UNIVERSIDADE FEDERAL DE MINAS GERAIS

INSTITUTO DE CIÊNCIAS EXATAS

CHARACTERISTICALLY SIMPLE SUBGROUPS OF QUASIPRIMITIVE PERMUTATION GROUPS

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Belo Horizonte - MG

2018

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A thesis submitted for the degree of Doctor of Philosophy of the Universidade Federal de Minas Gerais.

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Universidade Federal de Minas Gerais
March 2018

Agradecimentos

O correr da vida embrulha tudo, a vida é assim: esquenta e esfria, aperta e daí afrouxa, sossega e depois desinquieta. O que ela quer da gente é coragem. O que Deus quer é ver a gente aprendendo a ser capaz de ficar alegre a mais, no meio da alegria, e inda mais alegre ainda no meio da tristeza! Só assim de repente, na horinha em que se quer, de propósito – por coragem. Será? Era o que eu às vezes achava. Ao clarear do dia.

João Guimarães Rosa

Quando eu terminei o mestrado, o doutorado não foi um passo imediato. Ansiava por começar logo a colocar em prática as coisas que aprendi. Contudo, ao perceber que poderia simultaneamente fazer pesquisa e lecionar no departamento, encontrei o meu lugar.

O doutorado se mostrou um desafio muito mais difícil quando comparado ao mestrado, pois me foi (e ainda é) exigido a todo instante desenvolver qualidades de pesquisador, o que torna o trabalho por vezes muito solitário e estressante.

Gostaria de expressar então a minha gratidão a todos que contribuíram para a elaboração dessa tese, seja com conversas úteis ou inúteis, ou idas ao cinema, ou como companhias para o vôlei ou para dançar. Afinal de contas, nas entrelinhas, escoando das minhas mãos para o teclado, houve muito chá, cinema e samba. E sempre haverá, amém.

Bem, nada seria possível sem o meu orientador, Prof. Csaba Schneider, que me introduziu no estudo dos grupos de permutações, e que sempre me apoiou, pressionou, e compreendeu nos momentos difíceis. Obrigado por sua paciência e atenção, por sempre me manter em movimento, e por sua dedicação ao nosso projeto.

Agradeço aos professores e funcionários da UFMG, em especial à Profa. Ana Cristina Vieira, que mescla carinho e firmeza, e que sempre acreditou em mim, às meninas Eliane Kelli e Eliane Andréa, que me aguentam na secretaria e fora dela, aos Profs. Rafael Bezerra, John MacQuarrie, Rémy Sanchis e Michel Spira, obrigado pelas conversas e pela amizade.

Agradeço aos membros da banca, Profa. Ana Cristina Vieira (UFMG), Profa. Carmela Sica (UFBA), Profa. Cristina Acciarri (UnB), Prof. John MacQuarrie

(UFMG) e Prof. Martino Garonzi (UnB).

Agradeço à CAPES e ao CNPq, o apoio financeiro.

Agradeço carinhosamente aos meus amigos, à Lilian Batista, Lorena Oliveira, Rôney Rachide, Daniel Ungaretti, Marina Muniz e Jeanne Amaral, que bem sabem o que é ser matemático, ao Pedro Oliveira, Renata Soares e Thalis Diniz, que desconhecem a medida do tempo, à Kênia Ribeiro, Frederico Santiago e Luana Ribeiro, que estão sempre de prontidão, e a todos os outros que irão brigar, com razão, por não terem seus nomes aqui explicitados. Vocês fazem da minha vida uma jornada muito alegre e mais leve. Guardo vocês com muito carinho.

Ao Esdras Mellemberg, que chegou tarde e sem avisar, bagunçou tudo, e ilumina os meus segundos. A você dedico o meu amor.

Finalmente, agradeço aos meus pais, meus irmãos e meus familiares, que tanto me incentivaram ("Pra que foi caçar fazer matemática. Vê se pode, minino!") e torceram, mesmo sem entender. A vocês agradeço os cafunés, as alegrias e preocupações de todos os dias, que fazem eu me sentir mais vivo e comprometido.

Muito, muito obrigado...

Abstract

We say that $G \leq \mathrm{Sym}(\Omega)$ is **transitive** if G has just one orbit on Ω, namely $\{\Omega\}$. If G is transitive on Ω and the only partitions of Ω preserved by G are $\{\Omega\}$ and $\{\{\alpha\}: \alpha \in \Omega\}$, then we say that G is **primitive**.

The O'Nan-Scott Theorem [25] classifies the finite primitive permutation groups by dividing them into classes, according to the structure of their minimal normal subgroups. An important result in this classification is that every permutation group admits at most two distinct transitive minimal normal subgroups [8, Lemma 5.1].

A permutation group is **quasiprimitive** if all its nontrivial normal subgroups are transitive. For example, all primitive permutation groups are quasiprimitive. Finite quasiprimitive groups were characterized by Cheryl Praeger [28], who showed that they can be classified similarly to the O'Nan-Scott classification of finite primitive permutation groups.

The **inclusion problem** for a permutation group H asks to determine the possible (primitive or quasiprimitive) subgroups of the symmetric group that contain H. In other words, given a permutation group $H < \mathrm{Sym}(\Omega)$, we are asking about its overgroups. For instance, it is a common situation in algebraic combinatorics that we know a part of the group of automorphisms of a combinatorial structure (for example, a Cayley graph) and we wish to determine a larger automorphism group which may be primitive or quasiprimitive.

In this work we describe all inclusions $H \leq G$ such that H is a transitive nonabelian characteristically simple group and G is a finite primitive or quasiprimitive permutation group with nonabelian socle. The study of these inclusions is possible since we have detailed information concerning **factorizations** of finite nonabelian simple groups. For this reason, many of the results presented here rely on the classification of finite simple groups, specially chapters 4 and 7.

Resumo

Seja Ω um conjunto finito não vazio e considere G um grupo de permutações de Ω, isto é, $G \leq \mathrm{Sym}(\Omega)$. Dizemos que G é **transitivo em** Ω se dado qualquer par de pontos em Ω, existir uma permutação em G que associa estes pontos. Por outro lado, dizemos que G é **primitivo em** Ω se G for transitivo e não preservar uma partição não trivial de Ω – por partição trivial queremos dizer a partição formada por subconjuntos de um único ponto e a partição composta somente por Ω. Por exemplo, o grupo simétrico e o grupo alternado de Ω em suas ações naturais são ambos primitivos.

