-n+2s$. The first one corresponds to $-n + 2s \in (-n, -1]$. In this range, note that the function Θ, defined above Lemma 3.1.3, is decreasing for all $r > 0$, with a singularity at the origin if $-n + 2s \in (-n, -1)$ and bounded if $-n + 2s = -1$. For $0 < \varepsilon < R_0/2$, we define the function

$$w(r) = \begin{cases} \Theta(\varepsilon) & \text{if } 0 < r \leq \varepsilon \\ \Theta(r) & \text{if } \varepsilon < r \end{cases}.$$

Using Lemma 3.1.3, for any $r \geq R_0$ and x with $|x| = r$, we have

$$\begin{aligned} (-\Delta)^s w(r) &\leq \int\limits_{B_\varepsilon(x)} \frac{\log(1+|x-y|)|x-y|^{-n+2s} - \log(1+\varepsilon)\varepsilon^{-n+2s}}{|y|^{n+2s}} dy + \frac{C}{r^n} \\ &\leq C\frac{\varepsilon^{2s}}{r^{n+2s}} + \frac{C}{r^n} \leq \frac{C}{r^n} \end{aligned}$$

for all $r \geq R_0$ and some constant $C > 0$ independent of r.

Let φ be defined as above for $R_1 > R_0$. Again, we have $\varphi(x) \leq u(x)$ for all x with $|x| \leq R_0$ or $|x| \geq R_1$. Moreover,

$$(-\Delta)^s \varphi(x) \leq \frac{C}{|x|^n} \tag{3.17}$$

for all x with $R_0 < |x| < R_1$. From (3.14), one also has

$$(-\Delta)^s u(x) \geq C(m_u(R_0))^{pq} r^{(-n+2s)pq+2tp} = \frac{C}{|x|^n} \tag{3.18}$$

for $r \geq R_0$. By Silvestre's maximum principle (Lemma 2.3.6), we derive $u(x) \geq \varphi(x)$ for all $R_0 < |x| < R_1$. Letting $R_1 \to +\infty$ in this inequality, one obtains

$$u(x) \geq C\frac{\log(1+|x|)}{|x|^{n-2s}}\,.$$

On the other hand, using (3.12) and the fact that $(\frac{2s}{p}+2t)\frac{p}{pq-1} = n-2s$, one gets

$$C_1\frac{\log(1+|x|)}{|x|^{n-2s}} \leq m_u(r) \leq C_2\frac{1}{|x|^{n-2s}}$$

for all x with $|x| = r$ large enough. But this contradicts the positivity of u.

It still remains the situation when $-n+2s \in (-1,0)$. In this case, the function $\Theta(r)$ is increasing near the origin and decreasing for r large, with exactly one maximum point, say at $r_0 > 0$. Consider the function

$$w(r) = \begin{cases} \Theta(r_0) & \text{if } 0 < r \leq r_0 \\ \Theta(r) & \text{if } r_0 < r \end{cases}.$$

Again, one defines the comparison function for $R_0 > 1$ and $R_0/2 > r_0$ as in Lemma 3.1.2

$$\varphi(x) = m_u(R_0)\frac{w(r)-w(R_1)}{w(r_0)-w(R_1)}$$

for $|x| < R_1$ and $\varphi(x) = 0$ for $|x| \geq R_1$, where $R_1 > R_0$. It is clear that $\varphi(x) \leq u(x)$ for all x with $|x| \leq R_0$ or $|x| \geq R_1$. In addition,

$$(-\Delta)^s\varphi(x) \leq \frac{C}{|x|^n}$$

for all x with $R_0 < |x| < R_1$. Lastly, using Lemma 3.1.3 and the fact that Θ is increasing in $(0, r_0)$ and decreasing for $r \geq r_0$, the proof proceeds exactly as before and again we achieve the contradiction $u = 0$. This concludes the proof of sufficiency. ∎

Proof of the necessity of (4). Assume that the condition (4) fails. In other words, we have

$$\left(\frac{2s}{p}+2t\right)\frac{p}{pq-1} < n-2s \quad\text{and}\quad \left(\frac{2t}{q}+2s\right)\frac{q}{pq-1} < n-2t\,. \tag{3.19}$$

Consider the functions

$$u(x) = \frac{A}{(1+|x|)^{2sk_1}} \text{ and } v(x) = \frac{B}{(1+|x|)^{2tk_2}}\,, \tag{3.20}$$

where

$$k_1 = \frac{t+sp}{t(pq-1)} \text{ and } k_2 = \frac{s+tq}{s(pq-1)}.$$

The basic idea is to prove that (u,v) is a positive radial supersolution of (6) with $G = \mathbb{R}^n$ for a suitable choice of positive constants A and B.

Firstly, we assert that the inequalities

$$\frac{1}{(1-a+|ae_1+y|)^{2sk_1}} + \frac{1}{(1-a+|ae_1-y|)^{2sk_1}} \le \frac{1}{|e_1+y|^{2sk_1}} + \frac{1}{|e_1-y|^{2sk_1}} \tag{3.21}$$

and

$$\frac{1}{(1-a+|ae_1+y|)^{2tk_2}} + \frac{1}{(1-a+|ae_1-y|)^{2tk_2}} \le \frac{1}{|e_1+y|^{2tk_2}} + \frac{1}{|e_1-y|^{2tk_2}} \tag{3.22}$$

hold for all $a \in [0,1)$, $b \ge 0$ and $y \in \mathbb{R}$. In fact, consider the function $f(a,b,y)$ given by

$$f(a,b,y) = (1-a+((a+b)^2+y^2)^{1/2})^{-2\alpha} + (1-a+(a-b)^2+y^2)^{1/2})^{-2\alpha}$$

$$-((1+b)^2+y^2)^{-\alpha} - ((1-b)^2+y^2)^{-\alpha}$$

where $\alpha > 0$. One easily checks that

$$\begin{aligned}\frac{\partial f}{\partial a}(a,b,y) \quad = \quad & \frac{-2\alpha}{(1-a+(a+b)^2+y^2)^{1/2})^{2\alpha+1}}\left(-1+\frac{a+b}{((a+b)^2+y^2)^{1/2}}\right)\\ &+\frac{-2\alpha}{(1-a+(a-b)^2+y^2)^{1/2})^{2\alpha+1}}\left(-1+\frac{a-b}{((a-b)^2+y^2)^{1/2}}\right) \ge 0\end{aligned}$$

and $f(1,b,y) = 0$ for all $a \in [0,1)$, $b \ge 0$ and $y \in \mathbb{R}$. In particular, $f(a,b,y) \le 0$ for all $a \in [0,1)$, $b \ge 0$ and $y \in \mathbb{R}$.

For $a = r/(1+r)$ and x with $r = |x|$, we then have

$$\begin{aligned}&\frac{1}{(1+|x+y|)^{2\alpha}} + \frac{1}{(1+|x-y|)^{2\alpha}} - \frac{2}{(1+|x|)^{2\alpha}}\\ = \quad & \frac{1}{(1+|x|)^{2\alpha}}\left\{\frac{1}{(1-a+|ae_1+\overline{y}|)^{2\alpha}} + \frac{1}{(1-a+|ae_1-\overline{y}|)^{2\alpha}} - 2\right\}\\ \le \quad & \frac{1}{(1+|x|)^{2\alpha}}\left\{\frac{1}{|e_1+\overline{y}|^{2\alpha}} + \frac{1}{|e_1-\overline{y}|^{2\alpha}} - 2\right\},\end{aligned}$$

where $\overline{y} = \frac{1}{1+r}Py$, being P an appropriate rotation matrix.

With the choice $\alpha = sk_1$ and $\alpha = tk_2$, we derive (3.21) and (3.22), respectively. Using these inequalities, we find

$$\begin{aligned}
(-\Delta)^s u(x) &= -\frac{1}{2}\int\limits_{\mathbb{R}^n} \frac{A}{(1+|x-y|)^{2sk_1}|y|^{n+2s}} + \frac{A}{(1+|x+y|)^{2sk_1}|y|^{n+2s}} \\
&\quad -\frac{2A}{(1+|x|)^{2sk_1}|y|^{n+2s}}\,dy \\
&\geq -\frac{1}{2}\frac{A}{(1+|x|)^{2s(k_1+1)}}\int\limits_{\mathbb{R}^n} \frac{|e_1+y|^{-2sk_1}+|e_1-y|^{-2sk_1}-2}{|y|^{n+2s}}\,dy \\
&= \frac{c_1 A}{(1+|x|)^{2s(k_1+1)}}
\end{aligned}$$

and

$$\begin{aligned}
(-\Delta)^t v(x) &= -\frac{1}{2}\int\limits_{\mathbb{R}^n} \frac{B}{(1+|x-y|)^{2tk_2}|y|^{n+2t}} + \frac{B}{(1+|x+y|)^{2tk_2}|y|^{n+2t}} \\
&\quad -\frac{2B}{(1+|x|)^{2tk_2}|y|^{n+2t}}\,dy \\
&\geq -\frac{1}{2}\frac{B}{(1+|x|)^{2t(k_2+1)}}\int\limits_{\mathbb{R}^n} \frac{|e_1+y|^{-2tk_2}+|e_1-y|^{-2tk_2}-2}{|y|^{n+2t}}\,dy \\
&= \frac{c_2 B}{(1+|x|)^{2t(k_2+1)}}\,.
\end{aligned}$$

Since $pq > 1$, there exist constants k_1 and k_2 such that $2s(k_1+1) = 2tk_2p$ and $2t(k_2+1) = 2sk_1q$. Thanks to (3.19), it readily follows that k_1 and k_2 are positive, $2sk_1 < n-2s$ and $2tk_2 < n-2t$. These last two conditions guarantee the positivity of the above constants c_1 and c_2.

On the other hand, we have

$$(-\Delta)^s u(x) - v^p(x) \geq \frac{c_1 A}{(1+|x|)^{2s(k_1+1)}} - \frac{B^p}{(1+|x|)^{2tk_2p}} = \frac{c_1 A - B^p}{(1+|x|)^{2s(k_1+1)}}$$

and

$$(-\Delta)^t v(x) - u^q(x) \geq \frac{c_2 B}{(1+|x|)^{2t(k_2+1)}} - \frac{A^q}{(1+|x|)^{2sk_1q}} = \frac{c_2 B - A^q}{(1+|x|)^{2t(k_2+1)}}$$

for all $x \in \mathbb{R}^n$. Finally, the assumption $pq > 1$ also allows us to choose $A = (c_1 c_2^p)^{\frac{1}{pq-1}} > 0$ and $B = (c_1^q c_2)^{\frac{1}{pq-1}} > 0$ so that the right-hand side of the above inequalities are equal to zero. This concludes the proof of Theorem 0.0.4. ■

3.3 Proof of Theorem 0.0.5

The first tool to be used in the proof of Theorem 0.0.5 is the following result whose proof is based on the method of moving plane.

Proposition 3.3.1. *Let (u, v) be a positive viscosity bounded solution of*

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \mathbb{R}^n_+ \\ (-\Delta)^t v = u^q & \text{in } \mathbb{R}^n_+ \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}^n_+ \end{cases} \tag{3.23}$$

Assume $p, q \geq 1$. Then, u and v are strictly increasing in x_n-direction.

Proof of Proposition 3.3.1. Let $\Sigma_\mu := \{(\overline{x}, x_n) \in \mathbb{R}^n_+ \mid \ 0 < x_n < \mu\}$ and $T_\mu := \{(\overline{x}, x_n) \in \mathbb{R}^n_+ \mid \ x_n = \mu\}$. For $x = (\overline{x}, x_n) \in \mathbb{R}^n$, we denote $u_\mu(x) = u(x_\mu)$, $w_{\mu,u}(x) = u_\mu(x) - u(x)$, $v_\mu(x) = v(x_\mu)$ and $w_{\mu,v}(x) = v_\mu(x) - v(x)$, where $\mu > 0$ and $x_\mu = (\overline{x}, 2\mu - x_n)$ for all $(\overline{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. For any subset A of $\mathbb{R}^n$, we write $A_\mu = \{x_\mu \mid \ x \in A\}$, the reflection of A with respect to T_μ.

We next divide the proof into two steps.

