UNIVERSIDADE FEDERAL DE MINAS GERAIS
INSTITUTO DE CIÊNCIAS EXATAS
DEPARTAMENTO DE MATEMÁTICA

PhD Thesis

On the inhomogeneous nonlinear Schrödinger equation

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Adviser: Luiz Gustavo Farah

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Aos meus Pais,

Modesta e Antonio.

ATA DA SEPTUAGÉSIMA SEXTA DEFESA DE TESE DO ALUNO CARLOS MANUEL GUZMÁN JIMÉNEZ, REGULARMENTE MATRICULADO NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA, DO INSTITUTO DE CIÊNCIAS EXATAS, DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA NO DIA 15 DE JUNHO DE 2016.

Aos quinze dias do mês de junho de 2016, às 13h00, na sala 3060, reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese do aluno **Carlos Manuel Guzmán Jiménez,** intitulada: *"On the Inhomogeneous Nonlinear Schrodinger Equation"*, requisito final para obtenção do Grau de doutor em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Luiz Gustavo Farah Dias, que participou através de videoconferência, após dar conhecimento aos presentes o teor das normas regulamentares do trabalho final, passou a palavra ao aluno para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa do aluno. Após a defesa, os membros da banca examinadora reuniram-se sem a presença do aluno e do público, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: o aluno foi considerado aprovado, por unanimidade. O resultado final foi comunicado publicamente ao aluno pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 15 de junho de 2016.

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Abstract

The purpose of this work is to investigate some questions about the initial value problem (IVP) for the inhomogeneous nonlinear Schrödinger equation (INLS)

$$iu_t + \Delta u + \lambda |x|^{-b}|u|^{\alpha}u = 0,$$

where $\lambda = \pm 1$, α and $b > 0$.

First, we consider the local and global well-posedness of the (IVP) for the (INLS) with initial data in $H^s(\mathbb{R}^N)$, $0 \leq s \leq 1$. We study this problem using the standard fixed point argument based on the Strichartz estimates related to the linear problem. These results are showed in Chapter 2.

In the sequel, in Chapter 3, we study scattering for the (INLS) in $H^1(\mathbb{R}^N)$ for the focusing case ($\lambda = 1$), with radial initial data. The method employed here is parallel to the approach developed by Kenig-Merle [26] in their study of the energy-critical NLS, Roudenko-Holmer [23] and Fang-Xie-Cazenave [11] (see also Guevara [22]) for the mass-supercritical and energy-subcritical NLS.

Keywords

Global well-posedness. Inhomogeneous nonlinear Schrödinger. Local well-posedness. Scattering.

Introduction

In this work, we study the initial value problem (IVP), also called the Cauchy problem for the inhomogenous nonlinear Schrödinger equation (INLS)

$$\begin{cases} i\partial_t u + \Delta u + \lambda |x|^{-b}|u|^{\alpha}u = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^N, \\ \qquad\quad u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $u = u(t,x)$ is a complex-valued function in space-time $\mathbb{R}\times\mathbb{R}^N$, $\lambda = \pm 1$ and $\alpha, b > 0$. The equation is called "focusing INLS" when $\lambda = +1$ and "defocusing INLS" when $\lambda = -1$.

The case $b = 0$ is the classical nonlinear Schrödinger equation (NLS) and is named in honor of the Austrian physicist Erwin Schrödinger who was one of the first researchers of Quantum Mechanics. It is a prototypical dispersive nonlinear partial differential equation (PDE) that has been derived in many areas of physics and analyzed mathematically for over 40 years. It appears as a model in hydrodynamics, nonlinear optics, quantum condensates, heat pulses in solids and various other nonlinear instability phenomena, see, for instance, Newell [36] and Scott-Chu-McLaughlin [38].

In the end of the last century, it was suggested that stable high power propagation can be achieved in a plasma by sending a preliminary laser beam that creates a channel with a reduced electron density, and thus reduces the nonlinearity inside the channel, see Gill [18] and Liu-Tripathi [34]. In this case, the beam propagation can be modeled by the inhomogeneous nonlinear

Schrödinger equation in the following form:

$$i\partial_t u + \Delta u + K(x)|u|^\alpha u = 0.$$

This model has been investigated by several authors, see, for instance, Merle [35] and Raphaël-Szeftel [37], for $k_1 < K(x) < k_2$ with $k_1, k_2 > 0$, and Fibich-Wang [13], for $K(\varepsilon|x|)$ with ε small and $K \in C^4(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. However, in these works $K(x)$ is bounded which is not verified in our case.

Notice that if $u(t,x)$ is solution of (1) so is $u_\delta(t,x) = \delta^{\frac{2-b}{\alpha}} u(\delta^2 t, \delta x)$, with initial data $u_{0,\delta}(x)$ for all $\delta > 0$. Computing the homogeneous Sobolev norm we get

$$\|u_{0,\delta}\|_{\dot{H}^s} = \delta^{s-\frac{N}{2}+\frac{2-b}{\alpha}} \|u_0\|_{\dot{H}^s}.$$

Thus, the scale-invariant Sobolev norm is $H^{s_c}(\mathbb{R}^N)$, where $s_c = \frac{N}{2} - \frac{2-b}{\alpha}$ (critical Sobolev index). Note that, if $s_c = 0$ (alternatively $\alpha = \frac{4-2b}{N}$) the problem is known as the mass-critical or L^2-critical; if $s_c = 1$ (alternatively $\alpha = \frac{4-2b}{N-2}$) it is called energy-critical or $\dot{H}^1$-critical, finally the problem is known as mass-supercritical and energy-subcritical if $0 < s_c < 1$. That is,

$$\begin{cases} \frac{4-2b}{N} < \alpha < \infty, & N = 1, 2 \\ \frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}, & N \geq 3. \end{cases} \tag{2}$$

On the other hand, the inhomogeneous nonlinear Schrödinger equation has the following conserved quantities: Mass $\equiv M[u(t)] = M[u_0]$ and Energy $\equiv E[u(t)] = E[u_0]$, where

$$M[u(t)] = \int_{\mathbb{R}^N} |u(t,x)|^2 dx \tag{3}$$

and

$$E[u(t)] = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dx - \frac{\lambda}{\alpha+2} \left\| |x|^{-b} |u|^{\alpha+2} \right\|_{L^1_x}. \tag{4}$$

Moreover, since

$$\|u_\delta\|_{L^2_x} = \delta^{-s_c} \|u\|_{L^2_x}, \quad \|\nabla u_\delta\|_{L^2_x} = \delta^{1-s_c} \|\nabla u\|_{L^2_x} \tag{5}$$

and

$$\left\| |x|^{-b} |u_\delta|^{\alpha+2} \right\|_{L^1_x} = \delta^{2(1-s_c)} \left\| |x|^{-b} |u|^{\alpha+2} \right\|_{L^1_x},$$

the following quantities enjoy a scaling invariant property, indeed

$$E[u_\delta]^{s_c} M[u_\delta]^{1-s_c} = E[u]^{s_c} M[u]^{1-s_c}, \quad \|\nabla u_\delta\|^{s_c}_{L^2_x} \|u_\delta\|^{1-s_c}_{L^2_x} = \|\nabla u\|^{s_c}_{L^2_x} \|u\|^{1-s_c}_{L^2_x}. \tag{6}$$

These quantities were introduced in [23] in the context of mass-supercritical and energy-subcritical NLS (equation (1) with $b = 0$), and they were used to understand the dichotomy between blowup/global regularity.

By Duhamel's Principle the solution of (1) is equivalent to

$$u(t,x) = U(t)u_0(x) + i\lambda \int_0^t U(t-t') \left(|x|^{-b} |u(t',x)|^\alpha u(t',x) \right) dt', \tag{7}$$

where $U(t)$ denotes the unitary group associated with the linear problem $i\partial_t u + \Delta u = 0$, with initial data u_0, defined by

$$U(t)u_0 = u_0 * (e^{-it|\xi|^2})^\vee.$$

Concerning the local and global well-posedness question, several results have been obtained for (1). Hereafter, we refer to the expression "well-posedness theory" in the sense of Kato according to the following definition.

Definition 0.1. We say that the IVP (1) is locally well-posed if for any $u_0 \in H^s(\mathbb{R}^N)$, there exist a time $T > 0$, a closed subspace X of $C([-T,T]; H^s(\mathbb{R}^N))$ and a unique solution u such that

1. u is solution of the integral equation (7),
2. $u \in X$ (Persistence),
3. the solution varies continuously depending upon the initial data (Continuous Dependence).

Global well-posedness requires that the same properties hold for all time $T > 0$.

The well-posedness theory for the INLS equation (1) was studied for many authors in recent years. Let us briefly recall the best results available in the literature. Cazenave [2] studied the well-posedness in $H^1(\mathbb{R}^N)$ using an abstract theory. To do this, he analyzed (1) in the sense of distributions, that is, $i\partial_t u + \Delta u + |x|^{-b}|u|^\alpha u = 0$ in $H^{-1}(\mathbb{R}^N)$ for almost all $t \in I$. Therefore, using some results of Functional Analysis and Semigroups of Linear Operators, he proved that it is appropriate to seek solutions of (1) satisfying

$$u \in C\left([0,T); H^1(\mathbb{R}^N)\right) \cap C^1\left([0,T); H^{-1}(\mathbb{R}^N)\right) \text{ for some } T > 0.$$

It was also proved that for the defocusing case ($\lambda = -1$) any local solution of the IVP (1) with $u_0 \in H^1(\mathbb{R}^N)$ extends globally in time.
Other authors like Genoud-Stuart [15] (see also references therein) also studied this problem for the focusing case ($\lambda = 1$). Using the abstract theory developed by Cazenave [2], they showed that the IVP (1) is locally well-posed in $H^1(\mathbb{R}^N)$ if $0 < \alpha < 2^*$, where

$$2^* := \begin{cases} \frac{4-2b}{N-2} & N \geq 3, \\ \infty & N = 1, 2. \end{cases} \tag{8}$$

Recently, using some sharp Gagliardo-Nirenberg inequalities, Genoud [14] and Farah [12] extended for the focusing INLS equation (1) some global well-posedness results obtained, respectively, by Weinstein [42] for the L^2-critical NLS equation and by Holmer-Roudenko [23] for the L^2-supercritical and H^1-subcritical case. These authors proved that the solution u of the Cauchy problem (1) is globally defined in $H^1(\mathbb{R}^N)$ quantifying the smallness condition in the initial data.

However, the abstract theory developed by Cazenave and later used by Genoud-Stuart [15] to show well-posedness for (1), does not give sufficient tools to study other interesting questions, for instance, scattering and blow up investigated by Kenig-Merle [26], Duyckaerts-Holmer-Roudenko [10] and others, for the NLS equation. To study these problems, the authors rely on the Strichartz estimates for NLS equation and the classical fixed point argument combining with the concentration-compactness and rigidity techniques.

Inspired by these papers and working toward the proof of scattering for the INLS equation, our first main goal here is to establish local and global results for the Cauchy problem (1) in $H^s(\mathbb{R}^N)$, with $0 \leq s \leq 1$, applying Kato's method. Indeed, we construct a closed subspace of $C\left([-T,T]; H^s(\mathbb{R}^N)\right)$ such that the operator defined by

$$G(u)(t) = U(t)u_0 + i\lambda \int_0^t U(t-t')|x|^{-b}|u(t')|^{\alpha}u(t')dt', \tag{9}$$

is stable and contractive in this space, thus by the contraction mapping principle we obtain a unique fixed point. The fundamental tools to prove these results are the classic Strichartz estimates satisfied by the solution of the linear Schrödinger equation. These results are presented in Chapter 2.

In the sequel, we consider the scattering problem for (1) in $H^1(\mathbb{R}^N)$. First, we need the following definition

Definition 0.2. A global solution $u(t)$ to the Cauchy problem (1) scatters forward in time in $H^1(\mathbb{R}^N)$, if there exists $\phi^+ \in H^1(\mathbb{R}^N)$ such that

$$\lim_{t\to+\infty} \|u(t) - U(t)\phi^+\|_{H^1_x} = 0.$$

Also, we say that $u(t)$ scatters backward in time if there exists $\phi^- \in H^1(\mathbb{R}^N)$ such that

$$\lim_{t\to-\infty} \|u(t) - U(t)\phi^-\|_{H^1_x} = 0.$$

Similarly, we can define scattering in $H^s(\mathbb{R}^N)$.

For the $3D$ defocusing NLS equation, scattering has been established for all H^1 solutions (regardless of size) by Ginibre-Velo [21] using a Morawetz inequality. This proof was simplified by Colliander-Keel-Staffilani-Takaoka-Tao [7] using a new interaction Morawetz inequality they discovered. Other authors like Killip-Tao-Visan [30], Tao-Visan-Zhang [40] and Killip-Visan-Zhang [32] extended this result for arbitrary dimension $N \geq 1$, showing scattering for the L^2-critical NLS in the defocusing case.

Regarding the focusing case, Kenig-Merle [26] developed a powerful method to study scattering and blow-up for the energy-critical NLS equation, which is commonly referred as the concentration-compactness and rigidity technique. The concentration-compactness method previously appeared in the context of the Wave equation in Gérard [16] and for the NLS equation in Keraani [28]. The rigidity argument (estimates on a localized variance) is the technique introduced by Merle in mid 1980's. Years later, Killip and Visan [31] extended Kenig-Merle's result for $N \geq 5$. Several authors also applied the concentration compactness and rigidity approach to study the L^2-supercritical and H^1-subcritical focusing NLS, see for instance [23], [10], [22] and [11]. They showed that, if $u_0 \in H^1(\mathbb{R}^N)$, $E(u_0)^{s_c^*} M(u_0)^{1-s_c^*} < E(Q)^{s_c^*} M(Q)^{1-s_c^*}$ and $\|\nabla u_0\|_{L^2}^{s_c^*} \|u_0\|_{L^2}^{1-s_c^*} < \|\nabla Q\|_{L^2}^{s_c^*} \|Q\|_{L^2}^{1-s_c^*}$, then the solution u scatters in $H^1(\mathbb{R}^N)$. Here, the critical Sobolex index is given by $s_c^* = \frac{N}{2} - \frac{2}{\alpha}$ and Q is the ground state solution of the following equation

$$-Q + \Delta Q + |Q|^{\alpha} Q = 0.$$

In the spirit of Holmer-Roudenko [23], we prove scattering with radial data for the Cauchy Problem (1) in the case $0 < s_c < 1$, i.e. L^2-supercritical and H^1-subcritical. This result is showed in Chapter 3.

Chapter 1

Preliminaries

In this first chapter, we introduce some general notations and give basic results that will be used along the work.

1.1 Notations

- We use c to denote various constants that may vary line by line.
- $C_{p,q}$ denotes a constant depending on p and q.
- Given any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c that $a \leq cb$.
- Given a set $A \subset \mathbb{R}^N$ then $A^C = \mathbb{R}^N \backslash A$ denotes the complement of A.
- Given $x, y \in \mathbb{R}^N$ then $x \cdot y$ denotes the inner product of x and y on $\mathbb{R}^N$.
- B denotes the unite ball in $\mathbb{R}^N$ defined by $B(0,1) = \{x \in \mathbb{R}^N : |x| \leq 1\}$.
- For $s \in \mathbb{R}$, J^s and D^s denote the Bessel and the Riesz potentials of order s, given via Fourier transform by the formulas

$$\widehat{J^s f}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) \quad \text{and} \quad \widehat{D^s f} = |\xi|^s \widehat{f}(\xi),$$

where the Fourier transform of $f(x)$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix.\xi} f(x)dx.$$

- We denote the support of a function f, by

$$supp(f) = \overline{\{f : \mathbb{R}^N \to \mathbb{C} : f(x) \neq 0\}}.$$

- $C_0^\infty(\mathbb{R}^N)$ denotes the space of functions with continuous derivatives of all orders and compact support in $\mathbb{R}^N$.

- We use $\|.\|_{L^p}$ to denote the $L^p(\mathbb{R}^N)$ norm with $p \geq 1$. If necessary, we use subscript to inform which variable we are concerned with.

1.2 Functional spaces

We start with the definition of the well-known Sobolev spaces and the mixed "space-time" Lebesgue spaces.

Definition 1.1. Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$. The homogeneous Sobolev space and the inhomogeneous Sobolev space are defined, respectively, as the completion of $\mathcal{S}(\mathbb{R}^N)$ with respect to the norms

$$\|f\|_{H^{s,r}} := \|J^s f\|_{L^r} \quad \text{and} \quad \|f\|_{\dot{H}^{s,r}} := \|D^s f\|_{L^r}.$$

If $r = 2$ we denote $H^{s,2}(\mathbb{R}^N)$ (or $\dot{H}^{s,2}(\mathbb{R}^N)$) simply by $H^s(\mathbb{R}^N)$ (or $\dot{H}^s(\mathbb{R}^N)$).

Definition 1.2. Let $1 \leq q, r \leq \infty$ and $T > 0$, the $L^q_{[0,T]}L^r_x$ and $L^q_T L^r_x$ spaces are defined, respectively, by

$$L^q_{[0,T]}L^r_x = \left\{ f : [0,T] \times \mathbb{R}^N \to \mathbb{C} : \|f\|_{L^q_{[0,T]}L^r_x} = \left(\int_0^T \|f(t,.)\|^q_{L^r_x} dt \right)^{\frac{1}{q}} < +\infty \right\}$$

$$L^q_T L^r_x = \left\{ f : [T,+\infty) \times \mathbb{R}^N \to \mathbb{C} : \|f\|_{L^q_T L^r_x} = \left(\int_T^{+\infty} \|f(t,.)\|^q_{L^r_x} dt \right)^{\frac{1}{q}} < +\infty \right\}.$$

Remark 1.3. In the case when $I=[0,T]$ and we restrict the x-integration to a subset $A\subset\mathbb{R}^N$ then the mixed norm will be denoted by $\|f\|_{L^q_IL^r_x(A)}$.

In the same way, we also define

$$L^q_IH^s_x=\left\{f:I\times\mathbb{R}^N\to\mathbb{C}:\|f\|_{L^q_IH^s_x}=\left(\int_I\|f(t,.)\|^q_{H^s_x}dt\right)^{\frac{1}{q}}<+\infty\right\},$$

where $s\in\mathbb{R}$.

Remark 1.4. When $f(t,x)$ is defined for every time $t\in\mathbb{R}$, we shall consider the notations $\|f\|_{L^q_tL^r_x}$ and $\|f\|_{L^q_tH^s_x}$.

Next we recall some Strichartz norms. We begin with the following definitions:

Definition 1.5. The pair (q,r) is called L^2-admissible if it satisfies the condition

$$\frac{2}{q}=\frac{N}{2}-\frac{N}{r},$$

where

$$\begin{cases}2\leq & r\leq\frac{2N}{N-2} \quad \text{if} \quad N\geq 3,\\ 2\leq & r<+\infty \quad \text{if} \quad N=2,\\ 2\leq & r\leq+\infty \quad \text{if} \quad N=1.\end{cases} \tag{1.1}$$

Remark 1.6. We included in the above definition the improvement, due to M. Keel and T. Tao [25], to the limiting case for Strichartz's inequalities.

Definition 1.7. We say the pair (q,r) is $\dot{H}^s$-admissible if[1]

$$\frac{2}{q}=\frac{N}{2}-\frac{N}{r}-s, \tag{1.2}$$

[1]It is worth mentioning that the pair $\left(\infty,\frac{2N}{N-2s}\right)$ also satisfies the relation (1.2), however, in our work we will not make use of this pair when we estimate the nonlinearity $|x|^{-b}|u|^\alpha u$.

where

$$\begin{cases} \frac{2N}{N-2s} < & r \leq \left(\frac{2N}{N-2}\right)^{-} & \text{if } N \geq 3, \\ \frac{2}{1-s} < & r \leq \left(\left(\frac{2}{1-s}\right)^{+}\right)' & \text{if } N = 2, \\ \frac{2}{1-2s} < & r \leq +\infty & \text{if } N = 1. \end{cases} \tag{1.3}$$

Here, a^- is a fixed number slightly smaller than a ($a^- = a - \varepsilon$ with $\varepsilon > 0$ small enough) and, in a similar way, we define a^+. Moreover, $(a^+)'$ is the number such that

$$\frac{1}{a} = \frac{1}{(a^+)'} + \frac{1}{a^+}, \tag{1.4}$$

that is $(a^+)' := \frac{a^+ . a}{a^+ - a}$. Finally we say that (q, r) is $\dot{H}^{-s}$-admissible if

$$\frac{2}{q} = \frac{N}{2} - \frac{N}{r} + s,$$

where

$$\begin{cases} \left(\frac{2N}{N-2s}\right)^{+} \leq & r \leq \left(\frac{2N}{N-2}\right)^{-} & \text{if } N \geq 3, \\ \left(\frac{2}{1-s}\right)^{+} \leq & r \leq \left(\left(\frac{2}{1+s}\right)^{+}\right)' & \text{if } N = 2, \\ \left(\frac{2}{1-2s}\right)^{+} \leq & r \leq +\infty & \text{if } N = 1. \end{cases} \tag{1.5}$$

Given $s \in \mathbb{R}$, let $\mathcal{A}_s = \{(q, r);\ (q, r) \text{ is } \dot{H}^s - \text{admissible}\}$ and (q', r') is such that $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$ for $(q, r) \in \mathcal{A}_s$. We define the following Strichartz norm

$$\|u\|_{S(\dot{H}^s)} = \sup_{(q,r)\in\mathcal{A}_s} \|u\|_{L_t^q L_x^r}$$

and the dual Strichartz norm

$$\|u\|_{S'(\dot{H}^{-s})} = \inf_{(q,r)\in\mathcal{A}_{-s}} \|u\|_{L_t^{q'} L_x^{r'}}.$$

Remark 1.8. Note that, if $s = 0$ then $\mathcal{A}_0$ is the set of all L^2-admissible pairs. Moreover, if $s = 0$, $S(\dot{H}^0) = S(L^2)$ and $S'(\dot{H}^0) = S'(L^2)$. We just write $S(\dot{H}^s)$ or $S'(\dot{H}^{-s})$ if the mixed norm is evaluated over $\mathbb{R} \times \mathbb{R}^N$. To indicate a restriction to a time interval $I \subset (-\infty, \infty)$ and a subset A of $\mathbb{R}^N$, we will consider the notations $S(\dot{H}^s(A); I)$ and $S'(\dot{H}^{-s}(A); I)$.

1.3 Basic estimates

In this section we list (without proving) some well known estimates associated to the linear Schrödinger propagator.

Lemma 1.9. *If $t \neq 0$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $p' \in [1,2]$, then $U(t) : L^{p'}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is continuous and*

$$\|U(t)f\|_{L^p_x} \lesssim |t|^{-\frac{N}{2}(\frac{1}{p'}-\frac{1}{p})}\|f\|_{L^{p'}}.$$

Proof. See Linares-Ponce [33, Lemma 4.1]. □

Lemma 1.10. ***(Sobolev embedding)*** *Let $s \in (0,+\infty)$ and $1 \le p < +\infty$.*

(i) If $s \in \left(0, \frac{N}{p}\right)$ then $H^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$ where $s = \frac{N}{p} - \frac{N}{r}$. Moreover,

$$\|f\|_{L^r} \le c\|D^s f\|_{L^p}. \tag{1.6}$$

(ii) If $s = \frac{N}{2}$ then $H^s(\mathbb{R}^N) \subset L^r(\mathbb{R}^N)$ for all $r \in [2,+\infty)$. Furthermore,

$$\|f\|_{L^r} \le c\|f\|_{H^s}. \tag{1.7}$$

Proof. See Bergh-Löfström [1, Theorem 6.5.1] (see also Linares-Ponce [33, Theorem 3.3] and Demenguel-Demenguel [9, Proposition 4.18]). □

Remark 1.11. Using (i), with $p = 2$, we have that $H^s(\mathbb{R}^N)$, with $s \in (0, \frac{N}{2})$, is continuously embedded in $L^r(\mathbb{R}^N)$ and

$$\|f\|_{L^r} \le c\|f\|_{H^s}, \tag{1.8}$$

where $r \in [2, \frac{2N}{N-2s}]$.

Lemma 1.12. ***(Fractional product rule)*** *Let* $s \in (0,1]$ *and* $1 < r, r_1, r_2, p_1, p_2 < +\infty$ *are such that* $\frac{1}{r} = \frac{1}{r_i} + \frac{1}{p_i}$ *for* $i = 1, 2$. *Then,*

$$\|D^s(fg)\|_{L^r} \leq c\|f\|_{L^{r_1}}\|D^s g\|_{L^{p_1}} + c\|D^s f\|_{L^{r_2}}\|g\|_{L^{p_2}}.$$

Proof. See Kenig-Ponce-Vega [27]. □

Lemma 1.13. ***(Fractional chain rule)*** *Suppose* $G \in C^1(\mathbb{C})$, $s \in (0,1]$, *and* $1 < r, r_1, r_2 < +\infty$ *are such that* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. *Then,*

$$\|D^s G(u)\|_{L^r} \leq c\|G^{'}(u)\|_{L^{r_1}}\|D^s u\|_{L^{r_2}}.$$

Proof. See Kenig-Ponce-Vega [27]. □

The main tool to show the local and global well-posedness are the well-known Strichartz estimates. See for instance Linares-Ponce [33] and Kato [24] (see also Holmer-Roudenko [23] and Guevara [22]).

Lemma 1.14. *The following statements hold.*

(i) (Linear estimates).

$$\|U(t)f\|_{S(L^2)} \leq c\|f\|_{L^2}, \tag{1.9}$$

$$\|U(t)f\|_{S(\dot{H}^s)} \leq c\|f\|_{\dot{H}^s}. \tag{1.10}$$

(ii) (Inhomogeneous estimates).

$$\left\|\int_{\mathbb{R}} U(t-t')g(.,t')dt'\right\|_{S(L^2)} + \left\|\int_0^t U(t-t')g(.,t')dt'\right\|_{S(L^2)} \leq c\|g\|_{S'(L^2)}, \tag{1.11}$$

$$\left\|\int_0^t U(t-t')g(.,t')dt'\right\|_{S(\dot{H}^s)} \leq c\|g\|_{S'(\dot{H}^{-s})}. \tag{1.12}$$

The relations (1.11) and (1.12) will be very useful to perform estimates on the nonlinearity $|x|^{-b}|u|^\alpha u$.

We end this section with three important remarks.

Remark 1.15. Let $F(x,z) = |x|^{-b}|z|^{\alpha}z$, and $f(z) = |z|^{\alpha}z$. The complex derivative of f is

$$f_z(z) = \frac{\alpha+2}{2}|z|^{\alpha} \quad \text{and} \quad f_{\bar{z}}(z) = \frac{\alpha}{2}|z|^{\alpha-2}z^2.$$

For $z, w \in \mathbb{C}$, we have

$$f(z) - f(w) = \int_0^1 \left[f_z(w+\theta(z-w))(z-w) + f_{\bar{z}}(w+\theta(z-w))\overline{(z-w)} \right] d\theta.$$

Thus,

$$|F(x,z) - F(x,w)| \lesssim |x|^{-b} \left(|z|^{\alpha} + |w|^{\alpha} \right) |z-w|. \tag{1.13}$$

Now we are interested in estimating $\nabla \left(F(x,z) - F(x,w) \right)$. A simple computation gives

$$\nabla F(x,z) = \nabla(|x|^{-b}) f(z) + |x|^{-b} \nabla f(z) \tag{1.14}$$

where

$$\nabla f(z) = f'(z)\nabla z = f_z(z)\nabla z + f_{\bar{z}}(z)\overline{\nabla z}.$$

First we estimate $|\nabla(f(z) - f(w))|$. Note that

$$\nabla(f(z) - f(w)) = f'(z)(\nabla z - \nabla w) + (f'(z) - f'(w))\nabla w. \tag{1.15}$$

So, since (the proof of the following estimate can be found in Cazenave-Fang-Han [3, Remark 2.3])

$$|f_z(z) - f_z(w)| \lesssim \begin{cases} (|z|^{\alpha-1} + |w|^{\alpha-1})|z-w| & \text{if } \alpha > 1, \\ |z-w|^{\alpha} & \text{if } 0 < \alpha \leq 1 \end{cases}$$

and

$$|f_{\bar{z}}(z) - f_{\bar{z}}(w)| \lesssim \begin{cases} (|z|^{\alpha-1} + |w|^{\alpha-1})|z-w| & \text{if } \alpha > 1, \\ |z-w|^{\alpha} & \text{if } 0 < \alpha \leq 1, \end{cases}$$

we get by (1.15)

$$|\nabla(f(z)-f(w))| \lesssim |z|^{\alpha}|\nabla(z-w)| + (|z|^{\alpha-1}+|w|^{\alpha-1})|\nabla w||z-w| \ \text{ if } \alpha>1$$

and

$$|\nabla(f(z)-f(w))| \lesssim |z|^{\alpha}|\nabla(z-w)| + |z-w|^{\alpha}|\nabla w| \quad \text{if } \ 0<\alpha\leq 1.$$

Therefore, by (1.14), (1.13) and the two last inequalities we obtain

$$|\nabla\left(F(x,z)-F(x,w)\right)| \lesssim |x|^{-b-1}(|z|^{\alpha}+|w|^{\alpha})|z-w|+|x|^{-b}|z|^{\alpha}|\nabla(z-w)|+M, \tag{1.16}$$

where

$$M \ \lesssim \ \begin{cases} |x|^{-b}\left(|z|^{\alpha-1}+|w|^{\alpha-1}\right)|\nabla w||z-w| & \text{if } \ \alpha>1 \\ |x|^{-b}|\nabla w||z-w|^{\alpha} & \text{if } \ 0<\alpha\leq 1. \end{cases}$$

Remark 1.16. Let $B=B(0,1)=\{x\in\mathbb{R}^N; |x|\leq 1\}$ and $b>0$. If $x\in B^C$ then $|x|^{-b}<1$ and so

$$\left\||x|^{-b}f\right\|_{L_x^r} \leq \|f\|_{L_x^r(B^C)} + \left\||x|^{-b}f\right\|_{L_x^r(B)}.$$

The next remark provides a condition for the integrability of $|x|^{-b}$ on B and B^C.

Remark 1.17. Note that if $\frac{N}{\gamma}-b>0$ then $\||x|^{-b}\|_{L^\gamma(B)}<+\infty$. Indeed

$$\int_B |x|^{-\gamma b}dx = c\int_0^1 r^{-\gamma b}r^{N-1}dr = c_1\, r^{N-\gamma b}\Big|_0^1 < +\infty \ \text{ if } \ \frac{N}{\gamma}-b>0.$$

Similarly, we have that $\||x|^{-b}\|_{L^\gamma(B^C)}$ is finite if $\frac{N}{\gamma}-b<0$.

Chapter 2

Well-posedness theory

In this chapter, we study the well-posedness of the Cauchy problem (1). We obtain local and global results for initial data in $H^s(\mathbb{R}^N)$, with $0 \leq s \leq 1$. To this end, we use a contraction mapping argument based on the Strichartz estimates given in Lemma 1.14.

2.0.1 Introduction

As mentioned before, our goal here lies in establishing local and global results for the Cauchy problem (1) in $H^s(\mathbb{R}^N)$ using the Kato's method. That is, we construct a closed subspace of $C\left([-T,T]; H^s(\mathbb{R}^N)\right)$ such that the integral equation (9) is stable and contractive in this space. Then by the Banach Fixed Point Theorem we obtain a unique fixed point, which is the solution of the integral equation (7).

Applying this technique in the case $b = 0$ (classical nonlinear Schrödinger equation), the IVP (1) has been extensively studied over the three decades. The L^2-theory was obtained by Y. Tsutsumi [41] in the case $0 < \alpha < \frac{4}{N}$. The H^1-subcritical case was studied by Ginibre-Velo [19]-[20] and Kato [24] (these papers also consider nonlinearities much more general than a pure power).

Later, Cazenave-Weissler [4] treated the L^2-critical case and the H^1-critical case.

We summarize the well known well-posedness theory for the NLS equation in the following theorem (we refer, for instance, to Linares-Ponce [33] for a proof of these results).

Theorem 2.1. *Consider the Cauchy problem for the NLS equation* ((1) *with* $b = 0$)*. Then, the following statements hold*

1. *If* $0 < \alpha < \frac{4}{N}$*, then the IVP* (1) *with* $b = 0$ *is locally and globally well posed in* $L^2(\mathbb{R}^N)$*. Moreover if* $\alpha = \frac{4}{N}$*, it is globally well posed in* $L^2(\mathbb{R}^N)$ *for small initial data.*

2. *The IVP* (1) *with* $b = 0$ *is locally well posed in* $H^1(\mathbb{R}^N)$ *if* $0 < \alpha \leq \frac{4}{N-2}$ *for* $N \geq 3$ *or* $0 < \alpha < +\infty$*, for* $N = 1, 2$*. Also, it is globally well-posed in* $H^1(\mathbb{R}^N)$ *if*

   (i) $\lambda < 0$,

   (ii) $\lambda > 0$ *and* $0 < \alpha < \frac{4}{N}$,

   (iii) $\lambda > 0$, $\frac{4}{N} < \alpha < \frac{4}{N-2}$ *and small initial data,*

   (iv) $\lambda > 0$, $\alpha = \frac{4}{N-2}$ *and small initial data.*

In addition, Cazenave-Weissler [5] and recently Cazenave-Fang-Han [3] showed that the IVP for the NLS is locally well posed in $H^s(\mathbb{R}^N)$ if $0 < \alpha \leq \frac{4}{N-2s}$ and $0 < s < \frac{N}{2}$, moreover the local solution extends globally in time for small initial data.

Our main interest in this chapter is to prove similar results for the INLS equation. First, we show local-well posedness in $H^s(\mathbb{R}^N)$, with $0 \leq s \leq 1$. These results are presented in Section 2.2. Next, in Section 2.3, we establish the global theory.

2.1 Local well posedness

In this section we give the precise statements of our main local results. First, we consider the local well posedness of the IVP (1) in $L^2(\mathbb{R}^N)$.

Theorem 2.2. *Let* $0 < \alpha < \frac{4-2b}{N}$ *and* $0 < b < \min\{2, N\}$*, then for all* $u_0 \in L^2(\mathbb{R}^N)$ *there exist* $T = T(\|u_0\|_{L^2}, N, \alpha) > 0$ *and a unique solution* u *of the integral equation* (7) *satisfying*

$$u \in C\left([-T,T]; L^2(\mathbb{R}^N)\right) \cap L^q\left([-T,T]; L^r(\mathbb{R}^N)\right),$$

for any (q,r) *L^2-admissible. Moreover, the continuous dependence upon the initial data holds.*

It is worth mentioning that the last theorem is an extension of the result by Tsutsumi [41] (which asserts local well-posedness for the NLS equation, (1) with $b = 0$, when $0 < \alpha < \frac{4}{N}$) to the INLS model.

Next, we treat the local well posedness in $H^s(\mathbb{R}^N)$ for $0 < s \leq 1$. Before stating the theorem, we define the following numbers

$$\widetilde{2} := \begin{cases} \frac{N}{3} & \text{if } N = 1,2,3, \\ 2 & \text{if } N \geq 4 \end{cases} \quad \text{and} \quad \alpha_s := \begin{cases} \frac{4-2b}{N-2s} & \text{if } s < \frac{N}{2}, \\ +\infty & \text{if } s = \frac{N}{2}. \end{cases} \tag{2.1}$$

Theorem 2.3. *Assume* $0 < \alpha < \alpha_s$*,* $0 < b < \widetilde{2}$ *and* $\max\{0, s_c\} < s \leq \min\{\frac{N}{2}, 1\}$*. If* $u_0 \in H^s(\mathbb{R}^N)$ *then there exist* $T = T(\|u_0\|_{H^s}, N, \alpha) > 0$ *and a unique solution* u *of the integral equation* (7) *with*

$$u \in C\left([-T,T]; H^s(\mathbb{R}^N)\right) \cap L^q\left([-T,T]; H^{s,r}(\mathbb{R}^N)\right)$$

for any (q,r) *L^2-admissible. Moreover, the continuous dependence upon the initial data holds.*

Remark 2.4. Note that $\alpha < \frac{4-2b}{N-2s}$ is equivalent to $s_c < s$. On the other hand, if $0 < \alpha < \frac{4-2b}{N}$ then $s_c < 0$, for this reason we add the restriction $s > \max\{0, s_c\}$ (recalling that $s_c = \frac{N}{2} - \frac{2-b}{\alpha}$) in the above statement.

As an immediate consequence of Theorem 2.3, we have that the Cauchy problem (1) is locally well-posed in $H^1(\mathbb{R}^N)$.

Corollary 2.5. *Assume $N \geq 2$, $0 < \alpha < \alpha_s$ and $0 < b < \widetilde{2}$. If $u_0 \in H^1(\mathbb{R}^N)$ then the initial value problem* (1) *is locally well posed and*

$$u \in C\left([-T,T]; H^1(\mathbb{R}^N)\right) \cap L^q\left([-T,T]; H^{1,r}(\mathbb{R}^N)\right),$$

for any (q, r) L^2-admissible.

Remark 2.6. One important difference of the previous results and its counterpart for the NLS model (see Theorem 2.1-(2)) is that we do not treat the critical case here, i.e. $\alpha = \frac{4-2b}{N-2s}$ with $0 \leq s \leq 1$ and $N \geq 3$. It is still an open problem.

Our plain is the following: Subsection 2.2.1 will be devoted to prove Theorem 2.2 and in Subsection 2.2.2 we show Theorem 2.3 and Corollary 2.5.

2.1.1 L^2-Theory

We begin with the following lemma. It provides an estimate for the INLS model nonlinearity in the Strichartz spaces.

Lemma 2.7. *Let $0 < \alpha < \frac{4-2b}{N}$ and $0 < b < \min\{2, N\}$. Then,*

$$\left\| |x|^{-b} |u|^\alpha v \right\|_{S'(L^2;I)} \leq c(T^{\theta_1} + T^{\theta_2}) \|u\|^\alpha_{S(L^2;I)} \|v\|_{S(L^2;I)}, \tag{2.2}$$

where $I = [0,T]$ and $c, \theta_1, \theta_2 > 0$.

Proof. By Remark 1.16, we have

$$\begin{aligned}\left\||x|^{-b}|u|^{\alpha}v\right\|_{S'(L^2;I)} &\leq \||u|^{\alpha}v\|_{S'(L^2(B^C);I)} + \left\||x|^{-b}|u|^{\alpha}v\right\|_{S'(L^2(B);I)} \\ &\equiv A_1 + A_2.\end{aligned}$$

Note that in the norm A_1 we do not have any singularity, so we know that

$$A_1 \leq cT^{\theta_1}\|u\|^{\alpha}_{S(L^2;I)}\|v\|_{S(L^2;I)}, \tag{2.3}$$

where $\theta_1 > 0$. See Kato [24, Theorem 0] (also see Linares-Ponce [33, Theorem 5.2 and Corollary 5.1]).

On the other hand, we need to find an admissible pair to estimate A_2. In fact, using the Hölder inequality twice we obtain

$$\begin{aligned}A_2 &\leq \left\||x|^{-b}|u|^{\alpha}v\right\|_{L_I^{q'}L_x^{r'}(B)} \leq \left\|\||x|^{-b}\|_{L^{\gamma}(B)}\|u\|^{\alpha}_{L_x^{\alpha r_1}}\|v\|_{L_x^r}\right\|_{L_I^{q'}} \\ &\leq \||x|^{-b}\|_{L^{\gamma}(B)}T^{\frac{1}{q_1}}\|u\|^{\alpha}_{L_I^{\alpha q_2}L_x^{\alpha r_1}}\|v\|_{L_I^qL_x^r} \\ &\leq T^{\frac{1}{q_1}}\||x|^{-b}\|_{L^{\gamma}(B)}\|u\|^{\alpha}_{L_I^qL_x^r}\|v\|_{L_I^qL_x^r},\end{aligned}$$

if (q,r) L^2-admissible and

$$\begin{cases}\frac{1}{r'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r} \\ \frac{1}{q'} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q} \\ q = \alpha q_2 \ , \ r = \alpha r_1.\end{cases} \tag{2.4}$$

In order to have $\||x|^{-b}\|_{L^{\gamma}(B)} < +\infty$ we need $\frac{N}{\gamma} > b$, by Remark 1.17. Hence, in view of (2.4) (q,r) must satisfy

$$\begin{cases}\frac{N}{\gamma} = N - \frac{N(\alpha+2)}{r} > b \\ \frac{1}{q_1} = 1 - \frac{\alpha+2}{q}.\end{cases} \tag{2.5}$$

From the first equation in (2.5) we have $N-b-\frac{N(\alpha+2)}{r} > 0$, which is equivalent to

$$\alpha < \frac{r(N-b)-2N}{N}, \tag{2.6}$$

for $r > \frac{2N}{N-b}$. By hypothesis $\alpha < \frac{4-2b}{N}$, then setting r such that

$$\frac{r(N-b)-2N}{N} = \frac{4-2b}{N},$$

we get[1] $r = \frac{4-2b+2N}{N-b}$ satisfying (2.6). Consequently, since (q,r) is L^2-admissible we obtain $q = \frac{4-2b+2N}{N}$. Next, applying the second equation in (2.5) we deduce

$$\frac{1}{q_1} = \frac{4-2b-\alpha N}{4-2b+2N},$$

which is positive by the hypothesis $\alpha < \frac{4-2b}{N}$. Thus,

$$A_2 \leq cT^{\theta_2} \|u\|^{\alpha}_{S(L^2;I)} \|v\|_{S(L^2;I)}, \tag{2.7}$$

where $\theta_2 = \frac{1}{q_1}$. Therefore, combining (2.3) and (2.7) we prove (2.2). □

Our goal now is to show Theorem 2.2.

Proof of Theorem 2.2. We define

$$X = C\left([-T,T]; L^2(\mathbb{R}^N)\right) \bigcap L^q\left([-T,T]; L^r(\mathbb{R}^N)\right),$$

for any (q,r) L^2-admissible, and

$$B(a,T) = \{u \in X : \|u\|_{S(L^2;[-T,T])} \leq a\},$$

where a and T are positive constants to be determined later. We follow the standard fixed point argument to prove this result. It means that for appropriate values of a, T we shall show that

$$G(u)(t) = G_{u_0}(u)(t) = U(t)u_0 + i\lambda \int_0^t U(t-t')(|x|^{-b}|u|^{\alpha}u)(t')dt' \tag{2.8}$$

defines a contraction map on $B(a,T)$.

