Path coalgebra as a right adjoint functor

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Abstract

We define a certain variant of the category of quivers and construct the path coalgebra as a functor, the main tool for this process being the universal property of cotensor coalgebras. Then we construct a functor from the category of pointed coalgebras to this category of quivers, based on the Gabriel quiver of pointed coalgebras. With a relation on the morphisms of the category of pointed coalgebras we obtain an adjunction between these two functors.

Contents

Introduction

There is a well known and very useful correspondence between finite dimensional algebras and quivers.

A quiver $Q = (Q_0, Q_1)$ is an oriented graph with the set Q_0 of vertices and the set Q_1 of arrows. A path b in the quiver Q is the formal composition of arrows. For each vertex $i \in Q_0$ we associate a stationary path e_i of length 0. We denote by Q_l the set of all paths in Q of length l. For instance, the set of arrows Q_1 are paths of length 1.

For a fixed field k, the path algebra kQ of the quiver Q, is a graded k-algebra with direct sum decomposition

$$kQ = \bigoplus_{l \geq 0} kQ_l,$$

and the obvious addition. The multiplication is given by concatenation of the paths when it makes sense and 0 otherwise.

It is known that (see [ASS, II.1] for details)

- if Q_0 is finite, then the stationary paths form a complete set of primitive orthogonal idempotents of kQ;
- kQ has identity element if and only if Q_0 is finite. In this case $\sum_{i \in Q_0} e_i$ is the identity element of kQ;
- kQ is finite dimensional if and only if Q is finite and acyclic.

In the other direction, given a finite dimensional basic algebra, A, one can define a quiver (the Gabriel quiver of A). The vertices will be a complete set of primitive orthogonal idempotents, $Q_0 = \{e_1, e_2, \cdots, e_n\}$. The arrows between two vertices $e, f \in Q_0$ are a basis of the vector space $e\frac{J(A)}{J^2(A)}f$, where $J(A)$ is the Jacobson radical of A. In this way, $Q = (Q_0, Q_1)$ defines a quiver.

The path algebra over Q can be defined by a universal property similar to universal properties of free objects (see [ASS, Theorem II.1.8]). That suggests there is a stronger relationship between these two categories.

The operator that takes finite quivers to their path algebra is already a functor, but the Gabriel quiver construction is not, since one has many choices to make in the process. The first problem is that arrows of the Gabriel quiver correspond to a *choice* of basis and vertices correspond to a choice of a complete set of primitive orthogonal idempotents.

On [IM], Iusenko and MacQuarrie worked out a solution for this problem by considering a certain variant of the category of quivers, namely Vquivers, that instead of a set of arrows between vertices, we have vector spaces. Moreover, as an alternative for the choice of a complete set of primitive orthogonal idempotents we have a unique set of orbits of these elements. These techniques make it possible to construct functors between the category of finite dimensional pointed algebras and the category of finite Vquivers. Furthermore, under a specific relation on the morphisms of the category of algebras, the Path Algebra functor is a left adjoint of the Gabriel quiver functor.

In this dissertation, we dualize this theory for coalgebras. In a certain way we obtain a generalization, since there is no need for the restriction to finite dimensional coalgebras.

Chapter 0 contains well-known preliminary material: the basics of category theory, the universal property of quotient vector spaces and the First Isomorphism Theorem.

Chapters 1 and 2 contain standard definitions and results regarding coalgebras and related structures. Chapter 1 contains facts that are easily proved from the definitions, while Chapter 2 contains more powerful results.

Chapter 3 consists of the main results of this research. In this chapter we construct the path coalgebra and the Gabriel quiver as functors and under a quotient on the category of pointed coalgebras we obtain an adjunction between these functors.

Chapter 0

Some Category Theory

In this Chapter we will state standard definitions from Category Theory and well known results from Linear Algebra.

0.1 Standard definitions

Definition 0.1.1. A *category* $\mathbf{C}$ consists of a class of *objects* and for each pair of objects A, B a set $Hom_C(A, B)$ of *morphisms* from A to B satisfying

(i) the *composition law*: if $f \in Hom_C(A, B)$ and $g \in Hom_C(B, C)$, then $g \circ f \in Hom_C(A, C)$;

(ii) for every object A of $\mathbf{C}$ there is an *identity* morphism of A, $1_A : A \to A$;

(iii) the *associativity axiom*: if $f \in Hom_C(A, B)$, $g \in Hom_C(B, C)$ and $h \in Hom_C(C, D)$, then the following equality holds:

$$h \circ (g \circ f) = (h \circ g) \circ f;$$

(iv) the *unity law*: for every morphism $f \in Hom_C(A, B)$ the following equality holds:

$$f \circ 1_A = f = 1_B \circ f;$$

where $1_A : A \to A$ is the identity morphism of A and $1_B : B \to B$ is the identity morphism of B.

Definition 0.1.2. A *covariant functor* F from a category $\mathbf{C}$ to a category $\mathbf{D}$ is an assignment between objects and between morphisms such that

(i) $F(f : A \to B) = F(f) : F(A) \to F(B)$;

(ii) $F(g \circ f) = F(g) \circ F(f)$;

(iii) $F(1_A) = 1_{F(A)}$.

Definition 0.1.3. A functor $F : \mathbf{A} \to \mathbf{B}$ is an *isomorphism* if and only if there is a functor $G : \mathbf{B} \to \mathbf{A}$ for which both composites $G \circ F$ and $F \circ G$ are identity functors.

Definition 0.1.4. Given two functors $F, G : \mathbf{C} \rightrightarrows \mathbf{D}$, a *natural transformation* $\tau : F \to G$ is a function which assigns to each object $A \in \mathbf{C}$ a morphism $\tau_A : F(A) \to G(A)$ of $\mathbf{D}$ in such a way that every morphism $f : A \to B$ in $\mathbf{C}$ yields a commutative diagram

$$\begin{tikzcd} A \arrow[d, "f"] & F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\ B, & F(B) \arrow[r, "\tau_B"] & G(B) \end{tikzcd}$$

In this case, we say that $\tau_A : F(A) \to G(A)$ is *natural* in A.

Definition 0.1.5. [Mac, iv.1] Let $\mathbf{C}$ and $\mathbf{D}$ be categories. An *adjunction* from $\mathbf{C}$ to $\mathbf{D}$ is a triple $\langle F, G, \eta \rangle : \mathbf{C} \to \mathbf{D}$, where F and G are functors

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D},$$

while η is a function which assigns to each pair of objects $A \in \mathbf{C}$, $B \in \mathbf{D}$ a bijection of sets

$$\eta = \eta_{A,B} : Hom_{\mathbf{C}}(A, G(B)) \to Hom_{\mathbf{D}}(F(A), B)$$

which is natural in A and B. In this case, F is called a *left adjoint* of G and G is called a *right adjoint* of F.

0.2 Linear algebra

Now we will state and prove (even thought it is trivial) two basic theorems for vector spaces, which are the universal property of quotient spaces and the first isomorphism theorem. For what comes, there will be equivalent results for coalgebras, comodules and bicomodules.

Let k be a field, V a k-vector space and $W \subseteq V$ a subspace of V. Denote by $\overline{v} = v + W = \{v + w | w \in W\}$ the *coset* of W. Then, the quotient of V by W, $V/W = \{\overline{v} | v \in V\}$, is a k-vector space with the operations: $\overline{v_1} + \overline{v_2} = \overline{v_1 + v_2}$ and $\lambda\overline{v} = \overline{\lambda v}$, for $v, v_1, v_2 \in V$ and $\lambda \in k$. Define the *projection* $\pi : V \to V/W$ given by $\pi(v) = \overline{v}$.

Lemma 0.2.1 (The universal property of the quotient space)**.** *Let k, V, W and π be as above. Then, for any k-vector space Z and any k-linear map $\psi : V \to Z$ whose kernel contains W, there exists a unique k-linear map $\phi : V/W \to Z$ such that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, two heads, "\pi"] \arrow[dr, "\psi"'] & V/W \arrow[d, dashed, "\phi"] \\ & Z \end{tikzcd}$$

Proof. Let Z be a k-vector space and $\psi : V \to Z$ a k-linear map with $W \subseteq ker(\psi)$. Define $\phi : V/W \to Z$ to be the map $\overline{v} \mapsto \psi(v)$. Note that ϕ is well defined since if v_1 and v_2 are two representatives of $\overline{v}$, then there exists $w \in W$ such that $v_1 = v_2 + w$. Hence,

$$\psi(v_1) = \psi(v_2 + w) = \psi(v_2) + \psi(w) = \psi(v_2).$$

The linearity of ϕ is an immediate consequence of the linearity of ψ. Furthermore, it is clear from the definition that $\psi = \phi \circ \pi$. It remains to show that ϕ is unique. Suppose $\sigma : V/W \to Z$ is such that $\sigma \circ \pi = \psi = \phi \circ \pi$. Then $\sigma(\overline{v}) = \phi(\overline{v})$ for all $v \in V$. Since π is surjective, $\sigma = \phi$. □

Let V and W be two k-vector spaces and $f : V \to W$ a linear map. We write $im(f) = \{f(v) | v \in V\}$ the *image* of f and $ker(f) = \{v \in V | f(v) = 0\}$ the *kernel* of f. It is clear that $im(f)$ is a subspace of W and $ker(f)$ is a subspace of V. Let $V/ker(f)$ be the quotient space and write $\overline{v}$ for the coset of $ker(f)$. Define the map $\bar{f} : V/ker(f) \to im(f)$ given by $\overline{v} \mapsto f(v)$. Observe that $\bar{f}$ is well defined since for any two representatives v_1 and v_2 of $\overline{v}$, there exists a $\omega \in ker(f)$ such that $v_1 = v_2 + \omega$, and

$$f(v_1) = f(v_2 + \omega) = f(v_2) + f(\omega) = f(v_2).$$

Also $\bar{f}(V/ker(f)) = f(V) = im(f)$. We claim that $\bar{f}$ is a bijection. Surjectivity is direct from the last observation and injectivity is due to the following

$$ker(\bar{f}) = \{\overline{v} | \bar{f}(\overline{v}) = 0\} = \{\overline{v} | f(v) = 0\} = \{\overline{v} | v \in ker(f)\} = \overline{0}.$$

This gives us

Proposition 0.2.2 (The Fundamental Isomorphism Theorem for vector spaces)**.** *Given a linear map $f : V \to W$ of k-vector spaces, there exists a unique linear map $\bar{f} : V/ker(f) \to im(f)$ that makes the following diagram*

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, two heads, "\pi"'] & W \\
V/ker(f) \arrow[r, dashed, "\bar{f}"] & im(f) \arrow[u, hook, "\iota"']
\end{tikzcd}
$$

commutative, where $\pi : V \to V/ker(f)$ *is the canonical projection and* $\iota : im(f) \to W$ *is the inclusion.*

Proof. It remains to prove the uniqueness of $\bar{f}$. Suppose $g : V/ker(f) \to im(f)$ is a linear map such that $\iota \circ g \circ \pi = f$. Then injectivity of ι gives $g(\overline{v}) = \bar{f}(\overline{v})$, and surjectivity of π shows $g = \bar{f}$. $\square$

Chapter 1

On the structure of coalgebras: part 1

1.1 Coalgebras

Fix an algebraically closed field k. For now on, tensor products $\otimes$ are over k.

We define a k-coalgebra by dualizing the definition of a k-algebra (associative with identity) as follows:

Definition 1.1.1. A k-coalgebra $C = (C, \Delta, \varepsilon)$ is a k-vector space C together with two k-linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ satisfying the commutative diagrams:

$$
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "id\otimes\Delta"] \\
C \otimes C \arrow[r, "\Delta\otimes id"] & C \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
& C \arrow[dl, "\sim"'] \arrow[dr, "\sim"] \arrow[dd, "\Delta"] & \\
k \otimes C & & C \otimes k \\
& C \otimes C \arrow[ul, "\varepsilon\otimes id"] \arrow[ur, "id\otimes\varepsilon"'] &
\end{tikzcd}
$$

where $id : C \to C$ is the identity map and the maps from C to $k \otimes C$ and from C to $C \otimes k$ are the natural isomorphisms $c \mapsto 1 \otimes c$ and $c \mapsto c \otimes 1$, respectively. The left diagram is known as the coassociativity of the comultiplication Δ of C and the right diagram is known as the counitary property of the counity ε of C.

A subspace $S \subseteq C$ is a *subcoalgebra* of C if $\Delta(S) \subseteq S \otimes S$. In this case, $(S, \Delta|_S, \varepsilon|_S)$ is a k-coalgebra.

In order to simplify the notation, we will omit the k whenever there is no danger of confusion. By abuse of notation we will write $1 \otimes c = c$ and $c \otimes 1 = c$.

Definition 1.1.2. Let $C = (C, \Delta_C, \varepsilon_C)$ and $D = (D, \Delta_D, \varepsilon_D)$ be two coalgebras. A k-linear map $f : C \to D$ is a *coalgebra homomorphism* if the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f\otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\ & k & \end{tikzcd}$$

Sometimes we will use the *Sweedler notation* (or sigma notation) [Swe, Section 1.2] for computations, that is, if C is a coalgebra and $c \in C$, we write

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{i=1}^{n} c_{1i} \otimes c_{2i}$$

omitting the index i.

Lemma 1.1.3. *Let C and D be coalgebras and $f : C \to D$ a coalgebra homomorphism. If S is any subcoalgebra of C, then $f(S)$ is a subcoalgebra of D.*

Proof. If $s \in S$, then $\Delta_C(s) = \sum_{(s)} s_{(1)} \otimes s_{(2)} \subseteq S \otimes S$, and

$$\begin{aligned} \Delta_D(f(s)) =& (f \otimes f)\Delta_C(s) \\ =& (f \otimes f) \sum_{(s)} s_{(1)} \otimes s_{(2)} \\ =& \sum_{(s)} f(s_{(1)}) \otimes f(s_{(2)}) \subseteq f(S) \otimes f(S). \end{aligned}$$

□

Lemma 1.1.4. *Let A, B, and C be coalgebras and $f : A \to B$ and $g : B \to C$ be coalgebra homomorphisms. Then $g \circ f : A \to C$ is a coalgebra morphism.*

Proof. For any $a \in A$ we have

$$\begin{aligned} \Delta_C(g \circ f)(a) &= \Delta_C(g(f(a))) \\ &= (g \otimes g)(\Delta_B(f(a)) \\ &= (g \otimes g)(f \otimes f)(\Delta_A(a)) \\ &= (g \circ f \otimes g \circ f)(\Delta_A(a)) \end{aligned}$$

and

$$\begin{aligned}\varepsilon_C(g \circ f)(a) &= \varepsilon_C(g(f(a)) \\ &= \varepsilon_B(f(a)) \\ &= \varepsilon_A(a).\end{aligned}$$

Thus, the result follows. $\square$

Taking all k-coalgebras as objects and all coalgebra homomorphisms as morphisms we have a category, called **k-Cog**.

Let us see some examples to illustrate this definition.

Examples 1.1.5. (i) Let S be a set and kS be the k-vector space with basis S. Then kS is a coalgebra with comultiplication and counity defined by

$$\begin{aligned}\varDelta(s) &= s \otimes s \\ \varepsilon(s) &= 1\end{aligned}$$

extended linearly for all $s \in S$.

We must check that kS is indeed a coalgebra. Let $c \in kS$ and write $c = \sum_{s \in S} \lambda_s s$, where each $\lambda_s \in k$. Then

$$\begin{aligned}(id \otimes \varDelta)\varDelta(c) =&(id \otimes \varDelta)\varDelta\left(\sum_{s\in S}\lambda_s s\right) \\ =&(id \otimes \varDelta)\left(\sum_{s\in S}\varDelta(\lambda_s s)\right) \\ =&(id \otimes \varDelta)\left(\sum_{s\in S}\lambda_s\varDelta(s)\right) \\ =&(id \otimes \varDelta)\left(\sum_{s\in S}\lambda_s s \otimes s\right) \\ =&\sum_{s\in S}\lambda_s s \otimes \varDelta(s) \\ =&\sum_{s\in S}\lambda_s s \otimes s \otimes s \\ =&\sum_{s\in S}\lambda_s \varDelta(s) \otimes s \\ =&(\varDelta \otimes id)\left(\sum_{s\in S}\lambda_s s \otimes s\right)\end{aligned}$$

$$=(\varDelta \otimes id)\varDelta\left(\sum_{s\in S}\lambda_s s\right) = (\varDelta \otimes id)\varDelta(c),$$

and

$$\begin{aligned}(id \otimes \varepsilon)\varDelta(c) =&(id \otimes \varepsilon)\varDelta\left(\sum_{s\in S}\lambda_s s\right)\\ =&(id \otimes \varepsilon)\left(\sum_{s\in S}\lambda_s s \otimes s\right)\\ =&\sum_{s\in S}\lambda_s s \otimes \varepsilon(s)\\ =&\sum_{s\in S}\lambda_s s \otimes 1\\ =&c \otimes 1\\ =&c.\end{aligned}$$

Similarly one can show that $(\varepsilon \otimes id)\varDelta(c) = c$. Moreover, the only possible value for $\varepsilon(s)$ such that ε is a counity of kS is $\varepsilon(s) = 1$ for all $s \in S$.

We have picked an arbitrary element of kS to show the coassociativity of $\varDelta$ and the counitary property of ε, however by linearity of both maps it would be sufficient to show that it works for an arbitrary element of the basis.

(ii) Let H be a vector space with basis $\{g_i, d_i : i \in \mathbb{N}\}$. The comultiplication and counity given by:

$$\begin{aligned}\varDelta(g_i) &= g_i \otimes g_i\\ \varDelta(d_i) &= g_i \otimes d_i + d_i \otimes g_{i+1}\\ \varepsilon(g_i) &= 1\\ \varepsilon(d_i) &= 0\end{aligned}$$

extended linearly for all $d_i, g_i \in H$ defines a coalgebra H.