First step: We here prove that if $\mu > 0$ is small enough, then $w_{\mu,u} > 0$ and $w_{\mu,v} > 0$ in $\sum_\mu$. For this purpose, we define

$$\Sigma^-_{\mu,u} = \{x \in \Sigma_\mu \mid \ w_{\mu,u}(x) < 0\} \text{ and } \Sigma^-_{\mu,v} = \{x \in \Sigma_\mu \mid \ w_{\mu,v}(x) < 0\}.$$

We first show that $\Sigma^-_{\mu,u}$ is empty if μ is small enough. Indeed, assume for a contradiction that $\Sigma^-_{\mu,u}$ is not empty and define

$$w^1_{\mu,u}(x) = \begin{cases} w_{\mu,u}(x) & \text{if } \ x \in \Sigma^-_{\mu,u} \\ 0 & \text{if } \ x \in \mathbb{R}^n \setminus \Sigma^-_{\mu,u} \end{cases} \tag{3.24}$$

and

$$w^2_{\mu,u}(x) = \begin{cases} 0 & \text{if } \ x \in \Sigma^-_{\mu,u} \\ w_{\mu,u}(x) & \text{if } \ x \in \mathbb{R}^n \setminus \Sigma^-_{\mu,u} \end{cases} \tag{3.25}$$

It is clear that $w^1_{\mu,u}(x) = w_{\mu,u}(x) - w^2_{\mu,u}(x)$ for all $x \in \mathbb{R}^n$. For each $\mu > 0$, we now assert that

$$(-\Delta)^s w^2_{\mu,u}(x) \leq 0 \text{ for all } x \in \Sigma^-_{\mu,u}\,. \tag{3.26}$$

In fact, from the definition of $(-\Delta)^s$, we have

$$
\begin{aligned}
(-\Delta)^s w_{\mu,u}^2(x) &= \int\limits_{\mathbb{R}^n} \frac{w_{\mu,u}^2(x) - w_{\mu,u}^2(y)}{|x-y|^{n+2s}}\, dy = - \int\limits_{\mathbb{R}^n \setminus \Sigma_{\mu,u}^-} \frac{w_{\mu,u}^2(y)}{|x-y|^{n+2s}}\, dy \\
&= - \int\limits_{(\Sigma_\mu \setminus \Sigma_{\mu,u}^-) \cup (\Sigma_\mu \setminus \Sigma_{\mu,u}^-)_\mu} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}}\, dy \\
&\quad - \int\limits_{(\mathbb{R}^n \setminus \mathbb{R}^n_+) \cup (\mathbb{R}^n \setminus \mathbb{R}^n_+)_\mu} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}}\, dy - \int\limits_{(\Sigma_{\mu,u}^-)_\mu} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}}\, dy \\
&= -A_1 - A_2 - A_3
\end{aligned}
$$

for all $x \in \Sigma_{\mu,u}^-$.

We next estimate separately each of these integrals.

Firstly, note that $w_{\mu,u}(y_\mu) = -w_{\mu,u}(y)$ for all $y \in \mathbb{R}^n$ and $w_{\mu,u}^2(y) \geq 0$ in $\Sigma_\mu \setminus \Sigma_{\mu,u}^-$. Then,

$$
\begin{aligned}
A_1 &= \int\limits_{(\Sigma_\mu \setminus \Sigma_{\mu,u}^-) \cup (\Sigma_\mu \setminus \Sigma_{\mu,u}^-)_\mu} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}}\, dy \\
&= \int\limits_{\Sigma_\mu \setminus \Sigma_{\mu,u}^-} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}}\, dy + \int\limits_{\Sigma_\mu \setminus \Sigma_{\mu,u}^-} \frac{w_{\mu,u}(y_\mu)}{|x-y_\mu|^{n+2s}}\, dy \\
&= \int\limits_{\Sigma_\mu \setminus \Sigma_{\mu,u}^-} w_{\mu,u}(y) \left(\frac{1}{|x-y|^{n+2s}} - \frac{1}{|x-y_\mu|^{n+2s}} \right) dy \geq 0\,,
\end{aligned}
$$

since $|x-y_\mu| > |x-y|$ for all $x \in \Sigma_{\mu,u}^-$ and $y \in \Sigma_\mu \setminus \Sigma_{\mu,u}^-$.

In order to discover the sign of A_2 we observe that $u = 0$ in $\mathbb{R}^n \setminus \mathbb{R}^n_+$ and $u_\mu = 0$ in $(\mathbb{R}^n \setminus \mathbb{R}^n_+)_\mu$, so we have

$$
\begin{aligned}
A_2 &= \int\limits_{(\mathbb{R}^n \setminus \mathbb{R}^n_+) \cup (\mathbb{R}^n \setminus \mathbb{R}^n_+)_\mu} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}}\, dy \\
&= \int\limits_{\mathbb{R}^n \setminus \mathbb{R}^n_+} \frac{u_\mu(y)}{|x-y|^{n+2s}}\, dy - \int\limits_{(\mathbb{R}^n \setminus \mathbb{R}^n_+)_\mu} \frac{u(y)}{|x-y|^{n+2s}}\, dy \\
&= \int\limits_{\mathbb{R}^n \setminus \mathbb{R}^n_+} u_\mu(y) \left(\frac{1}{|x-y|^{n+2s}} - \frac{1}{|x-y_\mu|^{n+2s}} \right) dy \geq 0\,,
\end{aligned}
$$

since $u_\mu \geq 0$ in $\mathbb{R}^n \setminus \mathbb{R}^n_+$ and $|x-y_\mu| > |x-y|$ for all $x \in \Sigma_{\mu,u}^-$ and $y \in \mathbb{R}^n \setminus \mathbb{R}^n_+$.

Finally, since $w_{\mu,u} < 0$ in $\Sigma_{\mu,u}^-$, we have

$$
A_3 = \int\limits_{(\Sigma_{\mu,u}^-)_\mu} \frac{w_{\mu,u}(y)}{|x-y|^{n+2s}}\, dy = \int\limits_{\Sigma_{\mu,u}^-} \frac{w_{\mu,u}(y_\mu)}{|x-y_\mu|^{n+2s}}\, dy = - \int\limits_{\Sigma_{\mu,u}^-} \frac{w_{\mu,u}(y)}{|x-y_\mu|^{n+2s}}\, dy \geq 0\,.
$$

Hence, the claim (3.26) follows.

Using now (3.26), for any $x \in \Sigma^-_{\mu,u}$, one has

$$\begin{aligned}(-\Delta)^s w^1_{\mu,u}(x) &= (-\Delta)^s w_{\mu,u}(x) = (-\Delta)^s u_\mu(x) - (-\Delta)^s u(x)\\ &= v^p_\mu(x) - v^p(x) = \frac{v^p_\mu(x) - v^p(x)}{v_\mu(x) - v(x)} w_{\mu,v}(x)\,.\end{aligned}$$

Define

$$\varphi_v(x) = \frac{v^p_\mu(x) - v^p(x)}{v_\mu(x) - v(x)}$$

for $x \in \Sigma^-_{\mu,u}$.

Since $p \geq 1$, we have $\varphi_v \in L^\infty(\Sigma^-_{\mu,u})$ and $\varphi_v w_{\mu,v}$ is continuous. In addition, since $w^1_{\mu,u} = 0$ in $\mathbb{R}^n \setminus \Sigma^-_{\mu,u}$, by Theorem 2.3.2, one gets

$$\|w^1_{\mu,u}\|_{L^\infty(\Sigma^-_{\mu,u})} \leq CR(\Sigma^-_{\mu,u})^{2s}\|\varphi_v w_{\mu,v}\|_{L^\infty(\Sigma^-_{\mu,u})}\,, \tag{3.27}$$

where $R(\Sigma^-_{\mu,u})$ is the smallest positive constant R such that

$$|B_R(x) \setminus \Sigma^-_{\mu,u}| \geq \frac{1}{2}|B_R(x)|$$

for all $x \in \Sigma^-_{\mu,u}$. Besides, we have

$$\varphi_v w_{\mu,v}(x) = v^p(x) - v^p_\mu(x) \leq 0 \ \text{ in } \ \Sigma_\mu \setminus \Sigma^-_{\mu,v}$$

and

$$\varphi_v w_{\mu,v}(x) = v^p(x) - v^p_\mu(x) > 0 \ \text{ in } \ \Sigma^-_{\mu,v}\,.$$

Let $\Sigma^-_\mu = \Sigma^-_{\mu,u} \cap \Sigma^-_{\mu,v}$. Then, from (3.27), one derives

$$\begin{aligned}\|w^1_{\mu,u}\|_{L^\infty(\Sigma^-_{\mu,u})} &\leq CR(\Sigma^-_{\mu,u})^{2s}\|\varphi_v w_{\mu,v}\|_{L^\infty(\Sigma^-_\mu)}\\ &\leq CR(\Sigma^-_{\mu,u})^{2s}\|\varphi_v\|_{L^\infty(\Sigma^-_\mu)}\|w_{\mu,v}\|_{L^\infty(\Sigma^-_\mu)}\\ &\leq CR(\Sigma^-_{\mu,u})^{2s}\|w_{\mu,v}\|_{L^\infty(\Sigma^-_\mu)}\,,\end{aligned}$$

where in the last inequality we use the condition $p \geq 1$.

Similar to (3.24) and (3.25), we define

$$w^1_{\mu,v}(x) = \begin{cases} w_{\mu,v}(x) & \text{if } \ x \in \Sigma^-_{\mu,v}\\ 0 & \text{if } \ x \in \mathbb{R}^n \setminus \Sigma^-_{\mu,v}\end{cases} \tag{3.28}$$

and

$$w^2_{\mu,v}(x) = \begin{cases} 0 & \text{if } \ x \in \Sigma^-_{\mu,v} \\ w_{\mu,v}(x) & \text{if } \ x \in \mathbb{R}^n \setminus \Sigma^-_{\mu,v} \end{cases} \tag{3.29}$$

and argue in a completely analogous way with the aid of the assumption $q \geq 1$ to obtain

$$\|w^1_{\mu,v}\|_{L^\infty(\Sigma^-_{\mu,v})} \leq CR(\Sigma^-_{\mu,v})^{2t} \|w_{\mu,u}\|_{L^\infty(\Sigma^-_{\mu})} \,.$$

Thus,

$$\|w^1_{\mu,u}\|_{L^\infty(\Sigma^-_{\mu,u})} \leq C^2 R(\Sigma^-_{\mu,u})^{2s} R(\Sigma^-_{\mu,v})^{2t} \|w^1_{\mu,u}\|_{L^\infty(\Sigma^-_{\mu,u})}$$

and

$$\|w^1_{\mu,v}\|_{L^\infty(\Sigma^-_{\mu,v})} \leq C^2 R(\Sigma^-_{\mu,u})^{2s} R(\Sigma^-_{\mu,v})^{2t} \|w^1_{\mu,v}\|_{L^\infty(\Sigma^-_{\mu,v})} \,.$$

Now choosing μ small enough so that $C^2 R(\Sigma^-_{\mu,u})^{2s} R(\Sigma^-_{\mu,v})^{2t} < 1$, we conclude that $\|w^1_{\mu,u}\|_{L^\infty(\Sigma^-_{\mu,u})} = 0$, so $|\Sigma^-_{\mu,u}| = 0$. Since $\Sigma^-_{\mu,u}$ is open, we deduce that $\Sigma^-_{\mu,u}$ is empty, which is a contradiction. Therefore, we get $w_{\mu,u} \geq 0$ in Σ_μ for $\mu > 0$ small enough. Similarly, one gets $w_{\mu,v} \geq 0$ in Σ_μ for $\mu > 0$ small enough too. Moreover, since the functions u and v are positive in $\mathbb{R}^n_+$ and $u = v = 0$ in $\mathbb{R}^n \setminus \mathbb{R}^n_+$, it follows that $w_{\mu,u}$ and $w_{\mu,v}$ are positive in $\{x_n = 0\}$ and then, by continuity, $w_{\mu,u} \neq 0$ and $w_{\mu,v} \neq 0$ in Σ_μ.

In order to complete the proof of this step, we assert that if $w_{\mu,u} \geq 0$, $w_{\mu,v} \geq 0$, $w_{\mu,u} \neq 0$ and $w_{\mu,v} \neq 0$ in Σ_μ with $\mu > 0$, then $w_{\mu,u} > 0$ and $w_{\mu,v} > 0$ in Σ_μ. Indeed, we have

$$(-\Delta)^s w_{\mu,u}(x) = v^p_\mu(x) - v^p(x) \geq 0 \ \text{ in } \ \Sigma_\mu$$

and

$$(-\Delta)^t w_{\mu,v}(x) = u^q_\mu(x) - u^q(x) \geq 0 \ \text{ in } \ \Sigma_\mu \,.$$

Since $w_{\mu,u} \geq 0$, $w_{\mu,v} \geq 0$, $w_{\mu,u} \neq 0$ and $w_{\mu,v} \neq 0$ in Σ_μ, by the Silvestre's strong maximum principle, the conclusion follows.

Second step: Define

$$\mu^* = sup\{\mu > 0 \mid \ w_{\nu,u} > 0, \ w_{\nu,v} > 0 \ \text{ in } \ \Sigma_\nu \ \text{ for all } \ 0 < \nu < \mu\} \,.$$

It is clear that $\mu^* > 0$ and $w_{\mu,u} > 0$ and $w_{\mu,v} > 0$ in Σ_μ for all $0 < \mu < \mu^*$, so that u and v are strictly increasing in x_n-direction. Indeed, for $0 < x_n < \overline{x}_n < \mu^*$, let $\mu = \frac{x_n + \overline{x}_n}{2}$. Since $w_{\mu,u} > 0$ and $w_{\mu,v} > 0$ in Σ_μ, we have

$$0 < w_{\mu,u}(x', x_n) = u_\mu(x', x_n) - u(x', x_n) = u(x', \overline{x}_n) - u(x', x_n)$$

and

$$0 < w_{\mu,v}(x', x_n) = v_\mu(x', x_n) - v(x', x_n) = v(x', \overline{x}_n) - v(x', x_n)\,,$$

so that $u(x', \overline{x}_n) > u(x', x_n)$ and $v(x', \overline{x}_n) > v(x', x_n)$, as claimed. Thus, the proposition is proved if we are able to show that $\mu^* = +\infty$.

