FERNANDO LOURENÇO

# Baum-Bott residues for flags of foliations

Tese apresentada à Universidade Federal de Minas Gerais, como parte das exigências do Programa de Pós Graduação em Matemática, para obtenção do título de *Doctor*.

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*"Só no conhecimento de sua própria essência,*
*deixam de ser os homens,*
*um bando de macacos"*.
Aldoux Huxley.

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CONTENTS

**RESUMO**


Nesse trabalho de tese estudamos flags de folheações holomorfas singulares, formandos por 2 folheações. Estamos interessados em investigar classes características dessa estrutura e suas consequências. Desenvolvemos uma teoria de resíduos para esses flags. Para tal, provamos um teorema de anulamento do tipo Bott para flags e um teorema do tipo Baum-Bott para tais flags.

Analisamos também a conjectura de racionalidade de Bott para flags. Nesse sentido, definimos o resíduo de Nash para flags utilizando a construção de Nash adaptada para tal situação. Com isso, comparamos o resíduo de Nash para flags com o tal resíduo de Baum-Bott para flags, mostrando assim a racionalidade dos resíduos neste contexto.

Nesse último capítulo tratamos com folheações holomorfas. Nesse sentido, apresentamos uma maneira efetiva de calcular resíduos de folheações, quando a dimensão do conjunto singular da folheação é um a menos que a dimensão da folheação. Esse resultado generaliza o resultado de Bott, uma vez que retiramos hipóteses.

**ABSTRACT**

In this thesis we study flags of singular holomorphic foliations, formed by two foliations. We are interested in investigating characteristic classes for this structure and its consequences. In this work we develop a residue theory for these flags. Then, we prove a Bott vanishing theorem for flags. Next we proved a Baum-Bott type theorem for flags.

We treat also the Bott rationality conjecture for flags. In this sense we define the Nash residue for flag utilising Nash construction adapted for flags. With this we can do the comparison of the Bott residue and Nash residue for flags, which show the rationality of residues in this context.

In the last chapter we deal holomorphic foliations. For this purpose, we present an effective way to calculate residues of the foliations, when the dimension of singular set of the foliation is one less than the dimension of the foliation. This result generalizes the result of Bott.

# INTRODUCTION

A flag of singular holomorphic foliation on a complex manifold $M$, of dimension $n$, is a finite sequence of foliations $\mathcal{F} = (\mathcal{F}_1, ..., \mathcal{F}_k)$ such that, away from singular sets, each foliation $\mathcal{F}_{i+1}$ is tangent to the foliation $\mathcal{F}_i$ and $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ (we call it subfoliation) for each $i = 1, ..., k-1$.

When $k = 2$ we have the diagram

$$\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[dl] & & 0 \\
& \mathcal{F}_1 \arrow[dr] \arrow[dd] & & \mathcal{N}_2 \arrow[ul] \arrow[ur] & \\
& & \Theta_M \arrow[ur] \arrow[dr] & & \\
& \mathcal{F}_2 \arrow[ur] \arrow[dr] & & \mathcal{N}_1 \arrow[uu] \arrow[dr] & \\
0 \arrow[ur] & & \mathcal{N}_{1,2} \arrow[ur] \arrow[dr] & & 0 \\
& 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

Feigin started the study of characteristic classes of flags in 1975, see [14], where the author investigates an obstruction for existence of the flags integrably homotopic. Recently Mol in [22] studied the behavior of singularities of flags and its polar varieties. In the same sense, Corrêa and Soares study the Poincaré problem for flags in [12].

Flags of holomorphic foliations appear naturally in the theory of foliation. For example, a conjecture due to Marco Brunella says that a two-dimensional holomorphic foliation $\mathcal{F}_2$ on $\mathbb{P}^3$ either admits an invariant algebraic surface or it is a flag of holomorphic foliations, i.e.,

$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, where in this last case $\mathcal{F}_1$ is a foliation by algebraic curves on $\mathbb{P}^3$. We hope that a theorem of residues for flags can give important informations about the existence of this structure.

In this work we develop a residues theory for flags. The residues theory has been widely studied by Baum and Bott, see [3] and [2].

**Theorem** *Let $\mathcal{F}$ be an one-dimensional singular foliation on a compact complex manifold $M$ of dimensional $n$ and $\varphi$ a symmetric homogeneous polynomial of degree $d$ with $n-k < d \leq n$ and $Z \subset S(\mathcal{F})$. Then there exists a homology class $Res_\varphi(\mathcal{F}; Z) \in H_{2n-2d}(Z;\mathbb{C})$ such that*

$$\varphi(\mathcal{N}_{\mathcal{F}})[M] = \sum_Z Res_\varphi(\mathcal{F}; Z).$$

For $n - k + 1 < \deg(\varphi) \leq n$ we have the following

**Rationality conjecture of Baum-Bott:** *In the situation above, if $\varphi$ has rational coefficients, then*

$$Res_\varphi(\mathcal{F}; Z) \in H_*(Z;\mathbb{Q}).$$

Sertöz in [23] used Nash map to give a partial answer for this conjecture with certain hypothesis of regularity in the Nash modification. Brasselet and Suwa in [5] used characteristic classes on singular varieties to generalize the Sertöz's work and showed an answer to the aforementioned rationality conjecture.

**Theorem** *Let $\mathcal{F}$ be a $k$ dimensional holomorphic foliation on $M$. If $\varphi = c_{i_1}...c_{i_r}$ with $i_\nu > n-k$ for some $\nu$, then the $Res_\varphi(\mathcal{F}; Z)$ comes from an integral class, in particular it is a rational class, where $c_i$ denotes the i-th Chern class.*

Now, if $\deg\varphi = n - k + 1$ the residue can be computed, whenever the singular set of the foliation $S(\mathcal{F})$ satisfies certain conditions of non-degeneration. Baum and Bott in [3, Theorem 3 pg 285] showed that we have

$$\mathrm{Res}_\varphi(\mathcal{F}; Z) = \sum_i \lambda_i [Z_i],$$

where $\lambda_i$ is a Grothendieck residue, $Z_i$ is an irreducible complex analytic component of $Z \subset S(\mathcal{F})$ of dimension $k-1$ and $[Z_i]$ denote the fundamental class of $Z_i$. We prove the following result.

**Theorem** *Let $\mathcal{F}$ be a holomorphic foliation of codimension $k$ on a compact complex manifold $M$. For each irreducible component $Z$ of Sing$_{k+1}(\mathcal{F})$ there exists a complex number $BB(\mathcal{F}, \varphi; Z)$ which is determined by the local behavior of $\mathcal{F}$ near $Z$, and the residue is given by*

$$\text{Res}(\mathcal{F}, \varphi, Z) = BB(\mathcal{F}, \varphi; Z)[Z],$$

*where $[Z]$ denotes the fundamental class of $Z$ and $BB(\mathcal{F}, \varphi; Z)$ is the Grothendieck residue of $\mathcal{G}$ at $p$*

$$BB(\mathcal{F}, \varphi; Z) = \text{Res}_p\Big[\varphi(JX)\frac{dz_1 \wedge ... \wedge dz_{k+1}}{X_1...X_{k+1}}\Big]$$

*with $\mathcal{G}$ a one-dimension foliation on a disc $H$, $X = (X_1, ..., X_{k+1})$ the vector field that induces $\mathcal{G}$ around $p$ and $\varphi$ a homogeneous symmetric polynomials of degree $k+1$.*

We will work with flags formed by $2$ foliations $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$. The first result that we will show is the Bott vanishing theorem for this flag

**Theorem** *Let $M$ be a complex manifold of dimension $n$ and $E = E_1 \oplus E_2$ a vector bundle on $M$ with $E_1$ a $F_1$- bundle, $E_2$ a $F_2$-bundle with $F_1 \subset F_2 \subset TM$ regular foliations. Let $\varphi_1$ and $\varphi_2$ be homogeneous symmetric polynomials of degree $d_1$ and $d_2$, such that at least one of the inequalities*

$$d_1 > \mathit{corank}(F_1), \quad d_2 > \mathit{corank}(F_2) \quad \mathit{or} \quad d_1 + d_2 > \mathit{corank}(F_1) \tag{1}$$

*is satisfied, then $\varphi_1(E_1) \smile \varphi_2(E_2) \equiv 0$.*

Here, note that this theorem is more "fine" than Bott vanishing theorem for foliation, see remark 2.2.12. We obtain, by using characteristic classes via Chern-Weil theory with an approach of Lehmann and Suwa, a Baum-Bott type theorem for flags

**Theorem** *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a 2-flag of holomorphic foliations on a compact complex manifold $M$ of dimension $n$. Let $\varphi_1, \varphi_2$ be homogeneous symmetric polynomials , respectively of degree $d_1$ and $d_2$, satisfying (1). Then for each compact connected component $S$ of $S(\mathcal{F})$ there exists Res$_{\varphi_1,\varphi_2}(\mathcal{F}, \mathcal{N}_\mathcal{F}, S) \in H_{2n-2(d_1+d_2)}(S; \mathbb{C})$ such that*

$$\sum_\lambda (\iota_\lambda)_* \mathit{Res}_{\varphi_1,\varphi_2}(\mathcal{F}, \mathcal{N}_\mathcal{F}, S_\lambda) = (\varphi_1(\mathcal{N}_{12}).\varphi_2(\mathcal{N}_2)) \frown [M] \quad \mathit{in} \quad H_{2n-2(d_1+d_2)}(M; \mathbb{C}), \tag{2}$$

*where $\iota_\lambda$ denotes the embedding of $S_\lambda$ on $M$.*

This theorem is very general and it says that the characteristic class $\varphi(\mathcal{N}_\mathcal{F})$ localizes at the singular set $S(\mathcal{F}) := S(\mathcal{F}_1) \cup S(\mathcal{F}_2)$ of the flag. However we can refine this localization, i.e., if we request in (1) that

$$d_1 > \text{corank}(F_1) \ \ \textbf{and} \ \ d_2 > \text{corank}(F_2)$$

we have that the characteristic class $\varphi(\mathcal{N}_\mathcal{F})$ localizes on the intersection $S := S(\mathcal{F}_1) \cap S(\mathcal{F}_2)$.

How to calculate residues of flags in general? This answer is not simple, but we will give, in this thesis, a partial answer for some cases.

Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a flag on $M$ with codimension $(k_1, k_2)$. If the singular set of the flag $S(\mathcal{F})$ has codimension bigger than $k_1 + 1$, we have for each $0 \leq j \leq k_2$

**Theorem**

$$c_1^{k_1+1-j}(\mathcal{N}_{12}) c_1^j(\mathcal{N}_2) = \sum_Z BB^j(\mathcal{F}, Z)[Z],$$

*where $BB^j(\mathcal{F}, Z)$ is a complex number that depends of the singular component $Z$ such that* $\dim Z = k_1 + 1$.

For definition of $BB^j(\mathcal{F}, Z)$, see section 2.4.

We studied a relationship between flag's residues with residues of involved foliations as an immediate consequence.

**Corollary** *For each $Z \subset Sing_{k_1+1}(\mathcal{F})$ and hypothesis as above we have*

$$\sum_{j=0}^{k_2} \binom{k_1+1}{j} Res_{c_1^{k_1+1-j} c_1^j}(\mathcal{F}, \mathcal{N}_\mathcal{F}; Z) = Res_{c_1^{k_1+1}}(\mathcal{F}_1, \mathcal{N}_1; Z) \ \ in \ \ H_{2(n-k_1-1)}(M; \mathbb{C}).$$

In the third chapter we will study the Bott rationality conjecture for flags. For this we will develop the theory of Nash for flags. We will define the Nash modification, of the complex manifold $M$, with respect to the flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, denoted by $M^\nu$. With the projection map

$$\pi : M^\nu \longrightarrow M$$

Next, if $Z \subset S(\mathcal{F})$ we can do the pull-back $\pi^{-1}(Z) =: Z^{\nu}$ which we define

**Definition** *We have well-defined the class $Res_{\varphi_1,\varphi_2}(\mathcal{F}, N^{\nu}; Z^{\nu})$ in $H_{2(n-d_1-d_2)}(Z^{\nu};\mathbb{C})$ and we call it by Nash residue of the flag $\mathcal{F}$.*

The projection $\pi : M^{\nu} \longrightarrow M$ induces a homomorphism in homology level

$$\pi_* : H_{2(n-d_1-d_2)}(Z^{\nu};\mathbb{C}) \longrightarrow H_{2(n-d_1-d_2)}(Z;\mathbb{C})$$

With this we prove the following

**Theorem** *Let $\varphi = (\varphi_1, \varphi_2)$ be homogeneous symmetric polynomials, where $\varphi_i$ is of degree $d_i$ satisfying the condition (1). If $\varphi_i$ is with integral coefficients, then the difference*

$$\mathit{Res}_{\varphi_1,\varphi_2}(\mathcal{N}_{\mathcal{F}}, \mathcal{F}, S) - \pi_* \mathit{Res}_{\varphi_1,\varphi_2}(N^{\nu}, \mathcal{F}, S^{\nu})$$

*is in the image of the canonical homomorphism $H_{2n-2d}(S;\mathbb{Z}) \longrightarrow H_{2n-2d}(S;\mathbb{C})$, i.e., is a sum of integral classes.*

**Corollary** *If $\varphi_1 = c_{i_1}...c_{i_r}$ and $\varphi_2 = c_{j_1}...c_{j_t}$ with $i_{\nu} > codim\mathcal{F}_1$ for some $\nu \in [1, ..., r]$ or $i_s > codim\mathcal{F}_2$ for some $s \in [1, ..., t]$, then the Baum-Bott residue for the flag $\mathcal{F}$, $Res_{\varphi_1,\varphi_2}(\mathcal{N}_{\mathcal{F}}, \mathcal{F}, S)$, is a (sum of) integral class.*

# Chapter 1

# Basic material

## 1.1 Čech-de Rham cohomology and duality theorems

In this section, we present the theory of Čech-de Rham Cohomology and duality theorems. For the background on the Čech-de Rham cohomology on complex manifold, we refer to [25, 4].

Let $M$ be a $\mathcal{C}^\infty$ manifold of dimension $m$ and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open covering of $M$. Suppose that the index set $I$ is an ordered set with total order. We set

$$I^{(p)} = \{(\alpha_0, ..., \alpha_p) \ / \ \alpha_0 < ... < \alpha_p \ \text{ in } \ I\}.$$

We define $C^p(\mathcal{U}, A^q) = \displaystyle\prod_{(\alpha_0,...,\alpha_p)\in I^{(p)}} A^q(U_{\alpha_0...\alpha_p}),$

where $A^q(U_{\alpha_0...\alpha_p})$ is defined as the $q$-forms space.

It is possible to define the following coboundary operator, see [5, 25]

$$\delta : C^p(\mathcal{U}, A^q) \longrightarrow C^{p+1}(\mathcal{U}, A^q).$$

This operator together with the exterior derivation induces the following operator $D : A^\bullet(\mathcal{U}) \longrightarrow A^{\bullet+1}(\mathcal{U})$. Then $(A^\bullet(\mathcal{U}), D)$ is called the Čech-de-Rham complex and its cohomology, denoted by $H^r(A^\bullet(\mathcal{U}))$, the Čech-de-Rham cohomology associated to the covering $\mathcal{U}$.

**Proposition 1.1.1 (25, Theorem 3.3, pg 48)** *We have the following isomorphism*

$$H^r_{dR}(M;\mathbb{C}) \longrightarrow H^r(A^\bullet(\mathcal{U})).$$

We define the cup product

$$A^r(\mathcal{U}) \times A^s(\mathcal{U}) \longrightarrow A^{r+s}(\mathcal{U})$$

by assigning to $\sigma \in A^r(\mathcal{U})$ and $\tau \in A^s(\mathcal{U})$ the element $\sigma \smile \tau \in A^{r+s}(\mathcal{U})$ given by

$$(\sigma \smile \tau)_{\alpha_0 \dots \alpha_p} = \sum_{\nu=0}^{p} (-1)^{(r-\nu)(p-\nu)} \sigma_{\alpha_0 \dots \alpha_\nu} \wedge \tau_{\alpha_\nu \dots \alpha_p}.$$

Then $\sigma \smile \tau$ is linear in $\sigma$ and $\tau$ and we have

$$D(\sigma \smile \tau) = D\sigma \smile \tau + (-1)^r \sigma \smile D\tau.$$

Thus it induces the cup product in cohomology level

$$H^r(A^\bullet(\mathcal{U})) \times H^s(A^\bullet(\mathcal{U})) \longrightarrow H^{r+s}(A^\bullet(\mathcal{U})).$$

Now, we recall the integration on the Čech-de-Rham cohomology and duality theorems. For this let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of $M$ as above.

**Definition 1.1.2** *A system of honey-comb cells adapted to $\mathcal{U}$ is a collection $\{R_\alpha\}_{\alpha \in I}$ of $m$ dimensional manifolds $R_\alpha$ with piecewise $C^\infty$ boundary in $M$ satisfying the following conditions:*

*(a) $R_\alpha \subset U_\alpha$ and $M = \cup_\alpha R_\alpha$,*

*(b) int$R_\alpha \cap$ int$R_\beta = \emptyset$, if $\alpha \neq \beta$,*

*(c) If $U_{\alpha_0,\dots,\alpha_p} \neq \emptyset$, $R_{\alpha_0,\dots,\alpha_p} = \cap_{\nu=0}^{p} R_{\alpha_\nu}$ is a (m - p)-dimensional manifold with piecewise $C^\infty$ boundary,*

*(d) If the set $\{\alpha_0, \dots, \alpha_p\}$ is maximal, $R_{\alpha_0,\dots,\alpha_p}$ has no boundary.*

Also, let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of $M$ as above and $\{R_\alpha\}_{\alpha \in I}$ a system of honey-comb cells adapted to $\mathcal{U}$. Suppose $M$ is compact, each $R_\alpha$ is compact and we can define the integration

$$\int_M : A^m(\mathcal{U}) \longrightarrow \mathbb{C}$$

by the sum

$$\int_M \sigma = \sum_{p=0}^{m} \Big( \sum_{(\alpha_0,\dots,\alpha_p)\in I^{(p)}} \int_{R_{\alpha_0,\dots,\alpha_p}} \sigma_{\alpha_0,\dots,\alpha_p} \Big)$$

for $\sigma \in A^m(\mathcal{U})$. Then we say that

(1) if $D\sigma = 0$ then the sum does not depend on the choice of $\{R_\alpha\}$,

(2) if $\sigma = D\tau$, than $\displaystyle\int_M \sigma = 0$.