[1]Note that, since $0 < b < \min\{N,2\}$ the denominator of r is positive and $r > \frac{2N}{N-b}$. Moreover, by a simple computations we have $2 \leq r \leq \frac{2N}{N-2}$ if $N \geq 3$, and $2 \leq r < +\infty$ if $N = 1,2$, that is r satisfies (1.1). Therefore, the pair (q,r) above defined is L^2-admissible.

Without loss of generality we consider only the case $t > 0$. Applying Strichartz inequalities (1.9) and (1.11), we have

$$\|G(u)\|_{S(L^2;I)} \leq c\|u_0\|_{L^2} + c\||x|^{-b}|u|^{\alpha+1}\|_{S'(L^2;I)},$$

where $I = [0, T]$. Moreover, Lemma 2.7 yields

$$\begin{aligned} \|G(u)\|_{S(L^2;I)} &\leq c\|u_0\|_{L^2} + c(T^{\theta_1} + T^{\theta_2})\|u\|^{\alpha+1}_{S(L^2;I)} \\ &\leq c\|u_0\|_{L^2} + c(T^{\theta_1} + T^{\theta_2})a^{\alpha+1}, \end{aligned}$$

provided $u \in B(a, T)$. Hence,

$$\|G(u)\|_{S(L^2;[-T,T])} \leq c\|u_0\|_{L^2} + c(T^{\theta_1} + T^{\theta_2})a^{\alpha+1}.$$

Next, choosing $a = 2c\|u_0\|_{L^2}$ and $T > 0$ such that

$$ca^{\alpha}(T^{\theta_1} + T^{\theta_2}) < \frac{1}{4}, \tag{2.9}$$

we conclude $G(u) \in B(a, T)$.

Now we prove that G is a contraction. Again using Strichartz inequality (1.11) and (1.13), we deduce

$$\begin{aligned} \|G(u) - G(v)\|_{S(L^2;I)} &\leq c\left\||x|^{-b}(|u|^{\alpha}u - |v|^{\alpha}v)\right\|_{S'(L^2;I)} \\ &\leq c\left\||x|^{-b}|u|^{\alpha}|u - v|\right\|_{S'(L^2;I)} \\ &\quad + c\left\||x|^{-b}|v|^{\alpha}|u - v|\right\|_{S'(L^2;I)} \\ &\leq c(T^{\theta_1} + T^{\theta_2})\|u\|^{\alpha}_{S(L^2;I)}\|u - v\|_{S(L^2;I)} \\ &\quad + c(T^{\theta_1} + T^{\theta_2})\|v\|^{\alpha}_{S(L^2;I)}\|u - v\|_{S(L^2;I)}, \end{aligned}$$

where $I = [0, T]$. That is,

$$\begin{aligned} \|G(u) - G(v)\|_{S(L^2;I)} &\leq c(T^{\theta_1} + T^{\theta_2})\left(\|u\|^{\alpha}_{S(L^2;I)} + \|v\|^{\alpha}_{S(L^2;I)}\right)\|u - v\|_{S(L^2;I)} \\ &\leq 2c(T^{\theta_1} + T^{\theta_2})a^{\alpha}\|u - v\|_{S(L^2;I)}, \end{aligned}$$

provided $u, v \in B(a,T)$. Therefore, the inequality (2.9) implies that

$$\begin{aligned}\|G(u)-G(v)\|_{S(L^2;[-T,T])} &\leq 2c(T^{\theta_1}+T^{\theta_2})a^{\alpha}\|u-v\|_{S(L^2;[-T,T])}\\ &< \frac{1}{2}\|u-v\|_{S(L^2;[-T,T])},\end{aligned}$$

i.e., G is a contraction on $S(a,T)$.

The proof of the continuous dependence is similar to the one given above and it will be omitted. □

2.1.2 H^s-Theory

The aim of this subsection is to prove the local well posedness in $H^s(\mathbb{R}^N)$ with $0 < s \leq 1$ (Theorem 2.3) as well as Corollary 2.5. Before doing that we establish useful estimates for the nonlinearity $|x|^{-b}|u|^{\alpha}u$. First, we consider the nonlinearity in the space $S'(L^2)$ and in the sequel in the space $D^{-s}S'(L^2)$, that is, we estimate the norm $\left\||x|^{-b}|u|^{\alpha}u\right\|_{S'(L^2;I)}$ and $\left\|D^s(|x|^{-b}|u|^{\alpha}u)\right\|_{S'(L^2;I)}$.

We start this subsection with two remarks.

Remark 2.8. Since we will use the Sobolev embedding (Lemma 1.10), we divide our study in three cases: $N \geq 3$ and $s < \frac{N}{2}$; $N = 1,2$ and $s < \frac{N}{2}$; $N = 1,2$ and $s = \frac{N}{2}$. (see respectively Lemmas 2.10, 2.11 and 2.12 bellow).

Remark 2.9. Another interesting remark is the following claim

$$D^s(|x|^{-b}) = C_{N,b}|x|^{b-s}. \tag{2.10}$$

Indeed, we use the facts $\widehat{D^s f} = |\xi|^s \widehat{f}$ and $\widehat{(|x|^{-\beta})} = \frac{C_{N,\beta}}{|\xi|^{N-\beta}}$ for $\beta \in (0,N)$. Let $f(x) = |x|^{-b}$, we have

$$\widehat{D^s(|x|^{-b})} = |\xi|^s\widehat{(|x|^{-b})} = |\xi|^s\frac{C_{N,\beta}}{|\xi|^{N-b}} = \frac{C_{N,\beta}}{|\xi|^{N-(b+s)}}.$$

Since $0 < b < \widetilde{2}$ and $0 < s \leq \min\{\frac{N}{2}, 1\}$ then $0 < b+s < N$, so taking $\beta = s+b$, we get

$$D^s(|x|^{-b}) = \left(\frac{C_{N,\beta}}{|y|^{N-(b+s)}}\right)^{\vee} = C_{N,\beta}|x|^{b-s}.$$

Lemma 2.10. *Let $N \geq 3$ and $0 < b < \widetilde{2}$. If $s < \frac{N}{2}$ and $0 < \alpha < \frac{4-2b}{N-2s}$ then the following statements hold:*

(i) $\left\| |x|^{-b}|u|^{\alpha}v \right\|_{S'(L^2;I)} \leq c(T^{\theta_1} + T^{\theta_2})\|D^s u\|^{\alpha}_{S(L^2;I)}\|v\|_{S(L^2;I)}$

(ii) $\left\| D^s(|x|^{-b}|u|^{\alpha}u) \right\|_{S'(L^2;I)} \leq c(T^{\theta_1} + T^{\theta_2})\|D^s u\|^{\alpha+1}_{S(L^2;I)}$,

where $I = [0,T]$ and $c, \theta_1, \theta_2 > 0$.

Proof. (i) We divide the estimate in the regions B and B^C, indeed

$$\begin{aligned}\left\| |x|^{-b}|u|^{\alpha}v \right\|_{S'(L^2;I)} &\leq \left\| |x|^{-b}|u|^{\alpha}v \right\|_{S'(L^2(B^C);I)} + \left\| |x|^{-b}|u|^{\alpha}v \right\|_{S'(L^2(B);I)} \\ &\equiv B_1 + B_2.\end{aligned}$$

First, we consider B_1. Let (q_0, r_0) L^2-admissible given by[2]

$$q_0 = \frac{4(\alpha+2)}{\alpha(N-2s)} \quad \text{and} \quad r_0 = \frac{N(\alpha+2)}{N+\alpha s}. \tag{2.11}$$

If $s < \frac{N}{2}$ then $s < \frac{N}{r_0}$ and so using the Sobolev inequality (1.6) and the Hölder inequality twice, we get

$$\begin{aligned} B_1 &\leq \left\| |x|^{-b}|u|^{\alpha}v \right\|_{L_I^{q_0'}L_x^{r_0'}(B^C)} \leq \left\| \||x|^{-b}\|_{L^{\gamma}(B^C)} \|u\|^{\alpha}_{L_x^{\alpha r_1}} \|v\|_{L_x^{r_0}} \right\|_{L_I^{q_0'}} \\ &\leq \||x|^{-b}\|_{L^{\gamma}(B^C)} \left\| \|D^s u\|^{\alpha}_{L_x^{r_0}} \|v\|_{L_x^{r_0}} \right\|_{L_I^{q_0'}} \\ &\leq \||x|^{-b}\|_{L^{\gamma}(B^C)} T^{\frac{1}{q_1}} \|D^s u\|^{\alpha}_{L_I^{\alpha q_2}L_x^{r_0}} \|v\|_{L_I^{q_0}L_x^{r_0}} \\ &= \||x|^{-b}\|_{L^{\gamma}(B^C)} T^{\frac{1}{q_1}} \|D^s u\|^{\alpha}_{L_I^{q_0}L_x^{r_0}} \|v\|_{L_I^{q_0}L_x^{r_0}}, \end{aligned} \tag{2.12}$$

where

$$\begin{cases} \frac{1}{r_0'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r_0} \\ \frac{1}{q_0'} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_0} \\ q_0 = \alpha q_2 \ , \ s = \frac{N}{r_0} - \frac{N}{\alpha r_1}. \end{cases} \tag{2.13}$$

[2]It is not difficult to check that q_0 and r_0 satisfy the conditions of admissible pair, see (1.1).

In view of Remark 1.17 in order to show that the first norm in the right hand side of (2.12) is bounded we need $\frac{N}{\gamma} - b < 0$. Indeed, (2.13) is equivalent to

$$\begin{cases} \frac{N}{\gamma} = & N - \frac{2N}{r_0} - \frac{N\alpha}{r_0} + \alpha s \\ \frac{1}{q_1} = & 1 - \frac{\alpha+2}{q_0}, \end{cases}$$

which implies, by (2.11)

$$\frac{N}{\gamma} = 0 \quad \text{ and } \quad \frac{1}{q_1} = \frac{4-\alpha(N-2s)}{4}. \tag{2.14}$$

Therefore $\frac{N}{\gamma} - b < 0$ and $\frac{1}{q_1} > 0$, by our hypothesis $\alpha < \frac{4-2b}{N-2s}$. Therefore, setting $\theta_1 = \frac{1}{q_1}$ we deduce

$$B_1 \leq cT^{\theta_1} \|D^s u\|^{\alpha}_{S(L^2;I)} \|v\|_{S(L^2;I)}. \tag{2.15}$$

We now estimate B_2. To do this, we use similar arguments as the ones in the estimation of A_2 in Lemma 2.7. It follows from Hölder's inequality twice and Sobolev embedding (1.6) that

$$\begin{aligned} B_2 &\leq \left\| |x|^{-b} |u|^{\alpha} v \right\|_{L_I^{q'} L_x^{r'}(B)} \leq \left\| \||x|^{-b}\|_{L^\gamma(B)} \|u\|^{\alpha}_{L_x^{\alpha r_1}} \|v\|_{L_x^r} \right\|_{L_I^{q'}} \\ &\leq \left\| \||x|^{-b}\|_{L^\gamma(B)} \|D^s u\|^{\alpha}_{L_x^r} \|v\|_{L_x^r} \right\|_{L_I^{q'}} \\ &\leq \||x|^{-b}\|_{L^\gamma(B)} T^{\frac{1}{q_1}} \|D^s u\|^{\alpha}_{L_I^{\alpha q_2} L_x^r} \|v\|_{L_I^q L_x^r} \\ &= \||x|^{-b}\|_{L^\gamma(B)} T^{\frac{1}{q_1}} \|D^s u\|^{\alpha}_{L_I^q L_x^r} \|v\|_{L_I^q L_x^r} \end{aligned}$$

if (q,r) L^2-admissible and the following system is satisfied

$$\begin{cases} \frac{1}{r'} = & \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r} \\ s = & \frac{N}{r} - \frac{N}{\alpha r_1}, \quad s < \frac{N}{r} \\ \frac{1}{q'} = & \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q} \\ q = & \alpha q_2. \end{cases} \tag{2.16}$$

Similarly as in Lemma 2.7 we need to check that $\frac{N}{\gamma} > b$ (so that $\||x|^{-b}\|_{L^\gamma(B)}$ is finite) and $\frac{1}{q_1} > 0$ for a certain choice of (q,r) L^2-admissible pair. From (2.16) this is equivalent to

$$\begin{cases} \frac{N}{\gamma} = & N - \frac{2N}{r} - \frac{N\alpha}{r} + \alpha s > b \\ \frac{1}{q_1} = & 1 - \frac{\alpha+2}{q} > 0. \end{cases} \tag{2.17}$$

The first equation in (2.17) implies that $\alpha < \frac{(N-b)r-2N}{N-rs}$ (assuming $s < \frac{N}{r}$), then let us choose r such that

$$\frac{(N-b)r-2N}{N-rs} = \frac{4-2b}{N-2s}$$

since, by our hypothesis $\alpha < \frac{4-2b}{N-2s}$. Therefore r and q are given by[3]

$$r = \frac{2N[N-b+2(1-s)]}{N(N-2s)+4s-bN} \quad \text{and} \quad q = \frac{2[N-b+2(1-s)]}{N-2s}, \tag{2.18}$$

where we have used that (q,r) is a L^2-admissible pair to compute the value of q. Note that $s < \frac{N}{r}$ if, and only if, $b + 2s - N < 0$. Since $s \leq 1$, $b < \widetilde{2}$ (see (2.1)) and $N \geq 3$ it is easy to see that $s < \frac{N}{r}$ holds. In addition, from the second equation of (2.17) and (2.18) we also have

$$\frac{1}{q_1} = \frac{4-2b-\alpha(N-2s)}{2(N-b+2-2s)} > 0, \tag{2.19}$$

since $\alpha < \frac{4-2b}{N-2s}$.

Hence,

$$B_2 \leq cT^{\theta_2}\|D^s u\|^\alpha_{S(L^2;I)}\|v\|_{S(L^2;I)}, \tag{2.20}$$

where θ_2 is given by (2.19). Finally, collecting the inequalities (2.15) and (2.20) we obtain (i).

(ii) Observe that

$$\left\|D^s(|x|^{-b}|u|^\alpha u)\right\|_{S'(L^2;I)} \leq C_1 + C_2,$$

[3]It is easy to see that $r > 2$ if, and only if, $s < \frac{N}{2}$ and $r < \frac{2N}{N-2}$ if, and only if, $b < 2$. Therefore the pair (q,r) given in (2.18) is L^2-admissible.

where

$$C_1 = \left\| D^s(|x|^{-b}|u|^\alpha u)\right\|_{S'(L^2(B^C);I)} \ \text{ and } \ C_2 = \left\| D^s(|x|^{-b}|u|^\alpha u)\right\|_{S'(L^2(B);I)}.$$

We first consider C_1. To this end, we use the same admissible pair (q_0, r_0) used to estimate the term B_1 in item (i). Indeed, let

$$C_{11}(t) = \left\| D^s(|x|^{-b}|u|^\alpha u)\right\|_{L_x^{r_0'}(B^C)}$$

then Lemma 1.12 (fractional product rule), Lemma 1.13 (fractional chain rule) and Remark 2.9 yield

$$\begin{aligned} C_{11}(t) \leq & \||x|^{-b}\|_{L^\gamma(B^C)}\|D^s(|u|^\alpha u)\|_{L_x^\beta} + \|D^s(|x|^{-b})\|_{L^d(B^C)}\|u\|^{\alpha+1}_{L_x^{(\alpha+1)e}} \\ \leq & \||x|^{-b}\|_{L^\gamma(B^C)}\|u\|^\alpha_{\alpha r_1}\|D^s u\|_{L_x^{r_0}} + \||x|^{-b-s}\|_{L^d(B^C)}\|D^s u\|^{\alpha+1}_{L_x^{r_0}} \\ \leq & \||x|^{-b}\|_{L^\gamma(B^C)}\|D^s u\|^{\alpha+1}_{L_x^{r_0}} + \||x|^{-b-s}\|_{L^d(B^C)}\|D^s u\|^{\alpha+1}_{L_x^{r_0}}, \end{aligned} \tag{2.21}$$

where we also have used the Sobolev inequality (1.6) and (2.10). Moreover, we have the following relations

$$\begin{cases} \frac{1}{r_0'} = & \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e} \\ \frac{1}{\beta} = & \frac{1}{r_1} + \frac{1}{r_0} \\ s = & \frac{N}{r_0} - \frac{N}{\alpha r_1}; \quad s < \frac{N}{r_0} \\ s = & \frac{N}{r_0} - \frac{N}{(\alpha+1)e} \end{cases}$$

which implies that

$$\begin{cases} \frac{N}{\gamma} = & N - \frac{2N}{r_0} - \frac{\alpha N}{r_0} + \alpha s \\ \frac{N}{d} = & N - \frac{2N}{r_0} - \frac{\alpha N}{r_0} + \alpha s + s. \end{cases} \tag{2.22}$$

Note that, in view of (2.11) we have $\frac{N}{\gamma} - b < 0$ and $\frac{N}{d} - b - s < 0$. These relations imply that $\||x|^{-b}\|_{L^\gamma(B^C)}$ and $\||x|^{-b-s}\|_{L^d(B^C)}$ are bounded quantities (see Remark 1.17). Therefore, it follows from (2.21) that

$$C_{11}(t) \leq c\|D^s u\|^{\alpha+1}_{L_x^{r_0}}.$$

On the other hand, using $\frac{1}{q_0'} = \frac{1}{q_1} + \frac{\alpha+1}{q_0}$ and applying the Hölder inequality in the time variable we conclude

$$\|C_{11}\|_{L_I^{q_0'}} \leq cT^{\frac{1}{q_1}} \|D^s u\|^{\alpha+1}_{L_I^{q_0} L_x^{r_0}},$$

where $\frac{1}{q_1}$ is given in (2.14). The estimate of C_1 is finished since $C_1 \leq \|C_{11}\|_{L_I^{q_0'}}$.

Next, we consider C_2. Let $C_{22}(t) = \left\|D^s(|x|^{-b}|u|^\alpha u)\right\|_{L_x^{r'}(B)}$, we have $C_2 \leq \|C_{22}\|_{L_I^{q'}}$. Using the same arguments as in the estimate of C_{11} we obtain

$$C_{22}(t) \leq \||x|^{-b}\|_{L^\gamma(B)} \|D^s u\|^{\alpha+1}_{L_x^r} + \||x|^{-b-s}\|_{L^d(B)} \|D^s u\|^{\alpha+1}_{L_x^r}, \tag{2.23}$$

if (2.22) is satisfied replacing r_0 by r (to be determined later), that is

$$\begin{cases} \frac{N}{\gamma} = & N - \frac{2N}{r} - \frac{\alpha N}{r} + \alpha s \\ \frac{N}{d} = & N - \frac{2N}{r} - \frac{\alpha N}{r} + \alpha s + s. \end{cases} \tag{2.24}$$

In order to have that $\||x|^{-b}\|_{L^\gamma(B)}$ and $\||x|^{-b-s}\|_{L^d(B)}$ are bounded, we need $\frac{N}{\gamma} > b$ and $\frac{N}{d} > b + s$, respectively, by Remark 1.17. Therefore, since the first equation in (2.24) is the same as the first one in (2.17), we choose r as in (2.18). So we get $\frac{N}{\gamma} > b$, which also implies that $\frac{N}{d} - s > b$. Finally, (2.23) and the Hölder inequality in the time variable yield

$$\begin{aligned} C_2 &\leq cT^{\frac{1}{q_1}} \|D^s u\|^{\alpha+1}_{L_I^{(\alpha+1)q_2} L_x^r} \\ &= cT^{\frac{1}{q_1}} \|D^s u\|^{\alpha+1}_{L_I^q L_x^r}, \end{aligned}$$

where

$$\frac{1}{q'} = \frac{1}{q_1} + \frac{1}{q_2} \qquad q = (\alpha+1)q_2. \tag{2.25}$$

Notice that (2.25) is exactly to the second equation in (2.17), so $\frac{1}{q_1} > 0$ (see the relation (2.19)). This completes the proof of Lemma 2.10. □

Notice that Lemma 2.10 only holds for $N \geq 3$, since the admissible par (q,r) defined in (2.18) doesn't satisfy the condition $s < \frac{N}{r}$, for $N = 1,2$. In the next lemma we study these cases.

Lemma 2.11. *Let $N = 1,2$ and $0 < b < \widetilde{2}$. If $s < \frac{N}{2}$ and $0 < \alpha < \frac{4-2b}{N-2s}$ then*

(i) $\left\| |x|^{-b}|u|^{\alpha}v \right\|_{S'(L^2;I)} \leq c(T^{\theta_1} + T^{\theta_2}) \|D^s u\|^{\alpha}_{S(L^2;I)} \|v\|_{S(L^2;I)}$

(ii) $\left\| D^s(|x|^{-b}|u|^{\alpha}u) \right\|_{S'(L^2;I)} \leq c(T^{\theta_1} + T^{\theta_2}) \|D^s u\|^{\alpha+1}_{S(L^2;I)}$,

where $I = [0,T]$ and $c, \theta_1, \theta_2 > 0$.

Proof. (i) As before, we divide the estimate in B and B^C. The estimate on B^C is the same as the term B_1 in Lemma 2.10 (i), since (q_0, r_0) given in (2.11) is L^2-admissible for $s < \frac{N}{2}$ in all dimensions. Thus we only consider the estimate on B.

Indeed, set the L^2-admissible pair $(\bar{q}, \bar{r}) = (\frac{8}{2N-s}, \frac{4N}{s})$. We deduce from the Hölder inequality twice and Sobolev embedding (1.6)

$$\begin{aligned}
\left\| |x|^{-b}|u|^{\alpha}v \right\|_{L_I^{\bar{q}'} L_x^{\bar{r}'}(B)} &\leq \left\| \| |x|^{-b} \|_{L^{\gamma}(B)} \|u\|^{\alpha}_{L_x^{\alpha r_1}} \|v\|_{L_x^r} \right\|_{L_I^{q'}} \\
&\leq \| |x|^{-b} \|_{L^{\gamma}(B)} T^{\frac{1}{q_1}} \|D^s u\|^{\alpha}_{L_I^{\alpha q_2} L_x^r} \|v\|_{L_I^q L_x^r} \\
&= \| |x|^{-b} \|_{L^{\gamma}(B)} T^{\frac{1}{q_1}} \|D^s u\|^{\alpha}_{L_I^q L_x^r} \|v\|_{L_I^q L_x^r}
\end{aligned}$$

if (q,r) is L^2-admissible and the following system is satisfied

$$\begin{cases}
\frac{1}{\bar{r}'} = & \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{r} \\
s = & \frac{N}{r} - \frac{N}{\alpha r_1}; \quad s < \frac{N}{r} \\
\frac{1}{\bar{q}'} = & \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q} \\
q = & \alpha q_2.
\end{cases} \tag{2.26}$$

Using the values of $\bar{q}$ and $\bar{r}$ given above, the previous system is equivalent to

$$\begin{cases} \frac{N}{\gamma} = \frac{4(N-b)-s}{4} - \frac{N}{r} - \frac{\alpha(N-sr)}{r} + b \\ \frac{1}{q_1} = \frac{8-2N-s}{8} - \frac{\alpha+1}{q}. \end{cases} \tag{2.27}$$

From the first equation in (2.27) if $\alpha < \frac{r(4(N-b)-s)-4N}{N-sr}$ then $\frac{N}{\gamma} > b$, and so $|x|^{-b} \in L^{\gamma}(B)$. Now, in view of the hypothesis $\alpha < \frac{4-2b}{N-2s}$ we set r such that

$$\frac{r\,(4(N-b)-s)-4N}{4(N-sr)} = \frac{4-2b}{N-2s},$$

that is[4]

$$r = \frac{4N\,(N-2s+4-2b)}{4s(4-2b)+(N-2s)\,(4N-4b-s)}. \tag{2.28}$$

Note that, in order to satisfy the second equation in the system (2.26) we need to verify that $s < \frac{N}{r}$. A simple calculation shows that it is true if, and only if, $4b+5s < 4N$ and this is true since $b < \frac{N}{3}$ and $s < \frac{N}{2}$.

On the other hand, since we are looking for a pair (q,r) L^2-admissible we deduce

$$q = \frac{8(N-2s+4-2b)}{(8-2N+s)(N-2s)}. \tag{2.29}$$

Finally, from (2.29) the second equation in (2.27) is given by

$$\frac{1}{q_1} = \left(\frac{8-2N+s}{8}\right)\left(\frac{4-2b-\alpha(N-2s)}{N-2s+4-2b}\right). \tag{2.30}$$

which is positive, since $\alpha < \frac{4-2b}{N-2s}$, $s < \frac{N}{2}$ and $N = 1,2$.

(ii) Similarly as in item (i) we only consider the estimate on B. Let

$$D_2(t) = \left\| |x|^{-b}|u|^{\alpha}u \right\|_{L_x^{\bar{r}'}(B)}.$$

We use analogous arguments as the ones in the estimate of C_2 in Lemma 2.10 (ii). Lemmas 1.12-1.13, the Hölder inequality, the Sobolev embedding

[4] We claim that r satisfies (1.1). In fact, obviously $r < +\infty$. Moreover $r \geq 2$ if, and only if, $8-2N+s \geq 0$ and this is true since $s > 0$ and $N = 1,2$.

(1.6) and Remark 2.9 imply that

$$
\begin{aligned}
D_2(t) \leq & \||x|^{-b}\|_{L^\gamma(B)}\|D^s(|u|^\alpha u)\|_{L_x^\beta} + \|D^s(|x|^{-b})\|_{L^d(B)}\|u\|^{\alpha+1}_{L_x^{(\alpha+1)e}} \\
\leq & \||x|^{-b}\|_{L^\gamma(B)}\|u\|^\alpha_{\alpha r_1}\|D^s u\|_{L_x^r} + \||x|^{-b-s}\|_{L^d(B)}\|D^s u\|^{\alpha+1}_{L_x^r} \\
\leq & \||x|^{-b}\|_{L^\gamma(B)}\|D^s u\|^{\alpha+1}_{L_x^r} + \||x|^{-b-s}\|_{L^d(B)}\|D^s u\|^{\alpha+1}_{L_x^r}, \qquad (2.31)
\end{aligned}
$$

where

$$
\begin{cases}
\frac{1}{\bar{r}'} = & \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e} \\
\frac{1}{\beta} = & \frac{1}{r_1} + \frac{1}{r} \\
s = & \frac{N}{r} - \frac{N}{\alpha r_1}; \quad s < \frac{N}{r} \\
s = & \frac{N}{r} - \frac{N}{(\alpha+1)e},
\end{cases}
$$

which is equivalent to

$$
\begin{cases}
\frac{N}{\gamma} = & N - \frac{N}{\bar{r}} - \frac{(\alpha+1)N}{r} + \alpha s \\
\frac{N}{d} = & N - \frac{N}{\bar{r}} - \frac{(\alpha+1)N}{r} + \alpha s + s.
\end{cases} \qquad (2.32)
$$

Hence, setting again $(\bar{q}, \bar{r}) = (\frac{8}{2N-s}, \frac{4N}{s})$ the first equation in (2.32) the same as the first one in (2.27). Therefore choosing r as in (2.28) we have $\frac{N}{\gamma} > b$, which also implies $\frac{N}{d} > b + s$. Therefore, it follows from Remark 1.17 and (2.31) that

$$D_2(t) \leq c\|D^s u\|^{\alpha+1}_{L_x^r}.$$

Since, $\frac{1}{\bar{q}'} = \frac{1}{q_1} + \frac{\alpha+1}{q}$ (recall that q is given in (2.29)) and applying the Hölder inequality in the time variable, we conclude

$$\|D_2\|_{L_T^{\bar{q}'}} \leq cT^{\frac{1}{q_1}}\|D^s u\|^{\alpha+1}_{L_T^q L_x^r},$$

where $\frac{1}{q_1} > 0$ (see (2.30)). □

We finish the estimates for the nonlinearity considering the case $s = \frac{N}{2}$. Note that this case can only occur if $N = 1, 2$, since here we are interested in local (and global) results in $H^s(\mathbb{R}^N)$ for $\max\{0, s_c\} < s \leq \min\{\frac{N}{2}, 1\}$.

Lemma 2.12. *Let* $N = 1, 2$ *and* $0 < b < \frac{N}{3}$*. If* $s = \frac{N}{2}$ *and* $0 < \alpha < +\infty$ *then*

(i) $\left\| |x|^{-b}|u|^{\alpha}v \right\|_{S'(L^2;I)} \leq cT^{\theta_1}\|u\|^{\alpha}_{L^\infty_I H^s_x}\|v\|_{L^\infty_I L^2_x}$

(ii) $\left\| D^s(|x|^{-b}|u|^{\alpha}u) \right\|_{S'(L^2;I)} \leq cT^{\theta_1}\|u\|^{\alpha+1}_{L^\infty_I H^s_x}$,

where $I = [0, T]$ *and* $c, \theta_1 > 0$.

Proof. (i) First, we define the following numbers

$$r = \frac{N(\alpha+2)}{N-2b} \quad \text{and} \quad q = \frac{4(\alpha+2)}{N\alpha+4b}, \tag{2.33}$$

it is easy to check that (q, r) is L^2-admissible.

We divide the estimate in B and B^C. We first consider the estimate on B. From Hölder's inequality

$$\left\| |x|^{-b}|u|^{\alpha}v \right\|_{L^{r'}_x(B)} \leq \||x|^{-b}\|_{L^\gamma(B)}\|u\|^{\alpha}_{L^{\alpha r_1}_x}\|v\|_{L^2_x},$$

where

$$\frac{1}{r'} = \frac{1}{\gamma} + \frac{1}{r_1} + \frac{1}{2}. \tag{2.34}$$

In view of Remark 1.17 to show that $|x|^{-b} \in L^\gamma(B)$, we need $\frac{N}{\gamma} - b > 0$. So, the relations (2.33) and (2.34) yield

$$\frac{N}{\gamma} - b = \frac{\alpha(N-2b)}{2(\alpha+2)} - \frac{N}{r_1}. \tag{2.35}$$

If we choose $\alpha r_1 \in \left(\frac{2N(\alpha+2)}{N-2b}, +\infty\right)$ then the right hand side of (2.35) is positive. Therefore,

$$\left\| |x|^{-b}|u|^{\alpha}v \right\|_{L^{r'}_x(B)} \leq c\|u\|^{\alpha}_{L^{\alpha r_1}_x}\|v\|_{L^2_x}.$$

On the other hand, since $\frac{2N(\alpha+2)}{N-2b} > 2$ we can apply the Sobolev embedding (1.7) to obtain

$$\left\| |x|^{-b}|u|^{\alpha}v \right\|_{L^{r'}_x(B)} \leq c\|u\|^{\alpha}_{H^s}\|v\|_{L^2_x}. \tag{2.36}$$

Next, we consider the estimate on B^C. Using the same argument as in the first case, we get

$$\left\||x|^{-b}|u|^\alpha v\right\|_{L_x^{r'}(B^C)} \leq \||x|^{-b}\|_{L^\gamma(B^C)}\|u\|^\alpha_{L_x^{\alpha r_1}}\|v\|_{L_x^2},$$

where the relations (2.34) and (2.35) hold. Thus, choosing $\alpha r_1 \in \left(2, \frac{2N(\alpha+2)}{N-2b}\right)$ we have that $\frac{N}{\gamma} - b < 0$, which implies $|x|^{-b} \in L^\gamma(B^C)$, by Remark 1.17. Therefore, again by the Sobolev embedding (1.7), we obatin

$$\left\||x|^{-b}|u|^\alpha v\right\|_{L_x^{r'}(B^C)} \leq c\|u\|^\alpha_{H_x^s}\|v\|_{L_x^2}.$$

Finally, it follows from the Hölder inequality in time variable, (2.36) and the last inequality that

$$\left\||x|^{-b}|u|^\alpha v\right\|_{L_I^{q'}L_x^{r'}} \leq cT^{\theta_1}\|u\|^\alpha_{L_I^\infty H^s}\|v\|_{L_I^\infty L_x^2},$$

where $\theta_1 = \frac{1}{q'} > 0$, by (2.33).

(ii) Similarly as in the proof of item (i), we start setting

$$r = \frac{N(\alpha+2)}{N-b-s} \quad \text{and} \quad q = \frac{4(\alpha+2)}{\alpha N + 2b + 2s}. \tag{2.37}$$

Note that, since $s = \frac{N}{2}$ and $0 < b < \frac{N}{3}$ the denominator of r is a positive number. Furthermore it is easy to verify that (q,r) is L^2-admissible.

First, we consider the estimate on B. Lemma 1.13 together with the Hölder inequality and (2.10) imply

$$\begin{aligned} E_1(t) &\leq \||x|^{-b}\|_{L^\gamma(B)}\|D^s(|u|^\alpha u)\|_{L_x^\beta} + \|D^s(|x|^{-b})\|_{L^d(B)}\|u\|^{\alpha+1}_{L_x^{(\alpha+1)e}} \\ &\leq \||x|^{-b}\|_{L^\gamma(B)}\|u\|^\alpha_{L_x^{\alpha r_1}}\|D^s u\|_{L_x^2} + \||x|^{-b-s}\|_{L^d(B)}\|u\|^{\alpha+1}_{L_x^{(\alpha+1)e}}, \end{aligned}$$

where $E_1(t) = \left\|D^s(|x|^{-b}|u|^\alpha u)\right\|_{L_x^{r'}(B)}$ and

$$\begin{cases} \frac{1}{r'} = & \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e} \\ \frac{1}{\beta} = & \frac{1}{r_1} + \frac{1}{2}, \end{cases}$$

which implies

$$\begin{cases} \frac{N}{\gamma} = & \frac{N}{2} - \frac{N}{r} - \frac{N}{r_1} \\ \frac{N}{d} = & N - \frac{N}{r} - \frac{N}{e}. \end{cases} \tag{2.38}$$

Now, we claim that $\||x|^{-b}\|_{L^\gamma(B)}$ and $\||x|^{-b-s}\|_{L^d(B)}$ are bounded quantities for a suitable choice of r_1 and e. Indeed, using the value of r in (2.37), (2.38) and the fact that $s = \frac{N}{2}$, we deduce

$$\begin{cases} \frac{N}{\gamma} - b & = \frac{(\alpha+1)(N-2b)}{2(\alpha+2)} - \frac{N}{r_1} \\ \frac{N}{d} - b - s = & \frac{(\alpha+1)(N-2b)}{2(\alpha+2)} - \frac{N}{e}. \end{cases} \tag{2.39}$$

Note that, by Remark 1.17, if $r_1, e > \frac{2N(\alpha+2)}{(\alpha+1)(N-2b)}$ then the right hand side of both equations in (2.39) are positive, so $|x|^{-b} \in L^\gamma(B)$ and $|x|^{-b-s} \in L^d(B)$. Hence

$$E_1(t) \leq c\|u\|^\alpha_{L^{\alpha r_1}_x} \|D^s u\|_{L^2_x} + c\|u\|^{\alpha+1}_{L^{(\alpha+1)e}_x}.$$

Choosing r_1 and e as before, it is easy to see that[5] $\alpha r_1 > 2$ and $(\alpha+1)e > 2$, thus we can use the Sobolev inequality (1.7)

$$\begin{aligned} E_1(t) &\leq c\|u\|^\alpha_{H^s_x} \|D^s u\|_{L^2_x} + c\|u\|^{\alpha+1}_{H^s_x} \\ &\leq c\|u\|^{\alpha+1}_{H^s_x}. \end{aligned} \tag{2.40}$$

To complete the proof, we need to consider the estimate on B^C. Using the same arguments as before we have

$$E_2(t) \leq \||x|^{-b}\|_{L^\gamma(B^C)} \|u\|^\alpha_{L^{\alpha r_1}_x} \|D^s u\|_{L^2_x} + \||x|^{-b-s}\|_{L^d(B^C)} \|u\|^{\alpha+1}_{L^{(\alpha+1)e}_x},$$

where $E_2(t) = \left\|D^s(|x|^{-b}|u|^\alpha u)\right\|_{L^{r'}_x(B^C)}$ and (2.39) holds. Similarly as in item

[5] Increasing the value of r_1 if necessary.

(i), since[6] $\frac{2N\alpha(\alpha+2)}{(\alpha+1)(N-2b)}, \frac{2N(\alpha+2)}{N-b-s} > 2$, we can choose r_1 and e such that

$$\alpha r_1 \in \left(2, \frac{2N\alpha(\alpha+2)}{(\alpha+1)(N-2b)}\right) \quad \text{and} \quad (\alpha+1)e \in \left(2, \frac{2N(\alpha+2)}{N-2b}\right),$$

and thus we get from (2.39) that $\frac{N}{\gamma} - b < 0$ and $\frac{N}{d} - b - s < 0$. In other words, $\||x|^{-b}\|_{L^\gamma(B^C)}$ and $\||x|^{-b-s}\|_{L^d(B^C)}$ are bounded quantities for these choices of r_1 and e (see Remark 1.17). Furthermore, by the Sobolev inequality (1.7) we conclude

$$E_2(t) \leq c\|u\|^{\alpha+1}_{H^s_x}.$$

Finally, (2.40) and the last inequality lead to

$$\left\|D^s(|x|^{-b}|u|^\alpha u)\right\|_{L^{q'}_I L^{r'}_x} \leq cT^{\frac{1}{q'}}\|u\|^{\alpha+1}_{L^\infty_I H^s_x},$$

where $\frac{1}{q'} > 0$ by (2.37). □

We now have all tools to prove Theorem 2.3.

Proof of Theorem 2.3. We define

$$X = C\left([-T,T]; H^s(\mathbb{R}^N)\right) \bigcap L^q\left([-T,T]; H^{s,r}(\mathbb{R}^N)\right),$$

for any (q,r) L^2-admissible and

$$\|u\|_T = \|u\|_{S(L^2;[-T,T])} + \|D^s u\|_{S(L^2;[-T,T])}.$$

We shall show that $G = G_{u_0}$ defined in (2.8) is a contraction on the complete metric space

$$S(a,T) = \{u \in X : \|u\|_T \leq a\}$$

[6]Notice that, since $N = 1, 2$ and by hypothesis $\alpha > \frac{4-2b}{N}$ we have

$$\frac{2N\alpha(\alpha+2)}{(\alpha+1)(N-2b)} > \frac{2N\alpha}{N-2b} > \frac{2(4-2b)}{N-2b} > 2.$$

with the metric

$$d_T(u,v) = \|u-v\|_{S(L^2;[-T,T])},$$

for a suitable choice of a and T.

First, we claim that $S(a,T)$ with the metric d_T is a complete metric space. Indeed, the proof follows similar arguments as in [2] (see Theorem 1.2.5 and the proof of Theorem 4.4.1 page 94). Since $S(a,T) \subset X$ and X is a complete space, it suffices to show that $S(a,T)$, with the metric d_T, is closed in X. Let $u_n \in S(a,T)$ such that $d_T(u_n,u) \to 0$ as $n \to +\infty$, we want to show that $u \in S(a,T)$. If $u_n \in C\left([-T,T]; H^s(\mathbb{R}^N)\right)$ (see the definition of $S(a,T)$) we have, for almost all $t \in [-T,T]$, $u_n(t)$ bounded in $H^s(\mathbb{R}^N)$ and so (since $H^s(\mathbb{R}^N)$ is reflexive)

$$u_n(t) \rightharpoonup v(t) \ \text{ in } \ H^s(\mathbb{R}^N) \quad \text{and} \quad \|v(t)\|_{H^s} \le \liminf_{n\to+\infty} \|u_n\|_{H^s} \le a. \tag{2.41}$$

On the other hand, the hypothesis $d_T(u_n,u) \to 0$ implies that $u_n \to u$ in $L^q_I L^r_x$ for all (q,r) L^2-admissible. Since $(\infty,2)$ is L^2-admissible we get $u_n(t) \to u(t)$ in L^2, for almost all $t \in [-T,T]$. Therefore, by uniqueness of the limit we deduce that $u(t) = v(t)$. Moreover, we have from (2.41)

$$\|u(t)\|_{H^s_x} \le a.$$

That is, $u \in C\left([-T,T]; H^s(\mathbb{R}^N)\right)$.
From similar arguments, if $u_n \in L^q\left([-T,T]; H^{s,r}(\mathbb{R}^N)\right)$ we obtain $u \in S(a,T)$. This completes the proof of the claim.

Returning to the proof of the theorem, it follows from the Strichartz inequalities (1.9) and (1.11) that

$$\|G(u)\|_{S(L^2;[-T,T])} \le c\|u_0\|_{L^2} + c\|F\|_{S'(L^2;[-T,T])}$$

and

$$\|D^s G(u)\|_{S(L^2;[-T,T])} \le c\|D^s u_0\|_{L^2} + c\|D^s F\|_{S'(L^2;[-T,T])},$$

where $F(x,u) = |x|^{-b}|u|^\alpha u$. Similarly as in the proof of Theorem 2.2, without loss of generality we consider only the case $t > 0$. So, using Lemmas 2.10-2.11-2.12 and (2.1.2) we deduce

$$\|F\|_{S'(L^2;I)} \leq c(T^{\theta_1} + T^{\theta_2})\|u\|_I^{\alpha+1}$$

and

$$\|D^s F\|_{S'(L^2;I)} \leq c(T^{\theta_1} + T^{\theta_2})\|u\|_I^{\alpha+1},$$

where $I = [0,T]$ and $\theta_1, \theta_2 > 0$. Hence, if $u \in S(a,T)$ we get

$$\|G(u)\|_T \leq c\|u_0\|_{H^s} + c(T^{\theta_1} + T^{\theta_2})a^{\alpha+1}.$$

Now, choosing $a = 2c\|u_0\|_{H^s}$ and $T > 0$ such that

$$ca^\alpha(T^{\theta_1} + T^{\theta_2}) < \frac{1}{4}, \tag{2.42}$$

we obtain $G(u) \in S(a,T)$. Such calculations establish that G is well defined on $S(a,T)$.

On the other hand, using (1.13), an analogous argument as before yields

$$\begin{aligned} d_T(G(u),G(v)) &\leq c\|F(x,u) - F(x,v)\|_{S'(L^2;[-T,T])} \\ &\leq c(T^{\theta_1} + T^{\theta_2})\left(\|u\|_T^\alpha + \|v\|_T^\alpha\right) d_T(u,v), \end{aligned}$$

and so, taking $u, v \in S(a,T)$, the last inequality imply

$$d_T(G(u),G(v)) \leq c(T^{\theta_1} + T^{\theta_2})a^\alpha d_T(u,v).$$

Therefore, from (2.42), G is a contraction on $S(a,T)$ and by the Contraction Mapping Theorem we have a unique fixed point $u \in S(a,T)$ of G such that (7) holds. □

We finish this section noting that Corollary 2.5 follows directly from Theorem 2.3. It is worth to mention that Corollary 2.5 only holds for $N \geq 2$ since we assume $s \leq \min\{\frac{N}{2}, 1\}$ in Theorem 2.3.