O Teorema de O'Nan-Scott classifica os grupos de permutações primitivos finitos dividindo-os em classes, de acordo com a estrutura de seus subgrupos normais minimais [25]. Um importante resultado nesta classificação é que todo grupo de permutações admite no máximo dois subgrupos normais minimais transitivos distintos [8, Lemma 5.1].

Um grupo de permutações é dito **quase-primitivo** se todos os seus subgrupos normais não triviais são transitivos. Por exemplo, todo grupo de permutações primitivo é quase-primitivo, mas visto que um grupo simples transitivo é sempre quase-primitivo, mas nem sempre primitivo, a classe dos grupos quase-primitivos é estritamente maior que a classe dos grupos primitivos.

Grupos de permutações quase-primitivos finitos foram caracterizados por Praeger em [28]. Neste artigo Praeger mostrou que os grupos quase-primitivos podem ser classificados similarmente à classificação dos grupos primitivos.

O problema de inclusão para um grupo de permutações H almeja determinar os possíveis subgrupos (quase-primitivos ou primitivos) do grupo simétrico que contenham H. Tal problema possui um número importante de aplicações em teoria de grupos, combinatória algébrica e teoria algébrica de grafos. Por exemplo, é uma situação comum em combinatória algébrica sabermos uma parte do grupo de automorfismos de uma estrutura combinatória, por exemplo um grafo de Cayley, e queremos determinar um grupo de automorfismos maior que pode

ser quase-primitivo ou primitivo.

O estudo do problema de inclusão é possível visto que há informações detalhadas sobre as fatorações de grupos simples não abelianos finitos. Alguns resultados gerais sobre tais fatorações podem ser encontrados em [1, 26].

Neste trabalho descrevemos todas as inclusões $H \leq G$ tais que H é um grupo caracteristicamente simples, não abeliano e transitivo, e G é um grupo de permutações finito quase-primitivo com socle não abeliano. A este tipo de inclusão damos o nome de inclusão CharS-QP.

Tais inclusões ocorrem naturalmente, por exemplo quando tomamos um grupo quase-primitivo finito G que possui um subgrupo normal minimal não abeliano S. Se denotarmos por $\mathrm{soc}(G)$ o socle de G, tanto $S \leq G$ quanto $\mathrm{soc}(G) \leq G$ são inclusões CharS-QP.

Ao tratar esse problema, nosso primeiro passo foi imergir explicitamente, sob algumas hipóteses, um grupo de permutações quase-primitivo em um produto entrelaçado com a ação produto, de forma que tal imersão fosse permutacional.

Teorema 1. *(Teorema de Imersão) Sejam G um grupo de permutações quase-primitivo em Ω e $\alpha \in \Omega$. Assumamos as seguintes condições:*

1. *$S = Q_1 \times \cdots \times Q_r$ é um subgrupo normal minimal de G, em que cada Q_i é caracteristicamente simples e não abeliano, e $r \geq 2$.*

2. *G age transitivamente em $\{Q_1, \ldots, Q_r\}$ por conjugação.*

3. *Consideremos as projeções $\pi_i \colon S \to Q_i$, e assumamos que*

$$S_\alpha = (S_\alpha \pi_1) \times \cdots \times (S_\alpha \pi_r).$$

Se considerarmos $\Gamma := [Q_1 \colon (Q_1)_\alpha]$, então existe uma imersão permutacional $\psi \colon G \hookrightarrow \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_r$, em que consideramos o produto entrelaçado como um grupo de permutações agindo com a ação produto em Γ^r.

O resultado acima é um dos pontos-chave para demonstrar nosso segundo resultado.

Teorema 2. *(Teorema principal) Seja $H \leq G$ uma inclusão* CharS-QP*, tal que $\mathrm{soc}(G)$ é não abeliano. Então $H \leq \mathrm{soc}(G)$.*

Com o intuito de entender melhor as inclusões do teorema principal, atacamos o problema analisando separadamente cada uma das classes de O'Nan-Scott (Capítulo 7). Tais resultados dependem fortemente da fatoração de grupos simples, bem como do teorema de classificação dos grupos simples finitos.

Contents

Notation

$\mathrm{Sym}(\Omega)$	Symmetric group on a set Ω
$\mathrm{Alt}(\Omega)$	Alternating group on a set Ω
S_n	Symmetric group on $\{1, \ldots, n\}$
A_n	Alternating group on $\{1, \ldots, n\}$
$\underline{n}$	Short notation for $\{1, \ldots, n\}$
α^g	Image of α under g
α^G	Orbit of α under G
G_α	Stabilizer of α under G
G_Δ	Setwise stabilizer of Δ under G
$G_{(\Delta)}$	Pointwise stabilizer of Δ under G
G^Δ	Permutation group induced by G_Δ on Δ
$G^{(j)}$	j-th component of G
$\ker \mu$	Kernel of μ
$\mathrm{Im}\, \mu$	Image of μ
$\|X\|$	Number of elements of a set X
$C_G(H)$	Centralizer of H in G
$N_G(H)$	Normalizer of H in G
$[G : H]$	Set of the cosets of H in G
$\|G : H\|$	Number of cosets of H in G
$\langle S \rangle$	Subgroup of G generated by S
Ω^n	n-th Cartesian power of Ω
G^n	The direct product of n copies of a group G
$\mathrm{Aut}(\mathcal{G})$	Automorphism group of the graph $\mathcal{G}$
$\mathrm{Aut}(G)$	Automorphism group of a group G
$\mathrm{Inn}(G)$	Group of inner automorphisms of G
$\mathrm{Out}(G)$	Group of outer automorphisms of G
$K \rtimes H$	Semidirect product of K and H
$G \,\mathrm{wr}\, H$	Wreath product of G and H