Suppose for a contradiction that μ^* is finite. Now choose $\varepsilon_0 > 0$ small enough such that the operators $(-\Delta)^s - \varphi_v$ and $(-\Delta)^t - \varphi_u$ satisfies the strong maximum principle in the open $\Sigma_{\mu^*+\varepsilon_0} \setminus \Sigma_{\mu^*-\varepsilon_0}$, see [147]. Here we use that $\varphi_u(x) = \frac{u_\mu^q(x) - u^q(x)}{u_\mu(x) - u(x)}$ and $\varphi_v(x) = \frac{v_\mu^p(x) - v^p(x)}{v_\mu(x) - v(x)}$) can be taken small in the L^∞-norm, since $p, q > 1$. Therefore, $w_{\mu^*+\varepsilon_0,u} > 0$ and $w_{\mu^*+\varepsilon_0,v} > 0$ in $\Sigma_{\mu^*+\varepsilon_0}$, providing a contradiction. ∎

Proposition 3.3.2. *Let $p, q > 0$. If the system*

$$\begin{cases} (-\Delta)^s u = v^p & \text{in } \mathbb{R}^n_+ \\ (-\Delta)^t v = u^q & \text{in } \mathbb{R}^n_+ \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}^n_+ \end{cases} \tag{3.30}$$

has a positive viscosity bounded solution, then the same system has a positive viscosity solution in $\mathbb{R}^{n-1}$.

Proof of Proposition 3.3.2. Let (u, v) be a positive bounded solution of (3.30), that is there exists a constant M such that $0 < u \leq M$ and $0 < v \leq M$ in $\mathbb{R}^n_+$. In the strip $\Sigma_1 = \{x \in \mathbb{R}^n \mid 0 < x_n < 1\}$, we set

$$u_k(x', x_n) = u(x', x_n + k) \ \text{ and } \ v_k(x', x_n) = v(x', x_n + k)\,.$$

Note that (u_k, v_k) solves the system (3.30) in Σ_1 for each integer $k \geq 1$. In addition, $0 < u_k \leq M$ and $0 < v_k \leq M$ in Σ_1. Thus,

$$(-\Delta)^s u_k \leq M^p \ \text{ and } \ (-\Delta)^s u_k \geq 0 \ \text{ in } \Sigma_1\,,$$

$$(-\Delta)^t v_k \leq M^q \ \text{ and } \ (-\Delta)^t v_k \geq 0 \ \text{ in } \Sigma_1\,.$$

Then, by Theorem 2.3.9, for any $\Omega' \subset\subset \Sigma_1$ and $0 < \beta < 1$, there exists a constant $C > 0$ such that $u_k, v_k \in C^\beta(\Omega')$ and

$$\|u_k\|_{C^\beta(\Omega')} \le C\left\{\|u_k\|_{L^\infty(\Sigma_1)} + M^p\right\}$$

and

$$\|v_k\|_{C^\beta(\Omega')} \le C\left\{\|v_k\|_{L^\infty(\Sigma_1)} + M^q\right\} .$$

So, the sequences $\{u_k\}$ and $\{v_k\}$ are bounded in $C^\beta(\Omega')$ and then, up to a subsequence, $\{u_k\}$ and $\{v_k\}$ converge uniformly on compact subset of Σ_1 to functions $\overline{u}$ and $\overline{v}$, respectively. By Theorem 2.3.11, $(\overline{u},\overline{v})$ satisfies

$$\begin{cases} (-\Delta)^s\overline{u} = \overline{v}^q \text{ in } \Sigma_1 \\ (-\Delta)^t\overline{v} = \overline{u}^p \text{ in } \Sigma_1 \end{cases} \tag{3.31}$$

in the viscosity sense. The strict monotonicity provided in Proposition 3.3.1 guarantees that $(\overline{u},\overline{v})$ is positive and independent of the x_n-variable.

On the other hand, the definition of $(-\Delta)^s$ gives

$$\begin{aligned} (-\Delta)^s\overline{u}(x) &= \int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}} \frac{\overline{u}(x')-\overline{u}(y')}{(|x'-y'|^2+(x_n-y_n)^2)^{\frac{n+2s}{2}}}\,dy_n\,dy' \\ &= \int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}} \frac{\overline{u}(x')-\overline{u}(x'-y')}{(|y'|^2+(y_n)^2)^{\frac{n+2s}{2}}}\,dy_n\,dy' . \end{aligned}$$

Let $y_n = |y'|\tan\theta$, where $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$, then

$$\begin{aligned} (-\Delta)^s\overline{u}(x) &= \int\limits_{\mathbb{R}^{n-1}}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\overline{u}(x')-\overline{u}(x'-y')}{|y'|^{n-1+2s}}(\cos\theta)^{n-2+2s}\,d\theta\,dy' \\ &= \int\limits_{\mathbb{R}^{n-1}} \frac{\overline{u}(x')-\overline{u}(x'-y')}{|y'|^{n-1+2s}}\,dy' \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\theta)^{n-2+2s}\,d\theta \end{aligned}$$

and

$$\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\theta)^{n-2+2s}\,d\theta = 2\int\limits_{0}^{\frac{\pi}{2}}(\cos\theta)^{n-2+2s}\,d\theta < +\infty\,,$$

since $n-2+2s > 0$. This means that the n-dimension fractional Laplace operator is actually $(n-1)$-dimension, and we have

$$\begin{cases} (-\Delta)^s\overline{u} = \overline{v}^q \text{ in } \mathbb{R}^{n-1} \\ (-\Delta)^t\overline{v} = \overline{u}^p \text{ in } \mathbb{R}^{n-1} \end{cases} \tag{3.32}$$

■

Finally, Theorem 0.0.5 follows directly from Theorem 0.0.4 and Proposition 3.3.2. ■

3.4 Proof of Theorem 0.0.1

The proof of the part of existence is an application of degree theory for compact operators in cones. This theory, essentially developed by Krasnoselskii, has often been used to show that certain operators admit fixed points. We are going to use an extension of Krasnoselskii results (se for instance [146]). The applicability of this theory relies on a priori bounds in L^∞ of solutions of certain systems related to (1) to be obtained through blow-up techniques by invoking Theorems 0.0.4 and 0.0.5.

We begin by stating the above-mentioned abstract tool.

Proposition 3.4.1. *Let K be a closed cone with nonempty interior in a Banach space X and let $T : K \to K$ and $H : [0, \infty) \times K \to K$ be continuous compact operators such that $T(0) = 0$ and $H(0, x) = T(x)$ for all $x \in K$. Assume there exist $\theta_0 > 0$ and $0 < r < R$ such that*

(i) $x \neq \theta T(x)$ for all $0 \leq \theta \leq 1$ and $x \in K$ such that $\|x\| = r$,

(ii) $H(\theta, x) \neq x$ for all $\theta \geq \theta_0$ and $x \in K$ with $\|x\| \leq R$,

(iii) $H(\theta, x) \neq x$ for all $\theta \in [0, +\infty)$ and $x \in K$ with $\|x\| = R$.

Then, T has a fixed point $x_0 \in K$ such that $r \leq \|x_0\| \leq R$.

Here X denotes the Banach space $\{(u, v) \in C(\mathbb{R}^n) \times C(\mathbb{R}^n) \mid \ u, v = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$ endowed with the norm

$$\|(u, v)\| := \max\{\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}\}$$

and $K = \{u \in X \mid \ u, v \geq 0 \text{ in } \Omega\}$. It is clear that solving (1) is equivalent to finding a fixed point in K of the operator $T : K \to K$ given by

$$T(u, v)(x) := S(v^p, u^q)$$

for $x \in \Omega$, where for any $(f, g) \in K$ we define $S(f, g)$ as the solution of the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ (-\Delta)^t v = g & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{3.33}$$

Using that Ω is C^2 class, by Lemma 6.1 of [146], the operator S is well defined, linear, continuous and compact. Thus, one easily deduces that the operator T is well defined, continuous and compact. In addition, we have $T(0, 0) = 0$.

We also define $H : [0,\infty) \times K \to K$ as

$$H(\theta, u, v) = S((v+\theta)^p, (u+\theta)^q)\,.$$

Clearly, H is well defined, continuous and compact too.

First we show that the condition (i) of Proposition 3.4.1 is satisfied. This is the content of the following lemma:

Lemma 3.4.1. *Assume that $s, t \in (0,1)$ and $pq > 1$. Then, there exists a constant $r > 0$ such that for any $\theta \in [0,1]$, the system*

$$\begin{cases} (-\Delta)^s u = \theta v^p & \text{in } \Omega \\ (-\Delta)^t v = \theta u^q & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{3.34}$$

has no classical solution $(u, v) \in K$ with $\|(u,v)\| = r$.

Proof of Lemma 3.4.1. We argue by contradiction. Let $\{(\theta_k, u_k, v_k)\}_{k\in\mathbb{N}}$ be a sequence of triples with $\theta_k \in [0,1]$ and $(u_k, v_k) \in K$ satisfying (3.34) such that $\|u_k\|_{L^\infty(\Omega)}, \|v_k\|_{L^\infty(\Omega)} \to 0$ as $k \longrightarrow +\infty$. Since $pq > 1$, we choose γ such that

$$\frac{1}{q} < \gamma < p$$

and set $a_k = \|u_k\|_{L^\infty(\Omega)} + \|v_k\|^\gamma_{L^\infty(\Omega)}$. Define

$$z_k = \frac{u_k}{a_k} \quad \text{and} \quad w_k = \frac{v_k}{a_k^{1/\gamma}}\,.$$

We then have

$$(-\Delta)^s z_k = \frac{\theta_k}{a_k} v_k^p \quad \text{and} \quad (-\Delta)^t w_k = \frac{\theta_k}{a_k^{1/\gamma}} u_k^q\,.$$

Note that $\|z_k\|_{L^\infty(\Omega)} + \|w_k\|^\gamma_{L^\infty(\Omega)} = 1$,

$$\left|\frac{\theta_k}{a_k} v_k^p\right| \le \|v_k\|^{p-\gamma}_{L^\infty(\Omega)} \to 0 \quad \text{and} \quad \left|\frac{\theta_k}{a_k} u_k^q\right| \le \|u_k\|^{q-1/\gamma}_{L^\infty(\Omega)} \to 0$$

uniformly for $x \in \Omega$. So, one easily deduces that (z_k, w_k) converges uniformly to some couple (z, w) satisfying $\|z\|_{L^\infty(\Omega)} + \|w\|^\gamma_{L^\infty(\Omega)} = 1$ and

$$\begin{cases} (-\Delta)^s z = 0 & \text{in } \Omega \\ (-\Delta)^t w = 0 & \text{in } \Omega \\ z = w = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

But by uniqueness, we have $(z, w) = (0, 0)$, providing a contradiction. ■

The condition (ii) of Proposition 3.4.1 follows from the following lemma:

Lemma 3.4.2. *Assume that $s, t \in (0, 1)$, $p, q \geq 1$ and $pq > 1$. Then, there exists a constant $\theta_0 > 0$ such that for any $\theta \geq \theta_0$ the system*

$$\begin{cases} (-\Delta)^s u = (v+\theta)^p & \text{in } \Omega \\ (-\Delta)^t v = (u+\theta)^q & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{3.35}$$

has no classical solution $(u, v) \in K$.

Proof of Lemma 3.4.2. Firstly, we define

$$\lambda_1 := \inf\{\int_\Omega |(-\Delta)^{s/2}u|^2 + |(-\Delta)^{t/2}v|^2 \, dx \mid (u, v) \in H_0^s(\Omega) \times H_0^t(\Omega), \int_\Omega u^+ v^+ \, dx = 1\},$$

where $f^+ = \max\{f, 0\}$. As usual, it follows that λ_1 is positive and attained for some couple $(\varphi, \psi) \in H_0^s(\Omega) \times H_0^t(\Omega)$. Also, by the weak maximum principle, $\varphi, \psi \geq 0$ in Ω and $\varphi, \psi \neq 0$ and, moreover, (φ, ψ) satisfies

$$\begin{cases} (-\Delta)^s \varphi = \lambda_1 \psi & \text{in } \Omega \\ (-\Delta)^t \psi = \lambda_1 \varphi & \text{in } \Omega \\ \varphi = \psi = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

On the other hand, by assumption, $p > 1$ or $q > 1$. If the first situation occurs, then for $A \geq \lambda_1^2$ there exists $\theta_0 > 0$ such that

$$(y+\theta)^p \geq A(y+\theta) > Ay \quad \text{and} \quad (y+\theta)^p \geq (y+\theta) > y$$

for all $y \geq 0$ and $\theta \geq \theta_0$.