By the example above it suffices to confirm the coassociativity of $\varDelta$ and the counitary property of ε for an element of the base d_i, with $i \in \mathbb{N}$.

$$\begin{aligned}
(id \otimes \Delta)\Delta(d_i) &= (id \otimes \Delta)(g_i \otimes d_i + d_i \otimes g_{i+1}) \\
&= g_i \otimes \Delta(d_i) + d_i \otimes \Delta(g_{i+1}) \\
&= g_i \otimes g_i \otimes d_i + g_i \otimes d_i \otimes g_{i+1} + d_i \otimes g_{i+1} \otimes g_{i+1} \\
&= \Delta(g_i) \otimes d_i + \Delta(d_i) \otimes g_{i+1} \\
&= (\Delta \otimes id)(g_i \otimes d_i + d_i \otimes g_{i+1}) \\
&= (\Delta \otimes id)\Delta(d_i)
\end{aligned}$$

and

$$\begin{aligned}
(id \otimes \varepsilon)\Delta(d_i) &= (id \otimes \varepsilon)(g_i \otimes d_i + d_i \otimes g_{i+1}) \\
&= g_i \otimes \varepsilon(d_i) + d_i \otimes \varepsilon(g_{i+1}) \\
&= 0 + d_i \otimes 1 \\
&= d_i.
\end{aligned}$$

A similar computation shows that $(\varepsilon \otimes id)\Delta(d_i) = d_i$. Thus H is a coalgebra.

(iii) Let n be a positive integer and $M^C(n,k)$ a k-vector space of dimension n^2. Let $(e_{ij})_{1 \leq i,j \leq n}$ be a basis for $M^C(n,k)$. With comultiplication and counity defined by

$$\begin{aligned}
\Delta(e_{ij}) &= \sum_{l=1}^{n} e_{il} \otimes e_{lj} \\
\varepsilon(e_{ij}) &= \delta_{ij}
\end{aligned}$$

$M^C(n,k)$ becomes a coalgebra, which is called the *matrix coalgebra*.

Lets confirm the coassociativity and counitary properties for an arbitrary element of the basis, e_{ij}.

$$\begin{aligned}
(id \otimes \Delta)\Delta(e_{ij}) &= (id \otimes \Delta)(\sum_{l=1}^{n} e_{il} \otimes e_{lj}) \\
&= \sum_{l=1}^{n} e_{il} \otimes \Delta(e_{lj}) \\
&= \sum_{l=1}^{n} e_{il} \otimes (\sum_{p=1}^{n} e_{lp} \otimes e_{pj}) \\
&= \sum_{l,p=1}^{n} e_{il} \otimes e_{lp} \otimes e_{pj} \\
&= \sum_{p=1}^{n} (\sum_{l=1}^{n} e_{il} \otimes e_{lp}) \otimes e_{pj}) \\
&= \sum_{p=1}^{n} \Delta(e_{ip}) \otimes e_{pj} \\
&= (\Delta \otimes id)(\sum_{p=1}^{n} e_{ip} \otimes e_{pj}) \\
&= (\Delta \otimes id)\Delta(e_{ij})
\end{aligned}$$

and

$$\begin{aligned}
(id \otimes \varepsilon)\Delta(e_{ij}) &= (id \otimes \varepsilon)(\sum_{l=1}^{n} e_{il} \otimes e_{lj}) \\
&= \sum_{l=1}^{n} e_{il} \otimes \varepsilon(e_{lj}) \\
&= e_{ij} \otimes 1 \\
&= e_{ij}
\end{aligned}$$

Similarly $(id \otimes \varepsilon)\Delta(e_{ij}) = e_{ij}$ and, hence, $M^C(n,k)$ is a coalgebra.

(iv) Let V and W be sets and $f : V \to W$ an injective function. Define the map

$$\begin{aligned}
\bar{f} : kV &\to kW, \\
\sum_{v \in V} \lambda_v v &\mapsto \sum_{v \in V} \lambda_v f(v)
\end{aligned}$$

where each $\lambda_v \in k$ and kV and kW are coalgebras defined as in Example 1.1.5 (i). The following computations show that $\bar{f}$ satisfies the commutative diagrams for coalgebra homomorphisms:

$$
\begin{aligned}
\Delta_{kW}\left(\bar{f}\left(\sum_{v\in V}\lambda_v v\right)\right) =& \Delta_{kW}\left(\sum_{v\in V}\lambda_v f(v)\right)\\
=&\sum_{v\in V}\lambda_v \Delta_{kW}(f(v))\\
=&\sum_{v\in V}\lambda_v f(v)\otimes f(v)\\
=&(f\otimes f)\left(\sum_{v\in V}\lambda_v v\otimes v\right)\\
=&(f\otimes f)\left(\sum_{v\in V}\lambda_v \Delta_{kV}(v)\right)\\
=&(f\otimes f)\Delta_{kV}\left(\sum_{v\in V}\lambda_v v\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon_{kW}\left(\bar{f}\left(\sum_{v\in V}\lambda_v v\right)\right) =& \sum_{v\in V}\lambda_v \varepsilon_{kW}(f(v))\\
=&\sum_{v\in V}\lambda_v 1\\
=&\sum_{v\in V}\lambda_v \varepsilon_{kV}(v).
\end{aligned}
$$

Thus, $\bar{f}$ is a coalgebra homomorphism.

Definition 1.1.6. Let C be a coalgebra.

(i) If $c \in C$ satisfies $\Delta(c) = c\otimes c$ and $\varepsilon(c) = 1$, then we say that c is a *group-like element* of C. We write $G(C) := \{g \in C | \Delta(g) = g\otimes g \text{ and } \varepsilon(g) = 1\}$. The coalgebra kS in Example 1.1.5 (i) is called the *group-like coalgebra* of S. A special case of group-like coalgebra is the *group-like subcoalgebra* $kG(C)$ of C;

(ii) if $g, h \in G(C)$ and $c \in C$ is such that $\Delta(c) = c\otimes g + h\otimes c$, then we say that c is g, h*-primitive*. The set of all g, h-primitives is denoted by $P_{g,h}(C)$.

Proposition 1.1.7. *Let C be a coalgebra. Then the elements of $G(C)$ are linearly independent.*

Proof. See [DNR, Proposition 1.4.14], [Swe, Proposition 3.2.1] or [Abe, Theorem 2.1.2].

Suppose that $G(C)$ is not a linearly independent family.

Let n be the smallest natural number for which there exist distinct elements $g, g_1, \cdots, g_n \in G(C)$ such that $g = \sum_{i=1}^{n} \lambda_i g_i$, with $\lambda_i \in k, \forall i$. If $n = 1$, then $g = \lambda_1 g_1$ and

$$1 = \varepsilon(g) = \varepsilon(\lambda_1 g_1) = \lambda_1 \varepsilon(g_1) = \lambda_1.$$

Hence $g = g_1$. Thus $n \geq 2$. Applying Δ to $g = \sum_{i=1}^{n} \lambda_i g_i$ we obtain

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i g_i \otimes g_i = \Delta\left(\sum_{i=1}^{n} \lambda_i g_i\right) &= \Delta(g) = g \otimes g = \\ &= \left(\sum_{i=1}^{n} \lambda_i g_i\right) \otimes \left(\sum_{j=1}^{n} \lambda_j g_j\right) = \sum_{i,j=1}^{n} \lambda_i \lambda_j g_i \otimes g_j.\end{aligned}$$

Consequently

$$\begin{aligned}0 &= \sum_{i=1}^{n} \lambda_i g_i \otimes g_i - \sum_{i,j=1}^{n} \lambda_i \lambda_j g_i \otimes g_j \\ &= \sum_{i=1}^{n} (\lambda_i - \lambda_i \lambda_i) g_i \otimes g_i - \sum_{i \neq j} \lambda_i \lambda_j g_i \otimes g_j.\end{aligned}$$

Since $\{g_i\}_{1 \leq i \leq n}$ is a linearly independent set in C, it follows that $\{g_i \otimes g_j\}_{1 \leq i,j \leq n}$ is a linearly independent set in $C \otimes C$. Hence, the equality above shows that $\lambda_i \lambda_j = 0$ if $i \neq j$ and so $\lambda_i = 0$ or $\lambda_j = 0$, contradicting the minimality of n. $\square$

Lemma 1.1.8. *Let C be a coalgebra. Then, for $g, h, g', h' \in G(C)$, we have*

$$kG(C) \cap P_{g,h}(C) = k(h - g)$$

and

$$P_{g,h}(C) \cap P_{g',h'}(C) = \begin{cases} P_{g,h}(C), & \text{if } g' = g \text{ and } h' = h \\ k(h-g), & \text{if } g' = h \text{ and } h' = g \\ 0, & \text{otherwise} \end{cases}$$

Proof. Consider $c \in kG(C) \cap P_{g,h}(C)$ and write $c = \sum_{e \in G(C)} \lambda_e e$. Then

$$\begin{aligned}\left(\sum_{e \in G(C)} \lambda_e e\right) \otimes g + h \otimes \left(\sum_{e \in G(C)} \lambda_e e\right) &= \Delta\left(\sum_{e \in G(C)} \lambda_e e\right) \\ &= \sum_{e \in G(C)} \lambda_e e \otimes e,\end{aligned}$$

implies that

$$
\begin{aligned}
0 &= \left(\sum_{e\in G(C)} \lambda_e e\right) \otimes g + h \otimes \left(\sum_{e\in G(C)} \lambda_e e\right) - \sum_{e\in G(C)} \lambda_e e \otimes e \\
&= \sum_{e\in G(C)} \lambda_e e \otimes g + \sum_{e\in G(C)} h \otimes \lambda_e e - \sum_{e\in G(C)} \lambda_e e \otimes e \\
&= \sum_{e\in G(C)\setminus\{g,h\}} \lambda_e (e \otimes g + h \otimes e - e \otimes e) + (\lambda_h + \lambda_g) h \otimes g.
\end{aligned}
$$

Since $\{e \otimes f\}_{e,f\in G(C)}$ is a linearly independent set in $C \otimes C$, by Proposition 1.1.7, we must have

$$
\begin{aligned}
\lambda_e e \otimes e =&0, && \forall e \in G(C) \setminus \{g, h\} \\
\lambda_e e \otimes g =&0, && \forall e \in G(C) \setminus \{g, h\} \\
\lambda_e h \otimes e =&0, && \forall e \in G(C) \setminus \{g, h\} \\
(\lambda_h + \lambda_g) h \otimes g =&0.
\end{aligned}
$$

Hence, $\lambda_e = 0\ \forall e \in G(C) \setminus \{g, h\}$ and $\lambda_h = -\lambda_g$. Thus,

$$
kG(C) \cap P_{g,h}(C) = k(h - g).
$$

Consequently, by the linear independence of the set $G(C)$ and the equality $(h - g) = -(g - h)$, we have $P_{g,h}(C) \cap P_{h,g}(C) \cap kG(C) = k(h - g)$ and $P_{g,h}(C) \cap P_{g',h'}(C) \cap kG(C) = 0$, for any $(g', h') \neq (g, h)$ or (h, g).

To conclude our claim, it is enough to show that if $c \in P_{g,h}(C) \setminus kG(C)$, then $c \notin P_{g',h'}$ for any $g', h' \in G(C)$ with $g' \neq g$ or $h' \neq h$.

Suppose that $c \in P_{g',h'}$ and write $\Delta(c) = c' \otimes g' + h' \otimes c'$. Assume, without lost of generality that $g' \neq g$. The counitary property of ε give us

$$
\begin{aligned}
1 \otimes c = (\varepsilon \otimes id)\Delta(c) &= \varepsilon(c) \otimes g + 1 \otimes c \\
&= \varepsilon(c') \otimes g' + 1 \otimes c'.
\end{aligned}
$$

Thus, $\varepsilon(c) = 0$ and $c' = c - \varepsilon(c')g'$. Applying ε we obtain

$$
\varepsilon(c') = \varepsilon(c) - \varepsilon(c')\varepsilon(g') = -\varepsilon(c').
$$

Hence, $\varepsilon(c') = 0$ and $c' = c$. Now

$$
\begin{aligned}
0 = \Delta(c) - \Delta(c) =& c \otimes g + h \otimes c - c \otimes g' - h' \otimes c \\
=& c \otimes g + (h - h') \otimes c - c \otimes g'.
\end{aligned}
$$

But this is impossible, since $\{g, g', c\}$ are linearly independent. Thus $c \notin P_{g',h'}(C)$. $\square$

Lemma 1.1.9. *Let C and D be coalgebras and $f : C \to D$ a coalgebra homomorphism. Then*

(i) $f(G(C)) \subseteq G(D)$;

(ii) $f(P_{g,h}(C)) \subseteq P_{f(g),f(h)}(D)$. *Moreover, if f is injective and $c \in P_{g,h}(C) \setminus k(h-g)$, then $f(c) \in P_{f(g),f(h)}(D) \setminus k(f(h) - f(g))$.*

Proof. Let $g \in G(C)$. Then,

$$\begin{aligned} \Delta_D(f(g)) =& (f \otimes f)\Delta_C(g) \\ =& (f \otimes f)(g \otimes g) = f(g) \otimes f(g) \end{aligned}$$

and

$$\varepsilon_D(f(g)) = \varepsilon_C(g) = 1.$$

Thus $f(g) \in G(D)$ and (i) is done. Let $c \in P_{g,h}(C)$. Then,

$$\begin{aligned} \Delta_D(f(c)) =& (f \otimes f)\Delta_C(c) \\ =& (f \otimes f)(c \otimes g + h \otimes c) = f(c) \otimes f(g) + f(h) \otimes f(c). \end{aligned}$$

Since $f(g), f(h) \in G(D)$ by (i), we get $f(c) \in P_{f(g),f(h)}(D)$.

Consider now f injective and $c \in P_{g,h}(C) \setminus k(h-g)$. If $f(c) \in k(f(h) - f(g))$ then $f(c) = \lambda(f(h) - f(g))$ for some $\lambda \in k$, but

$$f(\lambda(h-g)) = \lambda(f(h) - f(g)) = f(c)$$

for $c \neq \lambda(h-g)$ by hypothesis, which contradicts the injectivity of f. Thus, we conclude (ii). $\square$

1.2 Comodules

Definition 1.2.1. Let $C = (C, \Delta, \varepsilon)$ be a coalgebra. We call a *right C-comodule* a pair (M, ρ), where M is a k-vector space, $\rho : M \to M \otimes C$ a morphism of k-vector spaces such that the following diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "id \otimes \Delta"] \\ M \otimes C \arrow[r, "\rho \otimes id"] & M \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[dr, "\sim"] \arrow[dd, "\rho"'] & \\ & M \otimes k \\ M \otimes C \arrow[ur, "id \otimes \varepsilon"'] & \end{tikzcd}$$

We say that $\rho : M \to M \otimes C$ is the *structure map* of the right C-comodule M.

A subspace $N \subseteq M$ is a *subcomodule* of M if $\rho(N) \subseteq N \otimes C$. In this case (N, ρ_N) is a right C-comodule, where $\rho_N : N \to N \otimes C$ is the restriction and corestriction of ρ to N and $N \otimes C$, respectively.

A *left C-comodule* is defined in a similar fashion.

Definition 1.2.2. Let C be a coalgebra, (M, ρ) and (N, ϕ) be right C-comodules. The k-linear map $f : M \to N$ is a *comodule homomorphism* if the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho"'] & N \arrow[d, "\phi"] \\ M \otimes C \arrow[r, "f \otimes id"] & N \otimes C \end{tikzcd}$$

Lemma 1.2.3. *Let C be a coalgebra, (M, ρ), (N, ϕ) and (P, ψ) be right C-comodules and $f : M \to N$, $g : N \to P$ comodule homomorphisms. Then $g \circ f : M \to P$ is a comodule homomorphism.*

Proof. For any $m \in M$ we have

$$\begin{aligned} \psi(g \circ f)(m) &= \psi(g(f(m))) \\ &= (g \otimes id)(\phi(f(m)) \\ &= (g \otimes id)(f \otimes id)(\rho(m)) \\ &= (g \circ f \otimes id)(\rho(m)) \end{aligned}$$

Thus the result follows. □

Let C be a coalgebra. Then, the category $\mathbf{M}^C$ having all right C-comodules as objects and all comodule homomorphisms as morphisms is well defined. The morphisms of $\mathbf{M}^C$ from M to N are usually denoted by $Com_{-C}(M, N)$.

Similarly, ${}^C\mathbf{M}$ denotes the category of all left C-comodules and comodule homomorphisms.

Proposition 1.2.4. *Let C be a coalgebra. Then the categories ${}^C\mathbf{M}$ and $\mathbf{M}^{C^{cop}}$ are isomorphic, where $C^{cop} = (C, \Delta^{cop}, \varepsilon)$ is the co-opposite coalgebra of C and $\Delta^{cop} = T \circ \Delta$, where $T : C \otimes C$ is the twist map given by $T(a \otimes b) = b \otimes a$.*

Proof. [DNR, Proposition 2.1.10]

Let $M \in {}^{C}\mathbf{M}$ with the structure map $\rho : M \to C \otimes M$, given by $\rho(m) = \sum_{(m)} m_{(-1)} \otimes m_{(0)}$, with all $m_{(-1)} \in C$ and all $m_{(0)} \in M$. Then, M becomes a right C^{cop}-comodule via the structure map $\rho' : M \to M \otimes C^{cop}$, given by $\rho'(m) = \sum_{(m)} m_{(0)} \otimes m_{(-1)}$.

$$
\begin{aligned}
(id \otimes T \circ \Delta)\rho'(m) =& (id \otimes id \circ \Delta) \sum_{(m)} m_{(0)} \otimes m_{(-1)} \\
=& \sum_{(m)} m_{(0)} \otimes (T \circ \Delta)(m_{(-1)}) \\
=& \sum_{(m)} m_{(0)} \otimes T(\Delta(m_{(-1)})) \\
=& (id \otimes T) \sum_{(m)} m_{(0)} \otimes \Delta(m_{(-1)}) \\
=& (id \otimes T)(T \otimes id(id \otimes T)) \circ (id \otimes T(T \otimes id)) \sum_{(m)} m_{(0)} \otimes \Delta(m_{(-1)}) \\
=& (id \otimes T)(T \otimes id(id \otimes T)) \sum_{(m)} \Delta(m_{(-1)}) \otimes m_{(0)} \\
=& (id \otimes T)(T \otimes id(id \otimes T))(\Delta \otimes id)\rho(m) \\
=& (id \otimes T)(T \otimes id(id \otimes T))(id \otimes \rho)\rho(m) \\
=& (id \otimes T)(T \otimes id(id \otimes T))(id \otimes \rho) \sum_{(m)} m_{(-1)} \otimes m_{(0)} \\
=& (id \otimes T)(T \otimes id(id \otimes T)) \sum_{(m)} m_{(-1)} \otimes \rho(m_{(0)}) \\
=& (id \otimes T)(T \otimes id) \sum_{(m)} m_{(-1)} \otimes \rho'(m_{(0)}) \\
=& \sum_{(m)} \rho'(m_{(0)}) \otimes m_{(-1)} = (\rho' \otimes id)\rho'(m).
\end{aligned}
$$

Moreover, if M and N are two left C-comodules and $f \in Com_{C-}(M, N)$, then $f \in Com_{-C^{cop}}(M, N)$, since

$$\begin{aligned}(\phi' \circ f)(m) =& ((T \circ \phi) \circ f)(m) \\ =& (T \circ (\phi \circ f))(m) \\ =& (T \circ ((id \circ f) \circ \rho))(m) \\ =& T\left(\sum_{(m)} m_{(-1)} \otimes f(m_{(0)})\right) \\ =& \sum_{(m)} f(m_{(0)}) \otimes m_{(-1)} = ((f \otimes id) \circ \rho')(m).\end{aligned}$$

This defines a functor $F : {}^C\mathbf{M} \to \mathbf{M}^{C^{cop}}$.

