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Subcanonical curves: on the loci of odd Weierstrass semigroups and the dimension of the loci for certain families of semigroups

Belo Horizonte

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Tese de Doutorado apresentada como requisito parcial para obtenção do título de Doutor em Matemática

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Departamento de Matemática

Programa de Pós-Graduação

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Abstract

A celebrated result of Kontsevich–Zorich ensures that the moduli space $\mathcal{G}_g$ of pointed curves of genus g whose marked point is subcanonical has three irreducible components. In this work we present an explicit method to construct a compactification of the loci which corresponds to a general point of an irreducible component of $\mathcal{G}_g$, namely the loci of pointed curves whose symmetric Weierstrass semigroup is odd. The construction is an extension of Stoehr's techniques using equivariant deformation of monomial curves given by Pinkham by exploring syzygies. As an application we prove the rationality of the loci for genus six. By fixing a family of semigroups of multiplicity 6, we also compute the dimension of the moduli space of pointed curve whose Weierstrass semigroup at the marked point belogns to the fixed family.

Key-words: Weierstrass point, Symmetric semigroup, Deformation, Syzygy, Moduli space.

Sumário

1 Introduction

It is very known that a billiard on a convex polygon with $2n$ sides induces on compact Riemann surface of genus $g := [(n-1)/2]$ an abelian differential with a zero of order $2g-2$ at a point P. One of the first to observe this was W. Veech in [Vee]. By the Riemann-Roch theorem the Weierstrass semigroup of the compact Riemann surfaces at P is symmetric, in terms of algebraic geometry, the compact Riemann surfaces with such abelian differential corresponds to a subcanonical algebraic curve. In a celebrated result, Kontsevich–Zorich [KZ] showed that the locus $\mathcal{G}_g$ of compact Riemann surfaces of genus g with a fixed abelian differential with a zero of order $2g-2$ has exactly three irreducible components, the locus $\mathcal{G}_g^{\mathrm{hyp}}$ of hyperelliptic points, the even $\mathcal{G}_g^{\mathrm{even}}$ and the odd $\mathcal{G}_g^{\mathrm{odd}}$ points. Ten years late E. Bullock [Bu] computed a general point of each component of $\mathcal{G}_g$.

Furthermore the sets $\mathcal{M}_{g,1}^{\mathcal{H}}$ which parametrizes pointed smooth genus g curves with Weierstrass semigroup $\mathcal{H}$ at the marked point form a stratification of the moduli $\mathcal{M}_{g,1}$ of pointed genus g curves. They are also important to obtain some classes in the Chow ring of $\mathbb{M}_{g,1}$, as can be seen in the work of Gatto–Ponza [GP].

Besides the above two applications of Weierstrass points we do not know general results, as we would like, about $\mathcal{M}_{g,1}^{\mathcal{H}}$. For example, when it is non empty? If no empty, what is this dimension? When are they rationals? Of coarse that are some beautiful works answering the above questions in suitable cases. We will talk about them in the Chapter one of this thesis, in order to set our work in the known literature.

We have to cite two relevant works for this thesis, the works of Stoehr [S] and Contiero–Stoehr [CS] on the construction of a compactification of $\mathcal{M}_{g,1}^{\mathcal{H}}$ when $\mathcal{H}$ is symmetric. We note that in both of these works there is a restriction on the symmetric semigroup, which is $3 < n_1 < g$. It is clear that if $n_1 = 2$ or $n_1 = 3$ the symmetric semigroup can be generated by less than 5 elements. Now, by the jacobian criterion and elimination theory, the moduli space $\mathscr{M}_{\mathcal{H}}$ is an open subspace of $\overline{\mathscr{M}_{g,1}^{\mathcal{H}}}$. If the symmetric semigroup $\mathcal{H}$ is generated by 4 elements, say $\mathcal{H} = <m_1, m_2, m_3, m_4>$, then by using Pinkham's equivariant deformation theory [P], complete intersection theory and a quasi-homogeneous version of Buchsbaum-Eisenbud's structure theorem for Gorenstein ideals of codimension 3 (see [BE, p. 466]), one can deduce that the affine monomial curve Spec $\mathbf{k}[\mathcal{H}] =$ Spec $\mathbf{k}[t^{m_1}, t^{m_2}, t^{m_3}, t^{m_4}]$ can be negatively smoothed without any obstructions (see [B], [W1] [W2, Satz 7.1]), hence $\dim \mathscr{M}_{\mathcal{H}} = \dim \mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}})$, and therefore

$$\overline{\mathscr{M}_{g,1}^{\mathcal{H}}} = \mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}),$$

and so $\mathscr{M}_{\mathcal{H}}$ is a dense open subvariety of $\overline{\mathscr{M}_{g,1}^{\mathcal{H}}}$.

So it remains to study the excluded case when the multiplicity of the symmetric semigroup is g. Thus $\mathcal{H} := \langle g, g+1, \ldots, 2g-2\rangle$ which corresponds to the odd general family of the Bullock's theorem, see Theorem (2.4.7). Let $\mathcal{C}$ be a trigonal curve, not necessarilly smooth, by taking a non-ramification nonsingular Weierstrass point P of $\mathcal{C}$ we show that if $\mathcal{H}$ symmetric, then the Weierstrass semigroup at P is $\mathcal{H}_P = \{0, g, g+1, \ldots 2g-2, 2g, 2g+1 \ldots\}$. This semigroups are negatively graded, see theorem (2.4.3), hence $\dim \mathcal{M}_{g,1}^{\mathcal{H}} = 2g+1$. Here we extend the Stoehr and Contiero–Stoehr techniques to deal with trigonal curves. We proof that the ideal of a gorenstein monomial curve which realizes a trigonal symmetric semigroup of genus g is generated by quadrics and cubic forms. By deform the ideal of the trigonal monomial curve and by exploring syzygies we get a rather explicity construction (3.2.9) of the moduli space $\mathcal{M}_{g,1}^{\mathcal{H}}$ with $\mathcal{H} := \langle g, g+1, \ldots, 2g-1 \rangle$. We also note that our construct can be applied for nontrigonal cases, just because it is a generalization of the Stoehr and Contiero–Stoehr results. With this we conclude the Chapter one.

In the chapter two we apply the our construction and we get explicity the moduli spaces $\mathcal{M}_{5,1}^{\mathcal{H}}$ and $\mathcal{M}_{6,1}^{\mathcal{H}}$ when $\mathcal{H}$ is a symmetric trigonal semigroup. For genus 5, since the trigonal symmetric semigroup $\mathcal{H} = \langle 5, 6, 7, 8 \rangle$ is negatively graded and genetared by less than 5 elements, we know that $\mathcal{M}_{g,1}^{\mathcal{H}} = \mathbb{P}^9$. Let us consider the trigonal symmetric semigroup $\mathcal{H} := \langle 6, 7, 8, 9, 10 \rangle$ of genus 6. Since it is negatively graded, $\dim \mathcal{M}_{g,1}^{\mathcal{H}} = 11$. Applying our construction we conclude that $\mathcal{M}_{g,1}^{\mathcal{H}}$ is irreducible in $\overline{\mathcal{M}_{6,1}^{\mathcal{H}}}$, locally given by three equations and therefore, we show that the variety $\mathcal{M}_{6,1}^{\mathcal{H}}$ is rational, while in Bullock [BUL] theorem 1.1, he only shows $\mathcal{M}_{g,1}^{\mathcal{H}}$ is irreducible.

A theorem of Deligne [D] ensures that $\dim \mathcal{M}_{g,1}^{\mathcal{H}} \leq 2g + \lambda - 2$, where $\lambda \geq 1$ stands for the number of gaps l such that $l + m$ is a nongap for each positive nongap m, whose proof involves an interplay between three different moduli spaces that in symmetric semigrous is equal to $2g-1$. Moreover, in [CS] is developed a method to calculate an upper bound of $\mathcal{M}_{g,1}^{\mathcal{H}}$, with $\mathcal{H}$ a symmetric semigroup, which consists in approximate the compactified moduli space $\overline{\mathcal{M}_{g,1}^{\mathcal{H}}}$ by an affine quadratic quasi-cone (see [CS], thm. 3.1). This upper bound improves the Deligne's upper bound in infinitely many exemples of symmetric semigroups. With this same approach and the lower bound obtained by Nathan in the theorem (2.4.8), in the chapter three we calculate the exact dimension of the moduli space $\mathcal{M}_{g,1}^{\mathcal{H}}$ for the family of symmetric semigroups

$$\mathcal{H} = \langle 6, 3+6\tau, 4+6\tau, 7+6\tau, 8+6\tau \rangle,$$

of genus $3+6\tau$. We found the upper bound $8\tau+7$ of $\mathcal{M}_{g,1}^{\mathcal{H}}$ which for each $\tau \geq 1$ it is better than the Deligne's upper bound $2g-1 = 12\tau+5$. On the other hand, by the theorem (2.4.8) the lower bound of $\mathcal{M}_{g,1}^{\mathcal{H}}$ is $8\tau+7$ which is equal to the upper bound.

2 Preliminaries

2.1 The Dualizing Sheaf

Let $\mathbf{k}$ be an algebraically closed field. A curve $\mathcal{C}$ is a reduzed complete integral scheme of dimension one defined over $\mathbf{k}$. We recall that a scheme X with rational function field K is complete if for each discrete valuation ring R of $K|_{\mathbf{k}}$ there is an unique $x \in X$ such that $\mathcal{O}_{x,X} \subseteq R \subseteq K$. Alternatively, fixed an algebraic function field of one variable $K|_{\mathbf{k}}$ a complete reduced integral curve $\mathcal{C}$ defined over $\mathbf{k}$ with field rational function K is the set $\{\mathcal{O}_P\}_{P\in\mathcal{C}}$ of local $\mathbf{k}$-algebras, properly contained in $\mathbf{k}(\mathcal{C})$ satisfying the follows properties:

i. For almost all $P \in \mathcal{C}$, the local ring $\mathcal{O}_{P,\mathcal{C}}$ is a discrete valuation ring.

ii. For each discrete valuation ring B of $\mathbf{k}(\mathcal{C})|_{\mathbf{k}}$ there is an unique $P \in \mathcal{C}$ such that $\mathcal{O}_{P,\mathcal{C}} \subseteq B \subseteq K$.

The first condition means that there is a finite number of singular points of $\mathcal{C}$. By the second condition we obtain a surjective map $\pi : \overline{\mathcal{C}} \longrightarrow \mathcal{C}$, where $\overline{\mathcal{C}}$ is called *normalization* of $\mathcal{C}$ defined to be the set of all discrete valuation rings of $\mathbf{k}(\mathcal{C})|_{\mathbf{k}}$. For $P \in \mathcal{C}$, the elements of the fiber $\pi^{-1}(P)$ are called *branches of $\mathcal{C}$ centered at P*. Since the branches over a point P are zeros of rational functions vanishing at P, the branches are finite.

For a singular point P of $\mathcal{C}$, let $Q_1, \ldots, Q_m \in \overline{\mathcal{C}}$ be the branches centered at P. The integral closure

$$\tilde{\mathcal{O}}_P = \mathcal{O}_{Q_1} \cap \ldots \cap \mathcal{O}_{Q_m}$$

of $\mathcal{O}_P$ is a principal ideal domain. By a Rosenlicht theorem ([R], Theorem 1), the dimension

$$\delta_P := \dim_{\mathbf{k}} \tilde{\mathcal{O}}_P/\mathcal{O}_P$$

is finite, called the *singularity degree of P*.

Let us recall the notion of dualizing sheaf of a curve $\mathcal{C}$. For any curve $\mathcal{C}$ with normalization $\nu : \overline{\mathcal{C}} \longrightarrow \mathcal{C}$, the *dualizing sheaf* $\omega_{\mathcal{C}}$ associates to each $U \subset \mathcal{C}$ the space of the rational one-forms λ on $\nu^{-1}(U) \subset \overline{\mathcal{C}}$ such that for each $P \in U$ and $f \in \mathcal{O}_{\mathcal{C},P}$,

$$\sum_{Q\in\nu^{-1}(P)} \operatorname*{Res}_Q(\nu^* f \cdot \lambda) = 0. \tag{2.1}$$

Alternatively, to introduce the concept of dualizing sheaf on curves defined from a fixed algebraic function field, we first recall the notion of *fractional ideal sheafs*. We say that a sheaf $\mathcal{F}$ is a fractional ideal sheaf over $\mathcal{C}$ if it is coherent and for each point P of $\mathcal{C}$, the stalk $\mathcal{F}_P$ is a fractional $\mathcal{O}_P$-ideal. Equivalently, $\mathcal{F}$ is a fractional ideal sheaf if

1. For every $P \in \mathcal{C}$

   a) $\mathcal{F}_P \subset \mathbf{k}(\mathcal{C})$;

   b) there is $f_P \in \mathbf{k}(\mathcal{C})$ such that $f_P\mathcal{F}_P$ is an ideal of $\mathcal{O}_P$.

2. $\mathcal{F}_P = \mathcal{O}_P$ for almost every $P \in \mathcal{C}$.

The dualizing sheaf can also be introduced as in [S2] as follows: for each $\eta \in \Omega^1_{\mathbf{k}(\mathcal{C})|_{\mathbf{k}}}$, let ω_η be the fractional ideal sheaf such that for all point $P \in \mathcal{C}$ the stalk $\omega_{\eta,P}$ is the largest fractional $\mathcal{O}_P$-ideal in $\mathbf{k}(\mathcal{C})$ such that satisfies the condition (2.1). Since the vector space $\Omega^1_{\mathbf{k}(\mathcal{C})|_{\mathbf{k}}}$ of differentials is one-dimensional over the function field $\mathbf{k}(\mathcal{C})$, we way assume that the dualizing sheaf $\omega_{\mathcal{C}} = \omega_\eta \cdot \eta$ for every $\eta \in \Omega^1_{\mathbf{k}(\mathcal{C})|_{\mathbf{k}}}$. We note that if $P \in \mathcal{C}$ is a smooth point then $\omega_{\eta,P} = t_P^{-v_P(\eta)} \cdot \mathcal{O}_P$.

Instead of canonical sheaf for smooth curves we use the dualizing sheaf to obtain the Riemann-Roch theorem. For $\mathcal{F}, \mathcal{G}$ fractional $\mathcal{O}_P$-ideal sheafs, let us consider the sheaf $\mathcal{H}\text{om}(\mathcal{F}, \mathcal{G})$ such that on a point $P \in \mathcal{C}$ the stalk

$$\mathcal{H}\text{om}(\mathcal{F}, \mathcal{G})_P = \hom(\mathcal{F}_P, \mathcal{G}_P) = (\mathcal{G}_P : \mathcal{F}_P) = \{f \in \mathbf{k}(\mathcal{C}) | f\mathcal{F}_P \subseteq \mathcal{G}_P\}.$$

Thus for $\eta \in \Omega_{\mathbf{k}(\mathcal{C})/\mathbf{k}}$ and $\mathcal{F}$ a fractional $\mathcal{O}_P$-ideal sheaf we can show that

$$h^0(\mathcal{F}) = \deg(\mathcal{F}) + 1 - g + h^0(\mathcal{H}\text{om}(\mathcal{F}, \omega_\eta)),$$

and applying the Serre duality

$$H^0(\mathcal{H}\text{om}(\mathcal{F}, \omega_\eta)) \cdot \eta = H^0(\mathcal{H}\text{om}(\mathcal{F}, \omega_{\mathcal{C}})) = \hom(\mathcal{F}, \omega_{\mathcal{C}}) \simeq H^1(\mathcal{F}),$$

and therefore

Theorem 2.1.1. *(Riemann-roch) For each fractional ideal sheaf $\mathcal{F}$ over a curve $\mathcal{C}$ the following relation is true*

$$h^0(\mathcal{F}) = \deg(\mathcal{F}) + 1 - g + h^1(\mathcal{F}).$$

Remark 2.1.2. By Riemann-Roch theorem we get the *genus formula*

$$g = \tilde{g} + \sum_{P \in \mathcal{C}} \delta_P,$$

where $\tilde{g}$ is the *geometric genus* of $\mathcal{C}$ which is defined as the arithmetical genus of the normalization $\overline{C}$.

Example 2.1.1. Let $\mathcal{C}$ be the curve given by the projective closure of $\text{Spec}(A)$, where $A = \mathbf{k}[t^3, t^4, t^5]$. The curve $\mathcal{C}$ has only singular point P which corresponds to $t = 0$, where $\mathcal{O}_P = A_{(t^3,t^4,t^5)}$. Since $t \in \mathbf{k}[t]$ is integral over A and $A \subseteq \mathbf{k}[t] \subseteq \mathbf{k}(t^3, t^4, t^5)$, we conclude that $\mathbf{k}[t]$ is the integral closure of A in its field fractions, hence the normalization of $\mathcal{C}$ is $\overline{\mathcal{C}} = \mathbb{P}^1$. We can see that the fiber over the point P is an unique point $\overline{P}$ and we have $\overline{\mathcal{O}}_P = \mathcal{O}_{\overline{P}} = \mathbf{k}[t]_{(t)}$ and therefore the genus of $\mathcal{C}$ is $g = \overline{g} + \dim(\overline{\mathcal{O}}_P/\mathcal{O}_P) = 0 + 2 = 2$, where $\overline{g}$ is the genus of $\overline{\mathcal{C}}$. As the dualizing sheaf does not depend on the chosen form, we take the form $\eta = \frac{dt}{t^3}$ and by what was written above $\omega_{\mathcal{C}} \simeq \omega_\eta$. In the singular point P we wish to show that $\omega_{\mathcal{C}} \simeq \omega_{\eta,P} = \mathcal{O}_P + t\mathcal{O}_P$. On the one hand, $t^2 \notin \omega_{\eta,P}$ because $\mathrm{Res}_{\overline{P}}(t^2\eta) = 1$ and the same is true for every $t^{-n}, n \geq -1$ since that $\omega_{\eta,P}$ is a $\mathcal{O}_P$-module and $t^n \in \mathcal{O}_P$ for $n \geq 3$, hence $\omega_{\eta,P} \subseteq \mathcal{O}_P + t\mathcal{O}_P$. The other inclusion it is immediate because $\mathrm{Res}_{\overline{P}}(f\eta) = 0$ for $f \in \mathcal{O}_P + t\mathcal{O}_P$. Now if Q is a nonsingular point of $\mathcal{C}$ given parametrically by $t = a$, then $\frac{1}{t^3}$ is an unity in $\mathcal{O}_P$ and so $\omega_{\eta,P} = \mathcal{O}_P$. Finally, in the infinite point P_∞ the local parameter is t^{-1} and can write $\eta = t^{-1}d(t^{-1})$ and we obtain $\omega_{\eta,P_\infty} = t\mathcal{O}_{P_\infty}$. This means that $H^0(\mathcal{C}, \omega_C) = \langle 1, t\rangle$.

A point $P \in \mathcal{C}$ is said to be a *Gorenstein* point if the stalk of the dualizing sheaf $\omega_{\mathcal{C},P}$ is a free $\mathcal{O}_P$-module. The curve $\mathcal{C}$ is *Gorenstein* if all of its points so, or equivalently, if ω is an invertible sheaf.

Let $\mathbf{f} = (\mathcal{O} : \tilde{\mathcal{O}})$ be the *conductor* of $\mathcal{C}$ such that for any $P \in \mathcal{C}$ its stalk is

$$(\mathcal{O} : \tilde{\mathcal{O}})_P = (\mathcal{O}_P : \tilde{\mathcal{O}}_P) = \{h \in \mathbf{k}(\mathcal{C}) | h\tilde{\mathcal{O}}_P \subseteq \mathcal{O}_P\},$$

which is the largest fractional $\tilde{\mathcal{O}}_P$-ideal smaller than or equal to $\mathcal{O}_P$. Now, let $x_0, \ldots, x_n$ be k-linear independent elements of $\mathbf{k}(\mathcal{C})$, so that for $n \geq 1$ we have the morphism

$$(x_0, \ldots, x_n) : \overline{\mathcal{C}} \longrightarrow \mathbb{P}^n,$$

whose image by the extension theorem of valuation theory is a projective algebraic curve (see [S3]). Thus we obtain a morphism $\mathcal{C} \longrightarrow \mathbb{P}^n$ such that the diagram

$$\begin{array}{ccc} \overline{\mathcal{C}} & \longrightarrow & \mathbb{P}^n \\ & \searrow^{\pi} & \uparrow \\ & & \mathcal{C} \end{array} \tag{2.2}$$

commutes if and only if the $\mathcal{O}_P$-ideal $\sum_{i=0}^{n} \mathcal{O}_P x_i$ is principal. Let η be a non-zero differential one form such that $\omega_{\mathcal{C}} = \omega_\eta \cdot \eta$. By choosing a basis $\eta_0, \ldots, \eta_{g-1}$ for the space of the regular differentials on $\mathcal{C}$, we can write

$$\eta_i = x_i\eta \ (i = 0, \ldots, g-1),$$

where $x_0, \ldots, x_{g-1}$ is a basis of $H^0(\mathcal{C}, \omega_{\mathcal{C}})$. In this way, we have

$$(\eta_0, \ldots, \eta_{g-1}) = (x_0, \ldots, x_{g-1}).$$

The following theorem is well known in the literature and we will reproduce its proof which can be found in [S2].

Theorem 2.1.3. *Let $\mathcal{C}$ be a curve of genus $g \geq 1$. For each $P \in \mathcal{C}$, we have*

$$\omega_{\mathcal{C},P} = \mathcal{O}_P x_0 + \ldots + \mathcal{O}_P x_{g-1}.$$

The morphism $(\eta_0, \ldots, \eta_{g-1}) : \overline{\mathcal{C}} \longrightarrow \mathbb{P}^{g-1}$ induces a morphism $\mathcal{C} \longrightarrow \mathbb{P}^{g-1}$ if only if the curve $\mathcal{C}$ is Gorenstein.

Demonstração. We know that a curve $\mathcal{C}$ is Gorenstein if and only if the dualizing sheaf $\omega_{\mathcal{C}}$ is locally principal which is equivalent to diagram (2.2) being commutative, and so we have a morphism $\mathcal{C} \longrightarrow \mathbb{P}^{g-1}$. In this manner, we will proof that the ideal $\omega_{\mathcal{C},P} = \mathcal{O}_P x_0 + \ldots + \mathcal{O}_P x_{g-1}$.

If $P \in \mathcal{C}$ is a singular point, then the conductor sheaf $\mathbf{f}_P \subseteq \mathbf{m}_P$ and by Nakayama lemma it is enough to observe that $\omega_{\mathcal{C},P} \subseteq \sum \mathcal{O}_P x_i + \mathbf{f}_P \cdot \omega_{\mathcal{C},P}$. On the other hand, if P is a nonsingular point of $\mathcal{C}$, then $\mathcal{O}_P$ is a discrete valuation ring and $\omega_{\mathcal{C},P}$ is a principal ideal. The idea is to consider the sheaf $\mathcal{F}$ on $\mathcal{C}$ such that

$$\mathcal{F}_P = \mathbf{m}_P \cdot \omega_{\mathcal{C},P} \text{ and } \mathcal{F}_Q = \omega_{\mathcal{C},Q} \text{ for } Q \in \mathcal{C} \setminus \{P\}.$$

We need to prove that if there is a function $s \in H^0(\mathcal{C}, \omega_{\mathcal{C}}) \backslash H^0(\mathcal{C}, \mathcal{F})$, then $\omega_{\mathcal{C},P} = \mathcal{O}_P \cdot s$. For this it is enouth to show that $h^0(\mathcal{C}, \mathcal{F}) < h^0(\mathcal{C}, \omega_{\mathcal{C}})$. By Riemann Roch theorem $h^0(\mathcal{C}, \mathcal{F}) = h^0(\mathcal{C}, \omega_{\mathcal{C}}) - 1$ that implies $h^0(\omega_{\mathcal{C}} : \mathcal{F}) \leq 1$ and still

$$(\omega_{\mathcal{C}} : \mathcal{F}) = \mathbf{m}_P^{-1} \prod_{Q \neq P} \mathcal{O}_P \geq \mathcal{O},$$

hence $H^0(\mathcal{C}, (\omega_{\mathcal{C}} : \mathcal{F})) \supseteq \mathbf{k} = H^0(\mathcal{C}, \mathcal{O})$. We suppose, by contradiction, that there is a non-constant function $z \in H^0(\mathcal{C}, (\omega_{\mathcal{C}} : \mathcal{F}))$. Note that function z has only a simple pole in P and does not other pole in $\tilde{X} \setminus \{P\}$. Thus $K = \mathbf{k}(z)$, and therefore the geometric genus $\tilde{g}$ of the curve $\mathcal{C}$ is zero. Now, for a point $Q \in \mathcal{C} - \{P\}$ let's consider c the value of the function z at this point Q. Since $z - c$ is zero at Q and P is the only simple pole of $z - c$ follows that $\dim_{\mathbf{k}} \mathcal{O}_Q/(z-c) = 1$, hence $\mathcal{O}_Q$ is not discret valuation ring, so the point Q is nonsingular. This means that $X = \tilde{X}$ has genus $g = 0$ that it is not possible. □

We will not prove the next result which can be found in [R].