Hence it induces the integration on the cohomology

$$\int_M : H^m(A^\bullet(\mathcal{U})) \longrightarrow \mathbb{C}.$$

We have a bilinear pairing

$$A^r(\mathcal{U}) \times A^{2n-r}(\mathcal{U}) \longrightarrow A^{2n}(\mathcal{U}) \longrightarrow \mathbb{C}$$

defined by composition of cup product and integration. We have the Poincaré duality

$$P : H^r_{dR}(M;\mathbb{C}) \simeq H^r(A^\bullet(\mathcal{U})) \longrightarrow H^{2n-r}(A^\bullet(\mathcal{U}))^* \simeq H_{2n-r}(M;\mathbb{C}).$$

Let us introduce the Alexander duality. Let $S \subset M$ be a closed subset and $U$ a neighborhood of $S$ in $M$ with $U\backslash S \subset M$. Denote $U\backslash S$ by $U_0$ and consider the covering $\mathcal{U} = \{U_0, U_1 = U\}$ of $U$. We have a canonical projection

$$\pi : A^r(\mathcal{U}) \longrightarrow A^r(U_0) \quad (\sigma_0, \sigma_1, \sigma_{01}) \longmapsto \sigma_0.$$

Denote by $A^r(\mathcal{U}, U_0)$ the kernel of this projection. Then, we have the exact sequence

$$0 \longrightarrow A^r(\mathcal{U}, U_0) \longrightarrow A^r(\mathcal{U}) \longrightarrow A^r(U_0) \longrightarrow 0.$$

We have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{r-1}(A^\bullet(U_0)) \arrow[r] \arrow[d] & H^r(A^\bullet(\mathcal{U},U_0)) \arrow[r] & H^r(A^\bullet(\mathcal{U})) \arrow[r] \arrow[d] & H^r(A^\bullet(U_0)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^{r-1}(U\setminus S;\mathbb{C}) \arrow[r] & H^r(U,U\setminus S;\mathbb{C}) \arrow[r] & H^r(U;\mathbb{C}) \arrow[r] & H^r(U\setminus S;\mathbb{C}) \arrow[r] & \cdots
\end{tikzcd}$$

Then, by the Five lemma, we have the isomorphism

$$H^r(A^\bullet(\mathcal{U},U_0)) \simeq H^r(U,U\backslash U_0;\mathbb{C}).$$

By the cup product in Čech-de-Rham cohomology in $A^r(\mathcal{U}) \times A^{2n-r}(\mathcal{U}) \longrightarrow A^{2n}(\mathcal{U})$ we have

$$(\sigma_0,\sigma_1,\sigma_{01}) \smile (\tau_0,\tau_1,\tau_{01}) = (\sigma_0\wedge\tau_0, \sigma_1\wedge\tau_1, (-1)^r\sigma_0\wedge\tau_{01}+\sigma_{01}\wedge\tau_1).$$

Now, suppose that $\sigma_0 = 0$, then the right hand side depends only on $\sigma_1,\sigma_{01}$ and $\tau_1$. Thus, we have a pairing

$$A^r(\mathcal{U},U_0)\times A^{2n-r}(U_1) \xrightarrow{\int_M} \mathbb{C}$$

This induces the Alexander duality

$$A : H^r(\mathcal{U},U\backslash S;\mathbb{C}) \simeq H^r(A^\bullet(\mathcal{U},U_0)) \longrightarrow H^{2n-r}(U_1,\mathbb{C})^* \simeq H_{2n-r}(S;\mathbb{C}). \tag{1.1}$$

**Proposition 1.1.3 (25, Proposition 3.11, pg 55)** *Let $S\subset M$ be a closed subset such that, let a neighborhood $U$ of $S$ we have $U_0 = U\backslash S \subset M$. Thus we have the commutative diagram*

$$\begin{tikzcd}
H^r(A^\bullet(\mathcal{U},U_0)) \simeq H^r(M,M\backslash S;\mathbb{C}) \arrow[r] \arrow[d, "A"'] & H^r(A^\bullet(\mathcal{U})) \simeq H^r(M;\mathbb{C}) \arrow[d, "P"] \\
H_{2n-r}(S;\mathbb{C}) \arrow[r, "i^*"] & H_{2n-r}(M;\mathbb{C}).
\end{tikzcd}$$

## 1.2 Characteristic classes via Chern-Weil theory

Let $M$ be a $\mathcal{C}^\infty$ manifold of dimension $m$. For an open set $U\subset M$ we denote by $A^0(U)$ the $\mathbb{C}$-algebra of $\mathcal{C}^\infty$-functions. Also for a $\mathcal{C}^\infty$ complex vector bundle $E$ of rank $r$ on $M$, we set

$A^p(U,E) := \mathcal{C}^\infty(U, \wedge^p(TM)^* \otimes E)$. Thus $A^0(U,E)$ is the $A^0(U)$- module of $\mathcal{C}^\infty$-module of $\mathcal{C}^\infty$-sections of $E$ and if it is a trivial line bundle, i.e., $E = M \times \mathbb{C}$, then $A^p(U,E)$ denotes the space of $p$-forms on $U$.

**Definition 1.2.1** *A connection for a complex vector bundle $E$ on $M$ is a $\mathbb{C}$-linear map*

$$\nabla : A^0(M,E) \longrightarrow A^1(M,E)$$

*that satisfies*

$$\nabla(f.s) = df \otimes s + f.\nabla(s) \ \textit{for} \ f \in A^0(M) \ \textit{and} \ s \in A^0(M,E).$$

**Lemma 1.2.2** *A connection $\nabla$ is a local operator i.e., if a section $s$ is identically zero on an open set $U$, so is $\nabla(s)$.*

**Proof:** See [25, Lemma 7.3, pg 67].

□

We say that a connection $\nabla$ is trivial on $U$ with respect to a non-vanishing section $s$ of $E$ if $\nabla(s) = 0$.

**Lemma 1.2.3** *Let $\nabla_1, ..., \nabla_k$ be connections for $E$ and $f_1, ..., f_k$ $\mathcal{C}^\infty$- functions on $M$ with $\sum f_i = 1$. Then $\sum f_i \nabla_i$ is a connection for $E$.*

**Lemma 1.2.4** *Given $E$ a vector bundle on $M$, there exists a connection $\nabla$ for $E$. In other words: every $\mathcal{C}^\infty$ vector bundle admits a connection.*

**Proof:** Let $\{U_\alpha\}$ be an open covering of $M$ that trivializes the vector bundles $TM$ and $E$. Choose a $k$-frame $s = \{s_1, ..., s_r\}$ of $E$ on $U_\alpha$. Let $\{\rho_\alpha\}$ be a partition of unity subordinative to the cover $\{U_\alpha\}$. Next define $\nabla^\alpha$ on $U_\alpha$ by $\nabla^\alpha(s_i^\alpha) = 0$ for all $i$ and extend $\nabla^\alpha$ to an arbitrary section on $U_\alpha$ using the above definition of connection. Thus $\nabla = \sum \rho_\alpha \nabla^\alpha$ is a connection for $E$.

□

If $\nabla$ is a connection for $E$, then it induces a $\mathbb{C}$-linear map

$$\nabla := \nabla^2 : A^1(M,E) \longrightarrow A^2(M,E)$$

satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s), \;\; \omega \in A^1(M), \;\; s \in A^0(M, E).$$

**Definition 1.2.5** *The composition* $K := \nabla \circ \nabla : A^0(M, E) \longrightarrow A^2(M, E)$ *is called the curvature of the connection* $\nabla$.

Here, note that the connection and the curvature are local operators. This allows us to get representatives of it.

If $\nabla$ denotes the curvature for a vector bundle $E$ of rank $r$ and $E$ is trivial on the open set $U$, i.e., $E|_U \simeq U \times \mathbb{C}^r$ and if $s = \{s_1, ..., s_r\}$ is a frame of $E$ on $U$, then we can write

$$\nabla(s_i) = \sum_{j=1}^{r} \theta_{ij} \otimes s_j \;\; ; \;\; \theta_{ij} \in A^1(U).$$

The connection matrix with respect to $s$ is $\theta = (\theta_{ij})$ . Also, using the curvature definition, we get

$$K(s_i) = \sum_{j=1}^{r} K_{ij} s_j, \quad \text{where} \quad K_{ij} = d\theta_{ij} - \sum_{k=1}^{r} \theta_{ik} \wedge \theta_{kj}.$$

The curvature matrix with respect to the frame $s$ is $K = (K_{ij})$. Now, to define the Chern class of a vector bundle $E$, we consider $\sigma_i, i = 1, ..., r$ the $i$-th elementary symmetric functions in the eigenvalues of the matrix $K$

$$\det(It + K) = 1 + \sigma_1(K)t + \sigma_2(K)t^2 + ... + \sigma_r(K)t^r.$$

Next, we define a $2i$-form of Chern $c_i$ on $U$ by

$$c_i(K) := \sigma_i(\frac{i}{2\pi}K).$$

In general, if $\varphi$ is a symmetric polynomial in $r$ variables of degree $d$, we can write $\varphi = P(c_1, ..., c_r)$ for some polynomial $P$. Then we can define

$$\varphi(K) := P(c_1(K), ..., c_r(K))$$

which is a closed form on $M$. Then, we have a cohomology class of $E$ on $M$, $\varphi(E) := \varphi(K) \in H^{2d}(M; \mathbb{C})$. If $I_r(\mathbb{C})$ denotes the graduate algebra of invariant polynomial and $E \longrightarrow M$ is a vector bundle of rank $r$, we get the homomorphism of algebras , called Weil homomorphism

$$\begin{array}{rccc} : & I_r(\mathbb{C}) & \longrightarrow & H^*(M;\mathbb{C}). \\ & \varphi & \longmapsto & \varphi(E) \end{array}$$

This theory is similarly developed for singular varieties, for this we refer to [4, 5].

## 1.3 General localization principle

In this section, we consider a general strategy for localization of characteristic classes. We first explore the Čech-de-Rham cohomology for two open sets, this is because it will be widely used in the whole of this thesis. We present the strategy of localization. We refer to [1, 27].

For $M$ a $\mathcal{C}^\infty$ manifold of dimension $m$ we let $\mathcal{U} = \{U_0, U_1\}$ be an open covering of $M$, where we use the notation $U_{01} := U_0 \cap U_1$. Now, define the vector space $A^p(\mathcal{U})$ by

$$A^p(\mathcal{U}) := A^p(U_0) \oplus A^p(U_1) \oplus A^{p-1}(U_{01}),$$

where $A^i(V)$ denote the space of $i$-forms in the open set $V$. Then, an element $\sigma \in A^p(\mathcal{U})$ is given by a triple

$$\sigma = (\sigma_0, \sigma_1, \sigma_{01})$$

with $\sigma_i$ a $p$-form in $U_i$ and $\sigma_{01}$ a $(p-1)$-form on $U_{01}$.

Define the following operator $D$ by

$$\begin{array}{rccc} D : & A^p(\mathcal{U}) & \longrightarrow & A^{p+1}(\mathcal{U}) \\ & \sigma = (\sigma_0, \sigma_1, \sigma_{01}) & \longmapsto & (d\sigma_0, d\sigma_1, \sigma_1 - \sigma_0 - d\sigma_{01}) \end{array}$$

it satisfies $D \circ D = 0$.

Then we have a complex that we call Čech-de-Rham complex and will denote by $(A^p(\mathcal{U}), D)$

$$\cdots \longrightarrow A^{p-1}(\mathcal{U}) \xrightarrow{D^{p-1}} A^p(\mathcal{U}) \xrightarrow{D^p} A^{p+1}(\mathcal{U}) \xrightarrow{D^{p+1}} \cdots$$

By simplicity we use the notation $D = D^p$ for all $p$.

Define, respectively, the closed forms and exact forms $Z^p(\mathcal{U}) = \ker D^p$ and $B^p(\mathcal{U}) = ImD^{p-1}$. We can define the $p$-th Čech-de Rham cohomology group with respect to the covering $\mathcal{U}$ by

$$H^p(\mathcal{U}) = \frac{Z^p(\mathcal{U})}{B^p(\mathcal{U})}.$$

**Theorem 1.3.1 (27, Theorem 2.1.1, pg 3)** *If $H^p(M)$ denote the $p$-th de Rham cohomology group of $M$ we have the natural isomorphism*

$$\alpha : H^p(M) \longrightarrow H^p(\mathcal{U}),$$

*where this is induced by the map $A^p(M) \ni \omega \longmapsto (\omega, \omega, 0) \in A^p(\mathcal{U})$.*

For the relative Čech-de-Rham cohomology we define the relative space

$$A^p(\mathcal{U}, U_0) := \{\sigma = (\sigma_0, \sigma_1, \sigma_{01}) \in A^p(\mathcal{U}) \ ; \ \sigma_0 = 0\}.$$

Observe that if $\sigma \in A^p(\mathcal{U}, U_0)$ then $D\sigma \in A^{p+1}(\mathcal{U}, U_0)$. Therefore, we have the relative complex $(A^p(\mathcal{U}, U_0), D)$

$$\cdots \longrightarrow A^{p-1}(\mathcal{U}, U_0) \xrightarrow{D^{p-1}} A^p(\mathcal{U}, U_0) \xrightarrow{D^p} A^{p+1}(\mathcal{U}, U_0) \xrightarrow{D^{p+1}} \cdots$$

Then, we can define the $p$-th relative Čech-de-Rham cohomology group with respect to $(\mathcal{U}, U_0)$ by

$$H^p(\mathcal{U}, U_0) = \frac{\ker D^p}{ImD^{p-1}}.$$

Now, we explain the strategy for localization that will be used in this thesis. For a complex manifold $M$ let $\varphi \in H^\bullet(M)$ be an element of its cohomology. Such a class might represent the obstruction to the existence of a certain global object.

Let $P : H^\bullet(M) \longrightarrow H_{2n-\bullet}(M)$ be the Poincaré homomorphism. For $S \subset M$ a closed subset we set $U = M - S$ and we have the exact sequence

$$\cdots \longrightarrow H^\bullet(M, U) \xrightarrow{f} H^\bullet(M) \xrightarrow{g} H^\bullet(U) \longrightarrow \cdots$$

Here assume that $g(\varphi) = 0 \in H^\bullet(U)$ (This hypothesis is verified by Bott vanishing theorem). As Im$(f) = \ker(g)$ there exists a class $\widehat{\varphi} \in H^\bullet(M, U)$. Consider the Alexander homomorphism

$$Al : H^{\bullet}(M, U) \longrightarrow H_{2n-\bullet}(S).$$

Then, we have the commutative diagram

$$\begin{tikzcd}
H^{\bullet}(M,U) \arrow[r] \arrow[d, "Al"'] & H^{\bullet}(M) \arrow[d, "P"] \\
H_{2n-\bullet}(S) \arrow[r, "i^*"] & H_{2n-\bullet}(M)
\end{tikzcd}$$

Therefore, one has a general theorem of "residues"

$$P(\varphi) = i^*(Al(\widehat{\varphi})).$$

## 1.4 Singular holomorphic foliations

Let us begin by recalling the basic material in holomorphic foliations. Let $M$ be a complex manifold of dimension $n$, $\Theta_M$ and $\Omega_M$ the sheaves of germs of holomorphic vector fields and of holomorphic 1-forms on $M$ respectively. We refer to [25, 17, 3, 7 ].

**Definition 1.4.1** *A singular holomorphic foliation $\mathcal{F}$ of dimension $k$ on $M$ is a coherent subsheaf of $\Theta_M$ of rank $k$, which satisfies the following integrability condition*

$$[\mathcal{F}_x, \mathcal{F}_x] \subset \mathcal{F}_x \qquad \textit{for all } x \in M$$

*such that, the normal sheaf, defined by $\mathcal{N}_{\mathcal{F}} := \Theta_M/\mathcal{F}$, is torsion free. ( It means that $\mathcal{F}$ is saturated).*

We have the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \Theta_M \longrightarrow \mathcal{N}_{\mathcal{F}} \longrightarrow 0.$$

The singular set $S(\mathcal{F})$ of the foliation $\mathcal{F}$ is defined by points in $M$, where the sheaf $\mathcal{N}_{\mathcal{F}}$ is not locally free, that is, $S(\mathcal{F}) := \text{Sing}(\mathcal{N}_{\mathcal{F}})$. Here, we suppose that Codim$S(\mathcal{F}) \geq 2$.

Now, we give a dual definition of singular foliation.

**Definition 1.4.2** *A singular holomorphic foliation $\mathcal{G}$, of codimension $k$, on $M$ is a coherent subsheaf of $\Omega_M$ of rank $k$, which satisfies the following integrability condition*

$$d\mathcal{G}_x \subset (\Omega_M \wedge \mathcal{G}_x) \quad \textit{for all } x \in M \backslash S(\mathcal{G}),$$

*where S($\mathcal{G}$) := sing($\Omega_M/\mathcal{G}$) is the singular set of $\mathcal{G}$.*

We consider only reduced foliations, then the two definitions 1.4.1 and 1.4.2 are equivalent by taking its annihilators, see [25 pg 178].

In case that $M = \mathbb{P}^n$ we have

**Proposition 1.4.3 (24, Proposition 4.1, pg 588)** *Let $\mathcal{F}$ be a holomorphic foliation of dimension $k$ on $\mathbb{P}^n$. Then $\mathcal{F}$ can be represented by a holomorphic section $s : \mathbb{P}^n \longrightarrow \wedge^{n-k} T^*\mathbb{P}^n \otimes \mathcal{O}(l)$ for some $l \in \mathbb{Z}$. In particular, in each affine coordinate domain $\mathbb{C}^n$, $\mathcal{F}$ can be represented by a polynomial $(n-k)$-form $\omega$.*

Given a holomorphic foliation $\mathcal{F}$ on the projective space $\mathbb{P}^n$, we can associate an integer number, denoted by $\deg(\mathcal{F}) = d$. The degree of the foliation. This number is defined as follows.

Choose a $(n-k)$-plane $H$ on the projective space $\mathbb{P}^n$. Set $\mathcal{F}_p$ the leaf of the foliation $\mathcal{F}$ through $p \in \mathbb{P}^n \backslash S(\mathcal{F})$. Now, the tangency set of $\mathcal{F}$ with $H$, denoted by $V(\mathcal{F}, H)$, is defined by the Zariski's closure of the tangency variety of $\mathcal{F}$ with $H$, $\mathcal{T}(\mathcal{F}, H) = \overline{\{p \in H / \dim(T_p\mathcal{F}_p \cap H) \geqslant 1\}}$.

**Definition 1.4.4** *The degree of $\mathcal{F}$ , denoted by $\deg(\mathcal{F})$, is defined by the degree of the tangency set $V(\mathcal{F}, H)$.*

This is well-defined and does not depend on the choice of the plane $H$, for details see [24]. It is possible to define the degree of a foliation in a more general case: in polarized projective varieties, for this see [10]. Now, note that, if $\mathcal{F}$ is a holomorphic foliation of one-dimension, then it is possible to represent it by a section $\sigma : \mathbb{P}^n \longrightarrow T\mathbb{P}^n \otimes \mathcal{O}(r)$, where in this case, the number $r$ is determined. By Proposition 1.4.3 $\mathcal{F}$ is given by section $s : \mathbb{P}^n \longrightarrow T^*\mathbb{P}^{n-1} \otimes \mathcal{O}(l)$, where locally it is represented by a polynomial $(n-1)$-form $\omega$. Then $s$ is a section such that it is represented locally by vector field $X$ that satisfies $i_X\omega = 0$. Moreover $r = \deg(\mathcal{F}) - 1$.

**Example 1.4.5** *Consider $\mathcal{F}$ an one-dimensional holomorphic foliation of degree 2 on $\mathbb{P}^3$ de-*

*fined locally by the following vector field*

$$X = (z_1^3 - z_2^2)\frac{\partial}{\partial z_1} + (z_1^3 z_2 - z_3^2)\frac{\partial}{\partial z_2} + (z_1^3 - z_1^2 z_3)\frac{\partial}{\partial z_3} \quad \textit{or}$$

$$X = z_1^2(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}) - z_2^2\frac{\partial}{\partial z_1} - z_3^2\frac{\partial}{\partial z_2}$$

*The annihilator of $X$ is the 2-form defined also locally by*

$$\omega = z_1^2 z_3 dz_1 \wedge dz_2 + (z_3^2 - z_1^2 z_2)dz_1 \wedge dz_3 + ({z_1}^3 - z_2^2)dz_2 \wedge dz_3$$

*i.e.,* $i_X\omega = 0$.

*On the other hand, consider the rational map $\varphi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ induced by the linear submersion*

$$\begin{array}{cccc} \varphi : & \mathbb{C}^5 & \longrightarrow & \mathbb{C}^4. \\ & (z_0, z_1, z_2, z_3, z_4) & \longmapsto & (z_1, z_2, z_3, z_4) \end{array}$$

*The pull-back of $\mathcal{F}$ to $\mathbb{P}^4$ by the rational map $\varphi$ is a two-dimensional foliation, denoted by $\mathcal{G} := \varphi^*\mathcal{F}$ whose singular set is $S(\mathcal{G}) = \{z_1 = z_2 = z_3 = 0\}$.*

**Example 1.4.6** *Another example of the foliation in this context is as follows: In particular $\varphi$ is a rational fibration for which the fiber at each point $p \in \mathbb{P}^3$ is the line in $\mathbb{P}^4$ through by $p$. Then, it induces an one-dimensional foliation $\mathcal{F}_1$ on $\mathbb{P}^4$, where these lines are the leaves of the foliation and the singular set of $\mathcal{F}_1$ is the degeneracy locus of $\varphi$, i.e., $S(\mathcal{F}_1) = \{[1:0:0:0:0]\}$. The last foliation has the particular property, that its leaves are contained in leaves of the foliation $\mathcal{G}$.*

# Chapter 2

# Characteristic classes of flags

In this chapter, we consider flags of holomorphic foliations on a complex manifold $M$ of dimension $n$. We study, in this context, a Baum-Bott type residue. Regular $\mathcal{C}^\infty$ flags were studied by Feigin in [14], where he proposed two constructions for characteristic classes of these flags in an attempt to answer a question about the obstruction for the existence of integrability foliations. Several authors studied characteristic classes, see [25, 27, 14]. R. Mol in [22] studied polar classes of flags of foliations.

