UNIVERSIDADE FEDERAL DE MINAS GERAIS
INSTITUTO DE CIÊNCIAS EXATAS
DEPARTAMENTO DE MATEMÁTICA

Tese de Doutorado

On superalgebras with graded involution

Rafael Bezerra dos Santos

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Tese apresentada ao corpo docente de Pós-Graduação em Matemática do Instituto de Ciências Exatas da Universidade Federal de Minas Gerais, como parte dos requisitos para a obtenção do título de Doutor em Matemática.

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À minha amada mãe Célia,
eterna em mim.

Aos meus afilhados,
Kauan, Matheus e William.

"E até lá

Vamos viver

Temos muito ainda por fazer

Não olhe pra trás

Apenas começamos

O mundo começa agora, ahh!

Apenas começamos."

Metal Contra as Nuvens, Legião Urbana

Universidade Federal de Minas Gerais
Departamento de Matemática
Programa de Pós-Graduação em Matemática

FOLHA DE APROVAÇÃO

"On superalgebras with graded involution"

RAFAEL BEZERRA DOS SANTOS

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Abstract

In this thesis, we extend classic results of PI-theory to a new class of algebras: the $*$-superalgebras, that is, algebras endowed with a graded involution. If a $*$-superalgebra A satisfies a non-trivial identity, then the sequence $\{c_n^{\mathrm{gri}}(A)\}_{n\geq 1}$ of $*$-graded codimensions of A is exponentially bounded and we study the $*$-graded exponent $\exp^{\mathrm{gri}}(A) := \lim_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gri}}(A)}$ of A. To this end, we prove a version of Wedderburn-Malcev theorem for $*$-superalgebras and classify the finite dimensional simple $*$-superalgebras over an algebraically closed field of characteristic zero. By using the representation theory of the symmetric group, we give an alternative proof of the existence of the $*$-graded exponent for any finite dimensional $*$-superalgebra over a field of characteristic zero and we characterize, in four equivalent ways, the finite dimensional $*$-superalgebras with polynomial growth of $*$-graded codimensions. Finally, we classify the finite dimensional $*$-superalgebras A such that $\exp^{\mathrm{gri}}(A) \geq 2$.

Keywords: polynomial identity, graded involution, $*$-graded codimension, cocharacter, exponential growth.

Resumo estendido

Nesta tese, trabalhamos com superálgebras sobre um corpo F de característica zero munidas de uma involução de modo que as componentes homogêneas são invariantes sob a involução. Mais precisamente, dizemos que uma superálgebra $A = A^{(0)} \oplus A^{(1)}$ munida de uma involução $*$ é uma $*$-*superálgebra* se $(A^{(0)})^* = A^{(0)}$ e $(A^{(1)})^* = A^{(1)}$. Neste caso, dizemos que $*$ é uma *involução graduada*.

Se A é uma álgebra sobre um corpo de característica zero, um método bem estabelecido para o estudo do crescimento do correspondente ideal de identidades polinomiais é através de uma sequência numérica associada à álgebra chamada de sequência de codimensões de A. Recentemente, vários resultados foram estabelecidos permitindo definir alguns invariantes que podem ser ligados a um determinado T-ideal (e.g., [5], [7], [6], [17], [23], [9]). Estes resultados têm sido estendidos para álgebras munidas de alguma estrutura adicional, por exemplo, superálgebras, ou mais geralmente, álgebras graduadas por um grupo, álgebras com involuções, etc., permitindo o estudo das correspondentes identidades (e.g., [8], [40], [15], [14]). Neste trabalho, introduzimos a teoria de identidades polinomiais $*$-graduadas em $*$-superálgebras e estendemos alguns destes resultados no contexto de $*$-superálgebras. Notamos que o estudo de identidades $*$-graduadas generalizam a teoria de $*$-identidades em álgebras com involução.

Vamos relembrar alguns fatos sobre PI-álgebras que serão importantes no desenvolvimento deste texto. É sabido que se uma álgebra A satisfaz uma identidade polinomial não-trivial, então sua sequência de codimensões $c_n(A)$, $n \geq 1$, é limitada exponencialmente, i.e. existem constantes $a, \alpha > 0$ tais que $c_n(A) \leq a\alpha^n$ para todo n (veja [34]). Nos últimos anos, vários autores têm estudado esta sequência no intuito de caracterizar variedades de álgebras var(A) através do comportamento assintótico de $c_n(A)$.

Uma das primeiras caracterizações foi dada por Kemer em [23]. Ele provou que a sequência $c_n(A)$ de codimensões de uma PI-álgebra A é polinomialmente limitada, i.e. para todo $n \geq 1$, $c_n(A) \leq an^t$ para algumas constantes a, t, se, e somente se, nem a álgebra de Grassmann de dimensão infinita $\mathcal{G}$ e nem a álgebra $UT_2(F)$ de matrizes triangulares superiores 2×2 pertencem à var(A). Em [24], Kemer deu uma caracterização na linguagem de S_n-caracteres: $c_n(A)$ é limitada polinomialmente se, e somente se, existe uma constante q, que depende somente de A, tal que os S_n-módulos irredutíveis não-triviais que aparecem na decomposição do S_n-módulo $P_n(A) := \dfrac{P_n}{P_n \cap Id(A)}$ correspondem a diagramas de Young que possuem no máximo q boxes abaixo da primeira linha, onde P_n denota o espaço dos polinômios multilineares de grau n e Id(A) é o ideal das identidades de A.

Também, em [17], Giambruno e Zaicev deram uma caracterização de variedades de álgebras var(A) tais que $c_n(A)$ é limitada polinomialmente que depende somente da estrutura da álgebra A. Eles provaram que a sequência de codimensões $c_n(A)$ de uma álgebra de dimensão finita é limitada polinomialmente se, e somente se, $\mathrm{Id}(A) = \mathrm{Id}(B_1 \oplus \cdots \oplus B_n)$, onde as álgebras $B_i's$ possuem certas propriedades.

Tais caracterizações foram estendidas para álgebras munidas de alguma estrutura adicional, e.g. álgebras com involução e álgebras G-graduadas, onde G é um grupo finito. Referimos ao leitor os artigos [17, 25, 32, 40].

Para dar exemplos de resultados importantes citados nas referências, recordaremos alguns detalhes. Se G é um grupo, dizemos que uma álgebra A é G-graduada se A pode ser escrita como a soma de subespaços $A = \bigoplus_{g \in G} A^{(g)}$ tais que $A^{(g)}A^{(h)} \subseteq A^{(gh)}$, para todos $g, h \in G$. Em particular, se $G = \mathbb{Z}_2$, dizemos que A é uma *superálgebra*. Como no caso ordinário, podemos definir a superálgebra livre associativa e a sequência de codimensões graduadas $c_n^{\mathrm{gr}}(A)$, $n \geq 1$, de uma superálgebra A. Em [15], Giambruno, Mishchenko e Zaicev caracterizaram supervariedades $\mathcal{V}$, i.e. variedades geradas por superálgebras, de crescimento polinomial através da exclusão de cinco superálgebras de $\mathcal{V}$ e discutiremos isso abaixo.

Seja $D = F \oplus F$. Denotamos por D^{gr} a álgebra D com graduação $D^{\mathrm{gr}} = F(1,1) \oplus F(1,-1)$.

Seja $UT_2(F) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}$ a álgebra matrizes triangulares superiores 2×2 sobre F. A álgebra $UT_2(F)$ tem, a menos de isomorfismos, apenas duas graduações: a graduação trivial e a graduação canônica $UT_2(F)^{(0)} = Fe_{11} + Fe_{22}$ e $UT_2(F)^{(1)} = Fe_{12}$, onde e_{ij} denota as matrizes elementares usuais. A álgebra $UT_2(F)$ com graduação canônica será denotada por $UT_2(F)^{\mathrm{gr}}$ e $UT_2(F)$ denota a álgebra $UT_2(F)$ com graduação trivial.

Denotamos por $\mathcal{G}$ a álgebra de Grassmann. A álgebra $\mathcal{G}$ é gerada por um conjunto infinito $\{e_1, e_2, \ldots\}$ sujeito às condições $e_ie_j = -e_je_i$, para todos i, j. A álgebra $\mathcal{G}$ pode ser munida de uma graduação $\mathcal{G} = \mathcal{G}^{(0)} \oplus \mathcal{G}^{(1)}$ onde

$$\mathcal{G}^{(0)} = \mathrm{span}\{e_{i_1}e_{i_2} \cdots e_{i_{2k}} : i_1 < i_2 < \cdots < i_{2k}, k \geq 0\}$$

e

$$\mathcal{G}^{(1)} = \mathrm{span}\{e_{i_1}e_{i_2} \cdots e_{i_{2k+1}} : i_1 < i_2 < \cdots < i_{2k+1}, k \geq 0\}.$$

$\mathcal{G}^{\text{gr}}$ denota a álgebra $\mathcal{G}$ com esta graduação e $\mathcal{G}$ denota a álgebra $\mathcal{G}$ com graduação trivial.

Theorem 0.1 ([15], Theorem 2)**.** *Seja $\mathcal{V}$ uma variedade de superálgebras. Então $\mathcal{V}$ tem crescimento polinomial se, e somente se, $\mathcal{G}$, $\mathcal{G}^{\text{gr}}$, $UT_2(F)$, $UT_2(F)^{\text{gr}}$,$D^{\text{gr}} \notin \mathcal{V}$.*

Uma *involução* em uma álgebra A é uma transformação linear $* : A \to A$ tal que $(ab)^* = b^*a^*$ and $(a^*)^* = a$, para todos $a, b \in A$. Como acima, podemos definir a álgebra livre associativa com involução e a sequência de $*$-codimensões $c_n^*(A)$, $n \geq 1$, de uma álgebra com involução A. Em [14], Giambruno e Mishchenko caracterizaram $*$-variedades $\mathcal{V}$, i.e. variedades geradas por álgebras com involução, de crescimento polinomial através da exclusão de duas álgebras com involução de $\mathcal{V}$ e discutiremos isso abaixo.

Como antes, denotaremos por $D = F \oplus F$. Denotaremos por D_* a álgebra D munida da involução $(a, b)^* = (b, a)$.

Agora, definimos M como sendo a seguinte subálgebra de $UT_4(F)$

$$M = \left\{ \begin{pmatrix} a & c & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & d \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d \in F \right\}.$$

Denotamos por M_* a álgebra M munida da involução reflexão, i.e. a involução obtida através da reflexão da matriz ao longo de sua diagonal secundária

$$\begin{pmatrix} a & c & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & d \\ 0 & 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & 0 & a \end{pmatrix}.$$

Theorem 0.2 ([14], Theorem 4.7)**.** *Seja $\mathcal{V}$ uma variedade de álgebras com involução. Então $\mathcal{V}$ tem crescimento polinomial se, e somente se, $D_*, M_* \notin \mathcal{V}$.*

Em geral, se G é um grupo e A é uma álgebra G-graduada munida de uma involução $*$, dizemos que $*$ é G-graduada (com respeito à G-graduação em A) se $(A^{(g)})^* = A^{(g)}$, para todo $g \in G$. Involuções G-graduadas em álgebras de matrizes apareceram nos trabalhos de Bahturin, Shestakov e Zaicev [1], Bahturin e Zaicev [2] e Bahturin e Giambruno em [3].

Nesta tese, trabalhamos com o caso particular em que $G = \mathbb{Z}_2$ e estudamos superálgebras munidas de involuções $\mathbb{Z}_2$-graduadas, ou seja, $*$-superálgebras. O objetivo principal é classificar os ideais de identidades $*$-graduadas $\mathrm{Id}^{\mathrm{gri}}(A)$ de uma $*$-superálgebra A cuja sequência de codimensões $*$-graduadas correspondente $c_n^{\mathrm{gri}}(A)$ cresce exponencialmente e possui a seguinte propriedade adicional: se $\mathrm{Id}^{\mathrm{gri}}(B)$ é um ideal de identidades $*$-graduadas tais que $\mathrm{Id}^{\mathrm{gri}}(A) \subsetneq \mathrm{Id}^{\mathrm{gri}}(B)$, então $c_n^{\mathrm{gri}}(B)$ é limitada polinomialmente. Na linguagem de variedades, nosso objetivo é classificar as variedades de $*$-superálgebras de crescimento quase polinomial. Nesta tese, atingimos este objetivo trabalhando com álgebras de dimensão finita. Além disso, estendemos outros resultados que são válidos para álgebras, álgebras com involução e superálgebras no contexto de $*$-superálgebras.

Esta tese é composta de quatro capítulos dispostos da seguinte maneira.

No Capítulo 1, estabelecemos as principais propriedades de $*$-superálgebras e demonstramos uma versão do teorema de Wedderburn-Malcev para $*$-superálgebras de dimensão finita. Também introduzimos o conceito de $*$-superálgebras simples e classificamos as $*$-superálgebras simples de dimensão finita sobre um corpo algebricamente fechado de característica zero.

No Capítulo 2, definimos a $*$-superálgebra livre associativa e introduzimos as identidades polinomiais $*$-graduadas em $*$-superálgebras. Também definimos o principal objeto de estudo desta tese: a sequência de codimensões $*$-graduadas $c_n^{\mathrm{gri}}(A)$ de uma $*$-superálgebra A e estudamos a ação do produto de quatro grupos simétricos sobre o espaço dos $(\mathbb{Z}_2, *)$-polinômios multilineares. Na terceira sessão deste capítulo, definimos o expoente $*$-graduado $\exp^{\mathrm{gri}}(A)$ de uma $*$-superálgebra A. A existência de $\exp^{\mathrm{gri}}(A)$ foi provada por Gordienko em [21], mas aqui damos uma demonstração alternativa de sua existência, para qualquer $*$-superálgebra de dimensão finita A, que não depende dos argumentos utilizados na demonstração de Gordienko. Na sessão final, caracterizamos $*$-superálgebras simples através do expoente $*$-graduado.

O Capítulo 3 é o capítulo principal desta tese. Neste capítulo, damos quatro caracterizações equivalentes de $*$-superálgebras de dimensão finita de crescimento polinomial das codimensões $*$-graduadas. Primeiro, caracterizamos $*$-superálgebras de crescimento polinomial através do expoente $*$-graduado. Na segunda caracterização, classificamos $*$-supervariedades de crescimento polinomial geradas por $*$-superálgebras de dimensão finita pela exclusão de cinco $*$-superálgebras da $*$-supervariedade. Como consequência, classificamos as $*$-supervariedades de crescimento quase polinomial geradas por $*$-superálgebras de dimensão finita. Após isso, provamos que se A é uma $*$-superálgebra de dimensão finita, então a sequência $c_n^{\mathrm{gri}}(A)$ é limitada polinomialmente se, e somente se, $\mathrm{Id}^{\mathrm{gri}}(A) = \mathrm{Id}^{\mathrm{gri}}(B_1 \oplus \cdots \oplus B_n)$ onde cada $B_i, i = 1, \ldots, n$, é uma $*$-superálgebra de dimensão finita tal que $\dim B_i/J(B_i) \leq 1$. Finalmente, usamos a teoria de representações do produto de quatro grupos simétricos $S_{\langle n \rangle} := S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ para provar que $c_n^{\mathrm{gri}}(A)$ é limitada polinomialmente se, e somente se, existe uma constante q, que depende somente de A, tal que os $S_{\langle n \rangle}$-módulos irredutíveis não-triviais que aparecem na decomposição de $P_n^{\mathrm{gri}}(A)$ são tais que o diagrama de Young correspondente à $\lambda(1)$, sem a primeira linha, junto com os diagramas de Young correspondentes à $\lambda(2), \lambda(3)$ e $\lambda(4)$ contém no máximo q boxes, onde $\lambda(i) \vdash n_i$, para $1 \leq i \leq 4$.

No Capítulo 4, estudamos $*$-superálgebras A tais que $\exp^{\mathrm{gri}}(A) \geq 2$. Construímos onze $*$-superálgebras $E_i, i = 1, \ldots, 11$, com a seguinte propriedade: $\exp^{\mathrm{gri}}(A) > 2$ se, e somente se, $E_i \in \mathrm{var}^{\mathrm{gri}}(A)$, para algum $i \in \{1, \ldots, 11\}$. Como consequência, caracterizamos as $*$-superálgebras A tais que $\exp^{\mathrm{gri}}(A) = 2$.

Os resultados desta tese já foram publicados em [10, 16, 37].

As principais técnicas utilizadas neste trabalho são métodos da teoria de representações do grupo simétrico S_n e o estudo do comportamento assintótico dos graus de S_n-representações irredutíveis. Sugerimos ao leitor o livro [22] para o estudo de S_n-representações e os livros [20] e [9] para mais informações sobre a teoria de PI-álgebras.

Introduction

In this thesis, we work with superalgebras over a field F of characteristic zero endowed with an involution such that the homogeneous components are invariant under the involution. More precisely, we say that a superalgebra $A = A^{(0)} \oplus A^{(1)}$ endowed with an involution $*$ is a $*$-*superalgebra* if $(A^{(0)})^* = A^{(0)}$ and $(A^{(1)})^* = A^{(1)}$. In this case, we say that $*$ is a *graded involution.*

If A is an algebra over a field of characteristic zero, a well-established method of studying the growth of the corresponding ideal of polynomial identities is through a numerical sequence called the sequence of codimensions of A. Several results have been established in recent years allowing to define some invariants that can be attached to a given T-ideal (e.g., [5], [7], [6], [17], [23], [9]). These results have been extended to algebras with an additional structure such as superalgebras, group graded algebras, algebras with involution, etc., allowing to study the corresponding identities (e.g. [8], [40], [15], [14]). Here, we introduce the theory of $*$-graded polynomial identities on $*$-superalgebras A and we extend some of those results in the setting of $*$-superalgebras. We notice that the study of $*$-graded identities on $*$-superalgebras generalize the theory of $*$-identities on algebras with involution.

Let us recall some facts about PI-algebras which will be important in the development of this text. It is well known that if an algebra A satisfies a non-trivial polynomial identity, then its sequence of codimensions $c_n(A)$, $n \geq 1$, is exponentially bounded, i.e. there exist constants $a, \alpha > 0$ such that $c_n(A) \leq a\alpha^n$ for all n (see [34]). In the last years, several authors have studied this sequence in order to characterize varieties of algebras $\mathrm{var}(A)$ through the asymptotic behavior of $c_n(A)$.

One of the first characterizations was given by Kemer in [23]. He proved that the sequence $c_n(A)$ is polynomially bounded, i.e. for all $n \geq 1$, $c_n(A) \leq an^t$ for some constants a, t, if and only if neither the infinite dimensional Grassmann algebra $\mathcal{G}$ nor the algebra $UT_2(F)$ of the 2×2 upper triangular matrices lie in $\mathrm{var}(A)$. In [24], Kemer gave such a characterization in the language of the S_n-characters: $c_n(A)$ is polynomially bounded if and only if there exists a constant q depending only on A such that the nonzero irreducible S_n-modules appearing in the decomposition of the S_n-module $P_n(A) := \dfrac{P_n}{P_n \cap Id(A)}$ correspond to Young diagrams having at most q boxes below the first row, where P_n denotes the space of multilinear polynomials of degree n and $\mathrm{Id}(A)$ is the ideal of identities of A.

Also, in [17], Giambruno and Zaicev gave a characterization of varieties of algebras $\mathrm{var}(A)$ such that $c_n(A)$ is polynomially bounded that depends only on the structure of the algebra A. They proved that the sequence $c_n(A)$ of a finite dimensional algebra A is polynomially bounded if and only if $\mathrm{Id}(A) = \mathrm{Id}(B_1 \oplus \cdots \oplus B_n)$, where the $B_i's$ are suitable algebras with some properties.

Such characterizations were extended to algebras with some additional structure, e.g. algebras with involution and G-graded algebras, where G is a finite group. We refer to the reader the papers [17, 25, 32, 40].

To give examples of important results in the cited references, we will recall some details. If G is a group, we say that an algebra A is G-graded if A can be written as a sum of subspaces $A = \bigoplus_{g \in G} A^{(g)}$ such that $A^{(g)}A^{(h)} \subseteq A^{(gh)}$, for all $g, h \in G$. In particular, if $G = \mathbb{Z}_2$, we say that A is a *superalgebra*. As in the ordinary case, we can define the free associative superalgebra and the sequence of graded codimensions $c_n^{\mathrm{gr}}(A)$, $n \geq 1$, of a superalgebra A. In [15], Giambruno, Mishchenko and Zaicev characterized supervarieties $\mathcal{V}$, i.e. varieties generated by superalgebras, with polynomial growth by excluding five superalgebras from $\mathcal{V}$ and we will discuss this below.

Let $D = F \oplus F$. We denote by D^{gr} the algebra D with grading $D^{\mathrm{gr}} = F(1,1) \oplus F(1,-1)$.

Let $UT_2(F) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}$ be the algebra of upper triangular matrices over F. The algebra $UT_2(F)$ has, up to isomorphism, only two gradings: the trivial grading and the natural grading $UT_2(F)^{(0)} = Fe_{11} + Fe_{22}$ and $UT_2(F)^{(1)} = Fe_{12}$, where e_{ij} denotes the usual elementary matrices. The algebra $UT_2(F)$ with the natural grading will be denoted by $UT_2(F)^{\mathrm{gr}}$ and $UT_2(F)$ denotes the algebra $UT_2(F)$ with trivial grading.

Let $\mathcal{G}$ denote the Grassmann algebra. The algebra $\mathcal{G}$ is generated by an infinite set $\{e_1, e_2, \ldots\}$ subject to the conditions $e_i e_j = -e_j e_i$, for all i, j. The algebra $\mathcal{G}$ can be endowed with grading $\mathcal{G} = \mathcal{G}^{(0)} \oplus \mathcal{G}^{(1)}$ where

$$\mathcal{G}^{(0)} = \mathrm{span}\{e_{i_1} e_{i_2} \cdots e_{i_{2k}} : i_1 < i_2 < \cdots < i_{2k}, k \geq 0\}$$

and

$$\mathcal{G}^{(1)} = \mathrm{span}\{e_{i_1} e_{i_2} \cdots e_{i_{2k+1}} : i_1 < i_2 < \cdots < i_{2k+1}, k \geq 0\}.$$

$\mathcal{G}^{\mathrm{gr}}$ denotes the algebra $\mathcal{G}$ with this grading and $\mathcal{G}$ denotes the algebra $\mathcal{G}$ with trivial grading.

Theorem 0.3 ([15], Theorem 2)**.** *Let $\mathcal{V}$ be a variety of superalgebras. Then $\mathcal{V}$ has polynomial growth if and only if $\mathcal{G}, \mathcal{G}^{\mathrm{gr}}, UT_2(F), UT_2(F)^{\mathrm{gr}}, D^{\mathrm{gr}} \notin \mathcal{V}$.*

An *involution* on an algebra A is a linear transformation $* : A \to A$ such that $(ab)^* = b^* a^*$ and $(a^*)^* = a$, for all $a, b \in A$. As above, we can define the free algebra with involution and the sequence of $*$-codimensions $c_n^*(A)$, $n \geq 1$, of an algebra with involution A. In [14], Giambruno and Mishchenko characterized $*$-varieties $\mathcal{V}$, i.e. varieties generated by algebras with involution, with polynomial growth by excluding two algebras with involution from $\mathcal{V}$ and we will discuss this below.

As before, we denote by $D = F \oplus F$. We denote by D_* the algebra D endowed with the involution $(a, b)^* = (b, a)$.

Next, we define M to be the following subalgebra of $UT_4(F)$

$$M = \left\{ \begin{pmatrix} a & c & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & d \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d \in F \right\}.$$

We denote by M_* the algebra M with reflection involution, i.e. the involution obtained by flipping the matrix along its secondary diagonal

$$\begin{pmatrix} a & c & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & d \\ 0 & 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & 0 & a \end{pmatrix}.$$

Theorem 0.4 ([14], Theorem 4.7)**.** *Let $\mathcal{V}$ be a variety of algebras with involution. Then $\mathcal{V}$ has polynomial growth if and only if $D_*, M_* \notin \mathcal{V}$.*

In general, if G is a group and A is a G-graded algebra endowed with an involution $*$, we say that the involution $*$ is G-graded (with respect to the G-grading on A) if $(A^{(g)})^* = A^{(g)}$, for all $g \in G$. G-Graded involutions on matrices algebras have appeared in the papers of Bahturin, Shestakov and Zaicev [1], Bahturin and Zaicev [2] and Bahturin and Giambruno [3].

In this thesis, we work in the particular case that $G = \mathbb{Z}_2$ and we study superalgebras endowed with $\mathbb{Z}_2$-graded involutions, that is, $*$-superalgebras.

The main goal of this thesis is to classify the ideals of $*$-graded identities $\mathrm{Id}^{\mathrm{gri}}(A)$ of a $*$-superalgebra A whose corresponding sequence of codimensions $c_n^{\mathrm{gri}}(A)$ grows exponentially and have the following further property: if $\mathrm{Id}^{\mathrm{gri}}(B)$ is an ideal of $*$-graded identities such that $\mathrm{Id}^{\mathrm{gri}}(A) \subsetneq \mathrm{Id}^{\mathrm{gri}}(B)$, then $c_n^{\mathrm{gri}}(B)$ is polynomially bounded. In the language of varieties, our aim is to classify the varieties of $*$-superalgebras of almost polynomial growth. We reach our goal in the setting of finite dimensional algebras. In addition, we extend other results which are valid for algebras, algebras with involution and for superalgebras to the set of $*$-superalgebras.

This thesis is composed by four chapters disposed in the following way.

In Chapter 1, we establish the principal properties of $*$-superalgebras and describe a Wedderburn-Malcev theorem for finite dimensional $*$-superalgebras. We also introduce the concept of simple $*$-superalgebras and classify all finite dimensional simple $*$-superalgebras over an algebraically closed field of characteristic zero.

In Chapter 2, we define the free associative $*$-superalgebra and introduce the $*$-graded polynomial identities on $*$-superalgebras. We also define the main object of study of this thesis: the $*$-graded codimensions $c_n^{\mathrm{gri}}(A)$ of a $*$-superalgebra A and study the action of the product of four symmetric groups on the space of multilinear $(\mathbb{Z}_2, *)$-polynomials. In the third section of this chapter, we define the $*$-graded exponent $\exp^{\mathrm{gri}}(A)$ of a $*$-superalgebra A. The existence of $\exp^{\mathrm{gri}}(A)$ was proved by Gordienko in [21], but here we give an alternative proof of its existence, for any finite dimensional $*$-superalgebra A, which does not depend on the arguments of Gordienko's proof. In the final section, we characterize simple $*$-superalgebras through the $*$-graded exponent.

Chapter 3 is the main chapter of this thesis. In this chapter, we give four characterizations of finite dimensional $*$-superalgebras with polynomial growth. First, we characterize finite dimensional $*$-superalgebras with polynomial growth through the $*$-graded exponent. In the second characterization, we classify the $*$-supervarieties of polynomial growth generated by finite dimensional $*$-superalgebras by the exclusion of five suitable $*$-superalgebras from the $*$-supervariety. As a consequence, we classify the $*$-supervarieties generated by finite dimensional $*$-superalgebras of almost polynomial growth. Next, we prove that if A is a finite dimensional $*$-superalgebra, then the sequence $c_n^{\text{gri}}(A)$ is polynomially bounded if and only if $\text{Id}^{\text{gri}}(A) = \text{Id}^{\text{gri}}(B_1 \oplus \cdots \oplus B_n)$ where each $B_i, i = 1, \ldots, n$, is a finite dimensional $*$-superalgebra such that $\dim B_i/J(B_i) \leq 1$. Finally, we use the representation theory of the product of four symmetric groups $S_{\langle n \rangle} := S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ to prove that $c_n^{\text{gri}}(A)$ is polynomially bounded if and only if there exists a constant q depending only on A such that the nonzero irreducible $S_{\langle n \rangle}$-modules appearing in the decomposition of $P_n^{\text{gri}}(A)$ are such that the Young diagram corresponding to $\lambda(1)$, without its first row, along with the Young diagrams corresponding to $\lambda(2), \lambda(3)$ and $\lambda(4)$ contain in all at most q boxes, where $\lambda(i) \vdash n_i$, for $1 \leq i \leq 4$.