Theorem 2.1.4. *Let $\mathcal{C}$ be a Gorenstein curve. The morphism $\mathcal{C} \longrightarrow \mathbb{P}^{g-1}$ is an isomorphism onto the image curve if only if $\mathcal{C}$ is non-hyperelliptic.*

2.2 Weierstrass Points

A *numerical semigroup* is a submonoide $\mathcal{H}$ of $\mathbb{N}$ (that is, a subset of $\mathbb{N}$ such that contains 0 and it is closed with respect to addition) such that $L = \mathbb{N} \setminus \mathcal{H}$ is finite.

The *genus* $g = g(\mathcal{H})$ of $\mathcal{H}$ is

$$g = g(\mathcal{H}) := |\mathbb{N} - \mathcal{H}|.$$

The elements $1 = l_1 < \ldots < l_g$ of L and the elements $0 = n_0 < n_1 < \ldots$ of $\mathcal{H}$ are called of *gaps* and *nongaps* of $\mathcal{H}$, respectively. If $n_1 = 2$ the semigroup is called *hyperelliptic*. For $n_1, \ldots, n_r$ relatively prime positive integers, we denote $\langle n_1, \ldots, n_r \rangle := \{a_1 n_1 + \ldots + a_r n_r | a_i \in \mathbb{N}\}$ the numerical semigroup generated by $n_1, \ldots, n_r$, and conversely all numerical semigroup can be generated by a finite number of elements. For example, the hyperelliptic semigroup is given by $\mathcal{H} = \langle 2, 2g+1 \rangle$.

Let $\mathcal{H}$ be a numerical semigroup. The least positive integer $c = c(\mathcal{H})$ such that $c+\mathbb{N} \subset \mathcal{H}$ is called the *conductor* of $\mathcal{H}$. We say that $\mathcal{H}$ is *symmetric* if there is an integer d such that $n \in \mathcal{H}$ if, and only if, $d-1-n \notin \mathcal{H}$, equivalently if $c(\mathcal{H}) = 2g(\mathcal{H})$. Alternatively, if we denote $\mathrm{End}(\mathcal{H}) = \{n \in \mathbb{N} | n + \mathcal{H}^+ \subset \mathcal{H}\}$ which is the set of the *translations* of $\mathcal{H}$, then $\mathcal{H}$ is symmetric when $\lambda = [\mathrm{End}(\mathcal{H}) : \mathcal{H}] = 1$. It's important to observe that $\mathrm{End}(\mathcal{H})$ is also a numerical semigroup which contains $\mathcal{H}$ and the largest gap l_g. We also note that $l_g = c-1$. To see this, given any non-negative interger x such that $l_g + (1+x) > l_g$ follows $l_g + 1 + x \in \mathcal{H}$, and by minimality of c, $c \leq l_g + 1$. If $c-1 < l_g$ then there is an integer $q \geq 1$ such that $l_g = c + (q-1) \in \mathcal{H}$ that is a contradiction. This means that $l_g = c-1$.

Example 2.2.1. The semigroups $\mathcal{H} = \langle 2, 5\rangle$ and $\mathcal{H}' = \langle 3, 4, 5\rangle$ are the only semigroups of genus 2 and both are symmetrical. There are four semigroups of genus 3: $\mathcal{H}_1 = \langle 2, 7\rangle$, $\mathcal{H}_2 = \langle 3, 5, 7\rangle$, $\mathcal{H}_3 = \langle 3, 4\rangle$, $\mathcal{H}_4 = \langle 4, 5\rangle$, of which $\mathcal{H}_1$ and $\mathcal{H}_3$ are symmetrical since $l_3 = 5$.

Proposition 2.2.1. *Let $\mathcal{C}$ be a monomial curve of genus g, see (2.4), P be a Weierstrass point of $\mathcal{C}$ having semigroup $\mathcal{H} = \langle n_0, \dots, n_{g-1}\rangle$ and c be the conductor of $\mathcal{H}$. The curve $\mathcal{C}$ is Gorenstein if and only if the semigroup $\mathcal{H}$ is symmetric.*

Demonstração. Initially we will show

$$\mathbf{f} = \{h \in \tilde{\mathcal{O}}_P |\, \mathrm{Ord_P}(h) \geq l_g + 1\}, \tag{2.3}$$

where $\mathbf{f}$ is the conductor of $\mathcal{C}$. Since c is the conductor of $\mathcal{H}$, we have $c = l_g + 1$. By definition of the conductor of $\mathcal{C}$, $\mathbf{f}$ must be contained in the set of the right side of the equality (2.3). Conversely, given $h \in \tilde{\mathcal{O}}_P$ with $\mathrm{Ord_P}(h) \geq c$ means that $\mathrm{Ord_P}(h) = \mathrm{Ord_P}(g)$ for some $g \in \mathcal{O}_P$. Thus there is an unity $e \in \mathcal{O}_P$ such that $\mathrm{Ord_P}(h - eg) > \mathrm{Ord_P}(h)$. By applying induction there is also an element g' in $\mathcal{O}_P$ satisfying $\mathrm{Ord_P}(h - g') \geq c'$ where c' is the last value of an element of $\mathbf{f}$. Each element $h' \in \tilde{\mathcal{O}}_P$ with $\mathrm{Ord_P}(h') \geq c'$ are in $\mathcal{O}_P$, so $h \in \mathcal{O}_P$. In particular, $h \in \mathbf{f}$.

We will suppose that $\mathcal{H}$ is a symmetric semigroup and $h \in \mathfrak{m}^{-1}, h \notin \mathcal{O}_P$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_P$. We must to show that the length of the $\mathcal{O}_P$-module $\mathfrak{m}^{-1}/\mathcal{O}_P$ is 1 (see [EK] to the equivalences of Gorenstein rings). If $\mathrm{Ord_P}(h) \in \mathcal{H}$ then we can find an element $g \in \mathcal{O}_P$ such that $\mathrm{Ord_P}(h - g) \notin \mathcal{H}$ and still $h - g \in \mathfrak{m}^{-1}$, and so we may assume that $\mathrm{Ord_P}(h) \notin \mathcal{H}$. If $\mathrm{Ord_P}(h) < l_g$, then since $\mathcal{H}$ is symmetric $l_g - \mathrm{Ord_P}(h) \in \mathcal{H}$. Take an element $g' \in \mathcal{O}_P$ with $\mathrm{Ord_P}(g') = l_g - \mathrm{Ord_P}(h)$, so that $\mathrm{Ord_P}(g'h) = l_g$ and hence $g'h \notin \mathcal{O}_P$, contradicting $h \in \mathfrak{m}^{-1}$. Thus $\mathrm{Ord_P}(h) = l_g$ that means $\mathfrak{m}^{-1}$ contains besides $\mathcal{O}_P$ only elements of order l_g. This implies that $\mathfrak{m}^{-1}/\mathcal{O}_P$ is a $\mathcal{O}_P$-module of length 1 and hence $\mathcal{O}_P$ is Gorenstein.

Conversely, we assume that the local ring $\mathcal{O}_P$ is Gorenstein. Define I_j as the set of all elements $h \in \mathcal{O}_P$ with $\mathrm{Ord_P}(h) \geq n_j$, $j = 0, \dots g-1$. We obtain the strictly decreasing sequence of ideals of $\mathcal{O}_P$

$$\mathcal{O}_P = I_0 \supset I_1 \supset \dots \supset I_{g-1} \supset \mathbf{f}.$$

Moreover, this sequence is maximal because if we adjoin to I_j an element $g \in \mathcal{O}_P$ of order n_{j-1}, then we get all of I_{j-1}. Therefore

$$2g = \mathrm{length}(\tilde{\mathcal{O}}_P/\mathbf{f}),$$

hence $c = \ell_g + 1 = 2g$ and therefore $\mathcal{H}$ is a symmetric semigroup. □

Let $\mathcal{C}$ be a curve and P a smooth point of $\mathcal{C}$. We have an ascending chain of $\mathbf{k}$-vector spaces

$$\mathbf{k} = H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(0 \cdot P)) \subseteq H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(1 \cdot P)) \subseteq \ldots \subseteq H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n \cdot P)) \subseteq \ldots,$$

where

$$H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n \cdot P)) = \{f \in \mathbf{k}(\mathcal{C}) | \operatorname{div}(f) + nP \geq 0\}.$$

By Riemann-Roch theorem (2.1.1) we obtain

$$\dim_{\mathbf{k}} H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}((2g-1)P)) = g$$

and

$$h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}((n+1)P)) - h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(nP)) \leq 1,$$

therefore there are precisely g integers $l_1 < \ldots < l_g$ between 0 and $2g-1$ for which there exist no rational function on $\mathbf{k}(\mathcal{C})$ with pole of order precisely l_i at P. These integers $l_1, \ldots, l_g$ are called gap sequence of $\mathcal{C}$ at P. We define the $\mathcal{H}_P$ to be the set of the pole orders of meromorphic functions of $\mathcal{C}$ which are regular away from P. Then

$$n \in \mathcal{H}_P \Longleftrightarrow H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}((n-1) \cdot P)) \subsetneq H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n \cdot P)).$$

Let $\mathcal{C}$ be a curve and P a smooth point of $\mathcal{C}$. We say that P is an *ordinary point* of $\mathcal{C}$ if $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(g \cdot P)) = 0$, that is, if $\mathcal{H}_P = \mathcal{H}_g = \{0, g+1, g+2, \ldots\}$. Otherwise P is called *Weierstrass point* of the curve $\mathcal{C}$. A semigroup $\mathcal{H}$ is called *hyperordinary* if $\mathcal{H} = m\mathbb{N} + \mathcal{H}_g$ where $\mathcal{H}_g$ is ordinary and $0 < m < g$.

For any Weierstrass point $P \in \mathcal{C}$, let $0 = n_0 < n_1 < \ldots$ be the nongaps of $\mathcal{C}$ at P. So for each n_i we can take a function $x_{n_i} \in H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n_i \cdot P)) \setminus H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}((n_i - 1) \cdot P))$ which the pole order at P is n_i, hence

$$H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n_i \cdot P)) = \mathbf{k}x_{n_0} \oplus \mathbf{k}x_{n_1} \oplus \ldots \oplus \mathbf{k}x_{n_i}.$$

In particular, $h^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n_i P)) = n_i + 1$.

Theorem 2.2.2. *Let $\mathcal{C}$ be a Gorenstein curve and $P \in \mathcal{C}$ a nonhyperelliptic Weierstrass point. If $\mathcal{H}$ is the Weierstrass semigroup of the pointed curve $(\mathcal{C}, P)$ and it is symmetric, then $\mathcal{C}$ can be viewed as a canonical curve (in $\mathbb{P}^{g-1}$ of degree $2g-2$ and genus g) and the integers $l_i - 1 (i = 1, \ldots, g)$ are the contact orders of $\mathcal{C}$ with the hyperplanes at $P = (0 : \ldots : 0 : 1)$. Conversely, every nonhyperelliptic symmetric semigroup $\mathcal{H}$ is a Weierstrass semigroup of some curve.*

Proof. Since $\mathcal{H}$ is a symmetric semigroup, $n_{g-1} = 2g-2$ and therefore the vector space $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}((2g-2) \cdot P))$ is generated by the g functions $x_{n_0}, x_{n_1}, \ldots, x_{n_{g-1}}$ where the sheaf $\mathcal{O}_{\mathcal{C}}((2g-2) \cdot P)$ has degree $2g-2$. Thus the sheaf $\mathcal{O}_{\mathcal{C}}((2g-2) \cdot P)$ is isomorphic to the dualizing sheaf ω_C. By applying the theorem (2.1.4), follows that $\mathcal{C}$ can be embedded

$$(x_{n_0}, x_{n_1}, \ldots, x_{n_{g-1}}) : \mathcal{C} \hookrightarrow \mathbb{P}^{g-1},$$

so $\mathcal{C}$ is a curve of degree $2g-2$. Moreover, if we consider the hyperplane corresponding to $X_{g-i}(i=1,\ldots,g)$ its order contact with the curve at P is

$$\begin{aligned}\mathrm{Ord_P}(X_{g-i}) = \; & v_P\left(\frac{x_{n_{g-i}}}{x_{n_{g-1}}}\right) = v_P(x_{n_{g-i}}) - v_P(x_{n_{g-1}}) = -n_{g-i} - (-n_{g-1}) \\ & = 2g-1-l_{g-(g-1)} - (2g-1-l_{g-(g-i)}) = l_i - l_1 = l_i - 1.\end{aligned}$$

Conversely, if $\mathcal{H} = \langle n_1, \ldots n_{g-1}\rangle$ is a nonhyperelliptic symmetric semigroup then we take the rational curve

$$\mathcal{C}^{(0)} := \left\{(a^{n_0}b^{l_g-1} : a^{n_1}b^{l_{g-1}-1} : \ldots : a^{n_{g-1}}b^{l_1-1}) | (a:b) \in \mathbb{P}^1\right\} \subset \mathbb{P}^{g-1}. \tag{2.4}$$

The symmetric semigroup $\mathcal{H}$ is realized as the Weierstrass semigroup of $\mathcal{C}^{(0)}$ at the smooth point $P = (0 : \ldots : 0 : 1)$. □

The curve $\mathcal{C}^{(0)}$ in (2.4) is called *canonical monomial curve*. The monomial curve $\mathcal{C}^{(0)}$ has an unique singular point, namely the unibranched point $Q = (1 : 0 : \ldots : 0)$ of multiplicity n_1 (see [S], pp 190). The point Q is the image of the only point $\overline{Q} = (0 : 1)$ under the normalization map π. This class of curves will be very important as a tool to construct the moduli space of the classes of the pairs $(\mathcal{C}, P)$ where $\mathcal{C}$ is a projective Gorenstein curve of arithmetical genus g and P is a smooth point of $\mathcal{C}$ whose Weierstrass semigroup is fixed.

2.3 A Moduli Problem

A moduli problem consists of two things: the first, a class of algebraic-geometric objects with a notion of what it means to have a family of these objects over a scheme B; the second, is to determine an equivalence relation $\sim$ on the set of all these families over each scheme B.

Example 2.3.1. The first exemple of moduli space is all of the lines subspaces of $\mathbb{R}^2$, which is the projective space $\mathbb{P}^1_{\mathbb{R}}$. So the projective space $\mathbb{P}^n_k$ over the field k is a moduli space.

Example 2.3.2. The Grassmanian $G_r(V)$, the collection of r-dimensional linear subspaces of V.

More precisely, if we consider a class $\mathcal{M}$ of algebraic varieties over a field $\mathbf{k}$ then a family will be a *flat* morphism $\pi : \mathcal{X} \longrightarrow S$ whose fibers $\mathcal{X}(s) = \pi^{-1}(s), s \in S$, are elements of $\mathcal{M}$. The spaces $\mathcal{X}$ and S are called the *total space* and the *parameter space* of the family π, respectively. When the space S is connected, we call π a *family of deformations* of $\mathcal{X}(s_0)$ for s_0 be a closed point in S. Now, for each scheme S over a field $\mathbf{k}$ we take two differents families over S:

$$\begin{array}{ccc} \mathcal{X} & & \mathcal{X}' \\ \downarrow^{\pi} & \text{and} & \downarrow^{\pi'} \\ S & & S \end{array},$$

we say that π and π' are isomorphics if there is an isomorphism $\varphi : \mathcal{X} \longrightarrow \mathcal{X}'$ such that the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\varphi} & \mathcal{X}' \\ & \underset{\pi}{\searrow} & \downarrow{\scriptstyle \pi'} \\ & & S \end{array}$$

commutes, i.e, $\pi = \varphi \circ \pi'$. Thus we can define a contravariant functor $F : (\text{Schemes} /\mathbf{k}) \longrightarrow (\text{Sets})$ as follows:

$$F(S) = \{\text{isomorphism classes of families over } S \text{ of objects of } \mathcal{M}\}.$$

Moreover if $f : T \longrightarrow S$ is a morphism, then we have a morphism

$$F(f) : F(S) \longrightarrow F(T),$$

induced by the pullback

$$F(f)([\mathcal{X} \to S]) = [T \times_S \mathcal{X} \to T].$$

We can ask whether the functor F is representable by a scheme M, that is, if there is an isomorphism of functors $\Psi : \text{Hom}(-, M) \longrightarrow F$, such isomorphism will be induced by an uniquely family $\upsilon : \mathcal{Y} \longrightarrow M$, called *universal family*. When this happens, M will be a *moduli space* or a fine moduli space in the strongest sense. The problem (or no for the us researchers) is that such moduli space very seldom exists, but in the most of the time $\mathcal{M}$ will have a weaker structure which corresponds to the structure of the morphism F.

It is interesting that we can study the structure of $\mathcal{M}$ even F not being representable. For this the idea is to consider an element $[X]$ of $\mathcal{M}$ and realize an infinitesimal study by constructing a family over $[X]$ parameterized by the spectrum of a local ring, getting informations in a neighborhood of this element $[X]$. Then we separate the *global moduli problem* from the *local moduli problem*. The moduli problem is studied through deformation theory.

According to Edoardo Sernesi,

"*Deformation theory* is the study of infinitesimal deformations as a tool to understand the local structure of the moduli space".

Definition 2.3.1. Let $\mathcal{M}_g$ be the moduli space of the **smooth** curves of arithmetical genus g. We also define the moduli space $\mathcal{M}_{g,n}$ of the smooth curves with n marked points.

Proposition 2.3.1. *The moduli space $\mathcal{M}_g$ has dimension $3g - 3$.*

2.4 Known results

Let $\mathcal{M}_{g,1}^{\mathcal{H}}$ be the moduli space of the smooth complete integral pointed algebraic curves with a Weierstrass point of semigroup $\mathcal{H}$ of genus g. There are many important questions about these spaces, namely: when $\mathcal{M}_{g,1}^{\mathcal{H}}$ is not empty? What is your dimension? What are their irreducible componets? When are they rationals or stably rational: In this thesis we will approach the questions about the dimension and global structure of $\mathcal{M}_{g,1}^{\mathcal{H}}$ when the semigroup $\mathcal{H}$ is symmetric.

Let $\mathcal{O}$ be a local ring of a projective curve defined over $\mathbf{k}$ and E be a component of the formal moduli of deformations of Spec($\mathcal{O}$). Assuming that the fiber over the generic point of $\mathcal{O}$ is smooth, P. Deligne [D] stablished, by analyzing three different moduli spaces, the following formula:

Theorem 2.4.1 ([D] Deligne, thm. 2.27)**.**

$$\dim\, E = 3\delta - m_1,$$

where

$$m_1 = [\overline{\tau} : \tau] := \dim_{\mathbf{k}}(\overline{\tau}/\tau \cap \overline{\tau}) - \dim_{\mathbf{k}}(\tau/\tau \cap \overline{\tau}),$$

and τ is the module of the differentials of the local ring $\mathcal{O}$ while $\overline{\tau}$ is the module of the differentials of the ring $\overline{\mathcal{O}}$.

By considering monomial curves and following the Pinkham's work [P] on equivariant deformation, Deligne's formula becomes

Theorem 2.4.2 (Deligne–Pinkham bound)**.**

$$\dim \mathcal{M}_{g,1}^{\mathcal{H}} \leq 2g + \lambda - 2,$$

where $\lambda = [\mathrm{End}(\mathcal{H}) : \mathcal{H}]$.

It is important to mention that Pinkham knew that this upper bound was attained for some exemples, eg. the hyperelliptic semigroup generated by 2 and $2g+1$. Pinkham constructs the moduli spaces $\mathcal{M}_{g,1}^{\mathcal{H}}$ by considering equivariant deformations, his basic idea is that the group action of the multiplicative group of the ground field extends to a group action on the space of deformations.

A numerical semigroup $\mathcal{H}$ is called *negatively graded* if the positively graded part of the first cohomology module of the cotangent complex of the semigroup ring $B_{\mathcal{H}}$ over $\mathbf{k}$ is zero. Rim and Vitulli in [RV] classified the negatively graded semigroups, as follows.

Theorem 2.4.3 ([RV], Theorem 4.7)**.** *Let $\mathcal{H}$ be a numerical semigroup of genus g and $\lambda = \lambda(\mathcal{H})$. Then $\mathcal{H}$ is negatively graded if and only if $\mathcal{H}$ is of one of the following types:*

i. $\mathcal{H}$ is ordinary;

ii. $\mathcal{H}$ is hyperordinary;

iii Excluding the ordinary and hyperordinary cases, given g and λ with $2 \leq \lambda \leq g-2$ there exists an unique negatively graded semigroup (denoted by $\mathcal{H}_{g,\lambda}$) of given g and λ. Namely,

$$\mathcal{H}_{g,\lambda} = \{0, g, \ldots, 2g-\lambda-1, \widehat{2g-\lambda}, 2g-\lambda+1, 2g-\lambda+2, \ldots\}.$$

If $\lambda = 1$ *we have two possibilities:*

$$\mathcal{H}_{g,1} = \{0, g, g+1, \ldots, 2g-2, \widehat{2g-1}, 2g, 2g+1, \ldots\}$$

or

$$\mathcal{H}_{g,1} = \{0, g-1, \widehat{g}, g+1, \ldots, 2g-2, \widehat{2g-1}, 2g, 2g+1, \ldots\}.$$

In the same work Rim and Vitulli [RV] showed that the upper bound $\dim \mathcal{M}_{g,1}^{\mathcal{H}} \leq 2g+\lambda-2$ is optimal whenever $\mathcal{H}$ is a negatively graded semigroup. In the case of symmetric semigroups it follows that $\lambda = 1$, because $\mathrm{End}(\mathcal{H}) = \mathcal{H} \cup \{l_g\}$.

In the late of 80's, using theory of limit linear series on algebraic curves, Eisenbud–Harris in [EH] computed an upper bound for the codimension of $\mathcal{M}_{g,1}^{\mathcal{H}}$ in $\mathcal{M}_{g,1}$.

Theorem 2.4.4 (Eisenbud–Harris bound)**.** *Let* X *be an irreducible component of* $\mathcal{M}_{g,1}^{\mathcal{H}}$*. Then*

$$\operatorname{codim} X \geq 3g - 2 - \mathrm{wt}(\mathcal{H})$$

where $\mathrm{wt}(\mathcal{H}) := \sum \ell_i - i$ *is the weight of the semigroup* $\mathcal{H}$*.*

This lower bound is attained for primitive Weierstrass semigroups whose weight is not bigger than $g-1$, see [EH]. However, if the weight is large, as in the case of symmetric semigroups, then their bound may be far from being sharp, and it may even be negative.

By considering symmetric semigroups Stoehr [S] constructs an explicit compactification of the moduli space $\mathcal{M}_{g,1}^{\mathcal{H}}$ by allowing Gorenstein singularities. His construction is done by deformations of a suitable monomial curve. Since his construction is in our particular interest we will described it breafly.

