FEDERAL UNIVERSITY OF MINAS GERAIS

MATHEMATICS DEPARTMENT

Doctoral Thesis

A COMPACTNESS THEOREM AND SOME GAP RESULTS FOR FREE BOUNDARY MINIMAL SURFACES

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Belo Horizonte - MG

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If only I had the theorems! Then I should find the proofs easily enough.

Bernhard Riemann

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RESUMO

Esta tese consiste de vários resultados sobre superfícies mínimas. Na primeira parte estudamos superfícies mínimas com bordo livre na bola Euclideana B^n. Nós provamos que se Σ^k é uma superfície minima com bordo livre e de dimensão k em B^n satisfazendo $|x^\perp|^2|A|^2 \leq \frac{k}{k-1}$, então Σ^k é difeomorfica ao disco $\mathbb{D}^k$ ou a $\mathbb{S}^1 \times \mathbb{D}^{k-1}$. Alem disso, obtemos a rigidez no caso especial onde Σ^2 é uma superfície de dimensão 2 em B^n, precisamente, $|x^\perp|^2|A(x)|^2 \equiv 0$ e Σ^2 é um disco equatorial $D^2 \subset B^n$ ou $|x_0^\perp|^2|A(x_0)|^2 = 2$ em algum ponto $x_0 \in \Sigma^2$ e Σ^2 é isométrica a catenoide crítica. Também provamos existência de um gap para a área of superfícies mínimas com bordo livre. Mais precisamente, existe $\varepsilon(k,n) > 0$ tal que se Σ^k é uma superfície mínima em B^n satisfazendo $\mathrm{Area}(\Sigma^k) < \mathrm{Area}(D^k) + \varepsilon(k,n)$, então Σ^k é um disco equatorial D^k. Para provar este resultado gap nós comparamos o excesso de superfícies mínimas com bordo livre com o excesso dos cones associados sobre o bordo e vértice na origem. Como consequência, nós mostramos que a única uma superfície mínima com bordo livre em B^n com bordo mínimo em ∂B^n é o disco equatorial D^k. Na segunda parte provamos dois resultados sobre superfícies mínimas fechadas em variedades tridimensionais. Mostramos que o espaço das superfícies mínimas mergulhadas com área limitada superiormente e raio de injetividade limitado inferiormente é compacto na topologia C^∞. Finalmente, provamos um resultado do tipo rigidez para variedades fechadas tridimensionais de curvatura positiva admitindo superfícies mínimas estavéis.

ABSTRACT

This thesis consists of several results about minimal surfaces. In the first part we study free boundary minimal surfaces in the Euclidean ball B^n. We prove that if Σ^k is a $k-$dimensional free boundary minimal surface in B^n satisfying $|x^\perp|^2|A|^2 \leq \frac{k}{k-1}$, then Σ^k is diffeomorphic to either $\mathbb{D}^k$ or to $\mathbb{S}^1 \times \mathbb{D}^{k-1}$. Further geometric information is given in the codimension one case. Moreover, in case Σ^2 is a 2-dimensional free boundary minimal surface, then either $|x^\perp|^2|A(x)|^2 \equiv 0$ and Σ^2 is an equatorial disk $D^2 \subset B^n$ or $|x_0^\perp|^2|A(x_0)|^2 = 2$ at a point $x_0 \in \Sigma^2$ and Σ^2 is isometric to a critical catenoid. We also prove the existence of a gap for the area of free boundary minimal surfaces in the ball. Namely, there exists $\varepsilon(k,n) > 0$ so that whenever Σ^k is a free boundary minimal surface in B^n satisfying $\text{Area}(\Sigma^k) < \text{Area}(D^k) + \varepsilon(k,n)$, then Σ^k is an equatorial disk D^k. To prove this gap result we compare the excess of free boundary minimal surfaces with the excess of the associated cones over the boundaries. As a corollary, we show that D^k is the only free boundary minimal surface in B^n whose boundary is minimal in ∂B^n. In the second part we prove two results about closed minimal surfaces in 3-manifolds. The main result is a compactness theorem for the space of minimal surfaces with area bounded from above and injective radius bounded from below. Finally, we prove a weak result for positively curved 3-manifolds with symmetries containing stable minimal surfaces.

CONTENTS

INTRODUCTION

The main objects of study in this work are k-dimensional free boundary minimal surfaces in the Euclidean unit ball $B^n \subset \mathbb{R}^n$. These surfaces arise as critical points to the area functional for surfaces in B^n with boundary in ∂B^n. It follows from the first variation formula that such surfaces intersect ∂B^n orthogonally. The simplest example is the equatorial disk $B^k \subset B^n$. The subject has been studied by many authors where the main themes are classification and existence results. Regarding the former we cite the classical work of Nitsche [27] on the characterization of the flat disk D^2 as the only free boundary minimal disk in B^3. For the existence problem we cite the works of Struwe [19] and Jost and Grüter [16], [17] on the existence of free boundary minimal disks in convex domains of $\mathbb{R}^3$. For the existence of minimal annuli as well as other topological types see recent work in [13] and [28], respectively. The interest of study free boundary minimal surfaces increased recently after the work of Fraser and Schoen on the relationship between metrics that maximize the first Steklov eigenvalues of surfaces with boundary and free boundary minimal surfaces in B^3, see [21], [22], and [23].

In this work we study geometric properties that in a way characterize the equatorial disk as well as others interesting surfaces. In this direction, we mention recent work of Ambrozio and Nunes [1] on a geometric characterization of the equatorial disk and the critical catenoid in terms of curvature and support function:

Theorem (Ambrozio-Nunes)**.** *Let Σ^2 be a compact free boundary minimal surface in B^3. Assume that for all points $x \in \Sigma$,*

$$\langle x, N(x)\rangle^2 |A(x)|^2 \leq 2$$

where $N(x)$ denotes a unit normal vector at the point $x \in \Sigma$ and A denotes the second fundamental form of Σ. Then

1. *$\langle x, N(x)\rangle^2 |A(x)|^2 \equiv 0$ and Σ is a flat equatorial disk;*
2. *$\langle x_0, N(x_0)\rangle^2 |A(x_0)|^2 = 2$ at some point $x_0 \in \Sigma$ and Σ is a critical catenoid.*

The authors in [1] raise the question if the above theorem can be generalized to higher ambient dimension and surface codimensions. The first part of this thesis is centered on these two questions. In the codimension one case we prove

Theorem. *Let Σ^n be a free boundary minimal hypersurface in $B_1^{n+1}(0)$. Assume that for every $x \in \Sigma^n$*

$$|A(x)|^2\,|x^\perp|^2 \leq \frac{n}{n-1}. \tag{0.1}$$

Then one of the following is true

1. *Σ is diffeomorphic to a disk $\mathbb{D}^n$.*
2. *Σ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^{n-1}$ and $C(\Sigma) := \{x \in \Sigma : |x| = d(0,\Sigma)\}$ is an equator in $\mathbb{S}^n(d(\Sigma, 0))$.*

Moreover, (0.1) becomes equality when $x \in C(\Sigma^n)$ and A is constant along $C(\Sigma)$ with only two principal curvatures: $\frac{-1}{d(\Sigma,0)}$ and $\frac{1}{(n-1)d(\Sigma,0)}$.

The key insight behind the proof of above theorem is the observation that inequality (0.1) implies that the Hessian of the square of the distance function to the origin is non-negative; from this many topological and geometric properties follow. In Chapter 2 we discuss some $SO(2)\times SO(n-1)$ invariant minimal surfaces in $\mathbb{R}^{n+1}$ which are candidates to satisfy inequality (0.1).

A natural question raised after previous theorem is if the equatorial free boundary minimal disk D^n is characterized by $\langle x, N\rangle^2|A(x)|^2 < \frac{n}{n-1}$. This is a non-trivial question as even for the 2-dimensional case one needs to invoke the aforementioned Nitsche's theorem to prove uniqueness. On the other hand, if one only asks for free boundary minimal surfaces with $\langle x, N\rangle^2|A(x)|^2$ sufficiently small, then we prove

Proposition. *There exists $\varepsilon(k) > 0$ such that whenever Σ^k is a free boundary minimal surface in $B_1^{n+1}(0)$ satisfying*

$$|x^\perp|^2|A(x)|^2 < \varepsilon(k),$$

where $A(x)$ is the second fundamental form of Σ^k, then Σ^k is the free boundary equatorial disk D^k.

Our next contribution addresses the case of 2-dimensional minimal surfaces in B^n satisfying an inequality of type (0.1). We remark that the Nitsche's uniqueness theorem mentioned earlier also holds in this setting, this was proved recently by Fraser and Schoen, see [20]. Our result gives the optimal characterization result:

Theorem. *If Σ^2 is a free boundary minimal surface in $B_1^{n+1}(0)$ satisfying*

$$|x^\perp|^2|A(x)|^2 \leq 2,$$

for every $x \in \Sigma^2$, then one of the following is true:

- $|x^\perp|^2|A(x)|^2 \equiv 0$ *and Σ^2 is a flat equatorial disk.*
- $|x_0^\perp|^2|A(x_0)|^2 = 2$ *at some point $x_0 \in \Sigma^2$ and Σ^2 is a critical catenoid in a 3-dimensional linear subspace.*

The second part of this thesis seek for a similar gap phenomenon for free boundary minimal surfaces in B^n in terms of the volume rather than curvature. The main result regarding the volume of free boundary minimal surfaces in B^n is due to S. Brendle who proved that the equatorial disk D^k has least volume among k-dimensional free boundary minimal surfaces in B^n. More precisely,

Theorem (S. Brendle)**.** *Let Σ^k be a k-dimensional free boundary minimal surface in B^n. Then*

$$|\Sigma^k| \geq |D^k|$$

Moreover, the equality holds if, and only if, Σ^k is contained in k-dimensional plane in $\mathbb{R}^n$.

Are there non trivial free boundary minimal surfaces with volume sufficiently close to the volume of D^k? This is a natural question in view of the previous theorem and is closely related to the problem of how a sequence of minimal surfaces can degenerate. We prove an Allard's regularity result for k-dimensional free boundary minimal surfaces in B^n whose volumes are sufficiently close to the volume of the equatorial disk D^k. As a result, we prove the existence of a gap for the volume of free boundary minimal surfaces in the ball.

Theorem. *There exists $\varepsilon(k,n) > 0$ so that whenever Σ^k is a k-dimensional free boundary minimal surface in B^n with $k \geq 3$, and satisfying*

$$|\Sigma^k| \,<\, |D^k| + \varepsilon(k,n),$$

then Σ^k is a free boundary equatorial disk D^k in B^n.

The 2-dimensional case is tis theorem was proved in [28]. Two important ingredients used there are Nitsche's uniqueness theorem and an excess inequality proved in [33]. The main difficulty when $k \geq 3$ is that none of these ingredients is available to use. We overcome these difficulty by considering a slightly more general quantity, originated in [32] and which also resembles an excess type formula, and compare it with that of the free boundary cone over the boundary to obtain curvature estimates for a sequence of free boundary minimal surfaces with volume sufficiently close to the volume of the disk of same dimension. Finally, we replace the use of Nitsche's theorem by an standard index of stability analysis.

The previous theorem is an application of the following general proposition that we prove in Chapter 2, Section 2.2 and which might be of independent interest:

Proposition. *Let Σ^k be a k-dimensional free boundary minimal surface in B^n and $C_1\partial\Sigma$ the cone with vertice at the origin and base $\partial\Sigma$. If $y \in \Sigma - \{0\}$, then*

$$\int_\Sigma \frac{|(x-y)^\perp|^2}{|x-y|^{k+2}} = \int_{C_1\partial\Sigma} \frac{|(x-y)^\perp|^2}{|x-y|^{k+2}} + \frac{1}{k}\int_{C_1\partial\Sigma} \langle \overrightarrow{H}_{C\partial\Sigma}, \frac{x-y}{|x-y|^k}\rangle - v(y)|D^k|,$$

where $v(y) = 1$ if $y \notin C_1\partial\Sigma$ and $v(y) = 0$ if $y \in C_1\partial\Sigma$.

As a consequence, we obtain the following unique continuation type result for minimal surfaces in the ball.

Corollary. *Let Σ^k be a smooth k-dimensional free boundary minimal surface in B^n such that $\partial\Sigma^k$ is a $(k-1)$-minimal surface in ∂B^n. Then Σ^k is the equatorial disk D^k.*

We also discuss in Chapter 2 the case of free boundary surfaces with constant mean curvature in B^3. We extend previous gap result to this setting by considering the Willmore energy of these surfaces instead of the volume, this is done in Section 2.3.

In Chapter 3 we give two rather independent results on the classical setting of closed minimal surfaces in 3-manifolds. The first one concerns a compactness theorem for the space of closed minimal surfaces in 3-manifolds with area bounded from above and injective radius bounded from below. The topic of compactness for minimal surfaces is a very attractive one in geometric analysis; the classical result in the subject is the famous theorem of Choi and Schoen [9] which asserts that the set of minimal sufaces with fixed genus in a closed 3-manifold with positive Ricci curvature is strongly compact in the C^∞ topology. The subject flourished again recently after important work of B. Sharp [31] who proved compactness for the space of minimal hypersurfaces with bounded area and bounded index in higher dimension closed manifolds with positive Ricci curvature. See also [2], [10], [11], and references therein for other related results. Our result is

Theorem. *Let M^3 be a closed 3-manifold and let $\mathcal{S}$ be the space of closed embedded minimal surfaces in M^3. Then the class*

$$C(A_0, i_0) := \{\Sigma \in \mathcal{S} : \; Area(\Sigma) \leq A_0, \; inj(\Sigma) \geq i_0\}$$

is compact in the C^∞ topology.

The example of a sequence of blow down of the minimal catenoid or the example of a sequence of blow down of the minimal helicoid shows that the assumptions on the area and injective radius in the above theorem are necessary to obtain compactness.

Finally, we mention the last result in this thesis in the context of manifolds with positive scalar curvature and stable minimal surfaces

Theorem. *Let (M^3, g) be a closed 3-manifold with positive scalar curvature $R_g > 0$ and admitting a Killing vector field V. If (M^3, g) contains an embedded stable minimal surface, then the universal cover of (M^3, g) is diffeomorphic to either $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$, and*

$$g = dr^2 + ds^2 + \varphi(r,s)\, d\theta^2 \quad \textit{and} \quad \varphi(r,s) = \frac{1}{2\pi}\int_0^{2\pi} |V(r,s,\theta)|\, d\theta.$$

1. PRELIMINARIES

In this chapter, we present the definitions of the objects of interest and state the basic facts which will be used throughout this work. In Section 1, we first list the definitions of the geometric objects related to a Riemannian manifold and its submanifolds. In Section 2, we recall the mean curvature equation for graphs with codimension and state a standard result on the zero set of a solution of a system of elliptic equations.