Now let $\theta \geq \theta_0$ and $(u, v) \in K$ be a classical solution of (3.35). Then, by the Silvestre's strong maximum principle, we have $u, v > 0$ in Ω and

$$\begin{cases} (-\Delta)^s u > Av & \text{in } \Omega \\ (-\Delta)^t v > u & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

Using the above equations satisfied by (φ, ψ), one obtains

$$\lambda_1 \int_\Omega u\psi \, dx > A \int_\Omega v\varphi \, dx \quad \text{and} \quad \lambda_1 \int_\Omega v\varphi \, dx > \int_\Omega u\psi \, dx \,,$$

so that $A < \lambda_1^2$, providing a contradiction. ■

Finally, the condition (iii) of Proposition 3.4.1 is a consequence of the following lemma:

Lemma 3.4.3. *Assume that Ω is of C^2 class, $s,t \in (0,1)$, $n > 2s+1$, $n > 2t+1$, $p,q \geq 1$, $pq > 1$ and (4) is satisfied. For each $\theta_0 > 0$ there exists a constant $C > 0$, depending only of s,t,p,q and Ω, such that for any classical solution $(u,v) \in K$ of the system (3.35) with $0 \leq \theta \leq \theta_0$, one has*

$$\|(u,v)\| \leq C\,.$$

Proof of Lemma 3.4.3. Suppose for a contradiction that there exists a sequence $(u_k,v_k) \in K$ of solutions of (3.35) with $\theta = \theta_k \in [0,\theta_0]$ such that at least one of the sequence (u_k) and (v_k) tends to infinity in the L^∞-norm.

Let $\beta_1 = \left(\frac{2s}{p} + 2t\right)\frac{p}{pq-1}$ and $\beta_2 = \left(\frac{2t}{q} + 2s\right)\frac{q}{pq-1}$. We set

$$\lambda_k = \|u_k\|_{L^\infty(\Omega)}^{-\frac{1}{\beta_1}}\,,$$

if $\|u_k\|_{L^\infty(\Omega)}^{\beta_2} \geq \|v_k\|_{L^\infty(\Omega)}^{\beta_1}$, up to a subsequence, and $\lambda_k = \|v_k\|_{L^\infty(\Omega)}^{-\frac{1}{\beta_2}}$, otherwise. It suffices to assume the first of these two situations.

Note that $\lambda_k \to 0$ as $k \to +\infty$. Let $x_k \in \Omega$ be a maximum point of u_k. The functions

$$z_k(x) = \lambda_k^{\beta_1} u_k(\lambda_k x + x_k) \quad\text{and}\quad w_k(x) = \lambda_k^{\beta_2} v_k(\lambda_k x + x_k)$$

are such that $z_k(0) = 1$ and $0 \leq z_k, w_k \leq 1$ in $\Omega_k := \frac{1}{\lambda_k}(\Omega - x_k)$. Also, one checks that the functions z_k and w_k satisfy

$$\begin{cases} (-\Delta)^s z_k = \left(\lambda_k^{(2s+\beta_1-p\beta_2)/p} w_k + \lambda_k^{(2s+\beta_1)/p}\theta_k\right)^p = \left(w_k + \lambda_k^{(2s+\beta_1)/p}\theta_k\right)^p \\ (-\Delta)^t w_k = \left(\lambda_k^{(2t+\beta_2-q\beta_1)/q} z_k + \lambda_k^{(2t+\beta_2)/q}\theta_k\right)^q = \left(z_k + \lambda_k^{(2t+\beta_2)/q}\theta_k\right)^q \end{cases} \tag{3.36}$$

in the open Ω_k.

By compactness, module a subsequence, (x_k) converges to some point $x_0 \in \overline{\Omega}$. Let

$$d_k = dist(x_k, \partial\Omega)\,.$$

Two cases may occur as $k \to +\infty$:

(a) $\frac{d_k}{\lambda_k} \to +\infty$, module a subsequence still denoted as before, or

(b) $\frac{d_k}{\lambda_k}$ is bounded.

If (a) occurs, then $\frac{1}{\lambda_k}B_{d_k}(0) \subset \Omega_k$ and $\frac{d_k}{\lambda_k} \to +\infty$ as $k \to +\infty$. So, (Ω_k) tends to $\mathbb{R}^n$ as $k \to +\infty$. We recall that $0 \le z_k, w_k \le 1$ in Ω_k. Thus, the right-hand side of (3.36) is bounded in $L^\infty(\Omega_k)$, so by compactness, we deduce that, up to a subsequence, (z_k, w_k) converges to some function (z, w) uniformly in compact sets of $\mathbb{R}^n$. By Theorem 2.3.11, (z, w) is a viscosity solution of (6) with $G = \mathbb{R}^n$. Note also that $z(0) = 1$, since $z_k(0) = 1$ for all k, and hence $(z, w) \neq (0, 0)$ and, by the Silvestre's strong maximum principle, $z, w > 0$ in $\mathbb{R}^n$. But this contradicts Theorem 0.0.4.

Assume now that (b) occurs, that is $\frac{d_k}{\lambda_k}$ is bounded. In this case, up to a subsequence, we may assume that

$$\frac{d_k}{\lambda_k} \to a \in [0, \infty)\,. \tag{3.37}$$

Assume for a moment that $a > 0$. After a suitable rotation of $\mathbb{R}^n$ for each fixed k, one concludes that (Ω_k) converge to the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > -a\}$. Again, we have $0 \le z_k, w_k \le 1$ in Ω_k and then, by compactness, (z_k, w_k) converges, module a subsequence, to some function (z, w) uniformly in compact sets of $\mathbb{R}^n_+$. As before, (z, w) is a viscosity bounded solution of (6) with $G = \mathbb{R}^n_+$. Furthermore, using that $a > 0$ and $z_k(0) = 1$ for all k, one gets $z(0) = 1$, so that again $z, w > 0$ in Ω and this contradicts Theorem 0.0.5.

The remainder of the proof consists in showing that $a > 0$. We argue by contradiction and assume that $a = 0$. The basic idea is to construct a barrier function h_k on Ω_k for z_k. For this purpose, we define

$$h_k(x) = (e^{-\frac{d_k}{\lambda_k}} - e^{x_n}) \sup_{\Omega_k} \frac{(w_k + \lambda_k^{(2s+\beta_1)/p}\theta_k)^p}{C_0}\,,$$

where C_0 is a positive constant such that

$$\begin{aligned}(-\Delta)^s e^{x_n} &= -\int_{\mathbb{R}^n} \frac{e^{(x_n+y_n)} + e^{(x_n-y_n)} - 2e^{x_n}}{|y|^{n+2s}}\,dy \\ &= -e^{x_n}\int_{\mathbb{R}^n} \frac{e^{y_n} + e^{-y_n} - 2}{|y|^{n+2s}} dy \le -C_0 < 0\end{aligned}$$

for all $-\frac{d_k}{\lambda_k} < x_n < 0$. Thus, from (3.36),

$$(-\Delta)^s(h_k - z_k) \ge C_0 \sup_{\Omega_k}\left(\frac{(w_k + \lambda_k^{(2s+\beta_1)/p}\theta_k)^p}{C_0}\right) - \frac{(w_k + \lambda_k^{(2s+\beta_1)/p}\theta_k)^p}{C_0} \ge 0$$

in Ω_k and $z_k \le h_k$ in $\mathbb{R}^n \setminus \Omega_k$. Then, the weak maximum principle gives $z_k \le h_k$ in Ω_k. In addition, there exist $C_1 > 0$ and $\delta > 0$ such that

$$|\nabla w_k(x)| \le C_1$$

for all $x \in \Omega_k \cap \{x \in \mathbb{R}^n \mid x_n + \frac{d_k}{\lambda_k} \leq \delta\}$. Since $x_k \in \Omega$, we have $0 \in \Omega_k \cap \{x \in \mathbb{R}^n \mid x_n + \frac{d_k}{\lambda_k} \leq \delta\}$ for k large enough. Finally,

$$1 = z_k(0) \leq h_k(0) \leq C_2 \left(e^{-\frac{d_k}{\lambda_k}} - 1 \right) \to 0$$

as $k \to \infty$, providing a contradiction. ■

Lastly, the conclusion of Theorem 0.0.1 follows readily from Lemmas 3.4.1, 3.4.2 and 3.4.3 applied to Proposition 3.4.1.

CAPÍTULO 4

Proof of variational contributions

4.1 Variational setting

Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. In order to inspire our formulation, assume that the couple (u, v) of nonnegative functions is roughly a solution of (8). From the first equation, we have $v = ((-\Delta)^s u)^{\frac{1}{p}}$. Plugging this equality into the second equation, we obtain

$$\left\{ \begin{array}{rcll} (-\Delta)^s \left((-\Delta)^s u\right)^{\frac{1}{p}} & = & u^q & \text{in } \Omega \\ u & = & 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{array} \right. . \tag{4.1}$$

On the other hand, nonnegative weak solutions of the above scalar problem can be seen as critical points of the functional $\Phi : E_p^s \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{p}{p+1} \int_\Omega |(-\Delta)^s u|^{\frac{p+1}{p}} dx - \frac{1}{q+1} \int_\Omega (u^+)^{q+1} dx\,, \tag{4.2}$$

where $E_p^s = W_0^{s,\frac{p+1}{p}}(\Omega) \cap W^{2s,\frac{p+1}{p}}(\Omega)$. Note that E_p^s is a reflexive Banach space.

In the case that E_p^s is continuously embedded in $L^{q+1}(\Omega)$, the Gateaux derivative of Φ at $u \in E_p^s$ in the direction $\varphi \in E_p^s$ is given by

$$\Phi'(u)\varphi = \int_\Omega |(-\Delta)^s u|^{\frac{1}{p}-1} (-\Delta)^s u (-\Delta)^s \varphi dx - \int_\Omega (u^+)^q \varphi dx\,.$$

Assume the couple (p, q) is below the critical hyperbole (11). In this case, the embedding $E_p^s \hookrightarrow L^{q+1}(\Omega)$ is continuous and compact. So, by Proposition 2.3.1, the problem

$$\left\{ \begin{array}{rcll} (-\Delta)^s v & = & (u^+)^q & \text{in } \Omega \\ v & = & 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{array} \right.$$

admits a unique nonnegative weak solution $v \in E_q^s$. Then, one easily checks that u is a nonnegative weak solution of the problem

$$\begin{cases} (-\Delta)^s u &= v^p \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

In short, starting from a critical point $u \in E_p^s$ of Φ, we have constructed a nonnegative weak solution $(u, v) \in E_p^s \times E_q^s$ of the problem (8). By the C^β regularity result to be proved in the next section (Proposition 4.2.1) and the Silvestre's strong maximum principle (see Lemma 2.3.6), we deduce that $(u, v) \in C^0(\mathbb{R}^n) \times C^0(\mathbb{R}^n)$ is a positive viscosity solution of (8), whenever $u \in E_p^s$ is a nonzero critical point of Φ.

4.2 Hölder regularity

In this section, we show that weak solutions of (8) are C^β viscosity solutions by assuming that (p, q) is below the hyperbole (11) if $n > 2s$.

Proposition 4.2.1. *Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. Let $(u, v) \in E_p^s \times E_q^s$ be a nonnegative weak solution of the problem (8). Assume that the couple (p, q) satisfies (9) in the case that $n > 2s$, then $(u, v) \in L^\infty(\Omega) \times L^\infty(\Omega)$ and, in addition, $(u, v) \in C^\beta(\mathbb{R}^n) \times C^\beta(\mathbb{R}^n)$ for any $0 < \beta < 1$.*

Proof. It suffices to prove the proposition for $n > 2s$, since the ideas involved in its proof are fairly similar when $n \leq 2s$.

We analyze separately some different cases depending on the values of p and q.

For $0 < p < \frac{2s}{n-2s}$, we have $W^{2s, \frac{p+1}{p}}(\Omega) \hookrightarrow L^\infty(\Omega)$ (see Theorem 2.1.6), so that $u \in L^\infty(\Omega)$, and thus $v \in L^\infty(\Omega)$, by Proposition 2.2.3.

For $\frac{2s}{n-2s} \leq p \leq 1$ and $q > 1$, we rewrite the problem (8) as follows

$$\begin{cases} (-\Delta)^s u = a(x) v^{p/2} & \text{in } \Omega \\ (-\Delta)^s v = b(x) u & \text{in } \Omega \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} . \tag{4.3}$$

Since $a \in L^{\frac{p+1}{p/2}}(\Omega)$, for any fixed $\varepsilon > 0$, we can construct functions $q_\varepsilon \in L^{\frac{p+1}{p/2}}(\Omega)$ and $f_\varepsilon \in L^\infty(\Omega)$ and a constant $K_\varepsilon > 0$ such that

$$v^p = q_\varepsilon(x) v^{p/2}(x) + f_\varepsilon(x)$$

and

$$\|q_\varepsilon\|_{L^{\frac{p+1}{p/2}}(\Omega)} < \varepsilon, \quad \|f_\varepsilon\|_{L^\infty(\Omega)} < K_\varepsilon \,.$$

In fact, consider the set

$$\Omega_k = \{x \in \Omega \mid |a| < k\} \,,$$

where k is chosen such that

$$\int_{\Omega_k^c} |a|^{\frac{p+1}{p/2}} dx < \frac{1}{2}\varepsilon \,.$$

This condition is clearly satisfied for k large enough.