Let $\mathcal{H}$ be a symmetric semigroup of genus g with *canonical system of generators* $n_0 = 0 < n_1 < \ldots < n_{g-1}$ and the gap sequence $1 = l_1 < 2 = l_2 < l_3 \ldots l_g = 2g-1$. Let us consider a complete irreducible Gorenstein curve $\mathcal{C}$ over $\mathbf{k}$ and P be a nonsingular point of $\mathcal{C}$ such that the Weierstrass semigroup of $\mathcal{C}$ at P is $\mathcal{H}$. By definition of the Weierstrass point P, there are functions $x_{n_0}, \ldots, x_{n_{g-1}}$ in $\mathbf{k}(\mathcal{C})$ whose pole orders at P is equal to $n_i, i = 0, \ldots, g-1$. Since $l_2 = 2$ the curve $\mathcal{C}$ is nonhyperelliptic and by theorem (2.1.4) the Gorenstein curve $\mathcal{C}$ can be embedded in $\mathbb{P}^{g-1}$. As in Petri's analysis, the Stoehr's idea is to construct a basis for the space of global sections $H^0(\mathcal{C}, r(2g-2)P)$, for $r \geq 1$ and calculating the ideal of $\mathcal{C}$. Conversely, making some considerations on the symmetric semigroup $\mathcal{H}$, Stöhr introduces homogeneous quadratic forms and asks for the conditions on their coefficients in order that the intersection that quadratic hypersurfaces in $\mathbb{P}^{g-1}$ is a complete irreducible Gorenstein curve whose Weierstrass semigroup at P is $\mathcal{H}$. We will now describe this procedure.

Let $I(\mathcal{C})$ be the canonical ideal of the Gorenstein curve $\mathcal{C} \subset \mathbb{P}^{g-1}$. Thus $I(\mathcal{C}) = \bigoplus_{n=2}^{\infty} I_r(\mathcal{C})$, where $I_r(\mathcal{C})$ is the vector space of n-forms vanishing identically at $\mathcal{C}$. Since the divisor $(2g-2)P$ is canonical and the functions $x_{n_0}, \ldots, x_{n_{g-1}}$ in $\mathbf{k}(\mathcal{C})$ are linearly independent, this functions form a basis for the space $H^0(\mathcal{C}, (2g-2)P)$. For a nongap $s \leq 4g-4$ we write all the partitions of s as sum of two generators of the symmetric semigroup $\mathcal{H}$, $s = a_s + b_s$ with $a_s \leq b_s \leq 2g-2$ so that the $3g-3$ products $x_{a_s}x_{b_s}$ form a P-hermitian basis of the space global sections $H^0(\mathcal{C},(4g-4)P)$ which allows to construct a P-hermitian basis of $H^0(\mathcal{C},r(2g-2)P)$, $r\geq 3$. So for each partition as sum of two nongaps $s=a_{si}+b_{si}$ we have $x_{a_{si}}x_{b_{si}}\in H^0(\mathcal{C},sP)$, hence

$$x_{a_{si}}x_{b_{si}}=\sum_{r=0}^{s}c_{sir}x_{a_{ri}}x_{b_{ri}},$$

where the summation only varies through nongaps. Multiplying the x_{n_i} by suitable constants we normalize $c_{sir}=1$ and so we obtain the $\frac{(g-2)(g-3)}{2}$ quadratic forms

$$F_{si}=X_{a_{si}}X_{b_{si}}-X_{a_s}X_{b_s}-\sum_{r=0}^{s-1}c_{sir}X_{a_{ri}}X_{b_{ri}}, \tag{2.5}$$

that vanish identically on the canonical curve $\mathcal{C}$. Thus they form a basis of the quadratic relations $I_2(\mathcal{C})$. To show that the quadratic forms F_{si} generate the ideal $I(\mathcal{C})$ stöhr [S], Contiero and Stöhr [CS], made the following assumptions on the semigroup $\mathcal{H}$:

$$3<n_1<g \text{ and } \mathbb{N}\neq\langle 4,5\rangle.$$

Conversely, Stöhr makes the same considerations on the symmetric semigroup $\mathcal{H}$, assumes that are given quadratic forms F_{si} as (2.5) and answers what are the conditions on their coefficients c_{sir} in order that the intersection of the quadratic hypersurfaces in $\mathbb{P}^{g-1}$ is a complete irreducible Gorenstein curve with gap sequence $l_1,\ldots,l_g$ at P. The key to answer this question is the lemma 2.3 in [S], whose proof involves Petry's analysis, and it is improved, by using only combinatorial facts in [CS]

Lemma 2.4.5. *(Syzygy Lemma 2.3, [CS])For each quadratic form* $F_{si}^{(0)}=X_{a_{si}}X_{b_{si}}-X_{a_s}X_{b_s}$ *different from* $F_{n_i+2g-2,1}^{(0)}(i=0,\ldots,g-3)$ *there is a linear isobaric syzygy of the form*

$$X_{2g-2}F_{s'i'}^{(0)}+\sum_{rsi}\epsilon_{rsi}^{(s'i')}X_rF_{si}^{(0)}=0, \tag{2.6}$$

where the coefficients $\epsilon_{nsi}^{(s'i')}$ *are integers equal to* $1,-1$ *or* 0, *and where the sum is take over the nongaps* $r<2g-2$.

Replacing the quadratic forms $F_{s'i'}^{(0)}$ and $F_{si}^{(0)}$ in (2.6) with the forms $F_{s'i'}$ and F_{si} and applying the division algorithm on the monomials that are not in the basis of $H^0(\mathcal{C},3(2g-2)P)$, Stöhr gets equations between the forms $F_{s'i'}$ and F_{si} which he imposes that are syzygies. By replacing $X_{n_i}\mapsto t^{n_i}$ in this syzygies we have the relations $\varrho_{s'i'r'}=0$ between the coefficients c_{sir}. After normalizing $\frac{1}{2}g(g-1)$ of the coefficients c_{sir} (see proposition 3.1 in [S]), the only freedon left to us is to transform $x_{n_i}\mapsto c^{n_i}x_{n_i}$ for some $c\in\mathbb{G}_m(\mathbf{k})=\mathbf{k}^*$. Finally we present the theorem that explains the construction of a compactification of the moduli space $\mathcal{M}_{g,1}^{\mathcal{H}}$.

Theorem 2.4.6 (Stöhr's Construction)**.** *The isomorphism classes of the projective irreducible pointed Gorenstein curves of arithmetical genus* g *and Weierstrass gap sequence* $l_1,\ldots,l_g$ *correspond bijectively to the orbits of the invariant* $\mathbb{G}_m(\mathbf{k})$*-action* $(z,c_{sir})\mapsto z^{r-s}c_{sir}$ *on the quasi-homogeneous affine algebraic set of the systems of constants* c_{sir} *normalized and satisfying the isobaric polynomial equations* $\varrho_{s'i'r'}=0$.

As we mentioned in the Introduction, E. Bullock, in a beautiful work due his PhD thesis, computed the general family of each component of the Kontsevich–Zorich space $\mathcal{G}_g$, namely

Theorem 2.4.7 (E. Bullock)**.** *If $g \geq 4$, then*

(a) a general point of $\mathcal{G}_g^{\text{hyp}}$ has Weierstrass gaps $\{1, 3, 5, \ldots, 2g-5, 2g-3, 2g-1\}$,

(b) a general point of $\mathcal{G}_g^{\text{odd}}$ has Weierstrass gaps $\{1, 2, 3, \ldots, g-2, g-1, 2g-1\}$, and

(c) a general point of $\mathcal{G}_g^{\text{even}}$ has Weierstrass gaps $\{1, 2, 3, \ldots, g-2, g, 2g-1\}$.

Thereafter Bullock [BUL] also made investigations on the structure of the moduli space $\mathcal{M}_{g,1}^{\mathcal{H}}$ for small genus. He showed that for genus $g \leq 6$, the moduli variety $\mathcal{M}_{g,1}^{\mathcal{H}}$ is irreducible and stably rational with the possible exceptions of the semigroups $\langle 5, 7, 8, 9, 11\rangle$ and $\langle 6, 7, 8, 9, 10\rangle$. Moreover, he shows that the existence of an irreducible component of the expected dimension for each semigroup. As an exemple of our tools we show the rationality of the moduli variety $\mathcal{M}_{g,1}^{\mathcal{H}}$ having Weierstrass semigroup $\langle 6, 7, 8, 9, 10\rangle$.

As noted above Contiero–Stoehr made a purely combinatorial proof of the syzygy lemma [CS, lemma 2.3] which provides an implementable algorithm to construct the space $\mathcal{M}_{g,1}^{\mathcal{H}}$ when $\mathcal{H}$ is symmetric. Furthermore, they created a method which allows to deal with families of symmetric semigroups, getting upper bounds for the dimension of $\mathcal{M}_{g,1}^{\mathcal{H}}$ which provides better bounds them Deligne–Pinkham's one. In the chapter 4 of this thesis we will apply this method to compute the dimension of some spaces.

In the last year N. Pflueger [PF1] improved the Eisenbud–Harrris bound. He introduced the effective weight of a numerical semigroup $\mathcal{H}$

$$\text{ewt}(\mathcal{H}) := \sum_{\text{gaps } l_i} (\#\text{ generators } n_j < l_i).$$

Alternatively, $\text{ewt}(\mathcal{H})$ is the number of pair (n_i, l_k) where $n_i \in \mathcal{H}$ and $l_k \notin \mathcal{H}$, so $\text{wt}(\mathcal{H}) - \text{ewt}(\mathcal{H})$ is equal to the number of pairs (n_i, l_k) where $n_i < l_k$, n_i is composite, and l_k is a gap. Therefore, $\text{wt}(\mathcal{H}) = \text{ewt}(\mathcal{H})$ if and only if $\mathcal{H}$ is primitive.

Theorem 2.4.8. *(Theorem 1.2, [PF1]) If $\mathcal{M}_{g,1}^{\mathcal{H}}$ in nonempty, and X is any irreducible component of it, then*

$$\dim X \geq \dim \mathcal{M}_{g,1} - \text{ewt}(\mathcal{H}).$$

Remark 2.4.9. We observe that for the semigroup $\mathcal{H} = \langle 6, 7, 8\rangle$ of genus 9 we have $\text{ewt}(\mathcal{H}) = 12$ while $\text{codim}(\mathcal{M}_{g,1}^{\mathcal{H}}) = 11$ (see [PF1], 2.6), and so $\text{codim}(\mathcal{M}_{g,1}^{\mathcal{H}}) < \text{ewt}(\mathcal{H})$. This bound is sharp (see [PF1], thm 1.3) whenever $\text{ewt}(\mathcal{H}) \leq g-2$.

Pflueger in [PF2] also studied a class of semigroups called Castelnuevo semigroups where his lower bound can not be attained, he produced the first examples where the moduli spaces $\mathcal{M}_{g,1}^{\mathcal{H}}$ is reducible and computed these components, see [PF2, thm 1.1].

3 Moduli space of curves with symmetric Weierstrass semigroup

3.1 Gorenstein subcanonical curves and Weierstrass points

Let $\mathcal{C}$ be a complete integral Gorenstein curve of arithmetical genus $g > 1$ defined over an algebraically closed field $\mathbf{k}$. Following the terminology introduced by Bullock [Bu], we assume that $\mathcal{C}$ is subcanonical, ie. there is a rational function on $\mathcal{C}$ with pole divisor $(2g-2)P$, where P is a nonsingular point of $\mathcal{C}$. By the Riemann–Roch theorem for singular curves, the dualizing sheaf ω is $\mathcal{O}_{\mathcal{C}}((2g-2)P)$. Hence, the vector space of its global sections is

$$H^0(\mathcal{C},\omega) = \mathbf{k}\cdot x_{n_0} \oplus \mathbf{k}\cdot x_{n_1} \oplus \cdots \oplus \mathbf{k}\cdot x_{n_{g-1}},$$

where x_{n_i} is a rational function on $\mathcal{C}$ whose pole divisor is n_iP, for $i \geq 1$, with $n_0 := 0$ and $n_{g-1} = 2g-2$. Equivalently, the base point $P \in \mathcal{C}$ is a Weierstrass point with gap sequence $1 = \ell_1 < \ell_2 < \cdots < \ell_g = 2g-1$, whose symmetric Weierstrass semigroup $\mathcal{H}$ of genus g is canonically generated by $\langle n_0, n_1, \ldots, n_{g-1}\rangle = \mathcal{H}$. We recall that a semigroup $\mathcal{H}$ of genus g is symmetric if its Frobenius number ℓ_g is the largest possible, namely $\ell_g = 2g-1$. Equivalently, $\mathcal{H}$ is symmetric if and only if $\ell_i = \ell_g - n_{g-i}$, for all $i = 1, \ldots, g$.

Let us assume that $\mathcal{C}$ is also non-hypereliptic, thus its dualizing sheaf ω is very ample and induces an embedding in the $(g-1)$-dimensional projective space $\mathbb{P}^{g-1}$

$$(x_{n_o} : \ldots : x_{n_{g-1}}) : \mathcal{C} \overset{\omega}{\hookrightarrow} \mathbb{P}^{g-1} = \mathbb{P}(H^0(\mathcal{C},\omega))$$

defined over $\mathbf{k}$, a rather general and beautiful approach on canonical models can be found in [KM], in particular theorem 4.3. Therefore, $\mathcal{C}$ can be identified with its image under the canonical embedding. Hence, $\mathcal{C} \subset \mathbb{P}^{g-1}$ is a projective curve of genus g and degree $2g-2$.

Conversely, every symmetric numerical semigroup $\mathcal{H}$ of genus $g > 1$ can be realized as a Weierstrass semigroup of a canonical Gorenstein curve. We just have to consider the canonical generators $0 = n_0 < n_1, \ldots, < n_{g-1} = 2g-2$ of $\mathcal{H}$ and take the induced monomial curve

$$\mathcal{C}_{\mathcal{H}} := \{(s^{n_0}t^{\ell_g-1} : s^{n_1}t^{\ell_{g-1}-1} : \ldots : s^{n_{g-2}}t^{\ell_2-1} : s^{n_{g-1}}t^{\ell_1-1}) \,|\, (s:t) \in \mathbb{P}^1\} \subset \mathbb{P}^{g-1}\,.$$

It can be checked that it has an unique singular point, namely $(1:0:\ldots:0)$, which is unibranch and has singularity degree g. Since the semigroup $\mathcal{H}$ is symmetric, $C_{\mathcal{H}}$ is a Gorenstein curve. The contact orders with hyperplanes at its unique point $P = (0:\ldots 0:1)$ at the infinity are exactly $\ell_i - 1$, $i = 1, \ldots, g$. Thus $\mathcal{C}_{\mathcal{H}}$ has degree $2g-2$ and its Weierstrass semigroup at P is $\mathcal{H}$. For a more detailed exposition on monomial curves we refer to [B].

We want to study the canonical ideal of $\mathcal{C}$. According to Enriques–Babbage's theorem [ACGH] for the smooth curves and the generalization to *Petri's singular curves* obtained by

Schreyer [FS] on homogeneous ideal of a smooth canonically-embedded curve, if we assume $\mathcal{C}$ not isomorphic to a plane quintic, then its ideal can be generated by quadratic forms, when it is non-trigonal, and by quadratic and cubic forms when it is trigonal, in the case of Petri's curves we also assume that it has a simple $(g-2)$-secant.

An extended version of Max Noether's theorem for complete integral non-hypereliptic curves, which is proven for uni and bi-branched points, see [Ma] and [ACM], states there is a surjective homomorphism

$$\mathrm{Sym}^r(H^0(\mathcal{C},\omega)) \longrightarrow H^0(\mathcal{C},\omega^r)$$

for all $r \geq 1$. In the following, we review a suitable proof of Max-Noether's theorem for subcanonical curves given by Stöhr in [S] , which is fundamental for this work.

Let $\mathcal{H}$ be a numerical symmetric semigroup of genus $g > 1$. Since $\mathcal{C}$ is non-hyperelliptic, we must to assume that the symmetric semigroup $\mathcal{H}$ is not hyperelliptic, ie. $2 \notin \mathcal{H}$, equivalently $\mathcal{H} \neq \langle 2, 2g+1 \rangle$. For each nongap $s \leq 4g-4$, we consider the partitions of s as sums of two nongaps as following

$$s = a_s + b_s, \ a_s \leq b_s \leq 2g-2,$$

with a_s the smallest possible nongap. It follows from Oliveira's work [O, theorem 1.3] that the $3g-3$ rational functions $x_{a_s}x_{b_s}$ of $\mathcal{C}$ form a P-hermitian basis for the space of the global sections of the bicanonical divisor $\omega^2 \cong \mathcal{O}_{\mathcal{C}}((4g-4)P)$. Now, for each integer $r \geq 3$ a P-hermitian basis for the space $H^0(\mathcal{C},\omega^r)$ is given by the r-monomials expressions

$$\begin{array}{rc} x_{n_0}^{r-1}x_{n_i} & (i=0,\ldots,g-1), \\ x_{n_0}^{r-2-i}x_{a_s}x_{b_s}x_{n_{g-1}}^{i} & (i=0,\ldots,r-2, \ s=2g,\ldots,4g-4), \\ x_{n_0}^{r-3-i}x_{n_1}x_{2g-n_1}x_{n_{g-2}}x_{n_{g-1}}^{i} & (i=0,\ldots,r-3). \end{array}$$

Let $I(\mathcal{C}) = \oplus_{r=2}^{\infty} I_r(\mathcal{C})$ be the homogeneous canonical ideal of $\mathcal{C} \subset \mathbb{P}^{g-1}$. As an immediate consequence of the existence of a above P-hermitian basis of r-monomials for the $\mathbf{k}$-vector space $H^0(\mathcal{C},\omega^r)$, the homomorphism

$$\mathbf{k}[X_{n_0},\ldots,X_{n_{g-1}}]_r \longrightarrow H^0(\mathcal{C},\omega^r)$$

induced by the substitutions $X_{n_i} \longmapsto x_{n_i}$ is surjective for each $r \geq 1$. Thus we get a proof of Max-Noether's theorem for subcanonical Gorenstein curves.

It is clear from Riemann's theorem that the codimension of $I_r(\mathcal{C})$ in the $\binom{r+g-1}{r}$-dimensional vector space $\mathbf{k}[X_{n_0},\ldots,X_{n_{g-1}}]_r$ of homogeneuos r-forms is equal to $(2r-1)(g-1)$, for each $r \geq 2$. An immediate consequence is that the vector space of quadratic and cubic relations have dimensions

$$\dim I_2(\mathcal{C}) = \frac{(g-2)(g-3)}{2} \ \text{ and } \ \dim I_3(\mathcal{C}) = \binom{g+2}{3} - (5g-5),$$

respectively.

For each $r \geq 2$, we define the vector subspace Λ_r of $\mathbf{k}[X_{n_0},\ldots,X_{n_{g-1}}]_r$ spanned by the lifting of the above P-hermitian r-monomial basis of $H^0(\mathcal{C},\omega^r)$. It is spanned by the r-monomials in $X_{n_0},\ldots,X_{n_{g-1}}$ whose weights are pairwise different between all the nongaps $n \le r(2g-2)$. Since $\Lambda_r \cap I_r(\mathcal{C}) = 0$ and

$$\dim \Lambda_r = \dim H^0(\mathcal{C},\omega^r) = \operatorname{codim} I_r(\mathcal{C}),$$

we obtain

$$\mathbf{k}[X_{n_0},\ldots,X_{n_{g-1}}]_r = I_r(\mathcal{C}) \oplus \Lambda_r, \text{ for each } r \ge 2.$$

Let $r\mathcal{H}$ be the set of all sums of r nongaps not bigger than $2g-2$. Oliveira showed, cf. [O, theorem 1.5], that each nongap smaller than or equal to $r(2g-2)$ belongs to $r\mathcal{H}$. Moreover, each sum of r nongaps $\le 2g-2$ is a nongap $\le r(2g-2)$. Consequently, $\#r\mathcal{H} = (2r-1)(g-1)$ and therefore

$$\#r\mathcal{H} = \dim H^0(\mathcal{C},\omega^r)\,.$$

In particular, for each nongap $s \le 4g-4$ we list all the partitions $s = a_{si} + b_{si} \in 2\mathcal{H}$, where

$$a_{si} \le b_{si} \le 2g-2 \ (i = 0,\ldots,\nu_s) \text{ and } a_s := a_{s0} < a_{s1} < a_{s2} < \ldots < a_{s\nu_s}\,.$$

Since $x_{a_{si}}x_{b_{si}} \in H^0(\mathcal{C}, sP)$ and $\{x_{a_s}x_{b_s}\}$ is the above fixed basis, we can write

$$x_{a_{si}}x_{b_{si}} = \sum_{n=0}^{s} c_{sin}x_{a_n}x_{b_n}\,,$$

for each $i = 0,\ldots,\nu_s$, where the coefficients c_{sir} are uniquely determined constants and the summation index only varies through nongaps. In the same way, for each nongap $\sigma \le 6g-6$ we consider the partitions $\sigma = a_{\sigma j} + b_{\sigma j} + c_{\sigma j} \in 3\mathcal{H}$ where $a_{\sigma j} \le b_{\sigma j} \le c_{\sigma j} \le 2g-2$ $(j = 0,\ldots,\nu_\sigma)$ with $a_\sigma := a_{\sigma 0} < a_{\sigma 1} < \ldots < a_{\sigma\nu_\sigma}$ and $b_\sigma := b_{\sigma 0} > b_{\sigma 1} > \ldots > b_{\sigma\nu_\sigma}$. Analogously, we may write

$$x_{a_{\sigma j}}x_{b_{\sigma j}}x_{c_{\sigma j}} = \sum_{n=0}^{\sigma} d_{\sigma jn}x_{a_n}x_{b_n}x_{c_n}\,,$$

for each integer $j = 0,\ldots,\nu_\sigma$, where the coefficients $d_{\sigma jn}$ are uniquely determined constants and the summation index only varies through nongaps.

Multiplying the functions $x_{n_0},\ldots,x_{n_{g-1}}$ by constants we do not change the P-hemitian property of the above basis, thus we can normalize the coefficients $c_{sis} = 1$ and $d_{\sigma j\sigma} = 1$. Therefore, by construction the $\binom{g+1}{2} - (3g-3) = \frac{1}{2}(g-3)(g-2)$ quadratic forms

$$F_{si} = X_{a_{si}}X_{b_{si}} - X_{a_s}X_{b_s} - \sum_{n=0}^{s-1} d_{sin}X_{a_n}X_{b_n} \tag{3.1}$$

and the $\binom{g+2}{3} - (5g-5)$ cubic forms

$$G_{\sigma j} = X_{a_{\sigma j}}X_{b_{\sigma j}}X_{c_{\sigma j}} - X_{a_\sigma}X_{b_\sigma}X_{c_\sigma} - \sum_{n=0}^{\sigma-1} d_{\sigma jn}X_{a_n}X_{b_n}X_{c_n}\,, \tag{3.2}$$

that vanish identically on the canonical curve $\mathcal{C}$. We attach to the variable X_n the weight n, to the coefficient c_{sin} the weight $s-n$ and to $d_{\sigma jn}$ the weight $\sigma - n$. Thus the above quadric and cubic forms seen as polynomial expressions in the variables X_n and the coefficients $c_{sin}, d_{\sigma jn}$ are also isobaric forms.

In the view of Henriques–Babbage's theorem for smooth canonical curves, cf. [ACGH], we want to assure that the canonical ideal of $\mathcal{C}$ is generated by the quadratic and cubic forms. This fact reflects on conditions on the symmetric semigroup. We assume that the non-hyperelliptic symmetric semigroup $\mathcal{H}$ is a non-trivial semigroup of genus $g > 1$, which is equivalent to assume that the multiplicity n_1 of $\mathcal{H}$ satisfies $2 < n_1 \leq g$.

By a theorem of Oliveira [O, theorem 1.7], if we consider $3 < n_1 < g$, then follows that there is at least one quadratic form, ie. $\nu_s \geq 1$, whenever $s = n_i + 2g - 2$ for $i = 0, \dots, g-3$. In this case Contiero–Stoehr [CS] gave an algorithmic proof that the canonical ideal of a Gorenstein curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ with Weierstrass semigroup $\mathcal{H}$ at the base point is generated by quadratic relations. If we assume that $3 \in \mathcal{H}$ then its genus has residue 1 or 0 module 3, hence $\mathcal{H} := \langle 3, g+1 \rangle$. In this case we already know that $\overline{\mathscr{M}_{g,1}^{\mathcal{H}}} = \mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}})$, as mentioned in the section Introduction of the present work. If $\mathcal{H} = \langle 4, 5 \rangle$ then $\mathcal{C}$ is isomorphic to a plane quintic where the quadric hypersurfaces contaning $\mathcal{C}$ is the Veronese surface.

