# UNIVERSIDADE FEDERAL DE MINAS GERAIS INSTITUTO DE CIÊNCIAS EXATAS <br> Departamento de Matemática 

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# NON-NILPOTENT LIE ALGEBRAS WITH NON-SINGULAR DERIVATIONS 

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Tese apresentada ao Departamento de Matemática da Universidade Federal de Minas Gerais, para a obtenção do grau de doutor em Matemática.

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## Agradecimentos

À Deus pelo sustento, saúde e redenção em Cristo. Por me permitir cursar o doutorado, dar ânimo durante as dificuldades e sabedoria durante todo o curso.

À minha esposa Helen pelo companheirismo e cumplicidade. Por partilhar a mesma fé, o mesmo projeto de vida, as alegrias, ansiedades, empolgações, tristezas, planos, carreira e muitas outras coisas. Por todo amor envolvido na nossa união.

Aos meus pais Almir e Dalva pelo apoio incondicional. Pelo suporte dado por meio das orações, telefonemas e abraços. Aos meus irmãos Filipe e Ana Paula pela união em todos os momentos.

À minha familia, amigos e irmãos em Cristo. À todos que de alguma forma me incentivaram e apoiaram durante o período do doutorado. Agradeço pelo tempo passado em comunhão juntos.

Ao meu orientador professor Csaba Schneider pela excelente competência como profissional. Pela paciência e cuidado que eu não consigo ter em ler e corrigir os meus textos. Obrigado e desculpe por essa falta de atenção.

Ao departamento DECEA da UFOP por permitir o meu afastamento pelo período integral do doutorado. Pelo suporte financeiro da UFOP, em especial à PROPP pelo auxilo para capacitação concedido desde o início do doutorado.

Aos docentes, alunos, técnicos e funcionários da UFMG pelo profissionalismo e ótimo ambiente de trabalho.

## Resumo

Seja $L$ um álgebra de Lie e $\delta$ uma derivação de $L$. A derivação $\delta$ é dita não-singular se for injetiva como transformação linear. Por um resultado bem conhecido de N. Jacobson, uma álgebra de Lie de dimensão finita, sobre um corpo de característica zero e com uma derivação não-singular é nilpotente. Embora saibamos que esse resultado não é válido em característica $p>0$, pouco se sabe sobre álgebras de Lie em característica $p>0$ com derivações nãosingulares. Neste texto, exploramos a estrutura das álgebras de Lie solúveis, não-nilpotentes e com derivação não-singular. Apresentamos um novo conceito para derivações, chamado Pares Compatíveis. Esse conceito é usado, por exemplo, para calcular as derivações de uma extensão de álgebra de Lie. Outra aplicação obtida é uma versão do teorema de Jacobson para as álgebras de Lie sobre corpos com característica $p>0$. Usando Pares Compatíveis foi possível obter uma caracterização para álgebras de Lie não-nilpotentes, com um ideal abeliano de codimensões 1 e derivação não-singular. Adicionalmente, construímos um exemplo de álgebra de Lie nãonilpotente, com derivação não-singular e classe de nilpotência arbitraria. Por fim, provamos que se $H$ é a álgebra de Heisenberg sobre um corpo de característica $p$ e $I$ um $H$-módulo, tais que a soma semi-direta de $H$ e $I$ é uma álgebra de Lie não-nilpotente com derivação não-singular, então a dimensão de $I$ é, no mínimo, $p+3$.

## Palavras-chave: Álgebras de Lie, Derivações Não-Singulares, Pares Compatíveis, Teorema de Jacobson


#### Abstract

Let $L$ be a Lie algebra and $\delta$ be a derivation of $L$. The derivation $\delta$ is non-singular if it is injective as linear transformation. By a well-known result of N. Jacobson, a Lie algebra of finite dimension over a field of characteristic zero having a non-singular derivation is nilpotent. Although we know that this result is not valid in characteristic $p>0$, little is known about Lie algebras in $p$ characteristic with non-singular derivations. In this text, we explore the structure of solvable, non-nilpotent Lie algebras with non-singular derivations. We present a new concept for derivations, called Compatible Pairs. This concept is used, for example, to calculate the derivations of an extension of Lie algebras. Another application obtained was a version of Jacobson's Theorem for Lie algebras over fields characteristic $p>0$. Using Compatible pairs it was possible to obtain a characterization of non-nilpotent Lie algebras, with an abelian deal of codimension 1 and non-singular derivations. Further, a new example of non-nilpotent Lie algebras, with non-singular derivations and arbitrarily nilpotency class was constructed. Finally, we prove that if $H$ is the Heisenberg algebra over a field of characteristic $p>0$, and $I$ is a $H$-module such that the semi-direct sum of $H$ and $I$, is a non-nilpotent Lie algebras with nonsingular derivation, then the dimension of $I$ is, at least, $p+3$.


Keywords: Lie algebras, Non-singular Derivation, Compatible Pairs, Jacobson's Theorem

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## 1 Introduction

Let $L$ be a Lie algebra and let $\delta$ be a derivation of $L$. The derivation $\delta$ is non-singular if it is injective as a linear transformation. We are interested in studying what information we can obtain about a Lie algebra if it has a non-singular derivation. Jacobson's Theorem [1] states that a finite-dimensional Lie algebra over a field of characteristic zero that admits a non-singular derivation must be nilpotent. It is well-known that this theorem is not valid when the characteristic is non-zero. Non-nilpotent and solvable examples were constructed by Shalev [2] and Mattarei [3], whereas the finite-dimensional simple Lie algebras with non-singular derivations were classified by Benkart and her collaborators in [4]. A significant application of Lie algebras with non-singular derivation in characteristic $p$ was presented by Shalev [5]. In his proof of the coclass conjectures of Leedham-Green and Newman for pro- $p$ groups, Shalev uses the fact that finite-dimensional Lie algebras over a field of characteristic $p>0$ with non-singular derivation $\delta$ such that $\delta^{p-1}=1$ must be nilpotent.

Despite the existing examples, little is known about non-nilpotent Lie algebras with non-singular derivations. In this text we propose to explore the structure of solvable, nonnilpotent Lie algebras with non-singular derivations. In order to study these algebras we develop a theory of derivations of Lie algebra extensions. We adopt the concept of a compatible pair of automorphisms utilized in [6] for derivations of Lie algebras.

In the rest of this introduction we state the main results presented in this thesis. Let us start by briefly reviewing some concepts that are studied in more details in Chapter 3.

Let $K$ and $I$ be Lie algebras such that $K$ acts on $I$. Denote by $\operatorname{Der}(K)$ the Lie algebra of derivations of $K$. Then we can define the subalgebra $\operatorname{Comp}(K, I)$ of compatible pairs of $\operatorname{Der}(K) \oplus \operatorname{Der}(I)$ as the set of derivations of $\operatorname{Der}(K) \oplus \operatorname{Der}(I)$ that are derivations of the semidirect sum $K \oplus I$. Formally,

$$
\operatorname{Comp}(K, I)=\{\alpha+\beta \in \mathfrak{g l}(K) \oplus \mathfrak{g l}(I) \mid \alpha+\beta \in \operatorname{Der}(K \oplus I)\} .
$$

The algebra $\operatorname{Der}(K)$ carries information about the multiplicative structure of $K$. Analogously, the algebra $\operatorname{Comp}(K, I)$ carries information about the action of $K$ on $I$.

We also adapt an algorithm presented by Bettina Eick [6] for calculating the automorphism group of solvable Lie algebras. A key step in the algorithm is the following. Let $L$ be a Lie algebra and let $I$ be an abelian ideal of $L$ such that $I$ is invariant under $\operatorname{Aut}(L)$.

Then there exists a homomorphism $\phi: \operatorname{Aut}(L) \rightarrow \operatorname{Aut}(L / I) \times \operatorname{Aut}(I)$ induced by the actions of $\operatorname{Aut}(L)$ on $L / I$ and $I$. The image of $\phi$ can be calculated using $\operatorname{Aut}(L / I)$, while $\operatorname{Ker}(\phi)$ is equal to $Z^{1}(K, I)$, the first cohomoly group of $K$ on $I$. Then the $\operatorname{group} \operatorname{Aut}(L)$ can be obtained applying the first isomorphism theorem to $\phi$. For derivation, the process is as follows.

Let $K$ be a Lie algebra and let $I$ be a $K$-module. Let $\mathrm{Z}^{2}(K, I)$ be the vector space of cocycles, let $\mathrm{B}^{2}(K, I)$ be the vector space of coboundaries and set $\mathrm{H}^{2}(K, I)=\mathrm{Z}^{2}(K, I) / \mathrm{B}^{2}(K, I)$. For $(\alpha, \beta) \in \operatorname{Comp}(K, I)$ and $\vartheta \in Z^{2}(K, I)$, define an action of $\operatorname{Comp}(K, I)$ on $Z^{2}(K, I)$ by

$$
(\alpha, \beta) \cdot \vartheta(h, k)=\beta(\vartheta(h, k))-\vartheta(\alpha(h), k)-\vartheta(h, \alpha(k)), \quad \text { for all } h, k \in K .
$$

The elements of the annihilator of $\vartheta$ under this action will be called induced pairs and we denote the set of induced pairs by $\operatorname{Indu}(K, I, \vartheta)$. Let $K_{\vartheta}$ be the Lie algebra extension of $K$ obtained as the extension by cocycle $\vartheta$. Suppose that $I$, as an ideal of $K_{\vartheta}$, is invariant under $\operatorname{Der}\left(K_{\vartheta}\right)$. Hence, each $d \in \operatorname{Der}\left(K_{\vartheta}\right)$ induces derivations $\alpha$ and $\beta$ of $K$ and $I$, respectively, and we can construct a Lie algebra homomorphism $\phi: \operatorname{Der}(L) \rightarrow \operatorname{Der}(L / I) \oplus \operatorname{Der}(I)$. Thus we obtain the following theorem, whose full proof will be presented in Chapter 3.

Theorem 3.3.1 Let $K$ be a Lie algebra and let I be a $K$-module. Let $\vartheta \in H^{2}(K, I)$ and suppose that $I$, as ideal of $K_{\vartheta}$, is invariant under derivations. Let $\phi: \operatorname{Der}\left(K_{\vartheta}\right) \rightarrow \operatorname{Der}(K) \oplus \operatorname{Der}(I)$ be defined as above. Then:

1. $\operatorname{Im}(\phi)=\operatorname{Indu}(K, I, \vartheta)$
2. $\operatorname{Ker}(\phi) \cong Z^{1}(K, I)$

The details of this construction can be seen in Chapter 3. There is a significant difference between the application of this approach to automorphisms and to derivations: calculating the automorphism groups of Lie algebras is usually a difficult task that may involve a large orbit-stabilizer calculation, while calculating the algebra of derivations can be done by solving a system of linear equations. Nevertheless, it is still interesting to see that derivations have properties similar to automorphisms.

Let $K$ be a finite-dimensional Lie a algebra and let $\psi: K \rightarrow \operatorname{Der}(I)$ be a Lie algebra representation. If $K$ is solvable and the base field has characteristic zero, we have a characterization of the matrices of the image of the representation $\psi$ using Lie's Theorem (see 2.3.4): there is a basis such that these matrices are all upper triangular. As this result is not true for
representations with base field of prime characteristic $p>0$, in Chapter 4 we explore some representations of solvable Lie algebra in prime characteristic. The existence of a compatible pair formed by non-singular derivations, guarantees that the image of $\psi: K \rightarrow \operatorname{Der}(I)$ with $K$ solvable and $\operatorname{dim} K<p$, must be formed by nilpotent matrices.

Theorem 4.1.6 Let $K$ and I be finite-dimensional Lie algebras over an algebraically closed field of characteristic $p>0$. Suppose that $K$ acts on I by the representation $\psi: K \rightarrow \operatorname{Der}(I)$. Let $(\alpha, \beta) \in \operatorname{Comp}(K, I)$ such that $\alpha$ is non-singular, and $\alpha$ has finite order. If $K$ is solvable and $\operatorname{dim} I<p$, then $\psi(k)$ is nilpotent, for all $k \in K$.

As a consequence, it is also possible to present a version of Jacobson's Theorem for Lie algebras over fields of characteristic $p>0$. This version considers a solvable Lie algebra $L$ and sets conditions for $L$ to be nilpotent.

Theorem 4.1.8 Let L be a solvable Lie algebra over a field $\mathbb{F}$ of characteristic $p>0$. Let $L>L^{(1)}>\cdots>L^{(k)}>L^{(k+1)}=0$ be the derived series of $L$. Suppose that $L$ has a non-singular derivation of finite order. If the dimension of $L^{(i)} / L^{(i+1)}<p$, for all $i$, then $L$ is nilpotent.

Further, we explore the structure of some finite-dimensional non-nilpotent Lie algebras with a non-singular derivation. Due to Jacobson's Theorem, these algebras can exist just over fields of prime characteristic. Let $K$ be a solvable Lie algebra over a field of prime characteristic $p>0$ and let $I$ be a $K$-module. Define the semidirect sum $L=K \oplus I$ and suppose that $L$ is solvable, non-nilpotent and with a non-singular derivation. We study some of these algebras in Chapters 5 and 6. In Chapter 5, we assume that $\operatorname{dim} K=1$ and that the center of $L$ is zero, that is $Z(L)=0$. With these hypotheses, it was possible to fully characterize such algebras, as presented in the following theorem. The concept of ( $\mathrm{x}, \mathrm{p}$ )-cyclic modules is presented in Section 5.1.

Theorem 5.1.18 Let L be a Lie algebra of derived length 2 over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$. Suppose that $\operatorname{dim}\left(L / L^{\prime}\right)=1$ and $Z(L)=0$. Let $x \in L \backslash L^{\prime}$. Then $L$ has a non-singular derivation of finite order if, and only if, $L^{\prime}$ can be written as a direct sum of ( $x, p$ )-cyclic modules.

In Chapter 6, we present some new examples of non-nilpotent Lie algebras with nonsingular derivations. Proposition 6.1.3 presents Lie algebras with arbitrarily large nilpotency class. Example 6.1.4 contains our first example of solvable and non-nilpotent Lie algebra, with a
non-singular derivation and derived length 3 . This example was obtained from a representation $\psi: H \rightarrow \operatorname{Der}(I)$ such that $H$ is the Heisenberg algebra. The thesis ends with a result on the representations of the Heisenberg algebra $H$. Suppose that a representation $\psi: H \rightarrow \operatorname{Der}(I)$ is faithful and $H \oplus I$ admits a non-singular derivation. Then, we have a condition on the dimension of $I$, as stated in the next theorem.

Theorem 6.1.5 Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \geqslant 3$. Let $H$ be the Heisenberg Lie algebra over $\mathbb{F}$. Let $\psi: H \rightarrow \mathfrak{g l}(I)$ be a faithful representation and suppose that $L=H \oplus I$ is non-nilpotent. Suppose that $I$, as ideal of $L$, is invariant under $\operatorname{Der}(L)$. If $L$ has a non-singular derivation of finite order, then $\operatorname{dim} I \geqslant p+3$.

In order facilitate the reading of the text and the references, we added Chapter 2 with results on the primary decomposition of vector spaces in relation to subalgebras of linear operators and a brief description of the main results of the articles used.

## 2 Basic Concepts

In this chapter we present some results on Lie algebra representations and non-singular derivations. A major reference for a decomposition of a vector space $V$ into $K$-modules such that $K$ is a subalgebra of $\mathfrak{g l}(V)$ was the book 'Lie algebra: Theory and Algorithms' [7] of W. A. de Graaf . We also present the main articles that motivated the initial study of nonsingular derivations. The purpose of this chapter is to speed up reading by including most of the references in the text itself.

### 2.1 Primary Decomposition

Let $K$ be a non-associative algebra over a field $\mathbb{F}$ and let us assume that the product of two elements $x, y \in K$ is denoted by $[x, y]$. Then $K$ is said to be a Lie algebra if satisfies the following properties:

1. $[x, x]=0$ for all $x \in K$,
2. $[x,[y, z]]+[y,[z, x]]+[z,[x, y]]=0$ for all $x, y, z \in K($ Jacobi identity $)$.

If $V$ is a vector space then $\operatorname{End}(V)$ denotes the associative algebra of endormorphisms of $V$ with product given by composition. For $f, g \in \operatorname{End}(V)$, set $[f, g]=f g-g f$. The bilinear map $(f, g) \mapsto[f, g]$ is called commutator, or Lie bracket. The space of linear maps from $V \rightarrow V$ together with the commutator is a Lie algebra. We denote it by $\mathfrak{g l}(V)$.

Let $K_{1}$ and $K_{2}$ be Lie algebras over the field $F$ such that the product in $K_{1}$ is [, ] $]_{1}$, and on $K_{2}$ is [, $]_{2}$. We can define a multiplication on the direct sum of vector spaces $K_{1}$ and $K_{2}$ by

$$
\begin{equation*}
\left[x_{1}+y_{1}, x_{2}+y_{2}\right]=\left[x_{1}, y_{1}\right]_{1}+\left[x_{2}, y_{2}\right]_{2}, \text { for } x_{1}, y_{1} \in K_{1} \text { and } x_{2}, y_{2} \in K_{2} . \tag{1}
\end{equation*}
$$

The multiplication defined in (1) makes the direct sum of $K_{1}$ and $K_{2}$ into a Lie algebra. This Lie algebra is called direct sum of Lie algebras $K_{1}$ and $K_{2}$, and will be denoted by $K_{1} \oplus K_{2}$. The symbol ' $\oplus$ ' will be used to denote the direct sum of algebras, while the direct sum of vector spaces will be denoted by ' $\dot{+}$ '. A linear transformation $\theta: K_{1} \rightarrow K_{2}$ is a homomorphism of Lie algebras if

$$
\theta\left([x, y]_{1}\right)=[\theta(x), \theta(y)]_{2}, \text { for all } x, y \in K_{1} .
$$

Let $K$ be a Lie algebra. A derivation of $K$ is an endomorphism $\delta: K \rightarrow K$ such that

$$
\delta([x, y])=[\delta(x), y]+[x, \delta(y)], \quad \text { for all } x, y \in K .
$$

The derivation $\delta$ is non-singular if it is bijective as linear transformation. The set of all derivations of $K$, denoted by $\operatorname{Der}(K)$, is a Lie subalgebra of $\mathfrak{g l}(K)$. For example, let $k \in K$ and define the map $\operatorname{ad}_{k}: K \rightarrow K$ by ad ${ }_{k}(x)=[k, x]$, for all $x \in K$. The endomorphism $\operatorname{ad}_{k}$ is a derivation.

Let $K$ and $I$ be a Lie algebras. A representation of $K$ on $I$ is a given Lie algebra homomorphism $\psi: K \rightarrow \operatorname{Der}(I)$. In this case, we say that $K$ acts on $I$. Additionally, if $I$ is an abelian Lie algebra, then $I$ is called $K$-module. The Lie algebra representation ad : $K \rightarrow$ $\operatorname{Der}(K)$ given by $k \mapsto \mathrm{ad}_{k}$, for all $k \in K$, is called the adjoint representation.

Let $K$ and $I$ be a Lie algebras such that $K$ acts on $I$, with action given by the representation $\psi: K \rightarrow \operatorname{Der}(I)$. To facilitate the reading of the text we will use different notations to represent the image $\psi(k)(v)$ for $k \in K$ and $v \in I$. Usually the element $\psi(k)(v)$ will be denoted by $[k, v]$. If $I$ is an ideal of $K$, then the image of $k$ under this action will be denoted by $\operatorname{ad}_{k}^{I}(v)$, or simply by $\operatorname{ad}_{k}(v)$ when the domain of the representation is clear from the context. To avoid an excess of brackets, we use the convention:

$$
\begin{equation*}
[k, \ldots,[k,[k, v]]]=[\underbrace{k, \ldots, k, k}_{n \text { times }}, v]=\left[k_{n}, v\right], \text { for all } k \in K \text { and } v \in I \text {. } \tag{2}
\end{equation*}
$$

Thus, for $v \in I$ and for $k \in K,\left(\operatorname{ad}_{k}^{l}\right)^{n}(v)=\left(\mathrm{ad}_{k}\right)^{n}(v)=\left[k_{n}, v\right]$ for all $n \geqslant 1$.
Example 2.1.1. Let $L$ be a Lie algebra with an abelian ideal $I$ and set $K=L / I$. Define the Lie algebra representation $\operatorname{ad}^{I}: K \rightarrow \operatorname{Der}(I)$ by $\operatorname{ad}_{x+I}^{I}(v)=[x, v]$ for all $x \in L$ and $v \in I$. This is well defined, since $I$ is abelian. Then $I$ is a $K$-module. In this case, we say that the action is induced by the adjoint representation.

Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$ and $x \in \operatorname{End}(V)$. Let $q \in \mathbb{F}[X]$ be a univariate polynomial and define

$$
\begin{equation*}
V_{0}(q(x))=\left\{v \in V \mid \text { there is an } m>0 \text { such that } q(x)^{m} v=0\right\} . \tag{3}
\end{equation*}
$$

The set $V_{0}(q(x))$ is a vector subspace of $V$ which is invariant under $x$. Now let $A$ be the associative subalgebra of $\operatorname{End}(V)$ with 1 generated by $x$. Let $q_{x}$ be the minimal polynomial of $x$ and suppose that

$$
q_{x}=q_{1}^{k_{1}} \cdots q_{r}^{k_{r}}
$$

is the factorization of $q_{x}$ into irreducible factors, such that $q_{i}$ has leading coefficient 1 and $q_{i} \neq q_{j}$ for $1 \leqslant i<j \leqslant r$. Then $V$ decomposes as a direct sum of subspaces

$$
V=V_{0}\left(q_{1}(x)\right)+\cdots+V_{0}\left(q_{r}(x)\right)
$$

with each space $V_{0}\left(q_{i}(x)\right)$ being invariant under $A$. Furthermore, the minimal polynomial of the restriction of $x$ to $V_{0}\left(q_{i}(x)\right)$ is $q_{i}^{k_{i}}$. A proof of this result can be found in [7] Lemma A.2.2.

We can generalize this decomposition to subalgebras of $\mathfrak{g l}(V)$ generated by more than one element.

Definition 2.1.2. Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$ and let $K \leqslant \mathfrak{g l}(V)$ be a subalgebra. A decomposition $V=V_{1} \dot{+} \cdots \dot{+} V_{s}$ of $V$ into $K$-modules $V_{i}$ is said to be primary if the minimal polynomial of the restriction of $x$ to $V_{i}$ is a power of an irreducible polynomial for all $x \in K$ and $1 \leqslant i \leqslant s$. The subspaces $V_{i}$ are called primary components. If for any two components $V_{i}$ and $V_{j}(i \neq j)$, there is an $x \in K$ such that the minimal polynomials of the restrictions of $x$ to $V_{i}$ and $V_{j}$ are powers of different irreducible polynomials, then the decomposition is called collected.

In general, a $K$-module $V$ will not have a primary (or collected primary) decomposition into $K$-modules, but such a decomposition is guaranteed to exist if $K$, as subalgebra of $\mathfrak{g l}(V)$, is nilpotent. Below we present some of these results that will be used in the text.

Proposition 2.1.3 ( [7], Propposition 3.1.7). Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$. Let $x, y \in \mathfrak{g l}(V)$ and $q \in \mathbb{F}[X]$ be a polynomial. Suppose that $\left[x_{n}, y\right]=0$, for some $n \geqslant 1$. Then $V_{0}(q(x))$ is invariant under $y$.

Proposition 2.1.3 implies the following corollary.
Corollary 2.1.4. Let $V$ be finite-dimensional vector space over a field $\mathbb{F}$. Let $K \leqslant \mathfrak{g l}(V)$ be a nilpotent subalgebra and let $q$ be a polynomial in $\mathbb{F}[X]$. Then $V_{0}(q(x))$ is a $K$-module for all $x \in K$.

Proposition 2.1.5 ([7], Theorem 3.1.10). Let $V$ be finite-dimensional vector space. Let $K \leqslant$ $\mathfrak{g l}(V)$ be a nilpotent subalgebra. Then $V$ has a unique collected primary decomposition relative to $K$

Let $K \leqslant \operatorname{gl}(V)$ be a nilpotent Lie algebra. By Proposition 2.1.5, $V$ has a unique collected primary decomposition $V=V_{1} \dot{+} V_{2} \dot{+} \cdots \dot{+} V_{r}$ into $K$-modules. The next proposition shows that for all $x \in K$ and all irreducible polynomials $q \in \mathbb{F}[X]$, that divide the minimal polynomial of $x$, the subspace $V_{0}(q(x))$ of $V$ can be written as the sum of some of these primary components.

Proposition 2.1.6. Let $K$ be a nilpotent Lie algebra and let $V$ be a finite-dimensional $K$-module. Let $V=V_{1}+\cdots \dot{+} V_{s}$ be the collected primary decomposition of $V$ into $K$-modules. Let $x \in K$ and let $q$ be an irreducible polynomial of $\mathbb{F}[X]$, such that $q$ divides the minimal polynomial of $x$. Then $V_{0}(q(x))=V_{j_{1}}+\cdots+V_{j_{t}}$, for some some primary components $V_{j_{1}}, V_{j_{2}}, \ldots, V_{j_{t}}$.

Proof. Let $U=\left\{V_{j_{1}}, V_{j_{2}}, \ldots, V_{j_{l}}\right\}$ be the set of all primary components such that the minimal polynomial of the restriction of $x$ to $V_{j_{i}}$ is a power of $q$. We claim that

$$
V_{0}(q(x))=V_{j_{1}}+V_{j_{2}}+\cdots+V_{j_{t}} .
$$

By definition, $V_{j_{i}} \leqslant V_{0}(q(x))$ for all $i$, and so $V_{j_{1}}+\cdots+V_{j_{l}} \subseteq V_{0}(q(x))$. Suppose now that $v \in V_{0}(q(x))$. Assume that $V_{k_{1}}, \ldots, V_{k_{s}}$ are the collected primary components of $V$ that are not elements of $U$. Then $v=v_{j_{1}}+\cdots+v_{j_{t}}+v_{k_{1}}+\cdots+v_{k_{s}}$ with $v_{j_{i}} \in V_{j_{i}}$ and $v_{k_{i}} \in V_{k_{i}}$. As $v_{j_{1}}+\ldots+v_{j_{t}} \in V_{0}(q(x))$, we obtain that $v_{k_{1}}+\cdots+v_{k_{s}} \in V_{0}(q(x))$, and we may assume without loss of generality that $v=v_{k_{1}}+\ldots+v_{k_{s}}$. Since $v \in V_{0}(q(x))$, there is some $m$ such that

$$
0=q(x)^{m}(v)=q(x)^{m}\left(v_{k_{1}}\right)+\cdots+q(x)^{m}\left(v_{k_{s}}\right)
$$

which implies that $q(x)^{m}\left(v_{k_{i}}\right)=0$ for all $i$. We claim that $v_{k_{i}}=0$ for all $i$. By the argument above, $q(x)^{m}\left(v_{k_{i}}\right)=0$. On the other hand, $v_{k_{i}} \in V_{k_{i}}$ and $V_{k_{i}} \notin U$, and hence there exists an irreducible polynomial $r(X)$ distinct from $q(X)$ such that $r(x)^{n}\left(v_{k_{i}}\right)=0$. Since $q(X)^{m}$ and $r(X)^{n}$ are coprime, there are polynomials $u(X)$ and $v(X)$ such that $u(X) q(X)^{m}+v(X) r(X)^{n}=1$. Therefore

$$
v_{k_{i}}=1(x)\left(v_{k_{i}}\right)=u(x) q(x)^{m}\left(v_{k_{i}}\right)+v(x) r(x)^{n}\left(v_{k_{i}}\right)=0 .
$$

Hence $v_{k_{i}}=0$, as claimed. Therefore $V_{0}(q(x)) \subseteq V_{j_{1}}+\ldots+V_{j_{t}}$, and also $V_{0}(q(x))=V_{j_{1}}+$ $\ldots+V_{j_{t}}$.