We now define

$$q_\varepsilon(x) = \begin{cases} \frac{1}{n} a(x) & \text{on } \Omega_k \\ a(x) & \text{on } \Omega_k^c \end{cases} \tag{4.4}$$

and

$$f_\varepsilon(x) = (a(x) - q_\varepsilon(x))\, v^{p/2}(x) \,.$$

Note that $f_\varepsilon(x) = 0$ for all $x \in \Omega_k^c$ and

$$\begin{aligned} \int_\Omega |q_\varepsilon|^{\frac{p+1}{p/2}} dx &= \int_{\Omega_k} |q_\varepsilon|^{\frac{p+1}{p/2}} dx + \int_{\Omega_k^c} |q_\varepsilon|^{\frac{p+1}{p/2}} dx \\ &= \left(\frac{1}{n}\right)^{\frac{p+1}{p/2}} \int_{\Omega_k} |a|^{\frac{p+1}{p/2}} dx + \int_{\Omega_k^c} |a|^{\frac{p+1}{p/2}} dx \\ &< \left(\frac{1}{n}\right)^{\frac{p+1}{p/2}} \int_{\Omega_k} |a|^{\frac{p+1}{p/2}} dx + \frac{1}{2}\varepsilon \,. \end{aligned}$$

So, for $n = n_\varepsilon > \left(\frac{2}{\varepsilon}\right)^{\frac{p/2}{p+1}} \|a\|_{L^{\frac{p+1}{p/2}}(\Omega)}$, we have

$$\|q_\varepsilon\|_{L^{\frac{p+1}{p/2}}(\Omega)} < \varepsilon \,.$$

Therefore, by construction, one obtains

$$\|f_\varepsilon\|_{L^\infty(\Omega)} = \left|1 - \frac{1}{n_\varepsilon}\right| k^2 < +\infty \,.$$

On the other hand, we have

$$v(x) = (-\Delta)^{-s}(bu)(x) \,,$$

where $b \in L^{\frac{q+1}{q-1}}(\Omega)$. Hence,

$$u(x) = (-\Delta)^{-s}\left[q_\varepsilon(x)((-\Delta)^{-s}(bu)(x))^{p/2}\right] + (-\Delta)^{-s} f_\varepsilon(x)\,.$$

By Proposition 2.2.3 and Hölder's inequality, we have the following properties for fixed $\gamma > 1$:

(i) The map $w \to b(x)w$ is bounded from $L^\gamma(\Omega)$ to $L^\beta(\Omega)$ for

$$\frac{1}{\beta} = \frac{q-1}{q+1} + \frac{1}{\gamma};$$

(ii) For any $\theta \geq 1$. in the case that $\beta \geq \frac{n}{2s}$, or for θ given by

$$2s = n\left(\frac{1}{\beta} - \frac{1}{\theta p/2}\right),$$

in the case that $\beta < \frac{n}{2s}$, there exists a constant $C > 0$, depending on β and θ, such that

$$\|((-\Delta)^s w)^{p/2}\|_{L^\theta(\Omega)} \leq C\|w\|^{p/2}_{L^\beta(\Omega)}$$

for all $w \in L^\beta(\Omega)$;

(iii) The map $w \to q_\varepsilon(x)w$ is bounded from $L^\theta(\Omega)$ to $L^\eta(\Omega)$ with norm given by $\|q_\varepsilon\|_{L^{\frac{p+1}{p/2}}(\Omega)}$, where $\theta \geq 1$ and η satisfies

$$\frac{1}{\eta} = \frac{p/2}{p+1} + \frac{1}{\theta};$$

(iv) For any $\delta \geq 1$, in the case that $\eta \geq \frac{n}{2s}$, or for δ given by

$$2s = n\left(\frac{1}{\eta} - \frac{1}{\delta}\right),$$

in the case that $\eta < \frac{n}{2s}$, the map $w \to (-\Delta)^{-s}w$ is bounded from $L^\eta(\Omega)$ to $L^\delta(\Omega)$.

Joining (i), (ii), (iii) and (iv) and using that (p, q) satisfies (9), one easily checks that $\gamma < \delta$ and, in addition,

$$\begin{aligned}
\|u\|_{L^\delta(\Omega)} &\leq \|(-\Delta)^{-s}\left[q_\varepsilon(x)\left((-\Delta)^{-s}(bu)\right)^{p/2}\right]\|_{L^\delta(\Omega)} + \|(-\Delta)^{-s} f_\varepsilon\|_{L^\delta(\Omega)} \\
&\leq C\left(\|q_\varepsilon\|_{L^{\frac{p+1}{p/2}}(\Omega)}\|u\|^{p/2}_{L^\delta(\Omega)} + \|f_\varepsilon\|_{L^\delta(\Omega)}\right).
\end{aligned}$$

Using now the fact that $\|q_\varepsilon\|_{L^{\frac{p+1}{p/2}}(\Omega)} < \varepsilon$ and $f_\varepsilon \in L^\infty(\Omega)$, one deduces that $\|u\|_{L^\delta(\Omega)} \leq C$ for some constant $C > 0$ independent of u. Proceeding inductively, one then gets $u \in$

$L^\delta(\Omega)$ for all $\delta \geq 1$. So, Proposition 2.2.3 implies that $v \in L^\infty(\Omega)$, and thus $u \in L^\infty(\Omega)$. Finally, the C^β regularity of u and v also follows from Proposition 2.2.3.

The other cases are treated in a similar way by writing $a(x) = v^{p-1}$ if $p > 1$ and $b(x) = u^{q/2}$ if $q \leq 1$ or $b(x) = u^{q-1}$ if $q > 1$. ■

4.3 Rellich variational identity

In this section, we deduce that positive viscosity solutions of (8) satisfy the following integral identity:

Proposition 4.3.1. *(Rellich identity) Let Ω be a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 1$ and $0 < s < 1$. Then, every positive viscosity solution (u, v) of the problem (8) satisfies*

$$\Gamma(1+s)^2 \int\limits_{\partial\Omega} \frac{u}{d^s}\frac{v}{d^s}(x \cdot \nu) d\sigma = \left(\frac{n}{q+1} + \frac{n}{p+1} - (n-2s)\right) \int\limits_{\Omega} u^{q+1} dx \,,$$

where ν denotes the unit outward normal to $\partial\Omega$ at x, Γ is the Gamma function, $d(x) = dist(x, \partial\Omega)$ and

$$\frac{u}{d^s}(x) := \lim_{\varepsilon \to 0^+} \frac{u(x - \varepsilon\nu)}{d^s(x - \varepsilon\nu)} > 0$$

for all $x \in \partial\Omega$.

It deserves mention that $u/d^s, v/d^s \in C^\alpha(\overline{\Omega})$ and $u/d^s, v/d^s > 0$ in $\overline{\Omega}$ (see [153] or Remark 2.3.16). So, the left-hand side of the Rellich identity is well defined.

Proof. Let (u, v) be a viscosity solution of (8). Then,

$$\left\{ \begin{array}{rcll} (-\Delta)^s(u+v) & = & v^p + u^q & \text{in } \Omega \\ u+v & = & 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{array} \right. \tag{4.5}$$

and

$$\left\{ \begin{array}{rcll} (-\Delta)^s(u-v) & = & v^p - u^q & \text{in } \Omega \\ u-v & = & 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{array} \right. \tag{4.6}$$

Applying the Pohozaev variational identity for semilinear problems involving the fractional Laplace operator (see Theorem 2.3.14), one gets

$$\begin{array}{rcl} -\int\limits_{\Omega} (x \cdot \nabla u + v)((-\Delta)^s u + (-\Delta)^s v) dx & = & -\dfrac{2s-n}{2} \int\limits_{\Omega} (u+v)(v^p + u^q) dx \\ & & +\dfrac{1}{2}\Gamma(1+s)^2 \int\limits_{\partial\Omega} \left(\dfrac{u+v}{d^s}\right)^2 (x \cdot \nu) d\sigma \end{array}$$

and

$$-\int\limits_{\Omega}(x\cdot\nabla u+v)((-\Delta)^s u-(-\Delta)^s v)dx = -\frac{2s-n}{2}\int\limits_{\Omega}(u-v)(v^p-u^q)dx + \frac{1}{2}\Gamma(1+s)^2\int\limits_{\partial\Omega}\left(\frac{u-v}{d^s}\right)^2(x\cdot\nu)d\sigma\,.$$

Now subtracting both identities, one obtains

$$2\int\limits_{\Omega}[(x\cdot\nabla u)(-\Delta)^s v+(x\cdot\nabla v)(-\Delta)^s u]dx = (2s-n)\int\limits_{\Omega}[u(-\Delta)^s v+v(-\Delta)^s u]dx - 2\Gamma(1+s)^2\int\limits_{\partial\Omega}\frac{u}{d^s}\frac{v}{d^s}(x\cdot\nu)d\sigma\,. \tag{4.7}$$

Because $v=0$ in $\mathbb{R}^n\setminus\Omega$, we have

$$\int\limits_{\Omega}(x\cdot\nabla u)(-\Delta)^s v dx=\int\limits_{\Omega}(x\cdot\nabla u)u^q dx=\frac{1}{q+1}\int\limits_{\Omega}(x\cdot\nabla u^{q+1})dx=-\frac{n}{q+1}\int\limits_{\Omega}u^{q+1}dx\,.$$

In a similar way,

$$\int\limits_{\Omega}(x\cdot\nabla v)(-\Delta)^s u dx=-\frac{n}{p+1}\int\limits_{\Omega}v^{p+1}dx\,.$$

Plugging these two identities into (4.7), we derive

$$2\Gamma(1+s)^2\int\limits_{\partial\Omega}\frac{u}{d^s}\frac{v}{d^s}(x\cdot\nu)d\sigma=\left(2s-n+\frac{2n}{q+1}\right)\int\limits_{\Omega}u^{q+1}dx+\left(2s-n+\frac{2n}{p+1}\right)\int_{\Omega}v^{p+1}dx\,.$$

Since every viscosity solution of (8) is also a bounded weak solution, one has

$$\int\limits_{\Omega}v^{p+1}dx=\int\limits_{\Omega}v(-\Delta)^s u dx=\int\limits_{\mathbb{R}^n}(-\Delta)^{s/2}u(-\Delta)^{s/2}v dx=\int\limits_{\Omega}u(-\Delta)^s v dx=\int\limits_{\Omega}u^{q+1}dx\,.$$

Thus, the desired conclusion follows directly from this equality. ■

4.4 Proof of Theorem 0.0.8

We organize the proof of Theorem 0.0.8 into two parts. We start by proving the existence of a positive viscosity solution. According to the variational framework described in the section 4.1, it suffices to show the existence of a nonzero critical point $u\in E^s_p$ of the functional Φ.

4.4.1 The existence part

We apply the direct method to the functional Φ on E^s_p.

In order to show the coercivity of Φ, note that $q+1 < \frac{p+1}{p}$ because $pq < 1$, so that the embedding $E^s_p \hookrightarrow L^{q+1}(\Omega)$ is continuous. So, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
\Phi(u) &= \frac{p}{p+1}\int_\Omega |(-\Delta)^s u|^{\frac{p+1}{p}} dx - \frac{1}{q+1}\int_\Omega |u|^{q+1} dx \\
&\geq \frac{C_1 p}{p+1}\|u\|_{E^s_p}^{\frac{p+1}{p}} - \frac{C_2}{q+1}\|u\|_{E^s_p}^{q+1} \\
&= \|u\|_{E^s_p}^{\frac{p+1}{p}} \left(\frac{C_1 p}{p+1} - \frac{C_2}{(q+1)\|u\|_{E^s_p}^{\frac{p+1}{p}-(q+1)}} \right)
\end{aligned}$$

for all $u \in E^s_p$. Therefore, Φ is lower bounded and coercive, that is, $\Phi(u) \to +\infty$ as $\|u\|_{E^s_p} \to +\infty$.

Let $(u_k) \subset E^s_p$ be a minimizing sequence of Φ. It is clear that (u_k) is bounded in E^s_p, since Φ is coercive. So, module a subsequence, we have $u_k \rightharpoonup u_0$ in E^s_p. Since E^s_p is compactly embedded in $L^{q+1}(\Omega)$ (see Theorem 2.1.8), we have $u_k \to u_0$ in $L^{q+1}(\Omega)$. Here, we again use the fact that $q+1 < \frac{p+1}{p}$. Thus,

$$\begin{aligned}
\liminf_{n\to\infty} \Phi(u_k) &= \liminf_{k\to\infty} \frac{p}{p+1}\|(-\Delta)^s u_k\|_{L^{\frac{p+1}{p}}(\Omega)}^{\frac{p+1}{p}} - \frac{1}{q+1}\|u_0\|_{L^{q+1}(\Omega)}^{q+1} \\
&\geq \frac{p}{p+1}\|(-\Delta)^s u_0\|_{L^{\frac{p+1}{p}}(\Omega)}^{\frac{p+1}{p}} - \frac{1}{q+1}\|u_0\|_{L^{q+1}(\Omega)}^{q+1} = \Phi(u_0)\,,
\end{aligned}$$

so that u_0 minimizers Φ on E^s_p. We just need to guarantee that u_0 is nonzero. But, this fact is clearly true since $\Phi(\varepsilon u_1) < 0$ for any nonzero nonnegative function $u_1 \in E^s_p$ and $\varepsilon > 0$ small enough, that is,

$$\Phi(\varepsilon u_1) = \frac{p\varepsilon^{\frac{p+1}{p}}}{p+1}\int_\Omega |(-\Delta)^s u_1|^{\frac{p+1}{p}} dx - \frac{\varepsilon^{q+1}}{q+1}\int_\Omega |u_1|^{q+1} dx < 0$$

for $\varepsilon > 0$ small enough. This ends the proof of existence. ■

4.4.2 The uniqueness part

The main tools in the proof of uniqueness are the Silvestre's strong maximum principle, a C^α regularity result up to the boundary and a Hopf's lemma adapted to fractional operators.