$\mathrm{Hol}(G)$	The semidirect product $G \rtimes \mathrm{Aut}(G)$
$\mathrm{soc}(G)$	Socle of G
$\mathcal{E}$	Cartesian decomposition
$G_{(\mathcal{E})}$	Pointwise stabilizer of $\mathcal{E}$ under G
π_i	i-th projection
$\mathrm{GL}_d(q)$	General linear group
$\mathrm{PSL}_d(q)$	Projective special linear group
$\mathrm{AGL}_d(q)$	Affine general linear group
$\mathrm{Sp}(d,q)$	Symplectic group
$\mathrm{PSp}(d,q)$	Projective symplectic group
M_{11}, M_{12}	Mathieu groups
$O_d(q)$, $O_d^+(q)$, $O_d^-(q)$	Orthogonal groups
$\Omega_d(q)$, $\Omega_d^+(q)$, $\Omega_d^-(q)$	Ω-groups
$\mathrm{P}\Omega_d(q)$, $\mathrm{P}\Omega_d^+(q)$, $\mathrm{P}\Omega_d^-(q)$	Projective Ω-groups
$G_2(q)$	Exceptional group of Lie type G_2
C_n	Cyclic group with order n
$\mathrm{Sp}(d,q)\cdot 2$	Extension of $\mathrm{Sp}(d,q)$ by C_2
$\mathrm{PSp}(d,q)\cdot 2$	Extension of $\mathrm{PSp}(d,q)$ by C_2

Chapter 1

Setting the scene

1.1 Conventions

Throughout this work, groups will be labeled with capital Roman letters and group elements will be written in lower case Roman letters. The only exception will be when working with permutations, which will be labeled with lower case Greek letters.

Sets on which groups act will be written in capital Greek letters and their elements will be written, exclusively, in lower case Greek letters α, β, γ, δ and ω. Other lower case Greek letters will denote functions. The letter π will always represent a projection.

The group of all permutations of a set Ω is the **symmetric group**, denoted by $\mathrm{Sym}(\Omega)$. A **permutation group** of Ω is a subgroup of $\mathrm{Sym}(\Omega)$.

We will use exponent notation for group actions, that is, given a point $\alpha \in \Omega$, we denote by α^g the image of α under the action of a group element g. Further,

$$\alpha^G := \{\alpha^g \colon g \in G\}$$

denotes the **orbit** and

$$G_\alpha := \{g \in G \colon \alpha^g = \alpha\}$$

denotes the **stabilizer** of a point α under a group G.

Let G act on Ω. If $\Delta \subseteq \Omega$, we define the **setwise** and the **pointwise stabilizer** of Δ in G as

$$G_\Delta := \{g \in G \colon \Delta^g = \Delta\},$$

$$G_{(\Delta)} := \{g \in G\colon \delta^g = \delta \text{ for all } \delta \in \Delta\},$$

respectively. If $G_\Delta = G$, that is, $\Delta^g = \Delta$ for all $g \in G$, then Δ is said to be G-**invariant**. The quotient $G_\Delta/G_{(\Delta)}$ is denoted by G^Δ, and is viewed as a permutation group on Δ in the natural way. Further, if $\mathcal{P}$ is a partition of Ω, we say that $\mathcal{P}$ is G-**invariant** if $\Gamma^g \in \mathcal{P}$ for all $\Gamma \in \mathcal{P}$. In this case we also say that G **preserves** $\mathcal{P}$. We say that $\mathcal{P}$ is **uniform** if all its parts have the same size.

Given any function f and a point α in its domain, we will denote by αf the image of α under f. Given a natural number n, the set $\{1, \ldots, n\}$ will be denoted by $\underline{n}$.

Given an action of G on a set Ω, we can obtain a corresponding representation $\mu\colon G \to \mathrm{Sym}(\Omega)$ via $\alpha^{g\mu} := \alpha^g$. On the other hand, given a representation $\mu\colon G \to \mathrm{Sym}(\Omega)$, we can define an action of G on Ω via $\alpha^g := \alpha^{g\mu}$. Thus, we will use freely expressions like "this action is faithful" meaning that the corresponding representation is faithful.

We will use indiscriminately some basic results in Group Theory, such as Isomorphisms Theorems, Correspondence Theorem, Jordan-Hölder Theorem. Excellent books about these topics are [35, 21]. However, the last one is in Portuguese.

In all this work we deal with finite quasiprimitive groups only, but some of the results are true also for infinite permutation groups. If this is the case, we will indicate that explicitly.

1.2 Introduction

Let G be a group acting on a set Ω. We say that G is **transitive** on Ω if given two points $\alpha, \beta \in \Omega$, there is an element $g \in G$ such that $\alpha^g = \beta$. This is equivalent to saying that G has just one orbit on Ω, that is, $\alpha^G = \Omega$ for all $\alpha \in \Omega$. As a simple example, you can consider the action of G on itself by **right multiplication**, that is, $g^h := gh$ for all $g, h \in G$. This action is clearly transitive, since fixed $g, h \in G$, we obtain that $g^{(g^{-1}h)} = h$. If G is not transitive, then it is called **intransitive**.

For a set Ω, we say that the partitions $\{\Omega\}$ and $\{\{\alpha\}\colon \alpha \in \Omega\}$ are the **trivial partitions** of Ω. If G is a transitive permutation group on Ω that preserves a nontrivial partition of Ω, then we say that G is **imprimitive**. Otherwise, G is **primitive**. For example, the symmetric and the alternating groups in their natural actions are both primitive.

Let G be a group and let H be a subgroup of G. Then H is said to be a **minimal normal subgroup** of G if H is a normal subgroup of G and the only normal subgroup of G properly contained in H is the identity subgroup.

The O'Nan-Scott Theorem classifies the finite primitive permutation groups by dividing them into classes, according to the structure of their minimal normal subgroups [25]. An important result in this classification is that every permutation group admits at most two distinct transitive minimal normal subgroups [8, Lemma 5.1].

A permutation group is **quasiprimitive** if all its nontrivial normal subgroups are transitive. For example, all primitive permutation groups are quasiprimitive (Lemma 2.5.2), but, since a transitive simple group is quasiprimitive, but not always primitive, the class of quasiprimitive permutation groups is strictly larger than the class of primitive permutation groups.

The class of quasiprimitive groups is often more suitable for combinatorial and graph theoretic applications than the class of primitive groups. For instance, quasiprimitive groups play a central role in understanding the structure of finite *non-bipartite 2-arc-transitive graphs* [28, 24, 30].

Finite quasiprimitive groups were characterized by Cheryl Praeger [28], who showed that they can be classified similarly to the O'Nan-Scott classification of finite primitive permutation groups. See Section 2.5 to see this classification.