In Chapter 4, we study $*$-superalgebras such that $\exp^{\text{gri}}(A) \geq 2$. We construct eleven $*$-superalgebras $E_i, i = 1, \ldots, 11$, with the following property: $\exp^{\text{gri}}(A) > 2$ if and only if $E_i \in \text{var}^{\text{gri}}(A)$, for some $i \in \{1, \ldots, 11\}$. As a consequence, we characterize the $*$-superalgebras A such that $\exp^{\text{gri}}(A) = 2$.

The results of this thesis have already been published in [10, 16, 37].

The main techniques employed in this work are methods of representation theory of the symmetric group S_n and computations of the asymptotics for the degrees of the irreducible S_n-representations. We refer the reader to the book [22] for the study of S_n-representations and the books [20] and [9] for more about the theory of PI-algebras.

Chapter 1

Superalgebras with graded involution

In this chapter we introduce the concept of $*$-superalgebras and their principal properties. So, we start by considering $A = A^{(0)} \oplus A^{(1)}$ a superalgebra over a field F of characteristic different from 2. We remind the reader that, if $A = A^{(0)} \oplus A^{(1)}$ is a superalgebra, then $\varphi \in \mathrm{Aut}(A)$ defined by $\varphi(a^{(0)} + a^{(1)}) = a^{(0)} - a^{(1)}$, where $a^{(0)} \in A^{(0)}$, $a^{(1)} \in A^{(1)}$, is an automorphism of order at most 2. Moreover, any automorphism $\varphi \in \mathrm{Aut}(A)$ of order at most 2 determines a structure of superalgebra on A by setting $A^{(0)} = \{a + \varphi(a) : a \in A\}$ and $A^{(1)} = \{a - \varphi(a) : a \in A\}$.

Recall that an involution on an algebra A is just an antiautomorphism on A of order at most 2 which we shall denote by $*$. We write $A^+ = \{a \in A : a^* = a\}$ and $A^- = \{a \in A : a^* = -a\}$ for the sets of symmetric and skew-symmetric elements of A, respectively. Clearly $A = A^+ \oplus A^-$, since $\mathrm{char}(F) \neq 2$.

Definition 1.1. Let $A = A^{(0)} \oplus A^{(1)}$ be a superalgebra over a field F of characteristic different from 2 and suppose that A is endowed with an involution

$*$. We say that the involution $*$ is a *graded involution* if $(A^{(0)})^* = A^{(0)}$ and $(A^{(1)})^* = A^{(1)}$. In this case, we say that A is a $*$-*superalgebra*.

It is clear that any algebra with involution $*$ endowed with trivial grading is a $*$-superalgebra and for a commutative superalgebra A, the identity map is a graded involution on A.

Next, we give important examples of $*$-superalgebras that will be useful along the thesis.

Example 1.2. Let $D = F \oplus F$. The algebra D can be endowed with the exchange involution, i.e. the involution defined as $(a,b)^* = (b,a)$. By considering D with trivial grading, D has a structure of $*$-superalgebra with this involution, that, with this structure of $*$-superalgebra, will be denoted by D_*. Now, let D^{gr} be the algebra D endowed with the grading $D^{\mathrm{gr}} = F(1,1) \oplus F(1,-1)$. If $*$ is the exchange involution, then $*$ is a graded involution on D^{gr}. The superalgebra D^{gr} with exchange involution will be denoted by D^{gri}. Also, since D^{gr} is a commutative superalgebra, the identity map is a graded involution on D^{gr}.

Example 1.3. Let M be the following subalgebra of $UT_4(F)$

$$M = \left\{ \begin{pmatrix} a & c & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & d \\ 0 & 0 & 0 & a \end{pmatrix} : a, b, c, d \in F \right\}.$$

We denote by M_* the algebra M with reflection involution, i.e. the involution obtained by flipping the matrix along its secondary diagonal

$$\begin{pmatrix} a & c & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & d \\ 0 & 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & d & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & 0 & a \end{pmatrix}.$$

By considering M_* with trivial grading, M_* is a $*$-superalgebra. Now, the algebra M can be endowed with the grading

$$\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix} \right).$$

If we consider the reflection involution, we have that $(M^{(0)})^* = M^{(0)}$, $(M^{(1)})^* = M^{(1)}$ and so the reflection involution is graded. Also, $(M^{(0)})^+ = M^{(0)}$, $(M^{(0)})^- = \{0\}$, $(M^{(1)})^+ = F(e_{12} + e_{34})$ and $(M^{(1)})^- = F(e_{12} - e_{34})$. The algebra M endowed with this grading and with this involution will be denoted by M^{gri}.

It is clear that there exist involutions on superalgebras that are not graded. For instance, consider the superalgebra

$$A = \left(\begin{pmatrix} 0 & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix} \right)$$

endowed with reflection involution. Then the reflection involution is not a graded involution on A.

The connection between the superstrucure and the involution on A is given in the next lemma.

Lemma 1.4. *Let A be a superalgebra over a field F of characteristic different from 2 endowed with an involution $*$ and φ the automorphism of order at most 2 determined by the superstructure. Then A is a $*$-superalgebra if and only if $* \circ \varphi = \varphi \circ *$.*

Proof. Suppose that $A = A^{(0)} \oplus A^{(1)}$ is a $*$-superalgebra and let $a = a^{(0)} + a^{(1)} \in A, a^{(0)} \in A^{(0)}, a^{(1)} \in A^{(1)}$. Then $\varphi(a^{(0)}) = a^{(0)}$ and $\varphi(a^{(1)}) = -a^{(1)}$. Since A is a $*$-superalgebra, we get that $(a^{(0)})^* \in A^{(0)}$ and $(a^{(1)})^* \in A^{(1)}$. Thus, $\varphi((a^{(0)})^*) = (a^{(0)})^*$ and $\varphi((a^{(1)})^*) = -(a^{(1)})^*$. Therefore, $\varphi(a^*) = \varphi((a^{(0)})^* + (a^{(1)})^*) = (a^{(0)})^* - (a^{(1)})^* = (\varphi(a))^*$, for all $a \in A$. Hence, $* \circ \varphi = \varphi \circ *$.

Conversely, suppose that $* \circ \varphi = \varphi \circ *$. We want to prove that if $a = a^{(0)} + a^{(1)} \in A, a^{(0)} \in A^{(0)}, a^{(1)} \in A^{(1)}$, then $(a^{(0)})^* \in A^{(0)}$ and $(a^{(1)})^* \in A^{(1)}$. We have that $\varphi((a^{(0)})^*) = (\varphi(a^{(0)}))^* = (a^{(0)})^*$ and $\varphi((a^{(1)})^*) = (\varphi(a^{(1)}))^* = -(a^{(1)})^*$. Hence, $(a^{(0)})^* \in A^{(0)}$, $(a^{(1)})^* \in A^{(1)}$ and A is a $*$-superalgebra. $\square$

Corollary 1.5. *Let A be a superalgebra over a field F of characteristic different from 2 endowed with an involution $*$. Then A is a $*$-superalgebra if and only the subspaces A^+ and A^- are graded subspaces. As a consequence, any $*$-superalgebra can be written as a sum of 4 subspaces*

$$A = (A^{(0)})^+ \oplus (A^{(1)})^+ \oplus (A^{(0)})^- \oplus (A^{(1)})^-.$$

In order to avoid confusion, we shall adopt the following notation: given a $*$-superalgebra A, we shall write A_* to denote the algebra A with involution $*$ and trivial grading. We also denote by A^{gr} the algebra with $\mathbb{Z}_2$-grading and trivial involution (notice that in this case A must be commutative).

1.1 The Wedderburn-Malcev theorem

In this section, we deal with finite dimensional $*$-superalgebras and we extend the Wedderburn's theorem on simple and semisimple algebras and the Wedderburn-Malcev theorem to the setting of finite dimensional $*$-superalgebras.

Definition 1.6. Let A be a $*$-superalgebra, φ the automorphism of order at most 2 determined by the superstructure and I an ideal of A. We say that I is a *$*$-graded ideal* if $I^\varphi = I$ and $I^* = I$. A $*$-superalgebra A is a *simple $*$-superalgebra* if $A^2 \neq \{0\}$ and A has no non-zero proper $*$-graded ideals.

Notice that, with this definition, if A is simple as an algebra or as an algebra with involution or as a superalgebra, then A is also simple as a $*$-superalgebra. On the other hand, the reverse is not true (cf. Theorem 1.12).

We start with the following result of independent interest. We recall that if A is a finite dimensional algebra then $J(A)$, the Jacobson radical of A, is a nilpotent ideal.

Proposition 1.7 ([16], Proposition 7.1)**.** *Let A be a finite dimensional algebra over a field with a Wedderburn-Malcev decomposition $A = B_1 \oplus \cdots \oplus B_k + J(A)$, where $B_1, \ldots, B_k$ are simple algebras. If B is a simple ideal of A then $B = B_i$, for some $i \in \{1, \ldots, k\}$, and $J(A)$ acts trivially on B by left and right multiplication.*

Proof. Let B be a simple ideal of A and write $J = J(A)$. Then $B \cap J$ is an ideal of B and, since B is simple, $B \cap J = \{0\}$ or $B \cap J = B$. Since J is a nilpotent ideal and B is not nilpotent we get that $B \cap J = \{0\}$.

We claim that $B \subseteq A_{ss} = B_1 \oplus \cdots \oplus B_k$ and $B = B_i$, for some $i \in \{1, \ldots, k\}$. In fact, since $B \not\subseteq J$, B is not a nil ideal and, so, there exists

$b \in B$ such that $b^n \neq 0$, for all $n \geq 1$. Now, we can write $b = x + y$, where $x \in A_{ss}$, $x \neq 0$ and $y \in J$. Since $BJ \subseteq B \cap J = \{0\}$, $b^2 = bx = xb$ and by induction we get $b^n = x^{n-1}b$, for all $n \geq 2$, and $b^n x^m = bx^{n+m-1} = b^{n+m}$, for all $n, m \geq 1$. Notice that $x^n \neq 0$, for all $n \geq 1$, since $b^n \neq 0$, for all $n \geq 1$. If q is the index of nilpotence of J, then $y^q = (b - x)^q = 0$ and we have

$$\begin{aligned} 0 = (b-x)^q &= \sum_{i=0}^{q} (-1)^i \binom{q}{i} b^{q-i} x^i \\ &= (-1)^q x^q + \sum_{i=0}^{q-1} (-1)^i \binom{q}{i} b^{q-i} x^i \\ &= (-1)^q x^q + \sum_{i=0}^{q-1} (-1)^i \binom{q}{i} b^q \\ &= (-1)^q x^q + (-1)^{q+1} b^q. \end{aligned}$$

It follows that $x^q = (-1)^{2(q+1)} b^q = b^q \in B$.

Let I be the ideal generated by x^q. Then I is a non-zero ideal of B. Since B is simple, we must have $I = B$. Notice that $Jx^q = x^q J = \{0\}$, since $x^q \in B$. Hence, $B = Bx^qB \subseteq Ax^qA = A_{ss}x^qA_{ss} \subseteq A_{ss}$ and, so, B is a simple ideal of A_{ss}. Being A_{ss} a semisimple algebra, $B = B_i$, for some $i \in \{1, \ldots, k\}$. Also $JB = BJ = \{0\}$ says that J acts trivially on B. □

Recall that an algebra A with an automorphism or antiautomorphism ψ is *ψ-simple* if $A^2 \neq \{0\}$ and A has no non-zero proper ideals I such that $I^\psi = I$.

Lemma 1.8 ([16], Lemma 7.2)**.** *Let A be an algebra with an automorphism or antiautomorphism ψ of order 2. If A is ψ-simple then either A is simple or $A = B \oplus B^\psi$, for some simple subalgebra B of A.*

Proof. If A is simple, we are done. Suppose that A is ψ-simple but not simple. Let B be a proper ideal of A. Then B^ψ is still an ideal of A and, since A is ψ-simple, we have that $B^\psi \neq B$. Now, $B + B^\psi$ is a ψ-ideal of A

and, since A is ψ-simple, $A = B + B^{\psi}$. Also, the ψ-simplicity of A implies that B is simple and $A = B \oplus B^{\psi}$. □

The next theorem is a generalization of the Wedderburn and Wedderburn-Malcev theorems.

Theorem 1.9 ([16], Theorem 7.3). *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero and let φ be the automorphism induced by the superstructure. Then:*

1. *$J(A)$ is a $*$-graded ideal;*
2. *If A is a simple $*$-superalgebra, then either A is simple or A is $*$-simple or $A = B \oplus B^{\varphi}$ for some $*$-simple ideal B;*
3. *If A is semisimple, then A is a finite direct sum of simple $*$-superalgebras;*
4. *If F is algebraically closed, then $A = A_1 \oplus \cdots \oplus A_m + J(A)$, where each algebra $A_i, i = 1, \ldots, m$, is a simple $*$-superalgebra.*

Proof. (1) Let $J = J(A)$. It is well known that if ψ is an automorphism or an antiautomorphism of order 2 then $J^{\psi} = J$, i.e. J is a $*$-ideal and a graded ideal. Hence, J is a $*$-graded ideal.

(2) Suppose that A is a simple $*$-superalgebra. Since $J(A)$ is a $*$-graded ideal and A is not a nilpotent algebra, we have that $J(A) = \{0\}$ and A is a semisimple algebra. Take I a minimal ideal of A. Then I is a simple algebra and either $I^* = I$ or $I \oplus I^*$ is a simple $*$-ideal of A. Hence A contains a simple $*$-ideal B. If $B = B^{\varphi}$, then B is a $*$-graded ideal of A and, since A is a simple $*$-superalgebra, we get that $A = B$. In case $B \neq B^{\varphi}$, $B \oplus B^{\varphi}$ is a $*$-graded ideal of A and so $A = B \oplus B^{\varphi}$ and we are done.

Part (3) follows from Wedderburn's theorem and (2). Part (4) follows from [38] and (3). □

1.2 Simple $*$-superalgebras

In this section, we classify the finite dimensional simple $*$-superalgebras over an algebraically closed field of characteristic zero.

We recall the classification of the simple superalgebras and of the $*$-simple algebras. Given $k \geq l \geq 0$, $k \geq 1$, $M_{k,l}(F)$ is the algebra $M_{k+l}(F)$ with grading $\left(\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}, \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right)$, where P, Q, R, S are $k \times k$, $k \times l$, $l \times k$ and $l \times l$ matrices, respectively. Also we consider the algebra $M_n(F + cF) = M_n(F) + cM_n(F)$, where $c^2 = 1$, with grading $(M_n(F), cM_n(F))$.

Theorem 1.10 ([20], Theorem 3.5.3)**.** *Let A be a finite dimensional simple superalgebra over an algebraically closed field F of characteristic zero. Then A is isomorphic either to $M_{k,l}(F), k \geq 1, k \geq l \geq 0$, or to $M_n(F) + cM_n(F), c^2 = 1$.*

Notice that, in light of Lemma 1.8, A is isomorphic to $M_{k,l}(F)$ when $B^\varphi = B$ and A is isomorphic to $M_n(F) + cM_n(F)$ when $B^\varphi \neq B$.

Theorem 1.11 ([36], Proposition 2.13.24)**.** *Let A be a finite dimensional $*$-simple algebra over an algebraically closed field F of characteristic zero. Then A is isomorphic to either $M_n(F)$ with transpose or symplectic involution or to $M_n(F) \oplus M_n(F)^{op}$ with exchange involution, where $M_n(F)^{op}$ denotes the opposite algebra of $M_n(F)$.*

We remark that, in light of Lemma 1.8, A is isomorphic to $M_n(F)$ with transpose or symplectic involution when $B^* = B$ and A is isomorphic to $M_n(F) \oplus M_n(F)^{op}$ with exchange involution when $B^* \neq B$.

In the next theorem, we classify the finite dimensional simple $*$-superalgebras over an algebraically closed field of characteristic zero. We remark that, if $M_n(F)$ is endowed with the symplectic involution, then n must be even.

Theorem 1.12 ([16], Theorem 7.6)**.** *Let A be a finite dimensional simple $*$-superalgebra over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following $*$-superalgebras:*

1. *$M_{k,l}(F)$, with $k \geq 1, k \geq l \geq 0$, with transpose or symplectic involution (the symplectic involution can occur only when $k = l$);*

2. *$M_{k,l}(F) \oplus M_{k,l}(F)^{op}$, with $k \geq 1$, $k \geq l \geq 0$, with induced grading and exchange involution;*

3. *$M_n(F) + cM_n(F)$, with involution given by $(a + cb)^\dagger = a^* - cb^*$, where $*$ denotes the transpose or symplectic involution;*

4. *$M_n(F) + cM_n(F)$, with involution given by $(a + cb)^\dagger = a^* + cb^*$, where $*$ denotes the transpose or symplectic involution;*

5. *$(M_n(F) + cM_n(F)) \oplus (M_n(F) + cM_n(F))^{op}$, with grading*
$$(M_n(F) \oplus M_n(F)^{op}, c(M_n(F) \oplus M_n(F)^{op}))$$
and exchange involution.

Proof. Let A be a simple $*$-superalgebra. By Theorem 1.9, (2), we have that either A is simple or A is $*$-simple or $A = B \oplus B^\varphi$ for some $*$-simple subalgebra B of A. If A is simple, by Theorem 1.10 and Theorem 1.11, we have (1).

Suppose that A is $*$-simple, but not simple. Then, by Theorem 1.11, $A = B \oplus B^{op}$, for some simple subalgebra B of A. If $B^\varphi = B$, then $(B^{op})^\varphi = B^{op}$ and, by Theorem 1.10, $B = M_{k,l}(F)$ and we have (2).

If $B^{\varphi} \neq B$, then $B^{\varphi} = B^{op}$. For every $(a, b) \in A$, write $\varphi(a, b) = (\varphi_0(b), \varphi_1(a))$ where $\varphi_0, \varphi_1 : B \to B$ are linear mappings. Denote by $\bar{*}$ the exchange involution. We have that

$$\varphi(a, b)^{\bar{*}} = (\varphi_0(b), \varphi_1(a))^{\bar{*}} = (\varphi_1(a), \varphi_0(b))$$

and

$$\varphi((a, b)^{\bar{*}}) = \varphi(b, a) = (\varphi_0(a), \varphi_1(b)).$$

Since φ commutes with $\bar{*}$, $\varphi_0 = \varphi_1$ and $\varphi(a, b) = (\varphi_0(b), \varphi_0(a))$, for every $(a, b) \in A$. Also, since φ is an automorphism, we have that, for every $(a_1, b_1), (a_2, b_2) \in A$, $\varphi((a_1, b_1)(a_2, b_2)) = \varphi(a_1, b_1)\varphi(a_2, b_2)$ and thus $\varphi_0(ab) = \varphi_0(b)\varphi_0(a)$, for every $a, b \in B$. Since $\varphi^2 = 1$, $\varphi_0^2 = 1$ and so φ_0 is an involution on B. Let $\varphi_0 = *$. Thus $\varphi(a, b) = (b^*, a^*)$.

Notice that $(a, b) = \frac{1}{2}(a + b^*, a^* + b) + \frac{1}{2}(a - b^*, -a^* + b)$. Recalling that $A^{(0)} = \{(a, b) + \varphi(a, b) : (a, b) \in A\}$ and $A^{(1)} = \{(a, b) - \varphi(a, b) : (a, b) \in A\}$, we can write $A^{(0)} = \{(a, a^*) : a \in B\}$ and $A^{(1)} = \{(a, -a^*) : a \in B\}$. Therefore $A = A^{(0)} \oplus A^{(1)}$ is a grading compatible with the exchange involution. Now, it is easily seen that $A^{(0)} \cong M_n(F)$, $A^{(1)} = (1, -1)A^{(0)} = cA^{(0)} \cong cM_n(F)$, $c^2 = 1$ and $c^{\bar{*}} = -c$. Hence, by Theorem 1.10 and Theorem 1.11, $A \cong M_n(F) + cM_n(F)$ with involution given by $(a + cb)^{\dagger} = a^* - cb^*$, where $*$ denotes the transpose or symplectic involution and we have (3).

Now, suppose that A is not $*$-simple. Then $A = B \oplus B^{\varphi}$ for some $*$-simple subalgebra B of A. If B is simple, then, by Theorem 1.11, $B \cong M_n(F)$ with transpose or symplectic involution. Hence, by Theorem 1.10, $A \cong M_n(F) + cM_n(F)$, with grading $(M_n(F), cM_n(F))$ and with involution given by $(a + cb)^{\dagger} = a^* + cb^*$, where $*$ denotes the transpose or symplectic involution and we have (4).

Finally, if B is not simple, then, by Theorem 1.11, $B = C \oplus C^{op}$, $C \cong$

$M_n(F)$, with exchange involution and

$$\begin{aligned} A &= (C \oplus C^{op}) \oplus (C \oplus C^{op})^{\varphi} \\ &= (C \oplus C^{\varphi}) \oplus (C^{op} \oplus (C^{op})^{\varphi}) \\ &\cong (M_n(F) + cM_n(F)) \oplus (M_n(F) + cM_n(F))^{op}, \end{aligned}$$

with grading

$$(M_n(F) \oplus M_n(F)^{op}, c(M_n(F) \oplus M_n(F)^{op}))$$

and exchange involution. The proof of the theorem is complete. □

As a consequence of Theorems 1.9 and 1.12 we get the following theorem.

Theorem 1.13. *Let A be a finite dimensional $*$-superalgebra over an algebraically closed field F of characteristic zero. Then $A = B + J(A)$ where B is a maximal semisimple $*$-superalgebra of A. Moreover, B is a finite direct sum of simple $*$-superalgebras each isomorphic to one of the $*$-superalgebras given in Theorem 1.12.*

Chapter 2

The $*$-graded exponent

It is well known that the sequence of codimensions of a PI-algebra is exponentially bounded [34] and also the exponent of a PI-algebra exists and is a non-negative integer [17, 18]. In this chapter, we extend the asymptotic methods developed in the context of algebras, superalgebras and algebras with involution to $*$-superalgebras in order to study the behavior of the sequence of $*$-graded codimensions of a $*$-superalgebra A. From now on, F will denote a field of characteristic zero.

Consider $\mathcal{F} = F\langle X|\mathbb{Z}_2, *\rangle$ the free $*$-superalgebra of countable rank on X. Such an algebra is defined by a universal property and can be explicitly described as follows. We write the set X as the disjoint union of four countable sets $X = Y_0 \cup Y_1 \cup Z_0 \cup Z_1$, where $Y_0 = \{y_{1,0}, y_{2,0}, \ldots\}$, $Y_1 = \{y_{1,1}, y_{2,1}, \ldots\}$, $Z_0 = \{z_{1,0}, z_{2,0}, \ldots\}$ and $Z_1 = \{z_{1,1}, z_{2,1}, \ldots\}$. We define a superstructure on $\mathcal{F}$ by requiring that the variables of $Y_0 \cup Z_0$ are homogeneous of degree 0 and those of $Y_1 \cup Z_1$ are homogeneous of degree 1. We also define an involution on $\mathcal{F}$ by requiring that the variables of $Y_0 \cup Y_1$ are symmetric and those of $Z_0 \cup Z_1$ are skew. If $\mathcal{F}^{(0)}$ is the vector space spanned by all monomials in the variables of X which have an even number of variables of degree 1 and $\mathcal{F}^{(1)}$

is the vector space spanned by all monomials in the variables of X which have an odd number of variables of degree 1 then $\mathcal{F} = \mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ has a structure of $*$-superalgebra, since clearly $(\mathcal{F}^{(0)})^* = \mathcal{F}^{(0)}$ and $(\mathcal{F}^{(1)})^* = \mathcal{F}^{(1)}$. The elements of $\mathcal{F}$ are called $(\mathbb{Z}_2, *)$-*polynomials*.

We remark that we can view the free algebra $F\langle X\rangle$, the free algebra with involution $F\langle X|*\rangle$ and the free superalgebra $F\langle X\rangle^{\mathrm{gr}}$ as embedded in $\mathcal{F}$ as follows: $F\langle X\rangle$ is the free algebra on the set $\{x_1, x_2, \ldots\}$, where $x_i = y_{i,0} + y_{i,1} + z_{i,0} + z_{i,1}$; $F\langle X|*\rangle$ is the free $*$-algebra on the symmetric elements $y_{i,0} + y_{i,1}$, and the skew elements $z_{i,0} + z_{i,1}$; $F\langle X\rangle^{\mathrm{gr}}$ is the free superalgebra on the elements $y_{i,0} + z_{i,0}$ of homogeneous degree 0 and on the elements $y_{i,1} + z_{i,1}$ of homogeneous degree 1.

Let A be a $*$-superalgebra and let

$$f = f(y_{1,0}, \ldots, y_{m,0}, y_{1,1}, \ldots, y_{n,1}, z_{1,0}, \ldots, z_{p,0}, z_{1,1}, \ldots, z_{q,1}) \in F\langle X|\mathbb{Z}_2, *\rangle$$

be a $(\mathbb{Z}_2, *)$-polynomial. We say that f is a $(\mathbb{Z}_2, *)$-*identity* for the algebra A, and we write $f \equiv 0$ on A, if

$$f(a^+_{1,0}, \ldots, a^+_{m,0}, a^+_{1,1}, \ldots, a^+_{n,1}, a^-_{1,0}, \ldots, a^-_{p,0}, a^-_{1,1}, \ldots, a^-_{q,1}) = 0,$$

for all $a^+_{1,0}, \ldots, a^+_{m,0} \in (A^{(0)})^+$, $a^+_{1,1}, \ldots, a^+_{n,1} \in (A^{(1)})^+$, $a^-_{1,0}, \ldots, a^-_{p,0} \in (A^{(0)})^-$ and $a^-_{1,1}, \ldots, a^-_{q,1} \in (A^{(1)})^-$. The set

$$\mathrm{Id}^{\mathrm{gri}}(A) := \{f \in F\langle X|\mathbb{Z}_2, *\rangle : f \equiv 0 \text{ on } A\}$$

is an ideal of $F\langle X|\mathbb{Z}_2, *\rangle$ called *the ideal of* $(\mathbb{Z}_2, *)$-*identities of* A.

Notice that $\mathrm{Id}^{\mathrm{gri}}(A)$ is a T_2^*-ideal of $F\langle X|\mathbb{Z}_2, *\rangle$, i.e. an ideal invariant under all endomorphisms of $F\langle X|\mathbb{Z}_2, *\rangle$ that preserve the superstructure and commute with the involution.

As in the ordinary case, since $\mathrm{char}(F) = 0$, $\mathrm{Id}^{\mathrm{gri}}(A)$ is determined by its multilinear polynomials and so we define

$$P_n^{\mathrm{gri}} := \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} : \sigma \in S_n, w_i = y_{i,g_i} \text{ or } w_i = z_{i,g_i}, g_i = 0, 1\},$$

the space of multilinear polynomials in the first n variables. Clearly

$$P_n^{\mathrm{gri}}(A) := \frac{P_n^{\mathrm{gri}}}{P_n^{\mathrm{gri}} \cap \mathrm{Id}^{\mathrm{gri}}(A)}$$

is the space of multilinear elements of degree n of the relative free $*$-superalgebra $F\langle X|\mathbb{Z}_2,*\rangle/\mathrm{Id}^{\mathrm{gri}}(A)$ and its dimension $c_n^{\mathrm{gri}}(A)$ is called *the nth $*$-graded codimension of A*.