2.2 Global well posedness

This section is devoted to study the global well-posedness of the Cauchy problem (1). Similarly as the local theory we use the fixed point theorem to prove our small data results in $H^s(\mathbb{R}^N)$. We start with a global result in $L^2(\mathbb{R}^N)$, which does not require any smallness assumption.

Theorem 2.13. *If* $0 < \alpha < \frac{4-2b}{N}$ *and* $0 < b < \min\{2, N\}$*, then for all* $u_0 \in L^2(\mathbb{R}^N)$ *the local solution* u *of the IVP* (1) *extends globally with*

$$u \in C\left(\mathbb{R}; L^2(\mathbb{R}^N)\right) \cap L^q\left(\mathbb{R}; L^r(\mathbb{R}^N)\right),$$

for any (q, r) L^2*-admissible.*

Next, we establish a small data global theory for the INLS model (1).

Theorem 2.14. *Let* $\frac{4-2b}{N} < \alpha < \alpha_s$ *with* $0 < b < \widetilde{2}$ *(see definition* (2.1)*),* $s_c < s \leq \min\{\frac{N}{2}, 1\}$ *and* $u_0 \in H^s(\mathbb{R}^N)$*. If* $\|u_0\|_{H^s} \leq A$ *then there exists* $\delta = \delta(A)$ *such that if* $\|U(t)u_0\|_{S(\dot{H}^{s_c})} < \delta$*, then the solution of* (7) *is globally defined. Moreover,*

$$\|u\|_{S(\dot{H}^{s_c})} \leq 2\|U(t)u_0\|_{S(\dot{H}^{s_c})}$$

and

$$\|u\|_{S(L^2)} + \|D^s u\|_{S(L^2)} \leq 2c\|u_0\|_{H^s}.$$

Remark 2.15. Note that in the last result we do not need the condition $s > \max\{0, s_c\}$ as in Theorem 2.3, since $\alpha > \frac{4-2b}{N}$ implies $s_c > 0$.

Remark 2.16. Also note that by the Strichartz estimates (1.10), the condition $\|U(t)u_0\|_{S(\dot{H}^{s_c})} < \delta$ is automatically satisfied if $\|u_0\|_{\dot{H}^{s_c}} \leq \frac{\delta}{c}$.

A similar small data global theory for the NLS model can be found in Cazenave-Weissler [6], Holmer-Roudenko [23] and Guevara [22].

2.2.1 L^2-Theory

The global well-posedness result in $L^2(\mathbb{R}^N)$ (see Theorem 2.13) is an immediate consequence of Theorem 2.2. Indeed, using (2.9) we have that $T(\|u_0\|_{L^2}) = \frac{C}{\|u_0\|^d_{L^2}}$ for some $C, d > 0$, then the conservation law (3) allows us to reapply Theorem 2.2 as many times as we wish preserving the lenght of the time interval to get a global solution.

2.2.2 H^s-Theory

In this subsection, we turn our attention to proof the Theorem 2.14 and again the heart of the proof is to establish good estimates on the nonlinearity $F(x,u) = |x|^{-b}|u|^\alpha u$. First, we estimate the norm $\|F(x,u)\|_{S'(\dot{H}^{-s_c})}$ (see Lemma 2.17 below), next we estimate $\|F(x,u)\|_{S'(L^2)}$ (see Lemma 2.18) and finally we consider the norm $\|D^sF(x,u)\|_{S'(L^2)}$ (see Lemmas 2.19, 2.21 and 2.23).

We start defining the following numbers (depending only on N, α and b)

$$\widehat{q} = \frac{4\alpha(\alpha+2-\theta)}{\alpha(N\alpha+2b)-\theta(N\alpha-4+2b)} \quad \widehat{r} = \frac{N\alpha(\alpha+2-\theta)}{\alpha(N-b)-\theta(2-b)} \tag{2.43}$$

and

$$\widetilde{a} = \frac{2\alpha(\alpha+2-\theta)}{\alpha[N(\alpha+1-\theta)-2+2b]-(4-2b)(1-\theta)} \quad \widehat{a} = \frac{2\alpha(\alpha+2-\theta)}{4-2b-(N-2)\alpha}, \tag{2.44}$$

where $\theta > 0$ sufficiently small[7]. It is easy to see that $(\widehat{q}, \widehat{r})$ is L^2-admissible,

[7] First note that, since $\theta > 0$ is sufficiently small, we have that the denominators of $\widehat{q}, \widehat{r}, \widehat{a}$ and $\widetilde{a}$ are all positive numbers. Moreover, it is easy to see that $\widehat{r}$ satisfies (1.3). In fact $\widehat{a}$ can be rewritten as $\widehat{a} = \frac{\alpha+2-\theta}{1-s_c}$ and since $\theta < \alpha$ we have $\widehat{a} > \frac{2}{1-s_c}$, which implies that $\widehat{r} < \frac{2N}{N-2}$, for $N \geq 3$. We also note that $\widehat{r} \leq ((\frac{2}{1+s_c})^+)'$, for $N = 2$. Indeed, the last inequality is

$(\widehat{a},\widehat{r})$ is $\dot{H}^{s_c}$-admissible[8] and $(\widetilde{a},\widehat{r})$ is $\dot{H}^{-s_c}$-admissible. Moreover, we observe that

$$\frac{1}{\widehat{a}}+\frac{1}{\widetilde{a}}=\frac{2}{\widehat{q}}. \tag{2.45}$$

Using the same notation of the previous section, we set $B=B(0,1)$ and we recall that $|x|^{-b}\in L^{\gamma}(B)$ if $\frac{N}{\gamma}>b$. Similarly, we have that $|x|^{-b}\in L^{\gamma}(B^C)$ if $\frac{N}{\gamma}<b$ (see Remark 1.17).

Our first result reads as follows.

Lemma 2.17. *Let* $\frac{4-2b}{N}<\alpha<\alpha_s$ *and* $0<b<\widetilde{2}$*. If* $s_c<s\leq\min\{\frac{N}{2},1\}$ *then the following statement holds*

$$\left\||x|^{-b}|u|^{\alpha}v\right\|_{S'(\dot{H}^{-s_c})}\leq c\|u\|^{\theta}_{L^\infty_tH^s_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|v\|_{S(\dot{H}^{s_c})}, \tag{2.46}$$

where $c>0$ *and* $\theta\in(0,\alpha)$ *is a sufficiently small number.*

Proof. The proof follows from similar arguments as the ones in the previous lemmas. We study the estimates in B and B^C separately.

We first consider the set B. From the Hölder inequality we deduce

$$\begin{aligned}\left\||x|^{-b}|u|^{\alpha}v\right\|_{L^{\widehat{r}'}_x(B)} &\leq \||x|^{-b}\|_{L^\gamma(B)}\|u\|^{\theta}_{L^{\theta r_1}_x}\|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)r_2}_x}\|v\|_{L^{\widehat{r}}_x}\\ &= \||x|^{-b}\|_{L^\gamma(B)}\|u\|^{\theta}_{L^{\theta r_1}_x}\|u\|^{\alpha-\theta}_{L^{\widehat{r}}_x}\|v\|_{L^{\widehat{r}}_x},\end{aligned} \tag{2.47}$$

where

$$\frac{1}{\widehat{r}'}=\frac{1}{\gamma}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{\widehat{r}}\ \text{ and }\ \widehat{r}=(\alpha-\theta)r_2. \tag{2.48}$$

---

equivalent to $\varepsilon\widehat{r}<(\frac{2}{1+s_c})^{+}(\frac{2}{1+s_c})$ (recall (1.4)) and this is true since $\varepsilon>0$ is a small enough number. For $N=1$, we see that $\widehat{r}<\infty$. Finally, we have $\widehat{r}>\frac{2N}{N-s_c}=\frac{N\alpha}{2-b}$. Indeed, this is equivalent to $(\alpha+2-\theta)(2-b)>\alpha(N-b)-\theta(2-b)\Leftrightarrow(\alpha+2)(2-b)>\alpha(N-b)\Leftrightarrow\alpha<\frac{4-2b}{N-2}$. So, since $\alpha<\frac{4-2b}{N-2s}$ and $s\leq1$ (our hypothesis), we have that $\alpha<\frac{4-2b}{N-2}$ holds, consequently $\widehat{r}>\frac{2N}{N-s_c}$.

[8] Recall that s_c is the critical Sobolev index given by $s_c=\frac{N}{2}-\frac{2-b}{\alpha}$.

Now, we make use of the Sobolev embedding (Lemma 1.10), so we consider two cases: $s = \frac{N}{2}$ and $s < \frac{N}{2}$.

Case $s = \frac{N}{2}$. Since $s \leq \min\{\frac{N}{2}, 1\}$, we only have to consider the cases where (N, s) is equal to $(1, \frac{1}{2})$ or $(2, 1)$. In order to have the norm $\||x|^{-b}\|_{L^\gamma(B)}$ bounded we need $\frac{N}{\gamma} > b$. In fact, observe that (2.48) implies

$$\frac{N}{\gamma} = N - \frac{N(\alpha + 2 - \theta)}{\widehat{r}} - \frac{N}{r_1},$$

and from (2.43) it follows that

$$\frac{N}{\gamma} - b = \frac{\theta(2-b)}{\alpha} - \frac{N}{r_1}. \tag{2.49}$$

Since $\alpha > \frac{4-2b}{N}$ then $\frac{N\alpha}{2-b} > 2$, therefore choosing

$$\theta r_1 \in \left(\frac{N\alpha}{2-b}, +\infty\right), \tag{2.50}$$

we have $\frac{N}{\gamma} > b$. Hence, inequality (2.47) and the Sobolev embedding (1.7) yield

$$\left\||x|^{-b}|u|^\alpha v\right\|_{L_x^{\widehat{r}'}(B)} \leq c\|u\|^\theta_{H^s_x}\|u\|^{\alpha-\theta}_{L_x^{\widehat{r}}}\|v\|_{L_x^{\widehat{r}}}. \tag{2.51}$$

Case $s < \frac{N}{2}$. In this case, we will also obtain the inequality (2.51). Indeed, we already have the relation (2.49), then the only change is the choice of θr_1 since we can not apply the Sobolev embedding (1.7) when $s < \frac{N}{2}$. In this case we set

$$\theta r_1 = \frac{2N}{N-2s}, \tag{2.52}$$

so

$$\frac{N}{\gamma} - b = \theta(s - s_c) > 0,$$

that is, the quantity $\||x|^{-b}\|_{L^\gamma(B)}$ is finite. Therefore by the Sobolev embedding (1.8) we obtain the desired inequality (2.51).

Next, we consider the set B^C. We claim that

$$\left\||x|^{-b}|u|^\alpha v\right\|_{L_x^{\widehat{r}'}(B^C)} \leq c\|u\|^\theta_{H^s_x}\|u\|^{\alpha-\theta}_{L_x^{\widehat{r}}}\|v\|_{L_x^{\widehat{r}}}. \tag{2.53}$$

Indeed, arguing in the same way as before we deduce

$$\left\| |x|^{-b}|u|^{\alpha}v \right\|_{L_x^{\widehat{r}'}(B^C)} \leq \| |x|^{-b}\|_{L^{\gamma}(B^C)} \|u\|^{\theta}_{L_x^{\theta r_1}} \|u\|^{\alpha-\theta}_{L_x^{\widehat{r}}} \|v\|_{L_x^{\widehat{r}}},$$

where the relation (2.49) holds. We first show that $\| |x|^{-b}\|_{L^{\gamma}(B^C)}$ is finite for a suitable of r_1. Similarly as before, we consider two cases: $s = \frac{N}{2}$ and $s < \frac{N}{2}$. In the first case, we choose r_1 such that

$$\theta r_1 \in \left(2, \frac{N\alpha}{2-b}\right) \tag{2.54}$$

then, from (2.49), $\frac{N}{\gamma} - b < 0$, so $|x|^{-b} \in L^{\gamma}(B^C)$. Thus, by the Sobolev inequality (1.7) and using the last inequality we deduce (2.53). Now if $s < \frac{N}{2}$, choosing again θr_1 as (2.54) we obtain $\frac{N}{\gamma} - b < 0$. In addition, since $\alpha < \frac{4-2b}{N-2s}$ we have $\frac{N\alpha}{2-b} < \frac{2N}{N-2s}$, therefore the Sobolev inequality (1.8) implies (2.53). This completes the proof of the claim.

Now, inequalities (2.51) and (2.53) yield

$$\left\| |x|^{-b}|u|^{\alpha}v \right\|_{L_x^{\widehat{r}'}} \leq c\|u\|^{\theta}_{H_x^s} \|u\|^{\alpha-\theta}_{L_x^{\widehat{r}}} \|v\|_{L_x^{\widehat{r}}} \tag{2.55}$$

and the Hölder inequality in the time variable leads to

$$\begin{aligned} \left\| |x|^{-b}|u|^{\alpha}v \right\|_{L_t^{\widetilde{a}'}L_x^{\widehat{r}'}} &\leq c\|u\|^{\theta}_{L_t^{\infty}H_x^s} \|u\|^{\alpha-\theta}_{L_t^{(\alpha-\theta)a_1}L_x^{\widehat{r}}} \|v\|_{L_t^{\widehat{a}}L_x^{\widehat{r}}} \\ &= c\|u\|^{\theta}_{L_t^{\infty}H_x^s} \|u\|^{\alpha-\theta}_{L_t^{\widehat{a}}L_x^{\widehat{r}}} \|v\|_{L_t^{\widehat{a}}L_x^{\widehat{r}}}, \end{aligned}$$

where

$$\frac{1}{\widetilde{a}'} = \frac{\alpha-\theta}{\widehat{a}} + \frac{1}{\widehat{a}}.$$

Since $\widehat{a}$ and $\widetilde{a}$ defined in (2.44) satisfy the last relation we conclude the proof of (2.46).[9] □

[9] Recall that $(\widehat{a}, \widehat{r})$ is $\dot{H}^{s_c}$-admissible and $(\widetilde{a}, \widehat{r})$ is $\dot{H}^{-s_c}$-admissible.

Lemma 2.18. *Let* $\frac{4-2b}{N} < \alpha < \alpha_s$ *and* $0 < b < \widetilde{2}$. *If* $s_c < s \leq \min\{\frac{N}{2}, 1\}$ *then*

$$\left\| |x|^{-b}|u|^{\alpha}v \right\|_{S'(L^2)} \leq c\|u\|^{\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|v\|_{S(L^2)}, \tag{2.56}$$

where $c > 0$ *and* $\theta \in (0, \alpha)$ *is a sufficiently small number.*

Proof. By the previous lemma we already have (2.55), then applying Hölder's inequality in the time variable we obtain

$$\left\| |x|^{-b}|u|^{\alpha}v \right\|_{L^{\widehat{q}'}_t L^{\widehat{r}'}_x} \leq c\|u\|^{\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{L^{\widehat{a}}_t L^{\widehat{r}}_x}\|v\|_{L^{\widehat{q}}_t L^{\widehat{r}}_x},$$

since

$$\frac{1}{\widehat{q}'} = \frac{\alpha-\theta}{\widehat{a}} + \frac{1}{\widehat{q}} \tag{2.57}$$

by (2.43) and (2.44). The proof is finished in view of $(\widehat{q}, \widehat{r})$ be L^2-admissible. □

We now estimate $\left\| D^s\left(|x|^{-b}|u|^{\alpha}u\right) \right\|_{S'(L^2)}$. We divide our study in three cases: $N \geq 4$, $N = 3$ and $N = 1, 2$.

Lemma 2.19. *Let* $N \geq 4$, $0 < b < \widetilde{2}$ *and* $\frac{4-2b}{N} < \alpha < \alpha_s$. *If* $s_c < s \leq 1$ *then the following statement holds*

$$\left\| D^s\left(|x|^{-b}|u|^{\alpha}u\right) \right\|_{S'(L^2)} \leq c\|u\|^{\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|D^s u\|_{S(L^2)}, \tag{2.58}$$

where $c > 0$ *and* $\theta \in (0, \alpha)$ *is a sufficiently small number.*

Proof. First note that we always have $s < \frac{N}{2}$ in this lemma, since we are assuming $N \geq 4$ and $s_c < s \leq 1$. Here, we also divide the estimate in B and B^C separately.

We begin estimating on B. The fractional product rule (Lemma 1.12) yields

$$\left\| D^s\left(|x|^{-b}|u|^{\alpha}u\right) \right\|_{L^{\widehat{r}'}_x(B)} \leq N_1(t, B) + N_2(t, B),$$

where

$$N_1(t,B) = \left\| |x|^{-b} \right\|_{L^\gamma(B)} \| D^s(|u|^\alpha u) \|_{L^\beta_x} \qquad N_2(t,B) = \left\| D^s(|x|^{-b}) \right\|_{L^d(B)} \| |u|^\alpha u \|_{L^e_x}$$

and

$$\frac{1}{\widehat{r}'} = \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e}. \tag{2.59}$$

First, we consider $N_1(t,B)$. It follows from the fractional chain rule (Lemma 1.13) and Hölder's inequality that

$$\begin{aligned} N_1(t,B) &\leq \| |x|^{-b} \|_{L^\gamma(B)} \|u\|^\theta_{L^{\theta r_1}_x} \|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)r_2}_x} \|D^s u\|_{L^{\widehat{r}}_x} \\ &= \| |x|^{-b} \|_{L^\gamma(B)} \|u\|^\theta_{L^{\theta r_1}_x} \|u\|^{\alpha-\theta}_{L^{\widehat{r}}_x} \|D^s u\|_{L^{\widehat{r}}_x}, \end{aligned} \tag{2.60}$$

where

$$\frac{1}{\beta} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\widehat{r}} \quad \text{and} \quad \widehat{r} = (\alpha-\theta) r_2. \tag{2.61}$$

Note that, the right hand side of (2.60) is the same as the right hand side of (2.47), with $v = D^s u$, so combining (2.59) and (2.61) we also have (2.48). Thus, arguing in the same way as in Lemma 2.17 we obtain (recall that (2.51) also holds when $s < \frac{N}{2}$)

$$N_1(t,B) \leq c \|u\|^\theta_{H^s_x} \|u\|^{\alpha-\theta}_{L^{\widehat{r}}_x} \|D^s u\|_{L^{\widehat{r}}_x}. \tag{2.62}$$

On the other hand, from (2.10), Hölder's inequality and the Sobolev emdebbing (1.6) we deduce

$$\begin{aligned} N_2(t,B) &\leq \| |x|^{-b-s} \|_{L^d(B)} \|u\|^\theta_{L^{\theta r_1}_x} \|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)r_2}_x} \|u\|_{L^{r_3}_x} \\ &= \| |x|^{-b-s} \|_{L^d(B)} \|u\|^\theta_{L^{\theta r_1}_x} \|u\|^{\alpha-\theta}_{L^{\widehat{r}}_x} \|D^s u\|_{L^{\widehat{r}}_x}, \end{aligned} \tag{2.63}$$

where

$$\begin{cases} \frac{1}{e} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} & \widehat{r} = (\alpha-\theta) r_2 \\ s = \frac{N}{\widehat{r}} - \frac{N}{r_3} & \text{with} \quad s < \frac{N}{\widehat{r}}\ , \end{cases} \tag{2.64}$$

which implies using (2.59) that

$$\frac{N}{d} - s = N - \frac{N(\alpha + 2 - \theta)}{\widehat{r}} - \frac{N}{r_1}$$

and so, by (2.43)

$$\frac{N}{d} - b - s = \frac{\theta(2-b)}{\alpha} - \frac{N}{r_1}. \tag{2.65}$$

Note that the right hand side of (2.65) is the same as the right hand side of (2.49). Hence, choosing θr_1 as in (2.52) (recall that $s < \frac{N}{2}$) we have $\frac{N}{d} - b - s > 0$, so the quantity $\| |x|^{-b-s} \|_{L^d(B)}$ is bounded, by Remark 1.17. Now, the Sobolev embedding (1.8) and (2.63) imply that

$$N_2(t,B) \leq c\|u\|^{\theta}_{H^s_x} \|u\|^{\alpha-\theta}_{L^{\widehat{r}}_x} \|D^s u\|_{L^{\widehat{r}}_x}.$$

Therefore, from the last inequality together with (2.62) we obtain

$$\left\| D^s \left(|x|^{-b} |u|^{\alpha} u \right) \right\|_{L^{\widehat{r}'}_x(B)} \leq c\|u\|^{\theta}_{H^s_x} \|u\|^{\alpha-\theta}_{L^{\widehat{r}}_x} \|D^s u\|_{L^{\widehat{r}}_x}.$$

Thus, applying Hölder's inequality in the time variable and recalling (2.57) we get

$$\begin{aligned} \left\| D^s \left(|x|^{-b} |u|^{\alpha} u \right) \right\|_{L^{\widehat{q}'}_t L^{\widehat{r}'}_x(B)} &\leq c\|u\|^{\theta}_{L^\infty_t H^s_x} \|u\|^{\alpha-\theta}_{L^{\widehat{a}}_t L^{\widehat{r}}_x} \|D^s u\|_{L^{\widehat{q}}_t L^{\widehat{r}}_x} \\ &\leq c\|u\|^{\theta}_{L^\infty_t H^s_x} \|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})} \|D^s u\|_{S(L^2)}. \end{aligned} \tag{2.66}$$

Next we consider the norm $\left\| D^s \left(|x|^{-b} |u|^{\alpha} u \right) \right\|_{L^{\widehat{r}'}_x(B^C)}$. Similarly as before, replacing B by B^C, we also get (2.60)-(2.61) and consequently, by the proof of Lemma 2.17, the inequality (2.62), that is

$$N_1(t,B^C) \leq c\|u\|^{\theta}_{H^s_x} \|u\|^{\alpha-\theta}_{L^{\widehat{r}}_x} \|D^s u\|_{L^{\widehat{r}}_x}.$$

We also have (replacing B by B^C)

$$N_2(t,B^C) \leq \| |x|^{-b-s} \|_{L^d(B^C)} \|u\|^{\theta}_{L^{\theta r_1}_x} \|u\|^{\alpha-\theta}_{L^{\widehat{r}}_x} \|D^s u\|_{L^{\widehat{r}}_x},$$

where the relation (2.65) holds, thus setting $\theta r_1 = 2$ we deduce

$$\frac{N}{d} - b - s = -\theta s_c < 0,$$

which implies that $|x|^{-b-s} \in L^d(B^C)$, by Remark 1.17. Now, the Sobolev embedding (1.8) yields

$$N_2(t, B^C) \leq c\|u\|^{\theta}_{H^s_x}\|u\|^{\alpha-\theta}_{L^{\widehat{r}}_x}\|D^s u\|_{L^{\widehat{r}}_x}.$$

Therefore,

$$\begin{aligned} \left\|D^s\left(|x|^{-b}|u|^{\alpha}u\right)\right\|_{L^{\widehat{r}'}_x(B^C)} &\leq N_1(t,B) + N_2(t,B^C) \\ &\leq c\|u\|^{\theta}_{H^s_x}\|u\|^{\alpha-\theta}_{L^{\widehat{r}}_x}\|D^s u\|_{L^{\widehat{r}}_x}. \end{aligned}$$

Finally, using Hölder's inequality in the time variable, the last inequality (recalling (2.57)) and the relation (2.66) we deduce the estimate (2.58). □

Remark 2.20. Notice that Lemma 2.19 does not hold in dimension three for every $\alpha < \alpha_s$ (recall (2.1)). In fact, the condition $s < \frac{N}{\widehat{r}}$ (used in (2.64)) is only true for $N \geq 4$. Indeed, since $s \leq 1$ it suffices to verify $1 < \frac{N}{\widehat{r}}$ and the last inequality is equivalent to

$$\theta(2-b) < \alpha(N-2-b+\theta-\alpha) \tag{2.67}$$

Now if $N = 3$, we have $\theta(2-b) < \alpha(1-b+\theta-\alpha)$, which cannot holds for every $\alpha < \frac{4-2b}{3-2s}$ (take $s = 1$ and $\alpha = 2$ for example).

On the other hand, if $N \geq 4$ we claim that the inequality (2.67) holds for $\theta > 0$ small enough. Indeed, in this case we have $N-2-b+\theta-\alpha \geq 2-b+\theta-\alpha$, so

$$\begin{aligned} \alpha(N-2-b+\theta-\alpha) - \theta(2-b) \geq & \ \alpha(2-b+\theta-\alpha) - \theta(2-b) \\ = & \ (\alpha-\theta)(2-b-\alpha). \end{aligned}$$

Since $\alpha > \theta$, (2.67) holds if

$$2 - b - \alpha > 0. \tag{2.68}$$

By our assumption $\alpha < \frac{4-2b}{N-2s}$ and the fact that $2 - b \geq \frac{4-2b}{N-2} > \frac{4-2b}{N-2s}$, for $N \geq 4$ and $s \leq 1$, we deduce (2.68). In the next lemma we consider the case $N = 3$.

Before stating the lemma, we define the following numbers:

$$k = \frac{4\alpha(\alpha+1-\theta)}{4-2b-\alpha} \qquad\qquad p = \frac{6\alpha(\alpha+1-\theta)}{(4-2b)(\alpha-\theta)+\alpha} \tag{2.69}$$

and

$$l = \frac{4\alpha(\alpha+1-\theta)}{\alpha(3\alpha-2+2b)-\theta(3\alpha-4+2b)}, \tag{2.70}$$

where $\theta \in (0, \alpha)$. It is not difficult to verify that (l, p) is L^2-admissible and (k, p) is $\dot{H}^{s_c}$-admissible[10].

We also define

$$m = \frac{4D}{D-\varepsilon} \qquad\qquad n = \frac{6D}{2D+\varepsilon} \tag{2.71}$$

and

$$a^* = \frac{4\theta}{2+\varepsilon-D} \qquad r^* = \frac{6\alpha\theta}{(4-2b)\theta-(2+\varepsilon-D)\alpha}, \tag{2.72}$$

where $D = \alpha - \theta + \mu$ with $\mu \in (b, 1)$ and ε is a sufficiently small number such that $\varepsilon < \mu - b$. Note that $2 < n < 3$ (n satisfies the condition (1.1) for $N = 3$) and (m, n) is L^2-admissible. Moreover, choosing $\theta = F\alpha$ with[11]

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[10] We claim $\frac{3\alpha}{2-b} = \frac{6}{3-2s_c} < p < 6$, i.e., p satisfies the condition (1.3) (and therefore (1.1), since $\frac{6}{3-2s_c} > 2$) for $N = 3$. Indeed, $\frac{3\alpha}{2-b} < p \Leftrightarrow (4-2b)(\alpha-\theta)+\alpha < (4-2b)(\alpha+1-\theta) \Leftrightarrow \alpha < 4-2b$. Moreover, $p < 6 \Leftrightarrow \alpha(\alpha+1-\theta) < (4-2b)(\alpha-\theta)+\alpha \Leftrightarrow \alpha(\alpha-\theta) < (4-2b)(\alpha-\theta) \Leftrightarrow \alpha < 4-2b$. Now $\alpha < 4-2b$ always holds under the assumptions $\alpha < \frac{4-2b}{3-2s}$ and $s \leq 1$.

[11] It is easy to check that $F \in \left(\frac{1}{2}, 1\right)$ if $\varepsilon < \mu - b$. Therefore, since $\theta = F\alpha$, we have $\theta < \alpha$.

$F = \frac{2-\varepsilon+\mu-2b}{4-2b}$ we claim that (a^*, r^*) is $\dot{H}^{s_c}$-admissible. We first show that the denominators of a^* and r^* are positive numbers. Indeed

$$2+\varepsilon-D = 2+\varepsilon-\mu+F\alpha-\alpha = 2+\varepsilon-\mu-\alpha(1-F) = 2+\varepsilon-\mu-\alpha\left(\frac{2+\varepsilon-\mu}{4-2b}\right),$$

so by the hypothesis $\alpha < \frac{4-2b}{3-2s}$ and since $s \leq 1$ we deduce $2+\varepsilon-D > 0$. We also have (using the value of F and the fact that $D > \mu$)

$$(4-2b)\theta - (2+\varepsilon-D)\alpha = \alpha\left((4-2b)F - 2 - \varepsilon + D\right) > \left(2(\mu-b) - 2\varepsilon\right),$$

which is positive setting $\varepsilon < \mu - b$.

Next, we show that r^* satisfies the condition (1.3), with $N = 3$. Note that r^* can be rewritten as $r^* = \frac{6\alpha F}{2(\mu-b-\varepsilon)+\alpha(1-F)}$. Hence, $r^* < 6$ is equivalent to

$$\alpha F < 2(\mu-b-\varepsilon) + \alpha(1-F) \;\Leftrightarrow\; \alpha < \frac{2(\mu-b-\varepsilon)}{2F-1} = 4-2b,$$

which is true since $\alpha < \frac{4-2b}{3-2s}$ and $s \leq 1$. In addition, $r^* > \frac{6}{3-2s_c} = \frac{3\alpha}{2-b}$ is equivalent to

$$(4-2b)F > 2(\mu-b-\varepsilon) + \alpha(1-F) \;\Leftrightarrow\; \alpha < 4-2b.$$

Finally, it is easy to see that (a^*, r^*) satisfy the condition (1.2).

The next lemma is concerned with the case $N = 3$.

Lemma 2.21. *Let $N = 3$, $\frac{4-2b}{3} < \alpha < \frac{4-2b}{3-2s}$ and $0 < b < 1$. If $s_c < s \leq 1$ then there exists $\mu \in (b, 1)$ such that*

$$\begin{aligned}\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{S'(L^2)} \;&\leq\; c\|u\|^\theta_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\left(\|D^s u\|_{S(L^2)} + \|u\|_{S(L^2)}\right)\\ &\quad + c\|u\|^{1-\mu}_{L^\infty_t H^s_x}\|u\|^\theta_{S(\dot{H}^{s_c})}\|D^s u\|^{\alpha-\theta+\mu}_{S(L^2)},\end{aligned} \tag{2.73}$$

where $c > 0$, $\theta = \alpha F$ with $F = \frac{2-\varepsilon+\mu-2b}{4-2b}$ and $\varepsilon > 0$ is a sufficiently small number.

Proof. Note that

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{S'(L^2)} \leq \left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{S'(L^2(B))}+\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{S'(L^2(B^C))}.$$

Let $A\subset\mathbb{R}^N$ that can be B or B^C. Since $(2,6)$ is L^2-admissible in 3D we have

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{S'(L^2(A))} \leq \left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L_t^{2'}L_x^{6'}(A)}.$$

As before, applying the fractional product rule (Lemma 1.12) we have

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L_x^{6'}(A)} \leq M_1(t,A)+M_2(t,A), \tag{2.74}$$

where

$$M_1(t,A)=\left\||x|^{-b}\right\|_{L^\gamma(A)}\|D^s(|u|^\alpha u)\|_{L_x^\beta}, \quad M_2(t,A)=\left\|D^s(|x|^{-b})\right\|_{L^d(A)}\||u|^\alpha u\|_{L_x^e}$$

and

$$\frac{1}{6'}=\frac{1}{\gamma}+\frac{1}{\beta}=\frac{1}{d}+\frac{1}{e}. \tag{2.75}$$

First, we estimate $M_1(t,A)$. It follows by the fractional chain rule (Lemma 1.13) and Hölder's inequality that

$$\begin{aligned} M_1(t,A) &\leq \||x|^{-b}\|_{L^\gamma(A)}\|u\|^\theta_{L_x^{\theta r_1}}\|u\|^{\alpha-\theta}_{L_x^{(\alpha-\theta)r_2}}\|D^s u\|_{L_x^p} \\ &= \||x|^{-b}\|_{L^\gamma(A)}\|u\|^\theta_{L_x^{\theta r_1}}\|u\|^{\alpha-\theta}_{L_x^p}\|D^s u\|_{L_x^p}, \end{aligned} \tag{2.76}$$

where

$$\frac{1}{\beta}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{p} \quad \text{and} \quad p=(\alpha-\theta)r_2. \tag{2.77}$$

Combining (2.75) and (2.77) we obtain

$$\frac{3}{\gamma}=\frac{5}{2}-\frac{3}{r_1}-\frac{3(\alpha+1-\theta)}{p},$$

which implies, by (2.69)

$$\frac{3}{\gamma}-b=\frac{\theta(2-b)}{\alpha}-\frac{3}{r_1}. \tag{2.78}$$

In to order to show that $\||x|^{-b}\|_{L^\gamma(A)}$ is finite we need to verify that $\frac{3}{\gamma}-b>0$ if $A=B$ and $\frac{3}{\gamma}-b<0$ if $A=B^C$, by Remark 1.17. Indeed if $\theta r_1=\frac{6}{3-2s}$, by (2.78), we have

$$\frac{3}{\gamma}-b=\theta(s-s_c)>0$$

and if $\theta r_1=2$ then

$$\frac{3}{\gamma}-b=-\theta s_c<0.$$

Therefore, the inequality (2.76) and the Sobolev embedding (1.8) yield

$$M_1(t,A)\leq c\|u\|^\theta_{H^s_x}\|u\|^{\alpha-\theta}_{L^p_x}\|D^su\|_{L^p_x}. \tag{2.79}$$

Next, we estimate $M_2(t,A)$. Let $A=B^C$, applying the Hölder inequality and (2.10) we have

$$\begin{aligned} M_2(t,B^C) &\leq \||x|^{-b-s}\|_{L^d(B^C)}\|u\|^\theta_{L^{\theta r_1}_x}\|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)r_2}_x}\|u\|_{L^p_x}\\ &\leq \||x|^{-b-s}\|_{L^d(B^C)}\|u\|^\theta_{L^{\theta r_1}_x}\|u\|^{\alpha-\theta}_{L^p_x}\|u\|_{L^p_x}, \end{aligned}$$

where

$$\frac{1}{e}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{p}\quad\text{and}\quad p=(\alpha-\theta)r_2.$$

The relation (2.75) and the last relation imply

$$\frac{3}{d}=\frac{5}{2}-\frac{3}{r_1}-\frac{3(\alpha+1-\theta)}{p}.$$

In view of (2.69) we deduce

$$\frac{3}{d}-b=\frac{\theta(2-b)}{\alpha}-\frac{3}{r_1}.$$

Setting $\theta r_1=2$ we have $\frac{3}{d}-b=-\theta s_c$, so $\frac{3}{d}-b-s=-\theta s_c-s<0$, i.e., $|x|^{-b-s}\in L^d(B^C)$. Thus, by the Sobolev inequality (1.8)

$$M_2(t,B^C)\leq c\|u\|^\theta_{H^s_x}\|u\|^{\alpha-\theta}_{L^p_x}\|u\|_{L^p_x}. \tag{2.80}$$

We now consider $M_2(t,B)$. From the Hölder inequality, the Sobolev embedding[12] (1.6) and (2.10), we deduce

$$\begin{aligned} M_2(t,B) &\leq \||x|^{-b-s}\|_{L^d(B)}\|u\|^{\theta}_{L_x^{\theta r_1}}\|u\|^{\alpha-\theta}_{L_x^{(\alpha-\theta)r_2}}\|u\|^{\mu}_{L_x^{\mu r_3}}\|u\|^{1-\mu}_{L_x^{(1-\mu)r_4}} \\ &\leq \||x|^{-b-s}\|_{L^d(B)}\|u\|^{\theta}_{L_x^{\theta r_1}}\|D^s u\|^{\alpha-\theta}_{L_x^{n}}\|D^s u\|^{\mu}_{L_x^{n}}\|u\|^{1-\mu}_{L_x^{(1-\mu)r_4}} \\ &= \||x|^{-b-s}\|_{L^d(B)}\|u\|^{\theta}_{L_x^{r^*}}\|D^s u\|^{\alpha-\theta+\mu}_{L_x^{n}}\|u\|^{1-\mu}_{L_x^{(1-\mu)r_4}}, \end{aligned}$$

if the following system is satisfied

$$\begin{cases} \frac{1}{e} = \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4} \\ s = \frac{3}{n}-\frac{3}{(\alpha-\theta)r_2} \qquad s=\frac{3}{n}-\frac{3}{\mu r_3} \\ r^* = \theta r_1. \end{cases}$$

It follows from (2.75) and the previous system that

$$\frac{3}{d} = \frac{5}{2} + sD - \frac{3\theta}{r^*} - \frac{3D}{n} - \frac{3}{r_4},$$

which implies by (2.71) and (2.72)

$$\frac{3}{d} = \frac{7}{2} + sD - \frac{(2-b)\theta}{\alpha} - \frac{3D}{2} - \frac{3}{r_4},$$

where $D = \alpha-\theta+\mu$. In view of Remark 1.17 to show that $\||x|^{-b-s}\|_{L^d(B)}$ is bounded we need $\frac{3}{d}-b-s>0$. In fact, choosing $(1-\mu)r_4 = \frac{6}{3-2s}$ we have

$$\begin{aligned} \frac{3}{d}-b-s &= 2-b-\frac{3\alpha}{2}+\frac{3\theta}{2}+s(\alpha-\theta)-\frac{(2-b)\theta}{\alpha} \\ &= -\alpha\left(\frac{3}{2}-\frac{2-b}{\alpha}\right)+\theta\left(\frac{3}{2}-\frac{2-b}{\alpha}\right)+s(\alpha-\theta) \\ &= (s-s_c)(\alpha-\theta), \end{aligned}$$

which is positive since $s>s_c$. So $|x|^{-b-s}\in L^d(B)$ and we have

$$M_2(t,B) \leq c\|u\|^{1-\mu}_{H_x^s}\|u\|^{\theta}_{L_x^{r^*}}\|D^s u\|^{\alpha-\theta+\mu}_{L_x^{n}}, \tag{2.81}$$

[12] We can use the Sobolev embedding (1.6) since $s \leq 1 < \frac{3}{n}$.

where we have used the Sobolev embedding (1.8).

Therefore, combining (2.74), (2.79) with $A = B^C$ and (2.80) we obtain

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L_x^{6'}(B^C)} \leq c\|u\|_{H_x^s}^{\theta}\|u\|_{L_x^p}^{\alpha-\theta}\|D^s u\|_{L_x^p} + c\|u\|_{H_x^s}^{\theta}\|u\|_{L_x^p}^{\alpha-\theta}\|u\|_{L_x^p}.$$

Moreover, by (2.79) with $A = B$ and (2.81) we have

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L_x^{6'}(B)} \leq c\|u\|_{H_x^s}^{\theta}\|u\|_{L_x^p}^{\alpha-\theta}\|u\|_{L_x^p} + c\|u\|_{H_x^s}^{1-\mu}\|u\|_{L_x^{r^*}}^{\theta}\|D^s u\|_{L_x^n}^{\alpha-\theta+\mu}.$$

Finally, since

$$\frac{1}{2'} = \frac{\alpha-\theta}{k} + \frac{1}{l}$$

and

$$\frac{1}{2'} = \frac{\theta}{a^*} + \frac{\alpha-\theta+\mu}{m},$$

we can use Hölder's inequality in the time variable in the last two inequalities to conclude

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L_t^{2'}L_x^{6'}(B^C)} \leq c\|u\|_{L_t^\infty H_x^s}^{\theta}\|u\|_{L_t^k L_x^p}^{\alpha-\theta}\left(\|D^s u\|_{L_t^l L_x^p} + \|u\|_{L_t^l L_x^p}\right)$$

and

$$\begin{aligned}\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L_t^{2'}L_x^{6'}(B)} \leq\ & c\|u\|_{L_t^\infty H_x^s}^{\theta}\|u\|_{L_t^k L_x^p}^{\alpha-\theta}\|D^s u\|_{L_t^l L_x^p} \\ & + c\|u\|_{L_t^\infty H_x^s}^{1-\mu}\|u\|_{L_t^{a^*}L_x^{r^*}}^{\theta}\|D^s u\|_{L_t^m L_x^n}^{\alpha-\theta+\mu}.\end{aligned}$$

The proof is completed recalling that (m,n) and (l,p) are L^2-admissible as well as (k,p) and (a^*,r^*) are $\dot{H}^{s_c}$-admissible. □

Remark 2.22. Note that in the previous lemma $\theta > 0$ is given by $\theta = F\alpha$ and since $F < 1$, we only have that $\theta < \alpha$ and it might be not true that θ is close to 0. In Lemma 3.12 below we show that if $s = 1$ we can actually choose θ to be a small number.

Before proving our global well-posedness results, we finish estimating the norm $\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{S'(L^2)}$ in dimensions $N=1,2$.

Lemma 2.23. *Let $N=1,2$ and $\frac{4-2b}{N}<\alpha<\alpha_s$ with $0<b<\widetilde{2}$. If $s_c<s\leq \min\{\frac{N}{2},1\}$ then*

$$\begin{aligned}\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{S'(L^2)} &\leq c\|u\|^\theta_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot H^{s_c})}\|D^s u\|_{S(L^2)}\\ &\quad + c\|u\|^{1+\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot H^{s_c})},\end{aligned} \tag{2.82}$$

where $c>0$ and $\theta\in(0,\alpha)$ is a sufficiently small number.

Proof. The proof follows from analogous arguments as the ones used in the previous lemmas. Let $A\subset\mathbb{R}^N$ that can be B or B^C and (q,r) any L^2-admissible pair. By the fractional product rule (Lemma 1.12) we get

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L^{r'}_x(A)} \leq P_1(t,A)+P_2(t,A), \tag{2.83}$$

where

$$P_1(t,A)=\left\||x|^{-b}\right\|_{L^\gamma(A)}\|D^s(|u|^\alpha u)\|_{L^\beta_x},\quad P_2(t,A)=\left\|D^s(|x|^{-b})\right\|_{L^d(A)}\||u|^\alpha u\|_{L^e_x} \tag{2.84}$$

and

$$\frac{1}{r'}=\frac{1}{\gamma}+\frac{1}{\beta}=\frac{1}{d}+\frac{1}{e}. \tag{2.85}$$

To estimate $P_1(t,A)$ and $P_2(t,A)$, we consider three cases: $N=1$ and $s<\frac12$; $N=2$ and $s<1$; $N=1,2$ and $s=\frac{N}{2}$.

Case $N=1$ and $s<\frac12$. We define the following numbers

$$k^*=\frac{4\alpha(\alpha+1-\theta)}{(4-2b)(\alpha-\theta+1)-\alpha}\qquad l^*=\frac{4(\alpha+1-\theta)}{\alpha-\theta}\qquad p^*=2(\alpha+1-\theta) \tag{2.86}$$

$$q_0=\frac{2\alpha}{\alpha b+\theta(2-b)}\quad\text{and}\quad r_0=\frac{2\alpha}{\alpha(1-2b)-\theta(4-2b)}. \tag{2.87}$$

It is straightforward to verify that, if $\theta > 0$ is a small enough number, the assumption $0 < b < \frac{1}{3}$ implies that the denominators of q_0, r_0, k^* and l^* are all positive numbers. Furthermore, (q_0, r_0), (l^*, p^*) are L^2-admissible[13] and (k^*, p^*) is $\dot{H}^{s_c}$-admissible.