In the excluded case $\mathcal{H} = \mathbb{N} \backslash \{1, 2, \dots, g-1, 2g-1\}$, the curve $\mathcal{C}$ is possibly trigonal, so its canonical ideal can be not generated by only quadratic relations. In the next section we investigate the Weierstrass semigroup of trigonal complete curves and then, we will give an algorithmic proof that the canonical ideal of a complete Gorenstein curve with symmetric Weierstrass semigroup

$$\mathcal{H} := \mathbb{N} \backslash \{1, 2, \dots, g-1, 2g-1\} = \langle 0, g, g+1, \dots, 2g-2 \rangle$$

at a smooth non-ramified point is generated by quadratic and cubics forms.

3.2 Trigonal subcanonical curves

Let $\mathcal{C}$ be a complete integral curve of arithmetic genus g defined over an algebraically closed field $\mathbf{k}$. A *linear system of dimension r on $\mathcal{C}$* is a set of the form

$$\mathscr{L} = \mathscr{L}(\mathscr{F}, V) := \{x^{-1}\mathscr{F} \mid x \in V \setminus 0\}$$

where $\mathscr{F}$ is a coherent fractional ideal sheaf on C and V is a vector subspace of $H^0(\mathcal{C}, \mathscr{F})$ of dimension $r+1$.

The notion of linear systems on curves presented here is characterized by interchanging bundles by torsion free sheaves of rank 1. This is a meaningful approach since they may possess *non-removable* base points, see Coppens [Cp].

The *degree* of the linear system $\mathscr{L}$ is the integer $\deg \mathscr{F} := \chi(\mathscr{F}) - \chi(\mathcal{O}_{\mathcal{C}})$, where χ denotes the Euler characteristic. Note, in particular, that if $\mathcal{O}_{\mathcal{C}} \subset \mathscr{F}$ then

$$\deg \mathscr{F} = \sum_{P \in C} \dim(\mathscr{F}_P / \mathcal{O}_{\mathcal{C},P}).$$

The notation g^r_d stands for a linear system of degree d and dimension r. The linear system is said to be *complete* if $V = H^0(\mathcal{C}, \mathscr{F})$, in this case one simply writes $\mathscr{L} = |\mathscr{F}|$. According to E.

Ballico's [Ba, p. 363, Dfn. 2.1 (3)], the gonality of $\mathcal{C}$ is the smallest d for which there exists a g_d^1 on $\mathcal{C}$, or equivalently, a torsion free sheaf $\mathscr{F}$ of rank 1 on $\mathcal{C}$ with degree d and $h^0(\mathcal{C}, \mathscr{F}) \geq 2$.

The following lemma is a straightforward generalization of a Kim's result [KIM, theorem 2.6] characterizing the Weierstrass semigroup associated to a non-ramification point of a trigonal curve.

Lemma 3.2.1. *Let $\mathcal{C}$ be a complete integral trigonal curve of arithmetical genus $g \geq 5$ and $P \in \mathcal{C}$ be a Weierstrass non-ramification point. The Weierstrass semigroup $\mathcal{H}$ of $\mathcal{C}$ at P is of the form*

$$\{0, m, m+1, m+2, \ldots, m+(s-g), s+2, s+3, s+4, \ldots\},$$

for some s and m such that $g \geq m \geq \left\lfloor \frac{s+1}{2} \right\rfloor + 1$. In particular, in the symmetric case we get

$$\mathcal{H} = \{0, g, g+1, \ldots, 2g-2, 2g, 2g+1, 2g+2, \ldots\}.$$

Proof. Let ℓ_g be the Frobenius number of the Weierstrass semigroup $\mathcal{H}$ associated to $P \in \mathcal{C}$. Equivalently, the integer $s := \ell_g - 1$ is the largest such that the divisor $D_0 = sP$ is special. Since P is a Weierstrass point, it is immediate that $g \leq \ell_g - 1 \leq 2g-2$. By the maximality of s

$$\dim |\mathcal{O}(D_0)| = s - g + 1.$$

Since D_0 is a special divisor, let be

$$\omega_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}(D_0 + P_1 + P_2 + \ldots + P_{2g-2-s})$$

the dualizing sheaf of $\mathcal{C}$ where $P_i \in \mathcal{C}$, $P_i \neq P$, with $i = 1, \ldots, 2g-2-s$. As P is not a ramification point, the first nongap m is greater than 3, and so $|mP|$ is not compounded of g_3^1. By considering the divisor

$$D := (s-m)P + P_1 + P_2 + \ldots + P_{2g-2-s}$$

the residual series to $|D|$ is compounded of g_3^1 because $\omega_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(mP) \otimes \mathcal{O}_{\mathcal{C}}(D)$.

Applying the Riemann-Roch theorem, $\dim |D| = g - m$, hence we can write $|D| = (g-m)g_3^1 + B$, where B is the base locus of $|D|$. For R be an element of g_3^1 we have $R = P + Q_1 + Q_2$, with $P \neq Q_1$ and $P \neq Q_2$ because P is not a ramification point of $\mathcal{C}$, thus

$$D = (g-m)(P + Q_1 + Q_2) + B = (s-m)P + P_1 + P_2 + \ldots + P_{2g-2-s},$$

and by the maximality of s

$$P_1 + P_2 + \ldots + P_{2g-2-s} \succeq (g-m)Q_1 + (g-m)Q_2,$$

implying $2(g-m) \leq 2g-2-s$. Therefore, $m \geq \left\lfloor \frac{s+1}{2} \right\rfloor + 1$.

On the other hand,

$$B \succeq (s-g)P,$$

that means $(s-g)P$ is contained in the base locus of $|D|$. Consequently, $(m+1)P, \ldots\ (m+(s-g))P$ are not in the base locus of the residual series $|mP|$ and by Riemann-Roch theorem

$$\dim |(m+i)P| = i + 1,\ (i = 1, \ldots, s-g).$$

Thus $m, m+1, \ldots, m+s-g$ are nongaps of $\mathcal{H}$. Now by definition of s and by Riemann-Roch theorem, $\dim|(s+1)P| = \dim|sP|$, which implies that $s+1$ is a gap of $\mathcal{H}$. As for each integer $r \geq s+2$, the divisor $(r-1)P$ is nonspecial follows that

$$\dim|rP| = r-g = \dim|(r-1)P|+1,$$

so each $r \geq s+2$ is a nongap. In this way the set

$$S = \{0, m, m+1, \ldots, m+(s-g), s+2, \ldots\}$$

is contained in $\mathcal{H}$ and the cardinality of $\mathbb{N} - S$ is g and follows the proof. □

Let us consider the *trigonal numerical semigroup* $\mathcal{H} := \langle 0, g, g+1, \ldots, 2g-2\rangle$ of genus $g \geq 5$. We now fix $\frac{1}{2}(g-3)(g-2)$ initial quadratic forms like in (3.1)

$$F_{si}^{(0)} := X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s}$$

and the $\binom{g+2}{3} - (5g-5)$ initial cubic forms

$$G_{\sigma j}^{(0)} := X_{a_{\sigma j}} X_{b_{\sigma j}} X_{c_{\sigma j}} - X_{a_\sigma} X_{b_\sigma} X_{c_\sigma}.$$

It is clear that a considerable amount of cubic forms are just multiplies of quadratic ones. In the next result we explicitly find them.

Proposition 3.2.2. *Let $\mathcal{H} := \langle 0, g, g+1, \ldots, 2g-2\rangle$. There are exactly $\wp = \binom{g+2}{3} - (5g-5) - \eta$, with*

$$\eta = (g-3)(g-2) + (g-2)\left\lfloor \frac{g-2}{2} \right\rfloor + \left\lfloor \frac{g-3}{2} \right\rfloor + \sum_{j=1}^{g-4} \left\lfloor \frac{g-2-j}{2} \right\rfloor$$

initial cubic forms that are not multiples of the quadratic ones.

Proof. Since the fixed basis for Λ_2 is $\{X_0^2, X_0 X_{g+i}, X_g X_{g+i}, X_{g+j} X_{2g-2}\}$ with $i = 0, \ldots, g-2$ and $j = 1, \ldots, g-2$, the initial quadratic forms are

$$F_{sl}^{(0)} = X_{a_{sl}} X_{b_{sl}} - X_g X_{g+i} \ \text{ and } \ F_{sl}^{(0)} = X_{a_{sl}} X_{b_{sl}} - X_{g+j} X_{2g-2},$$

where the 2-monomials nonbasis elements of Λ_2 are the products $X_{g+i} X_{g+j}$ where $1 \leq i \leq j = 1, \ldots, g-3$. While the fixed basis for Λ_3 is

$$\{X_0^2 X_i, X_0 X_{as} X_{bs}, X_{as} X_{bs} X_{2g-2}, X_g^2 X_{2g-3}\},$$

with $i = 0, g, g+1, \ldots, 2g-2$ and $\{X_{as} X_{bs}\}$ the above fixed basis for Λ_2. Set $F := F_{sl}^{(0)}$ for a initial quadratic form. It is clear that the $(g-3)(g-2)$ products $X_0 F$ and $X_{2g-2} F$ are cubic forms for every F. Since the monomials $X_{g+k} X_{g+i} X_{g+j} \notin \Lambda_3$ for $k = 0, \ldots, g-3$ and $i, j = 1, \ldots, g-3$, the product $X_{g+k} F$ defines a cubic form when $X_{g+k} X_g X_{g+i}$ or $X_{g+k} X_{g+j} X_{2g-2}$ are in Λ_3. In the first case, $X_{g+k} X_g X_{g+i} \in \Lambda_3$ just for $i = g-2$, $k = 0, \ldots, g-3$ and for $i = g-3, k = 0$. Hence we get the following $(g-2)\left\lfloor \frac{g-2}{2} \right\rfloor + \left\lfloor \frac{g-3}{2} \right\rfloor$ cubic forms

$$X_{g+k}\left(X_{a_{sl}} X_{b_{sl}} - X_g X_{2g-2}\right), \ \text{with } k = 0, \ldots, g-3$$

and

$$X_g\left(X_{a_{sl}}X_{b_{sl}} - X_g X_{2g-3}\right).$$

In the remaing case, $X_{g+k}X_{g+j}X_{2g-2} \in \Lambda_3$ just for $k = 0$, $j = 1, \ldots, g-2$. So we get the following initial cubic forms

$$X_g\left(X_{a_{sl}}X_{b_{sl}} - X_{g+j}X_{2g-2}\right),\ j = 1, \ldots, g-4,$$

whose amount is $\sum_{j=1}^{g-4}\left\lfloor \frac{g-2-j}{2}\right\rfloor$. □

It is straightforward that the quadratic $F_{si}^{(0)}$ and cubic forms $G_{\sigma j}^{(0)}$ vanish identically on the monomial curve $\mathcal{C}^{(0)}$. The next lemma show that they generate the ideal of $\mathcal{C}^{(0)}$.

Lemma 3.2.3. *The canonical ideal $I(\mathcal{C}^{(0)})$ is generated by the $\frac{1}{2}(g-2)(g-3)$ quadratic forms $F_{si}^{(0)}$ and by the $\wp$ cubic forms $G_{\sigma j}^{(0)}$.*

Proof. We first note that for smooth canonical curves, this is just Petri's theorem. Since the $I(\mathcal{C}^{(0)})$ is generated by homogeneous and isobaric forms, all we have to do is to show that for a homogeneous and isobaric form belongs to $I(\mathcal{C}^{(0)})$ if and only if belongs to the ideal $\mathcal{J}$ generated by the binomials $F_{si}^{(0)}$ and $G_{\sigma j}^{(0)}$. It is just obvious that $\mathcal{J} \subseteq I(\mathcal{C}^{(0)})$. For the opposite inclusion we order the monomials $\prod_{k=0}^{g-1} X_{n_k}^{i_k}$ according to the lexicographic ordering of the vectors $(\sum i_k, \sum n_k\, i_k, -i_0, -i_{g-1}, \ldots, -i_1)$. In this way the binomials $F_{si}^{(0)}$ and $G_{\sigma j}^{(0)}$ form a Groebner basis for $\mathcal{J}$. Now, for each homogenous form F of degree r which is also isobaric of weight ω we divide it by the Groebner basis getting a decomposition

$$F = \sum H_{si}F_{si}^{(0)} + T_{\sigma j}G_{\sigma j}^{(0)} + R$$

where $R \in \Lambda_r$ and H_{si} and $T_{\sigma j}$ are homogenous of degree $r-2$ and $r-3$ respectively, and weight $\omega - s$ and $\omega - \sigma$, respectively. The remainder R is the only monomial in Λ_r of weight ω whose coefficients is equal to the sum of the coefficients of F. Since $F \in I(\mathcal{C}^{(0)})$ the sum of its coefficients is equal to zero, then $R = 0$. □

A different proof of the above theorem can be found in [GSS, thm 1.1] by noting that the symmetric semigroup $\mathcal{H} = \langle 0, g, g+1, \ldots, 2g-2\rangle$ is generated by a generalized arithmetic sequence. So the ideal $I(\mathcal{C}^{(0)})$ of the monomial curve $\mathcal{C}^{(0)}$ is also generated by the 2×2 minors of suitable two matrices. We it can be seen immediately that the ideal given by this 2×2 minors is equal to the ideal generated by the binomials $F_{si}^{(0)}$ and $G_{\sigma j}^{(0)}$.

The following lemma is a generalization of result in [CS, Lemma 2.3], where due to the assumptions the authors just deal with the first syzygies of quadratic forms. Here we also deal with syzygies of cubic forms, getting non linear syzygies.

Syzygy Lemma 3.2.4. *For each of the $\frac{1}{2}(g-3)(g-4)$ quadratic forms $F_{s'i'}^{(0)}$ different from $F_{n_i+2g-2,1}^{(0)} (i = 1, \ldots, g-3)$ there is a syzygy of the form*

$$X_{2g-2}F_{s'i'}^{(0)} + \sum_{nsi} \epsilon_{nsi}^{(s'i')} X_n F_{si}^{(0)} = 0 \tag{3.3}$$

and for each cubic forms $G^{(0)}_{\sigma' j'}$ different from $G^{(0)}_{4g-4,1}$, there is a syzygy of the form

$$X_{2g-2}G^{(0)}_{\sigma' j'} + \sum_{q\sigma j} \rho^{(\sigma' j')}_{q\sigma j} X_q G^{(0)}_{\sigma j} = 0, \tag{3.4}$$

where the coefficients $\epsilon^{(s'i')}_{nsi}$, $\rho^{(\sigma' j')}_{q\sigma j}$ are integers equal to $1, -1$ or 0, and where the sum is take over the nongaps $n, q < 2g-2$, the double indices si with $s+n = 2g-2+s'$ and σj with $q+\sigma = 2g-2+\sigma'$.

Proof. Given a quadratic form $F = F^{(0)}_{s'i'}$ or $F = -F^{(0)}_{s'i'}$, we can write

$$F = X_m X_n - X_q X_r,$$

where m, n, q, r are nongaps satisfying $m+n = q+r$ and $q < m \leq n < r < 2g-2$. Now, we have to consider the follows cases: if $r+1$ is a gap then, by symmetry, $k := 2g-2-r+n$ is a nongap and we find the syzygy

$$X_{2g-2}(X_m X_n - X_q X_r) + X_r(X_q X_{2g-2} - X_m X_k) - X_m(X_n X_{2g-2} - X_r X_k) = 0,$$

The binomials in the brackets can be written as $F^{(0)}_{si} - F^{(0)}_{sj}$, $F^{(0)}_{si}$ or $-F^{(0)}_{sj}$. Analogously if $m+1$ is a gap then we take the nongap $k := 2g-2-m+r$ and we obtain a syzygy as above. Now we can assume that $r+1$ and $m+1$ are nongaps, hence we have the syzygy

$$\begin{aligned} &X_{2g-2}(X_m X_n - X_q X_r) + X_q(X_{2g-2}X_r - X_{2g-3}X_{r+1}) = \\ &X_{2g-3}(X_{m+1}X_n - X_q X_{r+1}) + X_n(X_m X_{2g-2} - X_{2g-3}X_{m+1}). \end{aligned}$$

For a cubic form, if we put $G = G^{(0)}_{\sigma j}$ or $G = -G^{(0)}_{\sigma j}$ then we can write

$$G = X_m X_n X_p - X_q X_r X_t,$$

where m, n, p, q, r, s are nongaps with $m+n+p = q+r+t$ and $q < m \leq n \leq r \leq p < t \leq 2g-2$. If $p+1$ is a gap then, by symmetry the integer $k := 2g-2-p+q$ is a nongap smaller than $2g-2$, hence we have the syzygy

$$\begin{aligned} X_{2g-2}(X_m X_n X_p - X_q X_r X_t) + X_r(X_{2g-2}X_t X_q - X_t X_p X_k) &= \\ X_p(X_{2g-2}X_m X_n - X_r X_t X_k)&, \end{aligned}$$

where the binomials in the brackets can be written as $G^{(0)}_{\sigma j} - G^{(0)}_{\sigma i}$, $G^{(0)}_{\sigma j}$ or $-G^{(0)}_{\sigma i}$. Analogously, if $r+1$ is a gap then $k := 2g-2-r+p$ is a nongap, and therefore we obtain the syzygy

$$\begin{aligned} X_{2g-2}(X_m X_n X_p - X_q X_r X_t) + X_m(X_k X_r X_n - X_{2g-2}X_p X_n) &= \\ X_r(X_k X_m X_n - X_{2g-2}X_t X_q)&. \end{aligned}$$

Now we can assume that $p+1$ and $r+1$ are the nongaps. We have the syzygy

$$\begin{aligned} &X_{2g-2}(X_m X_n X_p - X_q X_r X_t) + X_{2g-3}(X_{r+1}X_q X_t - X_{p+1}X_n X_m) = \\ &X_m(X_p X_{2g-2}X_n - X_{p+1}X_{2g-3}X_n) + X_q(X_{2g-3}X_{r+1}X_t - X_{2g-2}X_r X_t). \end{aligned}$$

□

Remark 3.2.5. The η syzygies corresponding to the cubic forms multiples of the quadratics are trivial, therefore we just to consider syzygies for the $\wp - 1$ cubic forms, however, these $\wp - 1$ syzygies are not necessarily linear.

Lemma 3.2.6. *Let I be the ideal generated by the $\frac{1}{2}(g-2)(g-3)$ quadratic forms F_{si} and by the $\wp$ cubic forms $G_{\sigma j}$. Then,*

$$\mathbf{k}[X_{n_0}, \ldots, X_{n_{g-1}}]_r = I_r + \Lambda_r, \text{ for each } r \geq 2.$$

Proof. Let F be a homogeneous polynomial of degree r and weight w. Let S be its quasi-homogeneous component of weight w and R the unique monomial in Λ_r of weight w whose coefficient is the sum of the coefficients of S. Thus, $S - R \in I(\mathcal{C}^{(0)})$ and by the lemma 3.2.4 we can write the expression

$$S - R = \sum_{si} S_{si} F_{si}^{(0)} + \sum_{\sigma j} H_{\sigma j} G_{\sigma j}^{(0)}. \tag{3.5}$$

Replacing each polynomial S_{si} and $H_{\sigma j}$ with its homogeneuos component of degree $r-2$ and $r-3$, respectively, we can take S_{si} and $H_{\sigma j}$ homogeneous of degree $r-2$ and $r-3$, respectively. Likewise, we can assume that S_{si} and $H_{\sigma j}$ are quasi-homogeneous of weight $w-s$ and $w-\sigma$, respectively. Then the polynomial

$$F - R - \sum_{si} S_{si} F_{si}^{(0)} - \sum_{\sigma j} H_{\sigma j} G_{\sigma j}^{(0)}$$

is homogeneous of degree r and weight smaller than w. Now, the proof follows by induction on w. □

Remark 3.2.7. We see that if the curve $\mathcal{C}$ is not trigonal, then the last summand in 3.5 does not appear because the ideal $I(\mathcal{C}^{(0)})$ is generated only by the $\frac{1}{2}(g-2)(g-3)$ quadratic forms $F_{si}^{(0)}$.

Let us now invert the situation on the previous section. Instead of take a pointed canonical gorenstein curve whose Weierstrass semigroup is $\mathcal{H} = \mathbb{N}\backslash\{1, \ldots, g-1, 2g-1\}$, we take $\mathcal{H}$ and the associated monomial curve $\mathcal{C}^{(0)}$ and deform it in order to get another gorenstein curve with a marked point whose Weierstrass semigroup is also $\mathcal{H}$. By lemma 3.2.3 the ideal of the monomial curve $\mathcal{C}^{(0)}$ is generated by the $\frac{1}{2}(g-2)(g-3)$ quadratic forms $F_{si}^{(0)}$ and by the $\wp$ cubic forms $G_{\sigma j}^{(0)}$. Let us consider a *forced deformation* of the ideal of $\mathcal{C}^{(0)}$ which is

$$F_{si} = X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s} - \sum_{n=0}^{s-1} c_{sin} X_{a_n} X_{b_n}$$

and

$$G_{\sigma j} = X_{a_{\sigma j}} X_{b_{\sigma j}} X_{c_{\sigma j}} - X_{a_\sigma} X_{b_\sigma} X_{c_\sigma} - \sum_{n=0}^{\sigma-1} d_{\sigma jn} X_{a_n} X_{b_n} X_{c_n},$$

where the coefficients c_{sin} and $d_{\sigma jn}$ belongs to the ground field $\mathbf{k}$. It is clear that we are looking for conditions on this coefficients such that this *forced deformation* is a *honest deformation*: a curve of degree $2g-2$ and genus g with a marked point whose Weierstrass semigroup is $\mathcal{H}$. The idea is to take the Syzygy Lemma and erase the superscript zeros of the quadratic and cubic forms and get conditions on the coefficients.

Replacing the left-hand side of the equation (3.3) of the Syzygy lemma the binomials $F^{(0)}_{s'i'}$ and $F^{(0)}_{si}$ with the quadratic forms $F_{s'i'}$ and F_{si} we obtain for each of the $\frac{1}{2}(g-3)(g-4)$ duble indices $s'i'$ a linear combination of cubic monomials of weight less than $s'+2g-2$, which by lemma 3.2.6 admits the decomposition

$$X_{2g-2}F_{s'i'} + \sum_{nsi} \epsilon^{(s'i')}_{nsi} X_n F_{si} = \sum_{nsi} \eta^{(s'i')}_{nsi} X_n F_{si} + R_{s'i'},$$

where the sum on the right-hand side is taken over all the nongaps $n \leq 2g-2$, the duble indices si with $n+s < s'+2g-2$, the coefficients $\epsilon^{(s'i')}_{nsi}, \eta^{(s'i')}_{nsi}$ are constants and where $R_{s'i'}$ is a linear combination of cubic monomialis of pairwise different weights less than $s'+2g-2$.

Repeating the above procedure for the equation (3.4) on the Syzygy Lemma, we obtain a decomposition

$$X_{2g-2}G_{\sigma'j'} + \sum_{q\sigma j} \rho^{(\sigma'j')}_{q\sigma j} X_q G_{\sigma j} = \sum_{mq\sigma j} \mu^{(\sigma'j')}_{mq\sigma j} X_m X_q F_{\sigma j} + \sum_{q\sigma j} \nu^{(\sigma'j')}_{q\sigma j} X_q G_{\sigma j} + R_{\sigma'j'},$$

where the sum on the right-hand side is taken over the nongaps $m, q \leq 2g-2$, the indices $mq\sigma$ and $q\sigma$ with $m+q+\sigma < 2g-2+\sigma'$ and $q+\sigma < 2g-2+\sigma'$, the coefficients $\mu^{(\sigma'j')}_{mq\sigma j}, \nu^{(\sigma'j')}_{q\sigma j}$ are constants and where $R_{\sigma'j'}$ is a linear combination of quartic monomials of pairwise different weights less than $2g-2+\sigma'$.

