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HOLOMORPHIC DISTRIBUTIONS ON FANO THREEFOLDS

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ATA DA CENTÉSIMA SEXTA DEFESA DE TESE DA ALUNA ALANA CAVALCANTE FELIPPE, REGULARMENTE MATRICULADA NO PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA, DO INSTITUTO DE CIÊNCIAS EXATAS, DA UNIVERSIDADE FEDERAL DE MINAS GERAIS, REALIZADA NO DIA 27 DE FEVEREIRO DE 2018.

Aos vinte e sete dias do mês de fevereiro de 2018, às 10h30, na sala 3060, reuniram-se os professores abaixo relacionados, formando a Comissão Examinadora homologada pelo Colegiado do Programa de Pós-Graduação em Matemática, para julgar a defesa de tese da aluna **Alana Cavalcante Felippe**, intitulada: *"Holomorphic distributions on Fano Threefolds"*, requisito final para obtenção do Grau de doutor em Matemática. Abrindo a sessão, o Senhor Presidente da Comissão, Prof. Maurício Barros Corrêa Júnior, após dar conhecimento aos presentes o teor das normas regulamentares do trabalho final, passou a palavra à aluna para apresentação de seu trabalho. Seguiu-se a arguição pelos examinadores com a respectiva defesa da aluna. Após a defesa, os membros da banca examinadora reuniram-se sem a presença da aluna e do público, para julgamento e expedição do resultado final. Foi atribuída a seguinte indicação: a aluna foi considerada aprovada sem ressalvas e por unanimidade. O resultado final foi comunicado publicamente à aluna pelo Senhor Presidente da Comissão. Nada mais havendo a tratar, o Presidente encerrou a reunião e lavrou a presente Ata, que será assinada por todos os membros participantes da banca examinadora. Belo Horizonte, 27 de fevereiro de 2018.

PROF. DR. MAURÍCIO BARROS CORRÊA JÚNIOR
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Contents

Resumo

Esta tese é dedicada ao estudo de distribuições holomorfas de dimensão e codimensão um em variedades Fano tridimensionais que são interseções completas com pesos e com número de Picard igual a um. Também estudamos o conjunto singular de distribuições holomorfas singulares nestas variedades. O objetivo deste trabalho é caracterizar estas distribuições cujos feixes tangentes e conormais são aritmeticamente Cohen-Macaulay (aCM), i.e. não têm cohomologia intermediária.

Palavras-chave: Distribuições holomorfas, variedades Fano, Feixes aCM.

Abstract

This thesis is devoted to the study of holomorphic distributions of dimension and codimension one on smooth weighted projective complete intersection Fano three dimensional manifolds, with Picard number equal to one. We also studied the singular set of singular holomorphic distributions in this manifolds. The goal of this work is to characterize this distributions whose tangent sheaf and conormal sheaf are arithmetically Cohen Macaulay (aCM), i.e. has no intermediate cohomology.

Palavras-chave: Holomorphic distributions, Fano manifolds, aCM sheaves.

Introduction

In the study of holomorphic distributions and foliations in complex projective manifolds, algebro-geometric techniques have been used. We are interested in analyzing when the tangent and conormal sheaves split, together with the properties of singular schemes of distributions.

In this thesis, we study holomorphic distributions on a smooth weighted projective complete intersection Fano threefold X with Picard number equal to one. The goal of this work is to characterize these distributions whose tangent sheaf and conormal sheaf are arithmetically Cohen Macaulay (aCM), i.e. has no intermediate cohomology. In addition, we study the properties of their singular schemes and we construct examples of codimension one distributions on X based on a result of O. Calvo-Andrade, M. Corrêa and M. Jardim [5].

Fano threefolds with rank one Picard group have been classified by Iskovskih [17, 18] and Mukai [23]. The *index* of X is the largest integer ι_X such that the canonical line bundle K_X is divisible by ι_X in $\mathrm{Pic}(X)$. In [20], Kobayashi and Ochiai showed that the index ι_X is at most $\dim(X)+1$ and $\iota_X = \dim(X)+1$, if and only if $X \simeq \mathbb{P}^n$. Moreover, $\iota_X = \dim(X)$, if and only if $X \simeq Q^n \subset \mathbb{P}^{n+1}$, where Q^n is a smooth quadric. By using this result, when X is a Fano threefold, it can have index $\iota_X = 4$ ($X \simeq \mathbb{P}^3$), $\iota_X = 3$ ($X \simeq Q^3$), $\iota_X = 2$ ($X \simeq X_3 \subset \mathbb{P}^4$ is either a cubic hypersurface, or $X \simeq X_{2,2} \subset \mathbb{P}^5$ is an intersection of two quadric hypersurface, or $X \simeq X_4 \subset \mathbb{P}(1,1,1,1,2)$ is a hypersurface of degree 4 in the weighted projective space, or $X \simeq X_6 \subset \mathbb{P}(1,1,1,2,3)$ is a hypersurface of degree 6 in the weighted projective space) and $\iota_X = 1$ ($X \simeq X_{2,3} \subset \mathbb{P}^5$ is either an intersection of a quadric and a cubic, or $X \simeq X_{2,2,2} \subset \mathbb{P}^6$ is an intersection of three quadrics, or X is an intersection of a quadratic cone and a hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,1,2)$).

L. Giraldo and A. J. Pan-Collantes showed in [13] that the tangent sheaf of a foliation of dimension 2 on $\mathbb{P}^3$ splits if and only if its singular scheme Z is aCM. More recently M. Corrêa, M. Jardim and R. Vidal Martins extended this result in [7], showing that the tangent sheaf of a codimension one locally free distribution on $\mathbb{P}^n$ splits as a sum of line bundles if and only if its singular scheme is aCM. We will extend this result for the others

Fano threefolds, see section 3.1. These theorems were the motivation for the main results of this thesis, which we divided into three chapters.

In Chapter 1, we recall some definitions and preliminary results. We begin with relevant concepts about vector bundles and sheaves. We also recall some concepts about arithmetically Cohen-Macaulay and arithmetically Buchsbaum schemes, some concepts about holomorphic distributions and finally we remember the definition of Weighted Projective Spaces. The main references used are [9], [10], [15], [16], [24], [25], and [26].

In Chapter 2, we present the classification of Fano threefold given by Iskovskih [17, 18] and Mukai [23]. We describe the most important facts, already known, about the cohomology of cotangent sheaf to these varieties. Moreover, we calculate the cohomology groups $H^p\big(X, \Omega_X^q(t)\big)$, $p, q \in \{1, 2\}$ and $p \neq q$, where X is a smooth weighted projective complete intersection Fano threefold with Picard number equal to one.

In Chapter 3, we study the tangent and conormal sheaves of holomorphic distributions of codimension one and dimension one, respectively, and some algebro-geometric properties of its singular schemes.

If $\iota_X = 3$, we characterize when the tangent sheaf of a distribution of dimension 2 on Q^3 is split or spinor. More precisely, we prove the following result.

Theorem A: *Let $\mathscr{F}$ be a distribution on Q^3 of codimension one such that the tangent sheaf $T_{\mathscr{F}}$ is locally free. If $T_{\mathscr{F}}$ either splits as a sum of line bundles or is a spinor bundle, then Z is arithmetically Buchsbaum, with $h^1(Q^3, I_Z(r-2)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$. Conversely, if Z is arithmetically Buchsbaum with $h^1(Q^3, I_Z(r-2)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$ and $h^2(T_{\mathscr{F}}(-2)) = h^2(T_{\mathscr{F}}(-1-c_1(T_{\mathscr{F}}))) = 0$, then $T_{\mathscr{F}}$ either split or is a spinor bundle.*

If $\iota_X = 2$, we characterize when the tangent sheaf of a distribution of dimension 2 on a smooth weighted projective complete intersection del Pezzo Fano threefold X, has no intermediate cohomology. More precisely, we prove the following result.

Theorem B: *Let $\mathscr{F}$ be a distribution of codimension one on a smooth weighted projective complete intersection del Pezzo Fano threefold X, such that the tangent sheaf $T_{\mathscr{F}}$ is locally free. If $T_{\mathscr{F}}$ has no intermediate cohomology, then $H^1(X, I_Z(r+t)) = 0$ for $t < -6$ and $t > 8$. Conversely, if $H^1(X, I_Z(r+t)) = 0$ for $t < -6$ and $t > 8$, and $H^2(X, T_{\mathscr{F}}(t)) = 0$ for $t \leq 8$ and $H^1(X, T_{\mathscr{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathscr{F}})$, then $T_{\mathscr{F}}$ has no intermediate cohomology.*

If $\iota_X = 1$, we characterize when the tangent sheaf of a distribution of dimension 2 on a smooth weighted projective complete intersection prime Fano threefold X, has no intermediate cohomology. More precisely, we prove the following result.

Theorem C: *Let $\mathscr{F}$ be a distribution of codimension one on a smooth weighted projective complete intersection prime Fano threefold X, such that the tangent sheaf $T_{\mathscr{F}}$ is locally free. If $T_{\mathscr{F}}$ has no intermediate cohomology, then $H^1(X, I_Z(r+t)) = 0$ for $t < -4$ and $t > 4$. Conversely, if $H^1(X, I_Z(r+t)) = 0$ for $t < -4$ and $t > 4$, and $H^2(X, T_{\mathscr{F}}(t)) = 0$ for $t \leq 4$ and $H^1(X, T_{\mathscr{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathscr{F}})$, then $T_{\mathscr{F}}$ has no intermediate cohomology.*

M. Corrêa, M. Jardim and R. Vidal Martins showed in [7] that the conormal sheaf $N^*_{\mathscr{F}}$ of a foliation of dimension one on $\mathbb{P}^n$ splits if and only if its singular scheme Z is arithmetically Buchsbaum with $h^1(\mathcal{I}_Z(d-1)) = 1$ being the only nonzero intermediate cohomology. We extend this result for the others Fano threefolds. More precisely, we have the following results.

Theorem D: *Let $\mathscr{F}$ be a distribution of dimension one on a smooth weighted projective complete intersection Fano threefold X, with index $\iota_X \in \{1, 2, 3, 4\}$. If $N^*_{\mathscr{F}}$ is arithmetically Cohen Macaulay, then Z is arithmetically Buchsbaum, with $h^1(X, I_Z(r)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$.*

Theorem E: *Let $\mathscr{F}$ be a distribution of dimension one on a smooth weighted projective complete intersection threefold X, with index $\iota_X \in \{1, 2, 3\}$. If Z is arithmetically Buchsbaum with $h^1(X, I_Z(r)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$, and $h^2(N^*_{\mathscr{F}}) = h^2(N^*_{\mathscr{F}}(-c_1(N^*_{\mathscr{F}}) - \iota_X)) = 0$, then $N^*_{\mathscr{F}}$ is arithmetically Cohen Macaulay.*

Chapter 1

Preliminaries

In this chapter, we recall some definitions and preliminary results. We begin with relevant concepts about vector bundles and sheaves, followed by the Spinor bundle construction on quadrics.

We also recall some concepts about arithmetically Cohen-Macaulay and arithmetically Buchsbaum schemes, some concepts about holomorphic distributions and finally, we remember the definition of Weighted Projective Spaces. The main references used are [9], [10], [15], [16], [24], [25], and [26].

1.1 Vector Bundles and Sheaves

Let X be a complex manifold of dimension n with $\mathrm{Pic}(X) \simeq \mathbb{Z}$, and denote by $\mathcal{O}_X(1)$ the ample generator of $\mathrm{Pic}(X)$.

We denote $E(t) = E \otimes_{\mathcal{O}_X} \mathcal{O}_X(t)$ for $t \in \mathbb{Z}$ when E is a vector bundle on X, and we denote by E^* the dual vector bundle of E. We denote by TX the tangent bundle of X. Thus, we have $TX \simeq (\Omega^1_X)^*$.

If F is a sheaf on X, we denote by $h^i(X, F)$ the dimension of the complex vector space $H^i(X, F)$.

Remark 1.1. For any holomorpic vector bundle E of rank r,

$$\bigwedge^k E \simeq \bigwedge^{r-k} E^* \otimes \det E.$$

In particular, if E is a rank 2 reflexive sheaf, then

$$E^* = E \otimes (\det E)^* \text{ and } (\det E)^* = \mathcal{O}_{\mathbb{P}^n}(-c_1(E)),$$

where $c_1(E)$ denotes the first Chern class of E.

Definition 1.2. A vector bundle E on X is called *globally generated*, if there exist global holomorphic sections $\sigma_1, \ldots, \sigma_N \in H^0(X, E)$ such that for all $x \in X$, $\sigma_1(x), \ldots, \sigma_N(x)$ span E_x. In other words, the sections $\sigma_1, \ldots, \sigma_N$ induce a surjection $\mathcal{O}_X^{\oplus N} \twoheadrightarrow E$.

Proposition 1.3. *[19] A compact manifold X is homogeneous if and only if its tangent bundle is globally generated.*

Remark 1.4. Since a quadric hypersurface $Q^3 \subset \mathbb{P}^4$ is a homogeneous variety, TQ^3 is globally generated.

Remark 1.5. Any quotient of a globally generated sheaf has the same property. Any tensor product of globally generated sheaves has the same property. The restriction of a globally generated sheaf to a subscheme has the same property.

Definition 1.6. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ be a basis of the vector space $H^0(X, L)$ of global sections of a line bundle L. A line bundle L is *very ample* if it satisfies the following two conditions:

1. L has no base points, that is, $\alpha_1, \alpha_2, \ldots, \alpha_N$ have no common zeroes,
2. the morphism
$$\begin{aligned} \Phi_L : X &\longrightarrow \mathbb{P}^{N-1} \\ p &\longmapsto (\alpha_1(p) : \alpha_2(p) : \ldots : \alpha_N(p)) \end{aligned}$$
is an embedding.

A line bundle L is *ample* if its suitable power $L^{\otimes m}, m > 0$, is very ample.

We have the following useful properties about sheaves.

Proposition 1.7. *[15, Proposition 1.1] A coherent sheaf E on X is reflexive if and only if it can be included into a locally free sheaf F with F/E torsion-free.*

The dual of any coherent sheaf is reflexive [15, Corollary 1.2]

Proposition 1.8. *[15, Proposition 2.6] The third Chern class $c_3(E)$ of a rank 2 reflexive sheaf E on a smooth projective threefold satisfies $c_3(E) \geq 0$, and it vanishes if and only if E is locally free.*

Definition 1.9. A bundle E is *arithmetically Cohen Macaulay (aCM)* if E has no intermediate cohomology, i.e.

$$H^i(X, E(t)) = 0 \text{ for all } t \in \mathbb{Z} \text{ and } 0 < i < \dim(X).$$

Theorem 1.10. *[14] [Griffiths Vanishing Theorem] If E is ample, then*

$$H^i(X, K_X \otimes S^m E \otimes \det E) = 0, \text{ for all } i > 0, m \geq 0,$$

where X is smooth irreducible complex projective variety, E is a vector bundle on X and $S^m E$ is the $m-th$ symmetric power of E.

1.1.1 Spinor bundles on quadrics

We recall the definition and some properties of spinor bundles on Q^n. For more details see [25, 26].

Let S_k be the spinor variety which parametrizes the family of $(k-1)$-planes in Q^{2k-1} or one of the two disjoint families of k-planes in Q^{2k}. We have that $\dim(S_k) = (k(k+1))/2$, $\mathrm{Pic}(S_k) = \mathbb{Z}$ and $h^0(S_k, \mathcal{O}(1)) = 2^k$.

When $n = 2k-1$ is odd, consider for all $x \in Q^{2k-1}$ the variety

$$\{\mathbb{P}^{k-1} \in Gr(k-1, 2k) \; / \; x \in \mathbb{P}^{k-1} \subset Q^{2k-1}\}.$$

This variety is isomorphic to S_{k-1} and we denote it by $(S_{k-1})_x$. Then we have a natural embedding

$$(S_{k-1})_x \xrightarrow{i_x} S_k.$$

Considering the linear spaces spanned by these varieties, we have a natural inclusion $H^0((S_{k-1})_x, \mathcal{O}(1))^* \to H^0(S_k, \mathcal{O}(1))^*$ for all $x \in Q^{2k-1}$ and then an embedding

$$s : Q^{2k-1} \to Gr(2^{k-1}-1, 2^k-1),$$

in the Grassmannian of $(2^{k-1}-1)$-subspaces of $\mathbb{P}^{2^k-1}$.

It is well known that $S_1 \simeq \mathbb{P}^1, \; S_2 \simeq \mathbb{P}^3$. The embedding $s : Q^3 \to Gr(1,3)$ corresponds to a hyperplane section.

Definition 1.11. Let U be the universal bundle of the Grassmannian $Gr(2^{k-1}-1, 2^k-1)$. We call $s^*U = S$ the *spinor bundle* on Q^{2k-1}. Its rank is 2^{k-1}.

The spinor bundle S on Q^3 is just the restriction of the universal sub-bundle on the 4-dimensional quadric.

Remark 1.12. The spinor bundle on Q^3 is a globally generated vector bundle of rank 2, by construction.

Theorem 1.13. *[25, Theorem 2.1] The spinor bundle on Q^3 is stable.*

We note that S is the unique indecomposable bundle of rank 2 on Q^3 with $c_1(S) = -1$ and $c_2(S) = 1$, (see [3]).

Theorem 1.14. *[25, Theorem 2.3] Let S a spinor bundle on Q^3. Then:*

1. $H^i(Q^3, S(t)) = 0$ *for i such that* $0 < i < 3$ *for all* $t \in \mathbb{Z}$*;*
2. $H^0(Q^3, S(t)) = 0$ *for* $t \leq 0$, $h^0(Q^3, S(1)) = 4$.

Theorem 1.15. *Let S be the spinor bundle on Q^3. We have a natural exact sequence:*

$$0 \to S \to \mathcal{O}_{Q^3}^{\oplus 4} \to S(1) \to 0$$

and an isomorphism $S^* \simeq S(1)$.

Proof: See [25], Theorem 2.8. □

1.2 aCM and aB schemes

In this section we recall some concepts about arithmetically Cohen-Macaulay and arithmetically Buchsbaum schemes.

A class of rings that is closed under the operations of localization, completion, adjoining polynomial and power series variables is the class of Cohen-Macaulay rings.

Definition 1.16. Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring and M a finite A-module. M is called *Cohen-Macaulay* (CM) if $M \neq 0$ and depth $M = \dim M$. If A is itself a Cohen-Macaulay module, we say that A is a *Cohen-Macaulay ring*.

Definition 1.17. A Noetherian ring A is said to be a CM ring if $A_{\mathfrak{m}}$ is a CM local ring for every maximal ideal $\mathfrak{m}$ of A.

Definition 1.18. A closed subscheme $Y \subset X$ is *arithmetically Cohen-Macaulay (aCM)* if its homogeneous coordinate ring $S(Y) = k[x_0, \ldots, x_n]/I(Y)$ is a Cohen-Macaulay ring.

Equivalently, Y is aCM if $H^p_*(\mathcal{O}_Y) = 0$ for $1 \leq p \leq dim(Y) - 1$ and $H^1_*(\mathcal{I}_Y) = 0$ (cf. [6]). For any coherent sheaf $\mathcal{F}$ we denote by $H^i_*(\mathcal{F})$ the sum $\oplus_{t \in \mathbb{Z}} H^i(\mathcal{F}(t))$. From the long exact sequence of cohomology associated to the short exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

one also deduces that Y is aCM if and only if $H^p_*(\mathcal{I}_Y) = 0$ for $1 \leq p \leq \dim(Y)$.

The concept of a Buchsbaum ring is a continuation of the concept of a Cohen-Macaulay ring. Let A be a local ring and let $\mathfrak{a} \subset A$ be an ideal of A. Denote by $U(\mathfrak{a})$ the intersection of the primary ideals $\mathfrak{q}$ belonging to $\mathfrak{a}$ with $\dim(\mathfrak{q}) = \dim(\mathfrak{a})$.

Definition 1.19. Let A be a local ring of dimension $d \geq 1$ with maximal ideal $\mathfrak{m}$. The following two conditions are equivalent:

(i) A is a Buchsbaum ring;

(ii) For each part $a_1, \ldots, a_k$ of a system of parameters of A we have

$$\mathfrak{m} \cdot U((a_1, \ldots, a_k)) \subseteq (a_1, \ldots, a_k), \quad \text{for every} \quad k = 0, \ldots, d-1.$$

Similarly, a closed subscheme in X is *arithmetically Buchsbaum (aB)* if its homogeneous coordinate ring is a Buchsbaum ring (see [31]).

1.3 Holomorphic Distributions

In this section we recall some concepts about holomorphic distributions.

Definition 1.20. Let X be a smooth complex manifold.

(i) A *codimension k distribution* $\mathscr{F}$ on X is an exact sequence

$$\mathscr{F}: \ 0 \longrightarrow T_{\mathscr{F}} \xrightarrow{\phi} TX \xrightarrow{\pi} N_{\mathscr{F}} \longrightarrow 0, \tag{1.1}$$

where $T_{\mathscr{F}}$ is a coherent sheaf of rank $r_{\mathscr{F}} := \dim(X) - k$, and $N_{\mathscr{F}} := TX/\phi(T_{\mathscr{F}})$ is a torsion free sheaf.

(ii) The sheaves $T_{\mathscr{F}}$ and $N_{\mathscr{F}}$ are called by the *tangent* and the *normal* sheaves of $\mathscr{F}$, respectively.

(iii) $\mathrm{Sing}(\mathscr{F}) = \{x \in X | (N_{\mathscr{F}})_x \text{ is not a free } \mathcal{O}_{X,x} - \text{module}\}$ is the *singular set* of the distribution $\mathscr{F}$.

A distribution $\mathscr{F}$ is said to be *locally free* if $T_{\mathscr{F}}$ is a locally free sheaf.

By definition, $\mathrm{Sing}(\mathscr{F})$ is the singular set of the sheaf $N_{\mathscr{F}}$. It is a closed analytic subvariety of X of codimension at least one.