First, we estimate $P_1(t, A)$ with $r = r_0$. The fractional chain rule (Lemma 1.13) and Hölder's inequality yield

$$\begin{aligned} P_1(t,A) &\leq \||x|^{-b}\|_{L^\gamma(A)} \|u\|^{\theta}_{L^{\theta r_1}_x} \|u\|^{\alpha-\theta}_{L_x^{(\alpha-\theta)r_2}} \|D^s u\|_{L_x^{p^*}} \\ &= \||x|^{-b}\|_{L^\gamma(A)} \|u\|^{\theta}_{L^{\theta r_1}_x} \|u\|^{\alpha-\theta}_{L_x^{p^*}} \|D^s u\|_{L_x^{p^*}}, \end{aligned}$$

where

$$\frac{1}{\beta} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{p^*} \quad \text{and} \quad p^* = (\alpha-\theta)r_2. \tag{2.88}$$

This implies

$$\frac{1}{\gamma} - b = \frac{\theta(2-b)}{\alpha} - \frac{1}{r_1},$$

where we have used (2.85), (2.88), (2.86) and (2.87). Now, if $A = B$ and setting $\theta r_1 = \frac{2}{1-2s}$ we get $\frac{1}{\gamma} - b = \theta(s - s_c) > 0$, furthermore, taking $A = B^C$ and choosing $\theta r_1 = 2$ one has $\frac{1}{\gamma} - b = -\theta s_c < 0$. Hence, from the Sobolev embedding[14] (1.8) and Remark 1.17 we deduce

$$P_1(t,A) \leq c\|u\|^{\theta}_{H^s_x} \|u\|^{\alpha-\theta}_{L_x^{p^*}} \|D^s u\|_{L_x^{p^*}}. \tag{2.89}$$

We now consider $P_2(t, A)$ with $r = r_0$. It follows from (2.84) and (2.10) that

$$P_2(t,A) \leq \||x|^{-b-s}\|_{L^d(A)} \|u\|^{\theta+1}_{L_x^{(\theta+1)e}} \|u\|^{\alpha-\theta}_{L^\infty_x} \tag{2.90}$$

and by (2.85)

$$\frac{1}{d} - b = \frac{1}{2} + \frac{\theta(2-b)}{\alpha} - \frac{1}{e}. \tag{2.91}$$

[13]Note that, $r_0 > 2$ (see (1.1) for $N = 1$). Moreover, since $0 < b < \frac{1}{3}$ we have $p^* \geq \frac{2}{1-2s_c} = \frac{\alpha}{2-b}$ (see (1.2) for $N = 1$).

[14]Since $\theta r_1 \in [2, \frac{2}{1-2s}]$ in both cases.

We claim that $\||x|^{-b-s}\|_{L^d(A)}$ is a finite quantity for a suitable choice of e. If $A=B$ we choose $(\theta+1)e=\frac{2}{1-2s}$, and if $A=B^C$ we set $(\theta+1)e=2$. In the first case we obtain

$$\frac{1}{d}-b-s=\theta(s-s_c)>0,$$

and in the second case we have

$$\frac{1}{\gamma}-b-s=-\theta s_c<0.$$

So, the Sobolev embedding (1.8), Remark 1.17 and (2.90) yield

$$P_2(t,A)\leq c\|u\|^{\theta+1}_{H^s_x}\|u\|^{\alpha-\theta}_{L^\infty_x}.$$

Therefore, the relations (2.83), (2.89) and the last inequality with $A=B$ and $A=B^C$ imply that

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L^{r_0'}_x(B)}\leq c\|u\|^{\theta}_{H^s_x}\|u\|^{\alpha-\theta}_{L^{p^*}_x}\|D^s u\|_{L^{p^*}_x}+c\|u\|^{\theta+1}_{H^s_x}\|u\|^{\alpha-\theta}_{L^\infty_x}$$

and

$$\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L^{r_0'}_x(B^C)}\leq c\|u\|^{\theta}_{H^s_x}\|u\|^{\alpha-\theta}_{L^{p^*}_x}\|D^s u\|_{L^{p^*}_x}+c\|u\|^{\theta+1}_{H^s_x}\|u\|^{\alpha-\theta}_{L^\infty_x}.$$

Finally since

$$\frac{1}{q_0'}=\frac{\alpha-\theta}{k^*}+\frac{1}{l^*}$$

we apply the Hölder inequality in the time variable to get (recalling (l^*,p^*) is L^2-admissible and (k^*,p^*) is $\dot{H}^{s_c}$-admissible)

$$\begin{aligned}\left\|D^s\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L^{q_0'}_tL^{r_0'}_x} &\leq c\|u\|^{\theta}_{L^\infty_tH^s_x}\|u\|^{\alpha-\theta}_{L^{k^*}_tL^{p^*}_x}\|D^s u\|_{L^{l^*}_tL^{p^*}_x}\\ &\quad+c\|u\|^{\theta+1}_{L^\infty_tH^s_x}\|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)q_0'}_tL^\infty_x}\\ &\leq c\|u\|^{\theta}_{L^\infty_tH^s_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|D^s u\|_{S(L^2)}\\ &\quad+c\|u\|^{\theta+1}_{L^\infty_tH^s_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}.\end{aligned}$$

where we have used the fact that $(\alpha-\theta)q_0' = \frac{4}{1-2s_c}$, by (2.87), and $(\frac{4}{1-2s_c}, \infty)$ is $\dot{H}^{s_c}$-admissible.

Case $N=2$ and $s<1$. We start defining

$$\widetilde{q} = \frac{2\alpha}{\alpha[b+2\varepsilon(\alpha-\theta)]+\theta(2-b)} \qquad \widetilde{r} = \frac{2\alpha}{\alpha[1-b-2\varepsilon(\alpha-\theta)]-\theta(2-b)}, \tag{2.92}$$

$$l_0 = \frac{2(\alpha+1-\theta)}{(\alpha-\theta)(1-2\varepsilon)} \qquad p_0 = \frac{2(\alpha+1-\theta)}{1+2\varepsilon(\alpha-\theta)} \tag{2.93}$$

and

$$k_0 = \frac{2\alpha(\alpha+1-\theta)}{\alpha[1-b-2\varepsilon(\alpha-\theta)]+(2-b)(1-\theta)} \tag{2.94}$$

Note that, $(\widetilde{q},\widetilde{r})$, (l_0,p_0) are L^2-admissible[15] and (k_0,p_0) is $\dot{H}^{s_c}$-admissible[16].

We first estimate $P_1(t,A)$ (recall (2.84)-(2.85)) with $r=\widetilde{r}$. Analogous as before, the fractional chain rule (Lemma 1.13) and Hölder's inequality lead to

$$\begin{aligned} P_1(t,A) &\leq \||x|^{-b}\|_{L^\gamma(A)}\|u\|^{\theta}_{L^{\theta r_1}_x}\|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)r_2}_x}\|D^s u\|_{L^{p_0}_x} \\ &= \||x|^{-b}\|_{L^\gamma(A)}\|u\|^{\theta}_{L^{\theta r_1}_x}\|u\|^{\alpha-\theta}_{L^{p_0}_x}\|D^s u\|_{L^{p_0}_x}, \end{aligned}$$

where

$$\frac{1}{\beta} = \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{p_0} \quad \text{and} \quad p_0 = (\alpha-\theta)r_2, \tag{2.95}$$

---

[15] The hypothesis $0<b<\frac{N}{3}$ with $N=2$ guarantee that the denominators of $\widetilde{q}$, $\widetilde{r}$, k_0, l_0 and p_0 are all positive numbers. Moreover, $\widetilde{r}>2$ is equivalent to $\alpha(b+2\varepsilon(\alpha-\theta)) > -\theta(2-b)$ which is true, therefore $\widetilde{r}$ satisfies (1.1) for $N=2$.

[16] We claim that $\frac{2\alpha}{2-b} = \frac{2}{1-s_c} \leq p_0 \leq ((\frac{2}{1-s_c})^+)'$. Indeed, the first inequality is equivalent to $\alpha(1-b)+(1-\theta)(2-b) \geq 2\varepsilon\alpha(\alpha-\theta)$ which holds true since $\varepsilon>0$ is a small enough number. On the other hand, the later inequality holds since $\varepsilon p_0 \leq (\frac{2}{1-s_c})^+(\frac{2}{1-s_c})$ (recall (1.4)) can be verified for $\varepsilon>0$ small enough.

so the relations (2.85), (2.95), (2.93) and (2.92) imply

$$\frac{2}{\gamma} - b = \frac{\theta(2-b)}{\alpha} - \frac{2}{r_1}.$$

As in the previous case, if $A = B$ we set $\theta r_1 = \frac{2}{1-s}$ and then $\frac{2}{\gamma} - b > 0$. On the other hand, if $A = B^C$, we set $\theta r_1 = 2$ and then $\frac{2}{\gamma} - b < 0$. Hence, the Sobolev embedding (1.8) and Remark 1.17 yield

$$P_1(t, A) \leq c\|u\|^{\theta}_{H^s_x} \|u\|^{\alpha-\theta}_{L^{p_0}_x} \|D^s u\|_{L^{p_0}_x}. \tag{2.96}$$

Next we estimate $P_2(t, A)$ with $r = \widetilde{r}$. An application of the Hölder inequality together with (2.84) and (2.10) imply

$$\begin{aligned} P_2(t, A) &\leq \||x|^{-b-s}\|_{L^d(A)} \|u\|^{\theta+1}_{L^{(\theta+1)r_1}_x} \|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)r_2}_x} \\ &\leq \||x|^{-b-s}\|_{L^d(A)} \|u\|^{\theta+1}_{L^{(\theta+1)r_1}_x} \|u\|^{\alpha-\theta}_{L^{\frac{1}{\varepsilon}}_x}, \end{aligned}$$

where

$$\frac{1}{e} = \frac{1}{r_1} + \frac{1}{r_2}, \qquad (\alpha - \theta) r_2 = \frac{1}{\varepsilon}. \tag{2.97}$$

We deduce from (2.97) and (2.85)

$$\begin{aligned} \frac{2}{d} &= 2 - \frac{2}{\widetilde{r}} - \frac{1}{r_1} - 2\varepsilon(\alpha - \theta) \\ &= 1 + b + \frac{\theta(2-b)}{\alpha} - \frac{2}{r_1}, \end{aligned}$$

where we have used (2.92). In addition, if $A = B$ and $(\theta+1)r_1 = \frac{2}{1-s}$ we get

$$\frac{2}{d} - b - s = \theta(s - s_c) > 0,$$

likewise if $A = B^C$ and $(\theta + 1)r_1 = 2$, we have

$$\frac{2}{d} - b - s = -\theta s_c - s < 0.$$

Thus

$$P_2(t, A) \leq c\|u\|^{\theta+1}_{H^s_x} \|u\|^{\alpha-\theta}_{L^{\frac{1}{\varepsilon}}_x},$$

where we have used the Sobolev inequality (1.8) and Remark 1.17.

Hence, the relations (2.83), (2.96) and the last inequality lead to

$$\left\| D^s \left(|x|^{-b}|u|^\alpha u\right)\right\|_{L_x^{\widetilde{r}}} \leq c\|u\|_{H_x^s}^{\theta}\|u\|_{L_x^{p_0}}^{\alpha-\theta}\|D^s u\|_{L_x^{p_0}} + c\|u\|_{H_x^s}^{\theta+1}\|u\|_{L_x^{\frac{1}{\varepsilon}}}^{\alpha-\theta}.$$

Finally, from (2.92) and (2.94), we have that

$$\frac{1}{\widetilde{q}'} = \frac{\alpha-\theta}{k_0} + \frac{1}{l_0},$$

so applying the Hölder inequality in the time variable one has

$$\begin{aligned}\left\| D^s \left(|x|^{-b}|u|^\alpha u\right)\right\|_{L_t^{\widetilde{q}'}L_x^{\widetilde{r}'}} &\leq c\|u\|_{L_t^\infty H_x^s}^{\theta}\|u\|_{L_t^{k_0}L_x^{p_0}}^{\alpha-\theta}\|D^s u\|_{L_t^{l_0}L_x^{p_0}} \\ &\quad + c\|u\|_{L_t^\infty H_x^s}^{\theta+1}\|u\|_{L_t^{(\alpha-\theta)\widetilde{q}'}L_x^{\frac{1}{\varepsilon}}}^{\alpha-\theta} \\ &\leq c\|u\|_{L_t^\infty H_x^s}^{\theta}\|u\|_{S(\dot{H}^{s_c})}^{\alpha-\theta}\|D^s u\|_{S(L^2)} \\ &\quad + c\|u\|_{L_t^\infty H_x^s}^{\theta+1}\|u\|_{S(\dot{H}^{s_c})}^{\alpha-\theta},\end{aligned}$$

where we have used the fact that $(\alpha-\theta)\widetilde{q}' = \frac{2\alpha}{2-b-2\varepsilon\alpha}$ and $\left(\frac{2\alpha}{2-b-2\varepsilon\alpha}, \frac{1}{\varepsilon}\right)$ is $\dot{H}^{s_c}$-admissible[17].

Case $N=1,2$ and $s=\frac{N}{2}$. As before, we start defining the following numbers

$$\bar{a} = \frac{2(\alpha+1-\theta)}{2-s_c} \qquad \bar{q} = \frac{2(\alpha+1-\theta)}{2+s_c(\alpha-\theta)} \tag{2.98}$$

$$\bar{r} = \frac{2N(\alpha+1-\theta)}{N(\alpha+1-\theta)-2s_c(\alpha-\theta)-4} \tag{2.99}$$

and

$$\bar{k} = \frac{2(\alpha+1-\theta)^2}{2(\alpha-\theta)(1-s_c)-s_c} \qquad \bar{l} = \frac{2(\alpha+1-\theta)^2}{2(\alpha-\theta)(1-s_c)+s_c\left((\alpha+1-\theta)^2-1\right)} \tag{2.100}$$

$$\bar{p} = \frac{2N(\alpha+1-\theta)^2}{(N-2s_c)(\alpha+1-\theta)^2-4(\alpha-\theta)(1-s_c)+2s_c}. \tag{2.101}$$

[17] Note that $\frac{1}{\varepsilon}$ satisfies assumption (1.3) with $N=2$. Also recall that (l_0,p_0) is L^2-admissible and (k_0,p_0) is $\dot{H}^{s_c}$-admissible.

Remark 2.24. We claim that the denominator of the numbers defined above is positive. Indeed, first it is easy to see that the denominators of $\bar{a}$ and $\bar{q}$ are positive numbers (since $s_c < 1$ and $\alpha > \theta$). We now show the denominators of $\bar{r}, \bar{k}, \bar{l}$ and $\bar{p}$ are also positive numbers for $\theta > 0$ sufficiently small.
Note that the denominator of $\bar{r}$ can be written as $N(1-\theta) - 2b + 2s_c\theta = N - 2b - \theta(N - 2s_c)$ and this is positive since $b < \frac{N}{3}$ and θ is small enough.
Moreover, since $\alpha > \frac{4-2b}{N}$, the denominator of $\bar{k}$ is given by $2\alpha - 2\alpha s_c - s_c - 2\theta(1-s_c) = 2\alpha - \alpha N + 4 - 2b - s_c - 2\theta(1-s_c) > \alpha(2-N) + 3 - 2b - 2\theta(1-s_c)$, (where we have used $s_c < 1$) which is positive since $N = 1, 2$; $b < \frac{N}{3}$ and θ is small enough.
It is clear that the denominator of $\bar{l}$ is a positive number since $s_c < 1$ and $\alpha > \theta$.
Finally, the denominator of $\bar{p}$ is positive. Indeed, $\bar{p}$ can be written as $\bar{p} = \frac{2N\alpha(\alpha+1-\theta)^2}{(4-2b)(\alpha+1-\theta)^2 - 2(\alpha-\theta)(4-2b-\alpha(N-2)) + N\alpha - (4-2b)}$.
If $N = 2$ we have $(4-2b)(\alpha+1-\theta)^2 - 2(\alpha-\theta)(4-2b) + 2\alpha - (4-2b) > (4-2b)\left((\alpha+1-\theta)^2 - 2(\alpha-\theta)\right) > 0$, where we have used the assumption $\alpha > 2 - b$ and the fact that $b < 2/3$.
Similarly if $N = 1$, we use $\alpha > 4 - 2b$ to obtain $(4-2b)(\alpha+1-\theta)^2 - 2(\alpha - \theta)(4-2b+\alpha) + \alpha - (4-2b) > (4-2b)\left((\alpha+1-\theta)^2 - 2(\alpha-\theta)\right) - 2\alpha(\alpha-\theta) = (4-2b)\left((\alpha-\theta)^2 + 1\right) - 2\alpha(\alpha-\theta) = (\alpha-\theta)\left((\alpha-\theta)(4-2b) - 2\alpha\right) + 4 - 2b > (\alpha-\theta)\left(2\alpha(1-b) - \theta(4-2b)\right)$, which is positive since θ is small enough and $b < 1/3$.

On the other hand, it is not difficult to check that $(\bar{q}, \bar{r})$ and $(\bar{l}, \bar{p})$ L^2-admissible and $(\bar{a}, \bar{r})$, $(\bar{k}, \bar{p})$ $\dot{H}^{s_c}$-admissible.[18]

First, we estimate $P_1(t, A)$ with $r = \bar{r}$. The fractional chain rule (Lemma

[18] We also have $\bar{r}, \bar{p} \geq \frac{2N}{N-2s_c} = \frac{N\alpha}{2-b}$. Indeed $\bar{r} = \frac{2N(\alpha+1-\theta)}{N-2b-\theta(N-2s_c)} \geq \frac{N\alpha}{2-b} \Leftrightarrow \alpha(4-N) + (1-\theta)(4-2b) > -\theta\alpha(N-2s_c)$ which is true since $N = 1, 2$ and $\theta < 1$. Moreover, $\bar{p} \geq \frac{N\alpha}{2-b}$

1.13) and Hölder's inequality lead to

$$\begin{aligned} P_1(t,A) &\leq \||x|^{-b}\|_{L^\gamma(A)}\|u\|^\theta_{L^{\theta r_1}_x}\|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)r_2}_x}\|D^s u\|_{L^{\bar{p}}_x} \\ &= \||x|^{-b}\|_{L^\gamma(A)}\|u\|^\theta_{L^{\theta r_1}_x}\|u\|^{\alpha-\theta}_{L^{\bar{p}}_x}\|D^s u\|_{L^{\bar{p}}_x}, \end{aligned} \tag{2.102}$$

where

$$\frac{1}{\beta}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{\bar{p}} \quad \text{and} \quad \bar{p}=(\alpha-\theta)r_2, \tag{2.103}$$

and so combining (2.85), (2.103) (2.99) and (2.101) we obtain

$$\begin{aligned} \frac{N}{\gamma}-b &= N-b-\frac{N}{r_1}-\frac{N}{\bar{r}}-\frac{N(\alpha+1-\theta)}{\bar{p}} \\ &= N-b-\frac{N}{r_1}-\left(\frac{(\alpha+1-\theta)(N-2s_c)+N-2(2-s_c)}{2}\right) \\ &= \frac{\theta(2-b)}{\alpha}-\frac{N}{r_1}. \end{aligned} \tag{2.104}$$

In order to have that the first norm in the right hand side of (2.102) is finite, we need to verify $\frac{N}{\gamma}-b>0$ if $A=B$ and $\frac{N}{\gamma}-b<0$ if $A=B^C$ for suitable choices of r_1. To this end, we set r_1 such that

$$\theta r_1 > \frac{N\alpha}{(2-b)} \ \text{(when } A=B) \quad \text{and} \quad 2<\theta r_1<\frac{N\alpha}{(2-b)} \ \text{(when } A=B^C) \tag{2.105}$$

Hence, the Sobolev embedding (1.7) and (2.102) yield

$$P_1(t,A) \leq c\|u\|^\theta_{H^s_x}\|u\|^{\alpha-\theta}_{L^{\bar{p}}_x}\|D^s u\|_{L^{\bar{p}}_x}. \tag{2.106}$$

We now consider $P_2(t,A)$ with $r=\bar{r}$. By the Hölder inequality and (2.84) we deduce

$$\begin{aligned} P_2(t,A) &\leq \||x|^{-b-s}\|_{L^d(A)}\|u\|^{\theta+1}_{L^{(\theta+1)r_1}_x}\|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)r_2}_x} \\ &= \||x|^{-b-s}\|_{L^d(A)}\|u\|^{\theta+1}_{L^{(\theta+1)r_1}_x}\|u\|^{\alpha-\theta}_{L^{\bar{r}}_x}, \end{aligned}$$

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is equivalent to $2(\alpha-\theta)(4-2b-\alpha(N-2))\geq N\alpha-(4-2b)$ so

$$\alpha\left(2(4-2b)-N-2\alpha(N-2)\right)+(4-2b)\geq 2\theta(4-2b-\alpha(N-2)),$$

this is true since θ small enough, $N=1,2$ and $b<\frac{N}{3}$.

where

$$\frac{1}{e} = \frac{1}{r_1} + \frac{1}{r_2} \quad \text{and} \quad \bar{r} = (\alpha - \theta) r_2. \tag{2.107}$$

The relations (2.85) and (2.107) as well as $\bar{r}$ defined in (2.99), yield (recall $s = \frac{N}{2}$)

$$\begin{aligned} \frac{N}{d} - b - s &= N - b - s - \frac{N}{r_1} - \frac{N(\alpha + 1 - \theta)}{\bar{r}} \\ &= \frac{N}{2} + (2 - b) - \frac{N}{r_1} - \frac{N(\alpha + 1 - \theta)}{2} + s_c(\alpha - \theta) \\ &= \frac{\theta(2 - b)}{\alpha} - \frac{N}{r_1}. \end{aligned} \tag{2.108}$$

Note that the right hand side of (2.108) is equal to the right hand side of (2.104), so choosing r_1 as in (2.105) and again applying the Sobolev inequality (1.7), we deduce

$$P_2(t, A) \leq c \|u\|^{\theta+1}_{H^s_x} \|u\|^{\alpha-\theta}_{L^{\bar{r}}_x}.$$

So the inequalities (2.83), (2.106) and the last inequality imply that

$$\left\| D^s \left(|x|^{-b} |u|^\alpha u \right) \right\|_{L^{\bar{r}'}_x} \leq c \|u\|^{\theta}_{H^s_x} \|u\|^{\alpha-\theta}_{L^{\bar{p}}_x} \|D^s u\|_{L^{\bar{p}}_x} + c \|u\|^{\theta+1}_{H^s_x} \|u\|^{\alpha-\theta}_{L^{\bar{r}}_x}.$$

Since

$$\frac{1}{\bar{q}'} = \frac{\alpha - \theta}{\bar{k}} + \frac{1}{\bar{l}}$$

we can apply the Hölder inequality in the time variable to deduce

$$\begin{aligned} \left\| D^s \left(|x|^{-b} |u|^\alpha u \right) \right\|_{L^{\bar{q}'}_t L^{\bar{r}'}_x} &\leq c \|u\|^{\theta}_{L^\infty_t H^s_x} \|u\|^{\alpha-\theta}_{L^{\bar{k}}_t L^{\bar{p}}_x} \|D^s u\|_{L^{\bar{l}}_t L^{\bar{p}}_x} \\ &\quad + c \, \|u\|^{\theta+1}_{L^\infty_t H^s_x} \|u\|^{\alpha-\theta}_{L^{(\alpha-\theta)\bar{q}'}_t L^{\bar{r}}_x} \\ &\leq c \|u\|^{\theta}_{L^\infty_t H^s_x} \|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})} \|D^s u\|_{S(L^2)} \\ &\quad + c \|u\|^{\theta+1}_{L^\infty_t H^s_x} \|u\|^{\alpha-\theta}_{L^{\bar{a}}_t L^{\bar{r}}_x}, \end{aligned}$$

where in the last equality we have used the fact that $\bar{a} = (\alpha - \theta)\bar{q}'$. This completes the proof since $(\bar{a}, \bar{r})$ $\dot{H}^{s_c}$-admissible. □

The next result follows directly from Lemmas 2.19, 2.21 and 2.23.

Corollary 2.25. *Assume* $\frac{4-2b}{N} < \alpha < \alpha_s$ *and* $0 < b < \widetilde{2}$. *If* $s_c < s \leq \min\{\frac{N}{2}, 1\}$ *then the following statement holds:*

$$\begin{aligned}\|D^sF\|_{S'(L^2)} &\leq c\|u\|^{\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\left(\|D^s u\|_{S(L^2)} + \|u\|_{S(L^2)} + \|u\|_{L^\infty_t H^s_x}\right)\\ &\quad + c\|u\|^{1-\mu}_{L^\infty_t H^s_x}\|u\|^{\theta}_{S(\dot{H}^{s_c})}\|D^s u\|^{\alpha-\theta+\mu}_{S(L^2)},\end{aligned}$$

where $F(x,u) = |x|^{-b}|u|^\alpha u$.

Now, we have all the tools to prove Theorem 2.14. Similarly as in the local theory, we use the contraction mapping principle.

Proof of Theorem 2.14. First, we define

$$B = \{u:\ \|u\|_{S(\dot{H}^{s_c})} \leq 2\|U(t)u_0\|_{S(\dot{H}^{s_c})} \text{ and } \|u\|_{S(L^2)} + \|D^s u\|_{S(L^2)} \leq 2c\|u_0\|_{H^s}\}.$$

We show that $G = G_{u_0}$ defined in (9) is a contraction on B equipped with the metric

$$d(u,v) = \|u-v\|_{S(L^2)} + \|u-v\|_{S(\dot{H}^{s_c})}.$$

Indeed, by the Strichartz inequalities (1.9), (1.10), (1.11) and (1.12), we deduce

$$\|G(u)\|_{S(\dot{H}^{s_c})} \leq \|U(t)u_0\|_{S(\dot{H}^{s_c})} + c\|F\|_{S'(\dot{H}^{-s_c})} \tag{2.109}$$

$$\|G(u)\|_{S(L^2)} \leq c\|u_0\|_{L^2} + c\|F\|_{S'(L^2)} \tag{2.110}$$

and

$$\|D^sG(u)\|_{S(L^2)} \leq c\|D^s u_0\|_{L^2} + c\|D^s F\|_{S'(L^2)}, \tag{2.111}$$

where $F(x,u) = |x|^{-b}|u|^\alpha u$. On the other hand, it follows from Lemmas 2.17 and 2.18 together with Corollary 2.25 that

$$\begin{aligned}\|F\|_{S'(\dot{H}^{-s_c})} &\leq c\|u\|^{\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|u\|_{S(\dot{H}^{s_c})}\\ \|F\|_{S'(L^2)} &\leq c\|u\|^{\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|u\|_{S(L^2)}\end{aligned}$$

and

$$\begin{aligned}\|D^sF\|_{S'(L^2)} &\leq c\|u\|^{\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot H^{s_c})}\left(\|D^s u\|_{S(L^2)}+\|u\|_{S(L^2)}+\|u\|_{L^\infty_t H^s_x}\right)\\ &\quad + c\|u\|^{1-\mu}_{L^\infty_t H^s_x}\|u\|^{\theta}_{S(\dot H^{s_c})}\|D^s u\|^{\alpha-\theta+\mu}_{S(L^2)}.\end{aligned}$$

In addition, combining (2.109)-(2.111) and the last inequalities, we get for $u \in B$

$$\begin{aligned}\|G(u)\|_{S(\dot H^{s_c})} &\leq \|U(t)u_0\|_{S(\dot H^{s_c})} + c\|u\|^{\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot H^{s_c})}\|u\|_{S(\dot H^{s_c})}\\ &\leq \|U(t)u_0\|_{S(\dot H^{s_c})} + 2^{\alpha+1}c^{\theta+1}\|u_0\|^{\theta}_{H^s}\|U(t)u_0\|^{\alpha-\theta+1}_{S(\dot H^{s_c})}.\end{aligned}$$

Also, setting $X = \|D^s u\|_{S(L^2)} + \|u\|_{S(L^2)} + \|u\|_{L^\infty_t H^s_x}$ we have

$$\begin{aligned}\|G(u)\|_{S(L^2)} + \|D^sG(u)\|_{S(L^2)} &\leq c\|u_0\|_{H^s} + c\|u\|^{\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot H^{s_c})}X\\ &\quad + c\|u\|^{1-\mu}_{L^\infty_t H^s_x}\|u\|^{\theta}_{S(\dot H^{s_c})}\|D^s u\|^{\alpha-\theta+\mu}_{S(L^2)}\\ &\leq c\|u_0\|_{H^s} + 2^{\alpha+2}c^{\theta+2}\|u_0\|^{\theta+1}_{H^s}\|U(t)u_0\|^{\alpha-\theta}_{S(\dot H^{s_c})}\\ &\quad +2^{\alpha+1}c^{\alpha-\theta+2}\|u_0\|^{\alpha-\theta+1}_{H^s}\|U(t)u_0\|^{\theta}_{S(\dot H^{s_c})},\end{aligned}$$

where we have used the fact that $X \leq 2^2c\|u_0\|_{H^s}$ since $u \in B$.

Now if $\|U(t)u_0\|_{S(\dot H^{s_c})} < \delta$ with

$$\delta \leq \min\left\{\sqrt[\alpha-\theta]{\frac{1}{2c^{\theta+1}2^{\alpha+1}A^{\theta}}},\ \sqrt[\alpha-\theta]{\frac{1}{4c^{\theta+1}2^{\alpha+2}A^{\theta}}},\ \sqrt[\theta]{\frac{1}{4c^{\alpha-\theta+1}2^{\alpha+1}A^{\alpha-\theta}}}\right\}, \tag{2.112}$$

where $A > 0$ is a number such that $\|u_0\|_{H^s} \leq A$, we get

$$\|G(u)\|_{S(\dot H^{s_c})} \leq 2\|U(t)u_0\|_{S(\dot H^{s_c})}$$

and

$$\|G(u)\|_{S(L^2)} + \|D^sG(u)\|_{S(L^2)} \leq 2c\|u_0\|_{H^s},$$

that is $G(u) \in B$.

Now we show that G is a contraction on B. From (1.13) and repeating the above computations, one has

$$\begin{aligned}
\|G(u)-G(v)\|_{S(\dot{H}^{s_c})} \leq & c\|F(x,u)-F(x,v)\|_{S(\dot{H}^{-s_c})} \\
\leq & c\left\||x|^{-b}|u|^{\alpha}|u-v|\right\|_{S(\dot{H}^{-s_c})}+\left\||x|^{-b}|v|^{\alpha}|u-v|\right\|_{S(\dot{H}^{-s_c})} \\
\leq & c\|u\|^{\theta}_{L^\infty_t H^s_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|u-v\|_{S(\dot{H}^{s_c})} \\
& + c\|v\|^{\theta}_{L^\infty_t H^s_x}\|v\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|u-v\|_{S(\dot{H}^{s_c})}
\end{aligned}$$

which implies, taking $u, v \in B$

$$\begin{aligned}
\|G(u)-G(v)\|_{S(\dot{H}^{s_c})} & \leq 2c(2c)^{\theta}\|u_0\|^{\theta}_{H^s}2^{\alpha-\theta}\|U(t)u_0\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|u-v\|_{S(\dot{H}^{s_c})} \\
& = 2^{\alpha+1}c^{\theta+1}\|u_0\|^{\theta}_{H^s}\|U(t)u_0\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|u-v\|_{S(\dot{H}^{s_c})}
\end{aligned}$$

By similar arguments we also obtain

$$\|G(u)-G(v)\|_{S(L^2)} \leq 2^{\alpha+1}c^{\theta+1}\|u_0\|^{\theta}_{H^s}\|U(t)u_0\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|u-v\|_{S(L^2)}.$$

Finally, from the last two inequalities and (2.112) we deduce

$$d(G(u),G(v)) \leq 2^{\alpha+1}c^{\theta+1}\|u_0\|^{\theta}_{H^s}\|U(t)u_0\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}d(u,v) \leq \frac{1}{2}d(u,v),$$

i.e., G is a contraction.

Therefore, by the Banach Fixed Point Theorem, G has a unique fixed point $u \in B$, which is a global solution of (7). □

Chapter 3

Scattering for INLS equation

3.1 Introduction

In this chapter, we consider the Cauchy problem for the focusing inhomogenous nonlinear Schrödinger equation, that is

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^\alpha u = 0, & t \in \mathbb{R},\ x \in \mathbb{R}^N, \\ \qquad u(0,x) = u_0(x), \end{cases} \tag{3.1}$$

Our principal aim here is to study scattering (recall Definition 0.2) for INLS equation in $\mathbb{R}^N$, $N \geq 2$, with radial data in $H^1(\mathbb{R}^N)$. We focus on the L^2-supercritical and H^1-subcritical case, which as explained in the introduction, corresponds to the cases where

$$\begin{cases} 2 - b < \alpha < \infty, & N = 2, \\ \frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}, & N \geq 3. \end{cases} \tag{3.2}$$

In the particular case $b = 0$, i.e., the classical nonlinear Schrödinger equation (NLS), this problem was already studied for many authors. Let us recall the best results available in the literature. The cubic NLS in 3D case with radial initial data was considered by Holmer-Roudenko [23], then Duyckaerts-Holmer-Roudenko [10] extended the same result for non radial initial data. It was later generalized, for arbitrary dimension $N \geq 1$ and all L^2-supercritical and H^1-subcritical NLS equations, by Fang-Xie-Cazenave [11] (see also Guevara [22]). All these works used the concentration-compactness method and rigidity technique introduced by Kenig-Merle [26] in their study of the energy critical NLS. Inspired in these works we show scattering for the INLS equation (3.1) under the assumption (3.2).

In a recent work, Farah [12] showed global well-posedness for the L^2-supercritical and H^1-subcritical INLS (3.1). More precisely, he obtained the following result:

Theorem 3.1. *Let $N \geq 1$, $\frac{4-2b}{N} < \alpha < 2^*$ and $0 < b < \min\{2, N\}$. Suppose that $u(t)$ is the solution of (3.1) with initial data $u_0 \in H^1(\mathbb{R}^N)$ satisfying*[1]

$$E[u_0]^{s_c} M[u_0]^{1-s_c} < E[Q]^{s_c} M[Q]^{1-s_c} \tag{3.3}$$

and

$$\|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}. \tag{3.4}$$

Then $u(t)$ is a global solution in $H^1(\mathbb{R}^N)$. Furthermore, for any $t \in \mathbb{R}$ we have

$$\|\nabla u(t)\|_{L^2_x}^{s_c} \|u(t)\|_{L^2_x}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}, \tag{3.5}$$

where Q is the unique smooth, radial and positive solution of the elliptic equation

$$-Q + \Delta Q + |x|^{-b} |Q|^{\alpha} Q = 0. \tag{3.6}$$

Remark 3.2. In [12, Teorema 1.6] was also showed that, if the condition (3.3) holds, $\|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} > \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}$ and u_0 has finite variance, i.e., $|x|u_0 \in L^2(\mathbb{R}^N)$. Then the solution u blows up in finite time. This is an extension to the INLS model of the result proved by Holmer-Roudenko [23] for the NLS equation.

[1] Recalling that $s_c = \frac{N}{2} - \frac{2-b}{\alpha}$.

The goal here is to prove scattering for the INLS equation (3.1) under the conditions (3.3)-(3.4). Before stating the main result we define

$$2_* := \begin{cases} \frac{4-2b}{N-2} & N \geq 4, \\ 3-2b & N = 3, \\ \infty & N = 2. \end{cases} \tag{3.7}$$

Note that, for $N \neq 3$ we have $2_* = 2^*$ (recalling that 2^* is given in (8)). For dimension $N = 3$, we need the condition $\alpha < 3 - 2b$ to have the exponent of $\|\nabla u\|_{S(L^2)}$ equal to 1, see Lemma 3.12 and also the footnote 3 below.

We now give the precise statement of our main result of this chapter.

Theorem 3.3. *Let $N \geq 2$, $u_0 \in H^1(\mathbb{R}^N)$ be radial and $\frac{4-2b}{N} < \alpha < 2_*$ with $0 < b < \min\{\frac{N}{3}, 1\}$. Suppose that (3.3) and (3.4) are satisfied then the solution u of (3.1) with initial data u_0 is global and scatters in $H^1(\mathbb{R}^N)$, i.e., there exists $\phi^{\pm} \in H^1(\mathbb{R}^N)$ such that*

$$\lim_{t \to \pm\infty} \|u(t) - U(t)\phi^{\pm}\|_{H^1_x} = 0. \tag{3.8}$$

Remark 3.4. Note that, for the scattering result we replace the condition $0 < b < \widetilde{2}$ by $0 < b < \min\{\frac{N}{3}, 1\}$. Recalling $\widetilde{2} = 2$ for $N \geq 4$ (see definition (2.1)), in the previous chapter we consider $b < 2$ for $N \geq 4$. However, in this chapter we assume the condition $b < 1$ when $N \geq 4$ (we need this condition to show the existence of the critical solution, see Proposition 3.28 and footnote 15 below).

Remark 3.5. It is worth to mention that although the above theorem does not hold for all L^2-supercritical and H^1-subcritical INLS equation (3.1), when $N = 3$, we still have scattering for the cubic INLS equation in 3D. Therefore, we were able to extend the result of Holmer-Roudenko [23] for INLS setting. Also, since the solutions of the INLS equation do not enjoy conservation of momentum, we can not use the ideas introduced by Duyckaerts-Holmer-Roudenko [10] to remove the radial assumption in Theorem 3.3.

Similarly as in the NLS model, the criteria to establish scattering is given by the following proposition (we will show it after the Proposition 3.14):

Proposition 3.6. **(H^1 scattering)** *Let $u(t)$ be a global solution of* (3.1) *with initial data $u_0 \in H^1(\mathbb{R}^N)$. If $\|u\|_{S(\dot{H}^{s_c})} < +\infty$ and $\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1_x} \leq B$. Then $u(t)$ scatters in $H^1(\mathbb{R}^N)$ as $t \to \pm\infty$ in the sense defined in* (3.8)*.*

The plan of this chapter is as follows: in Section 3.2, we give the idea of the proof of the main result (Theorem 3.3), assuming all the technical points. In section 3.3, we collect many preliminary results for the Cauchy problem (3.1). Next in Section 3.4, we recall some properties of the ground state and show the existence of the wave operator. In Section 3.5, we construct a critical solution denoted by u_c and show some of its properties (the key ingredient in this step is a profile decomposition result related to the linear flow). Finally, Section 3.6 is devoted to the rigidity theorem.

3.2 Sketch of the proof of the main result

Let $u(t)$ be the corresponding H^1 solution for the Cauchy problem (3.1) with radial data $u_0 \in H^1(\mathbb{R}^N)$ satisfying (3.3) and (3.4). We already know by Theorem 3.1 that the solution is globally defined and $\sup_{t\in\mathbb{R}} \|u(t)\|_{H^1} < \infty$. So, in view of Proposition 3.6, our goal is to show that

$$\|u\|_{S(\dot{H}^{s_c})} < +\infty. \tag{3.9}$$

The technique employed here to achieve the scattering property (3.9) combines the concentration-compactness with rigidity ideas introduced by Kenig-Merle [26]. It is also based on the works of Holmer-Roudenko [23], Fang-Xie-

Cazenave [11] and Guevara [22]. We describe it in the sequel, but first we need some preliminary definitions.

Definition 3.7. We shall say that $\mathrm{SC}(u_0)$ holds if the solution $u(t)$ with initial data $u_0 \in H^1(\mathbb{R}^N)$ is global and (3.9) holds.

Definition 3.8. For each $\delta > 0$ define the set A_δ to be the collection of all initial data in $H^1(\mathbb{R}^N)$ satisfying

$$A_\delta = \{u_0 \in H^1 : E[u_0]^{s_c} M[u_0]^{1-s_c} < \delta \text{ and } \|\nabla u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}\}$$

and define

$$\delta_c = \sup\{\ \delta > 0 :\ u_0\ \in A_\delta\ \Longrightarrow SC(u_0) \text{ holds}\} = \sup_{\delta>0} B_\delta. \tag{3.10}$$

First note that $B_\delta \neq \emptyset$. In fact, applying the Strichartz estimate (1.10), interpolation and Lemma 3.21 (i) below, we obtain

$$\begin{aligned} \|U(t)u_0\|_{S(\dot{H}^{s_c})} &\leq c\|u_0\|_{\dot{H}^{s_c}} \leq c\|\nabla u_0\|_{L^2}^{s_c}\|u_0\|_{L^2}^{1-s_c} \\ &\leq c\left(\frac{N\alpha + 2b}{\alpha s_c}\right)^{\frac{s_c}{2}} E[u_0]^{\frac{s_c}{2}} M[u_0]^{\frac{1-s_c}{2}}. \end{aligned}$$

So if $u_0 \in A_\delta$ we have

$$E[u_0]^{s_c} M[u_0]^{1-s_c} < \left(\frac{\alpha s_c}{N\alpha + 2b}\right)^{s_c} \delta'^2,$$

which implies

$$\|U(t)u_0\|_{S(\dot{H}^{s_c})} \leq c\delta'.$$

Then, by the small data theory (Proposition 3.14 below) we have that $SC(u_0)$ holds for $\delta' > 0$ small enough.

Next, we sketch the proof of Theorem 3.3. If $\delta_c \geq E[Q]^{s_c} M[Q]^{1-s_c}$ then we are done. Assume now, by contradiction, that $\delta_c < E[Q]^{s_c} M[Q]^{1-s_c}$.

Therefore, there exists a sequence of radial solutions u_n to (3.1) with H^1 initial data $u_{n,0}$ (rescale all of them to have $\|u_{n,0}\|_{L^2} = 1$ for all n) such that[2]

$$\|\nabla u_{n,0}\|_{L^2}^{s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c} \tag{3.11}$$

and

$$E[u_n]^{s_c} \searrow \delta_c \text{ as } n \to +\infty,$$

for which $\mathrm{SC}(u_{n,0})$ does not hold for any $n \in \mathbb{N}$. However, we already know by Theorem 3.1 that u_n is globally defined. Hence, we must have $\|u_n\|_{S(\dot{H}^{s_c})} = +\infty$. Then using a profile decomposition result (see Proposition 3.25 below) on the sequence $\{u_{n,0}\}_{n\in\mathbb{N}}$ we can construct a critical solution of (1), denoted by u_c, that lies exactly at the threshold δ_c, satisfies (3.11) (therefore u_c is globally defined again by Theorem 3.1) and $\|u_c\|_{S(\dot{H}^{s_c})} = +\infty$ (see Proposition 3.28 below). On the other hand, we prove that the critical solution u_c has the property that $K = \{u_c(t) : t \in [0, +\infty)\}$ is precompact in $H^1(\mathbb{R}^N)$ (see Proposition 3.29 below). Finally, the rigidity theorem (Theorem 3.32 below) will imply that such a critical solution is identically zero, which contradicts the fact that $\|u_c\|_{S(\dot{H}^{s_c})} = +\infty$.