For each nongap $m < s'+2g+2$ (resp. $r < \sigma'+2g+2$) let $\varrho_{s'i'm}$ (resp. $\vartheta_{\sigma'j'r}$) be the unique coefficient of $R_{s'i'}$ (resp. $R_{\sigma'j'}$) of weight m (resp. r). We do not lost information about the coefficients of $R_{s'i'}$ and $R_{\sigma'j'}$ replacing the variables X_n by powers t^n of an indeterminate t. Hence it is convenient to consider the polynomials

$$R_{s'i'}(t^{n_0}, \ldots, t^{n_{g-1}}) = \sum_{m=0}^{s'+2g-2} \varrho_{s'i'm} t^m$$

and

$$R_{\sigma'j'}(t^{n_0}, \ldots, t^{n_{g-1}}) = \sum_{r=0}^{\sigma'+2g-2} \vartheta_{\sigma'j'r} t^r.$$

We can asssume that the coefficients $\varrho_{s'i'm}$ are quasi-homogeneous polynomial expressions of weight $s'+2g-2-m$ in the constants c_{sin} while the coefficients $\vartheta_{\sigma'j'r}$ are quasi-homogeneous polynomial expressions of weight $\sigma'+2g-2-r$ in the constants $d_{\sigma jn}$.

Theorem 3.2.8. *Let $\mathcal{H}$ be a numerical symmetric semigroup of genus g satisfying $3 < n_1 \leq g$. Then the $\frac{1}{2}(g-2)(g-3)$ quadratic forms $F_{si} = F^{(0)}_{si} - \sum_{n=0}^{s-1} c_{sin} X_{a_{sin}} X_{b_{sin}}$ and the $\wp$ cubic forms $G_{\sigma j} = G^{(0)}_{\sigma j} - \sum_{n=0}^{\sigma} d_{\sigma jn} X_{a_n} X_{b_n} X_{c_n}$ cut out a canonical integral Gorenstein curve on $\mathbb{P}^{g-1}$ if and only if the coefficients $c_{sin}, d_{\sigma jn}$ satisfy the quasi-homogeneous equations $\varrho_{s'i'm} = 0$ and $\vartheta_{\sigma'j'r} = 0$. In this case, the point $P = (0 : \ldots : 0 : 1)$ is a smooth point of the cononical curve with Weierstrass semigroup $\mathcal{H}$.*

Proof. In the particular case $\mathcal{H} = \langle 4, 5\rangle$, we have that the intersection of the six quadric hypersurfaces $V(F_{si})$ is the Veronese surface in $\mathbb{P}^5$. We first assume that the $\frac{1}{2}(g-2)(g-3)$ quadratic forms F_{si} and the $\wp$ cubic forms $G_{\sigma j}$ cut out a canonical curve $\mathcal{C} \subset \mathbb{P}^{g-1}$. Since each $R_{s'i'}$ and $R_{\sigma' j'}$ belongs to the ideal I, follows that $R_{s'i'}(x_{n_0}, \dots, x_{n_{g-1}}) = R_{\sigma' j'}(x_{n_0}, \dots, x_{n_{g-1}}) = 0$ for each pair of index $s'i', \sigma' j'$. On the other hand,

$$R_{s'i'}(x_{n_0}, \dots, x_{n_{g-1}}) = \sum_{m=0}^{s'+2g-2} \varrho_{s'i'm} z_{s'i'm}$$

and

$$R_{\sigma' j'}(x_{n_0}, \dots, x_{n_{g-1}}) = \sum_{r=0}^{\sigma'+2g-2} \vartheta_{\sigma' j' r} z_{\sigma' j' r},$$

where the $z_{s'i'm}, z_{\sigma' j' r}$ are monomial expressions of weights m and r respectively in the projective coordinates functions $x_{n_0}, \dots, x_{n_{g-1}}$, and hence $z_{s'i'm}$ has pole divisor mP while $z_{\sigma' j' r}$ has pole divisor rP. Then we conclude that $\varrho_{s'i'm} = \vartheta_{\sigma' i' r} = 0$.

Now, we suppose that the coefficients $c_{sin}, d_{\sigma jn}$ satisfy the equations $\varrho_{s'i'm} = 0$, and $\vartheta_{\sigma' i' r} = 0$. Since the $g-3$ quadric hypersurfaces $V(F_{n_i+2g-2,1}) \subset \mathbb{P}^{g-1} (i = 1, \dots, g-3)$ and the cubic hypersurface $V(G_{4g-4,1})$ intersect transversaly at P, follows that in an open neighborhood of P, their intersection has an unique irreducible component that passes throuth P, and so this component is a projective integral algebraic curve, say $\mathcal{C}$, which is smooth at P and whose the tangent line at P is the intersection of their tangent hyperplanes $V(X_{n_i})(i = 0, \dots, g-3)$.

Let $y_{n_0}, \dots, y_{n_{g-1}}$ be the projective coordinate functions of $\mathcal{C}$ and we look for the affine open $y_{n_{g-1}} = 1$. Since the local coordinate ring of C at P is a discrete valuation ring and $n_{g-1} - n_{g-2} = l_2 - l_1 = 1$, we have that $t := y_{n_{g-2}}$ is a local parameter of $\mathcal{C}$ at P, and $y_{n_0}, \dots, y_{n_{g-3}}$ are the power series in t of order greater than 1. More precisely, comparing coefficients in the $g-3$ equations $F_{n_i+2g-2}(y_{n_0}, \dots, y_{n_{g-2}}, y_{n_{g-1}})(i = 1, \dots, g-3) = 0$ and $G_{4g-4}, 1(y_{n_0}, \dots, y_{n_{g-2}}, y_{n_{g-1}}) = 0$ one sees that

$$y_{n_i} = t^{n_{g-1}-n_i} + \text{ sum of higher orders terms} = t^{l_{g-i}-1} + \text{ sum of higher orders terms},$$

for each integer $i = 0, \dots, g-1$. This means that the g integers $l_i - 1$ $(i = 1, \dots, g)$ are the contact orders of the curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ with the hyperplanes at P. In particular, the curve $\mathcal{C}$ is not contained in any hyperplane.

Since by assumption the quasi-homogeneous equations $\varrho_{s'i'm} = 0$ and $\vartheta_{\sigma' j' r} = 0$ for each pair of duble indices $s'i'$ and $\sigma' j'$, respectively, we obtain the syzygies

$$X_{2g-2} F_{s'i'} + \sum_{nsi} \epsilon_{nsi}^{(s'i')} X_n F_{si} - \sum_{nsi} \eta_{nsi}^{(s'i')} X_n F_{si} = 0$$

and

$$X_{2g-2} G_{\sigma' j'} + \sum_{q\sigma j} \rho_{q\sigma j}^{(\sigma' j')} X_q G_{\sigma j} - \sum_{mq\sigma j} \mu_{mq\sigma j}^{(\sigma' j')} X_m X_q F_{\sigma j} - \sum_{q\sigma j} \nu_{q\sigma j}^{(\sigma' j')} X_q G_{\sigma j} = 0.$$

Replacing the variables $X_{n_0}, \dots, X_{n_{g-1}}$ by the projective coordinates functions $y_{n_0}, \dots, y_{n_{g-1}}$ we get two systems: a system with $\frac{1}{2}(g-3)(g-4)$ linear homogeneous equations in the $\frac{1}{2}(g-3)(g-4)$ functions $F_{s'i'}(y_{n_0}, \dots, y_{n_{g-1}})$ with the coefficients in the domain $k[[t]]$ of formal power series;

the second system is composed by $\wp - 1$ linear homogeneous equations in the $\wp - 1$ functions $G_{\sigma' j'}(y_{n_0}, \ldots, y_{n_{g-1}})$ with the coefficients in the domain $k[[t]]$ of formal power series. Since the triple indices nsi of the coefficients $\epsilon_{nsi}^{(s'i')}$, respectively, $\eta_{nsi}^{(s'i')}$, satisfy the inequalities $n < 2g - 2$ and $n + s = 2g - 2 + s'$, respectively, $n \leq 2g - 2$ and $n + s < 2g - 2 + s'$, the diagonal entries of the matrix of the system have constant terms 1, while the remaning entries have positive orders. Therefore, the matrix is invertible, and so the equation $F_{si}(y_{n_0}, \ldots, y_{n_{g-1}}) = 0$ holds for each duble index si. In the system second, the indices $q\sigma j, mq\sigma j$ and $n\sigma j$ of the coefficients $\rho_{q\sigma j}^{(\sigma' j')}, \mu_{mq\sigma j}^{(\sigma' j')}$ and $\nu_{n\sigma j}^{(\sigma' j')}$, respectively, are such that satisfy the inequalities $q < 2g - 2$ and $q + \sigma = 2g - 2 + \sigma'$, respectively, $m, q \leq 2g - 2$ and $m + q + \sigma < 2g - 2 + \sigma'$. So the diagonal entries of the matrix of the system have constant terms 1, while the remaining entries have positive orders, hence the matrix is invertible. This means that the equation $G_{\sigma j}(y_{n_0}, \ldots, y_{n_{g-1}}) = 0$ holds for each duble index σj. Therefore, we shown that $I \subset I(\mathcal{C})$, where I is the ideal generated by the $\frac{1}{2}(g - 2)(g - 3)$ quadratic forms F_{si} and by the $\wp$ cubic forms $G_{\sigma j}$.

By the lemma 3.2.6, $\operatorname{codim} I_r \leq \dim \Lambda_r$ for each $r \geq 2$. On the other hand, since $I_r(\mathcal{C}) \cap \Lambda_r = 0$ we deduce $\dim \Lambda_r \leq \operatorname{codim} I_r(\mathcal{C})$. Since $I \subseteq I(\mathcal{C})$, we obtain

$$\operatorname{codim} I_r(\mathcal{C}) = \operatorname{codim} I_r = \dim \Lambda_r = (2g - 2)r + 1 - g.$$

Thus $I(\mathcal{C}) = I$ and the curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ has Hilbert polynomial $(2g - 2)r + 1 - g$. Hence, $\mathcal{C}$ has degree $2g - 2$ and arithmetic genus equal to g.

Intersecting the curve $\mathcal{C}$ with the hyperplane $V(X_{2g-2})$ we obtain the divisor $D := (2g - 2)P$ of degree $2g - 2$, whose complete linear sistem $|D|$ has dimension at least $g - 1$, and so by Riemann-Roch theorem for complete integral (not necessarily smooth) curves the Cartier divisor D is canonical, and $\mathcal{C}$ is a canonical Gorenstein curve. □

Note that the P-hermitian basis $x_{n_0}, x_{n_1}, \ldots, x_{n_{g-1}}$ of $H^0(\mathcal{C}, (2g - 2)P)$ is uniquely determined up to a linear transformation $x_{n_i} \mapsto \sum_{j=i}^{g-1} c_{ij} x_{n_j}$, with $(c_{ij}) \in \mathrm{GL}_g(\mathbf{k})$ a upper triangular matrix whose diagonal entries are of the form $c_{ii} = c^{n_i}, i = 0, \ldots, g - 1$, for some non-zero constant c, because the normalizations $c_{sis} = 1$. We assume that the characteristic of the field of constants $\mathbf{k}$ is zero or a prime not dividing any of the differences $m - n$ with n, m nongaps such that $m < n \leq 2g - 2$. By Changing

$$X_n \mapsto X_n + \sum_{m=0}^{n-1} d_{nm} X_m,$$

where the coefficients d_{nm} are constants, we can normalize $\frac{1}{2}g(g - 1)$ of the coefficients c_{sin} to zero.

Due the normalizations and the normalization of the coefficients of weight zero, the only freedon left to us is to transform $x_{n_i} \mapsto c^{n_i} x_{n_i}, i = 0, \ldots, g - 1$ for some non-zero constant $c \in \mathbf{k}$. Therefore, we have showed the

Theorem 3.2.9. *Let $\mathcal{H}$ be a symmetric semigroup of genus g satisfying $3 < n_1 \leq g$. The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup*

$\mathcal{H}$ correspond bijectively to the orbits of the $\mathbb{G}_m(k)$-action

$$(c,\ldots,c_{sin},\ldots)\longmapsto(\ldots,c^{s-n}c_{sin},\ldots)$$

on the affine quasi-cone of the vectors whose coordinates are the coefficients $c_{sin}, d_{\sigma jn}$ of the normalized quadratic and cubic forms F_{si} and $G_{\sigma j}$ satisfying the quasi-homogeneous equations $\varrho_{s'i'm} = \vartheta_{\sigma' i' r} = 0$.

3.3 Explicit construction and rationality

3.3.1 The trigonal genus 5 case

Let $\mathcal{C}^{(0)}$ be the canonical monomial gorenstein curve of genus 5 associated to the trigonal symmetric semigroup of genus also 5. Up to change of coordinates can we write:

$$\mathcal{C}^{(0)} := \{(a^8 : a^3b^5 : a^2b^6 : a^1b^7 : b^8)\,|\,(a:b)\in\mathbb{P}^1\}\subseteq\mathbb{P}^4\,.$$

The symmetric Weierstrass semigroup of the smooth point $P = (0:0:0:0:1)$ is $\mathcal{H} := \langle 5,6,7,8\rangle$. Following lemma 3.2.3 the ideal of $\mathcal{C}^{(0)}$ can be generated by the following forms

$$\begin{array}{ll}
F_{12}^{(0)} = X_6^2 - X_5X_7 & F_{13}^{(0)} = X_6X_7 - X_5X_8,\\
F_{14}^{(0)} = X_7^2 - X_6X_8 & G_{15}^{(0)} = X_5^3 - X_0X_7X_8,\\
G_{16}^{(0)} = X_5^2X_6 - X_0X_8^2 & G_{18}^{(0)} = X_6^3 - X_5^2X_8,\\
G_{21}^{(0)} = X_7^3 - X_5X_8^2. &
\end{array}$$

Indeed, we have 15 cubic forms $G_{\sigma j}$ but the other eleven cubic forms are multiples of the quadratic ones.

For each nongap $n\in\mathcal{H}$ we have a rational function x_n and then we consider the monomial X_n of weight n. Writing each one of the rational functions $x_6^2, x_6x_7, x_7^2, x_5^3, x_5^2x_6, x_6^3$ and x_7^3 as linear combination of the basis elements of the vector spaces $H^0(\mathcal{C}, 2(2g-2))$ and $H^0(\mathcal{C}, 3(2g-2))$, respectively, we obtain in the variables X_0, X_5, X_6, X_7, X_8, the polynomials

$$F_i = F_i^{(0)} - \sum_{j=1}^{i} c_{ij}Z_{i-j},\ \ (i = 12, 13, 14),$$

and

$$G_i = G_i^{(0)} - \sum_{j=1}^{i} d_{ij}Z_{i-j},\ \ (i = 15, 16, 18, 21),$$

where the summation index j varies only through the integers such that $i - j \in \mathcal{H}$.

We can normalize the following ten coefficients

$$c_{13,1} = c_{13,2} = c_{13,3} = c_{13,8} = c_{12,1} = c_{12,2} = c_{12,7} = d_{16,1} = d_{6,6} = d_{21,5} = 0\,.$$

By applying the Syzygy Lemma 3.2.4 we obtain the following four syzygies of the canonical monomial curve $\mathcal{C}^{(0)}$

$$\begin{aligned}
&X_8F_{12}^{(0)} - X_7F_{13}^{(0)} + X_6F_{14}^{(0)} = 0,\\
&X_8G_{15}^{(0)} - X_5X_6F_{12}^{(0)} + X_5G_{18}^{(0)} - X_7G_{16}^{(0)} = 0,\\
&X_8G_{18}^{(0)} - X_5G_{21}^{(0)} + X_5X_7F_{14}^{(0)} - X_6X_8F_{12}^{(0)} = 0,\\
&X_8G_{21}^{(0)} - X_7X_8F_{14}^{(0)} + X_8^2F_{13}^{(0)} = 0.
\end{aligned}$$

Replacing each left-hand side of the above syzygies the binomials $F_{s,i}^{(0)}, F_{s',i'}^{(0)}, G_{\sigma,j}^{(0)}, G_{\sigma',j'}^{(0)}$ by the quadratic and cubic forms $F_{s,i}, F_{s',i'}, G_{\sigma,j}, G_{\sigma',j'}$, respectively, and applying the division algorithm recursively until all monomials of this equations belongs to basis of Λ_3 or Λ_4, we get the four polynomial equations

$$\begin{aligned}
&F_{12}X_8 - F_{13}X_7 + F_{14}X_6 = F_{12}\left(-c_{14,3}X_5 - c_{14,8}X_0\right) + F_{14}c_{13,6}X_0 - G_{16}c_{14,4}\\
&+F_{13}\left(c_{13,7}X_0 - c_{14,2}X_5 - c_{14,7}X_0\right)
\end{aligned}$$

$$\begin{aligned}
&X_8G_{15} - X_6G_{17} + X_5G_{18} - X_7G_{16} = -d_{18,1}X_0X_8 + c_{12,6}X_0X_5F_{12}\\
&+(-c_{14,3}d_{16,4} - c_{14,3}d_{15,3}d_{18,1} - d_{18,7})X_0G_{16} + (d_{16,5}X_5 + c_{12,5}X_5)X_0F_{13}\\
&+(d_{16,9}X_0 - d_{18,1}X_8 + d_{15,8}d_{18,1}X_0 + d_{15,3}d_{18,1}X_5 + d_{16,4}X_5)X_0F_{14}\\
&+(d_{16,10}X_0 + d_{15,9}d_{18,1}X_0 + d_{15,1}d_{18,1}X_8 + d_{15,4}d_{18,1}X_5 + d_{16,2}X_8)X_0F_{13}\\
&+(-c_{14,4}d_{16,4}X_0 - c_{14,4}d_{15,3}d_{18,1}X_0 - d_{18,1}X_7 - d_{18,8}X_0)G_{15}
\end{aligned}$$

$$\begin{aligned}
&-G_{21,1}X_8 + G_{21,2}X_8 - G_{22}X_7 = X_8(c_{14,3}X_5 + c_{14,8}X_0)F_{13}\\
&+X_8\left[(c_{14,2}X_5 + c_{14,7}X_0)F_{14} - c_{14,2}c_{14,4}G_{15} - c_{14,2}c_{14,3}G_{16}\right],
\end{aligned}$$

$$\begin{aligned}
&+X_8G_{18} - X_5G_{21} - X_6X_8F_{12} + X_7X_5F_{14}\\
&(-c_{14,3}^2d_{16,4} - c_{14,2}c_{14,3}d_{15,5} - c_{14,3}c_{14,4}d_{15,3} + c_{14,3}c_{14,7})G_{16}X_0\\
&+c_{14,3}d_{15,3}d_{14,4} + c_{14,2}c_{14,8} - c_{14,2}c_{14,4}d_{15,4} + c_{14,2}^2c_{14,3}d_{15,3})X_0G_{16}\\
&(+c_{14,4}d_{15,9}X_0 - d_{15,1}c_{14,2}^2X_8 - d_{15,4}d_{14,4}X_5 - c_{14,8}X_5 + c_{12,5}X_8\\
&+c_{14,2}c_{14,3}X_8 + c_{14,3}d_{16,2}X_8 - d_{15,4}c_{14,2}^2X_5 - c_{14,2}c_{14,3}d_{15,8}X_0\\
&+c_{14,4}d_{15,1}X_8 - d_{15,1}d_{14,4}X_8 - d_{15,9}d_{14,4}X_0 + c_{14,3}d_{16,10}X_0 + c_{14,3}d_{16,5}X_5\\
&+c_{14,4}d_{15,4}X_5 - d_{15,9}c_{14,2}^2X_0 - c_{14,2}c_{14,3}d_{15,3}X_5)X_0F_{13}\\
&(+d_{14,11}X_0 + d_{14,4}X_7 + d_{14,3}X_8 - c_{14,4}X_7 + c_{14,2}^2X_7 - c_{14,4}c_{14,3}d_{16,4}X_0\\
&+c_{14,1}c_{14,2}X_8 + d_{15,3}c_{14,2}^2c_{14,4}X_0 - c_{14,4}^2d_{15,3}X_0 + c_{14,4}d_{15,3}d_{14,4}X_0\\
&-c_{14,2}c_{14,4}d_{15,5}X_0 + c_{14,2}c_{14,9}X_0 + c_{14,4}c_{14,7}X_0 + c_{14,2}c_{14,4}X_5 + c_{14,2}c_{14,3}X_6)G_{15}\\
&(c_{14,2}^2X_0X_8 - c_{14,4}X_0X_8 + d_{14,4}X_0X_8 - c_{14,7}X_0X_5 - c_{14,2}X_5^2 + c_{14,3}d_{16,9}X_0^2\\
&+c_{14,4}d_{15,3}X_0X_5 - d_{15,3}d_{14,4}X_0X_5 + c_{14,3}d_{16,4}X_0X_5 - d_{15,8}d_{14,4}X_0^2\\
&+c_{14,4}d_{15,8}X_0^2 - c_{14,2}^2d_{15,3}X_0X_5 - c_{14,2}^2d_{15,8}X_0^2)F_{14} + (d_{14,2}X_8 - c_{14,3}X_7)G_{16}\\
&(+c_{12,6}X_8 - c_{14,2}c_{14,3}d_{15,9}X_0 - c_{14,2}c_{14,3}d_{15,4}X_5 - c_{14,2}c_{14,3}d_{15,1}X_8)X_0F_{12}.
\end{aligned}$$

The vanishing of the coefficients of the four combinations provides us with quasi-homogeneous equations between the coefficients c_{ij} and d_{ij} obtaining an explicit description of the compactified moduli $\overline{\mathscr{M}_{5,1}^{\mathcal{H}}}$ according to the theorem (3.2.9).

We first determine the weighted vector space $T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}$, which is (up to an isomorphism) the locus of the linearizations of the 4 equations, all we have to do is substituting by zero the right hand side of each equation. These four equations give rise to other 20 equations obtained by replacing $X_{n_i} \mapsto t^{n_i}$. We can solve this linear system as follows:

$$d_{16,10} = d_{15,10}, d_{16,9} = d_{15,9}, d_{16,8} = d_{15,8}, c_{14,7} = c_{13,7}, d_{18,7} = c_{13,7}, d_{15,7} = -c_{13,7},$$
$$d_{21,7} = 2c_{13,7}, c_{14,6} = -c_{12,6}, d_{21,6} = -c_{12,6}, d_{18,6} = c_{12,6}, d_{16,5} = d_{15,5},$$
$$c_{14,4} = -c_{12,4}, d_{16,4} = d_{15,4}, d_{21,4} = -c_{12,4}, d_{18,4} = c_{12,4}, d_{16,3} = d_{15,3}, d_{16,2} = d_{15,2}.$$

We can verify that the weighted vector space $T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}$ depends only on the ten coefficients $d_{15,10}, d_{15,9}, d_{15,8}, c_{13,7}, c_{12,6}, d_{15,5}, c_{12,4}, d_{15,2}, d_{15,3}, d_{15,4}$, which implies

$$\dim T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}} = 10.$$

More precisely, counting the coefficients of weight s, we obtain the dimension of the graded component of $T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}$ of negative weight $-s$:

$$\dim T^{1,-}_{s} = 1,\ (s = -10, -9, -8, -7, -6, -5, -3, -2) \text{ and } \dim T^{1,-}_{-4} = 2.$$

For the remainder integers, the dimension of $\dim T^{1,-}_{s}$ is zero. In particular, the compactified moduli space $\overline{\mathscr{M}^{\mathcal{H}}_{5,1}}$ has been realized as closed subspace of the 9-dimensional weighted projective space $\mathbb{P}\left(T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}\right)$.