1.1 Geometry of submanifolds

Let (M^n, g) be a Riemannian manifold of dimension n. The Levi-Civita connection associated to the metric g is denoted by $\overline{\nabla}$. The Riemannian curvature tensor, denoted by R, is the tensor defined as:

$$R(X, Y, Z) = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z,$$

for every $X, Y, Z \in \mathcal{X}(M)$. Here $\mathcal{X}(M)$ denotes the space of smooth vector fields on M. The sectional curvature of M at a point $x \in M$ in the direction of a 2-dimensional plane $\sigma \subset T_x M$ is given by:

$$K_M(\sigma, x) = g(R(e_1, e_2, e_1), e_2),$$

where $\{e_1, e_2\}$ is an orthonormal basis for σ. The Ricci tensor, denoted by Ric, is the symmetric two tensor defined by:

$$\mathrm{Ric}(X, Y)(x) = \sum_{i=1}^{n} g(R(X, e_i, Y), e_i)(x),$$

where $X, Y \in \mathcal{X}(M)$ and $x \in M$. Here, $\{e_1, \dots, e_n\}$ is an orthonormal basis for T_xM. Similarly, the scalar curvature of M, denoted by R_g, is the scalar function defined by:

$$R_g(x) = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i)(x),$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis for T_xM.

The Levi-Civita connection of a k-dimensional surface $\Sigma^k \subset M$ with the induced Riemannian metric of M is given by:

$$\nabla_X Y = (\overline{\nabla}_X Y)^\top,$$

where $X, Y \in \mathcal{X}(\Sigma)$. The second fundamental form of Σ, denoted by B, is then defined by:

$$B(X, Y) = (\overline{\nabla}_X Y)^\perp,$$

where $X, Y \in \mathcal{X}(\Sigma)$. The second fundamental form plays an important role in comparing the intrinsic curvatures of M^n and Σ^k as indicated in the Gauss equation:

Proposition 1 (Gauss Equation)**.** *Given $x \in \Sigma^k$ and σ a 2-dimensional plane in $T_x\Sigma$, then*

$$K_M(\sigma, x) - K_\Sigma(\sigma, x) = \langle B(e_1, e_1), B(e_2, e_2)\rangle - |B(e_1, e_2)|^2,$$

where $\{e_1, e_2\}$ is an orthonormal basis for σ.

The mean curvature vector of Σ^k, denoted by $\overrightarrow{H}$, is the normal vector defined as the trace of B, i.e.,

$$\overrightarrow{H}(x) = \frac{1}{k} B(e_i, e_i)(x),$$

where $x \in M$ and $e_1, \dots, e_k$ is an orthonormal basis for $T_x\Sigma$.

The geometric significance of the mean curvature vector comes from the first variation formula for the area. In order to make this statement precise we first recall what a smooth variation of a surface is. Let Σ^k be a surface in a domain $\Omega \subset M^n$ such that $\partial\Sigma \subset \partial\Omega$. A smooth variation of Σ inside Ω is just a smooth map $\varphi : \Sigma \times [0,\varepsilon) \to \Omega$ with the property that $\varphi(x,0) = x$ and $\varphi(x,t) \in \partial\Omega$ for every $x \in \partial\Sigma$.

Proposition 2 (First Variation Formula)**.**

$$\frac{d}{dt}\bigg|_{t=0} |\Sigma_t| = -k\int_\Sigma \left\langle \overrightarrow{H}, \frac{\partial\varphi}{\partial t}(x,0)\right\rangle d_\Sigma + \int_{\partial\Sigma}\left\langle \nu, \frac{\partial\varphi}{\partial t}(x,0)\right\rangle d\sigma,$$

where $|\Sigma_t|$ denotes the area of $\Sigma_t = \varphi(\Sigma,t)$ and ν is the exterior co-normal vector of $\partial\Sigma$ in Σ.

Proof. See Appendix. □

Definition 1. *A surface $\Sigma^k \subset \Omega \subset M$ with $\partial\Sigma \subset \partial\Omega$ is said to be a free boundary minimal surface if $\overrightarrow{H} = 0$ and if Σ intersects $\partial\Omega$ orthogonally.*

Proposition 3 (Second Variation Formula)**.** *If Σ^k is a k-dimensional free boundary minimal surface in Ω, then*

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} |\Sigma_t| = \int_\Sigma \Bigg(&|\nabla^\perp \frac{\partial\varphi}{\partial t}(x,0)|^2 - \sum_{i=1}^k g(R(\frac{\partial\varphi}{\partial t}(x,0), e_i, \frac{\partial\varphi}{\partial t}(x,0), e_i)) \\ &- \sum_{i,j}^k g(B(e_i,e_j), \frac{\partial\varphi}{\partial t}(x,0))^2\Bigg) d_\Sigma + \int_{\partial\Sigma} g(\overline{\nabla}_{\frac{\partial\varphi}{\partial t}(x,0)}\frac{\partial\varphi}{\partial t}(x,0), \nu)\, d\sigma.\end{aligned}$$

Proof. See Appendix. □

1.2 Mean curvature equation

Although the next two lemmas are standard, for the benefit of the reader we include their proofs.

Lemma 1. *Let Σ^k be a minimal submanifold in $\mathbb{R}^n$ given by the graph of the function $u : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ where $u(x) = (u_1(x), \ldots, u_{n-k}(x))$. Then for every $l = 1, \ldots, n-k$*

$$\frac{a_{ij}(\nabla u_1, \ldots, \nabla u_{n-k})}{\sqrt{1+|\nabla u_l|^2}} D_{ij}u_l = 0, \tag{1.1}$$

for some smooth functions $a_{ij}(\nabla u_1, \ldots, \nabla u_{n-k})$.

Proof. Parametrize Σ as $\varphi(x) = (x, u_1(x), \ldots, u_{n-k}(x))$. The coordinate basis for Σ is given by

$$D_{x_i}\varphi = (0, \ldots, 1, \ldots, D_{x_i}u_1, \ldots, D_{x_i}u_{n-k}),$$

for $i = 1, \ldots, k$. It follows that

$$g_{ij} = \delta_{ij} + \sum_{l=1}^{n-k} D_{x_i}u_l\, D_{x_j}u_l \quad \text{and} \quad g^{ij} = a_{ij}(\nabla u_1, \ldots, \nabla u_{n-k}). \tag{1.2}$$

Now we consider for each $l = 1, \ldots, n-k$ the unit normal vector

$$N_l = \frac{1}{\sqrt{1 + |\nabla\, u_l|^2}}(-D_{x_1}u_l, \ldots, -D_{x_k}u_l, 0, \ldots, 1, \ldots, 0).$$

A simple computation gives

$$\begin{aligned}(N_l)_{x_i} \;&=\; \left(\frac{1}{\sqrt{1 + |\nabla\, u_l|^2}}\right)_{x_i}\sqrt{1 + |\nabla u_l|^2}\, N_l + \\ &\quad \frac{1}{\sqrt{1 + |\nabla\, u_l|^2}}(-D^2_{x_1x_l}u_l, \ldots, -D^2_{x_kx_l}u_l, 0 \ldots, 0).\end{aligned}$$

Consequently,

$$(A_{N_l})_{ij} = \langle -dN_l(\varphi_{x_i}), \varphi_{x_j}\rangle = \frac{1}{\sqrt{1 + |\nabla\, u_l|^2}} D^2_{x_ix_j}u_l.$$

As Σ^k is minimal we have $0 = g^{ij}(A_{N_l})_{ij}$ and by (1.2) we obtain

$$\frac{a_{ij}(\nabla u_1, \ldots, \nabla u_{n-k})}{\sqrt{1 + |\nabla\, u_l|^2}} D_{ij}u_l \;=\; 0.$$

$\square$

Lemma 2. *Let $u, v : U \subset \mathbb{R}^k \to \mathbb{R}^p$ be smooth maps which satisfy (1.1). Then, the difference $\varphi = u - v$ satisfies for each $l = 1, \ldots, p$*

$$\frac{a_{ij}(\nabla u)}{\sqrt{1 + |\nabla u_l|^2}} D_{ij}(\varphi_l) + \sum_{m=1}^{p} b^m_j(\nabla u, \nabla v) D_j(\varphi_m) = 0.$$

for some smooth functions $a_{ij}(\nabla u)$ and $b^m_j(\nabla u, \nabla v)$.

Proof. As u_l and v_l satisfy equation (1.1), therefore

$$\begin{aligned}
0 &= \frac{a_{ij}(\nabla u)}{\sqrt{1+|\nabla u_l|^2}} D_{ij} u_l \quad - \quad \frac{a_{ij}(\nabla v)}{\sqrt{1+|\nabla v_l|^2}} D_{ij} v_l \\
&= \frac{a_{ij}(\nabla u)}{\sqrt{1+|\nabla u_l|^2}} D_{ij} u_l \quad - \quad \frac{a_{ij}(\nabla u)}{\sqrt{1+|\nabla u_l|^2}} D_{ij} v_l \\
&\qquad\qquad\qquad\qquad + \quad \frac{a_{ij}(\nabla u)}{\sqrt{1+|\nabla u_l|^2}} D_{ij} v_l - \frac{a_{ij}(\nabla v)}{\sqrt{1+|\nabla v_l|^2}} D_{ij} v_l \\
&= \frac{a_{ij}(\nabla u_l)}{\sqrt{1+|\nabla u_l|^2}} D_{ij}(\varphi_l) \quad + \quad \left(\frac{a_{ij}(\nabla u)}{\sqrt{1+|\nabla u_l|^2}} - \frac{a_{ij}(\nabla v)}{\sqrt{1+|\nabla v_l|^2}} \right) D_{ij} v_l.
\end{aligned}$$

Now, let $F_{ij} : \mathbb{R}^k \times \cdots \times \mathbb{R}^k \to \mathbb{R}$ be defined by

$$F_{ij}(z_1, \ldots, z_p) = \frac{a_{ij}(z_1, \ldots, z_p)}{\sqrt{1+|z_l|^2}}.$$

By the Fundamental Theorem of Calculus we can write

$$F_{ij}(\nabla u) - F_{ij}(\nabla v) = \left(\int_0^1 dF_{ij}(\nabla u + t(\nabla v - \nabla u))dt \right) \nabla(u-v).$$

The lemma follows by setting b_q^m to be

$$b_q^m = \left(\int_0^1 dF_{ij}(\nabla u + t(\nabla v - \nabla u))dt \right)_{qm} D_{ij} v_l \, D_q(u-v)_m.$$

□

The next lemma is a straightforward generalization of a result proved in [36] about the nodal set of a solution of an elliptic differential equation to the case of a system of elliptic equations.

Lemma 3. *Let $u : U \subset \mathbb{R}^n \to \mathbb{R}^p$ be a smooth map which satisfies for each $k = 1, \ldots, p$ an elliptic equation of the form:*

$$a_{ij}(x) D_{ij} u_k + \sum_{l=1}^{p} b_j^l(x) D_j u_l + \sum_{l=1}^{p} c_l(x)\, u_l = 0, \tag{1.3}$$

where a_{ij}, b_j, and c_l are smooth functions. Let's assume that a_{ij} is positive definite, and that $|b_j| \le C$ and $|c_l| \le C$ for some constant $C > 0$. If the order *of vanishing of u_l at $u_l^{-1}(0)$ is finite for each l and if $x_0 \in u^{-1}(0) \cap |Du|^{-1}(0)$, then*

$$u^{-1}(0) \cap |Du|^{-1}(0) \cap B_r(x_0)$$

decomposes into a countable union of subsets of a pairwise disjoint collection of $n-2$-dimensional smooth submanifolds.

Proof. We first define for each integer $q = 1, 2, \dots$ the set

$$S_q = \{x : D^\alpha u_l(x) = 0, \ \forall\, |\alpha| \le q, \ \ \forall l \text{ and } D^{q+1}u_{l_0}(x) \neq 0 \text{ for some } l_0\}. \quad (1.4)$$

Moreover, if $x \in u^{-1}(0) \cap |Du|^{-1}(0)$ and $r > 0$ is small enough, then

$$u^{-1}(0) \cap |Du|^{-1}(0) \cap B_r(x) = \cup_{q=1}^{d} S_q \cap B_r(x), \quad (1.5)$$

where $d-1$ is the order of vanishing of u at x. Now for each $x \in S_q$ we consider a multi-index β such that $|\beta| = q-1$ and $\mathrm{Hess}(D^\beta u_{l_0})(x) \neq 0$ for some l_0. Applying D^β to both sides of (1.3) with $k = l_0$ and recalling that $D^\alpha u_l(x) = 0$ for every multi-index α such that $|\alpha| \le q$ we obtain

$$a_{ij}(x) D_{ij}(D^\beta u_{l_0})(x) = 0.$$

Using that a_{ij} is positive definite and that $\mathrm{Hess}(D^\beta u_{l_0})(x) \neq 0$ we conclude that $\mathrm{rank}(\mathrm{Hess}(D^\beta u_{l_0})(x) \geq 2$. Thus there exist indexes i_1 and i_2 for which $\mathrm{grad}(D_{i_1} D^\beta u_{l_0})(x)$ and $\mathrm{grad}(D_{i_2} D^\beta u_{l_0})(x)$ are linearly independent. This implies that for small $r > 0$ that

$$B_r(x) \cap (D_{i_1} D^\beta u_{l_0})^{-1}(0) \cap (D_{i_2} D^\beta u_{l_0})^{-1}(0)$$

is a $n-2$-dimensional submanifold $\Sigma_{x,r,\beta}$ which contains $B_r(x) \cap S_q$. In view of (1.5) we conclude that for each $x \in u^{-1}(0) \cap |Du|^{-1}(0)$ there exist $r > 0$ and smooth $n-2$-dimensional submanifolds $\Sigma_{x,r,q_1}, \dots, \Sigma_{r,x,q_s}$ for which

$$B_r(x) \cap u^{-1}(0) \cap |Du|^{-1}(0) \subset \cup_{j=1}^{s} \Sigma_{x,r,q_j}. \quad (1.6)$$

The Lemma follows from (1.6). □

Lemma 4. *If Σ_1 and Σ_2 are 2-dimensional minimal surfaces in $\mathbb{R}^n$ having a tangential intersection of infinite order at $x_0 \in \Sigma_1 \cap \Sigma_2$, then $\Sigma_1 = \Sigma_2$.*

Proof. There exists a domain $\Omega \subset \mathbb{R}^2$ containing the origin and minimal maps $v_k : \Omega \to \mathbb{R}^n$, $k = 1, 2$, parameterizing neighborhoods of Σ_k and such that $v_k(0) = x_0$. We can assume that the coordinates $z = x + y\,i$ in Ω are isothermal for both v_1 and v_2. As v_k is minimal, each coordinate v_k^i, $i = 1, \dots, n$, is harmonic, which implies by the conformal invariance of the Laplacian that $\partial_{\bar{z}} \partial_z v_k^i = 0$. Hence, if we define $v(z) = v_1(z) - v_2(z)$, then each component of $\partial_z v$ is holomorphic, i.e., $\partial_{\bar{z}} \partial_z v^i = 0$. Since $z = 0$ is an infinite order zero of v, the analytic continuation property for holomorphic functions implies that $v \equiv 0$. Therefore, $\Sigma_1 = \Sigma_2$. $\square$

2. GAP RESULTS FOR FREE BOUNDARY MINIMAL HYPERSURFACES

In this Chapter we study k-dimensional minimal surfaces in the Euclidean unit ball B^n that meet ∂B^n orthogonally. We prove some gap theorems in terms of curvature in Section 2.1 and in terms of area in Section 2.2. Finally, in Section 2.3 we discuss some energy gap for surfaces with constant mean curvature in B^3.