3.3 Cauchy Problem

In this section we show a miscellaneous of results for the Cauchy problem (3.1). These results will be useful in the next sections. We start by stating the following two lemmas.

[2]We can rescale $u_{n,0}$ such that $\|u_{n,0}\|_{L^2} = 1$. Indeed, if $u_{n,0}^{\lambda}(x) = \lambda^{\frac{2-b}{\alpha}} u_{n,0}(\lambda x)$ then by (6) we have $E[u_{n,0}^{\lambda}]^{s_c} M[u_{n,0}^{\lambda}]^{1-s_c} < E[Q]^{s_c} M[Q]^{1-s_c}$ and $\|\nabla u_{n,0}^{\lambda}\|_{L^2}^{s_c} \|u_{n,0}^{\lambda}\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}$. Moreover, since $\|u_{n,0}^{\lambda}\|_{L^2} = \lambda^{-s_c} \|u_{n,0}\|_{L^2}$ by (5), setting $\lambda^{s_c} = \|u_{n,0}\|_{L^2}$ we have $\|u_{n,0}^{\lambda}\|_{L^2} = 1$.

Lemma 3.9. *Let* $N \geq 2$, $\frac{4-2b}{N} < \alpha < 2_*$ *and* $0 < b < \min\{\frac{N}{3}, 1\}$. *Then there exist* $c > 0$ *and* $\theta \in (0, \alpha)$ *sufficiently small such that*

$$\left\| |x|^{-b}|u|^{\alpha}v \right\|_{S'(\dot{H}^{-s_c})} \leq c\|u\|^{\theta}_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|v\|_{S(\dot{H}^{s_c})}.$$

Proof. See Lemma 2.17, with $s = 1$. □

Lemma 3.10. *Let* $N \geq 2$, $\frac{4-2b}{N} < \alpha < 2_*$ *and* $0 < b < \min\{\frac{N}{3}, 1\}$. *Then there exist* $c > 0$ *and* $\theta \in (0, \alpha)$ *sufficiently small such that*

$$\left\| |x|^{-b}|u|^{\alpha}v \right\|_{S'(L^2)} \leq c\|u\|^{\theta}_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|v\|_{S(L^2)}.$$

Proof. See Lemma 2.18, with $s = 1$. □

Remark 3.11. In the perturbation theory we use the following estimate for $\alpha > 1$

$$\left\| |x|^{-b}|u|^{\alpha-1}vw \right\|_{S'(L^2)} \leq c\|u\|^{\theta}_{L^\infty_t H^1_x}\|u\|^{\alpha-1-\theta}_{S(\dot{H}^{s_c})}\|v\|_{S(\dot{H}^{s_c})}\|w\|_{S(L^2)},$$

where $\theta \in (0, \alpha - 1)$ is a sufficiently small number.

Its proof follows from the ideas of Lemma 3.10, that is, we can repeat all the computations replacing $|u|^\alpha v$ by $|u|^{\alpha-1}vw$ or, to be more precise, replacing $|u|^\alpha v = |u|^\theta |u|^{\alpha-\theta} v$ by $|u|^{\alpha-1}vw = |u|^\theta |u|^{\alpha-1-\theta}vw$.

Similarly as in the proof of Theorem 2.14, to show the small data theory in H^1 (see Theorem 3.14 below), we need to estimate the nonlinearity $|x|^{-b}|u|^\alpha u$. We already have the estimates in the spaces $S'(\dot{H}^{s_c})$ and $S'(L^2)$ by the previous lemmas. To estimate $\|\nabla(|x|^{-b}|u|^\alpha u)\|_{S'(L^2)}$, when $N \neq 3$, the proof is the same as the one in Section 2.2, see Lemmas 2.19 and 2.23 with $s = 1$. In the next lemma we consider the case $N = 3$ separately. As it was mentioned before, we will need the exponent in the norm $\|\nabla u\|_{S(L^2)}$ that appears in the right hand side of (2.73) to be equal to 1, however in Lemma 2.21 we got the exponent $\alpha - \theta + \mu \neq 1$.

Lemma 3.12. *Let $N \geq 2$, $\frac{4-2b}{N} < \alpha < 2_*$ and $0 < b < \min\{\frac{N}{3}, 1\}$. There exist $c > 0$ and $\theta \in (0, \alpha)$ sufficiently small such that*

$$\left\|\nabla(|x|^{-b}|u|^\alpha u)\right\|_{S'(L^2)} \leq c\|u\|^\theta_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|\nabla u\|_{S(L^2)} + c\|u\|^{1+\theta}_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}.$$

Proof. For $N \neq 3$, the above inequality was already proved in Lemmas 2.19 and 2.23, with $s = 1$. Now, we only consider the case $N = 3$. We claim that

$$\left\|\nabla(|x|^{-b}|u|^\alpha u)\right\|_{S'(L^2)} \leq c\|u\|^\theta_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|\nabla u\|_{S(L^2)}.$$

Indeed, the proof follows from similar ideas as the ones in Lemma 2.21. Since $(2, 6)$ is L^2-admissible in 3D we deduce

$$\left\|\nabla\left(|x|^{-b}|u|^\alpha u\right)\right\|_{S'(L^2)} \leq \left\|\nabla\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L^{2'}_t L^{6'}_x(B)} + \left\|\nabla\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L^{2'}_t L^{6'}_x(B^C)}.$$

Let $A \subset \mathbb{R}^N$. Applying the product rule for derivatives and Hölder's inequality we have

$$\begin{aligned}\left\|\nabla\left(|x|^{-b}|u|^\alpha u\right)\right\|_{L^{6'}_x(A)} &\leq \left\|\nabla\left(|x|^{-b}\right)|u|^\alpha u\right\|_{L^{6'}_x(A)} + \left\||x|^{-b}\nabla\left(|u|^\alpha u\right)\right\|_{L^{6'}_x(A)} \\ &\leq M_1(t, A) + M_2(t, A),\end{aligned}$$

where

$$M_1(t, A) = \left\||x|^{-b}\right\|_{L^\gamma(A)}\|\nabla(|u|^\alpha u)\|_{L^\beta_x} \qquad M_2(t, A) = \left\|\nabla(|x|^{-b})\right\|_{L^d(A)}\||u|^\alpha u\|_{L^e_x}$$

and

$$\frac{1}{6'} = \frac{1}{\gamma} + \frac{1}{\beta} = \frac{1}{d} + \frac{1}{e}. \tag{3.12}$$

From the proof of Lemma 2.21 with $s = 1$ we already have

$$\|M_1(t, A)\|_{L^{2'}_t} \leq c\|u\|^\theta_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|\nabla u\|_{S(L^2)}. \tag{3.13}$$

To estimate $M_2(t,A)$ we use the pairs $(\bar{a},\bar{r})=\left(4(\alpha-2\theta),\frac{6\alpha(\alpha-2\theta)}{\alpha(3-2b)-2\theta(4-2b)}\right)$ $\dot{H}^{s_c}$-admissible and $(q,r)=\left(\frac{4(\alpha-2\theta)}{\alpha-3\theta},\frac{6(\alpha-2\theta)}{2\alpha-3\theta}\right)$ L^2-admissible.[3]
From the Hölder inequality, the Sobolev embedding (1.6) and (2.10) we obtain

$$\begin{aligned} M_2(t,A) &\leq \||x|^{-b-1}\|_{L^d(A)}\|u\|^{\theta}_{L_x^{\theta r_1}}\|u\|^{\alpha-\theta}_{L_x^{(\alpha-\theta)r_2}}\|u\|_{L_x^{r_3}} \\ &\leq \||x|^{-b-1}\|_{L^d(A)}\|u\|^{\theta}_{L_x^{\theta r_1}}\|u\|^{\alpha-\theta}_{L_x^{\bar{r}}}\|\nabla u\|_{L_x^{r}} \end{aligned} \qquad (3.14)$$

if

$$\begin{cases} \frac{1}{e}= & \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \\ 1= & \frac{3}{r}-\frac{3}{r_3} \\ \bar{r}= & (\alpha-\theta)r_2. \end{cases}$$

Note that the second equation is valid since $r<3$. On the other hand, in order to show that $\||x|^{-b-1}\|_{L^d(A)}$ is bounded, we need $\frac{3}{d}-b-1>0$ when A is the ball B and $\frac{3}{d}-b-1<0$ when $A=B^C$, by Remark 1.17. In fact, it follows from (3.12), the previous system and the values of q, r, $\bar{q}$ and $\bar{r}$ defined above that

$$\begin{aligned} \frac{3}{d}-b-1 &= \frac{5}{2}-b-\frac{3}{r_1}-\frac{3(\alpha-\theta)}{\bar{r}}-\frac{3}{r} \\ &= \frac{5}{2}-b-\frac{3}{r_1}-(\alpha-\theta)\left(\frac{2-b}{\alpha}-\frac{2}{\bar{a}}\right)-\frac{3}{2}+\frac{2}{q} \\ &= -1-\frac{3}{r_1}+\frac{\theta(2-b)}{\alpha}+\frac{2(\alpha-\theta)}{\bar{a}}+\frac{2}{q} \\ &= \frac{\theta(2-b)}{\alpha}-\frac{3}{r_1}. \end{aligned} \qquad (3.15)$$

[3]Note that $\frac{3\alpha}{2-b}=\frac{6}{3-2s_c}<\bar{r}<6$ (condition (1.3) with $N=3$), indeed $\bar{r}>\frac{3\alpha}{2-b}$ is equivalent to $2(\alpha-2\theta)(2-b)>\alpha(3-2b)-2\theta(4-2b)\Leftrightarrow\alpha>0$. Also, $\bar{r}<6\Leftrightarrow 2\theta(4-2b-\alpha)<\alpha(3-2b-\alpha)$, which is true by the assumption $\alpha<3-2b$ and $\theta>0$ is a small number. Moreover it is easy to see that $2<r<6$, i.e., r satisfies the condition of admissible pair (1.1) with $N=3$.

Now choosing r_1 such that

$$\theta r_1 > \frac{3\alpha}{2-b} \text{ when } A = B \quad \text{and} \quad \theta r_1 < \frac{3\alpha}{2-b} \text{ when } A = B^C$$

we get $\frac{3}{d} - b - 1 > 0$ when $A = B$ and $\frac{3}{d} - b - 1 < 0$ when $A = B^C$, so $|x|^{-b-1} \in L^d(A)$. In addition, we have by the Sobolev embedding (1.8) (since $2 < \frac{3\alpha}{2-b} < 6$) and (3.14)

$$M_2(t,A) \leq c\|u\|^{\theta}_{H^1_x}\|u\|^{\alpha-\theta}_{L^{\bar{r}}_x}\|\nabla u\|_{L^r_x}.$$

Therefore, using now Hölder's inequality in the time variable and the fact that

$$\frac{1}{2'} = \frac{\alpha-\theta}{\bar{a}} + \frac{1}{q}$$

we conclude

$$\|M_2(t,A)\|_{L^{2'}_t} \leq c\|u\|^{\theta}_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{L^{\bar{a}}_t L^{\bar{r}}_x}\|\nabla u\|_{L^q_t L^r_x}. \tag{3.16}$$

The proof is completed combining (3.13) and (3.16). □

Remark 3.13. A consequence of the previous lemma is the following estimate

$$\left\||x|^{-b-1}|u|^{\alpha}v\right\|_{S'(L^2)} \lesssim \|u\|^{\theta}_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\left(\|\nabla v\|_{S(L^2)} + \|v\|_{L^\infty_t H^1_x}\right).$$

Our first result in this section concerning the Cauchy problem (3.1) is the following

Proposition 3.14. (Small data global theory in H^1) *Let $N \geq 2$, $\frac{4-2b}{N} < \alpha < 2_*$ with $0 < b < \min\{\frac{N}{3}, 1\}$ and $u_0 \in H^1(\mathbb{R}^N)$. Assume $\|u_0\|_{H^1} \leq A$. There there exists $\delta = \delta(A) > 0$ such that if $\|U(t)u_0\|_{S(\dot{H}^{s_c})} < \delta$, then there exists a unique global solution u of the integral equation (7) such that*

$$\|u\|_{S(\dot{H}^{s_c})} \leq 2\|U(t)u_0\|_{S(\dot{H}^{s_c})}$$

and

$$\|u\|_{S(L^2)} + \|\nabla u\|_{S(L^2)} \leq 2c\|u_0\|_{H^1}.$$

Proof. The proof follows directly from Theorem 2.14 with[4] $s = 1$. □

Remark 3.15. It is worth mentioning that the previous results were proved in Chapter 2 under the condition $0 < b < \widetilde{2}$ (see definition (2.1)). Consequently, it is easy to see that they also hold for $0 < b < \min\{\frac{N}{3}, 1\}$.

We now show Proposition 3.6 (this result gives us the criterion to establish scattering).

Proof of Proposition 3.6. First, we claim that

$$\|u\|_{S(L^2)} + \|\nabla u\|_{S(L^2)} < +\infty. \tag{3.17}$$

Indeed, since $\|u\|_{S(\dot{H}^{s_c})} < +\infty$, given $\delta > 0$ we can decompose $[0, \infty)$ into n intervals $I_j = [t_j, t_{j+1})$ such that $\|u\|_{S(\dot{H}^{s_c}; I_j)} < \delta$ for all $j = 1, ..., n$. On the time interval I_j we consider the integral equation

$$u(t) = U(t - t_j)u(t_j) + i\int_{t_j}^{t_{j+1}} U(t-s)(|x|^{-b}|u|^{\alpha}u)(s)ds.$$

It follows from the Strichartz estimates (1.9) and (1.11) that

$$\|u\|_{S(L^2;I_j)} \leq c\|u(t_j)\|_{L^2_x} + c\left\||x|^{-b}|u|^{\alpha}u\right\|_{S'(L^2;I_j)} \tag{3.18}$$

and

$$\|\nabla u\|_{S(L^2;I_j)} \leq c\|\nabla u(t_j)\|_{L^2_x} + c\left\|\nabla(|x|^{-b}|u|^{\alpha}u)\right\|_{S'(L^2;I_j)}. \tag{3.19}$$

From Lemmas 3.10 and 3.12 we have

$$\left\||x|^{-b}|u|^{\alpha}u\right\|_{S'(L^2;I_j)} \leq c\|u\|^{\theta}_{L^{\infty}_{I_j}H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I_j)}\|u\|_{S(L^2;I_j)},$$

$$\|\nabla(|x|^{-b}|u|^{\alpha}u)\|_{S'(L^2;I_j)} \leq c\|u\|^{\theta}_{L^{\infty}_{I_j}H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I_j)}\left(\|\nabla u\|_{S(L^2;I_j)} + \|u\|_{L^{\infty}_{I_j}H^1_x}\right).$$

[4] In Theorem 2.14 we have the condition $s \leq \min\{\frac{N}{2}, 1\}$ and since $s = 1$ in this case, we deduce $N \geq 2$. For this reason, we study scattering in $H^1(\mathbb{R}^N)$ with $N \geq 2$.

Thus, using (3.18), (3.19) and the two last estimates, we get

$$\|u\|_{S(L^2;I_j)} \leq cB + cB^{\theta}\delta^{\alpha-\theta}\|u\|_{S(L^2;I_j)}$$

and

$$\|\nabla u\|_{S(L^2;I_j)} \leq cB + cB^{\theta+1}\delta^{\alpha-\theta} + cB^{\theta}\delta^{\alpha-\theta}\|\nabla u\|_{S(L^2;I_j)}, \tag{3.20}$$

where we have used the assumption $\sup_{t\in\mathbb{R}}\|u(t)\|_{H^1} \leq B$.
Taking $\delta > 0$ such that $cB^{\theta}\delta^{\alpha-\theta} < \frac{1}{2}$ we obtain[5]

$$\|u\|_{S(L^2;I_j)} + \|\nabla u\|_{S(L^2;I_j)} \leq cB,$$

and by summing over the n intervals, we conclude the proof of (3.17).

Returning to the proof of the proposition, let

$$\phi^+ = u_0 + i\int_0^{+\infty} U(-s)|x|^{-b}(|u|^{\alpha}u)(s)ds.$$

Note that, $\phi^+ \in H^1(\mathbb{R}^N)$. Indeed, by the same arguments as before, we deduce that

$$\|\phi^+\|_{L^2} \leq c\|u_0\|_{L^2} + c\|u\|^{\theta}_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|u\|_{S(L^2)}$$

and

$$\|\nabla\phi^+\|_{L^2} \leq c\|\nabla u_0\|_{L^2} + c\|u\|^{\theta}_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\left(\|\nabla u\|_{S(L^2)} + \|u\|_{L^\infty_t H^1_x}\right).$$

Therefore, (3.17) yields $\|\phi\|_{H^1} < +\infty$.

On the other hand, since u is a solution of (3.1) we get

$$u(t) - U(t)\phi^+ = -i\int_t^{+\infty} U(t-s)|x|^{-b}(|u|^{\alpha}u)(s)ds.$$

[5]Here, in order to prove that $\|\nabla u\|_{S(L^2;I_j)}$ is bounded we need the exponent on this norm to be equal to 1 since otherwise we can not absorb this term on the right-hand side of (3.20).

Similarly as before, we have

$$\|u(t)-U(t)\phi\|_{L^2_x}\leq c\|u\|^{\theta}_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c};[t,\infty))}\|u\|_{S(L^2)}$$

and

$$\|\nabla(u(t)-U(t)\phi)\|_{L^2_x} \quad\leq\quad c\|u\|^{\theta}_{L^\infty_t H^1_x}\|u\|^{\alpha-\theta}_{S(\dot{H}^{s_c};[t,\infty))}\left(\|\nabla u\|_{S(L^2)}+\|u\|_{L^\infty_t H^1_x}\right)$$

The proof is completed after using (3.17) and the fact that $\|u\|_{S(\dot{H}^{s_c};[t,\infty))}\to 0$ as $t\to+\infty$. □

Remark 3.16. In the same way we define

$$\phi^- = u_0 + i\int_0^{-\infty} U(-s)|x|^{-b}(|u|^\alpha u)(s)ds,$$

so that we have $\phi^-\in H^1$ and

$$u(t)-U(t)\phi^- = i\int_{-\infty}^{t} U(t-s)|x|^{-b}(|u|^\alpha u)(s)ds,$$

which also satisfies (using the same argument as before)

$$\|u(t)-U(t)\phi^-\|_{H^1_x}\to 0 \text{ as } t\to-\infty.$$

Next, we study the perturbation theory for the IVP (3.1) following the exposition in Killip-Kwon-Shao-Visan [29, Theorem 3.1]. We first obtain a short-time perturbation which can be iterated to obtain a long-time perturbation result.

Proposition 3.17. (Short-time perturbation theory for the INLS) *Let $I\subseteq\mathbb{R}$ be a time interval containing zero and let $\widetilde{u}$ defined on $I\times\mathbb{R}^N$ be a solution (in the sense of the appropriated integral equation) to*

$$i\partial_t\widetilde{u}+\Delta\widetilde{u}+|x|^{-b}|\widetilde{u}|^\alpha\widetilde{u}=e,$$

with initial data $\widetilde{u}_0 \in H^1(\mathbb{R}^N)$*, satisfying*

$$\sup_{t\in I} \|\widetilde{u}(t)\|_{H^1_x} \leq M \ \text{ and } \ \|\widetilde{u}\|_{S(\dot{H}^{s_c};I)} \leq \varepsilon, \tag{3.21}$$

for some positive constant M *and some small* $\varepsilon > 0$*.*

Let $u_0 \in H^1(\mathbb{R}^N)$ *such that*

$$\|u_0 - \widetilde{u}_0\|_{H^1} \leq M' \ \text{ and } \ \|U(t)(u_0 - \widetilde{u}_0)\|_{S(\dot{H}^{s_c};I)} \leq \varepsilon, \ \text{ for } \ M' > 0. \tag{3.22}$$

In addition, assume the following conditions

$$\|e\|_{S'(L^2;I)} + \|\nabla e\|_{S'(L^2;I)} + \|e\|_{S'(\dot{H}^{-s_c};I)} \leq \varepsilon. \tag{3.23}$$

There exists $\varepsilon_0(M, M') > 0$ *such that if* $\varepsilon < \varepsilon_0$*, then there is a unique solution* u *to* (3.1) *on* $I \times \mathbb{R}^N$ *with initial data* u_0*, at the time* $t = 0$*, satisfying*

$$\|u\|_{S(\dot{H}^{s_c};I)} \lesssim \varepsilon \tag{3.24}$$

and

$$\|u\|_{S(L^2;I)} + \|\nabla u\|_{S(L^2;I)} \lesssim c(M, M'). \tag{3.25}$$

Proof. We use the following claim (we will show it later): there exists $\varepsilon_0 > 0$ sufficiently small such that, if $\|\widetilde{u}\|_{S(\dot{H}^{s_c};I)} \leq \varepsilon_0$ then

$$\|\widetilde{u}\|_{S(L^2;I)} \lesssim M \quad \text{and} \quad \|\nabla\widetilde{u}\|_{S(L^2;I)} \lesssim M. \tag{3.26}$$

We may assume, without loss of generality, that $0 = \inf I$. Let us first prove the existence of a solution w for the following initial value problem

$$\begin{cases} i\partial_t w + \Delta w + H(x, \widetilde{u}, w) + e = 0, \\ w(0, x) = u_0(x) - \widetilde{u}_0(x), \end{cases} \tag{3.27}$$

where $H(x, \widetilde{u}, w) = |x|^{-b}\left(|\widetilde{u} + w|^\alpha(\widetilde{u} + w) - |\widetilde{u}|^\alpha\widetilde{u}\right)$.

To this end, let

$$G(w)(t) := U(t)w_0 + i\int_0^t U(t-s)(H(x,\widetilde{u},w)+e)(s)ds \tag{3.28}$$

and define

$$B_{\rho,K} = \{w \in C(I;H^1(\mathbb{R}^N)) :\ \|w\|_{S(\dot{H}^{s_c};I)} \leq \rho \text{ and } \|w\|_{S(L^2;I)} + \|\nabla w\|_{S(L^2;I)} \leq K\}.$$

For a suitable choice of the parameters $\rho > 0$ and $K > 0$, we need to show that G in (3.28) defines a contraction on $B_{\rho,K}$. Indeed, applying Strichartz inequalities (1.9), (1.10), (1.11) and (1.12), we have

$$\|G(w)\|_{S(\dot{H}^{s_c};I)} \lesssim \|U(t)w_0\|_{S(\dot{H}^{s_c};I)} + \|H(\cdot,\widetilde{u},w)\|_{S'(\dot{H}^{-s_c};I)} + \|e\|_{S'(\dot{H}^{-s_c};I)} \tag{3.29}$$

$$\|G(w)\|_{S(L^2;I)} \lesssim \|w_0\|_{L^2} + \|H(\cdot,\widetilde{u},w)\|_{S'(L^2;I)} + \|e\|_{S'(L^2;I)} \tag{3.30}$$

and

$$\|\nabla G(w)\|_{S(L^2;I)} \lesssim \|\nabla w_0\|_{L^2} + \|\nabla H(\cdot,\widetilde{u},w)\|_{S'(L^2;I)} + \|\nabla e\|_{S'(L^2;I)}. \tag{3.31}$$

On the other hand, since

$$||\widetilde{u}+w|^\alpha(\widetilde{u}+w) - |\widetilde{u}|^\alpha \widetilde{u}| \lesssim |\widetilde{u}|^\alpha |w| + |w|^{\alpha+1} \tag{3.32}$$

by (1.13), we get

$$\|H(\cdot,\widetilde{u},w)\|_{S'(\dot{H}^{-s_c};I)} \leq \||x|^{-b}|\widetilde{u}|^\alpha w\|_{S'(\dot{H}^{-s_c};I)} + \||x|^{-b}|w|^\alpha w\|_{S'(\dot{H}^{-s_c};I)},$$

which implies using Lemma 3.9 that

$$\|H(\cdot,\widetilde{u},w)\|_{S'(\dot{H}^{-s_c};I)} \lesssim \left(\|\widetilde{u}\|^\theta_{L^\infty_t H^1_x}\|\widetilde{u}\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)} + \|w\|^\theta_{L^\infty_t H^1_x}\|w\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)}\right)\|w\|_{S(\dot{H}^{s_c};I)}. \tag{3.33}$$

The same argument and Lemma 3.10 also yield

$$\|H(\cdot,\widetilde{u},w)\|_{S'(L^2;I)} \lesssim \left(\|\widetilde{u}\|^\theta_{L^\infty_t H^1_x}\|\widetilde{u}\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)} + \|w\|^\theta_{L^\infty_t H^1_x}\|w\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)}\right)\|w\|_{S(L^2;I)}. \tag{3.34}$$

Now, we estimate $\|\nabla H(\cdot,\widetilde{u},w)\|_{S'(L^2;I)}$. It follows from (1.16) and (3.32) that

$$|\nabla H(x,\widetilde{u},w)| \lesssim |x|^{-b-1}(|\widetilde{u}|^{\alpha}+|w|^{\alpha})|w| + |x|^{-b}(|\widetilde{u}|^{\alpha}+|w|^{\alpha})|\nabla w| + E,$$

where

$$E \lesssim \begin{cases} |x|^{-b}\left(|\widetilde{u}|^{\alpha-1}+|w|^{\alpha-1}\right)|w||\nabla \widetilde{u}| & \text{if} \quad \alpha>1 \\ |x|^{-b}|\nabla\widetilde{u}||w|^{\alpha} & \text{if} \quad \alpha\leq 1. \end{cases}$$

Thus, Lemma 3.10 and Remark 3.13 lead to

$$\begin{aligned} \|\nabla H(\cdot,\widetilde{u},w)\|_{S'(L^2;I)} \lesssim & \left(\|\widetilde{u}\|^{\theta}_{L^\infty_t H^1_x}\|\widetilde{u}\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)} + \|w\|^{\theta}_{L^\infty_t H^1_x}\|w\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)}\right)\|\nabla w\|_{S(L^2;I)} \\ & + \left(\|\widetilde{u}\|^{\theta}_{L^\infty_t H^1_x}\|\widetilde{u}\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)} + \|w\|^{\theta}_{L^\infty_t H^1_x}\|w\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)}\right)\|w\|_{L^\infty_t H^1_x} \\ & + \left(\|\widetilde{u}\|^{\theta}_{L^\infty_t H^1_x}\|\widetilde{u}\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)} + \|w\|^{\theta}_{L^\infty_t H^1_x}\|w\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)}\right)\|\nabla w\|_{S(L^2;I)} + E_1 \\ \lesssim & \left(\|\widetilde{u}\|^{\theta}_{L^\infty_t H^1_x}\|\widetilde{u}\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)} + \|w\|^{\theta}_{L^\infty_t H^1_x}\|w\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)}\right)\|\nabla w\|_{S(L^2;I)} \\ & + \left(\|\widetilde{u}\|^{\theta}_{L^\infty_t H^1_x}\|\widetilde{u}\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)} + \|w\|^{\theta}_{L^\infty_t H^1_x}\|w\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)}\right)\|w\|_{L^\infty_t H^1_x} + E_1. \end{aligned} \tag{3.35}$$

Moreover, using Remark 3.11,

$$E_1 \lesssim \begin{cases} \left(\|\widetilde{u}\|^{\theta}_{L^\infty_t H^1_x}\|\widetilde{u}\|^{\alpha-1-\theta}_{S(\dot{H}^{s_c};I)} + \|w\|^{\theta}_{L^\infty_t H^1_x}\|w\|^{\alpha-1-\theta}_{S(\dot{H}^{s_c};I)}\right)\|w\|_{S(\dot{H}^{s_c};I)}\|\nabla\widetilde{u}\|_{S(L^2;I)}, & \alpha>1 \\ \|w\|^{\theta}_{L^\infty_t H^1_x}\|w\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)}\|\nabla\widetilde{u}\|_{S(L^2;I)}\ , & \alpha\leq 1, \end{cases}$$

where $\theta\in(0,\alpha-1)$ if $\alpha>1$ or $\theta\in(0,\alpha)$ if $\alpha\leq 1$.

Hence, combining (3.33), (3.34) and if $u\in B(\rho,K)$, we have

$$\|H(\cdot,\widetilde{u},w)\|_{S'(\dot{H}^{-s_c};I)} \lesssim \left(M^{\theta}\varepsilon^{\alpha-\theta}+K^{\theta}\rho^{\alpha-\theta}\right)\rho \tag{3.36}$$

and

$$\|H(\cdot,\widetilde{u},w)\|_{S'(L^2;I)} \lesssim \left(M^{\theta}\varepsilon^{\alpha-\theta}+K^{\theta}\rho^{\alpha-\theta}\right)K. \tag{3.37}$$

Furthermore, (3.35) and (3.26) imply

$$\|\nabla H(\cdot,\widetilde{u},w)\|_{S'(L^2;I)} \lesssim \left(M^\theta \varepsilon^{\alpha-\theta}+K^\theta\rho^{\alpha-\theta}\right)K+E_1 \tag{3.38}$$

where

$$E_1 \lesssim \begin{cases} \left(M^\theta\varepsilon^{\alpha-1-\theta}+K^\theta\rho^{\alpha-1-\theta}\right)\rho M & \text{if } \alpha>1, \\ K^\theta\rho^{\alpha-\theta}M & \text{if } \alpha\leq 1. \end{cases}$$

Therefore, we deduce by (3.29)-(3.30) together with (3.36)- (3.37) that

$$\|G(w)\|_{S(\dot{H}^{s_c};I)} \leq c\varepsilon + cA\rho$$

and

$$\|G(w)\|_{S(L^2;I)} \leq cM' + c\varepsilon + cAK,$$

where we also used the hypothesis (3.22)-(3.23) and $A = M^\theta\varepsilon^{\alpha-\theta}+K^\theta\rho^{\alpha-\theta}$. We also have, using (3.31), (3.38), that if $\alpha>1$

$$\|\nabla G(w)\|_{S(L^2;I)} \leq cM' + c\varepsilon + cAK + cB\rho M,$$

where $B = M^\theta\varepsilon^{\alpha-1-\theta}+K^\theta\rho^{\alpha-1-\theta}$, and if $\alpha\leq 1$

$$\|\nabla G(w)\|_{S(L^2;I)} \leq cM' + c\varepsilon + cAK + K^\theta\rho^{\alpha-\theta}M.$$

Choosing $\rho = 2c\varepsilon$, $K = 3cM'$ and ε_0 sufficiently small such that

$$cA < \frac{1}{3} \quad \text{and} \quad c(\varepsilon + B\rho M + K^\theta\rho^{\alpha-\theta}M) < \frac{K}{3},$$

we obtain

$$\|G(w)\|_{S(\dot{H}^{s_c};I)} \leq \rho \quad \text{and} \quad \|G(w)\|_{S(L^2;I)} + \|\nabla G(w)\|_{S(L^2;I)} \leq K.$$

The above calculations establish that G is well defined on $B(\rho,K)$. The contraction property can be obtained by similar arguments. Hence, by the

Banach Fixed Point Theorem we obtain a unique solution w on $I \times \mathbb{R}^N$ such that

$$\|w\|_{S(\dot{H}^{s_c};I)} \lesssim \varepsilon \quad \text{and} \quad \|w\|_{S(L^2;I)} + \|w\|_{S(L^2;I)} \lesssim M'.$$

Finally, it is easy to see that $u = \widetilde{u} + w$ is a solution to (3.1) satisfying (3.24) and (3.25).

To complete the proof we now show (3.26). Indeed, we first show that

$$\|\nabla \widetilde{u}\|_{S(L^2;I)} \lesssim M. \tag{3.39}$$

Using the same arguments as before, we have

$$\|\nabla \widetilde{u}\|_{S(L^2;I)} \lesssim \|\nabla \widetilde{u}_0\|_{L^2} + \left\|\nabla(|x|^{-b}|\widetilde{u}|^{\alpha}\widetilde{u})\right\|_{S'(L^2;I)} + \|\nabla e\|_{S'(L^2;I)}.$$

Now, Lemma 3.12 leads to

$$\begin{aligned}\|\nabla \widetilde{u}\|_{S(L^2;I)} &\lesssim M + \|\widetilde{u}\|^{\theta}_{L^\infty_t H^1_x} \|\widetilde{u}\|^{\alpha-\theta}_{S(\dot{H}^{s_c};I)} \left(\|\nabla \widetilde{u}\|_{S(L^2;I)} + \|\widetilde{u}\|_{L^\infty_t H^1_x}\right) + \varepsilon \\ &\lesssim M + \varepsilon + M^{\theta+1}\varepsilon_0^{\alpha-\theta} + M^{\theta}\varepsilon_0^{\alpha-\theta}\|\nabla \widetilde{u}\|_{S(L^2;I)}.\end{aligned}$$

Therefore, choosing ε_0 sufficiently small the linear term $M^{\theta}\varepsilon_0^{\alpha-\theta}\|\nabla \widetilde{u}\|_{S(L^2;I)}$ may be absorbed by the left-hand term and we conclude the proof of (3.39). Similar estimates also imply $\|\widetilde{u}\|_{S(L^2;I)} \lesssim M$. □

Remark 3.18. From Proposition 3.17, we also have the following estimates:

$$\|H(\cdot, \widetilde{u}, w)\|_{S'(\dot{H}^{-s_c};I)} \leq C(M, M')\varepsilon \tag{3.40}$$

and

$$\|H(\cdot, \widetilde{u}, w)\|_{S'(L^2;I)} + \|\nabla H(\cdot, \widetilde{u}, w)\|_{S'(L^2;I)} \leq C(M, M')\varepsilon^{\alpha-\theta}, \tag{3.41}$$

with $\theta \in (0, \alpha)$.

Indeed, from (3.36), (3.37) and (3.38) we deduce

$$\|H(\cdot, \widetilde{u}, w)\|_{S'(\dot{H}^{-s_c};I)} \lesssim \left(M^{\theta}\varepsilon^{\alpha-\theta} + K^{\theta}\rho^{\alpha-\theta}\right)\rho,$$

$$\|H(\cdot,\widetilde{u},w)\|_{S'(L^2;I)} \lesssim \left(M^\theta \varepsilon^{\alpha-\theta} + K^\theta \rho^{\alpha-\theta}\right) K$$

and

$$\|\nabla H(\cdot,\widetilde{u},w)\|_{S'(L^2;I)} \lesssim E_1 + \left(M^\theta \varepsilon^{\alpha-\theta} + K^\theta \rho^{\alpha-\theta}\right) K,$$

where

$$E_1 \lesssim \begin{cases} \left(M^\theta \varepsilon^{\alpha-1-\theta} + K^\theta \rho^{\alpha-1-\theta}\right)\rho M & \text{if } \alpha > 1, \\ K^\theta \rho^{\alpha-\theta} M & \text{if } \alpha \leq 1. \end{cases}$$

Therefore, the choice $\rho = 2c\varepsilon$ and $K = 3cM'$ in Proposition 3.17 yield (3.40) and (3.41).

The long-time perturbation result for the mass-supercritical and energy-subcritical INLS will be obtained iteratively from the previous result.

Proposition 3.19. (Long-time perturbation theory for the INLS) *Let $I \subseteq \mathbb{R}$ be a time interval containing zero and let $\widetilde{u}$ defined on $I \times \mathbb{R}^N$ be a solution (in the sense of the appropriated integral equation) to*

$$i\partial_t \widetilde{u} + \Delta \widetilde{u} + |x|^{-b}|\widetilde{u}|^\alpha \widetilde{u} = e,$$

with initial data $\widetilde{u}_0 \in H^1(\mathbb{R}^N)$, satisfying

$$\sup_{t\in I} \|\widetilde{u}\|_{H^1_x} \leq M \ \text{ and } \ \|\widetilde{u}\|_{S(\dot{H}^{s_c};I)} \leq L, \tag{3.42}$$

for some positive constants M, L.

Let $u_0 \in H^1(\mathbb{R}^N)$ such that

$$\|u_0 - \widetilde{u}_0\|_{H^1} \leq M' \ \text{ and } \ \|U(t)(u_0 - \widetilde{u}_0)\|_{S(\dot{H}^{s_c};I)} \leq \varepsilon, \tag{3.43}$$

for some positive constant M' and some $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(M, M', L)$. Moreover, assume also the following conditions

$$\|e\|_{S'(L^2;I)} + \|\nabla e\|_{S'(L^2;I)} + \|e\|_{S'(\dot{H}^{-s_c};I)} \leq \varepsilon.$$

Then, there exists a unique solution u to (3.1) *on* $I \times \mathbb{R}^N$ *with initial data* u_0 *at the time* $t = 0$ *satisfying*

$$\|u - \widetilde{u}\|_{S(\dot{H}^{s_c};I)} \leq C(M, M', L)\varepsilon \qquad \text{and} \tag{3.44}$$

$$\|u\|_{S(\dot{H}^{s_c};I)} + \|u\|_{S(L^2;I)} + \|\nabla u\|_{S(L^2;I)} \leq C(M, M', L). \tag{3.45}$$

Proof. First observe that since $\|\widetilde{u}\|_{S(\dot{H}^{s_c};I)} \leq L$, given[6] $\varepsilon < \varepsilon_0(M, 2M')$ we can partition I into $n = n(L, \varepsilon)$ intervals $I_j = [t_j, t_{j+1})$ such that for each j, the quantity $\|\widetilde{u}\|_{S(\dot{H}^{s_c};I_j)} \leq \varepsilon$. Note that M' is being replaced by $2M'$, as the H^1-norm of the difference of two different initial data may increase in each iteration.

Again, we may assume, without loss of generality, that $0 = \inf I$. Let w be defined by $u = \widetilde{u} + w$, then w solves IVP (3.27) with initial time t_j. Thus, the integral equation in the interval $I_j = [t_j, t_{j+1})$ reads as follows

$$w(t) = U(t - t_j)w(t_j) + i \int_{t_j}^{t} U(t - s)(H(x, \widetilde{u}, w) + e)(s)ds,$$

where $H(x, \widetilde{u}, w) = |x|^{-b} \left(|\widetilde{u} + w|^{\alpha}(\widetilde{u} + w) - |\widetilde{u}|^{\alpha}\widetilde{u}\right)$.

Thus, choosing ε_1 sufficiently small (depending on n, M, and M'), we may apply Proposition 3.17 (Short-time Perturbation Theory) to obtain for each $0 \leq j < n$ and all $\varepsilon < \varepsilon_1$,

$$\|u - \widetilde{u}\|_{S(\dot{H}^{s_c};I_j)} \leq C(M, M', j)\varepsilon \tag{3.46}$$

and

$$\|w\|_{S(\dot{H}^{s_c};I_j)} + \|w\|_{S'(L^2;I_j)} + \|\nabla w\|_{S'(L^2;I_j)} \leq C(M, M', j) \tag{3.47}$$

provided we can show

$$\|U(t - t_j)(u(t_j) - \widetilde{u}(t_j))\|_{S(\dot{H}^{s_c};I_j)} \leq C(M, M', j)\varepsilon \leq \varepsilon_0 \tag{3.48}$$

[6] ε_0 is given by the previous result and ε to be determined later.

and

$$\|u(t_j) - \widetilde{u}(t_j)\|_{H^1_x} \leq 2M', \tag{3.49}$$

For each $0 \leq j < n$.

Indeed, by the Strichartz estimates (1.10) and (1.12), we have

$$\begin{aligned}\|U(t-t_j)w(t_j)\|_{S(\dot{H}^{s_c};I_j)} &\lesssim \|U(t)w_0\|_{S(\dot{H}^{s_c};I)} + \|H(\cdot,\widetilde{u},w)\|_{S'(\dot{H}^{-s_c};[0,t_j])} \\ &\quad + \|e\|_{S'(\dot{H}^{-s_c};I)},\end{aligned}$$

which implies by (3.40) that

$$\|U(t-t_j)(u(t_j) - \widetilde{u}(t_j))\|_{S(\dot{H}^{s_c};I_j)} \lesssim \varepsilon + \sum_{k=0}^{j-1} C(k,M,M')\varepsilon.$$

Similarly, it follows from Strichartz estimates (1.9), (1.11) and (3.41) that

$$\begin{aligned}\|u(t_j) - \widetilde{u}(t_j)\|_{H^1_x} &\lesssim \|u_0 - \widetilde{u}_0\|_{H^1} + \|e\|_{S'(L^2;I)} + \|\nabla e\|_{S'(L^2;I)} \\ &\quad + \|H(\cdot,\widetilde{u},w)\|_{S'(L^2;[0,t_j])} + \|\nabla H(\cdot,\widetilde{u},w)\|_{S'(L^2;[0,t_j])} \\ &\lesssim M' + \varepsilon + \sum_{k=0}^{j-1} C(k,M,M')\varepsilon^{\alpha-\theta}.\end{aligned}$$

Taking $\varepsilon_1 = \varepsilon(n,M,M')$ sufficiently small, we see that (3.48) and (3.49) hold and so, it implies (3.46) and (3.47).