We can now solve the four polynomial equations to obtain the equations of the moduli variety $\overline{\mathscr{M}^{\mathcal{H}}_{5,1}}$ by replacing $X_{n_i} \mapsto t^{n_i}$ and solving the polynomial equations. The compactified moduli space $\overline{\mathscr{M}^{\mathcal{H}}_{5,1}}$ is cut out by 70 equations which depends on 64 variables and we can solve in the following way:

$$c_{12,5} = 0, c_{13,5} = 0, c_{13,6} = 0, c_{14,1} = 0, c_{14,2} = 0, c_{14,3} = 0, d_{15,1} = 0, d_{16,11} = 0,$$

$$d_{18,1} = 0, d_{18,2} = 0, d_{18,3} = 0, d_{18,5} = 0, d_{18,8} = 0, d_{18,11} = 0, d_{21,1} = 0, d_{21,2} = 0,$$

$$c_{12,12} = -c_{12,4}d_{15,8}, c_{13,13} = c_{12,4}d_{15,9}, c_{14,4} = -c_{12,4}, c_{14,6} = -c_{12,4}d_{15,2} - c_{12,6},$$

$$c_{14,7} = -c_{12,4}d_{15,3} + c_{13,7}, c_{14,8} = -c_{12,4}d_{15,4}, c_{14,9} = -c_{12,4}d_{15,5},$$

$$c_{14,14} = -c_{12,4}d_{15,10}, d_{15,7} = -c_{13,7}, d_{15,15} = c_{12,6}d_{15,9} + c_{13,7}d_{15,8}, d_{16,2} = d_{15,2},$$

$$d_{16,4} = d_{15,4}, d_{16,5} = d_{15,5}, d_{16,9} = d_{15,9}, d_{16,8} = -c_{12,4}d_{15,4} + d_{15,8},$$

$$d_{16,10} = -c_{12,6}d_{15,4} - c_{13,7}d_{15,3} + d_{15,10}, d_{16,16} = -c_{12,4}d_{15,3}d_{15,9} + c_{12,4}d_{15,4}d_{15,8},$$

$$d_{18,4} = c_{12,4}, d_{18,6} = c_{12,6}, d_{18,7} = c_{13,7}, d_{18,12} = -c_{12,4}d_{15,8} - c_{12,6}^2,$$

$$d_{18,18} = c_{12,4}c_{12,6}d_{15,8}, d_{21,4} = -c_{12,4}, d_{21,6} = -c_{12,4}d_{15,2} - c_{12,6}, d_{18,13} = c_{12,4}d_{15,9},$$

$$d_{21,7} = -c_{12,4}d_{15,3} + 2\,c_{13,7}, d_{21,8} = -c_{12,4}d_{15,4}, d_{21,9} = -c_{12,4}d_{15,5},$$

$$d_{21,13} = -c_{12,4}^2 d_{15,2}d_{15,3} - c_{12,4}c_{12,6}d_{15,3} + c_{12,4}c_{13,7}d_{15,2} + c_{12,4}d_{15,9} + c_{12,6}c_{13,7},$$

$$d_{21,14} = -c_{12,4}^2 d_{15,3}^2 + 2\,c_{12,4}c_{13,7}d_{15,3} - c_{12,4}d_{15,10} - c_{13,7}^2, d_{21,10} = 0,$$

$$d_{21,15} = -c_{12,4}^2 d_{15,3}d_{15,4} + 2\,c_{12,4}c_{13,7}d_{15,4}, d_{21,11} = -{c_{12,4}}^2 d_{15,3} + c_{12,4}c_{13,7},$$

$$d_{21,16} = -c_{12,4}^2 d_{15,3}d_{15,5} + c_{12,4}c_{13,7}d_{15,5}, d_{18,10} = -c_{12,4}c_{12,6}, d_{21,3} = 0,$$

$$d_{21,21} = -c_{12,4}^2 d_{15,3}d_{15,10} + c_{12,4}^2 d_{15,4}d_{15,9} + c_{12,4}c_{13,7}d_{15,10}.$$

We note that there are no new conditions in the remaining constants which the vector space $T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}$ depends which means

$$\overline{\mathscr{M}^{\mathcal{H}}_{5,1}} = \mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}).$$

Therefore $\overline{\mathscr{M}^{\mathcal{H}}_{5,1}}$ is a weight projective space of dimension 9.

3.3.2 The trigonal genus 6 case

Let $\mathcal{C}^{(0)}$ be a trigonal canonical monomial gorenstein curve of genus six. Take $P = (0:0:0:0:0:1)$ in $\mathcal{C}^{(0)}$ a smooth point and by lemma (3.2.1) the symmetric Weierstrass semigroup of $\mathcal{C}^{(0)}$ at P is $\mathcal{H} = \langle 6,7,8,9,10\rangle$. Applying the lemma (3.2.3) we obtain that the generators of the ideal of $\mathcal{C}^{(0)}$ are the quadratic and cubic forms

$$\begin{array}{lll}
F^{(0)}_{14} = X_7^2 - X_6X_8 & F^{(0)}_{15} = X_7X_8 - X_6X_9 & F^{(0)}_{16} = X_8^2 - X_6X_{10}, \\
F^{(0)}_{16,1} = X_7X_9 - X_6X_{10} & F^{(0)}_{17} = X_8X_9 - X_7X_{10} & F^{(0)}_{18} = X_9^2 - X_8X_{10}, \\
G^{(0)}_{18} = X_6^3 - X_0X_8X_{10} & G^{(0)}_{19} = X_6^2X_7 - X_0X_9X_{10} & G^{(0)}_{20} = X_6^2X_8 - X_0X_{10}^2, \\
G^{(0)}_{20,1} = X_6X_7^2 - X_0X_{10}^2 & G^{(0)}_{21} = X_7^3 - X_6^2X_9 & G^{(0)}_{22} = X_7^2X_8 - X_6^2X_{10}, \\
G^{(0)}_{26} = X_8X_9^2 - X_6X_{10}^2 & G^{(0)}_{27} = X_9^3 - X_7X_{10}^2. &
\end{array}$$

As in the case of genus five writing each rational function, which corrreponds to the initial monomial of each quadratic and cubic form above, as combination of the elements of the basis of the vector spaces $H^0(\mathcal{C}, 2(2g-2))$ and $H^0(\mathcal{C}, 3(2g-2))$, respectively, we obtain in the variables $X_0, X_6, X_7, X_8, X_9, X_{10}$ the polynomials that vanish identically on the curve $\mathcal{C} \cap \mathbb{A}^5$

$$F_i = F_i^{(0)} - \sum_{j=1}^{i} c_{ij}Z_{i-j},\ (i = 14, \ldots, 18),$$

and

$$G_i = G_i^{(0)} - \sum_{j=1}^{i} d_{ij}Z_{i-j},\ (i = 18, \ldots, 22, 26, 27),$$

and the polynomials $F_{16,1}, G_{20,1}$, where Z_{i-j} is a polynomial of weight $i-j$, whenever $i-j$ is a nongap of $\mathcal{H}$. The freedom to change of coordinates on the variables $X_0, X_6, X_7, X_8, X_9, X_{10}$ allows us to normalize (in increasing weights) 15 coefficients as follows

$$c_{14,1} = c_{15,1} = c_{16,1,1} = d_{18,1} = d_{18,2} = c_{15,2} = c_{16,1,2} = c_{15,3} = 0,$$
$$c_{16,1,3} = c_{16,1,4} = c_{15,6} = c_{14,7} = c_{14,8} = c_{15,9} = c_{16,1,10} = 0.$$

The ten syzygies on the monomial curve $\mathcal{C}^{(0)}$ induced by the Syzygy Lemma (3.2.4) are

$$\begin{aligned}
&X_{10}F_{14}^{(0)} - X_8F_{16,1}^{(0)} + X_7F_{17}^{(0)} = 0,\\
&X_{10}F_{15}^{(0)} - X_9F_{16,1}^{(0)} + X_7F_{18}^{(0)} = 0,\\
&X_{10}F_{16}^{(0)} - X_{10}F_{16,1}^{(0)} - X_9F_{17}^{(0)} + X_8F_{18}^{(0)} = 0,\\
&X_{10}G_{18}^{(0)} - X_8G_{20}^{(0)} + X_6^2F_{16}^{(0)} = 0,\\
&X_{10}G_{19}^{(0)} - X_9G_{20,1}^{(0)} + X_6X_7F_{16,1}^{(0)} = 0,\\
&X_{10}G_{20}^{(0)} - X_{10}G_{20,1}^{(0)} + X_6X_{10}F_{14}^{(0)} = 0,\\
&X_{10}G_{21}^{(0)} - X_7X_{10}F_{14}^{(0)} - X_6X_{10}F_{15}^{(0)} = 0,\\
&X_{10}G_{22}^{(0)} - X_6X_{10}F_{16}^{(0)} - X_8X_{10}F_{14}^{(0)} = 0,\\
&X_{10}G_{26}^{(0)} - X_{10}^2F_{16,1}^{(0)} - X_9X_{10}F_{17}^{(0)} = 0,\\
&X_{10}G_{27}^{(0)} - X_{10}^2F_{17}^{(0)} - X_9X_{10}F_{18}^{(0)} = 0.
\end{aligned}$$

The 10 above syzygies of the monomial curve give rise to 10 polynomial equations. Again, we compute the locus of the linearizations of this ten equations, which is the weighted vector space $T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}$. We have to solve a linear system with 60 equations obtained by change of variables $X_{n_i} \to t^{n_i}$. By solving the system it depends only of the 15 coefficients $d_{18,12}$, $d_{18,11}$, $c_{15,8}$, $c_{16,1,9}$, $c_{16,1,8}$, $c_{15,7}$, $c_{14,6}$, $d_{18,6}$, $d_{18,10}$, $c_{14,5}$, $d_{18,5}$, $c_{14,4}$, $d_{18,4}$, $d_{18,3}$, $c_{14,2}$ and so

$$\dim T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}} = 15.$$

Thus we conclude that the compactified moduli space $\overline{\mathscr{M}^{\mathcal{H}}_{6,1}}$ has been realized as a closed subset of the 14-dimensional weighted projective space $\mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}})$. We already know that this semigroup $\mathcal{H}$ is negatively graded and thus the compactified moduli space $\overline{\mathscr{M}^{\mathcal{H}}_{6,1}}$ has codimension three in $\mathscr{M}_{6,1}$.

By changing the variables $d_{18,i} := b_i$ $(i = 3, 4, 5, 6, 10, 11, 12)$, $c_{14,j} := a_j$ $(j = 2, 4, 5, 6)$, and $c_{16,1,8} := b_8, c_{15,7} := a_7, c_{15,8} := a_8, c_{16,1,9} := a_9$ and making the substitutions $X_{n_i} \to t^{n_i}$ on the syzygies induced by the pre-syzygies of the monomial curve $\mathcal{C}^{(0)}$ we obtain 188 equations, and with a help of the Maple Software we can solve this system. The solution depends only on the 5 polynomial equations, namely

$$\begin{aligned}
&{a_2}^2{a_5}^3 + {a_2}^2{a_5}^2b_5 + a_2a_4{a_5}^2b_3 + a_2a_4a_5b_3b_5 - 4\,a_2{a_5}^2a_7 - 3\,a_2a_5a_7b_5 + b_8b_{11}\\
&+a_2a_5a_8b_4 + a_2a_5a_9b_3 + {a_4}^2a_5a_6 + {a_4}^2a_5b_6 + {a_4}^2a_6b_5 + {a_4}^2b_5b_6 + a_4{a_5}^3 - a_9b_{10}\\
&+2\,a_4{a_5}^2b_5 - 2\,a_4a_5a_7b_3 + a_4a_5{b_5}^2 - a_4a_7b_3b_5 + a_4a_8b_3b_4 + a_4a_9{b_3}^2 - a_2a_5b_{12}\\
&-a_2a_6b_{11} + a_4a_5b_{10} - a_4a_6a_9 - a_4a_9b_6 - a_4b_3b_{12} - a_4b_4b_{11} + a_4b_5b_{10} - {a_5}^2a_9\\
&+4\,a_5{a_7}^2 - a_5a_9b_5 - a_5b_3b_{11} + 2\,{a_7}^2b_5 - 2\,a_7a_8b_4 - 2\,a_7a_9b_3 + 2\,a_7b_{12} - a_8b_{11} = 0,
\end{aligned}$$

$$a_4a_5a_6 - a_2a_5b_8 + a_4a_5b_6 - a_4b_3b_8 + {a_5}^3 + {a_5}^2b_5 + a_4b_{11} + a_5b_{10} + 2\,a_7b_8 = 0,$$

$$-a_2{a_5}^3 - a_2{a_5}^2b_5 - a_4a_5b_8 - a_4b_5b_8 + 2\,{a_5}^2a_7 + a_5a_7b_5 - a_5a_8b_4 - a_5a_9b_3 +$$
$$a_5b_{12} - a_6b_{11} + a_9b_8 = 0,$$

$$2\,a_2a_5a_6 + {a_4}^2a_5 + {a_4}^2b_5 + a_4a_5b_4 + a_4a_6b_3 + {a_5}^2b_3 - a_4a_9 + a_5a_8 - a_5b_8 -$$
$$2\,a_6a_7 = 0,$$

$$a_2a_4{a_5}^2 + a_2a_4a_5b_5 - a_2a_6b_8 - 2\,a_4a_5a_7 - a_4{a_6}^2 - a_4a_6b_6 - a_4a_7b_5 + {b_8}^2$$
$$+a_4a_8b_4 + a_4a_9b_3 - a_4b_4b_8 - {a_5}^2a_6 - a_5a_6b_5 - a_5b_3b_8 - a_4b_{12} - a_6b_{10} - a_8b_8 = 0.$$

Thus the moduli space $\overline{\mathscr{M}^{\mathcal{H}}_{6,1}}$ is a projective variety in $\mathbb{P}^{14}$ give by the zero locus of the above 5 polynomials. Let us see this algebraic set on open affine chart $a_5 = 1$ of $\mathbb{P}^{15}$. Entering with $a_5 = 1$ into the 5 equations we get the only three simple equations

$$b_{10} = a_4b_3b_8 + a_2b_8 - a_4a_6 - a_4b_6 - a_4b_{11} - 2\,a_7b_8 - b_5 - 1$$
$$b_{12} = a_4b_5b_8 + a_2b_5 + a_4b_8 + a_6b_{11} - a_7b_5 + a_8b_4 + a_9b_3 - a_9b_8 + a_2 - 2\,a_7$$
$$b_8 = {a_4}^2b_5 + a_4a_6b_3 + 2\,a_2a_6 + {a_4}^2 - a_4a_9 + a_4b_4 - 2\,a_6a_7 + a_8 + b_3.$$

which gives a local parametrization of $\overline{\mathscr{M}^{\mathcal{H}}_{6,1}}$. Since $\mathscr{M}^{\mathcal{H}}_{6,1}$ is irreducible [Bu, thm 1.1], the moduli variety $\mathscr{M}^{\mathcal{H}}_{6,1}$ is rational of dimension 11.

4 The dimension of moduli spaces of curves with symmetric Weierstrass semigroup

4.1 A family of multiplicity six

In a view what we mentioned in the introduction about the moduli variety induced by symmetric semigroups generated by 4 elements, let us consider a family of symmetric semigroups of multiplicity six generated minimaly by five elements. We also to recall that a symmetric semigroup of multiplicity m can be generated by $m-1$ elements, to see this we just have to consider the Apery sequence. So, for each non negative integer τ set

$$\begin{aligned}\mathcal{H} &= \langle 6, 3+6\tau, 4+6\tau, 7+6\tau, 8+6\tau\rangle \\ &= 6\mathbb{N} \sqcup \bigsqcup_{j\in\{3,4,7,8\}} (j+6\tau+6\mathbb{N}) \sqcup (11+12\tau+6\mathbb{N}).\end{aligned}$$

We will apply the method developed in [CS], pg. 587-590, to obtain an upper bound for the dimension of the moduli space $\mathcal{M}^{\mathcal{H}}_{g,1}$. It consists in looking for the quadratic quasi-cone given by quadratic expressions of the equations of $\mathcal{M}^{\mathcal{H}}_{g,1}$, that contain this moduli space. Is much less expensive to obtain the equations and the dimension of the quadratic quasi-cone than the ones of the moduli variety $\mathcal{M}^{\mathcal{H}}_{g,1}$.

Couting the number of gaps of $\mathcal{H}$ and picking up the largest nongap, we see that

$$g = 3+6\tau \text{ and } l_g = 12\tau + 5 = 2g-1,$$

and so $\mathcal{H}$ is a symmetric semigroup. Let $\mathcal{C}$ be a complete integral Gorenstein curve and P be a smooth point of $\mathcal{C}$ whose Weierstrass semigroup at P is $\mathcal{H}$. For each $n \in \mathcal{H}$, let x_n be a rational function on $\mathcal{C}$ with pole divisor nP. We abbreviate

$$x := x_6 \text{ and } y_j := x_{j+6\tau} (j = 3,4,7,8)$$

and normalize

$$x_{6i} = x^i, x_{j+6\tau+6i} = x^i y_j,$$

for $i \geq 1$. The P-hermitian basis $\{x_{n_0}, x_{n_1}, \ldots, x_{n_{g-1}}\}$ for the vector space $H^0(\mathcal{C}, (2g-2)P)$ of the canonical divisor $(2g-2)P = (12\tau+4)P$ consists of the functions

$$\begin{aligned}&x^0, \ldots, x^{2\tau}, \\ &x^0 y_j, \ldots, x^{\tau} y_j \ (j = 3,4), \\ &x^0 y_j, \ldots, x^{\tau-1} y_j \ (j = 7,8).\end{aligned}$$

Since $l_2 = 2$, the complete integral Gorenstein curve $\mathcal{C}$ is nonhyperelliptic and by the theorem (2.1.4) it can be identified with its image under the canonical embedding

$$j := (x_{n_0} : x_{n_1} : \ldots : x_{n_{g-1}}) : \mathcal{C} \hookrightarrow \mathbb{P}^{g-1}.$$

By consider the normalizations, the projection map

$$(1 : x : y_3 : y_4 : y_7 : y_8) : \mathcal{C} \hookrightarrow \mathbb{P}^5$$

defines an isomorphim of the canonical curve $\mathcal{C}$ onto a curve $\mathcal{D} \subset \mathbb{P}^5$ of degree $6\tau + 5$. Now we will study the quadratic relation of the canonical curve $\mathcal{D}$. For this, we consider a P-hermitian basis of the vector space $H^0(\mathcal{C}, 2(2g-2)P)$ of the bicanonical divisor $24\tau + 8$ which consist of the $3g - 3$ functions

$$\begin{aligned}
&x^i \ (i = 0, 1, \ldots, 4\tau + 1),\\
&x^i y_j \ (i = 0, 1, \ldots, 3\tau, \ j = 3, 4, 7, 8),\\
&x^i y_3 y_8 \ (i = 0, 1, \ldots, 2\tau - 1).
\end{aligned}$$

For each $n \in \mathcal{H}$, having in mind the normalizations of the functions x_n, we define a monomial Z_n as follows

$$Z_{6i} = X^i, \ Z_{j+6\tau+6i} = Y_j X^i \text{ and } Z_{11+12\tau+6i} = Y_3 Y_8 X^i.$$

Let X, Y_3, Y_4, Y_7, Y_8 be the indeterminates whose weight we attached $6, 3+6\tau, 4+6\tau, 7+6\tau, 8+6\tau$, respectively. By writing the nine products $y_i y_j, (i, j) \neq (3, 8)$ as linear combination of the basis elements we obtain polynomials in the indeterminates X, Y_3, Y_4, Y_7, Y_8 that vanish identically on the affine curve $\mathcal{D} \cap \mathbb{A}^5$, say

$$\begin{aligned}
F_i &= F_i^{(0)} + \sum_{j=0}^{12\tau+i} f_{ij} Z_{12\tau+i-j} \quad (i = 6, 7, 11, 12, 14, 15)\\
G_i &= G_i^{(0)} + \sum_{j=0}^{12\tau+i} g_{ij} Z_{12\tau+i-j} \qquad (i = 8, 10, 16),
\end{aligned}$$

where

$$\begin{array}{lll}
F_6^{(0)} = Y_3^2 - X^{2\tau+1} & F_7^{(0)} = Y_3 Y_4 - X^\tau Y_7 & G_8^{(0)} = Y_4^2 - X^\tau Y_8,\\
G_{10}^{(0)} = Y_3 Y_7 - X^{\tau+1} Y_4 & F_{11}^{(0)} = Y_4 Y_7 - Y_3 Y_8 & F_{12}^{(0)} = Y_4 Y_8 - X^{2\tau+2},\\
F_{14}^{(0)} = Y_7^2 - X^{\tau+1} Y_8 & F_{15}^{(0)} = Y_7 Y_8 - X^{\tau+2} Y_3 & G_{16}^{(0)} = Y_8^2 - X^{\tau+2} Y_4,
\end{array}$$

and where the index j only varies through integers with $12\tau + i - j \in \mathcal{H}$.

Lemma 4.1.1. *If we denote $\mathcal{I}$ be the ideal generated by the quadratic forms F_i $(i = 6, 7, 11, 12, 14, 15)$ and G_i $(i = 8, 10, 16)$ then the ideal of the affine curve $\mathcal{D} \cap \mathbb{A}^5$ is equal to $\mathcal{I}$.*

Proof. Given a polynomial f in the variables X, Y_3, Y_4, Y_7, Y_8 we apply induction on degree in Y_3, Y_4, Y_7, Y_8 and we show that, module the ideal generated by the nine quadratic forms F_i, G_i, the monomials of this polynomial f are not divisible by the nine products $Y_i Y_j, (i, j) \neq (3, 8)$ and so the class of f is a sum $\sum c_n Z_n$ of monomials Z_n of pairwise different weight with $n \in \mathcal{H}$ and $c_n \in \mathbf{k}$. Thus the polynomial f belongs to the ideal of the curve $\mathcal{D} \cap \mathbb{A}^5$ if and only if the linear combination $\sum c_n Z_n$ vanishes identically on the curve $\mathcal{D} \cap \mathbb{A}^5$ and by taking the corresponding linear combination $\sum c_n x_n$ of rational functions on $\mathbf{k}(\mathcal{C})$ we have $c_n = 0$ for each $n \in \mathcal{H}$, hence f belongs to $\mathcal{I}$. □