2.1 Curvature gap for free boundary minimal surfaces

Lemma 5. *Let Σ^k be a free boundary minimal surface in $B_1^{n+1}(0)$ and f be the function $f : \Sigma^k \to \mathbb{R}$ by*

$$f(x) = \frac{|x|^2}{2}, \ x \in \Sigma^k.$$

Then $\nabla^\Sigma f = x^\top$ for every $x \in \Sigma$ and

$$Hess_\Sigma f(x)(X, Y) = \langle X, Y \rangle + \langle B(X, Y), \overrightarrow{x} \rangle. \tag{2.1}$$

Proof. Given $X \in \mathcal{X}(\Sigma)$, then

$$X(f) = \frac{1}{2} X \langle \overrightarrow{x}, \overrightarrow{x} \rangle = \langle X, \overrightarrow{x} \rangle = \langle X, x^\top \rangle.$$

This proves that $\nabla^\Sigma f(x) = x^\top$. In particular, $\nabla^\Sigma f(x) = x$ for every $x \in \partial\Sigma$ since Σ is a free boundary minimal surface. Given X and Y vector fields in

$\mathcal{X}(\Sigma)$ the $\mathrm{Hess}_\Sigma f\,(X,Y)$ is given by

$$\begin{aligned} Hess_\Sigma f\,(X,Y) &= \langle \nabla_X \nabla f, Y\rangle = \langle \overline{\nabla}_X \nabla f, Y\rangle = \langle \overline{\nabla}_X (x - x^\perp), Y\rangle \\ &= \langle X, Y\rangle - \langle \overline{\nabla}_X x^\perp, Y\rangle = \langle X, Y\rangle + \langle x^\perp, \overline{\nabla}_X Y\rangle \\ &= \langle X, Y\rangle + \langle B(X,Y), \overrightarrow{x}\rangle, \end{aligned}$$

where X and Y are vector fields in $\mathcal{X}(\Sigma)$. □

Definition 2. *Given a k-dimensional free boundary minimal surface Σ^k in $B_1^{n+1}(0)$ we define*

$$C(\Sigma) = \{x \in \Sigma : f(x) \;=\; m_0 \;:=\; \min_\Sigma \, f\}. \tag{2.2}$$

Lemma 6 (Chen [14])**.** *Let $a_1, \ldots, a_n$ and b be real numbers. If*

$$\sum_{i=1}^n a_i^2 \le \frac{(\sum_{i=1}^n a_i)^2}{n-1} - \frac{b}{n-1},$$

then $2\, a_i\, a_j > \frac{b}{n-1}$ for every $i, j \in \{1, \ldots, n\}$.

Proof. The proof is by induction on n. The assertion is clearly true when $n = 2$. Let $a_1, \ldots, a_{n+1}$ be a sequence of numbers satisfying

$$n \sum_{i=1}^{n+1} a_i^2 + b \le (\sum_{i=1}^{n+1} a_i)^2.$$

It follows that

$$(n-1)a_{n+1}^2 - 2(\sum_{i=1}^n a_i)a_{n+1} + n\sum_{i=1}^n a_i^2 - (\sum_{i=1}^n a_i)^2 + b \le 0.$$

This is a quadratic inequality on a_{n+1}. Hence, its discriminant is non-negative, i.e.:

$$4(\sum_{i=1}^n a_i)^2 - 4(n-1)n\sum_{i=1}^n a_i^2 + 4(n-1)(\sum_{i=1}^n a_i)^2 - 4(n-1)b \ge 0.$$

Above expression is equivalent to:

$$(n-1)\sum_{i=1}^n a_i^2 + \frac{(n-1)b}{n} \le (\sum_{i=1}^n a_i)^2.$$

By the induction argument we obtain that $2a_i a_j \ge \frac{b}{n}$ for every $i, j \in \{1, \ldots, n\}$. Since the choice of a_{n+1} was arbitrary, the lemma is proved. □

Theorem 7. *Let Σ^k be a k-dimensional free boundary minimal surface in the unit ball $B^n \subset \mathbb{R}^n$ and assume that $k \geq 3$. If*

$$|x^\perp|^2 \, |B(x)|^2 \leq \frac{k}{k-1} \tag{2.3}$$

for every $x \in \Sigma^k$, then one of the following is true:

1. *Σ^k is diffeomorphic to a disk $\mathbb{D}^k$.*
2. *Σ^k is diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^{k-1}$ and $C(\Sigma^k)$ is a closed geodesic.*

Proof. The first important observation is that (2.3) implies that $\operatorname{Hess}_\Sigma f$ is non-negative. Indeed, let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of eigenvectors of $\operatorname{Hess}_\Sigma f$ at $x \in \Sigma$ with respective eigenvalues $\overline{\lambda}_1, \ldots, \overline{\lambda}_k$. We want to show that $\overline{\lambda}_i \geq 0$ for every i. By Lemma 5, $\overline{\lambda}_i = 1 + \langle B(e_i, e_i), \overrightarrow{x} \rangle$ and this gives the following

$$\begin{aligned} \sum_{i=1}^k \overline{\lambda}_i^2 &= k + 2\sum_{i=1}^k \langle B(e_i, e_i), \overrightarrow{x} \rangle + \sum_{i=1}^k \langle B(e_i, e_i), \overrightarrow{x} \rangle^2 = k + \sum_{i=1}^k \langle B(e_i, e_i), \overrightarrow{x} \rangle^2 \\ &\leq k + |x^\perp|^2 \sum_{i=1}^k |B(e_i, e_i)|^2 \leq k + |x^\perp|^2 |B|^2. \end{aligned}$$

On the other hand, we have that $\left(\sum_{i=1}^k \overline{\lambda}_i\right)^2 = k^2$ since Σ^k is minimal. Hence,

$$k + |x^\perp|^2 |B|^2 \leq \frac{k^2}{k-1} \Rightarrow \sum_{i=1}^k \overline{\lambda}_i^2 \leq \frac{(\sum_{i=1}^k \overline{\lambda}_i)^2}{k-1}.$$

Therefore, (2.3) combined with Lemma 6, where $\overline{\lambda}_i = a_i$ and $b = 0$, imply that $2\overline{\lambda}_i \overline{\lambda}_j \geq 0$. Consequently, the eigenvalues $\overline{\lambda}_i$, $i = 1, \ldots, k$, have all the same sign. Since $\sum_{i=1}^k \overline{\lambda}_i = k$, we conclude that $\overline{\lambda_i} \geq 0$ for every i and the claim is proved.

The convexity of $\operatorname{Hess}_\Sigma f$ places strong restrictions on the set $C(\Sigma)$ as well as on the topology of Σ as we show below. We first prove that the set of critical points of $f : \Sigma \to \mathbb{R}$ coincides with $C(\Sigma^k)$. Indeed, let $\gamma(t)$ be a geodesic in Σ joining critical points x_0 and x_1 of f with x_0 in $C(\Sigma)$; such a geodesic exists since the geodesic curvature of $\partial\Sigma$ is positive by the free boundary condition and ∂B^n is convex. It follows that $(f\circ\gamma)''(t)\geq 0$, which implies that $(f\circ\gamma)'$ is non-decreasing. But since $(f\circ\gamma)'(0)=(f\circ\gamma)'(1)=0$, we conclude that $f\circ\gamma=\text{const}$, this implies that $f(x_1)=m_0$ and $x_1\in C(\Sigma^k)$. In particular, every geodesic segment with extremes at $C(\Sigma^k)$ is contained in $C(\Sigma^k)$, i.e. $C(\Sigma^k)$ is a totally convex set of Σ^k.

If $C(\Sigma^k)=\{x_0\}$ for some $x_0\in\Sigma^k$, then f has only one critical point, namely, $x_0\in\Sigma^k$. By standard Morse theory we conclude that Σ^k is diffeomorphic to a disk $\mathbb{D}^k$.

Let us now study the case where $C(\Sigma^k)$ contains more than one point. We begin by showing that $\dim(C(\Sigma^k))=1$. Indeed, if x_1 and x_2 are two distinct points in $C(\Sigma)$, then $C(\Sigma)$ contains the minimizing geodesic joining x_0 and x_1 since $C(\Sigma)$ is totally convex. Let γ be the maximal geodesic extending this minimizing geodesic segment and still contained in $C(\Sigma)$. If there exists a point $y\in C(\Sigma)-\gamma$, then $C(\Sigma)$ contains the cone obtained by the union of all geodesic segments with extremities in y and in γ. It follows that $\dim(\mathrm{Ker}(\mathrm{Hess}_\Sigma f))\geq 2$ at every point in this cone. Let e_1 and e_2 be two null eigenvectors of $\mathrm{Hess}_\Sigma f$, then $\langle B(e_1,e_1),\overrightarrow{x}\rangle=-1$ and $\langle B(e_2,e_2),\overrightarrow{x}\rangle=-1$, Consequently,

$$\frac{k}{k-1}\geq |B|^2|x^\perp|^2\geq\Bigg(|B(e_1,e_1)|^2+|B(e_2,e_2)|^2\Bigg)|x^\perp|^2\geq 2.$$

As this is a contradiction when $k\geq 3$, we conclude that $C(\Sigma)=\gamma$ and $\dim(C(\Sigma^k))=1$. If $C(\Sigma)$ fails to be a closed geodesic, then Σ is diffeormorphic to $\mathbb{D}^k$ as one can retract Σ to the boundary of a tubular neighborhood of $C(\Sigma)$ via the gradient flow of the function f and then extend the retraction to a single point as a tubular neighborhood of a line segment is contractible. The remaining case is that of $C(\Sigma)$ being a smooth closed geodesic. Standard Morse theory again guarantees a diffeomorphic retraction of Σ onto $C(\Sigma)$ and this forces Σ to be diffeomorphic to $\mathbb{S}^1\times\mathbb{D}^{k-1}$. $\square$

Corollary 1. *If Σ^n is a free boundary minimal hypersurface in $B_1^{n+1}(0)$ such that*

$$|B|^2(x)\,|x^\perp|^2 \leq \frac{n}{n-1} \tag{2.4}$$

for every $x \in \Sigma^n$, then one of the following is true

1. *Σ is diffeomorphic to a disk $\mathbb{D}^n$.*
2. *Σ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^{n-1}$ and $C(\Sigma)$ is an equator in $\mathbb{S}^n(2m_0)$, where m_0 is defined as in (2.2). Moreover, (2.4) becomes equality when $x \in C(\Sigma^n)$ and B is constant with only two principal curvatures: $\frac{-1}{2\,m_0}$ and $\frac{1}{2(n-1)m_0}$.*

Proof. As $C(\Sigma^n)$ is a geodesic in Σ^n and since Σ^n is tangent to $\mathbb{S}^n(2\,m_0)$ along $C(\Sigma)$ we conclude that

$$\nabla^{\mathbb{R}^3}_{\gamma'(t)}\gamma'(t) \in T\Sigma^\perp = T\mathbb{S}^n(2\,m_0)^\perp,$$

where $\gamma(t)$ is a parametrization of $C(\Sigma^n)$. This shows that $C(\Sigma^n)$ is also a geodesic in $\mathbb{S}^n(2\,m_0)$, hence, an equator. Now, recall that the eigenvectors of $\mathrm{Hess}_\Sigma f$ are also eigenvectors of A_N by formula (2.1). Using that $\overline{\lambda}_1 = 0$ over $C(\Sigma^n)$, we obtain

$$\left|\mathrm{Hess}_\Sigma\, f|_{C(\Sigma)^\perp}\right|^2 = \sum_{i=2}^n \overline{\lambda}_i = n + |x^\perp|^2\,|A_N|^2 \leq \frac{n^2}{n-1} = \frac{(\mathrm{trace}\,\mathrm{Hess}_\Sigma\, f|_{C(\Sigma)^\perp})^2}{n-1}.$$

Since $\dim C(\Sigma^n)^\perp = n-1$, we conclude that $\mathrm{Hess}_\Sigma\, f|_{C(\Sigma)^\perp} = \lambda\,\mathrm{Id}$. Moreover, since Σ^n is minimal, we have that $\mathrm{trace}\,\mathrm{Hess}_\Sigma\, f = n$, and this implies $\lambda = \frac{n}{n-1}$. Using that $\overline{\lambda}_i = 1 + \langle x, N\rangle\lambda_i$, that $\overline{\lambda}_1 = 0$ and $\overline{\lambda}_i = \frac{n}{n-1}$ for $i > 1$, item 2 follows. □

Proposition 4. *There exists $\varepsilon(k) > 0$ such that if Σ^k is a free boundary minimal surface in $B_1^{n+1}(0)$ satisfying*

$$|x^\perp|^2|B|^2 < \varepsilon(k),$$

then Σ^k is the free boundary equatorial disk B^k.

Proof. The proof is by a contradiction argument. Assume that Σ_i^k is a sequence of k-dimensional free boundary minimal surfaces such that

$$|x^{\perp}|^2|B_{\Sigma_i}|^2 \leq \frac{1}{i}$$

for every $x \in \Sigma_i^k$. We first show that the curvature of Σ_i is uniformly bounded. If this is not true, then choose a sequence of points $\{x_i\}_{i=1}$ with $x_i \in \Sigma_i$ and with the property that $|B_{\Sigma_i}(x_i)| = \max_{\Sigma_i} |B_{\Sigma_i}(x)|$. Define $\lambda_i = |B_{\Sigma_i}(x_i)|$ and consider the new surface $\hat{\Sigma}_i = \lambda_i(\Sigma_i - x_i)$ which is a free boundary minimal surface in $\lambda_i(B_1^{n+1}(0) - x_i)$. Up to a subsequence, $\hat{\Sigma}_i$ converge to either a complete without boundary k-dimensional minimal surface Σ_∞ in $\mathbb{R}^{n+1}$ or it is a free boundary minimal surface in a half space in $\mathbb{R}^{n+1}$, the convergence is smooth up to the boundary [5]. In any case, Σ_∞ has the property that $|B_{\Sigma_\infty}(0)| = 1$. On the other hand, for every $z \in \Sigma_\infty$ we have

$$\begin{aligned}
|z^{\perp}|^2\,|B_{\Sigma_\infty}(z)|^2 &= \lim_{i\to\infty} |z_i^{\perp}|^2|B_{\hat{\Sigma}_i}(z_i)|^2 = \lim_{i\to\infty} |y_i^{\perp} - x_i^{\perp}|^2\,|B_{\Sigma_i}(y_i)|^2 \\
&\leq \lim_{i\to\infty}\left(|y_i^{\perp}|^2\,|B_{\Sigma_i}(y_i)|^2 + |x_i^{\perp}|^2\,|B_{\Sigma_i}(y_i)|^2\right) \\
&\leq \lim_{i\to\infty}\left(|y_i^{\perp}|^2\,|B_{\Sigma_i}(y_i)|^2 + |x_i^{\perp}|^2\,|B_{\Sigma_i}(x_i)|^2\right) \\
&= 0.
\end{aligned}$$

Hence, Σ_∞ is totally geodesic. As this is a contradiction, we conclude that $\{\Sigma_i\}$ has uniformly bounded curvature. Therefore, Σ_i converges graphically with multiplicity one to an equatorial disk. As shown in the proof of Theorem **??** in Section 2.2, this is impossible unless Σ_i is an equatorial disk. $\square$

2.1.1 Equivariant minimal surfaces

We now look at minimal surfaces in $\mathbb{R}^{m+2}$ that are $SO(2)\times SO(m)$ invariant. They are natural candidates to satisfy Item 2 in Corollary 1. These surfaces were studied in [3] and [4]. Their constructions were later extended to the free boundary case in [15]. Following [3] and [15] we begin by recalling how these surfaces are constructed.