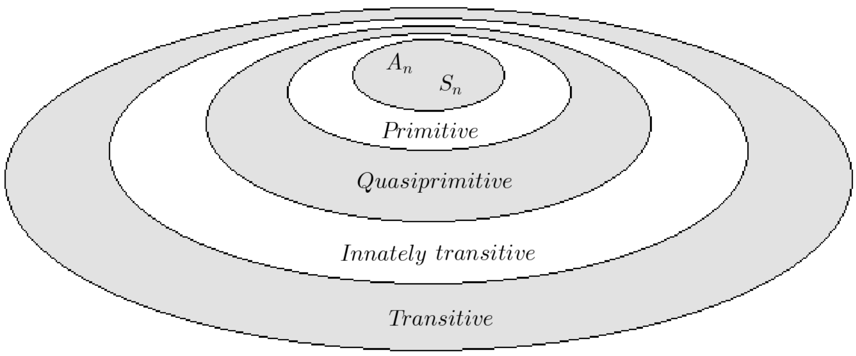

Figure 1.1: Subclasses of transitive groups.

Similarly to what was done for finite primitive and quasiprimitive permutation groups, Bamberg and Praeger [8] noticed that it was possible to describe, using an O'Nan-Scott classification, a strictly larger class of transitive permutation groups, the so-called innately transitive groups. We say that a permutation group is **innately transitive** if it has a transitive minimal normal subgroup, which is called a **plinth** of the group. It follows from the definition that any finite quasiprimitive group is innately transitive. The authors gave in [8, Proposition 5.3] necessary and sufficient conditions for an innately transitive group to be quasiprimitive.

The **inclusion problem** for a permutation group H asks to determine the possible (primitive or quasiprimitive) subgroups of the symmetric group that contain H. In other words, given a permutation group $H < \mathrm{Sym}(\Omega)$, we are asking about its overgroups.

The inclusion problem has a number of important applications in group theory and in algebraic combinatorics For instance, it is a common situation in algebraic combinatorics that we know a part of the group of automorphisms of a combinatorial structure (for example, a Cayley graph) and we wish to determine a larger automorphism group which may be primitive or quasiprimitive. For example in [13], Fang, Praeger and Wang described, for a finite nonabelian simple group G and for $\mathcal{G}$ a connected undirected Cayley graph for G, all the possible structures for the full automorphism group $\mathrm{Aut}(\mathcal{G})$. This result relies on a detailed study of the inclusion problem for G in $\mathrm{Sym}(G)$ under its Cayley representation. In turn, the study of this inclusion problem is possible since we have detailed information concerning factorizations of finite nonabelian simple groups.

If A and B are proper subgroups of a group G such that $G = AB$, then we call this expression a **factorization** of G. Some general results on factorizations of almost simple groups can be found in [26, 1].

Using the *blow-up construction* of a primitive group from smaller primitive groups introduced by Kovács, Praeger [27] solved the inclusion problem for finite primitive permutation groups by analyzing the O'Nan-Scott classes. Her paper contains a detailed description of all pairs (H, G) of finite primitive permutation groups such that $H \leq G$. The main result of this paper, which is too complex to reproduce here as a single theorem, states that each such inclusion is either natural, exceptional, a blow-up of exceptional inclusions, or a composition of such a blow-up and a natural inclusion.

The more general problem of describing inclusions $H \leq G$ in the case when either H is quasiprimitive or both H and G are quasiprimitive was addressed by [2] and [29]. The philosophy of the results in [2, 29] is similar to the results in [27], but the variety of possible inclusions is even richer than in the case when both H and G are primitive.

Several papers [10, 16, 22, 23, 3] addressed the inclusion problem in the special case when the overgroup G is a member of a particular O'Nan-Scott class (see Section 2.5 for the definition of the O'Nan-Scott classes). In a series of articles Baddeley, Praeger, and Schneider [3, 4, 32, 5] described inclusions $H \leq G$ assuming that H is innately transitive and G is a wreath product acting in product action (see Section 5.1 for the definition). In this more specialized case, the conclusions are often rather precise as in the following theorem, for instance. See Section 2.5 for the definition of *socle*.

Theorem 1.2.1. *[3, Theorem 1.1] Let S be a finite almost simple group with socle T such that S is a subgroup of $W := \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_l$ acting in product action on Γ^l, with $|\Gamma|, l \geq 2$. Then one of the following must hold.*

1. *T is intransitive.*
2. *T is isomorphic to one of the groups A_6, M_{12}, $\mathrm{PSp}(4, 2^a)$ or $\mathrm{P}\Omega_8^+(q)$. Further, in this case, $l = 2$ and T is in the base group $(\mathrm{Sym}(\Gamma))^2$ of W.*

In later papers Baddeley, Praeger and Schneider generalized Theorem 1.2.1 and described inclusions of other types of innately transitive and quasiprimitive groups into wreath products in product action. The proof of the following result can be found in [4, 6].

Theorem 1.2.2. *Let S be a quasiprimitive, almost simple permutation group acting on Γ, and for some $l \geq 2$ set $W := S \,\mathrm{wr}\, S_l$ acting on Γ^l in product action. Let U be the unique minimal normal subgroup of S and let $N := U_1 \times \cdots \times U_l \cong U^l$ be the unique minimal normal subgroup of W. Moreover, assume that G is an innately transitive subgroup of W with a nonabelian plinth $M := T_1 \times \cdots \times T_k$, where $T_1, \ldots, T_k$ are finite, nonabelian simple groups all isomorphic to a group T. If $\pi \colon W \to S_l$ is the natural projection map, then*

1. $\mathrm{soc}(G) \leq \mathrm{soc}(W) \cong (\mathrm{soc}(S))^l$.
2. *$G\pi$ has at most two orbits on $\underline{l}$.*
3. *If $G\pi$ has two orbits on $\underline{l}$, then T is isomorphic to one of the groups A_6, M_{12}, $PSp(4, 2^a)$ or $\mathrm{P}\Omega_8^+(q)$.*
4. *The O'Nan-Scott class of G is not* SD.
5. *Assume that $G\pi$ is transitive. Then exactly one of the following holds.*
   (a) *$k = l$: the T_i and U_i can be indexed so that $T_1 \leq U_1, \ldots, T_k \leq U_k$.*

(b) *$l = 2k$: T is isomorphic to one of the groups A_6, M_{12}, $\mathrm{PSp}(4, 2^a)$ or $\mathrm{P}\Omega_8^+(q)$. Moreover, the T_i and U_i can be indexed so that $T_1 \leq U_1 \times U_2$, $T_2 \leq U_3 \times U_4, \ldots, T_k \leq U_{2k-1} \times U_{2k}$.*

(c) *None of the case (a)-(b) holds and $U = \mathrm{Alt}(\Gamma)$.*

1.3 About this thesis

The purpose for this doctoral research is to extend Theorem 1.2.1 for characteristically simple groups. In particular we aim to describe all inclusions $H \leq G$ such that H is a transitive nonabelian characteristically simple group and G is a finite primitive or quasiprimitive permutation group.