Similarly, we can define a functor $G : \mathbf{M}^{C^{cop}} \to {}^C\mathbf{M}$ by associating to a right C^{cop}-comodule M with structure map $\mu : M \to M \otimes C^{cop}$, $\mu(m) = \sum m_{(0)} \otimes m_{(1)}$, a structure map of left C-comodule defined by $\mu' : M \to C \otimes M$, $\mu'(m) = \sum_{(m)} m_{(1)} \otimes m_{(0)}$. It is easy to see that $G \circ F$ is the identity functor, since for two left C-comodule (M, ρ) and (N, ϕ) and a comodule homomorphism $f : M \to N$ we have

$$(G \circ F)(M, \rho) = G(F((M, \rho))) = G((M, T \circ \rho)) = (M, T \circ T \circ \rho) = (M, \rho)$$

and

$$(G \circ F)(f) = G(F(f)) = G(f) = f.$$

A similar computation shows that $F \circ G$ is the identity functor in $\mathbf{M}^{C^{cop}}$. Thus the functors F and G define an isomorphism of categories. □

Remark 1.2.5. *Proposition 1.2.4 shows that any result for right C-comodules has an analogous result for left C-comodules.*

Examples 1.2.6. (i) A coalgebra C is a left and a right C-comodule with the structure map being in both cases the comultiplication of C;

(ii) Let C be a coalgebra, (M, ρ) be a right C-comodule and X a k-vector space. Then $X \otimes M$ becomes a right C-comodule with the structure map $id \otimes \rho : X \otimes M \to X \otimes M \otimes C$, since, for any $x \otimes m \in X \otimes M$, we have

$$\begin{aligned}(id \otimes id \otimes \Delta)(id \otimes \rho)(x \otimes m) =& (id \otimes id \otimes \Delta)(x \otimes \rho(m)) \\ =& x \otimes (id \otimes \Delta)\rho(m) \\ =& x \otimes (\rho \otimes id)\rho(m) \\ =& (id \otimes \rho \otimes id)(x \otimes \rho(m)) \\ =& (id \otimes \rho \otimes id)(id \otimes \rho)(x \otimes m)\end{aligned}$$

and

$$
\begin{aligned}
(id \otimes id \otimes \varepsilon)(id \otimes \rho)(x \otimes m) =&(id \otimes id \otimes \varepsilon)(x \otimes \rho(m)) \\
=&x \otimes (\mathrm{id} \otimes \varepsilon)\rho(m) \\
=&x \otimes m \otimes 1
\end{aligned}
$$

(iii) Let S be a non-empty set and kS the group-like coalgebra of S. Let $(M_s)_{s \in S}$ be a family of k-vector spaces and $M = \bigoplus_{s \in S} M_s$. Then M is a right kS-comodule with the structure map $\rho : M \to M \otimes kS$ defined by $\rho(m_s) = m_s \otimes s$ extended linearly for any $s \in S$ and $m_s \in M_s$. In order to check this, we need only analyze for any element of the basis:

$$
\begin{aligned}
(id \otimes \Delta)\rho(m_s) &= (id \otimes \Delta)m_s \otimes s \\
&= m_s \otimes \Delta(s) \\
&= m_s \otimes s \otimes s \\
&= \rho(m_s) \otimes s \\
&= (\rho \otimes id)(m_s \otimes s) \\
&= (\rho \otimes id)\rho(m_s)
\end{aligned}
$$

and

$$
\begin{aligned}
(id \otimes \varepsilon)\rho(m_s) &= (id \otimes \varepsilon)m_s \otimes s \\
&= m_s \otimes \varepsilon(s) \\
&= m_s \otimes 1 \\
&= m_s.
\end{aligned}
$$

Examples 1.2.7. (i) Let C be a coalgebra, (M, ρ) a right C comodule and N a subcomodule of M. Then the inclusion map $\iota : N \to M$, $\iota(n) = n$ for any $n \in N$ is a comodule homomorphism. Let us check this. We know that $(N, \rho|_N)$ is a right C-comodule. Then, for any $n \in N$, we have

$$
\begin{aligned}
\rho(\iota(n)) =&\rho(n) \\
=&\rho|_N(n) \\
=&(\iota \otimes id)\rho|_N(n),
\end{aligned}
$$

where the last equality comes from the fact that $\rho(N) \subseteq N \otimes C$ and the corestriction of ι to N is the identity;

(ii) If C and D are coalgebras and $f : C \to D$ is a coalgebra homomorphism, then $(C, (id \otimes f)\Delta_C)$ is a right D-comodule and $f : C \to D$ is a comodule homomorphism of D-comodules. First we must confirm that C is a right D-comodule. Consider any element $c \in C$. Thus,

$$\begin{aligned}
(id \otimes \Delta_D) \circ ((id \otimes f)\Delta_C)(c) =& (id \otimes \Delta_D \circ f)\Delta_C(c) \\
=& (id \otimes (f \otimes f)\Delta_C)\Delta_C(c) && (1) \\
=& (id \otimes f \otimes f)(id \otimes \Delta_C)\Delta_C(c) \\
=& (id \otimes f \otimes f)(\Delta_C \otimes id)\Delta_C(c) && (2) \\
=& ((id \otimes f)\Delta_C \otimes id)(id \otimes f)\Delta_C(c)
\end{aligned}$$

and

$$\begin{aligned}
(id \otimes \varepsilon_D) \circ (id \otimes f)\Delta_C(c) =& (id \otimes \varepsilon_D \circ f)\Delta_C(c) \\
=& (id \otimes \varepsilon_C)\Delta_C(c) && (3) \\
=& c, && (4)
\end{aligned}$$

where the steps (1) and (3) are due to f be a coalgebra homomorphism, (2) is because of the coassociativity of Δ_C, and (4) follows from the counitary property of ε_C. Hence $(C, (id \otimes f)\Delta_C)$ is a right D-comodule.

Now, for any $c \in C$ we have

$$\begin{aligned}
\Delta_D(f(c)) =& (f \otimes f)\Delta_C(c) && (5) \\
=& (f \otimes id)(id \otimes f)\Delta_C,
\end{aligned}$$

where (5) is due to f being a coalgebra homomorphism. This completes the proof.

Lemma 1.2.8. *Let V and W be two k-vector spaces and $X \subseteq V$, $Y \subseteq W$ subspaces. Then $(V \otimes Y) \cap (X \otimes W) = X \otimes Y$.*

Proof. [DNR, Lemma 1.4.5]. □

Corollary 1.2.9. *Let C be a coalgebra. If $M \subseteq C$ is a subcomodule of C which is a left and right C-comodule with the structure map Δ, then M is a subcoalgebra of C.*

Proof. If $c \in \Delta(M)$, then $c \in (C \otimes M) \cap (M \otimes C)$ and the result follows immediate from Lemma 1.2.8. □

Lemma 1.2.10. *Let V_1, V_2, W_1 and W_2 be k-vector spaces and $f : V_1 \to V_2$ and $g : W_1 \to W_2$ be linear maps. Then $ker(f \otimes g) = ker(f) \otimes W_1 + V_1 \otimes ker(g)$.*

Proof. [DNR, Lemma 1.4.8]. □

Proposition 1.2.11. *Let (M, ρ) and (N, ϕ) be two right C-comodules and $f : M \to N$ a comodule homomorphism. Then $Im(f)$ is a C-subcomodule of N and $Ker(f)$ is a C-subcomodule of M.*

Proof. [DNR, Proposition 2.1.16].

Since f is a comodule homomorphism we have $(f \otimes id)\rho = \phi \circ f$. Then

$$(f \otimes id)\rho(Ker(f)) = (\phi \circ f)(Ker(f)) = 0,$$

which shows that $\rho(Ker(f)) \subseteq Ker(f \otimes id) = Ker(f) \otimes C$, by Lemma 1.2.10, and hence $Ker(f)$ is a C-subcomodule of M.

Now

$$\phi(Im(f)) = (f \otimes id)\rho(M) \subseteq Im(f) \otimes C,$$

which shows that $Im(f)$ is a C-subcomodule of N. □

Now we are going to show that the isomorphism theorem works for comodules. First we need to define the quotient of comodules.

For what follows, consider C a coalgebra, (M, ρ) a right C-comodule and N a subcomodule of M. Let M/N be the quotient space, and $\pi : M \to M/N$ the canonical projection, $\pi(m) = \overline{m}$, where $\overline{m}$ is the coset of N.

Proposition 1.2.12. *There exists a unique structure of right C-comodule on M/N for which $\pi : M \to M/N$ is a comodule homomorphism.*

Proof. [DNR, Proposition 2.1.14] The composition $(\pi \otimes id)\rho : M \to M/N \otimes C$ is a linear map such that $N \subseteq ker((\pi \otimes id)\rho)$, since

$$(\pi \otimes id)\rho(N) \subseteq (\pi \otimes id)(N \otimes C) \subseteq \pi(N) \otimes C = 0.$$

By the universal property of the quotient space, Lemma 0.2.1, it follows that there exists a unique linear map $\bar{\rho} : M/N \to M/N \otimes C$ for which the diagram

$$\begin{tikzcd}
M \arrow[r, two heads, "\pi"] \arrow[d, "\rho"'] & M/N \arrow[d, dashed, "\bar{\rho}"] \\
M \otimes C \arrow[r, "\pi \otimes id"] & M/N \otimes C
\end{tikzcd}$$

is commutative. This map is defined by $\bar{\rho}(\overline{m}) = (\pi \otimes id)\rho(m)$ for any $m \in M$. Then $(M/N, \bar{\rho})$ is a right C-comodule, since

$$\begin{aligned}(id \otimes \Delta)\bar{\rho}(\overline{m}) =&(id \otimes \Delta)(\pi \otimes id)(\rho(m))\\ =&(\pi \otimes \Delta)(\rho(m))\\ =&(\pi \otimes id \otimes id)(id \otimes \Delta)(\rho(m))\\ =&(\pi \otimes id \otimes id)(\rho \otimes id)(\rho(m))\\ =&((\pi \otimes id)\rho \otimes id)(\rho(m))\\ =&(\bar{\rho} \circ \pi \otimes id)(\rho(m))\\ =&(\bar{\rho} \otimes id)(\pi \otimes id)(\rho(m))\\ =&(\bar{\rho} \otimes id)(\bar{\rho})(\overline{m}),\end{aligned}$$

and

$$\begin{aligned}(id \otimes \varepsilon)\bar{\rho}(\overline{m}) =&(id \otimes \varepsilon)(\pi \otimes id)\rho(m)\\ =&(\pi \otimes id) * id \otimes \varepsilon)\rho(m)\\ =&(\pi \otimes id)(m \otimes 1)\\ =&\overline{m} \otimes 1.\end{aligned}$$

If we would have a comodule structure on M/N given by $\omega : M/N \to M/N \otimes C$ such that π is a comodule homomorphism, then the diagram obtained by replacing $\bar{\rho}$ by ω in the above diagram should be also commutative. Then it would follow that $\omega = \bar{\rho}$ from the universal property of the quotient space. □

Remark 1.2.13. *The comodule M/N, with the structure given as in the above proposition is called the quotient comodule of M and N.*

Theorem 1.2.14 (The fundamental isomorphism theorem for comodules)**.** *Let $f : M \to N$ be a comodule homomorphism, $\pi : M \to M/Ker(f)$ the canonical projection, and $\iota : Im(f) \to N$ the inclusion. Then, there exists a unique comodule isomorphism $\bar{f} : M/Ker(f) \to Im(f)$ for which the diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, two heads, "\pi"'] & N \\ M/Ker(f) \arrow[r, dashed, "\bar{f}"] & Im(f) \arrow[u, hook, "\iota"'] \end{tikzcd}$$

is commutative.

Proof. [DNR, Theorem 2.1.17].

The existence of a unique linear map $\bar{f} : M/Ker(f) \to Im(f)$ making the diagram commutative follows from the fundamental isomorphism theorem for k-vector spaces, Proposition 0.2.2. We know that $\bar{f}$ is defined by $\bar{f}(\overline{m}) = f(m)$ for any $\overline{m} \in M/Ker(f)$. It remains to show that $\bar{f}$ is a comodule homomorphism. Denoting by $\omega = (\pi \otimes id)\rho : M/Ker(f) \to M/Ker(f) \otimes C$ and $\theta = \phi|_{im(f)} : Im(f) \to Im(f) \otimes C$ the maps giving the comodule structures, we have

$$\begin{aligned}(\bar{f} \otimes id)\omega(\overline{m}) =&(\bar{f} \otimes id)\omega(\pi(m))\\ =&(\bar{f} \otimes id)(\pi \otimes id)\rho(m)\\ =&(f \otimes id)\rho(m)\\ =&\phi(f(m))\\ =&\theta(f(m))\\ =&\theta(\bar{f}(\overline{m}))\end{aligned}$$

which shows that $\bar{f}$ is a comodule homomorphism. □

1.3 Bicomodules

Definition 1.3.1. Let C and D be two coalgebras. A k-vector space M is called a D-C-*bicomodule* if M has a left D-comodule structure $\mu : M \to D \otimes M$ and a right C-comodule structure $\rho : M \to M \otimes C$ such that $(\mu \otimes id)\rho = (id \otimes \rho)\mu$.

We call $N \subseteq M$ a *subbicomodule* of M if N is a subcomodule of (M, μ) and a subcomodule of (M, ρ).

If M and N are two D-C-bicomodules, then a *bicomodule homomorphism* from M to N is a linear map $f : M \to N$ which is a comodule homomorphism of left D-comodules and right C-comodules.

In this way we can define a category of D-C-bicomodules that we will denote by ${}^{D}\mathbf{M}^{C}$.

Examples 1.3.2. (i) Any coalgebra C is a C-C-bicomodule with the left and right comodule structures given by the comultiplication.

(ii) If C and D are coalgebras and $f : C \to D$ a coalgebra homomorphism, then $(C, (f \otimes id)\Delta_C, (id \otimes f)\Delta_C)$ is a D-D-bicomodule and $f : C \to D$ is a bicomodule homomorphism of D-D-bicomodules. Write $\mu = (f \otimes id)\Delta_C$ and $\rho = (id \otimes f)\Delta_C$. In Example 1.2.7 (ii) we have that (C, ρ) is a right D-comodule and $f : C \to D$

is a comodule homomorphism of right D-comodules . One can show that (C, μ) is a left D-comodule and $f : C \to D$ is a comodule homomorphism of left D-comodules in a very similar way done for its right version and we will not do it here. We must confirm that $(\mu \otimes id)\rho = (id \otimes \rho)\mu$. Let c be any element of C. Then

$$
\begin{aligned}
(\mu \otimes id)\rho(c) =& (((f \otimes id)\Delta_C) \otimes id)((id \otimes f)\Delta_C)(c) \\
=& (((f \otimes id)\Delta_C) \otimes f)\Delta_C(c) \\
=& (f \otimes id \otimes f)(\Delta_C \otimes id)\Delta_C(c) \\
=& (f \otimes id \otimes f)(id \otimes \Delta_C)\Delta_C(c) \qquad (1) \\
=& (id \otimes ((id \otimes f)\Delta_C))((f \otimes id)\Delta_C)(c) \\
=& (id \otimes \rho)\mu(c),
\end{aligned}
$$

where (1) is due to the coassociativity of Δ_C.

Lemma 1.3.3. *Let C be a coalgebra and for each pair $g, h \in G(C)$ let $P'_{g,h}(C)$ be a subspace of $P_{g,h}(C)$ such that*

$$P_{g,h}(C) = k(h - g) \oplus P'_{g,h}(C).$$

Define the linear maps

$$
\begin{aligned}
\mu' : P'_{g,h}(C) &\to kG(C) \otimes P'_{g,h}(C) \\
c &\mapsto h \otimes c
\end{aligned}
$$

and

$$
\begin{aligned}
\rho' : P'_{g,h}(C) &\to P'_{g,h}(C) \otimes kG(C). \\
c &\mapsto c \otimes g
\end{aligned}
$$

Then, the vector space $V = kG(C) \oplus \left(\bigoplus_{g,h \in G(C)} P'_{g,h}(C)\right)$ is a $kG(C)$-$kG(C)$-bicomodule with the structure maps given by

$$
\begin{aligned}
\mu : V &\to kG(C) \otimes V \\
v &\mapsto \sum_{g,h \in G(C)} \mu'(c_{g,h}) + \Delta(\omega)
\end{aligned}
$$

and

$$
\begin{aligned}
\rho : V &\to V \otimes kG(C) \\
v &\mapsto \sum_{g,h \in G(C)} \rho'(c_{g,h}) + \Delta(\omega)
\end{aligned}
$$

where $v = \sum_{g,h \in G(C)} c_{g,h} + \omega$, *with* $c_{g,h} \in P'_{g,h}(C)$, *for each* $g, h \in G(C)$, *and* $\omega \in kG(C)$.