If Σ is a minimal hypersurface in $\mathbb{R}^{m+2}$ and invariant by $SO(2)\times SO(m)$, then Σ can be parametrized as

$$X : I\times\mathbb{S}^1\times\mathbb{S}^{m-1}\to\mathbb{R}^{m+2}; \quad X(t,x,y)=(a(t)x,b(t)y).$$

The curve $\gamma(t)=(a(t),b(t))$ satisfies the following ODE:

Lemma 8.

$$\frac{b''a'-a''b'}{a'^2+b'^2}+\frac{b'}{a}-(m-1)\frac{a'}{b}=0.$$

Proof. A simple computation gives that $X_t=(a'x,b'y)$, $X_x=(a(t)\partial_x,0)$, and $X_y=(0,b(t)\partial_y)$. Hence, $g_{tt}=a'^2+b'^2$, $g_{xx}=a^2$, $g_{yy}=b^2$, and $g_{tx}=g_{ty}=g_{xy}=0$. Moreover, an unit normal vector N is given by

$$N=\frac{1}{\sqrt{a'^2+b'^2}}\Big(b'(t)x,-a'(t)y\Big).$$

It follows that $\langle N_t,X_t\rangle=\frac{b''a'-a''b'}{\sqrt{a'^2+b'^2}}$, $\langle N_x,X_x\rangle=\frac{1}{\sqrt{a'^2+b'^2}}ab'$, and $\langle N_y,X_y\rangle=-\frac{1}{\sqrt{a'^2+b'^2}}a'b$. A standard computation gives

$$0=(m+1)\,H=\frac{1}{\sqrt{a'^2+b'^2}}\left(\frac{b''a'-a''b'}{a'^2+b'^2}+\frac{b'a}{a^2}-(m-1)\frac{a'b}{b^2}\right).$$

From this the lemma follows. $\square$

Let $\gamma(s)$ be the curve $\gamma(s)=(a(s),b(s))$ in $\mathbb{R}^2$ with respect to a arc-length parametrization and $\varphi(s)$ and $\theta(s)$ be the functions defined by

1. $\gamma(s)=\sqrt{a^2+b^2}(\cos(\varphi(s)),\sin(\varphi(s)))$.

2. $\gamma'(s)=(\cos(\theta(s)),\sin(\theta(s)))$.

Using that $\varphi(s)=\arctan(b(s)/a(s))$, we obtain

$$\varphi'=\frac{\sin(\theta-\varphi)}{\sqrt{a^2+b^2}}.$$

To compute θ' we use Lemma 8

$$\theta' = \frac{2}{\sqrt{a^2+b^2}}\left(\frac{(m-1)\cos(\theta)\cos(\varphi)-\sin(\varphi)\sin(\theta)}{\sin(2\varphi)}\right).$$

The behavior of the surface Σ is understood in terms of the qualitative information given by the integral curves of the vector field (φ', θ') in the plane. This is obtained by studying the integral curves of the following vector field:

$$V(\varphi,\theta) = \Big(\sin(2\varphi)\sin(\theta-\varphi)\,,\; 2((m-1)\cos(\theta)\cos(\varphi)-\sin(\theta)\sin(\varphi))\Big).$$

A careful analysis of the zeros of V in the region $(0, \frac{\pi}{2}) \times (-\pi, \pi)$ is given in [3] and in [15]. When $m \leq 5$ there exists an integral curve $\{(\varphi(t), \theta(t) : t \in \mathbb{R}\}$ starting at the saddle point $(0, \frac{\pi}{2})$ and spiraling toward the focal point $(v_0, v_0))$ where $v_0 = \arctan(\sqrt{m-1})$, see Figure 2.1 below. This integral curve generates, by Lemma 8, a properly embedded minimal hypersurface $\Sigma \subset \mathbb{R}^{m+2}$, $m = 2, 3, 4,$ or 5, diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^m$ and asymptotic to the minimal cone

$$\{t\,(x, \sqrt{m}\,y) : \, t \geq 0,\; x \in \mathbb{S}^1,\; y \in \mathbb{S}^{m-1}\}.$$

The curve $(\varphi(t), \theta(t))$, $t \in (-\infty, \infty)$, starting at $(0, \frac{\pi}{2})$ intersects the line $\varphi = \theta$ infinitely often. Take t_0 to be the first t for which $\varphi(t) = \theta(t)$, i.e., the curve $\gamma(s(t))$ intersects the circle centered at the origin orthogonally when $s_0 = s(t_0)$. It is proved in [15] that $\gamma(s)$, $s \in [0, s_0]$, is contained inside the ball of radius $|\gamma(s_0)|$. Hence, the minimal surface obtained from $\gamma : [0, s_0] \to \mathbb{R}^2$ is a free boundary minimal surface in $B(0, |\gamma(s_0)|)$.

It is pointed out in [15] that when $m \geq 6$ the integral curves of $V(\varphi, \theta)$ no longer intersect the diagonal $\{\varphi = \theta\}$. Therefore, there exist no free boundary minimal surfaces in B^{m+2} which are $SO(2) \times SO(m)$ invariant when $m \geq 6$.

Example 1. Let's compute the quantity $|z^\perp|^2\,|B(z)|^2$ for the minimal surfaces $\Sigma^m \subset \mathbb{R}^{m+2}$ constructed above.

$$\begin{aligned} |z^\perp|^2 = \langle z, N\rangle^2 &= (-b'a + a'b)^2 = (a^2+b^2)(-\sin(\theta)\cos(\varphi)+\cos(\theta)\sin(\varphi))^2 \\ &= (a^2+b^2)\sin^2(\theta-\varphi). \end{aligned}$$

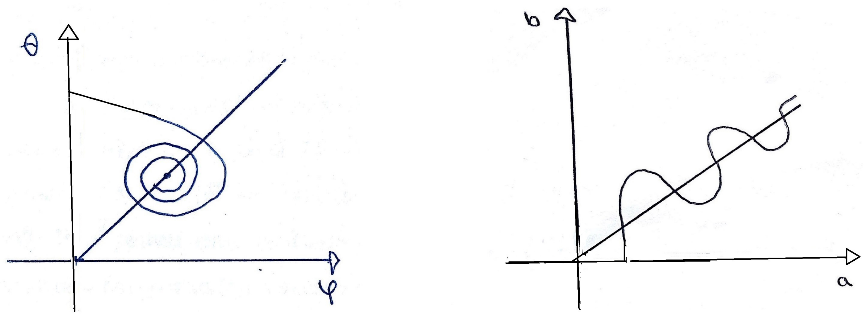

Fig. 2.1: An integral curve of $V(\varphi, \theta)$ and the curve γ.

Recall that $|B|^2 = |dN|^2$ and

$$|dN|^2 = \frac{\langle dN(X_s), X_s\rangle^2}{g_{ss}} + \frac{\langle dN(X_x), X_x\rangle^2}{g_{xx}} + \sum_{i=1}^{m-1} \frac{\langle dN(X_{y_i}), X_{y_i}\rangle^2}{g_{y_i y_i}}.$$

Using the expressions for $\langle dN(X_s), X_s\rangle$, $\langle dN(X_x), X_x\rangle$, and $\langle dN(X_{y_i}), X_{y_i}\rangle$ we obtain

$$\begin{aligned}
|B|^2 &= (b''a' - a''b')^2 + (\frac{b'}{a})^2 + (m-1)(\frac{-a'}{b})^2 \\
&= \left((m-1)\frac{a'}{b} - \frac{b'}{a}\right)^2 + (\frac{b'}{a})^2 + (m-1)(\frac{a'}{b})^2 \\
&= (m^2-m)(\frac{a'}{b})^2 + 2(\frac{b'}{a})^2 - 2(m-1)\frac{a'b'}{ab} \\
&= \frac{1}{a^2+b^2}\left((m^2-m)\frac{\cos^2(\theta)}{\sin^2(\varphi)} + 2\frac{\sin^2(\theta)}{\cos^2(\varphi)} - 2(m-1)\frac{\cos(\theta)\sin(\theta)}{\cos(\varphi)\sin(\varphi)}\right) \\
&= \frac{1}{a^2+b^2}\left(\frac{(m^2-m)\cos^2(\theta)\cos^2(\varphi) + 2\sin^2(\theta)\sin^2(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\right. \\
&\qquad \left. - \quad 2(m-1)\frac{\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|z^\perp|^2|B(z)|^2 &= \sin^2(\theta-\varphi)\left(\frac{(m^2-m)\cos^2(\theta)\cos^2(\varphi) + 2\sin^2(\theta)\sin^2(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\right. \\
&\qquad \left. - \quad 2(m-1)\frac{\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\right).
\end{aligned}$$

If we restrict to the case $m = 2$, which corresponds to a 3-dimensional minimal surface $\Sigma^3 \subset \mathbb{R}^4$, then we get

$$|z^{\perp}|^2|B(z)|^2 = 2\sin^2(\theta-\varphi)\Bigg(\frac{\cos^2(\theta)\cos^2(\varphi)+\sin^2(\theta)\sin^2(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)} - \frac{\cos(\theta)\sin(\theta)\cos(\varphi)\sin(\varphi)}{\cos^2(\varphi)\sin^2(\varphi)}\Bigg)$$

$$\begin{aligned} |z^{\perp}|^2|B(z)|^2 &= 2\sin^2(\theta-\varphi)\left(4\,\frac{\cos^2(\theta+\varphi)}{\sin^2(2\varphi)} + \frac{\sin(2\theta)\sin(2\varphi)}{\sin^2(2\varphi)}\right) \\ &= 2\frac{4\sin^2(\theta-\varphi)\cos^2(\theta+\varphi)}{\sin^2(2\varphi)} + 2\sin^2(\theta-\varphi)\left(\frac{\sin(2\theta)}{\sin(2\varphi)}\right) \\ &= 2\left(\left(\frac{\sin(2\theta)}{\sin(2\varphi)}-1\right)^2 + \sin^2(\theta-\varphi)\frac{\sin(2\theta)}{\sin(2\varphi)}\right). \end{aligned} \tag{2.5}$$

Let us show that $|z^{\perp}|^2|B(z)|^2 = \frac{3}{2}$ for every point $z \in C(\Sigma^3)$. First, note that $(a(0)x, b(0)y) = \lim_{t\to-\infty}(a(s(t)x, b(s(t)y) \in C(\Sigma^3)$. Let λ_1 be a eigenvalue of dN associated to the principal direction $C(\Sigma^3)$ and $z_0 \in C(\Sigma)$. It follows that $\langle z_0, N\rangle\lambda_1 = 1$ by (2.1). Using that $\gamma'(0) = |\gamma(0)|(1,0)$, $\gamma'(0) = (0,1)$ and applying the L'Hôpital's rule we obtain that the other two eigenvalues of dN are equal to $-a''(0)$ at $C(\Sigma^3)$. Since Σ^3 is minimal, we obtain that $2\,a''(0) = \lambda_1$. Therefore,

$$|z_0^{\perp}|^2|B(z_0)|^2 = \langle z, N\rangle^2(\lambda_1^2 + 2\,(a'')^2) = 1 + \frac{1}{2} = \frac{3}{2}. \tag{2.6}$$

In particular, if one can show that the right hand side of (2.5) is a monotone decreasing function on $[0, s_0]$, then Σ^3 will satisfy $|z^{\perp}|^2|B(z)|^2 \leq \frac{3}{2}$. Let's show that this is true for every $s \in [0,\delta)$ where δ is very small. By (2.6) and L'hôpital's rule we have that $\lim_{s\to 0}(\varphi'(s), \theta'(s)) = \frac{1}{\sqrt{a^2+b^2}}(1, -\frac{1}{2})$. Similarly, using the formulas for $\theta'(s)$ and $\varphi'(s)$ computed earlier we obtain that $\lim_{s\to 0}(\varphi''(s), \theta''(s)) = (0,0)$. Let h be the function $h(s) = \frac{\sin(2\theta(s))}{\sin(2\varphi(s))}$, then

$$h'(s) = 2\,\frac{\cos(2\theta)}{\sin(2\varphi)}\,\theta'(s) + \sin(2\theta)\left(\frac{1}{\sin(2\varphi)}\right)'\varphi'(s).$$

L'hôpital's rule once more gives that $h'(0) = 0$. The second derivative is

$$h''(s) = \left(-4\frac{\sin(2\theta)}{\sin(2\varphi)}\,\theta' + 2\cos(2\theta)\left(\frac{1}{\sin(2\varphi)}\right)'\varphi'\right)\theta' + 2\,\frac{\cos(2\theta)}{\sin(2\varphi)}\,\theta'' +$$

$$\left(2\cos(2\theta)\left(\frac{1}{\sin(2\varphi)}\right)'\theta' + \sin(2\theta)\left(\frac{1}{\sin(2\varphi)}\right)''\varphi'\right)\varphi' + \sin(2\theta)\left(\frac{1}{\sin(2\varphi)}\right)'\varphi''.$$

All quantities above have a limit as $s \to 0$ by L'hôpital's rule. Since $(\varphi'(0), \theta'(0)) = \frac{1}{\sqrt{a^2+b^2}}(1, -\frac{1}{2})$ and $(\varphi''(0), \theta''(0)) = (0, 0)$ we obtain

$$h''(0) = \left(\frac{\sin(2(\frac{\pi}{2} - \frac{1}{2}\frac{t}{\sqrt{a^2+b^2}}))}{\sin(2\frac{t}{\sqrt{a^2+b^2}})}\right)''(0) = \frac{1}{2(a^2+b^2)}.$$

As $(|z^\perp|^2|B(z)|^2)'(s) = 4(h-1)h' + 2\sin(2\theta - 2\varphi)(\theta - \varphi)'h + 2\sin^2(\theta - \varphi)h'$, we first obtain that $(|z^\perp|^2|B(z)|^2)'(0) = 0$. Similarly,

$$(|z^\perp|^2|B(z)|^2)''(s) = 4h'^2 + 4(h-1)h'' +$$

$$4\cos(2\theta - 2\varphi)(\theta-\varphi)'^2h + 2\sin(2\theta - 2\varphi)(\theta-\varphi)''h + \sin(2\theta-2\varphi)(\theta-\varphi)'h'$$

$$+ 2\sin(2\theta - 2\varphi)(\theta - \varphi)'h + 2\sin^2(\theta-\varphi)h''.$$

$$\left(|z^\perp|^2|B(z)|^2\right)''(0) = -\frac{1}{a^2+b^2} - \frac{9}{2(a^2+b^2)} + \frac{1}{a^2+b^2} = -\frac{9}{2(a^2+b^2)} < 0.$$

Hence, $(|z^\perp|^2|B(z)|^2)(s)$ is a decreasing function on $[0, \delta)$ for some $\delta > 0$ very small. This proves that Σ^3 satisfies the geometric inequality $|z^\perp|^2|B(z)|^2 \leq \frac{3}{2}$ near $C(\Sigma)$.