Definition 1.3.1. Let (H, G) be a pair of permutation groups on a set Ω. We say (H, G) is a CharS-QP inclusion if $H \leq G$, where H is a transitive nonabelian characteristically simple group, and G is a finite quasiprimitive permutation group. It is called a CharS-P inclusion if, in addition, G is primitive.

We observe that such inclusions occur rather naturally.

Example 1.3.2. If G is a quasiprimitive permutation group and S is a nonabelian minimal normal subgroup of G then S is a transitive nonabelian characteristically simple subgroup of G. Thus the socle $\mathrm{soc}(G)$ of G is also a transitive nonabelian characteristically simple subgroup of G. Hence (S, G) and $(\mathrm{soc}(G), G)$ are CharS-QP inclusions.

Example 1.3.3. Suppose that S is an almost simple group acting on Γ, and let $Q := \mathrm{soc}(S)$. Assume that Q is transitive and hence S is quasiprimitive. Suppose that T is a transitive nonabelian simple subgroup of Q. If $l \geq 2$, then T^l is a transitive characteristically simple subgroup of the quasiprimitive group $Q \,\mathrm{wr}\, S_l$ acting on Γ^l in product action. Hence $(T^l, Q \,\mathrm{wr}\, S_l)$ is a CharS-QP inclusion.

Example 1.3.4. Suppose that G is a quasiprimitive group of Sd type and let S be the the socle of G. Then $S = Q^k$ where Q is a nonabelian simple group. As noted in Example 1.3.2, (S, G) is a CharS-QP inclusion. However, in this case, if Q_i is a simple factor of $S = Q^k$, then $C_S(Q_i) = \prod_{j \neq i} Q_j$ is also a transitive nonabelian characteristically simple subgroup of G. Thus we obtain that $(C_S(Q_i), G)$ is also a CharS-QP inclusion.

Example 1.3.5. Suppose that T is one of the groups listed in part (ii) of Theorem 1.2.1 and consider T as a transitive subgroup of $W_0 := \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_2$. If $k \geq 1$, then T^k is a transitive characteristically simple subgroup of the primitive group $W_0 \,\mathrm{wr}\, S_k$. Thus we obtain the CharS-P inclusion $(T^k, W_0 \,\mathrm{wr}\, S_k)$.

Example 1.3.6. Let G be the affine transitive permutation group $\mathrm{AGL}_3(2)$, that is, $G = {F_2}^3 \rtimes \mathrm{GL}_3(2)$. According to [9], G contains a transitive subgroup T isomorphic to $\mathrm{GL}_3(2) \cong \mathrm{PSL}_3(2)$. Therefore, if $k \geq 1$, then the wreath product $G \,\mathrm{wr}\, S_k$ contains T^k as a transitive subgroup. Hence we obtain the CharS-P inclusion $(T^k, G \,\mathrm{wr}\, S_k)$.

Our objective in the proposed research is two-fold:

1. To find new kinds of CharS-QP and CharS-P inclusions that cannot be obtained from the previous examples.
2. To prove that all CharS-QP inclusions can be obtained by the constructions presented in Examples 1.3.2–1.3.6 and the constructions uncovered in the previous item.

Note that when (H, G) is a CharS-QP inclusion such that H is a minimal normal subgroup in $N := N_G(H)$, then N is an innately transitive subgroup of G and hence the pair (N, G) is described by the earlier work of Baddeley, Praeger, and Schneider. Our contribution to this project is to describe such pairs without assuming that H is a minimal normal subgroup of N.

The first step in the description is the following theorem.

Theorem 1.3.7. *(Main Theorem) Let* (H, G) *be a* CharS-QP *inclusion such that* $\mathrm{soc}(G)$ *is nonabelian. Then* $H \leq \mathrm{soc}(G)$.

In order to prove this theorem and study its consequences, the text of this thesis is divided into six chapters.

In Chapter 2 we recall some basic definitions and results on permutation groups and group actions, and give some results related to coset actions. The key result in this chapter is the O'Nan-Scott Theorem for finite quasiprimitive permutation groups according to the properties of their minimal normal subgroups (Theorem 2.5.4).

In Chapter 3 we have some properties about subgroups of direct products, and we state the Scott's Lemma (Lemma 3.1.4). Next we have results about minimal normal subgroups and stabilizers and about simple groups. Lastly, we give number theoretic results that will be used to prove the Main Theorem and to describe its consequences.

In Chapter 4 we present the concept of *full* and *strong multiple factorizations*, and we give some results regarding group factorizations. The last section presents an algorithm that characterizes the factorizations of finite nonsimple and nonabelian characteristically simple groups into subgroups whose factors are the product of pairwise disjoint full strips.

In Chapter 5 we present the concept of a cartesian decomposition and we state and demonstrate the Embedding Theorem (Theorem 5.4.2). This result says explicitly, under some hypotheses, how to embed a quasiprimitive permutation group into a wreath product in product action.

In Chapter 6 we have the proof of the Main Theorem by analyzing each O'Nan-Scott class.

Chapter 7 applies the Main Theorem to describe the CharS-QP inclusions (H, G) where G has a nonabelian plinth.

1.4 To be continued...

We observe that the Main Theorem says nothing about the groups with abelian socle. So the next natural step is to study these groups, trying to indicate the exceptions that do not satisfy the theorem. In [9] Baumeister determines all the maximal transitive subgroups of the primitive affine permutation groups. Since quasiprimitive groups with an abelian socle are primitive (of type HA), we want to use her description to analyze such groups.

In order to understand better the inclusions in the Main Theorem, another step to do is to describe explicitly the regular groups H that occur in such inclusions. In fact, we already obtained some results for the classes Tw, SD and HS (Chapter 7).