Other motivation of for the study of flags is a conjecture due to Brunella: any two-dimensional holomorphic foliation $\mathcal{F}_2$ on $\mathbb{P}^3$ either admits an invariant algebraic surface or it is a flag of holomorphic foliations, i.e., $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, where in this last case $\mathcal{F}_1$ is a foliation by algebraic curves on $\mathbb{P}^3$.

## 2.1 Flags of holomorphic foliations

Let $M$ be a complex manifold of dimension $n$. Let us denote by $\Theta_M$ the tangent sheaf of $M$ and $\Omega_M$ the sheaf of germs of holomorphic 1-forms on $M$.

**Definition 2.1.1** *Let $\mathcal{F}_1, ..., \mathcal{F}_t$ be $t$ holomorphic foliations on $M$ of dimensions $q = (q_1, ..., q_t)$. We say that $\mathcal{F} := (\mathcal{F}_1, ..., \mathcal{F}_t)$ is a flag of holomorphic foliations if for each $i = 1, ..., t-1$, $\mathcal{F}_i$ is a coherent sub $\mathcal{O}_M$-module of $\mathcal{F}_{i+1}$. We call $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ a subfoliation of $\mathcal{F}_{i+1}$.*

In the above definition, we say that $\mathcal{F}_i$ leaves $\mathcal{F}_j (i < j)$ invariant for each $i = 1, ..., t-1$. Note that, for $x \in M \setminus \cup_{i=1}^{t} S(\mathcal{F}_i)$ the inclusion relation $T_x\mathcal{F}_1 \subset ... \subset T_x\mathcal{F}_t$ holds, giving that the leaves of $\mathcal{F}_i$ are contained in leaves of $\mathcal{F}_j$ for $i < j$. Here $T\mathcal{F}_i$ is the tangent sheaf of the foliation $\mathcal{F}_i$. For simplicity we denote $T\mathcal{F}$ by $\mathcal{F}$. When $t = 2$, we have a diagram of exact sequences of sheaves, as in studies of Feigin for the real case, see [9, pg 64].

$$
\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[dl] & & 0 \\
& \mathcal{F}_1 \arrow[dd] \arrow[dr] & & \mathcal{N}_2 \arrow[ul] \arrow[ur] & \\
& & \Theta_M \arrow[ur] \arrow[dr] & & \\
& \mathcal{F}_2 \arrow[ur] \arrow[dr] & & \mathcal{N}_1 \arrow[uu] \arrow[dr] & \\
0 \arrow[ur] & & \mathcal{N}_{1,2} \arrow[ur] \arrow[dr] & & 0 \\
& 0 \arrow[ur] & & 0 &
\end{tikzcd}
$$

We define the singular set $S(\mathcal{F})$ of the flag $\mathcal{F}$ to be the analytic set $S(\mathcal{F}_1) \cup ... \cup S(\mathcal{F}_t)$ and $\mathcal{N}_{\mathcal{F}} = \mathcal{N}_{1,2} \oplus ... \oplus \mathcal{N}_{t-1,t} \oplus \mathcal{N}_t$ be the normal sheaf of the flag, where $\mathcal{N}_{i,j}$ is the quotient sheaf $\mathcal{F}_i/\mathcal{F}_j (i < j)$.

**Example 2.1.2** *A meromorphic map $\varphi : X \dashrightarrow Y$, where $X$ and $Y$ are complex manifolds, is a first integral of a foliation $\mathcal{F}$ on $X$, if the leaves of $\mathcal{F}$ are contained in the fibers of $\varphi$. Then, in this situation, $\mathcal{F}$ is a subfoliation of the meromorphic fibration induced by $\varphi$.*

**Example 2.1.3** *Let $\mathcal{F}_2$ be a foliation on a polarized smooth projective variety $(X, H)$ satisfying $\mu(T\mathcal{F}_2) > 0$ (slope, for definition see [19], 2.2 pg 7). If $T\mathcal{F}_2$ is not semi-stable then there exists a semi-stable foliation $\mathcal{F}_1$ such that $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ is a 2-flag satisfying $\mu(TF_1) > \mu(TF_2)$, see [21].*

**Example 2.1.4** *Let $\pi : X \longrightarrow Y$ be a surjective holomorphic map, where $X$ and $Y$ are complex manifolds. Given a regular holomorphic foliation $\mathcal{G}$ of codimension one on $Y$ one has that $\mathcal{F}_2 := \pi^*\mathcal{G}$ is a codimension one foliation on $X$. We set $\mathcal{F}_1$ the foliation induced by $\pi$. Then, we have that $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ is a flag on $X$ with $S(\mathcal{F}_1) = S(\mathcal{F}_2) = \{singular \;\; set \;\; of \;\; \pi\}$.*

**Example 2.1.5** *Let $X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial z_i}$ be a holomorphic vector field on $\big(\mathbb{C}^n, (z_1, ..., z_n)\big)$. Then, $X$ is tangent to a 1-form $\omega = \sum_{i=1}^{n} g_i dz_i$ if and only if we have $0 = i_X\omega$.*

## 2.2 Bott vanishing theorem and residues for flags

In this section, we prove a residues theorem for flags of holomorphic foliations by applying the localization theory of characteristic classes developed by D. Lehmann and T. Suwa. We will start with the review of the Chern-Weil theory of characteristic classes of vector bundles. For details we refer to [18] and [25].

**Definition 2.2.1** *A connection for a complex vector bundle $E$ on $M$ is a $\mathbb{C}$-linear map*

$$\nabla : A^0(M, E) \longrightarrow A^1(M, E)$$

*such that*

$$\nabla(fs) = df \otimes s + f\nabla(s) \quad \textit{for} \quad f \in A^0(M) \quad \textit{and} \quad s \in A^0(M, E).$$

If $H$ is a subbundle of the complexified tangent bundle $T^cM$, then its dual $H^*$ is canonically viewed as a quotient of $(T^cM)^*$. We denote by $\rho$ the canonical projection $(T^cM)^* \longrightarrow H^*$.

**Definition 2.2.2** *A partial connection for $E$ is a pair $(H, \delta)$ of a subbundle $H$ of $T^cM$ and a $\mathbb{C}$-linear map*

$$\delta : A^0(M, E) \longrightarrow A^0(M, H^* \otimes E)$$

*such that*

$$\delta(fs) = \rho(df) \otimes s + f\delta(s) \quad \textit{for} \quad f \in A^0(M) \quad \textit{and} \quad s \in A^0(M, E).$$

**Definition 2.2.3** *Let $(H, \delta)$ be a partial connection for $E$. We say that a connection $\nabla$ for $E$ extends $(H, \delta)$ if the following diagram is commutative*

$$\begin{tikzcd} A^0(M,E) \arrow[r, "\nabla"] \arrow[dr, "\delta"'] & A^0(M, T^cM^* \otimes E) \arrow[d, "\rho\otimes Id"] \\ & A^0(M, H^* \otimes E) \end{tikzcd}$$

**Lemma 2.2.4 (25, Lemma 9.3, pg 75)** *For an arbitrary partial connection for $E$, there is a connection that extends it.*

An important class of partial connections comes from "actions" of involutive subbundles of tangent bundle of manifolds.

**Definition 2.2.5** *Let $F \subset TM$ be a regular foliation on $M$. An action of $F$ on a vector bundle $E$ is a $\mathbb{C}$-bilinear map:*

$$\alpha : A^0(M, E) \times A^0(M, F) \longrightarrow A^0(M, E)$$

*satisfying the following conditions for $f \in A^0(M)$ $u, v \in A^0(M, F)$ $s \in A^0(M, E)$:*

*1) $\alpha([u, v], s) = \alpha(u, \alpha(v, s)) - \alpha(v, \alpha(u, s))$ ;*

*2) $\alpha(f.u, s) = f.\alpha(u, s)$ ;*

*3) $\alpha(u, f.s) = u(f).s + f\alpha(u, s)$ ;*

*4) $\alpha(u, s)$ is holomorphic whenever $u$ and $s$ are.*

**Lemma 2.2.6 (25, Lemma 9.8, pg 76)** *Let $\alpha$ be an action of $F$ on $E$ and let*

$$\delta_\alpha : A^0(M, E) \longrightarrow A^0(M, F^* \otimes E) \simeq A^0(M, Hom(F, E))$$

*be defined by $\delta_\alpha(s, u) = \alpha(u, s)$. Then the pair $(F, \delta_\alpha)$ is a partial connection for $E$.*

**Definition 2.2.7** *Let $\alpha$ be an action of $F$ on $E$. A $F$-connection for $E$ is a connection which extends the partial connection $(F \oplus \overline{TM}, \delta_\alpha \oplus \overline{\partial})$.*

Now, we will use the Chern-Weil theory of characteristic classes, in order to describe the Bott vanishing Theorem for flags. This is a holomorphic version of the vanishing theorem due to Cordero-Masa, see [9, Theorem 3.9, pg 71].

**Theorem 2.2.8** *Let M be a complex manifold of dimension $n$ and $E = E_1 \oplus E_2$ a vector bundle on $M$ with $E_1$ a $F_1$-bundle, $E_2$ a $F_2$-bundle with $F_1 \subset F_2 \subset TM$ regular foliations. Let $\varphi_1$ and $\varphi_2$ be homogeneous symmetric polynomials, of degrees $d_1$ and $d_2$, such that at least one of the inequalities*

$$d_1 > n - \textit{rank}(F_1) \quad \textit{or} \quad d_2 > n - \textit{rank}(F_2) \quad \textit{or} \quad d_1 + d_2 > n - \textit{rank}(F_1) \tag{2.1}$$

*is satisfied. Then $\varphi_1(E_1) \smile \varphi_2(E_2) = 0$.*

**Proof:** Let us denote rank$(F_1) = p_1$, rank$(F_2) = p_2$, rank$(E_1) = r_1$ and rank$(E_2) = r_2$.

Let $\alpha_i : A^\circ(M, F_i) \times A^\circ(M, E_i) \to A^\circ(M, E_i)$ be an action of $F_i$ in $E_i$, for $i = 1, 2$ and $\bigtriangledown^i$ a $F_i$-connection for $E_i$.

Now, let $\{U, (z_1, ..., z_n)\}$ be a coordinate neighborhood on $M$ such that $F_1$ and $F_2$ can be written(spanned) by:

$$F_1 =< v_1, ..., v_{p_1} > \quad \text{and} \quad F_2 =< v_1, ..., v_{p_1}, ..., v_{p_2} >, \quad \text{where} \quad v_i = \frac{\partial}{\partial z_i}.$$

It follows from [25] that there exist holomorphic frames $S^1 = (s^1_1, ..., s^1_{r_1})$ of $E_1|_U$ and $S^2 = (s^2_1, ..., s^2_{r_2})$ of $E_2|_U$ such that

$$\alpha_1(v_i, s^1_\nu) = 0 \quad \text{for} \quad i = 1, ..., p_1 \quad \text{and} \quad \nu = 1, ..., r_1. \tag{2.2}$$

$$\alpha_2(v_i, s^2_\nu) = 0 \quad \text{for} \quad i = 1, ..., p_2 \quad \text{and} \quad \nu = 1, ..., r_2. \tag{2.3}$$

Now, let $\Theta^1 = (\Theta^1_{\nu\mu})$ and $\Theta^2 = (\Theta^2_{\nu\mu})$ be the connection matrices of $\nabla^1$ and $\nabla^2$, respectively, i.e;

$$\nabla^1(s^1_\nu) = \sum_{\mu=1}^{r_1} \Theta^1_{\nu\mu} s^1_\mu \quad \text{and} \quad \nabla^2(s^2_\nu) = \sum_{\mu=1}^{r_2} \Theta^2_{\nu\mu} s^2_\mu.$$

It follows form (2.2) and (2.3) that

$$\nabla^1(s^1_\nu)(v_i) = \alpha_1(v_i, s^1_\nu) = 0 \quad \text{and} \quad \nabla^2(s^2_\nu)(v_i) = \alpha_2(v_i, s^2_\nu) = 0.$$

Then we have $0 = i_{\frac{\partial}{\partial z_i}} \Theta^1_{\nu\mu}$ for all $i = 1, ..., p_1$ and $\nu, \mu = 1, ..., r_1$. It implies that each $\Theta^1_{\nu\mu}$ is of the form $\displaystyle\sum_{i=p_1+1}^{n} f_i^{\nu\mu} dz_i$ with $f_i^{\nu\mu} \in \mathcal{O}(U)$. In particular, the curvature matrix has the following property

$$K^1 = (K^1_{\nu\mu}) \quad \text{with} \quad K^1_{\nu\mu} = \sum_{i=p_1+1}^{n} \eta_i^{\nu\mu} dz_i, \quad \text{where} \quad \eta_i^{\nu\mu} \in \Omega^1(U).$$

Similarly $\Theta^2_{\nu\mu} = \displaystyle\sum_{i=p_2+1}^{n} g_i^{\nu\mu} dz_i$ and $K^2_{\nu\mu} = \displaystyle\sum_{i=p_2+1}^{n} \omega_i^{\nu\mu} dz_i$. Then $\varphi(E) = \varphi_1(E_1) \smile \varphi_2(E_2) = \varphi_1(K^1) \smile \varphi_2(K^2)$.

Therefore, if either $d_1 > n - p_1$ or $d_2 > n - p_2$ or $d_1 + d_2 > n - p_1$ then $\varphi(E) = 0$.

$\square$

Now, we prove a Baum-Bott type residues theorem for flags of singular holomorphic foliations.

**Theorem 2.2.9** *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a 2-flag of holomorphic foliations on a compact complex manifold $M$ of dimension $n$. Let $\varphi_1, \varphi_2$ be homogeneous symmetric polynomials , respectively of degrees $d_1$ and $d_2$, satisfying (2.1). Then for each compact connected component $S$ of $S(\mathcal{F})$ there exists a class $Res_{\varphi_1,\varphi_2}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}, S) \in H_{2n-2(d_1+d_2)}(S;\mathbb{C})$ such that*

$$\sum_{\lambda} (\iota_\lambda)_* Res_{\varphi_1,\varphi_2}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}, S_\lambda) = (\varphi_1(\mathcal{N}_{12}).\varphi_2(\mathcal{N}_2)) \frown [M] \quad in \quad H_{2n-2(d_1+d_2)}(M;\mathbb{C}), \tag{2.4}$$

*where $\iota_\lambda$ denotes the embedding of $S_\lambda$ on $M$.*

**Proof:** Note that away from the singular set of the flag, $\mathcal{F}_1$ and $\mathcal{F}_2$ are free sheaves. So there exist vector bundles $F_1^0$ and $F_2^0$ on $M \setminus S(\mathcal{F})$ such that $\mathcal{O}(F_1^0) = \mathcal{F}_1$ and $\mathcal{O}(F_2^0) = \mathcal{F}_2$. Denoting $M \setminus S(\mathcal{F})$ by $M^0$ we have that $F_i^0 \subset TM^0$ are subbundles for $i = 1, 2$. Also, let $N_{F_2^0} = TM^0/F_2^0$ and $N_{12} = F_2^0/F_1^0$, then $\mathcal{N}_2 = \mathcal{O}(N_{F_2^0})$ and $\mathcal{N}_{12} := \mathcal{F}_2/\mathcal{F}_1 = \mathcal{O}(N_{12})$.

The exact sequences

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \Theta_M \longrightarrow \mathcal{N}_2 \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{N}_{12} \longrightarrow 0$$

induce, respectively, actions $\alpha_2$ of $F_2^0$ on $N_{F_2^0}$ and $\alpha_1$ of $F_1^0$ on $N_{12}$, see [3, 25].

Now, denote by $\nabla_{12}$ the $F_1^0$-connection for $N_{12}$ and $\nabla_2$ the $F_2^0$-connection for $N_{F_2^0}$. Let $S$ be a compact connected component of $S(\mathcal{F})$ and $U$ a relatively compact open neighborhood of $S$ on $M$ disjoint from the other components of $S(\mathcal{F})$. We set $U_0 = U \setminus S$ and $U_1 = U$ and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of $U$. We take resolutions of the normal sheaves $\mathcal{N}_{12}$ and $\mathcal{N}_2$ by real analytic vector bundles $E_i^{12}$ and $E_j^2$ on $U$

$$0 \longrightarrow A_U(E_q^{12}) \longrightarrow ... \longrightarrow A_U(E_0^{12}) \longrightarrow A_U \otimes \mathcal{N}_{12} \longrightarrow 0.$$

$$0 \longrightarrow A_U(E_r^2) \longrightarrow ... \longrightarrow A_U(E_0^2) \longrightarrow A_U \otimes \mathcal{N}_2 \longrightarrow 0.$$

Since the characteristic class $\varphi_1(\mathcal{N}_{12})$ is the characteristic class $\varphi_1(\xi^{12})$ of the virtual bundle $\xi^{12} = \sum_{i=0}^{q}(-1)^i E_i^{12}$ and $\varphi_2(\mathcal{N}_2) = \varphi_2(\xi^2)$ for $\xi^2 = \sum_{i=0}^{r}(-1)^i E_i^2$, we define the characteristic class $\varphi(\mathcal{N}_\mathcal{F})$, of the normal sheaf of the flag by $\varphi_1(\mathcal{N}_{12}) \smile \varphi_2(\mathcal{N}_2)$. On $U_0$ we have the exact sequences of vector bundles

$$0 \longrightarrow E_q^{12} \longrightarrow ... \longrightarrow E_0^{12} \longrightarrow \mathcal{N}_{12} \longrightarrow 0. \tag{2.5}$$

$$0 \longrightarrow E_r^{2} \longrightarrow ... \longrightarrow E_0^{2} \longrightarrow \mathcal{N}_{2} \longrightarrow 0. \tag{2.6}$$

There exist connections ${}^{12}\nabla_0^i$ on $U_0$ for each $E_i^{12}$ such that the family of connections $({}^{12}\nabla_0^q, ..., {}^{12}\nabla_0^0, \nabla_1)$ is compatible with (2.5). Analogously, there exists connections ${}^{2}\nabla_0^i$ on $M$ for each $E_i^2$ with the same property, see [3]. We denote ${}^{12}\nabla_0^\bullet$ by $({}^{12}\nabla_0^{(q)}, ..., {}^{12}\nabla_0^{(0)})$ and ${}^{2}\nabla_0^\bullet$ by $({}^{2}\nabla_0^{(r)}, ..., {}^{2}\nabla_0^{(0)})$. Then it follows from [25, Proposition 8.4, pg 73] that

$$\varphi_1({}^{12}\nabla_0^\bullet) = \varphi_1(\nabla^1) \text{ and } \varphi_2({}^{2}\nabla_0^\bullet) = \varphi_2(\nabla^2). \tag{2.7}$$

On $U_1$ we take an arbitrary family ${}^{12}\nabla_1^\bullet = ({}^{12}\nabla_1^{(q)}, ..., {}^{12}\nabla_1^{(0)})$ of connections, where each ${}^{12}\nabla_1^{(i)}$ is a connection for $E_i^{12}$ on $U_1$. Similarly, we take other arbitrary family ${}^{2}\nabla_1^\bullet = ({}^{2}\nabla_1^{(r)}, ..., {}^{2}\nabla_1^{(0)})$. Then the class $\varphi(\mathcal{N}_\mathcal{F}) = \varphi_1(\mathcal{N}_{12}) \smile \varphi_2(\mathcal{N}_2) = \varphi_1(\xi^{12}) \smile \varphi_2(\xi^2)$ in $H^{2(d_1+d_2)}(U;\mathbb{C})$ is represented in $A^{2(d_1+d_2)}(U)$ by the cocycle

$$\varphi({}_2^{12}\nabla_*^\bullet) = \Big(\varphi_1({}^{12}\nabla_0^\bullet), \varphi_1({}^{12}\nabla_1^\bullet), \varphi_1({}^{12}\nabla_0^\bullet, {}^{12}\nabla_1^\bullet)\big) \smile \big(\varphi_2({}^{2}\nabla_0^\bullet), \varphi_2({}^{2}\nabla_1^\bullet), \varphi_2({}^{2}\nabla_0^\bullet, {}^{2}\nabla_1^\bullet)\big) =$$

$$= \big(\varphi_1({}^{12}\nabla_0^\bullet) \wedge \varphi_2({}^{2}\nabla_0^\bullet), \varphi_1({}^{12}\nabla_1^\bullet) \wedge \varphi_2({}^{2}\nabla_1^\bullet), \varphi_1({}^{12}\nabla_0^\bullet) \wedge \varphi_2({}^{2}\nabla_0^\bullet, {}^{2}\nabla_1^\bullet) + \varphi_1({}^{12}\nabla_0^\bullet, {}^{12}\nabla_1^\bullet) \wedge \varphi_2({}^{2}\nabla_1^\bullet)\Big).$$

Then by Bott vanishing Theorem for flags (Theorem 2.2.8), $\varphi({}_2^{12}\nabla_*^\bullet) \in A^{2(d_1+d_2)}(U, U_0)$. Denoting $[\varphi({}_2^{12}\nabla_*^\bullet)] = \varphi_S(\mathcal{N}_\mathcal{F}, \mathcal{F}) \in H^{2(d_1+d_2)}(U, U \backslash S; \mathbb{C})$ we have the residue $\mathrm{Res}_{\varphi_1,\varphi_2}(\mathcal{N}_\mathcal{F}, \mathcal{F}; S) = A(\varphi_S(\mathcal{N}_\mathcal{F}, \mathcal{F})) \in H_{2n-2(d_1+d_2)}(S; \mathbb{C})$, where $A$ is Alexander duality.