Remark 1. The function $|z^\perp|^2|B(z)|^2(s)$ is not a decreasing function on the whole interval $[0, \infty)$ since $|z^\perp|^2(s)$ vanishes infinitely often. Regardless of that, we still expect Σ^3 to satisfy $|z^\perp|^2|B(z)|^2 \leq \frac{3}{2}$.

Example 2. The n-dimensional catenoid, denoted by Σ_c, is the minimal surface $SO(n)$-invariant in $\mathbb{R}^{n+1}$. By Lemma 8, Σ_c can be parametrized by

$\{(r, b(r)y) : r \in I,\, y \in \mathbb{S}^{n-1}\}$ where $b(s)$ satisfies

$$\frac{1}{1+b'^2}b'' - \frac{n-1}{b} = 0, \quad b(0) > 0, \quad b'(0) = 0.$$

Note that $\gamma(r) = (r, b(r))$ satisfies $\langle \gamma, \gamma'' \rangle \geq 0$. A simple computation will give the following:

$$\mathrm{Hess}_{\Sigma_c} f\,(\partial_s, \partial_s) = 1 + a\,a'' + b\,b'',$$

where $f(x) = \frac{1}{2}|x|^2$ and $\gamma(s) = (a(s), b(s)$ is a arc length parametrization of $\gamma(r) = (r, b(r))$. Since $\langle \gamma, \gamma'' \rangle \geq 0$, we conclude that $\mathrm{Hess}_{\Sigma_c} f \geq 0$.

Lemma 9. *If Σ^n is a free boundary minimal hypersurface in B^{n+1} satisfying Hess$_\Sigma$ $f \geq 0$, then*

$$|x^\perp|^2\, |B(x)|^2 \leq\, n(n-1).$$

Proof. Since $\mathrm{Hess}_\Sigma f$ is symmetric and non-negative definite, we conclude that

$$|\mathrm{Hess}_\Sigma f|^2 \leq (\mathrm{trace}(\mathrm{Hess}_\Sigma f))^2.$$

Using that $\mathrm{Hess}_\Sigma f(X, Y) = \langle X, Y \rangle + \langle B(X, Y), \overrightarrow{x} \rangle$ by Lemma 5 we have that $|\mathrm{Hess}_\Sigma f|^2 = n + |x^\perp|^2 |B|^2$. On the other hand, since $\mathrm{trace}(\mathrm{Hess}_\Sigma f) = n$, we obtain that $|x^\perp|^2\, |B|^2 \leq n^2 - n = n(n-1)$. □

Proposition 5. *Let Σ^n be a minimal hypersurface in $\mathbb{R}^{n+1}$ which satisfies Hess$_\Sigma$ $f \geq 0$. If dim $C(\Sigma) = n-1$, then Σ^n is isometric to the n-dimensional catenoid.*

Proof. Since $\mathrm{Hess}_\Sigma f \geq 0$ and $\dim(C(\Sigma)) = n-1$, we have that $C(\Sigma) \subset \mathbb{S}^{n-1}$. Without loss of generality, let us assume that $C(\Sigma) = \mathbb{S}^{n-1}$. Let $J_\theta : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be an one parameter family of isometries of $\mathbb{S}^{n-1}$, i.e., J_θ is a curve on $SO(n)$. Noting that J_θ is also a curve on $SO(n+1)$ also. Consider the function ϕ defined on Σ given by $\phi(x) = \langle \frac{dJ_\theta}{d\theta}(x), N(x) \rangle$. Since Σ is a minimal hypersurface, we have

$$\Delta\phi + |B|^2\,\phi = 0.$$

Note also that $\phi \equiv 0$ on $C(\Sigma)$. Let us now look at the gradient of ϕ on $C(\Sigma)$. If $v \in C(\Sigma) = \mathbb{S}^{n-1}$, then $v(\phi) = 0$ since $J_\theta \in SO(n)$. Assuming now that $v \in T_x\Sigma$ and is orthogonal to $C(\Sigma)$, then

$$v(\phi) = \langle \overline{\nabla}_v \frac{dJ_\theta}{d\theta}, N\rangle + \langle \frac{dJ_\theta}{d\theta}, dN(v)\rangle = 0 + 0 = 0,$$

since v is a principal direction of Σ by Lemma 2.1. By the results of Aronsajn [6], the function ϕ can only vanish to finite order on $C(\Sigma)$. On the other hand, since $\phi = \nabla\phi = 0$ on $C(\Sigma)$, we conclude by Lemma 1.5 that $\{\phi = 0\} \cap \{\nabla\phi = 0\}$ has Hausdorff dimension at most $n-2$. As this contradicts that $\dim(C(\Sigma)) = n-1$, we have that $\phi \equiv 0$. Hence, Σ is $SO(n)$ invariant, i.e. Σ is isometric to a n-dimensional catenoid. $\square$

2.1.2 2-dimensional minimal surfaces in B^n

Let $c : [a, b] \to N$ be a curve in a Riemannian manifold N. The set $\Delta(s) \subset T_{c(s)}N$ is called a distribution along c if for each $s \in [a, b]$ we have that $\Delta(s)$ is a j-dimensional subspace of $T_{c(s)}N$. Let $P : T_{c(a)}N \to T_{c(s)}N$ be the parallel transport map along c. We say that $\Delta(s)$ is parallel if $P(\Delta(a)) = \Delta(s)$ $\forall s \in [a, b]$.

Lemma 10. *If $\frac{DV}{dt}(t) \in \Delta(s)$ whenever V is a vector field in $\Delta(s)$, then $\Delta(s)$ is parallel along c.*

Proof. Choose $(V_1, \ldots, V_j)$ linearly independent vector fields along c and in $\Delta(s)$. Hence,

$$\frac{DV_i}{ds} = \sum_{l=1}^{j} a_{il} V_l.$$

We claim there exist functions $b_{ik}(s)$ for which

$$\frac{D}{ds}\sum_{k=1}^{j} b_{ik}(s) V_k(s) = 0, \quad i = 1, \ldots, j.$$

This equation is equivalent to

$$\begin{aligned} 0 &= \sum_{k=1}^{j} b'_{ik}(s)V_k(s) + \sum_{k=1}^{j} b_{ik}(s)\frac{DV_k}{ds} \\ &= \sum_{l=1}^{j} b'_{li}(s)V_l(s) + \sum_{l,k=1}^{j} b_{ik}(s)a_{lk}(s)V_l(s). \end{aligned}$$

Hence,

$$b'_{li}(s) = -\sum_{k=1}^{j} b_{ik}(s)a_{lk}(s), \quad i = 1, \dots, j.$$

This is a linear differential equation and so we can solve it in the whole interval $[a,b]$. Choosing $b_{ik}(0) = \delta_{ik}$ as initial conditions and defining

$$W_i(s) = \sum_{k=1}^{j} b_{ik}(s)V_k(s),$$

we obtain parallel vector fields which are linearly independent along c and that span $\Delta(s)$. This proves that $\Delta(s)$ is parallel along c. $\square$

Theorem 11. *If Σ^2 is a free boundary minimal surface in $B_1^{n+1}(0)$ satisfying*

$$|x^\perp|^2|B(x)|^2 \leq 2, \tag{2.7}$$

for every $x \in \Sigma^2$, then one of the following is true:

- $|x^\perp|^2|B(x)|^2 \equiv 0$ *and Σ^2 is a flat equatorial disk.*
- $|x_0^\perp|^2|B(x_0)|^2 = 2$ *at some point x_0 and Σ^2 is a critical catenoid.*

Proof. As in the proof of Theorem 7, inequality (2.7) implies that $\mathrm{Hess}_\Sigma f \geq 0$. Let us show that Σ is diffeomorphic to either a disk or an annulus. If Σ is simply connected, then Σ is topologically a disk. Hence, we can assume that $\pi_1(\Sigma, x) \neq \{0\}$ for some $x \in C(\Sigma)$. It follows that if we minimize the length in a nontrivial homotopy class $[\alpha] \in \pi_1(\Sigma, x)$ among closed loops passing through a fixed point $x \in C(\Sigma)$, then we obtain a geodesic loop $\gamma : [0,1] \to \Sigma$, where $\gamma(0) = \gamma(1) = x$; this is true since $\partial\Sigma$ is convex on

Σ by the free boundary condition. We claim that $\gamma'(0) = \gamma'(1)$, i.e., γ is smooth. If this is not true we can use the total convexity of $C(\Sigma)$ to find a disk spanning the curve γ, which is a contradiction since $[\alpha] \neq 0$. Using this information we prove that $\pi_1(\Sigma)$ is cyclic. If not, then we can find another smooth simple closed geodesic β in $C(\Sigma)$ such that $x = \gamma(0) = \beta(0)$. Hence, $[\gamma]$ and $[\beta]$ are trivial in homotopy since we can produce a geodesic spanning these curves by the total convexity property of $C(\Sigma)$ once again. Now is a standard fact that $\pi_1(\Sigma)$ being cyclic implies that Σ is an annulus.

If Σ^2 is a minimal disk, then it is proved by A. Fraser and R. Schoen in [20] that Σ^2 is an equatorial disk, in which case $|x^{\perp}|^2\,|B(x)|^2 \equiv 0$. If Σ^2 is an annulus, then $C(\Sigma)$ is a smooth simple closed geodesic and $\overline{\lambda} = 0$ is an eigenvalue of $\mathrm{Hess}_{\Sigma} f$ for every $x_0 \in C(\Sigma)$. Hence, we have that $(\overline{\lambda}_1^2 + \overline{\lambda}_2^2) = (\overline{\lambda}_1 + \overline{\lambda}_2)^2$. On the other hand,

$$\sum_{i=1} \overline{\lambda}_i^2 = 2 + \langle B(e_i, e_i), \overrightarrow{x}\rangle^2 \leq 2 + |x^{\perp}|^2\,|B|^2 \leq 4 = (\sum_{i=1} \overline{\lambda}_i)^2.$$

In particular, $\langle \sum_{i=1}^2 B(e_i, e_i), \overrightarrow{x}\rangle^2 = |B|^2 |x^{\perp}|^2 = 2$ for every $x \in C(\Sigma)$. Hence, by the Cauchy-Schwarz inequality we obtain

$$B(e_i, e_i) = \langle B(e_i, e_i), \frac{\overrightarrow{x}}{|x|}\rangle \frac{\overrightarrow{x}}{|x|}. \tag{2.8}$$

Consequently, if e_1 is tangent to $C(\Sigma)$, then

$$\overline{\nabla}_{e_1} e_1 = \langle B(e_i, e_i), \frac{\overrightarrow{x}}{|x|}\rangle \frac{\overrightarrow{x}}{|x|},$$

since $C(\Sigma)$ is a geodesic on Σ. Thus, $C(\Sigma)$ is also a geodesic in $\partial B^{n+1}_{2\,m_0}(0)$, i.e., a round circle. Now we consider the normal distribution E along $C(\Sigma)$ defined by

$$E = \{\xi : \xi \in \mathcal{X}^{\perp}(\Sigma)|_{C(\Sigma)} \quad \text{and} \quad \langle \xi, \overrightarrow{x}\rangle = 0\}.$$

By (2.8) we conclude that E has the property that for every $\xi \in E$ the following is true

$$\overline{\nabla}_{\gamma'(t)}\xi \in E.$$

Lemma 10 implies that the distribution E is parallel along $C(\Sigma)$. Hence, E is a constant $(n-2)$-dimensional plane throught the origin. Therefore, there exists a critical catenoid Σ_c which is tangent to Σ along $C(\Sigma)$. Near $x_0 \in C(\Sigma)$ we write Σ and Σ_c locally as a graph over $T_{x_0}\Sigma$. Hence, $\Sigma_c = \text{graph}(f_c)$ and

$$\text{div}\left(\frac{\nabla f_c}{\sqrt{1+|\nabla f_c|^2}}\right) = 0.$$

Similarly, $\Sigma = \text{graph}(u)$, where $u : \mathbb{R}^2 = T_{x_0}\Sigma \to \mathbb{R}^{n-1}$, and by Lemma 1

$$\frac{a_{ij}(\nabla u_1, \ldots, \nabla u_{n-1})}{\sqrt{1+|\nabla u_i|^2}} D_{ij} u_i = 0.$$

for every $i \in 1, \ldots, n-1$. Lemma 2 implies that the difference $v = u - f_c$ satisfies a linear PDE of the following form:

$$\frac{a_{ij}(\nabla u)}{\sqrt{1+|\nabla u_k|^2}} D_{ij} v_k + \sum_{l=1}^{n-1} b_j^l(\nabla u, \nabla f_c) D_j v_l = 0,$$

for each $k = 1, \ldots, n-1$. Note that since v vanishes on x_0, it follows that v vanishes to finite order at x_0 by Lemma 2. Therefore, $\mathcal{H}^1(v^{-1}(0) \cap |\nabla v|^{-1}(0) = 0$ by Lemma 3. As this contradicts the fact that Σ and Σ_c are tangent along $C(\Sigma)$ and $\dim C(\Sigma) = 1$, we conclude that $v \equiv 0$. The corollary now follows from standard analytic continuation property for minimal surfaces. $\square$

2.2 *Area gap for minimal surfaces in the unit ball*

2.2.1 *Introduction*

In these notes, we study the area of k-dimensional minimal surfaces in the Euclidean ball B^n that meet ∂B^n orthogonally. These surfaces are critical points of the area functional in the space of k-dimensional surfaces with boundary in ∂B^n. They are commonly known as free boundary minimal surfaces. The equatorial disk D^k is the simplest example. Brendle [32] proved that D^k is the least area free boundary minimal surface in B^n (see also [21] for the case of 2-dimensional free boundary surfaces). More precisely,

Theorem 12 (Brendle)**.** *Let Σ^k be a k-dimensional free boundary minimal surface in B^n. Then*

$$|\Sigma^k| \geq |D^k|$$

Moreover, the equality holds if, and only if, Σ^k is contained in a k-dimensional plane in $\mathbb{R}^n$.

This result is the free boundary analogue of a classical result about closed minimal surfaces in the round sphere $\mathbb{S}^n$. Namely,

Theorem 13. *There exists $\varepsilon(k,n) > 0$ so that whenever Σ^k is a k-dimensional minimal surface in $\mathbb{S}^n$ which is not totally geodesic, then*

$$|\Sigma^k| \geq |\mathbb{S}^k| + \varepsilon(k,n).$$

Despite the proofs of Theorem 1.1 and Theorem 1.2 both explore a monotonicity principle for minimal surfaces, they are quite different. Theorem 1.2, for instance, is only an application of the Monotonicity Formula for minimal surfaces together with the smooth version of Allard's Regularity Theorem:

Theorem 14 (Allard)**.** *There exist $\epsilon(k,n) > 0$, $C > 0$ and $r_0 > 0$ so that whenever Σ is a $k-$dimensional minimal surface in $\mathbb{R}^{n+1}$ satisfying*

$$\theta(x,r) \leq 1 + \epsilon(k,n)$$

for every $x \in \Sigma$ and every $r < r_0$, then

$$\sup_{\Sigma} |A_\Sigma| \leq C.$$

Indeed, let Σ_i be a sequence of $k-$dimensional minimal surfaces in $\mathbb{S}^{n+1}$ such that $\lim_{i\to\infty} |\Sigma_i| = \mathcal{A}(k,n)$, where $\mathcal{A}(k,n)$ is the infimum for the areas of free boundary minimal surfaces in $\mathbb{S}^n$. If $C\Sigma_i$ denotes the minimal cone over Σ_i with vertice at 0 and if $y_i \in \Sigma_i$, then

$$\frac{|\Sigma_i|}{|\mathbb{S}^k|} = \lim_{r\to\infty} \frac{|C\Sigma_i \cap B_r(y_i)|}{|B^{k+1}|r^{k+1}} \geq \frac{|C\Sigma_i \cap B_r(y_i)|}{|B^{k+1}|r^{k+1}} = \theta(C\Sigma_i, y_i, r) \geq 1,$$

with equality if, and only if, Σ_i is an equatorial sphere $\mathbb{S}^k$. The inequality follows from the monotonicity formula for minimal surfaces. Hence, $\mathcal{A}(k,n) = |\mathbb{S}^k|$ and from Theorem 1.3 we conclude that $|A_{\Sigma_i}| \leq C$. Standard compactness shows that $\Sigma_i \to \mathbb{S}^k$ graphically and with multiplicity one. As the round metric is analytic, we obtain that $|\Sigma_i| = |\mathbb{S}^k|$; thus, Σ_i is an equatorial sphere for i large enough.