Finally, we plan to apply our results to graph theory, generalizing some results in [13]. The idea is to study quasiprimitive subgroups of $\mathrm{Aut}(\mathcal{G})$, where $\mathcal{G}$ is a connected undirected Cayley graph of a nonabelian group G. However, unlike the article, which treats the case when G is nonabelian and simple, we will study the more general case in which G is nonabelian and characteristically simple.

Chapter 2

About quasiprimitive permutation groups

In this chapter we will present the O'Nan-Scott Theorem for quasiprimitive permutation groups according to the structure of their minimal normal subgroups. However, before stating the theorem, we need some preliminary definitions and results, given in the next sections.

2.1 Transitive groups, stabilizers and blocks

This section contains basic results in the theory of permutation groups and group actions. At the end, we present the close relation between blocks and stabilizers.

Let G be a group acting on a set Ω. We recall that G is **transitive** if for all $\alpha \in \Omega$ we have that $\alpha^G = \Omega$. We say that G is **regular** on Ω if G is transitive and $G_\alpha = 1$ for each $\alpha \in \Omega$. It is easy to prove that for a transitive group the stabilizers form a conjugacy class, so we can simply say that G is regular if G is transitive and $G_\alpha = 1$ for some $\alpha \in \Omega$. Further, a **block** (of imprimitivity) for a transitive group G is a nonempty subset Δ of Ω such that for all $g \in G$ we have either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \varnothing$.

A basic result related to transitive groups is the Orbit-Stabilizer Theorem [12, Theorem 1.4A], which says that given a transitive group G acting on a set Ω, then for each $\alpha \in \Omega$ we have that $|G\colon G_\alpha| = |\Omega|$. In particular, if Ω is finite, then G is regular if, and only if, G is transitive and $|G| = |\Omega|$.

We present below a useful result that relates stabilizers to transitive subgroups.

Lemma 2.1.1. *[12, Exercise 1.4.1] Let G be a transitive group on Ω, Y a subgroup of G and $\alpha \in \Omega$. Then Y is transitive on Ω if, and only if, $G = G_\alpha Y$. In particular, the only transitive subgroup of G that contains G_α is G itself.*

Proof. First we suppose that Y is transitive. Given $g \in G$, we will show that there is $y \in Y$ such that $gy \in G_\alpha$. Consider $\beta := \alpha^g$. Since Y is transitive, there is $x \in Y$ such that $\beta = \alpha^x$. Then

$$\alpha^{gx^{-1}} = (\alpha^g)^{x^{-1}} = \beta^{x^{-1}} = \alpha.$$

So taking $y := x^{-1}$, it follows that $gy \in G_\alpha$. Hence $g = (gy)y^{-1} \in G_\alpha Y$. Then $G = G_\alpha Y$.

Suppose now that $G = G_\alpha Y$. To see that Y is transitive, we will show that $\alpha^Y = \Omega$. So let $\beta \in \Omega$. Since G is transitive, there is $g = sy \in G$, where $s \in G_\alpha$ and $y \in Y$, such that

$$\beta = \alpha^g = \alpha^{sy} = \alpha^y,$$

which means that $\beta \in \alpha^Y$. Since β was arbitrary, we find that $\alpha^Y = \Omega$. Therefore Y is transitive, which concludes the proof. $\square$

Similarly to the Galois Correspondence for field extensions, in a transitive group there is a bijection between its blocks containing a fixed point $\alpha \in \Omega$ and the overgroups of G_α, as shown in the following lemma.

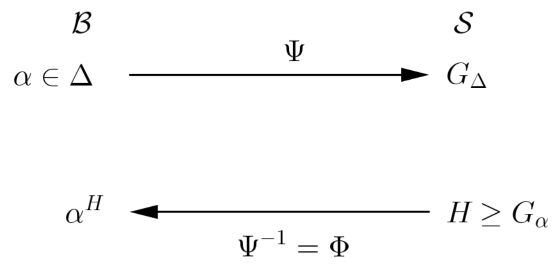

Figure 2.1: Blocks and stabilizers

Lemma 2.1.2. *[12, Theorem 1.5A] Let G be a transitive group on Ω and $\alpha \in \Omega$. Consider $\mathcal{B}$ the set of all blocks Δ for G with $\alpha \in \Delta$, and let $\mathcal{S}$ denote the set of all subgroups H of G with $G_\alpha \leq H$. Then there is a bijection Ψ of $\mathcal{B}$ onto $\mathcal{S}$ given by $(\Delta)\Psi := G_\Delta$, whose inverse mapping Φ is given by $(H)\Phi := \alpha^H$. Furthermore, the mapping Ψ is order-preserving, that is, if $\Delta, \Gamma \in \mathcal{B}$, then $\Delta \subseteq \Gamma$ if, and only if, $(\Delta)\Psi \leq (\Gamma)\Psi$.*

2.2 Equivalent representations

Algebraic structures are often studied up to isomorphism. For many group theoretic applications the concept of isomorphism between abstract groups is sufficiently strong. However, this is not the case for permutation groups. For example, we may consider the symmetric group S_3 acting on $\{1, 2, 3\}$ naturally, but also on $\{1, 2, 3, 4, 5, 6\}$ via its Cayley representation. Since these actions are very different, the resulting permutation groups must be considered as distinct. In order to make this distinction between permutation groups, we need the following definition.

Definition 2.2.1. Let G be a permutation group on a set Ω and let H be a permutation group on a set Ω'. Then G and H are **permutationally isomorphic** if there is an isomorphism $\psi\colon G \to H$ and a bijection $\lambda\colon \Omega \to \Omega'$ such that, for all $g \in G$ and $\alpha \in \Omega$, we have $(\alpha^g)\lambda = (\alpha\lambda)^{g\psi}$.

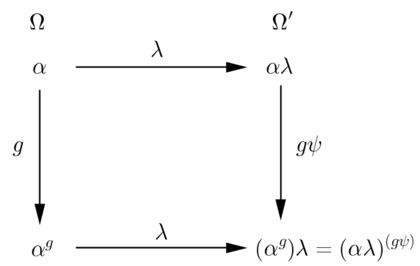

Figure 2.2: Permutational isomorphism from G on Ω to H on Ω'.

Essentially this means that the groups are "the same" except for the labeling of the points.

Analogously, given a group G acting on two sets Ω and Ω', we can compare the actions. That is, we say that these actions are **equivalent** if there is a bijection $\lambda\colon \Omega \to \Omega'$ such that, for all $g \in G$ and $\alpha \in \Omega$, we have $(\alpha^g)\lambda = (\alpha\lambda)^g$, with the respective actions. On the other hand, given a bijection between two sets, if a group acts on the first set, the following result says that the group acts on the second one too, and the actions are equivalent.