Let $K$ be a nilpotent Lie algebra and let $V$ be a $K$-module such that $V$ has a collected primary decomposition $V=V_{1} \dot{+} \cdots \dot{+} V_{s}$. For $x \in K$ and $1 \leqslant i \leqslant s$ define $q_{x, i} \in \mathbb{F}[X]$ to be the irreducible polynomial such that the minimal polynomial of $x$ restricted to $V_{i}$ is a power of $q_{x, i}$. Then we obtain the equality

$$
V_{i}=\left\{v \in V \mid \text { for all } x \in K \text { there is an } m>0 \text { such that } q_{x, i}(x)^{m} v=0\right\} .
$$

See [ [7], page 62].
If the base field of $V$ is algebraically closed, then all irreducible polynomials are of the form $q(X)=X-\lambda$, for some $\lambda \in \mathbb{F}$, and hence $q_{x, i}=X-\lambda_{i}(x)$, with $\lambda_{i}(x) \in \mathbb{F}$. Further, in this case, primary components are of the form

$$
V_{i}=\left\{v \in V \mid \text { for all } x \in K \text { there is an } m>0 \text { such that }\left(x-\lambda_{i}(x) I\right)^{m} v=0\right\}
$$

with $\lambda_{i} \in K^{*}$, where $K^{*}$ denotes the vector space of linear forms $K \rightarrow \mathbb{F}$. It is natural to give a name for this case.

Definition 2.1.7. Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$ and $K \leqslant \mathfrak{g l}(V)$ a subalgebra. Let $\lambda \in K^{*}$. Then

$$
V_{\lambda}=\left\{v \in V \mid \text { for all } x \in K \text { there is an } m>0 \text { such that }(x-\lambda(x) \cdot I)^{m} v=0\right\} .
$$

If $V_{\lambda} \neq 0$ then $V_{\lambda}$ is called a generalized eigenspace of $V$ associated to the generalized eigenvalue $\lambda \in K^{*}$.

Corollary 2.1.8. Let $L$ be a nilpotent Lie algebra over an algebraically closed field $\mathbb{F}$ and let $V$ be a finite-dimensional L-module. Then there exist generalized eigenvalues $\lambda_{1}, \ldots, \lambda_{k}$ of $L$ such that $V=V_{\lambda_{1}} \oplus \cdots \oplus V_{\lambda_{k}}$ where the $V_{\lambda_{i}}$ are the generalized eigenspaces as in Definition 2.1.7.

Another decomposition that can be derived from the primary decomposition is called the Fitting decomposition. It can be used to identify if $K$ acts nilpotent on some component of $V$. Assume as above that $K$ is a nilpotent $\mathbb{F}$-Lie algebra and $V$ is a finite-dimensional $K$-module and that $\mathbb{F}$ is algebraically closed. By Proposition $2.1 .5, V$ has a unique collected primary decomposition $V_{1} \dot{+} \cdots \dot{+} V_{r}$ into $K$-modules. For every $x \in K$ and $V_{i}, 1 \leqslant i \leqslant r$, we can describe if $x$ is nilpotent or non-singular on $V_{i}$ by looking at the minimal polynomial of the restriction of $x$ to $V_{i}$. Let $X$ be an indeterminate and $q_{x, i}$ be the unique irreducible factor of the minimal polynomial of $x$ on $V_{i}$. If $q_{x, i}(X)=X$ then $x$ acts nilpotently on $V_{i}$. Otherwise, if $q_{x, i}(X)=X-a, a \in \mathbb{F}, a \neq 0$. Then $x$ acts non-singularly on $V_{i}$ with $x V_{i}=V_{i}$. As the primary decomposition $V_{1}+\cdots \dot{+} V_{r}$ is collected, there is at most one component $V_{i}$ such that every element $x \in K$ is nilpotent. If such a component exists it will be denoted by $V_{0}(K)$. Let $V_{1}(K)$ be the sum of the remaining components and write $V=V_{0}(K) \dot{+} V_{1}(K)$. This can be summarized in the following definition.

Definition 2.1.9. Let $V$ be a finite-dimensional vector space over an algebraically closed field $\mathbb{F}$ and let $K \leqslant \mathfrak{g l}(V)$ be a nilpotent subalgebra. Let $V=V_{0}(K)+V_{1}(K)$ such that

$$
V_{0}=\left\{v \in V \mid \text { for all } x \in K \text { there is an } m>0 \text { such that } x^{m} v=0\right\},
$$

and $V_{1}(K)$ is the subspace defined above. This decomposition is called the Fitting decomposition, $V_{0}(K)$ and $V_{1}(K)$ are the Fitting-null and Fitting-one component of $V$ with respect to $K$.

### 2.2 Lie algebra extensions

An extension of a Lie algebra $K$ by a Lie algebra $I$ is an exact sequence

$$
\begin{equation*}
0 \rightarrow I \xrightarrow{i} L \xrightarrow{s} K \rightarrow 0 \tag{4}
\end{equation*}
$$

of Lie algebras. The Lie algebra $L$ in the middle of the exact sequence contains an ideal $\operatorname{Ker}(s)=\operatorname{Im} i \cong I$ such that $L / I \cong K$. We will write informally that ' $L$ is an extension of $K$ by $I^{\prime}$. The extension (4) splits if $L$ has a subalgebra $S$ such that $L=S+\operatorname{Ker}(s)$. The extension (4) is trivial if there exists an ideal $S$ of $L$ such that $L=S \oplus \operatorname{Ker}(s)$. The extension (4) is central if $\operatorname{Ker}(s)$ lies in the center $Z(L)$ of $L$.

Let $K$ be a Lie algebra over a field $\mathbb{F}$ and let $I$ be a $K$-module over $\mathbb{F}$. Denote by $\mathrm{C}^{2}(K, I)$ the vector space of alternating bilinear maps $\vartheta: K \times K \rightarrow I$. If $\vartheta \in \mathrm{C}^{2}(K, I)$ has the property that

$$
\begin{equation*}
\vartheta(x,[y, z])+\vartheta(y,[z, x])+\vartheta(z,[x, y])+[x, \vartheta(y, z)]+[y, \vartheta(z, x)]+[z, \vartheta(x, y)]=0 \tag{5}
\end{equation*}
$$

for all $x, y, z \in K$, then $\vartheta$ is said to be a cocycle. The vector space of cocycles is denoted by $\mathrm{Z}^{2}(K, I)$. Let $T: K \rightarrow I$ be a linear transformation and define, $\vartheta_{T}: K \times K \rightarrow I$ by

$$
\begin{equation*}
\vartheta_{T}(h, k)=T([h, k])+[k, T(h)]-[h, T(k)] \quad \text { for all } \quad h, k \in K . \tag{6}
\end{equation*}
$$

Then $\vartheta_{T} \in \mathrm{Z}^{2}(K, I)$ and such a cocycle $\vartheta_{T}$ is said to be a coboundary. The set of coboundaries is denoted by $\mathrm{B}^{2}(K, I)$. The set $\mathrm{B}^{2}(K, I)$ is a subspace of $\mathrm{Z}^{2}(K, I)$, and we set $\mathrm{H}^{2}(K, I)=$ $\mathrm{Z}^{2}(K, I) / \mathrm{B}^{2}(K, I)$ to be the quotient space. The first cohomology group of $K$ and $I$ is defined as

$$
Z^{1}(K, I)=\{v \in \operatorname{Hom}(K, I) \mid v([h, k])=[h, v(k)]-[k, v(h)] \text { for all } h, k \in K\} .
$$

The next result, whose proof can be found, for instance, in [8, Section 4.2], links Lie algebra extensions to cohomology.

Proposition 2.2.1. Let $K$ be a Lie algebra and let I be a $K$-module. Let $\vartheta \in Z^{2}(K, I)$ and define the Lie algebra $K_{\vartheta}=K+I$ with the product

$$
\begin{equation*}
[x+a, y+b]=[x, y]+\vartheta(x, y)+[a, y]-[b, x] \text { for all } x, y \in K \text { and } a, b \in I . \tag{7}
\end{equation*}
$$

The following hold for the Lie algebra $K_{\vartheta}$ :

1. $K_{\vartheta}$ is a Lie algebra extension of $K$ by $I$;
2. if $v \in B^{2}(K, I)$, then $K_{\vartheta}$ is isomorphic to $K_{\vartheta+v}$;
3. if $\vartheta \in B^{2}(K, I)$, then $K_{\vartheta}$ is a split extension of $K$ by $I$.

Conversely, let $L$ be a Lie algebra and let $J$ be an abelian ideal of $L$. Then there exists $\vartheta \in$ $Z^{2}(L / J, J)$ such that $L \cong(L / J)_{\vartheta}$.

The cocycle $\vartheta$ in last the statement of Proposition 2.2 .1 can be constructed as follows. Let $\pi: L \rightarrow L / I$ denote the natural projection, and let $\sigma: L / I \rightarrow L$ be a right inverse of $\pi$; that is, $\pi \sigma=\mathrm{id}_{L / I}$. Then, for $k+I, h+I \in L / I$, set

$$
\vartheta(h+I, k+I)=\sigma([h+I, k+I])-[\sigma(h+I), \sigma(k+I)] .
$$

Routine calculation shows that $\vartheta \in Z^{2}(L / I, I)$ and that $L \cong L_{\vartheta}$.

### 2.3 Representation of Lie Algebras

This section presents some general results about Lie algebras that will be used in this text. The following proposition will be used in the proof of Jacobson's Theorem in Section 2.4.

Proposition 2.3.1 ( [9], Proposition 5 of Chapter III). Let L be a Lie algebra over an algebraically closed field. Let $K$ be a subalgebra of $\operatorname{Der}(L)$. If $\lambda, \mu: K \rightarrow \mathbb{F}^{*}$ are generalized eigenvalues of $K$ then $\left[L_{\lambda}, L_{\mu}\right] \subseteq L_{\lambda+\mu}$ whenever $\lambda+\mu$ is also a generalized eigenvalue of $K$. Otherwise $\left[L_{\mu}, L_{\lambda}\right]=0$.

Proposition 2.3.2. Let L be a Lie algebra and let I be an ideal of $L$ such that $L / I$ is nilpotent. Let ad : $L \rightarrow \mathfrak{g l}(L)$ be the adjoint representation with $\operatorname{ad}_{x}(y)=[x, y]$ for all $x, y \in L$. If add is a nilpotent endomorphism of $I$, for all $x \in L$, then $L$ is nilpotent.

Proof. We claim that $\mathrm{ad}_{x}$ is a nilpotent endomorphism of $L$ for all $x$ in $L$. By Engel's Theorem (see [7], Theorem 2.1.5), this will imply that $L$ is a nilpotent Lie algebra as asserted by the proposition. Suppose that $x \in L$. Since $L / I$ is nilpotent, there exists $n>0$ such that $\left(\operatorname{ad}_{x}\right)^{n}(y)=$ $\left[x_{n}, y\right] \in I$, for all $y \in L$. Also, the restriction $\operatorname{ad}_{x}^{I}$ of $\operatorname{ad}_{x}$ to $I$ is a nilpotent endomorphism of $I$, and so there exists an $m$ such that $\left(\operatorname{ad}_{x}^{I}\right)^{m}(y)=0$ for all $y \in I$. Now let $y$ be an arbitrary element of $L$. Then, $\left(\operatorname{ad}_{x}\right)^{n}(y) \in I$, and so $\left(\operatorname{ad}_{x}\right)^{m+n}(y)=\left(\operatorname{ad}_{x}\right)^{m}\left(\operatorname{ad}_{x}\right)^{n}(y)=\left(\operatorname{ad}_{x}^{I}\right)^{m}\left(\left(\operatorname{ad}_{x}\right)^{n}(y)\right)=0$. Thus $\left(\mathrm{ad}_{x}\right)^{m+n}=0$, and hence $\operatorname{ad}_{x}$ is a nilpotent endomorphism of $L$, as claimed.

Theorem 2.3.3 ( [10], Theorem 4.1). Let V be a finite-dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic 0 and let $L$ be a solvable subalgebra of $\mathfrak{g l}(V)$. If $V \neq 0$, then $V$ contains a common eigenvector for all the endomorphism in $L$.

Theorem 2.3.4 (Lie's Theorem, see [10], Corollary A of Theorem 4.1). Let L be a finitedimensional solvable Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 . Let $\psi: L \rightarrow \mathfrak{g l}(V)$ be a finite-dimensional representation of $L$. Then there is a basis of $V$ relative to which the matrix of $\psi(x)$ is upper triangular, for all $x \in L$.

Theorems 2.3.3 and 2.3.4 are not true in prime characteristic, but as observed by G. Selligman in Chapter V Section 1 of [11] "some of the proofs referred to are still applicable when the degree of the matrices is less than the characteristic". J. E. Humphreys in [10] let an exercise (Exercise 2 of Section 4) to prove that Lie's Theorem can be adapted for prime characteristic $p$ if the dimension of the matrices is less than $p$. As I did not find the theorem stated in positive characteristic, I prefer to present this result here with a brief explanation of how to adapt the proof for this case.

Theorem 2.3.5. Let L be a finite-dimensional solvable Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$. Let $V$ be a vector space of dimension $n<p$. Let $\psi: L \rightarrow \mathfrak{g l}(V)$ be a representation of $L$. Then there is a basis of $V$ relative to which the matrix of $\psi(x)$ is upper triangular, for all $x \in L$.

The proof of Theorem 2.3.3 in prime characteristic under the additional condition that $\operatorname{dim} V<p$ goes through as in Humphreys' book [10] except for the last sentence. In the book's version, we have $n \lambda([x, y])=0$ and conclude that $\lambda([x, y])=0$, because the characteristic of $\mathbb{F}$ is 0 . In this case, since $p>\operatorname{dim} V=n$, we can still make the same conclusion, since $n$ will not be a zero divisor. The proof of Theorem 2.3.4 goes through exactly as in the book.

### 2.4 Jacobson's Theorem

The main objective of this thesis is to study what information can be obtained about a Lie algebra with a non-singular derivation. This study starts with a theorem of Nathan Jacobson, in the article A note on automorphism and derivations of Lie algebras [1]. Jacobson used a variation of Engel's Theorem for weakly closed sets to get sufficient conditions for a Lie algebra to be nilpotent. Next we present this theorem and some discussion about the subject. For more detailed results we recommend the reading of Sections 1 and 2 of Chapter 2 of Jacobson's book [9].

Let $A$ be an associative algebra with 1 over a field $\mathbb{F}$. A subset $S$ of $A$ is called weakly closed if for every ordered pair $(a, b) \in S \times S$, there is an element $\gamma(a, b) \in \mathbb{F}$ such that $a b+\gamma(a, b) b a \in S$. If $S$ is a subset of a Lie or associative algebra $X$, then $\langle S\rangle$ denotes the Lie or associative, respectively, subalgebra of $X$ generated by $S$. This notation may cause confusion when $X$ is an associative and Lie algebra in the same time, and in such cases we will denote by $\langle S\rangle_{A}$ and $\langle S\rangle_{L}$ the associative and the Lie algebra, respectively, generated by $S$. It is important clarify that, in this text, associative algebras may not have an identity.

Proposition 2.4.1 ( [9], Theorem 1 of Chapter II). Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$. Let $S \subseteq \operatorname{End}(V)$ be a weakly closed subset such that every $s \in S$ is associative nilpotent, that is, $s^{k}=0$, for some positive integer $k$. Then the associative subalgebra $\langle S\rangle \leqslant$ End $(V)$ is nilpotent.

With this result we can prove Jacobson's Theorem.

Theorem 2.4.2 ( [1], Theorem 3). Let L be a finite-dimensional Lie algebra over a field of characteristic 0 and suppose that there exists a subalgebra $D$ of the algebra of derivations of $L$ such that

## 1. $D$ is nilpotent;

2. if there is $x \in L$ such that $\delta(x)=0$ for all $\delta \in D$ then $x=0$.

## Then $L$ is nilpotent.

Proof. Let $\overline{\mathbb{F}}$ be the algebraic closure of the base field $\mathbb{F}$. We can extend all derivations of $L$ to $\bar{L}=L \otimes \overline{\mathbb{F}}$. If we prove that $\bar{L}$ is nilpotent then $L$ is nilpotent. So we will assume without loss of
generality that $\mathbb{F}$ is algebraically closed. Let $L=L_{\gamma_{1}} \dot{+} \cdots \dot{+} L_{\gamma_{t}}$ be the decomposition of $L$ into generalized eigenspaces of $D$, given by Corollary 2.1.8. We claim that $\gamma_{i} \neq 0$ for all $i$. Indeed, if, for example, $\gamma_{1}=0$, then $L_{\gamma_{1}}$ is the Fitting-null component of $L$ with respect to $D$ (see Definition 2.1.9). In this case, every element $x \in D$ induces a nilpotent linear transformation on $L_{\gamma_{1}}$. Then there is a non-zero vector $v \in L_{\gamma_{1}}$ such that $x v=0$ for all $x \in D$ (see [Proposition 2.1.2, [7]]). Therefore $\gamma_{i} \neq 0$ for all $i$, as claimed. By Proposition 2.3.1, we have $\left[L_{\gamma_{i}}, L_{\gamma_{j}}\right] \subseteq L_{\gamma_{i}+\gamma_{j}}$ if $\gamma_{i}+\gamma_{j}$ is a generalized eigenvalue of $D$ and $\left[L_{\gamma_{i}}, L_{\gamma_{j}}\right]=0$ otherwise. For a subset $Y \subseteq L$, we let $\operatorname{ad}_{Y}$ denote the set of adjoint mappings induced by elements of $Y$. Then the inclusion just noted shows that the set $S=\bigcup \operatorname{ad}_{L_{\gamma_{j}}}$ is a weakly closed set of linear transformations of End $(V)$. Let $y \in L_{\gamma_{j}}$ and $z \in L_{\gamma_{i}}$. Then $\left(\operatorname{ad}_{y}\right)^{s}(z) \in L_{\gamma_{i}+s \gamma_{j}}$, for all $s \geqslant 0 .(*)$ The generalized eigenvalue $\gamma_{j}$ is non-zero and $\mathbb{F}$ has characteristic 0 . Thus, $\gamma_{i}+s \gamma_{j}$, are pairwise distinct for all $s>0$. As $L$ has finite-dimension, for some $r$ large enough $\gamma_{i}+r \gamma_{j}$ is not an eigenvalue and $\left(\mathrm{ad}_{y}\right)^{r}(z)=0$. It follows that $\mathrm{ad}_{y}$ is a nilpotent linear transformation. Hence, every element of $S$ is nilpotent. By Proposition 2.4.1, the associative subalgebra $\langle S\rangle_{A} \leqslant \operatorname{End}(V)$ is nilpotent. Observe that the Lie subalgebra $\langle S\rangle_{L}$ is a subset of $\langle S\rangle_{A}$, and so $\langle S\rangle_{L}$ is nilpotent. However, $\langle S\rangle_{L}=\operatorname{ad}_{L}$ implies that $L$ is a nilpotent Lie algebra.

In the proof of Theorem 2.4.2 the hypothesis of zero characteristic is essential to prove that every element in a homogeneous component is nilpotent. As the following examples show, Theorem 2.4.2 fails to hold in characteristic $p>0$.

Example 2.4.3. ( [4], page 895) Let $m \geqslant 2$ and let $\mathbb{F}$ be the field of $2^{m}$ elements. Let $L$ be the vector space over $\mathbb{F}$ such that

$$
L=\left\langle v_{a} \mid a \in \mathbb{F}, a \neq 0\right\rangle
$$

with a basis $\left\{v_{a} \mid a \in \mathbb{F}, a \neq 0\right\}$ labeled by the nonzero elements of the field $\mathbb{F}$ under the multiplication $\left[v_{a}, v_{b}\right]=(a+b) v_{a+b}$. Then $L$ is a simple Lie algebra and the map $\delta \in \operatorname{End}(L)$ given by $\delta\left(v_{a}\right)=a v_{a}$ is a non-singular derivation. This example and a classification of simple Lie algebras with non-singular derivations can be found in [4] on pages 895 and 916.

Example 2.4.4. ( [3], Theorem 2.1) Let $V$ be a vector space over a field $\mathbb{F}$ of characteristic $p>0$. Let $B=\left\{v_{0}, v_{2}, \cdots, v_{p-1}\right\}$ be a basis of $V$. Define the linear map $x \in \mathfrak{g l}(V)$ by $x\left(v_{i}\right)=v_{i+1}$ for $0 \leqslant i \leqslant p-2$ and $x\left(v_{p-1}\right)=v_{0}$. Let $K$ be the abelian Lie algebra generated by $\left\{x, x^{2}, \cdots, x^{p-1}\right\}$. Then $V$ can be considered as $K$-module with the standard action of $\mathfrak{g l}(V)$ on $V$. Let $L$ be the semidirect sum $L=K \oplus V$ (see Section 3.1 for the definition). Then $L$ is an
solvable non-nilpotent Lie algebra of derived length 2 . Let $a, b \in \mathbb{F}$ both non-zero and $a \neq s b$, for all $s \in \mathbb{F}_{p}$. The linear map $\delta: L \rightarrow L$ defined by

$$
\delta: \begin{cases}x^{j} \mapsto j a x^{j}, & 1 \leqslant j \leqslant p-1 ; \\ v_{i} \mapsto(b+(i-1) a) v_{i}, & 0 \leqslant i \leqslant p-1,\end{cases}
$$

is a non-singular derivation of $L$.

Another question is whether the converse of Jacobson's Theorem is true, that is: is it true that all finite-dimensional nilpotent Lie algebras admit non-singular derivation? The answer is no. By Dixmier and Lister [12], there are nilpotent Lie algebras admitting only nilpotent derivations. Below we present the example of Dixmier and Lister of such an algebra.

Example 2.4.5. Let $\mathbb{F}$ be a field of characteristic 0 and $L=\left\langle v_{1}, v_{2}, \cdots, v_{8}\right\rangle$ be a Lie algebra over $\mathbb{F}$ with dimension 8 and multiplication table

$$
\begin{aligned}
& {\left[\begin{array}{lllll}
{\left[v_{1}, v_{2}\right]=v_{5}} & {\left[v_{1}, v_{3}\right]=v_{6}} & {\left[v_{1}, v_{4}\right]=v_{7}} & {\left[v_{1}, v_{5}\right]=-v_{8}} & {\left[v_{2}, v_{3}\right]=v_{8}}
\end{array} \quad\left[v_{2}, v_{4}\right]=v_{6}\right.} \\
& {\left[v_{2}, v_{6}\right]=-v_{7}}
\end{aligned} \quad\left[v_{3}, v_{4}\right]=-v_{5} \quad\left[v_{3}, v_{5}\right]=-v_{7} \quad\left[v_{4}, v_{6}\right]=-v_{8} \quad\left[v_{i}, v_{j}\right]=-\left[v_{j}, v_{i}\right] . . ~ l
$$

Moreover, $\left[v_{i}, v_{j}\right]=0$ if it is not in the table above. Then $L$ is nilpotent with $L^{3} \neq 0$, $L^{4}=0$ and every derivation of $L$ is nilpotent.

As the examples above show, Jacobson's Theorem is in general not true in characteristic $p>0$. However, we have the follow weaker result.

Theorem 2.4.6. Let L be a Lie algebra over a field of characteristic $p>0$ and suppose that there exists a subalgebra $D \leqslant \operatorname{Der}(L)$ such that

1. $D$ is nilpotent;
2. if there is $x \in L$ such that $\delta(x)=0$ for all $\delta \in D$ then $x=0$.

If $D$ has at most $p-1$ generalized eigenvalues then $L$ is nilpotent.

Proof. The proof of this theorem is identical to proof of Theorem 2.4.2 up to the point marked by (*). The generalized eigenvalue $\gamma_{j} \neq 0$ and the set $\left\{\gamma_{i}, \gamma_{i}+\gamma_{j}, \cdots, \gamma_{i}+(p-1) \gamma_{j}\right\}$ has $p$ distinct elements. As $D$ has at most $p-1$ generalized eigenvalues, for some $r, 0<r \leqslant p-1$, $\left(\gamma_{i}+r \gamma_{j}\right)$ is not an eigenvalue. It follows that $\mathrm{ad}_{y}$ is a nilpotent linear transformation, for every $a \in L_{\gamma_{i}}$. Thus every element of $S$ is a nilpotent. By Proposition 2.4.1, the associative subalgebra $\langle S\rangle_{A} \leqslant \operatorname{End}(V)$ is nilpotent and hence $\operatorname{ad}_{L}$ is nilpotent. Therefore $L$ is a nilpotent Lie algebra.

### 2.5 The orders of non-singular derivations

An interesting approach by Shalev in article [2] is to study the possible orders of nonsingular derivations, establishing conditions for a Lie algebra over a field of characteristic $p>0$ with non-singular derivations to be nilpotent. Later, Mattarei in [3] showed that this set of orders of non-singular derivations corresponds to the set of solutions of some polynomial equation over a field of characteristic $p>0$. Below we present some results of these articles.

Lemma 2.5.1. Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic $p>0$. If $\delta \in \operatorname{Der}(L)$ then $\delta^{p^{m}} \in \operatorname{Der}(L)$, for all $m \geqslant 1$.

Proof. Let $\delta \in \operatorname{Der}(L)$ and $x, y \in L$. For a natural number $n$, we have that

$$
\begin{equation*}
\delta^{n}([x, y])=\sum_{k=0}^{n}\binom{n}{k}\left[\delta^{k}(x), \delta^{n-k}(y)\right], \text { for all } n>0 \tag{8}
\end{equation*}
$$

Equation (8) is known as Leibniz's Formula; see equation (1.11) on page 23 of [7]. As the field $\mathbb{F}$ has characteristic $p>0$, setting $n=p^{m}$ Leibniz's formula is reduced to

$$
\delta^{p^{m}}([x, y])=\left[\delta^{p^{m}}(x), y\right]+\left[x, \delta^{p^{m}}(y)\right] .
$$

Therefore $\delta^{p^{m}} \in \operatorname{Der}(L)$ as claimed.
An endomorphism $\alpha$ of a finite-dimensional vector space $V$ is said to be diagonalizable if $V$ admits a basis in which the matrix of $\alpha$ is diagonal. For a non-singular linear transformation $\alpha$ of finite order, let $|\alpha|$ denote the order of $\alpha$.

Lemma 2.5.2. Suppose that L is a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic $p \geqslant 0$ and let $\delta$ be a non-singular derivation of $L$ with finite order. Then there exists an extension field $\mathbb{F}_{0}$ such that one of the following is valid.

- $p=0$ and $\delta$ is diagonalizable over $\mathbb{F}_{0}$;
- $p$ is a prime, $|\delta|=n p^{t}$ with $p \nmid n$, and $\delta^{p^{t}}$ is a non-singular derivation that is diagonalizable over $\mathbb{F}_{0}$.

Proof. First we proof this lemma for $p>0$, then we explain how the proof can be adapted for $p=0$. Suppose that $\delta$ is a non-singular derivation of $L$ with finite order. Suppose that $\delta^{m}=I d$, $p>0$ and write $m=n p^{t}$ with $t \geqslant 0$ and $\operatorname{gcd}(n, p)=1$. By Lemma 2.5.1, we have that $\delta^{p^{t}}$
is a derivation whose order is $n$. Let $\alpha=\delta^{p^{t}}$ and let $\mathbb{F}_{0}$ be the splitting field of the minimal polynomial of $\alpha$. Note that $\mathbb{F}_{0}$ is an extension field of $\mathbb{F}$ and set $L_{0}=L \otimes \mathbb{F}_{0}$. Now the matrix of $\alpha$, considered as a non-singular derivation of $L_{0}$, is in upper triangular Jordan normal form in a suitable basis of $L_{0}$. Let us identify the endomorphism $\alpha$ with this matrix in Jordan normal form. Hence, we may write $\alpha=\alpha_{S}+\alpha_{N}$, where $\alpha_{S}$ is a diagonal matrix and $\alpha_{N}$ is a nilpotent matrix such that $\alpha_{S}$ and $\alpha_{N}$ commute. As $\alpha^{n}=I d$,

$$
\begin{equation*}
I d=\alpha^{n}=\left(\alpha_{S}+\alpha_{N}\right)^{n}=\alpha_{S}^{n}+\binom{n}{1} \alpha_{S}^{n-1} \alpha_{N}+\binom{n}{2} \alpha_{S}^{n-2} \alpha_{N}^{2}+\cdots+\binom{n}{n-1} \alpha_{S} \alpha_{N}^{n-1}+\alpha_{N}^{n} \tag{9}
\end{equation*}
$$

The identity matrix on the left-hand side of the last equation is diagonal, while the summands, with the exception of the first summand, on the right-hand side are nilpotent. Further, if $\alpha_{N} \neq 0$, then the second summand $n \alpha_{S}^{n-1} \alpha_{N}$ in non-zero, since $p \nmid n$, and it is the only summand that contains a non-zero entry in a positions $(i, i+1)$ with $i>0$. However, this implies that $\alpha^{n}$ must contain a non-zero entry in a position $(i, i+1)$, which is a contradiction, as $\alpha^{n}=I d$. Hence $\alpha_{N}=0$ and $\alpha$ is diagonalizable. Therefore, $\alpha$ is a non-singular diagonalizable derivation of $L_{0}$.