□

**Definition 2.2.10** *We call the class* $\mathrm{Res}_{\varphi_1,\varphi_2}(\mathcal{N}_\mathcal{F}, \mathcal{F}; S)$ *by the Baum-Bott residue for the flag* $\mathcal{F}$ *with respect to* $\varphi_1$ *and* $\varphi_2$.

**Example 2.2.11** *Let* $\mathbb{P}^n$ *be the complex projective space* $(n \geq 3)$ *with homogeneous coordinates* $[z_0 : ... : z_n]$. *Consider an one-dimensional holomorphic foliation* $\mathcal{F}_1$ *induced by the vector field* $X = \dfrac{\partial}{\partial z_3}$. *Consider the codimension one holomorphic foliation, denoted by* $\mathcal{F}_2$

*induced by 1-form* $\omega = z_0 dz_1 - z_1 dz_0$. *Note that* $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ *in fact is a flag, since* $\omega(X) = 0$ .

*The singular set of* $\mathcal{F}_1$ *is the set of dependence of vector field* $X$ *with the radial vector field* $R = \sum_0^n z_i \frac{\partial}{\partial z_i}$, *i.e.,*

$$S(\mathcal{F}_1) = \Big\{ p \in \mathbb{P}^n;\ \frac{\partial}{\partial z_3} \wedge \sum z_i \frac{\partial}{\partial z_i} = 0 \Big\} = \Big\{ p = [0:0:0:1:0:...:0] \Big\}.$$

*On the other hand, the singular set of* $\mathcal{F}_2$ *is given by* $S(\mathcal{F}_2) = S = \{z_0 = z_1 = 0\}$. *We remark that* $S(\mathcal{F}_1) \subset S(\mathcal{F}_2)$. *Therefore,* $S$ *is the singular set of the flag* $\mathcal{F}$. *Now, we calculate the residues of this flag.*

*We have the following*

$\deg(\mathcal{F}_2) = \deg(\mathcal{F}_1) = 0$, $\quad \mathcal{F}_1 = \mathcal{O}(1)$, *then* $\quad c_1(\mathcal{F}_1) = c_1(\mathcal{O}(1)) = 1h$, *where* $h$ *is the hyperplane class.*

*We know that* $c_1(\mathcal{N}_2) = (2 + \deg(\mathcal{F}_2))h = 2h$ *and from the exact sequence*

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow T\mathbb{P}^n \longrightarrow \mathcal{N}_2 \longrightarrow 0$$

*we have* $c_1(\mathcal{F}_2) = (\dim(\mathcal{F}_2) - \deg(\mathcal{F}_2))h = (n-1)h$. *Then*

$$c_1(\mathcal{N}_{12}) = c_1(\mathcal{F}_2) - c_1(\mathcal{F}_1) = (n-1)h - 1h = (n-2)h.$$

*By Theorem 2.2.9 (Baum-Bott for flags) one has for each* $j = 0, ..., n-1$

$$\mathit{Res}_{c_1^{n-1-j} c_1^{1+j}}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; S) = \int_{\mathbb{P}^n} c_1^{n-1-j}(\mathcal{N}_{12}) c_1^{1+j}(\mathcal{N}_2) = \int_{\mathbb{P}^n} (n-2)^{n-1-j} 2^{1+j} h^n$$

$$= (n-2)^{n-1-j} 2^{1+j}.$$

**Remark 2.2.12** *Note that the Theorem 2.2.8 is legitime of the flag and more "fine" than Bott vanishing Theorem, see condition (2.1). Observe that, with this theorem we can compute the classes:*

$$\varphi(\mathcal{N}_{\mathcal{F}}) = \varphi_1(\mathcal{N}_{12}) \varphi_1(\mathcal{N}_2)$$

*with $d_i \leq codim(F_i)$ for $i = 1, 2$ but with $d_1 + d_2 > codim(\mathcal{F}_1)$. An important fact is that for these polynomials it is not possible to apply the classical Bott vanishing Theorem. Then in this case, the residue $Res_{\varphi_1,\varphi_2}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}, S)$ is a specific residue of the flag.*

**Remark 2.2.13** *Observe that if we consider $\varphi_1$ = "constant polynomial" then the residue $Res_{\varphi_1,\varphi_2}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}, S)$ is exactly the residue of $\mathcal{F}_2$. But it is not clear in general the relationship between flag residue and foliation residue involved in the flag. We will do this in the section 2.4, see Corollary 2.4.3.*

Now, we study a refinement of Theorem 2.2.9. It is because for some polynomials we can detect superfluous components, i.e., components that do not participate of the sum in (2.4).

**Theorem 2.2.14** *The characteristic class $\varphi(\mathcal{N}_{\mathcal{F}}) = \varphi_1(\mathcal{N}_{12}).\varphi_2(\mathcal{N}_2)$ is localized at the intersection $S := S(\mathcal{F}_1) \cap S(\mathcal{F}_2)$ if $d_1 > codim\mathcal{F}_1$* **and** *$d_2 > codim\mathcal{F}_2$.*

**Proof:** Consider $S(\mathcal{F}) = S \cup \overline{S}(\mathcal{F}_1) \cup \overline{S}(\mathcal{F}_2)$, where $\overline{S}(\mathcal{F}_i)$ are irreducible components only of $\mathcal{F}_i$ and $U_1$ a neighborhood of $S(\mathcal{F})$. We set $U_0 := U_1 \backslash S := U_0^1 \cup U_0^2$, where $U_0^1 := U_1 \backslash S \cup \overline{S}(\mathcal{F}_2)$ represents a neighborhood of the components only of $\mathcal{F}_1$ and $U_0^2$ is defined in the same way. Then the characteristic class $\varphi(\mathcal{N}_{\mathcal{F}}) \in H^{2(d_1+d_2)}(M; \mathbb{C})$ is represented by the cocycle

$$\varphi({}_2^{12}\nabla_*^\bullet) = \Big(\varphi_1({}^{12}\nabla_0^\bullet) \wedge \varphi_2({}^{2}\nabla_0^\bullet), \varphi_1({}^{12}\nabla_1^\bullet) \wedge \varphi_2({}^{2}\nabla_1^\bullet), \varphi_1({}^{12}\nabla_0^\bullet) \wedge \varphi_2({}^{2}\nabla_0^\bullet, {}^{2}\nabla_1^\bullet) +$$
$$+\varphi_1({}^{12}\nabla_0^\bullet, {}^{12}\nabla_1^\bullet) \wedge \varphi_2({}^{2}\nabla_1^\bullet)\Big).$$

We claim that $\varphi_1({}^{12}\nabla_0^\bullet) \wedge \varphi_2({}^{2}\nabla_0^\bullet) = \varphi({}_2^{12}\nabla_*^\bullet)|_{U_0} = 0$. In fact as $U_0 := U_0^1 \cup U_0^2$, we can represent this form in Čech-de-Rham cohomology in the open $U_0$ with covering $\{U_0^1, U_0^2\}$

$$\varphi_1({}^{12}\nabla_0^\bullet) \wedge \varphi_2({}^{2}\nabla_0^\bullet) = \Big(\varphi_1({}^{12}\nabla_0^\bullet) \wedge \varphi_2({}^{2}\nabla_0^\bullet)|_{U_0^1}, \varphi_1({}^{12}\nabla_0^\bullet) \wedge \varphi_2({}^{2}\nabla_0^\bullet)|_{U_0^2}, \varphi_1({}^{12}\nabla_0^\bullet) \wedge \varphi_2({}^{2}\nabla_0^\bullet)|_{U_0^1 \cap U_0^2}\Big)$$

Finally, we can see that

$\varphi_2({}^{2}\nabla_0^\bullet)|_{U_0^1} = 0$, $\varphi_1({}^{12}\nabla_0^\bullet)|_{U_0^2} = 0$ and $\varphi_2({}^{2}\nabla_0^\bullet) = \varphi_1({}^{12}\nabla_0^\bullet)|_{U_0^1 \cap U_0^2} = 0$ by Bott vanishing theorem. Now, the remainder is as in the proof of the Theorem 2.2.9.

□

**Corollary 2.2.15** *Given a 2-flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ on $\mathbb{P}^n$ with $n \geq 5$ such that $codim(\mathcal{F}_1) + codim(\mathcal{F}_2) < n - 1$ . If $S(\mathcal{F}_1) \cap S(\mathcal{F}_2) = \emptyset$ then $\dim(\mathcal{F}_2) + \deg(F_1) = \dim(\mathcal{F}_1) + \deg(F_2)$.*

**Proof:** Consider $\varphi_1 := c_1^{d_1}$ and $\varphi_2 := c_1^{d_2}$ polynomials, where $d_1 = codim(\mathcal{F}_1) + r$ and $d_2 = codim(\mathcal{F}_2) + 1$, for any $r \in \mathbb{Z}_+$ such that $d_1 + d_2 = n$. Note that, this is possible since $codim(\mathcal{F}_1) + codim(\mathcal{F}_2) < n - 1$. Then, by Theorem 2.2.14 we have that

$$\int_{\mathbb{P}^n} c_1^{d_1}(\mathcal{N}_{12}) c_1^{d_2}(\mathcal{N}_2) = \sum_{S \in S(\mathcal{F}_1) \cap S(\mathcal{F}_2)} Res_{c_1^{d_1}, c_1^{d_2}}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; S). \tag{2.8}$$

On the other hand

$c_1(\mathcal{F}_1) = (\dim(\mathcal{F}_1) - \deg(\mathcal{F}_1))h$ and $c_1(\mathcal{F}_2) = (\dim(\mathcal{F}_2) - \deg(\mathcal{F}_2))h$, where $h$ is the hyperplane class. Then, by exact sequence

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \Theta_{\mathbb{P}^n} \longrightarrow \mathcal{N}_2 \longrightarrow 0$$

we have

$$c_1(\mathcal{N}_2) = (n+1)h - (\dim(\mathcal{F}_2) - \deg(\mathcal{F}_2))h, \quad \text{with} \quad n + 1 - (\dim(\mathcal{F}_2) - \deg(\mathcal{F}_2)) \neq 0$$

$$c_1(\mathcal{N}_{12}) = c_1(\mathcal{F}_2) - c_1(\mathcal{F}_1) = \Big(\dim(\mathcal{F}_2) - \deg(\mathcal{F}_2) - \dim(\mathcal{F}_1) + \deg(\mathcal{F}_1)\Big)h$$

Now, by equation (2.8) and hypothesis $S(\mathcal{F}_1) \cap S(\mathcal{F}_2) = \emptyset$, we have the result $\Big(\dim(\mathcal{F}_2) - \deg(\mathcal{F}_2) - \dim(\mathcal{F}_1) + \deg(\mathcal{F}_1)\Big) = 0$.

□

Now, we quote the following conjecture

**Rationality conjecture for flags 2.2.16** *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a 2-flag of holomorphic foliations on a complex manifold $M$. Also let $S$ be a compact connected component of the singular set of the flag and $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_i$ is a homogeneous symmetric polynomial of degree $d_i$ satisfying (2.1). If $\varphi_i$ is with rational coefficients, then*

$$Res_{\varphi_1, \varphi_2}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}, S) \in H_{2n-2(d_1+d_2)}(S; \mathbb{Q}).$$

Next section, we will give a partial answer for this conjecture.

## 2.3 Residues formula

In this part of the work we will show a formula that calculates some residues of a flag $\mathcal{F}$. Naturally appears, as a consequence, a relationship between flag residue and residues of the involved foliations. For a basic reference, see [11, 22, 13, 6, 18].

Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a flag on a compact complex manifold $M$ of dimension $n$. We denote by $(k_1, k_2)$ the codimension of this flag and by $\mathrm{Sing}_{k_i+1}(\mathcal{F}_i)$ the set of irreducible components of $S(\mathcal{F}_i)$ of pure codimension $k_i + 1$.

Let us fix some notation: Let $S(\mathcal{F}_i) := \mathrm{Sing}(\mathcal{N}_i)$ be the singular set of the foliation $\mathcal{F}_i$. Recall that the singular set of flag is defined by $S(\mathcal{F}) := S(\mathcal{F}_1) \cup S(\mathcal{F}_2)$ and the relative normal sheaf by $\mathcal{N}_{12} := \mathcal{F}_2/\mathcal{F}_1$.

**Proposition 2.3.1** *Given a 2-flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ on a complex manifold $M$. On $M_0 := M \backslash S(\mathcal{F}_2)$ we have Sing$(\mathcal{N}_1) \cap M_0 =$ Sing$(\mathcal{N}_{12}) \cap M_0$.*

**Proof:** We recall the exact sequence

$$0 \longrightarrow \mathcal{N}_{12} \longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{N}_2 \longrightarrow 0. \tag{2.9}$$

Away from the singular set of $\mathcal{F}_2$, i.e., for $p \in M \setminus S(\mathcal{F}_2)$ one has that the stalk at $p$ $\mathcal{N}_{2,p}$ is $\mathcal{O}_{M,p}$ - free. The sequence (2.9) induces the exact sequence of $\mathcal{O}_{M,p}$-modules

$$0 \longrightarrow \mathcal{N}_{12,p} \longrightarrow \mathcal{N}_{1,p} \longrightarrow \mathcal{N}_{2,p} \longrightarrow 0. \tag{2.10}$$

Since $\mathcal{N}_{2,p}$ is a free module it implies that, by the splitting lemma see [15, pg 147], the sequence (2.10) splits (here $\mathcal{O}_{M,p}$ is a local ring, then projective and free modules are equivalent):

$$\mathcal{N}_{1,p} = \mathcal{N}_{12,p} \oplus \mathcal{N}_{2,p}$$

in which the module $\mathcal{N}_{1,p}$ is free, if and only if, $\mathcal{N}_{12,p}$ is free.

$\square$

**Corollary 2.3.2** *If the sheaf $\mathcal{N}_{12} = \dfrac{\mathcal{F}_2}{\mathcal{F}_1}$ is locally-free then we have $S(\mathcal{F}_1) \subset S(\mathcal{F}_2)$.*

**Proof:** Apply the Proposition 2.3.1.

□

**Example 2.3.3** *Let $\mathcal{F}$ be the foliation in $\mathbb{P}^3$ induced by the polynomial vector field*

$$X = \lambda_1 z_1 \frac{\partial}{\partial z_1} + \lambda_2 z_2 \frac{\partial}{\partial z_2} + \lambda_3 z_3 \frac{\partial}{\partial z_3} \quad with \quad \lambda_i \neq 0 \quad for \quad all \quad i.$$

*Consider the osculating planes distribution $\mathcal{F}_2$ associate to $X$, generated by $X$ and $Y := DX.X$. It is integrable and also given by the logarithmic 1-form*

$$\omega = \frac{\lambda_3 - \lambda_2}{\lambda_1}\frac{dz_1}{z_1} + \frac{\lambda_1 - \lambda_3}{\lambda_2}\frac{dz_2}{z_2} + \frac{\lambda_2 - \lambda_1}{\lambda_3}\frac{dz_3}{z_3}.$$

*We have that, in fact, $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ is a flag, since a simple calculation shows that $\omega(X) = 0$. For this we have the following*

$$S(\mathcal{F}_1) = \Big\{[1:0:0:0], [0:1:0:0], [0:0:1:0], [0:0:0:1]\Big\}.$$

$$S(\mathcal{F}_2) = S = \bigcup S_{ij} \quad for \quad i = 0,1,2, \ j = 1,2,3 \quad and \quad i \neq j,$$

*where $S_{ij} := \{z_i = z_j = 0\}$.*

*We observe that $S(\mathcal{F}_1) \subset S(\mathcal{F}_2)$ and that the relative normal sheaf $\mathcal{N}_{12} := \mathcal{F}_2/\mathcal{F}_1$ is locally-free, since $\mathcal{F}_1 = \mathcal{O}_{\mathbb{P}^3} \subset \mathcal{F}_2 = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}$ .*

**Example 2.3.4** *Let $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the rational map given in homogeneous coordinates by $[z_0 : z_1 : z_2 : z_3] \longmapsto [z_0 : z_1 : z_2]$. This is a rational fibration which induces an one-dimensional foliation on $\mathbb{P}^3$, we call it $\mathcal{F}_1$. The singular set of $\mathcal{F}_1$ is $S(\mathcal{F}_1) = \{[0:0:0:1]\}$.*

*On the other hand, let $\mathcal{G}$ be a codimension one foliation on $\mathbb{P}^2$ of degree $d$ with singular set given by $S(\mathcal{G}) = \{p_1, ..., p_l\}$. Now, consider the pull-back of $\mathcal{G}$ by $\pi$ and denote it by $\mathcal{F}_2 = \pi^*\mathcal{G}$. We have that $S(\mathcal{F}_2) = \bigcup\limits_{p_i \in S(\mathcal{G})} \pi^{-1}(p_i)$.*

*Note that, we have $\mathcal{F}_1 = \mathcal{O}_{\mathbb{P}^3}(1)$ since the degree of $\mathcal{F}_1$ is 0 and $\mathcal{G} = \mathcal{O}_{\mathbb{P}^2}(1-d)$, then $\mathcal{F}_2 = \mathcal{O}_{\mathbb{P}^3}(1-d) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$. Then, the relative sheaf $\mathcal{N}_{12}$ is $\mathcal{O}_{\mathbb{P}^3}(1-d)$, in particular it is locally free. Moreover one has $S(\mathcal{F}_1) \subset S(\mathcal{F}_2)$.*

**Proposition 2.3.5** *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a flag on a complex manifold $M$ with dim$(\mathcal{F}_1) =$ codim$(\mathcal{F}_2) = 1$. Then $\mathcal{F}_1$ has no isolated singularities in $M \diagdown S(\mathcal{F}_2)$.*

**Proof:** The situation is local. Suppose that $p$ is an isolated singularity of $\mathcal{F}_1$ and pick a neighborhood $\{U, (z_1, ..., z_n)\}$ of $p$, where $\mathcal{F}_2|_U$ is regular. On this open subset we can consider $\mathcal{F}_2$ induced by the 1-form $\omega = dz_1$ and $\mathcal{F}_1$ by the vector field $X = \sum_{i=1}^{n} f_i dz_i$.