Finally, summing this over all subintervals I_j, we obtain

$$\|u - \widetilde{u}\|_{S(\dot{H}^{s_c};I)} \leq C(M,M',L)\varepsilon$$

and

$$\|w\|_{S(\dot{H}^{s_c};I)} + \|w\|_{S'(L^2;I)} + \|\nabla w\|_{S'(L^2;I)} \leq C(M,M',L).$$

This completes the proof. □

3.4 Properties of the ground state, energy bounds and wave operator

In this section, we recall some properties that are related to our problem. In [12] Farah proved the following Gagliardo-Nirenberg inequality

$$\left\| |x|^{-b}|u|^{\alpha+2} \right\|_{L^1_x} \leq C_{GN} \|\nabla u\|_{L^2_x}^{\frac{N\alpha+2b}{2}} \|u\|_{L^2_x}^{\frac{4-2b-\alpha(N-2)}{2}}, \tag{3.50}$$

with the sharp constant

$$C_{GN} = \frac{2(\alpha+2)}{N\alpha+2b}\left(\frac{4-2b-\alpha(N-2)}{N\alpha+2b}\right)^{\alpha s_c/2} \frac{1}{\|Q\|_{L^2}^{\alpha}} \tag{3.51}$$

where Q is the ground state solution of (3.6). Moreover, Q satisfies the following relations

$$\|\nabla Q\|_{L^2}^2 = \frac{N\alpha+2b}{4-2b-\alpha(N-2)}\|Q\|_{L^2}^2 \tag{3.52}$$

and

$$\left\| |x|^{-b}|Q|^{\alpha+2} \right\|_{L^1} = \frac{2(\alpha+2)}{N\alpha+2b}\|\nabla Q\|_{L^2}^2. \tag{3.53}$$

Note that, combining (3.51), (3.52) and (3.53) we obtain

$$C_{GN} = \frac{2(\alpha+2)}{(N\alpha+2b)\|\nabla Q\|_{L^2}^{\alpha s_c}\|Q\|_{L^2}^{\alpha(1-s_c)}}, \tag{3.54}$$

where $s_c = \frac{N}{2} - \frac{2-b}{\alpha}$ is the critical Sobolev index. On the other hand, we also have

$$E[Q] = \frac{1}{2}\|\nabla Q\|_{L^2}^2 - \frac{1}{\alpha+2}\left\| |x|^{-b}|Q|^{\alpha+2} \right\|_{L^1} = \frac{\alpha s_c}{N\alpha+2b}\|\nabla Q\|_{L^2}^2. \tag{3.55}$$

We now show the radial Sobolev Gagliardo-Nirenberg inequality in N dimension. The proof follows the ideas introduced by Strauss [39].

Lemma 3.20. *Let $N \geq 2$, $R > 0$ and $f \in H^1(\mathbb{R}^N)$ a radial function. Then the following inequality holds*

$$\sup_{|x|\geq R} |f(x)| \leq \frac{1}{R^{\frac{N-1}{2}}} \|f\|_{L^2}^{\frac{1}{2}} \|\nabla f\|_{L^2}^{\frac{1}{2}}. \tag{3.56}$$

Proof. Since f is radial we deduce

$$\begin{aligned}
\sup_{|x|\geq R} |f(x)|^2 &= \sup_{|x|\geq R} \frac{1}{2} \int_{|x|}^{+\infty} \partial_r (f^2) dr \\
&\leq \int_R^{+\infty} f \partial_r f dr \\
&\leq \left(\int_R^{+\infty} |f|^2 dr \right)^{\frac{1}{2}} \left(\int_R^{+\infty} |\partial_r f|^2 dr \right)^{\frac{1}{2}},
\end{aligned}$$

where we have used that f has to vanish at infinite and the Cauchy-Schwarz inequality. On the other hand, the fact that $|x| \geq R$ (or $r \geq R$) implies $1 \leq \frac{r}{R}$ so

$$\begin{aligned}
\sup_{|x|\geq R} |f(x)|^2 &\leq \left(\int_R^{+\infty} |f|^2 \left(\frac{r}{R}\right)^{N-1} \right)^{\frac{1}{2}} \left(\int_R^{+\infty} |\partial_r f|^2 \left(\frac{r}{R}\right)^{N-1} dr \right)^{\frac{1}{2}} \\
&\leq \frac{1}{R^{\frac{N-1}{2}}} \left(\int_R^{+\infty} |f|^2 r^{2(N-1)} \right)^{\frac{1}{2}} \frac{1}{R^{\frac{N-1}{2}}} \left(\int_R^{+\infty} |\partial_r f|^2 r^{2(N-1)} dr \right)^{\frac{1}{2}} \\
&= \frac{1}{R^{N-1}} \left(\int_R^{+\infty} |f|^2 dx \right)^{\frac{1}{2}} \left(\int_R^{+\infty} |\nabla f|^2 dx \right)^{\frac{1}{2}} \\
&\leq \frac{1}{R^{N-1}} \|f\|_{L^2} \|\nabla f\|_{L^2},
\end{aligned}$$

where in the third line we have used the fact that $|\partial_r f| = |\nabla f|$ for radial functions. We finish the proof taking the square root on both sides. □

The next lemma provides some estimates that will be needed for the compactness and rigidity results.

Lemma 3.21. *Let $v \in H^1(\mathbb{R}^N)$ such that*

$$\|\nabla v\|_{L^2}^{s_c} \|v\|_{L^2}^{1-s_c} \leq \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}. \tag{3.57}$$

Then, the following statements hold

(i) $\frac{\alpha s_c}{N\alpha+2b}\|\nabla v\|_{L^2}^2 \leq E(v) \leq \frac{1}{2}\|\nabla v\|_{L^2}^2$,

(ii) $\|\nabla v\|_{L^2}^{s_c}\|v\|_{L^2}^{1-s_c} \leq w^{\frac{1}{2}}\|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c}$,

(iii) $16AE[v] \leq 8A\|\nabla v\|_{L^2}^2 \leq 8\|\nabla v\|_{L^2}^2 - \frac{4(N\alpha+2b)}{\alpha+2}\left\||x|^{-b}|v|^{\alpha+2}\right\|_{L^1}$,

where $w = \frac{E[v]^{s_c}M[v]^{1-s_c}}{E[Q]^{s_c}M[Q]^{1-s_c}}$ *and* $A = (1 - w^{\frac{\alpha}{2}})$.

Proof. (i) The second inequality is immediate from the definition of Energy (4). The first one is obtained by observing that

$$\begin{aligned}
E[v] &\geq \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{C_{GN}}{\alpha+2}\|\nabla v\|_{L^2}^{\frac{N\alpha+2b}{2}}\|v\|_{L^2}^{\frac{4-2b-\alpha(N-2)}{2}} \\
&= \frac{1}{2}\|\nabla v\|_{L^2}^2\left(1 - \frac{2C_{GN}}{\alpha+2}\|\nabla v\|_{L^2}^{\alpha s_c}\|v\|_{L^2}^{\alpha(1-s_c)}\right) \\
&\geq \frac{1}{2}\|\nabla v\|_{L^2}^2\left(1 - \frac{2C_{GN}}{\alpha+2}\|\nabla Q\|_{L^2}^{\alpha s_c}\|Q\|_{L^2}^{\alpha(1-s_c)}\right) \\
&= \frac{N\alpha-(4-2b)}{2(N\alpha+2b)}\|\nabla v\|_{L^2}^2 \\
&= \frac{\alpha s_c}{N\alpha+2b}\|\nabla v\|_{L^2}^2,
\end{aligned}$$

where we have used (3.50), (3.54) and (3.57).

(ii) The first inequality in (i) yields $\|\nabla v\|_{L^2}^2 \leq \frac{N\alpha+2b}{\alpha s_c}E(v)$, multiplying it by $M[v]^\sigma = \|v\|_{L^2}^{2\sigma}$, where $\sigma = \frac{1-s_c}{s_c}$, we have

$$\begin{aligned}
\|\nabla v\|_{L^2}^2\|v\|_{L^2}^{2\sigma} &\leq \frac{N\alpha+2b}{\alpha s_c}E[v]M[v]^\sigma \\
&= \frac{N\alpha+2b}{\alpha s_c}\frac{E[v]M[v]^\sigma}{E[Q]M[Q]^\sigma}E[Q]M[Q]^\sigma \\
&= w\|\nabla Q\|^2\|Q\|_{L^2}^{2\sigma},
\end{aligned}$$

where we have used (3.55).

(iii) The first inequality obviously holds. Next, let $B = 8\|\nabla v\|_{L^2}^2 - \frac{4(N\alpha+2b)}{\alpha+2}\left\||x|^{-b}|v|^{\alpha+2}\right\|_{L^1}$. Applying the Gagliardo-Niremberg inequality (3.50) and item (ii) we obtain

$$\begin{aligned} B &\geq 8\|\nabla v\|_{L^2}^2 - \frac{4(N\alpha+2b)C_{GN}}{\alpha+2}\|\nabla v\|_{L^2}^{\frac{N\alpha+2b}{2}}\|v\|_{L^2}^{\frac{4-2b-\alpha(N-2)}{2}} \\ &\geq \|\nabla v\|_{L^2}^2\left(8 - \frac{4(N\alpha+2b)}{\alpha+2}C_{GN}w^{\frac{\alpha}{2}}\|\nabla Q\|_{L^2}^{\alpha s_c}\|Q\|_{L^2}^{\alpha(1-s_c)}\right) \\ &= \|\nabla v\|_{L^2}^2 8(1-w^{\frac{\alpha}{2}}), \end{aligned}$$

where in the last equality, we have used (3.54). □

Now, using the ideas introduced by Côte [8] for the KdV equation (see also Guevara [22] Proposition 2.18), we show the existence of the Wave Operator. Before stating our result, we define

$$p^* = \frac{2N}{N-2} \text{ if } N \geq 3 \text{ and } p^* = \infty \text{ if } N = 2. \tag{3.58}$$

Moreover, we prove the following lemma.

Lemma 3.22. *Let* $\frac{4-2b}{\alpha} < \alpha < 2^*$ *and* $0 < b < \widetilde{2}$. *If* f *and* $g \in H^1(\mathbb{R}^N)$ *then*

(i) $\big\||x|^{-b}|f|^{\alpha+1}g\big\|_{L^1} \leq c\|f\|_{L^{\alpha+2}}^{\alpha+1}\|g\|_{L^{\alpha+2}} + c\|f\|_{L^r}^{\alpha+1}\|g\|_{L^r}$

(ii) $\big\||x|^{-b}|f|^{\alpha+1}g\big\|_{L^1} \leq c\|f\|_{H^1}^{\alpha+1}\|g\|_{H^1}$

(iii) $\lim\limits_{|t|\to+\infty}\big\||x|^{-b}|U(t)f|^{\alpha+1}g\big\|_{L^1_x} = 0.$

where[7] $2 < \frac{N(\alpha+2)}{N-b} < r < p^*$.

Proof. (i) We divide the estimate in B^C and B. Applying the Hölder inequality, since $1 = \frac{\alpha+1}{\alpha+2} + \frac{1}{\alpha+2}$, we obtain

$$\begin{aligned} \big\||x|^{-b}|f|^{\alpha+1}g\big\|_{L^1} &\leq \big\||x|^{-b}|f|^{\alpha+1}g\big\|_{L^1(B^C)} + \big\||x|^{-b}|f|^{\alpha+1}g\big\|_{L^1(B)} \\ &\leq \|f\|_{L^{\alpha+2}}^{\alpha+1}\|g\|_{L^{\alpha+2}} + \||x|^{-b}\|_{L^\gamma(B)}\|f\|_{L^{(\alpha+1)\beta}}^{\alpha+1}\|g\|_{L^r} \\ &= \|f\|_{L^{\alpha+2}}^{\alpha+1}\|g\|_{L^{\alpha+2}} + \||x|^{-b}\|_{L^\gamma(B)}\|f\|_{L^r}^{\alpha+1}\|g\|_{L^r}, \end{aligned} \tag{3.59}$$

[7]Note that, the hypothesis $0 < \alpha < 2_*$ (recall (3.7)) implies $\frac{N(\alpha+2)}{N-b} < p^*$.

where

$$1 = \frac{1}{\gamma} + \frac{1}{\beta} + \frac{1}{r} \quad \text{and} \quad r = (\alpha+1)\beta. \tag{3.60}$$

To complete the proof we need to check that $\||x|^{-b}\|_{L^\gamma(B)}$ is bounded, i.e., $\frac{N}{\gamma} > b$ (see Remark 1.17). In fact, we deduce from (3.60)

$$\frac{N}{\gamma} = N - \frac{N(\alpha+2)}{r},$$

and thus, since $r > \frac{N(\alpha+2)}{N-b}$ we obtain the desired result ($\frac{N}{\gamma} - b > 0$).

(ii) By the Sobolev inequality (1.7) (for $N = 2$ and $s = 1$) and (1.8) (for $N \geq 3$ and $s = 1$), it is easy to see that $H^1 \hookrightarrow L^{\alpha+2}$ and $H^1 \hookrightarrow L^r$ (where $2 < \frac{N(\alpha+2)}{N-b} < r < p^*$), then using (3.59) we get (ii).

(iii) Similarly as (i) and (ii), we obtain

$$\left\||x|^{-b}|U(t)f|^{\alpha+1}g\right\|_{L^1_x} \leq c\|U(t)f\|^{\alpha+1}_{L^{\alpha+2}}\|g\|_{H^1} + c\|U(t)f\|^{\alpha+1}_{L^r}\|g\|_{H^1}, \tag{3.61}$$

for $2 < \frac{N(\alpha+2)}{N-b} < r < p^*$.

We now show that $\|U(t)f\|_{L^r_x}$ and $\|U(t)f\|_{L^{\alpha+2}_x} \to 0$ as $|t| \to +\infty$. Indeed, since r and $\alpha+2$ belong to $(2, p^*)$ then it suffices to show

$$\lim_{|t|\to+\infty} \|U(t)f\|_{L^p_x} = 0, \tag{3.62}$$

where $2 < p < p^*$. Let $\widetilde{f} \in H^1 \cap L^{p'}$, the Sobolev embedding (1.7) if $N = 2$ or (1.8) if $N \geq 3$ and Lemma 1.9 yield

$$\|U(t)f\|_{L^p_x} \leq c\|f - \widetilde{f}\|_{H^1} + c|t|^{-\frac{N(p-2)}{2p}}\|\widetilde{f}\|_{L^{p'}}.$$

Since $p > 2$ then the exponent of $|t|$ is negative and so approximating f by $\widetilde{f} \in C_0^\infty$ in H^1, we deduce (3.62). $\square$

Proposition 3.23. (Existence of Wave Operator) *Suppose* $\phi \in H^1(\mathbb{R}^N)$ *and, for some*[8] $0 < \lambda \leq (\frac{2\alpha s_c}{N\alpha+2b})^{\frac{s_c}{2}}$,

$$\|\nabla\phi\|^{2s_c}_{L^2}\|\phi\|^{2(1-s_c)}_{L^2} < \lambda^2\left(\frac{N\alpha+2b}{\alpha s_c}\right)^{s_c} E[Q]^{s_c}M[Q]^{1-s_c}. \tag{3.63}$$

[8] Note that $(\frac{2\alpha s_c}{N\alpha+2b})^{\frac{s_c}{2}} < 1$.

Then, there exists $u_0^+ \in H^1(\mathbb{R}^N)$ *such that* u *solving* (3.1) *with initial data* u_0^+ *is global in* $H^1(\mathbb{R}^N)$ *with*

(i) $M[u] = M[\phi]$,

(ii) $E[u] = \frac{1}{2}\|\nabla\phi\|_{L^2}^2$,

(iii) $\lim_{t\to+\infty} \|u(t) - U(t)\phi\|_{H^1} = 0$,

(iv) $\|\nabla u(t)\|_{L^2}^{s_c}\|u(t)\|_{L^2}^{1-s_c} \leq \lambda\|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c}$.

Proof. We will divide the proof in two parts. First, we construct the wave operator for large time. Indeed, let $I_T = [T, +\infty)$ for $T \gg 1$ and define

$$G(w)(t) = -i\int_t^{+\infty} U(t-s)(|x|^{-b}|w + U(t)\phi|^\alpha(w + U(t)\phi)(s)ds, \quad t \in I_T$$

and

$$B(T,\rho) = \{w \in C\left(I_T; H^1(\mathbb{R}^N)\right) : \|w\|_T \leq \rho\},$$

where

$$\|w\|_T = \|w\|_{S(\dot{H}^{s_c};I_T)} + \|w\|_{S(L^2;I_T)} + \|\nabla w\|_{S(L^2;I_T)}.$$

Our goal is to find a fixed point for G on $B(T,\rho)$.

Applying the Strichartz estimates (1.11) (1.12) and Lemmas 3.9-3.10-3.12, we deduce

$$\|G(w)\|_{S(\dot{H}^{s_c};I_T)} \lesssim \|w + U(t)\phi\|_{L_T^\infty H_x^1}^\theta \|w + U(t)\phi\|_{S(\dot{H}^{s_c};I_T)}^{\alpha-\theta} \|w + U(t)\phi\|_{S(\dot{H}^{s_c};I_T)} \tag{3.64}$$

$$\|G(w)\|_{S(L^2;I_T)} \lesssim \|w + U(t)\phi\|_{L_T^\infty H_x^1}^\theta \|w + U(t)\phi\|_{S(\dot{H}^{s_c};I_T)}^{\alpha-\theta} \|w + U(t)\phi\|_{S(L^2;I_T)} \tag{3.65}$$

and

$$\|\nabla G(w)\|_{S(L^2;I_T)} \lesssim \|w+U(t)\phi\|^{\theta}_{L^\infty_T H^1_x}\|w+U(t)\phi\|^{\alpha-\theta}_{S(\dot H^{s_c};I_T)}\|\nabla(w+U(t)\phi)\|_{S(L^2;I_T)}$$
$$+\|w+U(t)\phi\|^{1+\theta}_{L^\infty_T H^1_x}\|w+U(t)\phi\|^{\alpha-\theta}_{S(\dot H^{s_c};I_T)}. \qquad (3.66)$$

Thus,

$$\begin{aligned}\|G(w)\|_T &\lesssim \|w+U(t)\phi\|^{\theta}_{L^\infty_T H^1_x}\|w+U(t)\phi\|^{\alpha-\theta}_{S(\dot H^{s_c};I_T)}\|w+U(t)\phi\|_T\\ &\quad+\|w+U(t)\phi\|^{\alpha-\theta}_{S(\dot H^{s_c};I_T)}\|w+U(t)\phi\|^{\theta+1}_T.\end{aligned}$$

Since[9]

$$\|U(t)\phi\|_{S(\dot H^{s_c};I_T)}\to 0 \qquad (3.67)$$

as $T\to+\infty$, we can find $T_0>0$ large enough and $\rho>0$ small enough such that G is well defined on $B(T_0,\rho)$. The same computations show that G is a contraction on $B(T_0,\rho)$. Therefore, G has a unique fixed point, which we denote by w.

On the other hand, from (3.64) and since

$$\|w+U(t)\phi\|_{L^\infty_T H^1_x}\le \|w\|_{H^1}+\|\phi\|_{H^1}<+\infty,$$

one has (recalling $G(w)=w$)

$$\begin{aligned}\|w\|_{S(\dot H^{s_c};I_T)} &\lesssim \|w+U(t)\phi\|^{\alpha-\theta}_{S(\dot H^{s_c};I_T)}\|w+U(t)\phi\|_{S(\dot H^{s_c};I_T)}\\ &\lesssim A\|w\|_{S(\dot H^{s_c};I_T)}+A\|U(t)\phi\|_{S(\dot H^{s_c};I_T)}\end{aligned}$$

where $A=\|w+U(t)\phi\|^{\alpha-\theta}_{S(\dot H^{s_c};I_T)}$. In addition, if ρ has been chosen small enough and since $\|U(t)\phi\|_{S(\dot H^{s_c};I_T)}$ is also sufficiently small for T large, we deduce

$$A\le c\|w\|^{\alpha-\theta}_{S(\dot H^{s_c};I_T)}+c\|U(t)\phi\|^{\alpha-\theta}_{S(\dot H^{s_c};I_T)}<\frac{1}{2},$$

[9]Note that (3.67) is possible not true using the norm $L^\infty_{I_T}L_x^{\frac{2N}{N-2s_c}}$ and for this reason we remove the pair $\left(\infty,\frac{2N}{N-2s_c}\right)$ in the Definition 1.7.

and so (using the last two inequalities)

$$\frac{1}{2}\|w\|_{S(\dot{H}^{s_c};I_T)} \lesssim A\|U(t)\phi\|_{S(\dot{H}^{s_c};I_T)},$$

which implies,

$$\|w\|_{S(\dot{H}^{s_c};I_T)} \to 0 \quad \text{as} \quad T \to +\infty. \tag{3.68}$$

Hence, (3.65), (3.66) and (3.68) also yield that[10]

$$\|w\|_{S(L^2;I_T)}\ ,\ \|\nabla w\|_{S(L^2;I_T)} \to 0 \quad \text{as} \quad T \to +\infty,$$

and finally

$$\|w\|_T \to 0 \ \text{ as } \ T \to +\infty. \tag{3.69}$$

Next, we claim that $u(t) = U(t)\phi + w(t)$ satisfies (3.1) in the time interval $[T_0, \infty)$. To do this, we need to show that

$$u(t) = U(t-T_0)u(T_0) + i\int_{T_0}^{t} U(t-s)(|x|^{-b}|u|^{\alpha}u)s ds, \tag{3.70}$$

for all $t \in [T_0, \infty)$. Indeed, since

$$w(t) = -i\int_t^{\infty} U(t-s)|x|^{-b}|w+U(t)\phi|^{\alpha}(w+U(t)\phi)(s)ds,$$

then

$$\begin{aligned} U(T_0-t)w(t) &= -i\int_t^{\infty} U(T_0-s)|x|^{-b}|w+U(t)\phi|^{\alpha}(w+U(t)\phi)(s)ds \\ &= i\int_{T_0}^{t} U(T_0-s)|x|^{-b}|w+U(t)\phi|^{\alpha}(w+U(t)\phi)(s)ds + w(T_0), \end{aligned}$$

and so applying $U(t-T_0)$ on both sides, we get

$$w(t) = U(t-T_0)w(T_0) + i\int_{T_0}^{t} U(t-s)|x|^{-b}|w+U(t)\phi|^{\alpha}(w+U(t)\phi)(s)ds.$$

[10] Observe that $\|w+U(t)\phi\|_{S(\dot{H}^{s_c};I_T)} \leq \|w\|_{S(\dot{H}^{s_c};I_T)} + \|U(t)\phi\|_{S(\dot{H}^{s_c};I_T)} \to 0$ as $T \to +\infty$ by (3.68) and $\|w+U(t)\phi\|^{\theta}_{L^\infty_T H^1_x}, \|w+U(t)\phi\|_{S(L^2;I_T)}, \|\nabla(w+U(t)\phi)\|_{S(L^2;I_T)} < \infty$ since $w \in B(T,\rho)$ and $\phi \in H^1(\mathbb{R}^N)$.

Finally, adding $U(t)\phi$ in both sides of the last equation, we deduce (3.70).

Now we show relations (i)-(iv). Since $u(t) = U(t)\phi + w$ then

$$\|u(t) - U(t)\phi\|_{L^\infty_T H^1_x} = \|w\|_{L^\infty_T H^1_x} \leq c\|w\|_{S(L^2;I_T)} + c\|\nabla w\|_{S(L^2;I_T)} \leq c\|w\|_T \tag{3.71}$$

and so from (3.65) we obtain (iii). Furthermore, using (3.71) it is clear that

$$\lim_{t\to\infty} \|u(t)\|_{L^2_x} = \|\phi\|_{L^2}. \tag{3.72}$$

and

$$\lim_{t\to\infty} \|\nabla u(t)\|_{L^2_x} = \|\nabla\phi\|_{L^2}. \tag{3.73}$$

By the mass conservation (3) we have $\|u(t)\|_{L^2} = \|u(T_0)\|_{L^2}$ for all t, so from (3.72) we deduce $\|u(T_0)\|_{L^2} = \|\phi\|_{L^2}$, i.e., item (i) holds. On the other hand, it follows from Lemma 3.22 (ii)

$$\begin{aligned} \left\||x|^{-b}|u(t)|^{\alpha+2}\right\|_{L^1_x} &\leq c\left\||x|^{-b}|u(t) - U(t)\phi|^{\alpha+2}\right\|_{L^1_x} + c\left\||x|^{-b}|U(t)\phi|^{\alpha+2}\right\|_{L^1_x} \\ &\leq c\,\|u(t) - U(t)\phi\|^{\alpha+2}_{H^1_x} + c\left\||x|^{-b}|U(t)\phi|^{\alpha+2}\right\|_{L^1_x}, \end{aligned}$$

which goes to zero as $t \to +\infty$, by item (iii) and Lemma 3.22 (iii), i.e.

$$\lim_{t\to\infty} \left\||x|^{-b}|u(t)|^{\alpha+2}\right\|_{L^1_x} = 0. \tag{3.74}$$

Combining (3.73) and (3.74), it is easy to deduce (ii).

Next, in view of (3.63), (i) and (ii) we have

$$E[u]^{s_c}M[u]^{1-s_c} = \frac{1}{2^{s_c}}\|\nabla\phi\|^{2s_c}_{L^2}\|\phi\|^{2(1-s_c)}_{L^2} < \lambda^2\left(\frac{N\alpha + 2b}{2\alpha s_c}\right)^{s_c} E[Q]^{s_c}M[Q]^{1-s_c}$$

and by our choice of λ we conclude

$$E[u]^{s_c}M[u]^{1-s_c} < E[Q]^{s_c}M[Q]^{1-s_c}.$$

Moreover, from (3.72), (3.73) and (3.63)

$$\begin{aligned}\lim_{t\to\infty}\|\nabla u(t)\|_{L_x^2}^{2s_c}\|u(t)\|_{L_x^2}^{2(1-s_c)} &= \|\nabla\phi\|_{L^2}^{2s_c}\|\phi\|_{L^2}^{2(1-s_c)}\\ &< \lambda^2\left(\frac{N\alpha+2b}{\alpha s_c}\right)^{s_c}E[Q]^{s_c}M[Q]^{1-s_c}\\ &= \lambda^2\|\nabla Q\|_{L^2}^{2s_c}\|Q\|_{L^2}^{2(1-s_c)}\end{aligned}$$

where we have used (3.55). Thus, one can take $T_1>0$ sufficiently large such that

$$\|\nabla u(T_1)\|_{L_x^2}^{s_c}\|u(T_1)\|_{L_x^2}^{1-s_c}<\lambda\|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c}.$$

Therefore, since $\lambda<1$, we deduce that relations (3.3) and (3.4) hold with $u_0=u(T_1)$ and so, by Theorem 3.1, we have in fact that $u(t)$ constructed above is a global solution of (3.1). □

Remark 3.24. A similar Wave Operator construction also holds when the time limit is taken as $t\to-\infty$ (backward in time).

3.5 Existence and compactness of a critical solution

The goal of this section is to construct a critical solution (denoted by u_c) of (3.1). We divide the study in two parts. First, we establish a profile decomposition result and also an Energy Pythagorean expansion for such a decomposition. In the sequel, using the results of the first part we construct u_c and discuss some of its properties.

We start this section recalling some elementary inequalities (see Gérard [17] inequality (1.10) and Guevara [22] page 217). Let $(z_j)\subset\mathbb{C}^M$ with

$M \geq 2$. For all $q > 1$ there exists $C_{q,M} > 0$ such that

$$\left| \left| \sum_{j=1}^{M} z_j \right|^q - \sum_{j=1}^{M} |z_j|^q \right| \leq C_{q,M} \sum_{j \neq k}^{M} |z_j||z_k|^{q-1}, \tag{3.75}$$

and for $\beta > 0$ there exists a constant $C_{\beta,M} > 0$ such that

$$\left| \left| \sum_{j=1}^{M} z_j \right|^\beta \sum_{j=1}^{M} z_j - \sum_{j=1}^{M} |z_j|^\beta z_j \right| \leq C_{\beta,M} \sum_{j=1}^{M} \sum_{1 \leq j \neq k \leq M} |z_j|^\beta |z_k|. \tag{3.76}$$

3.5.1 Profile decomposition

This subsection contains the profile decomposition and energy Pythagorean expansion results. We use similar arguments as the ones in Holmer-Roudenko [23, Lemma 5.2] (see also Fang-Xie-Cazenave [11, Theorem 5.1] and Guevara [22, Proposition 3.4]) and, for the sake of completeness, we provide the details here.

Proposition 3.25. (Profile decomposition) *Let $\phi_n(x)$ be a radial uniformly bounded sequence in $H^1(\mathbb{R}^N)$. Then for each $M \in \mathbb{N}$ there exists a subsequence of ϕ_n (also denoted by ϕ_n), such that, for each $1 \leq j \leq M$, there exist a profile ψ^j in $H^1(\mathbb{R}^N)$, a sequence t_n^j of time shifts and a sequence W_n^M of remainders in $H^1(\mathbb{R}^N)$, such that*

$$\phi_n(x) = \sum_{j=1}^{M} U(-t_n^j)\psi^j(x) + W_n^M(x) \tag{3.77}$$

with the properties:

- *Pairwise divergence for the time sequences. For $1 \leq k \neq j \leq M$,*

$$\lim_{n \to +\infty} |t_n^j - t_n^k| = +\infty. \tag{3.78}$$

- *Asymptotic smallness for the remainder sequence*

$$\lim_{M\to+\infty}\left(\lim_{n\to+\infty}\|U(t)W_n^M\|_{S(\dot H^{s_c})}\right)=0. \tag{3.79}$$

- *Asymptotic Pythagoream expansion. For fixed $M\in\mathbb{N}$ and any $s\in[0,1]$, we have*

$$\|\phi_n\|_{\dot H^s}^2=\sum_{j=1}^{M}\|\psi^j\|_{\dot H^s}^2+\|W_n^M\|_{\dot H^s}^2+o_n(1) \tag{3.80}$$

where $o_n(1)\to 0$ as $n\to+\infty$.

Proof. Let $C_1>0$ such that $\|\phi_n\|_{H^1}\le C_1$. For every (a,r) $\dot H^{s_c}$-admissible we can define $r_1=2r$ and $a_1=\frac{4r}{r(N-2s_c)-N}$. Note that (a_1,r_1) is also $\dot H^{s_c}$-admissible, then combining the interpolation inequality with $\eta=\frac{N}{r(N-2s_c)-N}\in(0,1)$ and the Strichartz estimate (1.10), we have

$$\begin{aligned}\|U(t)W_n^M\|_{L_t^aL_x^r} &\le \|U(t)W_n^M\|_{L_t^{a_1}L_x^{r_1}}^{1-\eta}\|U(t)W_n^M\|^{\eta}_{L_t^\infty L_x^{\frac{2N}{N-2s_c}}}\\ &\le \|W_n^M\|_{\dot H^{s_c}}^{1-\eta}\|U(t)W_n^M\|^{\eta}_{L_t^\infty L_x^{\frac{2N}{N-2s_c}}}.\end{aligned} \tag{3.81}$$

Since we will have $\|W_n^M\|_{\dot H^{s_c}}\le C_1$, then we need to show that the second norm in the right hand side of (3.81) goes to zero as n and M go to infinity, that is

$$\lim_{M\to+\infty}\left(\limsup_{n\to+\infty}\|U(t)W_n^M\|_{L_t^\infty L_x^{\frac{2N}{N-2s_c}}}\right)=0. \tag{3.82}$$

First we construct ψ_n^1, t_n^1 and W_n^1. Let

$$A_1=\limsup_{n\to+\infty}\|U(t)\phi_n\|_{L_t^\infty L_x^{\frac{2N}{N-2s_c}}}.$$

If $A_1=0$, the proof is complete with $\psi^j=0$ for all $j=1,\dots,M$. Assume that $A_1>0$. Passing to a subsequence, we may consider $A_1=\lim_{n\to+\infty}\|U(t)\phi_n\|_{L_t^\infty L_x^{\frac{2N}{N-2s_c}}}$. We claim that there exist a time sequence t_n^1 and ψ^1 such that $U(t_n^1)\phi_n\rightharpoonup\psi^1$ and

$$\beta C_1^{\frac{N-2s_c}{2s_c(1-s_c)}}\|\psi^1\|_{\dot H^{s_c}}\ge A_1^{\frac{N-2s_c^2}{2s_c(1-s_c)}}, \tag{3.83}$$

where $\beta > 0$ is independent of C_1, A_1 and ϕ_n. Indeed, let $\zeta \in C_0^\infty(\mathbb{R}^N)$ a real-valued and radially symmetric function such that $0 \leq \zeta \leq 1$, $\zeta(\xi) = 1$ for $|\xi| \leq 1$ and $\zeta(\xi) = 0$ for $|\xi| \geq 2$. Given $r > 0$, define χ_r by $\widehat{\chi_r}(\xi) = \zeta(\frac{\xi}{r})$. From the Sobolev embedding (1.6) and since the operator $U(t)$ is an isometry in H^{s_c}, we deduce (recalling $0 < s_c < 1$)

$$\begin{aligned}
\|U(t)\phi_n - \chi_r * U(t)\phi_n\|^2_{L_t^\infty L_x^{\frac{2N}{N-2s_c}}} &\leq c\|U(t)\phi_n - \chi_r * U(t)\phi_n\|^2_{L_t^\infty H_x^{s_c}} \\
&\leq c\int |\xi|^{2s_c}|(1-\widehat{\chi_r})^2|\widehat{\phi}_n(\xi)|^2 d\xi \\
&\leq c\int_{|\xi|>r} |\xi|^{-2(1-s_c)}|\xi|^2|\widehat{\phi}_n(\xi)|^2 d\xi \\
&\leq cr^{-2(1-s_c)}\|\phi\|^2_{\dot{H}^1} \leq cr^{-2(1-s_c)}C_1^2.
\end{aligned}$$

Choosing

$$r = \left(\frac{4\sqrt{c}C_1}{A_1}\right)^{\frac{1}{1-s_c}} \tag{3.84}$$

and for n large enough we have

$$\|\chi_r * U(t)\phi_n\|_{L_t^\infty L_x^{\frac{2N}{N-2s_c}}} \geq \frac{A_1}{2}. \tag{3.85}$$

Note that, from the standard interpolation in Lebesgue spaces

$$\begin{aligned}
\|\chi_r * U(t)\phi_n\|^N_{L_t^\infty L_x^{\frac{2N}{N-2s_c}}} &\leq \|\chi_r * U(t)\phi_n\|^{N-2s_c}_{L_t^\infty L_x^2}\|\chi_r * U(t)\phi_n\|^{2s_c}_{L_t^\infty L_x^\infty} \\
&\leq C_1^{N-2s_c}\|\chi_r * U(t)\phi_n\|^{2s_c}_{L_t^\infty L_x^\infty},
\end{aligned} \tag{3.86}$$

thus inequalities (3.85) and (3.86) lead to

$$\|\chi_r * U(t)\phi_n\|_{L_t^\infty L_x^\infty} \geq \left(\frac{A_1}{2C_1^{\frac{N-2s_c}{N}}}\right)^{\frac{N}{2s_c}}.$$

It follows from the radial Sobolev Gagliardo-Nirenberg inequality (3.56) (since all ϕ_n are radial functions and so are $\chi_r * U(t)\phi_n)$ that[11]

$$\begin{aligned}\|\chi_r * U(t)\phi_n\|_{L^\infty_t L^\infty_x(|x|\geq R)} &\leq \frac{1}{R^{\frac{N-1}{2}}}\|\chi_r * U(t)\phi_n\|^{\frac{1}{2}}_{L^2_x}\|\nabla(\chi_r * U(t)\phi_n)\|^{\frac{1}{2}}_{L^2_x}\\ &\leq \frac{C_1}{R^{\frac{N-1}{2}}},\end{aligned}$$

which implies for $R > 0$ sufficiently large

$$\|\chi_r * U(t)\phi_n\|_{L^\infty_t L^\infty_x(|x|\leq R)} \geq \frac{1}{2}\left(\frac{A_1}{2C_1^{\frac{N-2s_c}{N}}}\right)^{\frac{N}{2s_c}},$$

where we have used the two last inequalities. Now, let t^1_n and x^1_n, with $|x^1_n| \leq R$, be sequences such that for each $n \in \mathbb{N}$

$$\left|\chi_r * U(t^1_n)\phi_n(x^1_n)\right| \geq \frac{1}{4}\left(\frac{A_1}{2C_1^{\frac{N-2s_c}{N}}}\right)^{\frac{N}{2s_c}}$$

or

$$\frac{1}{4}\left(\frac{A_1}{2C_1^{\frac{N-2s_c}{N}}}\right)^{\frac{N}{2s_c}} \leq \left|\int \chi_r(x^1_n - y)U(t^1_n)\phi_n(y)dy\right|. \tag{3.87}$$

On the other hand, since $\|U(t^1_n)\phi_n\|_{H^1} = \|\phi_n\|_{H^1} \leq C_1$ then $U(t^1_n)\phi_n$ converges weakly in H^1 (since $U(t^1_n)\phi_n$ is a bounded sequence a Hilbert space), i.e., there exists ψ^1 a radial function such that (up to a subsequence) $U(t^1_n)\phi_n \rightharpoonup \psi^1$ in H^1 and $\|\psi^1\|_{H^1} \leq \limsup\limits_{n\to+\infty}\|\phi_n\|_{H^1} \leq C_1$. In addition, $x^1_n \to x^1$ (also up to a subsequence) since x^1_n is bounded. Hence the inequality (3.87), the Plancherel formula and the Cauchy-Schwarz inequality yield

$$\frac{1}{8}\left(\frac{A_1}{2C_1^{\frac{N-2s_c}{N}}}\right)^{\frac{N}{2s_c}} \leq \left|\int \chi_r(x^1 - y)\psi^1(y)dy\right| \leq \|\chi_r\|_{\dot{H}^{-s_c}}\|\psi^1\|_{\dot{H}^{s_c}},$$

which implies

$$\frac{1}{8}\left(\frac{A_1}{2C_1^{\frac{N-2s_c}{N}}}\right)^{\frac{N}{2s_c}} \leq cr^{\frac{N-2s_c}{2}}\|\psi^1\|_{\dot{H}^{s_c}},$$

[11] Note the radial Gagliardo-Nirenberg inequality only holds for dimensions $N \geq 2$. As pointed out in Holmer-Roudenko [23] page 466 this is probably an obstruction to extend the scattering result stated in Theorem 3.3 for 1D.

where we have used

$$\|\chi_r\|_{\dot H^{-s_c}} = \left(\int_{0<|\xi|<2r} |\xi|^{-2s_c}|\widehat{\chi_r}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \le c\left(\int_0^{2r} \rho^{-2s_c}\rho^{N-1}d\rho\right)^{\frac{1}{2}} \le cr^{\frac{N-2s_c}{2}}.$$

Therefore in view of our choice of r (see (3.84)) we obtain (3.83), concluding the claim.

Next, define $W_n^1 = \phi_n - U(-t_n^1)\psi^1$. It is easy to see that, for any $0 \le s \le 1$,

- $U(t_n^1)W_n^1 \rightharpoonup 0$ in H^1 (since $U(t_n^1)\phi_n \rightharpoonup \psi^1$),
- $\langle \phi_n, U(-t_n^1)\psi^1\rangle_{\dot H^s} = \langle U(t_n^1)\phi_n, \psi^1\rangle_{\dot H^s} \to \|\psi^1\|^2_{\dot H^s}$,
- $\|W_n^1\|^2_{\dot H^s} = \|\phi_n\|^2_{\dot H^s} - \|\psi^1\|^2_{\dot H^s} + o_n(1)$.

The last item, with $s = 0$ and $s = 1$, implies $\|W_n^1\|_{H^1} \le C_1$.

Next, let $A_2 = \limsup_{n\to+\infty} \|U(t)W_n^1\|_{L_t^\infty L_x^{\frac{2N}{N-2s}}}$. If $A_2 = 0$ the result follows taking $\psi^j = 0$ for all $j = 2, \dots, M$.. Let $A_2 > 0$, repeating the above argument with ϕ_n replaced by W_n^1 we obtain a sequence t_n^2 and a function ψ^2 such that $U(t_n^2)W_n^1 \rightharpoonup \psi^2$ in H^1 and

$$\beta C_1^{\frac{N-2s_c}{2s_c(1-s_c)}} \|\psi^2\|_{\dot H^{s_c}} \ge A_2^{\frac{N-2s_c^2}{2s_c(1-s_c)}}.$$

We now prove that $|t_n^2 - t_n^1| \to +\infty$. In fact, if we suppose (up to a subsequence) $t_n^2 - t_n^1 \to t^*$ finite, then

$$U(t_n^2 - t_n^1)\left(U(t_n^1)\phi_n - \psi^1\right) = U(t_n^2)\left(\phi_n - U(-t_n^1)\psi^1\right) = U(t_n^2)W_n^1 \rightharpoonup \psi^2.$$

On the other hand, since $U(t_n^1)\phi_n \rightharpoonup \psi^1$, the left side of the above expression converges weakly to 0, and thus $\psi^2 = 0$, a contradiction. Define $W_n^2 =$

$W_n^1 - U(-t_n^2)\psi^2$. For any $0 \le s \le 1$, since $|t_n^1 - t_n^2| \to +\infty$, we deduce

$$\begin{aligned}
\langle \phi_n, U(-t_n^2)\psi^2\rangle_{\dot H^s} &= \langle U(t_n^2)\phi_n, \psi^2\rangle_{\dot H^s} \\
&= \langle U(t_n^2)\left(W_n^1 + U(-t_n^1)\psi^1\right), \psi^2\rangle_{\dot H^s} \\
&= \langle U(t_n^2)W_n^1, \psi^2\rangle_{\dot H^s} + \langle U(t_n^2 - t_n^1)\psi^1, \psi^2\rangle_{\dot H^s} \\
&\to \|\psi^2\|_{\dot H^s}^2.
\end{aligned}$$

In addition, the definition of W_n^2 implies that

$$\|W_n^2\|_{\dot H^s}^2 = \|W_n^1\|_{\dot H^{s_c}}^2 - \|\psi^2\|_{\dot H^s}^2 + o_n(1)$$

and $\|W_n^2\|_{H^1} \le C_1$.

By induction we can construct ψ^M, t_n^M and W_n^M such that $U(t_n^M)W_n^{M-1} \rightharpoonup \psi^M$ in H^1 and

$$\beta C_1^{\frac{N-2s_c}{2s_c(1-s_c)}}\|\psi^M\|_{\dot H^{s_c}} \ge A_M^{\frac{N-2s_c^2}{2s_c(1-s_c)}}, \tag{3.88}$$

where $A_M = \lim\limits_{n\to+\infty} \|U(t)W_n^{M-1}\|_{L_t^\infty L_x^{\frac{2N}{N-2s_c}}}$.

Next, we show (3.78). Suppose $1 \le j < M$, we prove that $|t_n^M - t_n^j| \to +\infty$ by induction assuming $|t_n^M - t_n^k| \to +\infty$ for $k = j+1, \dots, M-1$. Indeed, let $t_n^M - t_n^j \to t_0$ finite (up to a subsequence) then it is easy to see

$$U(t_n^M - t_n^j)\left(U(t_n^j)W_n^{j-1} - \psi^j\right) - U(t_n^M - t_n^{j+1})\psi^{j+1} - \dots - U(t_n^M - t_n^{M-1})\psi^{M-1}$$

$$= U(t_n^M)W_n^{M-1} \rightharpoonup \psi^M.$$

Since the left side converges weakly to 0, we have $\psi^M = 0$, a contradiction.