To simplify the notation, let us write $Z := \mathrm{Sing}(\mathscr{F})$ and we suppose that $\operatorname{codim} \mathrm{Sing}(\mathscr{F}) \geq 2$.

Definition 1.21. A *foliation* is an integrable distribution, that is a distribution

$$\mathscr{F}: \ 0 \longrightarrow T_{\mathscr{F}} \xrightarrow{\phi} TX \xrightarrow{\pi} N_{\mathscr{F}} \longrightarrow 0$$

whose tangent sheaf is closed under the Lie bracket of vector fields, i.e. $[\phi(T_{\mathscr{F}}), \phi(T_{\mathscr{F}})] \subset \phi(T_{\mathscr{F}})$. Clearly, every distribution of codimension $\dim(X) - 1$ is integrable.

When $k = 1$, the normal sheaf, being a torsion free sheaf of rank 1, must be a twisted ideal sheaf $I_{Z/X} \otimes \det(TX) \otimes \det(T_{\mathscr{F}})^*$ of a closed subscheme $Z \subset X$ of codimension at least 2, which is precisely the singular scheme of $\mathscr{F}$.

1.3.1 Codimension one distributions and differential forms

We will consider codimension one distributions on a smooth weighted projective complete intersection Fano threefold X with $\operatorname{Pic}(X) = \mathbb{Z}$. Thus, the sequence (1.1) simplifies to

$$\mathscr{F}: \ 0 \longrightarrow T_{\mathscr{F}} \xrightarrow{\phi} TX \xrightarrow{\pi} I_{Z/X}(r) \longrightarrow 0, \tag{1.2}$$

where $T_{\mathscr{F}}$ is a rank 2 reflexive sheaf and r is integer such that $r = c_1(TX) - c_1(T_{\mathscr{F}})$. Observe that $N_{\mathscr{F}} = I_{Z/X}(r)$ where Z is the singular scheme of $\mathscr{F}$.

A codimension one distribution on X can also be represented by a section

$$\omega \in H^0(\Omega^1_X(r)),$$

given by the dual of the morphism $\pi : TX \to N_{\mathscr{F}}$. On the other hand, such section yields a sheaf map $\omega : \mathcal{O}_X \to \Omega^1_X(r)$. Taking duals, we get a cosection

$$\omega^* : (\Omega^1_X(r))^* = TX(-r) \to \mathcal{O}_X$$

whose image is the ideal sheaf $I_{Z/X}$ of the singular scheme. The kernel of ω^* is the tangent sheaf $\mathscr{F}$ of the distribution twisted by $\mathcal{O}(-r)$.

Remark 1.22. From this point of view, the integrability condition is equivalent to

$$\omega \wedge d\omega = 0.$$

Definition 1.23. Let $\mathscr{F} \subset TX$ be a codimension one distribution on a complex projective manifold X, and consider the associated twisted 1−form $\omega_{\mathscr{F}} \in H^0(X, \Omega^1_X \otimes \mathcal{L}_{\mathscr{F}})$, where $\mathcal{L}_{\mathscr{F}} = \det(N_{\mathscr{F}})$. For every integer $i \geq 0$, there is a well defined twisted $(2i+1)$-form

$$\omega_{\mathscr{F}} \wedge (d\omega_{\mathscr{F}})^i \in H^0(X, \Omega^{2i+1}_X \otimes \mathcal{L}_{\mathscr{F}}^{\otimes(i+1)}).$$

The *class* of $\mathscr{F}$ is the unique non negative integer $k = k(\mathscr{F}) \in \{0, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ such that

$$\omega \wedge (d\omega)^k \neq 0 \ \text{ and } \ \omega \wedge (d\omega)^{k+1} = 0.$$

By Frobenius theorem, a codimension one distribution is a foliation if and only if $k(\mathscr{F}) = 0$.

1.4 Weighted Projective Spaces

In this section we recall the definition of weighted projective spaces. We are considering that $\mathrm{Proj}(S(a_0,\ldots,a_n))$ has only closed points.

Let $a_0,\ldots,a_n$ be positive integers, and assume that $\gcd(a_0,\ldots,\hat{a}_i,\ldots a_n) = 1$ for every $i \in \{0,\ldots,n\}$. Denote by $S(a_0,\ldots,a_n)$ the polynomial ring $\mathbb{C}[z_0,\ldots,z_n]$ graded by $\deg z_i = a_i$, and set $\mathbb{P} = \mathbb{P}(a_0,\ldots,a_n) = \mathrm{Proj}(S(a_0,\ldots,a_n))$-weighted projective space. For each $t \in \mathbb{Z}$, let $\mathcal{O}_{\mathbb{P}}(t)$ be the $\mathcal{O}_{\mathbb{P}}$-module associated to the graded S-module $S(t)$.

In others words, consider the action in $\mathbb{C}^{n+1} \setminus \{0\}$:

$$\begin{array}{ccc} \mathbb{C}^* \times (\mathbb{C}^{n+1} \setminus \{0\}) & \longrightarrow & \mathbb{C}^{n+1} \setminus \{0\} \\ (\lambda, (z_0,\ldots,z_n)) & \longmapsto & (\lambda^{a_0} z_0,\ldots,\lambda^{a_n} z_n) \end{array}$$

We will denote $a = (a_0,\ldots,a_n)$; $|a| = a_0 + \ldots + a_n$.

Definition 1.24. We defined the *weighted projective space* in weights $a_0,\ldots,a_n$ by

$$\mathbb{P} = \mathbb{P}^n_a = \mathbb{P}(a_0,\ldots,a_n) = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\sim}$$

Note that:

- If $a_0 = \ldots = a_n = 1$, then $\mathbb{P} = \mathbb{P}^n_{\mathbb{C}}$;
- Sing $(\mathbb{P}) = \{(1:0:\ldots:0)_a,\ldots,(0:\ldots:0:1)_a\}$.

From the Euler sequence for weighted projective spaces ([10]), it follows that a nonzero twisted 1-form $\omega \in H^0(\mathbb{P}, \Omega^1_{\mathbb{P}}(r))$ can be written as:

$$\omega \;=\; \sum_{i=0}^{N} F_i dz_i, \tag{1.3}$$

with F_i weighted homogeneous of degree $(r - a_i)$, and such that $\sum_{i=0}^{N} a_i z_i F_i = 0$.

In [1], C. Araujo, M. Corrêa, A. Massarenti showed the following result.

Theorem 1.25. *[1, Theorem 1.4] Let $X \subset \mathbb{P}^n$ be a smooth complete intersection. Then*

1. $H^0(X, \Omega^1_X(1)) = 0$
2. *Let*

$$D_k \subseteq \mathbb{P}(H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2)))$$

be the subvariety parametrizing distributions of class $\leq k$ on $\mathbb{P}^n$, and let

$$\overline{D}_k \subseteq \mathbb{P}(H^0(X, \Omega^1_X(2)))$$

be the subset parametrizing distributions of class $\leq k$ *on* X*. Then there is a natural restriction isomorphism* $H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2)) \cong H^0(X, \Omega^1_X(2))$ *that maps* D_k *isomorphically onto* $\overline{D}_k$ *for any* $k < \left\lfloor \frac{dim(X)-1}{2} \right\rfloor$.

Chapter 2

Fano threefolds with rank one Picard group

Throughout this chapter, unless otherwise noted, X denotes a smooth weighted projective complete intersection Fano threefold. We assume that the Picard Group of X is $\mathbb{Z}$. As usual, we denote the Picard number of X by $\rho(X) = \text{rank Pic}(X)$.

2.1 Fano manifolds

In this section we recall some facts about Fano manifolds.

Definition 2.1. A compact complex manifold X is *Fano* if its anticanonical line bundle $\mathcal{O}_X(-K_X) \simeq \bigwedge^n TX$ is ample.

Smooth weighted projective complete intersection Fano threefold of Picard number one group have been classified by Iskovskikh [17, 18] and Mukai [23]. They are:

(a) the projective space $\mathbb{P}^3$;

(b) a quadric hypersurface $Q^3 = X_2 \subset \mathbb{P}^4$;

(c) a cubic hypersurface $X_3 \subset \mathbb{P}^4$;

(d) an intersection $X_{2,2}$ of two quadric hypersurfaces in $\mathbb{P}^5$;

(e) a hypersurface of degree 4 in the weighted projective space $X_4 \subset \mathbb{P}(1,1,1,1,2)$;

(f) a hypersurface of degree 6 in the weighted projective space $X_6 \subset \mathbb{P}(1,1,1,2,3)$;

(g) an intersection $X_{2,3}$ of a quadric and a cubic in $\mathbb{P}^5$;

(h) an intersection $X_{2,2,2}$ of three quadrics in $\mathbb{P}^6$;

(i) an intersection of a quadratic cone and a hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,1,2)$.

A basic invariant of a Fano manifold is its index.

Definition 2.2. The *index* of X is the maximal integer $\iota_X > 0$ dividing $-K_X$ in $\mathrm{Pic}(X)$, i.e. $-K_X = \iota_X \cdot H$, with H ample.

Definition 2.3. We say that $X \subset \mathbb{P}(a_0, ..., a_N)$ is a smooth n-dimensional *weighted complete intersection* in a weighted projective space, when X is the scheme-theoretic zero locus of $c = N - n$ weighted homogeneous polynomials $f_1, ..., f_c$ of degrees $d_1, ..., d_c$.

By [10, Theorem 3.3.4],

$$K_X \cong \mathcal{O}_X\left(\sum_{j=1}^{c} d_j - \sum_{i=0}^{N} a_i\right) \tag{2.1}$$

In particular, when X is Fano, its index is

$$\iota_X := \sum_{i=0}^{N} a_i - \sum_{j=1}^{c} d_j. \tag{2.2}$$

Let S_t be the t-th graded part of $S/(f_1, ..., f_c)$. By [8, Lemma 7.1],

$$H^i(X, \mathcal{O}_X(t)) \cong \begin{cases} S_t & \text{if} \quad i = 0; \\ 0 & \text{if} \quad 1 \le i \le n-1; \\ S_{-t+\iota_X} & \text{if} \quad i = n. \end{cases} \tag{2.3}$$

Theorem 2.4. *[20] Let X be Fano,* $\dim(X) = n$. *Then the index ι_X is at most $n+1$; moreover, if $\iota_X = n+1$, then $X \simeq \mathbb{P}^n$, and if $\iota_X = n$, then X is a quadric hypersurface $Q^n \subset \mathbb{P}^{n+1}$.*

By Theorem above, a Fano threefold X can have $\iota_X \in \{1, 2, 3, 4\}$. Then, $\iota_X = 4$ implies $X = \mathbb{P}^3$, while $\iota_X = 3$, implies that X is a smooth quadric hypersurface Q^3 in $\mathbb{P}^4$. In case, $\iota_X = 2$ the variety X is called a del Pezzo threefold, while $\iota_X = 1$, the variety X is called a prime Fano threefold.

Remark 2.5. From of classification given by Iskovskikh and Mukai and using the formula 2.2 , we have that the varieties with $\iota_X = 2$ are $(c), (d), (e)$ and (f), while the varieties with $\iota_X = 1$ are $(g), (h)$ e (i).

2.2 Cohomology of cotangent sheaf

The vanishing theorem for the cohomology of $\Omega^q(t)$ for projective spaces (Bott Theorem) can be found in [4]. For quadric hypersurfaces $X = Q^n$ in $\mathbb{P}^{n+1}$, we use [29]. In [10], Dolgachev generalized the Bott theorem on the cohomology of twisted sheaves of differentials to the case of weighted projective spaces ($\overline{\Omega}^q_{\mathbb{P}}(t)$). Finally, for a weighted complete intersection we use a result due to Flenner [11].

Let p, q and t be integers, with p and q non-negative. The values of $h^p\big(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(t)\big)$ are given by the Bott formula:

$$h^p\big(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(t)\big) = \begin{cases} \binom{t+n-q}{t}\binom{t-1}{q} & \text{for } p = 0, 0 \le q \le n \text{ and } t > q, \\ 1 & \text{for } t = 0 \text{ and } 0 \le p = q \le n, \\ \binom{-t+q}{-t}\binom{-t-1}{n-q} & \text{for } p = n, 0 \le q \le n \text{ and } t < q-n, \\ 0 & \text{otherwise.} \end{cases}$$

Throughout the work, we refer to this formula as classical Bott's formula.

In [29], Snow showed a vanishing theorem for the cohomology of $\Omega^q(t)$ for quadric hypersurfaces $X = Q^n$ in $\mathbb{P}^{n+1}$ and for a Grassmann manifold $X = Gr(s, m)$ of s-dimensional subspaces of $\mathbb{C}^m$.

Theorem 2.6. *[29] [Bott's formula for Quadric] Let X be a nonsingular quadric hypersurface of dimension n.*

1. *If $-n + q \le k \le q$ and $k \ne 0$ and $k \ne -n + 2q$, then $H^p(X, \Omega^q(k)) = 0$ for all p;*
2. *$H^p(X, \Omega^q) \ne 0$ if and only if $p = q$;*
3. *$H^p(X, \Omega^q(-n + 2q)) \ne 0$ if and only if $p = n - q$;*
4. *If $k > q$, then $H^p(X, \Omega^q(k)) \ne 0$ if and only if $p = 0$;*
5. *If $k < -n + q$, then $H^p(X, \Omega^q(k)) \ne 0$ if and only if $p = n$.*

Let Y be an n-dimensional Fano manifold with $\rho(Y) = 1$, and denote by $\mathscr{O}_Y(1)$ the ample generator of $\mathrm{Pic}(Y)$. Let $X \in \big|\mathscr{O}_Y(d)\big|$ be a smooth divisor. We have the following exact sequences:

$$0 \to \Omega^q_Y(t-d) \to \Omega^q_Y(t) \to \Omega^q_Y(t)|_X \to 0, \tag{2.4}$$

and

$$0 \to \Omega^{q-1}_X(t-d) \to \Omega^q_Y(t)|_X \to \Omega^q_X(t) \to 0. \tag{2.5}$$

Let $\mathbb{P} = \mathbb{P}(a_0, \dots, a_N) = \mathrm{Proj}\,\big(S(a_0, \dots, a_N)\big)$ be as in Section 1.4.

Theorem 2.7. *[10, Section 1.4] Let $A = \{a_0, \ldots, a_n\}$ be a finite set of positive integers and $|A| = a_0 + \ldots + a_n$. For all $t \in \mathbb{Z}$, follows that*

(i) $\alpha_t : S_t \; \to \; H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t))$ *is bijective;*

(ii) $H^p(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t)) = 0$ *for* $p \neq 0, n$*;*

(iii) $H^n(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t)) \simeq S_{-t-|A|}$.

In [10], Dolgachev generalized the Bott's theorem for the cohomology of twisted sheaves of differentials to the case of weighted projective spaces ($\overline{\Omega}^q_{\mathbb{P}}(t)$) as follows:

Consider the sheaves of $\mathscr{O}_{\mathbb{P}}$-modules $\overline{\Omega}^q_{\mathbb{P}}(t)$ defined in [10, Section 2.1.5] for $q, t \in \mathbb{Z}$, $q \geq 0$. If $U \subset \mathbb{P}$ denotes the smooth locus of $\mathbb{P}$, and $\mathscr{O}_U(t)$ is the line bundle obtained by restricting $\mathscr{O}_{\mathbb{P}}(t)$ to U, then $\overline{\Omega}^q_{\mathbb{P}}(t)_{|U} = \Omega^q_U \otimes \mathscr{O}_U(t)$. The cohomology groups $H^p\big(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t)\big)$ are described in theorem below:

Theorem 2.8. *[10, Section 2.3.2] Let $h^p\big(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t)\big) = \dim H^p\big(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t)\big)$. Then:*

- $h^0\big(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t)\big) = \sum_{i=0}^{q} \Big((-1)^{i+q} \sum_{\#J=i} \dim_{\mathbb{C}}\big(S_{t-a_J}\big)\Big)$, *where* $J \subset \{0, \ldots, N\}$ *and* $a_J := \sum_{i \in J} a_i$*;*
- $h^0\big(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t)\big) = 0$ *if* $t < \min\{\sum_{j \in J} a_{i_j} \mid \#J = q\}$*;*
- $h^p\big(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t)\big) = 0$ *if* $p \notin \{0, q, N\}$*;*
- $h^p\big(\mathbb{P}, \overline{\Omega}^p_{\mathbb{P}}(t)\big) = 0$ *if* $t \neq 0$ *and* $p \notin \{0, N\}$.

In particular, if $q \geq 1$, then

$$h^0(\mathbb{P}, \Omega^q_{\mathbb{P}}(t)) = 0 \quad \text{for any} \quad t \leq q. \tag{2.6}$$

When $\mathbb{P}(a_0, \ldots, a_N) = \mathbb{P}^N$ is a projective space we have the classical Bott's formulas.

Now assume that $\mathbb{P}$ has only isolated singularities, let $d > 0$ be such that $\mathscr{O}_{\mathbb{P}}(d)$ is a line bundle generated by global sections, and $X \in \big|\mathscr{O}_{\mathbb{P}}(d)\big|$ a smooth hypersurface. We will use the cohomology groups $H^p\big(\mathbb{P}, \overline{\Omega}^q_{\mathbb{P}}(t)\big)$ to compute some cohomology groups $H^p\big(X, \Omega^q_X(t)\big)$. Note that X is contained in the smooth locus of $\mathbb{P}$, so we have an exact sequence as in (2.5):

$$0 \to \; \Omega^{q-1}_X(t-d) \; \to \; \overline{\Omega}^q_{\mathbb{P}}(t)|_X \; \to \; \Omega^q_X(t) \to 0. \tag{2.7}$$

Tensoring the sequence

$$0 \to \; \mathscr{O}_{\mathbb{P}}(-d) \; \to \; \mathscr{O}_{\mathbb{P}} \; \to \; \mathscr{O}_X \; \to 0.$$

with the sheaf $\overline{\Omega}^q_{\mathbb{P}}(t)$, and noting that $\overline{\Omega}^q_{\mathbb{P}}(t) \otimes \mathscr{O}_{\mathbb{P}}(-d) \cong \overline{\Omega}^q_{\mathbb{P}}(t-d)$, we get an exact sequence as in (2.4):

$$0 \to \overline{\Omega}^q_{\mathbb{P}}(t-d) \ \to \ \overline{\Omega}^q_{\mathbb{P}}(t) \ \to \ \overline{\Omega}^q_{\mathbb{P}}(t)|_X \to 0. \tag{2.8}$$

Now, let X be as in Definition 2.3. The next Theorem in terms of cohomology of X, is due to Flenner:

Theorem 2.9. *[11, Satz 8.11] We have the following formulas for the cohomology of X:*

- $h^q(X, \Omega^q_X) = 1$ *for* $0 \leq q \leq n$, $q \neq \frac{n}{2}$.
- $h^p\big(X, \Omega^q_X(t)\big) = 0$ *in the following cases*
  - $0 < p < n$, $p + q \neq n$ *and either* $p \neq q$ *or* $t \neq 0$;
  - $p + q > n$ *and* $t > q - p$;
  - $p + q < n$ *and* $t < q - p$.

2.2.1 Calculations of $H^2(X, \Omega^1_X)$

In this subsection we present a machinery necessary for the proofs of results of the chapter three.

By using the cohomology formulas above, we compute the cohomology groups

$$H^p\big(X, \Omega^q_X(t)\big), \text{ with } p, q \in \{1, 2\} \text{ and } p \neq q,$$

where X is each one of the varieties described in Section 2.1.

For $\mathbb{P}^3$ and Q^3, the classical Bott's formula and the Bott's formula for quadric, respectively, are enough.

Thus, we begin the calculations considering X a cubic hypersurface.

(c) $X_3 \subset \mathbb{P}^4$.

By using the sequences (2.4) and (2.5), we have the exact sequences:

$$0 \ \to \ \Omega^1_{\mathbb{P}^4}(t-3) \ \to \ \Omega^1_{\mathbb{P}^4}(t) \ \to \ \Omega^1_{\mathbb{P}^4}(t)|_{X_3} \ \to \ 0, \tag{2.9}$$

$$0 \ \to \ \mathscr{O}_{X_3}(t-3) \ \to \ \Omega^1_{\mathbb{P}^4}(t)|_{X_3} \ \to \ \Omega^1_{X_3}(t) \ \to \ 0. \tag{2.10}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \ \to \ H^2(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(t)) \ \to \ H^2(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(t)|_{X_3}) \ \to \ H^3(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(t-3)) \ \to \cdots, \tag{2.11}$$

$$\cdots \to H^2(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(t)|_{X_3}) \to H^2(X_3, \Omega^1_{\mathbb{P}^4}(t)|_{X_3}) \to H^3(X_3, \mathcal{O}_{X_3}(t-3)) \to \cdots . \quad (2.12)$$

By classical Bott's formula we have $H^2(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(t)) = H^3(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(t-3)) = 0$, for all t. Thus, by sequence (2.11) we obtain $H^2(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(t)|_{X_3}) = 0$. Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^4}(t-6) \to \mathcal{O}_{\mathbb{P}^4}(t-3) \to \mathcal{O}_{X_3}(t-3) \to 0,$$

we get the following exact sequence

$$\cdots \to H^3(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t-3)) \to H^3(X_3, \mathcal{O}_{X_3}(t-3)) \to H^4(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t-6)) \to \cdots .$$

Again by classical Bott's formula we have that $H^3(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t-3)) = 0$, for all t and $H^4(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t-6)) = 0$ if $t > 1$. Thus, follow that $H^3(X_3, \mathcal{O}_{X_3}(t-3)) = 0$ if $t > 1$, and therefore we conclude that, $H^2(X_3, \Omega^1_{X_3}(t)) = 0$ for $t > 1$.

(d) $X_{2,2} \subset \mathbb{P}^5$.

Recall that $X_{2,2}$ is an intersection of two quadric hypersurfaces $X_2 \simeq Q^4$ in $\mathbb{P}^5$. Thus, $X_{2,2} \subset X_2 \subset \mathbb{P}^5$. For $X_2 \subset \mathbb{P}^5$ we use the Bott's formula for quadric (Theorem 2.6).