Proof. By Lemma 1.1.8, V is well defined. Since the maps μ and ρ are variations of the structure map of the right kS-comodule M from Example 1.2.6 (iii), we just need to show that $(id \otimes \rho)\mu = (\mu \otimes id)\rho$. Let $v = \sum_{g,h \in G(C)} c_{g,h} + \omega$ and $\omega = \sum_{e \in G(C)} \lambda_e e$.

$$\begin{aligned}
(id \otimes \rho)\mu(v) =&(id \otimes \rho)\mu\left(\sum_{g,h \in G(C)} c_{g,h} + \omega\right)\\
=&(id \otimes \rho)\left(\sum_{g,h \in G(C)} \mu'(c_{g,h}) + \Delta(\omega)\right)\\
=&(id \otimes \rho)\left(\sum_{g,h \in G(C)} h \otimes c_{g,h} + \sum_{e \in G(C)} \lambda_e e \otimes e\right)
\end{aligned}$$

$$\begin{aligned}
=& \sum_{g,h \in G(C)} h \otimes \rho'(c_{g,h}) + \sum_{e \in G(C)} \lambda_e e \otimes \Delta(e)\\
=& \sum_{g,h \in G(C)} h \otimes c_{g,h} \otimes g + \sum_{e \in G(C)} \lambda_e e \otimes e \otimes e\\
=& \sum_{g,h \in G(C)} \mu'(c_{g,h}) \otimes g + \sum_{e \in G(C)} \lambda_e \Delta(e) \otimes e
\end{aligned}$$

$$\begin{aligned}
=&(\mu \otimes id)\left(\sum_{g,h \in G(C)} c_{g,h} \otimes g + \sum_{e \in G(C)} \lambda_e e \otimes e\right)\\
=&(\mu \otimes id)\left(\sum_{g,h \in G(C)} \rho'(c_{g,h}) + \Delta(\omega)\right)\\
=&(\mu \otimes id)\rho(v).
\end{aligned}$$

Thus, (V, μ, ρ) is a $kG(C)$-$kG(C)$-bicomodule. □

Lemma 1.3.4. *Let* C *be a coalgebra,* (M, μ, ρ) *a* C-C*-bicomodule and* $N \subseteq M$ *a subbicomodule of* M*. Then there exists a unique bicomodule structure on the quotient* M/N *for which the canonical projection* $\pi : M \to M/N$ *is a bicomodule homomorphism.*

Proof. By Proposition 1.2.12, we already have unique left and right comodule structure maps on M/N for which π is a left and right comodule homomorphism. It remains to check that M/N is a C-C-bicomodule. Let $\bar{\mu}$ and $\bar{\rho}$ be the structure maps of M/N. Then for any $m \in M$ we have $\bar{\mu} \circ \pi(m) = (id \otimes \pi)\mu(m)$ and $\bar{\rho} \circ \pi(m) = (\pi \otimes\ id)\rho(m)$. Thus, for $\overline{m} \in M/N$ and any representative $m \in M$ we have

$$\begin{aligned}
(id \otimes \bar{\rho})\bar{\mu}(\overline{m}) =& (id \otimes \bar{\rho})\bar{\mu}(\pi(m)) \\
=& (id \otimes \bar{\rho})(id \otimes \pi)\mu(m) && (1) \\
=& (id \otimes \bar{\rho} \circ \pi)\mu(m) \\
=& (id \otimes (\pi \otimes id)\rho)\mu(m) && (2) \\
=& (id \otimes \pi \otimes id)(id \otimes \rho)\mu(m) \\
=& (id \otimes \pi \otimes id)(\mu \otimes id)\rho(m) && (3) \\
=& (id \otimes \pi)\mu \otimes id)\rho(m) \\
=& (\bar{\mu} \circ \pi \otimes id)\rho(m) && (4) \\
=& (\bar{\mu} \otimes id)(\pi \otimes id)\rho(m) \\
=& (\bar{\mu} \otimes id)\bar{\rho} \circ \pi(m) && (5) \\
=& (\bar{\mu} \otimes id)\bar{\rho}(\overline{m}),
\end{aligned}$$

where steps (1) and (4) are due to the definition of $\bar{\mu}$, steps (2) and (5) are due to the definition of $\bar{\rho}$, and step (3) is because (M, μ, ρ) is a C-C-bicomodule. Hence $(M/N, \bar{\mu}, \bar{\rho})$ is a C-C-bicomodule. $\square$

Definition 1.3.5. Let C be a coalgebra, M a right C-comodule with comodule structure $\rho_M : M \to M \otimes C$, and N a left C-comodule with comodule structure $\rho_N : N \to C \otimes N$. We denote by $M \square_C N$ the kernel of the morphism

$$\rho_M \otimes id - id \otimes \rho_N : M \otimes N \to M \otimes C \otimes N.$$

Then $M \square_C N$ is a k-subspace of $M \otimes N$ which is called the *cotensor product* of the comodules M and N.

Note that, if (M, ρ_l, ρ_r) is a D-C-bicomodule and (N, μ_l, μ_r) is a C-E-bicomodule, then $M \square_C N$ becomes a D-E-bicomodule with the structure maps $\rho_l \otimes id : M \square_C N \to D \otimes M \square_C N$ and $id \otimes \mu_r : M \square_C N \to M \square_C N \otimes E$.

We will just check that $(\rho_l \otimes id \otimes id)(id \otimes \mu_r) = (id \otimes id \otimes \mu_r)(\rho_l \otimes id)$, since from Example 1.2.6 (ii) we have that $(M \square_C N, id \otimes \mu_r)$ is a right E-comodule and similarly $(M \square_C N, \rho_l \otimes \mathrm{id})$ is a left D-comodule. Consider any $m \otimes n \in M \square_C N$. Then

$$\begin{aligned}
(\rho_l \otimes id \otimes id)(id \otimes \mu_r)(m \otimes n) =& (\rho_l \otimes \mu_r)(m \otimes n) \\
=& (id \otimes id \otimes \mu_r)(\rho_l \otimes id)(m \otimes n).
\end{aligned}$$

Thus $M\square_C N \in {}^D\mathbf{M}^E$ whenever $M \in {}^D\mathbf{M}^C$ and $N \in {}^C\mathbf{M}^E$.

What follows are extracts from [DNR, Chapter 2.3] that show some properties of the cotensor product.

Proposition 1.3.6. *(i) If $M \in \mathbf{M}^C$ and $N \in {}^C\mathbf{M}$, then $M\square_C C \cong M$ as right C-comodules and $C\square_C N \cong N$ as left C-comodules;*

(ii) If $M \in \mathbf{M}^C$ and $N \in {}^C\mathbf{M}$, then $M\square_C N \cong N\square_{C^{cop}} M$ as linear spaces;

(iii) If C and D are two coalgebras and $M \in {}^C\mathbf{M}^D$, $L \in \mathbf{M}^C$ and $N \in {}^D\mathbf{M}$, then we have a natural isomorphism $(L\square_C M)\square_D N \cong L\square_C(M\square_D N)$.

Proof. [DNR, Proposition 2.3.6]. □

Let C and D be coalgebras and $\phi : C \to D$ a coalgebra homomorphism. If $(M,\rho) \in \mathbf{M}^C$, then the map $(id \otimes \phi)\rho : M \to M \otimes D$ gives M a structure of right D-comodule (see Example 1.2.7 (ii) and treat Δ_C as a right C-comodule structure map). We denote by M_ϕ the space M regarded with this structure of right D-comodule. In this way we construct a left exact functor (since the tensor functor of vector spaces is exact and the cotensor is a kernel)

$$\begin{aligned}(-)_\phi : \mathbf{M}^C &\to \mathbf{M}^D\\ M &\mapsto M_\phi\end{aligned}$$

If $N \in \mathbf{M}^D$, we can define the right C-comodule $N^\phi = N\square_D C$, and in this way we have a left exact functor

$$\begin{aligned}(-)^\phi : \mathbf{M}^D &\to \mathbf{M}^C\\ N &\mapsto N^\phi\end{aligned}$$

Proposition 1.3.7. *Let $\phi : C \to D$ be a coalgebra homomorphism. Then, the functor $(-)_\phi : \mathbf{M}^C \to \mathbf{M}^D$ is a left adjoint to the functor $(-)^\phi : \mathbf{M}^D \to \mathbf{M}^C$.*

Proof. See [DNR, Proposition 2.3.8] or [Woo, Proposition 1.10]. □

With the bicomodule and the cotensor product concepts, we can construct a coalgebra in a similar way that is done for the tensor algebra (see [CHZ, 1.4] or [Woo, 4.1]).

Definition 1.3.8. Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra and (M, ρ_l, ρ_r) a C-C-bicomodule. Write $\rho_l(m) = \sum_{(m)} m_{(-1)} \otimes m_{(0)}$ and $\rho_r(m) = \sum m_{(0)} \otimes m_{(1)}$ for every $m \in M$, where each $m_{(0)}$ belongs to M and each $m_{(-1)}$ and $m_{(1)}$ belongs to C. Define $M^{\square_0} = C$, $M^{\square_1} = M$ and $M^{\square_n} = (M^{\square_{n-1}})\square_C M$, for any $n \geq 2$. If $m^1 \otimes \cdots \otimes m^n \in M^{\square_n}$, we write it as $m^1 \square \cdots \square m^n$.

We define the *cotensor coalgebra* $Cot_C(M)$ as the vector space

$$Cot_C(M) = \bigoplus_{i=0}^{\infty} M^{\square_i},$$

with counit ε given by

$$\varepsilon(\omega) = \begin{cases} \varepsilon_C(\omega), & \omega \in C, \\ 0, & \text{otherwise} \end{cases}$$

and comultiplication Δ given as follows:

- for any $c \in C$, we have
$$\Delta(c) = \Delta_C(c);$$
- and for any $m^1 \square \cdots \square m^n \in M^{\square_n}$, with $(n \geq 1)$, we have
$$\begin{aligned} \Delta(m^1 \square \cdots \square m^n) = &\sum_{(m^1)} ((m^1)_{(-1)}) \otimes ((m^1)_{(0)} \square m^2 \square \cdots \square m^n) \\ &+ \sum_{i=1}^{n-1} (m^1 \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^n) \\ &+ \sum_{(m^n)} (m^1 \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes ((m^n)_{(1)}). \end{aligned}$$

  where, if $n = 1$, we have
$$\Delta(m) = \sum_{(m)} m_{(-1)} \otimes m_{(0)} + \sum_{(m)} m_{(0)} \otimes m_{(1)} = \rho_l(m) + \rho_r(m);$$

In what follows, we will do the standard verification of a coalgebra for $Cot_C(M)$. Unfortunately, the computations are quite big. However, the reader could feel free to skip this calculation in regard of [CHZ, 1.4] and [Woo, 4.1] assert that $Cot_C(M)$ is indeed a coalgebra. As usual, we just need to check the coassociativity and the counitary property for the elements of a basis. For $c \in C$, $\Delta(c) = \Delta_C(c)$ and

$\varepsilon(c) = \varepsilon_C(c)$, so it is done. For $m^1\Box\cdots m^n \in M^{\Box_n}$, with $n \geq 1$, we have

$$
\begin{aligned}
(id\otimes\Delta)\Delta(m^1\Box\cdots\Box m^n) =&(id\otimes\Delta)\sum_{(m^1)}(m^1)_{(-1)}\otimes((m^1)_{(0)}\Box m^2\Box\cdots\Box m^n)\\
&+(id\otimes\Delta)\sum_{i=1}^{n-1}(m^1\Box\cdots\Box m^i)\otimes(m^{i+1}\Box\cdots\Box m^n)\\
&+(id\otimes\Delta)\sum_{(m^n)}(m^1\Box\cdots\Box m^{n-1}\Box(m^n)_{(0)})\otimes(m^n)_{(1)}
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{(m^1)}(m^1)_{(-1)}\otimes\Delta((m^1)_{(0)}\Box m^2\Box\cdots\Box m^n)\\
&+\sum_{i=1}^{n-1}(m^1\Box\cdots\Box m^i)\otimes\Delta(m^{i+1}\Box\cdots\Box m^n)\\
&+\sum_{(m^n)}(m^1\Box\cdots\Box m^{n-1}\Box(m^n)_{(0)})\otimes\Delta((m^n)_{(1)}
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{(m^1)}\sum_{((m^1)_{(0)})}(m^1)_{(-1)}\otimes((m^1)_{(0)})_{(-1)}\otimes(((m^1)_{(0)})_{(0)}\Box m^2\Box\cdots\Box m^n)\\
&+\sum_{(m^1)}\sum_{i=1}^{n}(m^1)_{(-1)}\otimes((m^1)_{(0)}\Box\cdots\Box m^i)\otimes(m^{i+1}\Box\cdots\Box m^n)\\
&+\sum_{(m^1)}\sum_{(m^n)}(m^1)_{(-1)}\otimes((m^1)_{(0)}\Box\cdots\Box m^{n-1}\Box(m^n)_{(0)})\otimes(m^n)_{(1)}\\
&+\sum_{i=1}^{n-1}\sum_{(m^{i+1})}(m^1\Box\cdots\Box m^i)\otimes(m^{i+1})_{(-1)}\otimes((m^{i+1})_{(0)}\Box\cdots\Box m^n)\\
&+\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}(m^1\Box\cdots\Box m^i)\otimes(m^{i+1}\Box\cdots\Box m^j)\otimes(m^{j+1}\Box\cdots\Box m^n)\\
&+\sum_{i=1}^{n-1}\sum_{(m^n)}(m^1\Box\cdots\Box m^i)\otimes(m^{i+1}\Box\cdots\Box m^{n-1}\Box(m^n)_{(0)})\otimes(m^n)_{(1)}\\
&+\sum_{(m^n)}(m^1\Box\cdots\Box m^{n-1}\Box(m^n)_{(0)})\otimes\Delta_C((m^n)_{(1)})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{(m^1)} \varDelta_C((m^1)_{(-1)}) \otimes ((m^1)_{(0)} \square m^2 \square \cdots \square m^n) && (1)\\
&\quad + \sum_{(m^1)} \sum_{i=1}^{n} (m^1)_{(-1)} \otimes ((m^1)_{(0)} \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^n)\\
&\quad + \sum_{(m^1)} \sum_{(m^n)} (m^1)_{(-1)} \otimes ((m^1)_{(0)} \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes (m^n)_{(1)}\\
&\quad + \sum_{i=1}^{n-1} \sum_{(m^{i+1})} (m^1 \square \cdots \square m^i) \otimes ((m^{i+1})_{(-1)}) \otimes ((m^{i+1})_{(0)} \square \cdots \square m^n)\\
&\quad + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} (m^1 \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^j) \otimes (m^{j+1} \square \cdots \square m^n)\\
&\quad + \sum_{i=1}^{n-1} \sum_{(m^n)} (m^1 \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes (m^n)_{(1)}\\
&\quad + \sum_{(m^n)} \sum_{((m^n)_(0))} (m^1 \square \cdots \square m^{n-1} \square ((m^n)_{(0)})_{(0)}) \otimes ((m^n)_{(0)})_{(1)} \otimes (m^n)_{(1)}, && (2)
\end{aligned}
$$

where on step (1) we use the identity $(id \otimes \rho_l)\rho_l = (\varDelta_C \otimes id)\rho_l$ and on step (2) we use the identity $(id \otimes \rho_r)\rho_r = (\rho_r \otimes id)\rho_r$,

$$
\begin{aligned}
&= \sum_{(m^1)} \varDelta_C((m^1)_{(-1)}) \otimes ((m^1)_{(0)} \square m^2 \square \cdots \square m^n)\\
&\quad + \sum_{(m^1)} \sum_{i=1}^{n} (m^1)_{(-1)} \otimes ((m^1)_{(0)} \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^n)\\
&\quad + \sum_{i=2}^{n-1} \sum_{j=1}^{i-1} (m^1 \square \cdots \square m^j) \otimes (m^{j+1} \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^n)\\
&\quad + \sum_{i=1}^{n-1} \sum_{(m^{i+1})} (m^1 \square \cdots \square m^i) \otimes ((m^{i+1})_{(-1)}) \otimes ((m^{i+1})_{(0)} \square \cdots \square m^n)\\
&\quad + \sum_{(m^n)} \sum_{(m^1)} (m^1)_{(-1)} \otimes ((m^1)_{(0)} \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes (m^n)_{(1)}\\
&\quad + \sum_{(m^n)} \sum_{i=1}^{n-1} (m^1 \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes (m^n)_{(1)}\\
&\quad + \sum_{(m^n)} (m^1 \square \cdots \square m^{n-1} \square ((m^n)_{(0)})_{(0)}) \otimes ((m^n)_{(0)})_{(1)} \otimes (m^n)_{(1)}
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{(m^1)} \Delta((m^1)_{(-1)}) \otimes ((m^1)_{(0)} \square m^2 \square \cdots \square m^n) \\
&+ \sum_{i=1}^{n-1} \Delta(m^1 \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^n) \\
&+ \sum_{(m^n)} \Delta(m^1 \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes (m^n)_{(1)}
\end{aligned}
$$

$$
\begin{aligned}
=&(\Delta \otimes id) \sum_{(m^1)} (m^1)_{(-1)} \otimes ((m^1)_{(0)} \square m^2 \square \cdots \square m^n) \\
&+ (\Delta \otimes id) \sum_{i=1}^{n-1} (m^1 \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^n) \\
&+ (\Delta \otimes id) \sum_{(m^n)} (m^1 \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes (m^n)_{(1)} \\
=&(\Delta \otimes id) \Delta(m^1 \square \cdots \square m^n).
\end{aligned}
$$

Therefore, the coassociativity is satisfied. Now we will check one side of the counitary property.

$$
\begin{aligned}
(\varepsilon \otimes id) \Delta(m^1 \square \cdots \square m^n) =&(\varepsilon \otimes id) \sum_{(m^1)} (m^1)_{(-1)} \otimes ((m^1)_{(0)} \square m^2 \square \cdots \square m^n) \\
&+ (\varepsilon \otimes id) \sum_{i=1}^{n-1} (m^1 \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^n) \\
&+ (\varepsilon \otimes id) \sum_{(m^n)} (m^1 \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes (m^n)_{(1)}
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{(m^1)} \varepsilon((m^1)_{(-1)}) \otimes ((m^1)_{(0)} \square m^2 \square \cdots \square m^n) \\
&+ \sum_{i=1}^{n-1} \varepsilon(m^1 \square \cdots \square m^i) \otimes (m^{i+1} \square \cdots \square m^n) \\
&+ \sum_{(m^n)} \varepsilon(m^1 \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes (m^n)_{(1)} \\
=&(m^1) \square m^2 \square \cdots \square m^n,
\end{aligned}
$$

where the last step is because of the identity $(\varepsilon \otimes id)\rho_l(m^1) = m^1$. The other way round goes in a similar fashion.

Hence, $(Cot_C(M), \Delta, \varepsilon)$ is indeed a coalgebra.