The study of the sequence $\{c_n^{\mathrm{gri}}(A)\}_{n\geq 1}$ and its growth is the main object of study of this thesis. Such growth is the growth of the $*$-supervariety generated by the $*$-superalgebra A.

In what follows we shall make use of several other sets of polynomials that here we recall. We let P_n be the space of multilinear polynomials in the first n variables of $F\langle X\rangle$, P_n^* the space of multilinear $*$-polynomials in the first n variables of $F\langle X|*\rangle$ and P_n^{gr} the space of multilinear graded polynomials in the first n variables of $F\langle X\rangle^{\mathrm{gr}}$. If A is an algebra (a $*$-algebra or a superalgebra) we denote by $\mathrm{Id}(A)$ ($\mathrm{Id}^*(A)$, $\mathrm{Id}^{\mathrm{gr}}(A)$, resp.) the ideal of identities ($*$-identities, graded identities, resp.) of A. We also write $c_n(A), c_n^*(A)$ and $c_n^{\mathrm{gr}}(A)$ for the nth ordinary codimension, $*$-codimension and graded codimension of A, respectively.

Since we can identify in a natural way P_n, P_n^* and P_n^{gr} with suitable subspaces of P_n^{gri}, in what follows we shall consider $\mathrm{Id}(A) \subseteq \mathrm{Id}^*(A) \subseteq \mathrm{Id}^{\mathrm{gri}}(A)$ and $\mathrm{Id}(A) \subseteq \mathrm{Id}^{\mathrm{gr}}(A) \subseteq \mathrm{Id}^{\mathrm{gri}}(A)$. Similarly we have $P_n \cap \mathrm{Id}(A) = P_n \cap \mathrm{Id}^{\mathrm{gri}}(A)$, $P_n^* \cap \mathrm{Id}^*(A) = P_n^* \cap \mathrm{Id}^{\mathrm{gri}}(A)$ and $P_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(A) = P_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gri}}(A)$.

If A is a $*$-superalgebra, we can consider its identities, $*$-identities and graded identities. The relation among the corresponding codimensions is given in the following lemma whose proof can be easily derived from the literature (see [20]).

Lemma 2.1 ([16], Lemma 3.1)**.** *Let A be a $*$-superalgebra. Then for any $n \geq 1$, we have*

1. $c_n(A) \le c_n^*(A) \le c_n^{\mathrm{gri}}(A)$;

2. $c_n(A) \le c_n^{\mathrm{gr}}(A) \le c_n^{\mathrm{gri}}(A)$;

3. $c_n^{\mathrm{gri}}(A) \le 4^n c_n(A)$.

By [34], an algebra A is a PI-algebra if and only if $c_n(A)$ is exponentially bounded. Thus, as an immediate consequence of the previous lemma, we have the following corollary.

Corollary 2.2 ([16], Corollary 3.2)**.** *Let A be a $*$-superalgebra. Then A is a PI-algebra if and only if its sequence of $*$-graded codimensions $\{c_n^{\mathrm{gri}}(A)\}_{n\ge 1}$ is exponentially bounded.*

Since any finite dimensional algebra A is a PI-algebra, we have the following corollary.

Corollary 2.3. *Let A be a finite dimensional $*$-superalgebra. Then the sequence of $*$-graded codimensions $\{c_n^{\mathrm{gri}}(A)\}_{n\ge 1}$ is exponentially bounded.*

2.1 The $S_{\langle n\rangle}$-action and the $\langle n\rangle$-cocharacter

For an integer number $n \ge 1$, we write $n = n_1 + n_2 + n_3 + n_4$ as a sum of four non-negative integers and write $\langle n\rangle = (n_1, n_2, n_3, n_4)$. We define $P_{\langle n\rangle}$ to be the space of multilinear $(\mathbb{Z}_2, *)$-polynomials in which the first n_1 variables are symmetric of homogeneous degree 0, the next n_2 variables are symmetric of homogeneous degree 1, the next n_3 variables are skew of homogeneous degree 0 and the next n_4 variables are skew of homogeneous degree 1.

We can notice that for any choice of $\langle n\rangle = (n_1, n_2, n_3, n_4)$ there are $\binom{n}{\langle n\rangle}$ subspaces isomorphic to $P_{\langle n\rangle}$ where $\binom{n}{\langle n\rangle} = \binom{n}{n_1,n_2,n_3,n_4}$ denotes the multinomial coefficient and it is clear that $P_{\langle n\rangle}$ is embedded into P_n^{gri}. Also we have that

$$P_n^{\text{gri}} \cong \bigoplus_{\langle n\rangle} \binom{n}{\langle n\rangle} P_{\langle n\rangle}.$$

Let us consider

$$P_{\langle n\rangle}(A) := \frac{P_{\langle n\rangle}}{P_{\langle n\rangle} \cap \text{Id}^{\text{gri}}(A)} \quad \text{and} \quad c_{\langle n\rangle}(A) := \dim_F P_{\langle n\rangle}(A).$$

By the above, it is also clear that

$$c_n^{\text{gri}}(A) = \sum_{\langle n\rangle} \binom{n}{\langle n\rangle} c_{\langle n\rangle}(A).$$

The representation theory of the product of four symmetric groups $S_{\langle n\rangle} := S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ will be used to prove our results about the $*$-graded codimensions of a $*$-superalgebra A. We refer [9] for the study of S_n-representations.

Recall that there is an one-to-one correspondence between the irreducible S_n-characters and the partitions of n. We denote by χ_λ the irreducible S_n-character corresponding to the partition $\lambda \vdash n$ and d_λ denotes the degree of χ_λ, given by the hook formula.

A multipartition $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(4)) \vdash n$ is such that $\lambda(i) = (\lambda(i)_1, \lambda(i)_2, \dots) \vdash n_i$, for $1 \le i \le 4$, and it is well known that the irreducible $S_{\langle n\rangle}$-characters are the outer tensor products of irreducible characters of $S_{n_1}, \dots, S_{n_4}$, respectively. So we denote by

$$\chi_{\langle\lambda\rangle} = \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$$

the irreducible $S_{\langle n\rangle}$-character corresponding to $\langle\lambda\rangle$ and by

$$d_{\langle\lambda\rangle} = d_{\lambda(1)} \cdots d_{\lambda(4)}$$

its degree.

Now we consider the natural left action of $S_{\langle n\rangle}$ on $P_{\langle n\rangle}$ by permuting four sets of variables separately, that is, for $f \in P_{\langle n\rangle}$ and $(\sigma_1,\dots,\sigma_4) \in S_{\langle n\rangle}$ we have

$$(\sigma_1,\dots,\sigma_4)f(y_{1,0},\dots,y_{n_1,0},y_{1,1},\dots,y_{n_2,1},z_{1,0},\dots,z_{n_3,0},z_{1,1},\dots,z_{n_4,1}) =$$

$$f(y_{\sigma_1(1),0},\dots,y_{\sigma_1(n_1),0},y_{\sigma_2(1),1},\dots,y_{\sigma_2(n_2),1},z_{\sigma_3(1),0},\dots,z_{\sigma_3(n_3),0},z_{\sigma_4(1),1},\dots,z_{\sigma_4(n_4),1})$$

and so $P_{\langle n\rangle}$ is a $S_{\langle n\rangle}$-module.

Furthermore, $P_{\langle n\rangle}(A)$ also inherits a structure of $S_{\langle n\rangle}$-module, since T_2^*-ideals are invariant under the given action. By complete reducibility, we can write the character $\chi_{\langle n\rangle}(A)$ of $P_{\langle n\rangle}(A)$ as

$$\chi_{\langle n\rangle}(A) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}, \tag{2.1}$$

where $m_{\langle\lambda\rangle}$ are the corresponding multiplicities. We call $\chi_{\langle n\rangle}(A)$ the *nth $\langle n\rangle$-cocharacter* of A.

Remark 2.4. Let A be a $*$-superalgebra. By [4, Theorem 13(b) and the remark after Theorem 14], if the $\langle n\rangle$-cocharacter of A has the decomposition as in (2.1), then there exist constants α and t such that $m_{\langle\lambda\rangle} \le \alpha n^t$, for all $\langle\lambda\rangle \vdash n$.

Given a partition $\lambda \vdash n$, we denote by T_λ the Young tableau of shape λ and by e_{T_λ} the minimal essential idempotent of FS_n associated to T_λ. If $\langle\lambda\rangle$ is a multipartition of n, we denote by $T_{\langle\lambda\rangle} = (T_{\lambda(1)},\dots,T_{\lambda(4)})$ the multitableau of shape $\langle\lambda\rangle$. It is well known that we can make the identification $FS_{\langle n\rangle} \equiv FS_{n_1}\otimes\cdots\otimes FS_{n_4}$. Thus, if $e_{T_{\lambda(i)}}$ is the minimal essential idempotent of FS_{n_i} associated to $T_{\lambda(i)}$ then $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}}\otimes\cdots\otimes e_{T_{\lambda(4)}}$ is the minimal essential idempotent of $FS_{\langle n\rangle}$ associated to $T_{\langle\lambda\rangle}$. Furthermore, if A is a $*$-superalgebra and the $\langle n\rangle$-cocharacter of A has the decomposition as in (2.1), then $m_{\langle\lambda\rangle} = 0$ if and only if for any multitableau $T_{\langle\lambda\rangle}$ of shape $\langle\lambda\rangle$ and for any polynomial $f \in P_{\langle n\rangle}$ we have that $e_{T_{\langle\lambda\rangle}}f \in \mathrm{Id}^{\mathrm{gri}}(A)$.

2.2 The $*$-graded exponent

Let A be a finite dimensional $*$-superalgebra. By Corollary 2.3, the sequence of $*$-graded codimension $\{c_n^{\mathrm{gri}}(A)\}_{n\geq 1}$ is exponentially bounded. This is a motivation for the next definition.

Definition 2.5. Let A be a finite dimensional $*$-superalgebra. We define $\underline{\exp^{\mathrm{gri}}(A)} := \liminf_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gri}}(A)}$ and $\overline{\exp^{\mathrm{gri}}(A)} := \limsup_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gri}}(A)}$. In case of equality,

$$\exp^{\mathrm{gri}}(A) := \lim_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gri}}(A)}$$

is called the $*$-*graded exponent of* A.

The existence of the $*$-graded exponent of a finite dimensional $*$-superalgebra was proved by Gordienko [21] in another context. Here, we present an alternative and independent proof and we use the $*$-graded exponent to characterize simple $*$-superalgebras and $*$-superalgebras having polynomial growth of $*$-graded codimensions.

Throughout this section, A denotes a $*$-superalgebra over an algebraically closed field F of characteristic zero. By Theorem 1.9, we can write $A = B + J(A)$, where $B = B_1 \oplus \cdots \oplus B_m$ and each $B_i, i = 1, \ldots, m$, is a simple $*$-superalgebra. Consider all possible non-zero products of the type

$$C_1JC_2J\cdots JC_{k-1}JC_k \neq \{0\},$$

where $C_1, \ldots, C_k$ are distinct $*$-superalgebras taken from the set $\{B_1, \ldots, B_m\}$, $k \geq 1$. If $k = 1$, we take $C_1 = B_i$, for some $i \in \{1, \ldots, m\}$. We define

$$d = d(A) = \max \dim(C_1 \oplus \cdots \oplus C_k),$$

where $C_1, \ldots, C_k \in \{B_1, \ldots, B_m\}$ are distinct and satisfy

$$C_1JC_2J\cdots JC_{k-1}JC_k \neq \{0\}.$$

The main goal of this section is to show that $\exp^{\mathrm{gri}}(A) = d$. We start with the following lemma.

Lemma 2.6. *Suppose that $C_1, \dots C_k$ are simple $*$-superalgebras from the set $\{B_1, \dots, B_m\}$, not necessarily distinct, and $C_1 J C_2 J \cdots J C_{k-1} J C_k \neq \{0\}$. Then* $\dim(C_1 \oplus \cdots \oplus C_k) \leq d$.

Proof. If in the product $C_1 J C_2 J \cdots J C_{k-1} J C_k \neq \{0\}$ some simple $*$-superalgebra C_i appears more than once then, since $J C_i J \subseteq J$, we can reduce this product to get a non-zero product of the type $C_1 J C_2 J \cdots J C_{l-1} J C_l \neq \{0\}$ where the C_i's are all distinct. □

Throughout this section, we shall use the following notations.

We denote by $S = A^+$ and $K = A^-$. If $C \subseteq A$, we denote by $S_C = C^+$ and $K_C = C^-$ the sets of symmetric and skew elements of C, respectively. Since A is a $*$-superalgebra, by Corollary 1.5, S and K are graded subspaces and

$$A = S^{(0)} \oplus K^{(0)} \oplus S^{(1)} \oplus K^{(1)}.$$

If $\dim_F(S^{(0)}) = s_0, \dim_F(S^{(1)}) = s_1, \dim_F(K^{(0)}) = k_0$ and $\dim_F(K^{(1)}) = k_1$ then $\dim_F(A) = s_0 + s_1 + k_0 + k_1$.

We denote by $W_{\langle\lambda\rangle}$ the irreducible $S_{\langle n\rangle}$-module associated to the multipartition $\langle\lambda\rangle \vdash n$. Thus, $W_{\langle\lambda\rangle} \cong F S_{\langle n\rangle} e_{T_{\langle\lambda\rangle}}$.

Given variables $w_{i,j} \in X$, we denote $w_{(n,j)} = (w_{1,j}, \dots, w_{n,j})$. When there is no danger of confusing, $w_{(n,j)}$ also denotes the elements $w_{1,j}, \dots, w_{n,j}$.

We denote by $\lambda'(i) = (\lambda'(i)_1, \lambda'(i)_2, \dots)$ the conjugate partition of $\lambda(i)$. Then $h(\lambda(i)) = \lambda'(i)_1$ is the height of the correspondent Young tableau.

Lemma 2.7. *Let $t \geq 0, m+n+p+q > d$ and let $f(y_{(m,0)}, y_{(n,1)}, z_{(p,0)}, z_{(q,1)}, x_{(t)})$ be a multilinear polynomial alternating on $\{y_{(m,0)}\}$, $\{y_{(n,1)}\}$, $\{z_{(p,0)}\}$ and on*

$\{z_{(q,1)}\}$. *If* $\bar{y}_{(m,0)} \in S_B^{(0)}$, $\bar{y}_{(n,1)} \in S_B^{(1)}$, $\bar{z}_{(p,0)} \in K_B^{(0)}$, $\bar{z}_{(q,1)} \in S_B^{(1)}$ *and* $\bar{x}_{(t)} \in A$, *then* $f(\bar{y}_{(m,0)}, \bar{y}_{(n,1)}, \bar{z}_{(p,0)}, \bar{z}_{(q,1)}, \bar{x}_{(t)}) = 0$.

Proof. Let $A = B_1 \oplus \cdots \oplus B_k + J(A)$ be a Wedderburn-Malcev decomposition of A, $\mathcal{B}_i$ be a $*$-graded basis of $B_i, i = 1, \ldots, k$, $\mathcal{B}_i = S_{\mathcal{B}_i}^{(0)} \cup S_{\mathcal{B}_i}^{(1)} \cup K_{\mathcal{B}_i}^{(0)} \cup K_{\mathcal{B}_i}^{(1)}$ and $\mathcal{J}$ be a $*$-graded basis of $J = J(A)$. Then $\mathcal{B} = \bigcup_{i=1}^{k} \mathcal{B}_i$ is a $*$-graded basis of $B_1 \oplus \cdots \oplus B_k$ and $\mathcal{A} = \mathcal{B} \cup \mathcal{J}$ is a $*$-graded basis of A. Since f is a multilinear polinomial, it is enough to evaluate f on the $*$-graded basis $\mathcal{A}$. If we evaluate f on the elements of $\mathcal{B}$, since $B_iB_j = \{0\}$, if $i \neq j$, we will get a zero value unless all elements come from one single B_i. In this case, since $\dim_F(B_i) < d$ and $m + n + p + q > d$, we get that either $m > |S_{\mathcal{B}_i}^{(0)}|$ or $n > |S_{\mathcal{B}_i}^{(1)}|$ or $p > |K_{\mathcal{B}_i}^{(0)}|$ or $q > |K_{\mathcal{B}_i}^{(1)}|$. Since f is alternating on the sets $\{y_{(m,0)}\}$, $\{y_{(n,1)}\}$, $\{z_{(p,0)}\}$ and on $\{z_{(q,1)}\}$, the value of f will still be zero. Therefore, in order to get a non-zero value of f we must evaluate at least one element of J. In this case, any monomial of f takes values in a subspace of one of the following types:

$$B_{i_1}JB_{i_2}J\cdots JB_{i_{d+l-1}}JB_{i_{d+l}}, \quad JB_{i_1}JB_{i_2}J\cdots JB_{i_{d+l-1}}JB_{i_{d+l}},$$
$$B_{i_1}JB_{i_2}J\cdots JB_{i_{d+l-1}}JB_{i_{d+l}}J, \quad JB_{i_1}JB_{i_2}J\cdots JB_{i_{d+l-1}}JB_{i_{d+l}}J,$$

for some $l \geq 1$, where the B_{i_j}'s are not necessarily distinct. Thus, $\dim_F(B_{i_1} + \cdots + B_{i_{d+l}}) \geq d + l$ and all the above products are equal to zero. Hence, f takes zero value on these elements. $\square$

Lemma 2.8. *Let* $\langle\lambda\rangle \vdash n$ *and* $W_{\langle\lambda\rangle} \subseteq P_{\langle n\rangle}$ *be an irreducible* $S_{\langle n\rangle}$*-module. Then there exists* $f \in W_{\langle\lambda\rangle}$ *such that* $f \neq 0$ *and* f *is alternating on each one of the sets of variables* $\{y^i_{(\lambda'(1)_i,0)}\}$, $\{y^j_{(\lambda'(2)_j,0)}\}$, $\{z^k_{(\lambda'(3)_k,0)}\}$ *and* $\{z^l_{(\lambda'(4)_l,0)}\}$, $1 \leq i \leq \lambda(1)_1$, $1 \leq j \leq \lambda(2)_1$, $1 \leq k \leq \lambda(3)_1$, $1 \leq l \leq \lambda(4)_1$.

Proof. Let $g \in W_{\langle\lambda\rangle}$ be a non-zero $(\mathbb{Z}_2, *)$-polynomial. Then there exists a multitableau $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, T_{\lambda(2)}, T_{\lambda(3)}, T_{\lambda(4)})$ and a polynomial h such that

$g = e_{T_{\langle\lambda\rangle}} h \neq 0$. Consider

$$f = \sum_{i=1}^{4} \sum_{\sigma_i \in C_{T_{\lambda(i)}}} \left(\prod_{j=1}^{4} \mathrm{sgn}(\sigma_j) \right) g.$$

Then f is a polynomial with the prescribed property. $\square$

Lemma 2.9. *Let $\langle\lambda\rangle \vdash n$ and $W_{\langle\lambda\rangle} \subseteq P_{\langle n\rangle}$ be an irreducible $S_{\langle n\rangle}$-module. If $W_{\langle\lambda\rangle} \not\subseteq P_{\langle n\rangle} \cap \mathrm{Id}^{\mathrm{gri}}(A)$, then $\lambda'(1)_1 \leq \dim_F(S^{(0)})$, $\lambda'(2)_1 \leq \dim_F(S^{(1)})$, $\lambda'(3)_1 \leq \dim_F(K^{(0)})$, $\lambda'(4)_1 \leq \dim_F(K^{(1)})$ and $\sum_{i=1}^{4} \lambda'(i)_{l+1} \leq d$, where $J^{l+1} = \{0\}$. Moreover, $\dim_F(W_{\langle\lambda\rangle}) \leq n^a \prod_{i=1}^{4} (\lambda'(i)_{l+1})^{n_i}$, for some $a \geq 1$.*

Proof. Let f be a polynomial as in Lemma 2.8. Since $W_{\langle\lambda\rangle}$ is an irreducible $S_{\langle n\rangle}$-module, we have that $W_{\langle\lambda\rangle} = FS_{\langle n\rangle} f$. Since f is alternating on the set $\{y^1_{(\lambda'(1)_1,0)}\}$, it follows that $h(\lambda(1)) = \lambda'(1)_1 \leq \dim_F(S^{(0)})$. Similarly, $\lambda'(2)_1 \leq \dim_F(S^{(1)})$, $\lambda'(3)_1 \leq \dim_F(K^{(0)})$ and $\lambda'(4)_1 \leq \dim_F(K^{(1)})$. Suppose, by contradiction, that $\sum_{i=1}^{4} \lambda'(i)_{l+1} > d$. Then $\sum_{i=1}^{4} \lambda'(i)_j > d$, for all $j = 1, \ldots, l$. Since, by hypothesis, $f \notin \mathrm{Id}^{\mathrm{gri}}(A)$, by Lemma 2.7, in each one of the sets $\{y^i_{(\lambda'(1)_i,0)}, y^i_{(\lambda'(2)_i,1)}, z^i_{(\lambda'(3)_i,0)}, z^i_{(\lambda'(4)_i,1)}\}$, $i = 1, \ldots, l+1$, there exists at least one variable which will be evaluated on one element of J. Since $J^{l+1} = \{0\}$, we have that f vanishes on A, a contradiction. Hence, $\sum_{i=1}^{4} \lambda'(i)_{l+1} \leq d$. Now, if $\chi_{\langle\lambda\rangle} = \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ is the irreducible $S_{\langle n\rangle}$-character associated to $\langle\lambda\rangle$, we have, by hook formula, that

$$\chi_{\lambda(1)}(1) \leq n^{ls_0} (\lambda'(1)_{l+1})^{n_1}, \quad \chi_{\lambda(2)}(1) \leq n^{ls_1} (\lambda'(2)_{l+1})^{n_2}$$
$$\chi_{\lambda(3)}(1) \leq n^{lk_0} (\lambda'(3)_{l+1})^{n_3}, \quad \chi_{\lambda(4)}(1) \leq n^{lk_1} (\lambda'(4)_{l+1})^{n_4}.$$

Hence, $\dim_F(W_{\langle\lambda\rangle}) = \prod_{i=1}^{4} \chi_{\lambda(i)}(1) \leq n^a \prod_{i=1}^{4} (\lambda'(i)_{l+1})^{n_i}$, where $a = l \dim_F(A)$. $\square$

Proposition 2.10. *$c_n^{\mathrm{gri}}(A) \le C_2 n^t d^n$, for some constants C_2, t.*

Proof. Write $\Lambda'_{l+1} = \lambda'(1)_{l+1} + \cdots + \lambda'(4)_{l+1}$. By Lemma 2.9, we have that

$$\chi_{\langle n\rangle}(A) = \sum_{\Lambda'_{l+1}\le d}\sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}.$$

Thus,

$$c_{\langle n\rangle}(A) \le \sum_{\Lambda'_{l+1}\le d}\sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle} n^a \prod_{i=1}^{4}(\lambda'(i)_{l+1})^{n_i},$$

for some constant a. Since, by Remark 2.4, the multiplicities $m_{\langle\lambda\rangle}$ are polynomially bounded, we have, by Lemma 2.9,

$$\begin{aligned}
c_n^{\mathrm{gri}}(A) &= \sum_{\langle n\rangle}\binom{n}{\langle n\rangle} c_{\langle n\rangle}(A)\\
&\le \alpha n^t \sum_{\Lambda'_{l+1}\le d}\sum_{\langle\lambda\rangle\vdash n}\binom{n}{\langle n\rangle}\prod_{i=1}^{4}(\lambda'(i)_{l+1})^{n_i}\\
&= \alpha n^t \sum_{\Lambda'_{l+1}\le d}(\Lambda'_{l+1})^n\\
&\le C_2 n^t d^n,
\end{aligned}$$

where $C_2 = \alpha d^4$. $\square$

The existence of central alternating polynomials in $M_n(F)$ was conjectured by Regev (see [35]) and proved by Formanek (see [11]).

Theorem 2.11 ([11], Theorem 16)**.** *The polynomial*

$$\mathsf{F}_n(x_1,\ldots,x_{n^2};y_1,\ldots,y_{n^2}) =$$
$$\sum_{\sigma,\tau\in S_{n^2}}(\mathrm{sgn}(\sigma\tau))x_{\sigma(1)}y_{\tau(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}y_{\tau(2)}y_{\tau(3)}y_{\tau(4)}\cdots x_{\sigma(n^2-2n+2)}\cdots x_{\sigma(n^2)}$$
$$y_{\tau(n^2-2n+2)}\cdots y_{\tau(n^2)}$$

is central in $M_n(F)$ and is not an identity on $M_n(F)$.

Notice that the polynomial $\mathsf{F}_n(x_1,\ldots,x_{n^2};y_1,\ldots,y_{n^2})$ is alternating on $\{x_1,\ldots,x_{n^2}\}$ and on $\{y_1,\ldots,y_{n^2}\}$.