Since that $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ is a flag, we have

$$0 = \iota_X \omega = f_1.$$

But this show that $S(\mathcal{F}_1)|_U = \{f_2 = ... = f_n = 0\}$ is not an isolated singularity.

□

**Corollary 2.3.6** *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a flag on a complex manifold $M$ with $\dim(\mathcal{F}_1) = codim(\mathcal{F}_2) = 1$ . If $S_0(\mathcal{F}_i)$ denotes the isolated singularities of the foliation $\mathcal{F}_i$, for $i = 1, 2$, we have that $S_0(\mathcal{F}_1) = S_0(\mathcal{F}_2)$.*

**Proof:** See Proposition 2.3.5 and [22, Corollary 1, pg 778].

□

**Proposition 2.3.7** *For a flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ on $M$ with $\dim(\mathcal{F}_1) = codim(\mathcal{F}_2) = 1$ and $S(\mathcal{F}_1) \cap S(\mathcal{F}_2)$ admitting isolated singularities (only) we have*

$$Res_{c_n}(\mathcal{F}_2, \mathcal{N}_2, p) = (-1)^n (n-1)! Res_{c_n}(\mathcal{F}_1, \mathcal{N}_1, p),$$

*where the residues involved are of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$.*

**Proof:** Let $p \in S(\mathcal{F}_1) \cap S(\mathcal{F}_2)$ be an isolated singulary. We know that near $p$ we can consider $\mathcal{F}_1$ as induced by a vector field $X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial z_i}$ and $\mathcal{F}_2$ by a 1-form $\eta = \sum_{i=1}^{n} g_i dz_i$. Then, $\mathrm{Res}_{c_n}(\mathcal{F}_1, \mathcal{N}_1; p) = \mu(f; p)$ is the Milnor number of $f = (f_1, ..., f_n)$ at $p$. On the other hand, we have $\mathrm{Res}_{c_n}(\mathcal{F}_2, \mathcal{N}_2; p) = (-1)^n (n-1)! \mu(g; p)$, where $g = (g_1, ..., g_n)$ with $n = \dim_{\mathbb{C}} M$, see Suwa [26, Proposition 3.12, pg 41]. Since $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ is a flag we have

$$0 = \iota_X \eta = \sum f_i g_i = 0. \tag{2.11}$$

We claim that $(f_1, ..., f_n) = (g_1, ..., g_n)$ as generated ideals.

In fact, consider the exact Koszul complex of regular sequence $(f_1, ..., f_n)$,

$$0 \longrightarrow \bigwedge^n \mathcal{O}^n \longrightarrow \cdots \longrightarrow \bigwedge^2 \mathcal{O}^n \xrightarrow{r} \mathcal{O}^n \xrightarrow{s} \mathcal{O} \longrightarrow 0,$$

where $r(e_i \wedge e_j) = f_i e_j - f_j e_i$ and $s(e_i) = f_i$. From (2.11) one has that $(g_1, ..., g_n) \in \mathrm{Ker}(s) = \mathrm{Im}(r)$, then

$$r(\sum P_{ij} e_i \wedge e_j) = \sum P_{ij}(f_i e_j - f_j e_i) = \sum g_i e_i.$$

This implies that $(g_1, ..., g_n) \subset (f_1, ..., f_n)$. If we consider the Koszul complex of $(g_1, ..., g_n)$ we have the equality of ideals.

Therefore $\mu(f; p) = \mu(g; p)$ and $\mathrm{Res}_{c_n}(\mathcal{F}_2, \mathcal{N}_2; p) = (-1)^n (n-1)! \mathrm{Res}_{c_n}(\mathcal{F}_1, \mathcal{N}_1; p)$.

□

The next example is inspired by the example of Izawa in [29, Example 5, pg 907].

**Example 2.3.8** *Let $Y := \mathbb{P}^5 \times \mathbb{P}^1$ with homogeneous coordinates $\Big([x_0 : x_1 : x_2 : x_3 : x_4 : x_5]; [y_0 : y_1]\Big)$. We consider a regular foliation on $Y$ given by $\widetilde{\mathcal{G}} := \pi^{-1}\Omega_{\mathbb{P}^1}$, where $\pi$ is the standard projection of $\mathbb{P}^5 \times \mathbb{P}^1$ in $\mathbb{P}^1$. Let*

$$X := V(x_0^l + x_1^l + x_2^l + x_3^l + x_4^l + x_5^l) \cap V(x_0 y_0 + x_1 y_1) \quad l \in \mathbb{Z}_+.$$

*This is a regular sub-manifold of $Y$. We consider the inclusion map $i : X \longrightarrow Y$. Put $\mathcal{F}_2 = i^{-1}\widetilde{\mathcal{G}}$, the inverse image of $\widetilde{\mathcal{G}}$, which defines a singular foliation of codimension one on $X$. In this case, the non-transversal locus of $i$ to $\widetilde{\mathcal{G}}$ determines $S(\mathcal{F}_2)$, the singular set of the foliation $\mathcal{F}_2$. To see the non-transversal points, we take the inhomogeneous coordinates over $x_0 \neq 0$ and $y_0 \neq 0$ as $(s, x, y, w, t) = (\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0}, \frac{x_5}{x_0})$ and $z = (\frac{y_1}{y_0})$. With these coordinates we can express, locally, $X$ by*

$$X = \{(s, x, y, w, t; z); 1 + x^l + y^l + w^l + t^l = 0 \text{ and } 1 + sz = 0\}.$$

*With this we have that $z = -(-1)^{\frac{-1}{l}}(1 + x^l + y^l + w^l + t^l)^{\frac{-1}{l}}$. We know that $\mathcal{F}_2$ is given by the 1-form $\omega = dz$, i.e.,*

$$\omega = dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial w}dw + \frac{\partial z}{\partial t}dt.$$

*Here, we use the following notation for coordinates of the 1-form that induces $\mathcal{F}_2$*

$$\varphi_1 = \frac{\partial z}{\partial x} = (-1)^{\frac{-1}{l}} x^{l-1}(1 + x^l + y^l + w^l + t^l)^{\frac{-l-1}{2}}$$

$$\varphi_2 = \frac{\partial z}{\partial y} = (-1)^{\frac{-1}{l}} y^{l-1}(1 + x^l + y^l + w^l + t^l)^{\frac{-l-1}{2}}$$

$$\varphi_3 = \frac{\partial z}{\partial w} = (-1)^{\frac{-1}{l}} w^{l-1}(1 + x^l + y^l + w^l + t^l)^{\frac{-l-1}{2}}$$

$$\varphi_4 = \frac{\partial z}{\partial t} = (-1)^{\frac{-1}{l}} t^{l-1}(1 + x^l + y^l + w^l + t^l)^{\frac{-l-1}{2}}.$$

*Since the $z$-axis is a transversal direction for the leaves of $\mathcal{F}_2$, the non-transversal conditions are given by $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0$ such that $(x, y, w, t) = (0, 0, 0, 0)$. Then, with the defining equations, we see that the non-transversal points are given by*

$$(s, x, y, w, t; z) = (\omega_k, 0, 0, 0, 0; -\omega_{l-k-1})_{k=0,\ldots,l-1},$$

*where we denote by $\omega_k$ the $l$-roots of $-1$. Therefore, the singular set of $\mathcal{F}_2$ is given by these points. Consider the one-dimensional foliation on $X$, denoted by $\mathcal{F}_1$, given locally by the following vector field $X = (X_1, X_2, X_3, X_4)$, where*

$$X_1 = (-1)^{\frac{-1}{l}}(-y^{l-1})(1 + x^l + y^l + w^l + t^l)^{\frac{-l-1}{2}} = -\varphi_2$$

$$X_2 = (-1)^{\frac{-1}{l}} x^{l-1}(1 + x^l + y^l + w^l + t^l)^{\frac{-l-1}{2}} = \varphi_1$$

$$X_3 = (-1)^{\frac{-1}{l}}(-t^{l-1})(1 + x^l + y^l + w^l + t^l)^{\frac{-l-1}{2}} = -\varphi_4$$

$$X_4 = (-1)^{\frac{-1}{l}} w^{l-1}(1 + x^l + y^l + w^l + t^l)^{\frac{-l-1}{2}} = \varphi_3.$$

*Note that $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ is in fact a flag, since the following holds*

$$i_X\omega = \omega(X) = X_1\varphi_1 + X_2\varphi_2 + X_3\varphi_3 + X_4\varphi_4 = 0.$$

*Observe that* $S(\mathcal{F}_1) = S(\mathcal{F}_2)$. *Now, using the local coordinates of the vector field and the 1-form as above, we have for each* $p \in S(\mathcal{F}_2)$

$$\textit{Res}_{c_4}(\mathcal{F}_2, \mathcal{N}_2; p) = (-1)^4 3! (\frac{1}{2\pi i})^4 \int_T \frac{d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 \wedge d\varphi_4}{\varphi_1.\varphi_2.\varphi_3.\varphi_4}$$

$$= \int_T \Big( (l-1)^2 + (l^2-1) \frac{x^l + y^l + w^l + t^l}{1 + x^l + y^l + w^l + t^l} \Big) \frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dw}{w} \wedge \frac{dt}{t} = 6.(l-1)^2,$$

*where* $T$ *is given by* $\{|x| = |y| = |w| = |t| = \epsilon\}$. *On the other hand, as we have*

$$\frac{dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4}{X_1.X_2.X_3.X_4} = \frac{d(-\varphi_2) \wedge d(\varphi_1) \wedge d(-\varphi_4) \wedge d(\varphi_3)}{(-\varphi_2)\varphi_1(-\varphi_4)\varphi_3} = \frac{d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 \wedge d\varphi_4}{\varphi_1.\varphi_2.\varphi_3.\varphi_4}.$$

*It follow that*

$$\textit{Res}_{c_4}(\mathcal{F}_1, \mathcal{N}_1; p) = (\frac{1}{2\pi i})^4 \int_T \frac{dX_1 \wedge dX_2 \wedge dX_3 \wedge dX_4}{X_1.X_2.X_3.X_4} = (l-1)^2.$$

*Therefore, we have*

$$Res_{c_4}(\mathcal{F}_2, \mathcal{N}_2; p) = (-1)^4 3!(l-1)^2 = (-1)^4 3! \textit{Res}_{c_4}(\mathcal{F}_1, \mathcal{N}_1; p).$$

## 2.4 Determination of certain Baum-Bott residues for flags

In this section, we will consider the Baum-Bott theorem for a flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ with codimension $(k_1, k_2)$. We will denote by $\text{Sing}_{k_i+1}(\mathcal{F}_i)$ the union of irreducible components of $S(\mathcal{F}_i)$ of pure codimension $k_i + 1$ for $i = 1, 2$. Next, we will show that the characteristic classes $c_1^{k_1-j+1}(\mathcal{N}_{12}) c_1^j(\mathcal{N}_2)$ can be localized at $\text{Sing}_{k_1+1}(\mathcal{F}_1)$. We consider the following notation

$$S_*(\mathcal{F}) := \text{Sing}_{k_1+1}(\mathcal{F}_1) \cup \text{Sing}_{k_2+1}(\mathcal{F}_2), \quad M^0 := M \setminus S(\mathcal{F}) \ \text{ e } \ M^* := M \setminus S_*(\mathcal{F}).$$

In the regular case (on $M^0$) there exist locally forms $\omega_\alpha^2$ and $\omega_\alpha^{12}$, where $\omega_\alpha^2$ is a $k_2$-form that induces $\mathcal{F}_2$ and $\omega_\alpha^{12}$ is a $(k_1 - k_2)$-form such that $\omega_\alpha^1 := \omega_\alpha^2 \wedge \omega_\alpha^{12}$ induces $\mathcal{F}_1$ satisfying the following two conditions

1) These forms are decomposable

$$\omega^2_\alpha = \eta^\alpha_1 \wedge ... \wedge \eta^\alpha_{k_2} \text{ and } \omega^{12}_\alpha = \eta^\alpha_{k_2+1} \wedge ... \wedge \eta^\alpha_{k_1}.$$

2) Integrability condition: There are matrices of 1-forms $(\theta^\alpha_{uv}), (\theta^\alpha_{av})$ and $(\theta^\alpha_{ab})$ with $1 \leq u, v \leq k_2$ and $k_2 + 1 \leq a, b \leq k_1$ such that

$$d\eta^\alpha_u = \sum_{v=1}^{k_2} \theta^\alpha_{uv} \wedge \eta^\alpha_v \text{ and } d\eta^\alpha_a = \sum_{v=1}^{k_2} \theta^\alpha_{av} \wedge \eta^\alpha_v + \sum_{b=k_2+1}^{k_1} \theta^\alpha_{ab} \wedge \eta^\alpha_b.$$

We define $\theta^2_\alpha = \sum_{u=1}^{k_2} (-1)^{u+1}\theta^\alpha_{uu}$, $\theta^{12}_\alpha = \sum_{a=k_2+1}^{k_1} (-1)^{a+1}\theta^\alpha_{aa}$ and put $\theta^1_\alpha := \theta^2_\alpha + \theta^{12}_\alpha$.

We define $\gamma^2_{\alpha\beta} := dg^2_{\alpha\beta}/g^2_{\alpha\beta} - \theta^2_\beta + \theta^2_\alpha$ and $\gamma^{12}_{\alpha\beta} := dg^{12}_{\alpha\beta}/g^{12}_{\alpha\beta} - \theta^{12}_\beta + \theta^{12}_\alpha$, where $\omega^2_\alpha = g^2_{\alpha\beta}\omega^2_\beta$, $\omega^1_\alpha = g^1_{\alpha\beta}\omega^1_\beta$ with $g^{12}_{\alpha\beta} := g^1_{\alpha\beta}/g^2_{\alpha\beta}$. The cocycle of 1-forms $\{\gamma^{12}_{\alpha\beta}\}$ corresponds to a cohomology class in $H^1(M^0, \mathcal{N}^*_{12})$. Analogously the cocycle $\{\gamma^2_{\alpha\beta}\}$ corresponds to a class in $H^1(M^0, \mathcal{N}^*_2)$.

We will consider now the Baum-Bott theorem for flags. For this we consider the local generators as above $\omega_2 = \eta_1 \wedge ... \wedge \eta_{k_2}$ and $\omega_{12} = \eta_{k_2+1} \wedge ... \wedge \eta_{k_1}$ with $\omega_1 = \omega_2 \wedge \omega_{12}$. Take smooth sections of $\mathcal{N}^*_{12}$ and $\mathcal{N}^*_2$ instead of holomorphic ones. Then, the cohomology groups $H^1(B^*_p, \mathcal{N}^*_{12})$ and $H^1(B^*_p, \mathcal{N}^*_2)$ are trivial. It is possible to find matrices of (1,0)-forms $(\theta_{uv}), (\theta_{av})$ and $(\theta_{ab})$ such that

$$d\eta_u = \sum \theta_{uv} \wedge \eta_v \text{ and } d\eta_a = \sum \theta_{av} \wedge \eta_v + \sum \theta_{ab} \wedge \eta_b.$$

We define $\theta^2 = \sum(-1)^{u+1}\theta_{uu}$ and $\theta^{12} = \sum(-1)^{a+1}\theta_{aa}$. Now, observe that the following forms for $0 \leq j \leq k_2$

$$\psi_j := (2\pi i)^{-k_1-1}\theta^{12} \wedge (d\theta^2)^j \wedge (d\theta^{12})^{k_1-j}$$

$$\varphi := (2\pi i)^{-k_2-1}\theta^2 \wedge (d\theta^2)^{k_2}$$

$$\tau := (2\pi i)^{-k_1-1}\theta^2 \wedge \theta^{12} \wedge (d\theta^2)^j \wedge (d\theta^{12})^{k_1-j}$$

are closed in de Rham cohomology, see Dominguez [13, Théorème 5.2, pg 830]. These forms correspond to cohomology classes in $H^*(B^*_p, \mathbb{C})$.

Take now an irreducible component $Z \subset \text{Sing}_{k_1+1}(\mathcal{F}_1)$ and a generic point $p \in Z$. Pick $B_p$ a small ball centered at $p$ such that $S(B_p) \subset B_p$ is a sub-ball of dimension $n - k_1 - 1$(same dimension than the component $Z$). The de Rham class can be integrated over an oriented $(2k_1 + 1)$-sphere $L_p \subset B_p^*$

$$BB^j(\mathcal{F}, Z) := (2\pi i)^{-k_1-1} \int_{L_p} \theta^{12} \wedge (d\theta^2)^j \wedge (d\theta^{12})^{k_1-j} \quad \text{for each} \quad 0 \le j \le k_2.$$

**Theorem 2.4.1 (Baum-Bott for flags)** *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a 2-flag of codimension $(k_1, k_2)$ on a compact complex manifold $M$. If codim $S(\mathcal{F}) \ge k_1 + 1$, then for each $0 \le j \le k_2$ we have*

$$c_1^{k_1-j+1}(\mathcal{N}_{12}) \smile c_1^j(\mathcal{N}_2) = \sum_{Z \,\subset\, Sing_{k_1+1}(\mathcal{F}_1) \cup Sing_{k_1+1}(\mathcal{F}_2)} \lambda_Z^j(\mathcal{F})[Z],$$

*where $\lambda_Z^j(\mathcal{F}) = BB^j(\mathcal{F}, Z)$.*

**Proof:** The flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ can be locally defined on open an subset $U_\alpha$ by $\omega_2 = \eta_1 \wedge \ldots \wedge \eta_{k_2}$, $\omega_{12} = \eta_{k_2+1} \wedge \ldots \wedge \eta_{k_1}$ and $\omega_1 = \omega_2 \wedge \omega_{12}$ as above. Then, we can find matrices of (1,0)-forms $(\theta^\alpha_{uv}), (\theta^\alpha_{av})$ and $(\theta^\alpha_{ab})$ with $\theta^\alpha_{ij} \in A^{1,0}(B_p^*)$ such that

$$d\eta_u = \sum_{v=1}^{k_2} \theta^\alpha_{uv} \wedge \eta_v \;\text{ and }\; d\eta_a = \sum_{v=1}^{k_2} \theta^\alpha_{av} \wedge \eta_v + \sum_{b=k_2+1}^{k_1} \theta^\alpha_{ab} \wedge \eta_b.$$

Roughly speaking, we say that $\nabla = \begin{pmatrix} \theta^\alpha_{uv} & 0 \\ \theta^\alpha_{av} & \theta^\alpha_{ab} \end{pmatrix}$ represents the curvature matrix of the flag $\mathcal{F}$. Let us fix a neighborhood $V$ of $S_*(\mathcal{F})$, then we can find $\widehat{\theta}^2_\alpha = \sum_u^{k_2+1} (-1)^{u+1} \widehat{\theta}^\alpha_{uu}$ and $\widehat{\theta}^{12}_\alpha = \sum_{a=k_2+1}^{k_1} (-1)^{a+1} \widehat{\theta}^\alpha_{aa}$, where $\widehat{\theta}^\alpha_{ij}$ is a suitable modification of $\theta^\alpha_{ij}$, for more details, see [11, 6].