We can introduce a more appropriate notation for the constants f_{ij}, g_{ij} with the wish to help in the moment of normalize this coefficients as follows

$$
\begin{aligned}
F_6 = {} & Y_3^2 - X^{2\tau+1} - \\
& - \sum_{i=0}^{\tau} f_{6,2+6i} X^{\tau-i} Y_4 - \\
& - \sum_{i=0}^{\tau} f_{6,3+6i} X^{\tau-i} Y_3 - \\
& - \sum_{i=0}^{\tau-1} f_{6,4+6i} X^{\tau-1-i} Y_8 - \\
& - \sum_{i=0}^{\tau-1} f_{6,5+6i} X^{\tau-1-i} Y_7 - \\
& - \sum_{i=0}^{2\tau} f_{6,6+6i} X^{2\tau-i}
\end{aligned}
\qquad
\begin{aligned}
F_7 = {} & Y_3 Y_4 - X^{\tau} Y_7 - \\
& - \sum_{i=0}^{2\tau+1} f_{7,1+6i} X^{2\tau+1-i} - \\
& - \sum_{i=0}^{\tau} f_{7,3+6i} X^{\tau-i} Y_4 - \\
& - \sum_{i=0}^{\tau} f_{7,4+6i} X^{\tau-i} Y_3 - \\
& - \sum_{i=0}^{\tau-1} f_{7,5+6i} X^{\tau-1-i} Y_8 - \\
& - \sum_{i=0}^{\tau-1} f_{7,6+6i} X^{\tau-1-i} Y_7
\end{aligned}
\qquad
\begin{aligned}
G_8 = {} & Y_4^2 - X^{\tau} Y_8 - \\
& - \sum_{i=0}^{\tau} g_{8,1+6i} X^{\tau-i} Y_7 - \\
& - \sum_{i=0}^{2\tau+1} g_{8,2+6i} X^{2\tau+1-i} - \\
& - \sum_{i=0}^{\tau} g_{8,4+6i} X^{\tau-i} Y_4 - \\
& - \sum_{i=0}^{\tau} g_{8,5+6i} X^{\tau-i} Y_3 - \\
& - \sum_{i=0}^{\tau-1} g_{8,6+6i} X^{\tau-1-i} Y_8
\end{aligned}
$$

$$
\begin{aligned}
G_{10} = {} & Y_3 Y_7 - X^{\tau+1} Y_4 - \\
& - \sum_{i=0}^{\tau+1} g_{10,1+6i} X^{\tau+1-i} Y_3 - \\
& - \sum_{i=0}^{\tau} g_{10,2+6i} X^{\tau-i} Y_8 - \\
& - \sum_{i=0}^{\tau} g_{10,3+6i} X^{\tau-i} Y_7 \\
& - \sum_{i=0}^{2\tau+1} g_{10,4+6i} X^{2\tau+1-i} - \\
& - \sum_{i=0}^{\tau} g_{10,6+6i} X^{\tau-i} Y_4
\end{aligned}
\qquad
\begin{aligned}
F_{11} = {} & Y_4 Y_7 - Y_3 Y_8 - \\
& - \sum_{i=0}^{\tau+1} f_{11,1+6i} X^{\tau+1-i} Y_4 - \\
& - \sum_{i=0}^{\tau+1} f_{11,2+6i} X^{\tau+1-i} Y_3 - \\
& - \sum_{i=0}^{\tau} f_{11,3+6i} X^{\tau-i} Y_8 - \\
& - \sum_{i=0}^{\tau} f_{11,4+6i} X^{\tau-i} Y_7 - \\
& - \sum_{i=0}^{2\tau+1} f_{11,5+6i} X^{2\tau+1-i}
\end{aligned}
\qquad
\begin{aligned}
F_{12} = {} & Y_4 Y_8 - X^{2\tau+2} - f_{12,1} Y_3 Y_8 - \\
& - \sum_{i=0}^{\tau+1} f_{12,2+6i} X^{\tau+1-i} Y_4 - \\
& - \sum_{i=0}^{\tau+1} f_{12,3+6i} X^{\tau+1-i} Y_3 - \\
& - \sum_{i=0}^{\tau} f_{12,4+6i} X^{\tau-i} Y_8 - \\
& - \sum_{i=0}^{\tau} f_{12,5+6i} X^{\tau-i} Y_7 - \\
& - \sum_{i=0}^{2\tau+1} f_{12,6+6i} X^{2\tau+1-i}
\end{aligned}
$$

$$
\begin{aligned}
F_{14} = {} & Y_7^2 - X^{\tau+1} Y_8 \\
& - \sum_{i=0}^{\tau+1} f_{14,1+6i} X^{\tau+1-i} Y_7 - \\
& - \sum_{i=0}^{2\tau+2} f_{14,2+6i} X^{2\tau+2-i} - \\
& - \sum_{i=0}^{\tau+1} f_{14,4+6i} X^{\tau+1-i} Y_4 - \\
& - \sum_{i=0}^{\tau+1} f_{14,5+6i} X^{\tau+1-i} Y_3 - \\
& - \sum_{i=0}^{\tau} f_{14,6+6i} X^{\tau-i} Y_8 \\
& - f_{14,3} Y_3 Y_8
\end{aligned}
\qquad
\begin{aligned}
F_{15} = {} & Y_7 Y_8 - X^{\tau+2} Y_3 \\
& - \sum_{i=0}^{\tau+1} f_{15,1+6i} X^{\tau+1-i} Y_8 - \\
& - \sum_{i=0}^{\tau+1} f_{15,2+6i} X^{\tau+1-i} Y_7 - \\
& - \sum_{i=0}^{2\tau+2} f_{15,3+6i} X^{2\tau+2-i} - \\
& - \sum_{i=0}^{\tau+1} f_{15,5+6i} X^{\tau+1-i} Y_4 - \\
& - \sum_{i=0}^{\tau+1} f_{15,6+6i} X^{\tau+1-i} Y_3 \\
& - f_{15,4} Y_3 Y_8
\end{aligned}
\qquad
\begin{aligned}
G_{16} = {} & Y_8^2 - X^{\tau+2} Y_4 \\
& - \sum_{i=0}^{\tau+2} g_{16,1+6i} X^{\tau+2-i} Y_3 - \\
& - \sum_{i=0}^{\tau+1} g_{16,2+6i} X^{\tau+1-i} Y_8 - \\
& - \sum_{i=0}^{\tau+1} g_{16,3+6i} X^{\tau+1-i} Y_7 - \\
& - \sum_{i=0}^{2\tau+2} g_{16,4+6i} X^{2\tau+2-i} - \\
& - \sum_{i=0}^{\tau+1} g_{16,6+6i} X^{\tau+1-i} Y_4 \\
& - g_{16,5} Y_3 Y_8 .
\end{aligned}
$$

For normalize some constants of the f_{ij}, g_{ij}, we assume that the characteristic of the field of contants is zero. We observe that the rational functions $x_n, n \leq 2g-2$ nongap, are not uniquely determined by their pole divisor nP, instead we have just the follows freedon to transform

$$\begin{aligned}
&x \mapsto x + c_6\\
&y_3 \mapsto y_3 + \sum_{i=0}^{\tau} c_{3+6\tau} x^{\tau-i}\\
&y_4 \mapsto y_4 + c_1 y_3 + \sum_{i=0}^{\tau} c_{4+6\tau} x^{\tau-i}\\
&y_7 \mapsto y_7 + c_3 y_4 + c_4 y_3 + \sum_{i=0}^{\tau+1} c_{1+6\tau} x^{\tau+1-i}\\
&y_8 \mapsto y_8 + c_1' y_7 + c_4' y_4 + c_5 y_3 + \sum_{i=0}^{\tau+1} c_{2+6\tau} x^{\tau+1-i},
\end{aligned}$$

where $c_1, c_1', c_3, c_4, c_4', c_5$ and c_6 are constants with weight $1, 3, 4, 5$ and 6, respectively. By making this change, we normalize the only coefficients with $i - j \equiv 5 \mod 6$

$$f_{12,1} = 0,\ f_{14,3} = 0,\ f_{15,4} = 0,\ g_{16,5} = 0,$$

and besides these

$$f_{7,3+6i} = f_{11,4+6i} = 0\ (i = 0, \ldots, \tau),$$

$$g_{10,1+6i} = f_{11,2+6i} = 0\ (i = 0, \ldots, \tau + 1).$$

and

$$g_{8,1} = 0, g_{8,4} = 0, g_{16,6} = 0.$$

Due the normalizations of these constants and the ones such that $c_{sir} = 1$, the only freedon left us is to transform $x_{n_i} \mapsto c^{n_i} x_{n_i} (i = 1, \ldots, g-1)$, where $c \in \mathbf{k}^* = \mathbb{G}_m(\mathbf{k})$. By theorem (3.2.9) the isomorphism classes of pointed Gorenstein curves $(\mathcal{C}, P)$ determine uniquely the coefficients up to $\mathbb{G}_m(\mathbf{k})$-action

$$g_{ij} \mapsto c^j g_{ij} \text{ e } f_{ij} \mapsto c^j f_{ij},$$

where $c \in \mathbf{k}^*$. We attach to constants f_{ij}, g_{ij} the weight j, and applying the syzygy lemma we get the six syzygies of the monomial curve $\mathcal{D}^{(0)} \cap \mathbb{A}^5$

$$\begin{aligned}
&Y_4 F_6^{(0)} - Y_3 F_7^{(0)} + X^\tau G_{10}^{(0)} = 0\\
&X Y_4 F_7^{(0)} - Y_7 G_{10}^{(0)} + Y_3 F_{14}^{(0)} - X Y_3 G_8^{(0)} = 0\\
&Y_4 F_{11}^{(0)} - Y_7 G_8^{(0)} + Y_8 F_7^{(0)} = 0\\
&Y_4 F_{12}^{(0)} - Y_8 G_8^{(0)} - X^\tau G_{16}^{(0)} = 0\\
&Y_4 F_{14}^{(0)} - Y_8 G_{10}^{(0)} - Y_7 F_{11}^{(0)} = 0\\
&Y_4 F_{15}^{(0)} - Y_8 F_{11}^{(0)} - Y_3 G_{16}^{(0)} = 0.
\end{aligned}$$

The six syzygies of the affine monomial curve $\mathcal{D}^{(0)} \cap \mathbb{A}^5$ give rise to six syzygies of the curve $\mathcal{D} \cap \mathbb{A}^5$:

$$Y_4F_6 - Y_3F_7 + X^\tau G_{10} = -\sum_{i=0}^{\tau-1} X^{\tau-1-i}\left(f_{6,4+6i}F_{12} + f_{6,5+6i}F_{11} - f_{7,6+6i}G_{10}\right)$$
$$-\sum_{i=0}^{\tau} X^{\tau-i}\left(f_{6,2+6i}G_8 + (f_{6,3+6i} - f_{7,3+6i})F_7 - f_{7,4+6i}F_6\right),$$

$$XY_4F_7 - Y_7G_{10} + Y_3F_{14} - XY_3G_8 = -\sum_{i=0}^{\tau-1} X^{\tau-i}\left(f_{7,5+6i}F_{12} + f_{7,6+6i}F_{11}\right)$$
$$-\sum_{i=0}^{\tau+1} X^{\tau+1-i}\left(f_{14,1+6i}G_{10} + f_{14,4+6i}F_7 + f_{14,5+6i}F_6\right)$$
$$+\sum_{i=0}^{\tau} X^{\tau-i}(g_{8,1+6i}XG_{10} + g_{10,2+6i}F_{15} + g_{10,3+6i}F_{14})$$
$$+\sum_{i=0}^{\tau} X^{\tau-i}(g_{8,5+6i}XF_6 + (g_{8,4+6i} - f_{7,4+6i})XF_7 + g_{10,6+6i}F_{11}),$$

$$Y_4F_{11} - Y_7G_8 + Y_8F_7 = -\sum_{i=0}^{\tau+1} X^{\tau+1-i}(f_{11,1+6i}G_8 + f_{11,2+6i}F_7)$$
$$-\sum_{i=0}^{\tau-1} X^{\tau-1-i}(f_{7,5+6i}G_{16} + (f_{7,6+6i} - g_{8,6+6i})F_{15})$$
$$-\sum_{i=0}^{\tau} X^{\tau-i}(-g_{8,1+6i}F_{14} + (f_{11,3+6i} + f_{7,3+6i})F_{12} + (f_{11,4+6i} - g_{8,4+6i})F_{11} - g_{8,5+6i}G_{10}),$$

$$Y_4F_{12} - Y_8G_8 - X^\tau G_{16} =$$
$$-\sum_{i=0}^{\tau} X^{\tau-i}\left(-g_{8,1+6i}F_{15} + (f_{12,4+6i} - g_{8,4+6i})F_{12} + f_{12,5+6i}F_{11}\right)$$
$$+\sum_{i=0}^{\tau-1} g_{8,6+6i}X^{\tau-1-i}G_{10} - \sum_{i=0}^{\tau+1} X^{\tau+1-i}\left(f_{12,2+6i}G_8 + f_{12,3+6i}F_7\right),$$

$$Y_4F_{14} - Y_8G_{10} - Y_7F_{11} = \sum_{i=0}^{\tau} X^{\tau-i}(g_{10,6+6i} - f_{14,6+6i})F_{12})$$
$$\sum_{i=0}^{\tau} X^{\tau-i}\left(g_{10,2+6i}G_{16} + (g_{10,3+6i} + f_{11,3+6i})F_{15} + f_{11,4+6i}F_{14}\right)$$
$$-\sum_{i=0}^{\tau+1} X^{\tau+1-i}\left(f_{14,1+6i} - f_{11,1+6i})F_{11} - f_{11,2+6i}G_{10} + f_{14,4+6i}G_8 + f_{14,5+6i}F_7\right),$$

$$Y_4F_{15} - Y_8F_{11} - Y_3G_{16} = \sum_{i=0}^{\tau} X^{\tau-i}\left(f_{11,3+6i}G_{16} + f_{11,4+4i}F_{15}\right)$$
$$-\sum_{i=0}^{\tau+1} X^{\tau+1-i}\left((f_{15,1+6i} - f_{11,1+6i})F_{12} + (f_{15,6+6i} - g_{16,6+6i})\,F_7\right)$$
$$-\sum_{i=0}^{\tau+1} X^{\tau+1-i}(f_{15,5+6i}G_8 - g_{16,3+6i}G_{10} + f_{15,2+6i}F_{11})$$
$$+\sum_{i=0}^{\tau+2} g_{16,1+6i}X^{\tau+2-i}F_6.$$

We observe that each right-hand side differs from the corresponding left-hand side by a linear combination of elements of the vector space Λ_3, which are lifting of the elements of the

P-basis of $H^0(\mathcal{C}, 3(2g-2)P)$ that vanish identically on the curve $\mathcal{D} \cap \mathbb{A}^5$, hence is identically zero. The vanishing of the coefficients of the six linear combinations gives the homogeneous equations between the coefficients f_{ij} and g_{ij}. For express these equations in a concise manner we introduce the polynomials in only one variable

$$f_i := \sum_{r=1}^{12\tau+i} F_i(t^{-6}, t^{-6-3\tau}, t^{-6-4\tau}, t^{-6-7\tau}, t^{-8-3\tau})t^{i+12\tau} \ (i = 6, 7, 11, 12, 14, 15),$$

and we write each one as the sum of its partial polynomials

$$f_i^{(j)} = \sum_{r \equiv j \mod 6} f_{ir}t^r, \ (j = 1, \ldots, 6),$$

where are defined by collecting every terms whose exponents are in the same residue class module 6. Analogously we define the polynomials g_j and the partial polynomials $g_i^{(j)}$. Due the normalizations in the constants f_{ij}, g_{ij}, we may express each g_j and f_j in terms of 41 partial polynomials as follows

$$\begin{array}{rlrl} f_6 = & f_6^{(2)} + f_6^{(3)} + f_6^{(4)} + f_6^{(5)} + f_6^{(6)}, & f_7 = & f_7^{(1)} + f_7^{(4)} + f_7^{(5)} + f_7^{(6)}, \\ g_8 = & g_8^{(1)} + g_8^{(2)} + g_8^{(4)} + g_8^{(5)} + g_8^{(6)}, & g_{10} = & g_{10}^{(2)} + g_{10}^{(3)} + g_{10}^{(4)} + g_{10}^{(6)}, \\ f_{15} = & f_{15}^{(1)} + f_{15}^{(2)} + f_{15}^{(3)} + f_{15}^{(5)} + f_{15}^{(6)}, & f_{11} = & f_{11}^{(1)} + f_{11}^{(3)} + f_{11}^{(5)}, \\ f_{12} = & f_{12}^{(2)} + f_{12}^{(3)} + f_{12}^{(4)} + f_{12}^{(5)} + f_{12}^{(6)} & f_{14} = & f_{14}^{(1)} + f_{14}^{(2)} + f_{14}^{(4)} + f_{14}^{(5)} + f_{14}^{(6)}, \\ g_{16} = & g_{16}^{(1)} + g_{16}^{(2)} + g_{16}^{(3)} + g_{16}^{(4)} + g_{16}^{(6)}. & & \end{array}$$

We see that the formal degree of the partial polynomials with $i = j$ and $i - j = 6$ that is $f_6^{(6)}$, and $f_7^{(1)}, g_8^{(2)}, g_{10}^{(4)}, f_{11}^{(5)}, f_{12}^{(6)}$ and $f_{12}^{(4)}, f_{15}^{(3)}, g_{16}^{(4)}$ is $i + 12\tau$. The partial polynomials $f_6^{(4)}, f_6^{(5)}, f_7^{(5)}, f_7^{(6)}, g_8^{(1)}$ and $g_{16}^{(1)}$ have formal degree $j + 6(\tau - 1)$, and $13 + 6\tau$, respectively. Of the remaining 26 polynomials, 13 these partial polynomials have formal degree $j + 6\tau$ and the other 13 partial palynomials have formal degree $j + 6(\tau + 1)$. Therefore, the number of the coefficients that are still involved is equal to

$$(2\tau + 1) + 5(2\tau + 2) + 3(2\tau + 3) + 6\tau + 3 + 13(\tau + 1) + 13(\tau + 2) - 3 = 50\tau + 59,$$

where the subtraction by three corresponds to the normalizations $g_{8,1} = g_{8,4} = g_{16,6} = 0$. By applying the theorem (3.2.9) we find an explicit construction of the compactified moduli space $\overline{\mathscr{M}_{g,1}^{\mathcal{H}}}$.

Theorem 4.1.2. *Let $\mathcal{H}$ be the semigroup generated by $6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau$ and $8 + 6\tau$ where τ is a positive integer. The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $\mathcal{H}$ correspond bijectively to the orbtis of the $\mathbb{G}_m$-action on the quasi-cone of the vectors of length $50\tau + 59$ whose coordinates are the coefficients g_{ij}, f_{ij} of the 41 partial polynomials that satisfy the six equations:*

$$\begin{aligned}
f_6 - f_7 + g_{10} = \quad & -f_6^{(2)} g_8 - (f_6^{(3)} - f_7^{(3)}) f_7 + f_7^{(4)} f_6 - f_6^{(4)} f_{12} \\
& -f_6^{(5)} f_{11} + f_7^{(6)} g_{10}, \\
f_7 - g_{10} + f_{14} - g_8 \quad & = (g_8^{(1)} - f_{14}^{(1)}) g_{10} + g_{10}^{(2)} f_{15} + g_{10}^{(3)} f_{14} + (g_{10}^{(6)} - f_7^{(6)}) f_{11} \\
& +(g_8^{(4)} - f_7^{(4)} - f_{14}^{(4)}) f_7 + (g_8^{(5)} - f_{14}^{(5)}) f_6 - f_7^{(5)} f_{12}, \\
f_{11} - g_8 + f_7 = \quad & g_8^{(5)} g_{10} - f_{11}^{(1)} g_8 - f_{11}^{(2)} f_7 - f_7^{(5)} g_{16} + (g_8^{(6)} - f_7^{(6)}) f_{15} \\
& +g_8^{(1)} f_{14} - (f_{11}^{(3)} + f_7^{(3)}) f_{12} - (f_{11}^{(4)} - g_8^{(4)}) f_{11}, \\
f_{12} - g_8 - g_{16} = \quad & g_8^{(1)} f_{15} - (f_{12}^{(4)} - g_8^{(4)}) f_{12} - f_{12}^{(5)} f_{11} + g_8^{(6)} g_{10} \\
& -f_{12}^{(2)} g_8 - f_{12}^{(3)} f_7, \\
f_{14} - g_{10} - f_{11} = \quad & (f_{11}^{(1)} - f_{14}^{(1)}) f_{11} + f_{11}^{(2)} g_{10} - f_{14}^{(4)} g_8 - f_{14}^{(5)} f_7 + g_{10}^{(2)} g_{16} \\
& +(g_{10}^{(3)} + f_{11}^{(3)}) f_{15} + f_{11}^{(4)} f_{14} - (f_{14}^{(6)} - g_{10}^{(6)}) f_{12}, \\
f_{15} - f_{11} - g_{16} = \quad & (f_{11}^{(1)} - f_{15}^{(1)}) f_{12} - (f_{15}^{(6)} - g_{16}^{(6)}) f_7 + g_{16}^{(3)} g_{10} - f_{15}^{(5)} g_8 \\
& -f_{15}^{(2)} f_{11} + g_{16}^{(1)} f_6 + f_{11}^{(3)} g_{16} + f_{11}^{(4)} f_{15}.
\end{aligned}$$

This means that the compactified moduli space $\overline{\mathscr{M}_{g,1}^{\mathcal{H}}}$ can be embedded into a weighted projective space of dimensional $50\tau + 58$. Now the key is diminish the dimension of the ambient space by projecting this space onto space of lower dimension. Initially, we take the six equations of the moduli space given by theorem (4.1.2) and rewritten this equations in terms of 36 polynomial equations between 41 partial polynomials. Among this equations, there are six linear equations between the partial polynomials

$$f_7^{(5)} = f_6^{(5)}, f_{14}^{(5)} = g_8^{(5)} - f_6^{(5)}, g_8^{(4)} = f_7^{(4)}, f_{12}^{(5)} = g_8^{(5)}, f_{14}^{(1)} = f_{11}^{(1)}, g_{16}^{(1)} = f_{15}^{(1)} - f_{11}^{(1)}.$$

With this normalizations we diminish the dimension of the ambient space to $44\tau + 50$. By analizing the formal degree in the remaining 30 equations we can eliminate more partial polynomials, until the remaining quasi-homogeneuos equations do not admit linear terms. However, this procedure is very long. As seen in the method developed by A. Contiero and Sthör in [CS], we will calculate in an explicity way the equations of the quadratic quasi-cone $\mathfrak{Q}_{\mathcal{H}}$ and its dimension.

First we determine the vector space $T^{1,-}_{\mathbf{k}[\mathcal{H}]\mathbf{k}}$ which is, up to an isomorphism, the locus of the linearizations of the 36 equations between the partial polynomials. Solving this system we obtain

$$\begin{aligned}
& f_7^{(1)} = f_{15}^{(1)} = 0,\ f_{11}^{(1)} = g_8^{(1)},\ f_{14}^{(1)} = g_8^{(1)},\ g_{16}^{(1)} = -g_8^{(1)}; \\
& g_8^{(2)} = 0,\ g_{10}^{(2)} = f_6^{(2)},\ f_{14}^{(2)} = f_6^{(2)},\ f_{15}^{(2)} = f_{12}^{(2)},\ g_{16}^{(2)} = f_{12}^{(2)}; \\
& f_6^{(3)} = f_{11}^{(3)} = g_{10}^{(3)} = 0,\ f_{15}^{(3)} = f_{12}^{(3)}, g_{16}^{(3)} = f_{12}^{(3)}; \\
& g_{16}^{(4)} = 0,\ f_7^{(4)} = g_8^{(4)},\ g_{10}^{(4)} = f_6^{(4)} - g_8^{(4)},\ f_{12}^{(4)} = g_8^{(4)},\ f_{14}^{(4)} = f_6^{(4)} - g_8^{(4)}; \\
& f_7^{(5)} = f_6^{(5)},\ f_{14}^{(5)} = f_{11}^{(5)} = -f_6^{(5)} + g_8^{(5)},\ f_{12}^{(5)} = g_8^{(5)},\ f_{15}^{(5)} = -f_6^{(5)} + g_8^{(5)}; \\
& f_6^{(6)} = f_7^{(6)} + g_{10}^{(6)},\ g_8^{(6)} = f_7^{(6)},\ f_{12}^{(6)} = f_7^{(6)} + g_{16}^{(6)},\ f_{14}^{(6)} = g_{10}^{(6)},\ f_{15}^{(6)} = g_{16}^{(6)}.
\end{aligned}$$

Thus we conclude that the vector space $T^{1,-}_{\mathbf{k}[\mathcal{H}]\mathbf{k}}$ can be identified with the space which entries are the coefficients of the remaining partial polynomials

$$g_8^{(1)},\ f_6^{(2)},\ f_{12}^{(2)},\ f_{12}^{(3)},\ g_8^{(4)},\ f_6^{(4)},\ f_6^{(5)},\ g_8^{(5)},\ f_7^{(6)},\ g_{10}^{(6)},\ g_{16}^{(6)}.$$

By couting the coefficients of this partial polynomials we have $11\tau + 11$ coefficients, and discounting the conditions corresponding to the three normalizations

$$g_{8,1} = g_{8,4} = g_{16,6} = 0,$$

we obtain

$$\dim T^{1,-}_{\mathbf{k}[\mathcal{H}]\mathbf{k}} = 11\tau + 8.$$

We conclude that the compactified moduli space $\overline{\mathscr{M}^{\mathcal{H}}_{g,1}}$ has been realized as a closed subspace of the $11\tau + 8$-dimensional weighted projective space $\mathbb{P}(T^{1,-}_{\mathbf{k}[\mathcal{H}]\mathbf{k}})$.