Lemma 2.12. *Let C be a finite dimensional simple $*$-superalgebra over an algebraically closed field of characteristic zero, $p=\dim_F(S_C^{(0)})$, $q=\dim_F(S_C^{(1)})$, $r=\dim_F(K_C^{(0)})$ and $s=\dim_F(K_C^{(1)})$. For each $m\geq 1$, there exists a multilinear polynomial*

$$f=f(y^1_{(p,0)},\ldots,y^{2m}_{(p,0)},y^1_{(q,1)},\ldots,y^{2m}_{(q,1)},z^1_{(r,0)},\ldots,z^{2m}_{(r,0)},z^1_{(s,1)},\ldots,z^{2m}_{(s,1)})$$

such that:

1. *f is alternating on each set of variables $\{y^i_{(p,0)}\}$, $\{y^i_{(q,1)}\}$, $\{z^i_{(r,0)}\}$ and $\{z^i_{(s,1)}\}$, $i=1,\ldots,2m$;*
2. *There exist $\bar{y}^i_{(p,0)}\in S_C^{(0)}$, $\bar{y}^i_{(q,1)}\in S_C^{(1)}$, $\bar{z}^i_{(r,0)}\in K_C^{(0)}$, $\bar{z}^i_{(s,1)}\in K_C^{(1)}$, $i=1,\ldots,2m$, such that*

$$f(\bar{y}^1_{(p,0)},\ldots,\bar{y}^{2m}_{(p,0)},\bar{y}^1_{(q,1)},\ldots,\bar{y}^{2m}_{(q,1)},\bar{z}^1_{(r,0)},\ldots,\bar{z}^{2m}_{(r,0)},\bar{z}^1_{(s,1)},\ldots,\bar{z}^{2m}_{(s,1)})=1_C.$$

Proof. By Theorem 1.9, we have that either C is simple (and hence isomorphic to $M_n(F)$, for some $n\geq 1$) or $C=C_1\oplus C_1^*$ or $C=C_1\oplus C_1^\varphi$ or $C=C_1\oplus C_1^*\oplus C_1^\varphi\oplus(C_1^*)^\varphi$, for some simple algebra C_1 (and hence isomorphic to $M_n(F)$, for some $n\geq 1$), where φ denotes the automorphism of order 2 determined by the superstructure. Let F_n be the Regev's polynomial given in the previous theorem. If C is simple, F_n is alternating in two distinct sets of variables of order $p+q+r+s=\dim C$. By taking the product of $m\geq 1$ of such polynomials in distinct sets of variables, we obtain the existence of a multilinear polynomial f, alternating in each of the $2m$ sets of variables and f is a central polynomial for C. It is clear that f can be viewed as alternating on $2m$ disjoint sets of symmetric and skew variables of homogeneous degree

0 and 1. If C is not simple, let f be the polynomial obtained above. Then the polynomial

$$\bar{f} = f(y^1_{(p,0)},\ldots,y^{2m}_{(p,0)})f(y^1_{(q,1)},\ldots,y^{2m}_{(q,1)})f(z^1_{(r,0)},\ldots,z^{2m}_{(r,0)})f(z^1_{(s,1)},\ldots,z^{2m}_{(s,1)})$$

is the required one. $\square$

Lemma 2.13. *Let $C_1JC_2J\cdots JC_{k-1}JC_k \neq \{0\}$, where $C_1,\ldots,C_k$ are distinct simple $*$-graded subalgebras of A and $C = C_1+\cdots+C_k = C_1\oplus\cdots\oplus C_k$. If $p = \dim_F(S^{(0)}_C)$, $q = \dim_F(S^{(1)}_C)$, $r = \dim_F(K^{(0)}_C)$ and $s = \dim_F(K^{(1)}_C)$, then, for each $m \geq 1$, there exists a multilinear polynomial*

$$f = f(y^1_{(p,0)},\ldots,y^{2m}_{(p,0)},y^1_{(q,1)},\ldots,y^{2m}_{(q,1)},z^1_{(r,0)},\ldots,z^{2m}_{(r,0)},z^1_{(s,1)},\ldots,z^{2m}_{(s,1)},$$
$$y_{(k_1,0)},y_{(k_2,1)},z_{(k_3,0)},z_{(k_4,1)}),$$

where $k_1+\cdots+k_4 = 2k-1$, such that:

1. *f is alternating on each set of variables $\{y^i_{(p,0)}\}$, $\{y^i_{(q,1)}\}$, $\{z^i_{(r,0)}\}$ and $\{z^i_{(s,1)}\}$, $i = 1,\ldots,2m$;*
2. *f does not vanish on A.*

Proof. For every $i = 1,\ldots,k$, let $p_i = \dim_F(S^{(0)}_{C_i})$, $q_i = \dim_F(S^{(1)}_{C_i})$, $r_i = \dim_F(K^{(0)}_{C_i})$ and $s_i = \dim_F(K^{(1)}_{C_i})$ and let

$$f_i = f_i(y^{1,i}_{(p_i,0)},\ldots,y^{2m,i}_{(p_i,0)},y^{1,i}_{(q_i,1)},\ldots,y^{2m,i}_{(q_i,1)},z^{1,i}_{(r_i,0)},\ldots,z^{2m,i}_{(r_i,0)},z^{1,i}_{(s_i,1)},\ldots,z^{2m,i}_{(s_i,1)})$$

be the polynomial constructed in Lemma 2.12. Let

$$\hat{f} = \mathcal{A}^{y_0}_{(2m)}\mathcal{A}^{y_1}_{(2m)}\mathcal{A}^{z_0}_{(2m)}\mathcal{A}^{z_1}_{(2m)}x_1f_1\tilde{x}_1x_2f_2\tilde{x}_2\cdots x_{k-1}f_{k-1}\tilde{x}_{k-1}x_kf_k,$$

where $\mathcal{A}^w_{(n)} = \mathcal{A}^w_1\cdots\mathcal{A}^w_n, w \in \{y_0,y_1,z_0,z_1\}$, and $\mathcal{A}^{y_0}_j$ means alternation on the p variables $y^{j,1}_{(p_1,0)},\ldots,y^{j,k}_{(p_k,0)}$, $\mathcal{A}^{y_1}_j$ means alternation on the q variables

$y^{j,1}_{(q_1,1)}, \ldots, y^{j,k}_{(q_k,1)}$, $\mathcal{A}^{z_0}_j$ means alternation on the r variables $z^{j,1}_{(r_1,0)}, \ldots, z^{j,k}_{(r_k,0)}$ and $\mathcal{A}^{z_1}_j$ means alternation on the s variables $z^{j,1}_{(s_1,1)}, \ldots, z^{j,k}_{(s_k,1)}$. Notice that each polynomial f_i corresponds to a multitableau $(T^i_1, T^i_2, T^i_3, T^i_4)$ where T^i_1 is a $p_i \times 2m$ rectangle, T^i_2 is a $q_i \times 2m$ rectangle, T^i_3 is a $r_i \times 2m$ rectangle and T^i_4 is a $s_i \times 2m$ rectangle and the variables in each column of $T^i_j, j = 1, 2, 3, 4$, are alternating. Then, $\hat{f}$ corresponds to the multitableau (T_1, T_2, T_3, T_4) where $T_j, j = 1, 2, 3, 4$, is obtain by gluing the rectangles $T^i_j, i = 1, \ldots, k$, one on top of the other and by alternating the variables. Hence, $\mathcal{A}^{y_0}_j$ is alternation on the variables in the jth column of T_1, $\mathcal{A}^{y_1}_j$ is alternation on the variables in the jth column of T_2, $\mathcal{A}^{z_0}_j$ is alternation on the variables in the jth column of T_3 and $\mathcal{A}^{z_1}_j$ is alternation on the variables in the jth column of T_4. Since $C_1JC_2J \cdots JC_{k-1}JC_k \neq \{0\}$, there exist $c_i \in C_i, i = 1, \ldots, k$, $b_1, \ldots, b_{k-1} \in J$ such that $c_1b_1c_2b_2 \cdots b_{k-1}c_k \neq 0$. For every $i = 1, \ldots, k$, let $\bar{y}^{t,i}_{(p_i,0)} \in S^{(0)}_{C_i}$, $\bar{y}^{t,i}_{(q_i,1)} \in S^{(1)}_{C_i}$, $\bar{z}^{t,i}_{(r_i,0)} \in K^{(0)}_{C_i}$, $\bar{z}^{t,i}_{(s_i,1)} \in K^{(1)}_{C_i}$, $t = 1, \ldots, 2m$, such that $f_i(\bar{y}^{1,i}_{(p_i,0)}, \ldots, \bar{y}^{2m,i}_{(p_i,0)}, \bar{y}^{1,i}_{(q_i,1)}, \ldots, \bar{y}^{2m,i}_{(q_i,1)}, \bar{z}^{1,i}_{(r_i,0)}, \ldots, \bar{z}^{2m,i}_{(r_i,0)}, \bar{z}^{1,i}_{(s_i,1)}, \ldots, \bar{z}^{2m,i}_{(s_i,1)}) = 1_{C_i}$. Notice that, since $C_iC_j = \{0\}$, for $i \neq j$, alternation on the columns of $T_i, i = 1, \ldots, 4$, can be replaced with alternation on the columns of each $T^j_i, j = 1, \ldots k$, respectively. Hence,

$$\hat{f}(\bar{y}^{1,1}_{(p_1,0)}, \ldots, \bar{y}^{2m,k}_{(p_k,0)}, \bar{y}^{1,1}_{(q_1,1)}, \ldots, \bar{y}^{2m,k}_{(q_k,1)}, \bar{z}^{1,1}_{(r_1,0)}, \ldots, \bar{z}^{2m,k}_{(r_k,0)}, z^{1,1}_{(s_1,1)}, \ldots, \bar{z}^{2m,k}_{(s_k,1)},$$
$$c_1, \ldots, c_k, b_1, \ldots, b_{k-1}) =$$
$$(p_1! \cdots p_k! q_1! \cdots q_k! r_1! \cdots r_k! s_1! \cdots s_k!)^{2m} c_1b_1c_2b_2 \cdots b_{k-1}c_k \neq 0.$$

We may assume that $c_1, \ldots, c_k, b_1 \ldots, b_{k-1} \in S^{(0)} \cup S^{(1)} \cup K^{(0)} \cup K^{(1)}$. Suppose that k_1 of them belong to $S^{(0)}$, k_2 of them belong to $S^{(1)}$, k_3 of them belong to $K^{(0)}$ and k_4 of them belong to $K^{(1)}$, $k_1 + k_2 + k_3 + k_4 = 2k - 1$. Then

$$f = \hat{f}(y^{1,1}_{(p_1,0)}, \ldots, y^{2m,k}_{(p_k,0)}, y^{1,1}_{(q_1,1)}, \ldots, y^{2m,k}_{(q_k,1)}, z^{1,1}_{(r_1,0)}, \ldots, z^{2m,k}_{(r_k,0)}, z^{1,1}_{(s_1,1)}, \ldots, z^{2m,k}_{(s_k,1)},$$
$$y_{(k_1,0)}, y_{(k_2,1)}, z_{(k_3,0)}, z_{(k_4,1)})$$

does not vanish on A and is the desired polynomial. $\square$

Remark 2.14. For every $a, b, c \in \mathbb{N}$, $\dfrac{(a+b+c)!}{a!(b+c)!} \geq \dfrac{(a+b)!}{a!b!}$.

Remark 2.15 ([20], Lemma 6.2.5). Let $\lambda = (n^m)$. Then $d_\lambda \simeq a(nm)^b m^{nm}, n \to \infty$, for some non-zero constants a and b.

Theorem 2.16. *Let A be a finite dimensional $*$-superalgebra over an algebraically closed field of characteristic zero. Then*

$$C_1 n^{t_1} d^n \leq c_n^{\mathrm{gri}}(A) \leq C_2 n^{t_2} d^n,$$

for some non-zero constants C_1, C_2, t_1, t_2. Hence, $\exp^{\mathrm{gri}}(A) = d$.

Proof. The upper bound for $c_n^{\mathrm{gri}}(A)$ was obtained in Lemma 2.10. Now we will obtain the lower bound. Let $A = B_1 \oplus \cdots \oplus B_m + J(A)$ be a Wedderburn-Malcev decomposition of A and let $C_1, \ldots, C_k \in \{B_1, \ldots, B_m\}$ be distinct $*$-superalgebras such that

$$C_1 J C_2 J \cdots C_{k-1} J C_k \neq \{0\}.$$

Write $C = C_1 + \cdots + C_k$ and let $p = \dim_F(S_C^{(0)})$, $q = \dim_F(S_C^{(1)})$, $r = \dim_F(K_C^{(0)})$, $s = \dim_F(K_C^{(1)})$ and $d = p+q+r+s$. Let $n \geq 2d+k_1+k_2+k_3+k_4$, where $k_1, \ldots, k_4$ are as in Lemma 2.13, and divide $n - (k_1+k_2+k_3+k_4)$ by $2d$. Then we can write

$$n = 2m(p+q+r+s) + (k_1+k_2+k_3+k_4) + t,$$

for some m, t where $0 \leq t < 2d$. Set $n_1 = 2mp + k_1 + t$, $n_2 = 2mq + k_2$, $n_3 = 2mr + k_3$ and $n_4 = 2ms + k_4$. Let f be the polynomial constructed in Lemma 2.13 of degree $2m(p+q+r+s) + (k_1+k_2+k_3+k_4)$ and set $g = f y_{k_1+1,0} \cdots y_{k_1+t,0} \in P_{\langle n \rangle}$. We have that g does not vanish on A, since f does not vanish on A and we may evaluate $y_{k_1+1,0} = \cdots = y_{k_1+t,0} = 1_C$. The group $H = S_{2mp} \times S_{2mq} \times S_{2mr} \times S_{2ms} \leq S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4} = G$ acts on f in a natural way and then H acts on g. Let M be the H-submodule of $P_{\langle n \rangle}$ generated by g. Then M contains an irreducible H-submodule of the form $W_{\langle \lambda \rangle} = FHe_{T_{\langle \lambda \rangle}}g$, where $\langle \lambda \rangle \vdash 2md$, $\langle \lambda \rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$, $\lambda(1) \vdash 2mp$, $\lambda(2) \vdash 2mq$, $\lambda(3) \vdash 2mr$, $\lambda(4) \vdash 2ms$ and $e_{T_{\langle \lambda \rangle}} = e_{T_{\lambda(1)}} \otimes \cdots \otimes e_{T_{\lambda(4)}}$. Now, for all $\sigma \in S_{2mp}$, $\sigma(g)$ is still alternating on $2m$ disjoint sets of variables $\{y^i_{(p,0)}\}, i = 1, \ldots, 2m$, and $\displaystyle\sum_{\sigma \in R_{T_{\lambda(1)}}} \sigma$ acts on g by symmetrizing $\lambda(1)_1$ variables. Thus, if $\lambda(1)_1 > 2m$, we get $e_{T_{\lambda(1)}}g = 0$, a contradiction. Similarly for $\lambda(i)_1, i = 2, 3, 4$. Hence, $\lambda(i)_1 \leq 2m, i = 1, 2, 3, 4$. Thus, $\lambda'(1)_1 \geq p$, $\lambda'(2)_1 \geq q$, $\lambda'(3)_1 \geq r$ and $\lambda'(4)_1 \geq s$. Suppose that either $\lambda'(1)_1 > p$ or $\lambda'(2)_1 > q$ or $\lambda'(3)_1 > r$ or $\lambda'(4)_1 > s$ and the total number of boxes out of the first p rows of the diagram $D_{\lambda(1)}$, out of the first q rows of the diagram $D_{\lambda(2)}$, out of the first r rows of the diagram $D_{\lambda(3)}$ and out of the first s rows of the diagram $D_{\lambda(4)}$ is at least $l + 1$, where $J^{l+1} = \{0\}$. Since $FHe_{T_{\langle \lambda \rangle}}$ is a minimal left ideal of FH, then $FH\bar{C}_{T_{\langle \lambda \rangle}}e_{T_{\langle \lambda \rangle}} = FHe_{T_{\langle \lambda \rangle}}$, where $\bar{C}_{T_{\langle \lambda \rangle}} = \bar{C}_{T_{\lambda(1)}} \otimes \cdots \otimes \bar{C}_{T_{\lambda(4)}}$, $\bar{C}_{T_{\lambda(i)}} = \displaystyle\sum_{\sigma \in C_{T_{\lambda(i)}}} (\mathrm{sgn}(\sigma))\sigma, i = 1, 2, 3, 4$. Set $\bar{g} = \bar{C}_{T_{\langle \lambda \rangle}}e_{T_{\langle \lambda \rangle}}g$. Let

$$\lambda'(1) = (p + p_1, \ldots, p + p_a, \lambda'(1)_{a+1}, \ldots, \lambda'(1)_{m_1}),$$

$$\lambda'(2) = (q + q_1, \ldots, q + q_b, \lambda'(2)_{b+1}, \ldots, \lambda'(2)_{m_2}),$$

$$\lambda'(3) = (r + r_1, \ldots, r + r_c, \lambda'(3)_{c+1}, \ldots, \lambda'(3)_{m_3}),$$

and

$$\lambda'(4) = (s + s_1, \ldots, s + s_d, \lambda'(4)_{d+1}, \ldots, \lambda'(4)_{m_4}),$$

where

$$p_1 + \cdots + p_a + q_1 + \cdots + q_b + r_1 + \cdots + r_c + s_1 + \cdots + s_d \geq l + 1,$$

$$\lambda'(1)_{a+1}, \ldots, \lambda'(1)_{m_1} \leq p,$$

$$\lambda'(2)_{b+1}, \ldots, \lambda'(2)_{m_2} \leq q,$$

$$\lambda'(3)_{c+1}, \ldots, \lambda'(3)_{m_3} \leq r,$$

and

$$\lambda'(4)_{d+1}, \ldots, \lambda'(4)_{m_4} \leq s.$$

Now, the polynomial $\bar{g}$ is alternating on each one of the a sets of variables $y_{i,0}$ of order $p+p_1, \ldots, p+p_a$, on each one of the b sets of variables $y_{i,1}$ of order $q+q_1, \ldots, q+q_b$, on each one of the c sets of variables $z_{i,0}$ of order $r+r_1, \ldots, r+r_c$ and on each one of the d sets of variables $z_{i,1}$ of order $s+s_1, \ldots, s+s_d$. If we substitute on any of these sets of variables only elements from C, we would get zero, since $p = \dim_F(S_C^{(0)})$, $q = \dim_F(S_C^{(1)})$, $r = \dim_F(K_C^{(0)})$ and $s = \dim_F(K_C^{(1)})$. It follows that we have to substitute into these sets of variables at least

$$p_1 + \cdots + p_a + q_1 + \cdots + q_b + r_1 + \cdots + r_c + s_1 + \cdots + s_d \geq l+1$$

elements from the Jacobson radical J. Since $J^{l+1} = \{0\}$, we get that $\bar{g}$ vanishes on A, a contradiction. Hence, $\lambda(1)$ must contain the rectangle $\mu(1) = ((2m-l)^p)$, $\lambda(2)$ must contain the rectangle $\mu(2) = ((2m-l)^q)$, $\lambda(3)$ must contain the rectangle $\mu(3) = ((2m-l)^r)$ and $\lambda(4)$ must contain the rectangle $\mu(4) = ((2m-l)^s)$. By Remark 2.15, when $m \to \infty$,

$$\begin{aligned} \prod_{i=1}^{4} \deg(\mu(i)) &\simeq a((2m-l)p)^{b_1}((2m-l)q)^{b_2}((2m-l)r)^{b_3}((2m-l)s)^{b_4} \\ &\quad p^{(2m-l)p} q^{(2m-l)q} r^{(2m-l)r} s^{(2m-l)s} \\ &\geq n^{t_2} p^{2mp} q^{2mq} r^{2mr} s^{2ms}, \end{aligned}$$

for some constants $a, b_1, \ldots, b_4, t_2$. Notice that the constant t_2 is possibly non-positive. Hence, since

$$\dim_F(W_{\langle\lambda\rangle}) = \prod_{i=1}^{4} \deg(\lambda(i)) \geq \prod_{i=1}^{4} \deg(\mu(i)),$$

we obtain

$$c_{n_1,\dots,n_4}(A) \geq \dim_F(W_{\langle\lambda\rangle}) \geq n^{t_2} p^{2mp} q^{2mq} r^{2mr} s^{2ms}.$$

Therefore,

$$\begin{aligned} c_n^{\mathrm{gri}}(A) &= \sum_{\tilde{n}_1+\cdots+\tilde{n}_4=n} \binom{n}{\tilde{n}_1,\dots,\tilde{n}_4} c_{\tilde{n}_1,\dots,\tilde{n}_4}(A) \\ &\geq \binom{n}{n_1,\dots,n_4} c_{n_1,\dots,n_4}(A) \\ &\geq \binom{n}{n_1,\dots,n_4} n^{t_2} p^{2mp} q^{2mq} r^{2mr} s^{2ms}. \end{aligned}$$

Now, $n_1 = 2mp + k_1 + t$, $n_2 = 2mq + k_2$, $n_3 = 2mr + k_3$, $n_4 = 2ms + k_4$ and $n = 2m(p+q+r+s) + (k_1+k_2+k_3+k_4) + t$. Thus, by Remark 2.14,

$$\binom{n}{n_1,\dots,n_4} \geq \frac{(2mp+2mq+2mr+2ms)!}{(2mp)!(2mq)!(2mr)!(2ms)!}.$$

Now, recalling the Stirling formula

$$n! \simeq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n > \left(\frac{n}{e}\right)^n,$$

we get that

$$\frac{(2mp+2mq+2mr+2ms)!}{(2mp)!(2mq)!(2mr)!(2ms)!} \geq \frac{(2mp+2mq+2mr+2ms)^{(2mp+2mq+2mr+2ms)}}{(2mp)^{2mp}(2mq)^{2mq}(2mr)^{2mr}(2ms)^{2ms}}.$$

Hence

$$\begin{aligned} c_n^{\mathrm{gri}}(A) &\geq n^{t_2} \frac{(2mp+2mq+2mr+2ms)^{(2mp+2mq+2mr+2ms)}}{(2mp)^{2mp}(2mq)^{2mq}(2mr)^{2mr}(2ms)^{2ms}} p^{2mp} q^{2mq} r^{2mr} s^{2ms} \\ &= n^{t_2} d^{2md} \\ &= C_1 n^{t_2} d^n, \end{aligned}$$

where $C_1 = d^{-(k_1+\cdots+k_4+t)}$ is constant. $\square$

Corollary 2.17. *Let A be a finite dimensional $*$-superalgebra over an algebraically closed field F of characteristic zero and let B be a maximal semisimple $*$-graded subalgebra of A. Then*

$$\exp^{\mathrm{gri}}(A) = \max_i \dim_F(C_1^{(i)} + \cdots + C_k^{(i)}),$$

where $C_1^{(i)}, \ldots, C_k^{(i)}$ *are distinct simple* $*$*-graded subalgebras of* B *and*

$$C_1^{(i)} J C_2^{(i)} J \cdots J C_{k-1}^{(i)} J C_k^{(i)} \neq \{0\}.$$

The next lemma will be useful in the proofs of next results. It can be easily checked and the proof will be omitted.

Lemma 2.18. *Let* F *be a field of characteristic zero,* $\bar{F}$ *its algebraic closure and* A *a* $*$*-superalgebra over* F*. Then the algebra* $\bar{A} = A \otimes_F \bar{F}$ *has an induced structure of* $*$*-superalgebra,* $c_n^{\mathrm{gri}}(A) = c_n^{\mathrm{gri}}(\bar{A})$ *and* $\exp^{\mathrm{gri}}(A) = \exp^{\mathrm{gri}}(\bar{A})$*. Furthermore,* $\mathrm{Id}^{\mathrm{gri}}(A) = \mathrm{Id}^{\mathrm{gri}}(\bar{A})$*, viewed as* $*$*-superalgebras over* F*.*

Corollary 2.19. *Let* A *be a finite dimensional* $*$*-superalgebra over a field of characteristic zero. Then* $\exp^{\mathrm{gri}}(A)$ *exists, is a non-negative integer and* $\exp^{\mathrm{gri}}(A) \leq \dim_F(A)$*.*

2.3 A characterization of simple $*$-superalgebras

In this section, we characterize finite dimensional simple $*$-superalgebras by using the $*$-graded exponent.

For a $*$-superalgebra A, let $\mathcal{Z} = \mathcal{Z}(A)$ be the center of A. We start with the following result about $\mathcal{Z}(A)$.

Lemma 2.20 ([37], Lemma 9)**.** *Let* A *be a* $*$*-superalgebra and consider* $\mathcal{Z} = \mathcal{Z}(A)$*. Then:*

1. $\mathcal{Z}$ *is a* $*$*-graded subalgebra of* A*. As a consequence,*

$$\mathcal{Z} = (\mathcal{Z}^{(0)})^+ \oplus (\mathcal{Z}^{(1)})^+ \oplus (\mathcal{Z}^{(0)})^- \oplus (\mathcal{Z}^{(1)})^-;$$

2. *If A is a finite dimensional $*$-superalgebra, simple as an algebra, then $\mathcal{Z} = \mathfrak{Z}(\alpha, \beta)$, where $\alpha \in (\mathcal{Z}^{(0)})^-, \beta \in \mathcal{Z}^{(1)}$ and $\mathfrak{Z} = (\mathcal{Z}^{(0)})^+$. As a consequence, $[\mathcal{Z} : \mathfrak{Z}] \leq 4$.*

Proof. 1. Let φ be the automorphism of order two determined by the $\mathbb{Z}_2$-grading and let $a \in \mathcal{Z}$. If $b \in A$, then there exists $c \in A$ such that $c^\varphi = b$ and

$$a^\varphi b = a^\varphi c^\varphi = (ac)^\varphi = (ca)^\varphi = ba^\varphi.$$

Hence, $\mathcal{Z}^\varphi = \mathcal{Z}$. Analogously, $\mathcal{Z}^* = \mathcal{Z}$ and hence $\mathcal{Z}$ is a $*$-graded subalgebra of A.

2. Let A be a finite dimensional $*$-superalgebra, simple as an algebra. Then $\mathcal{Z}$ is a field and $F \subseteq \mathfrak{Z} \subseteq \mathcal{Z}^{(0)} \subseteq \mathcal{Z}$ are fields extensions, where $\mathfrak{Z} = (\mathcal{Z}^{(0)})^+$. Let $\alpha \in (\mathcal{Z}^{(0)})^-$. Then α and $-\alpha$ are the roots of $f(x) = x^2 - \alpha^2 \in \mathfrak{Z}[x]$ and so $f(x)$ is irreducible in $\mathfrak{Z}[x]$. Hence $[\mathcal{Z}^{(0)} : \mathfrak{Z}] = 2$ and $\mathcal{Z}^{(0)} = \mathfrak{Z}(\alpha)$. Analogously, $\mathcal{Z} = \mathcal{Z}^{(0)}(\beta)$, where $\beta \in \mathcal{Z}^{(1)}$ and $[\mathcal{Z} : \mathcal{Z}^{(0)}] = 2$. Hence, $\mathcal{Z} = \mathfrak{Z}(\alpha, \beta)$ and $[\mathcal{Z} : \mathfrak{Z}] \leq 4$.

□

We are in condition to prove the main theorem of this section.