Lemma 1.3.9 (The universal property of the cotensor coalgebra)**.** *Let C and D be coalgebras and M a C-C-bicomodule. Given a coalgebra map $f_0 : D \to C$, and a C-C-bicomodule map $f_1 : D \to M$ with the property that f_1 vanishes on the coradical D_0 of D, where the C-C-bicomodule structure D is given via f_0. Then there exists a unique coalgebra map*

$$F : D \to Cot_C(M)$$

with $\pi_i \circ F = f_i$ for $i \in \{0, 1\}$, where each $\pi_i : Cot_C(M) \to M^{\square i}$ is the canonical projection.

Proof. [CHZ, Lemma 3.2].

□

1.4 Coideals

Definition 1.4.1. [DNR, Definition 1.4.3 (ii)] Let C be a coalgebra and $I \subset C$ a subspace of C. We say I is a *coideal* of C if $\Delta(I) \subseteq I \otimes C + C \otimes I$ and $\varepsilon(I) = 0$.

Definition 1.4.2. Let C be a coalgebra.

(i) The right C-comodule M is *simple* if there is no non-zero proper subcomodule of M;

(ii) C is *simple* if there is no non-zero proper subcoalgebra of C.

Proposition 1.4.3. *Let C and D be coalgebras and $f : C \to D$ a coalgebra homomorphism. Then $im(f)$ is a subcoalgebra of D and $ker(f)$ is a coideal of C.*

Proof. [DNR, Proposition 1.4.9].

Since C is a subcoalgebra of C, then $f(C)$ is a subcoalgebra of D by Lemma 1.1.3. Note that $\Delta_D(f(ker(f)) = 0$ implies $(f \otimes f)\Delta_C(ker(f)) = 0$. Thus

$$\Delta_C(ker(f)) \subseteq \ker(f \otimes f) = ker(f) \otimes C + C \otimes ker(f)$$

by Lemma 1.2.10. Since f is a coalgebra homomorphism, we have

$$\varepsilon_C(ker(f)) = \varepsilon_D(f(ker(f))) = 0.$$

Hence $ker(f)$ is a coideal of C. □

Theorem 1.4.4 (The quotient coalgebra)**.** *Let C be a coalgebra, I a coideal of C and $\pi : C \to C/I$ the canonical projection. Then there exists a unique coalgebra structure on C/I such that π is a coalgebra homomorphism. Moreover, if D is a coalgebra and $f : C \to D$ is a coalgebra homomorphism with $I \subseteq ker(f)$, then there exists a unique coalgebra homomorphism $\bar{f} : C/I \to D$ for which $\bar{f} \circ \pi = f$.*

Proof. [DNR, Theorem 1.4.10]. □

Corollary 1.4.5 (The fundamental isomorphism theorem for coalgebras)**.** *Let C and D be coalgebras and $f : C \to D$ be a coalgebra homomorphism. Then there exists a canonical isomorphism of coalgebras between $C/ker(f)$ and $im(f)$.*

Proof. We have the following diagram

$$\begin{tikzcd} C \arrow[r, two heads, "\pi"] \arrow[dr, two heads, "f'"'] & C/ker(f) \arrow[d, dashed, "\bar{f}"] \\ & im(f), \end{tikzcd}$$

where f' is the corestriction of f to its image, π is the canonical projection and $\bar{f}$ is the unique coalgebra homomorphism for which π is a coalgebra homomorphism and the diagram commutes, by Theorem 1.4.4. We will show that $\bar{f}$ is a bijection.

Surjectivity is immediate since any element in $im(f)$ is of the form $f(c)$ for some $c \in C$ and surjectivity of π give us $\bar{f} \circ \pi(c) = f'(c) = f(c)$.

Consider now two elements $c, d \in C$. If $f(c) = f(d)$, then $f(c-d) = 0$ and, hence, c and d belong to the same coset. Thus $\bar{f}$ is an isomorphism of coalgebras. □

The next two theorems show a property of coalgebras which is not shared with algebras that is any finitely generated coalgebra is finite dimensional.

Theorem 1.4.6. *[The Fundamental Theorem of comodules] Let V be a right C-comodule. Any element $v \in V$ belongs to a finite dimensional subcomodule of V.*

Proof. [DNR, Theorem 2.1.7].

Let $\{c_i\}_{i \in I}$ be a basis for C. Denote by $\rho : V \to V \otimes C$ the structure map of V and write

$$\rho(v) = \sum_{i \in I} v_i \otimes c_i,$$

where almost all of the v_i's are zero. Then the subspace W generated by the v_i's is finite dimensional. For each $i \in I$, write

$$\Delta(c_i) = \sum_{j,l \in I} \lambda_{ijl} c_j \otimes c_l.$$

Thus, the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "id\otimes\Delta"] \\
M \otimes C \arrow[r, "\rho\otimes id"] & M \otimes C \otimes C
\end{tikzcd}$$

gives

$$\begin{aligned}
\sum_{i \in I} \rho(v_i) \otimes c_i =& (\rho \otimes id)\rho(v) \\
=& (id \otimes \Delta)\rho(v) \\
=& \sum_{i \in I} v_i \otimes \sum_{j,l \in I} \lambda_{ijl} c_j \otimes c_l \\
=& \sum_{i,j,l \in I} v_i \otimes \lambda_{ijl} c_j \otimes c_l.
\end{aligned}$$

Since the c_i's are linearly independent, we must have

$$\sum_{i,j \in I} (\rho(v_l) - v_i \otimes \lambda_{ijl} c_j) \otimes c_l = 0,$$

for each $l \in I$. Consequently, $\rho(v_l) = \sum_{i,j \in I} v_i \otimes \lambda_{ijl} c_j \subseteq W \otimes C$ and so W is a finite dimensional subcomodule of V. Moreover, $v = (id \otimes \varepsilon)\rho(v) \in W$ and the theorem is proved. □

Theorem 1.4.7 (The Fundamental Theorem of coalgebras)**.** *Let C be a coalgebra. Given any finite subset $\{c_i\} \subset C$ there exists a finite dimensional subcoalgebra D of C such that $c_i \in D, \forall i$.*

Proof. [Mon, Theorem 5.1.1.2].

Since C is a C-C-bicomodule, by Theorem 1.4.6 the given $\{c_i\}$ are contained in a finite dimensional subspace V with $\Delta(V) \subseteq V \otimes C$. Let $\{v_j\}$ be a basis of V with $\Delta(v_j) = \sum_{i \in I} v_i \otimes c_{ij}$, for I a finite index set. Then the coassociativity of the comultiplication gives

$$\begin{aligned}\sum_{i\in I} v_i \otimes \Delta(c_{ij}) =&(id \otimes \Delta)\Delta(v_j)\\ =&(\Delta \otimes id)\Delta(v_j)\\ =&\sum_{i\in I} \Delta(v_i) \otimes c_{ij}\\ =&\sum_{i\in I}\left(\sum_{l\in I} v_l \otimes c_{li}\right) \otimes c_{ij}\\ =&\sum_{i,l\in I} v_l \otimes c_{li} \otimes c_{ij}.\end{aligned}$$

Since the v_i's are linearly independent, we must have

$$\sum_{t,\in I} v_i \otimes (\Delta(c_{ij}) - c_{it} \otimes c_{tj}) = 0.$$

Consequently, for each $i, j \in I$ we have $\Delta(c_{ij}) = \sum_{t\in I} c_{it} \otimes c_{tj}$. Thus the span D of $\{v_j\}$ and $\{c_{ij}\}$ is finite dimensional and satisfies $\Delta(D) \subseteq D \otimes D$. Since $V \subseteq D$ by construction, the theorem is proved. □

Corollary 1.4.8. *Let C be a coalgebra. Then*

(i) every simple subcoalgebra of C is finite dimensional;

(ii) every simple C-comodule is finite dimensional.

Proof. Immediately from Theorem 1.4.6 and Theorem 1.4.7. □

Chapter 2

On the structure of coalgebras: part 2

2.1 The coradical filtration

The theorems 2.1.2 and 2.1.7 and the Lemma 2.1.5 of this Section, and Theorem 2.2.3 of the next Section, will not be proved, since the proof involves explicitly the duality between algebras and coalgebras that we have decided not treat it here (see [DNR, Chapter 1.3] for more details).

Definition 2.1.1. Let C be a coalgebra.

(i) C is *cosemisimple* if it is a direct sum of simple coalgebras;

(ii) The *coradical* C_0 of C is the sum of all simple subcoalgebra of C.

The following definitions were taken from [Abe, Chapter 2.4.1]

Let $I = \{0, 1, 2, \cdots\}$ be the set of all non-negative integers. Given a coalgebra C, if a family $\{A_i\}_{i \in I}$ of k-linear subspaces of C satisfies the conditions

$$A_i \subset A_{i+1} \quad (i \in I), \quad C = \bigcup_{i \in I} A_i$$

$$\Delta(A_n) \subset \sum_{i=0}^{n} A_i \otimes A_{n-i} \quad (n \in I)$$

then C is called a *filtered coalgebra*, and $\{A_i\}$ is said to be a *filtration* on C. By definition, A_i $(i \in I)$ are subcoalgebras of C.

If there exists a family of subspaces $\{A_{(i)}\}_{i\in I}$ of C such that

$$C=\bigoplus_{i\in I}A_{(i)},\quad \varepsilon(A_{(n)})=0\quad (n\neq 0),$$
$$\Delta(A_{(n)})\subset\sum_{i=0}^{n}A_{(i)}\otimes A_{(n-i)}\quad (n\in I)$$

then C is called a *graded coalgebra*.

If $C=\bigoplus_{i\in I}A_{(i)}$ is a graded coalgebra and we set $A_n=\bigoplus_{i\leq n}A_{(i)}$, then $\{A_n\}_{n\in I}$ becomes a filtration on C and, hence, C is a filtered coalgebra.

On the other hand, if C is a filtered coalgebra with filtration $\{A_i\}_{i\in I}$ then, setting $A_{(i)}=A_i/A_{i-1}$ for $i\geq 1$ and $A_{(0)}=A_0$, we obtain a graded coalgebra $grC=\bigoplus_{i\in I}A_{(i)}$, called the *associated graded coalgebra of the filtered coalgebra C*.

Now, let C be a coalgebra and C_0 its coradical. Define inductively

$$C_n=\Delta^{-1}(C\otimes C_{n-1}+C_0\otimes C). \tag{2.1}$$

Then

Theorem 2.1.2. *$\{C_n\}_{n\in I}$ is a filtration on C.*

Proof. [Mon, Theorem 5.2.2]. □

We call $\{C_n\}_{n\in I}$, as defined in (2.1), the *coradical filtration* of C.

Let $C=\bigoplus_{i\in I}C_{(i)}$ be a graded coalgebra with coradical filtration $\{C_j\}_{j\in I}$. If $C_0=C_{(0)}$ and $C_1=C_{(0)}\oplus C_{(1)}$, then we say that C is *coradically graded*.

Lemma 2.1.3. *If C is coradically graded, then*

$$C_j=\bigoplus_{i\leq j}C_{(i)}.$$

Proof. See [CM, Lemma 2.2]. □

Lemma 2.1.4. *The cotensor coalgebra $Cot_C(M)$, as in Definition 1.3.8, is a graded coalgebra with grading $\{M^{\square_i}\}_{i\in I}$. Moreover, if C is cosemisimple, then $Cot_C(M)$ is coradically graded.*

Proof. [Woo, Lemma 4.4]. □

Lemma 2.1.5. *If D is a subcoalgebra of C, then $D_n=D\cap C_n$, for all $n\geq 0$.*

Proof. [Mon, Lemma 5.2.12] and [Mon, Lemma 5.1.9]. □

The proof of the next theorem is too long and we will not prove it here.

Theorem 2.1.6 (Heyneman-Radford)**.** *Let C and D be coalgebras and $f : C \to D$ a coalgebra homomorphism. Then f is injective if and only if $f|_{C_1} : C_1 \to D$ is injective, where C_1 is the subcoalgebra of the coradical filtration of C as defined in (2.1).*

Proof. [Mon, Theorem 5.3.1]. □

The next theorem is the dual version of the Principal Theorem of Wedderburn [Abe, Theorem 1.4.9]. It was originally stated for coalgebras with separable coradical (see [Mon, Theorem 5.4.2]). However, since every k-coalgebra with k algebraically closed has separable coradical, we will omit this term.

Theorem 2.1.7 (Dual Wedderburn-Malcev theorem)**.** *Let C be a coalgebra. Then, there exists a coideal I such that $C = C_0 \oplus I$ (as vector spaces).*

Proof. See [Mon, Theorem 5.4.2] or [Abe, Theorem 2.3.11]. □

Remark 2.1.8. *As a consequence of the above theorem, we have a projection $\pi_I : C \to C/I$ that is a coalgebra homomorphism (see Theorem 1.4.4), where I is a coideal of the coalgebra $C = C_0 \oplus I$. However, the coideal I is not uniquely determined.*

Note that if $c \in C$, we can write $c = c_0 + c_I$, where $c_0 \in C_0$ and $c_I \in I$. Then

$$\pi_I(c) = \overline{c} = c + I = (c_0 + c_I) + I = c_0 + I = \overline{c_0}.$$

Thus, there exists a bijection between the cosets of I in C and the elements of C_0

$$\begin{aligned} \sigma_I : C/I &\to C_0 \\ \overline{c_0} &\mapsto c_0 \end{aligned}$$

It is easy to see that σ_I is a coalgebra homomorphism. The composition $\sigma_I \circ \pi_I$ restricted to C_0 is the identity map of C_0, for any decomposition $C = C_0 \oplus I$. We will call $\pi_0 : C \to C_0$ the canonical projection of coalgebras*, where, for any fixed decomposition $C = C_0 \oplus I$, $\pi_0 = \sigma_I \circ \pi_I$.*

2.2 Pointed coalgebras

Definition 2.2.1. C is *pointed* if every simple subcoalgebra of C is one dimensional.

Remark 2.2.2. *Necessarily a one-dimensional subcoalgebra is of the form kg, for $g \in G(C)$, since if $\{c\}$ is any basis for C then*

$$\Delta(c) = \lambda_1 c \otimes \lambda_2 c$$

for some $\lambda_1, \lambda_2 \in k$. The counitary property gives

$$c \otimes 1 = (id \otimes \varepsilon)\Delta(c) = \lambda_1 c \otimes \varepsilon(\lambda_2 c) = c \otimes \varepsilon(\lambda_1\lambda_2 c).$$

Thus, $c \otimes (\varepsilon(\lambda_1\lambda_2 c) - 1) = 0$ implies $\varepsilon(\lambda_1\lambda_2 c) = 1$. Moreover,

$$\Delta(\lambda_1\lambda_2 c) = \lambda_1\lambda_2\Delta(c) = \lambda_1\lambda_2\lambda_1 c \otimes \lambda_2 c = \lambda_1\lambda_2 c \otimes \lambda_1\lambda_2 c.$$

Hence $\lambda_1\lambda_2 c \in G(C)$ and $C = k(\lambda_1\lambda_2 c)$.

Thus C is pointed iff $C_0 = kG(C)$. Furthermore, since $G(C)$ is a linearly independent set by Proposition 1.1.7, a sum of simple subcoalgebra is in fact a direct sum. Thus a pointed coalgebra C is cosemisimple iff $C = C_0$.

Theorem 2.2.3. *Let C be a pointed coalgebra. Then*

(i) $C_1 = kG(C) \oplus \left(\bigoplus_{g,h\in G(C)} P'_{g,h}(C)\right)$;

(ii) for any $n \geq 1$ and $c \in C_n$,

$$c = \sum_{g,h\in G(C)} c_{g,h}, \text{ where } \Delta(c_{g,h}) = c_{g,h} \otimes g + h \otimes c_{g,h} + \omega$$

for some $\omega \in C_{n-1} \otimes C_{n-1}$.

Proof. [Mon, Theorem 5.4.1]. □

Corollary 2.2.4. *Let C and D be pointed coalgebras and $f : C \to D$ be a coalgebra homomorphism. Then $f(C_1) \subseteq D_1$ and $f(C_0) \subseteq D_0$.*

Proof. It is immediate from the description of C_0 and C_1 for pointed coalgebras and Lemma 1.1.9. □

Lemma 2.2.5. *Let C be a pointed coalgebra. Then there exists a unique C_0-C_0-bicomodule structure map on the quotient $\bar{P}_{g,h}(C) = P_{g,h}(C)/k(h-g)$ such that the canonical projection $\pi : P_{g,h}(C) \to P_{g,h}(C)/k(h-g)$ is a bicomodule homomorphism.*

Proof. By Lemma 1.3.3, C_1 is a C_0-C_0-bicomodule. It is clear that $P_{g,h}(C)$ is a subbicomodule of C_1 and $k(h-g)$ is a subbicomodule of $P_{g,h}(C)$. Thus, the result follows from Lemma 1.3.4. $\square$

Proposition 2.2.6. *Let C be a pointed coalgebra. Then C_1/C_0 is a C_0-C_0-bicomodule and $C_1/C_0 \cong \bigoplus_{g,h\in G(C)} \bar{P}_{g,h}(C)$.*

Proof. By Lemma 1.3.3, C_1 is a C_0-C_0-bicomodule. Since C_0 is a subbicomodule of C_1, we have that C_1/C_0 is a C_0-C_0-bicomodule by Lemma 1.3.4. It remains to prove the second claim.