Let $(u_1,v_1),(u_2,v_2) \in C^0(\mathbb{R}^n)\times C^0(\mathbb{R}^n)$ be two positive viscosity solutions of (8). Define

$$S=\{s\in(0,1] \mid u_1-tu_2,\ v_1-tv_2\geq 0 \ \text{ in } \overline{\Omega} \ \text{ for all } \ t\in[0,s]\}\,.$$

By Theorem 2.2.7, we have $u_i/d^s, v_i/d^s \in C^\alpha(\overline{\Omega})$ and both quotients are positive on $\overline{\Omega}$, by Hopf's lemma (see [153] or Remark 2.3.16). So, $(u_1-tu_2)/d^s,(v_1-tv_2)/d^s>0$ on $\partial\Omega$ for $t>0$ small enough and thus the set S is no empty.

Let $s_*=\sup S$ and assume that $s_*<1$.

Clearly,

$$u_1-s_*u_2,\ v_1-s_*v_2\geq 0 \ \text{ in } \ \overline{\Omega}\,. \tag{4.8}$$

By (4.8) and the integral representation in terms of the Green function G_Ω of $(-\Delta)^s$ (see [21, 111]), we have

$$\begin{aligned} u_1(x) &= \int_\Omega G_\Omega(x,y)v_1^p(y)dy \geq \int_\Omega G_\Omega(x,y)s_*^p v_2^p(y)dy \\ &= s_*^p\int_\Omega G_\Omega(x,y)v_2^p(y)dy = s_*^p u_2(x) \end{aligned}$$

for all $x\in\overline{\Omega}$. In a similar way, one gets $v_1\geq s_*^q v_2$ in $\overline{\Omega}$.

Using the assumption $pq<1$ and the fact that $s_*<1$, we derive

$$\begin{cases} (-\Delta)^s(u_1-s_*u_2)=v_1^p-s_*v_2^p\geq (s_*^{pq}-s_*)v_2^p>0 \\ (-\Delta)^s(v_1-s_*v_2)=u_1^q-s_*u_2^q\geq (s_*^{pq}-s_*)u_2^q>0 \end{cases} \quad \text{in } \Omega \tag{4.9}$$

So, by the Silvestre's strong maximum principle (see Lemma 2.3.6), one has $u_1-s_*u_2, v_1-s_*v_2>0$ in Ω. Again, arguing as above, we easily deduce that $(u_1-s_*u_2)/d^s,(v_1-s_*v_2)/d^s>0$ on $\partial\Omega$, so that $u_1-(s_*+\varepsilon)u_2, v_1-(s_*+\varepsilon)v_2>0$ in Ω for $\varepsilon>0$ small enough, contradicting the definition of s_*. Therefore, $s_*\geq 1$ and, by (4.8), $u_1-u_2, v_1-v_2\geq 0$ in $\overline{\Omega}$. A similar reasoning also produces $u_2-u_1, v_2-v_1\geq 0$ in $\overline{\Omega}$. This ends the proof of uniqueness. ■

4.5 Proof of Theorem 0.0.9

Assume $p,q>0$, $pq>1$ and the assumption (9). The proof consists in applying the classical mountain pass theorem of Ambrosetti and Rabinowitz in our variational setting. Firstly, by well-known embedding theorems (see Theorem 2.1.8), (9) implies that E_p^s is compactly embedded in $L^{q+1}(\Omega)$. We now assert that Φ has a local minimum in the origin. Consider the set $\Gamma := \left\{u \in E_p^s \mid \|u\|_{E_p^s} = \rho\right\}$. Then, on Γ, we have

$$\begin{aligned}\Phi(u) &= \frac{p}{p+1}\int_\Omega |(-\Delta)^s u|^{\frac{p+1}{p}}dx - \frac{1}{q+1}\int_\Omega |u|^{q+1}dx \\ &\geq C_1\frac{p}{p+1}\|u\|_{E_p^s}^{\frac{p+1}{p}} - \frac{C_2}{q+1}\|u\|_{E_p^s}^{q+1} = \rho^{\frac{p+1}{p}}\left(C_1\frac{p}{p+1} - \frac{C_2}{q+1}\rho^{q+1-\frac{p+1}{p}}\right) \\ &> 0 = \Phi(0)\end{aligned}$$

for fixed $\rho > 0$ small enough, so that the origin $u_0 = 0$ is a local minimum point. In particular, $\inf_\Gamma \Phi > 0 = \Phi(u_0)$.

Note that Γ is a closed subset of E_p^s and decomposes E_p^s into two connected components, namely $\left\{u \in E_p^s \mid \|u\|_{E_p^s} < \rho\right\}$ and $\left\{u \in E_p^s \mid \|u\|_{E_p^s} > \rho\right\}$.

Let $u_1 = t\overline{u}$, where $t > 0$ and $\overline{u} \in E_p^s$ is a nonzero nonnegative function. Since $pq > 1$, we can choose t sufficiently large so that

$$\Phi(u_1) = \frac{pt^{\frac{p+1}{p}}}{p+1}\int_\Omega |(-\Delta)^s\overline{u}|^{\frac{p+1}{p}}dx - \frac{t^{q+1}}{q+1}\int_\Omega (\overline{u}^+)^{q+1}dx < 0\,.$$

It is clear that $u_1 \in \left\{u \in E_p^s \mid \|u\|_{E_p^s} > \rho\right\}$. Moreover, $\inf_\Gamma \Phi > \max\{\Phi(u_0), \Phi(u_1)\}$, so that the mountain pass geometry is satisfied.

Finally, we show that Φ fulfills the Palais-Smale condition (PS). Let $(u_k) \subset E_p^s$ be a (PS)-sequence, that is,

$$|\Phi(u_k)| \leq C_0$$

and

$$|\Phi'(u_k)\varphi| \leq \varepsilon_k\|\varphi\|_{E_p^s}$$

for all $\varphi \in E_p^s$, where $\varepsilon_k \to 0$ as $k \to +\infty$.

From these two inequalities, we deduce that

$$\begin{aligned}C_0 + \varepsilon_k\|u_k\|_{E_p^s} &\geq |(q+1)\Phi(u_k) - \Phi'(u_k)u_k| \\ &\geq \left(\frac{(q+1)p}{p+1} - 1\right)\int_\Omega |(-\Delta)^s u_k|^{\frac{p+1}{p}}dx \\ &\geq C\|u_k\|_{W^{2s,\frac{p+1}{p}}(\Omega)}^{\frac{p+1}{p}} \geq C\|u_k\|_{E_p^s}^{\frac{p+1}{p}}\end{aligned}$$

and thus (u_k) is bounded in E_p^s. Thanks to the compactness of the embedding $E_p^s \hookrightarrow L^{q+1}(\Omega)$, one easily checks that (u_k) converges strongly in E_p^s. So, by the mountain pass theorem, we obtain a nonzero critical point $u \in E_p^s$. This ends the proof.

4.6 Proof of Theorem 0.0.10

It suffices to assume that Ω is star-shaped with respect to the origin, that is, $(x \cdot \nu) > 0$ for any $x \in \partial\Omega$, where ν is the unit outward normal to $\partial\Omega$ at x.

Let (u, v) be a positive viscosity solution of the problem (8). Then, on the one hand, we have

$$2\Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{d^s} \frac{v}{d^s} (x \cdot \nu) d\sigma > 0 \,.$$

On the other hand, the assumption (10) is equivalent to $\frac{n}{q+1} + \frac{n}{p+1} - (n-2s) \leq 0$, and thus we arrive at a contradiction. Hence, the problem (8) admits no positive viscosity solution and we end the proof. ■

APÊNDICE A

Appendix

Theorem A.0.1. *(Method of direct minimization) Let E be a Banach space and $\phi : E \to \mathbb{R} \cup \{+\infty\}$ sequentially weakly lower semicontinuous. If E is reflexive and ϕ is coercive, then*

(i) ϕ is bounded below.

(ii) the smallest is reached.

Proof. See [61] . ■

Theorem A.0.2. *(Mountain Pass Theorem) Let E be a Banach space and $\phi : E \to \mathbb{R}$ be a Fréchet-differentiable functional satisfying the geometry of the mountain pass. Consider*

$$X := \{g \in C([0,1], \mathbb{R}) \mid g(0) = u_0, g_1 = u_1\} \text{ and } C := \inf_{g \in X} \max_{t \in [0,1]} \phi(g(t)).$$

Then:

(i) $C > \max\{\phi(u_0), \phi(u_1)\}$.

(ii) $\exists\ (u_n) \subset E$ such that $\phi(u_n) \to C, \phi'(u_n) \to 0$ in E'.

In particular, if ϕ is C^1 and satisfies $(PS)_C$, then C is critical value.

Proof. See [3, 61]. ■

Referências Bibliográficas

[1] N. Abatangelo - *Large s-harmonic functions and boundary blow-up solutions for the fractional Laplacian*, arXiv: 1310.3193, 2013.

[2] R.A. Adams - *Sobolev Spaces*, Academic Press, New York, 1975.

[3] A. Ambrosetti, P. Rabinowitz - *Dual variational methods in critical points theory and applications*, J. Funct. Anal. 14 (1972), 349-381.

[4] C.J. Amick, J.F. Toland - *Uniqueness and related analytic properties for the Benjamin-Ono equation*, Acta Math. 167 (1991), 107-126.

[5] D. Applebaum - *Lévy processes – from probability to finance and quantum groups*, Notices Amer. Math. Soc. 51 (2004), 1336-1347.

[6] I. Athanasopoulos, L. Caffarelli - *Optimal regularity of lower dimensional obstacle problems*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 310 (2004), 49-66.

[7] I. Athanasopoulos, L. Caffarelli, S. Salsa - *The structure of the free boundary for lower dimensional obstacle problems*, Amer. J. Math. 130 (2008), 485-498.

[8] G. Barles, E. Chasseigne, C. Imbert - *The Dirichlet problem for second-order elliptic integro-differential equations*, Indiana Univ. Math. J. 57 (2008), 213-146.

[9] B. Barrios, E. Colorado, A. de Pablo, U. Sánchez - *On some critical problems for the fractional Laplacian operator*, J. Diff. Eq. 252 (2012), 6133-6162.

[10] I. Birindelli, E. Mitidieri - *Liouville theorems for elliptic inequalities and applications*, Proc. Roy. Soc. Ed. A 128 (1998), 1217-1247.

[11] M. Birkner, J.A. Lópes-Mimbela, A. Wakolbinger - *Comparison results and steady states for the Fujita equation with fractional Laplacian*, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), 83-97.

[12] R.M. Blumenthal, R.K. Getoor - *The asymptotic distribution of the eigenvalues for a class of Markov operators*, Pacific J. Math. 9 (1959), 399-408.

[13] R.M. Blumenthal, R.K. Getoor - *Some theorems on stable processes*, Trans. Amer. Math. Soc. 95 (1960), 263-273.

[14] R.M. Blumenthal, R.K. Getoor, D.B. Ray - *On the distribution of first hits for the symmetric stable processes*, Trans. Amer. Math. Soc. 99 (1961), 540-554.

[15] K. Bogdan - *The boundary Harnack principle for the fractional Laplacian*, Studia Math. 123 (1997), 43-80.

[16] K. Bogdan, T. Grzywny, M. Ryznar - *Heat kernel estimates for the fractional Laplacian with Dirichlet conditions*, Ann. of Prob. 38 (2010), 1901-1923.

[17] K. Bogdan, T. Kulczycki, M. Kwaśnicki - *Estimates and structure of α-harmonic functions*, Probab. Theory Related Fields 140 (2008), 345-381.

[18] K. Bogdan, T. Kumagai, M. Kwaśnicki - *Boundary Harnack inequality for Markov processes with jumps*, Trans. Amer. Math. Soc. 367 (2015), 477-517.

[19] K. Bogdan, P. Sztonyk - *Harnack's inequality for stable Lévy processes*, Potential Anal. 22 (2005), 133-150.

[20] K. Bogdan, P. Sztonyk - *Estimates of the potential kernel and Harnack's inequality for the anisotropic fractional Laplacian*, Studia Math. 181 (2007), 101-123.