Theorem 2.21 ([37], Theorem 10)**.** *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero and $\mathfrak{Z} = (\mathcal{Z}(A)^{(0)})^+$.*

1. *If A is a simple $*$-superalgebra, then $\exp^{\text{gri}}(A) = \dim_{\mathfrak{Z}}(A)$;*

2. *If A is a semisimple $*$-superalgebra and $A = A_1 \oplus \cdots \oplus A_m$ is a decomposition of A into simple $*$-superalgebras, then $\exp^{\text{gri}}(A) = \max_{1 \leq i \leq m} \dim_{\mathfrak{Z}_i}(A_i)$, where $\mathfrak{Z}_i = (\mathcal{Z}(A_i)^{(0)})^+$;*

3. *$\exp^{\text{gri}}(A) = \dim_F(A)$ if and only if A is a simple $*$-superalgebra and $F = \mathfrak{Z}$.*

Proof. 1. Let A be a simple $*$-superalgebra. By Theorem 1.9, we have that either A is simple or A is $*$-simple or $A = B \oplus B^{\varphi}$, where B is a simple $*$-ideal of A. First, suppose that A is simple. Then, by Lemma 2.20, $\mathcal{Z}$ is a field and either $\mathcal{Z} = \mathfrak{Z}$ or $\mathcal{Z} = \mathfrak{Z}(\alpha), \alpha \in (\mathcal{Z}^{(0)})^{-}$, or $\mathcal{Z} = \mathfrak{Z}(\beta), \beta \in \mathcal{Z}^{(1)}$, or $\mathcal{Z} = \mathfrak{Z}(\alpha, \beta), \alpha \in (\mathcal{Z}^{(0)})^{-}, \beta \in \mathcal{Z}^{(1)}$. If $\mathcal{Z} = \mathfrak{Z}$, then

$$\begin{aligned} \mathfrak{Z} \otimes_F \bar{F} &\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} F_i \otimes_F \bar{F} \\ &\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} \bar{F}_i, \end{aligned}$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathfrak{Z} : F]$. Therefore

$$\begin{aligned} A \otimes_F \bar{F} &\cong A \otimes_{\mathfrak{Z}} \mathfrak{Z} \otimes_F \bar{F} \\ &\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} (A \otimes_{\mathfrak{Z}} \bar{F}_i), \end{aligned}$$

where $A \otimes_{\mathfrak{Z}} \bar{F}_i$ is a central simple algebra over $\bar{F}$ with induced structure of $*$-superalgebra. Moreover,

$$[\mathfrak{Z} : F] \dim_{\mathfrak{Z}}(A) = \dim_F(A) = \dim_{\bar{F}}(A \otimes_F \bar{F}) = [\mathfrak{Z} : F] \dim_{\bar{F}}(A \otimes_{\mathfrak{Z}} \bar{F}_i).$$

Thus, $\dim_{\mathfrak{Z}}(A) = \dim_{\bar{F}}(A \otimes_{\mathfrak{Z}} \bar{F}_i)$ and, by Theorem 2.16, we have that

$$\dim_{\mathfrak{Z}}(A) = \dim_{\bar{F}}(A \otimes_{\mathfrak{Z}} \bar{F}_i) = \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) = \exp^{\mathrm{gri}}(A).$$

Now, suppose that $\mathcal{Z} = \mathfrak{Z}(\alpha), \alpha \in (\mathcal{Z}^{(0)})^-$. Then

$$\begin{aligned}
\mathcal{Z} \otimes_F \bar{F} &\cong \mathcal{Z} \otimes_{\mathfrak{Z}} \mathfrak{Z} \otimes_F \bar{F} \\
&\cong \mathcal{Z} \otimes_{\mathfrak{Z}} \left(\bigoplus_{i=1}^{[\mathfrak{Z}:F]} F_i \otimes_F \bar{F} \right) \\
&\cong \mathcal{Z} \otimes_{\mathfrak{Z}} \left(\bigoplus_{i=1}^{[\mathfrak{Z}:F]} \bar{F}_i \right) \\
&\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} (\mathfrak{Z} \oplus \mathfrak{Z}) \otimes_{\mathfrak{Z}} \bar{F}_i \\
&\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} \bar{F}_i \oplus \bar{F}_i,
\end{aligned}$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathfrak{Z} : F]$. Therefore

$$\begin{aligned}
A \otimes_F \bar{F} &\cong A \otimes_{\mathcal{Z}} \mathcal{Z} \otimes_F \bar{F} \\
&\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} A \otimes_{\mathcal{Z}} (\bar{F}_i \oplus \bar{F}_i) \\
&\cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i).
\end{aligned}$$

On each $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$, φ acts as $(a_1 \otimes f_1 + a_2 \otimes f_2)^\varphi = a_1^\varphi \otimes f_1 + a_2^\varphi \otimes f_2$ and $*$ acts as $(a_1 \otimes f_1 + a_2 \otimes f_2)^* = a_1^* \otimes f_2 + a_2^* \otimes f_1$. Hence, $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$ and, as in the previous case, it follows that

$$\dim_{\mathfrak{Z}}(A) = \dim_{\bar{F}}((A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)) = \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) = \exp^{\mathrm{gri}}(A).$$

If $\mathcal{Z} = \mathfrak{Z}(\beta), \beta \in \mathcal{Z}^{(1)}$, then, as in the previous case,

$$\mathcal{Z} \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} \bar{F}_i \oplus \bar{F}_i$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathfrak{Z} : F]$, and

$$A \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i).$$

On each $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$, φ acts as $(a_1 \otimes f_1 + a_2 \otimes f_2)^{\varphi} = a_1^{\varphi} \otimes f_2 + a_2^{\varphi} \otimes f_1$ and $*$ acts as $(a_1 \otimes f_1 + a_2 \otimes f_2)^* = a_1^* \otimes f_1 + a_2^* \otimes f_2$, if $\beta \in (\mathcal{Z}^{(1)})^+$, and as $(a_1 \otimes f_1 + a_2 \otimes f_2)^* = a_1^* \otimes f_2 + a_2^* \otimes f_1$, if $\beta \in (\mathcal{Z}^{(1)})^-$. In any case, $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$ and, as in the previous case, it follows that

$$\dim_{\mathfrak{Z}}(A) = \dim_{\bar{F}}((A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)) = \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) = \exp^{\mathrm{gri}}(A).$$

Finally, if $\mathcal{Z} = \mathfrak{Z}(\alpha, \beta), \alpha \in (\mathcal{Z}^{(0)})^-, \beta \in (\mathcal{Z}^{(1)})$ then, as before,

$$\mathcal{Z} \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} \bar{F}_i \oplus \bar{F}_i \oplus \bar{F}_i \oplus \bar{F}_i$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathfrak{Z} : F]$, and

$$A \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathfrak{Z}:F]} (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i).$$

On each $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$, φ acts as

$$(a_1 \otimes f_1 + a_2 \otimes f_2 + a_3 \otimes f_3 + a_4 \otimes f_4)^{\varphi} = a_1^{\varphi} \otimes f_2 + a_2^{\varphi} \otimes f_1 + a_3^{\varphi} \otimes f_4 + a_4^{\varphi} \otimes f_3$$

and $*$ acts as

$$(a_1 \otimes f_1 + a_2 \otimes f_2 + a_3 \otimes f_3 + a_4 \otimes f_4)^* = a_1^* \otimes f_3 + a_2^* \otimes f_1 + a_4^* \otimes f_1 + a_4^* \otimes f_2,$$

if $\beta \in (\mathcal{Z}^{(1)})^+$ and as

$$(a_1 \otimes f_1 + a_2 \otimes f_2 + a_3 \otimes f_3 + a_4 \otimes f_4)^* = a_1^* \otimes f_4 + a_2^* \otimes f_3 + a_4^* \otimes f_2 + a_4^* \otimes f_1,$$

if $\beta \in (\mathcal{Z}^{(1)})^-$. In any case, $(A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$ and hence

$$\begin{aligned} \dim_{\mathfrak{Z}}(A) &= \dim_{\bar{F}}((A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i) \oplus (A \otimes_{\mathcal{Z}} \bar{F}_i)) \\ &= \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) \\ &= \exp^{\mathrm{gri}}(A). \end{aligned}$$

This proves (1) in case A is a simple algebra.

Now, suppose that A is $*$-simple but not simple. Then $A \cong C \oplus C^*$ where C is a simple algebra. Notice that the map $\psi : C\oplus C^* \to C\oplus C^{op}$, where C^{op} denotes de opposite algebra of C, defined by $\psi(a, b^*) = (a, b)$ is an isomorphism of algebras with involution, when $C\oplus C^{op}$ is endowed with the exchange involution. If $C^\varphi = C$, then $(C^{op})^\varphi \cong C^{op}$ and $\mathfrak{Z} \cong \mathcal{Z}(C)^{(0)}$. If $\mathcal{Z}(C)^{(0)} = \mathcal{Z}(C)$, then

$$\mathcal{Z}(C) \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} \bar{F}_i,$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathcal{Z}(C) : F]$. Therefore,

$$\begin{aligned} A \otimes_F \bar{F} &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} (A \otimes_{\mathcal{Z}(C)} \bar{F}_i) \\ &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} (C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i) \end{aligned}$$

and $(C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$. Thus, as before, we get

$$\begin{aligned} \dim_{\mathfrak{Z}}(A) &= \dim_{\bar{F}}((C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i)) \\ &= \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) \\ &= \exp^{\mathrm{gri}}(A). \end{aligned}$$

If $\mathcal{Z}(C)^{(0)} \neq \mathcal{Z}(C)$, then $\mathcal{Z}(C) \cong \mathfrak{Z}(\gamma)$, where $\gamma \in \mathcal{Z}(C)^{(1)}$. We have that

$$\mathcal{Z}(C) \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} \bar{F}_i \oplus \bar{F}_i,$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathcal{Z}(C) : F]$ and

$$\begin{aligned} A \otimes_F \bar{F} &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} A \otimes_{\mathcal{Z}(C)} (\bar{F}_i \oplus \bar{F}_i) \\ &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} ((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus ((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i). \end{aligned}$$

Each summand $((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus ((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$ and

$$\begin{aligned} \dim_{\mathfrak{Z}}(A) &= \dim_{\bar{F}}(((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus ((C \oplus C^*) \otimes_{\mathcal{Z}(C)} \bar{F}_i)) \\ &= \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) \\ &= \exp^{\mathrm{gri}}(A). \end{aligned}$$

If $C^\varphi \neq C$, then $C^\varphi \cong C^{op}$ and $\mathfrak{Z} \cong \mathcal{Z}(C)$. Thus

$$\mathcal{Z}(C) \otimes_F \bar{F} \cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} \bar{F}_i,$$

where $\bar{F}_i \cong \bar{F}$, for all $i = 1, \ldots, [\mathcal{Z}(C) : F]$. Therefore,

$$\begin{aligned} A \otimes_F \bar{F} &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} (A \otimes_{\mathcal{Z}(C)} \bar{F}_i) \\ &\cong \bigoplus_{i=1}^{[\mathcal{Z}(C):F]} (C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i) \end{aligned}$$

and $(C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i)$ is a simple $*$-superalgebra over $\bar{F}$. Thus, as before, we get

$$\begin{aligned} \dim_{\mathfrak{Z}}(A) &= \dim_{\bar{F}}((C \otimes_{\mathcal{Z}(C)} \bar{F}_i) \oplus (C^* \otimes_{\mathcal{Z}(C)} \bar{F}_i)) \\ &= \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) \\ &= \exp^{\mathrm{gri}}(A). \end{aligned}$$

Finally, suppose that $A = B \oplus B^\varphi$, where B is a simple $*$-algebra. In this case, $\mathfrak{Z} \cong \mathcal{Z}(B)^+$. If B is a simple algebra, then, as in the previous case, we get that $\dim_{\mathfrak{Z}}(A) = \exp^{\mathrm{gri}}(A)$. If B is not simple, then $B = C \oplus C^*$, where C is a simple algebra, $\mathfrak{Z} \cong \mathcal{Z}(B)^+ \cong \mathcal{Z}(C)$ and, as before, $\dim_{\mathfrak{Z}}(A) = \exp^{\mathrm{gri}}(A)$. This proves (1).

2. Suppose that A is a semisimple $*$-superalgebra. Then $A = A_1 \oplus \cdots \oplus A_m$, where each A_i, $i = 1, \ldots, m$, is a simple $*$-superalgebra. Thus,

$$A \otimes_F \bar{F} \cong \bigoplus_{i=1}^{m} A_i \otimes_F \bar{F}.$$

Now, by part (1), for each $i = 1, \dots, m$,

$$A_i \otimes_F \bar{F} \cong B_{i1} \oplus \cdots \oplus B_{it_i},$$

where $t_i = [\mathfrak{Z}_i : F]$, $\bar{F}_j \cong \bar{F}$, $j = 1, \dots, t_i$, $B_{i1} \cong \cdots \cong B_{it_i}$, each B_{ij} is a simple $*$-superalgebra over $\bar{F}$ and $\dim_{\bar{F}}(B_{ij}) = \dim_{\mathfrak{Z}_i}(A_i)$, $j = 1, \dots, t_i$. Hence, by Theorem 2.16,

$$\begin{aligned} \exp^{\mathrm{gri}}(A) &= \exp^{\mathrm{gri}}(A \otimes_F \bar{F}) \\ &= \max_{1 \le i \le m} \dim_{\bar{F}}(B_{i1}) \\ &= \max_{1 \le i \le m} \dim_{\mathfrak{Z}_i}(A_i). \end{aligned}$$

3. In order to prove (3), by part (1), we only need to show that $\exp^{\mathrm{gri}}(A) = \dim_F(A)$ implies that A is a simple $*$-superalgebra and $F = \mathfrak{Z}$. Let $\bar{A} = A \otimes_F \bar{F}$. Then $\dim_F(A) = \dim_{\bar{F}}(\bar{A}) = \exp^{\mathrm{gri}}(\bar{A}) = \exp^{\mathrm{gri}}(A)$. If $\bar{A}$ is nilpotent, then $\exp^{\mathrm{gri}}(\bar{A}) = 0$, a contradiction. Thus $\bar{A}$ contains a maximal semisimple $*$-superalgebra $B = B_1 \oplus \cdots \oplus B_m$ and $\dim_{\bar{F}}(\bar{A}) = \exp^{\mathrm{gri}}(\bar{A}) = \dim_{\bar{F}}(C)$, where C is a suitable $*$-graded subalgebra of B. Hence, A is a semisimple $*$-superalgebra and, by part (2), $\dim_{\bar{F}}(\bar{A}) = \dim_{\bar{F}} B_i$, for some $i \in \{1, \dots, m\}$, and so A is a simple $*$-superalgebra. Hence, by part (1), $\dim_F(A) = \exp^{\mathrm{gri}}(A) = \dim_{\mathfrak{Z}}(A)$ implies that $F = \mathfrak{Z}$. This complete the proof of the theorem.

$\square$

Chapter 3

$*$-Superalgebras of polynomial growth

Let $\mathcal{V}$ be a variety of $*$-superalgebras. We write $\mathcal{V} = \mathrm{var}^{\mathrm{gri}}(A)$ in case $\mathcal{V}$ is generated by a $*$-superalgebra A. We also write $c_n^{\mathrm{gri}}(\mathcal{V}) = c_n^{\mathrm{gri}}(A)$ and the growth of $\mathcal{V}$ is the growth of the sequence $c_n^{\mathrm{gri}}(\mathcal{V}), n \geq 1$.

We say that a $*$-supervariety $\mathcal{V}$ has *polynomial growth* if there exist constants α, t such that $c_n^{\mathrm{gri}}(\mathcal{V}) \leq \alpha n^t$, for all $n \geq 1$. We say that $\mathcal{V}$ has *almost polynomial growth* if $c_n^{\mathrm{gri}}(\mathcal{V})$ cannot be bounded by any polynomial function but any proper subvariety of $\mathcal{V}$ has polynomial growth.

This chapter is mainly devoted to the characterization of $*$-supervarieties of polynomial growth. The main references are [10, 16, 37]. We shall characterize $*$-supervarieties in four ways: through the $*$-graded exponent, the exclusion of $*$-superalgebras from the $*$-supervariety, T_2^*-equivalence and through the decomposition of the $\langle n \rangle$-cocharacter.

In what follows, given a $*$-superalgebra A, we shall denote by $\mathrm{var}(A)$ the variety of algebras (with no additional structure) generated by A, by $\mathrm{var}^*(A)$ the variety of $*$-algebras generated by A as an algebra with involution and by $\mathrm{var}^{\mathrm{gr}}(A)$ the variety of superalgebras generated by A as a superalgebra.

3.1 Through the $*$-graded exponent

The main result of this section characterizes finite dimensional $*$-superalgebras with polynomial growth in terms of the $*$-graded exponent.

Let m, n be positive integers. We denote by $\mathbf{P}(n,m)$ the number of partitions of n in no more than m parts. Notice that when $m \geq n$, $\mathbf{P}(n,m) = \mathbf{P}(n)$, the number of partitions of n. We have the following technical lemma.

Lemma 3.1 ([16], Lemma 8.2)**.** *Let $n \geq 1$ be an integer and write $n = n_1 + n_2 + n_3 + n_4$, a sum of four non-negative integers. If $n - n_1 = n_2 + n_3 + n_4 < q$, then*

$$\sum_{n-n_1<q} \binom{n}{n_1, n_2, n_3, n_4} \leq 6\mathbf{P}(q,3)n^q.$$

Proof. If we write $n = (n-k)+k$, with $1 \leq k < q$, then k can be written as $k = n_2 + n_3 + n_4$ in at most $3!\mathbf{P}(k,3) = 6\mathbf{P}(k,3)$ different ways. Hence,

$$\sum_{n-n_1<q} \binom{n}{n_1, n_2, n_3, n_4} \leq \sum_{k=0}^{q-1} 6\mathbf{P}(k,3)\frac{n!}{(n-k)!} \leq 6\mathbf{P}(q,3)n^q.$$

□

In the next lemma we provide a connection between the $\langle n \rangle$-codimensions and the ordinary codimensions of a $*$-superalgebra A.

Lemma 3.2 ([16], Remark 4.1)**.** *Let A be a $*$-superalgebra. Then $c_{\langle n \rangle}(A) \leq c_n(A)$, for all $n \geq 1$.*

Proof. If $f(x_1, \ldots, x_n) \in P_n \cap \mathrm{Id}(A)$, then

$$f(y_{1,0}, \ldots, y_{n_1,0}, y_{1,1}, \ldots, y_{n_2,1}, z_{1,0} \ldots, z_{n_3,0}, z_{1,1}, \ldots z_{n_4,1}) \in P_{\langle n \rangle} \cap \mathrm{Id}^{\mathrm{gri}}(A).$$

Since $P_n \cap \mathrm{Id}(A)$ can be canonically embedded in $P_{\langle n\rangle} \cap \mathrm{Id}^{\mathrm{gri}}(A)$ we derive that $c_{\langle n\rangle}(A) \leq c_n(A)$. $\square$

Next we characterize the finite dimensional $*$-superalgebras over an algebraically closed field of characteristic zero whose sequence of $*$-graded codimensions is polynomially bounded.

Theorem 3.3 ([16], Theorem 8.3). *Let A be a finite dimensional $*$-superalgebra over an algebraically closed field F of characteristic zero. Then $c_n^{\mathrm{gri}}(A)$ is polynomially bounded if and only if*

1. *$c_n(A)$ is polynomially bounded;*
2. *$A = B + J(A)$, where B is a maximal semisimple subalgebra of A with trivial induced $\mathbb{Z}_2$-grading and trivial induced involution.*

Proof. Suppose that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded. Since, by Lemma 2.1, $c_n(A) \leq c_n^{\mathrm{gri}}(A)$, we have that $c_n(A)$ is also polynomially bounded. Let $A = B + J(A)$ be a Wedderburn-Malcev decomposition of A where B is a maximal semisimple $*$-graded subalgebra. Now, since, by Lemma 2.1, $c_n^*(A) \leq c_n^{\mathrm{gri}}(A)$, the $*$-codimensions are polynomially bounded. Hence, by regarding A as an algebra with involution, by [19], we get that $B = B_1 \oplus \cdots \oplus B_k$, where $B_i \cong F$ for all $i = 1, \dots, k$, and $*$ is the identity map on B.

If we now regard B as a superalgebra, since, by Lemma 2.1, $c_n^{\mathrm{gr}}(A) \leq c_n^{\mathrm{gri}}(A)$, by [20, Theorem 11.9.3], we get that B has trivial $\mathbb{Z}_2$-grading.

Conversely, suppose that (1) and (2) are satisfied. Then $c_n(A)$ is polynomially bounded, $A = B_1 \oplus \cdots \oplus B_m + J(A)$ and, for all $i = 1, \dots, m$, $B_i \cong F, B_i^* = B_i$ and $B_i^{\varphi} = B_i$, where φ is the automorphism of order 2 determined by the superstructure. In order to prove that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded, we shall make use of $c_{\langle n\rangle}(A)$. Now, by Lemma 3.2,

$c_{\langle n\rangle}(A) \leq c_n(A)$ and since, by hypothesis, $c_n(A)$ is polynomially bounded, we get that $c_{\langle n\rangle}(A) \leq \alpha n^t$, for some $\alpha, t \geq 0$. Let q be the index of nilpotence of $J(A)$. Since $B^{(1)} = B^- = \{0\}$, we have that $A^{(1)} \subseteq J(A)$ and $A^- \subseteq J(A)$. This says that, whenever $n - n_1 = n_2 + n_3 + n_4 \geq q$, we have that $P_{\langle n\rangle} \cap \mathrm{Id}^{\mathrm{gri}}(A) = P_{\langle n\rangle}$ and so $c_{\langle n\rangle}(A) = 0$. Thus, for all n such that $n - n_1 < q$, by Lemma 3.1, we get

$$\begin{aligned} c_n^{\mathrm{gri}}(A) &= \sum_{\langle n\rangle} \binom{n}{\langle n\rangle} c_{\langle n\rangle}(A) \\ &\leq \alpha n^t \sum_{n-n_1<q} \binom{n}{n_1, n_2, n_3, n_4} \\ &\leq 6\alpha \mathbf{P}(q,3) n^{t+q}. \end{aligned}$$

This says that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded and the proof is complete. □

Finally, we have the our first characterization.

Theorem 3.4 ([37], Theorem 3.7)**.** *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero. Then* $\exp^{\mathrm{gri}}(A) \leq 1$ *if and only if A has polynomial growth.*

Proof. By Lemma 2.18, we may assume that the field F is algebraically closed. It is clear that if A has polynomial growth, then $\exp^{\mathrm{gri}}(A) \leq 1$. Conversely, suppose that $\exp^{\mathrm{gri}}(A) \leq 1$. Let $A = A_1 \oplus \cdots \oplus A_k + J(A)$ be a Wedderburn-Malcev decomposition of A. By Theorem 2.16, we have that, for any $i, j \in \{1, \ldots, k\}, i \neq j$, $A_i J(A) A_j = \{0\}$ and $\dim(A_i) = 1$, for every $i = 1, \ldots, k$. This says that $A_i \cong F$, for every $i = 1, \ldots, k$, and $B = A_1 \oplus \cdots \oplus A_k$ has trivial induced involution and grading. Now, since $\exp(A) \leq \exp^{\mathrm{gri}}(A) \leq 1$, we get that $c_n(A)$ is polynomially bounded and, by Theorem 3.3, we have that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded. This proves the theorem. □

3.2 Through the exclusion of $*$-superalgebras from $\mathrm{var}^{\mathrm{gri}}(A)$

In this section, we classify the $*$-supervarieties of almost polynomial growth generated by finite dimensional $*$-superalgebras.

We have seen in Theorems 0.3 and 0.4 that $\mathrm{var}^{\mathrm{gr}}(D^{\mathrm{gr}})$, $\mathrm{var}^*(D_*)$ and $\mathrm{var}^*(M_*)$ are varieties of almost polynomial growth. We have the following.

Theorem 3.5 ([16], Theorem 5.1)**.** $\mathrm{var}^{\mathrm{gri}}(D_*), \mathrm{var}^{\mathrm{gri}}(M_*)$ *and* $\mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gr}})$ *are* $*$*-supervarieties of almost polynomial growth.*

Proof. Since the grading on D_* is trivial, we have that

$$\mathrm{Id}^{\mathrm{gri}}(D_*) = \langle \mathrm{Id}^*(D_*), y_{1,1}, z_{1,1}\rangle_{T_2^*},$$

the T_2^*-ideal generated by $\mathrm{Id}^*(D_*), y_{1,1}, z_{1,1}$. Also $c_n^{\mathrm{gri}}(D_*) = c_n^*(D_*)$. Let $\mathcal{U}$ be a proper subvariety of $\mathrm{var}^{\mathrm{gri}}(D_*)$. Since $\mathcal{U} \subset \mathrm{var}^{\mathrm{gri}}(D_*)$, $y_{1,1}, z_{1,1} \in \mathrm{Id}^{\mathrm{gri}}(\mathcal{U})$. Hence $\mathrm{Id}^{\mathrm{gri}}(\mathcal{U}) = \langle \mathrm{Id}^*(\mathcal{U}), y_{1,1}, z_{1,1}\rangle_{T_2^*}$, $c_n^{\mathrm{gri}}(\mathcal{U}) = c_n^*(\mathcal{U})$ and $c_n^{\mathrm{gri}}(\mathcal{U})$ is polynomially bounded. Analogously, $\mathrm{var}^{\mathrm{gri}}(M_*)$ and $\mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gr}})$ are $*$-supervarieties of almost polynomial growth. □

Now, we will study two others $*$-superalgebras that appear in this characterization: the $*$-superalgebra M^{gri} and the $*$-superalgebra D^{gri}.

3.2.1 The $*$-superalgebra M^{gri}

Recall that we denote by M^{gri} the algebra M endowed with the grading

$$\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)$$

and endowed with the reflection involution. We have that $(M^{(0)})^+ = M^{(0)}$, $(M^{(0)})^- = \{0\}$, $(M^{(1)})^+ = F(e_{12}+e_{34})$ and $(M^{(1)})^- = F(e_{12}-e_{34})$. Notice that $z_{1,0}$ and $x_{1,1}x_{2,1}$ are identities of M^{gri}, where $x_{i,1} = y_{i,1}$ or $x_{i,1} = z_{i,1}$. Let us denote by I the T_2^*-ideal generated by the polynomials $z_{1,0}$ and $x_{1,1}x_{2,1}$.

Remark 3.6. For any polynomial $f \in F\langle X|\mathbb{Z}_2, *\rangle$ we have that $x_{1,1}fx_{2,1} \in I$.

Proof. We may clearly assume that f is a monomial of homogeneous degree 0. Since $[x_{1,1}, f] \in F\langle X|, \mathbb{Z}_2, *\rangle^{(1)}$, we get

$$x_{1,1}fx_{2,1} = [x_{1,1}, f]x_{2,1} + fx_{1,1}x_{2,1} \equiv 0 \pmod{I}.$$

□

Remark 3.7. For any $\sigma \in S_n$, we have $y_{\sigma(1),0}\cdots y_{\sigma(n),0} \equiv y_{1,0}\cdots y_{n,0} \pmod{I}$.

Proof. Notice that $[y_{i,0}, y_{j,0}] \equiv 0 \pmod{I}$. Hence $y_{i,0}y_{j,0} \equiv y_{j,0}y_{i,0} \pmod{I}$ and the conclusion is clear. □

Theorem 3.8 ([16], Theorem 6.3)**.** $\mathrm{Id}^{\mathrm{gri}}(M^{\mathrm{gri}}) = \langle z_{1,0}, x_{1,1}x_{2,1}\rangle_{T_2^*}$. *Moreover,* $c_n^{\mathrm{gri}}(M^{\mathrm{gri}})$ *grows exponentially.*

Proof. Since, by Lemma 2.1, $c_n^*(M_*) \le c_n^{\mathrm{gri}}(M^{\mathrm{gri}})$ and $c_n^*(M_*)$ grows exponentially, we get that $c_n^{\mathrm{gri}}(M^{\mathrm{gri}})$ grows exponentially. Let $I = \langle z_{1,0}, x_{1,1}x_{2,1}\rangle_{T_2^*}$. By the discussion above, $I \subseteq \mathrm{Id}^{\mathrm{gri}}(M^{\mathrm{gri}})$.

We shall prove that if $f \in \mathrm{Id}^{\mathrm{gri}}(M^{\mathrm{gri}})$, then $f \equiv 0 \pmod{I}$. To this end, we may clearly assume that f is a multilinear polynomial of degree, say, n. Then, by Remark 3.6 and Remark 3.7, we get that either $f \equiv \alpha y_{1,0}\cdots y_{n,0} \pmod{I}$, for some $\alpha \in F$, or f can be written $\pmod{I}$ as a linear combination of monomials of the type

$$y_{i_1,0}\cdots y_{i_t,0}x_{1,1}y_{i_{t+1},0}\cdots y_{i_{n-1},0},$$

where $0 \leq t \leq n-1$, $i_1 < \cdots < i_t$ and $i_{t+1} < \cdots < i_{n-1}$.

In the first case, by making the evaluation $y_{i,0} = 1$, for $i = 1, \dots, n$, we get $\alpha = 0$ and so $f \in I$ as wished.

In the second case, write

$$f \equiv \sum_{t=0}^{n-1} \sum_{1 \leq i_1 < \cdots < i_t \leq n-1} \alpha_{i_1,\dots,i_t} y_{i_1,0} \cdots y_{i_t,0} x_{1,1} y_{i_{t+1},0} \cdots y_{i_{n-1},0} \pmod{I},$$

with $\alpha_{i_1,\dots,i_t} \in F$. If for some $i_1 < \cdots < i_t$, $\alpha_{i_1,\dots,i_t} \neq 0$, we make the evaluation $y_{i_1,0} = \cdots = y_{i_t,0} = e_{11} + e_{44}$, $y_{i_{t+1},0} = \cdots = y_{i_{n-1},0} = e_{22} + e_{33}$ and $x_{1,1} = e_{12} + e_{34}$, in case $x_{1,1}$ is symmetric, or $x_{1,1} = e_{12} - e_{34}$, in case $x_{1,1}$ is skew. It is easily seen that f evaluates to $\alpha_{i_1,\dots,i_t}(e_{11} + e_{44})(e_{12} \pm e_{34})(e_{22} + e_{33}) = \alpha_{i_1,\dots,i_t} e_{12} \pmod{I}$ and $\alpha_{i_1,\dots,i_t} = 0$. Thus $f \in I$ and the proof is complete. □

In the next theorem, we will use the representation theory of the general linear group. We refer [9] for more details.

Theorem 3.9 ([16], Theorem 6.4)**.** *M^{gri} generates a $*$-supervariety of almost polynomial growth.*

Proof. By Theorem 3.8, $c_n^{\mathrm{gri}}(M^{\mathrm{gri}})$ grows exponentially. Hence, M^{gri} generates a $*$-supervariety of exponential growth.