By using the sequences (2.4) and (2.5) for $X_{2,2} \subset X_2$, we have the exact sequences:

$$0 \to \Omega^1_{X_2}(t-2) \to \Omega^1_{X_2}(t) \to \Omega^1_{X_2}(t)|_{X_{2,2}} \to 0, \quad (2.13)$$

$$0 \to \mathcal{O}_{X_{2,2}}(t-2) \to \Omega^1_{X_2}(t)|_{X_{2,2}} \to \Omega^1_{X_{2,2}}(t) \to 0. \quad (2.14)$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(X_2, \Omega^1_{X_2}(t)) \to H^2(X_2, \Omega^1_{X_2}(t)|_{X_{2,2}}) \to H^3(X_2, \Omega^1_{X_2}(t-2)) \to \cdots , \quad (2.15)$$

$$\cdots \to H^2(X_{2,2}, \Omega^1_{X_2}(t)|_{X_{2,2}}) \to H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t)) \to H^3(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-2)) \to \cdots . \quad (2.16)$$

By Theorem 2.6 we have $H^2(X_2, \Omega^1_{X_2}(t)) = 0$, for all t and $H^3(X_2, \Omega^1_{X_2}(t-2)) = 0$ for $t \neq 0$. Thus, by sequence (2.15), we obtain $H^2(X_2, \Omega^1_{X_2}(t)|_{X_{2,2}}) = 0$ for $t \neq 0$. Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{X_2}(t-4) \to \mathcal{O}_{X_2}(t-2) \to \mathcal{O}_{X_{2,2}}(t-2) \to 0,$$

we get

$\cdots \to H^3(X_2, \mathcal{O}_{X_2}(t-2)) \to H^3(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-2)) \to H^4(X_2, \mathcal{O}_{X_2}(t-4)) \to \cdots$.

Again by Theorem 2.6 we have that

$$H^3(X_2, \mathcal{O}_{X_2}(t-2)) = 0 \text{ for all } t, \text{ and } H^4(X_2, \mathcal{O}_{X_2}(t-4)) = 0 \text{ if } t > 0.$$

Then, $H^3(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-2)) = 0$ for $t > 0$, and therefore we conclude that $H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t)) = 0$ for $t > 0$.

(e) $X_4 \subset \mathbb{P}(1,1,1,1,2) = \mathbb{P}$.

Recall that X_4 is a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1,1,1,1,2)$. By using the sequences (2.8) and (2.7), we have the exact sequences:

$$0 \to \overline{\Omega}^1_{\mathbb{P}}(t-4) \to \overline{\Omega}^1_{\mathbb{P}}(t) \to \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_4} \to 0. \tag{2.17}$$

$$0 \to \mathscr{O}_{X_4}(t-4) \to \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_4} \to \Omega^1_{X_4}(t) \to 0. \tag{2.18}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)) \to H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_4}) \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)) \to \cdots, \tag{2.19}$$

$$\cdots \to H^2(X_4, \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_4}) \to H^2(X_4, \Omega^1_{X_4}(t)) \to H^3(X_4, \mathscr{O}_{X_4}(t-4)) \to \cdots. \tag{2.20}$$

By Theorem 2.8 we have $H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)) = H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)) = 0$, for all t. Thus, by sequence (2.19) we obtain $H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_4}) = 0$. Now, taking cohomology of the sequence

$$0 \to \mathscr{O}_{\mathbb{P}}(t-8) \to \mathscr{O}_{\mathbb{P}}(t-4) \to \mathscr{O}_{X_4}(t-4) \to 0,$$

we get $\cdots \to H^3(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t-4)) \to H^3(X_4, \mathscr{O}_{X_4}(t-4)) \to H^4(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t-8)) \to \cdots$.

By Theorem 2.7 we have that $H^3(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t-4)) = 0$ for all t, and $H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-8)) = 0$ for $t > 2$. Then, $H^3(X_4, \mathscr{O}_{X_4}(t-4)) = 0$ for $t > 2$, and therefore we conclude that $H^2(X_4, \Omega^1_{X_4}(t)) = 0$ for $t > 2$.

(f) $X_6 \subset \mathbb{P}(1,1,1,2,3) = \mathbb{P}$.

Recall that X_6 is a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,2,3)$. By using the sequences (2.8) and (2.7), we have the exact sequences:

$$0 \to \overline{\Omega}^1_{\mathbb{P}}(t-6) \to \overline{\Omega}^1_{\mathbb{P}}(t) \to \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_6} \to 0. \tag{2.21}$$

$$0 \to \mathscr{O}_{X_6}(t-6) \to \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_6} \to \Omega^1_{X_6}(t) \to 0. \tag{2.22}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)) \to H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_6}) \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)) \to \cdots, \tag{2.23}$$

$$\cdots \to H^2(X_6, \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_6}) \to H^2(X_6, \Omega^1_{X_6}(t)) \to H^3(X_6, \mathscr{O}_{X_6}(t-6)) \to \cdots. \tag{2.24}$$

By Theorem 2.8 we get $H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)) = H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)) = 0$, for all t. Thus, by sequence (2.23) we obtain that $H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_6}) = 0$. Now, taking cohomology of the sequence

$$0 \to \mathscr{O}_{\mathbb{P}}(t-12) \to \mathscr{O}_{\mathbb{P}}(t-6) \to \mathscr{O}_{X_6}(t-6) \to 0,$$

we have $\cdots \to H^3(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t-6)) \to H^3(X_6, \mathscr{O}_{X_6}(t-6)) \to H^4(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t-12)) \to \cdots$.

By Theorem 2.7 we have that $H^3(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t-4)) = 0$ for all t, and $H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-12)) = 0$ for $t > 4$. Thus, follow that $H^3(X_6, \mathscr{O}_{X_6}(t-6)) = 0$ for $t > 4$, and therefore we conclude that $H^2(X_6, \Omega^1_{X_6}(t)) = 0$ for $t > 4$.

(g) $X_{2,3} \subset \mathbb{P}^5$.

Recall that $X_{2,3}$ is an intersection of a quadric X_2 and a cubic X_3 in $\mathbb{P}^5$.

We have $X_{2,3} \simeq X_2 \cap X_3 \subset X_2 \subset \mathbb{P}^5$ and $X_{2,3} \simeq X_2 \cap X_3 \subset X_3 \subset \mathbb{P}^5$. Let's consider the two cases:

$$(\bullet)\ X_{2,3} \subset X_2 \subset \mathbb{P}^5, \quad (\bullet\bullet)\ X_{2,3} \subset X_3 \subset \mathbb{P}^5.$$

($\bullet$) For $X_2 \subset \mathbb{P}^5$ we use the Bott's formula for quadric (Theorem 2.6). By using the sequences (2.4) and (2.5) for $X_{2,3} \subset X_2$, we have the exact sequences:

$$0 \to \Omega^1_{X_2}(t-3) \to \Omega^1_{X_2}(t) \to \Omega^1_{X_2}(t)|_{X_{2,3}} \to 0, \tag{2.25}$$

$$0 \to \mathcal{O}_{X_{2,3}}(t-3) \to \Omega^1_{X_2}(t)|_{X_{2,3}} \to \Omega^1_{X_{2,3}}(t) \to 0. \tag{2.26}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(X_2, \Omega^1_{X_2}(t)) \to H^2(X_2, \Omega^1_{X_2}(t)|_{X_{2,3}}) \to H^3(X_2, \Omega^1_{X_2}(t-3)) \to \cdots, \tag{2.27}$$

$$\cdots \to H^2(X_{2,3}, \Omega^1_{X_2}(t)|_{X_{2,3}}) \to H^2(X_{2,3}, \Omega^1_{X_{2,3}}(t)) \to H^3(X_{2,3}, \mathcal{O}_{X_{2,3}}(t-3)) \to \cdots. \tag{2.28}$$

By Theorem 2.6 we have $H^2(X_2, \Omega^1_{X_2}(t)) = 0$ for all t and $H^3(X_2, \Omega^1_{X_2}(t-3)) = 0$ for $t \neq 1$. Thus, by sequence (2.27) we obtain that $H^2(X_2, \Omega^1_{X_2}(t)|_{X_{2,3}}) = 0$ for $t \neq 1$. Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{X_2}(t-6) \to \mathcal{O}_{X_2}(t-3) \to \mathcal{O}_{X_{2,3}}(t-3) \to 0,$$

we have

$$\cdots \to H^3(X_2, \mathcal{O}_{X_2}(t-3)) \to H^3(X_{2,3}, \mathcal{O}_{X_{2,3}}(t-3)) \to H^4(X_2, \mathcal{O}_{X_2}(t-6)) \to \cdots.$$

Again by Theorem 2.6 we get that

$$H^3(X_2, \mathcal{O}_{X_2}(t-3)) = 0 \text{ for all } t, \text{ and } H^4(X_2, \mathcal{O}_{X_2}(t-6)) = 0 \text{ for } t > 2.$$

Then we get $H^3(X_{2,3}, \mathcal{O}_{X_{2,3}}(t-3)) = 0$ for $t > 2$, and therefore we conclude that $H^2(X_{2,3}, \Omega^1_{X_{2,3}}(t)) = 0$ for $t > 2$.

(••) Since we don't have a Bott's formula for a cubic hypersurface, we first calculate the cohomology considering $X_3 \subset \mathbb{P}^5$ and after this, we will calculate the cohomology considering $X_{2,3} \subset X_3$.

By using the sequences (2.4) and (2.5) for $X_3 \subset \mathbb{P}^5$, we have the exact sequences:

$$0 \to \Omega^1_{\mathbb{P}^5}(t-3) \to \Omega^1_{\mathbb{P}^5}(t) \to \Omega^1_{\mathbb{P}^5}(t)|_{X_3} \to 0, \tag{2.29}$$

$$0 \to \mathcal{O}_{X_3}(t-3) \to \Omega^1_{\mathbb{P}^5}(t)|_{X_3} \to \Omega^1_{X_3}(t) \to 0. \tag{2.30}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t)) \to H^2(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t)|_{X_3}) \to H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-3)) \to \cdots, \tag{2.31}$$

$$\cdots \to H^2(X_3, \Omega^1_{\mathbb{P}^5}(t)|_{X_3}) \to H^2(X_3, \Omega^1_{X_3}(t)) \to H^3(X_3, \mathcal{O}_{X_3}(t-3)) \to \cdots. \tag{2.32}$$

By classical Bott's formula we have $H^2(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t)) = H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-3)) = 0$, for all t. Thus, by sequence (2.31) we obtain $H^2(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t)|_{X_3}) = 0$, for all t. Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^5}(t-6) \to \mathcal{O}_{\mathbb{P}^5}(t-3) \to \mathcal{O}_{X_3}(t-3) \to 0,$$

we have $\cdots \to H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-3)) \to H^3(X_3, \mathcal{O}_{X_3}(t-3)) \to H^4(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-6)) \to \cdots.$

Again by classical Bott's formula we get $H^3(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-3)) = H^4(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-6)) = 0$, for all t. Then, $H^3(X_3, \mathcal{O}_{X_3}(t-3)) = 0$, for all t and by sequence (2.32) we obtain $H^2(X_3, \Omega^1_{X_3}(t)) = 0$, for all t.

Now, by using the sequences (2.4) and (2.5) for $X_{2,3} \subset X_3$, we have the exact sequences:

$$0 \to \Omega^1_{X_3}(t-2) \to \Omega^1_{X_3}(t) \to \Omega^1_{X_3}(t)|_{X_{2,3}} \to 0, \quad (2.33)$$

$$0 \to \mathcal{O}_{X_{2,3}}(t-2) \to \Omega^1_{X_3}(t)|_{X_{2,3}} \to \Omega^1_{X_{2,3}}(t) \to 0. \quad (2.34)$$

Consider the following piece of the long exact cohomology sequence obtained of (2.33):

$$\cdots \to H^2(X_3, \Omega^1_{X_3}(t)) \to H^2(X_3, \Omega^1_{X_3}(t)|_{X_{2,3}}) \to H^3(X_3, \Omega^1_{X_3}(t-2)) \to \cdots . \quad (2.35)$$

By development of the first part, we conclude that $H^2(X_3, \Omega^1_{X_2}(t)) = 0$, for all t. We will calculate $H^3(X_3, \Omega^1_{X_3}(t-2))$. Consider the exact sequences:

$$0 \to \Omega^1_{\mathbb{P}^5}(t-5) \to \Omega^1_{\mathbb{P}^5}(t-2) \to \Omega^1_{\mathbb{P}^5}(t-2)|_{X_3} \to 0, \quad (2.36)$$

$$0 \to \mathcal{O}_{X_3}(t-5) \to \Omega^1_{\mathbb{P}^5}(t-2)|_{X_3} \to \Omega^1_{X_3}(t-2) \to 0. \quad (2.37)$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-2)) \to H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-2)|_{X_3}) \to H^4(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-5)) \to \cdots , \quad (2.38)$$

$$\cdots \to H^3(X_3, \Omega^1_{\mathbb{P}^5}(t-2)|_{X_3}) \to H^3(X_3, \Omega^1_{X_3}(t-2)) \to H^4(X_3, \mathcal{O}_{X_3}(t-5)) \to \cdots . \quad (2.39)$$

By classical Bott's formula we have $H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-2)) = H^4(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-5)) = 0$, for all t.

Thus, by sequence (2.38) we obtain that $H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-2)|_{X_3}) = 0$, for all t. Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^5}(t-8) \to \mathcal{O}_{\mathbb{P}^5}(t-5) \to \mathcal{O}_{X_3}(t-5) \to 0,$$

we have $\cdots \to H^4(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-5)) \to H^4(X_3, \mathcal{O}_{X_3}(t-5)) \to H^5(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-8)) \to \cdots$.

By classical Bott's formula we have

$$H^4(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-5)) = 0, \text{ for all } t \text{ and } H^5(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-8)) = 0, \text{ for } t > 2.$$

Then, $H^4(X_3, \mathcal{O}_{X_3}(t-5)) = 0$ for $t > 2$. Returning in the sequence (2.39) we get $H^3(X_3, \Omega^1_{X_3}(t-2)) = 0$ for $t > 2$. Returning in the sequence (2.35) we have

$$H^2(X_3, \Omega^1_{X_3}(t)|_{X_{2,3}}) = 0 \text{ for } t > 2.$$

And finally, consider the following piece of the long exact cohomology sequence obtained of (2.34):

$$\cdots \to H^2(X_{2,3}, \Omega^1_{X_3}(t)|_{X_{2,3}}) \to H^2(X_{2,3}, \Omega^1_{X_{2,3}}(t)) \to H^3(X_{2,3}, \mathcal{O}_{X_{2,3}}(t-2)) \to \cdots . \tag{2.40}$$

By formula (2.3) we have $H^3(X_{2,3}, \mathcal{O}_{X_{2,3}}(t-2)) = 0$ for $t > 3$. Therefore we conclude that $H^2(X_{2,3}, \Omega_{X^1_{2,3}}(t)) = 0$ for $t > 3$.

(h) $X_{2,2,2} \subset \mathbb{P}^6$.

Recall that $X_{2,2,2}$ is an intersection of three quadrics in $\mathbb{P}^6$. Thus,

$$X_{2,2,2} \subset X_{2,2} \subset X_2 \subset \mathbb{P}^6.$$

For $X_2 \subset \mathbb{P}^6$ we use the Bott's formula for quadric (Theorem 2.6). We first calculate the cohomology considering $X_{2,2} \subset X_2$ and after this, we will calculate the cohomology considering $X_{2,2,2} \subset X_{2,2}$.

By using the sequences (2.4) and (2.5) for $X_{2,2} \subset X_2$, we have the exact sequences:

$$0 \to \Omega^1_{X_2}(t-2) \to \Omega^1_{X_2}(t) \to \Omega^1_{X_2}(t)|_{X_{2,2}} \to 0, \tag{2.41}$$

$$0 \to \mathcal{O}_{X_{2,2}}(t-2) \to \Omega^1_{X_2}(t)|_{X_{2,2}} \to \Omega^1_{X_{2,2}}(t) \to 0. \tag{2.42}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(X_2, \Omega^1_{X_2}(t)) \to H^2(X_2, \Omega^1_{X_2}(t)|_{X_{2,2}}) \to H^3(X_2, \Omega^1_{X_2}(t-2)) \to \cdots , \tag{2.43}$$

$$\cdots \to H^2(X_{2,2}, \Omega^1_{X_2}(t)|_{X_{2,2}}) \to H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t)) \to H^3(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-2)) \to \cdots . \tag{2.44}$$

By Theorem 2.6 we have that $H^2(X_2, \Omega^1_{X_2}(t)) = H^3(X_2, \Omega^1_{X_2}(t-2)) = 0$, for all t. Thus, by sequence (2.43) we obtain $H^2(X_2, \Omega^1_{X_2}(t)|_{X_{2,2}}) = 0$, for all t. Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{X_2}(t-4) \to \mathcal{O}_{X_2}(t-2) \to \mathcal{O}_{X_{2,2}}(t-2) \to 0,$$

we have

$$\cdots \to H^3(X_2, \mathcal{O}_{X_2}(t-2)) \to H^3(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-2)) \to H^4(X_2, \mathcal{O}_{X_2}(t-4)) \to \cdots .$$

Again by Theorem 2.6 we have that $H^3(X_2, \mathcal{O}_{X_2}(t-2)) = H^4(X_2, \mathcal{O}_{X_2}(t-4)) = 0$, for all t. Thus, we get $H^3(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-2)) = 0$, for all t, and by sequence (2.44), we have that $H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t)) = 0$, for all t.

Now, using the sequences (2.4) and (2.5) for $X_{2,2,2} \subset X_{2,2}$, we have the exact sequences:

$$0 \to \Omega^1_{X_{2,2}}(t-2) \to \Omega^1_{X_{2,2}}(t) \to \Omega^1_{X_{2,2}}(t)|_{X_{2,2,2}} \to 0, \tag{2.45}$$

$$0 \to \mathcal{O}_{X_{2,2,2}}(t-2) \to \Omega^1_{X_{2,2}}(t)|_{X_{2,2,2}} \to \Omega^1_{X_{2,2,2}}(t) \to 0. \tag{2.46}$$

Consider the following piece of the long exact cohomology sequence obtained of (2.45):

$$\cdots \to H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t)) \to H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t)|_{X_{2,2,2}}) \to H^3(X_{2,2}, \Omega^1_{X_{2,2}}(t-2)) \to \cdots. \tag{2.47}$$

By development above, we conclude that $H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t)) = 0$, for all t. We compute $H^3(X_{2,2}, \Omega^1_{X_{2,2}}(t-2))$. Consider the exact sequences:

$$0 \to \Omega^1_{X_2}(t-4) \to \Omega^1_{X_2}(t-2) \to \Omega^1_{X_2}(t-2)|_{X_{2,2}} \to 0, \tag{2.48}$$

$$0 \to \mathcal{O}_{X_{2,2}}(t-4) \to \Omega^1_{X_2}(t-2)|_{X_{2,2}} \to \Omega^1_{X_{2,2}}(t-2) \to 0. \tag{2.49}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^3(X_2, \Omega^1_{X_2}(t-2)) \to H^3(X_2, \Omega^1_{X_2}(t-2)|_{X_{2,2}}) \to H^4(X_2, \Omega^1_{X_2}(t-4)) \to \cdots, \tag{2.50}$$

$$\cdots \to H^3(X_{2,2}, \Omega^1_{X_2}(t-2)|_{X_{2,2}}) \to H^3(X_{2,2}, \Omega^1_{X_{2,2}}(t-2)) \to H^4(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-4)) \to \cdots. \tag{2.51}$$

By Bott's formula for quadric (Theorem 2.6) we have $H^3(X_2, \Omega^1_{X_2}(t-2)) = 0$ for all t, and $H^4(X_2, \Omega^1_{X_2}(t-4)) = 0$ for $t \neq 1$. Thus, by sequence (2.50) we have

$$H^3(X_2, \Omega^1_{X_2}(t-2)|_{X_{2,2}}) = 0 \text{ for } t \neq 1.$$

Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{X_2}(t-6) \to \mathcal{O}_{X_2}(t-4) \to \mathcal{O}_{X_{2,2}}(t-4) \to 0,$$

we have

$$\cdots \to H^4(X_2, \mathcal{O}_{X_2}(t-4)) \to H^4(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-4)) \to H^5(X_2, \mathcal{O}_{X_2}(t-6)) \to \cdots.$$

By Bott's formula for quadric we have

$$H^4(X_2, \mathcal{O}_{X_2}(t-4)) = 0 \text{ for all } t, \text{ and } H^5(X_2, \mathcal{O}_{X_2}(t-6)) = 0 \text{ for } t > 1.$$

Thus, we obtain that $H^4(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-4)) = 0$ for $t > 1$. Returning in the sequence (2.51) we get $H^3(X_{2,2}, \Omega^1_{X_{2,2}}(t-2)) = 0$ for $t > 1$. Returning in the sequence (2.47) we get $H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t)|_{X_{2,2,2}}) = 0$ for $t > 1$. And finally, consider the following piece of the long exact cohomology sequence obtained of (2.46):

$$\cdots \to H^2(X_{2,2,2}, \Omega^1_{X_{2,2}}(t)|_{X_{2,2,2}}) \to H^2(X_{2,2,2}, \Omega^1_{X_{2,2,2}}(t)) \to H^3(X_{2,2,2}, \mathcal{O}_{X_{2,2,2}}(t-2)) \to \cdots . \tag{2.52}$$

By formula (2.3) we have that $H^3(X_{2,2,2}, \mathcal{O}_{X_{2,2,2}}(t-2)) = 0$ for $t > 3$. Therefore, we conclude that $H^2(X_{2,2,2}, \Omega^1_{X_{2,2,2}}(t)) = 0$ for $t > 3$.

(i) $Y = C_2 \cap X_4 \subset \mathbb{P}(1,1,1,1,1,2) = \mathbb{P}$.