[21] M. Bonforte, V. J. Luis - *A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains*, Arch. Ration. Mech. Anal. 218 (2015), 317-362.

[22] M. Bonforte, Y. Sire, J.L. Vázquez - *Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains*, arXiv: 1401-6195, 2014.

[23] J. Bourgain, H. Brezis, P. Mironescu - *Another look at Sobolev spaces*, in: J.L. Menaldi, E. Rofman, A. Sulem (Eds.), Optimal Control and Partial Differential Equations, IOS Press, Amsterdam, (2001), 439-455. A volume in honor of A. Bensoussan's 60th birthday.

[24] C. Brändle, E. Colorado, A. de Pablo, U. Sánchez - *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A 143 (2013), 39-71.

[25] H. Brezis - *How to recognize constant functions. Connections with Sobolev spaces*, Uspekhi Mat. Nauk 57 (4) (2002), 59-74.

[26] H. Brezis, P. Mironescu - *Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces*, J. Evol. Equ. 1 (2001), 387-404.

[27] A. Buades, B. Coll, J-M. Morel - *On image denoising methods*, SIAM Multiscale Modeling and Simulation 4 (2005), 490-530.

[28] J. Busca, R. Manásevich - *A Liouville-type theorem for Lane-Emden system*, Indiana Univ. Math. J. 51 (2002), 37-51.

[29] X. Cabré, E. Cinti - *Energy estimates and 1D symmetry for nonlinear equations involving the half-Laplacian*, Disc. Cont. Dyn. Syst. 28 (2010), 1179-1206. A special issue Dedicated to Louis Nirenberg on the Occasion of his 85th Birthday.

[30] X. Cabré, E. Cinti - *Sharp energy estimates for nonlinear fractional diffusion equations*, Calc. Var. Partial Differential Equations 49 (2014), 233-269.

[31] X. Cabré, J.M. Roquejoffre - *The influence of fractional diffusion on Fisher-KPP equations*, Comm. Math. Phys. 320 (2013), 679-22.

[32] X. Cabré, Y. Sire - *Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (1) (2014), 23-53.

[33] X. Cabré, Y. Sire - *Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions*, Trans. Amer. Math. Soc. 367 (2015), 911-941.

[34] X. Cabré, J. Solà-Morles - *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. 58 (2005), 1678-1732.

[35] X. Cabré, J. Tan - *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. 224 (2010), 2052-2093.

[36] L. Caffarelli, C.H. Chan - *Regularity theory for parabolic nonlinear integral operators*, J. Amer. Math. Soc. 24 (2011), 849-869.

[37] L. Caffarelli, A. Figalli - *Regularity of solutions to the parabolic fractional obstacle problem*, J. Reine Angew. Math. 680 (2013), 191-233.

[38] L. Caffarelli, J.M. Roquejoffre, Y. Sire - *Variational problems in free boundaries for the fractional Laplacian*, J. Eur. Math. Soc. 12 (2010), 1151-1179.

[39] L. Caffarelli, J.M. Roquejoffre, O. Savin - *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. 63 (2010), 1111-1144.

[40] L. Caffarelli, S. Salsa, L. Silvestre - *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. 171 (2008), 425-461.

[41] L. Caffarelli, L. Silvestre - *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32 (2007), 1245-1260.

[42] L. Caffarelli, L. Silvestre - *Regularity theory for fully nonlinear integro-differential equations*, Commun. Pure Appl. Math. 62 (5) (2009), 597-638.

[43] L. Caffarelli, E. Valdinoci - *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations 41 (2011), 203-240.

[44] L. Caffarelli, E. Valdinoci - *Regularity properties of nonlocal minimal surfaces via limiting arguments*, Adv. Math. 248 (2013), 843-871.

[45] L. Caffarelli, A. Vasseur - *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. 171 (2010), 1903-1930.

[46] L. Caffarelli, A. Vasseur - *The De Giorgi method for regularity of solutions of elliptic equations and its applications to uid dynamics*, Discrete Contin. Dyn. Syst. Ser. S 3 (2010), 409-427.

[47] L. Caffarelli, J.L. Vázquez - *Nonlinear Porous Medium Flow with Fractional Potential Pressure*, Arch. Rat. Mech. Anal. 202 (2011), 537-565.

[48] A. Capella, J. D'avila, L. Dupaigne, Y. Sire - *Regularity of radial extremal solutions for some non-local semilinear equations*, Comm. Partial Differential Equations 36 (2011), 1353-1384.

[49] J.A. Carrillo, A. Figalli, M. Di Francesco, T. Laurent, D. Slepcev - *Global in time measure-valued solutions and finite-time aggregation for nonlocal interaction equations*, Duke Math. J. 156 (2011), 229-271.

[50] S.-Y. A. Chang, M. González - *Fractional Laplacian in conformal geometry*, Adv. Math. 226 (2011), 1410-1432.

[51] S.J. Chapman, J. Rubinstein, M. Schatzman - *A mean-field model for superconducting vortices*, Eur. J. Appl. Math. 7 (1996), 97-111.

[52] Z.-Q. Chen, R. Song - *Estimates on Green functions and Poisson kernels for symmetric stable processes*, Math. Ann. 312 (1998), 465-501.

[53] W. Chen, Y. Fang - *Semilinear equations involving the fractional Laplacian on domains*, arXiv: 1309.7499, 2013.

[54] Z. Chen, P. Kim, R. Song - *Heat kernel estimates for the Dirichlet fractional Laplacian*, J. Eur. Math. Soc. 12 (2010), 1307-1329.

[55] W. Choi - *On strongly indefinite systems involving the fractional Laplacian*, Nonlinear Anal. 120 (2015), 127-153.

[56] W. Choi, S. Kim, K. Lee - *Asymptotic behavior of solutions for nonlinear elliptic problems with the fractional Laplacian*, J. Funct. Anal. 266 (2014), 6531-6598.

[57] E. Cinti - *Saddle-shaped solutions of bistable elliptic equations involving the half-Laplacian*, Ann. Scuola Norm. Sup. Pisa, 12 (2013), 623-664.

[58] Ph. Clément, D.G. de Figueiredo, E. Mitidieri - *Positive solutions of semilinear elliptic systems*, Comm. Partial Differential Equations 17 (1992), 923-940.

[59] P. Constantin - *Euler equations, Navier-Stokes equations and turbulence*, In Mathematical foundation of turbulent viscous flows, Lecture Notes in Math. pages 1-43. Springer, Berlin, 2006.

[60] R. Cont, P. Tankov - *Financial Modelling With Jump Processes, Financial Mathematics Series*, Chapman e Hall/CRC, Boca Raton, FL, 2004.

[61] D.G. Costa - *An Invitation to Variational Methods in Differential Equations*, Birkhauser Boston (2007).

[62] A.-L. Dalibard, D. Gérard-Varet - *On shape optimization problems involving the fractional Laplacian*, ESAIM Control Optim. Calc. Var. 19 (2013), 976-1013.

[63] D.G. de Figueiredo - *Semilinear elliptic systems*, Nonl. Funct. Anal. Appl. Diff. Eq. World Sci. Publishing, River Edge (1998), 122-152.

[64] D.G. de Figueiredo, P. Felmer - *On superquadratic elliptic systems*, Trans. Amer. Math. Soc. 343 (1994), 99-116.

[65] D.G. de Figueiredo, P. Felmer - *A Liouville-type theorem for elliptic systems*, Ann. Sc.Norm. Sup. Pisa 21 (1994), 387-397.

[66] D.G. Figueiredo, B. Ruf - *Elliptic systems with nonlinearities of arbitrary growth*, Mediterr. J. Math. 1 (2004), 417-431.

[67] D.G. de Figueiredo, B. Sirakov - *Liouville type theorems, monotonicity results and a priori bounds for positive solutions of elliptic systems*, Math. Ann. 333 (2005), 231-260.

[68] D.G. de Figueiredo, B. Sirakov - *On the Ambrosetti-Prodi problem for non-variational elliptic systems*, J. Diff. Eq. 240 (2007), 357-374.

[69] D. del-Castillo-Negrete - *Truncation effects in superdiffusive front propagation with Lévy flights*, Physical Review 79 (2009), 1-10.

[70] F. Demengel, G. Demengel - *Functional Spaces for the Theory of Elliptic Partial Differential Equations*, (Universitext)-Springer (2012).

[71] D. De Silva, O. Savin, Y. Sire - *A one-phase problem for the fractional Laplacian: regularity of at free boundaries*, Bull. Inst. Math. Acad. Sin. 9 (2014), 111-145.

[72] E. DiBenedetto - *Real Analysis*, Birkhäuser Boston (2012).

[73] E. Di Nezza, G. Palatucci, E. Valdinoci - *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (5) (2012), 521-573.

[74] S. Dipierro, G. Palatucci, E. Valdinoci - *Dislocation dynamics in crystals: A macroscopic theory in fractional Laplace setting*, Comm. Math. Phys. 2 (2015), 1061-1105.

[75] H. Dong, D. Kim - *On L_p-estimates for a class of nonlocal elliptic equations*, J. Funct. Anal. 262 (2012), 1166-1199.

[76] W. J. Drugan - *Two Exact Micromechanics-Based Nonlocal Constitutive Equations for Random Linear Elastic Composite Materials*, Journal of the Mechanics and Physics of Solids 51 (2003), 1745-1772.

[77] B. Dyda - *A fractional order Hardy inequality*, Ill. J. Math, vol. 48, no. 2, (2004) 575-588.

[78] A. C. Eringen - *Linear theory of nonlocal elasticity and dispersion of plane waves*, Int. J. Engng Sci 10 (1972), 425-435.

[79] M.M. Fall, V. Felli - *Unique continuation property and local asymptotics of solutions to fractional elliptic equations*, Comm. Partial Differential Equations 39 (2014), 1-44.

[80] M.M. Fall, S. Jarohs - *Overdetermined problems with fractional Laplacian*, arXiv: 1311.7549, 2013.

[81] M.M. Fall, T. Weth - *Nonexistence results for a class of fractional elliptic boundary value problems*, J. Funct. Anal. 263 (2012), 2205-2227.

[82] C.L. Fefferman, R. de la Llave - *Relativistic stability of matter (i)*, Revista Matematica Iberoamericana 2 (1986), 119-213.

[83] P. Felmer, S. Martínez - *Existence and uniqueness of positive solutions to certain differential systems*, Adv. Differential Equations 4 (1998), 575-593.

[84] P. Felmer, A. Quaas - *Boundary blow up solutions for fractional elliptic equations*, Asymptot. Anal. 78 (2012), 123-144.

[85] P. Felmer, A. Quaas - *Fundamental solutions and Liouville type theorems for nonlinear integral operators*, Adv. Math. 226 (2011), 2712-2738.

[86] M. Felsinger, M. Kassmann, P. Voigt - *The Dirichlet problem for nonlocal operators*, Math. Z. 279 (2015), 779-809.

[87] A. Figalli, E. Valdinoci - *Regularity and Bernstein-type results for nonlocal minimal surfaces*, arXiv: 1307.0234, 2013.

[88] A. Fiscella - *Saddle point solutions for nonlocal elliptic operators*, Topol. Methods Nonlinear Anal. 44 (2014), 527-538.

[89] A. Fiscella, E. Valdinoci - *A critical Kirchhoff type problem involving a nonlocal operator*, Nonlinear Anal. 94 (2014), 156-170.

[90] P. Francesco - *Some remarks on the duality method for integro-differential equations with measure data*, arXiv: 1409.8463, 2014.

[91] R. Frank, E. Lenzmann - *Uniqueness and nondegeneracy of ground states for* $(-\Delta)^s Q + Q - Q^{\alpha+1} = 0$ *in* $\mathbb{R}$, Acta Math. 210 (2013), 261-318.

[92] R. Frank, E. Lenzmann, L. Silvestre - *Uniqueness of radial solutions for the fractional Laplacian*, arXiv: 1302.2652, 2013.

[93] G. Franzina, G. Palatucci - *Fractional p-eigenvalues*, Riv. Math. Univ. Parma 5 (2014), 373-386.

[94] P. Germain, N. Masmoudi, J. Shatah - *Global solutions for the gravity surface water waves equation in dimension 3*, Annals of Math. 175 (2012), 691-754.

[95] R.K. Getoor - *Markov operators and their associated semigroups*, Pacific J. Math. 9 (1959), 449-472.

[96] R.K. Getoor - *First passage times for symmetric stable processes in space*, Trans. Am. Math. Soc. 101 (1961), 75-90.

[97] G. Giacomin, J. L. Lebowitz - *Phase segregation dynamics in particle systems with long range interaction I. Macroscopic limits*, J. Stat. Phys. 87 (1997), 37-61.

[98] B. Gidas, W.M. Ni, L. Nirenberg - *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68 (1979), 209-243.

[99] B. Gidas, J. Spruck - *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. 34 (1981), 525-598.

[100] B. Gidas, J. Spruck - *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations 6 (1981), 883-901.

[101] D. Gilbarg, N.S. Trudinger - *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag (1983).