We start by computing the decomposition of the $\langle n \rangle$-cocharacter of M^{gri} into irreducible characters. Let

$$\chi_{\langle n \rangle}(M^{\mathrm{gri}}) = \sum_{\langle \lambda \rangle \vdash n} m_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle} \tag{3.1}$$

be the decomposition of the $\langle n \rangle$-cocharacter of M^{gri}.

Now, since $z_{1,0}$ is an identity of M^{gri}, if $\chi_{\langle \lambda \rangle}$ appears with non-zero multiplicity in (3.1), we must have $\lambda(3) = 0$. Moreover, by Remark 3.6, two variables of homogeneous degree 1 cannot appear in any non-zero monomial $\pmod{\mathrm{Id}^{\mathrm{gri}}(M^{\mathrm{gri}})}$. Thus $m_{\langle\lambda\rangle} \neq 0$ in (3.1) implies that either $\langle\lambda\rangle = (\lambda(1), (1), \varnothing, \varnothing)$ or $\langle\lambda\rangle = (\lambda(1), \varnothing, \varnothing, (1))$ or $\langle\lambda\rangle = (\lambda(1), \varnothing, \varnothing, \varnothing)$. Since $\dim_F((M^{(0)})^+) = 2$, any polynomial alternating on three symmetric variables of homogeneous degree 0 vanishes in M^{gri}. By standard arguments (see [20]) this says that $m_{\langle\lambda\rangle} \neq 0$ implies that $\lambda(1) = (p+q, p)$, where $p \geq 0, q \geq 0$, is a partition with at most two parts.

By Remark 3.7, symmetric variables of homogeneous degree 0 commute $\pmod{\mathrm{Id}^{\mathrm{gri}}(M^{\mathrm{gri}})}$. Hence we have that $m_{\langle\lambda\rangle} \neq 0$ implies that either $\langle\lambda\rangle = ((n), \varnothing, \varnothing, \varnothing)$ or $\langle\lambda\rangle = ((p+q, p), \varnothing, \varnothing, (1))$ or $\langle\lambda\rangle = ((p+q, p), (1), \varnothing, \varnothing)$, where $p \geq 0$, $q \geq 0$ and $n = 2p + q + 1$.

We claim that $m_{((p+q,p),\varnothing,\varnothing,(1))} = m_{((p+q,p),(1),\varnothing,\varnothing)} = q+1$. To this end, we follow closely the proof of [32] (or [39]), taking into account the due changes.

Define, for $0 \leq i \leq q$, the polynomials

$$a^{(i)}_{p,q}(y_{1,0}, y_{2,0}, x_{1,1}) = y^{i}_{1,0}\, \underbrace{\bar{y}_{1,0} \cdots \tilde{y}_{1,0}}_{p}\, x_{1,1}\, \underbrace{\bar{y}_{2,0} \cdots \tilde{y}_{2,0}}_{p}\, y^{q-i}_{1,0},$$

where $-$ and $\sim$ mean alternation on the corresponding variables and $x_{1,1} = y_{1,1}$ or $x_{1,1} = z_{1,1}$.

As in the proof of [32] (or [39]), we can use the representation theory of the general linear group and the following can be shown: the polynomials $a^{(i)}_{p,q}$ are highest weight vectors corresponding to Young tableaux and they are linearly independent $\pmod{\mathrm{Id}^{\mathrm{gri}}(M^{\mathrm{gri}})}$. Hence $m_{((p+q,p),\varnothing,\varnothing,(1))} = m_{((p+q,p),(1),\varnothing,\varnothing)} = q+1$ as claimed. Also, through an obvious evaluation, it is clear that $m_{((n),\varnothing,\varnothing,\varnothing)} = 1$, for all $n \geq 1$.

Now, let $\mathcal{U}$ be a proper subvariety of $\mathrm{var}^{\mathrm{gri}}(M^{\mathrm{gri}})$. Then, if

$$\chi_{\langle n\rangle}(\mathcal{U}) = \sum_{\langle\lambda\rangle \vdash n} m'_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}$$

is its $\langle n\rangle$-cocharacter, by comparing with (3.1) we must have that for some $(n_1,\dots,n_4)$ and for some $\langle\lambda\rangle \vdash (n_1,\dots,n_4)$, $m'_{\langle\lambda\rangle} < m_{\langle\lambda\rangle}$.

Let $\tilde{I} = \mathrm{Id}^{\mathrm{gri}}(\mathcal{U})$. If $\langle\lambda\rangle = ((n),\varnothing,\varnothing,\varnothing)$, this means that $y_{1,0}\cdots y_{n,0}$ is an identity of $\mathcal{U}$. But then also $a^{(i)}_{p,q} \in \tilde{I}$ as soon as $p \geq n$. It follows that $m'_{\langle\lambda\rangle} = 0$ for all $\langle\lambda\rangle \vdash (n_1,\dots,n_4)$ with $n_1+\cdots+n_4 \geq 2n+1$ and $\mathcal{U}$ has polynomial growth.

In case $\langle\lambda\rangle = ((p+q,q),(1),\varnothing,\varnothing)$ or $\langle\lambda\rangle = ((p+q,q),\varnothing,\varnothing,(1))$, then $m'_{\langle\lambda\rangle} < m_{\langle\lambda\rangle}$ says that the corresponding polynomials $a^{(i)}_{p,q}$, $0 \leq i \leq q$, are linearly dependent $\pmod{\tilde{I}}$. Notice that $\bar{y}_{1,0}z_{1,1}\bar{y}_{2,0}$ is symmetric of homogeneous degree 1 and $\bar{y}_{1,0}y_{1,1}\bar{y}_{2,0}$ is skew of homogeneous degree 1. Hence, by substituting $y_{1,1}$ with $\bar{y}_{1,0}z_{1,1}\bar{y}_{2,0}$ we get that if

$$\sum_{i=0}^{q} \alpha_i a^{(i)}_{p,q}(y_{1,0},y_{2,0},y_{1,1}) \equiv 0 \pmod{\tilde{I}}$$

then

$$\sum_{i=0}^{q} \alpha_i a^{(i)}_{p+1,q}(y_{1,0},y_{2,0},z_{1,1}) \equiv 0 \pmod{\tilde{I}}.$$

Similarly, from

$$\sum_{i=0}^{q} \alpha_i a^{(i)}_{p,q}(y_{1,0},y_{2,0},z_{1,1}) \equiv 0 \pmod{\tilde{I}}$$

we get

$$\sum_{i=0}^{q} \alpha_i a^{(i)}_{p+1,q}(y_{1,0},y_{2,0},y_{1,1}) \equiv 0 \pmod{\tilde{I}}.$$

As in the proof of [32] (or [39]) one deduces that for $N = 3p+q-1$ and suitable $M < N$,

$$y^M_{1,0}x_{1,1}y^{N-M}_{1,0} \equiv \sum_{i<M} \alpha_i y^i_{1,0}x_{1,1}y^{N-i}_{1,0} \pmod{\tilde{I}}.$$

By proceeding as in that proof, we finally get that

$$m_{(((N+1)^2),(1),\varnothing,\varnothing)} = m_{(((N+1)^2),\varnothing,\varnothing,(1))} = 1.$$

The outcome of this is that if $\lambda = (\lambda_1, \lambda_2) \vdash n-1$ is such that $\lambda_2 \geq N+1$, then $m_{(\lambda,(1),\varnothing,\varnothing)} = m_{(\lambda,\varnothing,\varnothing,(1))} = 0$. Thus

$$\begin{aligned}
\chi^{gri}_{n,0,0,0}(\mathcal{U}) &= \chi_{((n),\varnothing,\varnothing,\varnothing)},\\
\chi^{gri}_{n-1,1,0,0}(\mathcal{U}) &= \sum_{\substack{(\lambda_1,\lambda_2)\vdash n-1\\ \lambda_2\leq N}} (\lambda_1-\lambda_2+1)\chi_{((\lambda_1,\lambda_2),(1),\varnothing,\varnothing)},\\
\chi^{gri}_{n-1,0,0,1}(\mathcal{U}) &= \sum_{\substack{(\lambda_1,\lambda_2)\vdash n-1\\ \lambda_2\leq N}} (\lambda_1-\lambda_2+1)\chi_{((\lambda_1,\lambda_2),\varnothing,\varnothing,(1))}
\end{aligned}$$

and $\chi_{\langle n\rangle}(\mathcal{U}) = 0$ for all other $\langle\lambda\rangle \vdash n$. Now, for a partition $\lambda = (\lambda_1, \lambda_2) \vdash n-1$ such that $\lambda_2 < N$, it is easily seen, by the hook formula, that $\deg(\chi_\lambda) = \chi_\lambda(1) = \binom{n}{\lambda_2}\frac{n-2\lambda_2}{n} \leq n^N$. Hence, $c_{\langle n\rangle}(\mathcal{U}) \leq 1 + 2n^N$ and $c_n^{\mathrm{gri}}(\mathcal{U})$ is polynomially bounded. $\square$

3.2.2 The $*$-superalgebra D^{gri}

Recall that we denote by D^{gri} the algebra $D = F \oplus F$, with grading $(F(1,1), F(1,-1))$ and exchange involution. Notice that $D^{\mathrm{gri}} \cong F + cF$, with $c^2 = 1$ and $c^* = -c$. We have that $(D^{(0)})^+ = D^{(0)}$, $(D^{(0)})^- = \{0\}$, $(D^{(1)})^+ = \{0\}$ and $(D^{(1)})^- = D^{(1)}$. Hence, $z_{1,0}$ and $y_{1,1}$ are $*$-graded identities of D^{gri}.

Theorem 3.10. $\mathrm{Id}^{\mathrm{gri}}(D^{\mathrm{gri}}) = \langle z_{1,0}, y_{1,1}\rangle_{T_2^*}$. *Moreover,* $c_n^{\mathrm{gri}}(D^{\mathrm{gri}})$ *grows exponentially.*

Proof. Since, by Lemma 2.1, $c_n^*(D_*) \leq c_n^{\mathrm{gri}}(D^{\mathrm{gri}})$ and $c_n^*(D_*)$ grows exponentially, we get that $c_n^{\mathrm{gri}}(D^{\mathrm{gri}})$ grows exponentially. Let us denote by I the T_2^*-ideal generated by the polynomials $z_{1,0}$ and $y_{1,1}$. By the above, $I \subseteq \mathrm{Id}^{\mathrm{gri}}(D^{\mathrm{gri}})$. Notice that $[y_{1,0}, y_{2,0}] \equiv 0 \pmod{I}$, $[z_{1,1}, z_{2,1}] \equiv 0 \pmod{I}$ and $[y_{1,0}, z_{1,1}] \equiv 0 \pmod{I}$.

We shall prove that if $f \in \mathrm{Id}^{\mathrm{gri}}(D^{\mathrm{gri}})$, then $f \equiv 0 \pmod{I}$. To this end, we may clearly assume that f is multilinear polynomial of degree say, n. Since $[y_{1,0}, y_{2,0}] \equiv 0 \pmod{I}$, $[z_{1,1}, z_{2,1}] \equiv 0 \pmod{I}$ and $[y_{1,0}, z_{1,1}] \equiv 0 \pmod{I}$, we get that f can be written $\pmod{I}$ as a linear combination of monomials of the type

$$y_{i_1,0} \cdots y_{i_t,0} z_{i_{t+1},1} \cdots z_{i_n,1},$$

where $0 \leq t \leq n$, $i_1 < \cdots < i_t$ and $i_{t+1} < \cdots < i_n$.

Write

$$f \equiv \sum_{t=0}^{n} \sum_{1 \leq i_1 < \cdots < i_t \leq n-1} \alpha_{i_1,\ldots,i_n} y_{i_1,0} \cdots y_{i_t,0} z_{i_{t+1},1} \cdots z_{i_n,1} \pmod{I},$$

with $\alpha_{i_1,\ldots,i_n} \in F$. If for some $i_1 < \cdots < i_n$, $\alpha_{i_1,\ldots,i_n} \neq 0$, we make the evaluation $y_{i_1,0} = \cdots = y_{i_t,0} = 1$ and $z_{i_{t+1},1} = \cdots = z_{i_n,1} = c$. It is easily seen that f evaluates to $\alpha_{i_1,\ldots,i_n} c \pmod{I}$ and $\alpha_{i_1,\ldots,i_n} = 0$. Thus, $f \in I$ and the proof is complete. $\square$

Theorem 3.11. *Let A be a $*$-superalgebra. Then $\mathrm{Id}^{\mathrm{gri}}(A) \not\subseteq \mathrm{Id}^{\mathrm{gri}}(D^{\mathrm{gri}})$ if and only if $z_{1,1}^d \in \mathrm{Id}^{\mathrm{gri}}(A)$, for some $d \geq 1$.*

Proof. Since $z_{1,1}^d \notin \mathrm{Id}^{\mathrm{gri}}(D^{\mathrm{gri}})$, for every $d \geq 1$, we have that $\mathrm{Id}^{\mathrm{gri}}(A) \not\subseteq \mathrm{Id}^{\mathrm{gri}}(D^{\mathrm{gri}})$.

Suppose that $\mathrm{Id}^{\mathrm{gri}}(A) \not\subseteq \mathrm{Id}^{\mathrm{gri}}(D^{\mathrm{gri}})$. Let $f \in \mathrm{Id}^{\mathrm{gri}}(A)$, $f \notin \mathrm{Id}^{\mathrm{gri}}(D^{\mathrm{gri}})$, a multilinear polynomial of degree n. Since $f \notin \mathrm{Id}^{\mathrm{gri}}(D^{\mathrm{gri}})$,

$$f = f(y_{1,0}, \ldots, y_{r,0}, z_{1,1} \ldots, z_{n-r,1})$$

and f does not vanish on a basis of D^{gri}. We have that $\{1, c\}$ is a $*$-graded basis of D^{gri} and by the above,

$$f(1, \ldots, 1, c \ldots, c) = f(c^2, \ldots c^2, c, \ldots, c) = \alpha c^{n+r},$$

with $\alpha \neq 0$. Since $z_{1,1}^2$ is symmetric of homogeneous degree 0, we get that $f(z_{1,1}^2, \ldots, z_{1,1}^2, z_{1,1}, \ldots, z_{1,1}) = \alpha z_{1,1}^{n+r} \in \mathrm{Id}^{\mathrm{gri}}(A)$. $\square$

In the proof of the next theorem, as in Theorem 3.9, we will use the representation theory of the general linear group.

Theorem 3.12. $\chi_{\langle n\rangle}(D^{\mathrm{gri}}) = \sum_{r=0}^{n} \chi_{(n-r),\varnothing,\varnothing,r}$.

Proof. Let

$$\chi_{\langle n\rangle}(D^{\mathrm{gri}}) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle} \tag{3.2}$$

be the decomposition of the $\langle n\rangle$-cocharacter of D^{gri}.

Since $z_{1,0}$ and $y_{1,1}$ are identities of D^{gri}, if $\chi_{\langle\lambda\rangle}$ appears with non-zero multiplicity in (3.2), we must have $\lambda(2) = \lambda(3) = 0$. Thus, $m_{\langle\lambda\rangle} \neq 0$ in (3.2) implies that $\langle\lambda\rangle = ((n-r), \varnothing, \varnothing, (r))$. Since $\dim(D^{(0)})^+ = \dim(D^{(1)})^- = 1$, any polynomial alternating on two symmetric variables of homogeneous degree 0 or on two skew variables of homogeneous degree 1 vanishes in D^{gri}. The commutativity of D^{gri} implies that $m_{\langle\lambda\rangle} \leq 1$. Also, since $y_{1,0}^{n-r}z_{1,1}^r$ does not vanish in D^{gri}, we get that $m_{\langle\lambda\rangle} = 1$. $\square$

Corollary 3.13. $c_n^{\mathrm{gri}}(D^{\mathrm{gri}}) = 2^n$.

Theorem 3.14. D^{gri} *generates a $*$-supervariety of almost polynomial growth.*

Proof. By Theorem 3.10, we have that $c_n^{\mathrm{gri}}(D^{\mathrm{gri}})$ grows exponentially. Let $\mathcal{U} = \mathrm{var}^{\mathrm{gri}}(A)$ be a proper subvariety of $\mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gri}})$. By Theorem 3.12, we have that

$$\chi_{\langle n\rangle}(\mathcal{U}) = \sum_{r=0}^{n} m_{\langle\lambda\rangle}\chi_{(n-r),\varnothing,\varnothing,r},$$

with $m_{\langle\lambda\rangle} \in \{0,1\}$. Since, by Theorem 3.11, $z_{1,1}^d \in \mathrm{Id}^{\mathrm{gri}}(A)$, for some $d \geq 1$, it follows that $m_{\langle\lambda\rangle} = 0$ as soon as $r \geq d$. This means that $c_n^{\mathrm{gri}}(\mathcal{U}) =$

$\displaystyle\sum_{r=0}^{d-1}\binom{n}{r} \leq n^d$ and $\mathcal{U}$ has polynomial growth. This complete the proof of the theorem. $\square$

In order to prove the main theorem of this section, we need of two technical lemmas.

Lemma 3.15 ([16], Lemma 8.4)**.** *Let A and B be $*$-superalgebras. If B has trivial grading and $B \notin \mathrm{var}^{\mathrm{gri}}(A)$, then $B \notin \mathrm{var}^*(A^{(0)})$.*

Proof. Clearly, $\mathrm{Id}^{\mathrm{gri}}(A^{(0)}) = \langle \mathrm{Id}^*(A^{(0)}), y_{1,1}, z_{1,1}\rangle_{T_2^*}$ and also $\mathrm{Id}^{\mathrm{gri}}(B) = \langle \mathrm{Id}^*(B), y_{1,1}, z_{1,1}\rangle_{T_2^*}$. Hence, if $B \in \mathrm{var}^*(A^{(0)})$, then $B \in \mathrm{var}^{\mathrm{gri}}(A^{(0)})$. Since $A^{(0)}$ is a subalgebra of A, $\mathrm{var}^{\mathrm{gri}}(A^{(0)}) \subseteq \mathrm{var}^{\mathrm{gri}}(A)$ which says that $B \in \mathrm{var}^{\mathrm{gri}}(A)$. $\square$

Lemma 3.16 ([16], Lemma 8.5)**.** *Let A be a finite dimensional $*$-superalgebra over an algebraically closed field of characteristic zero. Let $A = A_1 \oplus \cdots \oplus A_k + J$ be a Wedderburn-Malcev decomposition of A, where $A_1, \ldots, A_k$ are simple $*$-superalgebras. If for some $i, l \in \{1, \ldots, k\}, i \neq l$, we have that $A_i^{(0)} J^{(1)} A_l^{(0)} \neq \{0\}$, then $M^{\mathrm{gri}} \in \mathrm{var}^{\mathrm{gri}}(A)$.*

Proof. Suppose that there exist $i, l \in \{1, \ldots, k\}, i \neq l$, such that $A_i^{(0)} J^{(1)} A_l^{(0)} \neq \{0\}$ and let $a \in A_i^{(0)}, b \in A_l^{(0)}, j' \in J^{(1)}$ such that $aj'b \neq 0$. If e_1 and e_2 are the unit elements of $A_i^{(0)}$ and $A_l^{(0)}$, respectively, then $e_1 a j' b e_2 \neq 0$ and if we set $aj'b = j$, we have that $e_1 j e_2 \neq 0$ with $j \in J^{(1)}$.

Let $k \geq 1$ be the largest integer such that $e_1 J e_2 \subseteq J^k$ and let $A' = A/J^{k+1}$. Since J is a $*$-graded ideal, A' is a $*$-superalgebra and $A' \in \mathrm{var}^{\mathrm{gri}}(A)$.

Let $\bar{e_1}, \bar{e_2}, \bar{j}$ be the images of e_1, e_2, j in A', respectively. Since $\bar{J} = J(A') = J/J^{k+1}$, we have that $\bar{e_1}\bar{J}\bar{e_2} \neq \{0\}$. Let $C = \mathrm{span}\{\bar{e_1}, \bar{e_2}, \overline{e_1 j e_2}, \overline{e_2 j^* e_1}\}$. Since e_1 and e_2 are orthogonal idempotents and $e_1 J e_2 J, e_2 J e_1 J \subseteq J^{k+1}$ we get that C is a subalgebra of A'. Moreover, C is a $*$-superalgebra and $(C^{(0)})^+ = \mathrm{span}\{\bar{e_1}, \bar{e_2}\}$, $(C^{(0)})^- = \{0\}$, $(C^{(1)})^+ = \mathrm{span}\{\overline{e_1 j e_2} + \overline{e_2 j^* e_1}\}$ and

$(C^{(1)})^- = \mathrm{span}\{\overline{e_1 j e_2} - \overline{e_2 j^* e_1}\}$. Recalling the multiplication table of M^{gri} we obtain that the map $\psi : C \to M^{\mathrm{gri}}$ defined by $\bar{e_1} \mapsto e_{11} + e_{44}$, $\bar{e_2} \mapsto e_{22} + e_{33}$, $\overline{e_1 j e_2} \mapsto e_{12}$, $\overline{e_2 j^* e_1} \mapsto e_{34}$ is an isomorphism of $*$-superalgebras. Hence $M^{\mathrm{gri}} \in \mathrm{var}^{\mathrm{gri}}(C) \subseteq \mathrm{var}^{\mathrm{gri}}(A') \subseteq \mathrm{var}^{\mathrm{gri}}(A)$ and we are done. $\square$

In the following theorem we characterize varieties of almost polynomial growth which are generated by finite dimensional $*$-superalgebras.

Theorem 3.17 ([16], Theorem 8.6)**.** *Let A be a finite dimensional $*$-superalgebra over a field of characteristic zero. Then $c_n^{\mathrm{gri}}(A)$ is polynomially bounded if and only if M_*, D_*, D^{gr}, D^{gri}, $M^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$.*

Proof. By Lemma 2.18, we may assume that the field F is algebraically closed. Suppose that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded. Since, by Theorem 3.5, by Theorem 3.8 and by Theorem 3.10, the $*$-graded codimensions of M_*, D_*, D^{gr}, D^{gri} and M^{gri} grow exponentially we get that M_*, D_*, D^{gr}, D^{gri} $M^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$.

Conversely, suppose that $M_*, D_*, D^{\mathrm{gr}}, D^{\mathrm{gri}}, M^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$. Let $A = B + J$ be a Wedderburn-Malcev decomposition of A, where B is a maximal semisimple $*$-superalgebra. Write $B = A_1 \oplus \cdots \oplus A_k$, where the $A_i's$ are simple $*$-superalgebras. Then

$$A^{(0)} = B^{(0)} + J^{(0)} = A_1^{(0)} \oplus \cdots \oplus A_k^{(0)} + J^{(0)}$$

is an algebra with involution and with trivial grading. Since, by Lemma 3.15, $M_*, D_* \notin \mathrm{var}^{\mathrm{gri}}(A^{(0)})$, we have, by [14], that $c_n^*(A^{(0)}) = c_n^{\mathrm{gri}}(A^{(0)})$ is polynomially bounded. Also, by [19], $A_i^{(0)} \cong F$, for all $i = 1, \ldots, k$, and $*$ is the identity map on $B^{(0)}$. Since $c_n(A^{(0)}) \leq c_n^*(A^{(0)})$ is polynomially bounded, $\exp(A^{(0)}) \leq 1$ and so $A_i^{(0)} J^{(0)} A_l^{(0)} = \{0\}$, for all $i, l \in \{1, \ldots, k\}, i \neq l$.

Since $A_i^{(0)} \cong F$, for all $i = 1, \ldots, k$, we must have either $A_i^{(1)} \cong \{0\}$ or $A_i^{(1)} \cong F$, for each $i = 1, \ldots, k$. If $A_i^{(1)} \cong F$, for some $i = 1, \ldots, k$, then either D^{gr} or $D^{\mathrm{gri}} \in \mathrm{var}^{\mathrm{gri}}(A)$, a contradiction. Thus $A_i^{(1)} \cong \{0\}$ for every $i = 1, \ldots, k$, and B has trivial grading and trivial involution.

Now, suppose that there exist $i, l \in \{1, \ldots, k\}, i \neq l$, such that $A_i J A_l = A_i^{(0)} J^{(1)} A_l^{(0)} \neq \{0\}$. Then, by Lemma 3.16, $M^{\mathrm{gri}} \in \mathrm{var}^{\mathrm{gri}}(A)$, a contradiction. Therefore, we have that, for all $i, l \in \{1, \ldots, k\}, i \neq l, A_i J A_l = \{0\}$. By the properties of $\exp(A)$, we have that $\exp(A) \leq 1$ and $c_n(A)$ is polynomially bounded. Hence, by Theorem 3.3, $c_n^{gri}(A)$ is polynomially bounded and this completes the proof of the theorem. □

As an immediately consequence of the above theorem, we have the following two corollaries.

Corollary 3.18 ([16], Corollary 8.7). *Let A be a finite dimensional $*$-superalgebra over a field of characteristic zero. Then the sequence $c_n^{\mathrm{gri}}(A)$, $n \geq 1$, is either polynomially bounded or grows exponentially.*

Corollary 3.19 ([16], Corollary 8.8). $\mathrm{var}^{\mathrm{gri}}(M_*)$, $\mathrm{var}^{\mathrm{gri}}(D_*)$, $\mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gr}})$, $\mathrm{var}^{\mathrm{gri}}(D^{\mathrm{gri}})$ *and* $\mathrm{var}^{\mathrm{gri}}(M^{\mathrm{gri}})$ *are the only $*$-supervarieties of almost polynomial growth generated by finite dimensional $*$-superalgebras.*

3.3 Through T_2^*-equivalence

In this section, our goal is to prove that a finite dimensional $*$-superalgebra A has polynomial growth if and only if any finite dimensional $*$-superalgebra B such that $\mathrm{Id}^{\mathrm{gri}}(A) = \mathrm{Id}^{\mathrm{gri}}(B)$ has an explicit decomposition into suitable subalgebras with induced graded involution $*$. The next result will be useful to this end.

Lemma 3.20 ([10], Lemma 3.4). *Let $\bar{F}$ be the algebraic closure of the field F of characteristic zero and let A be a finite dimensional $*$-superalgebra over $\bar{F}$.*

Then $\mathrm{var}^{\mathrm{gri}}(A) = \mathrm{var}^{\mathrm{gri}}(B)$*, where* B *is a finite dimensional* $*$*-superalgebra over* F *such that* $\dim_{\bar{F}} A/J(A) = \dim_F B/J(B)$*.*

Proof. By Theorem 1.9, we may write $A = A_1 \oplus \cdots \oplus A_m + J$, where each A_i is a simple $*$-superalgebra, $i = 1, \ldots, m$, and $J = J(A)$ is the Jacobson radical of A. By Theorem 1.12, the structure constants of each A_i are rational. Let $\mathcal{A}_i$ be a basis of A_i consisting of symmetric and skew elements of homogeneous degree 0 and 1 and let B_i be the $*$-superalgebra generated by $\mathcal{A}_i$ over F. Since $\mathrm{char}(F) = 0$, $\mathbb{Q} \subset F$ and so B_i is finite dimensional over F. Let $\mathcal{J}$ be a basis of J over $\bar{F}$ consisting of symmetric and skew elements of homogeneous degree 0 and 1 and let B be the algebra generated by $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m \cup \mathcal{J}$ over F. Since J is nilpotent, B is finite dimensional over F. Moreover,

$$\dim_{\bar{F}} A/J(A) = \dim_{\bar{F}}(A_1 \oplus \cdots \oplus A_m) = \dim_F(B_1 \oplus \cdots \oplus B_m) = \dim_F B/J(B).$$

Now, it is clear that $\mathrm{Id}^{\mathrm{gri}}(A) \subseteq \mathrm{Id}^{\mathrm{gri}}(B)$. On the other hand, let $f \in \mathrm{Id}^{\mathrm{gri}}(B)$, that we may suppose multilinear. Then f vanishes on the set $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m \cup \mathcal{J}$, which is a basis of A over $\bar{F}$. Hence $\mathrm{Id}^{\mathrm{gri}}(B) \subseteq \mathrm{Id}^{\mathrm{gri}}(A)$ and $\mathrm{var}^{\mathrm{gri}}(A) = \mathrm{var}^{\mathrm{gri}}(B)$. $\square$

Now we present the main result of this section.