In view of Theorem 1.1 and Theorem 1.2, it is natural to expect similar gap phenomena for the area of free boundary minimal surfaces in B^n as well. In contrast with Theorem 1.2, the smooth free boundary version of Allard's regularity theorem does not readily apply to this end. It can be proved, however, that it follows from the strong Allard's regularity theorem, proved by Grüter and Jost [16], together with the analysis developed in [32], which we also use here. Our main result is a direct and simpler proof of this fact:

Theorem 15. *There exists $\varepsilon(k,n) > 0$ such that whenever Σ^k is a k-dimensional free boundary minimal surface in B^n satisfying*

$$|\Sigma^k| < |D^k| + \varepsilon(k,n),$$

then Σ^k is, up to ambient isometries, the equatorial disk D^k.

The 2-dimensional case in Theorem 15 was proved by Ketover [28]. The key ingredients in the proof there are an excess inequality for 2-dimensional free boundary surfaces in B^n, proved by Vokmann in [33], and the classical Nitsche's Uniqueness Theorem for free boundary minimal disks in B^3 (see also [20], for the generalization of this result to high codimension). The excess inequality is particularly important in proving curvature estimates for a sequence of free boundary minimal surfaces with area sufficiently close to the area of the equatorial disk. The main difficulty in implementing the arguments of [28] to k-dimensional surfaces in B^n is that neither the excess inequality in the form used in [28] nor Nitsche's Theorem is available when $k \geq 3$. To get around these difficulties, we consider a slightly more general quantity, originated in [32] and which also resemble an excess type formula, and compare it with that of the free boundary cones over the boundaries to obtain the necessary curvature estimates. Finally, we replace the use of Nitsche's theorem by an standard index of stability analysis. These ideas leads us to the result below for which Theorem 15 follows in view of the arguments in [28]. More precisely,

Proposition 6. *Let Σ^k be a k-dimensional free boundary minimal surface in B^n and $C_1\partial\Sigma$ the cone with vertice at the origin and base $\partial\Sigma$. If $y \in \Sigma - \{0\}$, then*

$$\int_{\Sigma} \frac{|(x-y)^{\perp}|^2}{|x-y|^{k+2}} = \int_{C_1\partial\Sigma} \frac{|(x-y)^{\perp}|^2}{|x-y|^{k+2}} + \frac{1}{k}\int_{C_1\partial\Sigma} \langle \overrightarrow{H}_{C\partial\Sigma}, \frac{x-y}{|x-y|^k}\rangle - v(y)|D^k|, \tag{2.9}$$

where $v(y) = 1$ if $y \notin C_1\partial\Sigma$ and $v(y) = 0$ if $y \in C_1\partial\Sigma$.

As a consequence, we obtain the following unique continuation type result for minimal surfaces in the ball.

Corollary 2. *If Σ^k is a k-dimensional free boundary minimal surface in B^n such that $\partial\Sigma$ is a $(k-1)$-minimal surface in $\mathbb{S}^{n-1}$, then Σ^k is an equatorial disk.*

2.2.2 *Higher dimension free boundary minimal surfaces*

We start by recalling an excess inequality for free boundary minimal surfaces in the ball proved in [32]. More precisely, if Σ is a k-dimensional free boundary minimal surface in B^n and if $y \in \partial\Sigma$, then

$$\int_{\Sigma^k} \frac{|(x-y)^{\perp}|^2}{|x-y|^{k+2}}\, d_{\Sigma} \leq |\Sigma^k| - |D^k|. \tag{2.10}$$

This inequality, which implies Theorem 12, follows from a monotonicity argument obtained by an application of the Divergence Theorem to the vector field $W_{t_0,y}(x)$ defined on $B^n - \{y\}$ and given by

$$W_{t_0,y}(x) = \frac{x}{2} - \frac{x-y}{|x-y|^k} - \frac{k-2}{2}\int_{t_0}^{|y|^2} \frac{tx-y}{|tx-y|}dt.$$

We will need a formula similar to (2.10) for when y is not necessarily at the boundary. For this, we need to recall the techniques in [32] behind the proof of (2.10).

Lemma 16. *Let Σ^k a free boundary surface in B^n and $y \in \Sigma$. For r sufficiently small, we have*

$$2 \int_{\Sigma\setminus B_r(y)} \frac{|(x-y)^{\perp}|^2}{|x-y|^{k+2}}\, d_\Sigma + (k-2)\int_{\Sigma\setminus B_r(y)} \int_{t_0}^{|y|^2} \frac{t\,|(tx-y)^{\perp}|^2}{|tx-y|^{k+2}}\, dt\, d_\Sigma$$

$$= |\Sigma\setminus B_r(y)| - \frac{2}{k}\int_{\Sigma\cap\partial B_r(y)} \langle W_{t_0,y}(x), \nu(x)\rangle\, d\sigma$$

$$-\frac{2}{k}\int_{\partial\Sigma} \langle W_{t_0,y}, x\rangle d\sigma + \frac{2}{k}\int_{\Sigma - B_r(y)} \langle \overrightarrow{H}, W_{t_0,y}\rangle d_\Sigma. \tag{2.11}$$

Proof. See Section 2 in [32]. □

The next lemma deals with the second term in the right hand side of (2.11):

Lemma 17. *Let Σ^k be a free boundary minimal surface in B^n and let $\varphi(y) = 1$ if $y \in \partial\Sigma$ and $\varphi(y) = 2$ if $y \in \Sigma\setminus\partial\Sigma$. Then*

$$\lim_{r\to 0} \frac{2}{k}\int_{\Sigma\cap\partial B_r(y)} \langle W_{t_0,y}(x), \nu(x)\rangle = \varphi(y)\,|D^k|.$$

Proof. See Section 2 in [32]. □

Lemma 18. *If $y \in \partial\Sigma$, then $\langle W_{0,y}(x), x\rangle = 0$ for every $x \in \partial\Sigma$.*

Proof. See Section 2 in [32]. □

Applying Lemmas 16, 17, and 18, we obtain the inequality (2.10).

Proof of Proposition 6. For this proposition we choose $t_0 = |y|^2$. Hence, th vector field $W_{t_0,y}$ becomes

$$W_y = \frac{x}{2} - \frac{x-y}{|x-y|^k}.$$

Applying Lemma 16 and Lemma 17 we obtain

$$2\int_{\Sigma} \frac{|(x-y)^{\perp}|^2}{|x-y|^{k+2}}\, d_{\Sigma} = |\Sigma| - 2|D^k| - \frac{2}{k}\int_{\partial\Sigma} \langle W_y(x), x\rangle d\sigma. \tag{2.12}$$

Now we look at the last term in (2.12). Let $C_1\partial\Sigma$ be the free boundary cone over $\partial\Sigma$ and vertice at 0. By assumption $y \notin C_1\partial\Sigma$. Applying Lemma 16 to $C_1\partial\Sigma$ and observing that $C_1\partial\Sigma_i$ is not a minimal surface, we obtain:

$$\begin{aligned} 2\int_{C_1\partial\Sigma\setminus B_r(0)} \frac{|(x-y_i)^{\perp}|^2}{|x-y_i|^{k+2}} &= |C_1\partial\Sigma \setminus B_r(0)| - \frac{2}{k}\int_{C_1\partial\Sigma\cap\partial B_r(0)} \langle W_y, \nu\rangle\, d\sigma \\ &\quad - \frac{2}{k}\int_{\partial\Sigma} \langle W_y(x), x\rangle d\sigma + \int_{C_1\partial\Sigma\setminus B_r(0)} \langle \overrightarrow{H}_{C_1\partial\Sigma}, W_y\rangle\, d_{C_1\partial\Sigma}. \end{aligned}$$

Taking the limit as $r \to 0$ in above expression, we obtain

$$\begin{aligned} 2\int_{C_1\partial\Sigma} \frac{|(x-y)^{\perp}|^2}{|x-y|^{k+2}}\, d_{C\Sigma} = |C_1\partial\Sigma| \quad &- \quad \frac{2}{k}\int_{\partial\Sigma} \langle W_y(x), x\rangle d\sigma \\ &+ \quad \int_{C_1\partial\Sigma} \langle \overrightarrow{H}_{C_1\partial\Sigma}, W_y\rangle\, d_{C_1\partial\Sigma}. \end{aligned} \tag{2.13}$$

Plugging (2.13) into (2.12), we obtain

$$\begin{aligned} 2\int_{\Sigma} \frac{|(x-y_i)^{\perp}|^2}{|x-y_i|^{k-2}}\, d_{\Sigma} \quad &= \quad |\Sigma| - |C_1\partial\Sigma| + 2\int_{C_1\partial\Sigma} \frac{|(x-x_i)^{\perp}|^2}{|x-x_i|^{k+2}} \\ &\quad - \quad \frac{2}{k}\int_{C_1\partial\Sigma} \langle \overrightarrow{H}_{C_1\partial\Sigma}, W_y\rangle\, d_{C_1\partial\Sigma} - 2|D^k|. \end{aligned} \tag{2.14}$$

The free boundary condition of Σ combined with the Divergence Theorem applied to the position vector $X = \overrightarrow{x}$ give

$$k|C_1\partial\Sigma| = |\partial\Sigma| - \int_{C_1\partial\Sigma} \langle \overrightarrow{H}_{C_1\partial\Sigma}, x\rangle d_{\Sigma} = |\partial\Sigma| = k|\Sigma|.$$

This completes the proof of the proposition. □

2.2.3 Proof of Theorem 1.4

Lemma 19. *If Σ^k is a free boundary minimal surface in B^n which is not totally geodesic, then Index$(\Sigma^k) \geq (k+2)(n-k)$.*

Proof. Following [35] we have for $X \in \mathcal{X}(\mathbb{R}^{n+1})$ the following expression for the second variation of area Σ in the direction of X

$$\delta^2\Sigma(X,X) = \int_\Sigma \left(|D^\perp X|^2 - |\langle B, X\rangle|^2 \right) d_\Sigma + \int_{\partial\Sigma} \langle D_X X, \nu\rangle d\sigma.$$

Given $v \in \mathbb{R}^n$ we consider, for each $i = 1, \dots, n-k$, the vector field $X_i = \langle x, v\rangle N_i$. As Σ^k is minimal, we have that $\Delta_\Sigma(\langle v, x\rangle + t) = 0$, for every $(v,t) \in \mathbb{R}^{n+2}$. Moreover, the free boundary condition implies that $\frac{d}{d\nu}\langle v, x\rangle = \langle v, x\rangle$. Putting these facts together we obtain

$$\delta^2\Sigma(X_i, X_i) = -\int_\Sigma |B|^2 |X_i|^2 \, d_\Sigma \; < \; 0.$$

Similarly, if we consider for each $j = 1, \dots, n-k$ the vector field $Y_j = N_j$, then

$$\delta^2\Sigma(Y_j, Y_j) = -\int_\Sigma |B|^2 - \int_{\partial\Sigma} d\sigma < 0.$$

Using that $\frac{d}{d\nu}(\langle v, x\rangle + t) = \langle v, x\rangle$ and also $\int_{\partial\Sigma} \frac{d}{d\nu}\langle v, x\rangle \, d\sigma = 0$, one can check that $\delta^2\Sigma < 0$ in the space generated by $\{X_i, Y_j\}$. Therefore, index of Σ^k is at least $(k+2)(n-k)$. □

Proof of Theorem 15. The proof is by contradiction, we assume that $\{\Sigma_i\}$ is a sequence of k-dimensional free boundary minimal surfaces in B^n satisfying

$$|\Sigma_i| \to |D^k| \tag{2.15}$$

Following the strategy in [28], we first show that (2.15) implies curvature estimates for Σ_i.

Lemma 20. *Let A_{Σ_i} be the second fundamental form of Σ_i. Then, there exists $C > 0$ such that*

$$\sup_{x \in \Sigma_i} |A_{\Sigma_i}(x)| \leq C. \tag{2.16}$$

Let us show that Lemma 20 together with the index estimate of Lemma 19 imply the theorem:

By Lemma 20 the second fundamental form of $\{\Sigma_i\}$ is uniformly bounded. Hence, Theorem 6.1 in [34] (see also [5]) implies that Σ_i converges smoothly up to the boundary to a free boundary surface Σ_∞ and $|\Sigma_\infty| = |D^k|$. By Theorem 12, Σ_∞ is an equatorial disk and consequently Σ_i is, for i large enough, diffeomorphic to a k-dimensional disk. On the other hand, by Lemma 19, $\mathrm{Index}(\Sigma_i^k) \geq (k+2)(n-k)$ since Σ_i is assumed to be not totally geodesic. Thus, exist $(k+2)(n-k)$ mutually orthonormal eigenvectors of the Jacobi operator defined on $\mathcal{X}^\perp(\Sigma_i)$ each satisfying

$$\Delta^\perp X \quad + \quad \sum_{jl} \langle B(e_j, e_l), X\rangle B(e_j, e_l) + \lambda_X\, X = 0, \tag{2.17}$$

$$(D_\nu X - D_X \nu)^{T\partial B^n} = 0, \quad \text{and} \quad \lambda_X < 0.$$

As $i \to \infty$, these eigenvectors converge to eigenvectors of $B^k \subset B^n$. Hence, (2.17) reduces to an scalar equation of form

$$\Delta\phi + \lambda_\phi\, \phi = 0, \quad \lambda_\phi \leq 0, \quad \text{and} \quad \frac{\partial \phi}{\partial \nu} = \phi,$$

since Σ_∞ is totally geodesic. The respective eigenvectors are of form $X = \phi e_l$, where $\{e_{k+1}, \ldots, e_n\}$ being the parallel orthonormal base for $\Sigma_\infty^\perp$. Since $\mathrm{Index}(\Sigma_\infty) = n - k$, we we obtain $k+1$ orthonormal eigenfunctions for the Steklov eigenvalue problem:

$$\Delta u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = u$$

on Σ_∞. This is a contradiction since the multiplicity for the first Steklov eigenvalue of the k-dimensional equatorial disk is k. □