In case $p=0$, suppose that $|\delta|=n$ and set $\alpha=\delta$. Let $\mathbb{F}_{0}$ be the splitting field of $\alpha$ and set $L_{0}=L \otimes \mathbb{F}_{0}$. Following the same steps as the previous case, we can consider a suitable bases for $L_{0}$ such that the matrix of $\alpha$ is in upper triangular Jordan normal form. Write $\alpha=\alpha_{S}+\alpha_{N}$ and suppose that $\alpha_{N} \neq 0$. By equation $9, n \alpha_{S}^{n-1} \alpha_{N}$ is non-zero, since $n \neq 0$, which is a contradiction. Hence $\alpha_{N}=0$ and $\alpha$ is a non-singular diagonalizable derivation of $L_{0}$.

Remark 2.5.3. If the field $\mathbb{F}$ in Lemma 2.5.2 is finite, then every non-singular endomorphism of Lhas finite order, and hence Lemma 2.5.2 is valid in this case without the additional assumption that $\delta$ has finite order. It is also clear considering the proof of Lemma 2.5.2 that if $\delta$ is a nonsingular derivation of $L$ with finite order such that the degree is coprime to $p$, then $\delta$ itself is diagonalizable over a suitable finite order extension field $\mathbb{F}_{0}$. In particular, if $\delta$ is a non-singular derivation of $L$ with order coprime to $p$, then $\delta$ is diagonalizable over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$.

For a field $\mathbb{F}$, let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$.
Proposition 2.5.4 ( [2], Lemma 2.2). Let L be a finite-dimensional Lie algebra in characteristic $p>0$ which admits a non-singular derivation $\delta$ whose finite order $n$ is coprime to $p$. Suppose that $L$ is not nilpotent. Then there exists $a \in \overline{\mathbb{F}}$ such that $(a+b)^{n}=1$ for all $b \in \mathbb{F}_{p}$.

Proof. Let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$ and consider $\delta$ as a derivation of $\bar{L}=L \otimes \overline{\mathbb{F}}$. By Lemma 2.5.2, $\delta$ is diagonalizable (see also Remark 2.5.3). Let $\bar{L}=L_{a_{1}} \dot{+} \cdots \dot{+} L_{a_{r}}$ be the decomposition of $\bar{L}$ into eigenspaces of $\delta$. The set $S=\bigcup \operatorname{ad}_{L_{a_{j}}}$ is weakly closed with $\gamma\left(\mathrm{ad}_{x}, \mathrm{ad}_{y}\right)=-1$ for all $x \in L_{a_{i}}, y \in L_{a_{j}}$. If each $\operatorname{ad}_{x}$ is nilpotent then the associative subalgebra $\langle S\rangle \leqslant \operatorname{End}(\bar{L})$ is nilpotent by Proposition 2.4.1. Hence $\operatorname{ad}_{\bar{L}}$ is a nilpotent Lie algebra and $\bar{L}$ is nilpotent. As $L$ is non-nilpotent by hypothesis, there are $x \in L_{a_{j}}$ and $y \in L_{a_{i}}$ such that $\left(\operatorname{ad}_{x}\right)^{n}(y) \neq 0$, for all $1 \leqslant n \leqslant p$. However this implies $a_{i}+b a_{j}$ are eigenvalues of $\delta$ for $1 \leqslant b \leqslant p$. Since $|\delta|=n$ the order of each eigenvalue of $\delta$ divides $n$. Thus $\left(a_{i}+b a_{j}\right)^{n}=1$, for all $b \in \mathbb{F}_{p}$ and $a_{j}^{n}=1$. Hence, $a_{j}^{-n}=1$. Thus $1=\left(a_{i}+b a_{j}\right)^{n} a_{j}^{-n}=\left(a_{i} a_{j}^{-1}+b\right)^{n}$. Set $a=a_{i} a_{j}^{-1}$. Then $(a+b)^{n}=1$ for all $b \in \mathbb{F}_{p}$.

Corollary 2.5.5 ( [2], Corollary 2.3). Let L be a finite-dimensional non-nilpotent Lie algebra in characteristic $p>0$ which admits a non-singular derivation $\delta$ whose order $n$ is coprime to $p$. Then there is an element $c \in \overline{\mathbb{F}}$ such that $X^{p}-X-c$ divides $X^{n}-1$ as elements of the polynomial ring $\overline{\mathbb{F}}[X]$.

Proof. Let $a \in \overline{\mathbb{F}}$ as in Proposition 2.5.4. Let $R=\left\{x \in \overline{\mathbb{F}} \mid x^{n}=1\right\}$ be the set of the $n$-th roots of unity in $\overline{\mathbb{F}}$. Write the polynomial

$$
X^{n}-1=\prod_{x \in R}(X-x) \in \mathbb{F}[X] .
$$

For all $b \in \mathbb{F}_{p}, a+b \in R$, and so $\prod_{b \in \mathbb{F}_{p}}(X-a-b)$ divides $X^{n}-1$. But

$$
\prod_{b \in \mathbb{F}_{p}}(X-a-b)=(X-a)^{p}-(X-a)=X^{p}-X-c,
$$

where $c=a^{p}-a$. The first equation of the last display can be seen by observing that the elements $a+b$ with $b \in \mathbb{F}_{p}$ are exacty the $p$ roots of the polynomial $(X-a)^{p}-(X-a)$. Let $g(X)=X^{p}-X-c$. Then $g(X)$ divides $X^{n}-1$.

Lemma 2.5.6 ( [2], Lemma 2.4). Let L be a finite-dimensional non-nilpotent Lie algebra in characteristic $p>0$ which admits a non-singular derivation $\delta$ whose order $n$ is coprime to $p$. Then $n \geqslant p^{2}-1$.

Proof. Suppose $n<p^{2}-1$ and write $n=a+b p$ where $0 \leqslant a \leqslant p-1$. Observe that $b$ can be at most $p-1$ and when $b=p-1$, then $a<p-1$. It follows, $a+b \leqslant 2 p-3$. Let $c \in \overline{\mathbb{F}}$ as in Corollary 2.5 .5 and let $g(X)=X^{p}-X-c$. Then, by Corollary 2.5.5, $X^{n}$ is congruent to 1
modulo $g$. Working modulo $g$, we have

$$
X^{n}=X^{a}\left(X^{p}\right)^{b} \equiv X^{a}(X+c)^{b} .
$$

Then

$$
X^{n} \equiv X^{a+b}+b c X^{a+b-1}+\cdots+\binom{b}{i} c^{i} X^{a+b-i}+\cdots+c^{b} X^{a}
$$

If $a+b<p$ then the above polynomial is not congruent to 1 modulo $g$. Thus $a+b \geqslant p$. Write $a+b=p+e$ where $0 \leqslant e \leqslant p-3$. It follows that

$$
\begin{gathered}
X^{n} \equiv X^{p+e}+b c X^{p+e-1}+\cdots+\binom{b}{i} c^{i} X^{p+e-i}+\cdots+c^{b} X^{a} \\
=X^{p} \underbrace{\left(X^{e}+b c X^{e-1}+\cdots+\binom{b}{e} c^{e}\right)}_{A}+\underbrace{\binom{b}{e+1} c^{e+1} X^{p-1}+\cdots+c^{b} X^{a}}_{B} .
\end{gathered}
$$

Thus $X^{n} \equiv X^{p} A+B \equiv(X+c) A+B$. Note that the polynomial $(X+c) A$ has degree at most $p-2$. On the other hand, $\binom{b}{e+1} c^{e+1} \neq 0$ since $c \neq 0$ and $e+1 \leqslant b<p$, so $B$ has degree $p-1$. Therefore the polynomial $(X+c) A+B$ has degree $p-1$, and is the residue of $X^{n}$ modulo $g$. We see that $X^{n} \not \equiv 1$ modulo $g$, a contradiction.

Now we can prove the following theorem.
Theorem 2.5.7 ( [2], Theorem 1.1). Let L be a finite-dimensional Lie algebra in characteristic $p>0$ which admits a non-singular derivation $\delta$ of order $n$. Write $n=p^{s} m$ where $m$ is coprime to $p$. Suppose $m<p^{2}-1$. Then L is nilpotent.

Proof. The derivation $\delta^{p^{s}}$ has order $m$. Suppose that $L$ is not nilpotent. Then by Lemma 2.5.6 we have $m \geqslant p^{2}-1$.

Mattarei completed this result in [3].
Proposition 2.5.8. Let $p$ be a prime number and let $n$ be a positive integer, prime to $p$. The following statements are equivalent:

1. there exists a finite-dimensional non-nilpotent Lie algebra of characteristic $p$ with a nonsingular derivation of order n;
2. there exists an element $a \in \overline{\mathbb{F}}_{p}$ such that $(a+b)^{n}=1$ for all $b \in \mathbb{F}_{p}$
3. there exists an element $c \in \overline{\mathbb{F}}_{p}^{*}$ such that $X^{p}-X-c$ divides $X^{n}-1$ as elements of the polynomial ring $\overline{\mathbb{F}}_{p}[X]$.

## 3 Derivations of Lie algebra extensions

In [6] Eick utilized compatible pairs to compute automorphisms of solvable groups and solvable Lie algebras. This is related to the method for groups proposed by Robinson [13] and Smith [14]. We adapt the concept for derivations of Lie algebras. That is, below we present a process to lift a derivation from a Lie algebra $K$ to an extension $K_{\vartheta}$ where $\vartheta$ is a cocycle.

### 3.1 Compatible pairs and derivations of semidirect sums

Let $K$ and $I$ be Lie algebras such that $K$ acts on $I$ via the homomorphism $\psi: K \rightarrow$ $\operatorname{Der}(I)$. We define the semidirect sum $K \oplus_{\psi} I$ as the vector space $K \dot{+} I$ with the product operation given as

$$
\left[\left(k_{1}, v_{1}\right),\left(k_{2}, v_{2}\right)\right]=\left(\left[k_{1}, k_{2}\right],\left[k_{1}, v_{2}\right]-\left[k_{2}, v_{1}\right]+\left[v_{1}, v_{2}\right]\right) .
$$

When the $K$-action on $I$ is clear from the context, then we usually suppress the homomorphism ' $\psi$ ' from the notation and write simply $K \oplus I$. If $L$ is a Lie algebra, such that $L$ has an ideal $I$, and a subalgebra $K$ in such a way that $L=K \dot{+} I$, then $L \cong K \oplus_{\psi} I$ where $\psi$ is the restriction of $\operatorname{ad}_{I}$ to $K$. In a semidirect sum $K \oplus I$, an element $(k, v) \in K+I$ will usually be written as $k+v$.

Suppose that $K$ and $I$ are as in the previous paragraph. Let $\operatorname{Der}(K) \oplus \operatorname{Der}(I)$ be the direct sum of $\operatorname{Der}(K)$ and $\operatorname{Der}(I)$. An element $(\alpha, \beta) \in \operatorname{Der}(K) \oplus \operatorname{Der}(I)$ is said to be a compatible pair if

$$
\begin{equation*}
\beta([k, v])=[\alpha(k), v]+[k, \beta(v)] \quad \text { for all } \quad k \in K, v \in I . \tag{10}
\end{equation*}
$$

We let $\operatorname{Comp}(K, I)$ denote the set of compatible pairs in $\operatorname{Der}(K) \oplus \operatorname{Der}(I)$. Using the representation $\psi: K \rightarrow \operatorname{Der}(I)$ associated to the $K$-action on $I$, we can write equation (10) in another form as follows. Writing $[k, v]$ as $\psi(k)(v)$, we have that $(\alpha, \beta) \in \operatorname{Comp}(K, I)$ if, and only if, the equation

$$
\beta \psi(k)=\psi(\alpha(k))+\psi(k) \beta
$$

holds in $\operatorname{Der}(I)$ for all $k \in K$. Using commutator, this is equivalent to

$$
\begin{equation*}
[\beta, \psi(k)]=\psi(\alpha(k)), \quad \text { for all } \quad k \in K . \tag{11}
\end{equation*}
$$

Letting ad : $\operatorname{Der}(I) \rightarrow \operatorname{Der}(I)$ denote the adjoint representation of $I$, equation (11) can be rewritten as

$$
\begin{equation*}
\operatorname{ad}_{\beta} \psi(k)=\psi(\alpha(k)), \quad \text { for all } \quad k \in K . \tag{12}
\end{equation*}
$$

Therefore, $(\alpha, \beta) \in \operatorname{Comp}(K, I)$ if, and only if, the following diagram commutes:


A compatible pair $(\alpha, \beta) \in \operatorname{Der}(K) \oplus \operatorname{Der}(I)$ will usually be written as $\alpha+\beta$. If $\alpha+\beta \in$ $\operatorname{Der}(K) \oplus \operatorname{Der}(I)$ as above, then $\alpha+\beta$ can be considered as an element of $\mathfrak{g l}(K \oplus I)$ by letting $(\alpha+\beta)(k+v)=\alpha(k)+\beta(v)$ for all $a \in I$ and $k \in K$.

Proposition 3.1.1. Using the notation above, we have that

$$
\operatorname{Comp}(K, I)=\{\alpha+\beta \in \mathfrak{g l}(K) \oplus \mathfrak{g l}(I) \mid \alpha+\beta \in \operatorname{Der}(K \oplus I)\} .
$$

In particular $\operatorname{Comp}(K, I)$ is a Lie subalgebra of $\operatorname{Der}(K \oplus I)$.

Proof. Suppose that $\alpha+\beta \in \operatorname{Comp}(K, I)$ is a compatible pair and let $k+v, k^{\prime}+v^{\prime} \in K \oplus I$. Then

$$
\begin{aligned}
&(\alpha+\beta)\left[k+v, k^{\prime}+v^{\prime}\right]=(\alpha+\beta)\left(\left[k, k^{\prime}\right]+\left(\left[k, v^{\prime}\right]-\left[k^{\prime}, v\right]+\left[v, v^{\prime}\right]\right)\right) \\
&=\alpha\left(\left[k, k^{\prime}\right]\right)+\beta\left(\left[k, v^{\prime}\right]-\left[k^{\prime}, v\right]+\left[v, v^{\prime}\right]\right) \\
&=\left[\alpha(k), k^{\prime}\right]+\left[k, \alpha\left(k^{\prime}\right)\right]+\left[\alpha(k), v^{\prime}\right]-\left[\alpha\left(k^{\prime}\right), v\right] \\
&+\left[\beta(v), v^{\prime}\right]+\left[k, \beta\left(v^{\prime}\right)\right]-\left[k^{\prime}, \beta(v)\right]+\left[v, \beta\left(v^{\prime}\right)\right] .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& {\left[(\alpha+\beta)(k+v), k^{\prime}+v^{\prime}\right]+\left[k+v,(\alpha+\beta)\left(k^{\prime}+v^{\prime}\right)\right]=} \\
& {\left[\alpha(k), k^{\prime}\right]+\left[\alpha(k), v^{\prime}\right]+\left[\beta(v), k^{\prime}\right]+\left[\beta(v), v^{\prime}\right]} \\
& +
\end{aligned}
$$

Thus, $\alpha+\beta \in \operatorname{Der}(K \oplus I)$. Conversely, let $\alpha+\beta \in \mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$ such that $\alpha+\beta$ is a derivation of $K \oplus I$. Then $\left.(\alpha+\beta)\right|_{K}=\alpha$ and $\left.(\alpha+\beta)\right|_{I}=\beta$, and so $\alpha \in \operatorname{Der}(K)$ and $\beta \in \operatorname{Der}(I)$. Further, if $k \in K$ and $v \in I$, then $[k, v] \in I$, and so

$$
\beta([k, v])=(\alpha+\beta)[k, v]=[(\alpha+\beta)(k), v]+[k,(\alpha+\beta)(v)]=[\alpha(k), v]+[k, \beta(v)] .
$$

Thus, $\alpha+\beta \in \operatorname{Comp}(K, I)$, as required. The fact that $\operatorname{Comp}(K, I)$ is a Lie subalgebra of $\operatorname{Der}(K \oplus I)$ follows from the fact that $\operatorname{Comp}(K, I)$ is the intersection of two Lie algebras; namely,

$$
\operatorname{Comp}(K, I)=(\mathfrak{g l}(K) \oplus \mathfrak{g l}(I)) \cap \operatorname{Der}(K \oplus I)
$$

Lemma 3.1.2. Let $K$ and I be Lie algebras over a field $\mathbb{F}$ of characteristic $p>0$. Suppose that $K$ acts on I. If $(\alpha, \beta) \in \operatorname{Comp}(K, I)$, then $(\alpha, \beta)^{p^{t}}=\left(\alpha^{p^{t}}, \beta^{p^{t}}\right) \in \operatorname{Comp}(K, I)$ for all $t \geqslant 1$.

Proof. Let $L=K \oplus I$ be the semidirect sum of $K$ and $I$. By Proposition 3.1.1, $(\alpha, \beta) \in \operatorname{Der}(L)$. Let $\delta=(\alpha, \beta)$. By Lemma (2.5.1), $(\alpha, \beta)^{p^{t}} \in \operatorname{Der}(L)$ for all $t \geqslant 1$. Thus, by Proposition 3.1.1, $(\alpha, \beta)^{p^{t}}=\left(\alpha^{p^{t}}, \beta^{p^{t}}\right) \in \operatorname{Comp}(K, I)$, for all $t \geqslant 1$.

Let $K$ and $I$ be vector spaces. Consider the Lie algebra $\mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$ and the vector space $\operatorname{Hom}(K, \mathfrak{g l}(I))$. Let ad : $\mathfrak{g l}(I) \rightarrow \mathfrak{g l}(I)$ be the adjoint representation of $\mathfrak{g l}(I)$ such that $\operatorname{ad}_{\beta}\left(\beta^{\prime}\right)=\left[\beta, \beta^{\prime}\right]$ for every $\beta, \beta^{\prime} \in \mathfrak{g l}(I)$. Then define the action of $\mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$ on $\operatorname{Hom}(K, \mathfrak{g l}(I))$ by setting

$$
\begin{equation*}
(\alpha, \beta) \cdot T=\operatorname{ad}_{\beta} T-T \alpha, \tag{13}
\end{equation*}
$$

for all $(\alpha, \beta) \in \mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$ and for all $T \in \operatorname{Hom}(K, \mathfrak{g l}(I))$. Let us show that this in fact defines a Lie algebra action. Notice that $(\alpha, \beta) \cdot T \in \operatorname{Hom}(K, \mathfrak{g l}(I))$, since it is linear combination of compositions of linear maps. Let us check that the action is compatible with Lie brackets. Let $(\alpha, \beta),\left(\alpha^{\prime}, \beta^{\prime}\right) \in \mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$. By definition $\left(\alpha^{\prime}, \beta^{\prime}\right) \cdot T=\operatorname{ad}_{\beta^{\prime}} T-T \alpha^{\prime}$. Thus,

$$
(\alpha, \beta) \cdot\left(\left(\alpha^{\prime}, \beta^{\prime}\right) \cdot T\right)=\operatorname{ad}_{\beta} \operatorname{ad}_{\beta^{\prime}} T-\operatorname{ad}_{\beta^{\prime}} T \alpha-\operatorname{ad}_{\beta} T \alpha^{\prime}+T \alpha^{\prime} \alpha .
$$

In the same way,

$$
\left(\alpha^{\prime}, \beta^{\prime}\right) \cdot((\alpha, \beta) \cdot T)=\operatorname{ad}_{\beta^{\prime}} \operatorname{ad}_{\beta} T-\operatorname{ad}_{\beta} T \alpha^{\prime}-\operatorname{ad}_{\beta^{\prime}} T \alpha+T \alpha \alpha^{\prime} .
$$

Hence,

$$
\begin{aligned}
(\alpha, \beta) \cdot\left(\left(\alpha^{\prime}, \beta^{\prime}\right) \cdot T\right)-\left(\alpha^{\prime}, \beta^{\prime}\right) \cdot((\alpha, \beta) \cdot T) & =\operatorname{ad}_{\beta} \operatorname{ad}_{\beta^{\prime}} T-\operatorname{ad}_{\beta^{\prime}} \mathrm{ad}_{\beta} T-T \alpha \alpha^{\prime}+T \alpha^{\prime} \alpha \\
& =\left[\operatorname{ad}_{\beta}, \operatorname{ad}_{\beta^{\prime}}\right] T-T\left[\alpha, \alpha^{\prime}\right] .
\end{aligned}
$$

Therefore,

$$
\left[(\alpha, \beta),\left(\alpha^{\prime}, \beta^{\prime}\right)\right] \cdot T=\left(\left[\alpha, \alpha^{\prime}\right],\left[\beta, \beta^{\prime}\right]\right) \cdot T
$$

Now, if $K$ and $I$ are Lie algebras such that $K$ acts on $I$, then there is a corresponding homomorphism $\psi \in \operatorname{Hom}(K, \operatorname{Der}(I))$. Suppose that $\alpha+\beta \in \mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$ such that $\alpha+\beta \in$ $\operatorname{Der}(K) \oplus \operatorname{Der}(I)$. Then, for $k \in K$, we have $\operatorname{ad}_{\beta} T(k)+T \alpha(k)$ is a derivation of $I$, since $\operatorname{ad}_{\beta} T(k), T \alpha(k) \in \operatorname{Der}(I)$.

If $X$ is a subalgebra of $\operatorname{Der}(K) \oplus \operatorname{Der}(I)$, then the annihilator $\operatorname{Ann}_{X}(\psi)$ of $\psi$ in $X$ is defined as

$$
\operatorname{Ann}_{X}(\psi)=\{(\alpha, \beta) \in X \mid(\alpha, \beta) \cdot \psi=0\} .
$$

Computing the annihilator of $\psi$ in $\operatorname{Der}(K) \oplus \operatorname{Der}(I)$ explicitly, we obtain

$$
\begin{aligned}
& \operatorname{Ann}_{\operatorname{Der}(K) \oplus \operatorname{Der}(I)}(\psi)=\{(\alpha, \beta) \in \operatorname{Der}(K) \oplus \operatorname{Der}(I) \mid(\alpha, \beta) \cdot \psi=0\} \\
& \quad=\left\{(\alpha, \beta) \in \operatorname{Der}(K) \oplus \operatorname{Der}(I) \mid \operatorname{ad}_{\beta} \psi-\psi \alpha=0\right\}=\operatorname{Comp}(K, I) .
\end{aligned}
$$

The last equality follows from (12). Hence, we have proved the following proposition.
Proposition 3.1.3. Let $K$ and I be Lie algebras such that $K$ acts in I via the representation $\psi \in$ $\operatorname{Hom}(K, \operatorname{Der}(I))$. Then $\operatorname{Comp}(K, I)=\operatorname{Ann}_{\operatorname{Der}(K) \oplus \operatorname{Der}(I)}(\psi)$, where the action of $\operatorname{Der}(K) \oplus \operatorname{Der}(I)$ on $\operatorname{Hom}(K, \operatorname{Der}(I))$ is given by (13).

### 3.2 An action of $\mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$ on $\mathbf{C}^{2}(K, I)$

This is a technical section where we define, for two vector spaces $K$ and $I$, an action of $\mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$ on the vector space of alternating bilinear maps $\mathrm{C}^{2}(K, I)$ and we show that this is a well defined. We also present the necessary lemmas for the main result of this chapter.

Let $K$ and $I$ be vector spaces. Let $(\alpha, \beta)$ be an element of the Lie algebra $\mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$ and let $\vartheta \in \mathrm{C}^{2}(K, I)$. Define an action of $\mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$ on $\mathrm{C}^{2}(K, I)$ by setting, for $\vartheta \in \mathrm{C}^{2}(K, I)$,

$$
\begin{equation*}
(\alpha, \beta) \cdot \vartheta(h, k)=\beta(\vartheta(h, k))-\vartheta(\alpha(h), k)-\vartheta(h, \alpha(k)), \quad \text { for all } h, k \in K . \tag{14}
\end{equation*}
$$

Let $\left(\alpha^{\prime}, \beta^{\prime}\right) \in \mathfrak{g l}(K) \oplus \mathfrak{g l}(I)$. Then

$$
\begin{equation*}
(\alpha, \beta) \cdot\left(\left(\alpha^{\prime}, \beta^{\prime}\right) \cdot \vartheta(h, k)\right)=(\alpha, \beta) \cdot\left(\beta^{\prime}(\vartheta(h, k))-\vartheta\left(\alpha^{\prime}(h), k\right)-\vartheta\left(h, \alpha^{\prime}(k)\right)\right) . \tag{15}
\end{equation*}
$$

Applying the action in each summand of the right-hand of equation (15), we have

$$
\begin{array}{r}
(\alpha, \beta) \cdot \beta^{\prime}\left(\vartheta(h, k)=\beta \beta^{\prime} \vartheta(h, k)\right)-\beta^{\prime} \vartheta(\alpha(h), k)-\beta^{\prime} \vartheta(h, \alpha(k)), \\
\left.(\alpha, \beta) \cdot \vartheta\left(\alpha^{\prime}(h), k\right)=\beta \vartheta\left(\alpha^{\prime}(h), k\right)\right)-\vartheta\left(\alpha^{\prime} \alpha(h), k\right)-\vartheta\left(\alpha^{\prime}(h), \alpha(k)\right), \\
(\alpha, \beta) \cdot \vartheta\left(h, \alpha^{\prime}(k)\right)=\beta \vartheta\left(h, \alpha^{\prime}(k)\right)-\vartheta\left(\alpha(h), \alpha^{\prime}(k)\right)-\vartheta\left(h, \alpha^{\prime} \alpha(k)\right) .
\end{array}
$$

Thus,

$$
\begin{aligned}
& \left.(\alpha, \beta) \cdot\left(\left(\alpha^{\prime}, \beta^{\prime}\right) \cdot \vartheta(h, k)\right)=\beta \beta^{\prime} \vartheta(h, k)\right)-\beta^{\prime} \vartheta(\alpha(h), k)-\beta^{\prime} \vartheta(h, \alpha(k)) \\
& \left.-\beta \vartheta\left(\alpha^{\prime}(h), k\right)\right)+\vartheta\left(\alpha^{\prime} \alpha(h), k\right)+\vartheta\left(\alpha^{\prime}(h), \alpha(k)\right) \\
& -\beta \vartheta\left(h, \alpha^{\prime}(k)\right)+\vartheta\left(\alpha(h), \alpha^{\prime}(k)\right)+\vartheta\left(h, \alpha^{\prime} \alpha(k)\right) .
\end{aligned}
$$

We obtain similarly that

$$
\begin{aligned}
& \left.\left(\alpha^{\prime}, \beta^{\prime}\right) \cdot((\alpha, \beta) \cdot \vartheta(h, k))=\beta^{\prime} \beta \vartheta(h, k)\right)-\beta \vartheta\left(\alpha^{\prime}(h), k\right)-\beta \vartheta\left(h, \alpha^{\prime}(k)\right) \\
& \left.-\beta^{\prime} \vartheta(\alpha(h), k)\right)+\vartheta\left(\alpha \alpha^{\prime}(h), k\right)+\vartheta\left(\alpha(h), \alpha^{\prime}(k)\right) \\
& \quad-\beta^{\prime} \vartheta(h, \alpha(k))+\vartheta\left(\alpha^{\prime}(h), \alpha(k)\right)+\vartheta\left(h, \alpha \alpha^{\prime}(k)\right) .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
{\left[(\alpha, \beta),\left(\alpha^{\prime}, \beta^{\prime}\right)\right] \cdot \vartheta(h, k) } & =\left[\beta, \beta^{\prime}\right] \vartheta(h, k)-\vartheta\left(\left[\alpha, \alpha^{\prime}\right](h), k\right)-\vartheta\left(h,\left[\alpha, \alpha^{\prime}\right](k)\right) \\
& =\left(\left[\alpha, \alpha^{\prime}\right],\left[\beta, \beta^{\prime}\right]\right) \cdot \vartheta(h, k) .
\end{aligned}
$$