Lemma 2.2.2. *Let $\lambda\colon \Omega \to \Omega'$ be a bijection between two sets Ω and Ω', and suppose that G is a group acting on Ω. Then G has an action on Ω' given by*

$$\omega^g := (\omega\lambda^{-1})^g\lambda, \tag{2.1}$$

where $\omega \in \Omega'$ and $g \in G$. Furthermore, the actions of G on Ω and Ω' are equivalent.

Proof. Clearly $\omega^g \in \Omega'$. In order to verify that the relation above defines an action, we consider $g, h \in G$ and $\omega \in \Omega'$. Then

$$\omega^1 = (\omega\lambda^{-1})\lambda = \omega,$$

$$(\omega^g)^h = (\omega^g\lambda^{-1})^h\lambda = (((\omega\lambda^{-1})^g\lambda)\lambda^{-1})^h\lambda = (\omega\lambda^{-1})^{gh}\lambda = \omega^{gh}.$$

This shows that (2.1) defines an action. Now, to see that the actions of G on Ω and Ω' are equivalent, let $\alpha \in \Omega$ and $g \in G$. We will show that $(\alpha^g)\lambda = (\alpha\lambda)^g$. From (2.1) we have that

$$(\alpha\lambda)^g = ((\alpha\lambda)\lambda^{-1})^g\lambda = (\alpha^g)\lambda.$$

Therefore the actions are equivalent, which concludes the proof. □

In other words, the result above says that if $\lambda\colon \Omega \to \Omega'$ is a bijection and $\mu : G \to \mathrm{Sym}(\Omega)$ is a representation of G on Ω, we can extend λ to obtain an isomorphism $\lambda'\colon \mathrm{Sym}(\Omega) \to \mathrm{Sym}(\Omega')$, defined by $\sigma \mapsto \lambda^{-1}\sigma\lambda$, in a way that $\mu\lambda'\colon G \to \mathrm{Sym}(\Omega')$ is a representation of G on Ω' equivalent to μ. Therefore we conclude that the following diagram is commutative.

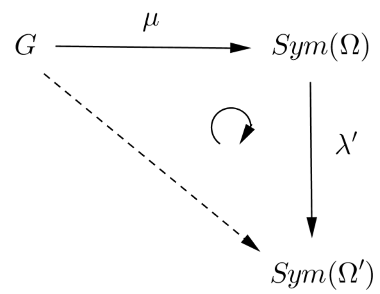

Figure 2.3: Equivalent actions of G

When a group G has two transitive actions, there is a simple method to verify if the actions are equivalent, as shown in the result below.

Lemma 2.2.3. *[12, Lemma 1.6B]) Let G be a transitive group on two sets Ω and Ω', and let P be a stabilizer of a point in the first action. Then the actions of G on Ω and Ω' are equivalent if, and only if, P is the stabilizer of some point in the second action.*

2.3 Coset actions

Let G be a group and $K \leq G$. There is a transitive action on the right cosets of K in G defined by $(Kg_1)^{g_2} := Kg_1g_2$, for all $g_1, g_2 \in G$. We call this the **right coset action** of G on K. In this section we present a result that gives us an action that extends the right coset action of a normal subgroup to the whole group. This result is a slight adaptation of Lemma 4.8 in [8], of Bamberg and Praeger. We will use it later to prove the Embedding Theorem (Theorem 5.4.2), and despite the statement being technical, there is a particular case given by Corollary 2.3.2 which interests us immediately and that explains in part our approach.

Lemma 2.3.1. *Let G be a group and let K and Y be subgroups of G such that $K \trianglelefteq G$ and $G = KY$. Further, suppose that P is a subgroup of K normalized by Y such that $K \cap Y \leq P$. Then G has an action on the coset space $\Omega' := [K\colon P]$ given by*

$$(Px)^{ky} := P(y^{-1}xky), \tag{2.2}$$

where $x, k \in K$ and $y \in Y$. Moreover, $G_P = PY$. Thus the action of G extends the action of K on Ω' via right multiplication.

Proof. First we will show that relation (2.2) is independent of the coset representatives. Consider elements $x_1, x_2, k_1, k_2 \in K$ and $y_1, y_1 \in Y$ such that

$$Px_1 = Px_2, \tag{2.3}$$

$$k_1y_1 = k_2y_2. \tag{2.4}$$

We have to verify that $P(y_1^{-1}x_1k_1y_1) = P(y_2^{-1}x_2k_2y_2)$, which is equivalent to show that $(y_1^{-1}x_1k_1y_1)(y_2^{-1}x_2k_2y_2)^{-1} \in P$. But from (2.4) it follows that

$$\begin{aligned}(y_1^{-1}x_1k_1y_1)(y_2^{-1}x_2k_2y_2)^{-1} &= y_1^{-1}x_1(k_2y_2)y_2^{-1}k_2^{-1}x_2^{-1}y_2\\ &= y_1^{-1}x_1x_2^{-1}y_2\\ &= (y_1^{-1}y_2)(y_2^{-1}(x_1x_2^{-1})y_2). \end{aligned} \tag{2.5}$$

Observe that we divided the expression above into two factors: $y_1^{-1}y_2$ and $y_2^{-1}(x_1x_2^{-1})y_2$. We work with the first factor. Since Y normalizes $K \cap Y$ and by (2.4) $y_1y_2^{-1} \in K$, we have that

$$(y_1^{-1}y_2)^{-1} = y_1^{-1}(y_1y_2^{-1})y_1 \in K \cap Y \leq P,$$

that is, $y_1^{-1}y_2 \in P$. We analyze now the second factor. Since Y normalizes P, and $x_1x_2^{-1} \in P$ by (2.3), we have that $y_2^{-1}(x_1x_2^{-1})y_2 \in P$. Thus both factors of (2.5) are elements of P, and we conclude that (2.2) is independent of the coset representatives.