Recall that Y is an intersection of a quadratic cone and a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1,1,1,1,1,2)$. Thus, we have that $Y = C_2 \cap X_4 \subset C_2 \subset \mathbb{P}$ and $Y = C_2 \cap X_4 \subset X_4 \subset \mathbb{P}$. We first calculate the cohomology considering $C_2 \subset \mathbb{P}$ and after this, we will calculate the cohomology considering $Y \subset C_2$. Similarly, we calculate the cohomology considering $X_4 \subset \mathbb{P}$ and after this, we will calculate the cohomology considering $Y \subset X_4$.

By using the sequences (2.8) and (2.7) for $C_2 \subset \mathbb{P}$, we have the exact sequences:

$$0 \to \overline{\Omega}^1_{\mathbb{P}}(t-2) \to \overline{\Omega}^1_{\mathbb{P}}(t) \to \overline{\Omega}^1_{\mathbb{P}}(t)|_{C_2} \to 0. \tag{2.53}$$

$$0 \to \mathscr{O}_{C_2}(t-2) \to \overline{\Omega}^1_{\mathbb{P}}(t)|_{C_2} \to \Omega^1_{C_2}(t) \to 0. \tag{2.54}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)) \to H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)|_{C_2}) \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-2)) \to \cdots , \tag{2.55}$$

$$\cdots \to H^2(C_2, \overline{\Omega}^1_{\mathbb{P}}(t)|_{C_2}) \to H^2(C_2, \Omega^1_{C_2}(t)) \to H^3(C_2, \mathscr{O}_{C_2}(t-2)) \to \cdots . \tag{2.56}$$

By Theorem 2.8 we have that $H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)) = H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-2)) = 0$, for all t. Thus, by sequence (2.55) we obtain $H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)|_{C_2}) = 0$, for all t. Now, taking cohomology of the sequence

$$0 \to \mathscr{O}_{\mathbb{P}}(t-4) \to \mathscr{O}_{\mathbb{P}}(t-2) \to \mathscr{O}_{C_2}(t-2) \to 0,$$

we have $\cdots \to H^3(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t-2)) \to H^3(C_2, \mathscr{O}_{C_2}(t-2)) \to H^4(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t-4)) \to \cdots$.

By Theorem 2.7 we get that $H^3(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t-2)) = H^4(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(t-4)) = 0$, for all t. Thus, we have $H^3(C_2, \mathscr{O}_{C_2}(t-2)) = 0$ for all t, and therefore, $H^3(C_2, \mathscr{O}_{C_2}(t-2)) = 0$ for all t. Returning in the sequence (2.56) we get $H^2(C_2, \Omega^1_{C_2}(t)) = 0$ for all t.

Now, by using the sequences (2.4) and (2.5) for $Y \subset C_2$, we have the exact sequences:

$$0 \to \Omega^1_{C_2}(t-4) \to \Omega^1_{C_2}(t) \to \Omega^1_{C_2}(t)|_Y \to 0, \tag{2.57}$$

$$0 \to \mathcal{O}_Y(t-4) \to \Omega^1_{C_2}(t)|_Y \to \Omega^1_Y(t) \to 0. \tag{2.58}$$

Consider the following piece of the long exact cohomology sequence obtained of (2.57):

$$\cdots \to H^2(C_2, \Omega^1_{C_2}(t)) \to H^2(C_2, \Omega^1_{C_2}(t)|_Y) \to H^3(C_2, \Omega^1_{C_2}(t-4)) \to \cdots. \tag{2.59}$$

By development of the first part, we conclude that $H^2(C_2, \Omega^1_{C_2}(t)) = 0,$ for all t. We will calculate $H^3(C_2, \Omega^1_{C_2}(t-4))$. Consider the exact sequences:

$$0 \to \overline{\Omega}^1_{\mathbb{P}}(t-6) \to \overline{\Omega}^1_{\mathbb{P}}(t-4) \to \overline{\Omega}^1_{\mathbb{P}}(t-4)|_{C_2} \to 0, \tag{2.60}$$

$$0 \to \mathcal{O}_{C_2}(t-6) \to \overline{\Omega}^1_{\mathbb{P}}(t-4)|_{C_2} \to \Omega^1_{C_2}(t-4) \to 0. \tag{2.61}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)) \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)|_{C_2}) \to H^4(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)) \to \cdots, \tag{2.62}$$

$$\cdots \to H^3(C_2, \overline{\Omega}^1_{\mathbb{P}}(t-4)|_{C_2}) \to H^3(C_2, \Omega^1_{C_2}(t-4)) \to H^4(C_2, \mathcal{O}_{C_2}(t-6)) \to \cdots. \tag{2.63}$$

By Theorem 2.8 we have that $H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)) = H^4(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)) = 0$, for all t. Thus, by sequence (2.62) we get $H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)|_{C_2}) = 0$, for all t. Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(t-8) \to \mathcal{O}_{\mathbb{P}}(t-6) \to \mathcal{O}_{C_2}(t-6) \to 0,$$

we have $\cdots \to H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-6)) \to H^4(C_2, \mathcal{O}_{C_2}(t-6)) \to H^5(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-8))$.

By Theorem 2.7 we have $H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-6)) = 0$, for all t and $H^5(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-8)) = 0$ for $t > 1$. Thus we get $H^4(C_2, \mathcal{O}_{C_2}(t-6)) = 0$ for $t > 1$. Returning in the sequence (2.63) we get $H^3(C_2, \Omega^1_{C_2}(t-4)) = 0$ for $t > 1$. Returning in the sequence (2.59) we get $H^2(C_2, \Omega^1_{C_2}(t)|_Y) = 0$ for $t > 1$. And finally, consider the following piece of the long exact cohomology sequence obtained of (2.58):

$$\cdots \to H^2(Y, \Omega^1_{C_2}(t)|_Y) \to H^2(Y, \Omega^1_Y(t)) \to H^3(Y, \mathcal{O}_Y(t-4)) \to \cdots. \tag{2.64}$$

By formula (2.3) we have that $H^3(Y, \mathcal{O}_Y(t-4)) = 0$ for $t > 5$. Then, $H^2(Y, \Omega^1_Y(t)) = 0$ for $t > 5$.

By using the sequences (2.8) and (2.7) for $X_4 \subset \mathbb{P}$, we have the exact sequences:

$$0 \to \overline{\Omega}^1_{\mathbb{P}}(t-4) \to \overline{\Omega}^1_{\mathbb{P}}(t) \to \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_4} \to 0, \tag{2.65}$$

$$0 \to \mathcal{O}_{X_4}(t-4) \to \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_4} \to \Omega^1_{X_4}(t) \to 0. \tag{2.66}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)) \to H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_4}) \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)) \to \cdots, \tag{2.67}$$

$$\cdots \to H^2(X_4, \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_4}) \to H^2(X_4, \Omega^1_{X_4}(t)) \to H^3(X_4, \mathcal{O}_{X_4}(t-4)) \to \cdots. \tag{2.68}$$

By Theorem 2.8 we have that $H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t)) = H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)) = 0$, for all t. Thus, by sequence (2.67) we get $H^2(X_2, \overline{\Omega}^1_{\mathbb{P}}(t)|_{X_4}) = 0,\ \forall t$. Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(t-8) \to \mathcal{O}_{\mathbb{P}}(t-4) \to \mathcal{O}_{X_4}(t-4) \to 0,$$

we have $\cdots \to H^3(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-4)) \to H^3(X_4, \mathcal{O}_{X_4}(t-4)) \to H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-8)) \to \cdots$.

By Theorem 2.7 we have $H^3(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-4)) = H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-8)) = 0$, for all t. Thus, we have $H^3(X_4, \mathcal{O}_{X_4}(t-4)) = 0$ for all t, and therefore, $H^2(X_4, \Omega^1_{X_4}(t)) = 0$, for all t.

Now, using the sequences (2.4) and (2.5) for $Y \subset X_4$, we have the exact sequences:

$$0 \to \Omega^1_{X_4}(t-2) \to \Omega^1_{X_4}(t) \to \Omega^1_{X_4}(t)|_Y \to 0, \tag{2.69}$$

$$0 \to \mathcal{O}_Y(t-2) \to \Omega^1_{X_4}(t)|_Y \to \Omega^1_Y(t) \to 0. \tag{2.70}$$

Consider the following piece of the long exact cohomology sequence obtained of (2.69):

$$\cdots \to H^2(X_4, \Omega^1_{X_4}(t)) \to H^2(X_4, \Omega^1_{X_4}(t)|_Y) \to H^3(X_4, \Omega^1_{X_4}(t-2)) \to \cdots. \tag{2.71}$$

By development of the first part, we conclude that $H^2(X_4, \Omega^1_{X_4}(t)) = 0$. We will calculate $H^3(X_4, \Omega^1_{X_4}(t-2))$. Consider the exact sequences:

$$0 \to \overline{\Omega}^1_{\mathbb{P}}(t-6) \to \overline{\Omega}^1_{\mathbb{P}}(t-2) \to \overline{\Omega}^1_{\mathbb{P}}(t-2)|_{X_4} \to 0, \tag{2.72}$$

$$0 \to \mathcal{O}_{X_4}(t-6) \to \overline{\Omega}^1_{\mathbb{P}}(t-2)|_{X_4} \to \Omega^1_{X_4}(t-2) \to 0. \tag{2.73}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-2)) \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-2)|_{X_4}) \to H^4(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)) \to \cdots, \quad (2.74)$$

$$\cdots \to H^3(X_4, \overline{\Omega}^1_{\mathbb{P}}(t-2)|_{X_4}) \to H^3(X_4, \Omega^1_{X_4}(t-2)) \to H^4(X_4, \mathcal{O}_{X_4}(t-6)) \to \cdots. \quad (2.75)$$

By Theorem 2.8 we have that $H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-2)) = H^4(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)) = 0$, for all t. Thus, by sequence (2.74), $H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-2)|_{X_4})) = 0$, for all t. Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(t-10) \to \mathcal{O}_{\mathbb{P}}(t-6) \to \mathcal{O}_{X_4}(t-6) \to 0,$$

we have $\cdots \to H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-6)) \to H^4(X_4, \mathcal{O}_{X_4}(t-6)) \to H^5(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-10)) \to \cdots$.

By Theorem 2.7 we have $H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-6)) = 0$, for all t and $H^5(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-10)) = 0$ for $t > 3$. Thus, we get $H^4(X_4, \mathcal{O}_{X_4}(t-6)) = 0$ for $t > 3$. Returning in the sequence (2.75) we get $H^3(X_4, \Omega^1_{X_4}(t-2)) = 0$ for $t > 3$. Returning in the sequence (2.71) we get $H^2(X_4, \Omega^1_{X_4}(t)|_Y) = 0$ for $t > 3$. And finally, consider the following piece of the long exact cohomology sequence obtained of (2.70):

$$\cdots \to H^2(Y, \Omega^1_{X_4}(t)|_Y) \to H^2(Y, \Omega^1_Y(t)) \to H^3(Y, \mathcal{O}_Y(t-2)) \to \cdots. \quad (2.76)$$

By formula (2.3) we have that $H^3(Y, \mathcal{O}_Y(t-2)) = 0$ for $t > 3$. Therefore, we conclude that $H^2(Y, \Omega^1_Y(t)) = 0$ for $t > 3$.

Proposition 2.10. *The above computations are summarized in the following table:*

X	ι_X	$H^2(X, \Omega^1_X(t))$	t
X_3	2	0	$t > 1$
$X_{2,2}$	2	0	$t > 0$
X_4	2	0	$t > 2$
X_6	2	0	$t > 4$
$X_{2,3}$	1	0	$t > 2$ if $X_{2,3} \subset X_2$ and $t > 3$ if $X_{2,3} \subset X_3$
$X_{2,2,2}$	1	0	$t > 3$
Y	1	0	$t > 5$ if $Y \subset C_2$ and $t > 3$ if $Y \subset X_4$

Table 2.1: Values of t for which $H^2(X, \Omega^1_X(t)) = 0$.

2.2.2 Calculations of $H^1(X,\Omega^2_X)$

As in the previous subsection, we will calculate the groups of cohomology $H^1(X,\Omega^2_X)$ for each one of the varieties described in Section 2.1, except for $\mathbb{P}^3$ and Q^3.

Then, we start the calculations considering X a cubic hypersurface.

(c) $X_3 \subset \mathbb{P}^4$.

By using the sequences (2.4) and (2.5), we have the exact sequences:

$$0 \;\to\; \Omega^2_{\mathbb{P}^4}(t-3) \;\to\; \Omega^2_{\mathbb{P}^4}(t) \;\to\; \Omega^2_{\mathbb{P}^4}(t)|_{X_3} \;\to\; 0, \tag{2.77}$$

$$0 \;\to\; \Omega^1_{X_3}(t-3) \;\to\; \Omega^2_{\mathbb{P}^4}(t)|_{X_3} \;\to\; \Omega^2_{X_3}(t) \;\to\; 0. \tag{2.78}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \;\to\; H^1(\mathbb{P}^4,\Omega^2_{\mathbb{P}^4}(t)) \;\to\; H^1(\mathbb{P}^4,\Omega^2_{\mathbb{P}^4}(t)|_{X_3}) \;\to\; H^2(\mathbb{P}^4,\Omega^2_{\mathbb{P}^4}(t-3)) \;\to \cdots, \tag{2.79}$$

$$\cdots \;\to\; H^1(X_3,\Omega^2_{\mathbb{P}^4}(t)|_{X_3}) \;\to\; H^2(X_3,\Omega^2_{X_3}(t)) \;\to\; H^2(X_3,\Omega^1_{X_3}(t-3)) \;\to\; \cdots. \tag{2.80}$$

By classical Bott's formula we have that

$$H^1(\mathbb{P}^4,\Omega^2_{\mathbb{P}^4}(t)) = 0 \text{ for all } t, \text{ and } H^2(\mathbb{P}^4,\Omega^2_{\mathbb{P}^4}(t-3)) = 0 \text{ for } t \neq 3.$$

Thus, by sequence (2.79) we obtain $H^1(\mathbb{P}^4,\Omega^2_{\mathbb{P}^4}(t)|_{X_3}) = 0$ for $t \neq 3$. By subsection 2.2.1, we have that $H^2(X_3,\Omega^1_{X_3}(t)) = 0$ for $t > 1$.

Thus, we get $H^2(X_3,\Omega^1_{X_3}(t-3)) = 0$ for $t > 4$. Returning in the sequence (2.80), we have that $H^1(X_3,\Omega^2_{X_3}(t)) = 0$ for $t > 4$.

(d) $X_{2,2} \subset \mathbb{P}^5$.

By using the sequences (2.4) and (2.5), we have the exact sequences:

$$0 \;\to\; \Omega^2_{\mathbb{P}^5}(t-2) \;\to\; \Omega^2_{\mathbb{P}^5}(t) \;\to\; \Omega^2_{\mathbb{P}^5}(t)|_{X_{2,2}} \;\to\; 0, \tag{2.81}$$

$$0 \;\to\; \Omega^1_{X_{2,2}}(t-2) \;\to\; \Omega^2_{\mathbb{P}^5}(t)|_{X_{2,2}} \;\to\; \Omega^2_{X_{2,2}}(t) \;\to\; 0. \tag{2.82}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \;\to\; H^1(\mathbb{P}^5,\Omega^2_{\mathbb{P}^5}(t)) \;\to\; H^1(\mathbb{P}^5,\Omega^2_{\mathbb{P}^5}(t)|_{X_{2,2}}) \;\to\; H^2(\mathbb{P}^5,\Omega^2_{\mathbb{P}^5}(t-2)) \;\to \cdots, \tag{2.83}$$

$$\cdots \to H^1(X_{2,2}, \Omega^2_{\mathbb{P}^5}(t)|_{X_{2,2}}) \to H^2(X_{2,2}, \Omega^2_{X_{2,2}}(t)) \to H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t-2)) \to \cdots . \tag{2.84}$$

By classical Bott's formula we have that

$$H^1(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t)) = 0 \text{ for all } t, \text{ and } H^2(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t-2)) = 0 \text{ for } t \neq 2.$$

Thus, by sequence (2.83) we get $H^1(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t)|_{X_{2,2}}) = 0$ for $t \neq 2$. By subsection 2.2.1, we have that $H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t)) = 0$ for $t > 0$. Then, we get $H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t-2)) = 0$ for $t > 2$. Returning in the sequence (2.84), we obtain $H^1(X_{2,2}, \Omega^2_{X_{2,2}}(t)) = 0$ for $t > 2$.

(e) $X_4 \subset \mathbb{P}(1,1,1,1,2) = \mathbb{P}$.

By using the sequences (2.8) and (2.7), we have the exact sequences:

$$0 \to \overline{\Omega}^2_{\mathbb{P}}(t-4) \to \overline{\Omega}^2_{\mathbb{P}}(t) \to \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_4} \to 0. \tag{2.85}$$

$$0 \to \overline{\Omega}^1_{X_4}(t-4) \to \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_4} \to \Omega^2_{X_4}(t) \to 0. \tag{2.86}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)) \to H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_4}) \to H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-4)) \to \cdots , \tag{2.87}$$

$$\cdots \to H^1(X_4, \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_4}) \to H^1(X_4, \Omega^2_{X_4}(t)) \to H^2(X_4, \Omega^1_{X_4}(t-4)) \to \cdots . \tag{2.88}$$

By Theorem 2.8 we have that $H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)) = 0$ for all t, and $H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-4)) = 0$ for $t \neq 4$. Thus, by sequence (2.87) we obtain $H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_4}) = 0$, for $t \neq 4$. By subsection 2.2.1, we have that $H^2(X_4, \Omega^1_{X_4}(t)) = 0$ for $t > 2$. Thus, we get $H^2(X_4, \Omega^1_{X_4}(t-4)) = 0$ for $t > 6$. Returning in the sequence (2.88), we obtain that $H^1(X_4, \Omega^2_{X_4}(t)) = 0$ for $t > 6$.

(f) $X_6 \subset \mathbb{P}(1,1,1,2,3) = \mathbb{P}$.

By using the sequences (2.8) and (2.7), we have the exact sequences:

$$0 \to \overline{\Omega}^2_{\mathbb{P}}(t-6) \to \overline{\Omega}^2_{\mathbb{P}}(t) \to \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_6} \to 0. \tag{2.89}$$

$$0 \to \overline{\Omega}^1_{X_6}(t-6) \to \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_6} \to \Omega^2_{X_6}(t) \to 0. \tag{2.90}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \rightarrow H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)) \rightarrow H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_6}) \rightarrow H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-6)) \rightarrow \cdots, \tag{2.91}$$

$$\cdots \rightarrow H^1(X_6, \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_6}) \rightarrow H^1(X_6, \Omega^2_{X_6}(t)) \rightarrow H^2(X_6, \Omega^1_{X_6}(t-6)) \rightarrow \cdots. \tag{2.92}$$

By Theorem 2.8 we have that $H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)) = 0$ for all t, and $H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-6)) = 0$ for $t \neq 6$. Thus, by sequence (2.91), we get $H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_6}) = 0$, for $t \neq 6$. By subsection 2.2.1, we have that $H^2(X_6, \Omega^1_{X_6}(t)) = 0$ for $t > 4$. Then, we get $H^2(X_6, \Omega^1_{X_6}(t-6)) = 0$ for $t > 10$. Returning in the sequence (2.92), we obtain that $H^1(X_6, \Omega^2_{X_6}(t)) = 0$ for $t > 10$.

(g) $X_{2,3} \subset \mathbb{P}^5$.

As in the previous subsection, we will consider two cases.

(•) $X_{2,3} \subset X_2 \subset \mathbb{P}^5$.

For $X_2 \subset \mathbb{P}^5$ we use the Bott's formula for quadric (Theorem 2.6). by using the sequences (2.4) and (2.5) for $X_{2,3} \subset X_2$, we have the exact sequences:

$$0 \rightarrow \Omega^2_{X_2}(t-3) \rightarrow \Omega^2_{X_2}(t) \rightarrow \Omega^2_{X_2}(t)|_{X_{2,3}} \rightarrow 0. \tag{2.93}$$

$$0 \rightarrow \Omega^1_{X_{2,3}}(t-3) \rightarrow \Omega^2_{X_2}(t)|_{X_{2,3}} \rightarrow \Omega^2_{X_{2,3}}(t) \rightarrow 0. \tag{2.94}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \rightarrow H^1(X_2, \Omega^2_{X_2}(t)) \rightarrow H^1(X_2, \Omega^2_{X_2}(t)|_{X_{2,3}}) \rightarrow H^2(X_2, \Omega^2_{X_2}(t-3)) \rightarrow \cdots, \tag{2.95}$$

$$\cdots \rightarrow H^1(X_{2,3}, \Omega^2_{X_2}(t)|_{X_{2,3}}) \rightarrow H^1(X_{2,3}, \Omega^2_{X_{2,3}}(t)) \rightarrow H^2(X_{2,3}, \Omega^1_{X_{2,3}}(t-3)) \rightarrow \cdots. \tag{2.96}$$

By Theorem 2.6 we get $H^1(X_2, \Omega^2_{X_2}(t)) = 0$ for all t, and $H^2(X_2, \Omega^2_{X_2}(t-3)) = 0$ for $t \neq 3$. Thus, by sequence (2.95) we obtain $H^1(X_2, \Omega^2_{X_2}(t)|_{X_{2,3}}) = 0$, for $t \neq 3$. By subsection 2.2.1, we have that $H^2(X_{2,3}, \Omega^1_{X_{2,3}}(t)) = 0$ for $t > 2$. Thus, we get $H^2(X_{2,3}, \Omega^1_{X_{2,3}}(t-3)) = 0$ for $t > 5$. Returning in the sequence (2.96), we obtain that $H^1(X_{2,3}, \Omega^2_{X_{2,3}}(t)) = 0$ for $t > 5$.

(••) $X_{2,3} \subset X_3 \subset \mathbb{P}^5$.

Since we don't have a Bott's formula for a cubic hypersurface, we first calculate the cohomology considering $X_3 \subset \mathbb{P}^5$ and after this, we will calculate the cohomology considering $X_{2,3} \subset X_3$.