Therefore, the action presented in (14) is well defined.
Our goal now is to study the action of compatible pairs $\operatorname{Comp}(K, I)$ on the subspaces $\mathrm{Z}^{2}(K, I)$ and $\mathrm{B}^{2}(K, I)$ of $\mathrm{C}^{2}(K, I)$. For this, assume that $K$ is a Lie algebra and $I$ is a $K$-module. Then for all $h, k, l \in K,(\alpha, \beta) \in \operatorname{Comp}(K, I)$ and $\vartheta \in \mathrm{Z}^{2}(K, I)$, we have

$$
\begin{aligned}
(\alpha, \beta) \cdot \vartheta(k,[h, l]) & =\beta(\vartheta(k,[h, l]))-\vartheta(\alpha(k),[h, l])-\vartheta(k, \alpha([h, l])) \\
& =\beta(\vartheta(k,[h, l]))-\vartheta(\alpha(k),[h, l])-\vartheta(k,[\alpha(h), l])-\vartheta(k,[h, \alpha(l)])
\end{aligned}
$$

If

$$
X=(\alpha, \beta) \cdot \vartheta(k,[h, l])+(\alpha, \beta) \cdot \vartheta(h,[l, k])+(\alpha, \beta) \cdot \vartheta(l,[k, h]),
$$

then

$$
\begin{aligned}
X=\beta(\vartheta(k,[h, l]))+ & \beta(\vartheta(h,[l, k]))+\beta(\vartheta(l,[k, h])) \\
& -\vartheta(\alpha(k),[h, l])-\vartheta(\alpha(h),[l, k])-\vartheta(\alpha(l),[k, h]) \\
& -\vartheta(k,[\alpha(h), l])-\vartheta(h,[\alpha(l), k])-\vartheta(l,[\alpha(k), h]) \\
& -\vartheta(k,[h, \alpha(l)])-\vartheta(h,[l, \alpha(k)])-\vartheta(l,[k, \alpha(h)]) .
\end{aligned}
$$

Using that $\beta$ is linear and the definition of cocycles in (5), we have

$$
\begin{aligned}
X=-\beta([k, \vartheta(h, l)]) & -\beta([h, \vartheta(l, k)])-\beta([l, \vartheta(k, h)]) \\
& +[\alpha(k), \vartheta(h, l)]+[\alpha(h), \vartheta(l, k)]+[\alpha(l), \vartheta(k, h)] \\
& +[k, \vartheta(\alpha(h), l)]+[h, \vartheta(\alpha(l), k)]+[l, \vartheta(\alpha(k), h)] \\
& +[k, \vartheta(h, \alpha(l))]+[h, \vartheta(l, \alpha(k))]+[l, \vartheta(k, \alpha(h))] .
\end{aligned}
$$

Since $(\alpha, \beta)$ is a compatible pair, we have by (10)

$$
\begin{aligned}
& \beta([k, \vartheta(h, l)])=[\alpha(k), \vartheta(h, l)]+[k, \beta(\vartheta(h, l))] ; \\
& \beta([h, \vartheta(l, k)])=[\alpha(h), \vartheta(l, k)]+[h, \beta(\vartheta(l, k))] ; \\
& \beta([l, \vartheta(k, h)])=[\alpha(l), \vartheta(k, h)]+[l, \beta(\vartheta(k, h))] .
\end{aligned}
$$

Hence, we obtain combining the last two displayed systems of equations

$$
\begin{aligned}
X=-[k, \beta(\vartheta(h, l))] & -[h, \beta(\vartheta(l, k))]-[l, \beta(\vartheta(k, h))] \\
& +[k, \vartheta(\alpha(h), l)]+[h, \vartheta(\alpha(l), k)]+[l, \vartheta(\alpha(k), h)] \\
& +[k, \vartheta(h, \alpha(l))]+[h, \vartheta(l, \alpha(k))]+[l, \vartheta(k, \alpha(h))] .
\end{aligned}
$$

Again, by the definition of the action in (14)

$$
X=-[k,(\alpha, \beta) \cdot \vartheta(h, l)]-[h,(\alpha, \beta) \cdot \vartheta(l, k)]-[l,(\alpha, \beta) \cdot \vartheta(k, h)] .
$$

So $(\alpha, \beta) \cdot \vartheta \in \mathbf{Z}^{2}(K, I)$.
Now suppose that $\vartheta \in \mathrm{B}^{2}(K, I)$. By definition (6), there is a linear map $T: K \rightarrow I$ such that $\vartheta=\vartheta_{T}$. Hence,

$$
\begin{equation*}
\vartheta_{T}(h, k)=T([h, k])+[k, T(h)]-[h, T(k)] . \tag{16}
\end{equation*}
$$

Let $Y=(\alpha, \beta) \cdot \vartheta_{T}(h, k)$. By (16), we have

$$
\begin{equation*}
Y=\beta\left(\vartheta_{T}(h, k)\right)-\vartheta_{T}(\alpha(h), k)-\vartheta_{T}(h, \alpha(k)) . \tag{17}
\end{equation*}
$$

Using the definition of $\vartheta_{T}$, we have

$$
\begin{align*}
\beta\left(\vartheta_{T}(h, k)\right) & =\beta T([h, k])+\beta[k, T(h)]-\beta[h, T(k)],  \tag{18}\\
\vartheta_{T}(\alpha(h), k) & =T([\alpha(h), k])+[k, T \alpha(h)]-[\alpha(h), T(k)], \\
\vartheta_{T}(h, \alpha(k)) & =T([h, \alpha(k)])+[\alpha(k), T(h)]-[h, T \alpha(k)] .
\end{align*}
$$

We can use that $(\alpha, \beta)$ is a compatible pair in equation (18) to write

$$
\beta\left(\vartheta_{T}(h, k)\right)=\beta T([h, k])+[\alpha(k), T(h)]+[k, \beta T(h)]-[\alpha(h), T(k)]-[h, \beta T(k)] .
$$

Then

$$
\begin{aligned}
Y=\beta T([h, k])+[\alpha(k), & T(h)]+[k, \beta T(h)]-[\alpha(h), T(k)]-[h, \beta T(k)] \\
& -T([\alpha(h), k])-[k, T \alpha(h)]+[\alpha(h), T(k)] \\
& -T([h, \alpha(k)])-[\alpha(k), T(h)]+[h, T \alpha(k)] .
\end{aligned}
$$

Making the cancellations, $Y$ can be written as

$$
\begin{aligned}
Y=\beta T([h, k])-T([\alpha(h), k])-T( & {[h, \alpha(k)]) } \\
& +[k, \beta T(h)]-[k, T \alpha(h)]+[h, T \alpha(k)]-[h, \beta T(k)] .
\end{aligned}
$$

Now we use that $T$ and the action are linear to obtain

$$
Y=\beta T([h, k])-T([\alpha(h), k]+[h, \alpha(k)])+[k, \beta T(h)-T \alpha(h)]-[h, \beta T(k)-T \alpha(k)] .
$$

Hence,

$$
Y=(\beta T-T \alpha)([h, k])+[k,(\beta T-T \alpha)(h)]-[h,(\beta T-T \alpha)(k)] .
$$

If $U=\beta T-T \alpha: K \rightarrow I$, then

$$
(\alpha, \beta) \cdot \vartheta_{T}(h, k)=U([h, k])+[k, U(h)]-[h, U(k)] .
$$

Therefore, $(\alpha, \beta) \cdot \vartheta_{T} \in \mathrm{~B}^{2}(K, I)$. We just proved
Proposition 3.2.1. Let $K$ be a Lie algebra and let I be a $K$-module. Consider the action of $\operatorname{Comp}(K, I)$ on $C^{2}(K, I)$ defined in (14). Then the vector spaces $Z^{2}(K, I)$ and $B^{2}(K, I)$ are invariants under this action.

This result allows us to define an action of $\operatorname{Comp}(K, I)$ on $\mathrm{H}^{2}(K, I)$ : let $\vartheta \in \mathrm{Z}^{2}(K, I)$ and $(\alpha, \beta) \in \operatorname{Comp}(K, I)$. Define the action

$$
\begin{equation*}
(\alpha, \beta) \cdot\left(\vartheta+\mathrm{B}^{2}(K, I)\right)=((\alpha, \beta) \cdot \vartheta)+\mathrm{B}^{2}(K, I) . \tag{19}
\end{equation*}
$$

This is well defined by Proposition 3.2.1.
Definition 3.2.2. Let $K$ be a Lie algebra and let $I$ be a $K$-module. Let $\vartheta \in Z^{2}(K, I)$ and consider the action of $\operatorname{Comp}(K, I)$ on $\mathrm{H}^{2}(K, I)$ defined in (19). Define the set of induced pairs of $\operatorname{Comp}(K, I)$ by

$$
\operatorname{Indu}(K, I, \vartheta)=\operatorname{Ann}_{\operatorname{Comp}(K, I)}\left(\vartheta+\mathrm{B}^{2}(K, I)\right)
$$

Now we have the tools needed to describe the Lie algebra $\operatorname{Der}\left(K_{\vartheta}\right)$ from the Lie algebra $\operatorname{Der}(K)$. We will define a homomorphism $\phi: \operatorname{Der}\left(K_{\vartheta}\right) \rightarrow \operatorname{Der}(K)$, whose kernel is known and the image coincides with the set of induced pairs defined above. So, using the First Isomorphism Theorem for Lie algebras we have $\operatorname{Der}\left(K_{\vartheta}\right)$ is isomorphic to $\operatorname{Ker}(\phi)+\operatorname{Im}(\phi)$. These subspaces correspond to the structures: $\operatorname{Ker}(\phi) \cong Z^{1}(K, I)$ and $\operatorname{Im}(\phi) \cong \operatorname{Indu}(K, I, \vartheta)$. Therefore, this method will allow us to study some properties of derivations of Lie algebra extensions by cocycles. First we define $\phi$.

Let $K$ be a Lie algebra and let $I$ be a $K$-module. Let $\vartheta \in \mathrm{H}^{2}(K, I)$ and $\delta \in \operatorname{Der}\left(K_{\vartheta}\right)$. Suppose that $I$, as an ideal of $K_{\vartheta}$, is invariant under $\delta$. Recall that $K_{\vartheta}=K \oplus I$ and let $\pi_{K}$ : $K_{\vartheta} \rightarrow K$ and $\pi_{I}: K_{\vartheta} \rightarrow I$ be the natural vector space projections of $K_{\vartheta}$ onto $K$ and $K_{\vartheta}$ onto $I$ respectively. Then define the maps

- $\alpha: K \rightarrow K$ by $\alpha(k)=\pi_{K} \delta(k)$, for all $k \in K$;
- $\beta: I \rightarrow I$ by $\beta(v)=\delta(v)$, for all $v \in I$;
- $\eta: K \rightarrow I$ by $\eta(k)=\pi_{I} \delta(k)$, for all $k \in K$.

For each $k+v \in K_{\vartheta}$, we have

$$
\begin{equation*}
\delta(k+v)=\alpha(k)+\eta(k)+\beta(v) \text { for all } k \in K \text { and } v \in I . \tag{20}
\end{equation*}
$$

We can see that $\beta$ is a derivation of $I$ because it is the restriction of $\delta$ to $I$. To see that $\alpha \in \operatorname{Der}(K)$, let $x, y \in K$. To make our calculation more clear, we will denote by $[\cdot, \cdot]_{K}$ the product in $K$, and by $[,, \cdot]_{\vartheta}$ the product in $K_{\vartheta}$. Then by product definition on $K_{\vartheta}$

$$
\delta\left([h, k]_{\vartheta}\right)=\delta\left([h, k]_{K}+\vartheta(h, k)\right) .
$$

By the decomposition showed in (20),

$$
\begin{equation*}
\delta\left([h, k]_{\vartheta}\right)=\alpha\left([h, k]_{K}\right)+\eta\left([h, k]_{K}\right)+\beta(\vartheta(h, k)) . \tag{21}
\end{equation*}
$$

We can calculate

$$
\begin{equation*}
[\delta(h), k]_{\vartheta}+[h, \delta(k)]_{\vartheta}=[\alpha(h)+\eta(h), k]_{\vartheta}+[h, \alpha(k)+\eta(k)]_{\vartheta}, \tag{22}
\end{equation*}
$$

and use the definition of the product in equation (22) to get

$$
\begin{align*}
{[\delta(h), k]_{\vartheta}+[h, \delta(k)]_{\vartheta}=[\alpha(h), k]_{K}+\vartheta(\alpha(h), k) } & -[k, \eta(h)]_{\vartheta} \\
& +[h, \alpha(k)]_{K}+\vartheta(h, \alpha(k))+[h, \eta(k)]_{\vartheta} \tag{23}
\end{align*}
$$

Comparing the components of $K$ in (21) and (23) we have

$$
\alpha\left([h, k]_{K}\right)=[\alpha(h), k]_{K}+[h, \alpha(k)]_{K},
$$

and $\alpha \in \operatorname{Der}(K)$.
Now it is possible define our homomorphism $\phi$. Let $K$ be a Lie algebra and let $I$ be a $K$-module. Let $\vartheta \in \mathrm{H}^{2}(K, I)$ and suppose that $I$, as an ideal of $K_{\vartheta}$, is invariant under derivations. For all $k+v \in K_{\vartheta}$ and $\delta \in \operatorname{Der}\left(K_{\vartheta}\right)$ write $\delta(k+v)=\alpha(k)+\eta(k)+\beta(v)$ with $\alpha \in \operatorname{Der}(K)$ and $\beta \in \operatorname{Der}(I)$. Define $\phi: \operatorname{Der}\left(K_{\vartheta}\right) \rightarrow \operatorname{Der}(K) \oplus \operatorname{Der}(I)$ by

$$
\begin{equation*}
\phi(\delta)=(\alpha, \beta) . \tag{24}
\end{equation*}
$$

The following calculation will check that $\phi$ is a Lie algebra homomorphism. Let $\delta, \delta^{\prime} \in$ $\operatorname{Der}\left(K_{\vartheta}\right)$ such that

$$
\begin{aligned}
\delta(k+v) & =\alpha(k)+\eta(k)+\beta(v) \\
\delta^{\prime}(k+v) & =\alpha^{\prime}(k)+\eta^{\prime}(k)+\beta^{\prime}(v)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\delta \delta^{\prime}(k) & =\delta\left(\alpha^{\prime}(k)+\eta^{\prime}(k)+\beta^{\prime}(v)\right) \\
& =\alpha \alpha^{\prime}(k)+\eta\left(\alpha^{\prime}(k)\right)+\beta\left(\eta^{\prime}(k)+\beta^{\prime}(v)\right) .
\end{aligned}
$$

Hence, $\pi_{K} \delta \delta^{\prime}(k)=\alpha \alpha^{\prime}(k)$. Analogously, $\pi_{K} \delta^{\prime} \delta(k)=\alpha^{\prime} \alpha(k)$. So $\pi_{K}\left[\delta, \delta^{\prime}\right]=\left[\alpha, \alpha^{\prime}\right]$. As $\beta$ and $\beta^{\prime}$ are defined by restriction of $\delta$ and $\delta^{\prime}$ to $I$, respectively, $\pi_{I}\left[\delta, \delta^{\prime}\right]=\left[\beta, \beta^{\prime}\right]$. Therefore,

$$
\phi\left(\left[\delta, \delta^{\prime}\right]\right)=\left(\left[\alpha, \alpha^{\prime}\right],\left[\beta, \beta^{\prime}\right]\right)=\left[(\alpha, \beta),\left(\alpha^{\prime}, \beta^{\prime}\right)\right]=\left[\phi(\delta), \phi\left(\delta^{\prime}\right)\right],
$$

and $\phi$ is indeed a Lie algebra homomorphism.
The next result presents the first connection between compatible pairs and the homomorphism $\phi$.

Lemma 3.2.3. Let $K$ be a Lie algebra and let I be a $K$-module. Let $\vartheta \in H^{2}(K, I)$ and suppose that $I$, as an ideal of $K_{\vartheta}$, is invariant under derivations. Let $\phi: \operatorname{Der}\left(K_{\vartheta}\right) \rightarrow \operatorname{Der}(K) \oplus \operatorname{Der}(I)$ given by $\phi(\delta)=(\alpha, \beta)$, defined in (24). Then $\operatorname{Im}(\phi) \leqslant \operatorname{Comp}(K, I)$.

Proof. Let $(\alpha, \beta) \in \operatorname{Im}(\phi)$. Then, there is $\delta \in \operatorname{Der}\left(K_{\vartheta}\right)$ such that $\phi(\delta)=(\alpha, \beta)$. If $h \in K$ and
$a \in I$, then

$$
\begin{aligned}
\beta\left([h, a]_{\vartheta}\right) & =\delta\left([h, a]_{\vartheta}\right) & & (\text { since }[h, a] \in I) \\
& =[\delta(h), a]_{\vartheta}+[h, \delta(a)]_{\vartheta} & & \left(\delta \in \operatorname{Der}\left(K_{\vartheta}\right)\right) \\
& =[\alpha(h)+\eta(h), a]_{\vartheta}+[h, \beta(a)]_{\vartheta} & & \\
& =[\alpha(h), a]_{\vartheta}+[h, \beta(a)]_{\vartheta} & & \text { (since } I \text { is abelian). }
\end{aligned}
$$

We set for further reference

$$
\begin{align*}
\phi: \operatorname{Der}\left(K_{\vartheta}\right) & \rightarrow \quad \operatorname{Comp}(K, I)  \tag{25}\\
\delta & \mapsto \quad(\alpha, \beta) .
\end{align*}
$$

### 3.3 Derivations of $K_{\vartheta}$

Now we present the main theorem of this chapter. We describe the derivations of an extension $K_{\vartheta}$ presented in Proposition 2.2 .1 from the derivations of the Lie algebra $K$. As we will see, this theorem sets conditions which guarantee that a derivation of $K$ can be lifted to a derivation of $K_{\vartheta}$. Recall that for a Lie algebra $K$, for a $K$-module $I$, and for $\vartheta \in \mathrm{Z}^{2}(K, I)$, $\operatorname{Indu}(K, I, \vartheta)$ was defined in Definition 3.2.2.

Theorem 3.3.1. Let $K$ be a Lie algebra and let I be a K-module. Let $\vartheta \in H^{2}(K, I)$ and suppose that $I$, as ideal of $K_{\vartheta}$, is invariant under derivations. Let $\phi: \operatorname{Der}\left(K_{\vartheta}\right) \rightarrow \operatorname{Der}(K) \oplus \operatorname{Der}(I)$ be defined as above. Then:

1. $\operatorname{Im}(\phi)=\operatorname{Indu}(K, I, \vartheta)$
2. $\operatorname{Ker}(\phi) \cong Z^{1}(K, I)$

Proof. In this proof we will denote the product in $K_{\vartheta}$ of $h \in K$ and $a \in I$ just by the action $[h, a]$ of $K$ on $I$, since $[h, a]_{\vartheta}=[h, a]$.

1) Let $(\alpha, \beta) \in \operatorname{Indu}(K, I, \vartheta)$. By definition,

$$
(\alpha, \beta) \cdot \vartheta \in \mathrm{B}^{2}(K, I) .
$$

Then there is a linear map $T: K \rightarrow I$, such that, for all $h, k \in K$,

$$
\begin{equation*}
\beta(\vartheta(h, k))-\vartheta(\alpha(h), k)-\vartheta(h, \alpha(k))=T\left([h, k]_{K}\right)+[k, T(h)]-[h, T(k)] . \tag{26}
\end{equation*}
$$

Let $h \in K, a \in I$ and define the linear map $(\alpha, \beta)^{*}: K_{\vartheta} \rightarrow K_{\vartheta}$ by

$$
\begin{equation*}
(\alpha, \beta)^{*}(h+a)=\alpha(h)-T(h)+\beta(a) . \tag{27}
\end{equation*}
$$

Let us check that $(\alpha, \beta)^{*}$ is a derivation of $K_{\vartheta}$. Let $k+b \in K_{\vartheta}$. If

$$
X=(\alpha, \beta)^{*}\left([h+a, k+b]_{\vartheta}\right)
$$

then

$$
\begin{aligned}
X & =(\alpha, \beta)^{*}\left([h, k]_{K}+\vartheta(h, k)+[h, b]-[k, a]\right) \\
& =\alpha\left([h, k]_{K}\right)-T\left([h, k]_{K}\right)+\beta(\vartheta(h, k))+\beta([h, b])-\beta([k, a]) .
\end{aligned}
$$

Now, let

$$
Y=\left[(\alpha+\beta)^{*}(h+a), k+b\right]_{\vartheta}+\left[h+a,(\alpha+\beta)^{*}(k+b)\right]_{\vartheta} .
$$

By equation (27),

$$
\left[(\alpha+\beta)^{*}(h+a), k+b\right]_{\vartheta}=[\alpha(h)-T(h)+\beta(a), k+b]_{\vartheta} .
$$

Hence, by the definition of the product in (7),

$$
[\alpha(h)-T(h)+\beta(a), k+b]_{\vartheta}=[\alpha(h), k]_{K}+\vartheta(\alpha(h), k)+[\alpha(h), b]-[k,-T(h)+\beta(a)]
$$

and

$$
\left[(\alpha+\beta)^{*}(h+a), k+b\right]_{\vartheta}=[\alpha(h), k]_{K}+\vartheta(\alpha(h), k)+[\alpha(h), b]-[k,-T(h)+\beta(a)] .
$$

Analogously,

$$
\left[h+a,(\alpha+\beta)^{*}(k+b)\right]_{\vartheta}=[h, \alpha(k)]_{K}+\vartheta(h, \alpha(k))+[h,-T(k)+\beta(b)]-[\alpha(k), a] .
$$

It follows that,

$$
\begin{aligned}
& Y=[\alpha(h), k]_{K}+[h, \alpha(k)]_{K}+\vartheta(\alpha(h), k)+\vartheta(h, \alpha(k)) \\
& \quad+[\alpha(h), b]+[h, \beta(b)]-[k, \beta(a)]-[\alpha(k), a]-[h, T(k)]+[k, T(h)] .
\end{aligned}
$$

We can use that $(\alpha, \beta) \in \operatorname{Comp}(K, I)$ to write $Y$ as

$$
\begin{aligned}
Y=\alpha\left([h, k]_{K}\right)+\vartheta(\alpha(h), k)+\vartheta(h, \alpha(k))+\beta([h, b])-\beta([k, a]) & \\
& -[h, T(k)]+[k, T(h)] .
\end{aligned}
$$

By equation (26),

$$
\vartheta(\alpha(h), k)+\vartheta(h, \alpha(k))=\beta(\vartheta(h, k))-T([h, k])-[k, T(h)]+[h, T(k)] .
$$

Then

$$
\begin{aligned}
Y=\alpha\left([h, k]_{K}\right)+\beta(\vartheta(h, k))-T\left([h, k]_{K}\right)- & {[k, T(h)]+[h, T(k)] } \\
& +\beta([h, b])-\beta([k, a])-[h, T(k)]+[k, T(h)] .
\end{aligned}
$$

As $X=Y,(\alpha, \beta)^{*}$ is a derivation.
Also observe that $\pi_{K}(\alpha, \beta)^{*}=\alpha$ and $\pi_{I}(\alpha, \beta)^{*}=\beta$. Hence, $\phi\left((\alpha+\beta)^{*}\right)=(\alpha, \beta)$ and $\operatorname{Indu}(K, I, \vartheta) \subseteq \operatorname{Im}(\phi)$.

Suppose now that $(\alpha, \beta) \in \operatorname{Im}(\phi)$. Then, there is $\delta \in \operatorname{Der}\left(K_{\vartheta}\right)$, such that

$$
\phi(\delta)=(\alpha, \beta)
$$

By Theorem 3.2.3, $\operatorname{Im}(\phi) \subseteq \operatorname{Comp}(K, I)$. Then, it is enough to show that there is a linear map $T: K \rightarrow I$, such that equation (26) is satisfied.

For each $h+a \in K_{\vartheta}$, we can use the decomposition defined in (20) to write

$$
\delta(h+a)=\alpha(h)+\eta(h)+\beta(a) .
$$

Thus,

$$
[\delta(h+a), k+b]_{\vartheta}=[\alpha(h)+\eta(h)+\beta(a), k+b]_{\vartheta} .
$$

By the definition of the product in (7), we get

$$
[\alpha(h)+\eta(h)+\beta(a), k+b]_{\vartheta}=[\alpha(h), k]_{K}+\vartheta(\alpha(h), k)+[\alpha(h), b]-[k, \eta(h)+\beta(a)] .
$$

Hence,

$$
[\delta(h+a), k+b]_{\vartheta}=[\alpha(h), k]_{K}+\vartheta(\alpha(h), k)+[\alpha(h), b]-[k, \eta(h)+\beta(a)] .
$$

Analogously,

$$
[h+a, \delta(k+b)]_{\vartheta}=[h, \alpha(k)]_{K}+\vartheta(h, \alpha(k))+[h, \eta(k)+\beta(b)]-[\alpha(k), a] .
$$

Therefore,

$$
\begin{align*}
{[\delta(h+a), k+b]_{\vartheta} } & +[h+a, \delta(k+b)]_{\vartheta}=[\alpha(h), k]_{K}+[h, \alpha(k)]_{K}+\vartheta(\alpha(h), k)+\vartheta(h, \alpha(k)) \\
& +[\alpha(h), b]+[h, \beta(b)]-[\alpha(k), a]-[k, \beta(a)]-[k, \eta(h)]+[h, \eta(k)] . \tag{28}
\end{align*}
$$

We can use that $(\alpha, \beta) \in \operatorname{Comp}(K, I)$ in the last displayed equation to write

$$
\begin{aligned}
{[\delta(h+a), k+b]_{\vartheta}+[h+a, \delta(k+b)]_{\vartheta}=\alpha } & \left([h, k]_{K}\right)+\vartheta(\alpha(h), k)+\vartheta(h, \alpha(k)) \\
& +\beta([h, b])-\beta([k, a])-[k, \eta(h)]+[h, \eta(k)] .
\end{aligned}
$$

Now we will calculate $\delta\left([k+a, h+b]_{\vartheta}\right)$. By the definition of the product,

$$
\delta\left([h+a, k+b]_{\vartheta}\right)=\delta\left([h, k]_{K}+\vartheta(h, k)+[h, b]-[k, a]\right) .
$$

Hence,
$\delta\left([h, k]_{K}+\vartheta(h, k)+[h, b]-[k, a]\right)=\alpha\left([h, k]_{K}\right)+\eta\left([h, k]_{K}\right)+\beta(\vartheta(h, k))+\beta([h, b])-\beta([k, a])$.
As $\delta$ is a derivation, we have the equality

$$
\delta\left([h+a, k+b]_{\vartheta}\right)=[\delta(h+a), k+b]_{\vartheta}+[h+a, \delta(k+b)]_{\vartheta} .
$$

It follows that,

$$
\vartheta(\alpha(h), k)+\vartheta(h, \alpha(k))-[k, \eta(h)]+[h, \eta(k)]=\eta\left([h, k]_{K}\right)+\beta(\vartheta(h, k)) .
$$

We can rearrange the last displayed equation to get

$$
\left.(-\eta)\left([h, k]_{K}\right)+[k,(-\eta)(h)]-[h,(-\eta)(k)]\right)=\beta(\vartheta(h, k))-\vartheta(\alpha(h), k)-\vartheta(h, \alpha(k)) .
$$

Therefore, $T=-\eta$ satisfies the equation (26) e $\operatorname{Im}(\phi) \subseteq \operatorname{Indu}(K, I, \vartheta)$. This concludes the proof of assertion 1 .
2) Let $\delta \in \operatorname{Ker}(\phi)$. The decomposition showed in (20) provides us

$$
\delta(h)=\eta(h), \quad \text { for all } h \in K
$$

Let $h, k \in K$. By the definition of a derivation,

$$
\begin{equation*}
\delta\left([h, k]_{\vartheta}\right)=[\delta(h), k]_{\vartheta}+[h, \delta(k)]_{\vartheta} . \tag{29}
\end{equation*}
$$

We can use the definition of the product in $K_{\vartheta}$ to write

$$
\delta\left([h, k]_{\vartheta}\right)=\delta\left([h, k]_{K}+\vartheta(h, k)\right) .
$$

Since $\delta \in \operatorname{Ker}(\phi)$,

$$
\delta\left([h, k]_{\vartheta}\right)=\eta\left([h, k]_{K}\right) .
$$

On other hand,

$$
[\delta(h), k]_{\vartheta}+[h, \delta(k)]_{\vartheta}=[\eta(h), k]_{\vartheta}+[h, \eta(k)]_{\vartheta} .
$$

Then, (29) can be written as

$$
\eta\left([k, h]_{K}\right)=[k, \eta(h)]-[h, \eta(k)],
$$

and $\eta \in \mathbf{Z}^{1}(K, I)$. Observe that $\eta$ is the restriction of $\delta$ to $K$. Denote the restriction of $\delta$ to $K$ by $\left.\delta\right|_{K}$. Therefore, if $\delta \in \operatorname{Ker}(\phi)$, then $\left.\delta\right|_{K} \in Z^{1}(K, I)$.