Now, let us consider $\Theta^2 := (2\pi i)^{-1} d\widehat{\theta}^2_\alpha$ and $\Theta^{12} := (2\pi i)^{-1} d\widehat{\theta}^{12}_\alpha$ globally defined closed forms which represent in de Rham cohomology the Chern classes of $\mathcal{N}_2$ and $\mathcal{N}_{12}$ respectively. Therefore $(\Theta^2)^j \wedge (\Theta^{12})^{k_1-j+1}$ represent $c_1^{k_1-j+1}(\mathcal{N}_{12}) \smile c_1^j(\mathcal{N}_2)$ and moreover, by Bott vanishing theorem for flags, see Theorem 2.2.8, we have

$$\text{Supp}\Big(c_1^{k_1-j+1}(\mathcal{N}_{12}) \smile c_1^j(\mathcal{N}_2)\Big) \subset \overline{V}.$$

Take $T \subset M$ a ball of real dimension $2(k_1+1)$ intersecting transversally $\text{Sing}_{k_1+1}(\mathcal{F}_1)$ at a single point $p \in Z$, with $V \cap T \Subset T$. Then by Stokes formula

$$BB^j(\mathcal{F}, Z) = (2\pi i)^{-k_1-1} \int_{\partial T} \widehat{\theta}_\alpha^{12} \wedge (d\widehat{\theta}_\alpha^2)^j \wedge (d\theta_\alpha^{12})^{k_1-j} =$$

$$= (2\pi i)^{-k_1-1} \int_T (d\widehat{\theta}_\alpha^2)^j \wedge (d\widehat{\theta}_\alpha^{12})^{k_1-j+1}. \tag{2.12}$$

This means that the $2(k_1+1)$-form $(\Theta^2)^j \wedge (\Theta^{12})^{k_1-j+1} = (d\widehat{\theta}_\alpha^2)^j \wedge (d\widehat{\theta}_\alpha^{12})^{k_1-j+1}$ is cohomologous, as a current, to the integration current over $BB^j(\mathcal{F}, Z)[Z]$, i.e.,

$$c_1^{k_1-j+1}(\mathcal{N}_{12}) \smile c_1^j(\mathcal{N}_2) = \sum_Z BB^j(\mathcal{F}, Z)[Z].$$

□

This theorem answers, partially, to the question: *How to calculate residues to flags?* As above is the Baum-Bott theorem for flags, we have

$$\text{Res}_{c_1^{k_1-j+1} c_1^j}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; Z) = \alpha_*\big(BB^j(\mathcal{F}; Z)[Z]\big),$$

where $\alpha_*$ is the Poincaré duality isomorphism

$$H^{2(k_1+1)}(M; \mathbb{C}) \xrightarrow{\alpha_*} H_{2(n-k_1-1)}(M; \mathbb{C}).$$

**Corollary 2.4.2** *If either $\mathcal{N}_{12}$ or $\mathcal{N}_2$ is ample then, there exist at least one irreducible component $Z \subset Sing_{k_1+1}(\mathcal{F}_1)$ of codimension $k_1+1$.*

**Proof:** By hypothesis either $\mathcal{N}_{12}$ or $\mathcal{N}_2$ is ample then, we have that either $c_1(\mathcal{N}_{12})$ or $c_1(\mathcal{N}_2)$ is non zero. Using Theorem 2.4.1 one has the result.

□

We prove a formula that compares the (sum) flag's residues with residues of the involved foliations.

**Corollary 2.4.3** *For each $Z \subset Sing_{k_1+1}(\mathcal{F}_1)$ and the above hypotheses we have*

$$\sum_{j=0}^{k_2} \binom{k_1+1}{j} BB^j(\mathcal{F}, Z) = BB(\mathcal{F}_1, Z), \tag{2.13}$$

*where the term in the right side of (2.13) is defined in [11, pg 6] and [6, pg 300] for $k_1 = 1$. Note also that if $k_1 = n-1$ then $\sum_{j=0}^{k_2} \binom{k_1+1}{j} BB^j(\mathcal{F}, Z) =$ "Grothendieck Residue".*

**Proof:** By Dominguez [13, Remarque 1, pg 830], we have

$$\sum_{j=0}^{k_2} \binom{k_1+1}{j} [\theta^{12} \wedge (d\theta^2)^j \wedge (d\theta^{12})^{k_1-j}] = [\theta^1 \wedge (d\theta^1)^{k_1}]$$

in the de Rham cohomology, where $\theta^1 = \theta^2 + \theta^{12}$. Thus

$$\sum_{j=0}^{k_2} \binom{k_1+1}{j} \theta^{12} \wedge (d\theta^2)^j \wedge (d\theta^{12})^{k_1-j} - \theta^1 \wedge (d\theta^1)^{k_1} = d\sigma$$

for some form $\sigma$. Now, integrating over a sphere $\partial T$ as above, we have

$$\sum_{j=0}^{k_2} \binom{k_1+1}{j} BB^j(\mathcal{F}, Z) = BB(\mathcal{F}_1, Z).$$

Therefore the corollary is proved.

□

**Corollary 2.4.4** *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be a flag such that $\dim(\mathcal{F}_1) = codim(\mathcal{F}_2) = 1$ and the singular set of the flag is composed of isolated singularities (only). Then, we have*

$$Res_{c_1^n}(\mathcal{F}, \mathcal{N}_{12}; p) = Res_{c_1^n}(\mathcal{F}_1, \mathcal{N}_1; p),$$

*where* $p \in S(\mathcal{F}) = S(\mathcal{F}_1) = S(\mathcal{F}_2)$.

**Proof:** By Corollary 2.4.3 and the hypothesis that $k_1 = n-1$ and $k_2 = 1$ we have

$$BB^0(\mathcal{F}; p) + nBB^1(\mathcal{F}; p) = BB(\mathcal{F}_1; p).$$

Since the singularities are isolated, we have

$$\mathrm{Res}_{c_1^n}(\mathcal{F}, \mathcal{N}_{12}; p) + \mathrm{Res}_{c_1^{n-1}, c_1}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; p) = \mathrm{Res}_{c_1^n}(\mathcal{F}_1, \mathcal{N}_1; p),$$

where

$$\mathrm{Res}_{c_1^{n-1}, c_1}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; p) = (\frac{1}{2\pi i})^n \int_{L_p} \theta^{12} \wedge (d\theta^2)^1 \wedge (d\theta^{12})^{n-2}$$

with $\theta^2$ is a (1,0)-form such that if $\omega$ is the 1-form that induces locally $\mathcal{F}_2$, we have

$$d\omega = \theta^2 \wedge \omega.$$

By Malgrange, see [20, Théorème 0.I, pg 163], we have that $\omega$ admits an integral factor, i.e., there are holomorphic functions $f$ and $g$ with $f(p) \neq 0$ such that $\omega = f dg$. This implies that

$$d\omega = df \wedge dg = \frac{df}{f} \wedge (f.dg) = \frac{df}{f} \wedge \omega.$$

Then, we can consider $\theta^2 = \dfrac{df}{f} = d(\log f)$. Since this is an exact form, we have $d\theta^2 = 0$ and $\mathrm{Res}_{c_1^{n-1}, c_1}(\mathcal{F}, \mathcal{N}_{\mathcal{F}}; p) = 0$. Therefore, the result is proved.

$\square$

**Example 2.4.5** *Let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ be the flag on the manifold $X \subset \mathbb{P}^5 \times \mathbb{P}^1$ of the Example 2.3.8. By Corollary 2.4.4 we have*

$$Res_{c_1^n}(\mathcal{F}, \mathcal{N}_{12}; p) = Res_{c_1^n}(\mathcal{F}_1, \mathcal{N}_1; p),$$

*where Res*$_{c_1^n}(\mathcal{F}_1, \mathcal{N}_1; p) = (\frac{1}{2\pi i})^4 \int_T tr(JX)^4 \frac{dx \wedge dy \wedge dw \wedge dt}{X_1 X_2 X_3 X_4}$.

*We can check that* $tr(JX) = 0$. *Therefore we have the flag's residue*

$$Res_{c_1^4}(\mathcal{F}, \mathcal{N}_{12}; p) = 0.$$

# Chapter 3

# Nash residues and comparison of residues

In this chapter, we propose to analyze the rationality of the Baum-Bott residues for flags. We consider the Nash modification $M^{\nu}$ of a complex manifold $M$ with respect to a flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ and we will give a partial answer for this conjecture. In the foliation context, Sertöz in [23, Theorem V 1, pg 242] studied this conjecture with the hypothesis that $M^{\nu}$ is non-singular and he gave a partial answer to Baum-Bott conjecture. In [5, Theorem 4.1, pg 44] Brasselet and Suwa generalized the work of Sertöz, where they use characteristic classes on singular varieties. For characteristic classes in singular varieties, we refer to [4].

## 3.1 Nash residues for flags

Let $M$ be a complex manifold of dimension $n$ and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ a 2-flag of singular holomorphic foliations of dimension $(q_1, q_2)$ on $M$. Then for each point $x \in M$, we set

$$F_i(x) = \{v(x) \ / \ v \in \mathcal{F}_{i,x}\} \subset T_xM.$$

This is a $q_i$-dimensional subspace if and only if $x \notin S(\mathcal{F})$, for $i = 1, 2$. Thus we have a flag of subspaces $F_1(x) \subset F_2(x) \subset T_xM$ for each point $x \in M \setminus S(\mathcal{F})$. We will consider the flag bundle using the Grassmann bundle of $q_i$-planes.

Let $\widetilde{\pi}_2 : G_{q_2}(TM) \longrightarrow M$ be the Grassmann bundle of $q_2$-planes in $TM$. We have the Nash modification of $M$ with respect to $\mathcal{F}_2$, $\ M_2^{\nu} = \overline{Im\sigma_2}$, where $\sigma_2$ is a natural section induced by

$\mathcal{F}_2$. We have the exact sequence on $M_2^\nu$

$$0 \longrightarrow T_2^\nu \longrightarrow \pi_2^* TM \longrightarrow N_2^\nu \longrightarrow 0. \tag{3.1}$$

Analogously, we consider the Grassmann bundle of $q_1$-planes in $TM$ denoted by $\widetilde{\pi}_1 : G_{q_1}(TM) \longrightarrow M$ and we obtain the Nash modification of $M$ with respect to $\mathcal{F}_1$, $M_1^\nu = \overline{Im\sigma_1}$ and the exact sequence on $M_1^\nu$

$$0 \longrightarrow T_1^\nu \longrightarrow \pi_1^* TM \longrightarrow N_1^\nu \longrightarrow 0. \tag{3.2}$$

Now, if we consider the Grassmann bundle of $(n-q_2)$-planes in $TM$, i.e.,

$$\widetilde{\pi}_{n-q_2} : G_{n-q_2}(TM) \longrightarrow M,$$

then we have the exact sequence

$$0 \longrightarrow \widetilde{T}_{n-q_2}^\nu \longrightarrow \widetilde{\pi}_{n-q_2}^* TM \longrightarrow \widetilde{N}_{n-q_2}^\nu \longrightarrow 0.$$

**Remark 3.1.1** *The fiber of the fibre bundle $N_{n-q_2}^\nu \longrightarrow G_{n-q_2}(TM)$ over a $(n-q_2)$-plane $E_{n-q_2} \in G_{n-q_2}(TM)$ is the $q_2$-plane*

$$(\widetilde{N}_{n-q_2}^\nu)_{E_{n-q_2}} \simeq \frac{T_x M}{E_{n-q_2}} \simeq E_{q_2},$$

*where $\widetilde{\pi}_{n-q_2}(E_{n-q_2}) = x$.*

If we let $\widetilde{\pi}_{q_1} : G_{q_1}(\widetilde{N}_{n-q_2}^\nu) \longrightarrow G_{n-q_2}(TM)$ be the Grassmann bundle of $p_1$-planes in $\widetilde{N}_{n-q_2}^\nu$, we have the flag bundle $\widetilde{\pi} : F_{q_1,q_2}(TM) \longrightarrow M$ of $(q_1, q_2)$-planes in $TM$, where $\widetilde{\pi} = \widetilde{\pi}_{n-q_2} \circ \widetilde{\pi}_{q_1}$.

**Remark 3.1.2** *A point of $F_{q_1,q_2}(TM)$ over $x \in M$ means first a $q_2$-plane $E_{q_2}$ in $T_x M$ and then a $q_1$-plane $E_{q_1}$ in $E_{q_2}$; this is a flag in $T_x M$.*

*For details see [16].*

**Definition 3.1.3** *We define the Nash modification of $M$ with respect of the flag $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ by*

$$M^{\nu} = \overline{Im\sigma},$$

*where the closure is taken in the fibre bundle $F_{q_1,q_2}(TM)$ and $\sigma$ is the natural section induced by the flag $\mathcal{F}$.*

If we consider the projections $\widetilde{p}_i : F_{q_1,q_2}(TM) \longrightarrow G_{q_i}(TM);\ \ i = 1,2$, then we can take the pull-back of the exact sequences (3.1) and (3.2) to $M^{\nu}$.

$$0 \longrightarrow p_1^* T_1^{\nu} \longrightarrow p_1^* \pi_1^* TM \longrightarrow p_1^* N_1^{\nu} \longrightarrow 0. \tag{3.3}$$

$$0 \longrightarrow p_2^* T_2^{\nu} \longrightarrow p_2^* \pi_2^* TM \longrightarrow p_2^* N_2^{\nu} \longrightarrow 0. \tag{3.4}$$

**Proposition 3.1.4** *The following diagram*

$$\begin{tikzcd}
 & M^{\nu} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "\pi"] & \\
M_1^{\nu} \arrow[dr, "\pi_1"'] & & M_2^{\nu} \arrow[dl, "\pi_2"] \\
 & M &
\end{tikzcd}$$

*is commutative.*

**Proposition 3.1.5** *On $M^{\nu}$ we have the exact sequences*

$$0 \longrightarrow N_{12}^{\nu} \longrightarrow p_1^* N_1^{\nu} \longrightarrow p_2^* N_2^{\nu} \longrightarrow 0 \tag{3.5}$$

$$0 \longrightarrow p_1^* T_1^{\nu} \longrightarrow p_2^* T_2^{\nu} \longrightarrow N_{12}^{\nu} \longrightarrow 0, \tag{3.6}$$

*where $N_{12}^{\nu} := p_2^* T_2^{\nu} / p_1^* T_1^{\nu}$.*

It follows from the Proposition 3.1.4 and Proposition 3.1.5, that $p_1^* \pi_1^* TM = p_2^* \pi_2^* TM = \pi^* TM$. Therefore, we have the following diagram on $M^{\nu}$.

$$
\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[dl] & & 0 \\
& p_1^*T_1^\nu \arrow[dr] \arrow[dd] & & p_2^*N_2^\nu \arrow[ul] \arrow[ur] & \\
& & \pi^*TM \arrow[ur] \arrow[dr] & & \\
& p_2^*T_2^\nu \arrow[ur] \arrow[dr] & & p_1^*N_1^\nu \arrow[uu] \arrow[dr] & \\
0 \arrow[ur] & & N_{12}^\nu \arrow[ur] \arrow[dr] & & 0 \\
& 0 \arrow[ur] & & 0 &
\end{tikzcd}
$$

We define the normal bundle $N^\nu$ over $M^\nu$ by $N_{12}^\nu \oplus p_2^* N_2^\nu$ and also we define

$$\varphi(N^\nu) := \varphi_1(N_{12}^\nu) \smile \varphi_2(p_2^* N_2^\nu),$$

where $\varphi_i$ is a homogeneous symmetric polynomial of degree $d_i$.

Let $S$ be a compact connected component of $S(\mathcal{F})$ and let $S^\nu = \pi^{-1}(S)$. Also, let $U^\nu$ be a neighborhood of $S^\nu$ in $M^\nu$ disjoint from the other components of $S(\mathcal{F})^\nu$. Let $\widetilde{U}_1^\nu$ be a regular neighborhood of $S^\nu$ in $F_{q_1,q_2}(TM)$ with $\widetilde{U}_1^\nu \cap M^\nu \subset U^\nu$ and $\widetilde{U}_0^\nu$ be a tubular neighborhood of $U_0^\nu = U^\nu \setminus S^\nu$ in $F_{q_1,q_2}(TM)$ with the projection $\rho$. We consider the covering $\widetilde{\mathcal{U}}^\nu = \{\widetilde{U}_0^\nu, \widetilde{U}_1^\nu\}$ of $\widetilde{U}^\nu = \widetilde{U}_0^\nu \cup \widetilde{U}_1^\nu$. The characteristic class $\varphi(N^\nu)$ is represented by the cocycle

$$\varphi({}_2^{12}\nabla_*^\nu) = \varphi_1({}^{12}\nabla_*^\nu) \smile \varphi_2({}^{2}\nabla_*^\nu) \in A^{2(d_1+d_2)}(\widetilde{\mathcal{U}}^\nu),$$

where

$$\varphi_1({}^{12}\nabla_*^\nu) = (\varphi_1({}^{12}\nabla_0^\nu), \varphi_1({}^{12}\nabla_1^\nu), \varphi_1({}^{12}\nabla_0^\nu, {}^{12}\nabla_1^\nu))$$

and

$$\varphi_2({}^{2}\nabla_*^\nu) = (\varphi_2({}^{2}\nabla_0^\nu), \varphi_2({}^{2}\nabla_1^\nu), \varphi_2({}^{2}\nabla_0^\nu, {}^{2}\nabla_1^\nu)).$$

Here ${}^{12}\nabla_0^\nu$ and ${}^{12}\nabla_1^\nu$ are connections on $N_{12}^\nu$ over $\widetilde{U}_0^\nu$ and $\widetilde{U}_1^\nu$, respectively, and ${}^{2}\nabla_0^\nu$ and ${}^{2}\nabla_1^\nu$ are connections on $p_2^* N_2^\nu$ over $\widetilde{U}_0^\nu$ and $\widetilde{U}_1^\nu$, respectively.

If we set $U = \pi(U^\nu)$, i.e., a neighborhood of $S$ on $M$, then $\pi$ induces a biholomorphic map $U_0^\nu \longrightarrow U_0 = U \setminus S$. On $U_0$ we have basic (Bott) connections $\nabla_{12}$ and $\nabla_2$ a $N_{12}$ and $N_{F_2^0}$ respectively. We take as ${}^{12}\nabla_0^\nu$ the connection for $N_{12}^\nu$ given by ${}^{12}\nabla_0^\nu = \pi^*(\nabla_{12})$ and analogously ${}^{2}\nabla_0^\nu = \pi^*(\nabla_2)$ for $p_2^* N_2^\nu$ then

$$\varphi({}_2^{12}\nabla_0^\nu) = \varphi_1({}^{12}\nabla_0^\nu) \smile \varphi_2({}^{2}\nabla_0^\nu) = 0.$$

The cocycle $\varphi({}_2^{12}\nabla_*^\nu) \in A^{2(d_1+d_2)}(\widetilde{\mathcal{U}}^\nu, \widetilde{U}_0^\nu)$ defines a class $\varphi_{S^\nu}(N^\nu; \mathcal{F}) \in H^{2(d_1+d_2)}(U^\nu, U^\nu \setminus S^\nu; \mathbb{C})$. We denote its image in $H_{2(n-d_1-d_2)}(S^\nu; \mathbb{C})$ by Alexander homomorphism by $\mathrm{Res}_{\varphi_1,\varphi_2}(N^\nu, \mathcal{F}, S^\nu)$.

**Definition 3.1.6** *We call the class $Res_{\varphi_1,\varphi_2}(N^\nu, \mathcal{F}, S^\nu)$ the Nash residue of the flag $\mathcal{F}$ with respect to $\varphi = (\varphi_1, \varphi_2)$ at $S^\nu$.*

## 3.2 Comparison of Baum-Bott and Nash residues for flags

After the definition of the Nash residue for flags above, we can compare it with the Baum-Bott residue for flags. The result is analogous to [5]. This comparison gives a partial answer to the Rationality conjecture for flags, see (2.2.16).