We now consider

$$W_n^M = \phi_n - U(-t_n^1)\psi^1 - U(-t_n^2)\psi^2 - \dots - U(-t_n^M)\psi^M.$$

Similarly as before, by (3.78) we get for any $0 \le s \le 1$

$$\langle \phi_n, U(-t_n^M)\psi^M\rangle_{\dot H^s} = \langle U(t_n^M)W_n^{M-1}, \psi^M\rangle_{\dot H^s} + o_n(1),$$

and so $\langle \phi_n, U(-t_n^M)\psi^M\rangle_{\dot{H}^s} \to \|\psi^M\|^2_{\dot{H}^s}$. Thus expanding $\|W_n^M\|^2_{\dot{H}^s}$ we deduce that (3.80) also holds.

Finally, the inequality (3.88) together with the relation (3.80) yield

$$\sum_{M\geq 1}\left(\frac{A_M^{\frac{N-2s_c^2}{s_c(1-s_c)}}}{\beta^2 C_1^{\frac{N-2s_c}{s_c(1-s_c)}}}\right) \leq \lim_{n\to+\infty}\|\phi_n\|^2_{\dot{H}^{s_c}} < +\infty,$$

which implies that $A_M \to 0$ as $M \to +\infty$ i.e., (3.82) holds[12]. Therefore, from (3.81) we get (3.79). This completes the proof. □

Remark 3.26. It follows from the proof of Proposition 3.25 that

$$\lim_{M,n\to\infty}\|W_n^M\|_{L^p} = 0, \tag{3.89}$$

where $2 < p < p^*$ (recalling p^* is defined in (3.58)). Indeed, first we show

$$\lim_{M\to+\infty}\left(\lim_{n\to+\infty}\|U(t)W_n^M\|_{L_t^\infty L_x^p}\right) = 0. \tag{3.90}$$

Note that, $\dot{H}^s \hookrightarrow L^p$ where $s = \frac{N}{2} - \frac{N}{p}$ (see inequality (1.6)). Since $2 < p < p^*$ then $0 < s < 1$, thus repeating the argument used for showing (3.82) with $\frac{2N}{N-2s_c}$ replaced by p and s_c by s, we obtain (3.90). On the other hand, (3.89) follows directly from (3.90) and the inequality

$$\|W_n^M\|_{L_x^p} \leq \|U(t)W_n^M\|_{L_t^\infty L_x^p},$$

since $W_n^M = U(0)W_n^M$.

Proposition 3.27. (Energy Pythagoream Expansion) *Under the hypothesis of Proposition 3.25 we obtain*

$$E[\phi_n] = \sum_{j=1}^{M} E[U(-t_n^j)\psi^j] + E[W_n^M] + o_n(1). \tag{3.91}$$

[12]Note that $N - 2s_c^2 > 0$ since $s_c < \min\{1, N/2\}$.

Proof. By definition of $E[u]$ and (3.80) with $s=1$, we have

$$E[\phi_n]-\sum_{j=1}^{M}E[U(-t_n^j)\psi^j]-E[W_n^M]=-\frac{A_n}{\alpha+2}+o_n(1),$$

where

$$A_n=\left\||x|^{-b}|\phi_n|^{\alpha+2}\right\|_{L^1}-\sum_{j=1}^{M}\left\||x|^{-b}|U(-t_n^j)\psi^j|^{\alpha+2}\right\|_{L^1_x}-\left\||x|^{-b}|W_n^M|^{\alpha+2}\right\|_{L^1}.$$

For a fixed $M\in\mathbb{N}$, if $A_n\to 0$ as $n\to+\infty$ then (3.91) holds. To prove this fact, pick $M_1\geq M$ and rewrite the last expression as

$$\begin{aligned}A_n &= \int\left(|x|^{-b}|\phi_n|^{\alpha+2}-\sum_{j=1}^{M}|x|^{-b}|U(-t_n^j)\psi^j|^{\alpha+2}-|x|^{-b}|W_n^M|^{\alpha+2}\right)dx\\ &= I_n^1+I_n^2+I_n^3,\end{aligned}$$

where

$$\begin{aligned}I_n^1 &= \int|x|^{-b}\left[|\phi_n|^{\alpha+2}-|\phi_n-W_n^{M_1}|^{\alpha+2}\right]dx\\ I_n^2 &= \int|x|^{-b}\left[|W_n^{M_1}-W_n^M|^{\alpha+2}-|W_n^M|^{\alpha+2}\right]dx\end{aligned}$$

and

$$I_n^3 = \int|x|^{-b}\left[|\phi_n-W_n^{M_1}|^{\alpha+2}-\sum_{j=1}^{M}|U(-t_n^j)\psi^j|^{\alpha+2}-|W_n^{M_1}-W_n^M|^{\alpha+2}\right]dx.$$

We first estimate I_n^1. Combining (3.75) and Lemma 3.22 (i)-(ii) we have

$$\begin{aligned}|I_n^1| &\lesssim \int|x|^{-b}\left(|\phi_n|^{\alpha+1}|W_n^{M_1}|+|\phi_n||W_n^{M_1}|^{\alpha+1}+|W_n^{M_1}|^{\alpha+2}\right)dx\\ &\lesssim \left(\|\phi_n\|_{L^r}^{\alpha+1}\|W_n^{M_1}\|_{L^r}+\|\phi_n\|_{L^r}\|W_n^{M_1}\|_{L^r}^{\alpha+1}+\|W_n^{M_1}\|_{L^r}^{\alpha+2}\right)+\\ &\quad\left(\|\phi_n\|_{L^{\alpha+2}}^{\alpha+1}\|W_n^{M_1}\|_{L^{\alpha+2}}+\|\phi_n\|_{L^{\alpha+2}}\|W_n^{M_1}\|_{L^{\alpha+2}}^{\alpha+1}+\|W_n^{M_1}\|_{L^{\alpha+2}}^{\alpha+2}\right)\\ &\lesssim \|\phi_n\|_{H^1}^{\alpha+1}\|W_n^{M_1}\|_{L^r}+\|\phi_n\|_{H^1}\|W_n^{M_1}\|_{L^r}^{\alpha+1}+\|W_n^{M_1}\|_{L^r}^{\alpha+2}+\\ &\quad\|\phi_n\|_{H^1}^{\alpha+1}\|W_n^{M_1}\|_{L^{\alpha+2}}+\|\phi_n\|_{H^1}\|W_n^{M_1}\|_{L^{\alpha+2}}^{\alpha+1}+\|W_n^{M_1}\|_{L^{\alpha+2}}^{\alpha+2},\end{aligned}$$

where $\frac{N(\alpha+2)}{N-b} < r < p^*$ (recall that p^* is defined in (3.58)). In view of inequality (3.89) and since $\{\phi_n\}$ is uniformly bounded in H^1, we conclude that[13]

$$I_n^1 \to +\infty \text{ as } n, M_1 \to +\infty.$$

Also, by similar arguments (replacing ϕ_n by W_n^M) we have

$$I_n^2 \to +\infty \text{ as } n, M_1 \to +\infty,$$

where we have used that W_n^M is uniformly bounded by (3.80).

Finaly we consider the term I_n^3. Since,

$$\phi_n - W_n^{M_1} = \sum_{j=1}^{M_1} U(-t_n^j)\psi^j$$

and

$$W_n^M - W_n^{M_1} = \sum_{j=M+1}^{M_1} U(-t_n^j)\psi^j,$$

we can rewrite I_n^3 as

$$I_n^3 = \int |x|^{-b}\left(\left|\sum_{j=1}^{M_1} U(-t_n^j)\psi^j\right|^{\alpha+2} - \sum_{j=1}^{M_1} |U(-t_n^j)\psi^j|^{\alpha+2}\right) dx$$

$$- \int |x|^{-b}\left(\left|\sum_{j=M+1}^{M_1} U(-t_n^j)\psi^j\right|^{\alpha+2} - \sum_{j=M+1}^{M_1} |U(-t_n^j)\psi^j|^{\alpha+2}\right) dx.$$

To complete the prove we make use of the following claim.

Claim. For a fixed $M_1 \in \mathbb{N}$ and for some $j_0 \in \mathbb{N}$ ($j_0 < M_1$), we get

$$D_n = \left\| |x|^{-b}\left|\sum_{j=j_0}^{M_1} U(-t_n^j)\psi\right|^{\alpha+2}\right\|_{L_x^1} - \sum_{j=j_0}^{M_1} \left\| |x|^{-b}|U(-t_n^j)\psi^j|^{\alpha+2}\right\|_{L_x^1} \to 0,$$

as $n \to +\infty$.

[13] We can apply Remark 3.26 since r and $\alpha + 2 \in (2, p^*)$.

Indeed, it is clear that the last limit implies that $I_n^3 \to 0$ as $n \to +\infty$ completing the proof of relation (3.91).

To prove the claim note that (3.75) implies

$$D_n \leq \sum_{j \neq k}^{M_1} \int |x|^{-b} |U(-t_n^j)\psi^j| |U(-t_n^k)\psi^k|^{\alpha+1} dx.$$

Thus, from Lemma 3.22 (i) one has

$$E_n^{j,k} \leq c\|U(-t_n^k)\psi^k\|_{L_x^{\alpha+2}}^{\alpha+1} \|U(-t_n^j)\psi^j\|_{L_x^{\alpha+2}} + c\|U(-t_n^k)\psi^k\|_{L_x^r}^{\alpha+1} \|U(-t_n^j)\psi^j\|_{L_x^r},$$

where $2 < \frac{N(\alpha+2)}{N-b} < r < p^*$ and $E_n^{j,k} = \int |x|^{-b} |U(-t_n^j)\psi^j| |U(-t_n^k)\psi^k|^{\alpha+1} dx$. In view of (3.78) we can consider that t_n^k, t_n^j or both go to infinite as n goes to infinite. If $t_n^j \to +\infty$ as $n \to +\infty$, so it follow from the last inequality and since $H^1 \hookrightarrow L^{\alpha+2}$ and $H^1 \hookrightarrow L^r$ that

$$\begin{aligned} E_n^{j,k} &\leq c\|\psi^k\|_{H^1}^{\alpha+1} \|U(-t_n^j)\psi^j\|_{L_x^{\alpha+2}} + c\|\psi^k\|_{H^1}^{\alpha+1} \|U(-t_n^j)\psi^j\|_{L_x^r} \\ &\leq c\|U(-t_n^j)\psi^j\|_{L_x^{\alpha+2}} + c\|U(-t_n^j)\psi^j\|_{L_x^r}, \end{aligned}$$

where in the last inequality we have used that $(\psi^k)_{k\in\mathbb{N}}$ is a uniformly bounded sequence in H^1. Hence, if $n \to +\infty$ we have $t_n^j \to +\infty$ and using (3.62) with $t = t_n^j$ and $f = \psi^j$ we conclude that $E_n^{j,k} \to 0$ as $n \to +\infty$. Similarly for the case $t_n^k \to +\infty$ as $n \to +\infty$, we have

$$\begin{aligned} E_n^{j,k} &\leq c\|U(-t_n^k)\psi^k\|_{L_x^{\alpha+2}}^{\alpha+1} \|\psi^j\|_{H^1} + c\|U(-t_n^k)\psi^k\|_{L_x^r}^{\alpha+1} \|\psi^j\|_{H^1} \\ &\leq c\|U(-t_n^k)\psi^k\|_{L_x^{\alpha+2}}^{\alpha+1} + c\|U(-t_n^k)\psi^k\|_{L_x^r}^{\alpha+1}, \end{aligned}$$

which implies that $E_n^{j,k} \to 0$ as $n \to +\infty$ by (3.62) with $t = t_n^k$ and $f = \psi^k$. Finally, since D_n is a finite sum of terms in the form of $E^{j,k}$ we deduce $D_n \to 0$ as $n \to +\infty$. □

3.5.2 Critical solution

In this subsection we study a critical solution of (3.1). First, assuming that $\delta_c < E[u]^{s_c} M[u]^{1-s_c}$ (see (3.10)), we construct a global solution called u_c of (3.1) with infinite Strichartz norm $\|\cdot\|_{S(\dot{H}^{s_c})}$ satisfying

$$E[u_c]^{s_c} M[u_c]^{1-s_c} = \delta_c.$$

After that, we show that the flow associated to this critical solution is pre-compact in $H^1(\mathbb{R}^N)$.

Proposition 3.28. (Existence of a critical solution) *Let* $0 < b < \min\{\frac{N}{3}, 1\}$*. If*

$$\delta_c < E[Q]^{s_c} M[Q]^{1-s_c},$$

then there exists a radial function $u_{c,0} \in H^1(\mathbb{R}^N)$ *such that the corresponding solution* u_c *of the IVP* (3.1) *is global in* $H^1(\mathbb{R}^N)$*. Moreover the following properties hold*

(i) $M[u_c] = 1$,

(ii) $E[u_c]^{s_c} = \delta_c$,

(iii) $\|\nabla u_{c,0}\|_{L^2}^{s_c} \|u_{c,0}\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}$,

(iv) $\|u_c\|_{S(\dot{H}^{s_c})} = +\infty$.

Proof. Recall from Subsection 3.2 that there exists a sequence of solutions u_n to (3.1) with H^1 initial data $u_{n,0}$, with $\|u_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$, such that

$$\|\nabla u_{n,0}\|_{L^2}^{s_c} < \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c} \tag{3.92}$$

and

$$E[u_n] \searrow \delta_c^{\frac{1}{s_c}} \quad \text{as} \quad n \to +\infty.$$

Moreover

$$\|u_n\|_{S(\dot{H}^{s_c})} = +\infty \tag{3.93}$$

for every $n \in \mathbb{N}$. Note that, in view of the assumption $\delta_c < E[Q]^{s_c} M[Q]^{1-s_c}$, there exists $a \in (0,1)$ such that

$$E[u_n] \leq aE[Q]M[Q]^{\sigma}, \tag{3.94}$$

where $\sigma = \frac{1-s_c}{s_c}$. Furthermore, (3.92) implies by Lemma 3.21 (ii) that

$$\|\nabla u_{n,0}\|_{L^2}^2 \leq w^{\frac{1}{s_c}} \|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{2\sigma},$$

where $w = \frac{E[u_n]^{s_c} M[u_n]^{1-s_c}}{E[Q]^{s_c} M[Q]^{1-s_c}}$, thus we deduce from (3.94) and $\|u_n\|_{L^2} = 1$ that $w^{\frac{1}{s_c}} \leq a$ which implies

$$\|\nabla u_{n,0}\|_{L^2}^2 \leq a\|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{2\sigma}. \tag{3.95}$$

On the other hand, the linear profile decomposition (Proposition 3.25) applied to $u_{n,0}$, which is a uniformly bounded sequence in $H^1(\mathbb{R}^N)$ by (3.95), yields

$$u_{n,0}(x) = \sum_{j=1}^{M} U(-t_n^j)\psi^j(x) + W_n^M(x), \tag{3.96}$$

where M will be taken large later. It follows from the Pythagorean expansion (3.80), with $s = 0$, that for all $M \in \mathbb{N}$

$$\sum_{j=1}^{M} \|\psi^j\|_{L^2}^2 + \lim_{n\to+\infty} \|W_n^M\|_{L^2}^2 \leq \lim_{n\to+\infty} \|u_{n,0}\|_{L^2}^2 = 1, \tag{3.97}$$

this implies that

$$\sum_{j=1}^{M} \|\psi^j\|_{L^2}^2 \leq 1. \tag{3.98}$$

In addition, another application of (3.80), with $s = 1$, and (3.95) lead to

$$\sum_{j=1}^{M} \|\nabla\psi^j\|_{L^2}^2 + \lim_{n\to+\infty} \|\nabla W_n^M\|_{L^2}^2 \leq \lim_{n\to+\infty} \|\nabla u_{n,0}\|_{L^2}^2 \leq a\|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^{2\sigma}, \tag{3.99}$$

and so

$$\|\nabla\psi^j\|_{L^2}^{s_c} \leq a^{\frac{s_c}{2}}\|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c}, \;\; j = 1, \dots, M. \tag{3.100}$$

Let $\{t_n^j\}_{n\in\mathbb{N}}$ be the sequence given by Proposition 3.25. From (3.98), (3.100) and the fact that $U(t)$ is an isometry in $L^2(\mathbb{R}^N)$ and $\dot{H}^1(\mathbb{R}^N)$ we deduce

$$\|U(-t_n^j)\psi^j\|_{L_x^2}^{1-s_c}\|\nabla U(-t_n^j)\psi^j\|_{L_x^2}^{s_c} \leq a^{\frac{s_c}{2}}\|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c}.$$

Now, Lemma 3.21 (i) yields

$$E[U(-t_n^j)\psi^j] \geq c(N, b, \alpha)\|\nabla\psi^j\|_{L^2} \geq 0 \tag{3.101}$$

A complete similar analysis also gives, for all $M \in \mathbb{N}$,

$$\lim_{n\to+\infty}\|W_n^M\|_{L^2}^2 \leq 1,$$

$$\lim_{n\to+\infty}\|\nabla W_n^M\|_{L^2}^{s_c} \leq a^{\frac{s_c}{2}}\|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c},$$

and for n large enough (depending on M)

$$E[W_n^M] \geq 0. \tag{3.102}$$

The energy Pythagorean expansion (Proposition 3.27) allows us to deduce

$$\sum_{j=1}^{M}\lim_{n\to+\infty}E[U(-t_n^j)\psi^j] + \lim_{n\to+\infty}E[W_n^M] = \lim_{n\to+\infty}E[u_{n,0}] = \delta_c^{\frac{1}{s_c}},$$

which implies, by (3.101) and (3.102), that

$$\lim_{n\to\infty}E[U(-t_n^j)\psi^j] \leq \delta_c^{\frac{1}{s_c}}, \text{ for all } \; j = 1, ..., M. \tag{3.103}$$

Now, if more than one $\psi^j \neq 0$, we show a contradiction and thus the profile expansion given by (3.96) is reduced to the case that only one profile is nonzero. In fact, if more than one $\psi^j \neq 0$, then by (3.97) we must have $M[\psi^j] < 1$ for each j. Passing to a subsequence, if necessary, we have two cases to consider:

Case 1. If for a given j, $t_n^j \to t^*$ finite (at most only one such j exists by (3.78)), then the continuity of the linear flow in $H^1(\mathbb{R}^N)$ yields

$$U(-t_n^j)\psi^j \to U(-t^*)\psi^j \quad \text{strongly in } H^1. \tag{3.104}$$

Let us denote the solution of (3.1) with initial data ψ by INLS$(t)\psi$. Set $\widetilde{\psi}^j = \text{INLS}(t^*)(U(-t^*)\psi^j)$ so that $\text{INLS}(-t^*)\widetilde{\psi}^j = U(-t^*)\psi^j$. Since the set

$$\mathcal{K} := \left\{ u_0 \in H^1(\mathbb{R}^N) : \text{ relations (3.3) and (3.4) hold } \right\}$$

is closed in $H^1(\mathbb{R}^N)$ then $\widetilde{\psi}^j \in \mathcal{K}$ and therefore INLS$(t)\widetilde{\psi}^j$ is a global solution by Theorem 3.1. Moreover from (3.28), (3.103) and the fact that $M[\psi^j] < 1$ we have

$$\|\widetilde{\psi}^j\|_{L^2_x}^{1-s_c} \|\nabla \widetilde{\psi}^j\|_{L^2_x}^{s_c} \leq \|\nabla Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{1-s_c}$$

and

$$E[\widetilde{\psi}^j]^{s_c} M[\widetilde{\psi}^j]^{1-s_c} < \delta_c.$$

So, the definition of δ_c (see (3.10)) implies

$$\|\text{INLS}(t)\widetilde{\psi}^j\|_{S(\dot{H}^{s_c})} < +\infty. \tag{3.105}$$

Finally, from (3.104) it is easy to see

$$\lim_{n\to+\infty} \|\text{INLS}(-t_n^j)\widetilde{\psi}^j - U(-t_n^j)\psi^j\|_{H^1_x} = 0. \tag{3.106}$$

Case 2. If $|t_n^j| \to +\infty$ then by Lemma 3.22 (iii)

$$\left\| |x|^{-b} |U(-t_n^j)\psi^j|^{\alpha+2} \right\|_{L^1_x} \to 0,$$

and thus, by the definition of Energy (4) and the fact that $U(t)$ is an isometry in $\dot{H}^1(\mathbb{R}^N)$, we deduce

$$\left(\frac{1}{2}\|\nabla\psi^j\|_{L^2}^2\right)^{s_c} = \lim_{n\to\infty} E[U(-t_n^j)\psi^j]^{s_c} \leq \delta_c < E[Q]^{s_c}M[Q]^{1-s_c}, \qquad (3.107)$$

where we have used (3.103). Therefore, by the existence of wave operator, Proposition 3.23 with $\lambda = (\frac{2\alpha s_c}{N\alpha+2b})^{\frac{s_c}{2}} < 1$ (see also Remark 3.24), there exists $\widetilde{\psi}^j \in H^1(\mathbb{R}^N)$ such that

$$M[\widetilde{\psi}^j] = M[\psi^j] \quad \text{and} \quad E[\widetilde{\psi}^j] = \frac{1}{2}\|\nabla\psi^j\|_{L^2}^2, \qquad (3.108)$$

$$\|\nabla \text{INLS}(t)\widetilde{\psi}^j\|_{L_x^2}^{s_c}\|\widetilde{\psi}^j\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c} \qquad (3.109)$$

and (3.106) also holds in this case.

Since $M[\psi^j] < 1$ and using (3.107)-(3.108), we get $E[\widetilde{\psi}^j]^{s_c}M[\widetilde{\psi}^j]^{1-s_c} < \delta_c$. Hence, the definition of δ_c together with (3.109) also lead to (3.105).

To sum up, in either case, we obtain a new profile $\widetilde{\psi}^j$ for the given ψ^j such that (3.106) (3.105) hold.

Next, we define

$$u_n(t) = \text{INLS}(t)u_{n,0},$$

$$v^j(t) = \text{INLS}(t)\widetilde{\psi}^j,$$

$$\widetilde{u}_n(t) = \sum_{j=1}^{M} v^j(t-t_n^j),$$

and

$$\widetilde{W}_n^M = \sum_{j=1}^{M}\left[U(-t_n^j)\psi^j - \text{INLS}(-t_n^j)\widetilde{\psi}^j\right] + W_n^M. \qquad (3.110)$$

Then $\widetilde{u}_n(t)$ solves the following equation

$$i\partial_t\widetilde{u}_n + \Delta\widetilde{u}_n + |x|^{-b}|\widetilde{u}_n|^\alpha\widetilde{u}_n = e_n^M, \qquad (3.111)$$

where

$$e_n^M = |x|^{-b}\left(|\widetilde{u}_n|^\alpha\widetilde{u}_n - \sum_{j=1}^{M}|v^j(t-t_n^j)|^\alpha v^j(t-t_n^j)\right). \qquad (3.112)$$

Also note that by definition of $\widetilde{W}_n^M$ in (3.110) and (3.96)we can write

$$u_{n,0} = \sum_{j=1}^{M} \text{INLS}(-t_n^j)\widetilde{\psi}^j + \widetilde{W}_n^M,$$

so it is easy to see $u_{n,0} - \widetilde{u}_n(0) = \widetilde{W}_n^M$, then combining (3.110) and the Strichartz inequality (1.10), we estimate

$$\|U(t)\widetilde{W}_n^M\|_{S(\dot{H}^{s_c})} \leq c\sum_{j=1}^{M} \|\text{INLS}(-t_n^j)\widetilde{\psi}^j - U(-t_n^j)\psi^j\|_{H^1} + \|U(t)W_n^M\|_{S(\dot{H}^{s_c})},$$

which implies

$$\lim_{M\to+\infty}\left[\lim_{n\to+\infty}\|U(t)(u_{n,0}-\widetilde{u}_{n,0})\|_{S(\dot{H}^{s_c})}\right]=0, \tag{3.113}$$

where we used (3.79) and (3.106).

The idea now is to approximate u_n by $\widetilde{u}_n$. Therefore, from the long time perturbation theory (Proposition 3.19) and (3.105) we conclude

$$\|u_n\|_{S(\dot{H}^{s_c})} < +\infty,$$

for n large enough, which is a contradiction with (3.93). Indeed, we assume the following two claims to conclude the proof.

Claim 1. For each M and $\varepsilon > 0$, there exists $n_0 = n_0(M,\varepsilon)$ such that

$$n > n_0 \;\;\Rightarrow\;\; \|e_n^M\|_{S'(\dot{H}^{-s_c})} + \|e_n^M\|_{S'(L^2)} + \|\nabla e_n^M\|_{S'(L^2)} \leq \varepsilon. \tag{3.114}$$

Claim 2. There exist $L > 0$ and $S > 0$ independent of M such that for any M, there exists $n_1 = n_1(M)$ such that

$$n > n_1 \;\;\Rightarrow\;\; \|\widetilde{u}_n\|_{S(\dot{H}^{s_c})} \leq L \;\text{ and }\; \|\widetilde{u}_n\|_{L_t^\infty H_x^1} \leq S. \tag{3.115}$$

Note that by (3.113), there exists $M_1 = M_1(\varepsilon)$ such that for each $M > M_1$ there exists $n_2 = n_2(M)$ such that

$$n > n_2 \;\;\Rightarrow\;\; \|U(t)(u_{n,0}-\widetilde{u}_{n,0})\|_{S(\dot{H}^{s_c})} \leq \varepsilon,$$

with $\varepsilon < \varepsilon_1$ as in Proposition 3.19. Thus, if the two claims hold true, by Proposition 3.19, for M large enough and $n > \max\{n_0, n_1, n_2\}$, we obtain $\|u_n\|_{S(\dot{H}^{s_c})} < +\infty$, reaching the desired contradiction .

Up to now, we have reduced the profile expansion to the case where $\psi^1 \neq 0$ and $\psi^j = 0$ for all $j \geq 2$. We now begin to show the existence of a critical solution. From the same arguments as the ones in the previous case (the case when more than one $\psi^j \neq 0$), we can find $\widetilde{\psi}^1$ such that

$$u_{n,0} = \mathrm{INLS}(-t_n^1)\widetilde{\psi}^1 + \widetilde{W}_n^M,$$

with

$$M[\widetilde{\psi}^1] = M[\psi^1] \leq 1 \tag{3.116}$$

$$E[\widetilde{\psi}^1]^{s_c} = \left(\frac{1}{2}\|\nabla\psi^1\|_{L^2}^2\right)^{s_c} \leq \delta_c \tag{3.117}$$

$$\|\nabla \mathrm{INLS}(t)\widetilde{\psi}^1\|_{L^2_x}^{s_c}\|\widetilde{\psi}^1\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c} \tag{3.118}$$

and

$$\lim_{n\to+\infty}\|U(t)(u_{n,0} - \widetilde{u}_{n,0})\|_{S(\dot{H}^{s_c})} = \lim_{n\to+\infty}\|U(t)\widetilde{W}_n^M\|_{S(\dot{H}^{s_c})} = 0. \tag{3.119}$$

Let $\widetilde{\psi}^1 = u_{c,0}$ and u_c be the global solution to (3.1) (in view of Theorem 3.1 and inequalities (3.116)-(3.118)) with initial data $\widetilde{\psi}^1$, that is, $u_c(t) = \mathrm{INLS}(t)\widetilde{\psi}^1$. We claim that

$$\|u_c\|_{S(\dot{H}^{s_c})} = +\infty. \tag{3.120}$$

Indeed, suppose, by contradiction, that $\|u_c\|_{S(\dot{H}^{s_c})} < +\infty$. Let,

$$\widetilde{u}_n(t) = \mathrm{INLS}(t - t_n^j)\widetilde{\psi}^1,$$

then

$$\|\widetilde{u}_n(t)\|_{S(\dot{H}^{s_c})} = \|\mathrm{INLS}(t-t_n^j)\widetilde{\psi}^1\|_{S(\dot{H}^{s_c})} = \|\mathrm{INLS}(t)\widetilde{\psi}^1\|_{S(\dot{H}^{s_c})} = \|u_c\|_{S(\dot{H}^{s_c})} < +\infty.$$

Furthermore, it follows from (3.116)-(3.119) that

$$\sup_{t\in\mathbb{R}}\|\widetilde{u}_n\|_{H^1_x}=\sup_{t\in\mathbb{R}}\|u_c\|_{H^1_x}<+\infty.$$

and

$$\|U(t)(u_{n,0}-\widetilde{u}_{n,0})\|_{S(\dot{H}^{s_c})}\leq\varepsilon,$$

for n large enough. Hence, by the long time perturbation theory (Proposition 3.19) with $e=0$, we obtain $\|u_n\|_{S(\dot{H}^{s_c})}<+\infty$, which is a contradiction with (3.93).

On the other hand, the relation (3.120) implies $E[u_c]^{s_c}M[u_c]^{1-s_c}=\delta_c$ (see (3.10)). Thus, we conclude from (3.116) and (3.117)

$$M[u_c]=1 \quad\text{and}\quad E[u_c]^{s_c}=\delta_c.$$

Also note that (3.118) implies (iii) in the statement of the Proposition 3.28.

To complete the proof it remains to establish Claims 1 and 2 (see (3.115) and (3.114)). To show these claims we use the same admissible pairs already used in Subsection 2.2.2.

$$\widehat{q}=\frac{4\alpha(\alpha+2-\theta)}{\alpha(N\alpha+2b)-\theta(N\alpha-4+2b)},\quad \widehat{r}\;=\;\frac{N\alpha(\alpha+2-\theta)}{\alpha(N-b)-\theta(2-b)},$$

and

$$\widetilde{a}\;=\;\frac{2\alpha(\alpha+2-\theta)}{\alpha[N(\alpha+1-\theta)-2+2b]-(4-2b)(1-\theta)},\quad \widehat{a}=\frac{2\alpha(\alpha+2-\theta)}{4-2b-(N-2)\alpha}.$$

Recall that $(\widehat{q},\widehat{r})$ is L^2-admissible, $(\widehat{a},\widehat{r})$ is $\dot{H}^{s_c}$-admissible and $(\widetilde{a},\widehat{r})$ is $\dot{H}^{-s_c}$-admissible.

Proof of Claim 1. First, we show that for each M and $\varepsilon>0$, there exists $n_0=n_0(M,\varepsilon)$ such that $\|e_n^M\|_{S'(\dot{H}^{-s_c})}<\frac{\varepsilon}{3}$. From (3.112) and (3.76) we deduce

$$\|e_n^M\|_{S'(\dot{H}^{-s_c})}\leq C_{\alpha,M}\sum_{j=1}^{M}\sum_{1\leq j\neq k\leq M}\left\||x|^{-b}|v^k|^{\alpha}|v^j|\right\|_{L_t^{\widetilde{a}'}L_x^{\widehat{r}'}}. \tag{3.121}$$

We claim that the norm in the right hand side of (3.121) goes to 0 as $n \to +\infty$. Indeed, using (2.55) with $s=1$ we have

$$\left\| |x|^{-b}|v^k|^{\alpha}|v^j| \right\|_{L_t^{\widetilde{a}'}L_x^{\widehat{r}'}} \leq c\|v^k\|^{\theta}_{L_t^{\infty}H_x^1}\left\| \|v^k(t-t_n^k)\|^{\alpha-\theta}_{L_x^{\widehat{r}}}\|v^j(t-t_n^j)\|_{L_x^{\widehat{r}}}\right\|_{L_t^{\widetilde{a}'}}. \tag{3.122}$$

Fix $1 \leq j \neq k \leq M$. Note that, $\|v^k\|_{H^1_x} < +\infty$ (see (3.108) - (3.109)) and by (3.105) v^j, $v^k \in S(\dot{H}^{s_c})$ and , so we can approximate v^j by functions of $C_0^{\infty}(\mathbb{R}^{N+1})$. Hence, defining

$$g_n(t) = \|v^k(t)\|^{\alpha-\theta}_{L_x^{\widehat{r}}}\|v^j(t-(t_n^j-t_n^k))\|_{L_x^{\widehat{r}}},$$

we deduce

(i) $g_n \in L_t^{\widetilde{a}'}$. Indeed, applying the Hölder inequality since $\frac{1}{\widetilde{a}'} = \frac{\alpha-\theta}{\widehat{a}} + \frac{1}{\widehat{a}}$ we get

$$\|g_n\|_{L_t^{\widetilde{a}'}} \leq \|v^k\|^{\alpha-\theta}_{L_t^{\widehat{a}}L_x^{\widehat{r}}}\|v^j\|_{L_t^{\widehat{a}}L_x^{\widehat{r}}} \leq \|v^k\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|v^j\|_{S(\dot{H}^{s_c})} < +\infty.$$

Furthermore, (3.78) implies that $g_n(t) \to 0$ as $n \to +\infty$.

(ii) $|g_n(t)| \leq K I_{supp(v^j)}\|v^k(t)\|^{\alpha-\theta}_{L_x^{\widehat{r}}} \equiv g(t)$ for all n, where $K > 0$ and $I_{supp(v^j)}$ is the characteristic function of $supp(v^j)$. Similarly as (i), we obtain

$$\|g\|_{L_t^{\widetilde{a}'}} \leq \|v^k\|^{\alpha-\theta}_{L_t^{\widehat{a}}L_x^{\widehat{r}}}\|I_{supp(v^j)}\|_{L_t^{\widehat{a}}L_x^{\widehat{r}}} < +\infty.$$

That is, $g \in L_t^{\widetilde{a}'}$.

Then, the dominated convergence theorem yields $\|g_n\|_{L_t^{\widetilde{a}'}} \to 0$ as $n \to +\infty$, and so combining this result with (3.122) we conclude the proof of the first estimate.

Next, we prove $\|e_n^M\|_{S'(L^2)} < \frac{\varepsilon}{3}$. Using again the elementary inequality (3.76) we estimate

$$\|e_n^M\|_{S'(L^2)} \leq C_{\alpha,M}\sum_{j=1}^{M}\sum_{1\leq j\neq k\leq M}\left\| |x|^{-b}|v^k|^{\alpha}|v^j| \right\|_{L_t^{\widehat{q}'}L_x^{\widehat{r}'}}.$$

On the other hand, we have (see proof of Lemma 2.18 with $s=1$)

$$\begin{aligned}\left\||x|^{-b}|v^k|^\alpha|v^j\right\|_{L_t^{\widehat{q}'}L_x^{\widehat{r}'}} &\leq c\|v^k\|^\theta_{L_t^\infty H_x^1}\left\|\|v^k(t-t_n^k)\|^{\alpha-\theta}_{L_x^{\widehat{r}}}\|v^j(t-t_n^j)\|_{L_x^{\widehat{r}}}\right\|_{L_t^{\widehat{q}'}}\\ &\leq c\|v^k\|^\theta_{L_t^\infty H_x^1}\|v^k\|^{\alpha-\theta}_{L_t^{\widehat{a}}L_x^{\widehat{r}}}\|v^j\|_{L_t^{\widehat{q}}L_x^{\widehat{r}}}\\ &\leq c\|v^k\|^\theta_{L_t^\infty H_x^1}\|v^k\|^{\alpha-\theta}_{S(\dot{H}^{s_c})}\|v^j\|_{S(L^2)}.\end{aligned}$$

Since $v^j\in S(\dot{H}^{s_c})$ then by (3.17) the norms $\|v^j\|_{S(L^2)}$ and $\|\nabla v^j\|_{S(L^2)}$ are bounded quantities. This implies that the right hand side of the last inequality is finite. Therefore, using the same argument as in the previous case we get

$$\left\|\|v^k(t-t_n^k)\|^{\alpha-\theta}_{L_x^{\widehat{r}}}\|v^j(t-t_n^j)\|_{L_x^{\widehat{r}}}\right\|_{L_t^{\widehat{q}'}}\to 0,$$

as $n\to+\infty$, which lead to $\left\||x|^{-b}|v^k|^\alpha|v^j\right\|_{L_t^{\widehat{q}'}L_x^{\widehat{r}'}}\to 0$.

Finally, we prove $\|\nabla e_n^M\|_{S'(L^2)}<\frac{\varepsilon}{3}$. Note that

$$\begin{aligned}\nabla e_n^M &= \nabla(|x|^{-b})\left(f(\widetilde{u}_n)-\sum_{j=1}^M f(v^j)\right)+|x|^{-b}\nabla\left(f(\widetilde{u}_n)-\sum_{j=1}^M f(v^j)\right)\\ &\equiv R_n^1+R_n^2, \qquad (3.123)\end{aligned}$$

where $f(v)=|v|^\alpha v$. First, we consider R_n^1. The estimate (3.76) yields

$$\|R_n^1\|_{S'(L^2)}\leq c\,C_{\alpha,M}\sum_{j=1}^M\sum_{1\leq j\neq k\leq M}\left\||x|^{-b-1}|v^k|^\alpha|v^j|\right\|_{L_t^{\widehat{q}'}L_x^{\widehat{r}'}}$$

and by Remark 3.13 we deduce that $\left\||x|^{-b-1}|v^k|^\alpha|v^j|\right\|_{L_t^{\widehat{q}'}L_x^{\widehat{r}'}}$ is finite, then by the same argument as before we have

$$\left\||x|^{-b-1}|v^k(t-t_n^k)|^\alpha|v^j(t-t_n^j)|\right\|_{L_t^{\widehat{q}'}L_x^{\widehat{r}'}}\to 0 \text{ as } n\to+\infty.$$

Therefore, the last two relations yield $\|R_n^1\|_{S'(L^2)}\to 0$ as $n\to+\infty$.

On the other hand, observe that

$$\begin{aligned}\nabla(f(\widetilde{u}_n)-\sum_{j=1}^M f(v^j)) &= f'(\widetilde{u}_n)\nabla\widetilde{u}_n-\sum_{j=1}^M f'(v^j)\nabla v^j\\ &= \sum_{j=1}^M(f'(\widetilde{u}_n)-f'(v^j))\nabla v^j. \qquad (3.124)\end{aligned}$$

Since (by Remark 1.15)

$$|f'(\widetilde{u}_n) - f'(v^j)| \leq C_{\alpha,M} \sum_{1\leq k\neq j\leq M} |v^k|(|v^j|^{\alpha-1} + |v^k|^{\alpha-1}) \quad \text{if} \quad \alpha > 1$$

and

$$|f'(\widetilde{u}_n) - f'(v^j)| \leq C_{\alpha,M} \sum_{1\leq k\neq j\leq M} |v^k|^{\alpha} \quad \text{if} \quad \alpha \leq 1,$$

we deduce using the last two relations together with (3.123) and (3.124)

$$\|R_n^2\|_{S'(L^2)} \lesssim \sum_{j=1}^{M} \sum_{1\leq k\neq j\leq M} \left\| |x|^{-b} |v^k| (|v^j|^{\alpha-1} + |v^k|^{\alpha-1}) |\nabla v^j| \right\|_{S'(L^2)} \quad \text{if} \quad \alpha > 1,$$

and

$$\|R_n^2\|_{S'(L^2)} \lesssim \sum_{j=1}^{M} \sum_{1\leq k\neq j\leq M} \left\| |x|^{-b} |v^k|^{\alpha} |\nabla v^j| \right\|_{S'(L^2)} \quad \text{if} \quad \alpha \leq 1.$$

Therefore, from Lemma 3.10 (see also Remark 3.11) we have that the right hand side of the last two inequalities are finite quantities and, by an analogous argument as before, we conclude that

$$\|R_n^2\|_{S'(L^2)} \to 0 \quad \text{as} \quad n \to +\infty.$$

This completes the proof of Claim 1.

Proof of Claim 2. First, we show that $\|\widetilde{u}_n\|_{L_t^\infty H_x^1}$ and $\|\widetilde{u}_n\|_{L_t^\gamma L_x^\gamma}$ are bounded quantities where $\gamma = \frac{2(N+2)}{N}$. Indeed, we already know (see (3.98) and (3.99)) that there exists C_0 such that

$$\sum_{j=1}^{\infty} \|\psi^j\|_{H_x^1}^2 \leq C_0,$$

then we can choose $M_0 \in \mathbb{N}$ large enough such that

$$\sum_{j=M_0}^{\infty} \|\psi^j\|_{H_x^1}^2 \leq \frac{\delta}{2}, \tag{3.125}$$

where $\delta > 0$ is a sufficiently small.

Fix $M \geq M_0$. From (3.106), there exists $n_1(M) \in \mathbb{N}$ where for all $n > n_1(M)$, we obtain

$$\sum_{j=M_0}^{M} \|\text{INLS}(-t_n^j)\widetilde{\psi}^j\|^2_{H^1_x} \leq \delta,$$

where we have used (3.125). This is equivalent to

$$\sum_{j=M_0}^{M} \|v^j(-t_n^j)\|^2_{H^1_x} \leq \delta. \tag{3.126}$$

Therefore, by the Small Data Theory (Proposition 3.14)[14]

$$\sum_{j=M_0}^{M} \|v^j(t-t_n^j)\|^2_{L^\infty_t H^1_x} \leq c\delta \;\text{ for } n \geq n_1(M).$$

Note that,

$$\left\|\sum_{j=M_0}^{M} v^j(t-t_n^j)\right\|^2_{H^1_x} = \sum_{j=M_0}^{M} \|v^j(t-t_n^j)\|^2_{H^1_x} + 2\sum_{M_0 \leq l \neq k \leq M} \langle v^l(t-t_n^l), v^k(t-t_n^k)\rangle_{H^1_x},$$

so, for $l \neq k$ we deduce from (3.78) that (see [11, Corollary 4.4] for more details)

$$\sup_{t\in\mathbb{R}} |\langle v^l(t-t_n^l), v^k(t-t_n^k)\rangle_{H^1_x}| \to 0 \;\text{ as }\; n \to +\infty.$$

Hence, since $\|v^j\|_{L^\infty_t H^1_x}$ is bounded (see (3.108) - (3.109)), by definition of $\widetilde{u}_n$ there exists $S > 0$ (independent of M) such that

$$\sup_{t\in\mathbb{R}} \|\widetilde{u}_n\|^2_{H^1_x} \leq S \;\text{ for }\; n > n_1(M). \tag{3.127}$$

We now show $\|\widetilde{u}_n\|_{L^\gamma_t L^\gamma_x} \leq L_1$. Using again (3.126) with δ small enough and the Small Data Theory (noting that (γ, γ) is L^2-admissible and $\gamma > 2$), we have

$$\sum_{j=M_0}^{M} \|v^j(t-t_n^j)\|^\gamma_{L^\gamma_t L^\gamma_x} \leq c\sum_{j=M_0}^{M} \|v^j(-t_n^j)\|^\gamma_{H^1_x} \leq c\sum_{j=M_0}^{M} \|v^j(-t_n^j)\|^2_{H^1_x} \leq c\delta, \tag{3.128}$$

[14] Recall that the pair $(\infty, 2)$ is L^2-admissible (see (1.1)).

for $n \geq n_1(M)$.