Theorem 3.21 ([10], Theorem 3.5)**.** *Let* A *be a finite dimensional* $*$*-superalgebra over a field* F *of characteristic zero. Then* $c_n^{\mathrm{gri}}(A)$ *is polynomially bounded if and only if* $\mathrm{var}^{\mathrm{gri}}(A) = \mathrm{var}^{\mathrm{gri}}(B_1 \oplus \cdots \oplus B_m)$*, where each* B_i *is a finite dimensional* $*$*-superalgebra over* F *such that* $\dim_F B_i/J(B_i) \leq 1$*, for all* $i = 1, \ldots, m$*.*

Proof. Suppose that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded. First, suppose that F is an algebraically closed field. Then, by Theorem 1.9, $A = B + J$, where B is a maximal semisimple $*$-graded subalgebra of A and $J = J(A)$ is the

Jacobson radical of A. Since $c_n^{\mathrm{gri}}(A)$ is polynomially bounded, by Theorem 3.4, $B = A_1 \oplus \cdots \oplus A_m$ where $A_i \cong F$, for all $i = 1, \ldots, m$, and $A_i A_k = A_i J A_k = \{0\}$, for all $i \neq k$.

Let $E_i = A_i + J$, $i = 1, \ldots, m$. Then $A = E_1 + \cdots + E_m$ and $J_i = J(E_i) = J \subseteq E_i$ is the Jacobson radical of E_i. We claim that

$$\begin{aligned}\mathrm{Id}^{\mathrm{gri}}(A) &= \mathrm{Id}^{\mathrm{gri}}(E_1 + \cdots + E_m) \\ &= \mathrm{Id}^{\mathrm{gri}}(E_1) \cap \cdots \cap \mathrm{Id}^{\mathrm{gri}}(E_m) \cap \mathrm{Id}^{\mathrm{gri}}(J).\end{aligned}$$

This completes the proof in this case, since

$$\mathrm{Id}^{\mathrm{gri}}(E_1) \cap \cdots \cap \mathrm{Id}^{\mathrm{gri}}(E_m) \cap \mathrm{Id}^{\mathrm{gri}}(J) = \mathrm{Id}^{\mathrm{gri}}(E_1 \oplus \cdots \oplus E_m \oplus J)$$

and $\dim_F E_i/J_i = 1$.

In fact, it is clear that

$$\mathrm{Id}^{\mathrm{gri}}(A) \subseteq \mathrm{Id}^{\mathrm{gri}}(E_1) \cap \cdots \cap \mathrm{Id}^{\mathrm{gri}}(E_m) \cap \mathrm{Id}^{\mathrm{gri}}(J).$$

Let $f \in \mathrm{Id}^{\mathrm{gri}}(E_1) \cap \cdots \cap \mathrm{Id}^{\mathrm{gri}}(E_m) \cap \mathrm{Id}^{\mathrm{gri}}(J)$ multilinear and suppose that $f \notin \mathrm{Id}^{\mathrm{gri}}(A)$. Let $\mathcal{B}$ and $\mathcal{J}$ be basis consisting of symmetric and skew elements of homogeneous degree 0 and 1 of B and J, respectively. Then $\mathcal{A} = \mathcal{B} \cup \mathcal{J}$ is a basis of A and it is enough to evaluate f on this basis.

Since $f \notin \mathrm{Id}^{\mathrm{gri}}(A)$, there exist $s_1, \ldots, s_r \in \mathcal{A}$ such that $f(s_1, \ldots, s_r) \neq 0$. Since $f \in \mathrm{Id}^{\mathrm{gri}}(J)$, there exists at least one element, say s_k, that does not belong to J. Then $s_k \in A_i$, for some $i \in \{1, \ldots, m\}$. Recalling that $A_i A_k = A_i J A_k = \{0\}$, for all $i \neq k$, we have that $s_1, \ldots, s_r \in A_i \cup J$, otherwise $f \in \mathrm{Id}^{\mathrm{gri}}(A)$. Thus $s_1, \ldots, s_r \in A_i + J = E_i$, a contradiction, since $f \in \mathrm{Id}^{\mathrm{gri}}(E_i)$. This proves the claim.

If F is arbitrary, we consider the algebra $\bar{A} = A \otimes_F \bar{F}$, where $\bar{F}$ is the algebraic closure of F. Since $\dim_F A = \dim_{\bar{F}} \bar{A}$, by the first part, $\mathrm{Id}^{\mathrm{gri}}(\bar{A}) = \mathrm{Id}^{\mathrm{gri}}(B_1 \oplus \cdots \oplus B_m)$ where each B_i is a finite dimensional $*$-superalgebra over $\bar{F}$ and $\dim_{\bar{F}} B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$. By Lemma 3.20, for all $i = 1, \dots, m$, $\mathrm{Id}^{\mathrm{gri}}(B_i) = \mathrm{Id}^{\mathrm{gri}}(C_i)$, where C_i is a finite dimensional $*$-superalgebra over F such that $\dim_F C_i/J(C_i) = \dim_{\bar{F}} B_i/J(B_i) \leq 1$. Since $\mathrm{Id}^{\mathrm{gri}}(\bar{A}) = \mathrm{Id}^{\mathrm{gri}}(A)$, viewed as $*$-superalgebras over F, we get that

$$\begin{aligned}\mathrm{Id}^{\mathrm{gri}}(\bar{A}) &= \mathrm{Id}^{\mathrm{gri}}(B_1 \oplus \cdots \oplus B_m)\\ &= \mathrm{Id}^{\mathrm{gri}}(B_1) \cap \cdots \cap \mathrm{Id}^{\mathrm{gri}}(B_m)\\ &= \mathrm{Id}^{\mathrm{gri}}(C_1) \cap \cdots \cap \mathrm{Id}^{\mathrm{gri}}(C_m)\\ &= \mathrm{Id}^{\mathrm{gri}}(C_1 \oplus \cdots \oplus C_m)\\ &= \mathrm{Id}^{\mathrm{gri}}(A).\end{aligned}$$

Conversely, suppose that $\mathrm{var}^{\mathrm{gri}}(A) = \mathrm{var}^{\mathrm{gri}}(B_1 \oplus \cdots \oplus B_m)$, where each B_i is a finite dimensional $*$-superalgebra such that $\dim_F B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$. Then either B_i is nilpotent or $B_i = C_i + J_i$, where $C_i \cong F$ and $J_i = J(B_i)$. Since $B_iB_k = \{0\}$, if $i \neq k$, it follows that $J = J_1 + \cdots + J_m$ is a nilpotent ideal, $B_1 \oplus \cdots \oplus B_m = C_1 \oplus \cdots \oplus C_m + J$ and $C_iJC_k = \{0\}$, for $i \neq k$. Therefore, by Theorem 3.4, $c_n^{\mathrm{gri}}(B_1 \oplus \cdots \oplus B_m)$ is polynomially bounded. Hence, $c_n^{\mathrm{gri}}(A)$ is polynomially bounded and the proof of the theorem is complete. □

3.4 Through the decomposition of the $\langle n\rangle$-cocharacter

The main result of this section characterizes finite dimensional $*$-superalgebras having polynomial growth in terms of the decomposition of its $\langle n\rangle$-cocharacter. We start with the following technical lemma.

Lemma 3.22 ([10], Lemma 4.1). *Let $\delta \in \mathbb{N}$ and let $\langle\lambda\rangle$ be a multipartition of n. If $n_i - \lambda(i)_1 \leq \delta$, for all $i = 1, 2, 3, 4$, then $d_{\langle\lambda\rangle} \leq n^{4\delta}$.*

Proof. Let $i \in \{1,2,3,4\}$. Since $n_i - \lambda(i)_1 \leq \delta$, by hook formula, we get that $d_{\lambda(i)} \leq \frac{n_i!}{\lambda(i)_1!} \leq \frac{n_i!}{(n_i-\delta)!} \leq n_i^{\delta}$. Hence, $d_{\langle\lambda\rangle} = d_{\lambda(1)} \cdots d_{\lambda(4)} \leq (n_1 \cdots n_4)^{\delta} \leq n^{4\delta}$. □

In order to prove the next proposition, we use the Lemmas 3.1 and 3.22.

Proposition 3.23 ([10], Proposition 4.4)**.** *Let A be a finite dimensional $*$-superalgebra and let*

$$\chi_{\langle n\rangle}(A) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}$$

be its $\langle n\rangle$-cocharacter. If, for some positive integer δ, $m_{\langle\lambda\rangle} = 0$ whenever $n - \lambda(i)_1 > \delta$, for some $i \in \{1,2,3,4\}$, then $c_n^{\mathrm{gri}}(A)$ is polynomially bounded.

Proof. Suppose that there exists a constant δ that satisfies the hypothesis of the proposition. Then $m_{\langle\lambda\rangle} \neq 0$ implies that $n - \lambda(i)_1 \leq \delta$, for all $i \in \{1,2,3,4\}$. Notice that, since $n-\lambda(i)_1 \leq \delta$, we have that $n-n_i \leq n-\lambda(i)_1 \leq \delta$, for all $i \in \{1,2,3,4\}$. Then, by Lemma 3.22, $d_{\langle\lambda\rangle} \leq n^{4\delta}$.

By Remark 2.4, we have that $m_{\langle\lambda\rangle} \leq \alpha n^t$, for some constants α and t, for all $\langle\lambda\rangle \vdash n$. Hence

$$\begin{aligned} c_{\langle n\rangle}(A) &= \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}d_{\langle\lambda\rangle} \\ &= \sum_{\substack{n-\lambda(i)_1\leq\delta \\ 1\leq i\leq 4}} m_{\langle\lambda\rangle}d_{\langle\lambda\rangle} \\ &\leq \alpha n^{t+4\delta} \sum_{\substack{n-\lambda(i)_1\leq\delta \\ 1\leq i\leq 4}} 1 \\ &\leq \alpha C n^{t+4\delta}, \end{aligned}$$

where C is a constant that depends only on δ.

Thus, by Lemma 3.1, we have that

$$\begin{aligned} c_n^{\mathrm{gri}}(A) &= \sum_{\langle\lambda\rangle\vdash n} \binom{n}{\langle n\rangle} c_{\langle n\rangle}(A) \\ &= \sum_{\substack{n-n_i\leq\delta \\ 1\leq i\leq 4}} \binom{n}{\langle n\rangle} c_{\langle n\rangle}(A) \\ &\leq \alpha C n^{t+4\delta} \sum_{\substack{n-n_i\leq\delta \\ 1\leq i\leq 4}} \binom{n}{\langle n\rangle} \\ &\leq \beta n^k, \end{aligned}$$

where $\beta = 6\alpha C\mathbf{P}(\delta,3)$ and $k = t+5\delta$. Hence $c_n^{\mathrm{gri}}(A)$ is polynomially bounded.

□

Now we are in condition to prove the main result of this section.

Theorem 3.24 ([10], Theorem 4.5)**.** *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero. Then $c_n^{\mathrm{gri}}(A)$ is polynomially bounded if and only if*

$$\chi_{\langle n\rangle}(A) = \sum_{\substack{\langle\lambda\rangle\vdash n \\ n-\lambda(1)_1<q}} m_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}$$

where q is such that $J(A)^q = \{0\}$.

Proof. By Lemma 2.18, we may assume that the field F is algebraically closed. Suppose that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded. Then, by Theorem 3.3, $A = A_1\oplus\cdots\oplus A_m+J$, where $J = J(A)$ is the Jacobson radical of A. Also, by Theorem 3.4, $A_i \cong F$, for all $i = 1,\ldots,m$, and $A_iA_k = A_iJA_k = \{0\}$, for all $i \neq k$.

Let $\langle\lambda\rangle$ be a multipartition of n such that $n-\lambda(1)_1 \geq q$ and suppose by contradiction that $m_{\langle\lambda\rangle} \neq 0$. Then there exist a multitableau $T_{\langle\lambda\rangle}$ and $f \in P_{\langle n\rangle}$ such that $g = e_{T_{\langle n\rangle}} f \notin \mathrm{Id}^{\mathrm{gri}}(A)$. Notice that g is, in particular, a linear combination of alternating polynomials in $\lambda(1)_1$ sets of symmetric variables of homogeneous degree 0.

Let h be a summand of g. We shall prove that $h \in \mathrm{Id}^{\mathrm{gri}}(A)$ and so $g \in \mathrm{Id}^{\mathrm{gri}}(A)$, a contradiction. Since $A_iA_k = A_iJA_k = \{0\}$, if $i \neq k$, in order to get a non-zero value of h, we must evaluate its variables in elements of J and in elements of one simple component, say A_i. Since $\dim(A_i) = 1$, we can substitute at most one element in each alternating set of symmetric variables of homogeneous degree 0. Thus, we can evaluate at most $\lambda(1)_1$ elements of A_i and at least $n - \lambda(1)_1$ elements of J. Since $n - \lambda(1)_1 \geq q$ and $J^q = \{0\}$, we get that $h \equiv 0$ and hence $g \in \mathrm{Id}^{\mathrm{gri}}(A)$. This contradiction proves that $m_{\langle\lambda\rangle} = 0$ for all $\langle\lambda\rangle \vdash n$ such that $n - \lambda(1)_1 \geq q$.

Conversely, suppose that the $\langle n \rangle$-cocharacter of A has the decomposition

$$\chi_{\langle n \rangle}(A) = \sum_{\substack{\langle\lambda\rangle \vdash n \\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle},$$

where q is such that $J(A)^q = \{0\}$. In this case, $m_{\langle\lambda\rangle} = 0$ whenever $n - \lambda(1)_1 \geq q$. Then, by Proposition 3.23, we have that $c_n^{\mathrm{gri}}(A)$ is polynomially bounded. The proof of the theorem is complete. □

By Theorems 3.4, 3.17, 3.21 and 3.24, we have the following equivalent characterizations of finite dimensional $*$-superalgebras with polynomial growth.

Theorem 3.25. *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero. The following conditions are equivalent:*

1. $\exp^{\mathrm{gri}}(A) \leq 1$;
2. $c_n^{\mathrm{gri}}(A) \leq \alpha n^t$, *for some constants α and t;*
3. $M_*, D_*, D^{\mathrm{gr}}, D^{\mathrm{gri}}, M^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$;

4. $\mathrm{var}^{\mathrm{gri}}(A) = \mathrm{var}^{\mathrm{gri}}(B_1 \oplus \cdots \oplus B_m)$, *where each* B_i *is a finite dimensional* $*$*-superalgebra such that* $\dim_F B_i/J(B_i) \leq 1$, *for all* $i = 1, \ldots, m$;

5. $\chi_{\langle n\rangle}(A) = \displaystyle\sum_{\substack{\langle\lambda\rangle\vdash n\\ n-\lambda(i)_1<q}} m_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}$ *where* q *is such that* $J(A)^q = \{0\}$.

Chapter 4

$*$-Superalgebras with $\exp^{\mathrm{gri}}(A) \geq 2$

In the previous chapter, we classified finite dimensional $*$-superalgebras such that $\exp^{\mathrm{gri}}(A) \leq 1$. In this chapter, we shall classify finite dimensional $*$-superalgebras such that $\exp^{\mathrm{gri}}(A) \geq 2$. The main reference for this chapter is [37].

Recall that the algebra UT_n of upper triangular matrices of order n can be endowed with the involution $(a_{ij})^* = a_{n+1-j,n+1-i}$, called reflection involution. This involution is obtained by flipping the matrix along its secondary diagonal. Any subalgebra of UT_n, for some $n \geq 1$, appearing in this chapter will be endowed with this involution.

Consider the following $*$-superalgebras:

1. $E_1 = \left\{ \begin{pmatrix} a & d & e & 0 & 0 & 0 \\ 0 & b & f & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & g & h \\ 0 & 0 & 0 & 0 & b & i \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} : a,b,c,d,e,f,g,h,i \in F \right\}$ with trivial grading and reflection involution;

2. $E_2 = \left\{ \begin{pmatrix} a & d & e & 0 & 0 & 0 \\ 0 & b & f & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & g & h \\ 0 & 0 & 0 & 0 & b & i \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} : a,b,c,d,e,f,g,h,i \in F \right\}$ with grading
$$\left(\begin{pmatrix} a & d & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & b & i \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right)$$
and reflection involution;

3. $E_3 = \left\{ \begin{pmatrix} a & d & e & 0 & 0 & 0 \\ 0 & b & f & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & g & h \\ 0 & 0 & 0 & 0 & b & i \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} : a,b,c,d,e,f,g,h,i \in F \right\}$ with grad-

ing

$$\left(\begin{pmatrix} a & 0 & e & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & h \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right)$$

and reflection involution;

4. $E_4 = \left\{ \begin{pmatrix} a & d & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & e \\ 0 & 0 & 0 & c \end{pmatrix} : a, b, c, d, e \in F \right\}$ with trivial grading and reflection involution;

5. $E_5 = \left\{ \begin{pmatrix} a & d & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & e \\ 0 & 0 & 0 & c \end{pmatrix} : a, b, c, d, e \in F \right\}$ with grading

$$\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \begin{pmatrix} 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)$$

and reflection involution;

6. $E_6 = \left\{ \begin{pmatrix} a+\alpha b & e+\alpha f & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & g+\alpha h \\ 0 & 0 & 0 & a+\alpha b \end{pmatrix} : a, b, d, e, f, g \in F, \alpha^2 = 1 \right\}$

with grading

$$\left(\begin{pmatrix} a & e & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & g \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} \alpha b & \alpha f & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha h \\ 0 & 0 & 0 & \alpha b \end{pmatrix}\right)$$

and reflection involution;

7. $E_7 = M_2(F)$ with trivial grading and transpose involution;

8. $E_8 = M_2(F)$ with trivial grading and symplectic involution;

9. $E_9 = M_{1,1}(F)$ with transpose involution;

10. $E_{10} = M_{1,1}(F)$ with symplectic involution;

11. $E_{11} = (F+cF)\oplus(F+cF)$ with grading $(F+F, c(F+F))$ and exchange involution.

In the next 4 lemmas, by Lemma 2.18, we assume that the field F is algebraically closed.

Lemma 4.1 ([37], Lemma 13)**.** $\exp^{\mathrm{gri}}(E_i) = 3, i = 1, 2, 3.$

Proof. We have that the Wedderburn-Malcev decompositions as $*$-superalgebras of $E_i, i = 1, 2, 3$, are the same: $E_i = A_1 \oplus A_2 \oplus A_3 + J(E_i), i = 1, 2, 3,$ where $A_1 = F(e_{11} + e_{66}), A_2 = F(e_{22} + e_{55}), A_3 = F(e_{33} + e_{44})$ and

$$J(E_i) = Fe_{12} \oplus Fe_{13} \oplus Fe_{23} \oplus Fe_{45} \oplus Fe_{46} \oplus Fe_{56}.$$

We get that $A_1J(E_i)A_2J(E_i)A_3 \neq \{0\}$ and, by Theorem 2.16, it follows that $\exp^{\mathrm{gri}}(E_i) = 3, i = 1, 2, 3.$ □

Lemma 4.2 ([37], Lemma 14)**.** $\exp^{\mathrm{gri}}(E_i) = 3, i = 4, 5.$

Proof. We have that the Wedderburn-Malcev decompositions as $*$-superalgebras of $E_i, i = 4, 5$, are the same: $E_i = A_1 \oplus A_2 + J(E_i), i = 4, 5$, where $A_1 = Fe_{11} \oplus F_{44}$, $A_2 = F(e_{22} + e_{33})$ and $J(E_i) = Fe_{12} \oplus Fe_{34}$. We get that $A_1 J(E_i) A_2 \neq \{0\}$ and, by Theorem 2.16, it follows that $\exp^{\mathrm{gri}}(E_i) = 3, i = 4, 5$. □

Lemma 4.3 ([37], Lemma 15)**.** $\exp^{\mathrm{gri}}(E_6) = 3$.

Proof. We have that the Wedderburn-Malcev decomposition as $*$-superalgebra of E_6 is $E_6 = A_1 \oplus A_2 + J(E_6)$, where $A_1 = F\mathbb{Z}_2(e_{11} + e_{44}), A_2 = F(e_{22}+e_{33})$ and $J(E_6) = F\mathbb{Z}_2(e_{12}) \oplus F\mathbb{Z}_2(e_{34})$. We get that $A_1 J(E_6) A_2 \neq \{0\}$ and, by Theorem 2.16, it follows that $\exp^{\mathrm{gri}}(E_6) = 3$. □

Lemma 4.4 ([37], Lemma 16)**.** $\exp^{\mathrm{gri}}(E_i) = 4, i = 7, \ldots, 11$.

Proof. The result follows from Theorems 1.12 and 2.21. □

We remind the reader that we denote by D_* the algebra $D = F \oplus F$ with trivial grading and exchange involution and by D^{gr} the algebra $D = F \oplus F$ with grading $F(1, 1) \oplus F(1, -1) \cong F + cF, c^2 = 1$, and trivial involution.

From now on F will be a field of characteristic zero and A a finite dimensional $*$-superalgebra over F. By Theorem 1.9, if F is an algebraically closed field, we can write $A = A_1 \oplus \cdots \oplus A_m + J$, where each algebra A_i, $i = 1, \ldots, m$, is a simple $*$-superalgebra and $J = J(A)$ is the Jacobson radical of A.

Lemma 4.5 ([37], Lemma 17)**.** *Suppose that F is algebraically closed and $\exp^{\mathrm{gri}}(A) > 2$. If there exist three distinct $*$-graded simple components $A_i \cong A_k \cong A_l \cong F$ such that $A_i J A_k J A_l \neq \{0\}$, then $E_i \in \mathrm{var}^{\mathrm{gri}}(A)$ for some $i \in \{1, 2, 3\}$.*

Proof. Let e_1, e_2, e_3 be the unit elements of A_i, A_k and A_l, respectively. Then $e_n^2 = e_n$, $e_n \in A_n^{(0)}$, $e_n^* = e_n$ and $e_r e_s = \delta_{rs} e_r$ for $r, s = 1, 2, 3$ and $n \in \{i, k, l\}$.

Since $e_1 J e_2 J e_3 \neq \{0\}$, let $m \geq 1$ be the greatest integer such that $J^m \neq \{0\}$ and $e_a J e_b J e_c \subseteq J^m$, for all permutations (a, b, c) of $(1, 2, 3)$. Let $\bar{A} = A/J^{m+1}$. Then $\bar{A}$ is a $*$-superalgebra and $\bar{A} \in \mathrm{var}^{\mathrm{gri}}(A)$. Let $\bar{e}_i = e_i + J^{m+1}, i = 1, 2, 3$. Then $\bar{e}_i, i = 1, 2, 3$, are orthogonal idempotents of $\bar{A}$ such that, by eventually renaming the idempotents, $\bar{e}_1 \bar{J} \bar{e}_2 \bar{J} \bar{e}_3 \neq \{0\}$, where $\bar{J} = J(\bar{A})$ is the Jacobson radical of $\bar{A}$. Also, $\bar{e}_a \bar{J} \bar{e}_b \bar{J} \bar{e}_c \bar{J} = \bar{J} \bar{e}_a \bar{J} \bar{e}_b \bar{J} \bar{e}_c = \{0\}$, for all permutations (a, b, c) of $(1, 2, 3)$. Hence, we may assume that in A we have $e_1 J e_2 J e_3 \neq \{0\}$ and $J e_a J e_b J e_c = e_a J e_b J e_c J = \{0\}$ for all permutations (a, b, c) of $(1, 2, 3)$.

Let I be the ideal of A generated by $\{e_n J e_m J e_n : m, n \in \{1, 2, 3\}, m \neq n\}$. Since the idempotents $e_i, i = 1, 2, 3$, are symmetric and have homogeneous degree 0, we get that I is a $*$-graded ideal of A, $e_1 J e_2 J e_3 \not\subset I$ and $A/I \in \mathrm{var}^{\mathrm{gri}}(A)$. Hence, we may assume that in A we have $e_1 J e_2 J e_3 \neq \{0\}$ and $e_m J e_n J e_m = \{0\}$, $m, n \in \{1, 2, 3\}, m \neq n$.

Since $e_1 J e_2 J e_3 \neq \{0\}$, there exist $j_1 = j_1^{(0)} + j_1^{(1)}$, $j_2 = j_2^{(0)} + j_2^{(1)} \in J$, with $j_1^{(0)}, j_2^{(0)} \in J^{(0)}$, $j_1^{(1)}, j_2^{(1)} \in J^{(1)}$ such that

$$\begin{aligned} e_1 j_1 e_2 j_2 e_3 &= e_1 (j_1^{(0)} + j_1^{(1)}) e_2 (j_2^{(0)} + j_2^{(1)}) e_3 \\ &= e_1 j_1^{(0)} e_2 j_2^{(0)} e_3 + e_1 j_1^{(0)} e_2 j_2^{(1)} e_3 + e_1 j_1^{(1)} e_2 j_2^{(0)} e_3 + e_1 j_1^{(1)} e_2 j_2^{(1)} e_3 \\ &\neq 0. \end{aligned}$$

Therefore, one of the following inequalities must hold:

1. $e_1 j_1^{(0)} e_2 j_2^{(0)} e_3 \neq 0$;
2. $e_1 j_1^{(0)} e_2 j_2^{(1)} e_3 \neq 0$;
3. $e_1 j_1^{(1)} e_2 j_2^{(0)} e_3 \neq 0$;

4. $e_1 j_1^{(1)} e_2 j_2^{(1)} e_3 \neq 0$.

Suppose that (1) holds. Then $e_1 j_1^{(0)} e_2 \neq 0$ and $e_2 j_2^{(0)} e_3 \neq 0$. Let U_1 be the $*$-superalgebra linearly generated by the elements e_1, e_2, e_3, $e_1 j_1^{(0)} e_2$, $e_2 (j_1^{(0)})^* e_1$, $e_2 j_2^{(0)} e_3$, $e_3 (j_2^{(0)})^* e_2$, $e_1 j_1^{(0)} e_2 j_2^{(0)} e_3$, $e_3 (j_2^{(0)})^* e_2 (j_1^{(0)})^* e_1$. Notice that U_1 has trivial induced $\mathbb{Z}_2$-grading. Then, the map $\psi_1 : U_1 \to E_1$ defined by

$$
\begin{aligned}
&e_1 \mapsto e_{11} + e_{66}, && e_2 \mapsto e_{22} + e_{55},\\
&e_3 \mapsto e_{33} + e_{44}, && e_1 j_1^{(0)} e_2 \mapsto e_{12},\\
&e_2 (j_1^{(0)})^* e_1 \mapsto e_{56}, && e_2 j_2^{(0)} e_3 \mapsto e_{23},\\
&e_3 (j_2^{(0)})^* e_2 \mapsto e_{45}, && e_1 j_1^{(0)} e_2 j_2^{(0)} e_3 \mapsto e_{13},\\
&e_3 (j_2^{(0)})^* e_2 (j_1^{(0)})^* e_1 \mapsto e_{46}
\end{aligned}
$$

is an isomorphism of $*$-superalgebras. Hence, $E_1 \in \mathrm{var}^{\mathrm{gri}}(A)$.