Let $M$ be a complex manifold of dimension $n$ and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ a 2-flag of singular holomorphic foliations of dimension $(q_1, q_2)$ on $M$. Also let $S \subset S(\mathcal{F})$ be a compact connected component and $S^\nu = \pi^{-1}(S)$ as above. Then, there is a canonical homomorphism

$$\pi_* : H_{2n-2d}(S^\nu; \mathbb{C}) \longrightarrow H_{2n-2d}(S; \mathbb{C}).$$

**Theorem 3.2.1** *Let $\varphi = (\varphi_1, \varphi_2)$ where $\varphi_i$ is a homogeneous symmetric polynomial of degree $d_i$ satisfying the condition of the Bott vanishing theorem for flags (2.1). If $\varphi_i$ has integral coefficients, then the difference*

$$Res_{\varphi_1,\varphi_2}(\mathcal{N}_{\mathcal{F}}, \mathcal{F}, S) - \pi_* Res_{\varphi_1,\varphi_2}(N^\nu, \mathcal{F}, S^\nu)$$

*is in the image of the canonical homomorphism $H_{2n-2d}(S; \mathbb{Z}) \longrightarrow H_{2n-2d}(S; \mathbb{C})$, i.e., it is a (sum of) integral class.*

**Proof:** Take analytic resolutions of the sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$

$$0 \longrightarrow A_U(E_q^{12}) \longrightarrow ... \longrightarrow A_U(E_1^{12}) \longrightarrow A_U \otimes \mathcal{F}_1 \longrightarrow 0$$

$$0 \longrightarrow A_U(E_r^{2}) \longrightarrow ... \longrightarrow A_U(E_1^{2}) \longrightarrow A_U \otimes \mathcal{F}_2 \longrightarrow 0.$$

The exact sequences

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] \arrow[d] & \mathcal{N}_{12} \arrow[r] & 0 \\ & & \Theta_M \arrow[d] & & \\ & & \mathcal{N}_2 \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

provide a resolution of the sheaves $\mathcal{N}_{12}$ and $\mathcal{N}_2$.

$$0 \longrightarrow A_U(E_q^{12}) \xrightarrow{\eta_q^{12}} ... \longrightarrow A_U(E_1^{1}) \xrightarrow{\eta_1^{12}} A_U(F_2^0) \longrightarrow A_U \otimes \mathcal{N}_{12} \longrightarrow 0$$

$$0 \longrightarrow A_U(E_q^{2}) \xrightarrow{\eta_r^{2}} ... \longrightarrow A_U(E_1^{2}) \xrightarrow{\eta_1^{2}} A_U(TM) \longrightarrow A_U \otimes \mathcal{N}_{2} \longrightarrow 0.$$

Then, we have exact sequences of vector bundles on $U_0$.

$$0 \longrightarrow E_q^{12} \longrightarrow ... \longrightarrow E_1^{12} \longrightarrow F_2^0 \longrightarrow N_{12} \longrightarrow 0 \tag{3.7}$$

$$0 \longrightarrow E_r^{2} \longrightarrow ... \longrightarrow E_1^{2} \longrightarrow TM \longrightarrow N_{F_2^0} \longrightarrow 0. \tag{3.8}$$

The sheaves homomorphisms $\eta_j^{12}$ and $\eta_i^2$ induce bundles homomorphisms on $U$ and $U^\nu$

$$h_j^{12} : E_j^{12} \longrightarrow E_{j-1}^{12}$$

$$h_i^2 : E_i^2 \longrightarrow E_{i-1}^2.$$

$$\pi^* h_j^{12} : \pi^* E_j^{12} \longrightarrow \pi^* E_{j-1}^{12}$$

$$\pi^* h_i^2 : \pi^* E_i^2 \longrightarrow \pi^* E_{i-1}^2.$$

We claim that

$$Im(\pi^* h_1^2) \subset p_2^* T_2^\nu \ \text{ and } \ Im(\pi^* h_1^{12}) \subset p_1^* T_1^\nu \ \text{ on } \ U^\nu. \tag{3.9}$$

In fact, away from the singular set we have equivalent sequences

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{N}_{12} \longrightarrow 0.$$

$$0 \longrightarrow p_1^* T_1^\nu \longrightarrow p_2^* T_2^\nu \longrightarrow N_{12}^\nu \longrightarrow 0.$$

Note that $T_1^\nu = \pi_1^* F_1^0$ (on $M_1^\nu$ ) implies that $p_1^* T_1^\nu = \pi^* F_1^0$. Analogously we have $p_2^* T_2^\nu = \pi^* F_2^0$.

Then, we have the exact sequences

$$\begin{tikzcd}
 & & \vdots \arrow[d] & & \\
 & & \pi^* E_1^{12} \arrow[d, "\pi^* h_1^{12}"] & & \\
0 \arrow[r] & \pi^* F_1^0 \arrow[r] & \pi^* F_2^0 \arrow[r] & \pi^* N_{1,2} \arrow[r] & 0.
\end{tikzcd}$$

Therefore, away from singular sets, which is dense in $U^\nu$, we have the equalities in (3.9). Then, by the continuity arguments we have the inequalities of (3.9) in $U^\nu$. We have two complexes of vector bundles on $U^\nu$, which are exact on $U_0^\nu$.

$$0 \longrightarrow \pi^*(E_q^{12}) \longrightarrow \ldots \longrightarrow \pi^*(E_1^{12}) \longrightarrow \pi^* F_2^0 \longrightarrow N_{12}^\nu \longrightarrow 0. \tag{3.10}$$

$$0 \longrightarrow \pi^*(E_r^2) \longrightarrow \ldots \longrightarrow \pi^*(E_1^2) \longrightarrow \pi^* TM \longrightarrow p_2^* N_2^\nu \longrightarrow 0. \tag{3.11}$$

We consider the virtual bundles $\widetilde{\varepsilon}_{12} = \widetilde{\pi}^*(\xi^{12}) - N_{12}^\nu$ and $\widetilde{\varepsilon}_2 = \widetilde{\pi}^*(\xi^2) - p_2^* N_2^\nu$ or

$\widetilde{\pi}^*(\xi^{12}) = N_{12}^{\nu} + \widetilde{\varepsilon}_{12}$ and $\widetilde{\pi}^*(\xi^2) = p_2^* N_2^{\nu} + \widetilde{\varepsilon}_2$.

By classical properties of characteristic classes we can write

$$\varphi_1(\widetilde{\pi}^*(\xi^{12})) = \varphi_1(N_{12}^{\nu}) + \sum \varphi_1^i(N_{12}^{\nu})\psi_1^i(\widetilde{\varepsilon}_{12}), \tag{3.12}$$

where the $\varphi_1^i$ are symmetric polynomials with integral coefficients and $\psi_1^i$ are symmetric polynomials with integral coefficients without constant term. Analogously

$$\varphi_2(\widetilde{\pi}^*(\xi^2)) = \varphi_2(p_2^* N_2^{\nu}) + \sum \varphi_2^i(p_2^* N_2^{\nu})\psi_2^i(\widetilde{\varepsilon}_2). \tag{3.13}$$

By taking the cap product of (3.12) with (3.13) we have

$\varphi_1(\widetilde{\pi}^*\xi^{12}).\varphi_2(\widetilde{\pi}^*\xi^2) =$

$= \varphi_1(N_{12}^{\nu}).\varphi_2(p_2^* N_2^{\nu}) + \varphi_1(N_{12}^{\nu}).\sum \varphi_2^i(p_2^* N_2^{\nu})\psi_2^i(\widetilde{\varepsilon}_2) +$

$+ \sum \varphi_1^i(N_{12}^{\nu})\psi_1^i(\widetilde{\varepsilon}_1).\varphi_2(p_2^* N_2^{\nu}) + \sum \varphi_1^i(N_{12}^{\nu})\psi_1^i(\widetilde{\varepsilon}_{12}).\varphi_2^i(p_2^* N_2^{\nu})\psi_2^i(\widetilde{\varepsilon}_2).$

on $H^{2(d_1+d_2)}(U^{\nu})$.

We claim that we have a good localization, i.e., in $A^*(\widetilde{U}^{\nu}, \widetilde{U}_0^{\nu})$ we have

$\varphi(\widetilde{\pi}^*({}^{12}_{2}\nabla_*^{\bullet})) = \varphi_1(\widetilde{\pi}^*({}^{12}\nabla_*^{\bullet})).\varphi_2(\widetilde{\pi}^*({}^{2}\nabla_*^{\bullet})) =$

$= \varphi_1({}^{12}\nabla_*^{\nu}).\varphi_2({}^{2}\nabla_*^{\nu}) + \varphi_1({}^{12}\nabla_*^{\nu}).\sum \varphi_2^i({}^{2}\nabla_*^{\nu})\psi_2^i({}^{2}\nabla_*^{\varepsilon}) +$

$+ \sum \varphi_1^i({}^{12}\nabla_*^{\nu})\psi_1^i({}^{12}\nabla_*^{\varepsilon}).\varphi_2({}^{2}\nabla_*^{\nu}) + \sum \varphi_1^i({}^{12}\nabla_*^{\nu})\psi_1^i({}^{12}\nabla_*^{\varepsilon}).\varphi_2^i({}^{2}\nabla_*^{\nu})\psi_2^i({}^{2}\nabla_*^{\varepsilon}) + D\tau,$

where $\tau = (0, 0, \tau_{01})$

with $\tau_{01} = \varphi_1({}^{12}\nabla_0^{\nu}).{}^{2}\tau_{01} +^{12} \tau_{01}.\varphi_2({}^{2}\nabla_1^{\nu}) +^{12} \tau_{01}.\sum \varphi_2^i({}^{2}\nabla_1^{\nu}).\psi_2^i({}^{2}\nabla_1^{\varepsilon})$

For further details of the ${}^{2}\tau_{01}$ and ${}^{1}\tau_{01}$, we refer to [5, pg 46].

The above claim shows that we have in $H^{2(d_1+d_2)}(U^{\nu}, U^{\nu} \backslash S^{\nu}, \mathbb{C})$

$\pi^*\varphi_S(\mathcal{N}_{\mathcal{F}}, \mathcal{F},) = \varphi_{S^{\nu}}(N^{\nu}, \mathcal{F}) + \sum \varphi_1(N_{12}^{\nu}).\varphi_2^i(p_2^* N_2^{\nu}).\psi_{2,S}^i(\varepsilon_2) +$

$+ \sum \varphi_1^i(N_{12}^{\nu}).\psi_{1,S}^i(\varepsilon_{12}).\varphi_2(p_2^* N_2^{\nu}) + \sum \varphi_1^i(N_{12}^{\nu}).\psi_{1,S}^i(\varepsilon_{12}).\varphi_2^i(p_2^* N_2^{\nu}).\psi_{2,S}^i(\varepsilon_2).$

Thus, by the commutative diagram

$$\begin{tikzcd} H^{2(d_1+d_2)}(U^\nu, U^\nu \setminus S^\nu, \mathbb{C}) \arrow[d, "A"] & H^{2(d_1+d_2)}(U, U \setminus S, \mathbb{C}) \arrow[l, "\pi^*"'] \arrow[d, "A"] \\ H_{2n-2(d_1+d_2)}(S^\nu, \mathbb{C}) \arrow[r, "\pi^*"] & H_{2n-2(d_1+d_2)}(S, \mathbb{C}) \end{tikzcd}$$

we obtain that the difference between these residues in $H_{2n-2(d_1+d_2)}(S, \mathbb{C})$ is a sum of integral classes.

$\square$

**Corollary 3.2.2** *If $\varphi_1 = c_{i_1}...c_{i_r}$ and $\varphi_2 = c_{j_1}...c_{j_t}$ with $i_\nu > codim(\mathcal{F}_1)$ for some $\nu \in [1, ..., r]$ or $i_s > codim(\mathcal{F}_2)$ for some $s \in [1, ..., t]$, then the Baum-Bott residue for the flag $\mathcal{F}$, $Res_{\varphi_1,\varphi_2}(\mathcal{N}_\mathcal{F}, \mathcal{F}, S)$, is a (sum of) integral class.*

# Chapter 4

# Determination of Baum-Bott residues of the foliations

The purpose of this chapter is twofold. First, we give a generalization of a construction of Brunella-Perrone in [6] and Corrêa-Pérez in [11, Theorem 4.1, pg 6], for any polynomial $\varphi$ of degree $k+1$; and second, we show that, in this theorem, the complex number $\mathrm{BB}(\mathcal{F}, Z)$ can be calculated as a Grothendieck residue.

Let $\mathcal{F}$ be a holomorphic foliation of codimension $k$ on a complex manifold $M$ with $\dim M = n$. Assume that $\mathcal{F}$ is induced by $\omega \in H^0(M, \Omega_M^k \otimes \mathcal{N})$. Denote by $\mathrm{Sing}_{k+1}(\mathcal{F})$, the union of the irreducible components of $S(\mathcal{F})$ of pure codimension $k+1$. Assume that

$$\text{Codim S}(\mathcal{F}) \geq k+1.$$

We can consider $\omega$ decomposable and integrable, i. e., locally $\omega$ is given by a product of $k$ 1-forms $\eta_1 \wedge ... \wedge \eta_k$. Then, it is possible to find a matrix of $(1,0)$-forms $(\theta_{ls})$ such that

$$d\eta_l = \sum_{s=1}^{k} \theta_{ls} \wedge \eta_s \quad \forall \quad l = 1, ..., k.$$

Set $\theta := \sum_{l=1}^{k} (-1)^{l+1} \theta_{ll}$. Observe that the smooth $(2k+1)$-form

$$(\frac{1}{2\pi i})^{k+1} \theta \wedge \underbrace{d\theta \wedge ... \wedge d\theta}_{k-th}$$

is closed. Its the de Rham cohomology class in $H^{2k+1}(B_p^*; \mathbb{C})$ does not depend on the choice of $\omega$ and $\theta$.

Take now an irreducible component $Z \subset \mathrm{Sing}_{k+1}(\mathcal{F})$ and a generic point $p \in Z$. Pick $B_p$ a small ball centered at $p$ such that $S(B_p) \subset B_p$ is a sub-ball of dimension $n - k - 1$. The de-Rham class can be integrated over an oriented $(2k+1)$-sphere $L_p \subset B_p^*$

$$BB(\mathcal{F}, Z) := (\frac{1}{2\pi i})^{k+1} \int_{L_p} \theta \wedge (d\theta)^k.$$

Corrêa and Pérez in [11, Theorem 4.1, pg 6] give a new proof of the Baum-Bott theorem and presented an effective way (different of Baum-Bott) to calculate residues of a foliations, when the dimension of the singular set of the foliation is one less than the dimension of the foliation.

**Theorem 4.0.3** *Let $\mathcal{F}$ be a holomorphic foliation of codimension $k$ on a complex manifold $M$. Then the following hold:*

*(i) for each irreducible component $Z$ of Sing$_{k+1}(\mathcal{F})$ there exists a complex number $\lambda_Z(\mathcal{F})$ which is determined by the local behavior of $\mathcal{F}$ near $Z$.*

*(ii) If $M$ is compact*

$$c_1^{k+1}(\mathcal{N}_{\mathcal{F}}) = \sum_Z \lambda_Z(\mathcal{F})[Z],$$

*where the sum is done over all irreducible components of Sing$_{k+1}(\mathcal{F})$. We will show $\lambda_Z(\mathcal{F}) = BB(\mathcal{F}, Z)$.*

We will show the following result

**Corollary 4.0.4** *Let $\mathcal{F}$ be a holomorphic foliation of codimension one on $M$ induced by $\omega \in H^0(M, \Omega^1 \otimes \det(\mathcal{N}_{\mathcal{F}}))$. Consider $Z \subset$ Sing$_2(\mathcal{F})$. If $d\omega \equiv 0$ in a neighborhood of $Z$ then*

$$\mathit{Res}_{c_1^2}(\mathcal{F}; Z) = 0.$$

**Proof:** By Theorem 4.0.3 one has

$$\mathrm{Res}_{c_1^2}(\mathcal{F}; Z) = \alpha_*(BB(\mathcal{F}, Z)[Z]),$$

where $\alpha_*$ is the Poincaré duality isomorphism

$$H^{2(2)}(M;\mathbb{C}) \xrightarrow{\alpha_*} H_{2(n-2)}(M;\mathbb{C}).$$

I will show that $BB(\mathcal{F}, Z) = 0$.

Recall the definition of this complex number $BB(\mathcal{F}, Z)$, see [6]. There is a $(1,0)$-form $\beta \in A^{(1,0)}(B_p^*)$ such that

$$d\omega = \beta \wedge \omega,$$

where $B_p^*$ is defined as follow.

Take a point $p \in \mathrm{Sing}_2(\mathcal{F})$. We consider a ball $B_p \subset M$ centered at $p$, next consider $S(B_p) = \mathrm{Sing}_2(\mathcal{F}) \cap B_p$ and $B_p^* = B_p \backslash S(B_p)$.

By hypothesis $d\omega|_Z = \beta \wedge \omega|_Z = 0$ then, by the division lemma, there is a holomorphic function $f$ such that

$$\beta = f\omega.$$

On the other hand, we have

$$BB(\mathcal{F}, Z) = (\frac{1}{2\pi i})^2 \int_{L_p} \beta \wedge d\beta,$$

where, $\beta \wedge d\beta = f\omega \wedge df \wedge \omega = 0$.

$\square$

**Example 4.0.5** *Let $\mathcal{F}$ be the logarithmic foliation on $\mathbb{P}^3$ induced, locally in $(\mathbb{C}^3, (x, y, z))$ by the 1-form*

$$\omega = yzdx + xzdy + xydz.$$

*In this chart, the singular set of $\omega$ is the union of the irreducible compoents $Z_1, Z_2$ and $Z_3$, where $Z_1 = \{x = y = 0\}$; $Z_2 = \{x = z = 0\}$ and $Z_3 = \{y = z = 0\}$. Note that $d\omega|_{Z_i} = 0$*

*for* $i = 1, 2, 3$. *Therefore,* $BB(\mathcal{F}; Z_i) = 0$ *and, we have*

$$Res_{c_1^2}(\mathcal{F}; Z_i) = \alpha_*(BB(\mathcal{F}; Z_i)[Z_i]) = 0.$$

We show that the Theorem 4.0.3, with the construction that appears in [11], is valid for any polynomial $\varphi = c_1^{\alpha_1} c_2^{\alpha_2} ... c_k^{\alpha_k}$ with $1\alpha_1 + 2\alpha_2 + ... + k\alpha_k = k + 1$, where $c_i$ denotes the $i$-th Chern class.

Given a multi-index $\alpha = (\alpha_1, ..., \alpha_k)$ with $\alpha_j \geq 0$ for $j = 1, ..., k$, we can associate a homogeneous symmetric polynomial of degree $k + 1$, $\varphi = c_1^{\alpha_1} c_2^{\alpha_2} ... c_k^{\alpha_k}$ with $1\alpha_1 + 2\alpha_2 + ... + k\alpha_k = k + 1$. Denote by $\theta$ the Bott connection matrix of the foliation $\mathcal{F}$ and $K$ its curvature matrix. Next, consider the unique complete polarization of the polynomial $\varphi$, denoted by $\widetilde{\varphi}$, that is, $\widetilde{\varphi}$ is a symmetric $k$-linear map that satisfies

$$\widetilde{\varphi}(K, ..., K) = \varphi(K) = c_1^{\alpha_1}(K) c_2^{\alpha_2}(K) ... c_k^{\alpha_k}(K).$$

Define the polynomial $\varphi_j$ for $j = 1, ..., k$ as follow

$$\varphi_j(\theta, K) := \widetilde{\varphi}(\theta, \underbrace{-2\theta \wedge \theta, ..., -2\theta \wedge \theta}_{j-1}, \underbrace{K, ..., K}_{k-j})$$

$$= c_1^{\alpha_1}(\theta) c_2^{\alpha_2}(-2\theta \wedge \theta) ... c_k^{\alpha_k}(K).$$

Now, we consider the $(2k + 1)$- form

$$\varphi_\alpha(\theta, K) = \sum_{j=0}^{k-1} (-1)^j \frac{(k-1)!}{2^j (k-j-1)!(k+j)!} \varphi_{j+1}(\theta, K).$$

Note that $\varphi(K) = c_1^{\alpha_1}(K) c_2^{\alpha_2}(K) ... c_k^{\alpha_k}(K)$ represents, in de Rham sense, the characteristic class $\varphi(\mathcal{N}_\mathcal{F})$. It follows from the Bott vanishing theorem, see [25, Theorem 9.11, pg 76], that $\varphi(K) = 0$ outside $V$, where $V$ is a small neighborhood of $\text{Sing}_{k+1}(\mathcal{F})$.