[102] G. Gilboa, S. Osher - *Nonlocal operators with applications to image processing*, Multiscale Model. Simul. 7 (2008), 1005-1028.

[103] C.R. Graham, M. Zworski - *Scattering matrix in conformal geometry*, Invent. Math. 152 (2003), 89-118.

[104] P. Grisvard - *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston (1985).

[105] H. Hajaiej, X. Yu, Z. Zhai - *Fractional Gagliardo-Nirenberg and Hardy inequalities under Lorentz norms*, J. Math. Anal. Appl. 396 (2012), 569-577.

[106] A. Hildebrandt, R. Blossey, S. Rjasanow, O. Kohlbacher, H.P. Lenhof - *Electrostatic potentials of proteins in water: a structured continuum approach*, Bioinformatics (Oxford, England), 23 (2007), 99-103.

[107] J. Hulshof, R. van der Vorst - *Diferential systems with strongly indefinite variational structure*, J. Funct. Anal. 114 (1993), 32-58.

[108] N.E. Humphries et al. - *Environmental context explains Levy and Brownian movement patterns of marine predators*, Nature, 465 (2010), 1066-1069.

[109] L. Ilcewicz, A. Narasimhan, J. Wilson - *An experimental verification of nonlocal fracture criterion*, Engineering Fracture Mechanics 14 (1981), 801-808.

[110] R. Ishizuka, S.H. Chong, F. Hirata - *An integral equation theory for inhomogeneous molecular uids: the reference interaction site model approach*, The Journal of chemical physics, 128 (2008).

[111] T. Jakubowski - *The estimates for the Green function in Lipschitz domains for the symmetric stable processes*, Probab. Math. Statist. 22 (2002), 419-441.

[112] T. Jin, Y. Li, J. Xiong - *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions*, J. Eur. Math. Soc. 16 (2014), 1111-1171.

[113] M. Kac, H. Pollard - *Partial sums of independent random variables*, Canad. J. Math 11 (1950), 375-384.

[114] V. Katkovnik, A. Foi, K. Egiazarian, J. Astola - *From local kernel to nonlocal multiple-model image denoising*, Int. J. Computer Vision, 2009.

[115] E. Kroner - *Elasticity theory of materials with long range cohesive forces*, Int. J. Solids and Structures 3 (1967), 731-742.

[116] T. Kulczycki - *Properties of Green function of symmetric stable processes*, Probab. Math. Statist. 17 (1997), 339-364.

[117] K. Kyeong-Hun, K. Panki - *An L_p-theory of stochastic parabolic equations with the random fractional Laplacian driven by Lévy processes*, arXiv: 1111.4712, 2011.

[118] N.S. Landkof - *Foundations of modern potential theory*, Berlin: Springer-Verlaq, 1972.

[119] R.B. Lehoucq, K. Zhou, Q. Du, M. Gunzburger - *A non-local vector calculus, non-local volume-constrained problems, and non-local balance laws*, Mathematical Models and Methods in Applied Sciences, 2011.

[120] E.J.F. Leite, M. Montenegro - *A priori bounds and positive solutions for non-variational fractional elliptic systems*, arXiv: 1409.6060, 2014.

[121] E.J.F. Leite, M. Montenegro - *On positive viscosity solutions of fractional Lane-Emden systems*, arXiv: 1509.01267, 2015.

[122] S.Z. Levendorski - *Pricing of the American put under Lévy processes*, Int. J. Theor. Appl. Finance 7 (2004), 303-335.

[123] E.H. Lieb - *Sharp constants in the Hardy-Littlewood-Sobolev inequalities and related inequalities*, Ann. of Math. 118 (1983), 349-374.

[124] E. Lindgren and P. Lindqvist - *Fractional eigenvalues*, Calc. Var. Partial Differential Equations 49 (2014), 795-826.

[125] G. Lu - *The Peierls-Nabarro model of dislocations: a venerable theory and its current development*, In Handbook of Materials Modeling, Springer Netherlands (2005), 793-811.

[126] R. Mancinelli, D. Vergni, A. Vulpiani - *Front propagation in reactive systems with anomalous diffusion*, Phys. D 185 (2003), 175-195.

[127] V. Maz'ya, T. Shaposhnikova - *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funcy. Anal. 195 (2002), 230-238.

[128] R. Merton - *Option pricing when the underlying stock returns are discontinuous*, J. Finan. Econ. 5 (1976), 125-144.

[129] R. Metzler, J. Klafter - *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. 339 (2000), 1-77.

[130] Z. Mingfeng - *The Fractional Sobolev Spaces and the Fractional Laplacians*, University of Connecticut, 2012.

[131] E. Mitidieri - *A Rellich type identity and applications*, Comm. Partial Differential Equations 18 (1993), 125-151.

[132] E. Mitidieri - *Nonexistence of positive solutions of semilinear elliptic systems in* $\mathbb{R}^N$, Differential and Integral Equations 9 (1996), 465-479.

[133] G. Molica, P. Pizzimenti - *Sequences of weak solutions for non-local elliptic problems with Dirichlet boundary condition*, Proc. Edinb. Math. Soc. (2) 57 (2014), 779-809.

[134] I. Monetto, W.J. Drugan - *A Micromechanics-Based Nonlocal Constitutive Equation for Elastic Composites Containing randomly oriented spheroidal Heterogeneities*, Journal of the Mechanics and Physics of Solids 52 (2004), 359-393.

[135] M. Montenegro - *Criticalidade, superlinearidade e sublinearidade para sistemas elípticos semilineares*, Ph.D Thesis (1997) Unicamp.

[136] M. Montenegro - *The construction of principal spectra curves for Lane-Emden systems and applications*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 29 (4) (2000), 193-229.

[137] R. Musina, A.I. Nazarov - *On fractional Laplacians*, Comm. Partial Differential Equations 39 (2014), 1780-1790.

[138] A. Niang - *Fractional Elliptic Equations*, (2014) African Institute for Mathematical Sciences, Senegal.

[139] B. Oksendal, A. Sulem - *Applied Stochastic Control Of Jump Diffusions*, Springer-Verlag, Berlin, 2005.

[140] H. Pham - *Optimal stopping, free boundary, and American option in a jump-diffusion model*, Appl. Math. Optim. 35 (1997), 145-164.

[141] S. I. Pohozaev - *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR 165 (1965), 1408-1411.

[142] P. Polácik, P. Quittner, Ph. Souplet - *Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part I: Elliptic systems*, Duke Math. J. 139 (2007), 555-579.

[143] W.E. Pruitt, S.J. Taylor - *The potential kernel and hitting probabilities for the general stable process in* $\mathbb{R}^N$, Trans. Amer. Math. Soc. 146 (1969), 299-321.

[144] P. Pucci, V. Radulescu - *The impact of the mountain pass theory in nonlinear analysis: a mathematical survey*, Boll. Unione Mat. Ital. (9) 3 (2010), 543-584.

[145] Y. Qin - *Nonlinear Parabolic-Hyperbolic Coupled Systems and Their Attractors*, Birkhäuser Verlag (2008).

[146] A. Quaas, B. Sirakov - *Existence and non-existence results for fully nonlinear elliptic systems*, Indiana Univ. Math. J. 58 (2009), 751-788.

[147] A. Quaas, A. Xia - *Liouville type theorems for nonlinear elliptic equations and systems involving fractional Laplacian in the half space*, Calc. Var. Partial Differential Equations 526 (2014), 1-19.

[148] P.H. Rabinowitz - *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math., vol. 65, American Mathematical Society, Providence, RI, 1986.

[149] A.M. Reynolds, C.J. Rhodes - *The Lévy flight paradigm: Random search patterns and mechanisms*, Ecology 90 (2009), 877-887.

[150] S. Ridgway, B. Mercedes - *Continuum equations for dielectric response to macromolecular assemblies at the nano scale*, Journal of Physics A: Mathematical and General, 37 (2004).

[151] M. Riesz - *Integrales de Riemann-Liouville et potentiels*, Acta Sci. Math. Szeged, 1938.

[152] X. Ros-Oton - *Integro-differential equations: Regularity theory and Pohozaev identities*, Ph.D Thesis (2014) Universitat Politècnica de Catalunya.

[153] X. Ros-Oton, J. Serra - *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. 101 (2014), 275-302.

[154] X. Ros-Oton, J. Serra - *The extremal solution for the fractional Laplacian*, Calc. Var. Partial Differential Equations 50 (2014), 723-750.

[155] X. Ros-Oton, J. Serra - *The Pohozaev identity for the fractional Laplacian*, Arch. Ration. Mech. Anal. 213 (2014), 587-628.

[156] O. Savin, E. Valdinoci - Γ-*convergence for nonlocal phase transitions*, Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), 479-500.

[157] O. Savin, E. Valdinoci - *Regularity of nonlocal minimal cones in dimension 2*, Calc. Var. Partial Differential Equations 48 (2013), 33-39.

[158] W. Schoutens - *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley, New York 2003.

[159] S. Serfaty, J.L. Vázquez - *A mean field equation as limit of nonlinear diffusions with fractional Laplacian operators*, Calc. Var. Partial Differential Equations 49 (2014), 1091-1120.

[160] J. Serrin - *A symmetry problem in potential theory*, Arch. Rat. Mech. Anal. 43 (1971), 304-318.

[161] J. Serrin, H. Zou - *Existence of positive entire solutions of elliptic Hamiltonian systems*, Comm. Partial Differential Equations 23 (1998), 577-599.

[162] J. Serrin, H. Zou - *Non-existence of positive solutions of Lane-Emden systems*, Differential and Integral Equations 9 (1996), 635-653.

[163] R. Servadei, E. Valdinoci - *On the spectrum of two different fractional operators*, Proc. Roy. Soc. Edinburgh Sect. A 144 (2014), 831-855.

[164] R. Servadei, E. Valdinoci - *Mountain pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. 389 (2012), 887-898.

[165] R. Servadei, E. Valdinoci - *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. 33 (2013), 2105-2137.

[166] R. Servadei, E. Valdinoci - *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal. 12 (2013), 2445-2464.

[167] R. Servadei, E. Valdinoci - *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. 367 (2015), 67-102.

[168] R. Servadei, E. Valdinoci - *Weak and viscosity solutions of the fractional Laplace equation*, Publ. Mat 58 (2014), 133-154.

[169] L. Silvestre - *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Commun. Pure Appl. Math. 60 (2007), 67-112.

[170] B. Sirakov - *Existence results and a priori bounds for higher order elliptic equations and systems*, J. Math. Pures Appl. 89 (2008), 114-133.

[171] P. Souplet - *The proof of the Lane-Emden conjecture in four space dimensions*, Advances in Mathematics 221 (2009), 1409-1427.

[172] M.A.S. Souto - *A priori estimates and and existence of positive solutions of nonlinear cooperative elliptic systems*, Differential and Integral Equations 8 (1995), 1245-1258.

[173] E. Stein - *Singular Integrals and Differentiability Properties of Functions*, New York: Princeton Univ. Press, 1970.

[174] P. Sztonyk - *Boundary potential theory for stable Lévy processes*, Colloq. Math. 95 (2003), 191-206.

[175] P. Sztonyk - *Regularity of harmonic functions for anisotropic fractional Laplacians*, Math. Nachr. 283 (2010), 289-311.

[176] J. Tan - *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations 42 (2011), 21-41.

[177] J. Tan - *Positive solutions for non local elliptic problems*, Discrete Contin. Dyn. Syst. 33 (2013), 837-859.

[178] L. Tartar - *An introduction to Sobolev spaces and interpolation spaces*, Lecture Notes of the Unione Matematica Italiana, 3. Springer, Berlin; UMI, Bologna, 2007.

[179] S. Terracini, G. Verzini, A. Zilio - *Uniform Hölder bounds for strongly competing systems involving the square root of the laplacian*, arXiv: 1211.6087, 2012.

[180] S. Terracini, G. Verzini, A. Zilio - *Uniform Hölder regularity with small exponent in competition-fractional diffusion systems*, Disc. Cont. Dyn. Syst. 34 (2014), 2669-2691.

[181] J.F. Toland - *The Peierls-Nabarro and Benjamin-Ono equations*, J. Funct. Anal. 145 (1997), 136-150.

[182] G.M. Viswanathan et al. - *Lévy flight search patterns of wandering albatrosses*, Nature 381 (1996), 413-415.

[183] M. Webb - *Analysis and Approximation of a Fractional Differential Equation*, PhD thesis, Masters Thesis, Oxford University, Oxford, 2012.

[184] E. Weinan - *Dynamics of vortex-liquids in Ginzburg-Landau theories with applications to superconductivity*, Phys. Rev. B 50 (1994), 1126-1135.

[185] A. Xia, J. Yang - *Regularity of nonlinear equations for fractional Laplacian*, Proc. Amer. Math. Soc. 141 (2013), 2665-2672.

[186] L.P. Yaroslavsky - *Digital Picture Processing, an Introduction*, Springer-Verlag, Berlin, 1985.

[187] H. Zou - *A priori estimates and existence for strongly coupled semilinear elliptic systems*, Comm. Partial Differential Equations 31 (2006), 735-773.