By using the sequences (2.4) and (2.5) for $X_3 \subset \mathbb{P}^5$, we have the exact sequences:

$$0 \to \Omega^2_{\mathbb{P}^5}(t-3) \to \Omega^2_{\mathbb{P}^5}(t) \to \Omega^2_{\mathbb{P}^5}(t)|_{X_3} \to 0, \tag{2.97}$$

$$0 \to \Omega^1_{X_3}(t-3) \to \Omega^2_{\mathbb{P}^5}(t)|_{X_3} \to \Omega^2_{X_3}(t) \to 0. \tag{2.98}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^1(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t)) \to H^1(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t)|_{X_3}) \to H^2(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t-3)) \to \cdots, \tag{2.99}$$

$$\cdots \to H^1(X_3, \Omega^2_{\mathbb{P}^5}(t)|_{X_3}) \to H^1(X_3, \Omega^2_{X_3}(t)) \to H^2(X_3, \Omega^1_{X_3}(t-3)) \to \cdots. \tag{2.100}$$

By classical Bott's formula we have that

$$H^1(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t)) = 0 \text{ for all } t, \text{ and } H^2(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t-3)) = 0 \text{ for } t \neq 3.$$

Thus, by sequence (2.99) we obtain $H^1(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t)|_{X_3}) = 0$ for $t \neq 3$. We will calculate $H^2(X_3, \Omega^1_{X_3}(t-3))$. Consider the exact sequences:

$$0 \to \Omega^1_{\mathbb{P}^5}(t-6) \to \Omega^1_{\mathbb{P}^5}(t-3) \to \Omega^1_{\mathbb{P}^5}(t-3)|_{X_3} \to 0, \tag{2.101}$$

$$0 \to \mathcal{O}_{X_3}(t-6) \to \Omega^1_{\mathbb{P}^5}(t-3)|_{X_3} \to \Omega^1_{X_3}(t-3) \to 0. \tag{2.102}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-3)) \to H^2(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-3)|_{X_3}) \to H^2(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-6)) \to \cdots, \tag{2.103}$$

$$\cdots \to H^2(X_3, \Omega^1_{\mathbb{P}^5}(t-3)|_{X_3}) \to H^2(X_3, \Omega^1_{X_3}(t-3)) \to H^3(X_3, \mathcal{O}_{X_3}(t-6)) \to \cdots. \tag{2.104}$$

By classical Bott's formula we get $H^2(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-3)) = H^2(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-6)) = 0$, for all t. Thus, by sequence (2.103) we get $H^2(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-3)|_{X_3}) = 0$, for all t.

Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^5}(t-9) \to \mathcal{O}_{\mathbb{P}^5}(t-6) \to \mathcal{O}_{X_3}(t-6) \to 0,$$

we have $\cdots \to H^3(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-6)) \to H^3(X_3, \mathcal{O}_{X_3}(t-6)) \to H^4(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-9)) \to \cdots$.

By classical Bott's formula we have $H^3(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-6)) = H^4(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-9)) = 0$, for all t. Thus, we get $H^3(X_3, \mathcal{O}_{X_3}(t-6)) = 0$, for all t. Returning in the sequence (2.104), we get $H^2(X_3, \Omega^1_{X_3}(t-3)) = 0$, for all t and returning in the sequence (2.100), we obtain $H^1(X_3, \Omega^2_{X_3}(t)) = 0$ for $t \neq 3$.

Now, by using the sequences (2.4) and (2.5) for $X_{2,3} \subset X_3$, we have the exact sequences:

$$0 \to \Omega^2_{X_3}(t-2) \to \Omega^2_{X_3}(t) \to \Omega^2_{X_3}(t)|_{X_{2,3}} \to 0, \tag{2.105}$$

$$0 \to \Omega^1_{X_{2,3}}(t-2) \to \Omega^2_{X_3}(t)|_{X_{2,3}} \to \Omega^2_{X_{2,3}}(t) \to 0. \tag{2.106}$$

Consider the following piece of the long exact cohomology sequence obtained of (2.105):

$$\cdots \to H^1(X_3, \Omega^2_{X_3}(t)) \to H^1(X_3, \Omega^2_{X_3}(t)|_{X_{2,3}}) \to H^2(X_3, \Omega^2_{X_3}(t-2)) \to \cdots. \tag{2.107}$$

By development of the first part, we conclude that $H^1(X_3, \Omega^2_{X_2}(t)) = 0$ for $t \neq 3$. We will calculate $H^2(X_3, \Omega^2_{X_3}(t-2))$. Consider the exact sequences:

$$0 \to \Omega^2_{\mathbb{P}^5}(t-5) \to \Omega^2_{\mathbb{P}^5}(t-2) \to \Omega^2_{\mathbb{P}^5}(t-2)|_{X_3} \to 0, \tag{2.108}$$

$$0 \to \Omega^1_{X_3}(t-5) \to \Omega^2_{\mathbb{P}^5}(t-2)|_{X_3} \to \Omega^2_{X_3}(t-2) \to 0. \tag{2.109}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t-5)) \to H^2(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t-2)|_{X_3}) \to H^3(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t-5)) \to \cdots, \tag{2.110}$$

$$\cdots \to H^2(X_3, \Omega^2_{\mathbb{P}^5}(t-2)|_{X_3}) \to H^2(X_3, \Omega^2_{X_3}(t-2)) \to H^3(X_3, \Omega^1_{X_3}(t-5)) \to \cdots. \tag{2.111}$$

By classical Bott's formula we have

$$H^2(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t-5)) \text{ for } t \neq 5, \text{ and } H^3(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t-5)) = 0 \text{ for all } t.$$

Thus, by sequence (2.110) we obtain that $H^2(\mathbb{P}^5, \Omega^2_{\mathbb{P}^5}(t-2)|_{X_3}) = 0$ for $t \neq 5$.

We will calculate $H^3(X_3, \Omega^1_{X_3}(t-5))$. Consider the exact sequences:

$$0 \to \Omega^1_{\mathbb{P}^5}(t-8) \to \Omega^1_{\mathbb{P}^5}(t-5) \to \Omega^1_{\mathbb{P}^5}(t-5)|_{X_3} \to 0, \tag{2.112}$$

$$0 \ \to \ \mathcal{O}_{X_3}(t-8) \ \to \ \Omega^1_{\mathbb{P}^5}(t-5)|_{X_3} \ \to \ \Omega^1_{X_3}(t-5) \ \to \ 0. \tag{2.113}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \ \to \ H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-5)) \ \to \ H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-5)|_{X_3}) \ \to \ H^4(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-8)) \ \to \ \cdots, \tag{2.114}$$

$$\cdots \ \to \ H^3(X_3, \Omega^1_{\mathbb{P}^5}(t-5)|_{X_3}) \ \to \ H^3(X_3, \Omega^1_{X_3}(t-5)) \ \to \ H^4(X_3, \mathcal{O}_{X_3}(t-8)) \ \to \ \cdots. \tag{2.115}$$

By classical Bott's formula we get $H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-5)) = H^4(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-8)) = 0$, for all t. Thus, by sequence (2.114) we get $H^3(\mathbb{P}^5, \Omega^1_{\mathbb{P}^5}(t-5)|_{X_3}) = 0$, for all t.

Now, taking cohomology of the sequence

$$0 \ \to \ \mathcal{O}_{\mathbb{P}^5}(t-11) \ \to \ \mathcal{O}_{\mathbb{P}^5}(t-8) \ \to \ \mathcal{O}_{X_3}(t-8) \ \to \ 0,$$

we have

$$\cdots \ \to \ H^4(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-8)) \ \to \ H^4(X_3, \mathcal{O}_{X_3}(t-8)) \ \to \ H^5(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-11)) \ \to \ \cdots.$$

By classical Bott's formula we have

$$H^4(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-8)) = 0 \text{ for all } t, \text{ and } H^5(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(t-11)) = 0 \text{ for } t > 5.$$

Then, we obtain $H^4(X_3, \mathcal{O}_{X_3}(t-8)) = 0$ for $t > 5$. Returning in the sequence (2.115), we get $H^3(X_3, \Omega^1_{X_3}(t-5)) = 0$ for $t > 5$. Returning in the sequence (2.111), we get $H^2(X_3, \Omega^2_{X_3}(t-2)) = 0$ for $t > 5$. Returning in the sequence (2.107), we have $H^1(X_3, \Omega^2_{X_3}(t)|_{X_{2,3}}) = 0$ for $t > 5$. And finally, consider the following piece of the long exact cohomology sequence obtained of (2.106):

$$\cdots \ \to \ H^1(X_{2,3}, \Omega^2_{X_3}(t)|_{X_{2,3}}) \ \to \ H^1(X_{2,3}, \Omega^2_{X_{2,3}}(t)) \ \to \ H^2(X_{2,3}, \Omega^1_{X_{2,3}}(t-2)) \ \to \ \cdots. \tag{2.116}$$

By subsection 2.2.1, we have that $H^2(X_{2,3}, \Omega^1_{X_{2,3}}(t)) = 0$ for $t > 3$. Thus, we get $H^2(X_{2,3}, \Omega^1_{X_{2,3}}(t-2)) = 0$ for $t > 5$. Therefore, we conclude that $H^1(X_{2,3}, \Omega^2_{X_{2,3}}(t)) = 0$ for $t > 5$.

(h)$X_{2,2,2} \subset \mathbb{P}^6$.

For $X_2 \subset \mathbb{P}^6$ we use the Bott's formula for quadric (Theorem 2.6). We first calculate the cohomology considering $X_{2,2} \subset X_2$ and after this, we will calculate the cohomology considering $X_{2,2,2} \subset X_{2,2}$.

By using the sequences (2.4) and (2.5) for $X_{2,2} \subset X_2$, we have the exact sequences:

$$0 \to \Omega^2_{X_2}(t-2) \to \Omega^2_{X_2}(t) \to \Omega^2_{X_2}(t)|_{X_{2,2}} \to 0, \tag{2.117}$$

$$0 \to \Omega^1_{X_{2,2}}(t-2) \to \Omega^2_{X_2}(t)|_{X_{2,2}} \to \Omega^2_{X_{2,2}}(t) \to 0. \tag{2.118}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^1(X_2, \Omega^2_{X_2}(t)) \to H^1(X_2, \Omega^2_{X_2}(t)|_{X_{2,2}}) \to H^2(X_2, \Omega^2_{X_2}(t-2)) \to \cdots, \tag{2.119}$$

$$\cdots \to H^1(X_{2,2}, \Omega^2_{X_2}(t)|_{X_{2,2}}) \to H^1(X_{2,2}, \Omega^2_{X_{2,2}}(t)) \to H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t-2)) \to \cdots. \tag{2.120}$$

By Theorem 2.6 we have that $H^1(X_2, \Omega^2_{X_2}(t)) = 0$, for all t and $H^2(X_2, \Omega^2_{X_2}(t-2)) = 0$ for $t \neq 2$. Thus, by sequence (2.119), we obtain $H^1(X_2, \Omega^2_{X_2}(t)|_{X_{2,2}}) = 0$ for $t \neq 2$.

We will calculate $H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t-2))$. Consider the exact sequences:

$$0 \to \Omega^1_{X_2}(t-4) \to \Omega^1_{X_2}(t-2) \to \Omega^1_{X_2}(t-2)|_{X_{2,2}} \to 0, \tag{2.121}$$

$$0 \to \mathcal{O}_{X_{2,2}}(t-4) \to \Omega^1_{X_2}(t-2)|_{X_{2,2}} \to \Omega^1_{X_{2,2}}(t-2) \to 0. \tag{2.122}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(X_2, \Omega^1_{X_2}(t-2)) \to H^2(X_2, \Omega^1_{X_2}(t-2)|_{X_{2,2}}) \to H^3(X_2, \Omega^1_{X_2}(t-4)) \to \cdots, \tag{2.123}$$

$$\cdots \to H^2(X_{2,2}, \Omega^1_{X_2}(t-2)|_{X_{2,2}}) \to H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t-2)) \to H^3(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-4)) \to \cdots. \tag{2.124}$$

By Bott's formula for quadric, we get $H^2(X_2, \Omega^1_{X_2}(t-2)) = H^3(X_2, \Omega^1_{X_2}(t-4)) = 0$, for all t. Thus, by sequence (2.123) we have $H^2(X_2, \Omega^1_{X_2}(t-2)|_{X_{2,2}}) = 0$, for all t.

Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{X_2}(t-6) \to \mathcal{O}_{X_2}(t-4) \to \mathcal{O}_{X_{2,2}}(t-4) \to 0,$$

we have

$$\cdots \to H^3(X_2, \mathcal{O}_{X_2}(t-4)) \to H^3(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-4)) \to H^4(X_2, \mathcal{O}_{X_2}(t-6)) \to \cdots.$$

Again by Theorem 2.6, we get $H^3(X_2, \mathcal{O}_{X_2}(t-4)) = H^4(X_2, \mathcal{O}_{X_2}(t-6)) = 0$, for all t. Thus, $H^3(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-4)) = 0$ for all t, and by sequence (2.124), we have that $H^2(X_{2,2}, \Omega^1_{X_{2,2}}(t-2)) = 0$, for all t.

Returning in the sequence (2.120), we get $H^1(X_{2,2}, \Omega^2_{X_{2,2}}(t)) = 0$, for $t \neq 2$.

Now, by using the sequences (2.4) and (2.5) for $X_{2,2,2} \subset X_{2,2}$, we have the exact sequences:

$$0 \to \Omega^2_{X_{2,2}}(t-2) \to \Omega^2_{X_{2,2}}(t) \to \Omega^2_{X_{2,2}}(t)|_{X_{2,2,2}} \to 0, \tag{2.125}$$

$$0 \to \Omega^1_{X_{2,2,2}}(t-2) \to \Omega^2_{X_{2,2}}(t)|_{X_{2,2,2}} \to \Omega^2_{X_{2,2,2}}(t) \to 0. \tag{2.126}$$

Consider the following piece of the long exact cohomology sequence obtained of (2.125):

$$\cdots \to H^1(X_{2,2}, \Omega^2_{X_{2,2}}(t)) \to H^1(X_{2,2}, \Omega^2_{X_{2,2}}(t)|_{X_{2,2,2}}) \to H^2(X_{2,2}, \Omega^2_{X_{2,2}}(t-2)) \to \cdots. \tag{2.127}$$

By development of the first part, we conclude that $H^1(X_{2,2}, \Omega^2_{X_{2,2}}(t)) = 0$, for $t \neq 2$. We will calculate $H^2(X_{2,2}, \Omega^2_{X_{2,2}}(t-2))$. Consider the exact sequences:

$$0 \to \Omega^2_{X_2}(t-4) \to \Omega^2_{X_2}(t-2) \to \Omega^2_{X_2}(t-2)|_{X_{2,2}} \to 0, \tag{2.128}$$

$$0 \to \Omega^1_{X_{2,2}}(t-4) \to \Omega^2_{X_2}(t-2)|_{X_{2,2}} \to \Omega^2_{X_{2,2}}(t-2) \to 0. \tag{2.129}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(X_2, \Omega^2_{X_2}(t-2)) \to H^2(X_2, \Omega^2_{X_2}(t-2)|_{X_{2,2}}) \to H^3(X_2, \Omega^1_{X_2}(t-4)) \to \cdots, \tag{2.130}$$

$$\cdots \to H^2(X_{2,2}, \Omega^2_{X_2}(t-2)|_{X_{2,2}}) \to H^2(X_{2,2}, \Omega^2_{X_{2,2}}(t-2)) \to H^3(X_{2,2}, \Omega^2_{X_{2,2}}(t-4)) \to \cdots. \tag{2.131}$$

By Bott's formula for quadric (Theorem 2.6), we get $H^2(X_2, \Omega^2_{X_2}(t-2)) = 0$, for $t \neq 2$ and $H^3(X_2, \Omega^2_{X_2}(t-4)) = 0$ for $t \neq 3$. Thus, by sequence (2.130), we obtain that $H^2(X_2, \Omega^2_{X_2}(t-2)|_{X_{2,2}}) = 0$ for $t \neq \{2, 3\}$.

We will calculate $H^3(X_{2,2}, \Omega^1_{X_{2,2}}(t-4))$. Consider the exact sequences:

$$0 \to \Omega^1_{X_2}(t-6) \to \Omega^1_{X_2}(t-4) \to \Omega^1_{X_2}(t-4)|_{X_{2,2}} \to 0, \tag{2.132}$$

$$0 \to \mathcal{O}X_{2,2}(t-6) \to \Omega^1_{X_2}(t-4)|_{X_{2,2}} \to \Omega^1_{X_{2,2}}(t-4) \to 0. \tag{2.133}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^3(X_2, \Omega^1_{X_2}(t-4)) \to H^3(X_2, \Omega^1_{X_2}(t-4)|_{X_{2,2}}) \to H^4(X_2, \Omega^1_{X_2}(t-6)) \to \cdots, \tag{2.134}$$

$$\cdots \to H^3(X_{2,2}, \Omega^1_{X_2}(t-4)|_{X_{2,2}}) \to H^3(X_{2,2}, \Omega^1_{X_{2,2}}(t-4)) \to H^4(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-6)) \to \cdots . \tag{2.135}$$

By Bott's formula for quadric (Theorem 2.6), we get

$$H^3(X_2, \Omega^1_{X_2}(t-4)) = 0 \text{ for all } t, \text{ and } H^4(X_2, \Omega^1_{X_2}(t-6)) = 0 \text{ for } t \neq 3.$$

Thus, by sequence (2.134), we have $H^3(X_2, \Omega^1_{X_2}(t-4)|_{X_{2,2}}) = 0$ for $t \neq 3$.

Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{X_2}(t-8) \to \mathcal{O}_{X_2}(t-6) \to \mathcal{O}_{X_{2,2}}(t-6) \to 0,$$

we have

$$\cdots \to H^4(X_2, \mathcal{O}_{X_2}(t-6)) \to H^4(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-6)) \to H^5(X_2, \mathcal{O}_{X_2}(t-8)) \to \cdots .$$

By Bott's formula for quadric we have

$$H^4(X_2, \mathcal{O}_{X_2}(t-6)) = 0 \text{ for all } t, \text{ and } H^5(X_2, \mathcal{O}_{X_2}(t-8)) = 0 \text{ for } t > 3.$$

Thus, $H^4(X_{2,2}, \mathcal{O}_{X_{2,2}}(t-6)) = 0$ for $t > 3$. Returning in the sequence (2.135), we get $H^3(X_{2,2}, \Omega^1_{X_{2,2}}(t-4)) = 0$ for $t > 3$. Returning in the sequence (2.131), we get $H^2(X_{2,2}, \Omega^2_{X_{2,2}}(t-2)) = 0$ for $t > 3$. Returning in the sequence (2.127), we get $H^1(X_{2,2}, \Omega^1_{X_{2,2}}(t)|_{X_{2,2,2}}) = 0$ for $t > 3$. And finally, consider the following piece of the long exact cohomology sequence obtained of (2.126):

$$\cdots \to H^1(X_{2,2,2}, \Omega^2_{X_{2,2}}(t)|_{X_{2,2,2}}) \to H^1(X_{2,2,2}, \Omega^2_{X_{2,2,2}}(t)) \to H^2(X_{2,2,2}, \Omega^1_{X_{2,2,2}}(t-2)) \to \cdots . \tag{2.136}$$

By subsection 2.2.1, we have that $H^2(X_{2,2,2}, \Omega^1_{X_{2,2,2}}(t)) = 0$ for $t > 3$. Thus, we have $H^2(X_{2,2,2}, \Omega^1_{X_{2,2,2}}(t-2)) = 0$ for $t > 5$.

Therefore, we conclude that $H^1(X_{2,2,2}, \Omega^2_{X_{2,2,2}}(t)) = 0$ for $t > 5$.

(i) $Y = C_2 \cap X_4 \subset \mathbb{P}(1,1,1,1,1,2) = \mathbb{P}$.

We first calculate the cohomology considering $C_2 \subset \mathbb{P}$ and after this, we will calculate the cohomology considering $Y \subset C_2$. Similarly, we calculate the cohomology considering $X_4 \subset \mathbb{P}$ and after this, we will calculate the cohomology considering $Y \subset X_4$.

By using the sequences (2.8) and (2.7) for $C_2 \subset \mathbb{P}$, we have the exact sequences:

$$0 \to \overline{\Omega}^2_{\mathbb{P}}(t-2) \to \overline{\Omega}^2_{\mathbb{P}}(t) \to \overline{\Omega}^2_{\mathbb{P}}(t)|_{C_2} \to 0. \tag{2.137}$$

$$0 \to \overline{\Omega}^1_{C_2}(t-2) \to \overline{\Omega}^2_{\mathbb{P}}(t)|_{C_2} \to \Omega^2_{C_2}(t) \to 0. \tag{2.138}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \;\to\; H^1(\mathbb{P},\overline{\Omega}^2_{\mathbb{P}}(t)) \;\to\; H^1(\mathbb{P},\overline{\Omega}^2_{\mathbb{P}}(t)|_{C_2}) \;\to\; H^2(\mathbb{P},\overline{\Omega}^2_{\mathbb{P}}(t-2)) \;\to\; \cdots, \tag{2.139}$$

$$\cdots \;\to\; H^1(C_2,\overline{\Omega}^2_{\mathbb{P}}(t)|_{C_2}) \;\to\; H^1(C_2,\Omega^2_{C_2}(t)) \;\to\; H^2(C_2,\Omega^1_{C_2}(t-2)) \;\to\; \cdots. \tag{2.140}$$

By Theorem 2.8, we have $H^1(\mathbb{P},\overline{\Omega}^2_{\mathbb{P}}(t)) = 0$ for all t, and $H^2(\mathbb{P},\overline{\Omega}^2_{\mathbb{P}}(t-2)) = 0$ for $t \neq 2$. Thus, by sequence (2.139), we get $H^1(\mathbb{P},\overline{\Omega}^2_{\mathbb{P}}(t)|_{C_2}) = 0$, for $t \neq 2$.