Let $\delta \in \operatorname{Ker}(\phi)$ and define $\sigma: \operatorname{Ker}(\phi) \rightarrow\left(\mathrm{Z}^{1}(K, I),+\right)$ by $\sigma(\delta)=\left.\delta\right|_{K}$. The argument above shows that $\sigma$ is well defined, in the sense that $\sigma(\delta) \in \mathbf{Z}^{1}(K, I)$. The map $\sigma$ is clearly linear. Further, $\sigma$ is injective, since if $\delta \in \operatorname{Ker}(\sigma)$, then $\delta=0$. Now, to prove that $\sigma$ is onto, let $\eta \in \mathrm{Z}^{1}(K, I)$ and define a linear map $\delta: K_{\vartheta} \rightarrow K_{\vartheta}$ by

$$
\delta(h+a)=\eta(h), h \in K, a \in I .
$$

We will show that $\delta$ is a derivation. Observe that, for all $h+a, k+b \in K_{\vartheta}$, we have

$$
\delta\left([h+a, k+b]_{\vartheta}\right)=\delta\left([h, k]_{K}+\vartheta(h, k)+[h, b]-[k, a]\right)=\eta\left([h, k]_{K}\right) .
$$

On the other hand,

$$
\begin{gathered}
{[\delta(h+a), k+b]_{\vartheta}+[h+a, \delta(k+b)]_{\vartheta}=[\eta(h), k+b]_{\vartheta}+[h+a, \eta(k)]_{\vartheta}} \\
=-[k, \eta(h)]+[h, \eta(k)] .
\end{gathered}
$$

Since $\eta \in \mathrm{Z}^{1}(K, I), \delta\left([h+a, k+b]_{\vartheta}\right)=[\delta(h+a), k+b]_{\vartheta}+[h+a, \delta(k+b)]_{\vartheta}$, hence $\delta \in \operatorname{Der}\left(K_{\vartheta}\right)$. It is immediate that $\phi(\delta)=0$. So $\delta \in \operatorname{Ker}(\phi)$. As, by definition, $\sigma(\delta)=\eta, \sigma$ is onto and, therefore, it is an isomorphism.

Example 3.3.2. Let $L$ be a Lie algebra and let $I$ be an abelian ideal of $L$. Suppose that $I$ is invariant under derivations. Set $K=L / I$. By Proposition 2.2.1, there is a $\vartheta \in Z^{2}(K, I)$ such that
$L \cong K_{\vartheta}$. Then, we can apply the map $\phi: \operatorname{Der}(L) \rightarrow \operatorname{Der}(L / I) \oplus \operatorname{Der}(I)$ defined in Theorem 3.3.1. Further, if $\delta \in \operatorname{Der}(L)$, then $\phi(\delta)=(\alpha, \beta) \in \operatorname{Comp}(L / I, I)$. Hence, each derivation of $L$ gives rise to a pair of derivations $\alpha \in \operatorname{Der}(L / I)$ and $\beta \in I$. In particular, if $\delta$ is non-singular, then $\alpha$ and $\beta$ are non-singulars.

## 4 Applications of Compatible pairs

### 4.1 Compatible pairs and Jacobson's Theorem

In this chapter we present some examples of the use of compatible pairs in the study of non-singular derivations.

Example 4.1.1. Let $K$ and $I$ be finite-dimensional Lie algebras over an algebraically closed field $\mathbb{F}$. Suppose that $K$ acts on $I$ via the representation $\psi: K \rightarrow \operatorname{Der}(I)$. Let $D \leqslant \operatorname{Comp}(K, I)$ be a subalgebra. Define $L=K \oplus I$. By Proposition 3.1.1, $D \leqslant \operatorname{Der}(L)$. Suppose that $D$ is nilpotent. By Corollary 2.1.8, $L$ has a decomposition into generalized eigenspaces of $D$. This decomposition induces decompositions on $K$ and on $I$, since $K$ and $I$ are invariant under $D$. Hence, $L=K_{\lambda_{1}} \oplus \cdots \oplus K_{\lambda_{r}} \oplus I_{\mu_{1}} \cdots \oplus I_{\mu_{s}}$. In particular, $\left[K_{\lambda_{i}}, I_{\mu_{j}}\right] \subseteq I_{\lambda_{i}+\mu_{j}}$ if $\lambda_{i}+\mu_{j}$ is an eigenvalue of $D$ in $I$. Otherwise $\left[K_{\lambda_{i}}, I_{\mu_{j}}\right]=0$.

From this example we can state the following result:
Proposition 4.1.2. Let $K$ and I be finite-dimensional Lie algebras over an algebraically closed field $\mathbb{F}$. Suppose that $K$ acts on I by representation $\psi: K \rightarrow \operatorname{Der}(I)$. Let $D \leqslant \operatorname{Comp}(K, I)$ be a nilpotent subalgebra. Suppose that 0 is not a generalized eigenvalue of $D$. Then if either the characteristic of $\mathbb{F}$ is zero or the characteristic of $\mathbb{F}$ is $p$ and $D$ has at most $p-1$ generalized eigenvalues, then the Lie subalgebra $\psi(K) \leqslant \mathfrak{g l}(I)$ is nilpotent.

Proof. Let $L=K_{\lambda_{1}} \dot{+} \cdots+K_{\lambda_{r}}+I_{\mu_{1}} \cdots \dot{+} I_{\mu_{s}}$ be the generalized eigenspace decomposition presented in Example 4.1.1. Suppose that 0 is not a generalized eigenvalue of $D$. Let $E_{K}=$ $\left\{\lambda_{1}, \cdots, \lambda_{r}\right\}$ and $E_{I}=\left\{\mu_{1}, \cdots, \mu_{s}\right\}$ be the generalized eigenvalues of $D$ in $K$ and $I$, respectively. Let $k \in K_{\lambda_{j}}, a \in I_{\mu_{i}}$ then

$$
\left\{\begin{array}{ccc}
\psi^{n}(k)(a) \in I_{\mu_{i}+n \lambda_{j}} & \text { if } & \mu_{i}+n \lambda_{j} \in E_{I} \\
\psi^{n}(k)(a)=0 & \text { if } & \mu_{i}+n \lambda_{j} \notin E_{I} .
\end{array}\right.
$$

- If the characteristic of $\mathbb{F}$ is 0 , then the linear functions $\mu_{i}+\lambda_{j}, \mu_{i}+2 \lambda_{j}, \ldots, \mu_{i}+n \lambda_{j} \ldots$ are all distinct since $\lambda_{j} \neq 0$. Since $\operatorname{dim} I$ is finite, $\mu_{i}+n \lambda_{j} \notin E_{I}$ for some $n>0$. Hence, $\psi(k)^{n}=0$.
- If the characteristic of $\mathbb{F}$ is $p>0$ and $s<p$, then the linear forms $\left\{\mu_{i}+\lambda_{j}, \mu_{i}+\right.$ $\left.2 \lambda_{j}, \cdots, \mu_{i}+(p-1) \lambda_{j}, \mu_{i}\right\}$ cannot be all non-trivial, and $\mu_{i}+n \lambda_{j}=0$ for some $1 \leqslant n \leqslant p$. Thus, $\psi^{n}(k)=0$, for some $n$ with $1 \leqslant n \leqslant p$.

In both cases, $\psi(k)$ is nilpotent for all $k \in K_{\lambda_{j}}, 1 \leqslant j \leqslant r$. Let $S=\bigcup \psi\left(K_{\lambda_{j}}\right)$. Since $S$ is a weakly closed set such that each element is nilpotent, the associative subalgebra $\langle S\rangle_{A} \leqslant \operatorname{End}(I)$ is nilpotent, by Proposition 2.4.1. Therefore, the Lie algebra $\langle S\rangle_{L}=\psi(K) \leqslant \mathfrak{g l}(I)$ is nilpotent.

For our next example we need a result about traces of matrices. Let $n>0$ be a integer number and let $\mathbb{F}$ be a field, and denote by $\mathrm{M}(n, \mathbb{F})$ the set of $n \times n$ matrices over $\mathbb{F}$. The statement of the following proposition is well-known in characteristic 0 , see for instance [ [15], Theorem 24.2.1, pg 110].

Proposition 4.1.3. Let $\mathbb{F}$ be a field of characteristic $p \geqslant 0$ and suppose that $A \in M(n, \mathbb{F})$. Assume that either $n<p$ or $p=0$. Then $A$ is nilpotent if, and only if, the trace of the matrices $A^{r}$ is zero, for all $1 \leqslant r \leqslant n$.

Proof. Suppose without loss of generality that $\mathbb{F}$ is algebraically closed. Then a square matrix $A$ over $\mathbb{F}$ is nilpotent if and only if 0 is the only eigenvalue of $A$. Assume, without loss of generality, that $A$ is in Jordan normal form. Hence, $A$ is a block-diagonal matrix where each block is formed by grouping the Jordan blocks associated to the same eigenvalue. Let $\lambda_{1}, \cdots, \lambda_{k}$ be the non-zero eigenvalues of $A$. Denote by $A_{t}$ the diagonal block in $A$ associated with eigenvalue $\lambda_{t}$, and assume that $A_{t}$ is an $n_{t} \times n_{t}$-matrix. Then

$$
\begin{equation*}
\operatorname{tr}\left(A^{r}\right)=n_{1} \lambda_{1}^{r}+\cdots+n_{k} \lambda_{k}^{r} . \tag{30}
\end{equation*}
$$

Suppose that $A$ is nilpotent. Then zero is the only eigenvalue of $A$, and also of $A^{r}$ for all $r \geqslant 1$, and by equation (30) we have $\operatorname{tr}\left(A^{r}\right)=0$ for $1 \leqslant r \leqslant n$. Conversely, suppose that $\operatorname{tr}\left(A^{r}\right)=0$ for $1 \leqslant r \leqslant n$. Since $k \leqslant n$, from equation (30) we can extract the system

$$
\begin{equation*}
n_{1} \lambda_{1}^{r}+\cdots+n_{k} \lambda_{k}^{r}=0, \quad 1 \leqslant r \leqslant k, \tag{31}
\end{equation*}
$$

of linear equations in the variables $n_{1}, \cdots, n_{k}$ over $\mathbb{F}$, considering each $n_{j}$ as the element $n_{j} \cdot 1$ in $\mathbb{F}$, whose matrix of coefficients is

$$
C=\left[\begin{array}{cccc}
\lambda_{1} & \lambda_{2} & \cdots & \lambda_{k} \\
\lambda_{1}^{2} & \lambda_{2}^{2} & \cdots & \lambda_{k}^{2} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{1}^{k} & \lambda_{2}^{k} & \cdots & \lambda_{k}^{k}
\end{array}\right] .
$$

Denote by $m_{i}(\lambda)$ the operation that multiplies row $i$ of a matrix by $\lambda$, and $A^{t}$ the transposed matrix of $A$. So, we can write

$$
C^{t}=m_{1}\left(\lambda_{1}\right) \cdot m_{2}\left(\lambda_{2}\right) \cdots m_{k}\left(\lambda_{k}\right) \cdot V,
$$

where

$$
V=\left[\begin{array}{cccc}
1 & \lambda_{1} & \lambda_{1}^{2} \cdots & \lambda_{1}^{k-1} \\
1 & \lambda_{2} & \lambda_{2}^{2} \cdots & \lambda_{2}^{k-1} \\
\vdots & \vdots & \vdots & \vdots \\
1 & \lambda_{k} & \lambda_{k}^{2} \cdots & \lambda_{k}^{k-1}
\end{array}\right]
$$

is the Vandermonde matrix in the variables $\lambda_{1}, \lambda_{2}, \cdots, \lambda_{k}$, whose determinant is

$$
\operatorname{det} V=\prod_{1 \leqslant i<j \leqslant n}\left(\lambda_{j}-\lambda_{i}\right) ;
$$

see, [16] Fact 5.16.3, pag. 354. As the $\lambda_{i}$ are pairwise distinct, $\operatorname{det} V$ is non-zero. Thus, the determinant of $C$ is $\lambda_{1} \cdot \lambda_{2} \cdots \lambda_{k}$. det $V$. As we assume that $\lambda_{i} \neq 0,1 \leqslant i \leqslant k, C$ is non-singular. It follows that the system (31) has only the trivial solution. Therefore, considered as an element of $\mathbb{F}$, each $n_{j}$ is zero. If $p=0$, then zero is the only eigenvalue of $A$. If $p>0$, then, since we assume that $n<p$, we also have that $n_{j}<p$ for all $j$. Hence, the fact that $n_{j}=0$ in $\mathbb{F}$, implies that $n_{j}=0$ as a natural number. We conclude that zero is the only eigenvalue of $A$, and so $A$ is nilpotent as explained in the beginning of the proof.

Proposition 4.1.4 ( [16], Fact 3.17.13). Let $\mathbb{F}$ be a field of characteristic $p \geqslant 0$. Let $A, B, C \in$ $M(n, \mathbb{F})$ and assume that either $p=0$ or $n<p$. If $[A, B]=C+\lambda B$, for some $\lambda \in \mathbb{F}$ and $[B, C]=0$, then $\left[A, B^{r}\right]=r B^{r-1} C+\lambda r B^{r}$ for all $r \geqslant 1$. In particular, if $\lambda \neq 0$ and $C$ is nilpotent, then $B$ is nilpotent.

Proof. We prove the first statement of this result by induction on $r$. The case $r=1$ follows from the conditions. Suppose that result is valid for $(r-1)$. Then,

$$
\left[A, B^{r-1}\right]=(r-1) B^{r-2} C+\lambda(r-1) B^{r-1}
$$

We can rewrite this equation as

$$
\lambda(r-1) B^{r-1}=A B^{r-1}-B^{r-1} A-(r-1) B^{r-2} C .
$$

Multiplying the last equation on the right by $B$, we have

$$
\lambda(r-1) B^{r}=A B^{r}-B^{r-1}(A B)-(r-1) B^{r-2}(C B) .
$$

By the conditions, we can write $A B=B A+C+\lambda B$ and $C B=B C$. Replacing these terms above we obtain

$$
\lambda(r-1) B^{r}=A B^{r}-B^{r} A-B^{r-1} C-\lambda B^{r}-(r-1) B^{r-1} C .
$$

Therefore,

$$
A B^{r}-B^{r} A=\lambda r B^{r}+r B^{r-1} C
$$

This proves the first assertion. For the second statement, suppose $\lambda \neq 0$ and $C$ is nilpotent with nilpotency index $m$. Using the first assertion, we have

$$
B^{r}=(1 / \lambda r)\left[A, B^{r}\right]-(1 / \lambda) B^{r-1} C, \text { for all } r \geqslant 1 .
$$

Since, $B$ and $C$ commute, $\left(B^{r-1} C\right)^{m}=\left(B^{r-1}\right)^{m}(C)^{m}=0$, Hence, for all $r \geqslant 1 B^{r-1} C$ is nilpotent and, by Proposition 4.1.3, has trace zero. As the trace of commutators is always zero, $\operatorname{tr}\left(\left[A, B^{r}\right]\right)=0$ for all $r \geqslant 1$. It follows that $\operatorname{tr}\left(B^{r}\right)=0$ for all $r \geqslant 1$ and again, by Proposition 4.1.3, we conclude that $B$ is nilpotent.

Lemma 4.1.5. Let $K$ and I be finite-dimensional Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$ such that $K$ acts on $I$.

1. Let $(\alpha, \beta) \in \operatorname{Comp}(K, I)$ be a compatible pair of non-singular derivations such that both $\alpha$ and $\beta$ have finite order and suppose that $|\alpha|=n_{\alpha} p^{t_{\alpha}}$ and $|\beta|=n_{\beta} p^{t_{\beta}}$ such that $n_{\alpha}$ and $n_{\beta}$ are coprime to $p$. Let $t=\max \left\{t_{\alpha}, t_{\beta}\right\}$. Then $\left(\alpha^{p^{t}}, \beta^{p^{t}}\right)$ is a compatible pair of non-singular diagonalizable derivations such that the orders of $\alpha^{\prime}=\alpha^{p^{t}}$ and $\beta^{\prime}=\beta^{p^{t}}$ are coprimes to p.
2. If $L=K \oplus I$ and $\delta$ is a non-singular derivation of $L$ with finite order such that $\delta(I)=I$, then there exists a non-singular derivation $\delta^{\prime}$ of $L$ with finite order such that $\delta^{\prime}(I)=I$, $\delta^{\prime}(K)=K$, and the restrictions of $\delta^{\prime}$ to I and to $K$ are diagonalizable and have orders that are coprime to $p$.

Proof.
(1) Let $(\alpha, \beta) \in \operatorname{Comp}(K, I)$ be a a compatible pair of non-singular derivations as in the lemma. Let $\alpha^{\prime}=\alpha^{p^{t}}$ and $\beta^{\prime}=\beta^{p^{t}}$. By Lemma 3.1.2, $\left(\alpha^{\prime}, \beta^{\prime}\right)$ is a compatible pair, and
by Lemma 2.5.2, $\alpha^{\prime}$ and $\beta^{\prime}$ are non-singular derivations, such that the orders of $\alpha^{\prime}$ and $\beta^{\prime}$ are coprimes to $p$. Further, as $\mathbb{F}$ is an algebraically closed field, $\alpha^{\prime}$ and $\beta^{\prime}$ are diagonalizable nonsingular derivations of $L$.
(2) Suppose that $\delta$ is as in the lemma, and let $\phi: \operatorname{Der}(L) \rightarrow \operatorname{Der}(K) \oplus \operatorname{Der}(I)$ defined in 24. By Lemma 3.2.3, $\phi(\delta)=(\alpha, \beta) \in \operatorname{Comp}(K, I)$. Let $|\alpha|=n_{\alpha} \phi^{t_{\alpha}}$ and $|\beta|=n_{\beta} p^{t_{\beta}}$, and set $t=\max \left\{t_{\alpha}, t_{\beta}\right\}$. Then $\left(\alpha^{\prime}, \beta^{\prime}\right)=\left(\alpha^{p^{t}}, \beta^{p^{t}}\right)$ is a compatible pair of non-singular derivations such that $n_{\alpha}$ and $n_{\beta}$ are coprime to $p$. By Proposition 3.1.1, the compatible pair $(\alpha, \beta)$ determines a derivation $\delta^{\prime} \in \operatorname{Der}(L)$ defined by $\delta^{\prime}(k+a)=\alpha(k)+\beta(a)$, and $\delta^{\prime}$ is as claimed.

Now we can present a result similar to the Proposition 4.1.2, but with a new proof using compatible pairs.

Theorem 4.1.6. Let $K$ and I be finite-dimensional Lie algebras over an algebraically closed field of characteristic $p>0$. Suppose that $K$ acts on I by the representation $\psi: K \rightarrow \operatorname{Der}(I)$. Let $(\alpha, \beta) \in \operatorname{Comp}(K, I)$ such that $\alpha$ is non-singular, and $\alpha$ has finite order. If $K$ is solvable and $\operatorname{dim} I<p$, then $\psi(k)$ is nilpotent, for all $k \in K$.

Proof. Let $|\alpha|=n_{\alpha} p^{t_{\alpha}}$ be the order of $\alpha$, with $p \nmid n_{\alpha}$. By Lemma 2.5.2, $\alpha^{p^{p_{\alpha}}}$ is non-singular and diagonalizable. By Lemma 3.1.2, ( $\left.\alpha^{p^{t_{\alpha}}}, \beta^{p^{t_{x}}}\right)$ is a compatible pair. Then we may assume without loss of generality that $(\alpha, \beta)$ is a compatible pair with $\alpha$ non-singular and diagonalizable. Let $x_{1}, \ldots, x_{s}$ be a basis of $K$ such that $\alpha\left(x_{i}\right)=\lambda_{i} x_{i}$. Let $B$ be a basis for $I$ and denote by $\llbracket a \rrbracket$ the matrix of the endomorphism $a$, for all $a \in \mathfrak{g l}(I)$, in $B$. Then, by equation (11),

$$
\left[\llbracket \beta \rrbracket, \llbracket \psi\left(x_{i}\right) \rrbracket\right]=\llbracket \psi\left(\alpha\left(x_{i}\right)\right) \rrbracket .
$$

It follows that

$$
\left[\llbracket \beta \rrbracket, \llbracket \psi\left(x_{i}\right) \rrbracket\right]=\lambda_{i} \llbracket \psi\left(x_{i}\right) \rrbracket .
$$

We can apply Proposition 4.1.4 to this last equation with $A=\llbracket \beta \rrbracket, B=\llbracket \psi\left(x_{i}\right) \rrbracket, C=0$ and $\lambda=\lambda_{i} \neq 0$ to conclude that $\llbracket \psi\left(x_{i}\right) \rrbracket$ is nilpotent, for $1 \leqslant i \leqslant s$. Now, we observe that $K$ is a solvable Lie algebra and $\operatorname{dim} I<p$, and so, by Theorem 2.3.5, there is a basis of $I$ such that the image of $\psi$ lies in the subalgebra of $\mathfrak{g l}(I)$ formed by upper triangular matrices. So let us work in this basis. Since $\llbracket \psi\left(x_{i}\right) \rrbracket$ is nilpotent and upper triangular, it must be strictly upper triangular (that is, it contains zeros in the diagonal), for all $i$. Then, all $\llbracket \psi(k) \rrbracket$, for all $k \in K$, are also strictly upper triangular matrices, since they are linear combinations of the $\llbracket \psi\left(x_{i}\right) \rrbracket$. Hence every $\psi(k)$ is nilpotent.

Remark 4.1.7. The proof presented for Theorem 4.1 .6 is still valid if the characteristic of the field $\mathbb{F}$ is zero. Suppose that we have the same conditions as in Theorem 4.1.6, except we do not assume that char $(\mathbb{F})>0$ and that $\operatorname{dim} I<p$. Using the same notation as in the proof, we can again apply Proposition 4.1.4 in zero characteristic to guarantee that $\llbracket \psi\left(x_{i}\right) \rrbracket$ is nilpotent, for $1 \leqslant i \leqslant s$. Since $K$ is solvable, by Theorem 2.3.4, there is a basis of I such that the image of $\psi$ lies in the subalgebra of $\mathfrak{g l}(I)$ formed by upper triangular matrices. Then we proceed as in the proof of Theorem 4.1.6 to conclude that $\psi(k)$, for all $k \in K$, is nilpotent.

As a consequence of Theorem 4.1.6, we can present a version of Jacobson's Theorem, in prime characteristic, for solvable Lie algebras.

Theorem 4.1.8. Let L be a solvable Lie algebra over a field $\mathbb{F}$ of characteristic $p>0$. Let $L>L^{(1)}>\cdots>L^{(k)}>L^{(k+1)}=0$ be the derived series of $L$. Suppose that $L$ has a non-singular derivation of finite order. If the dimension of $L^{(i)} / L^{(i+1)}<p$, for all $i$, then $L$ is nilpotent.

Proof. Since the solvability of $L$ and the dimensions of the quotients $L^{(i)} / L^{(i+1)}$ do not change over extension fields, we may assume that $\mathbb{F}$ is algebraically closed. Suppose that $L>L^{(1)}>$ $\cdots>L^{(k)}>L^{(k+1)}=0$ is the derived series of $L$. We prove this result by induction on $k$. When $k=0$, then $L$ is clearly nilpotent, as it is actually abelian. Suppose that the result holds for Lie algebras of derived length $k$ and assume that $L$ has derived length $k+1$. Then $I=L^{(k)}$ is an abelian ideal of $L$. Setting $K=L / I$, we have that $K$ acts on $I$ by the adjoint representation ad $: K \rightarrow \operatorname{Der}(I)$ such that for all $k \in K$ and $a \in I$ we have $\operatorname{ad}_{k}(a)=[k, a]$. Further, since the terms of the derived series are invariant under derivations, a non-singular derivation $\delta \in \operatorname{Der}(L)$ gives rise to a compatible pair $(\alpha, \beta) \in \operatorname{Comp}(K, I)$ as in the definition of $\phi$ in (25). Since $\delta$ is non-singular, so are $\alpha \in \operatorname{Der}(K)$ and $\beta \in \operatorname{Der}(I)$. Note that $K$ is solvable of solvable length $k$ and $K^{(i)} / K^{(i+1)} \cong L^{(i)} / L^{(i+1)}$ for all $i \leqslant k-1$. Hence the induction hypothesis is valid for $K$ and we obtain that $K$ is nilpotent. Besides that, since $\operatorname{dim} I<p$ we can set ad $=\psi$ in Theorem 4.1.6 to conclude that $\mathrm{ad}_{k}$ is nilpotent for all $k \in K$. Therefore, $L / I$ is nilpotent and $\mathrm{ad}_{x}: I \rightarrow I$ is nilpotent for all $x \in L$. It follows from Proposition 2.3.2 that $L$ is nilpotent.

In Example 2.4.4 we saw a solvable Lie algebra $L$ of derived length 2, defined over a field of positive characteristic which admits a non-singular derivation. The algebra $L$ it is not nilpotent, and indeed $\operatorname{dim} L^{(1)} / L^{(2)}=p$. This example shows that the condition that $\operatorname{dim} L^{(i)} / L^{(i+1)}<p$ for all $i$ cannot be weakened in Theorem 4.1.8.

## 5 Lie algebras with an abelian ideal of codimension 1

A Lie algebra $L$ is said to be metabelian if it is a solvable Lie algebra of derived length 2 , that is, the derived series is $L>L^{\prime}>0$. Then $L$ can be regarded as an extension by cocycles of the Lie algebra $K=L / L^{\prime}$ by the abelian Lie algebra $L^{\prime}$. In this chapter we study non-singular derivations of such Lie algebras. When $\operatorname{dim} K=1$, the vector space $\mathrm{Z}^{2}(K, I)$ has only the trivial cocycle, and the extension $L$ is the semidirect sum $L=K \oplus I$. So using the concept of cyclic modules, we can characterize the Lie algebras with $\operatorname{dim} K=1$, whose center is trivial and admit a non-singular derivation. The decomposition presented in this chapter can be applied to any Lie algebra defined as a semidirect-sum $\langle x\rangle \oplus I$, but if $L$ has a non-singular derivation, then the cyclic modules on which $x$ acts non-singularly have dimension divisible by $p$. This will give us information about the degree of the minimal polynomial of $x$ and about the isomorphism classes of the Lie algebras of the form $\langle x\rangle \oplus \subseteq$ with non-singular derivation.

## 5.1 ( $x, p$ )-cyclic modules

Let $K$ be a Lie algebra and let $I$ be a faithful $K$-module. Hence, $K$ is a subalgebra of $\mathfrak{g l}(I)$, and we can define the semidirect sum $L=K \oplus I$. Thus, every element $x \in K$ may simultaneosly be considered as an endomorphism in $\mathfrak{g l}(I)$ and as an element of $L$. Let $v \in I$. We denote by $x(v)$ the action of $x$ viewed as an endomorphism of $\mathfrak{g l}(I)$ on $v$, and by $[x, v]$ the product of $x$ and $v$ as elements of $L$. Then $[x, v]=x(v)$.

We start with an example that will serve as a model.
Example 5.1.1. ([3], Theorem 2.1) Let $\mathbb{F}$ be a field of prime characteristic $p>0$, such that there is $a, b \in \mathbb{F}$ with $a b^{-1} \notin\{0,1, \ldots, p-1\}$. Let $I$ be a $p$-dimensional vector space over $\mathbb{F}$ with basis $v_{0}, \ldots, v_{p-1}$. Define $x \in \mathfrak{g l}(I)$ by $x\left(v_{i}\right)=v_{i+1}$ for $i=0, \cdots, p-2$ and $x\left(v_{p-1}\right)=v_{0}$. Then the semidirect sum $L=\langle x\rangle \oplus I$ is a non-nilpotent, solvable Lie algebra of derived length 2. Let $\delta: L \rightarrow L$ be defined by $\delta\left(v_{i}\right)=(a+i b) v_{i}$ and $\delta(x)=b x$. This linear transformation is a non-singular derivation, such that the basis vectors $x, v_{0}, \ldots, x_{p-1}$ are eigenvectors of $\delta$ with eigenvalues $b, a, a+b, a+2 b, \ldots, a+(p-1) b$ respectively.