Proof of Lemma 20. Arguing by contradiction, we assume that

$$\mathrm{Area}(\Sigma_i) \to |D^k| \quad \text{and} \quad \lambda_i = \sup_{x \in \Sigma_i} |A_i|^2(x) \to \infty.$$

For each i choose $x_i \in \Sigma_i$ with the property that $\sup_{\Sigma_i} |A_i|^2 = |A_i|^2(x_i)$. Note that $\lim_{i\to\infty} |x_i| = 1$. Indeed, the excess inequality (2.10) implies that

Σ_i converges with multiplicity one to $\mathbb{D}^k$ as a varifold. Hence, in $B^n(R)$, $0 < R < 1$, the surface Σ_i satisfy $\theta(\Sigma_i, x, r) \leq 1 + \varepsilon$ for every i large enough and r small enough. If $\lim_{i\to\infty} |x_i| < 1$, then we would get a contradiction with Allard's regularity theorem. Now we consider the surface

$$\hat{\Sigma}_i = \lambda_i(\Sigma_i - x_i).$$

One can check that $\hat{\Sigma}_i$ satisfies

$$\sup_{x\in\hat{\Sigma}_i} |A|(x) \leq 1 \quad \text{and} \quad |A_{\hat{\Sigma}_i}|(0) = 1 \tag{2.18}$$

and it is a free boundary minimal surface in $\lambda_i(B_1^{n+1}(0) - x_i)$. It follows from Theorem 6.1 in [34](see also [5]) that, after passing to a subsequence, $\hat{\Sigma}_i$ converges smoothly and locally uniformly to Σ_∞. Σ_∞ is either complete without boundary minimal surface or it is a free boundary minimal surface in a half space. Moreover, (2.18) implies that

$$|A_{\Sigma_\infty}|(0) = 1. \tag{2.19}$$

On the other hand, by the scale invariance of the excess, we have that

$$\int_{\Sigma_\infty} \frac{|z^\perp|^2}{|z|^{k+2}}\, d_{\Sigma_\infty} \leq \liminf_{i\to\infty} \int_{\Sigma_i} \frac{|z^\perp|^2}{|z|^{k+2}}\, d_{\Sigma_i} = \liminf_{i\to\infty} \int_{\Sigma_i} \frac{|(x-x_i)^\perp|^2}{|x-x_i|^{k+2}}\, d_{\Sigma_i}$$

We want to prove that the last term above goes to zero as $i \to \infty$. If a subsequence x_i lies in $\partial\Sigma$, then, by (2.10),

$$\int_{\Sigma_\infty} \frac{|z^\perp|^2}{|z|^{k+2}}\, d_{\Sigma_\infty} \leq \liminf_{i\to\infty} (|\Sigma_i| - |B^k|) = 0.$$

Hence, Σ_∞ is a half plane which is in contradiction with (2.19). Therefore, $y_i \in \Sigma_i - \partial\Sigma_i$ and, without loss of generality, we can also assume that $y_i \notin C_1\partial\Sigma_i$. Applying Proposition 6,

$$\int_{\Sigma_\infty} \frac{|z^\perp|^2}{|z|^{k+2}}\, d_{\Sigma_\infty}| \quad \leq$$

$$\int_{C_1\partial\Sigma_i} \frac{|(x-y_i)^\perp|^2}{|x-y|^{k+2}}\, d_{C\partial\Sigma_i} \quad + \quad \frac{1}{k}\int_{C\partial\Sigma_i} \langle \overrightarrow{H}_{C_1\partial\Sigma_i}, \frac{x-y_i}{|x-y_i|^k}\rangle\, d_{C_1\partial\Sigma_i} - |D^k|.$$

On the other hand, we showed that $A_{\Sigma_i}(y)$ is uniformly bounded at the boundary, i.e.,

$$|A_{\Sigma_i}|(y) \leq C$$

for every $y \in \partial\Sigma_i$. In particular, the second fundamental form of $\partial\Sigma_i$ in $\mathbb{R}^{n+1}$ is uniformly bounded. Thus, up to subsequence, $\partial\Sigma_i$ converges in the $C^{1,\alpha}$ topology to $\partial D^k \subset \mathbb{S}^{n-1}$. Equivalently, $C_1\partial\Sigma_i$ converges in the $C^{1,\alpha}$ topology to D^k.

Lemma 21.
$$\lim_{i\to\infty}\left(\int_{C_1\partial\Sigma_i}\frac{|(x-y_i)^\perp|^2}{|x-y|^{k+2}}+\frac{1}{k}\int_{C\partial\Sigma_i}\langle\overrightarrow{H}_{C_1\partial\Sigma_i},\frac{x-y_i}{|x-y_i|^k}\rangle\right)\leq|D^k|.$$

Proof. First note that

$$\lim_{i\to\infty}\int_{C_1\partial\Sigma_i-B_s(y)}\frac{|(x-y_i)^\perp|^2}{|x-y|^{k+2}}+\frac{1}{k}\int_{C\partial\Sigma_i-B_s(y)}\langle\overrightarrow{H}_{C_1\partial\Sigma_i},\frac{x-y_i}{|x-y_i|^k}\rangle=0$$

since $C_1\partial\Sigma_i\to D^k$ in the $C^{1,\alpha}$ topology. Hence, it is enough to focus on $\Sigma_i\cap B_s(y)$. Let us assume that $y_i\to y\in\partial\Sigma$. The convergence $C_1\partial\Sigma_i\to D^k$ also implies that we can choose $s<1$ very small so that $T_xC_1\partial\Sigma_i$ is uniformly close to T_yD^k for every $x\in C_1\partial\Sigma_i\cap B_s(y)$. Let $z_i\in T_xC_1\partial\Sigma_i$ be a point which realizes the distance $r_i=d(T_xC_1\partial\Sigma_i,y_i)$ and let $t_i=d(T_yD^k,y_i)$.Hence, $|(x-y_i)^\perp|^2=r_i^2$ and $|x-x_i|^2=|x-z_i|^2+r_i^2$ for every $x\in C_1\partial\Sigma_i\cap B_s(y_i)$. Therefore,

$$\lim_{i\to\infty}\int_{C_1\partial\Sigma_i\cap B_s(y)}\frac{|(x-y_i)^\perp|^2}{|x-y_i|^{k+2}}=\lim_{i\to\infty}\int_{C_1\partial\Sigma_i\cap B_s(y)}\frac{r_i^2}{\sqrt{|x-z_i|^2+r_i^2}^{k+2}}=$$

$$\lim_{i\to\infty}\int_{C_1\partial\Sigma_i\cap B_s(y)-z_i}\frac{r_i^2t_i^{-k-2}}{\sqrt{|\frac{x}{t_i}|^2+\frac{r_i^2}{t_i^2}}^{k+2}}=\lim_{i\to\infty}\int_{\frac{1}{t_i}(C_1\partial\Sigma_i\cap B_s(y)-z_i)}\frac{\frac{r_i^2}{t_i^2}}{\sqrt{|y|^2+\frac{r_i^2}{t_i^2}}^{k+2}}$$

$$=\int_{P_1}\frac{1}{(|y|^2+1)^{\frac{k+2}{2}}}\leq\int_{\mathbb{R}^k}\frac{1}{(|y|^2+1)^{\frac{k+2}{2}}}=\int_0^\infty\int_{\partial D^k}\frac{s^{k-1}}{(s^2+1)^{\frac{k+2}{2}}}ds$$

$$=|\partial D^k|\int_0^\infty\frac{s^{k-1}}{(s^2+1)^{\frac{k+2}{2}}}\,ds=\frac{|\partial D^k|}{k}\;=\;|D^k|,$$

where P_1 is either $\mathbb{R}^k$ or a half space $\mathbb{R}^k_a = \{x \in \mathbb{R}^k : \langle x, e_k\rangle \le a\}$. Similarly,

$$\lim_{i\to\infty} \int_{C_1\partial\Sigma_i\cap B_s(y)} \langle \overrightarrow{H}_{C_1\partial\Sigma_i}, \frac{(x-y_i)}{|x-y_i|^k} d_{C_1\partial\Sigma_i} \le$$

$$\lim_{i\to\infty} \sup_{C_1\partial\Sigma_i\cap B_s(y)} \left|\left\langle \overrightarrow{H}_{C_1\partial\Sigma_i}, \frac{(x-y_i)}{|x-y_i|}\right\rangle\right| \int_{C_1\partial\Sigma_i\cap B_s(y)} \frac{1}{|x-y_i|^{k-1}} d_{C_1\partial\Sigma_i}$$

$$\begin{aligned}
&= \lim_{i\to\infty} o(s) \int_{C_1\partial\Sigma_i\cap B_s(y)} \frac{1}{\sqrt{|x-z_i|^2+r_i^2}^{\,k-1}} + o(s) \\
&= \lim_{i\to\infty} o(s) \int_{C_1\partial\Sigma_i\cap B_s(y)-z_i} \frac{t_i^{1-k}}{\sqrt{|\frac{x}{t_i}|^2+\frac{r_i^2}{t_i^2}}^{\,k-1}} + o(s) \\
&= \lim_{i\to\infty} o(s) \int_{\frac{1}{t_i}(C\partial\Sigma_i\cap B_s(y)-z_i)} \frac{t_i}{\sqrt{|y|^2+\frac{r_i^2}{t_i^2}}^{\,k-1}} + o(s) \\
&\le \lim_{i\to\infty} o(s) \int_{\frac{1}{t_i}(C_1\partial\Sigma_i\cap B_s(y)-z_i)} \frac{t_i}{|y|^{k-1}} + o(s) \\
&= \lim_{i\to\infty} o(s) \int_0^{\frac{s}{t_i}} t_i \frac{s^{k-1}}{s^{k-1}} ds + o(s) = o(s).
\end{aligned}$$

Making $s \to 0$, we conclude the proof of the claim. □

Using the Claim above, we obtain that

$$\int_{\Sigma_\infty} \frac{|z^\perp|^2}{|z|^{k+2}} d_{\Sigma_\infty} \le |D^k| - |D^k| = 0.$$

Since this contradicts (2.19), the lemma is proved. □

Proof of Corollary 2. Since $\partial\Sigma$ is a $(k-1)$-dimensional minimal surface in ∂B^n, $C\partial\Sigma$ is a k-dimensional minimal surface in $\mathbb{R}^n$. Arguing by contradiction, let $\{y_i\}$ be a sequence of points in $C_1\partial\Sigma - \Sigma$ such that $\lim_{i\to\infty} y_i = 0$.

$$\begin{aligned}
|\Sigma| - |D^k| &= |C_1\partial\Sigma| - |D^k| = \lim_{R\to\infty} \frac{|C\partial\Sigma \cap B_{R+1}|}{(R+1)^k}\frac{(R+1)^k}{R^k} - |D^k| \\
&\geq \lim_{R\to\infty} \frac{|C\partial\Sigma \cap B_R(y_i)|}{R^k} - |D^k| = \lim_{i\to\infty} \int_{C\partial\Sigma} \frac{|(x-y_i)^\perp|^2}{|x-y_i|^{k+2}} \\
&= \lim_{i\to\infty} \int_{C_1\partial\Sigma} \frac{|(x-y_i)^\perp|^2}{|x-y_i|^{k+2}} = \lim_{i\to\infty} \int_{\Sigma} \frac{|(x-y_i)^\perp|^2}{|x-y_i|^{k+2}} + |D^k| \\
&= |\Sigma| + |D^k|.
\end{aligned} \tag{2.20}$$

The third equality in (2.20) follows from the Monotonicity Formula for minimal submanifolds in $\mathbb{R}^n$, see [12, Proposition 1.12]. In the last equality we used similar analysis as in the proof of Lemma 21 and the Monotonicity Formula again. Since (2.20) is contradictory, $\Sigma = C_1\partial\Sigma$ and the result follows. $\square$

2.2.4 ε regularity for free boundary cmc surfaces

Finally, we observe that the 2-dimensional case in Theorem 15 proved in [28] extends naturally to surfaces with constant mean curvature in B^3. The quantity to consider in this case is the Willmore energy instead of area.

Definition 3. *If Σ^2 is a surface with boundary in $\mathbb{R}^3$, the Willmore energy $\mathcal{W}(\Sigma)$ is defined as*

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 \, d_\Sigma + \int_{\partial\Sigma} k_g \, d\sigma.$$

Theorem 22. *There exists $\varepsilon > 0$ such that whenever Σ is a free boundary surface with constant mean curvature in B^3 and satisfying*

$$\mathcal{W}(\Sigma) < 2\pi + \varepsilon,$$

then Σ is either an equatorial disk or a spherical cap. The constant ε is independent of the value of the mean curvature.

The key in proving Theorem 22 is the following excess inequality proved by Vokmann [33]:

$$\mathcal{W}(\Sigma) - 2\pi \geq \int_{\Sigma} \left| \overrightarrow{H} - \frac{(x-x_0)^{\perp}}{|x-x_0|^2} \right|^2 d_{\Sigma}. \tag{2.21}$$

The equality $\mathcal{W}(\Sigma) = 2\pi$ if, and only if, Σ is a spherical cap or a flat disk.

Proof of Theorem 22. Assume for every i that Σ_i is a free boundary surface with constant mean curvature H_i and

$$\mathcal{W}(\Sigma_i) \to 2\pi. \tag{2.22}$$

We start showing that (2.22) implies that $|\Sigma_i| \leq C_0$ for some $C_0 > 0$. Indeed, the free boundary condition and the 1ª Minkowski formula gives that

$$2|\Sigma_i| = |\partial\Sigma_i| - \int_{\Sigma_i} \langle H_i, x\rangle d_{\Sigma_i} \leq |\partial\Sigma_i| + \left(\int_{\Sigma_i} H_i^2 d_{\Sigma_i} \right)^{\frac{1}{2}} |\Sigma_i|^{\frac{1}{2}}. \tag{2.23}$$

Following [28], we show that (2.22) and (2.21) imply curvature estimates for Σ_i.

Lemma 23. *Either Σ_i is totally umbilical or there is $C > 0$ such that*

$$\sup_{\Sigma_i} |A_{\Sigma_i}| \leq C. \tag{2.24}$$

Assuming Lemma 23, one can finish the proof of the theorem. Indeed, by Lemma 23, the sequence Σ_i converges to a weakly embedded free boundary surface Σ_∞ in B^3 with constant mean curvature and satisfying $\mathcal{W}(\Sigma_\infty) = 2\pi$. By Theorem 4.1 in [33], Σ_∞ is either a spherical cap or a flat disk. Therefore, Σ_i is topologically a disk for i large enough and the result follows from Nitsche's Theorem. □

Proof of Lemma 23. Define $\lambda_i = \sup_{x\in\Sigma_n} |A_i|^2(x)$ and assume that $\lim_{i\to\infty} \lambda_i = \infty$. For each i choose $x_i \in \Sigma_i$ with the property that $\sup_{\Sigma_i} |A_i|^2 = |A_i|^2(x_i)$ and consider the new surface $\hat{\Sigma}_i = \lambda_i(\Sigma_i - x_i)$ which satisfies

$$\sup_{x\in\hat{\Sigma}_i} |A|(x) \leq 1 \quad \text{and} \quad |A_{\hat{\Sigma}_i}|(0) = 1. \tag{2.25}$$

The surface $\hat{\Sigma}_i$ is a free boundary surface with constant mean curvature $\hat{H}_i$ in the region $\lambda_i(B_1^3(0) - x_i)$.