Since $P_{g,h}(C)$ is a subbicomodule of C_1 and $P_{g,h}(C) \cap C_0 = k(h-g)$ (Lemma 1.1.8), we have that $\bar{P}_{g,h}(C)$ is a subbicomodule of C_1/C_0. By Theorem 2.2.3, $C_1 = C_0 \oplus \left(\bigoplus_{g,h\in G(C)} P'_{g,h}(C)\right)$. Then, if $\pi : C_1 \to C_1/C_0$ is the canonical projection, we have

$$\pi\left(C_0 + \sum_{g,h\in G(C)} P_{g,h}(C)\right) = \sum_{g,h\in G(C)} \bar{P}_{g,h}(C) = C_1/C_0. \tag{2.2}$$

By Lemma 1.1.8, $P_{g,h}(C) \cap P_{g',h'}(C) \subseteq k(h-g)$ whenever $g' \neq g$ or $h' \neq h$. Thus, the sum in 2.2 is actually a direct sum. $\square$

Remark 2.2.7. *C_1/C_0 does not depend on the choice of $P'_{g,h}(C)$, since for any other decomposition of $P_{g,h}(C)$, say $P_{g,h}(C) = k(h-g) \oplus P''_{g,h}(C)$, write*

$$c = \sum_{g,h\in G(C)} c_{g,h} + \omega = \sum_{g,h\in G(C)} c'_{g,h} + \omega',$$

where $c_{g,h} \in P'_{g,h}(C)$ and $c'_{g,h} \in P''_{g,h}(C)$, for each $g, h \in G(C)$, and $\omega, \omega' \in C_0$. Write $\omega = \sum_{e\in G(C)} \lambda_e e$ and $\omega' = \sum_{e\in G(C)} \lambda'_e e$. Then

$$\begin{aligned}
\sum_{g,h\in G(C)} h \otimes \overline{c_{g,h}} \otimes g =& (\bar{\mu} \otimes id)\bar{\rho}(\pi(c)) \\
=& (\bar{\mu} \otimes id)(\pi \otimes id)\rho(c) \\
=& (\bar{\mu} \circ \pi \otimes id)\rho(c) \\
=& ((id \otimes \pi)\mu \otimes id)\left(\sum_{g,h\in G(C)} c'_{g,h} \otimes g + \sum_{e\in G(C)} \lambda'_e e \otimes e\right) \\
=& \sum_{g,h\in G(C)} h \otimes \pi(c'_{g,h}) \otimes g + \sum_{e\in G(C)} \lambda'_e e \otimes \pi(e) \otimes e \\
=& \sum_{g,h\in G(C)} h \otimes \overline{c'_{g,h}} \otimes g.
\end{aligned}$$

Since the set $G(C)$ is linearly independent (Proposition 1.1.7), we have that, for each $g \in G(C)$,

$$\sum_{h \in G(C)} h \otimes \overline{c_{g,h}} = \sum_{h \in G(C)} h \otimes \overline{c'_{g,h}},$$

and so, for each $h \in G(C)$,

$$\overline{c_{g,h}} = \overline{c'_{g,h}}$$

Thus, the right C_0-comodule C_1/C_0 is independent of the decomposition of C_1. The same is true if we view C_1/C_0 as a left C_0-comodule. Thus the C_1/C_0 bicomodule does not depend on the decomposition of C_1.

Examples 2.2.8. Let C be a pointed coalgebra. By the Proposition above we have that C_1/C_0 is a C_0-C_0-bicomodule. Define the cotensor coalgebra $Cot_{C_0}(C_1/C_0)$ as in Definition 1.3.8. By lemma 2.1.4, $Cot_{C_0}(C_1/C_0)$ is coradically graded. Thus $Cot_{C_0}(C_1/C_0)_0 = C_0$ and so $Cot_{C_0}(C_1/C_0)$ is a pointed coalgebra. Moreover, $Cot_{C_0}(C_1/C_0)_1 = C_0 \oplus C_1/C_0$.

Theorem 2.2.9. *Let C be a coalgebra (with separable coradical C_0). Then there exists a coalgebra embedding*

$$\iota : C \hookrightarrow Cot_{C_0}(C_1/C_0)$$

with $\iota(C_1) = C_0 \oplus C_1/C_0$.

Proof. We will rewrite the proof of this theorem given on [CHZ, Theorem 3.1] because we need some conclusions within this proof.

By the dual Wedderburn-Malcev theorem, Theorem 2.1.7, there exists a coideal I of C such that $C = C_0 \oplus I$. Thus, we have a canonical projection $f_0 : C \to C_0$ such that $f_0|_{C_0} = id$. Note that C becomes a C_0-C_0-bicomodule via f_0, Example 1.3.2 (ii), and I is a C_0-C_0-subbicomodule of C. Set $C_{(1)} = C_1 \cap I$. Then $C_1 = C_0 \oplus C_{(1)}$. Note that $C_{(1)}$ is a C_0-C_0-subbicomodule of I and the canonical vector space isomorphism $\theta : C_{(1)} \cong C_1/C_0$ is a C_0-C_0-bicomodule map.

View I as a $C_0 \otimes C_0^{cop}$-comodule and $C_{(1)}$ its subcomodule. Since C_0 is separable, it follows there exists a $C_0 \otimes C_0^{cop}$-comodule decomposition $I = C_{(1)} \oplus J$. Thus we have a C_0-C_0-bicomodule projection $p : I \to C_{(1)}$ such that $p|_{C_{(1)}} = id$. Define a map $f_1 = \theta \circ p \circ f'_0$ from C to C_1/C_0, where $f'_0 : C \to I$ is the canonical projetion. Clearly $f_1 : C \to C_1/C_0$ is a C_0-C_0-bicomodule map vanishing on C_0. Thus, by Lemma 1.3.9 we obtain a unique coalgebra map $\iota : C \to Cot_{C_0}(C_1/C_0)$ such that $\pi_0 \circ \iota = f_0$ and $\pi_1 \circ \iota = f_1$. Clearly $\iota(C_1) = C_0 \oplus C_1/C_0$. By Theorem 2.1.6, ι is injective. This completes the proof. □

Corollary 2.2.10. *Let C be a pointed coalgebra and $\iota : C \hookrightarrow Cot_{C_0}(C_1/C_0)$ be the coalgebra homomorphism as in the theorem above. Then, $\pi_0 \circ \iota|_{C_0} : C_0 \to C_0$ is the identity of C_0 and, for any decomposition $C = C_0 \oplus I$, there exists an isomorphism $\theta_I : C_1 \cap I \to C_1/C_0$ such that*

$$\theta_I^{-1} \circ \pi_1 \circ \iota|_{C_1 \cap I} : C_1 \cap I \to C_1 \cap I$$

is the identity map.

Proof. Within the proof of the Theorem 2.2.9. $\square$

2.3 Quivers and path coalgebras

Recall that a quiver $Q = (Q_0, Q_1)$ is an oriented graph with a set of vertices Q_0 and a set of arrows Q_1. For each arrow $\alpha \in Q_1$ we associate a pair of vertices $i, j \in Q_0$ that we call the *source* of α and the *target* of α, respectively. In this case, we write α as $\alpha : i \to j$ and say that α is an arrow from i to j. A *path* b in Q is the formal composition of arrows in Q_1 such that the target of an arrow coincides with the source of the next arrow. We say that a path b has *length* the number of arrows in the sequence that determine b. For instance, if $b = \alpha_n \alpha_{n-1} \cdots \alpha_1$ is a path in Q, with each $\alpha_l \in Q_1$, then b has length n and for each pair $\alpha_l, \alpha_l + 1$ we must have that the target of α_l is equal to the source of $\alpha_l + 1$. For each $i \in Q_0$ we associate a *stationary path* e_i of length 0 and source and target i. We can compose paths in a similar way as done for arrows. We say that a quiver Q is connected if its underlying graph is connected.

Definition 2.3.1. [Sim, Description 4.12] For a given pointed coalgebra C we define the *left Gabriel quiver* ${}_CQ = ({}_CQ_0, {}_CQ_1)$ by identifying the set of vertices ${}_CQ_0$ with the set $G(C)$ of group-like elements of C and, given two vertices $g, h \in G(C)$, we identify the arrows from g to h with a k-basis of the quotient space $\bar{P}_{g,h}(C) = P_{g,h}(C)/k(h-g)$.

Definition 2.3.2. [Woo, Definition 4.10] For a given quiver Q, we define the *path coalgebra* $k^{\square}Q$ of Q as the vector space with basis all paths in Q and, for each path $b \in Q$, the comultiplication and counity given by

$$\begin{aligned} \Delta(b) =& \sum_{b=b_2 b_1} b_2 \otimes b_1, \\ \varepsilon(b) =& \delta_{|b|\,0}, \end{aligned}$$

where the pairs b_1, b_2 are all possible paths in Q whose composition gives the path b. $|b|$ denotes the length of b and

$$\delta_{|b|\,0} = \begin{cases} 1, & \text{if } |b| = 0, \\ 0, & \text{otherwise} \end{cases} .$$

Write

$$(k^{\square}Q)_m = \bigoplus_{l \leq m} kQ_l$$

where kQ_l are all paths of Q of length l.

Proposition 2.3.3. *Let Q be a connected quiver and $k^{\square}Q$ its path coalgebra. Then*

(i) $k^{\square}Q$ is pointed, $G(k^{\square}Q) = \{e_i | i \in Q_0\}$, $(k^{\square}Q)_0 = kQ_0$, and $k^{\square}Q$ is coradically graded with coradical filtration $\{(k^{\square}Q)_m\}_{m\in\mathbb{N}}$;

(ii) Q is isomorphic to the left Gabriel quiver of $k^{\square}Q$.

Proof. See [Sim, Proposition 7.7]. It also follows from the canonical isomorphism $k^{\square}Q \cong Cot_{kQ_0}(span\{Q_1\})$ stated in [Woo] right after Definition 4.10. $\square$

Examples 2.3.4. (i) Consider the quiver Q^1 given by

$$\circ_1 \xrightarrow{\quad\alpha\quad} \circ_2$$

The path coalgebra $k^{\square}Q^1$ is the vector space with basis $\{e_1, e_2, \alpha\}$, together with the comultiplication Δ_1 given by

$$\Delta_1(e_i) = e_i \otimes e_i, \text{ for } i \in \{1, 2\}; \qquad \Delta_1(\alpha) = \alpha \otimes e_1 + e_2 \otimes \alpha,$$

and the counity ε_1 given by

$$\varepsilon_1(e_i) = 1, \text{ for } i \in \{1, 2\}; \qquad \varepsilon_1(\alpha) = 0.$$

(ii) Consider the quiver Q^2 given by

$$\begin{tikzcd} \circ_1 \arrow[loop above, "\alpha"] \end{tikzcd}$$

The path coalgebra $k^{\square}Q^2$ is the vector space with basis $\{e_1 = \alpha^0, \alpha, \alpha^2, \cdots\}$, together with the comultiplication Δ_2 given by

$$\Delta_2(\alpha^n) = \sum_{i=0}^{n} \alpha^i \otimes \alpha^{n-i}, \text{ for } n \geq 0,$$

and the counity ε_2 given by

$$\varepsilon_2(\alpha^n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1 \end{cases} .$$

This coalgebra is sometimes known as the *divided power coalgebra* (see [DNR, 1.1.4,2])

(iii) Consider the quiver Q^3 given by

$$\begin{tikzcd} \circ_1 \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & \circ_3 \\ & \circ_2 \arrow[ur, "\gamma"'] & \end{tikzcd}$$

The path coalgebra $k^{\square}Q^3$ is the vector space with basis $\{e_1, e_2, \alpha, \beta, \gamma, \gamma\beta\}$, together with the comultiplication Δ_3 given by

$$\Delta_3(b) = \begin{cases} e_i \otimes e_i, & \text{if } b = e_i, \text{ for } i \in \{1,2,3\} \\ \alpha \otimes e_1 + e_3 \otimes \alpha, & \text{if } b = \alpha \\ \beta \otimes e_1 + e_2 \otimes \beta, & \text{if } b = \beta \\ \gamma \otimes e_2 + e_3 \otimes \gamma, & \text{if } b = \gamma \\ \gamma\beta \otimes e_1 + \gamma \otimes \beta + e_3 \otimes \gamma\beta, & \text{if } b = \gamma\beta \end{cases}$$

and the counity ε_3 given by

$$\varepsilon_3(b) = \begin{cases} 1, & \text{if } b = e_i, \text{ for } i \in \{1,2,3\} \\ 0, & \text{otherwise} \end{cases} .$$

(iv) Let n be a positive integer and $L^C(n,k)$ be the *lower triangular matrix coalgebra*, which is the matrix coalgebra $M^C(n,k)$ with all entries $e_{ij} = 0$, for $i > j$ (see Example 1.1.5 (iii)). Then, the set of group-like elements of $L^C(n,k)$ is

$$G(L^C(n,k)) = \{e_{ii} | i \in \{1, 2, \cdots, n\}\} \tag{2.3}$$

and the set of e_{ii}, e_{jj}-primitive elements are given by

$$P_{e_{ii},e_{jj}}(L^C(n,k)) = \begin{cases} \{\lambda e_{ij} + \kappa(e_{jj} - e_{ii}) \,|\, \lambda, \kappa \in k\}, & \text{if } j = i+1, \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

Hence, the Gabriel quiver of $L^C(n,k)$ is the quiver ${}_{L^C(n,k)}Q$ given by

$$e_{11} \xrightarrow{\alpha_1} e_{22} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} e_{nn}$$

where α_i is an element of a basis of the quotient space $\bar{P}_{e_{ii},e_{i+1i+1}}(L^C(n,k))$.

For instance, if $n = 3$ and $k = \mathbb{C}$, then a basis for $L^C(3,\mathbb{C})$ is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\},$$

the comultiplication is given by

$$\begin{aligned} \Delta\left(\begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ f & e & c \end{bmatrix}\right) =& a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &+ b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &+ d\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &+ e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &+ f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &+ f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

and the counit is given by

$$\varepsilon\left(\begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ f & e & c \end{bmatrix}\right) = a + b + c.$$

Write e_{ij} for the matrix with entry 1 at the row i and column j and zero for all other entries. Then, $G(L^C(3,\mathbb{C}))$ and $P_{e_{ii},e_{jj}}(L^C(3,\mathbb{C}))$ are given as in (2.3) and (2.4), respectively. Moreover, $\bar{P}_{e_{11},e_{22}}(L^C(3,\mathbb{C})) =< \overline{e_{21}} >$ and $\bar{P}_{e_{22},e_{33}}(L^C(3,\mathbb{C})) =< \overline{e_{32}} >$, where $\overline{e_{ij}} = \{e_{ij} + \lambda(e_{jj} - e_{ii}) \,|\, \lambda \in \mathbb{C}\}$. Write $\alpha_1 = \overline{e_{21}}$ and $\alpha_2 = \overline{e_{32}}$. Then, the left Gabriel quiver of $L^C(3,\mathbb{C})$ is given by

$$e_{11} \xrightarrow{\alpha_1} e_{22} \xrightarrow{\alpha_2} e_{33}$$

Chapter 3

The path coalgebra and the adjunction

3.1 Categories

Definition 3.1.1. (i) A *Vquiver*, $VQ = (VQ_0, VQ_1)$ is a set of vertices $VQ_0 = \{e_1, e_2, ...\}$, together with a direct sum of vector spaces $VQ_1 = \bigoplus_{e,f \in VQ_0} VQ_{e,f}$. We call VQ_0 the *vertex set* of VQ and VQ_1 the *arrow* set of VQ. A Vquiver $VS = (VS_0, VS_1)$ is said to be a *subVquiver* of VQ if $VS_0 \subseteq VQ_0$ and for each pair $e, f \in VS_0$, $VS_{e,f} \subseteq VQ_{e,f}$.

(ii) A *map of Vquivers* $\varphi : VQ \to VR$ consists of an injective map $\varphi_0 : VQ_0 \to VR_0$, called the *vertex map*, and a linear map $\varphi_{e,f} : VQ_{e,f} \to VR_{\varphi_0(e),\varphi_0(f)}$ for each pair $e, f \in VQ_0$, called *arrow maps*. We say that φ is injective if each $\varphi_{e,f}$ is injective.

If $\varphi : VQ \to VR$ is an injective map of Vquivers, then $\varphi(VQ)$ is a subVquiver of VR. Moreover, if $\sigma : VR \to VS$ is an injective map of Vquivers, then $\sigma \circ \varphi : VQ \to VS$ is an injective map of Vquivers. Hence, taking all Vquivers as objects and all injective maps of Vquivers as morphisms, we obtain a category that we will denote by **IVquiv**.

We have a correspondence between quivers and Vquivers that is actually functorial from the first to the second, but not the other way round. The following diagram illustrates this correspondence

$$\begin{tikzcd}
\circ_1 \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\delta"'] \arrow[dr, "\beta"'] & & \circ_3 \arrow[r, rightsquigarrow] & \circ_1 \arrow[rr, "<\alpha,\delta>"] \arrow[dr, "<\beta>"'] & & \circ_3 \\
& \circ_2 \arrow[ur, "\gamma"'] & & & \circ_2 \arrow[ur, "<\gamma>"'] &
\end{tikzcd}$$

Denote by **IPCog** the category of pointed coalgebras and injective coalgebra homomorphisms.

Define the following congruence relations on the morphisms of $Hom_{\mathbf{IPCog}}(C, D)$. For $\rho, \gamma \in Hom_{\mathbf{IPCog}}(C, D)$ we write $\rho \sim \gamma$ if

$$\begin{cases} (\rho - \gamma)(C_0) = 0 \\ (\rho - \gamma)(C_1) \subseteq D_0 \end{cases}$$

Lemma 3.1.2. $\sim$ *is indeed a congruence relation.*

Proof. It is obvious that $\sim$ is reflexive and symmetric. Let us check that $\sim$ is transitive and that it preserves composition. Let $\rho, \gamma, \sigma \in Hom_{\mathbf{IPCog}}(C, D)$ be such that $\rho \sim \gamma$ and $\gamma \sim \sigma$. Then

$$\begin{aligned}(\rho - \sigma)(C_0) =& (\rho - \gamma + \gamma - \sigma)(C_0) \\ =& (\rho - \gamma)(C_0) + (\gamma - \sigma)(C_0) = 0\end{aligned}$$

and

$$\begin{aligned}(\rho - \sigma)(C_1) =& (\rho - \gamma + \gamma - \sigma)(C_1) \\ =& (\rho - \gamma)(C_1) + (\gamma - \sigma)(C_1) \\ \subseteq& D_0 + D_0 = D_0\end{aligned}$$

Now consider $\rho_1, \rho_2 \in Hom_{\mathbf{IPCog}}(A, B)$ and $\gamma_1, \gamma_2 \in Hom_{\mathbf{IPCog}}(B, C)$ such that $\rho_1 \sim \rho_2$ and $\gamma_1 \sim \gamma_2$. The following computation shows that $\gamma_1 \circ \rho_1 \sim \gamma_2 \circ \rho_2$:

$$\begin{aligned}(\gamma_1 \circ \rho_1 - \gamma_2 \circ \rho_2)(A_1) =& (\gamma_1 \circ \rho_1 - \gamma_1 \circ \rho_2 + \gamma_1 \circ \rho_2 - \gamma_2 \circ \rho_2)(A_1) \\ =& (\gamma_1 \circ \rho_1 - \gamma_1 \circ \rho_2)(A_1) + (\gamma_1 \circ \rho_2 - \gamma_2 \circ \rho_2)(A_1) \\ =& \gamma_1(\rho_1 - \rho_2)(A_1) + (\gamma_1 - \gamma_2)(\rho_2(A_1)) \\ \subseteq& \gamma_1(B_0) + (\gamma_1 - \gamma_2)(B_1) \\ \subseteq& C_0 + C_0 = C_0\end{aligned}$$

where the two last steps are due to the congruence $\sim$ and Corollary 2.2.4. and

$$\begin{aligned}(\gamma_1 \circ \rho_1 - \gamma_2 \circ \rho_2)(A_0) =& (\gamma_1 \circ \rho_1 - \gamma_1 \circ \rho_2 + \gamma_1 \circ \rho_2 - \gamma_2 \circ \rho_2)(A_0) \\ =& (\gamma_1 \circ \rho_1 - \gamma_1 \circ \rho_2)(A_0) + (\gamma_1 \circ \rho_2 - \gamma_2 \circ \rho_2)(A_0) \\ =& \gamma_1(\rho_1 - \rho_2)(A_0) + (\gamma_1 - \gamma_2)(\rho_2(A_0)) \\ \subseteq& (\gamma_1 - \gamma_2)(B_0) = 0\end{aligned}$$

where the two last steps in both equations are due to the congruence $\sim$ and Corollary 2.2.4. $\square$

Lemma 3.1.3. **IPCog**$_\sim$=**IPCog**$/\!\sim$ *is a category.*

Proof. See [Mac, Chapter 8] or [Awo, Chapter 3.4] $\square$

Examples 3.1.4. Consider the path coalgebra $C = k^\square Q$ of the quiver

$$\begin{tikzcd}
\circ_1 \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & \circ_3 \\
& \circ_2 \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

as in Example 2.3.4 (iii). Routine computations show that the maps $f, f' : C \to C$ defined on an element b of the basis $\{e_1, e_2, e_3, \alpha, \beta, \gamma, \gamma\beta\}$ of C by

$$f(b) = \begin{cases} b, & \text{if } b \neq \gamma\beta \\ \gamma\beta + \alpha, & \text{if } b = \gamma\beta \end{cases}$$

and

$$f'(b) = \begin{cases} b, & \text{if } b \neq \alpha \\ \alpha + e_3 - e_1, & \text{if } b = \alpha \end{cases}$$

are injective coalgebra homomorphisms.