Let $I$ be a vector space and $x \in \mathfrak{g l}(I)$. Next we will present some results on the decomposition of $I$ as subspaces invariant under $x$. This decomposition is similar to the primary
decomposition, but it is specialized to one-dimensional algebras. We start with the definition of cyclic modules.

Definition 5.1.2. Let $I$ be a finite-dimensional vector space over a field $\mathbb{F}$ and $x \in \mathfrak{g l}(I)$. A vector subspace $U$ of $I$ is $x$-cyclic if there is $v \in U$ such that $\left\{v, x(v), x^{2}(v), x^{3}(v), \ldots\right\}$ is a basis for $U$.

Let $I$ be a vector space and let $x \in \mathfrak{g l}(I)$. Set $K=\langle x\rangle \leqslant \mathfrak{g l}(I)$ and consider $I$ as $K$ module with the natural action. Let $v \in I$ and denote by $\langle v\rangle_{K}=\{q(x) v \mid q \in \mathbb{F}[X]\}$. We say that $\langle v\rangle_{K}$ is the $K$-submodule of $I$ generated by $v$. It is easy to see that $\langle v\rangle_{K}$ is generated, as vector space, by $\left\{v, x(v), x^{2}(v), x^{3}(v), \ldots\right\}$. Hence, by Definition 5.1.2, $\langle v\rangle_{K}$ is $x$-cyclic. The vector space $I$ can be decomposed as a direct sum of $K$-modules, such that each module is $x$-cyclic. A complete theorem regarding this subject can be found in Theorem 7.6 of [17]. Below we present the item concerning cyclic decompositions.

Theorem 5.1.3. Let I be a finite-dimensional vector space and $x \in \mathfrak{g l}(I)$. Then I can be decomposed into a direct sum of $x$-cyclic subspaces

$$
I=I_{1} \dot{+} I_{2} \dot{+} \cdots \dot{+} I_{r}
$$

If the vector space $I$ is $x$-cyclic, then it is possible obtain more results about the matrix of the operator $x \in \mathfrak{g l}(I)$ and its minimal polynomial. Next we define companion matrices and characterize the matrix of $x$ on an $x$-cyclic space $I$.

Definition 5.1.4. Let $q$ be the monic polynomial $q(X)=X^{n}-c_{n-1} X^{n-1}-\cdots-c_{2} X^{2}-c_{1} X-c_{0} \in$ $\mathbb{F}[X]$, with $n \geqslant 1$. Then the matrix

$$
\left[\begin{array}{ccccc}
0 & 0 & \ldots & 0 & c_{0} \\
1 & 0 & \ldots & 0 & c_{1} \\
0 & 1 & \ldots & 0 & c_{2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & c_{n-1}
\end{array}\right]
$$

is called the companion matrix of $q$. We denote this matrix by CompMatrix $\left(c_{0}, c_{1}, \cdots, c_{n-1}\right)$.
Lemma 5.1.5. Let I be a vector space and let $x \in \mathfrak{g l}(I)$. The following are equivalent:
2. $\operatorname{dim} \operatorname{Ker}(c I d-x) \leqslant 1$, for all $c \in \mathbb{F}$;
3. the matrix of the operator $x \in \mathfrak{g l}(I)$ is similar to the companion matrix of the minimal polynomial of $x$;
4. the minimal polynomial of $x$ coincides with the characteristic polynomial of $x$;
5. for each eigenvalue of $x$, the Jordan normal form of $x$ has only one block associated to this eigenvalue.

A proof of this lemma can be found in [18, Theorem 1.5.8 and Corollary 1.5.14].
We will use the equivalence presented in Lemma 5.1.5 to introduce a new concept related to cyclic modules. This concept will be the main tool for the characterization presented in this section.

Definition 5.1.6. Let $I$ be a vector space over a field $\mathbb{F}$ of characteristic $p>0$ and let $x \in \mathfrak{g l}(I)$. The vector space $I$ is $(x, p)$-cyclic if the following hold:

1. $I$ is $x$-cyclic;
2. $p$ divides the dimension of $I$;
3. if CompMatrix $\left(c_{0}, c_{1}, \cdots, c_{n-1}\right)$ is the companion matrix of the minimal polynomial of $x$, then $c_{0} \neq 0$ and $c_{i}=0$ for all $i>0$ such that $p \nmid i$.

The following Proposition follows from Definition 5.1.6 and Lemma 5.1.5.

Proposition 5.1.7. Let I be a vector space and $x \in \mathfrak{g l}(I)$. The vector space I is $(x, p)$-cyclic if, and only if, I is $x$-cyclic and the minimal polynomial of $x$ is of the form

$$
\begin{equation*}
q_{x}(X)=X^{p n}-c_{p(n-1)} X^{p(n-1)}-\cdots-c_{2 p} X^{2 p}-c_{p} X^{p}-c_{0} \in \mathbb{F}[X] \tag{32}
\end{equation*}
$$

where $c_{0} \neq 0$ and $n \geqslant 1$.

If $X^{p}=S$ in equation (32), then $q_{x}(S)$ is a polynomial of degree $n$ in $\mathbb{F}[S]$. This means that the polynomial $q_{x}$ can be obtained from a polynomial with non-zero constant term. In general, this is a method for generating $(x, p)$-cyclic vector spaces. Let $q \in \mathbb{F}[X]$ be a polynomial in $\mathbb{F}[X]$ with non-zero constant term of degree $n \geqslant 1$. Let $M$ be the companion matrix of $r(X)=q\left(X^{p}\right)$. Let $I$ be a vector space of dimension $p n$ and let $x$ be the endomorphism
determined by $M$ on $I$. Hence, $I$ is an $(x, p)$-cyclic vector space as we can see it in the next example.

Example 5.1.8. Let $p$ be a prime and let $\mathbb{F}$ be a field of characteristic $p$ of order greater than or equal to $p^{2}$. Let $a \in \mathbb{F} \backslash\{0\}$. Define $q(X)=X^{p^{n}}-a^{p^{n}}$, for $n \geqslant 1$. Let $I$ be a vector space of dimension $p^{n}$. Then the companion matrix of $q$ defines an endomorphism $x \in \mathfrak{g l}(I)$. Let $B=\left\{v_{0}, \ldots, v_{p^{n}-1}\right\}$ be a basis of $I$ such that the action of $x$ on $I$ is given by CompMatrix $\left(a^{p^{n}}, 0, \ldots, 0\right)$. By Definition 5.1.6, $I$ is an $(x, p)$-cyclic module. Let $L$ be the Lie algebra defined by $L=\langle x\rangle \oplus I$. Then $L$ is solvable of derived length 2, non-nilpotent and has trivial center. Suppose that $b \in \mathbb{F} \backslash\{0,1, \ldots, p-1\}$. Then the endomorphism $\delta: L \rightarrow L$ given by $\delta(x)=x$ and $\delta\left(v_{i}\right)=(i+b) v_{i}, 0 \leqslant i \leqslant p^{n}-1$ is a non-singular derivation.

Note that, if $q \in \mathbb{F}[X]$ is the polynomial $q(X)=X^{p}-1$ in Example 5.1.8, then the Lie algebra constructed is the same as in Example 5.1.1. In both examples we took non-zero constant term because we restrict our study to Lie algebras with trivial center.

A Lie algebra $L$ defined by $L=\langle x\rangle \oplus I$, where $I$ is a vector space considered as a Lie algebra under the null multiplication, is completely determined by the action of $x$ on $I$. Further, if $I$ is $x$-cyclic, then the $x$-action on $I$ is determined by the minimal polynomial of $x$. Our next results will link these Lie algebras, defined from the semidirect sum of $(x, p)$-cyclic modules, to the existence of non-singular derivations. First, we need a lemma that gives us information concerning the action of $x$ on the eigenvectors of a non-singular derivation.

Lemma 5.1.9. Let I be a vector space and $x \in \mathfrak{g l}(I)$. Let $K=\langle x\rangle$ be a Lie algebra and set $L=\langle x\rangle \oplus$ I. Assume that $Z(L)=0$. If $\delta \in \operatorname{Der}(L)$ is a derivation and $v \in I$ is an eigenvector of $\delta$, then $[x, v]$ is an eigenvector of $\delta$.

Proof. Let $K=\langle x\rangle$ be a Lie algebra of dimension 1 that acts on $I$. The center of $L$ is the kernel of $x$ viewed as an endomorphism of $I$. Since we assume that $Z(L)=0$, we obtain that $x$ is an invertible endomorphism. This also implies that $L^{\prime}=I$, and so $I$ is invariant under $\operatorname{Der}(L)$, and we can consider $\phi: \operatorname{Der}(L) \rightarrow \operatorname{Der}(K) \oplus \operatorname{Der}(I)$ defined in 25. Suppose that $\phi(\delta)=(\alpha, \beta)$. Then $\alpha(\langle x\rangle)=\langle x\rangle$ and $\beta(u)=\delta(u)$, for all $u \in I$. It follows that, $x$ and $v$ are eigenvectors of $\alpha$ and $\beta$, respectively. Suppose that $x, v$ are associated to eigenvalues $a, b \in \mathbb{F}$, respectively. By the definition of compatible pairs

$$
\delta([x, v])=\beta([x, v])=[\alpha(x), v]+[x, \beta(v)]=(a+b)[x, v] .
$$

Since, $Z(L)=0$, the endomorphism of $I$ induced by $x$ is non-singular, and so $[x, v] \neq 0$. Therefore $[x, v]$ is an eigenvector of $\delta$ associated to eigenvalue $(a+b)$.

Lemma 5.1.10. Let I be a vector space over a field of characteristic $p>0$ and $x \in \mathfrak{g l}(I)$. Let $K=\langle x\rangle$ be a Lie algebra and set $L=K \oplus I$. Also assume that $Z(L)=0$. Let $\delta \in \operatorname{Der}(L)$ be a derivation such that $\delta(x)=x$ and let $E_{a}$ be the $\delta$-eigenspace associated to eigenvalue $a$. Then the vector space $E=E_{a}+E_{a+1}+\cdots+E_{a+p-1}$ is $x$-invariant.

Proof. Let $v \in E_{a+i}$ with $0 \leqslant i \leqslant p-1$. Then

$$
\delta([x, v])=[\delta(x), v]+[x, \delta(v)]=(a+i+1)[x, v] .
$$

Hence, $x\left(E_{a+i}\right) \subseteq E$. As $E_{a_{i}}$ generates $E, E$ is $x$-invariant.
Before stating the next lemma, recall from Section 2.1 that for a polynomial $q \in \mathbb{F}[X]$ and $x$ in $\operatorname{End}(I), I_{0}(q(x))$ is defined as

$$
I_{0}(q(x))=\left\{v \in V \mid \text { there is an } m>0 \text { such that } q(x)^{m} v=0\right\} .
$$

Proposition 5.1.11. Let $K$ be a nilpotent Lie algebra over a field $\mathbb{F}$ of characteristic $p>0$ and let I be a finite-dimensional $K$-module. Let $L=K \oplus I, x \in K$ and $q(X)=X-a$, with $a \in \mathbb{F}$. Suppose that I, as an ideal of $L$, is invariant under $\operatorname{Der}(L)$. Let $\delta \in \operatorname{Der}(L)$ such that $\delta(x)=b x$. Then $I_{0}(q(x))$ is $\delta$-invariant.

Proof. Let $\delta \in \operatorname{Der}(L), x \in K$ given by the hypothesis and let $w \in I_{0}(q(x))$. Hence, there is $m>0$ such that $(x-a \cdot I d)^{p^{m}} \cdot w=0$. As $\operatorname{char}(\mathbb{F})=p$, we have

$$
\begin{equation*}
(x-a \cdot I d)^{p^{m}} \cdot w=\left(x^{p^{m}}-a^{p^{m}} \cdot I d\right) \cdot w=x^{p^{m}} \cdot w-a^{p^{m}} w=0 . \tag{33}
\end{equation*}
$$

As $\delta \in \operatorname{Der}(L)$, using the right-normed convention introduced in equation (2.1),

$$
\begin{align*}
\delta\left(x^{p^{m}} \cdot w\right) & =\delta\left(\left[x_{p^{m}}, w\right]\right) \\
& =[\delta(x), \ldots, x, w]+[x, \delta(x), \ldots, x, w]+[x, \ldots, \delta(x), w]+[x, \ldots, x, \delta(w)] \\
& =[a x, \ldots, x, w]+[x, a x, \ldots, x, w]+[x, \ldots, a x, w]+[x, \ldots, x, \delta(w)] \\
& =p^{m} \cdot a \cdot\left[x_{p^{m}}, w\right]+\left[x_{p^{m}}, \delta(w)\right] \\
& =x^{p^{m}} \cdot \delta(w) . \tag{34}
\end{align*}
$$

Combining (34) and (33) we obtain

$$
0=\delta(0)=\delta\left(x^{p^{m}} \cdot w-a^{p^{m}} w\right)=\delta\left(x^{p^{m}} \cdot w\right)-a^{p^{m}} \delta(w)=x^{p^{m}} \cdot \delta(w)-a^{p^{m}} \delta(w) .
$$

Hence,

$$
(x-a I)^{p^{m}} \cdot \delta(w)=x^{p^{p^{m}}} \cdot \delta(w)-a^{p^{m}} \delta(w)=0
$$

and $\delta(w) \in I_{0}(q(x))$.
Lemma 5.1.12. Let $K=\langle x\rangle$ be a Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$. Let I be a finite-dimensional $K$-module such that $x$ induces an invertible endomorphism of $I$ and set $L=K \oplus I$. Assume that $\delta$ is a non-singular derivation of $L$ such that $\delta(x)=x$ and $\left.\delta\right|_{I}$ is diagonalizable. Suppose that $v \in I$ is an eigenvector of $\delta$. Then the $K$-submodule $\langle v\rangle_{K}$ is $(x, p)$-cyclic.

Proof. We will verify that the vector space $\langle v\rangle_{K}$ satisfies the conditions of Proposition 5.1.7. Define the sequence: $v_{0}=v$ and $v_{i+1}=\left[x, v_{i}\right], i \geqslant 0$. Then the set $\left\{v_{0}, v_{1}, \ldots,\right\}$ generates $\langle v\rangle_{K}$ and $\langle v\rangle_{K}$ is $x$-cyclic. As $I$ has finite dimension, there is a $k>0$ such that $\left\{v_{0}, v_{1}, \ldots, v_{k-1}\right\}$ is linearly independent and $\left\{v_{0}, v_{1}, \ldots, v_{k}\right\}$ is linearly dependent. By Lemma 5.1.9, each $v_{i}$ is an eigenvector of $\delta$. In this case, $v_{i}$ is associated to the eigenvalue $a+i$, where $a$ is the eigenvalue associated with $v$. Note that $a, a+1, \ldots, a+(p-1)$ are distinct eigenvalues and the set $\left\{v_{0}, v_{1}, \ldots, v_{p-1}\right\}$ is linearly independent. Hence, $k \geqslant p$. If the eigenvectors $v_{i}$ and $v_{j}$ are associated with eigenvalues $a+i$ and $a+j$, then $v_{i}$ and $v_{j}$ are associated with the same eigenvalue if, and only if, $i \equiv j(\bmod p)$. Suppose that $k=r p+t, 0 \leqslant t \leqslant p-1$. Since a set of eigenvectors associated to pairwise distinct eigenvalues is linearly independent, the eigenvector $v_{k}$ must be a linear combination of the eigenvectors $v_{i}$, for $i \leqslant k-1$, that have the same eigenvalue as $v_{k}$, which is $a+t$. Hence,

$$
\begin{equation*}
v_{k}=c_{0} v_{t}+c_{1} v_{p+t}+c_{2} v_{2 p+t}+\cdots+c_{r-1} v_{(r-1) p+t} \tag{35}
\end{equation*}
$$

If $t \neq 0$, then we can replace every $v_{i}$ by $\left[x, v_{i-1}\right]$ in equation (35) and obtain that:

$$
\begin{equation*}
\left[x, v_{k-1}\right]=c_{0}\left[x, v_{t-1}\right]+c_{1}\left[x, v_{p+t-1}\right]+c_{2}\left[x, v_{2 p+t-1}\right]+\cdots+c_{r-1}\left[x, v_{(r-1) p+t-1}\right] . \tag{36}
\end{equation*}
$$

If $L=\langle x\rangle \oplus \in$ with some non-trivial endomorphism $x$, then $Z(L)=\operatorname{Ker}(x)$ and $L^{\prime}=\operatorname{Im}(x)$. Since $Z(L)=0$, we have that $x$ induces an injective endomorphism on $I$. Thus,

$$
v_{k-1}=c_{0} v_{t-1}+c_{1} v_{p+t-1}+c_{2} v_{2 p+t-1}+\cdots+c_{r-1} v_{(r-1) p+t-1}
$$

This contradicts to the assumption that $\left\{v_{0}, v_{1}, \ldots, v_{k-1}\right\}$ is linearly independent. Thus, $t=0$ and $k=r p$. Equation (35) implies also that $v_{k}=c_{0} v_{0}+c_{1} v_{p}+\cdots+c_{r-1} v_{(r-1) p}$ and so, the
characteristic polynomial of $x$ is

$$
q_{x}(X)=X^{r p}-c_{(r-1)} X^{p(r-1)}-\cdots-c_{2} X^{2 p}-c_{1} X^{p}-c_{0} .
$$

If $c_{0}=0$, then replacing $v_{i}$ with $\left[x, v_{i-1}\right]$ as above implies that $\left\{v_{0}, v_{1}, \ldots, v_{k-1}\right\}$ is linearly dependent. Therefore $c_{0} \neq 0$. As $\langle v\rangle_{K}$ is $x$-cyclic, by Lemma 5.1.5, the minimal polynomial of $x$ restricted to $\langle v\rangle_{K}$ is $q_{x}$ and, by Proposition 5.1.7, $\langle v\rangle_{K}$ is $(x, p)$-cyclic.

Recall that for an endomorphism $x$ of a vector space $I, q_{x}$ denotes the minimal polynomial of $x$. When we want to emphasize the domain of $x$, we use the notation $q_{x, I}$. If $v \in I$, then $q_{x, v}$ denotes the minimal polynomial of $x$ with respect to $v$. That is, $q_{x, v}$ is the smallest degree, non-zero, monic polynomial such that $q_{x, v}(x)(v)=0$. It is well-known that $q_{x, v} \mid q_{x, I}$ for all $v \in I$. The proof of the following theorem was inspired by the proof of Theorem 6.6 in [19].

Lemma 5.1.13. Let $K=\langle x\rangle$ be a Lie algebra of dimension 1 over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$. Let I be a finite-dimensional $K$-module such that $x$ induces an invertible endomorphism of I and set $L=K \oplus I$. Assume that $\delta$ is a non-singular derivation of $L$ such that $\delta(x)=x$ and $\left.\delta\right|_{I}$ is diagonalizable. Assume, further, that $m_{x, I}(X)=(X-\lambda)^{m}$ with some $\lambda \in \mathbb{F}$ and $m \geqslant 1$ and that the $\delta$-eigenvalues on $I$ are $a, a+1, \ldots, a+p-1$ with some $a \in \mathbb{F}$. Then I is the direct sum of $(x, p)$-cyclic subspaces, each of which is generated by a $\delta$-eigenvector.

Proof. We prove this lemma by induction on $\operatorname{dim} I$. By Lemma 5.1.12, $\operatorname{dim} I \geqslant p$, and so the base case of the induction is when $\operatorname{dim} I=p$. In this case, if $v \in I$ is a $\delta$-eigenvector, then $\langle v\rangle_{K}$ is $(x, p)$-cyclic of dimension greater than or equal to $p$, and hence $I=\langle v\rangle_{K}$. Thus the lemma is valid when $\operatorname{dim} I=p$.

Suppose now that $\operatorname{dim} I \geqslant p+1$ and that the lemma is valid for spaces of dimension less than $\operatorname{dim} I$. By our conditions, $I=E_{a} \dot{+} \cdots \dot{+} E_{a+p-1}$ where $E_{b}$ denotes the $b$-eigenspace of $\delta$ in $I$. Since $\bigcup_{b} E_{b}$ generates $I$ as a vector space, there is some eigenvector $v_{0} \in I$ such that $q_{x, v_{0}}(X)=q_{x, I}(X)=(X-\lambda)^{m}$. Let $I_{0}$ be the $K$-module generated by $v_{0}, I_{0}=\left\langle v_{0}\right\rangle_{K}$, and let $J=I / I_{0}$. Since $v_{0}$ is a $\delta$-eigenvector, $I_{0}$ is $(x, p)$-cyclic by Lemma 5.1.12, and hence $p \mid m$. In particular, $q_{x, v_{0}}(X)=(X-\lambda)^{m}=(X-\lambda)^{m_{0} p}$, where $m_{0} \geqslant 1$. Note that $I_{0}$ is an ideal of $L$ that is invariant under $\delta$. Considering $J$ as a $K$-module, we can consider the Lie algebra $K \oplus J \cong L / I_{0}$ that satisfies the conditions of the lemma. Since $\operatorname{dim} J<\operatorname{dim} I$, the induction hypothesis applies to $J$, and we may write $J=J_{1} \dot{+} \cdots \dot{+} J_{k}$ where the $J_{i}$ are $(x, p)$-cyclic subspaces of $J$ and each
$J_{i}$ is generated by a $\delta$-eigenvector, $w_{i}+I_{0}$, say. Since $I_{0}$ has a basis consisting of $\delta$-eigenvectors, the $\delta$-eigenvalues in $J=I / I_{0}$ are $a, a+1, \ldots, a+p-1$. We claim that $w_{i}$ can be chosen such that $w_{i}$ is a $\delta$-eigenvector in $I$. We may assume that $w_{i}+I_{0}$ is associated to the eigenvalue $a$. As $\delta\left(w_{i}\right)+I_{0}=a w_{i}+I_{0}, \delta\left(w_{i}\right)-a w_{i}=u \in I_{0}$. Since $\delta$ is diagonalizable on $I$, we may write $w_{i}=z_{a}+z_{a+1}+\cdots+z_{a+p-1}$, with $z_{b} \in E_{b}$. Further, since $I_{0}$ is spanned by $\delta$-eigenvectors, we have that $I_{0}=\left(I_{0} \cap E_{a}\right) \dot{+}+\left(I_{0} \cap E_{a+p-1}\right)$. Thus we have $u=u_{a}+u_{a+1}+\cdots+u_{a+p-1}$, with $u_{b} \in E_{b} \cap I_{0}$. Hence,

$$
\begin{aligned}
\delta\left(w_{i}\right)-a w_{i}=a z_{a}+(a+1) z_{a+1}+\cdots+(a+p-1) z_{a+p-1} & -a\left(z_{a}+z_{a+1}+\cdots+z_{a+p-1}\right) \\
& =u_{a}+u_{a+1}+\cdots+u_{a+p-1}
\end{aligned}
$$

Since eigenvectors with different eigenvalues are linearly independent, $u_{a+j}=j \cdot z_{a+j}$ for all $j \geqslant 0$. This implies that $z_{a+j}=j^{-1} u_{a+j} \in I_{0}$ holds for all $j \geqslant 1$. Therefore $w_{i}=z_{a}+u_{a+1}+$ $2^{-1} u_{a+2}+\cdots(p-1)^{-1} u_{a+p-1} \in z_{a}+I_{0}$. Therefore we may replace $w_{i}$ by $z_{a}$ and so, we may assume without loss of generality that $w_{i}$ is a $\delta$-eigenvector in $I$. In fact we assume that $w_{i} \in E_{a}$. Since $J_{i}$ is $(x, p)$-cyclic, $q_{x, J_{i}}(X)=(X-\lambda)^{m_{i} p}$ with some $m_{i} \geqslant 1$. We claim, for all $i=1, \ldots, k$, that there is some $v_{i} \in E_{a} \cap\left(w_{i}+I_{0}\right)$ such that

$$
q_{x, v_{i}}(X)=q_{x, J_{i}}(X)=(X-\lambda)^{m_{i} p} .
$$

We prove this claim for $i=1$. Since $q_{x, J_{1}}(X)=(X-\lambda)^{m_{1} p}$, we have $(x-\lambda)^{m_{1} p}\left(w_{1}+I_{0}\right)=0$, and so $(x-\lambda)^{m_{1} p}\left(w_{1}\right) \in I_{0}$. Thus, there is some polynomial $h \in \mathbb{F}[X]$ with $\operatorname{deg} h<m$ and $(x-\lambda)^{m_{1} p}\left(w_{1}\right)=h(x)\left(v_{0}\right)$. On the other hand, $w_{1} \in E_{a}$, and hence

$$
(x-\lambda)^{m_{1} p}\left(w_{1}\right)=\left[(x-\lambda)^{p}\right]^{m-1}\left(w_{1}\right)=\left[\left(x^{p}-\lambda^{p}\right)\right]^{m-1}\left(w_{1}\right) \in E_{a},
$$

which gives $h(x)\left(v_{0}\right) \in E_{a}$, since $x^{p}$ fixes each eigenspace $E_{b}$. Write $h(X)=h_{0}(X)+h_{1}(X)+$ $\cdots+h_{p-1}(X)$ such that

$$
h_{j}(X)=a_{j}+a_{p+j} X^{p+j}+a_{2 p+j} X^{2 p+j}+\ldots,
$$

for all $0 \leqslant j \leqslant p-1$. Suppose that $h_{j} \neq 0$ for some $j>0$. Observe that $h_{j}(x)\left(v_{0}\right) \in E_{a+j}$. As $h(x)\left(v_{0}\right) \in E_{a}$ and eigenvectors associated to different eigenvalues are linearly independent, we have $h_{j}(x)\left(v_{0}\right)=0$. Thus, $q_{x, v_{0}} \mid h_{j}$. On the other hand, $\operatorname{deg} h_{i}<m=\operatorname{deg} q_{x, v_{0}}$, which implies that $h_{j}=0$, for all $j>0$. Hence, we can assume that $h=\bar{h}^{p}$ with some $\bar{h} \in \mathbb{F}[X]$. Now observe that

$$
0=(x-\lambda)^{m}\left(w_{1}\right)=(x-\lambda)^{m-m_{1} p}(x-\lambda)^{m_{1} p}\left(w_{1}\right)=(x-\lambda)^{m-m_{1} p} h(x)\left(v_{0}\right) .
$$

Since $q_{x, v_{0}}(X)=(X-\lambda)^{m}$, we have that $(X-\lambda)^{m} \mid(X-\lambda)^{m-m_{1} p} h(X)$, and so $(X-\lambda)^{m_{1} p} \mid h(X)$. Therefore there is some $q \in \mathbb{F}[X]$ such that $q(X)(X-\lambda)^{m_{1} p}=h(X)=\bar{h}(X)^{p}$. This also implies that $q=\bar{q}^{p}$ with some $\bar{q}$. Now set $v_{1}=w_{1}-q(x)\left(v_{0}\right)$. Since $q(x)\left(v_{0}\right) \in I_{0}$, we have $v_{1} \in w_{1}+I_{0}$. Further, $q(x)\left(v_{0}\right)=\bar{q}(x)^{p}\left(v_{0}\right) \in E_{a}$, and hence $v_{1} \in E_{a}$. This implies also that $q_{x, J_{1}} \mid q_{x, v_{1}}$. On the other hand,

$$
(x-\lambda)^{m_{1} p}\left(v_{1}\right)=(x-\lambda)^{m_{1} p}\left(w_{1}-q(x)\left(v_{0}\right)\right)=(x-\lambda)^{m_{1} p}\left(w_{1}\right)-(x-\lambda)^{m_{1} p} q(x)\left(v_{0}\right)=0 .
$$

Thus $q_{x, v_{1}}(X)=(X-\lambda)^{m_{1} p}=q_{x, J_{1}}(X)$, as claimed. For $i=1, \ldots, k$, let $I_{i}=\left\langle v_{i}\right\rangle_{K}$. We claim that $I=I_{0}+\cdots+I_{k}$. First,

$$
J_{i}=\left\langle w_{i}+I_{0}\right\rangle_{K}=\left\langle v_{i}+I_{0}\right\rangle_{K}=\left(I_{i}+I_{0}\right) / I_{0}
$$

and so

$$
I / I_{0}=\left(I_{1}+I_{0}\right) / I_{0} \dot{+} \cdots \dot{+}\left(I_{k}+I_{0}\right) / I_{0}
$$

This implies that $I=I_{0}+I_{1}+\cdots+I_{k}$. Further, the direct decomposition of $I / I_{0}$ also implies that $\operatorname{dim} I_{0}+\sum_{i} \operatorname{dim}\left(I_{i}+I_{0}\right) / I_{0}=\operatorname{dim} I$. On the other hand, since $q_{x, v_{i}}=q_{x, J_{i}}$, we also obtain that $\operatorname{dim} I_{i}=\operatorname{dim} J_{i}=\operatorname{dim}\left(I_{i}+I_{0}\right) / I_{0}$. Therefore

$$
\operatorname{dim} I_{0}+\operatorname{dim} I_{1}+\cdots+\operatorname{dim} I_{k}=\operatorname{dim} I .
$$

Hence the decomposition $I=I_{0} \dot{+} I_{1} \dot{+} \cdots I_{k}$ is valid.
Theorem 5.1.14. Let $K=\langle x\rangle$ be a Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$. Let I be a finite-dimensional $K$-module and set $L=K \oplus$ I. Assume that $Z(L)=0$. If $L$ has a non-singular derivation of finite order, then $I$ is the direct sum of $(x, p)$-cyclic modules.