Property (2.25) implies that $\hat{\Sigma}_i \to \Sigma_\infty$ graphically in the interior. Σ_∞ is either weakly embedded without boundary or it is weakly embedded in a half space and with boundary contained in a plane. By regularity results for free boundary surfaces, see [5], we also have smooth convergence up to the boundary. It follows from (2.25) that

$$|A_{\Sigma_\infty}|(0) = 1 \quad \text{and} \quad \lim_{i\to\infty} \hat{H}_i = H_\infty.$$

By the excess formula (2.21), we conclude that

$$\int_{\Sigma_\infty} \left| \overrightarrow{H}_\infty - \frac{x^\perp}{|x|^2} \right|^2 d_{\Sigma_\infty} \leq$$

$$\liminf_{i\to\infty} \int_{\hat{\Sigma}_i} \left| \overrightarrow{\hat{H}}_i - \frac{x^\perp}{|x|^2} \right|^2 d_{\Sigma_\infty} = \liminf_{i\to\infty} \int_{\Sigma_i} \left| \overrightarrow{H}_i - \frac{(x-x_i)^\perp}{|x-x_i|^2} \right|^2 \leq$$

$$\liminf_{i\to\infty} \mathcal{W}(\Sigma_i) - 2\pi = 0.$$

Hence, $\overrightarrow{H}_\infty = \frac{x^\perp}{|x|^2}$. If Σ_∞ is non-compact, then $\overrightarrow{H}_\infty = 0$ and this implies that Σ_∞ is a plane, a contradiction since $A_{\Sigma_\infty}(0) = 1$. Hence, Σ_∞ is compact and $\overrightarrow{H}_\infty = \frac{x^\perp}{|x|^2}$. Applying Proposition 4.1 in [33], we conclude that Σ_∞ is a spherical cap or an equatorial disk. The strong convergence implies that Σ_i is a topologically a disk and, hence, a spherical cap by Nitsche's Theorem. $\square$

3. COMPACTNESS FOR MINIMAL SURFACES

In this chapter we prove two results about closed minimal surfaces in three manifolds. The first result is a compactness theorem for minimal surfaces. The second result is a rigidity type theorem for positively curved three manifolds with symmetries admitting stable minimal surfaces.

3.1 Compactness

Example 3 (Catenoid)**.** Let Σ_i be the minimal surface obtained from the catenoid Σ_c via a scaling by $\lambda_i = \frac{1}{i}$, i.e., $\Sigma_i = \frac{1}{i}\Sigma_c$. For every point $x \in \Sigma_i$ the following is true

$$|x^{\perp}|^2 \, |A_{\Sigma_i}(x)|^2 \leq 2.$$

In particular, $C(\Sigma_i)$ is pinching off at the origin and $\lim_{i\to\infty} |A_{\Sigma_i}(x)| = \infty$ for every $x \in C(\Sigma_i)$. In fact, Σ_i converge weakly, as varifolds, to the plane with multiplicity two.

Example 4 (Helicoid)**.** The Helicoid is a complete simply connected minimal surface Σ_h in $\mathbb{R}^3$ defined by

$$\varphi(u, v) = (u\cos(v), u\sin(v), v), \quad (u, v) \in \mathbb{R}^2.$$

Let's consider the sequence of minimal surfaces Σ_i defined by $\Sigma_i = \frac{1}{i}\Sigma_h$. Note that, whereas Σ_h is invariant by vertical translations by $2\pi m$, the surfaces Σ_i is invariant by vertical translations by $2\pi\frac{m}{i}$. It can be proved that Σ_i converge smoothly away from the vertical axis to a foliation by planes. The curvature of the sequence blows up at points in the vertical axis.

The sequence of minimal surfaces considered in the examples above do not admit subsequence which having graphical convergence. In the first example the compactness fails due to a pinch off of the catenoid neck. In the second example the compactness fails due to lack of local area bounds. Our next result shows that those are the only obstructions to obtain compactness for minimal surfaces in three manifolds.

Theorem 24. *Let M^3 be a closed 3-manifold and let $\mathcal{S}$ be the space of closed embedded minimal surfaces in M^3. Then the class*

$$C(A_0, i_0) := \{\Sigma \in \mathcal{S} : \ \mathit{Area}(\Sigma) \leq A_0, \ \mathit{inj}(\Sigma) \geq i_0\}$$

is compact in the C^∞ topology.

Proof. Let Σ_n^2 be a sequence of closed minimal surfaces with

$$\mathrm{Area}(\Sigma_n^2) \leq A_0 \quad \text{and} \quad \mathrm{inj}(\Sigma_n^2) \geq i_0.$$

We want to prove the existence of a constant $C > 0$ such that $\sup_{\Sigma_i} |A_{\Sigma_i}| \leq C$. Arguing by contradiction, let λ_n be given by $\lambda_n = \sup_{\Sigma_n^2} |A_n|$ and assume that $\limsup_{n\to\infty} \lambda_n = \infty$. Pick base points $p_n \in \Sigma_n^2$ for which $\sup_{\Sigma_n^2} |A_n| = |A_n|(p_n)$ and consider the sequence of minimal surfaces $\Sigma_n' = \lambda_n \Sigma_n$ in $(B_{\frac{i_0}{2}}(p_n), \lambda_n^2 g)$. One can check that the sequence $\{\Sigma_n'\}$ satisfies $|A_{\Sigma_n'}| \leq 1$ and $A_{\Sigma_n'}(y_n) = 1$. If the sequence satisfies local area bounds, i.e., $|\Sigma_n' \cap B_R(x_n)| \leq C_1$, then it follows that Σ_n' converges to a properly embedded minimal surface $\Sigma_0 \subset \mathbb{R}^3$ and such that

$$\sup_{\Sigma} |A_{\Sigma_0}| = |A_{\Sigma_0}|(0) = 1. \tag{3.1}$$

Otherwise, Σ_n' converges to minimal lamination $\Sigma \subset \mathbb{R}^3$ containing a minimal leaf Σ_0 which also satisfies (3.1). In either case, Σ_0 is topologically a disk as $\mathrm{inj}_0(\Sigma) = \lim_{n\to} \lambda_n \mathrm{inj}_{y_p}(\Sigma_n) = \infty$. Moreover, Σ_0 is also properly embedded by a result of H. Rosemberg, see remark in the end of [30]. Let us prove that

Σ_0 has bounded total curvature. First, note that the geodesic ball $B^{\Sigma_n}_{\frac{i_0}{2}}(p_n)$ is topologically a disk. Hence, following Chapter 2 in [12], we have

$$\frac{d}{dt}\int_{\partial B^{\Sigma_n}_t} = \int_{\partial B^{\Sigma_n}_t} k_g = 2\pi - \int_{B^{\Sigma_n}_t} K_{\Sigma_n}.$$

Integrating above formula from 0 to ρ implies that

$$|\partial B^{\Sigma_N}_\rho| - 2\pi\,\rho \;=\; -\int_0^\rho \int_{B^{\Sigma_n}_t} K_{\Sigma_n}. \tag{3.2}$$

Integrating (3.2) from 0 to i_0 and applying the Coarea Formula we obtain

$$|B^{\Sigma_n}_{i_0}(p_n)| - \pi\, i_0^2 = -\int_0^{i_0}\int_0^\rho \int_{B^{\Sigma_n}_t} K_{\Sigma_n}.$$

Since Σ_n is minimal, the Gauss equation, Proposition 1, gives that

$$\overline{K}_M(T\Sigma_n) = K_{\Sigma_n} + \frac{1}{2}|A_n|^2.$$

This implies that

$$\begin{aligned} |B^{\Sigma_n}_{i_0}(p_n)| - \pi\, i_0^2 &= -\int_0^{i_0}\int_0^\rho \int_{B^{\Sigma_n}_t} \overline{K}_M(T\Sigma_n) + \frac{1}{2}\int_0^{i_0}\int_0^\rho \int_{B^{\Sigma_n}_t} |A_n|^2 \\ &\geq -\int_0^{i_0}\int_0^\rho \int_{B^{\Sigma_n}_t} \overline{K}_M(T\Sigma_n) + \frac{i_0^2}{8}\int_{B^{\Sigma_n}_{\frac{i_0}{2}}(p_n)} |A_n|^2. \end{aligned}$$

Since M^3 is compact, there exists $K_0 > 0$ for which $|K_M| \leq K_0$. Moreover, since $\text{Area}(\Sigma_n) \leq A_0$, we obtain

$$\frac{i_0^2}{8}\int_{B^{\Sigma_n}_{\frac{i_0}{2}}(p_n)} |A_n|^2 \leq A_0 - \pi\, i_0^2 + K_0\,\frac{i_0^2}{2}\,A_0.$$

Hence, there exists $C_1 = C_1(i_0, A_0, K_0)$ for which

$$\int_{B^{\Sigma_n}_{\frac{i_0}{2}}(p_n)} |A_n|^2 \leq C_1. \tag{3.3}$$

The left hand side in (3.3) is scale invariant and consequently we conclude that

$$\int_{\Sigma_0} |A_\Sigma|^2 < \infty.$$

By a theorem of F. López and A. Ros [25], the only properly embedded minimal surface in $\mathbb{R}^3$ of genus 0 and with bounded total curvature is the totally geodesic plane. Hence, Σ_0 is totally geodesic and this contradicts (3.1). Therefore, $\sup_{\Sigma_i} |A_{\Sigma_i}| \leq C$ for some constant $C > 0$ and this proves the compactness of $C(A_0, i_0)$. □

The proof of Theorem 24 explore crucially that the surfaces have dimension two as the arguments relies heavily on the Gauss-Bonnet Theorem as well as in the classification result of López and Ros [25]. A natural question worth investigating is weather Theorem 24 holds true in higher dimensions.

3.2 Stable minimal surfaces and symmetries

Theorem 25. *Let (M^3, g) be a closed 3-manifold with positive scalar curvature $R_g > 0$ and let V be a Killing vector field in $\mathcal{X}(M^3)$. If (M^3, g) contains an embedded stable minimal surface, then the universal cover of (M^3, g) is diffeomorphic to either $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$, and*

$$g = dr^2 + ds^2 + \varphi(r,s)\, d\theta^2 \quad \textit{and} \quad \varphi(r,s) = \frac{1}{2\pi}\int_0^{2\pi} |V(r,s,\theta)|\, d\theta.$$

Proof. Let Σ^2 be a embedded stable minimal surface in (M^3, g). Since $R_M > 0$ we have that Σ is either a sphere or projective plane. Let $\hat{\Sigma}$ be the lift of Σ in the universal cover $(\hat{M}^3, g)$. Hence, each component of $\hat{\Sigma}$ is a stable minimal sphere. Let us work with a connected component of $\hat{\Sigma}$ in $\hat{M}$ which we also denote by Σ. Let $\phi_t : M \to M$ be the one parameter family of diffeomorphism generated by the vector field V, i.e., $\frac{d}{dt}\Big|_{t=0} \phi(x,t) = V(x)$. Since V is Killing, ϕ_t is an isometry for every t. In particular, $\mathrm{Area}(\phi_t(\Sigma)) = \mathrm{Area}(\Sigma)$ for every t. Hence,

$$0 = \frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Area}(\phi_t(\Sigma)) = -\int_\Sigma f\, L\, f,$$

where $f = \langle V, N\rangle$ and N is the unit normal to Σ. The function f is a Jacobi function, i.e., $L\,f = \Delta\, f + (\mathrm{Ric}_M(N,N) + |A|^2)\, f = 0$, and $\lambda = 0$ is the first eigenvalue of L since Σ is assumed to be stable. It follows that f is a first eigenfunction of L and, hence, positive or identically zero. If Ω is a tubular neighborhood of Σ in $\hat{M}$, then by the Divergence theorem we have

$$0 = \int_{\Omega} \mathrm{div}_M V \; = \; \int_{\partial\Omega} \langle V, N \rangle.$$

If $f > 0$, then V is transversal to Σ which contradicts above formula if Ω is very small. Hence, $f \equiv 0$ and V is tangent to Σ. Moreover, V is Killing in Σ:

$$\begin{aligned} g(\nabla^{\Sigma}_X V, Y) + g(X, \nabla^{\Sigma}_Y V) &= g(\nabla_X V - B(X,V), Y) + g(X, \nabla_Y V - B(Y,V)) \\ &= g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0, \end{aligned}$$

where $X, Y \in \mathcal{X}(\Sigma)$ and B is the second fundamental form of Σ. Let $x_0 \in \Sigma$ such that $V(x_0) = 0$, the existence of x_0 is guaranteed by the Poincare-Hopf Index Theorem and since $g(\Sigma) = 0$. Following an argument in [7] we conclude that $(\phi_t)_* : T_{x_0}(\Sigma) \to T_{x_0}\Sigma$ induces a homomorphism $\phi_* : \mathbb{R} \to SO(2) = \mathbb{S}^1$. In particular, there exists $t_0 \in \mathbb{R}$ for which $\phi_*(0) = \varphi_*(t_0)$. It follows that $(\phi_t)_* : T_{x_0}M \to T_{x_0}M$ shares similar properties, i.e. $\phi_* : \mathbb{R} \to SO(3)$ satisfies $\phi_*(0) = \varphi_*(t_0)$ and $(\phi_t)_*(N) = N$. Using that ϕ_t is determined by $(\phi_t)_*$ we conclude that $\phi_{t+t_0} = \phi_t$. Hence, M is $\mathbb{S}^1$ invariant and Σ is rotationally symmetric, i.e., $g_\Sigma = ds^2 + \varphi(s)\, d\theta^2$, $\varphi(0) = \varphi(1) = 0$, and

$$\varphi(s) = \frac{1}{2\pi} \int_0^{2\pi} |V(s,\theta)|\, d\theta.$$

Let $F : \Sigma \times (0, \varepsilon) \to M$ the exponential map $F(x,t) = \exp_x(rN(x))$. Since V is a Killing vector field we have that $V \in \mathcal{X}(F(\Sigma))$. Indeed, if $\gamma(r) = F(x,r)$, then

$$\frac{d}{dr}\langle \gamma'(r), V \rangle = \langle \nabla^M_{\gamma'(r)} \gamma'(r), V \rangle + \langle \gamma'(r), \nabla^M_{\gamma'(r)} V \rangle = 0 + 0 = 0.$$

Since, $\langle V, \gamma'(0) \rangle = 0$ we conclude that $\langle \gamma'(r), V \rangle = 0$. Hence, $\Sigma_r = F(\Sigma, r)$ is also rotationally symmetric. Therefore, there exists a maximal coordinate system $X : (r_0, r_1) \times [0,1] \times \mathbb{S}^1 \to M^3$ where

$$g = dr^2 + ds^2 + \varphi(r,s)\, d\theta^2 \quad \text{and} \quad \varphi(r,s) = \frac{1}{2\pi}\int_0^{2\pi} |V(r,s,\theta)|\, d\theta.$$

If $\varphi(r_0, s) = \varphi(r_1, s) = 0$ for every $s \in [0,1]$, then M^3 is diffeomorphic to $\mathbb{S}^3$. Otherwise, $r_0 = -\infty$ and $r_1 = \infty$ and M^3 is diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$. $\square$

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