Let $id : C \to C$ be the identity map of C. Then, $(f - id)(C_1) = 0$ implies that $f \sim id$. Furthermore, since

$$(f' - id)(\alpha) = f'(\alpha) - \alpha = \alpha + e_3 - e_1 - \alpha = e_3 - e_1 \in C_0$$

We have that $(f' - id)(C_1) \subseteq C_0$ and $(f' - id)(C_0) = 0$. Thus $f' \sim id$.

Note that not all coalgebra automorphism is congruent to the identity, since the coalgebra homomorphism that fix all paths but send α to $\lambda\alpha$, with $\lambda \notin \{0, 1\}$, is an example of such coalgebra automorphism.

3.2 The Path Coalgebra functor

Denote by $(kVQ_0, \Delta_0, \varepsilon_0)$ the group-like coalgebra of VQ_0 (as in Example 1.1.5 (i)), and by (VQ_1, ρ_l, ρ_r) the direct sum $VQ_1 = \bigoplus_{e,f\in VQ_0} VQ_{e,f}$ treated as a kVQ_0-kVQ_0-bicomodule with structure maps:

$$\rho_l\left(\sum_{e,f\in VQ_0} m_{e,f}\right) = \sum_{e,f\in VQ_0} f \otimes m_{e,f}$$

and

$$\rho_r\left(\sum_{e,f\in VQ_0} m_{e,f}\right) = \sum_{e,f\in VQ_0} m_{e,f} \otimes e,$$

$\forall m_{e,f} \in VQ_{e,f}$ (see Example 1.2.6 (iii) and Lemma 1.3.3).

Define the path coalgebra $k^{\Box}[VQ]$ as the cotensor coalgebra $Cot_{kVQ_0}(VQ_1)$, as in Definition 1.3.8.

For a given $\gamma \in Hom_{\mathbf{IVquiv}}(VQ, VR)$, we will construct a coalgebra homomorphism $f \in Hom_{\mathbf{IPCog}}(k^{\Box}[VQ], k^{\Box}[VR])$.

Let $\pi_0' : Cot_{kVQ_0}(VQ_1) \to kVQ_0$ be the canonical projection of coalgebras (see Remark 2.1.8 and lemma 2.1.4).

Define the map $\bar{\gamma}_0 : kVQ_0 \to kVR_0$ as the linear extension of the vertex map $\gamma_0 : VQ_0 \to VR_0$ of γ. Then $\bar{\gamma}_0$ is a coalgebra homomorphism (see Example 1.1.5 (iv)).

Define

$$\begin{aligned} f_0 : Cot_{kVQ_0}(VQ_1) &\to kVR_0 \\ c &\mapsto (\bar{\gamma}_0 \circ \pi_0')(c) \end{aligned}$$

f_0 is a coalgebra homomorphism since it is the composition of coalgebra homomorphisms.

Then $Cot_{kVQ_0}(VQ_1)$ becomes a kVR_0-kVR_0-bicomodule via f_0 and a kVQ_0-kVQ_0-bicomodule via π_0' (see Example 1.3.2 (ii)).

Now consider $\pi_1' : Cot_{kVQ_0}(VQ_1) \to VQ_1$ the canonical projection. We must check that π_1' is a bicomodule homomorphism.

We will show that π_1' is a comodule homomorphism of right kVQ_0-comodules. Consider $\rho : VQ_1 \to VQ_1 \otimes kVQ_0$ the structure map of the right kVQ_0-comodule VQ_1. For any $m \in VQ_1$, write $\rho(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$, where each $m_{(0)} \in VQ_1$ and $m_{(1)} \in kVQ_0$.

Then, for any element of the basis $m^1 \Box \cdots \Box m^n \in VQ_1^{\Box n}$ $(n \geq 1)$ we have

$$\begin{aligned} &(\pi_1' \otimes id)(id \otimes \pi_0')\Delta(m^1 \Box \cdots \Box m^n) = \\ &\qquad = (\pi_1' \otimes \pi_0') \sum_{(m^1)} ((m^1)_{(-1)}) \otimes ((m^1)_{(0)} \Box m^2 \Box \cdots \Box m^n) \\ &\qquad + (\pi_1' \otimes \pi_0') \sum_{i=1}^{n-1} (m^1 \Box \cdots \Box m^i) \otimes (m^{i+1} \Box \cdots \Box m^n) \\ &\qquad + (\pi_1' \otimes \pi_0') \sum_{(m^n)} (m^1 \Box \cdots \Box m^{n-1} \Box (m^n)_{(0)}) \otimes ((m^n)_{(1)}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{(m^1)} \pi_1'((m^1)_{(-1)}) \otimes \pi_0'((m^1)_{(0)} \square m^2 \square \cdots \square m^n) \\
&\quad + \sum_{i=1}^{n-1} \pi_1'(m^1 \square \cdots \square m^i) \otimes \pi_0'(m^{i+1} \square \cdots \square m^n) \\
&\quad + \sum_{(m^n)} \pi_1'(m^1 \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes \pi_0'((m^n)_{(1)}) \\
&= \sum_{(m^n)} \pi_1'(m^1 \square \cdots \square m^{n-1} \square (m^n)_{(0)}) \otimes ((m^n)_{(1)}) \\
&= \rho(\pi_1'(m^1 \square \cdots \square m^n)),
\end{aligned}
$$

where the last equality comes from the fact that $\pi_1'(m^1 \square \cdots \square m^n) \neq 0$ only if $n = 1$. In the case of $m^0 \in kVQ_0$, then

$$
\begin{aligned}
(\pi_1' \otimes id)(id \otimes \pi_0')\Delta(m^0) &= \pi_1'(m^0) \otimes \pi_0'(m^0) \\
&= 0 = \rho(\pi_1'(m^0)).
\end{aligned}
$$

A similar computation shows that π_1' is a comodule homomorphism of left kVQ_0-comodules. Thus π_1' is a bicomodule homomorphism of kVQ_0-kVQ_0-bicomodules.

Now, for any $c \in Cot_{kVQ_0}(VQ_1)$, write $\pi_1'(c) = \sum_{e,f \in VQ_0} \overline{c}_{e,f}$, where each $\overline{c}_{e,f} \in VQ_{e,f}$. Then, define the map

$$
\begin{aligned}
f_1 : Cot_{kVQ_0}(VQ_1) &\to VR_1 \\
c &\mapsto \sum_{e,f \in VQ_0} \gamma_{e,f}(\overline{c}_{e,f})
\end{aligned}
$$

where $\gamma_{e,f}$ are the arrow maps of γ. Hence, f_1 is a bicomodule homomorphism of kVR_0-kVR_0-bicomodules with $f_1(kVQ_0) = 0$.

The universal property of cotensor, Lemma 1.3.9, gives a unique coalgebra homomorphism $f : Cot_{kVQ_0}(VQ_1) \to Cot_{kVR_0}(VR_1)$ such that $\pi_i \circ f = f_i$ for $i \in \{0, 1\}$.

Lemma 3.2.1. *The coalgebra homomorphism $f : Cot_{kVQ_0}(VQ_1) \to Cot_{kVR_0}(VR_1)$ as constructed above is injective.*

Proof. By the Heyneman-Radford Theorem, Lemma 2.1.6 (and Example 2.2.8), it suffices to show that $f|_{kVQ_0 \oplus VQ_1}$ is injective. Since $Cot_{kVQ_0}(VQ_1)$ and $Cot_{kVR_0}(VR_1)$ are pointed coalgebras, by Corollary 2.2.4, we have that $f(kVQ_0 \oplus VQ_1) \subseteq kVR_0 \oplus VR_1$. Thus, for any $c \in kVQ_0 \oplus VQ_1$, we have

$$
f(c) = \pi_0 \circ f(c) + \pi_1 \circ f(c) = f_0(c) + f_1(c).
$$

Write $c = c_0 + c_1$, with $c_0 = \sum_{e \in VQ_0} \lambda_e e \in kVQ_0$ and $c_1 = \sum_{e,f \in VQ_0} c_{e,f}$ where each $c_{e,f} \in VQ_{e,f}$. Then

$$\begin{aligned}
f(c) = f_0(c) + f_1(c) =& f_0(c_0) + f_0(c_1) + f_1(c_0) + f_1(c_1) \\
=& (\bar{\gamma}_0 \circ \pi_0')(c_0) + (\bar{\gamma}_0 \circ \pi_0')(c_1) \\
&+ \left(\sum_{e,f \in VQ_0} \gamma_{e,f} \circ \pi_1' \right)(c_0) + \left(\sum_{e,f \in VQ_0} \gamma_{e,f} \circ \pi_1' \right)(c_1) \\
=& \bar{\gamma}_0(c_0) + \left(\sum_{e,f \in VQ_0} \gamma_{e,f} \right)(c_1) \\
=& \sum_{e \in VQ_0} \lambda_e \gamma_0(e) + \sum_{e,f \in VQ_0} \gamma_{e,f}(c_{e,f}).
\end{aligned}$$

Since $\bar{\gamma}_0 : kVQ_0 \to kVR_0$ is injective and $\gamma_{e,f} : VQ_{e,f} \to VR_{\gamma_0(e),\gamma_0(f)}$ is injective in each $VQ_{e,f}$, the result follows. $\square$

Define $k^{\square}[\gamma] = f$.

Proposition 3.2.2. *The above construction defines the covariant functors:*

$$\begin{aligned}
k^{\square}[-] :& \mathbf{IVquiv} \to \mathbf{IPCog} \\
\mathscr{K}^{\square}[-] = \Pi_{\sim} \circ k^{\square}[-] :& \mathbf{IVquiv} \to \mathbf{IPCog}_{\sim},
\end{aligned}$$

where $\Pi_{\sim} : \mathbf{IPCog} \to \mathbf{IPCog}_{\sim}$ *is the quotient functor.*

3.3 The Gabriel Vquiver functor

Let $C \in \mathbf{IPCog}$ be a coalgebra. Define the *Vquiver* $GQ(C)$ of C, $GQ(C) = (GQ(C)_0, GQ(C)_1)$, as follows:

$$\begin{aligned}
GQ(C)_0 =& G(C) \\
GQ(C)_1 = C_1/C_0 =& \bigoplus_{g,h \in G(C)} \bar{P}_{g,h}(C),
\end{aligned}$$

(see Proposition 2.2.6), where, for each $g, h \in GQ(C)_0$, we have $GQ(C)_{g,h} = \bar{P}_{g,h}(C)$.

If $\rho \in Hom_{\mathbf{IPCog}}(C, D)$, then define the maps

$$\begin{aligned}
\varphi_0 : G(C) &\to G(D) \\
g &\mapsto \rho(g)
\end{aligned}$$

and

$$\varphi_{g,h} : \bar{P}_{g,h}(C) \to \bar{P}_{\varphi_0(g),\varphi_0(h)}(D),$$
$$\bar{c} \mapsto \overline{\rho(c)}$$

where $c \in P_{g,h}(C)$ is any representative of $\bar{c} = c + k(h-g)$. Both maps are well defined injective maps (see Lemma 1.1.9). Hence, the map $\varphi : GQ(C) \to GQ(D)$ whose vertex map is $\varphi_0 : G(C) \to G(D)$ and arrow maps $\varphi_{g,h} : \bar{P}_{g,h}(C) \to \bar{P}_{\varphi_0(g),\varphi_0(h)}(D)$ for each pair of vertices $g, h \in G(C)$ defines a map of Vquivers.

Define $GQ(\rho) = \varphi$.

Proposition 3.3.1. *The above construction defines the covariant functor:*

$$GQ(-) : \mathbf{IPCog} \to \mathbf{IVquiv}$$

Proof. We need to check that if $\gamma \in Hom_{\mathbf{IPCog}}(B, C)$ and $\rho \in Hom_{\mathbf{IPCog}}(C, D)$, then $GQ(\rho \circ \gamma) = GQ(\rho) \circ GQ(\gamma)$. We have as follows

$$(GQ(\rho) \circ GQ(\gamma))_0 : G(B) \to G(D)$$
$$g \mapsto \rho(\gamma(g)) = (\rho \circ \gamma)(g)$$

and

$$(GQ(\rho) \circ GQ(\gamma))_{g,h} : \bar{P}_{g,h}(B) \to \bar{P}_{\rho(\gamma(g)),\rho(\gamma(h))}(D).$$
$$\bar{c} \mapsto \overline{\rho(\gamma(c))} = \overline{(\rho \circ \gamma)(c)}$$

Thus $GQ(\rho) \circ GQ(\gamma) = GQ(\rho \circ \gamma)$ $\square$

Now suppose $\rho, \gamma \in Hom_{\mathbf{IPCog}}(C, D)$ are such that $\rho \sim \gamma$, as in Lemma 3.1.2. We will show that $GQ(\rho) = GQ(\gamma)$. Since $(\rho - \gamma)(C_0) = 0$ and $C_0 = kG(C)$, the vertex maps of $GQ(\rho)$ and $GQ(\gamma)$ have to coincide. Write this map as $\varphi_0 : G(C) \to G(D)$. Moreover, the relation $(\rho - \gamma)(C_1) \subseteq D_0$ implies that for any $c \in P_{g,h}(C)$ we have $(\rho - \gamma)(c) \in k(\varphi_0(h) - \varphi_0(g))$. Thus, for each pair $g, h \in G(C)$ the arrow maps from $\bar{P}_{g,h}(C)$ to $\bar{P}_{\varphi_0(g),\varphi_0(h)}(D)$ of $GQ(\rho)$ and $GQ(\gamma)$ are identical. Hence $GQ(\rho) = GQ(\gamma)$.

Thus, the assignment $\mathscr{GQ}(C) = GQ(C)$ and $\mathscr{GQ}([\rho]) = GQ(\rho)$ for any coalgebra $C \in \mathbf{IPCog}$ and any morphism $[\rho] \in Hom_{\mathbf{IPCog}_\sim}(C, D)$, where $\rho \in Hom_{\mathbf{IPCog}}(C, D)$ is any representative of $[\rho]$, define a covariant functor

$$\mathscr{GQ}(-) : \mathbf{IPCog}_\sim \to \mathbf{IVquiv},$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathbf{IPCog} \arrow[r, "GQ(-)"] \arrow[d, "\Pi_\sim"'] & \mathbf{IVquiv} \\
\mathbf{IPCog}_\sim \arrow[ur, "\mathscr{G}\mathscr{Q}(-)"'] &
\end{tikzcd}$$

3.4 Adjunction

In this Section, we will prove that $\mathscr{G}\mathscr{Q}(-)$ is left adjoint to $\mathscr{K}^\square[-]$.

Consider the function

$$\eta_{C,VQ} : Hom_{\mathbf{IPCog}_\sim}(C, \mathscr{K}^\square[VQ]) \to Hom_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(C), VQ)$$
$$[\rho] \mapsto \varphi$$

where φ is given by:

$$\varphi_0 : G(C) \to VQ_0$$
$$g \mapsto \rho(g)$$

and for each pair $g, h \in G(C)$,

$$\varphi_{g,h} : \bar{P}_{g,h}(C) \to VQ_{\varphi_0(g),\varphi_0(h)},$$
$$\overline{c} \mapsto (\pi_1 \circ \rho)(c)$$

where $\overline{c} = c + k(h-g)$ is the coset of $k(h-g)$ in $P_{g,h}(C)$, $\pi_1 : Cot_{kVQ_0}(VQ_1) \to VQ_1$ is the canonical projection and ρ is any representative of $[\rho]$.

Lemma 3.4.1. *$\eta_{C,VQ}$ is well defined.*

Proof. Since $Cot_{kVQ_0}(VQ_1)$ is a pointed coalgebra (see Example 2.2.8) and

$$(Cot_{kVQ_0}(VQ_1))_0 = (kVQ_0)_0 = kVQ_0 = kG(Cot_{kVQ_0}(VQ_1)),$$

we have that $G(Cot_{kVQ_0}(VQ_1)) = VQ_0$. Thus, by Lemma 1.1.9, for each $g \in G(C)$ we have $\rho(g) \in VQ_0$.