Proof. Let $\delta \in \operatorname{Der}(L)$ be a non-singular derivation and let $\phi$ be the transformation defined in (25). It follows that, $\phi(\delta)=(\alpha, \beta) \in \operatorname{Comp}(K, I)$ is a compatible pair of non-singular derivations. By Proposition 3.1.1, $(\alpha, \beta) \in \operatorname{Der}(L)$. Hence, by Lemma 4.1.5, $L$ has a nonsingular derivation $\delta$ such that $\delta(\langle x\rangle)=\langle x\rangle, \delta(I)=I$ and that the restriction of $\delta$ to $I$ is diagonalizable. By multiplying $\delta$ with a scalar, we may assume without loss of generality that $\delta(x)=x$. Let $q_{x, I}(X)=\left(X-\lambda_{1}\right)^{m_{1}} \cdots\left(X-\lambda_{k}\right)^{m_{k}}$ be the minimal polynomial of $x$ as an element of $\mathfrak{g l}(I)$. As $K$ is one-dimensional, the collected primary decomposition of $I$ into $K$-modules is $I=I_{0}\left(\left(x-\lambda_{1}\right)^{m_{1}}\right) \dot{+} \cdots \dot{+} I_{0}\left(\left(x-\lambda_{k}\right)^{m_{k}}\right)$, and Proposition 5.1.11 implies that the $I_{0}\left(x-\lambda_{i}\right)^{m_{i}}$ are $\delta$ invariant. Hence we may assume without loss of generality that $k=1$ and $m_{x, I}(X)=(X-\lambda)^{m}$.

Further, $I$ can be decomposed as $I=\bar{E}_{a_{1}} \dot{+}+\dot{+} \bar{E}_{a_{s}}$ where, for $a_{i} \in \mathbb{F}, \bar{E}_{a_{i}}$ is the sum of the eigenspaces of $\delta$ that correspond to the eigenvalues $a_{i}, a_{i}+1, \ldots, a_{i}+p-1$. By a Lemma 5.1.10, the $\bar{E}_{a_{i}}$ are $x$-invariant. Therefore we may assume that $I=\bar{E}_{a}$ with some $a \in \mathbb{F}$. Now the theorem follows from Lemma 5.1.13.

It is interesting to note that cyclic modules appear in an arbitrary non-nilpotent Lie algebra with a non-singular derivation. In the paper [2] by Shalev, the proof of Lemma 2.2 uses the existence of a subalgebra isomorphic to $L=\langle x\rangle \oplus I$, with $I$ being a $(x, p)$-cyclic module. This subalgebra is generated by the eigenvectors of a non-singular derivation, and it is used to show that the eigenvalues are roots of the polynomial $q(X)=X^{n}-1$.

As consequence of Theorem 5.1.14, we can get some information of the matrix and the minimal polynomial of $x \in \operatorname{End}(L)$ in the Lie algebra $L=\langle x\rangle \oplus I$.

Corollary 5.1.15. Let L be a finite-dimensional Lie algebra with derived length 2 over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$. Suppose that $L$ has a non-singular derivation of finite order. Let $K=L /[L, L]$ and $I=[L, L]$. If $\operatorname{dim}(K)=1$ with $K=\langle x\rangle$ and $Z(L)=0$, then there is a basis $B$ of $I$ such that the matrix of $x$ in this basis is

$$
[x]=\left[\begin{array}{cccc}
{\left[x_{1}\right]} & 0 & \ldots & 0 \\
0 & {\left[x_{2}\right]} & \ldots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \ldots & {\left[x_{n}\right]}
\end{array}\right]
$$

such that each $\left[x_{i}\right]$ is in the form CompMatrix $\left(c_{0}^{i}, c_{1}^{i}, \cdots, c_{n-1}^{i}\right)$ with $c_{0}^{i} \neq 0$ and $c_{j}^{i}=0$ for each $j$ such that $p \nmid j$. In particular, there is $d_{1}, \ldots, d_{r} \in \mathbb{F}$, pairwise distinct, and positive integers $e_{1}, \ldots, e_{r}$ such that the minimal polynomial of $x$ is in the form

$$
q_{x}(X)=\left(X-d_{1}\right)^{e_{1} p}\left(X-d_{2}\right)^{e_{2} p} \cdots\left(X-d_{r}\right)^{e_{r} p} .
$$

Corollary 5.1.16. Let L be a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$, with derived length 2 with $\operatorname{dim}(L /[L, L])=1$ and $Z(L)=0$. Let $\delta \in$ $\operatorname{Der}(L)$ be a non-singular derivation of finite order and suppose that $x \in L \backslash[L, L]$. If a and $b$ are eigenvalues of $\delta$ associated to $v \in I$ and $x$, respectively, then $\{a, a+b, a+2 b, \cdots, a+(p-1) b\}$ are eigenvalues of $\delta$. In particular, the number of eigenvalues of $\delta$ is congruent 1 modulo $p$.

In Proposition 5.1.14, we showed that if a metabelian Lie algebra $L=\langle x\rangle \oplus I$ has a non-singular derivation, then $I$ is the direct sum of $(x, p)$-cyclic subspaces. Now we show the converse.

Proposition 5.1.17. Let $K$ be a one-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$ with $K=\langle x\rangle$ and $I_{1}, I_{2}, \ldots, I_{s}(x, p)$-cyclic $K$-modules. Then the Lie algebra L given by the semidirect sum,

$$
L=K \oplus\left(I_{1}+I_{2}+\cdots+I_{s}\right),
$$

has a non-singular derivation with $s p+1$ distinct eigenvalues.

Proof. Recall that $\mathbb{F}$ is algebraically closed and so we can choose $b, a_{1}, \ldots, a_{s} \in \mathbb{F}$ such that $a_{j} b^{-1} \notin \mathbb{F}_{p}$, for all $1 \leqslant j \leqslant s$ and

$$
\mid\left\{a_{j}+i b \mid 1 \leqslant j \leqslant s \text { and } 0 \leqslant i \leqslant p-1\right\} \mid=p s
$$

By assumption, $I_{j}$ is $(x, p)$-cyclic, for $1 \leqslant j \leqslant s$, and so there is a basis $B_{j}=\left\{v_{0}^{j}, v_{1}^{j}, \ldots, v_{r_{j} p-1}^{j}\right\}$ of $I_{j}$ such that the matrix of $x$ in $B_{j}$ is CompMatrix $\left(c_{0}^{j}, c_{1}^{j}, \ldots, c_{r_{j} p-1}^{j}\right)$ with $c_{i}^{j}=0$ whenever $p \nmid i$. By our definition, this implies that for $1 \leqslant j \leqslant s$

$$
\begin{aligned}
{\left[x, v_{i}^{j}\right] } & =v_{i+1}^{j}, \quad \text { for } 0 \leqslant i<r_{j} p-1, \\
{\left[x, v_{r_{j} p-1}^{j}\right] } & =\sum_{i=0}^{r_{j}-1} c_{i p}^{j} v_{i p}^{j} .
\end{aligned}
$$

Define the endomorphism $\delta \in \mathfrak{g l}(L)$ by $\delta(x)=b x$ and $\delta\left(v_{i}^{j}\right)=\left(a_{j}+i b\right) v_{i}^{j}$. Then $\delta$ is nonsingular with eigenvalues $a_{j}+i b$ for $1 \leqslant j \leqslant s$ and $0 \leqslant i \leqslant p-1$. Let us check that $\delta$ is a derivation of $L$. We are required to show that $\delta\left(\left[x, v_{i}^{j}\right]\right)=\left[\delta(x), v_{i}^{j}\right]+\left[x, \delta\left(v_{i}^{j}\right)\right]$. Suppose that $i \neq r_{j} p-1$. On the one hand,

$$
\delta\left(\left[x, v_{i}^{j}\right]\right)=\delta\left(v_{i+1}^{j}\right)=\left(a_{j}+(i+1) b\right) v_{i+1}^{j}=\left(a_{j}+(i+1) b\right)\left[x, v_{i}^{j}\right] .
$$

On the other hand,

$$
\left[\delta(x), v_{i}^{j}\right]+\left[x, \delta\left(v_{i}^{j}\right)\right]=\left[b x, v_{i}^{j}\right]+\left[x,\left(a_{j}+i b\right) v_{i}^{j}\right]=\left(a_{j}+(i+1) b\right)\left[x, v_{i}^{j}\right] .
$$

Therefore,

$$
\delta\left(\left[x, v_{i}^{j}\right]\right)=\left[\delta(x), v_{i}^{j}\right]+\left[x, \delta\left(v_{i}^{j}\right)\right] .
$$

For $i=r_{j} p-1$ we have,

$$
\begin{aligned}
& \delta\left(\left[x, v_{r_{j} p-1}^{j}\right]\right)=\delta\left(\sum_{i=0}^{r_{j}-1} c_{i p}^{j} v_{i p}^{j}\right)=\sum_{i=0}^{r_{j}-1} c_{i p}^{j} \delta\left(v_{i p}^{j}\right)=\left(\sum_{i=0}^{r_{j}-1}\left(a_{j}+i p b\right) c_{i p}^{j} v_{i p}^{j}\right) \\
&=a_{j}\left(\sum_{i=0}^{r_{j}-1} c_{i p}^{j} v_{i p}^{j}\right)=a_{j}\left[x, v_{r_{j} p-1}^{j}\right],
\end{aligned}
$$

and

$$
\left[\delta(x), v_{r_{j} p-1}\right]+\left[x, \delta\left(v_{r_{j} p-1}\right)\right]=b\left[x, v_{r_{j} p-1}\right]+\left[x,\left(a_{j}+(r p-1) b\right) v_{r_{j} p-1}^{j}\right]=a_{j}\left[x, v_{r_{j}-1}^{j}\right] .
$$

Therefore,

$$
\delta\left(\left[x, v_{r_{j}-1}^{j}\right]\right)=\left[\delta(x), v_{r_{j}-1}^{j}\right]+\left[x, \delta\left(v_{r_{j}-1}^{j}\right)\right] .
$$

Thus, $\delta$ is a non-singular derivation, as claimed.
We can combine Proposition 5.1.17 and Corollary 5.1.14 in one result and state the main theorem of this chapter:

Theorem 5.1.18. Let L be a Lie algebra of derived length 2 over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$. Suppose that $\operatorname{dim}\left(L / L^{\prime}\right)=1$ and $Z(L)=0$. Let $x \in L \backslash L^{\prime}$. Then $L$ has a non-singular derivation of finite order if, and only if, $L^{\prime}$ can be written as a direct sum of ( $x, p$ )-cyclic modules.

Using Theorem 5.1.18 we can construct more examples of Lie algebras with nonsingular derivations from polynomials.

Example 5.1.19. Let $p$ be a prime number and let $\mathbb{F}$ be a field of characteristic $p$. Let $q(X)=$ $\left(X-a_{1}\right)^{p}\left(X-a_{2}\right)^{p} \cdots\left(X-a_{n}\right)^{p}$ be a polynomial in $\mathbb{F}[X]$ with $a_{1}, \ldots, a_{n} \in \mathbb{F}$. Let $I_{j}, 1 \leqslant j \leqslant n$ be a vector space over $\mathbb{F}$ of dimension $p$. Fix a basis $B_{j}=\left\{v_{0}^{j}, v_{1}^{j}, \cdots, v_{p-1}^{j}\right\}$ of $I_{j}$. Let $x_{j}$ be the endomorphism of $I_{j}$ given by CompMatrix $\left(a_{j}^{p}, 0, \ldots, 0\right)$ in basis $B_{j}$. By construction, each $I_{j}$ is $\left(x_{j}, p\right)$-cyclic. Now let $I=I_{1}+I_{2} \dot{+} \cdots+I_{n}$ and

$$
[x]=\left[\begin{array}{cccc}
{\left[x_{1}\right]} & 0 & \ldots & 0 \\
0 & {\left[x_{2}\right]} & \ldots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \ldots & {\left[x_{n}\right] .}
\end{array}\right]
$$

Then $L=\langle x\rangle \oplus I$ is a Lie algebra with non- singular derivation.
In Example 5.1.19 we can also see that the correspondence between polynomials and Lie algebras is not one-to-one. Let $a \in \mathbb{F}, a \neq 0$ and define $q_{1}(X)=\left(X-a_{1}\right)^{p}\left(X-a_{2}\right)^{p}$, with $a_{1}=a_{2}=a$. Then, define $L_{1}=\left\langle x_{1}\right\rangle \oplus\left(I_{1}+I_{2}\right)$ using polynomial $q_{1}$ as in Example 5.1.18. It follows that, the minimal polynomial of $x_{1}$ is $q_{x_{1}}(X)=(X-a)^{p}$. If we define $L_{2}=\left\langle x_{2}\right\rangle \oplus I$ using the polynomial $q_{2}(X)=(X-a)^{p}$, then the minimal polynomial of $x_{2}$ is also $q_{x_{2}}(X)=(X-a)^{p}$. But $L_{1}$ and $L_{2}$ have different dimensions.

### 5.2 Decomposition of $(x, p)$-cyclic modules

Next we will identify minimal $(x, p)$-cyclic modules, that is, we will present conditions for an ( $x, p$ )-cyclic module $I$ that imply that $I$ cannot be decomposed as a sum of smaller $(x, p)$ cyclic modules. First we note that given an $x$-cyclic module $I$, the decomposition induced by the minimal polynomial, which is the primary decomposition for the one-dimensional Lie algebra $\langle x\rangle$, also decomposes $(x, p)$-cyclic modules into the sum of $(x, p)$-cyclic modules. We present an example to introduce this result.

Example 5.2.1. Let $I$ be a vector space over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_{3}$ of dimension 6 and let $B=\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right\}$ be a basis of $I$. Let $x \in \mathfrak{g l}(I)$ be given by the companion matrix CompMatrix $(1,0,0,0,0,0)$. Define $L=\langle x\rangle \oplus I$. Then $I$ is $(x, 3)$-cyclic and, by Proposition 5.1.17, $L$ has a non-singular derivation with 4 eigenvalues. Let $B^{\prime}$ the basis given by

$$
\begin{array}{lll}
u_{1}=v_{1}+v_{4}, & u_{2}=v_{2}+v_{5}, & u_{3}=v_{3}+v_{6}, \\
u_{4}=v_{1}-v_{4}, & u_{5}=v_{2}-v_{5}, & u_{6}=v_{3}-v_{6} .
\end{array}
$$

Then $I_{1}=\left\langle u_{1}, u_{2}, u_{3}\right\rangle$ and $I_{2}=\left\langle u_{4}, u_{5}, u_{6}\right\rangle$ are $(x, 3)$-cyclic modules and, $L$ can be written as $\langle x\rangle \oplus\left(I_{1}+I_{2}\right)$. Hence, again by Proposition 5.1.17, $L$ has a non-singular derivation with 7 eigenvalues. Therefore, the $(x, 3)$-cyclic vector space $I$ could be decomposed into the direct sum of two smaller $(x, 3)$-cyclic modules.

Lemma 5.2.2. Let I be a finite-dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$, and let $x \in \mathfrak{g l}(I)$ be such that $I$ is $x$-cyclic. Assume that $I=I_{1} \dot{+} I_{2} \dot{+} \cdots+I_{r}$ is the collected primary decomposition of I into $x$-modules. Then

1. $I_{i}$ is $x$-cyclic;
2. Let $v \in I$ such that $\left\{v, x(v), x^{2}(v), \cdots, x^{n}(v)\right\}$ is a basis of $I$ and $v=v_{1}+v_{2}+\cdots+v_{r}$ with $v_{i} \in I_{i}$. Then $\left\{v_{i}, x\left(v_{i}\right), x^{2}\left(v_{i}\right), \cdots, x^{n_{i}}\left(v_{i}\right)\right\}$ is a basis of $I_{i}$, for some $n_{i} \leqslant n$.
3. If I is $(x, p)$-cyclic, then $I_{i}$ is $(x, p)$-cyclic, for $1 \leqslant i \leqslant r$.

Proof. Itens 1 and 2 follow from Theorem 6.4 of [17]. To prove item 3 we will verify the conditions in Proposition 5.1.7. By item 1 of this proposition, $I_{i}$ is $x$-cyclic. Let $x_{i}$ be the restriction of $x$ to $I_{i}$ and let $q_{x_{i}}$ be the minimal polynomial of $x_{i}$. By the definition of collected
primary decomposition, there is $e_{1}, \ldots, e_{r} \in \mathbb{F}$, pairwise distinct, such that

$$
q_{x_{i}}(X)=\left(X-e_{i}\right)^{t_{i}}, \text { for some } t_{i} \geqslant 1 .
$$

Suppose that the minimal polynomial of $x$ is in the form

$$
q_{x}(X)=X^{p m}-c_{p(m-1)} X^{p(m-1)}-\cdots-c_{2 p} X^{2 p}-c_{p} X^{p}-c_{0}, \quad c_{0} \neq 0 .
$$

Define $S=X^{p}$. Then we can write

$$
q_{x}(S)=S^{m}-c_{p(m-1)} S^{m-1}-\cdots-c_{2 p} S^{2}-c_{p} S-c_{0}
$$

As $\mathbb{F}$ is algebraically closed, there are $d_{1}, \cdots, d_{r} \in \mathbb{F}$ (non-zero because $c_{0} \neq 0$ ) and $m_{i} \geqslant 1$, $1 \leqslant i \leqslant r$, such that

$$
q_{x}(S)=\left(S-d_{1}\right)^{m_{1}} \cdots\left(S-d_{r}\right)^{m_{r}}
$$

Replacing $S$ by the variable $X$ we have

$$
q_{x}(X)=\left(X^{p}-d_{1}\right)^{m_{1}} \cdots\left(X^{p}-d_{r}\right)^{m_{r}} .
$$

By assumption, $\mathbb{F}$ has prime characteristic, and so

$$
q_{x}(X)=\left(X-e_{1}\right)^{p m_{1}} \cdots\left(X-e_{m}\right)^{p m_{r}}
$$

where $e_{i}$ is such that $e_{i}^{p}=d_{i}$, for all $i$. Thus, we can assume that the minimal polynomial of the restriction of $x$ to $I_{i}$ is

$$
q_{x_{i}}(X)=\left(X-e_{i}\right)^{p m_{i}}, \text { with } e_{i} \neq 0
$$

By Proposition 5.1.7, $I_{i}$ is $(x, p)$-cyclic, $1 \leqslant i \leqslant r$.
Now we can characterize a type of irreducibility for $(x, p)$-cyclic modules. The Lie algebra using these modules, over an algebraically closed field, have non-singular derivations with exactly $p+1$ eigenvalues.

Proposition 5.2.3. Let $K$ be a Lie algebra of dimension 1 over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$, and let $I$ be a $K$-module of dimension $n$. Let $x \in K \backslash\{0\}$ and $L=\langle x\rangle \oplus \subseteq$ such that $Z(L)=0$. Suppose that $L$ has a non-singular derivation of finite order. Then I can be written as direct sum of $(x, p)$-cyclic modules $I=I_{1}+\cdots+I_{r}$, with $r>1$, if, and only if, the minimal polynomial of $x$, viewed as an element of $\mathfrak{g l}(I)$, is either $q_{x}(X)=(X-a)^{m}$, with $a \in \mathbb{F}$ and $m<n$, or $q_{x}(X)=\left(X-a_{1}\right)^{m_{1}} \ldots\left(X-a_{s}\right)^{m_{s}}$, where $a_{1}, \ldots, a_{s} \in \mathbb{F}$ are distinct and $s>1$.

Proof. Suppose that $I$ can be write as $I=I_{1} \dot{+} \cdots \dot{+} I_{r}$, for $r>1$, such that $I_{i}$ is $(x, p)$-cyclic module, $1 \leqslant i \leqslant r$. Let $x_{i}$ be the restriction of $x$ to $I_{i}$ and let $q_{x_{i}}$ be the minimal polynomial of $x_{i}$. If $q_{x_{i}}(X)=(X-a)^{m_{i}}, 1 \leqslant i \leqslant r$, for some $a \in \mathbb{F}$ and $m_{i} \geqslant 1$, then $q_{x}(X)=(X-a)^{m}$, such that $m=\max \left\{m_{1}, \ldots, m_{r}\right\}$. As $r>1, \operatorname{dim} I_{i}<n$ and $m_{i}<n$, for all $i$. Thus, $m<n$. If there is distinct $a_{i}, a_{j} \in \mathbb{F}$ such that $q_{x_{i}}(X)=\left(X-a_{i}\right)^{m_{i}}$ and $q_{x_{j}}(X)=\left(X-a_{j}\right)^{m_{j}}$, then $\left(X-a_{i}\right)^{m_{i}}\left(X-a_{j}\right)^{m_{j}} \mid q_{x}(X)$. Let us prove the other direction,

- Suppose that $q_{x}(X)=\left(X-a_{1}\right)^{m_{1}} \ldots\left(X-a_{s}\right)^{m_{s}}$, where $a_{1}, \ldots, a_{s} \in \mathbb{F}$ are distinct and $s>1$. As $L$ has a non-singular derivation of finite order, by Theorem 5.1.13, $I$ can be written as a direct sum of $(x, p)$-cyclic modules, $I=I_{1} \dot{+} \cdots \dot{+} I_{r}$. If $r>1$, then the result is verified. If $r=1$, then $I$ is $(x, p)$-cyclic. Let $I=J_{1} \dot{+} \cdots \dot{+} J_{s}$ be the collected primary decomposition of $I$ into $x$-modules. Thus, by item 3 of Proposition 5.2.2, $J_{i}$ is $(x, p)$-cyclic and the result is verified.
- If $q_{x}(X)=(X-a)^{m}$, with $a \in \mathbb{F}$ and $m<n$, let $I=I_{1} \dot{+} \cdots \dot{+} I_{r}$ be the decomposition into $(x, p)$-cyclic modules presented by Theorem 5.1.13. If $r=1$, then $I$ is $(x, p)$-cyclic and, by item 4 of Lemma 5.1.5, $m=n$, which is a contradiction. Then $r>1$ and the result is verified.


## 6 More examples of Lie algebra with non-singular derivation

This chapter is dedicated to present new examples of Lie algebras with non-singular derivations. The examples explored in the previous chapters were defined by the semidirect sum $K \oplus I$, with $K=\langle x\rangle$. In this section we will mostly consider the case when $K$ is a nilpotent Lie algebra of higher dimension. In Proposition 6.1.3, we present an example of a non-nilpotent Lie algebra $L=H \oplus I$, such that $H$ has arbitrarily large nilpotency class and $L$ has a non-singular derivation. Then we suppose that $H$ is the Heisenberg algebra. In this case, we present an example of a representation $\psi: H \rightarrow \mathfrak{g l}(I)$ such that $L=H \oplus I$ is solvable, non-nilpotent with non-singular derivation and set some conditions for this type of representations.

### 6.1 Examples with derived length 3

Recall the notation of adjoint representation presented in Section 2.1. Let ad : $L \rightarrow$ $\operatorname{Der}(L)$ be the adjoint representation of $L$ given by $x \mapsto \operatorname{ad}_{x}$, for all $x \in L$, such that, $\operatorname{ad}_{x}(y)=$ $[x, y]$, for all $y \in L$.

Lemma 6.1.1. ( [20] Corollary 5.2.7) Let L be a finite-dimensional Lie algebra graded by some abelian group. Suppose L satisfies the Engel condition $\left(\operatorname{ad}_{x}\right)^{n}(y)=0$ for some $n \geqslant 1$ and all homogeneous elements $x, y \in L$. Then $L$ is nilpotent.

Lemma 6.1.2. ([21] Proposition 1.3) Let I be a vector space over a field $\mathbb{F}$. If $x, y \in \operatorname{End}(I)$, then

$$
\left(\operatorname{ad}_{x}\right)^{n}(y)=\sum_{i=0}^{n}(-1)^{i}\binom{n}{i} x^{n-i} y x^{i}, \text { for all } n \geqslant 1 .
$$

In the next proposition we use an action of a Lie algebra $H$ on a vector space $I$ to define a Lie algebra $L=H \oplus I$. As we want to emphasize that the action of $H$ on $I$ corresponds to the product in $L$, we will replace the notation used in the Lemma 6.1.2 by Lie brackets. Hence, if $x, v \in L$, then

$$
\left(\operatorname{ad}_{x}\right)^{n}(v)=\left[x_{n}, v\right] \text { for all } n \geqslant 1 .
$$

Proposition 6.1.3. Let I be a vector space over a field $\mathbb{F}$ of characteristic $p>0$. Suppose that I has dimension $2 p$ and let $B=\left\{v_{1}, \cdots, v_{2 p}\right\}$ be a basis of I. Define the elements $x, y \in \mathfrak{g l}(I)$ with the following rules

$$
x: v_{k p+1} \mapsto v_{k p+2}, v_{k p+2} \mapsto v_{k p+3}, \ldots v_{k p+p-1} \mapsto v_{k p+p}, v_{k p+p} \mapsto v_{k p+1}, \text { for } k=0,1
$$

and

$$
y: v_{p+1} \mapsto v_{1} \text { and } v_{i} \mapsto 0 \text { if } i \neq p+1
$$

Then

1. $\left[x_{p-1}, y\right] \neq 0$.
2. $\left[x_{p}, y\right]=0$.
3. The Lie algebra $H$, generated by $x$ and $y$ in $\mathfrak{g l}(I)$, is nilpotent, with nilpotent class $p$ and dimension $p+1$.
4. The Lie algebra $L=H \oplus I$ is not nilpotent and if $|\mathbb{F}| \geqslant p^{2}$, then $L$ has a non-singular derivation.