On the other hand, in view of (3.75)

$$\left\| \sum_{j=M_0}^{M} v^j(t-t_n^j) \right\|_{L_t^\gamma L_x^\gamma}^{\gamma} \leq \sum_{j=M_0}^{M} \|v^j\|_{L_t^\gamma L_x^\gamma}^{\gamma} + C_M \sum_{M_0 \leq j \neq k \leq M} \int_{\mathbb{R}^{N+1}} |v^j||v^k||v^k|^{\gamma-2}$$

for all $M > M_0$. Observe that, given j such that $M_0 \leq j \neq k \leq M$, the Hölder inequality yields

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} |v^j||v^k||v^k|^{\gamma-2} &\leq \|v^k(t-t_n^k)\|_{L_t^\gamma L_x^\gamma} \left(\int_{\mathbb{R}^{N+1}} |v^j|^{\frac{\gamma}{2}} |v^k|^{\frac{\gamma}{2}} \right)^{\frac{2}{\gamma}} \\ &\leq c\|v^j(-t_n^j)\|_{H_x^1} \left(\int_{\mathbb{R}^{N+1}} |v^j|^{\frac{\gamma}{2}} |v^k|^{\frac{\gamma}{2}} \right)^{\frac{2}{\gamma}}. \quad (3.129) \end{aligned}$$

Since v^j and $v^k \in L_t^\gamma L_x^\gamma$ we have that the right hand side of (3.129) is bounded and so by similar arguments as in the previous claim, we deduce from (3.78) that the integral in the right hand side of the previous inequality goes to 0 as $n \to +\infty$ (another proof of this fact can be found in [11, Lemma 4.5]). This implies that there exists L_1 (independent of M) such that

$$\|\widetilde{u}_n\|_{L_t^\gamma L_x^\gamma} \leq \sum_{j=1}^{M_0} \|v^j\|_{L_t^\gamma L_x^\gamma} + \left\| \sum_{j=M_0}^{M} v^j \right\|_{L_t^\gamma L_x^\gamma} \leq L_1 \quad \text{for } n \geq n_1(M), \quad (3.130)$$

where we have used (3.128).

To complete the proof of the Claim 2 we show the following inequalities

$$\left\| |x|^{-b} |\widetilde{u}_n|^\alpha \widetilde{u}_n \right\|_{L_t^{\bar{a}'} L_x^{\bar{r}'}} \leq c\|\widetilde{u}_n\|_{L_t^\infty H_x^1}^{\theta} \|\widetilde{u}_n\|_{L_t^a L_x^r}^{\alpha-\theta+1} \quad (3.131)$$

and

$$\|\widetilde{u}_n\|_{L_t^a L_x^r} \leq \|\widetilde{u}_n\|_{L_t^\infty H_x^1}^{1-\frac{\gamma}{a}} \|\widetilde{u}_n\|_{L_t^\gamma L_x^\gamma}^{\frac{\gamma}{a}}, \quad (3.132)$$

where $\theta \in (0, \alpha)$ is a small enough number and the pairs $(\bar{a}, \bar{r})$ and (a, r) are $\dot{H}^{-s_c}$-admissible and $\dot{H}^{s_c}$-admissible, respectively.

Assuming the last two inequalities for a moment let us conclude the proof of the Claim 2. Indeed combining (3.127) and (3.130) we deduce from (3.132) that

$$\|\widetilde{u}_n\|_{L^a_tL^r_x} \leq S^{1-\frac{\gamma}{a}} L_1^{\frac{\gamma}{a}} = L_2, \quad \text{for } n \geq n_1(M).$$

Then, since $\widetilde{u}_n$ satisfies the perturbed equation (3.111) we can apply the Strichartz estimates (Lemma 1.14) and (3.131) to the integral formulation and conclude (using also Claim 1)

$$\begin{aligned} \|\widetilde{u}_n\|_{S(\dot{H}^{s_c})} &\leq c\|\widetilde{u}_{n,0}\|_{H^1_x} + c\left\||x|^{-b}|\widetilde{u}_n|^\alpha \widetilde{u}_n\right\|_{L^{\bar{a}'}_tL^{\bar{r}'}_x} + \|e_n^M\|_{S'(\dot{H}^{-s_c})} \\ &\leq cS + cL_2 + \varepsilon = L, \end{aligned}$$

for $n \geq n_1(M)$, which completes the proof of the Claim 2.

To prove the inequalities (3.131) and (3.132) we divide in two cases: $N \geq 3$ and $N = 2$, since we will make use of the Sobolev embeddings in Lemma 1.10.

Case $N \geq 3$: We begin defining

$$a = \frac{4\alpha(N+2)}{ND} \qquad\qquad r = \frac{2\alpha N(N+2)}{(4-2b)(N+2) - ND}$$

$$\bar{a} = \frac{4\alpha(N+2)}{4\alpha(N+2) - (\alpha+1-\theta)ND}$$

and

$$\bar{r} = \frac{2\alpha N(N+2)}{2(N+2)\left(\alpha(N-2) - (2-b)\right) + ND(\alpha+1-\theta)}$$

where $D = 4 - 2b - \alpha(N-2)$ and $\theta \in (0, \alpha)$ to be chosen below.

Note that $\bar{r}$ satisfies the condition (1.5), that is $\frac{2N}{N-2s_c} < \bar{r} < \frac{2N}{N-2}$. Indeed $\bar{r} < \frac{2N}{N-2}$ is equivalent to

$$\alpha(N+2)(N-2) < 2(N+2)\left(\alpha(N-2) - (2-b)\right) + ND(\alpha+1-\theta) \Leftrightarrow$$

$$(N+2)D < ND(\alpha+1-\theta) \Leftrightarrow$$

$$N(\alpha+1-\theta) > N+2 \;\Leftrightarrow\; \alpha N - 2 - \theta N > 0.$$

Since $\alpha > (4-2b)/N$ we have $\alpha N - 2 - \theta N > 4 - 2b - 2 - \theta N = 2(1-b) - \theta N$ and this is positive choosing $\theta < \frac{2(1-b)}{N}$ (here we use the condition $0 < b < \min\{\frac{N}{3}, 1\}$ to guarantee that θ can be chosen to be a positive number). Therefore, since $\alpha N - 2 - \theta N > 0$ one gets $\bar{r} < \frac{2N}{N-2}$. On the other hand, $\bar{r} > \frac{2N}{N-2s_c} = \frac{N\alpha}{2-b}$ is equivalent to

$$(N+2)(4-2b) > 2(N+2)\left(\alpha(N-2)-(2-b)\right) + ND(\alpha+1-\theta) \Leftrightarrow$$

$$2(N+2)D > ND(\alpha+1-\theta) \;\Leftrightarrow\; \alpha < \frac{N+4+\theta N}{N}.$$

Since $\alpha < 2_*$ (defined in (3.7)) we need to verify that $\frac{4-2b}{N-2} \leq \frac{N+4+\theta N}{N}$ for $N \geq 4$ and $3 - 2b \leq \frac{7+3\theta}{3}$ for $N = 3$. The first inequality is equivalent to $N(4-2b) \leq (N+4+\theta N)(N-2)$ and this is always true since $N \geq 4$. The second case is also true choosing[15] $\theta > \max\left\{0, \frac{2(1-3b)}{3}\right\}$.

Moreover, it is not difficult to see that (a, r) is $\dot{H}^{s_c}$-admissible[16] and $(\bar{a}, \bar{r})$ is $\dot{H}^{-s_c}$-admissible.

We first show the inequality (3.132). Indeed, by interpolation we have

$$\|\widetilde{u}_n\|_{L_t^a L_x^r} \leq \|\widetilde{u}_n\|^{1-\frac{\gamma}{a}}_{L_t^\infty L_x^p} \|\widetilde{u}_n\|^{\frac{\gamma}{a}}_{L_t^\gamma L_x^\gamma},$$

where

$$\frac{1}{r} = \left(1 - \frac{\gamma}{a}\right)\left(\frac{1}{p}\right) + \frac{1}{a},$$

[15] In the particular case when $N = 3$, we need to choose $\theta > 0$ such that $\max\left\{0, \frac{2(1-3b)}{3}\right\} < \theta < \frac{2(1-b)}{3}$, since also need $\theta < \frac{2(1-b)}{N}$ to obtain $\bar{r} < \frac{2N}{N-2}$.

[16] We notice that r satisfies (1.3), that is $\frac{2N}{N-2s_c} < r < \frac{2N}{N-2}$. Indeed $r < \frac{2N}{N-2}$ is equivalent to $\alpha(N^2-4) < 2(4-2b) + \alpha N(N-2) \Leftrightarrow \alpha < \frac{4-2b}{N-2}$. Moreover, $r > \frac{2N}{N-2s_c} = \frac{N\alpha}{2-b}$ is equivalent to $(N+2)(4-2b) > 2(4-2b) + \alpha N(N-2) \Leftrightarrow \alpha < \frac{4-2b}{N-2}$.

which is equivalent to (recall that $\gamma = \frac{2(N+2)}{N}$)

$$\begin{aligned}\left(1-\frac{\gamma}{a}\right)\left(\frac{1}{p}\right) &= \frac{1}{r}-\frac{1}{a}\\ \frac{2\alpha - D}{p} &= \frac{2(4-2b)-ND}{2N}\\ p &= \frac{2N}{N-2}.\end{aligned}$$

Hence, since $H^1 \hookrightarrow L^{\frac{2N}{N-2}}$ (see inequality (1.8) with $s=1$) we obtain the desired result. On the other hand, the proof of inequality (3.131) follows from similar ideas as the ones used in the previous chapter. We divide the estimate in B and B^C. Let $A \subset \mathbb{R}^N$ that can be the ball B or B^C. From the Hölder inequality we deduce

$$\begin{aligned}\left\||x|^{-b}|\widetilde{u}_n|^{\alpha}\widetilde{u}_n\right\|_{L_t^{\bar{a}'}L_x^{\bar{r}'}(A)} &\leq \left\|\||x|^{-b}\|_{L^d(A)}\|\widetilde{u}_n\|^{\theta}_{L_x^{\theta r_1}}\|\widetilde{u}_n\|^{\alpha+1-\theta}_{L_x^{(\alpha+1-\theta)r_2}}\right\|_{L_t^{\bar{a}'}}\\ &\leq \||x|^{-b}\|_{L^d(A)}\|\widetilde{u}_n\|^{\theta}_{L_x^{\theta r_1}}\|\widetilde{u}_n\|^{\alpha+1-\theta}_{L_t^{(\alpha+1-\theta)\bar{a}'}L_x^{(\alpha+1-\theta)r_2}}\\ &= \||x|^{-b}\|_{L^d(A)}\|\widetilde{u}_n\|^{\theta}_{L_x^{\theta r_1}}\|\widetilde{u}_n\|^{\alpha-\theta+1}_{L_t^aL_x^r},\end{aligned}$$

where

$$\frac{1}{\bar{r}'} = \frac{1}{d}+\frac{1}{r_1}+\frac{1}{r_2} \qquad r=(\alpha+1-\theta)r_2 \qquad a=(\alpha+1-\theta)\bar{a}'.$$

Using the values of a and $\bar{a}$ above defined, it is easy to check $a=(\alpha+1-\theta)\bar{a}'$. Moreover, to show that $\||x|^{-b}\|_{L^d(A)}$ is a bounded quantity we need $\frac{N}{d}-b>0$ for $A=B$ and $\frac{N}{d}-b<0$ for $A=B^C$, see Remark 1.17. Indeed, the last relation implies

$$\begin{aligned}\frac{N}{d}-b &= N-b-\frac{N}{r_1}-\frac{N}{\bar{r}}-\frac{N(\alpha+1-\theta)}{r}\\ &= N-b-\frac{N}{r_1}-\left(N-\frac{2-b}{\alpha}-\frac{2}{\bar{a}}\right)-(\alpha+1-\theta)\left(\frac{2-b}{\alpha}-\frac{2}{a}\right)\\ &= -b-\frac{N}{r_1}+\frac{2-b}{\alpha}+\frac{2}{\bar{a}}-(\alpha+1-\theta)\frac{2-b}{\alpha}+\frac{2(\alpha+1-\theta)}{a}\\ &= -2-\frac{N}{r_1}+\frac{\theta(2-b)}{\alpha}+\frac{2}{\bar{a}}+\frac{2}{\bar{a}'}\\ &= \frac{\theta(2-b)}{\alpha}-\frac{N}{r_1}.\end{aligned}$$

Choosing $\theta r_1 = 2$ we have $\frac{N}{d}-b=-\theta s_c<0$, so $|x|^{-b}\in L^d(B^C)$ and if $\theta r_1=\frac{2N}{N-2}$ then $\frac{N}{d}-b=\theta(1-s_c)>0$, i.e., $|x|^{-b}\in L^d(B)$. Therefore, since in both cases $\theta r_1\in\left[2,\frac{2N}{N-2}\right]$ by the Sobolev embedding (1.8) we complete the proof of the inequality (3.131).

Case $N=2$. In this case we use the following numbers

$$a=\frac{2\alpha(\alpha+1-\theta)}{2-b+\varepsilon}\qquad\qquad r=\frac{2\alpha(\alpha+1-\theta)}{(2-b)(\alpha-\theta)-\varepsilon}$$

and

$$\bar{a}=\frac{2\alpha}{2\alpha-(2-b)-\varepsilon}\qquad\qquad \bar{r}=\frac{2\alpha}{\varepsilon},$$

where $\theta\in(0,\alpha)$ and $\varepsilon>0$ are sufficiently small numbers. A simple computation shows that (a,r) is $\dot{H}^{s_c}$-admissible and $(\bar{a},\bar{r})$ is $\dot{H}^{-s_c}$ admissible.[17]

The interpolation inequality implies (in this case $\gamma=4$)

$$\|\widetilde{u}_n\|_{L^a_tL^r_x}\leq\|\widetilde{u}_n\|^{1-\frac{\gamma}{a}}_{L^\infty_tL^p_x}\|\widetilde{u}_n\|^{\frac{\gamma}{a}}_{L^\gamma_tL^\gamma_x},$$

[17]Note that $\bar{r}$ satisfies assumption (1.5) with $N=2$, that is $\frac{2}{1-2s}=\frac{2\alpha}{2-b}<\bar{r}\leq\left((\frac{2}{1+s_c})^+\right)'$. The first inequality is equivalent to $\frac{2\alpha}{\varepsilon}>\frac{2\alpha}{2-b}$ and this holds since $2-b-\varepsilon>0$. On the other hand by the definition of $\left((\frac{2}{1+s_c})^+\right)'$ (see (1.4)) we conclude $\bar{r}=\frac{2\alpha}{\varepsilon}\leq\left((\frac{2}{1+s_c})^+\right)'$.

where

$$\frac{1}{r} = \left(1 - \frac{\gamma}{a}\right)\left(\frac{1}{p}\right) + \frac{1}{a}.$$

This is equivalent to

$$\begin{aligned}\left(1-\frac{4}{a}\right)\left(\frac{2}{p}\right) &= \frac{2}{r} - \frac{2}{a} \\ &= \frac{2-b}{\alpha} - \frac{4}{a} \\ &= \frac{(2-b)(\alpha-\theta+1) - 2(2-b-\varepsilon)}{\alpha(\alpha-\theta+1)}.\end{aligned}$$

So we obtain

$$p = 2\frac{\alpha(\alpha-\theta+1) - 2[(2-b)-\varepsilon]}{(2-b)(\alpha+1-\theta) - 2[(2-b)-\varepsilon]}.$$

Since we are assuming $\alpha > 2-b$ we have $p > 2$, thus by the Sobolev embedding $H^1 \hookrightarrow L^p$ (see (1.7) with $N = 2$) the inequality (3.132) holds. To show the inequality (3.131) we use the same argument as the previous case, that is

$$\begin{aligned}\left\| |x|^{-b} |\widetilde{u}_n|^\alpha \widetilde{u}_n \right\|_{L_t^{\bar{a}'} L_x^{\bar{r}'}(A)} &\leq \| |x|^{-b} \|_{L^d(A)} \|\widetilde{u}_n\|^{\theta}_{L_x^{\theta r_1}} \|\widetilde{u}_n\|^{\alpha+1-\theta}_{L_t^{(\alpha+1-\theta)\bar{a}'} L_x^{(\alpha+1-\theta)r_2}} \\ &= \| |x|^{-b} \|_{L^d(A)} \|\widetilde{u}_n\|^{\theta}_{L_x^{\theta r_1}} \|\widetilde{u}_n\|^{\alpha-\theta+1}_{L_t^a L_x^r},\end{aligned}$$

where $A = B$ or B^C and

$$\frac{1}{\bar{r}'} = \frac{1}{d} + \frac{1}{r_1} + \frac{1}{r_2} \qquad r = (\alpha+1-\theta)r_2 \qquad a = (\alpha+1-\theta)\bar{a}'.$$

Moreover, we obtain

$$\begin{aligned}\frac{2}{d} - b &= 2 - b - \frac{2}{r_1} - \frac{2}{\bar{r}} - \frac{2(\alpha+1-\theta)}{r} \\ &= \frac{\theta(2-b)}{\alpha} - \frac{2}{r_1}.\end{aligned}$$

If we choose $\theta r_1 \in \left(2, \frac{2\alpha}{2-b}\right)$ then $\frac{2}{d} - b < 0$ (so $|x|^{-b} \in L^d(B^C)$) and if $\theta r_1 \in \left(\frac{2\alpha}{2-b}, +\infty\right)$ we have $\frac{2}{d} - b < 0$ (so $|x|^{-b} \in L^d(B)$). Therefore $|x|^{-b} \in L^d(A)$

and so by the Sobolev inequality (1.7) with $s = 1$, we complete the proof of the inequality (3.131).

□

In the next proposition, we prove the precompactness of the flow associated to the critical solution u_c. The argument is very similar to Holmer-Roudenko [23, Proposition 5.5].

Proposition 3.29. (Precompactness of the flow of the critical solution) *Let u_c be as in Proposition 3.28 and define*

$$K = \{u_c(t) \ : \ t \in [0, +\infty)\} \subset H^1.$$

Then K is precompact in $H^1(\mathbb{R}^N)$.

Proof. Let $\{t_n\} \subseteq [0, +\infty)$ a sequence of times and $\phi_n = u_c(t_n)$ be a uniformly bounded sequence in $H^1(\mathbb{R}^N)$. We need to show that $u_c(t_n)$ has a subsequence converging in $H^1(\mathbb{R}^N)$. If $\{t_n\}$ is bounded, we can assume $t_n \to t^*$ finite, so by the continuity of the solution in $H^1(\mathbb{R}^N)$ the result is clear. Next, assume that $t_n \to +\infty$.

The linear profile expansion (Proposition 3.25) implies the existence of profiles ψ^j and a remainder W_n^M such that

$$u_c(t_n) = \sum_{j=1}^{M} U(-t_n^j)\psi^j + W_n^M,$$

with $|t_n^j - t_n^k| \to +\infty$ as $n \to +\infty$ for any $j \neq k$. Then, by the energy Pythagorean expansion (Proposition 3.27), we get

$$\sum_{j=1}^{M} \lim_{n\to+\infty} E[U(-t_n^j)\psi^j] + \lim_{n\to+\infty} E[W_n^M] = E[u_c] = \delta_c, \tag{3.133}$$

where we have used Proposition 3.28 (ii). This implies that

$$\lim_{n\to+\infty} E[U(-t_n^j)\psi^j] \leq \delta_c \quad \forall\, j,$$

since each energy in (3.133) is nonnegative by Lemma (3.21) (i).
Moreover, by (3.80) with $s = 0$ we obtain

$$\sum_{j=1}^{M} M[\psi^j] + \lim_{n\to+\infty} M[W_n^M] = M[u_c] = 1, \tag{3.134}$$

by Proposition 3.28 (i).

If more than one $\psi^j \neq 0$, similar to the proof in Proposition 3.28, we have a contradiction with the fact that $\|u_c\|_{S(\dot{H}^{s_c})} = +\infty$. Thus, we address the case that only $\psi^j = 0$ for all $j \geq 2$, and so

$$u_c(t_n) = U(-t_n^1)\psi^1 + W_n^M. \tag{3.135}$$

Also as in the proof of Proposition 3.28, we obtain that

$$M[\psi^1] = M[u_c] = 1 \quad \text{and} \quad \lim_{n\to+\infty} E[U(-t_n^1)\psi^1] = \delta_c, \tag{3.136}$$

and using (3.133), (3.134) together with (3.136), we deduce that

$$\lim_{n\to+\infty} M[W_n^M] = 0 \quad \text{and} \quad \lim_{n\to+\infty} E[W_n^M] = 0. \tag{3.137}$$

Thus, Lemma 3.21 (i) yields

$$\lim_{n\to+\infty} \|W_n^M\|_{H^1} = 0. \tag{3.138}$$

We claim now that t_n^1 converges to some finite t^* (up to a subsequence). In this case, since $U(-t_n^1)\psi^1 \to U(-t^*)\psi^1$ in $H^1(\mathbb{R}^N)$ and (3.138) holds, the relation (3.135) implies that $u_c(t_n)$ converges in $H^1(\mathbb{R}^N)$, concluding the proof.

Assume by contradiction that $|t_n^1| \to +\infty$, then we have two cases to consider. If $t_n^1 \to -\infty$, by (3.135)

$$\|U(t)u_c(t_n)\|_{S(\dot{H}^{s_c};[0,+\infty))} \leq \|U(t-t_n^1)\psi^1\|_{S(\dot{H}^{s_c};[0,+\infty))} + \|U(t)W_n^M\|_{S(\dot{H}^{s_c};[0,+\infty))}.$$

Next, note that since $t_n^1 \to -\infty$ we obtain

$$\|U(t-t_n^1)\psi^1\|_{S(\dot{H}^{s_c};[0,+\infty))} \le \|U(t)\psi^1\|_{S(\dot{H}^{s_c};[-t_n^j,+\infty))} \le \frac{1}{2}\delta,$$

and also

$$\|U(t)W_n^M\|_{S(\dot{H}^{s_c})} \le \frac{1}{2}\delta,$$

given $\delta > 0$ for n, M sufficiently large, where in the last inequality we have used (1.10) and (3.138). Hence,

$$\|U(t)u_c(t_n)\|_{S(\dot{H}^{s_c};[0,+\infty))} \le \delta.$$

Therefore, choosing $\delta > 0$ sufficiently small, by the small data theory (Proposition 3.14) we get that

$$\|u_c\|_{S(\dot{H}^{s_c})} \le 2\delta,$$

which is a contradiction with Proposition 3.28 (iv).

On the other hand, if $t_n^1 \to +\infty$, the same arguments also give that for n large,

$$\|U(t)u_c(t_n)\|_{S(\dot{H}^{s_c};(-\infty,0])} \le \delta,$$

and again the small data theory (Proposition 3.14) implies

$$\|u_c\|_{S(\dot{H}^{s_c};(-\infty,t_n])} \le 2\delta.$$

Since $t_n \to +\infty$ as $n \to +\infty$, from the last inequality we get $\|u_c\|_{S(\dot{H}^{s_c})} \le 2\delta$, which is also a contradiction. Thus, t_n^1 must converge to some finite t^*, completing the proof of Proposition 3.29.

□

3.6 Rigidity theorem

The main result of this section is a rigidity theorem, which implies that the critical solution u_c constructed in Section 3.5.2 must be identically zero and so reaching a contradiction in view of Proposition 3.28 (iv). Before proving this result, we begin showing some preliminary results that will help us in the proof.

Proposition 3.30. (Precompactness of the flow implies uniform localization) *Let u be a solution of* (3.1) *such that*

$$K = \{u(t) \;:\; t \in [0, +\infty)\}$$

is precompact in $H^1(\mathbb{R}^N)$. Then for each $\varepsilon > 0$, there exists $R > 0$ so that

$$\int_{|x|>R} |\nabla u(t,x)|^2 dx \leq \varepsilon, \text{ for all } 0 \leq t < +\infty. \tag{3.139}$$

Proof. The proof is similar to that in Holmer-Roudenko [23, Lemma 5.6]. If (3.139) does not hold, then there exists $\varepsilon > 0$ and a sequence $t_n \to +\infty$ such that, for each $n \in \mathbb{N}$,

$$\int_{|x|>n} |\nabla u(t_n, x)|^2 dx \geq 2\varepsilon. \tag{3.140}$$

The fact that K is precompact yields that there exists some $\phi \in H^1$ such that, up to a subsequence of t_n, $u(t_n) \to \phi$ in H^1, which implies

$$\int |\nabla(u(t_n) - \phi)|^2 dx < \frac{1}{4}\varepsilon. \tag{3.141}$$

On the other hand, since $\phi \in H^1$, taking n sufficiently large we can get

$$\int_{|x|>n} |\nabla \phi|^2 dx \leq \frac{1}{4}\varepsilon. \tag{3.142}$$

Thus, (3.141) and (3.142) lead to

$$\int_{|x|>n} |\nabla u(t,x)|^2 dx \leq 2\int |\nabla(u(t_n) - \phi)|^2 dx + 2\int_{|x|>n} |\nabla \phi|^2 dx < \varepsilon,$$

which is a contradiction with (3.140). □

We will also need the following local virial identity.

Proposition 3.31. (Virial identity) *Let $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \geq 0$ and $T > 0$. For $R > 0$ and $t \in [0, T]$ define*

$$z_R(t) = \int_{\mathbb{R}^N} R^2 \phi\left(\frac{x}{R}\right) |u(t,x)|^2 dx,$$

where u is a solution of (3.1)*. Then we have*

$$z_R'(t) = 2RIm \int_{\mathbb{R}^N} \nabla\phi\left(\frac{x}{R}\right) \cdot \nabla u \bar{u} dx \tag{3.143}$$

and

$$\begin{aligned} z_R''(t) = {} & 4\sum_{j,k} Re \int \frac{\partial u}{\partial_{x_k}} \frac{\partial \bar{u}}{\partial_{x_j}} \frac{\partial^2 \phi}{\partial x_k \partial x_j} \left(\frac{x}{R}\right) dx - \frac{1}{R^2} \int |u|^2 \Delta^2 \phi\left(\frac{x}{R}\right) dx \\ & - \frac{2\alpha}{\alpha+2} \int |x|^{-b} |u|^{\alpha+2} \Delta\phi\left(\frac{x}{R}\right) dx + \frac{4R}{\alpha+2} \int \nabla(|x|^{-b}) \cdot \nabla\phi\left(\frac{x}{R}\right) |u|^{\alpha+2} dx. \end{aligned} \tag{3.144}$$

Proof. We first compute z_R'. Note that

$$\partial_t |u|^2 = 2Re(u_t \bar{u}) = 2Im(iu_t \bar{u}).$$

Since u satisfies (3.1) and using integration by parts, we have

$$\begin{aligned} z_R'(t) &= 2Im \int R^2 \phi\left(\frac{x}{R}\right) iu_t \bar{u} dx \\ &= -2Im \int R^2 \phi\left(\frac{x}{R}\right) \left(\Delta u \bar{u} + |x|^{-b} |u|^{\alpha+2}\right) dx \\ &= -2Im \int R^2 \phi\left(\frac{x}{R}\right) \nabla \cdot (\nabla u \bar{u}) dx \\ &= 2RIm \int \nabla\phi\left(\frac{x}{R}\right) \cdot \nabla u \bar{u} dx. \end{aligned}$$

On the other hand, using again integration by parts and the fact that $z - \bar{z} = 2iImz$, we obtain

$$\begin{aligned}
z_R''(t) &= 2RIm\int \nabla\phi\left(\frac{x}{R}\right)\cdot(\bar{u}_t\nabla u+\bar{u}\nabla u_t)\,dx\\
&= 2RIm\left\{\sum_j\int \bar{u}_t\partial_{x_j}u\partial_{x_j}\phi\left(\frac{x}{R}\right)dx - u_t\partial_{x_j}\left(\bar{u}\partial_{x_j}\phi\left(\frac{x}{R}\right)\right)dx\right\}\\
&= 2RIm\left\{\sum_j 2iIm\int \bar{u}_t\partial_{x_j}u\partial_{x_j}\phi\left(\frac{x}{R}\right)dx - \int\frac{1}{R}u_t\bar{u}\partial_{x_j}^2\phi\left(\frac{x}{R}\right)dx\right\}\\
&= 4RI_1+2I_2,
\end{aligned}$$

where

$$I_1 = Im\sum_j\int \bar{u}_t\partial_{x_j}u\partial_{x_j}\phi\left(\frac{x}{R}\right) \quad\text{and}\quad I_2=-Im\sum_j\int u_t\bar{u}\partial_{x_j}^2\phi\left(\frac{x}{R}\right)dx.$$

We start considering I_2. Since u is a solution of (3.1) we get

$$\begin{aligned}
I_2 &= -Im\left\{\sum_{j,k}\int i\partial_{x_k}^2u\bar{u}\partial_{x_j}^2\phi\left(\frac{x}{R}\right)dx\right\}-\sum_j\int |x|^{-b}|u|^{\alpha+2}\partial_{x_j}^2\phi\left(\frac{x}{R}\right)dx\\
&= Im\left\{\sum_{j,k}\int i\left(|\partial_{x_k}u|^2\partial_{x_j}^2\phi\left(\frac{x}{R}\right)+\frac{1}{R}\partial_{x_k}u\bar{u}\frac{\partial^3\phi}{\partial x_k\partial x_j^2}\left(\frac{x}{R}\right)\right)dx\right\}\\
&\quad -\int |x|^{-b}|u|^{\alpha+2}\Delta\phi\left(\frac{x}{R}\right)dx\\
&= \int\left(|\nabla u|^2-|x|^{-b}|u|^{\alpha+2}\right)\Delta\phi\left(\frac{x}{R}\right)dx+\frac{1}{R}\sum_{j,k}Re\int\partial_{x_k}u\bar{u}\frac{\partial^3\phi}{\partial x_k\partial x_j^2}\left(\frac{x}{R}\right)dx,
\end{aligned}$$

where we have used integration by parts and the fact that $Im(iz)=Re(z)$. Furthermore, since $\partial_{x_k}|u|^2=2Re\left(\partial_{x_k}u\bar{u}\right)$ another integration by parts yields

$$\begin{aligned}
I_2 &= \int\left(|\nabla u|^2-|x|^{-b}|u|^{\alpha+2}\right)\Delta\phi\left(\frac{x}{R}\right)dx-\frac{1}{2R^2}\sum_{j,k}\int|u|^2\frac{\partial^4\phi}{\partial x_k^2\partial x_j^2}\left(\frac{x}{R}\right)dx\\
&= \int\left(|\nabla u|^2-|x|^{-b}|u|^{\alpha+2}\right)\Delta\phi\left(\frac{x}{R}\right)dx-\frac{1}{2R^2}\int|u|^2\Delta^2\phi\left(\frac{x}{R}\right)dx. \qquad (3.145)
\end{aligned}$$

Next, we deduce using the equation (3.1) and $Im(z)=-Im(\bar{z})$ that

$$
\begin{aligned}
I_1 &= -Im \sum_j u_t \partial_{x_j} \bar{u} \partial_{x_j} \phi\left(\frac{x}{R}\right) dx \\
&= -Imi \sum_j \left\{ \int \left(\Delta u + |x|^{-b}|u|^{\alpha} u\right) \partial_{x_j} \bar{u} \partial_{x_j} \phi\left(\frac{x}{R}\right) dx \right\} \\
&= -Re \sum_{j,k} \int \partial^2_{x_k} u \partial_{x_j} \bar{u} \partial_{x_j} \phi\left(\frac{x}{R}\right) dx - \sum_j \int |x|^{-b} \partial_{x_j} \phi\left(\frac{x}{R}\right) |u|^{\alpha} Re(\partial_{x_j} \bar{u} u) dx \\
&= -Re \sum_{j,k} \int \partial^2_{x_k} u \partial_{x_j} \bar{u} \partial_{x_j} \phi\left(\frac{x}{R}\right) dx - \frac{1}{\alpha+2} \sum_j \int |x|^{-b} \partial_{x_j} \phi\left(\frac{x}{R}\right) \partial_{x_j}(|u|^{\alpha+2}) dx \\
&\equiv A + B,
\end{aligned}
$$

where we have used $Im(iz) = Re(z)$ and $\partial_{x_j}(|u|^{\alpha+2}) = (\alpha+2)|u|^{\alpha} Re(\partial_{x_j}\bar{u}u)$. Moreover, since $\partial_{x_j}|\partial_{x_k} u|^2 = 2Re\left(\partial_{x_k} u \frac{\partial^2 \bar{u}}{\partial x_k \partial x_j}\right)$ and using integration by parts twice, we get

$$
\begin{aligned}
A &= Re \sum_{j,k} \left\{ \int \left(\partial_{x_j} \phi\left(\frac{x}{R}\right) \partial_{x_k} u \frac{\partial^2 \bar{u}}{\partial x_k \partial x_j} + \frac{1}{R} \partial_{x_k} u \partial_{x_j} \bar{u} \frac{\partial^2 \phi}{\partial x_j \partial x_k}\left(\frac{x}{R}\right) \right) dx \right\} \\
&= -\sum_{j,k} \frac{1}{2R} \int |\partial_{x_k} u|^2 \partial^2_{x_j} \phi\left(\frac{x}{R}\right) dx + \frac{1}{R} \sum_{i,j} Re \int \partial_{x_k} u \partial_{x_j} \bar{u} \frac{\partial^2 \phi}{\partial x_j \partial x_k}\left(\frac{x}{R}\right) dx \\
&= -\frac{1}{2R} \int |\nabla u|^2 \Delta\phi\left(\frac{x}{R}\right) dx + \frac{1}{R} \sum_{i,j} Re \int \partial_{x_k} u \partial_{x_j} \bar{u} \frac{\partial^2 \phi}{\partial x_j \partial x_k}\left(\frac{x}{R}\right) dx.
\end{aligned}
$$

Similarly, integrating by parts

$$
\begin{aligned}
B &= \frac{1}{\alpha+2} \sum_j \left(\int \partial_{x_j} \phi\left(\frac{x}{R}\right) \partial_{x_j}(|x|^{-b}) |u|^{\alpha+2} dx + \frac{1}{R} \int \partial^2_{x_j} \phi\left(\frac{x}{R}\right) |x|^{-b} |u|^{\alpha+2} dx \right) \\
&= \frac{1}{\alpha+2} \int \nabla\phi\left(\frac{x}{R}\right) \cdot \nabla(|x|^{-b}) |u|^{\alpha+2} dx + \frac{1}{R(\alpha+2)} \int \Delta\phi\left(\frac{x}{R}\right) |x|^{-b} |u|^{\alpha+2} dx.
\end{aligned}
$$

Therefore,

$$I_1 = -\frac{1}{2R}\int |\nabla u|^2 \Delta\phi\left(\frac{x}{R}\right)dx + \frac{1}{R}\sum_{i,j} Re\int \partial_{x_k} u \partial_{x_j}\bar{u}\frac{\partial^2\phi}{\partial x_j \partial x_k}\left(\frac{x}{R}\right)dx$$
$$+\frac{1}{\alpha+2}\int \nabla\phi\left(\frac{x}{R}\right)\cdot\nabla(|x|^{-b})|u|^{\alpha+2}dx + \frac{1}{R(\alpha+2)}\int \Delta\phi\left(\frac{x}{R}\right)|x|^{-b}|u|^{\alpha+2}dx. \tag{3.146}$$

Finally it is easy to check that combining (3.145) and (3.146), we obatin (3.144), which completes the proof. □

Finally, we apply the previous results to prove the rigidity theorem.

Theorem 3.32. (Rigidity) *Let $u_0 \in H^1(\mathbb{R}^N)$ satisfying*

$$E[u_0]^{s_c}M[u_0]^{1-s_c} < E[Q]^{s_c}M[Q]^{1-s_c}$$

and

$$\|\nabla u_0\|_{L^2}^{s_c}\|u_0\|_{L^2}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c}.$$

If the global $H^1(\mathbb{R}^N)$-solution u with initial data u_0 satisfies

$$K = \{u(t) \,:\, t\in[0,+\infty)\} \text{ is precompact in } H^1(\mathbb{R}^N)$$

then u_0 must vanish, i.e., $u_0 = 0$.

Proof. By Theorem 3.1 we have that u is global in $H^1(\mathbb{R}^N)$ and

$$\|\nabla u(t)\|_{L^2_x}^{s_c}\|u(t)\|_{L^2_x}^{1-s_c} < \|\nabla Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{1-s_c}. \tag{3.147}$$

On the other hand, let $\phi \in C_0^\infty$ be radial, with

$$\phi(x) = \begin{cases} |x|^2 & \text{for } |x|\le 1 \\ 0 & \text{for } |x|\ge 2. \end{cases}$$

Then, using (3.143), the Hölder inequality and (3.147) we obtain

$$|z_R'(t)| \;\leq\; cR\int_{|x|<2R} |\nabla u(t)||u(t)|dx \leq cR\|\nabla u(t)\|_{L^2}\|u(t)\|_{L^2} \lesssim cR.$$

Hence,

$$|z_R'(t) - z_R'(0)| \leq |z_R'(t)| + |z_R'(0)| \leq 2cR, \;\text{ for all }\; t>0. \tag{3.148}$$

The idea now is to obtain a lower bound for $z_R''(t)$ strictly greater than zero and reach a contradiction. Indeed, from the local virial identity (3.144)

$$\begin{aligned} z_R''(t) &= 4\sum_{j,k} Re\int \partial_{x_k}u\partial_{x_j}\bar{u}\frac{\partial^2\phi}{\partial x_k\partial x_j}\left(\frac{x}{R}\right)dx - \frac{1}{R^2}\int |u|^2\Delta^2\phi\left(\frac{x}{R}\right)dx \\ &- \frac{2\alpha}{\alpha+2}\int |x|^{-b}|u|^{\alpha+2}\Delta\phi\left(\frac{x}{R}\right)dx + \frac{4R}{\alpha+2}\int \nabla(|x|^{-b})\cdot\nabla\phi\left(\frac{x}{R}\right)|u|^{\alpha+2}dx \\ &= 8\|\nabla u\|^2_{L^2_x} - \frac{4(N\alpha+2b)}{\alpha+2}\left\||x|^{-b}|u|^{\alpha+2}\right\|_{L^1_x} + R(u(t)), \end{aligned} \tag{3.149}$$

where

$$\begin{aligned} R(u(t)) &= 4\sum_j Re\int\left(\partial^2_{x_j}\phi\left(\frac{x}{R}\right)-2\right)|\partial_{x_j}u|^2 + 4\sum_{j\neq k} Re\int\frac{\partial^2\phi}{\partial x_k\partial x_j}\left(\frac{x}{R}\right)\partial_{x_k}u\partial_{x_j}\bar{u} \\ &- \frac{1}{R^2}\int |u|^2\Delta^2\phi\left(\frac{x}{R}\right) + \frac{4R}{\alpha+2}\int \nabla(|x|^{-b})\cdot\nabla\phi\left(\frac{x}{R}\right)|u|^{\alpha+2} \\ &+ \int\left(\frac{-2\alpha(\Delta\phi\left(\frac{x}{R}\right)-2N)+8b}{\alpha+2}\right)|x|^{-b}|u|^{\alpha+2}. \end{aligned}$$

Since $\phi(x)$ is radial and $\phi(x)=|x|^2$ if $|x|\leq 1$, the sum of all terms in the definition of $R(u(t))$ integrating over $|x|\leq R$ is zero. Indeed, for the first three terms this is clear by the definition of $\phi(x)$. In the fourth term we have

$$\frac{8}{\alpha+2}\int_{|x|\leq R}\nabla(|x|^{-b})\cdot x|u|^{\alpha+2}dx = \frac{8}{\alpha+2}\int_{|x|\leq R} -b|x|^{-b}|u|^{\alpha+2}dx,$$

and adding the last term (also integrating over $|x|\leq R$) we get zero since $\Delta\phi = 2N$, if $|x|\leq R$. Therefore, for the integration on the region $|x|>R$, we have the following bound

$$|R(u(t))| \;\leq\; c\int_{|x|>R}\left(|\nabla u(t)|^2 + \frac{1}{R^2}|u(t)|^2 + |x|^{-b}|u(t)|^{\alpha+2}\right)dx$$

$$\leq \quad c\int_{|x|>R}\left(|\nabla u(t)|^2+\frac{1}{R^2}|u(t)|^2+\frac{1}{R^b}|u(t)|^{\alpha+2}\right)dx, \qquad (3.150)$$

where we have used that all derivatives of ϕ are bounded and $|R\partial_{x_j}(|x|^{-b})| \leq c|x|^{-b}$ if $|x| > R$.

Next we use that K is precompact in $H^1(\mathbb{R}^N)$. By Proposition 3.30, given $\varepsilon > 0$ there exists $R_1 > 0$ such that $\int_{|x|>R_1} |\nabla u(t)|^2 \leq \varepsilon$. Furthermore, by mass conservation (3), there exists $R_2 > 0$ such that $\frac{1}{R_2^2}\int_{|x|>R_2} |u(t)|^2 \leq \varepsilon$. Finally, by the Sobolev embedding $H^1 \hookrightarrow L^{\alpha+2}$, there exists R_3 such that $\frac{1}{R_3^b}\int_{|x|>R_3} |u(t)|^{\alpha+2} \leq c\varepsilon$ (recall that $\|u(t)\|_{H^1_x}$ is uniformly bounded for all $t > 0$ by (3.147) and Mass conservation (3)). Taking $R = \max\{R_1, R_2, R_3\}$ the inequality (3.150) implies

$$|R(u(t))| \leq c\int_{|x|>R}\left(|\nabla u(t)|^2+\frac{1}{R^2}|u(t)|^2+\frac{1}{R^b}|u(t)|^{\alpha+2}\right)dx \leq c\varepsilon. \quad (3.151)$$

On the other hand, Lemma 3.21 (iii), (3.149) and (3.151) yield

$$z_R''(t) \geq 16AE[u] - |R(u(t))| \geq 16AE[u] - c\varepsilon,$$

where $A = 1 - w^{\frac{\alpha}{2}}$ and $w = \frac{E[v]^{s_c}M[v]^{1-s_c}}{E[Q]^{s_c}M[Q]^{1-s_c}}$.
Now, choosing $\varepsilon = \frac{8A}{c}E[u]$, with c as in (3.151) we have

$$z_R''(t) \geq 8AE[u].$$

Thus, integrating the last inequality from 0 to t we deduce that

$$z_R'(t) - z_R'(0) \geq 8AE[u]t. \qquad (3.152)$$

Now sending $t \to \infty$ the left hand of (3.152) also goes to $+\infty$, however from (3.148) it must be bounded. Therefore, we have a contradiction unless $E[u] = 0$ which implies $u \equiv 0$ by Lemma 3.21 (i). □

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