We will calculate $H^2(C_2,\Omega^1_{C_2}(t-2))$. Consider the exact sequences:

$$0 \;\to\; \overline{\Omega}^1_{\mathbb{P}}(t-4) \;\to\; \overline{\Omega}^1_{\mathbb{P}}(t-2) \;\to\; \overline{\Omega}^1_{\mathbb{P}}(t-2)|_{C_2} \;\to\; 0, \tag{2.141}$$

$$0 \;\to\; \mathcal{O}_{C_2}(t-4) \;\to\; \overline{\Omega}^1_{\mathbb{P}}(t-2)|_{C_2} \;\to\; \Omega^1_{C_2}(t-2) \;\to\; 0. \tag{2.142}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \;\to\; H^2(\mathbb{P},\overline{\Omega}^1_{\mathbb{P}}(t-2)) \;\to\; H^2(\mathbb{P},\overline{\Omega}^1_{\mathbb{P}}(t-2)|_{C_2}) \;\to\; H^3(\mathbb{P},\overline{\Omega}^1_{\mathbb{P}}(t-4)) \;\to\; \cdots, \tag{2.143}$$

$$\cdots \;\to\; H^2(C_2,\overline{\Omega}^1_{\mathbb{P}}(t-2)|_{C_2}) \;\to\; H^2(C_2,\Omega^1_{C_2}(t-2)) \;\to\; H^3(C_2,\mathcal{O}_{C_2}(t-4)) \;\to\; \cdots. \tag{2.144}$$

By Theorem 2.8, we get $H^2(\mathbb{P},\overline{\Omega}^1_{\mathbb{P}}(t-2)) = H^3(\mathbb{P},\overline{\Omega}^1_{\mathbb{P}}(t-4)) = 0$, for all t. Thus, by sequence (2.150), we get $H^2(\mathbb{P},\overline{\Omega}^1_{\mathbb{P}}(t-2))|_{C_2} = 0$, for all t.

Now, taking cohomology of the sequence

$$0 \;\to\; \mathscr{O}_{\mathbb{P}}(t-6) \;\to\; \mathscr{O}_{\mathbb{P}}(t-4) \;\to\; \mathscr{O}_{C_2}(t-4) \;\to\; 0,$$

we have $\cdots \;\to\; H^3(\mathbb{P},\mathscr{O}_{\mathbb{P}}(t-4)) \;\to\; H^3(C_2,\mathscr{O}_{C_2}(t-4)) \;\to\; H^4(\mathbb{P},\mathscr{O}_{\mathbb{P}}(t-6)) \;\to\; \cdots.$

By Theorem 2.7, we get $H^3(\mathbb{P},\mathscr{O}_{\mathbb{P}}(t-4)) = H^4(\mathbb{P},\mathscr{O}_{\mathbb{P}}(t-6)) = 0$, for all t. Then, we have $H^3(C_2,\mathscr{O}_{C_2}(t-4)) = 0$ for all t, and therefore, we get $H^2(C_2,\Omega^1_{C_2}(t-2)) = 0$ for all t. Returning in the sequence (2.140), we obtain $H^2(C_2,\Omega^2_{C_2}(t)) = 0$, for $t \neq 2$.

Now, by using the sequences (2.4) and (2.5) for $Y \subset C_2$, we have the exact sequences:

$$0 \;\to\; \Omega^2_{C_2}(t-4) \;\to\; \Omega^2_{C_2}(t) \;\to\; \Omega^2_{C_2}(t)|_Y \;\to\; 0, \tag{2.145}$$

$$0 \;\to\; \Omega^1_Y(t-4) \;\to\; \Omega^2_{C_2}(t)|_Y \;\to\; \Omega^2_Y(t) \;\to\; 0. \tag{2.146}$$

Consider the following piece of the long exact cohomology sequence obtained of (2.157):

$$\cdots \to H^1(C_2, \Omega^2_{C_2}(t)) \to H^1(C_2, \Omega^2_{C_2}(t)|_Y) \to H^2(C_2, \Omega^2_{C_2}(t-4)) \to \cdots. \quad (2.147)$$

By development of the first part, we conclude that $H^1(C_2, \Omega^2_{C_2}(t)) = 0$, for $t \neq 2$. We will calculate $H^2(C_2, \Omega^2_{C_2}(t-4))$. Consider the exact sequences:

$$0 \to \overline{\Omega}^2_{\mathbb{P}}(t-6) \to \overline{\Omega}^2_{\mathbb{P}}(t-4) \to \overline{\Omega}^2_{\mathbb{P}}(t-4)|_{C_2} \to 0, \quad (2.148)$$

$$0 \to \Omega^1_{C_2}(t-6) \to \overline{\Omega}^2_{\mathbb{P}}(t-4)|_{C_2} \to \Omega^2_{C_2}(t-4) \to 0. \quad (2.149)$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-4)) \to H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-4)|_{C_2}) \to H^3(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-6)) \to \cdots, \quad (2.150)$$

$$\cdots \to H^2(C_2, \overline{\Omega}^2_{\mathbb{P}}(t-4)|_{C_2}) \to H^2(C_2, \Omega^2_{C_2}(t-4)) \to H^3(C_2, \Omega^1_{C_2}(t-6)) \to \cdots. \quad (2.151)$$

By Theorem 2.8, we have $H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-4)) = 0$ for $t \neq 4$ and $H^3(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-6)) = 0$, for all t. Thus, by sequence (2.150), we get $H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-4)|_{C_2}) = 0$, for $t \neq 4$.

We will calculate $H^3(C_2, \Omega^1_{C_2}(t-6))$. Consider the exact sequences:

$$0 \to \overline{\Omega}^2_{\mathbb{P}}(t-8) \to \overline{\Omega}^1_{\mathbb{P}}(t-6) \to \overline{\Omega}^1_{\mathbb{P}}(t-6)|_{C_2} \to 0, \quad (2.152)$$

$$0 \to \mathscr{O}^1_{C_2}(t-8) \to \overline{\Omega}^1_{\mathbb{P}}(t-6)|_{C_2} \to \Omega^1_{C_2}(t-6) \to 0. \quad (2.153)$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)) \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)|_{C_2}) \to H^4(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-8)) \to \cdots, \quad (2.154)$$

$$\cdots \to H^3(C_2, \overline{\Omega}^1_{\mathbb{P}}(t-6)|_{C_2}) \to H^3(C_2, \Omega^1_{C_2}(t-6)) \to H^4(C_2, \mathscr{O}^1_{C_2}(t-8)) \to \cdots. \quad (2.155)$$

By Theorem 2.8, we get $H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)) = H^4(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-8)) = 0$, for all t. Thus, by sequence 2.154, we obtain $H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)|_{C_2}) = 0$, for all t.

Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(t-10) \to \mathcal{O}_{\mathbb{P}}(t-8) \to \mathcal{O}_{C_2}(t-8) \to 0,$$

we have $\cdots \to H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-8)) \to H^4(C_2, \mathcal{O}_{C_2}(t-8)) \to H^5(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-10)) \to \cdots$.

By Theorem 2.7, we get $H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-8)) = 0$, for all t and $H^5(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-10)) = 0$ for $t > 3$. Thus, we obtain $H^4(C_2, \mathcal{O}_{C_2}(t-8)) = 0$ for $t > 3$. Returning in the sequence (2.155), we get $H^3(C_2, \Omega^1_{C_2}(t-6)) = 0$ for $t > 3$. Returning in the sequence (2.151), we get $H^2(C_2, \Omega^2_{C_2}(t-4)) = 0$ for $t > 4$. Returning in the sequence (2.139), we have $H^1(C_2, \Omega^2_{C_2}(t)|_Y) = 0$ for $t > 4$. And finally, consider the following piece of the long exact cohomology sequence obtained of (2.138):

$$\cdots \to H^1(Y, \Omega^1_{C_2}(t)|_Y) \to H^1(Y, \Omega^2_Y(t)) \to H^2(Y, \Omega^1_Y(t-4)) \to \cdots . \tag{2.156}$$

By subsection 2.2.1, we have that $H^2(Y, \Omega^1_Y(t)) = 0$ for $t > 5$. Then, we get $H^2(Y, \Omega^1_Y(t-4)) = 0$ for $t > 9$. Therefore, we get $H^1(Y, \Omega^2_Y(t)) = 0$ for $t > 9$.

By using the sequences (2.8) and (2.7) for $X_4 \subset \mathbb{P}$, we have the exact sequences:

$$0 \to \overline{\Omega}^2_{\mathbb{P}}(t-4) \to \overline{\Omega}^2_{\mathbb{P}}(t) \to \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_4} \to 0, \tag{2.157}$$

$$0 \to \Omega^1_{X_4}(t-4) \to \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_4} \to \Omega^2_{X_4}(t) \to 0. \tag{2.158}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)) \to H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_4}) \to H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-4)) \to \cdots , \tag{2.159}$$

$$\cdots \to H^1(X_4, \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_4}) \to H^1(X_4, \Omega^2_{X_4}(t)) \to H^2(X_4, \Omega^1_{X_4}(t-4)) \to \cdots . \tag{2.160}$$

By Theorem 2.8, we get $H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)) = 0$, for all t and $H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-4)) = 0$ for $t \neq 4$. Thus, by sequence (2.159), we get $H^1(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t)|_{X_4}) = 0$, for $t \neq 4$.

We will calculate $H^2(X_4, \Omega^1_{X_4}(t-4))$. Consider the exact sequences:

$$0 \to \overline{\Omega}^1_{\mathbb{P}}(t-8) \to \overline{\Omega}^1_{\mathbb{P}}(t-4) \to \overline{\Omega}^1_{\mathbb{P}}(t-4)|_{X_4} \to 0, \tag{2.161}$$

$$0 \to \mathscr{O}^1_{X_4}(t-8) \to \overline{\Omega}^1_{\mathbb{P}}(t-4)|_{X_4} \to \Omega^1_{X_4}(t-4) \to 0. \tag{2.162}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)) \to H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)|_{X_4}) \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-8)) \to \cdots , \tag{2.163}$$

$$\cdots \to H^2(X_4, \overline{\Omega}^1_{\mathbb{P}}(t-4)|_{X_4}) \to H^2(X_4, \Omega^1_{X_4}(t-4)) \to H^3(X_4, \mathscr{O}_{X_4}(t-8)) \to \cdots . \tag{2.164}$$

By Theorem 2.8, we get $H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)) = H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-8)) = 0$, for all t. Thus, by sequence (2.163), we obtain $H^2(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-4)|_{X_4}) = 0$, for all t.

Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(t-12) \to \mathcal{O}_{\mathbb{P}}(t-8) \to \mathcal{O}_{X_4}(t-8) \to 0,$$

we have $\cdots \to H^3(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-8)) \to H^3(X_4, \mathcal{O}_{X_4}(t-8)) \to H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-12)) \to \cdots$.

By Theorem 2.7, we get $H^3(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-8)) = H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-12)) = 0$, for all t. Thus, we have $H^3(X_4, \mathcal{O}_{X_4}(t-8)) = 0$ for all t, and therefore, $H^2(X_4, \Omega^1_{X_4}(t-4)) = 0$, for all t. Returning in the sequence (2.160), we get $H^1(X_4, \Omega^2_{X_4}(t)) = 0$ for $t \neq 4$.

Now, by using the sequences (2.4) and (2.5) for $Y \subset X_4$, we have the exact sequences:

$$0 \to \Omega^2_{X_4}(t-2) \to \Omega^2_{X_4}(t) \to \Omega^2_{X_4}(t)|_Y \to 0, \tag{2.165}$$

$$0 \to \Omega^1_Y(t-2) \to \Omega^2_{X_4}(t)|_Y \to \Omega^2_Y(t) \to 0. \tag{2.166}$$

Consider the following piece of the long exact cohomology sequence obtained of (2.165):

$$\cdots \to H^1(X_4, \Omega^2_{X_4}(t)) \to H^1(X_4, \Omega^2_{X_4}(t)|_Y) \to H^2(X_4, \Omega^2_{X_4}(t-2)) \to \cdots. \tag{2.167}$$

By development of the first part, we conclude that $H^1(X_4, \Omega^2_{X_4}(t)) = 0$, for $t \neq 4$. We will calculate $H^2(X_4, \Omega^2_{X_4}(t-2))$. Consider the exact sequences:

$$0 \to \overline{\Omega}^2_{\mathbb{P}}(t-6) \to \overline{\Omega}^2_{\mathbb{P}}(t-2) \to \overline{\Omega}^2_{\mathbb{P}}(t-2)|_{X_4} \to 0, \tag{2.168}$$

$$0 \to \Omega^1_{X_4}(t-6) \to \overline{\Omega}^2_{\mathbb{P}}(t-2)|_{X_4} \to \Omega^2_{X_4}(t-2) \to 0. \tag{2.169}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-2)) \to H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-2)|_{X_4}) \to H^3(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-6)) \to \cdots, \tag{2.170}$$

$$\cdots \to H^2(X_4, \overline{\Omega}^2_{\mathbb{P}}(t-2)|_{X_4}) \to H^2(X_4, \Omega^2_{X_4}(t-2)) \to H^3(X_4, \Omega^1_{X_4}(t-6)) \to \cdots. \tag{2.171}$$

By Theorem 2.8, we get $H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-2)) = 0$ for $t \neq 2$ and $H^3(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-6)) = 0$, for all t. Thus, by sequence (2.170) we have $H^2(\mathbb{P}, \overline{\Omega}^2_{\mathbb{P}}(t-2)|_{X_4}) = 0$, for $t \neq 2$.

We will calculate $H^3(X_4, \Omega^1_{X_4}(t-6))$. Consider the exact sequences:

$$0 \to \overline{\Omega}^1_{\mathbb{P}}(t-10) \to \overline{\Omega}^1_{\mathbb{P}}(t-6) \to \overline{\Omega}^1_{\mathbb{P}}(t-6)|_{X_4} \to 0, \tag{2.172}$$

$$0 \to \mathscr{O}^1_{X_4}(t-6) \to \overline{\Omega}^1_{\mathbb{P}}(t-6)|_{X_4} \to \Omega^1_{X_4}(t-6) \to 0. \tag{2.173}$$

Taking cohomology of the above exact sequences we get:

$$\cdots \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)) \to H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)|_{X_4}) \to H^4(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-10)) \to \cdots, \tag{2.174}$$

$$\cdots \to H^3(X_4, \overline{\Omega}^1_{\mathbb{P}}(t-6)|_{X_4}) \to H^3(X_4, \Omega^1_{X_4}(t-6)) \to H^4(X_4, \mathscr{O}^1_{X_4}(t-6)) \to \cdots. \tag{2.175}$$

By Theorem 2.8 we have that $H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)) = H^4(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-10)) = 0$, for all t. Thus, by sequence (2.174), we get $H^3(\mathbb{P}, \overline{\Omega}^1_{\mathbb{P}}(t-6)|_{X_4}) = 0$, for all t.

Now, taking cohomology of the sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(t-10) \to \mathcal{O}_{\mathbb{P}}(t-6) \to \mathcal{O}_{X_4}(t-6) \to 0,$$

we have $\cdots \to H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-6)) \to H^4(X_4, \mathcal{O}_{X_4}(t-6)) \to H^5(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-10)) \to \cdots$.

By Theorem 2.7, we have $H^4(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-6)) = 0$ for all t, and $H^5(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(t-10)) = 0$ for $t > 3$. Thus, we obtain $H^4(X_4, \mathcal{O}_{X_4}(t-6)) = 0$ for $t > 3$. Returning in the sequence (2.175), we get $H^3(X_4, \Omega^1_{X_4}(t-6)) = 0$ for $t > 3$. Returning in the sequence (2.171), we get $H^2(X_4, \Omega^2_{X_4}(t-2)) = 0$ for $t > 3$. Returning the sequence (2.167), we get $H^1(X_4, \Omega^2_{X_4}(t)|_Y) = 0$ for $t > 4$. And finally, consider the following piece of the long exact cohomology sequence obtained of (2.166):

$$\cdots \to H^1(Y, \Omega^2_{X_4}(t)|_Y) \to H^1(Y, \Omega^2_Y(t)) \to H^2(Y, \Omega^1_Y(t-2)) \to \cdots. \tag{2.176}$$

By subsection 2.2.1, we have that $H^2(Y, \Omega^1_Y(t)) = 0$ for $t > 3$.

Thus, we get $H^2(Y, \Omega^1_Y(t-2)) = 0$ for $t > 5$.

Therefore, we conclude that $H^1(Y, \Omega^2_Y(t)) = 0$ for $t > 5$.

Proposition 2.11. *The above computations are summarized in the following table:*

X	ι_X	$H^1(X, \Omega^2_X(t))$	t
X_3	2	0	$t > 4$
$X_{2,2}$	2	0	$t > 2$
X_4	2	0	$t > 6$
X_6	2	0	$t > 10$
$X_{2,3}$	1	0	$t > 5$ if $X_{2,3} \subset X_2$ and $t > 5$ if $X_{2,3} \subset X_3$
$X_{2,2,2}$	1	0	$t > 5$
Y	1	0	$t > 9$ if $Y \subset C_2$ and $t > 5$ if $Y \subset X_4$

Table 2.2: Values of t for which $H^1(X, \Omega^2_X(t)) = 0$.

Chapter 3

Split distributions on Fano threefolds

In this Chapter we characterize the holomorphic distributions whose tangent sheaf and conormal sheaf are arithmetically Cohen Macaulay. Furthermore, we construct examples of codimension one distributions on a smooth weighted projective complete intersection Fano threedimensional.

3.1 Tangent sheaf vs. singular scheme

In this section we characterize when the tangent sheaf is split (i.e. direct sum of line bundles), in terms of the geometry of the singular scheme of the distribution. In addition we prove the Theorems A, B and C.

If $\iota_X = 4$, L. Giraldo and A. J. Pan-Collantes showed in [13] that the tangent sheaf of a foliation of dimension 2 on $\mathbb{P}^3$ splits if and only if its singular scheme Z is aCM:

Theorem 3.1. *[13, Theorem 3.3] The tangent sheaf $\mathscr{F}$ splits if and only if Z is an arithmetically Cohen-Macaulay curve.*

Definition 3.2. We say that E is a *split bundle* if it is (isomorphic to) the direct sum of two line bundles, i.e. $E = \mathcal{O}_X(a) \oplus \mathcal{O}_X(b)$ for suitable integers a and b.

Remark 3.3. Let E a bundle on Q^3. It is well known that if E splits on Q^3, then:

$$H^i(Q^3, E(t)) = 0 \ \text{ for } \ 0 < i < 3, \ \text{ for all } \ t \in \mathbb{Z}.$$

Lemma 3.4. *[1, Lemma 5.17] Let $X \subset \mathbb{P}$ be a weighted complete intersection of dimension $n \geq 3$, defined by weighted homogeneous polynomials $f_1, ..., f_c$ of degrees $d_1, ..., d_c$, with $2 \leq d_1 \leq d_2 \leq ... \leq d_c$. Then,*

$$H^0(X, \Omega_X^{q-1}(t)) = 0$$

for any $2 \leq q \leq n$ and $t \leq q - 1$.

Proposition 3.5. *Let $\mathscr{F}$ be a codimension one distribution on Q^3, with split tangent sheaf, such that $T_{\mathscr{F}} = \mathcal{O}_{Q^3}(a) \oplus \mathcal{O}_{Q^3}(b)$. Then $a, b < 1$.*

Proof: Indeed, we get

$$\begin{aligned} T_{\mathscr{F}} \hookrightarrow TQ^3 \in Hom(T_{\mathscr{F}}, TQ^3) &\simeq T^*_{\mathscr{F}} \otimes TQ^3 \\ &\simeq (\mathcal{O}_{Q^3}(-a) \otimes TQ^3) \oplus (\mathcal{O}_{Q^3}(-b) \otimes TQ^3). \end{aligned}$$

Thus, we have

$$H^0((\mathcal{O}_{Q^3}(-a) \otimes TQ^3) \oplus (\mathcal{O}_{Q^3}(-b) \otimes TQ^3)) = H^0(\mathcal{O}_{Q^3}(-a) \otimes TQ^3) \oplus H^0(\mathcal{O}_{Q^3}(-b) \otimes TQ^3),$$

and by Bott's fórmula for quadric, its has section when $a, b < 1$.

□

If $\iota_X = 3$, we characterize when the tangent sheaf of a distribution of dimension 2 on Q^3, is split or spinor. More precisely, we prove the following result:

Theorem 3.6. *Let $\mathscr{F}$ be a distribution on Q^3 of codimension one, such that the tangent sheaf $T_{\mathscr{F}}$ is locally free. If $T_{\mathscr{F}}$ either splits as a sum of line bundles or is a spinor bundle, then Z is arithmetically Buchsbaum, with $h^1(Q^3, I_Z(r-2)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$. Conversely, if Z is arithmetically Buchsbaum with $h^1(Q^3, I_Z(r-2)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$ and $h^2(T_{\mathscr{F}}(-2)) = h^2(T_{\mathscr{F}}(-1-c_1(T_{\mathscr{F}}))) = 0$, then $T_{\mathscr{F}}$ either split or is a spinor bundle.*

Proof: Suppose that $T_{\mathscr{F}}$ either split or is a spinor bundle. Consider, for each $t \in \mathbb{Z}$, the exact sequence

$$0 \to T_{\mathscr{F}}(t) \to TQ^3(t) \to \mathcal{I}_Z(r+t) \to 0, \tag{3.1}$$

where r is an integer such that $r = c_1(TQ^3) - c_1(T_{\mathscr{F}})$. Taking the long exact sequence of cohomology we get:

$$\begin{aligned} 0 \to H^0(Q^3, T_{\mathscr{F}}(t)) \to H^0(Q^3, TQ^3(t)) \to H^0(Q^3, \mathcal{I}_Z(r+t)) \to \\ \to H^1(Q^3, T_{\mathscr{F}}(t)) \to H^1(Q^3, TQ^3(t)) \to H^1(Q^3, \mathcal{I}_Z(r+t)) \to \\ \to H^2(Q^3, T_{\mathscr{F}}(t)) \to H^2(Q^3, TQ^3(t)) \to H^2(Q^3, \mathcal{I}_Z(r+t)) \to ... \end{aligned} \tag{3.2}$$

Since $T_{\mathscr{F}}$ has no intermediate cohomology, we have that

$$H^1(Q^3, T_{\mathscr{F}}(t)) = H^2(Q^3, T_{\mathscr{F}}(t)) = 0 \text{ for all } t \in \mathbb{Z}.$$

Thus, we get $H^1(Q^3, TQ^3(t)) \simeq H^1(Q^3, \mathcal{I}_Z(r+t))$.