Moreover,

$$(Cot_{kVQ_0}(VQ_1))_1 = kVQ_0 \oplus VQ_1 = kVQ_0 \oplus \left(\bigoplus_{e,f \in VQ_0} VQ_{e,f} \right)$$

and for any element $m \in VQ_{e,f}$ the comultiplication $\varDelta$ of $Cot_{kVQ_0}(VQ_1)$ gives

$$\varDelta(m) = m \otimes e + f \otimes m.$$

Hence, $P_{e,f}(Cot_{kVQ_0}(VQ_1)) \cap VQ_1 = VQ_{e,f}$ and, therefore,

$$P_{e,f}(Cot_{kVQ_0}(VQ_1)) = k(f-e) \oplus VQ_{e,f}.$$

Thus, there is an isomorphism between $\bar{P}_{e,f}(Cot_{kVQ_0}(VQ_1))$ and $VQ_{e,f}$ given by

$$\begin{aligned} \bar{P}_{e,f}(Cot_{kVQ_0}(VQ_1)) &\to VQ_{e,f}. \\ \bar{b} &\mapsto \pi_1(b) \end{aligned}$$

Since, by Lemma 1.1.9,

$$\rho(P_{g,h}(C)) \subseteq P_{\rho(g),\rho(h)}(Cot_{kVQ_0}(VQ_1)),$$

we have that $\varphi_{g,h}$ is well defined and so $\eta_{C,VQ}$ is well defined. $\square$

Lemma 3.4.2. *$\eta_{C,VQ}$ is a bijection.*

Proof. Injectivity.

Suppose $[\rho],[\sigma] \in Hom_{\mathbf{IPCog}/\sim}(C, k^{\square}[VQ])$, are such that $\eta_{C,VQ}([\rho]) = \varphi = \eta_{C,VQ}([\sigma])$.

For $c \in C_0$, write $c = \sum_{g\in G(C)} \lambda_g g$, with $\lambda_g \in k$ for each $g \in G(C)$. We have

$$\begin{aligned} \rho(c) = \rho\left(\sum_{g\in G(C)} \lambda_g g\right) &= \sum_{g\in G(C)} \lambda_g \rho(g) \\ &= \sum_{g\in G(C)} \lambda_g \varphi_0(g) \\ &= \sum_{g\in G(C)} \lambda_g \sigma(g) \\ &= \sigma\left(\sum_{g\in G(C)} \lambda_g g\right) = \sigma(c). \end{aligned}$$

Thus $(\rho - \sigma)(C_0) = 0$.

If $c \in C_1$, write $c = \sum_{g,h\in G(C)} c_{g,h} + \omega$, where for each $g, h \in G(C)$ we have

$c_{g,h} \in P_{g,h}(C)$ and $\omega \in C_0$, then

$$\begin{aligned}(\pi_1 \circ \rho)(c) =&(\pi_1 \circ \rho)\left(\sum_{g,h\in G(C)} c_{g,h} + \omega\right)\\ =&\sum_{g,h\in G(C)} (\pi_1 \circ \rho)(c_{g,h}) + (\pi_1 \circ \rho)(\omega)\\ =&\sum_{g,h\in G(C)} \varphi_{g,h}(c_{g,h}) + 0\\ =&\sum_{g,h\in G(C)} (\pi_1 \circ \sigma)(c_{g,h}) + (\pi_1 \circ \sigma)(\omega)\\ =&(\pi_1 \circ \sigma)\left(\sum_{g,h\in G(C)} c_{g,h} + \omega\right)\\ =&(\pi_1 \circ \sigma)(c)\end{aligned}$$

and hence $(\pi_1 \circ (\rho - \sigma))(C_1) = 0$. Since $\rho(C_1) \subseteq (k^\square[VQ])_1 = kVQ_0 \oplus VQ_1$ and $\pi_1 : k^\square[VQ] \to VQ_1$ is the canonical projection, we have that $(\rho - \sigma)(C_1) \subseteq kVQ_0 = (k^\square[VQ])_0$. Therefore, $\rho \sim \sigma$ and so $[\rho] = [\sigma]$.

Surjectivity.

Suppose $\varphi \in Hom_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(C), VQ)$ with the vertex map

$$\varphi_0 : G(C) \to VQ_0$$

and the arrow maps

$$\varphi_{g,h} : \bar{P}_{g,h}(C) \to VQ_{\varphi_0(g),\varphi_0(h)},$$

for each $g, h \in G(C)$.

Let $\pi'_0 : C \to C_0$ be the canonical projection of coalgebras (see Proposition 2.1.8) and $\bar{\varphi}_0 : C_0 \to kVQ_0$ the linear extension of φ_0.

The map

$$\begin{aligned} f_0 : C &\to kVQ_0\\ c &\mapsto (\bar{\varphi}_0 \circ \pi'_0)(c)\end{aligned}$$

is a coalgebra homomorphism (see the considerations before Lemma 3.2.1).

Let $\iota : C \to Cot_{C_0}(C_1/C_0)$ be the embedding as in Theorem 2.2.9. Thus $\iota(C_1) = C_0 \oplus C_1/C_0$ and the composition

$$\pi'_1 \circ \iota : C \to C_1/C_0,$$

is surjective, where $\pi_1' : Cot_{C_0}(C_1/C_0) \to C_1/C_0$ is the canonical projection.

For any $b \in Cot_{C_0}(C_1/C_0)$ write $\pi_1'(b) = \sum_{g,h\in G(C)} \overline{b}_{g,h}$, where each $\overline{b}_{g,h} \in \bar{P}_{g,h}(C)$. Define the map

$$\begin{aligned} f_1 : C &\to VQ_1 \\ c &\mapsto \sum_{g,h\in G(C)} \varphi_{g,h}(\overline{\iota(c)}_{g,h}) \end{aligned}$$

Then f_1 is a bicomodule homomorphism of kVQ_0-kVQ_0-bicomodules and $f_1(C_0) = 0$. Thus, by the universal property of the cotensor coalgebra, Lemma 1.3.9, there exists a unique morphism of coalgebras $f : C \to Cot_{kVQ_0}(VQ_1)$ such that $\pi_i \circ f = f_i$ for $i \in \{0, 1\}$, where $\pi_i : Cot_{kVQ_0}(VQ_1) \to VQ_1^{\square_i}$ is the canonical projection. Injectivity of f follows in a similar way done in Lemma 3.2.1.

Now, if $[f]$ is the congruence class of f in $Hom_{\mathbf{IPCog}/\sim}(C, Cot_{kVQ_0}(VQ_1))$, then $\eta_{C,VQ}([f]) = \phi$ is given by the maps

$$\begin{aligned} \phi_0 : G(C) &\to VQ_0 \\ g &\mapsto f(g) \end{aligned}$$

$$\begin{aligned} \phi_{g,h} : \bar{P}_{g,h}(C) &\to VQ_{\phi(g),\phi(h)} \\ \overline{c} &\mapsto (\pi_1 \circ f)(c) \end{aligned}$$

However, since $f(C_0) \subseteq kVQ_0$, for any $g \in G(C)$ we have

$$\begin{aligned} f(g) = (\pi_0 \circ f)(g) =& f_0(g) \\ =& (\bar{\varphi}_0 \circ \pi_0')(g) \\ =& \bar{\varphi}_0(\pi_0'(g)) \\ =& \bar{\varphi}_0(g) = \varphi_0(g) \end{aligned}$$

and for any $\overline{c} \in \bar{P}_{g,h}(C)$ we have

$$\iota(c) \in P_{\iota(g),\iota(h)}(Cot_{C_0}(C_1/C_0)) = P_{g,h}(Cot_{C_0}(C_1/C_0)) \subseteq C_0 \oplus \bar{P}_{g,h}(C).$$

Thus

$$\begin{aligned} (\pi_1 \circ f)(c) = f_1(c) =& \sum_{g',h'\in G(C)} \varphi_{g',h'}(\overline{\iota(c)}_{g',h'}) \\ =& \varphi_{g,h}(\overline{\iota(c)}) + \sum_{g',h'\in G(C)} \varphi_{g',h'}(\overline{0}) \\ =& \varphi_{g,h}(\overline{c}) \end{aligned}$$

Therefore, $\eta_{C,VQ}([f]) = \varphi$ □

Now the following two lemmas show naturality of η.

Lemma 3.4.3. *Fix a coalgebra C. The map*

$$\eta_{C,VQ} : Hom_{\mathbf{IPCog}_\sim}(C, \mathscr{K}^\square[VQ]) \to Hom_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(C), VQ)$$

is the component at VQ of a natural transformation

$$\eta_C : Hom_{\mathbf{IPCog}_\sim}(C, \mathscr{K}^\square[-]) \to Hom_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(C), -)$$

Proof. Let $\gamma \in Hom_{\mathbf{IVquiv}}(VQ, VR)$. We need to confirm that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{IPCog}_\sim}(C, \mathscr{K}^\square[VQ]) \arrow[r, "\eta_{C,VQ}"] \arrow[d, "\mathscr{K}^\square[\gamma]\circ-"'] & \mathrm{Hom}_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(C), VQ) \arrow[d, "\gamma\circ-"] \\
\mathrm{Hom}_{\mathbf{IPCog}_\sim}(C, \mathscr{K}^\square(VR)) \arrow[r, "\eta_{C,VR}"] & \mathrm{Hom}_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(C), VR)
\end{tikzcd}$$

commutes. Consider $[f] \in Hom_{\mathbf{IPCog}/\sim}(C, \mathscr{K}^\square[VQ])$ and f any representative.

We will show that the vertex map and the arrow maps of the two Vquiver maps $\eta_{C,VR}(\mathscr{K}^\square[\gamma] \circ [f])$ and $\gamma \circ \eta_{C,VQ}([f])$ are equal.

On one hand we have

$$\begin{aligned}
\eta_{C,VR}(\mathscr{K}^\square[\gamma] \circ [f])_0 : G(C) &\to VR_0 \\
g &\mapsto (k^\square[\gamma] \circ f)(g)
\end{aligned}$$

where, for $g \in G(C)$,

$$\begin{aligned}
(k^\square[\gamma] \circ f)(g) =& (\bar{\gamma}_0 \circ \pi'_0)(f(g)) \\
=& \bar{\gamma}_0(\pi'_0(f(g))) \\
=& \bar{\gamma}_0(f(g)) \\
=& \gamma_0(f(g)) \\
=& (\gamma_0 \circ f)(g).
\end{aligned}$$

On the other hand

$$\begin{aligned}
(\gamma \circ \eta_{C,VQ}([f]))_0 : G(C) &\to VR_0 \\
g &\mapsto (\gamma_0 \circ f)(g)
\end{aligned}$$

Thus the vertex maps coincide.

Now consider

$$\eta_{C,VQ}(\mathscr{K}^{\Box}[\gamma]\circ[f])_{g,h} : \bar{P}_{g,h}(C) \to VR_{\gamma_0(f(g)),\gamma_0(f(h))}$$
$$\bar{c} \mapsto \pi_1 \circ (k^{\Box}[\gamma]\circ f)(c)$$

where, for $\bar{c} \in \bar{P}_{g,h}(C)$, we have $f(c) \in kVQ_0 \oplus VQ_{f(g),f(h)}$ and so

$$\begin{aligned}\pi_1 \circ (k^{\Box}[\gamma]\circ f)(c) =& (\pi_1 \circ k^{\Box}[\gamma])(f(c)) \\ =& \sum_{g',h' \in VR_0} \gamma_{g',h'}(\overline{f(c)}) \\ =& \gamma_{f(g),f(h)}(\overline{f(c)}) \\ =& \gamma_{f(g),f(h)} \circ (\pi_1 \circ f)(c).\end{aligned}$$

However,

$$(\gamma \circ \eta_{C,VQ}([f]))_{g,h} : \bar{P}_{g,h}(C) \to VR_{\gamma_0(f(g)),\gamma_0(f(h))}.$$
$$\bar{c} \mapsto \gamma_{f(g),f(h)} \circ (\pi_1 \circ f)(c)$$

Thus $\eta_{C,VQ}(\mathscr{K}^{\Box}[\gamma]\circ[f]) = \gamma \circ \eta_{C,VR}([f])$. $\Box$

Lemma 3.4.4. *Fix a Vquiver VQ. The map*

$$\eta_{C,VQ} : Hom_{\mathbf{IPCog}_{\sim}}(C, \mathscr{K}^{\Box}[VQ]) \to Hom_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(C), VQ)$$

is the component at C of a natural transformation

$$\eta_{VQ} : Hom_{\mathbf{IPCog}_{\sim}}(-, \mathscr{K}^{\Box}[VQ]) \to Hom_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(-), VQ)$$

Proof. Let $[\rho] \in Hom_{\mathbf{IPCog}_{\sim}}(D, C)$ and ρ any representative. We need to confirm that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{IPCog}_{\sim}}(C, \mathscr{K}^{\Box}[VQ]) \arrow[r, "\eta_{C,VQ}"] \arrow[d, "-\circ\rho"'] & \mathrm{Hom}_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(C), VQ) \arrow[d, "-\circ\mathscr{G}\mathscr{Q}(\rho)"] \\ \mathrm{Hom}_{\mathbf{IPCog}_{\sim}}(D, \mathscr{K}^{\Box}(VQ)) \arrow[r, "\eta_{D,VQ}"] & \mathrm{Hom}_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(D), VQ) \end{tikzcd}$$

commutes. Consider $[f] \in Hom_{\mathbf{IPCog}/\sim}(C, \mathscr{K}^{\Box}[VQ])$ and f any representative.

We must check if $(\eta_{D,VQ}([f]\circ[\rho]))_0 = (\eta_{C,VQ}([f])\circ \mathscr{G}\mathscr{Q}(\rho))_0$ and for any $g, h \in G(C)$, $(\eta_{D,VQ}([f]\circ[\rho]))_{g,h} = (\eta_{C,VQ}([f])\circ\mathscr{G}\mathscr{Q}(\rho))_{g,h}$.

We have as follows

$$(\eta_{C,VQ}([f]) \circ \mathscr{G}\mathscr{Q}(\rho))_0 : G(C) \to VQ_0$$
$$g \mapsto f(\rho(g))$$

and

$$(\eta_{D,VQ}([f] \circ [\rho]))_0 : G(C) \to VQ_0$$
$$g \mapsto (f \circ \rho)(g)$$

Furthermore,

$$(\eta_{C,VQ}([f]) \circ \mathscr{G}\mathscr{Q}(\rho))_{g,h} : \bar{P}_{g,h}(C) \to VQ_{f(\rho(g)),f(\rho(h))}$$
$$\bar{c} \mapsto (\pi_1 \circ f)(\rho(c))$$

and

$$(\eta_{D,VQ}([f] \circ [\rho]))_{g,h} : \bar{P}_{g,h}(C) \to VQ_{f(\rho(g)),f(\rho(h))}$$
$$\bar{c} \mapsto (\pi_1 \circ (f \circ \rho))(c)$$

Thus $\eta_{C,VQ}([f]) \circ \mathscr{G}\mathscr{Q}(\rho) = \eta_{D,VQ}([f] \circ [\rho])$. □

This gives the following

Theorem 3.4.5. *The triple $\langle \mathscr{G}\mathscr{Q}, \mathscr{K}^{\square}, \eta \rangle$ is an adjunction between* **IPCog**$_{\sim}$ *and* **IVquiv**.

Examples 3.4.6. Consider the path coalgebra $C = k^{\square}Q$ of the quiver

$$a \xrightarrow{\delta} b$$

(see Example 2.3.4 (i)) and the Vquiver VQ given by $VQ_0 = \{e_1, e_2, e_3\}$ and $VQ_1 = VQ_{e_1,e_2} \oplus VQ_{e_2,e_3} \oplus VQ_{e_1,e_3}$, with $VQ_{e_i,e_j} = k$, for $j > i$. Write $VQ_{e_1,e_2} =< \beta >$, $VQ_{e_2,e_3} =< \gamma >$ and $VQ_{e_1,e_3} =< \alpha >$.

The Gabriel Vquiver of C, $\mathscr{G}\mathscr{Q}(C)$ is given by $\mathscr{G}\mathscr{Q}(C)_0 = \{a, b\}$ and $\mathscr{G}\mathscr{Q}(C)_1 = \mathscr{G}\mathscr{Q}(C)_{a,b} =< \overline{\delta} >$.

If $\varphi \in Hom_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(C), VQ)$ then the image of $\overline{\delta}$ by $\varphi_{a,b}$ must be one of the following

$$\varphi_{a,b}(\overline{\delta}) = \begin{cases} \lambda\alpha \\ \lambda\beta \\ \lambda\gamma \end{cases}$$

Note that φ_0 is completely determined by $\varphi_{a,b} : \bar{P}_{a,b}(\mathscr{G}\mathscr{Q}(C)) \to VQ_{\varphi_0(a),\varphi_0(b)}$.

Now, the Path Coalgebra of VQ, $\mathscr{K}^\square[VQ]$, is a pointed coalgebra with the set of group-like elements $G(\mathscr{K}^\square[VQ]) = \{e_1, e_2, e_3\}$ and the sets of primitive elements $P_{e_1,e_2}(\mathscr{K}^\square[VQ]) =< \beta, e_2 - e_1 >$, $P_{e_2,e_3}(\mathscr{K}^\square[VQ]) =< \gamma, e_3 - e_2 >$, and $P_{e_1,e_3}(\mathscr{K}^\square[VQ]) =< \alpha, e_3 - e_1 >$. There are no other primitive elements. Since injective coalgebra homomorphisms take non trivial primitive elements to non trivial primitive elements (see Lemma 1.1.9), for a given coalgebra homomorphism ρ such that $[\rho] \in Hom_{\mathbf{IPCog}_\sim}(C, \mathscr{K}^\square[VQ])$ we have that the image of δ by ρ must be one of the following

$$\rho(\delta) = \begin{cases} \lambda\alpha + \mu(e_3 - e_1) \\ \lambda\beta + \mu(e_2 - e_1) \\ \lambda\gamma + \mu(e_3 - e_2) \end{cases}$$

By Example 3.1.4, for any such ρ given by $\rho(\delta) = \lambda\theta + \mu(e_j - e_i)$, we have $\rho \sim \rho'$, where ρ' is given by $\rho'(\delta) = \lambda\theta$. Now the isomorphism $Hom_{\mathbf{IVquiv}}(\mathscr{G}\mathscr{Q}(C), VQ) \cong Hom_{\mathbf{IPCog}_\sim}(C, \mathscr{K}^\square[VQ])$ follows easily.

Remark 3.4.7. *An immediately conclusion one can take from Theorem 3.4.5 is that any pointed coalgebra C is isomorphic to a subcoalgebra of $\mathscr{K}^\square[\mathscr{G}\mathscr{Q}(C)]$.*

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