## Proof.

1. By Lemma (6.1.2),

$$
\begin{aligned}
{\left[x_{p-1}, y\right] v_{p+1} } & =\sum_{i=0}^{p-1}(-1)^{i}\binom{p-1}{i} x^{p-1-i} y x^{i} v_{p+1} \\
& =\sum_{i=0}^{p-1}(-1)^{i}\binom{p-1}{i} x^{p-1-i} y v_{p+1+i} \\
& =x^{p-1} y v_{p+1} \quad\left(\text { since } y v_{i}=0, \text { for } i \neq p+1\right) \\
& =x^{p-1} v_{1} \\
& =v_{p} .
\end{aligned}
$$

Therefore, $\left[x_{p-1}, y\right] \neq 0$ as claimed.
2. By definition, $x^{p}$ fixes the basis $B=\left\{v_{1}, \ldots, v_{p}, v_{p+1}, \ldots, v_{2 p}\right\}$ elementwise. Hence, $x^{p}$ acts as identity on $I$. Further, by Lemma (6.1.2), $\left[x_{p}, y\right]=x^{p} y-y x^{p}=0$.
3. First we claim that the only non-zero right-normed brackets in $H$ are of the form $\left[x_{n}, y\right]$, for $1 \leqslant n \leqslant p-1$. Let $I_{1}$ be the vector space generated by $\left\{v_{1}, \cdots, v_{p}\right\}$. By the definition of $x \in \operatorname{End}(I)$,

$$
\begin{equation*}
x(I)=I, \quad x\left(I_{1}\right)=I_{1}, \quad y(I) \subset I_{1} \quad \text { and } \quad y\left(I_{1}\right)=0 \tag{37}
\end{equation*}
$$

Let $\left[w_{1}, w_{2}, \ldots, w_{r}\right]$ be a right-normed bracket such that $w_{j} \in\{x, y\}$ for $1 \leqslant j \leqslant r$. Thus,

$$
\begin{equation*}
\left[w_{1}, w_{2}, \ldots, w_{r}\right]=\sum_{i=0}^{s} c_{i} w_{j_{1}} w_{j_{2}} \ldots w_{j_{t_{i}}} \tag{38}
\end{equation*}
$$

for some $s>0, t_{i}>0$ and $c_{i} \in \mathbb{F}$. If $y$ appears twice in the list $\left\{w_{1}, w_{2}, \ldots, w_{r}\right\}$, then each summand of the right side of equation (38) is of the form $c_{i} x^{l} y x^{j} y x^{k}$ for some $l, j, k \geqslant 0$. Thus, by equations in (37),

$$
x^{l} y x^{i} y x^{k}(I)=x^{l} y x^{i} y(I)=x^{l} y x^{i}\left(I_{1}\right)=x^{l} y\left(I_{1}\right)=0 .
$$

Hence, $\left[w_{1}, w_{2}, \ldots, w_{r}\right]=0$. Suppose now that $y$ appears exactly once in $\left\{w_{1}, w_{2}, \ldots, w_{r}\right\}$. If $w_{j}=y$ for $1 \leqslant j \leqslant r-2$, then, $\left[w_{1}, w_{2}, \ldots, w_{r}\right]=[x, x, \ldots, y, \ldots, x, x]=0$. If $y=w_{r-1}$, then $[x, x, \ldots, y, x]=-[x, x, \ldots, x, y]$, and we can say that all non-zero rightnormed brackets are of the form $\left[x_{n}, y\right]$ for $n \geqslant 1$. Observe that, by item 1 of this proposition, $\left[x_{n}, y\right] \neq 0$ for $1 \leqslant n<p$ and $\left[x_{p}, y\right]=0$. Thus, the only non-zero right-normed brackets in $H$ are of the form $\left[x_{n}, y\right]$, for $1 \leqslant n \leqslant p-1$, as we claimed. Let $z_{i}=\left[x_{i}, y\right]$ for $1 \leqslant i \leqslant p-1$. By the calculations in the last paragraph, the Lie algebra $H$ generated by $x, y \in \mathfrak{g l}(I)$ can be presented by the presentation

$$
\begin{aligned}
H=\left\langle x, y, z_{1}, \cdots, z_{p-1}\right|[x, y]=z_{1},\left[x, z_{i}\right]= & z_{i+1},\left[x, z_{p-1}\right]=\left[y, z_{l}\right]=\left[z_{l}, z_{j}\right]=0 \\
& \text { for } 1 \leqslant i \leqslant p-2 \text { and } 1 \leqslant j, l \leqslant p-1\rangle .
\end{aligned}
$$

Thus $H$ is nilpotent with nilpotency class $p$.
4. Let $a, b \in \mathbb{F} \backslash \mathbb{F}_{p}$ such that $a b^{-1} \notin \mathbb{F}_{p}$, and define $\delta: L \rightarrow L$ by

$$
\left\{\begin{aligned}
\delta(x) & =x \\
\delta(y) & =a y \\
\delta\left(z_{j}\right) & =(a+j) z_{j} \\
\delta\left(v_{k p+i}\right) & =(b-k a+i-1) v_{k p+i}, \quad 0 \leqslant k \leqslant 1, \quad 1 \leqslant i \leqslant p
\end{aligned}\right.
$$

By definition, $\delta$ is a non-singular endomorphism of $L$. We will check that $\delta$ satisfies the definition of derivations in each non-zero product: $[x, y],\left[x, v_{k p+i}\right],\left[y, v_{p+1}\right],\left[x, z_{j}\right]$ and $\left[z_{j}, v_{p+i}\right]$, for $k \in\{0,1\}, 1 \leqslant i \leqslant p$ and $1 \leqslant j<p$. First we compute

$$
\begin{aligned}
{[\delta(x), y]+[x, \delta(y)] } & =[x, y]+[x, a y] \\
& =(1+a)[x, y] \\
& =(1+a) z_{1} \\
& =\delta\left(z_{1}\right) \\
& =\delta([x, y]) .
\end{aligned}
$$

Suppose that $i \in\{1,2, \ldots, p-1\}$, then

$$
\begin{aligned}
{\left[\delta(x), v_{k p+i}\right]+\left[x, \delta\left(v_{k p+i}\right)\right] } & =\left[x, v_{k p+i}\right]+(b-k a+i-1)\left[x, v_{k p+i}\right] \\
& =(b-k a+i) v_{k p+i+1} \\
& =\delta\left(v_{k p+i+1}\right) \\
& =\delta\left(\left[x, v_{k p+i}\right]\right) .
\end{aligned}
$$

For $i=p$,

$$
\begin{aligned}
{\left[\delta(x), v_{k p+p}\right]+\left[x, \delta\left(v_{k p+p}\right)\right] } & =\left[x, v_{k p+p}\right]+(b-k a+p-1)\left[x, v_{k p+p}\right] \\
& =(b-k a) v_{k p+1} \\
& =\delta\left(v_{k p+1}\right) \\
& =\delta\left(\left[x, v_{k p+p}\right]\right) .
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
{\left[\delta(y), v_{p+1}\right]+\left[y, \delta\left(v_{p+1}\right)\right] } & =a\left[y, v_{p+1}\right]+(b-a)\left[y, v_{p+1}\right] \\
& =b\left[y, v_{p+1}\right] \\
& =b v_{1} \\
& =\delta\left(v_{1}\right) \\
& =\delta\left(\left[y, v_{p+1}\right]\right)
\end{aligned}
$$

and

$$
\begin{aligned}
{\left[\delta(x), z_{j}\right]+\left[x, \delta\left(z_{j}\right)\right] } & =\left[x, z_{j}\right]+(a+j)\left[x, z_{j}\right] \\
& =(a+j+1)\left[x, z_{j}\right] \\
& =\delta\left(z_{j+1}\right) \\
& =\delta\left(\left[x, z_{j}\right]\right) .
\end{aligned}
$$

Before we verify the identity of derivations for $\left[z_{j}, v_{p+i}\right]$, we need to perform some calculations. Observe that, $z_{j}=\left[x_{j}, y\right]=\left(\operatorname{ad}_{x}\right)^{j}(y)$. Hence, $\left[z_{j}, v_{i}\right]=\left(\operatorname{ad}_{x}\right)^{j}(y)\left(v_{i}\right)$. We will use the notation $\left(\operatorname{ad}_{x}\right)^{j}(y)$ to calculate the Lie bracket $\left[z_{j}, v_{p+i}\right]$, for $0 \leqslant j \leqslant p-1$ and $1 \leqslant i \leqslant p$.

By Lemma 6.1.2,

$$
\begin{equation*}
\left(\operatorname{ad}_{x}\right)^{j}(y) v_{p+i}=\sum_{s=0}^{j}(-1)^{s}\binom{j}{s} x^{j-s} y x^{s}\left(v_{p+i}\right) . \tag{39}
\end{equation*}
$$

The summand $x^{j-s} y x^{s}\left(v_{p+i}\right)$ is non-zero only for $x^{s}\left(v_{p+i}\right)=v_{p+1}$. Observe that $x$ acts as cyclic permutation modulo $p$ on the sets $\left\{v_{1}, \ldots, v_{p}\right\}$ and $\left\{v_{p+1}, \ldots, v_{2 p}\right\}$. Hence, $x^{s}\left(v_{p+i}\right)=v_{p+1}$ for $s+i=1$ modulo $p$. The solutions for this equation are $s=0$ for $i=1$ and $s=p-i+1$ for $i>1$.

- If $i>1$ and $j<p-i+1$, then $x^{s}\left(v_{p+i}\right) \neq v_{1}$ and $x^{j-s} y x^{s}\left(v_{p+i}\right)=0$.
- If $i>1$ and $j \geqslant p-i+1$, then

$$
\begin{aligned}
& \left(\operatorname{ad}_{x}\right)^{j}(y) v_{p+i}=\sum_{s=0}^{j}(-1)^{s}\binom{j}{s} x^{j-s} y x^{s}\left(v_{p+i}\right) \\
& \quad=(-1)^{p-i+1}\binom{j}{p-i+1} x^{j-(p-i+1)} y\left(v_{p+1}\right)=(-1)^{p-i+1}\binom{j}{p-i+1} v_{j-(p-i)} \\
& \quad=c_{j, i} v_{j-(p-i)}, \text { for } c_{j, i}=(-1)^{p-i+1}\binom{j}{p-i+1} .
\end{aligned}
$$

- If $i=1$, then $x^{s}\left(v_{p+1}\right)=v_{p+1}$ only for $s=0$. Hence,

$$
\left(\operatorname{ad}_{x}\right)^{j}(y) v_{p+1}=\sum_{s=0}^{j}(-1)^{s}\binom{j}{s} x^{j-s} y x^{s}\left(v_{p+1}\right)=v_{j+1}
$$

It follows that,

$$
\left[z_{j}, v_{p+i}\right]=\left\{\begin{array}{cll}
v_{j+1} & \text { if } \quad i=1 \\
c_{j, i} v_{j-(p-i)} & \text { if } \quad i>1 \text { and } j \geqslant p-i+1 . \\
0 & \text { if } \quad i>1 \text { and } j<p-i+1
\end{array}\right.
$$

Thus,

$$
\begin{aligned}
{\left[\delta\left(z_{j}\right), v_{p+1}\right]+\left[z_{j}, \delta\left(v_{p+1}\right)\right] } & =(a+j)\left[z_{j}, v_{p+1}\right]+(b-a)\left[z_{j}, v_{p+1}\right] \\
& =(b+j) v_{1+j} \\
& =\delta\left(v_{1+j}\right) \\
& =\delta\left(\left[z_{j}, v_{p+1}\right]\right) .
\end{aligned}
$$

For $i>1$ and $j \geqslant p-i+1$,

$$
\begin{aligned}
{\left[\delta\left(z_{j}\right), v_{p+i}\right]+\left[z_{j}, \delta\left(v_{p+i}\right)\right] } & =(a+j)\left[z_{j}, v_{p+i}\right]+(b-a+i-1)\left[z_{j}, v_{p+i}\right] \\
& =(b+j+i-1) c_{j, i} v_{j-(p-i)} \\
& =\delta\left(c_{j, i} v_{j-(p-i)}\right) \\
& =\delta\left(\left[z_{j}, v_{p+1}\right]\right) .
\end{aligned}
$$

Therefore, $\delta \in \operatorname{Der}(L)$. The derived series of $L=H \oplus I$ is

$$
L>L^{\prime}=\left\langle z_{1}, \ldots, z_{p}, v_{1}, \ldots, v_{2 p}\right\rangle>L^{(2)}=\left\langle v_{1}, \ldots, v_{2 p}\right\rangle>L^{(3)}=0
$$

and $L$ is solvable of derived length 3. Let ad : $L \rightarrow \operatorname{Der}(L)$ be the adjoint representation of $L$. For all $n \geqslant 1,\left(\operatorname{ad}_{x}\right)^{n}\left(v_{1}\right)=v_{1+n} \neq 0$. By Engel's Theorem, $L$ is not nilpotent.

Next we explore conditions on the existence of non-singular derivations. We study Lie algebras of the form $H \oplus I$, where $H$ is the Heisenberg Lie algebra over a field of prime characteristic, and $I$ is an $H$-module. First we present an example of such Lie algebras.

Example 6.1.4. Let $H=\langle x, y, z \mid[x, y]=z,[x, z]=[y, z]=0\rangle$ be the Heisenberg Lie algebra over a field $\mathbb{F}$ of characteristic $p>0$. Suppose that $|\mathbb{F}| \geqslant p^{2}$. Let $I$ be a vector space of dimension $2 p$ and let $B=\left\{v_{0}, v_{1}, \ldots, v_{2 p-1}\right\}$ be a basis of $I$. Denote by $m_{p}$ the unique positive integer between 0 and $p-1$ that is congruent to $m$ modulo $p$. Also, we can write a number $a \in\{0, \ldots, 2 p-1\}$ uniquely in the form $a_{1} p+a_{2}$ where $0 \leqslant a_{1} \leqslant 1$ and $0 \leqslant a_{2} \leqslant p-1$. Define the following representation $\psi: H \rightarrow \mathfrak{g l}(I)$, for $0 \leqslant k \leqslant 1$ and $0 \leqslant i \leqslant p-1$,

$$
\left\{\begin{aligned}
\psi(x)\left(v_{k p+i}\right) & =v_{k p+(i+1)_{p}} \\
\psi(y)\left(v_{p+i}\right) & =i v_{i} \\
\psi(z)\left(v_{p+i}\right) & =-v_{(i+1)_{p}}
\end{aligned}\right.
$$

and $\psi(y)\left(v_{i}\right)=\psi(z)\left(v_{i}\right)=0$. Observe that $(\psi(x) \psi(y)-\psi(y) \psi(x))\left(v_{i}\right)=0$ and

$$
\begin{aligned}
(\psi(x) \psi(y)-\psi(y) \psi(x))\left(v_{p+i}\right)=\psi(x)\left(i v_{i}\right)-\psi(y)\left(v_{p+(i+1)_{p}}\right)=i v_{(i+1)_{p}}-(i+1)_{p} v_{(i+1)_{p}} \\
=-v_{(i+1)_{p}}=\psi(z)\left(v_{p+i}\right) .
\end{aligned}
$$

Hence, $\psi$ is a representation of $H$. As $|\mathbb{F}| \geqslant p^{2}$, there are $a, b \in\left(\mathbb{F} \backslash \mathbb{F}_{p}\right)$ such that $b-k a+i-1 \neq$ 0 , for all $0 \leqslant k \leqslant 1$ and $0 \leqslant i \leqslant p-1$. Define $\delta: L \rightarrow L$ by

$$
\left\{\begin{aligned}
\delta(x) & =x \\
\delta(y) & =a y \\
\delta(z) & =(a+1) z \\
\delta\left(v_{k p+i}\right) & =(b-k a+i-1) v_{k p+i}, \quad 0 \leqslant k \leqslant 1, \quad 0 \leqslant i \leqslant p-1
\end{aligned}\right.
$$

By definition $\delta$ is non-singular. We will check that $\delta$ is a derivation. Indeed,

$$
\begin{aligned}
& {[\delta(x), y]+[x, \delta(y)]=[x, y]+[x, a y]=(1+a)[x, y]=(1+a) z=\delta(z)=\delta([x, y]),} \\
& \begin{array}{r}
{\left[\delta(x), v_{k p+i}\right]+\left[x, \delta\left(v_{k p+i}\right)\right]=\left[x, v_{k p+i}\right]+(b-k a+i-1)\left[x, v_{k p+i}\right]} \\
=(b-k a+i) v_{k p+(i+1)_{p}}=\delta\left(v_{\left.k p+(i+1)_{p}\right)}\right)=\delta\left(\left[x, v_{k p+i}\right]\right)
\end{array} \\
& \begin{array}{r}
{\left[\delta(y), v_{p+i}\right]+\left[y, \delta\left(v_{p+i}\right)\right]=a\left[y, v_{p+i}\right]+(b-a+i-1)\left[y, v_{p+i}\right]} \\
=(b+i-1)\left[y, v_{p+i}\right]=(b+i-1) i v_{i}=\delta\left(i v_{i}\right)=\delta\left(\left[y, v_{p+i}\right]\right)
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
{\left[\delta(z), v_{p+i}\right]+\left[z, \delta\left(v_{p+i}\right)\right]=} & (a+1)\left[z, v_{p+i}\right]+(b-a+i-1)\left[z, v_{p+i}\right] \\
& =(b+i)\left[z, v_{p+i}\right]=-(b+i) v_{(i+1)_{p}}=-\delta\left(v_{(i+1)_{p}}\right)=\delta\left(\left[z, v_{p+i}\right]\right) .
\end{aligned}
$$

Therefore, $L=H \oplus I$ is a solvable, non-nilpotent Lie algebra with non-singular derivation and derived length 3 .

Theorem 6.1.5. Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \geqslant 3$. Let $H$ be the Heisenberg Lie algebra over $\mathbb{F}$. Let $\psi: H \rightarrow \mathfrak{g l}(I)$ be a faithful representation and suppose that $L=H \oplus I$ is non-nilpotent. Suppose that $I$, as ideal of $L$, is invariant under $\operatorname{Der}(L)$. If $L$ has a non-singular derivation of finite order, then $\operatorname{dim} I \geqslant p+3$.

Proof. Let $\delta \in \operatorname{Der}(L)$ be a non-singular derivation of finite order. By Proposition 4.1.5, we can suppose that $\delta$ is diagonalizable, $\delta(I)=I$ and $\delta(H)=H$. For $a \in \mathbb{F}$ let $L_{a}$ be the $\delta$-eigenspace associated to eigenvalue $a$. By our conditions, there is $a_{1}, \ldots, a_{r} \in \mathbb{F} \backslash\{0\}$ such that

$$
L=L_{a_{1}}+L_{a_{2}} \dot{+} \cdots \dot{+} L_{a_{r}}
$$

If $a \in \mathbb{F}$ is not an eigenvalue of $\delta$, then define $L_{a}=0$. Hence,

$$
L=\dot{+}_{a \in \mathbb{F}} L_{a}, \text { with }\left[L_{a_{i}}, L_{a_{j}}\right] \leqslant L_{a_{i}+a_{j}}
$$

This turns $L$ into a Lie algebra graded by the additive group of $\mathbb{F}$. As $L$ is non-nilpotent, by Lemma 6.1.1, there are homogeneous eigenvectors $k, v \in L$ such that $\left[k_{m}, v\right] \neq 0$ for all $m>0$. Write $k=k_{H}+k_{I}$ and $v=v_{H}+v_{I}$, such that $k_{H}, v_{H} \in H$ and $k_{I}, v_{I} \in I$. Observe that, $k_{H}, v_{H}, k_{I}$ and $v_{I}$ are $\delta$-eigenvectors. In fact, if $\delta(k)=b k$, for some $b \in \mathbb{F}$, then

$$
\delta\left(k_{H}\right)+\delta\left(k_{I}\right)=\delta\left(k_{H}+k_{I}\right)=b\left(k_{H}+k_{I}\right) .
$$

Hence, as $H$ and $I$ are invariants under $\delta$ and $L=H \oplus I, \delta\left(k_{H}\right)=b k_{H}$ and $\delta\left(k_{I}\right)=b k_{I}$. Analogously, if $\delta(v)=c v$, for some $c \in \mathbb{F}$, then $\delta\left(v_{H}\right)=c v_{H}$ and $\delta\left(v_{I}\right)=c v_{I}$. We claim that there is $h \in H$ and $a \in I$ such that $\left[h_{m}, a\right] \neq 0$ for all $m \geqslant 1$. We have that,

$$
\left[k_{m}, v_{H}+v_{I}\right] \neq 0,
$$

for all $m \geqslant 1$, and so, $\left[k_{m}, v_{H}\right] \neq 0$ or $\left[k_{m}, v_{I}\right] \neq 0$, for all $m \geqslant 1$.

- If $\left[k_{m}, v_{H}\right] \neq 0$, for all $m \geqslant 1$, then

$$
\begin{aligned}
{\left[k_{2}, v_{H}\right] } & =\left[k_{H}+k_{I},\left[k_{H}+k_{I}, v_{H}\right]\right] \\
& =\left[k_{H}+k_{I},\left[k_{H}, v_{H}\right]+\left[k_{I}, v_{H}\right]\right] \\
& =\left[k_{H},\left[k_{H}, v_{H}\right]\right]+\left[k_{H},\left[k_{I}, v_{H}\right]\right]+\left[k_{I},\left[k_{H}, v_{H}\right]\right]+\left[k_{I},\left[k_{I}, v_{H}\right]\right] \\
& =\left[k_{H},\left[k_{I}, k_{H}\right]\right]+\left[k_{I},\left[k_{H}, v_{H}\right]\right], \text { since }\left[k_{H}, v_{H}\right] \in Z(H) \text { and } I \text { is abelian. }
\end{aligned}
$$

Let $a=\left[k_{2}, v_{H}\right]$ and let $h=k_{H}$. This means that $\left[k_{2}, v_{H}\right] \in I$. Therefore, as $I$ is abelian,

$$
\left[k_{3}, v_{H}\right]=\left[k,\left[k_{2}, v_{H}\right]\right]=\left[k_{H}+k_{I},\left[k_{2}, v_{H}\right]\right]=\left[k_{H},\left[k_{2}, v_{H}\right]\right]
$$

and easy induction shows that

$$
\left[\left(k_{H}\right)_{m},\left[k_{2}, v_{h}\right]\right]=\left[k_{m+2}, v_{H}\right] \neq 0
$$

holds for all $m \geqslant 1$. Hence the choice of $h=k_{H}$ and $a=\left[k_{2}, v_{H}\right]$ is as claimed.

- If $\left[k_{m}, v_{I}\right] \neq 0$, for all $m \geqslant 1$, then let $a=v_{I}$ and let $h=k_{H}$. Hence, an argument similar to the one on the previous case shows that $\left[h_{m}, a\right] \neq 0$, for all $m \geqslant 1$.

In both cases, there is $h \in H$ and $a \in I, \delta$-eigenvectors, such that $\left[h_{m}, a\right] \neq 0$, for all $m \geqslant 1$, as we claimed. Let $q$ be the minimal polynomial of $\psi(h)$ as element of $\operatorname{End}(I)$ and suppose that $q=q_{1}^{s_{1}} \ldots q_{r}^{s_{r}}$ is the factorization of $q$ into irreducible factors. Then, by Lemma A.2.2 of [7], $I$ can be written as the direct sum $I=I_{0}\left(q_{1}(h)\right) \dot{+} \ldots+I_{0}\left(q_{r}(h)\right)$. By Proposition 2.1.3, each $I_{0}\left(q_{i}(h)\right)$ is an $H$-module. Let $I_{1}$ be the sum of $I_{0}\left(q_{i}(h)\right)$ such that $q_{i}(X) \neq X$, and set $I_{0}=I_{0}(h)$. Thus,

$$
L=H \oplus\left(I_{0} \dot{+} I_{1}\right) .
$$

Also, $I_{1} \neq 0$, since $a \in I_{1}$. By Proposition 5.1.11, $I_{0}$ and $I_{1}$ are $H$-modules and $\delta$-invariant. It follows that, the Lie algebras $L_{0}=H \oplus I_{0}$ and $L_{1}=H \oplus I_{1}$ have a non-singular derivation. Observe that, by the construction of $L_{1}, h$ acts non-singularly in $I_{1}$. Hence, $L_{1}$ is non-nilpotent. Let $\delta_{1}$ be the restriction of $\delta$ to $L_{1}$. The derivation $\delta_{1} \in \operatorname{Der}\left(L_{1}\right)$ is non-singular and has finite order. As $h$ is an eigenvector of $\delta_{1}$ and $I$ is $\delta$-invariant, the Lie algebra $\langle h\rangle \oplus I$ is $\delta$-invariant, and so the restriction of $\delta$ to $\langle h\rangle \oplus \subseteq$ is a non-singular derivation of finite order. As the action of $h$ is non-singular on $I_{1}$, by Theorem 5.1.18, $I_{1}$ can be written as a direct sum of $(h, p)$-cyclic modules, and so

$$
\operatorname{dim} I_{1}=n p, \quad n \geqslant 1
$$

The action of $H$ must be faithful either on $I_{1}$ or on $I_{0}$. For, if $H$ were not faithful on $I_{0}$ and on $I_{1}$, then $Z(H)$ would act trivially on both $I_{1}$ and $I_{0}$, hence $Z(H)$ would act trivially on $I$. This contradicts the assumption that $I$ is a faithful $H$-module. As $\delta(H)=H, \delta(Z(H))=Z(H)$. Hence, if $z \in Z(H) \backslash\{0\}$, then $\delta(z)=d z$, since $\operatorname{dim}(Z(H))=1$. If $I_{1}$ is a faithful representation, then there is $u \in I_{1}$ an $\delta$-eigenvector associated to the eigenvalue $e \in \mathbb{F}$, such that $[z, u] \neq 0$. It follows that, $\delta([z, u])=(d+e)[z, u]$. Then, since $d \neq 0, u$ and $[z, u]$ are linearity independent.

If $\operatorname{dim} I_{1}=p$, then by Corollary 2.4 of [22] the representation is irreducible and there exists $f \in \mathbb{F}$ such that $[z, w]=f w$ for all $w \in I$. Which contradicts the fact that $u$ and $[z, u]$ are linearity independent, and so $\operatorname{dim} I_{1} \geqslant 2 p$. As $p \geqslant 3, \operatorname{dim} I_{1} \geqslant p+3$. If $I_{1}$ is not faithful, then $I_{0}$ is, and, by Theorem 3.1 of [22], $\operatorname{dim} I_{0} \geqslant 3$. In both cases,

$$
\operatorname{dim} I=\operatorname{dim} I_{0}+\operatorname{dim} I_{1} \geqslant p+3 .
$$

Proposition 6.1.6. Let $K$ be a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic $p>0$. Let I be a $K$-module such that $I$, as an ideal of the Lie algebra $L=K \oplus I$, is invariant under $\operatorname{Der}(L)$. Suppose that L has a non-singular derivation $\delta$ of finite order. Suppose that there is $x, y \in K$ eigenvectors of $\delta$ associated to eigenvalues $a, b \in \mathbb{F}$, respectively, such that $a \neq k b$, for $k \in \mathbb{F}_{p}$. If the matrices of the action of $x, y$ are non-singular then $\operatorname{dim} I \geqslant p^{2}$.

Proof. By Proposition 4.1.5, we can suppose that $\delta$ is diagonalizable. Let $v \in I$ be an eigenvector of $\delta$ such that $\delta(v)=c v$. Since $\delta$ is a derivation,

$$
\delta([x, v])=(a+c)[x, v]
$$

and $[x, v]$ is an eigenvector of $\delta$. Define $v_{0}=v$ and $v_{i+1}=\left[x, v_{i}\right]$ for $1 \leqslant i \leqslant p-1$. As $x$ is nonsingular, $B=\left\{v_{0}, \cdots, v_{p-1}\right\}$ is a set of $p$ non-zero $\delta$-eigenvectors associated to the eigenvalues $A=\{c, c+a, c+2 a, \cdots, c+(p-1) a\}$. As $\delta$ is non-singular, $c+k a \neq 0$, for $0 \leqslant k \leqslant p-1$, and all elements of $A$ are pairwise distinct. For each $v_{i} \in B$ we have that

$$
\delta\left(\left[y, v_{i}\right]\right)=(c+i a+b)\left[y, v_{i}\right] .
$$

Define $v_{i}^{0}=v_{i}$ and $v_{i}^{j+1}=\left[y, v_{i}^{j}\right]$, for $0 \leqslant j \leqslant p-1$. Let us check that elements of the set

$$
\{c+i a+j b \mid 0 \leqslant i, j \leqslant p-1\}
$$

are pairwise distinct. Suppose by contradiction that there is $i_{1}, i_{2}, j_{1}, j_{2}$, with $i_{1} \neq i_{2}$ or $j_{1} \neq j_{2}$, such that

$$
c+i_{1} a+j_{1} b=c+i_{2} a+j_{2} b
$$

Hence,

$$
\left(i_{1}-i_{2}\right) a=\left(j_{2}-j_{1}\right) b
$$

If $i_{1}-i_{2}=0$, then $b=0$ and $\delta$ is singular. Same conclusion for $j_{1}-j_{2}=0$. Thus $a=k b$, for $1 \leqslant k \leqslant p-1$, which contradicts the hypothesis. Then $v_{i}^{j}$ for $0 \leqslant i \leqslant p-1$ and $0 \leqslant j \leqslant p-1$
are $p^{2}$ eigenvectors of $\delta$ in $I$ associated to the distinct eigenvalues $(c+(i-1) a+(j-1) b)$. Therefore, $\operatorname{dim} I \geqslant p^{2}$.

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