By Bott's formula for quadric (Theorem 2.6), we have that $H^1(Q^3, TQ^3(t)) = 0$ for all $t \neq -2$, i.e. $H^1(Q^3, TQ^3(-2)) \neq 0$. Therefore, we conclude that $H^1(Q^3, \mathcal{I}_Z(r-2)) \neq 0$ and Z is arithmetically Buchsbaum.

Conversely, suppose that $h^2(T_{\mathscr{F}}(-2)) = h^2(T_{\mathscr{F}}(-1-c_1(T_{\mathscr{F}}))) = 0$ and that Z is arithmetically Buchsbaum, with $h^1(Q^3, I_Z(r-2)) = 1$ being the only nonzero intermediate cohomology.

Consider the long exact cohomology sequence (3.2) for all $t \neq -2$. By Theorem 2.6, we obtain that $H^1(Q^3, TQ^3(t)) = 0$, for all $t \neq -2$. Applying Serre duality and Bott's formula, respectively, we get $H^0(Q^3, \mathcal{I}_Z(r+t)) = H^3(\mathcal{O}_{Q^3}(-t-r-3)) = 0$, for all $r \neq 2$. Thus, we have $H^1(Q^3, T_{\mathscr{F}}(t)) = 0$.

By Serre duality, we conclude that

$$0 = H^1(Q^3, T_{\mathscr{F}}(t)) = H^2(Q^3, T_{\mathscr{F}}(s)), \text{ for all } s \neq -1 - c_1(T_{\mathscr{F}}),$$

where $s = -t-3-c_1(T_{\mathscr{F}})$. As by hypothesis $h^2(T_{\mathscr{F}}(-1-c_1(T_{\mathscr{F}}))) = 0$, we conclude that $H^2(Q^3, T_{\mathscr{F}}(s)) = 0$ for all s.

Now, consider the long exact cohomology sequence (3.2), with $t = -2$. By Bott's formula, we have that $H^0(Q^3, TQ^3(-2)) = 0$ and $H^2(Q^3, TQ^3(-2)) = 0$. By hypothesis, we get $h^2(T_{\mathscr{F}}(-2)) = 0$ and $h^1(Q^3, I_Z(r-2)) = 1$. Moreover, applying Serre duality and Bott's formula, respectively, we get $H^0(Q^3, \mathcal{I}_Z(r-2)) = H^3(\mathcal{O}_{Q^3}(-1-r)) = 0$, for all $r \neq 2$. So, from the exact sequence,

$$0 \to H^1(Q^3, T_{\mathscr{F}}(-2)) \to H^1(Q^3, TQ^3(-2)) \simeq \mathbb{C} \xrightarrow{\beta} H^1(Q^3, \mathcal{I}_Z(r-2)) \simeq \mathbb{C} \to 0,$$

we conclude that $H^1(Q^3, T_{\mathscr{F}}(-2)) = 0$, since β is injective and $ker(\beta) = H^1(Q^3, T_{\mathscr{F}}(-2))$. Therefore, $T_{\mathscr{F}}$ either split or is a spinor bundle, for all $t \in \mathbb{Z}$.

□

Remark 3.7. By Theorem (1.25) we have the isomorphism

$$\begin{array}{ccc} \mathbb{P}H^0(\mathbb{P}^4, \Omega^1_{\mathbb{P}^4}(2)) & \to & \mathbb{P}H^0(Q^3, \Omega^1_{Q^3}(2)) \\ \widetilde{\omega} & \to & \omega = \widetilde{\omega}_{|Q^3}. \end{array}$$

If the class, $k(\widetilde{\omega}) = 0$, then ω is the restriction of a degree zero distribution on $\mathbb{P}^4$. We get $\mathrm{Sing}(\widetilde{\omega}) = H \simeq \mathbb{P}^2 \subset \mathbb{P}^4$. Then, $\mathrm{Sing}(\omega) = Sing(\widetilde{\omega}_{|Q^3}) = \mathbb{P}^2 \cap Q^3 = Q^2$. Therefore, if $r = 2$, $\mathrm{Sing}(\omega)$ is ACM.

If $\iota_X = 2$, we characterize when the tangent sheaf of a distribution of dimension 2 on a smooth weighted projective complete intersection del Pezzo Fano threefold X, has no intermediate cohomology. More precisely, we prove the following results:

Lemma 3.8. *Let X be a smooth weighted projective complete intersection del Pezzo Fano threefold. Then, $H^2(X, \Omega^1_X(t)) = 0$ for $t > 4$, and $H^1(X, \Omega^2_X(t)) = 0$ for $t > 10$.*

Proof: By hypothesis, X is a smooth weighted projective complete intersection del Pezzo Fano threefold. Comparing the values of t for which $H^2(X, \Omega^1_X(t)) = 0$, with $\iota_X = 2$, in the table 2.10, we can see that the common vanishing of cohomology group for these varieties, occurs when $t > 4$. Similarly, comparing the values of t for which $H^1(X, \Omega^2_X(t)) = 0$, with $\iota_X = 2$, in the table 2.11, we can see that the common vanishing of cohomology group for these varieties, occurs when $t > 10$. □

Theorem 3.9. *Let $\mathscr{F}$ be a distribution of codimension one on a smooth weighted projective complete intersection del Pezzo Fano threefold X, such that the tangent sheaf $T_{\mathscr{F}}$ is locally free. If $T_{\mathscr{F}}$ has no intermediate cohomology, then $H^1(X, I_Z(r+t)) = 0$ for $t < -6$ and $t > 8$. Conversely, if $H^1(X, I_Z(r+t)) = 0$ for $t < -6$ and $t > 8$, and $H^2(X, T_{\mathscr{F}}(t)) = 0$ for $t \leq 8$ and $H^1(X, T_{\mathscr{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathscr{F}})$, then $T_{\mathscr{F}}$ has no intermediate cohomology.*

Proof: Suppose that $T_{\mathscr{F}}$ has no intermediate cohomology. Consider, for each $t \in \mathbb{Z}$, the exact sequence

$$0 \to T_{\mathscr{F}}(t) \to TX(t) \to \mathcal{I}_Z(r+t) \to 0, \tag{3.3}$$

where r is a integer such that $r = c_1(TX) - c_1(T_{\mathscr{F}})$. Taking the long exact sequence of cohomology we get:

$$\begin{aligned} 0 \to H^0(X, T_{\mathscr{F}}(t)) \to H^0(X, TX(t)) \to H^0(X, \mathcal{I}_Z(r+t)) \to \\ \to H^1(X, T_{\mathscr{F}}(t)) \to H^1(X, TX(t)) \to H^1(X, \mathcal{I}_Z(r+t)) \to \\ \to H^2(X, T_{\mathscr{F}}(t)) \to H^2(X, TX(t)) \to H^2(X, \mathcal{I}_Z(r+t)) \to ... \end{aligned} \tag{3.4}$$

Since $T_{\mathscr{F}}$ has no intermediate cohomology, we have that

$$H^1(X, T_{\mathscr{F}}(t)) = H^2(X, T_{\mathscr{F}}(t)) = 0 \text{ for all } t \in \mathbb{Z}.$$

Thus, we get $H^1(X, TX(t)) \simeq H^1(X, \mathcal{I}_Z(r+t))$.

By Remark 1.1, we have that $H^1(X, TX(t)) \simeq H^1(X, \Omega^2_X(t+2))$ and by Lemma 3.8, we get $H^1(X, TX(t)) = 0$, for $t > 8$. Moreover, by using Serre duality, we obtain $H^1(X, TX(t)) \simeq H^2(X, \Omega^1_X(-t-2))$ and thus, by Lemma 3.8, we get $H^1(X, TX(t)) = 0$, for $t < -6$.

Therefore, $H^1(X, \mathcal{I}_Z(r+t)) = 0$ for $t < -6$ and $t > 8$.

Conversely, suppose that $h^2(T_{\mathscr{F}}(t)) = 0$ for $t \leq 8$. Consider the long exact cohomology sequence (3.4). Applying Serre duality and Theorem 2.9, respectively, we get $H^2(X, TX(t)) = 0$ for $t > -2$. Since by hypothesis, $h^1(X, I_Z(r+t)) = 0$ for $t < -6$ and $t > 8$, we get $H^2(X, T_{\mathscr{F}}(t)) = 0$ for $t > 8$. Thus, $H^2(X, T_{\mathscr{F}}(t)) = 0$ for all $t \in \mathbb{Z}$.

By Serre Duality and by Remark 1.1, we conclude that

$$0 = H^2(X, T_{\mathscr{F}}(t)) = H^1(X, T_{\mathscr{F}}(s)),$$

where $s = -t - \iota_X - c_1(T_{\mathscr{F}})$. As by hipothesis $h^1(T_{\mathscr{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathscr{F}})$, we conclude that $H^1(X, T_{\mathscr{F}}(t)) = 0$ for all $t \in \mathbb{Z}$. Therefore, $T_{\mathscr{F}}$ has no intermediate cohomology.

□

If $\iota_X = 1$, we characterize when the tangent sheaf of a distribution of dimension 2 on a smooth weighted projective complete intersection prime Fano threefold X, has no intermediate cohomology. More precisely, we prove the following results:

Lemma 3.10. *Let X be a smooth weighted projective complete intersection prime Fano threefold. Then, $H^1(X, \Omega_X^2(t)) = 0$ for $t > 5$, and $H^2(X, \Omega_X^1(t)) = 0$ for $t > 3$.*

Proof: By hyphotesis, X is a smooth weighted projective complete intersection prime Fano threefold. Comparing the values of t for which $H^2(X, \Omega_X^1(t)) = 0$, with $\iota_X = 1$, in the table 2.10, we can see that these values coincide. Thus, we get $H^2(X, \Omega_X^1(t)) = 0$, for $t > 3$. Similarly, comparing the values of t for which $H^1(X, \Omega_X^2(t)) = 0$, with $\iota_X = 1$, in the table 2.11, we can see that these values also coincide. Thus, we have that $H^1(X, \Omega_X^2(t)) = 0$, for $t > 5$. □

Theorem 3.11. *Let $\mathscr{F}$ be a distribution of codimension one on a smooth weighted projective complete intersection prime Fano threefold X, such that the tangent sheaf $T_{\mathscr{F}}$ is locally free. If $T_{\mathscr{F}}$ has no intermediate cohomology, then $H^1(X, I_Z(r+t)) = 0$ for $t < -4$ and $t > 4$. Conversely, if $H^1(X, I_Z(r+t)) = 0$ for $t < -4$ and $t > 4$, and $H^2(X, T_{\mathscr{F}}(t)) = 0$ for $t \leq 4$ and $H^1(X, T_{\mathscr{F}}(s)) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathscr{F}})$, then $T_{\mathscr{F}}$ has no intermediate cohomology.*

Proof: Suppose that $T_{\mathscr{F}}$ has no intermediate cohomology. Consider, for each $t \in \mathbb{Z}$, the exact sequence (3.3) and the long exact sequence of cohomology (3.4). Since $T_{\mathscr{F}}$ has no intermediate cohomology,

$$H^1(X, T_{\mathscr{F}}(t)) = H^2(X, T_{\mathscr{F}}(t)) = 0 \text{ for all } t \in \mathbb{Z}.$$

Thus, we have $H^1(X, TX(t)) \simeq H^1(X, \mathcal{I}_Z(r+t))$.

By remark 1.1, we get that $H^1(X, TX(t)) \simeq H^1(X, \Omega^2_X(t+1))$ and by Lemma 3.10, we get $H^1(X, TX(t)) = 0$, for $t > 4$. Moreover, using Serre duality, we obtain that $H^1(X, TX(t)) \simeq H^2(X, \Omega^1_X(-t-1))$ and thus, by Lemma 3.10, we get $H^1(X, TX(t)) = 0$, for $t < -4$. Therefore, $H^1(X, \mathcal{I}_Z(r+t)) = 0$ for $t < -4$ and $t > 4$.

Conversely, suppose that $h^2(T_{\mathscr{F}}(t)) = 0$ for $t \leq 4$. Consider the long exact cohomology sequence (3.4). Applying Serre duality and Theorem 2.9, respectively, we get $H^2(X, TX(t)) = 0$ for $t > -1$. Since by hypothesis, $h^1(X, I_Z(r+t)) = 0$ for $t < -4$ and $t > 4$, we get $H^2(X, T_{\mathscr{F}}(t)) = 0$ for $t > 4$. Thus, we have $H^2(X, T_{\mathscr{F}}(t)) = 0 \; \forall \, t \in \mathbb{Z}$.

By Serre Duality and by Remark 1.1, we obtain that

$$0 = H^2(X, T_{\mathscr{F}}(t)) = H^1(X, T_{\mathscr{F}}(s)),$$

where $s = -t - \iota_X - c_1(T_{\mathscr{F}})$. As by hipothesis $h^1(T_{\mathscr{F}})(s) = 0$ for $s \neq -t - \iota_X - c_1(T_{\mathscr{F}})$, we conclude that $H^1(X, T_{\mathscr{F}}(t)) = 0$ for all $t \in \mathbb{Z}$. Therefore, $T_{\mathscr{F}}$ has no intermediate cohomology.

□

3.2 Foliations as subsheaves of the cotangent sheaf

In this section we prove the Theorems D and E.

Alternatively, we can define a foliation through a coherent subsheaf $N^*_{\mathscr{F}}$ of Ω^1_X such that

1. $N^*_{\mathscr{F}}$ is integrable ($dN^*_{\mathscr{F}} \subset N^*_{\mathscr{F}} \wedge \Omega^1_X$) and
2. the quocient $\Omega^1_X / N^*_{\mathscr{F}}$ is torsion free.

The codimension of $\mathscr{F}$ is the generic rank of $N^*_{\mathscr{F}}$.

M. Corrêa, M. Jardim and R. Vidal Martins, showed in [7] that the conormal sheaf of a foliation of dimension one on $\mathbb{P}^n$ splits if and only if its singular scheme is arithmetically Buchsbaum with $h^1(\mathcal{I}_Z(d-1)) = 1$ being the only nonzero intermediate cohomology:

Theorem 3.12. *[7, Theorem 5.2] Let $\mathscr{F}$ be an one-dimensional distribution on $\mathbb{P}^n$, of degree d, such that $\mathrm{cod}(\mathrm{Sing}(\mathscr{F})) = 2$. Suppose that $N^*_{\mathscr{F}}$ is locally free. Then $N^*_{\mathscr{F}}$ splits if and only if $\mathrm{Sing}(\mathscr{F}) = Z$ is arithmetically Buchsbaum with $h^1(\mathcal{I}_Z(d-1)) = 1$ being the only nonzero intermediate cohomology for $H^i_*(\mathcal{I}_Z)$ in the range $1 \leq i \leq n-2$.*

For all $\iota_X \in \{1, 2, 3, 4\}$ we prove the following result:

Theorem 3.13. *Let $\mathscr{F}$ be an one-dimensional distribution on a smooth weighted projective complete intersection Fano threefold X, with $\iota_X \in \{1,2,3,4\}$. If $N^*_{\mathscr{F}}$ is aCM, then Z is arithmetically Buchsbaum, with $h^1(X, I_Z(r)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$.*

Proof: Suppose that $N^*_{\mathscr{F}}$ is aCM and $\iota_X \in \{1,2,3,4\}$. For the case $\iota_X = 4$, i.e. $X \simeq \mathbb{P}^3$, the result follows from Theorem 3.12.

Consider, for each $t \in \mathbb{Z}$, the exact sequence

$$0 \to N^*_{\mathscr{F}}(t) \to \Omega^1_X(t) \to \mathcal{I}_Z(r+t) \to 0, \tag{3.5}$$

where r is a integer such that $r = c_1(\Omega^1_X) - c_1(N^*_{\mathscr{F}})$. Taking the long exact sequence of cohomology we get:

$$\begin{aligned} 0 \to H^0(X, N^*_{\mathscr{F}}(t)) \to H^0(X, \Omega^1_X(t)) \to H^0(X, \mathcal{I}_Z(r+t)) \to \\ \to H^1(X, N^*_{\mathscr{F}}(t)) \to H^1(X, \Omega^1_X(t)) \to H^1(X, \mathcal{I}_Z(r+t)) \to \\ \to H^2(X, N^*_{\mathscr{F}}(t)) \to H^2(X, \Omega^1_X(t)) \to H^2(X, \mathcal{I}_Z(r+t)) \to ... \end{aligned} \tag{3.6}$$

Since $N^*_{\mathscr{F}}$ has no intermediate cohomology, we have that

$$H^1(X, N^*_{\mathscr{F}}(t)) = H^2(X, N^*_{\mathscr{F}}(t)) = 0, \text{ for all } t \in \mathbb{Z}.$$

Thus, we get $H^1(X, \Omega^1_X(t)) \simeq H^1(X, \mathcal{I}_Z(r+t))$. By Theorem 2.9, $H^1(X, \Omega^1_X(t)) = 0$ for all $t \neq 0$, and $h^1(X, \Omega^1_X) = 1$. Then, we have $H^1(X, \mathcal{I}_Z(r)) \neq 0$ and Z is arithmetically Buchsbaum.

□

If X is a Fano threefold, meaning that $K_X^{-1} = \bigwedge^3 TX$ is ample, then the *Kodaira vanishing Theorem* shows that $H^q(X, \mathcal{O}_X) = 0$ and $H^q(X, \mathcal{O}(K_X^{-1})) = 0$ for $q > 0$, see [28]

If $\iota_X \in \{1,2,3\}$, we prove the following result:

Theorem 3.14. *Let $\mathscr{F}$ be an one-dimensional distribution on a smooth weighted projective complete intersection Fano threefold X, with index $\iota_X \in \{1,2,3\}$. If Z is arithmetically Buchsbaum with $h^1(X, I_Z(r)) = 1$ being the only nonzero intermediate cohomology for $H^i(I_Z)$, and $h^2(N^*_{\mathscr{F}}) = h^2(N^*_{\mathscr{F}}(-c_1(N^*_{\mathscr{F}}) - \iota_X)) = 0$, then $N^*_{\mathscr{F}}$ is aCM.*

Proof: Suppose that $h^2(N^*_{\mathscr{F}}) = h^2(N^*_{\mathscr{F}}(-c_1(N^*_{\mathscr{F}}) - \iota_X)) = 0$ and that Z is arithmetically Buchsbaum with $h^1(Q^3, I_Z(r)) = 1$ being the only nonzero intermediate cohomology.

Consider the long exact cohomology sequence (3.6), for all $t \neq 0$. By Theorem 2.9, we have that $H^1(X, \Omega^1_X(t)) = 0$, for all $t \neq 0$. Applying Serre duality and the Kodaira vanishing Theorem, respectively, we get

$$H^0(X, \mathcal{I}_Z(r+t)) = H^3(\mathcal{O}_X(-r-t-\iota_X)) = 0 \text{ for } r \neq -\iota_X.$$

Thus, we get $H^1(X, N^*_{\mathscr{F}}(t)) = 0$ for $t \neq 0$.

By Serre duality and by Remark 1.1, we conclude that

$$0 = H^1(X, N^*_{\mathscr{F}}(t)) = H^2(X, (N^*_{\mathscr{F}})^*(-t-\iota_X)) = H^2(X, N^*_{\mathscr{F}}(-t-\iota_X - c_1(N^*_{\mathscr{F}}))).$$

Let $s = -c_1(N^*_{\mathscr{F}}) - t - \iota_X$ and $t \neq 0$. Thus, we get

$$H^2(X, N^*_{\mathscr{F}}(s)) = 0 \text{ for } s \neq -\iota_X - c_1(N^*_{\mathscr{F}}).$$

Since by hypothesis $h^2(N^*_{\mathscr{F}}(s)) = 0$ for $s = -\iota_X - c_1(N^*_{\mathscr{F}})$, we conclude that $H^2(X, N^*_{\mathscr{F}}(t)) = 0$. Then, for all $t \neq 0$, $N^*_{\mathscr{F}}$ is aCM.

Now, consider the following piece of the long exact cohomology sequence (3.6), for $t = 0$:

$$\ldots \to H^1(X, N^*_{\mathscr{F}}) \to H^1(X, \Omega^1_X) \simeq \mathbb{C} \xrightarrow{\beta} H^1(X, \mathcal{I}_Z(r)) \simeq \mathbb{C} \to 0.$$

The map β is surjective and injective. Thus, $ker(\beta) = H^1(X, N^*_{\mathscr{F}})$ is trivial, i.e. $H^1(X, N^*_{\mathscr{F}}) = 0$. As by hypothesis, $h^2(N^*_{\mathscr{F}}) = 0$, we conclude that for $t = 0$, $N^*_{\mathscr{F}}$ is aCM. Therefore, $N^*_{\mathscr{F}}$ is aCM for all $t \in \mathbb{Z}$.

□

3.3 Indecomposable aCM bundles

If $X \simeq \mathbb{P}^n$ it is know that an aCM bundle must be a direct sum of line bundles (Horrock's Theorem). For $X \simeq Q^n \subset \mathbb{P}^{n+1}$, an aCM bundle is a direct sum of a line bundles and twisted spinor bundles [30].

The problem of classifying aCM bundles has been taken up only in some special cases. The case of smooth Fano threefolds X with Picard group $\mathbb{Z}$ has also been studied. When $\iota_X \in \{2, 1\}$ this classification is highly nontrivial since there are several deformation classes of these varieties [17]; [18].

3.3.1 Rank-two vector bundles

Let E be a rank 2 vector bundle on a smooth weighted projective complete intersection Fano threefold X with Picard number $\rho(X) = 1$.