Let $Z$ be an irreducible component of $\text{Sing}_{k+1}(\mathcal{F})$. Take a generic point $p \in Z$, that is, $p$ is a point where $Z$ is smooth and disjoint from the other singular component. Pick $B_p$ a ball centered at $p$ sufficiently small, such that $S(B_p) := Z \cap B_p$ is a subball of $B_p$ of dimension $n - k - 1$. Then, the de Rham class $\varphi_\alpha(\theta, K)$ can be integrated over an oriented $(2k+1)$-sphere $L_p \subset B_p^* := B_p \backslash S(B_p)$ positively linked with $S(B_p)$:

$$BB(\mathcal{F},\varphi;Z) := (\frac{1}{2\pi i})^{k+1}\int_{L_p}\varphi_\alpha(\theta,K). \tag{4.1}$$

**Theorem 4.0.6** *Let $\mathcal{F}$ be a holomorphic foliation of codimension $k$ on a complex manifold $M$. If Codim $S(\mathcal{F}) \geq k+1$, then the following hold:*

*(i) for each irreducible component $Z$ of Sing$_{k+1}(\mathcal{F})$ there exist a complex number $\lambda_Z(\mathcal{F})$ which is determined by the local behavior of $\mathcal{F}$ near $Z$.*

*(ii) If $M$is compact*

$$\varphi(\mathcal{N}_{\mathcal{F}}) = \sum_Z \lambda_Z(\mathcal{F};\varphi)[Z] \quad \text{in} \quad H^{2(k+1)}(M;\mathbb{C}),$$

*where the sum is done over all irreducible components of Sing$_{k+1}(\mathcal{F})$. We will show that $\lambda_Z(\mathcal{F};\varphi) = BB(\mathcal{F},\varphi;Z)$.*

**Proof:** Let us consider $L \subset M$ a $(k+1)$-ball intersecting transversally Sing$_{k+1}(\mathcal{F})$ at a single point $p \in Z$, with $V \cap T \Subset T$.

For the form $\varphi_\alpha(\theta,K)$, one has

$$d(\varphi_\alpha(\theta,K)) = \varphi(K).$$

See Vishik [28, Lemma 2.3, pg 5].

Then by Stokes theorem we have

$$BB(\mathcal{F},\varphi;Z) = (\frac{1}{2\pi i})^{k+1}\int_{\partial L}\varphi_\alpha(\theta,K)$$

$$= (\frac{1}{2\pi i})^{k+1}\int_{L} d(\varphi_\alpha(\theta,K)) = (\frac{1}{2\pi i})^{k+1}\int_{L}\varphi(K).$$

This means that the $2(k+1)$-form $d(\varphi_\alpha) = \varphi(K)$ is cohomologous, as a current, to the integration current over $BB(\mathcal{F},\varphi;Z)[Z]$, i.e.,

$$\varphi(\mathcal{N}_{\mathcal{F}}) = \sum_Z BB(\mathcal{F},\varphi;Z)[Z].$$

□

**Corollary 4.0.7** *For $k = n - 1$ we have*

$$BB(\mathcal{F}, \varphi; Z = q) = \textbf{\textit{Res}}_q \Big[ \varphi(JX) \frac{dz_1 \wedge ... \wedge dz_n}{X_1 ... X_n} \Big],$$

*where the right side is the Grothendieck residue of $\mathcal{F}$ around at the singular point $q$.*

Now, we will apply the "transversal disc method" of Baum-Bott and Vishik. However we do not use the hypothesis that at the singular set of foliation all $p \in S(\mathcal{F})$ are a Baum-Kupka type singularities, see [3, Theorem 3, pg 285]. We do not use also the non degeneration condition used by Vishik in [28, Theorem 2, pg 3].

For this, consider give a transversal disc $H \subset M$ of dimension $k+1$ such that $H \cap Z = \{p\}$, where $Z \subset \mathrm{Sing}_{k+1}(\mathcal{F})$. Taking local coordinates $z = (z_1, ..., z_{k+1})$ in $H$ around $p$, we can assume $p = 0$. Then, the restriction $\mathcal{F}|_H =: \mathcal{G}$ is an one-dimensional foliation on $H$ of which the singular set is given by $S(\mathcal{G}) = S(\mathcal{F}) \cap H$.

Given $Z \subset \mathrm{Sing}_{k+1}(\mathcal{F})$ an irreducible component. Let us denote by $[Z] \in H_{2(n-k-1)}(Z; \mathbb{C})$ its fundamental class and consider $\eta_Z$ its Poincaré dual in $H^{2(k+1)}(M; \mathbb{C})$. On the other hand, let $T_Z$ be the integration current associated to $Z$, that can be conveniently interpreted as a cohomology class in $M$, that is, $T_Z \in H^{2(k+1)}(M; \mathbb{C})$.

**Proposition 4.0.8** *$T_Z$ and $\eta_Z$ represent the same class in $H^{2(k+1)}(M; \mathbb{C})$.*

**Proof:** We will verify that the two $2(k+1)$-forms, seen as linear functional in $H^{2(n-k-1)}(M; \mathbb{C})$, act in the same way.

In fact, given $\omega$ a $2(n-k-1)$- form, we have by definition that $T_Z(\omega) = \displaystyle\int_Z \omega$. On the other hand, we recall the Poincaré duality

$$\Big(H^{2r}(M)\Big)^* \simeq H^{2(n-r)}(M).$$

We have the Poincaré dual, $\eta_Z$, associates a linear functional, denoted by ( by abuse of notation), $\eta_Z$ which satisfies $\eta_Z(\omega) = \displaystyle\int_Z i^*\omega = \int_Z \omega$, where $i$ denotes the inclusion map $Z \hookrightarrow M$. Therefore, $T_Z(\omega) = \eta_Z(\omega)$.

$\square$

**Theorem 4.0.9** *Let $\mathcal{F}$ be a holomorphic foliation of codimension $k$ on a compact complex manifold $M$. If Codim $S(\mathcal{F}) \geq k+1$ for each irreducible component $Z$ of Sing$_{k+1}(\mathcal{F})$ there exists a complex number $BB(\mathcal{F}, \varphi; Z)$ which is determined by the local behavior of $\mathcal{F}$ near $Z$, and is given by*

$$BB(\mathcal{F}, \varphi; Z) = \textit{Res}_p\Big[\varphi(JX)\frac{dz_1 \wedge ... \wedge dz_{k+1}}{X_1 ... X_{k+1}}\Big],$$

*where $X = (X_1, ..., X_{k+1})$ is the vector field that induces $\mathcal{G}$ around $p$ and $\varphi$ is a homogeneous symmetric polynomial of degree $k+1$.*

**Proof:** We have that, locally, there is a $k$-form $\omega$ that induces the foliation $\mathcal{F}$. Then, $\mathcal{G}$ is induced by restriction of this form to $H$, i.e.,

$$\widetilde{\omega} := \omega|_H.$$

We recall the isomorphism between $\Theta_U$ and $\Omega^k_U$ defined by the contraction by a vector field

$$i_{\frac{\partial}{\partial z_i}} dz_1 \wedge ... \wedge dz_{k+1} = (-1)^i dz_1 \wedge ... \wedge \widehat{dz_i} \wedge ... \wedge dz_{k+1}.$$

We can consider the vector field $X = (X_1, ..., X_{k+1})$ in $H$ dual to this $k$ - form $\widetilde{\omega}$ in $H$. If we denote by $\Theta = (\theta_{ls})$ the Bott connection matrix of $\mathcal{F}$, then $\widetilde{\Theta} := \Theta|_H = (\theta_{ls}|_H)$ represents the Bott connection matrix of $\mathcal{G}$ and we denote by $\widetilde{K}$ its curvature matrix to $\mathcal{G}$. The $(2k+1)$ - form $\varphi_\alpha(\widetilde{\theta}, \widetilde{K}) := \varphi_\alpha(\theta, K)|_H$ in $H$ satisfies

$$d\varphi_\alpha(\widetilde{\theta}, \widetilde{K}) = \varphi(\widetilde{K}),$$

where $\varphi(\widetilde{K})$ represents, in de Rham sense, the characteristic class $\varphi(\mathcal{N}_\mathcal{G})$.

We consider a $(2k+1)$- sphere $L_p \subset H \cap M$ then, we have by Corollary 4.0.7

$$\text{Res}_p\Big[\varphi(JX)\frac{dz_1 \wedge ... \wedge dz_{k+1}}{X_1 ... X_{k+1}}\Big] = BB(\mathcal{G}, \varphi; p).$$

By definition of the complex number $BB(\mathcal{G}, \varphi; p)$, see (4.1), one has

$$
\begin{aligned}
BB(\mathcal{G},\varphi;p) &= \left(\frac{1}{2\pi i}\right)^{k+1}\int_{L_p}\varphi_\alpha(\widetilde{\theta},\widetilde{K}) \\
&= \left(\frac{1}{2\pi i}\right)^{k+1}\int_{L_p}\varphi_\alpha(\theta,K)|_H \\
&= \left(\frac{1}{2\pi i}\right)^{k+1}\int_{L_p}\varphi_\alpha(\theta,K) = BB(\mathcal{F},\varphi,Z).
\end{aligned}
$$

□

**Corollary 4.0.10** *Considering the notations of Theorem 4.0.9, we have*

$$
Res(\mathcal{F},\varphi,Z) = Res_p\left[\varphi(JX)\frac{dz_1\wedge ...\wedge dz_{k+1}}{X_1...X_{k+1}}\right][Z],
$$

*where $[Z]$ denotes the fundamental class of $Z$ and $Res_p\left[\varphi(JX)\frac{dz_1\wedge ...\wedge dz_{k+1}}{X_1...X_{k+1}}\right]$ denotes the Grothendieck residue of $\mathcal{G}$ at $p$.*

**Example 4.0.11** *Recall the example 4.0.5, of the logarithmic foliation $\mathcal{F}$ on $\mathbb{P}^3$. In local coordinates $\{\mathbb{C}^3,(x,y,z)\}$, the singular set of $\mathcal{F}$ has one component $Z$ with 3 irreducible components $Z_1,Z_2,Z_3$. The Corollary 4.0.4 affirms that $BB(\mathcal{F},c_1^2;Z_i)=0$, for $i=1,2,3$. We will see this by applying the Corollary 4.0.10.*

*In fact, by Corollary 4.0.9, we have $BB(\mathcal{F},c_1^2;Z_i) = Res_{c_1^2}(\mathcal{G};p_i)$, where $\mathcal{G}$ is a foliation on $D_i$ with $D_i$ a 2-disc cutting transversally $Z_i$.*

*Consider $D_1$ given by $\{z=1\}$ then, we have*

$$
\omega|_{D_1} =: \omega_1 = ydx + xdy \quad \text{with dual vector field} \quad X_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.
$$

*Then, $D_1\cap Z_1 = \{p_1=(0,0,1)\}$. Now, a straightforward calculation shows that*

$$
JX_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{then,} \quad JX_1(p_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

*Thus,*

$$BB(\mathcal{F}, c_1^2; Z_1) = Res_{c_1^2}(\mathcal{G}; p_1) = \frac{c_1^2(JX_1(p_1))}{\det(JX_1(p_1))} = 0.$$

*The same holds for $Z_2$ and $Z_3$.*

The following example is due to D. Cerveau and A. Lins Neto, see [8]. It originates from the so-called exceptional component of the space of codimension one holomorphic foliations of degree 2 of $\mathbb{P}^n$.

**Example 4.0.12** *Consider $\mathcal{F}$ be a holomorphic foliation of codimension one on $\mathbb{P}^3$, given locally by the 1-form*

$$\omega = z(2y^2 - 3x)dx + z(3z - xy)dy - (xy^2 - 2x^2 + yz)dz.$$

*The singular set of this foliation has one connect component, denoted by $Z$, with 3 irreducible components, given by:*

*1) the twisted cubic* $\Gamma : \quad y \longmapsto (2/3y^2, y, 2/9y^3)$

*2) the quadric* $Q : \quad y \longmapsto (y^2/2, y, 0)$

*3) the line* $L : \quad y \longmapsto (0, y, 0)$.

*We consider the 2-plane $H$ given by $\{y = 1\}$ and we do the restriction of $\mathcal{F}$ to $H$. We have an one-dimensional holomorphic foliation, denoted by $\mathcal{G}$, given by the 1-form on $H$*

$$\widetilde{\omega} = (2z - 3xz)dx + (2x^2 - x - z)dz$$

*with dual vector field*

$$X = (2x^2 - x - z)\frac{\partial}{\partial x} + (-2z + 3xz)\frac{\partial}{\partial z}.$$

*The singular set of $\mathcal{G}$ is given by*

$$S(X) = \Big\{p_1 = (2/3, 1, 2/9); p_2 = (1/2, 1, 0); p_3 = (0, 1, 0)\Big\}.$$

*We know how to calculate the Grothendieck residue of the foliation $\mathcal{G}$*

$$Res_{c_1^2}(\mathcal{G};p_1) = \frac{c_1^2(JX(p_1))}{\det(JX(p_1))} = \frac{25}{6}$$

$$Res_{c_1^2}(\mathcal{G};p_2) = \frac{c_1^2(JX(p_2))}{\det(JX(p_2))} = -\frac{1}{2}$$

$$Res_{c_1^2}(\mathcal{G};p_3) = \frac{c_1^2(JX(p_3))}{\det(JX(p_3))} = \frac{9}{2}.$$

*Therefore, we get*

$$Res_{c_1^2}(\mathcal{F};Z) = Res_{c_1^2}(\mathcal{G};p_1)[\Gamma] + Res_{c_1^2}(\mathcal{G};p_2)[Q] + Res_{c_1^2}(\mathcal{G};p_3)[L]$$

$$= \frac{25}{6}[\Gamma] + (\frac{-1}{2})[Q] + \frac{9}{2}(\mathcal{G};p_3)[L],$$

*where* $[\Gamma]$ *denotes de fundamental class of the component* $\Gamma$*. By Baum-Bott theorem*

$$c_1^2(\mathcal{N}_{\mathcal{F}}) \frown [\mathbb{P}^3] = Res_{c_1^2}(\mathcal{F};Z)$$

$$(2+\deg(\mathcal{F}))^2 h^2 = Res_{c_1^2}(\mathcal{G};p_1)[\Gamma] + Res_{c_1^2}(\mathcal{G};p_2)[Q] + Res_{c_1^2}(\mathcal{G};p_3)[L]$$

$$16h^2 = \frac{25}{6}[\Gamma] + (\frac{-1}{2})[Q] + \frac{9}{2}(\mathcal{G};p_3)[L],$$

*where* $h$ *represent the hyperplane class. This exemple was considered by M. Soares in [24] with another calculations.*

The next example is very import, since we can use the Theorem 4.0.9 but we cannot use the Bott's Theorem in [3, Theorem 3, pg 285]. This confirms that our result is more general than Bott's result.

**Example 4.0.13** *We recall the logarithmic foliation* $\mathcal{F}$ *on* $\mathbb{P}^3$ *with homogeneous coordinates* $[X,Y,Z,T]$*, see examples 4.0.11 and 4.0.5, given locally by the following 1-form in the chart* $\{T=1\}$.

$$\omega = yzdx + xzdy + xydz.$$

*If we pull-back* $\omega$ *by the biholomorphism*

$$\begin{array}{rccc} \varphi : & \mathbb{P}^3 & \longrightarrow & \mathbb{C}^3 \\ & [X:Y:Z:T] & \longmapsto & (X/T, Y/T, Z/T) = (x,y,z) \end{array}$$

*we have the forms that defines globally $\mathcal{F}$ in homogeneous coordinates*

$$\widetilde{\omega} = YZTdX + XZTdY + XYTdZ - 3XYZdT.$$

*The singular set of $\mathcal{F}$ is the union of the lines $Z_1$, $Z_2$, $Z_3$, $Z_4 = \{T = X = 0\}$, $Z_5 = \{T = Y = 0\}$ and $Z_6 = \{T = X = 0\}$. Note that $d\omega|_{Z_i}$ is nowhere vanishing for $i = 4, 5, 6$ . We can use the process of the Theorem 4.0.9 to computing the residue of these components.*

*For $Z_4 = \{X = T = 0\}$ we can consider the local chart $U_y = \{Y = 1\}$. Then, we have,*

$$\omega_y := \widetilde{\omega}|_{U_y} = ztdx + xtdz - 3xzdt.$$

*Take, a 2-disc transversal to this component, for example, $D_2 = \{z = 1\}$.*

$$\omega_2 := \omega_y|_{D_2} = tdx - 3xdt.$$

*The dual vector field is $X_2 = -3x\dfrac{\partial}{\partial x} - t\dfrac{\partial}{\partial t}$ with singularity $Z_4 \cap D_2 = \{(0, 1, 0) =: p_4\}$.*

$$JX_2(p_4) = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}$$

*then, $Res_{c_1^2}(\mathcal{G}; p_4) = BB(\mathcal{F}, c_1^2; Z_4) = \dfrac{c_1^2(JX_2)(p_4)}{\det(JX_2)(p_4)} = \dfrac{16}{3}$.*

*An analogous calculation shows that*

$$Res_{c_1^2}(\mathcal{G}; p_5) = BB(\mathcal{F}, c_1^2; Z_5) = \frac{16}{3}.$$

$$Res_{c_1^2}(\mathcal{G}; p_6) = BB(\mathcal{F}, c_1^2; Z_6) = \frac{16}{3}.$$

*Therefore, Theorem 4.0.6 and Theorem 4.0.9 combine to imply*

$$c_1^2(\mathcal{N}_{\mathcal{F}}) \frown [\mathbb{P}^3] = \sum_{i=1}^{6} BB(\mathcal{F}, c_1^2; Z_i)[Z_i]$$

$$(2+\deg(\mathcal{F}))^2h^2 = \frac{16}{3}[Z_4] + \frac{16}{3}[Z_5] + \frac{16}{3}[Z_6]$$

$$16h^2 = \frac{16}{3}[Z_4] + \frac{16}{3}[Z_5] + \frac{16}{3}[Z_6],$$

*where $h$ represents the hyperplane class.*

Note that our result in Theorem 4.0.9 generalizes the Bott Theorem, because if we consider the hypothesis in [3], the Theorem 4.0.9 provides the Theorem 3 in [3].

Let $\mathcal{F}$ be a holomorphic foliation on $M$ of codimension $k$. We have that, in general, a connected irreducible component $Z$ of $\text{Sing}_{k+1}(\mathcal{F})$ comes endowed with a filtration. For given $p \in Z$ let us choose holomorphic vector fields $X_1, ...X_s$ defined on an open neighborhood $U_p$ of $p \in M$ and such that for all $x \in U_p$, the germs at $x$ of the holomorphic vector fields $X_1, ...X_s$ are in $\mathcal{F}_x$ and span $\mathcal{F}_x$ as a $\mathcal{O}_x$-module.

Define a subspace $V_p(\mathcal{F}) \subset T_pM$ by letting $V_p(\mathcal{F})$ be the subspace of $T_pM$ spanned by $X_1(p), ...X_s(p)$. We have

$$Z^{(i)} = \{p \in Z; \ \dim V_p(\mathcal{F}) \leq n-k-i\} \ \text{ for } \ i = 1, ..., n-k.$$

Then,

$$Z \supseteq Z^{(1)} \supseteq Z^{(2)} \supseteq ... \supseteq Z^{(n-k)}$$

is a filtration of $Z$.

If we assume that

$$\text{Codim } Z = k+1 \quad \text{and} \quad \text{Codim } Z^{(2)} < k+1$$

we have

**Corollary 4.0.14 (3, Theorem 3, pg 285)** *Let $\mathcal{F}$ be a holomorphic foliation of codimension $k$ on $M$. Then,*

$$\varphi(\mathcal{N}_{\mathcal{F}}) = \sum_{Z} BB(\mathcal{F}, \varphi; Z)[Z],$$

*where the sum is done over all irreducible components of* $\text{Sing}_{k+1}(\mathcal{F})$. *Then* $\alpha_*(BB(\mathcal{F}, \varphi; Z)[Z])$

*is the residue of* $\mathcal{F}$ *in* $Z$ *and moreover*

$$BB(\mathcal{F}, \varphi; Z) = Res_{\varphi}(\mathcal{G}; p)$$

*with* $Res_{\varphi}(\mathcal{G}; p)$ *representing the Grothendieck residue at* $p$ *of the foliation* $\mathcal{G}$*, i.e., of the restriction of the foliation* $\mathcal{F}$ *on a* $(k+1)$ *- disc* $H$ *with* $p = Z \cap H$*.*

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