To determine the quadratic quasi-cone $\mathfrak{Q}_{\mathcal{H}}$ is sufficient to enter with the solutions of the system of 36 linear equations in the quadratic terms of the 36 original equations of degree at most 2 and eliminate the same partial polynomials that the linear case. Thus the equations of the quadratic quasi-cone $\mathfrak{Q}_{\mathcal{H}}$ are the ones whose left hand-side degree is less than the right hand-side degree. So we obtain the five equations

$$\begin{array}{rcl}
f_{14}^{(1)} &=& -f_6^{(4)} f_{12}^{(3)} - f_6^{(5)} f_{12}^{(2)} + (g_{10}^{(6)} - f_7^{(6)}) g_8^{(1)} + g_8^{(1)} \\
g_{16}^{(1)} &=& g_8^{(1)} (f_7^{(6)} - g_{16}^{(6)}) + f_{12}^{(2)} g_8^{(5)} + f_{12}^{(3)} g_8^{(4)} - g_8^{(1)} \\
f_{11}^{(3)} &=& f_6^{(2)} g_8^{(1)} + f_6^{(4)} g_8^{(5)} - f_6^{(5)} g_8^{(4)} \\
f_{14}^{(4)} &=& f_6^{(2)} f_{12}^{(2)} - f_6^{(4)} (f_7^{(6)} - g_{16}^{(6)}) - g_8^{(4)} (g_{10}^{(6)} - f_7^{(6)}) + f_6^{(4)} - g_8^{(4)} \\
f_{14}^{(5)} &=& f_6^{(2)} f_{12}^{(3)} + f_6^{(5)} (f_7^{(6)} - g_{16}^{(6)}) + g_8^{(5)} (g_{10}^{(6)} - f_7^{(6)}) - f_6^{(5)} + g_8^{(5)}.
\end{array}$$

This means that the quadratic quasi-cone $\mathfrak{Q}_{\mathcal{H}}$ is a subvariety of $T^{1,-}_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}$ whose equations are

$$\begin{array}{l}
\pi_{7+6\tau}(-f_6^{(4)} f_{12}^{(3)} - f_6^{(5)} f_{12}^{(2)} + \tilde{g}_{10}^{(6)} g_8^{(1)}) = 0 \\
\pi_{13+6\tau}(-g_8^{(1)} \tilde{g}_{16}^{(6)} + f_{12}^{(2)} g_8^{(5)} + f_{12}^{(3)} g_8^{(4)}) = 0 \\
\pi_{3+6\tau}(f_6^{(2)} g_8^{(1)} + f_6^{(4)} g_8^{(5)} - f_6^{(5)} g_8^{(4)}) = 0 \\
\pi_{10+6\tau}(f_6^{(2)} f_{12}^{(2)} + f_6^{(4)} \tilde{g}_{16}^{(6)} - g_8^{(4)} \tilde{g}_{10}^{(6)}) = 0 \\
\pi_{11+6\tau}(f_6^{(2)} f_{12}^{(3)} - f_6^{(5)} \tilde{g}_{16}^{(6)} + g_8^{(5)} \tilde{g}_{10}^{(6)}) = 0,
\end{array}$$

where $\tilde{g}_{10}^{(6)} = g_{10}^{(6)} - f_7^{(6)}, \tilde{g}_{16}^{(6)} = g_{16}^{(6)} - f_7^{(6)}$ and π_i denotes the projection operator in t that annihilates the terms of degree not large than i. We note that the congruences above does not depend of the coefficients

$$f_{6,2}, f_{12,2}, f_{12,3}, g_{8,5}, \tilde{g}_{10,6}, \tilde{g}_{16,6}, f_{12,8}, f_{12,9}, \tilde{g}_{16,12} \text{ and } f_{7,6i}, i = 1, \ldots, \tau - 1.$$

These congruences depend only 10τ coefficients. They can be expressed in five equations between ten elements of the τ-dimensional artinian algebra

$$A := k[\epsilon] = \bigoplus_{j=0}^{\tau-1} k\epsilon^j, \text{ where } \epsilon^\tau = 0.$$

Theorem 4.1.3. *The quadratic quasi-cone $\mathfrak{Q}_{\mathcal{H}}$ is isomorphic to the direct product*

$$\mathfrak{Q}_{\mathcal{H}} = M \times N,$$

where M is the $(\tau+8)$-dimensional weighted space of weights $2,2,3,5,6,6,8,9,12$ and $6i, i = 1,\dots,\tau-1$, and N is the quadratic quasi-cone consisting of vectors

$$(\omega_1,\dots,\omega_{10}) = \left(\sum_{j=0}^{\tau-1}\omega_{1j}\epsilon^j,\dots,\sum_{j=0}^{\tau-1}\omega_{10,j}\epsilon^j\right),$$

such that satisfying the five equations

$$\begin{aligned}
\omega_4\omega_9-\omega_3\omega_7-\omega_2\omega_8&=0,\\
\omega_4\omega_{10}+\omega_6\omega_7+\omega_5\omega_8&=0,\\
\omega_1\omega_4+\omega_2\omega_6-\omega_3\omega_5&=0,\\
\omega_1\omega_7-\omega_2\omega_{10}-\omega_5\omega_9&=0,\\
\omega_3\omega_{10}+\omega_6\omega_9+\omega_1\omega_8&=0,
\end{aligned}$$

in the artinian algebra A.

Proof. We define

$$\omega_{1j}=f_{6,6\tau+2-6i},\omega_{2j}=f_{6,6\tau-2-6i},\omega_{3j}=f_{6,6\tau-1-6i},\omega_{4j}=f_{7,6\tau+6-6i},\omega_{5j}=g_{8,6\tau+1-6i},$$

$$\omega_{6j}=g_{8,6\tau+5-6i},\omega_{7j}=f_{12,6\tau+8-6i},\omega_{8j}=f_{12,6\tau+9-6i},\omega_{9j}=\tilde{g}_{10,6\tau+6-6i},\omega_{10,j}=\tilde{g}_{16,6\tau+6-6i},$$

and note that the conditions on the 10τ coefficients are equivalents to the five quadratic equations in the antinian algebra A. □

Corollary 4.1.4. *We have*

$$\dim \mathfrak{Q}_{\mathcal{H}} = 8\tau+8.$$

Proof. Since $\dim M=\tau+8$, we just have to show that $\dim N = 7\tau$. If W_i is the open subset of N defined by $\omega_{i0}\neq 0$, then ω_i is an unit in the local artinian algebra A. For exemple, if $(\omega_1,\dots,\omega_{10})$ is a vector belong to the open W_1 then we can eliminate ω_4,ω_7 and ω_8 from the third, forth and fifth quadratic equations and the remaining two equations become trivial. This means that W_1 has codimension 3τ in A^{10}, thus W_1 has dimension 7τ. In a similar way, we see that

$$\dim W_i = 7\tau (i=1,\dots,10).$$

If $\tau=1$, then $N=W_1\cup W_2\cup\ldots\cup W_{10}$ and therefore $\dim N=7$. We suppose that $\tau>1$. If a vector $(\omega_1,\dots,\omega_{10})\in N$ does not belong to the union $W_1\cup W_2\cup\ldots\cup W_{10}$ means that $\omega_{ij}=0$ whenever $j=0$, and then the ten coefficients ω_{ij} with $j=0$ do not enter into the five quadratic equations, and by induction we obtain

$$\dim(N\backslash(W_1\cup\ldots\cup W_{10})) = 7(\tau-2)+10 < 7\tau,$$

and therefore we have $\dim N = 7\tau$. □

Now apllying the theorem (3.2.9), we obtain an upper bound for the dimension of the moduli variety

$$\dim \bar{\mathscr{M}}_{\mathcal{H}} < 8\tau + 8,$$

which for every $\tau \geq 1$ is batter than Deligne's bound $2g - 1 = 5 + 12\tau$.

By applying theory of limit linear series Eisenbud and Harris in [EH] found a lower bound for the moduli space $\mathcal{M}^{\mathcal{H}}_{g,1}$. More precisely, a lower bound for the dimension of any irreducible component of $\mathcal{M}^{\mathcal{H}}_{g,1}$ is

$$\dim \mathcal{M}^{\mathcal{H}}_{g,1} \geq 3g - 2 - \mathrm{wt}(\mathcal{H}),$$

where $\mathrm{wt}(\mathcal{H}) = \sum_{i=1}^{g}(l_i - i)$ is the weight of the semigroup. This lower bound is attained whenever $\mathrm{wt}(\mathcal{H}) \leq g - 2$ and the Weierstrass semigroup $\mathcal{H}$ is *primitive* that is, the last gap is smaller than twice the first nongap. However for semigroups of weight large as symmetric semigroups the lower bound $3g - 2 - \mathrm{wt}(\mathcal{H})$ becomes far from sharp, may even be negative.

Nathan in [PF1] improves this lower bound by introduce the *effective weight* of a numerical semigroup $\mathcal{H}$

$$\mathrm{ewt}(\mathcal{H}) := \sum_{\text{gaps } l_i} (\#\text{ generators } n_j < l_i).$$

Alternatively, $\mathrm{ewt}(\mathcal{H})$ is the number of pairs (n_i, l_k) where n_i is a generator of $\mathcal{H}$ and $l_k \notin \mathcal{H}$ with $n_i < l_k$, and so $\mathrm{wt}(\mathcal{H}) - \mathrm{ewt}(\mathcal{H})$ is equal to the number of pairs (n_i, l_k) where $n_i < l_k$, n_i is composite, and l_k is a gap. Therefore,

$$\mathrm{wt}(\mathcal{H}) = \mathrm{ewt}(\mathcal{H}) \Longleftrightarrow \mathcal{H} \text{ is primitive.}$$

Theorem 4.1.5. *(Theorem 1.2, [PF1]) If $\mathcal{M}^{\mathcal{H}}_{g,1}$ in nonempty, and X is any irreducible component of it, then*

$$\dim X \geq \dim \mathcal{M}_{g,1} - \mathrm{ewt}(\mathcal{H}).$$

Moreover, this bound is sharp (see [PF1], thm 1.3) whenever $\mathrm{ewt}(\mathcal{H}) \leq g-2$. As exemple where the dimension of $\mathcal{M}^{\mathcal{H}}_{g,1}$ is strictly greater than $3g - 2 - \mathrm{ewt}(\mathcal{H})$ consider the symmetric semigroup $\mathcal{H} = \langle 6, 7, 8\rangle$ of genus 9. Its effective weight $\mathrm{ewt}(\mathcal{H}) = 12$ and by embbeding a curve $\mathcal{C}$ in $\mathbb{P}^3$ (see [PF1], pg. 12) follows that $\dim \mathcal{M}^{\mathcal{H}}_{g,1} = 11$.

We will calculate the dimension of $\mathcal{M}^{\mathcal{H}}_{g,1}$ when $\mathcal{H}$ is the symmetric semigroup family $\mathcal{H} = \langle 6, 3 + 6\tau, 4 + 6\tau, 7 + 6\tau, 8 + 6\tau\rangle$. By corollary (4.1.4) we have $\dim \mathcal{M}^{\mathcal{H}}_{g,1} < 8\tau + 8$. On the other hand, the gaps of $\mathcal{H}$ are

$$\begin{aligned}
&j + 6i, \quad i = 0, \ldots, \tau \text{ and } j = 1, 2\\
&j + 6i, \quad i = 0, \ldots, \tau - 1 \text{ and } j = 3, 4\\
&5 + 6i, \quad i = 0, \ldots, 2\tau.
\end{aligned}$$

Thus, $\operatorname{ewt}(\mathcal{H}) = 10\tau$ and by theorem (4.1.5) we obtain

$$\begin{aligned} \dim \mathcal{M}_{g,1}^{\mathcal{H}} &\geq \dim \mathcal{M}_{g,1} - \operatorname{ewt}(\mathcal{H}) \\ &= 3g - 2 - 10\tau \\ &= 8\tau + 7. \end{aligned}$$

Corollary 4.1.6. *If we take the symmetric semigroup family* $\mathcal{H} = \langle 6, 3+6\tau, 4+6\tau, 7+6\tau, 8+6\tau \rangle$, *then*

$$\dim \mathcal{M}_{g,1}^{\mathcal{H}} = 8\tau + 7.$$

We also consider the symmetric semigroup family worked by A. Contiero and Stöhr in [CS]

$$\mathcal{H} = \langle 6, 2+6\tau, 3+6\tau, 4+6\tau, 5+6\tau \rangle,$$

where $\tau \geq 1$ and the genus of $\mathcal{H}$ is $g = 1 + 6\tau$. Calculating the gaps of this semigroup

$$\begin{aligned} &j + 6i, \quad i = 0, \ldots, \tau - 1 \text{ and } j = 2, 3, 4, 5 \\ &1 + 6i, \quad i = 0, \ldots, 2\tau. \end{aligned}$$

Now observe that $\operatorname{ewt}(\mathcal{H}) = 10\tau - 4$. Therefore applying the theorem (4.1.5) we obtain the lower bound

$$\dim \mathcal{M}_{g,1}^{\mathcal{H}} \geq 3(1 + 6\tau) - 2 - (10\tau - 4) = 8\tau + 5.$$

On the other hand, follows the corollary 4.5 in [CS] an upper bound of the moduli space $\mathcal{M}_{g,1}^{\mathcal{H}}$

$$\dim \mathcal{M}_{g,1}^{\mathcal{H}} \leq 8\tau + 5.$$

Corollary 4.1.7. *If* $\mathcal{H} = \langle 6, 2+6\tau, 3+6\tau, 4+6\tau, 5+6\tau \rangle$, *then*

$$\dim \mathcal{M}_{g,1}^{\mathcal{H}} = 8\tau + 5.$$

4.2 Future Works

Through this thesis we can formulate a number of questions that we will try to solve in the next years.

The fisrt question was proposed my advisor A. Contiero which try to get a finer upper bound for the dimension of $\mathcal{M}_{g,1}^{\mathcal{H}}$.

Question 4.2.1 (A. Contiero)**.** *Is it true that*

$$\dim \mathcal{M}_{g,1}^{\mathcal{H}} \leq 2g - 2 + \lambda - \dim T^{1,+}(\mathbf{k}[\mathcal{H}])?$$

We can answer positively the truth of these questions for all semigroups of genus not bigger than 6 organizing in the following table following the notation: NP denotes the lower bound [1]; *Del* is the upper bound of Deline $2g - 1 - \lambda(\mathcal{H})$.

[1]The lower bounds in blue are better that the lower bound of Eisenbud and Harris in [EH].

gaps	NP	$\dim \mathcal{M}_{g,1}^{\mathcal{H}}$	$\lambda(\mathcal{H})$	*Del*	$\dim T^{1,+}$
1, 3	3	3	1	3	0
1, 2	4	4	2	4	0
1, 3, 5	5	5	1	5	0
1, 2, 4	6	6	2	6	0
1, 2, 5	5	5	1	5	0
1, 2, 3	7	7	3	7	0
1, 3, 5, 7	7	7	1	7	0
1, 2, 4, 5	8	8	2	8	0
1, 2, 4, 7	7	7	1	7	0
1, 2, 3, 5	9	9	3	9	0
1, 2, 3, 6	8	8	2	8	0
1, 2, 3, 7	7	7	1	7	0
1, 2, 3, 4	10	10	4	10	0
1, 3, 5, 7, 9	9	9	1	9	0
1, 2, 4, 5, 7	10	10	2	10	0
1, 2, 4, 5, 8	9	9	2	10	1
1, 2, 3, 5, 6	11	11	3	11	0
1, 2, 3, 5, 7	10	10	3	11	1
1, 2, 3, 5, 9	9	9	1	9	0
1, 2, 3, 6, 7	9	9	2	10	1
1, 2, 3, 4, 6	12	12	4	12	0
1, 2, 3, 4, 7	11	11	3	11	0
1, 2, 3, 4, 8	10	10	2	10	0
1, 2, 3, 4, 9	9	9	1	9	0
1, 2, 3, 4, 5	13	13	5	13	0
1, 3, 5, 7, 9, 11	11	11	1	11	0
1, 2, 4, 5, 7, 8	12	12	2	12	0
1, 2, 4, 5, 7, 10	11	11	2	12	1
1, 2, 4, 5, 8, 11	10	10	1	11	1
1, 2, 3, 5, 6, 7	13	13	3	13	0
1, 2, 3, 5, 6, 9	12	12	3	13	1
1, 2, 3, 5, 6, 10	11	11	2	12	1
1, 2, 3, 5, 7, 9	11	11	3	13	2
1, 2, 3, 5, 7, 11	10	10	1	11	1
1, 2, 3, 6, 7, 9	10	10	1	11	1
1, 2, 3, 4, 6, 7	14	14	4	14	0

gaps	NP	$\dim \mathcal{M}_{g,1}^{\mathcal{H}}$	$\lambda(\mathcal{H})$	*Del*	$\dim T^{1,+}$
1, 2, 3, 4, 6, 8	13	?	4	14	1
1, 2, 3, 4, 6, 9	12	12	3	13	1
1, 2, 3, 4, 6, 11	11	11	1	11	0
1, 2, 3, 4, 7, 8	12	12	3	13	1
1, 2, 3, 4, 7, 9	11	11	2	12	1
1, 2, 3, 4, 8, 9	10	10	2	12	2
1, 2, 3, 4, 5, 7	15	15	5	15	0
1, 2, 3, 4, 5, 8	14	14	4	14	0
1, 2, 3, 4, 5, 9	13	13	3	13	0
1, 2, 3, 4, 5, 10	12	12	2	12	0
1, 2, 3, 4, 5, 11	11	11	1	11	0
1, 2, 3, 4, 5, 6	16	16	6	16	0

From the work [CS] we can try to fix the multicplicity of a (symmetric) semigroup $\mathcal{H}$ (for example equal to 6), and try to use some of the Pflueger ideas to get an upper bound for $\dim \mathcal{M}_{g,1}^{\mathcal{H}}$.

As can be noted from this thesis, it can be really hard, or even impossible, try to study the structure of the spaces $\mathcal{M}_{g,1}^{\mathcal{H}}$ in an explicit way. So we can try to study some of Hilbert spaces, following again the Pflueger approach [PF2]. Here we can try also work with symmetric semigroups using facts on Hilbert schemes of canonical curves. In this way we can try to work on Buchsweitz's question [B].

Conjecture 4.2.1. *If $T^2(\mathbf{k}[\mathcal{H}])$ is equal to zero than $\mathcal{M}_{g,1}^{\mathcal{H}}$ is rational.*

An other question is about the minimal amount of cubic forms that generate the ideal of a trigonal Gorenstein curve. We have for a canonical curve the Petri's theorem in [ACGH] which prove that a certain amount of forms generate the ideal of the curve. Also in this thesis, we calculate the ideal of a trigonal curve of genus 6, in the subsection 3.3.2. In both cases the amount of cubic forms is not minimal.

We also want to answer what semigroups can be realized as Weierstrass semigroup of a Gorenstein or Kunz curve. Nivaldo in [M] answered this question for a class of semigroups called $(d_1, \ldots, d_m)$-symmetric semigroups. He proved that each semigroup of this class is realized as a Weierstrass semigroup of a Gorenstein curve which is an union of a monomial curve with curves of genus zero. We will try to work with necessary and sufficient conditions of a Gorenstein and Kunz local ring proved in [BAF].

References

[ACM] L. F. F. Abrantes, A. Contiero and R. V. Martins, ‘Max Noether’s theorem for integral curves’, *ARXIV* (2014).

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, ‘Geometry of algebraic curves’, *Grundlehren der Mathematischen Wissenschaften* 267 (Springer-Verlag, New York, 1985).

[Ba] E. Ballico, ‘Bril–Noether theory for rank 1 torsion free sheaves on singular projective cuves’, *J. Korean Math. Soc.* 37 (2000). 359–369.

[BAF] V. Barucci, M. D’Anna and R. Froeberg, ‘The semigroup of values of a one-dimension local ring with two minimal primes’, *Communications in algebra* (2000) 3607-3633.

[BE] D.A. Buchsbaum and D. Eisenbud, ‘Algebra structures for finite free resolutions, and some structure teorems for ideals of codimension 3’, *Amer. J. Math.* 99 (1977) 447–485.

[B] R.-O. Buchweitz, ‘On deformations of monomial curves’, *Lecture notes in Mathematics* 777 (1980) 205–220.

[BUL] E.M. Bullock, ‘Irreducibility and stable rationality of the loci of Weierstrass points on curves of genus at most six’, *Proc. Am. Math. Soc.* 365 (2014) 1121–1132.

[Bu] E.M. Bullock, ‘Subcanonical points on algebraic curves’, *Trans. Amer. Math. Soc.* 365 (2013) 99–122.

[CS] A. Contiero, K. -O. Stöhr, ’Upper bounds for the dimension of moduli spaces of curves with symmetric Weierstrass semigroups’, *J. London Math. Soc.* 88 (2013) 580-598.

[Cp] M. Coppens, ‘Free linear systems on integral Gorenstein curves’, *J. Algebra* 145 (1992) 209–218.

[D] P. Deligne, ‘Intersections sur les surfaces régulières’ (SGA 7, Exposé X), *Lectures notes in Mathematics* 340 (1973) 1–37.

[E] D. Eisenbud, ‘Commutative algebra with a view toward algebraic geometry’, *Graduate Texts in Mathematics* 150 (Springer-Verlag, Berlin, 1995).

[EH] D. Eisenbud and J. Harris, ‘Existence, decomposition, and limits of certain Weierstrass points’, *Invent. Math.* 87 (1987) 495–515.

[GP] L. Gatto and F. Ponza, ‘Derivatives of Wronskians with applications to families of special Weierstrass points’, *Trans. Am. Math. Soc.*351 (1999), 2233–2255.

[GSS] P. Gimenez, I. Sengupta, H. Srinivasan, ‘Minimal free resolutions for certain affine monomial curves’ *Commutative Algebra and Its connections to Geometry, Contemporary Mathematics*, 555 (2011).

[HM] J. Harris and I. Morrison, ’Moduli of Curves’, *Springer* (1998).

[H] R. Hartshorne, ’Graduate texts in Mathematics’, *Algebraic Geometry* (Springer-Verlag, New York, 1977).

[KIM] S.Kim, ‘On the existence of Weierstrass gap sequence on trigonal curves’, *Journal of Pure and Applied Algebra* (1990) 171–180.

[KM] S.L. Kleiman and R. Vidal Martins, ‘The canonical model of a singular curve’, *Geom. Dedicata* 139 (2009) 139–166.

[K] J. Komeda, ‘On primitive Schubert indices of genus g and weight $g-1$’, *J. Math. Soc. Japan* 43 (1991) 437–445.

[KZ] M. Kontsevich and A. Zorich, ‘Connected components of the moduli spaces of Abelian differentials with prescribed singularities’, *Invent. Math.* 153 (2003) 319–353.

[EK] E. Kunz, ‘The value-semigroup of a one dimensinal Gorenstein ring’, *proc. Amer. Math. Soc.(1970) 748-751* .

[Ma] R. V. Martins, ‘A generalization of Max Noether’s theorem’, *Proceeding of the American Mathematical Society*, 140, 377–391 (2012).

[M] N. Medeiros, ‘On canonical curves and osculating spaces’ *J. Pure and Appplied Algebra* (2002) 267–285.

[PF1] N. Pflueger,‘On non-primitive Weierstrass points’, *ARXIV* (2016).

[PF2] N. Pflueger,‘Weierstrass semigroups on Castelnuovo curves’, *ARXIV* (2016).

[O] G. Oliveira, ‘Weierstrass semigroups and the canonical ideal of nontrigonal curves’, *Manuscripta Math.* 71 (1991) 431–450.

[P] H. Pinkham, ‘Deformations of algebraic varieties with G_m-action’, *Astérisque 20* (1974).

[RV] D.S. Rim and M.A. Vitulli, ‘Weierstrass points and monomial curves’, *J. Algebra 48* (1977) 454–476.

[R] M.Rosenlicht, ’Equivalence relations on algebraic curves, *Ann.of math.*(1952), 228-240.

[FS] F. O. Schreyer, ‘A standart basis approach to syzygies of canonical curves’, *J. reine angew. Math.* 421 (1991) 83–123.

[S] K.-O. Stöhr, ‘On the moduli spaces of Gorenstein curves with symmetric Weierstrass semigroups’, *J. reine angew. Math.* 441 (1993) 189–213.

[S2] K.-O. Stöhr, 'On the poles of regular differentials of singular curves', *Bol. Soc. Bras. Math.* (1993) 105–136.

[S3] K.-O. Stöhr and J.F.Voloch, 'Weierstrass points and curves over finite fields', *Proc. London Math. Soc.* (3) (1986), 1-19 (1993) 105–136.

[Tes] B. Teissier, Appendice à *Le probléme des modules pour le branches planes*, cours donné par O. Zariski au Centre de Mathématiques de LÕÉcole Polytechnique, Paris, 1973.

[Vee] W. Veech, 'Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards' *Inv. Math.*, 97 (1989) 553–583.

[W1] R. Waldi, 'Deformation von Gorenstein-Singularitäten der Kodimension 3', *Math. Ann.* 242 (1979) 201–208.

[W2] R. Waldi, 'Äquivariante Deformation monomialer Kurven', *Regensburger Math. Schriften* 4 (1980) 1–88.