Definition 3.15. The bundle E is called *normalized* if it has first Chern $c_1 \in \{0, -1\}$. We define the *first relevant level* of E as the integer

$$\alpha = \alpha(E) := \min\{t \in \mathbb{Z}/h^0(X, E(t)) \neq 0\}.$$

Definition 3.16. Let E be a rank 2 vector bundle on X with first Chern class c_1 and first relevant level α. We say that E is *stable* if $2\alpha + c_1 > 0$, or equivalently, if $\alpha > 0$ when E is normalized. We say that E is *semistable* if $2\alpha + c_1 \geq 0$, or equivalently, if $\alpha \geq -c_1$ when E is normalized. We say that E is *non-stable* if $2\alpha + c_1 \leq 0$, or equivalently, if $\alpha \leq 0$ when E is normalized.

Remark 3.17. Obviously every stable bundle is semistable. Conversely, the only semistable bundles which are not stable are those with $c_1 = \alpha = 0$.

Definition 3.18. An (odd) *k-instanton* E on Q^3 is a rank–2 stable vector E with $c_1(E) = -1, c_2(E) = k$ and $H^1(Q^3, E(-1)) = 0$.

Remark 3.19. The spinor bundle S on Q^3 is a $1-$instanton bundle. Indeed, $c_1(S) = -1$ and $H^1(Q^3, S(-1)) = 0$ by Theorem 1.14.

Theorem 3.20. *[21, Theorem 2] Let $X \subset \mathbb{P}^4$ be a non singular hypersurface, of degree r; E be a rank 2 vector bundle on X with first Chern class c_1 and first relevant level α. If E is aCM, then E splits, unless $-r + 2 < 2\alpha + c_1 < r$.*

Corollary 3.21. *The only indecomposable, arithmetically Cohen-Macaulay, normalized, rank 2 vector bundles on a smooth quadric threefold Q^3 are stable with $c_1 = -1, c_2 = 1$ and $\alpha = 1$, i.e. they are the spinor bundles.*

Proof: See [3], Corollary 2.13. □

Corollary 3.22. *The only indecomposable, arithmetically Cohen-Macaulay, normalized, rank 2 vector bundles on a smooth cubic threefold X_3 are either stable with $c_1 = \{-1, 0\}$ and $\alpha = 1$, or semistable with $c_1 = 0$ and $\alpha = 0$.*

Proof: By Theorem 3.20, if E is an indecomposable, aCM, normalized, rank 2 vector bundle on X_3, then we must have $2\alpha + c_1 = \{0, 1, 2\}$. Thus, if $2\alpha + c_1 = 0$, then E is semistable with $c_1 = \alpha = 0$; if $2\alpha + c_1 = 1$, then E is stable with $c_1 = -1$ and $\alpha = 1$, and if $2\alpha + c_1 = 2$, then E is stable with $c_1 = 0$ and $\alpha = 1$. □

Arrondo and Costa in [2] also proved the Corollary 3.22 by using a completely different argument. In addition, they also proved this corollary for the case where X_d is a smooth intersection of the Grassmannian $G(1, 4) \subset \mathbb{P}^9$ with three hyperplanes.

Theorem 3.23. *[2, Theorem 3.4] An indecomposable rank-two vector bundle F on X_d with $d = 3, 4, 5$ has not intermediate cohomology if and only if it is a twist of one of the following:*

1. *a semistable vector bundle S_L fitting in an exact sequence*

$$0 \to \mathcal{O}_{X_d} \to S_L \to \mathcal{I}_L \to 0, \tag{3.7}$$

where $L \subset X_d$ is a line contained in X_d;

2. *a stable vector bundle S_C fitting in an exact sequence*

$$0 \to \mathcal{O}_{X_d}(-1) \to S_C \to \mathcal{I}_C \to 0, \tag{3.8}$$

where $C \subset X_d$ is a conic contained in X_d;

3. *a stable vector bundle S_E fitting in an exact sequence*

$$0 \to \mathcal{O}_{X_d}(-1) \to S_E \to \mathcal{I}_E(1) \to 0, \tag{3.9}$$

where $E \subset X_d$ is an elliptic curve of degree $d + 2$.

The next Theorem is the well-known regularity criterion of Castelnuovo-Mumford, and we will use it in the examples below.

Theorem 3.24 (Castelnuovo-Mumford criterion)**.** *Let $\mathcal{O}(1)$ be an ample invertible sheaf on a variety X which is generated by global sections. Let F be a vector bundle on X such that*

$$H^i(X, F(-i)) = 0 \text{ for } i > 0.$$

Then,

(i) F is generated by global sections;

(ii) $H^i(X, F(-i + j)) = 0$ for $i > 0,\ j \geq 0$.

Example 3.25. The rank-two vector bundle S_L has Chern classes $(c_1, c_2) = (0, 1)$.
It holds that S_L has only one section, whereas $S_L(1)$ is generated by global sections. The latter comes from the exact sequence (3.7), the fact that $\mathcal{O}_{X_d}(1)$ and $I_L(1)$ are generated by its sections and the vanishing of $H^1(\mathcal{O}_{X_d}(1))$.

Example 3.26. The rank-two vector bundle S_C is generated by its global sections. Indeed, by considering the short exact sequence 3.8 after tensoring with $\mathcal{O}_{X_d}(-1)$, and taking the long exact sequence of cohomology we get:

$$\cdots \to H^1(X_d, \mathcal{O}_{X_d}(-2)) \to H^1(X_d, S_C(-1)) \to H^1(X_d, \mathcal{I}_C(-1)) \to \cdots .$$

By the formula (2.3), $H^1(X_d, \mathcal{O}_{X_d}(-2)) = 0, \ \forall t$. Using Serre's duality and the formula 2.3, we have $H^1(X_d, \mathcal{I}_C(-1)) \simeq H^2(X_d, \mathcal{O}_{X_d}(-1)) = 0, \ \forall t$. Hence, since S_C has not intermediate cohomology, using Castelnuovo-Mumford criterion, we obtain that S_C is globally generated.
It has Chern classes $(c_1, c_2) = (-1, 2)$.

Example 3.27. The rank-two vector bundle $S_E(1)$ is generated by its global sections. Indeed, by considering the short exact sequence 3.9 after taking the long exact sequence of cohomology we get:

$$\cdots \to H^1(X_d, \mathcal{O}_{X_d}(-1)) \to H^1(X_d, S_C) \to H^1(X_d, \mathcal{I}_E(1)) \to \cdots .$$

By the formula 2.3, $H^1(X_d, \mathcal{O}_{X_d}(-1)) = 0, \ \forall t$. Using Serre's duality and the formula 2.3, we have $H^1(X_d, \mathcal{I}_E(1)) \simeq H^2(X_d, \mathcal{O}_{X_d}(-3)) = 0, \ \forall t$. Hence, since S_E has not intermediate cohomology, using Castelnuovo-Mumford criterion, we obtain that $S_E(1)$ is globally generated.
S_E is a vector bundle with Chern classes $(c_1, c_2) = (0, 2)$.

Definition 3.28. The *degree* $d = (-K_X)^3$ of a prime Fano threefold X is always even, The integer $\frac{1}{2}(-K_X)^3 + 1$ is called the *genus* of a Fano $3-$fold X.

For $\iota_X = 1$, a result due to Madonna [22] implies that if a rank-2 aCM bundle $\mathcal{E}$ is defined on X, it is characterized by:

Theorem 3.29. *[22, Main Theorem] Let $\mathcal{E}$ be a normalized aCM bundle on a prime Fano threefolds $X_{2g-2} := X$ of genus g. Then $\mathcal{E}$ is a twist of one of the bundles in the list below:*

1. *$c_1 = -1, \ c_2 = 1$ and $\mathcal{E}$ is associated to a line in X;*
2. *$c_1 = 0, \ c_2 = 2$ and $\mathcal{E}$ is associated to a conic in X;*
3. *$c_1 = 1$, either $c_2 = g+2$ and $\mathcal{E}$ is associated to a non-degenerate elliptic curve C^1_{g+2} of degree $g+2$ contained in X or $c_2 = d < g+2$ and C^1_d is degenerate;*
4. *$c_1 = 2, \ c_2 = 2+2g$ and $\mathcal{E}$ is associated to a curve C^{g+2}_{2g+2} of genus $g+2$ and degree $2g+2$ contained in X;*
5. *$c_1 = 3, \ c_2 = 5g-1$ and $\mathcal{E}$ is associated to a smooth 2-canonical curve C^{5g}_{5g-1} contained in X.*

3.3.2 Distributions and Globally Generated Sheaves

In this subsection we will construct examples of codimension one distributions on X, based on the following result due to O. Calvo-Andrade, M. Corrêa and M. Jardim [5], which is a generalization of Ottaviani's Bertini type Theorem [27, Teorema 2.8].

Theorem 3.30. *[5, Theorem 11.8] Let $\mathcal{G}$ be a globally generated reflexive sheaf on a projective variety X such that $rk(\mathcal{G}) \leq \dim(X) - 1 \geq 2$. If $TX \otimes L$ is globally generated, for some line bundle L, then $\mathcal{G}^* \otimes L^*$ is the tangent sheaf of a codimension one distribution on X.*

Corollary 3.31. *Let S be the spinor bundle on Q^3. Then $S(1-t)$ is the tangent sheaf of a codimension one distribution $\mathscr{F}$, for all t.*

Proof: Since the spinor bundle S is globally generated, $S(t)$ is globally generated, for all $t \geq 0$ (see remark 1.5). Moreover, its rank 2 reflexive sheaf on Q^3, and $S(t) \otimes TQ^3$ is also globally generated, since TQ^3 is globally generated (see remarks 1.5 and 1.4). Now, apply the Theorem 3.30 with $L = \mathcal{O}_{Q^3}$ to obtain the desired codimension one distribution

$$0 \to S^*(-t) \simeq S(1-t) \to TQ^3 \to \mathcal{I}_Z(r) \to 0.$$

Therefore, $S(1-t)$ is the tangent sheaf of a codimension one distribution $\mathscr{F}$.

□

Consider $\mathcal{E} \simeq S_L(1);\ S_C;\ S_E(1)$. By the examples in the subsection 3.3.1, we have to $\mathcal{E}$ is globally generated.

Corollary 3.32. *Let $\mathcal{E}$ be a globally generated rank 2 reflexive sheaf on X_d. Then $\mathcal{E}^*(-t)$ is the tangent sheaf of a codimension one distribution $\mathscr{F}$, for all t.*

Proof: Since $\mathcal{E}$ is globally generated, $\mathcal{E}(t)$ is globally generated, for all $t \geq 0$ (see remark 1.5). Moreover, its rank 2 reflexive sheaf on X_d, and $\mathcal{E}(t) \otimes TX_d$ is also globally generated, since TX_d is globally generated. Now, apply the Theorem 3.30 with $L = \mathcal{O}_{X_d}$ to obtain the desired codimension one distribution

$$0 \to \mathcal{E}^*(-t) \to TX_d \to \mathcal{I}_Z(r) \to 0.$$

Therefore, $\mathcal{E}^*(-t)$ is the tangent sheaf of a codimension one distribution $\mathscr{F}$. □

3.4 Properties of the singular locus of distributions

In this section we analyze the properties of their singular schemes of codimension one holomorphic distributions on X.

3.4.1 Numerical Invariants

Let $\mathscr{F}$ be a codimension one distribution on threefold X given as in the exact sequence (1.2), with tangent sheaf $T_{\mathscr{F}}$ and singular scheme Z.

Let $\mathcal{Q}$ be the maximal subsheaf of $\mathcal{O}_{Z/X}$ of codimension > 2, so that one has an exact sequence of the form

$$0 \to \mathcal{Q} \to \mathcal{O}_{Z/X} \to \mathcal{O}_{C/X} \to 0 \tag{3.10}$$

where $C \subset X$ is a (possibly empty) subscheme of pure codimension 2.

The quotient sheaf is the structure sheaf of a subscheme $C \subset Z \subset X$ of pure dimension 1.

Definition 3.33. If Z is a 1-dimensional subscheme, then Z has a maximal pure dimension 1 subscheme C defining a sequence

$$0 \to \mathcal{I}_Z \to \mathcal{I}_C \to \mathcal{Q} \to 0, \tag{3.11}$$

where $\mathcal{Q}$ is the maximal 0-dimensional subsheaf of $\mathcal{O}_Z$.

Kähler manifolds form an important class of complex manifolds.

Definition 3.34. A Kähler metric is an hermitian structure g for which the fundamental form ω is closed, i.e. $d\omega = 0$. In this case, the fundamental ω form is called the Kähler form. The complex manifold endowed with the Kähler structure is called a *Kähler manifold.*

If X is a Kähler manifold of dimension n, and $Z \subset X$ is an analytic subset of codimension k, then

$$c_k(\mathcal{I}_Z) = (-1)^k (k-1)![Z]. \tag{3.12}$$

See [12].

Theorem 3.35. *Let $\mathscr{F}$ be a codimension one distribution on a threefold X, with $\rho(X) = 1$, given as in the exact sequence (1.2), with tangent sheaf $T_{\mathscr{F}}$ and singular scheme Z. Then,*

$$c_2(T_{\mathscr{F}}) = c_2(TX) - r \cdot K_X^{-1} + r^2 - [C],$$

and

$$c_3(T_{\mathscr{F}}) = c_3(TX) - c_3(I_{Z/X}) + r \cdot [C] - K_X^{-1} \cdot [C] - r \cdot c_2(TX) + r^2 \cdot K_X^{-1} - r^3.$$

Proof: Considering the exact sequence (1.2), we use that $c(TX) = c(T_{\mathscr{F}}) \cdot c(I_{Z/X}(r))$ to obtain

$$\begin{aligned} c_1(TX) &= c_1(T_{\mathscr{F}}) + c_1(I_{Z/X}(r)), \\ c_2(TX) &= c_1(T_{\mathscr{F}}) \cdot c_1(I_{Z/X}(r)) + c_2(T_{\mathscr{F}}) + c_2(I_{Z/X}(r)), \\ c_3(TX) &= c_3(T_{\mathscr{F}}) + c_3(I_{Z/X}(r)) + c_1(T_{\mathscr{F}}) \cdot c_2(I_{Z/X}(r)) + c_2(T_{\mathscr{F}}) \cdot c_1(I_{Z/X}(r)). \end{aligned} \tag{3.13}$$

The first equation gives $c_1(T_{\mathscr{F}}) = c_1(TX) - r$. From the exact sequence 3.11, it follows that $c_2(I_{Z/X}(r)) = c_2(I_{C/X}(r)) = [C]$, thus substitution into the second equation yields

$$c_2(T_{\mathscr{F}}) = c_2(T_X) - r \cdot K_X^{-1} + r^2 - [C].$$

Moreover, the substituting the expressions for the first and second Chern classes into the third equation we obtain

$$c_3(TX) = c_3(T_{\mathscr{F}}) + c_3(I_{Z/X}(r)) + K_X^{-1} \cdot [C] - 2r[C] + r \cdot c_2(TX) - r^2 \cdot K_X^{-1} + r^3. \tag{3.14}$$

Note that

$$c_3(I_{Z/X}(r)) = c_3(I_{Z/X}) + r \cdot c_2(I_{Z/X}) + r^3, \tag{3.15}$$

while

$$c_2(I_{Z/X}) = [C] - r^2. \tag{3.16}$$

Substituting 3.16 into the equation 3.15, we obtain

$$c_3(I_{Z/X}(r)) = c_3(I_{Z/X}) + r \cdot [C], \tag{3.17}$$

and thus

$$c_3(T_{\mathscr{F}}) = c_3(TX) - c_3(I_{Z/X}) + r \cdot [C] - K_X^{-1} \cdot [C] - r \cdot c_2(TX) + r^2 \cdot K_X^{-1} - r^3. \tag{3.18}$$

□

3.4.2 Connectedness of the Singular locus

In [5], O. Calvo-Andrade, M. Corrêa and M. Jardim obtained the following generalization of [13, Theorem 3.2].

Lemma 3.36. *[5, Lemma 2.1] The tangent sheaf of a codimension one distribution is locally free if and only if its singular locus has pure codimension 2.*

Lemma 3.37. *Let X be a smooth weighted projective complete intersection Fano threefold with Picard number $\rho(X) = 1$. Then, $H^1(TX(-r)) = 0$ for $r > 6$ and $H^2(TX(-r)) = 0$ for $r \neq \iota_X$.*

Proof: By hypothesis, X is a smooth weighted projective complete intersection Fano threefold.

If $\iota_X = 4$, i.e. $X \simeq \mathbb{P}^3$, by classical Bott's formula we have that $H^1(TX(-r)) = 0$ for all r and $H^2(TX(-r)) = 0$ for $r \neq 2$.

If $\iota_X = 3$, i.e. $X \simeq Q^3$, by Bott's formula for quadric, we have that $H^1(TX(-r)) = 0$ for $r \neq 2$ and $H^2(TX(-r)) = 0$ for $r \neq 3$.

If $\iota_X = 2$, by using Serre duality we get that $H^1(TX(-r)) = H^2(\Omega^1_X(r-2))$. By Proposition 2.10, comparing the values of t for which $H^2(\Omega^1_X(t)) = 0$ with $\iota_X = 2$, we can see that the common vanishing of cohomology group for these varieties, occurs when $t > 4$. Then, $H^2(\Omega^1_X(r-2)) = 0$ for $r > 6$. And $H^2(TX(-r)) = 0$ for $r \neq 2$ by Theorem 2.9.

If $\iota_X = 1$, by using Serre duality we get $H^1(TX(-r)) = H^2(\Omega^1_X(r-1))$. By Proposition 2.11, comparing the values of t for which $H^2(\Omega^1_X(t)) = 0$ with $\iota_X = 1$, we can see that the common vanishing of cohomology group for these varieties, occurs when $t > 3$. Then, $H^2(\Omega^1_X(r-1)) = 0$ for $r > 4$. And $H^2(TX(-r)) = 0$ for $r \neq 1$ by Theorem 2.9.

Now, comparing the values of r for which $H^1(TX(-r)) = 0$, we can see that the common vanishing of cohomology group considering all indices of X, occurs when $r > 6$. And $H^2(TX(-r)) = 0$ for $r \neq \iota_X$.

□

Theorem 3.38. *Let $\mathscr{F}$ be a codimension one distribution with singular scheme Z and let X be a smooth weighted projective complete intersection Fano threefold with Picard number $\rho(X) = 1$. If $h^2(T_{\mathscr{F}}(-r)) = 0$ and $C \subset X$, $C \neq \emptyset$, then Z is connected and of pure dimension 1, so that $T_{\mathscr{F}}$ is locally free. Conversely, for $r \neq \iota_X$, if $Z = C$ is connected, then $T_{\mathscr{F}}$ is locally free and $h^2(T_{\mathscr{F}}(-r)) = 0$.*

Proof: Twisting the exact sequence (1.2) by $\mathcal{O}_X(-r)$ and passing to cohomology we obtain,

$$H^1(TX(-r)) \to H^1(I_{Z/X}) \to H^2(T_{\mathscr{F}}(-r)) \to H^2(TX(-r)).$$

By Lemma above, we get that $H^1(TX(-r)) = 0$ for $r > 6$.

If $h^2(T_{\mathscr{F}}(-r)) = 0$, then $h^1(I_{Z/X}) = 0$, for $r > 6$. It follows from the sequence

$$0 \to I_{Z/X} \to \mathcal{O}_X \to \mathcal{O}_{Z/X} \to 0$$

that

$$H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_{Z/X}) \to 0,$$

hence $h^0(\mathcal{O}_{Z/X}) = 1$. From the sequence (3.10), we get

$$0 \to H^0(\mathcal{Q}) \to H^0(\mathcal{O}_{Z/X}) \to H^0(\mathcal{O}_{C/X}) \to 0$$

Thus either $h^0(\mathcal{O}_{C/X}) = 1$, and $\mathcal{Q} = 0$ and C is connected, or length$(\mathcal{Q}) = 1$ and C is empty. This second possibility is not valid because by hipothesis $C \neq \emptyset$. It follows that $Z = C$ must be connect and of pure dimension 1, and thus, by Lemma 3.36, $T_{\mathscr{F}}$ is locally free.

Conversely, assume that $Z = C$ is connected. Thus Z must be of pure dimension 1, and Lemma 3.36 implies that $T_{\mathscr{F}}$ is locally free. It also follows that $h^1(I_{Z/X}) = 0$, using Serre duality and 2.3. Since $h^2(TX(-r)) = 0$ for $r \neq \iota_X$, we conclude that $h^2(T_{\mathscr{F}}(-r)) = 0$, as desired. □

Corollary 3.39. *If $\mathscr{F}$ is a codimension one distribution on X whose tangent sheaf splits as a sum of line bundles, then its singular scheme is connected.*

Proof: Assuming that $T_{\mathscr{F}} = \mathcal{O}_X(r_1) \oplus \mathcal{O}_X(r_2)$, then clearly $h^2(T_{\mathscr{F}}(-r)) = 0$, where $r = r_1 + r_2$. The result follows from Theorem 3.38. □

Corollary 3.40. *Let $\mathscr{F}$ be a codimension one distribution on X with locally free tangent sheaf. If $T^*_{\mathscr{F}}$ is ample, then its singular scheme is connected.*

Proof: We have, by Serre duality,

$$H^2(T_{\mathscr{F}}(-r)) \simeq H^1(T^*_{\mathscr{F}}(r) \otimes K_X) = H^1(T^*_{\mathscr{F}}(r - c_1(TX)) \otimes \mathcal{O}_X(c_1(TX)) \otimes K_X).$$

Observe that $T^*_{\mathscr{F}}(r - c_1(TX)) \otimes \mathcal{O}_X(c_1(TX)) \otimes K_X = T^*_{\mathscr{F}} \otimes \det(T^*_{\mathscr{F}}) \otimes \mathcal{O}_X(c_1(TX)) \otimes K_X$; since $T^*_{\mathscr{F}}$ and $\mathcal{O}_X(c_1(TX))$ are ample, then, by Theorem 1.10, we get

$$h^2(T_{\mathscr{F}}(-r)) = h^1(T^*_{\mathscr{F}} \otimes \det(T^*_{\mathscr{F}}) \otimes \mathcal{O}_X(c_1(TX)) \otimes K_X) = 0.$$

The result follows from Theorem 3.38. □

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