Now, suppose that (2) holds. Then $e_1 j_1^{(0)} e_2 \neq 0$ and $e_2 j_2^{(1)} e_3 \neq 0$. Let U_2 be the $*$-superalgebra linearly generated by the elements e_1, e_2, e_3, $e_1 j_1^{(0)} e_2$, $e_2 (j_1^{(0)})^* e_1$, $e_2 j_2^{(1)} e_3$, $e_3 (j_2^{(1)})^* e_2$, $e_1 j_1^{(0)} e_2 j_2^{(1)} e_3$, $e_3 (j_2^{(1)})^* e_2 (j_1^{(0)})^* e_1$. Notice that U_2 has induced $\mathbb{Z}_2$-grading $U_2 = (U_2^{(0)}, U_2^{(1)})$ where

$$U_2^{(0)} = \mathrm{span}_F\{e_1, e_2, e_3, e_1 j_1^{(0)} e_2, e_2 (j_1^{(0)})^* e_1\}$$

and

$$U_2^{(1)} = \mathrm{span}_F\{e_2 j_2^{(1)} e_3, e_3 (j_2^{(1)})^* e_2, e_1 j_1^{(0)} e_2 j_2^{(1)} e_3, e_3 (j_2^{(1)})^* e_2 (j_1^{(0)})^* e_1\}.$$

Then, the map $\psi_2 : U_2 \to E_2$ defined by

$$
\begin{aligned}
&e_1 \mapsto e_{11} + e_{66}, && e_2 \mapsto e_{22} + e_{55},\\
&e_3 \mapsto e_{33} + e_{44}, && e_1 j_1^{(0)} e_2 \mapsto e_{12},\\
&e_2 (j_1^{(0)})^* e_1 \mapsto e_{56}, && e_2 j_2^{(1)} e_3 \mapsto e_{23},\\
&e_3 (j_2^{(1)})^* e_2 \mapsto e_{45}, && e_1 j_1^{(0)} e_2 j_2^{(1)} e_3 \mapsto e_{13},\\
&e_3 (j_2^{(1)})^* e_2 (j_1^{(0)})^* e_1 \mapsto e_{46}
\end{aligned}
$$

is an isomorphism of $*$-superalgebras. Hence, $E_2 \in \mathrm{var}^{\mathrm{gri}}(A)$. Analogously, if (3) holds, then $E_2 \in \mathrm{var}^{\mathrm{gri}}(A)$.

Finally, suppose that (4) holds. Then $e_1 j_1^{(1)} e_2 \neq 0$ and $e_2 j_2^{(1)} e_3 \neq 0$. Let U_3 be the $*$-superalgebra linearly generated by the elements e_1, e_2, e_3, $e_1 j_1^{(1)} e_2$, $e_2 (j_1^{(1)})^* e_1$, $e_2 j_2^{(1)} e_3$, $e_3 (j_2^{(1)})^* e_2$, $e_1 j_1^{(1)} e_2 j_2^{(1)} e_3$, $e_3 (j_2^{(1)})^* e_2 (j_1^{(1)})^* e_1$. Notice that U_3 has induced $\mathbb{Z}_2$-grading $U_3 = (U_3^{(0)}, U_3^{(1)})$ where

$$U_3^{(0)} = \mathrm{span}_F\{e_1, e_2, e_3, e_1 j_1^{(1)} e_2 j_2^{(1)} e_3, e_3 (j_2^{(1)})^* e_2 (j_1^{(1)})^* e_1\}$$

and

$$U_3^{(1)} = \mathrm{span}_F\{e_1 j_1^{(1)} e_2, e_2 (j_1^{(1)})^* e_1, e_2 j_2^{(1)} e_3, e_3 (j_2^{(1)})^* e_2\}.$$

Then, the map $\psi_3 : U_3 \to E_3$ defined by

$$\begin{array}{ll}
e_1 \mapsto e_{11} + e_{66}, & e_2 \mapsto e_{22} + e_{55}, \\
e_3 \mapsto e_{33} + e_{44}, & e_1 j_1^{(1)} e_2 j_2^{(1)} e_3 \mapsto e_{13}, \\
e_3 (j_2^{(1)})^* e_2 (j_1^{(1)})^* e_1 \mapsto e_{46}, & e_1 j_1^{(1)} e_2 \mapsto e_{12}, \\
e_2 (j_1^{(1)})^* e_1 \mapsto e_{56}, & e_2 j_2^{(1)} e_3 \mapsto e_{23}, \\
e_3 (j_2^{(1)})^* e_2 \mapsto e_{45} &
\end{array}$$

is an isomorphism of $*$-superalgebras. Hence, $E_3 \in \mathrm{var}^{\mathrm{gri}}(A)$. This completes the proof. □

Lemma 4.6 ([37], Lemma 18). *Suppose that F is algebraically closed and $\exp^{\mathrm{gri}}(A) > 2$. If there exist two $*$-graded simple components $A_i \cong F$ and $A_k \cong D_*$ or D^{gri} such that either $A_i J A_k \neq \{0\}$ or $A_k J A_i \neq \{0\}$ then E_4 or $E_5 \in \mathrm{var}^{\mathrm{gri}}(A)$.*

Proof. Suppose first that $A_i J A_k \neq \{0\}$. Let e_i and e_k be the unit elements of A_i and A_k, respectively. Then $e_n^2 = e_n = e_n^*$, $e_n \in A_n^{(0)}$, $e_r e_s = \delta_{rs} e_r$ for $r, s, n \in \{i, k\}$.

Since $e_iJe_k \neq \{0\}$, let $m \geq 1$ be the greatest integer such that $J^m \neq \{0\}$ and $e_aJe_b \subseteq J^m$, $a, b \in \{i, k\}$. Let $\bar{A} = A/J^{m+1}$. Then $\bar{A}$ is a $*$-superalgebra and $\bar{A} \in \mathrm{var}^{\mathrm{gri}}(A)$. Let $\bar{e}_n = e_n + J^{m+1}, n = i, k$. Then $\bar{e}_n, n = i, k$, are orthogonal idempotents of $\bar{A}$ such that, by eventually renaming the idempotents, $\bar{e}_i\bar{J}\bar{e}_k \neq \{0\}$, where $\bar{J} = J(\bar{A})$ is the Jacobson radical of $\bar{A}$. Also, $\bar{e}_a\bar{J}\bar{e}_b\bar{J} = \bar{J}\bar{e}_a\bar{J}\bar{e}_b = \{0\}$, $a, b \in \{i, k\}$. Hence, we may assume that in A we have $e_iJe_k \neq \{0\}$ and $Je_aJe_b = e_aJe_bJ = \{0\}$, $a, b \in \{i, k\}$. Writing $e_i = e_1$ and $e_k = e_2 + e_3$, we have that $e_1^* = e_1$ and $e_2^* = e_3$.

Since $A_iJA_k \neq \{0\}$, there exists $j = j^{(0)} + j^{(1)} \in J$, $j^{(0)} \in J^{(0)}$, $j^{(1)} \in J^{(1)}$ such that

$$e_1(j^{(0)} + j^{(1)})(e_2 + e_3) = e_1j^{(0)}e_2 + e_1j^{(0)}e_3 + e_1j^{(1)}e_2 + e_1j^{(1)}e_3 \neq 0.$$

Therefore, one of the following inequalities must hold:

1. $e_1j^{(0)}e_2 \neq 0$;
2. $e_1j^{(0)}e_3 \neq 0$;
3. $e_1j^{(1)}e_2 \neq 0$;
4. $e_1j^{(1)}e_3 \neq 0$.

Suppose that (1) holds. Let H_1 be the $*$-superalgebra linearly generated by the elements $e_1, e_2, e_3, e_1j^{(0)}e_2, e_3(j^{(0)})^*e_1$. Notice that H_1 has trivial induced $\mathbb{Z}_2$-grading. Then, the map $\psi_1 : H_1 \to E_4$ defined by

$$\begin{array}{ll} e_1 \mapsto e_{22} + e_{33}, & e_2 \mapsto e_{44} \\ e_3 \mapsto e_{11} & e_1j^{(0)}e_2 \mapsto e_{34} \\ e_3(j^{(0)})^*e_1 \mapsto e_{12} & \end{array}$$

is an isomorphism of $*$-superalgebras. Hence, $E_4 \in \mathrm{var}^{\mathrm{gri}}(A)$. Analogously, if (2) holds, then $E_4 \in \mathrm{var}^{\mathrm{gri}}(A)$.

Suppose that (3) holds. Let H_2 be the $*$-superalgebra linearly generated by the elements $e_1, e_2, e_3, e_1 j^{(1)} e_2, e_3 (j^{(1)})^* e_1$. Notice that H_2 has induced $\mathbb{Z}_2$-grading $H_2 = (H_2^{(0)}, H_2^{(1)})$ where

$$H_2^{(0)} = \operatorname{span}_F\{e_1, e_2, e_3\}$$

and

$$H_2^{(1)} = \operatorname{span}_F\{e_1 j^{(1)} e_2, e_3 (j^{(1)})^* e_1\}.$$

Then, the map $\psi_2 : H_2 \to E_5$ defined by

$$\begin{array}{ll} e_1 \mapsto e_{22} + e_{33}, & e_2 \mapsto e_{44} \\ e_3 \mapsto e_{11} & e_1 j^{(1)} e_2 \mapsto e_{34} \\ e_3 (j^{(1)})^* e_1 \mapsto e_{12} & \end{array}$$

is an isomorphism of $*$-superalgebras. Hence, $E_5 \in \operatorname{var}^{\mathrm{gri}}(A)$. Analogously, if (4) holds, then $E_5 \in \operatorname{var}^{\mathrm{gri}}(A)$.

The case $A_k J A_i \neq \{0\}$ is analogous. $\square$

Lemma 4.7 ([37], Lemma 19)**.** *Suppose that F is algebraically closed and* $\exp^{\mathrm{gri}}(A) > 2$*. If there exist two $*$-graded simple components $A_i \cong F$ and $A_k \cong D^{\mathrm{gr}}$ such that either $A_i J A_k \neq \{0\}$ or $A_k J A_i \neq \{0\}$ then $E_6 \in \operatorname{var}^{\mathrm{gri}}(A)$.*

Proof. Let e_1 and e_2 be the unit elements of A_i and A_k, respectively. Then $e_n^2 = e_n$, $e_n \in A_n^{(0)}$, $e_n^* = e_n$ and $e_r e_s = \delta_{rs} e_r$ for $r, s = 1, 2$ and $n \in \{i, k\}$.

Since $e_1 J e_2 \neq \{0\}$, let $m \geq 1$ be the greatest integer such that $J^m \neq \{0\}$ and $e_a J e_b \subseteq J^m$, $a, b \in \{1, 2\}$. Let $\bar{A} = A/J^{m+1}$. Then $\bar{A}$ is a $*$-superalgebra and $\bar{A} \in \operatorname{var}^{\mathrm{gri}}(A)$. Let $\bar{e}_i = e_i + J^{m+1}, i = 1, 2$. Then $\bar{e}_i, i = 1, 2$, are orthogonal idempotents of $\bar{A}$ such that, by eventually renaming the idempotents, $\bar{e}_1 \bar{J} \bar{e}_2 \neq \{0\}$, where $\bar{J} = J(\bar{A})$ is the Jacobson radical of $\bar{A}$. Also, $\bar{e}_a \bar{J} \bar{e}_b \bar{J} = \bar{J} \bar{e}_a \bar{J} \bar{e}_b = \{0\}$, $a, b \in \{1, 2\}$. Hence, we may assume that in A we have $e_1 J e_2 \neq \{0\}$ and $J e_a J e_b = e_a J e_b J = \{0\}$, $a, b \in \{1, 2\}$.

Since $e_1Je_2 \neq \{0\}$, there exists $j = j^{(0)} + j^{(1)} \in J$, $j^{(0)} \in J^{(0)}, j^{(1)} \in J^{(1)}$ such that

$$e_1(j^{(0)} + j^{(1)})e_2 = e_1j^{(0)}e_2 + e_1j^{(1)}e_2 \neq 0.$$

Thus, we must have either $e_1j^{(0)}e_2 \neq 0$ or $e_1j^{(1)}e_2 \neq 0$. If $e_1j^{(1)}e_2 \neq 0$, by multiplying by c on the right, we may assume that $e_1j^{(0)}e_2 \neq 0$, for some $j^{(0)} \in J^{(0)}$.

Let H be the $*$-superalgebra linearly generated by the elements e_1, e_2, ce_2, $e_1j^{(0)}e_2$, $ce_1j^{(0)}e_2$, $e_2(j^{(0)})^*e_1$, $ce_2(j^{(0)})^*e_1$. Notice that H has induced $\mathbb{Z}_2$-grading $H = (H^{(0)}, H^{(1)})$ where

$$H^{(0)} = \text{span}_F\{e_1, e_2, e_1j^{(0)}e_2, e_2(j^{(0)})^*e_1\}$$

and

$$H^{(1)} = \text{span}_F\{ce_2, ce_1j^{(0)}e_2, ce_2(j^{(0)})^*e_1\}.$$

Then, the map $\psi : H \to E_6$ defined by

$$\begin{array}{ll} e_1 \mapsto e_{22} + e_{33} & e_2 \mapsto e_{11} + e_{44} \\ ce_2 \mapsto \alpha(e_{11} + e_{44}) & e_1j^{(0)}e_2 \mapsto e_{34} \\ ce_1j^{(0)}e_2 \mapsto \alpha e_{34} & e_2(j^{(0)})^*e_1 \mapsto e_{12} \\ ce_2(j^{(0)})^*e_1 \mapsto \alpha e_{12} & \end{array}$$

is an isomorphism of $*$-superalgebras. Hence $E_6 \in \text{var}^{\text{gri}}(A)$.

The case $A_kJA_i \neq \{0\}$ is analogous. $\square$

The next remark will be useful in the proof of the main theorem.

Remark 4.8. 1. If $M_{k,l}(F)$, with $k + l \geq 2, l \geq 0$, with transpose or symplectic involution lies in $\text{var}^{\text{gri}}(A)$, then either $M_2(F)$ with trivial grading or $M_{1,1}(F)$, with transpose or symplectic involution, lies in $\text{var}^{\text{gri}}(A)$;

2. If $M_{k,l}(F) \oplus M_{k,l}(F)^{op}$, with $k+l \geq 2, l \geq 0$, with induced grading and exchange involution lies in $\mathrm{var}^{\mathrm{gri}}(A)$, then either $M_2(F)$ with trivial grading or $M_{1,1}(F)$, with transpose involution, lies in $\mathrm{var}^{\mathrm{gri}}(A)$;

3. If $M_n(F) + cM_n(F)$, $n \geq 2$, with involution given by $(a+cb)^\dagger = a^* \pm cb^*$, where $*$ denotes the transpose or symplectic involution lies in $\mathrm{var}^{\mathrm{gri}}(A)$, then $M_n(F)$ with trivial grading and transpose or symplectic involution and $(F+cF) \oplus (F+cF)$ with grading $(F+F, c(F+F))$ and exchange involution lie in $\mathrm{var}^{\mathrm{gri}}(A)$. Hence, $M_2(F)$ with trivial grading and transpose or symplectic involution and $(F+cF) \oplus (F+cF)$ with grading $(F+F, c(F+F))$ and exchange involution lie in $\mathrm{var}^{\mathrm{gri}}(A)$.

4. If $(M_n(F) + cM_n(F)) \oplus (M_n(F) + cM_n(F))^{op}$, $n \geq 2$, with grading

$$(M_n(F) \oplus M_n(F)^{op}, c(M_n(F) \oplus M_n(F)^{op}))$$

   and exchange involution lies in $\mathrm{var}^{\mathrm{gri}}(A)$, then $M_n(F) + cM_n(F)$, with involution given by $(a+cb)^\dagger = a^* \pm cb^*$, where $*$ denotes the transpose or symplectic involution lies in $\mathrm{var}^{\mathrm{gri}}(A)$. Hence, $M_2(F)$ with trivial grading and transpose or symplectic involution lies in $\mathrm{var}^{\mathrm{gri}}(A)$.

Now we are in condition to proof the main theorem of this chapter.

Theorem 4.9 ([37], Theorem 20). *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero. Then* $\exp^{\mathrm{gri}}(A) > 2$ *if and only if* $E_i \in \mathrm{var}^{\mathrm{gri}}(A)$*, for some* $i \in \{1, \ldots, 11\}$.

Proof. By Lemma 2.18, we may assume that F is an algebraically closed field. If, for some $i \in \{1, \ldots, 11\}$, $E_i \in \mathrm{var}^{\mathrm{gri}}(A)$ then, by Lemmas 4.1, 4.2, 4.3 and 4.4, $\exp^{\mathrm{gri}}(A) > 2$.

Conversely, suppose that $\exp^{\mathrm{gri}}(A) > 2$. By Theorem 1.9, we can write $A = A_1 \oplus \cdots \oplus A_m + J$, where each algebra A_i, $i = 1, \ldots, m$, is a simple

$*$-superalgebra and $J = J(A)$ is the Jacobson radical of A. If, for some $i \in \{1, \dots, m\}$, A_i is isomorphic to one of the simple $*$-superalgebras given in Theorem 1.12 with $\dim_F(A_i) \geq 4$, then, by Remark 4.8, $E_i \in \mathrm{var}^{\mathrm{gri}}(A)$ for some $i \in \{7, 8, 9, 10, 11\}$.

Since $\exp^{\mathrm{gri}}(A) > 2$, by Theorem 2.16, there exist distinct $*$-graded simple components $A_{i_1}, \dots, A_{i_n}$ such that $A_{i_1} J \cdots J A_{i_n} \neq \{0\}$ and $\dim_F(A_{i_1} + \cdots + A_{i_n}) > 2$. By the above, we may assume that one of the following possibilities occurs:

1. there exist distinct A_i, A_k, A_l such that $A_i J A_k J A_l \neq \{0\}$ and $A_i \cong A_k \cong A_l \cong F$;
2. for some $i \neq k$, $A_i J A_k \neq \{0\}$ where $A_i \cong F$ and $A_k \cong D_*$;
3. for some $i \neq k$, $A_i J A_k \neq \{0\}$ where $A_i \cong F$ and $A_k \cong D^{\mathrm{gr}}$.

If (1) holds, then, by Lemma 4.5, $E_i \in \mathrm{var}^{\mathrm{gri}}(A)$, for some $i \in \{1, 2, 3\}$. If (2) holds, then, by Lemma 4.6, either E_4 or $E_5 \in \mathrm{var}^{\mathrm{gri}}(A)$. Finally, if (3) holds, then, by Lemma 4.7, $E_6 \in \mathrm{var}^{\mathrm{gri}}(A)$. The proof is complete. $\square$

We can notice that the above list of $*$-superalgebras cannot be reduced. In fact, we have the following proposition.

Proposition 4.10 ([37], Proposition 21)**.** *For all* $i, j \in \{1, \dots, 11\}, i \neq j, \mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$.

Proof. We shall prove the proposition into several steps by utilizing different arguments.

- it is clear that if $\mathrm{Id}^{\mathrm{gri}}(E_i) \subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, then $\exp^{\mathrm{gri}}(E_j) \leq \exp^{\mathrm{gri}}(E_i)$. Hence, by Lemmas 4.1, 4.2, 4.3 and 4.4, $\mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$ for $i \in \{1, 2, 3, 4, 5, 6\}$ and $j \in \{7, 8, 9, 10, 11\}$;

- the $*$-superalgebras $E_i, i \in \{1,4,7,8\}$ have trivial $\mathbb{Z}_2$-grading. Hence $\mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $i \in \{1,4,7,8\}$, $j \in \{2,3,5,6,9,10,11\}$;

- $z_{1,0}^2 \in \mathrm{Id}^{\mathrm{gri}}(E_i), i \in \{2,3,6\}$ and $z_{1,0}^2 \notin \mathrm{Id}^{\mathrm{gri}}(E_j), i \in \{1,4,5\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $i \in \{2,3,6\}$, $j \in \{1,4,5\}$;

- $z_{1,0}^3 \in \mathrm{Id}^{\mathrm{gri}}(E_1)$ and $z_{1,0}^3 \notin \mathrm{Id}^{\mathrm{gri}}(E_4)$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_1) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_4)$;

- $y_{1,1}^2 \in \mathrm{Id}^{\mathrm{gri}}(E_2)$ and $y_{1,1}^2 \notin \mathrm{Id}^{\mathrm{gri}}(E_j), j \in \{3,6\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_2) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{3,6\}$;

- $z_{1,0}y_{1,1} \in \mathrm{Id}^{\mathrm{gri}}(E_3)$ and $z_{1,0}y_{1,1} \notin \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{2,6\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_3) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{2,6\}$;

- $[y_{1,0}, y_{2,0}][y_{3,0}, y_{4,0}] \in \mathrm{Id}^{\mathrm{gri}}(E_4)$ and $[y_{1,0}, y_{2,0}][y_{3,0}, y_{4,0}] \notin \mathrm{Id}^{\mathrm{gri}}(E_1)$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_4) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_1)$;

- $[y_{1,0}, y_{2,0}] \in \mathrm{Id}^{\mathrm{gri}}(E_i), i \in \{5,8,10\}$ and $[y_{1,0}, y_{2,0}] \notin \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{1,2,3,4,6,7\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $i \in \{5,8,10\}$, $j \in \{1,2,3,4,6,7\}$;

- $z_{1,0}z_{1,1} \in \mathrm{Id}^{\mathrm{gri}}(E_6)$ and $z_{1,0}z_{1,1} \notin \mathrm{Id}^{\mathrm{gri}}(E_2)$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_6) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_2)$;

- $z_{1,1}^2 \in \mathrm{Id}^{\mathrm{gri}}(E_6)$ and $z_{1,1}^2 \notin \mathrm{Id}^{\mathrm{gri}}(E_3)$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_6) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_3)$;

- $[z_{1,0}, z_{2,0}] \in \mathrm{Id}^{\mathrm{gri}}(E_i)$, $i \in \{7,10\}$ and $[z_{1,0}, z_{2,0}] \notin \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{1,4,8\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_i) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $i \in \{7,10\}$, $j \in \{1,4,8\}$;

- $z_{1,0} \in \mathrm{Id}^{\mathrm{gri}}(E_9)$ and $z_{1,0} \notin \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{1,2,3,4,5,6,7,8,10,11\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_9) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{1,2,3,4,5,6,7,8,10,11\}$;

- $y_{1,1} \in \mathrm{Id}^{\mathrm{gri}}(E_{10})$ and $y_{1,1} \notin \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{2,3,5,6,9,11\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_{10}) \not\subset \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{2,3,5,6,9,11\}$;

- the $*$-superalgebra E_{11} is commutative and the $*$-superalgebras E_j are not, $j \in \{1, \ldots, 10\}$. Hence $\mathrm{Id}^{\mathrm{gri}}(E_{11}) \not\subseteq \mathrm{Id}^{\mathrm{gri}}(E_j)$, $j \in \{1, \ldots, 10\}$.

These facts prove the proposition. $\square$

Let $\mathcal{V}$ be a $*$-supervariety and k a positive integer. We say that $\mathcal{V}$ is a *minimal $*$-supervariety of $*$-graded exponent greater than k*, if $\exp^{\mathrm{gri}}(\mathcal{V}) > k$ and for every proper $*$-graded subvariety $\mathcal{U}$ of $\mathcal{V}$, $\exp^{\mathrm{gri}}(\mathcal{U}) \leq k$. If we denote by $\mathcal{V}_i, i = 1, \ldots, 11$, the $*$-supervariety generated by the $*$-superalgebra E_i, as a consequence of the previous proposition, we have the following corollary.

Corollary 4.11. *The $*$-supervarieties $\mathcal{V}_i, i = 1, \ldots, 11$, are the only minimal $*$-supervarieties of $*$-graded exponent greater than 2.*

As a consequence of Theorems 3.17 and 4.9, we have the following characterization of finite dimensional $*$-superalgebras A such that $\exp^{\mathrm{gri}}(A) = 2$.

Corollary 4.12 ([37], Corollary 22)**.** *Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero. Then $\exp^{\mathrm{gri}}(A) = 2$ if and only if $E_i \notin \mathrm{var}^{\mathrm{gri}}(A)$, for every $i \in \{1, \ldots, 11\}$, and either $D_*, D^{\mathrm{gr}}, M_*, D^{\mathrm{gri}}$ or $M^{\mathrm{gri}} \in \mathrm{var}^{\mathrm{gri}}(A)$.*

Final considerations

In this thesis, we study the theory of $*$-graded identities on finite dimensional $*$-superalgebras. It is clear that the results presented here generalize the results for algebras with involution. In fact, if A is a $*$-superalgebra with trivial involution, then $c_n^{\mathrm{gri}}(A) = c_n^*(A)$. Moreover, if B is a $*$-superalgebra with non-trivial grading, then $B \notin \mathrm{var}^{\mathrm{gri}}(A)$. Hence, for example, if A has trivial grading, then $D^{\mathrm{gr}}, D^{\mathrm{gri}}, M^{\mathrm{gri}} \notin \mathrm{var}^{\mathrm{gri}}(A)$ and Theorem 3.17 becomes Theorem 0.4 (in case A is finite dimensional).

Here, we have just started the study of $*$-graded identities on finite dimensional $*$-superalgebras. The next step is to extend classic results on PI-theory to this new class of algebras and to work with the problems listed below.

We characterized finite dimensional $*$-superalgebras A such that $c_n^{\mathrm{gri}}(A) \leq an^t$, for some constants a, t. Now, we would like to classify the types of polynomial growth, i.e. to give a classification in terms of the exponent t. Such a classification has already been given in the setting of algebras [12, 43], superalgebras [13] and algebras with involution [30].

We classified $*$-supervarieties generated by finite dimensional $*$-superalgebras of almost polynomial growth. Now, we would like to characterize the $*$-graded subvarieties of the $*$-supervarieties of almost polynomial growth. Such a characterization has already been given in the setting of algebras

[26], superalgebras [28] and algebras with involution [29].

For a finite dimensional $*$-superalgebra A, we consider its $\langle n\rangle$-cocharacter

$$\chi_{\langle n\rangle}(A)=\sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}$$

and we would like to classify finite dimensional $*$-superalgebras A such that the multiplicities $m_{\langle\lambda\rangle}$ are bounded by a constant K. Such a classification has already been given in the setting of algebras [31], superalgebras [33] and algebras with involution [42].

Another sequence that can be attached to a $*$-superalgebra A is the sequence of $*$-graded colength $l_n^{\mathrm{gri}}(A), n\geq 1$. This sequence is defined to be

$$l_n^{\mathrm{gri}}(A)=\sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}.$$

By Remark 2.4, we have that this sequence is polynomially bounded. We would like to classify finite dimensional $*$-superalgebras A such that $l_n^{\mathrm{gri}}(A)\leq K$, for specific values of a constant K. Such a classification has already been given in the setting of algebras [12, 27] and superalgebras [41].

Finally, we intend to work with algebras with G-graded involution, that is, G-graded algebras endowed with a G-graded involution $*$, where G is a group. In this case, we say that A is a $(G,*)$-*algebra*. It is possible to show that a G-graded algebra A endowed with an involution $*$ is a $(G,*)$-algebra if and only if the subspaces A^+ and A^- are G-graded. As we have done in this work, we may consider the sequence of $(G,*)$-codimensions $c_n^{(G,*)}(A), n\geq 1$, and prove that, if G is a finite group, then $c_n^{(G,*)}(A)\leq 2^n|G|^n c_n(A)$. Thus, we have that $c_n^{(G,*)}(A)$ is exponentially bounded if and only if A is a PI-algebra. In this case, we may define the $(G,*)$-exponent of A $\exp^{(G,*)}(A)$. We notice that we can apply the same arguments used in the proof of Theorem 2.16 to show that, if A is a finite dimensional $(G,*)$-algebra over a field of characteristic zero, then $\exp^{(G,*)}(A)$ exists and is a non-negative integer. It would be interesting to extend the results presented in this thesis and the problems presented above to $(G,*)$-algebras, where G is a finite group.

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