We show now that (2.2) indeed defines an action. Clearly the identity fixes all the elements of $[K\colon P]$. Let $x, k_1, k_2 \in K$ and $y_1, y_2 \in Y$. Then

$$(k_1y_1)(k_2y_2) = (k_1y_1k_2y_1^{-1})(y_1y_2),$$

where the first factor is an element of K and the second belongs to Y. Therefore,

$$(Px)^{(k_1y_1)(k_2y_2)} = P(y_1y_2)^{-1}x(k_1y_1k_2y_1^{-1})(y_1y_2).$$

Also we have

$$\begin{aligned}[(Px)^{k_1y_1}]^{k_2y_2} &= [P(y_1^{-1}xk_1y_1)]^{k_2y_2} \\ &= P[y_2^{-1}(y_1^{-1}xk_1y_1)k_2y_2] \\ &= P(y_1y_2)^{-1}x(k_1y_1k_2y_1^{-1})(y_1y_2).\end{aligned}$$

Then

$$[(Px)^{k_1y_1}]^{k_2y_2} = (Px)^{(k_1y_1)(k_2y_2)},$$

which shows that (2.2) defines an action.

In order to see that $G_P = PY$, consider $g = ky \in G$. We have that $g = ky$ and $P^g = P(y^{-1}ky)$. So $g \in G_P$ if, and only if, $P(y^{-1}ky) = P$, that is, if $y^{-1}ky \in P$. Since Y normalizes P, it is equivalent to say that $k \in yPy^{-1} = P$. Therefore, $g = ky \in G_P$ if and only if $k \in P$, which means that $G_P = PY$. By Dedekind's Modular Law [35, Proposition 1.3.14],

$$G_P \cap K = PY \cap K = P(Y \cap K) = P.$$

Hence the stabilizer in K of this action is P. Thus the G-action so defined extends the K-action on Ω'. This completes the proof. □

Corollary 2.3.2. *Let G be a group acting on Ω. Consider $\alpha \in \Omega$ and let S be a transitive normal subgroup of G. Thus writing $G = SG_\alpha$, we have that G has a transitive action on $\Omega' := [S\colon S_\alpha]$ given by*

$$(S_\alpha x)^{sy} := S_\alpha(y^{-1}xsy), \tag{2.6}$$

where $x, s \in S$ and $y \in G_\alpha$. That is, first S acts via right multiplication and then G_α acts via conjugation. Moreover, the G-actions on Ω and on Ω' are equivalent.

Proof. Applying Lemma 2.1.1, the transitivity of S allows us to write $G = SG_\alpha$. So to see that the relation above defines an action, it is sufficient to apply Lemma 2.3.1 considering $K = S$, $Y = G_\alpha$ and $P = S_\alpha = S \cap G_\alpha$. This action is transitive because the action of S via right multiplication is transitive on Ω'. Therefore, G is transitive on Ω'.

Hence we have that both actions of G are transitive. Thus, to show the equivalence between them, our strategy will be to use Lemma 2.2.3. Consider $P \in \Omega'$. We will show that, with the respective actions, $G_\alpha = G_P$. However, from the previous lemma, we obtain that $G_P = PY = S_\alpha G_\alpha = G_\alpha$, which completes the proof. □

2.4 Automorphism groups and the holomorph

Let S be a group. We denote by $\mathrm{Aut}(S)$ the group of all automorphisms of S, by $\mathrm{Inn}(S)$ the group of inner automorphisms of S, while $\mathrm{Out}(S)$ denotes the group $\mathrm{Aut}(S)/\mathrm{Inn}(S)$ of outer automorphisms.

The **holomorph** of S is the semidirect product of S with $\mathrm{Aut}(S)$:

$$\mathrm{Hol}(S) := S \rtimes \mathrm{Aut}(S),$$

where the action of $\mathrm{Aut}(S)$ on S is the natural one. We call S the **base group** of the holomorph.

The point we want to emphasize is that the holomorph can be viewed as a permutation group on its base group. In fact, the following action

$$x^{(s,\varphi)} := (xs)\varphi,$$

where $x, s \in S$ and $\varphi \in \mathrm{Aut}(S)$, is faithful. We call this the **base group action** of the holomorph.

The next result gives us a relation between the holomorph and the group of outer automorphisms.

Lemma 2.4.1. *Let S be a group. Then*

$$\frac{\mathrm{Hol}(S)}{S \rtimes \mathrm{Inn}(S)} \cong \mathrm{Out}(S).$$

Proof. Consider the homomorphism below.

$$\begin{array}{rccc} \varphi\colon & \mathrm{Hol}(S) & \to & \mathrm{Out}(S) \\ & (s,\sigma) & \mapsto & \mathrm{Inn}(S)\sigma, \end{array}$$

where $s \in S$ and $\sigma \in \mathrm{Aut}(S)$. Clearly φ is surjective. Still, $\ker\varphi = S \rtimes \mathrm{Inn}(S)$. Therefore, by the isomorphism theorem, we have that

$$\frac{\mathrm{Hol}(S)}{S \rtimes \mathrm{Inn}(S)} \cong \mathrm{Out}(S),$$

which completes the proof. □

2.5 The O'Nan-Scott Theorem

A **minimal normal subgroup** of a nontrivial group G is a nontrivial normal subgroup of G which does not contain properly any other nontrivial normal subgroup of G. A group G is a **characteristically simple group** if it has no proper and nontrivial characteristic subgroups; that is, its only subgroups invariant under $\mathrm{Aut}(G)$ are $\{1\}$ and G itself. In particular, when G is finite, this is equivalent to saying that G is a direct product of isomorphic simple groups [35, 3.3.15]. For example, any minimal normal subgroup S of a group G is characteristically simple, since every characteristic subgroup of S is a normal subgroup of G. The **socle** of a group G is the subgroup generated by the set of all minimal normal subgroups of G, and it is denoted by $\mathrm{soc}(G)$. In case G has no minimal normal subgroups, for instance G is an infinite cyclic group, by convention $\mathrm{soc}(G) = 1$.

Let G be a transitive permutation group on a set Ω. Recall that blocks were defined in Section 2.1. We say that G is **primitive** on Ω if its only blocks are Ω and the singleton sets $\{\alpha\}$ for $\alpha \in \Omega$. It is well known that this is equivalent to saying that all of the point-stabilizers are maximal subgroups of G [12, Corollary 1.5A]. In particular, by Lemma 2.1.2, a regular permutation group $G \leq \mathrm{Sym}(\Omega)$ is primitive if, and only if, $|\Omega|$ is prime. We say that G is **imprimitive** if it is not primitive.

If $\Delta \subseteq \Omega$ is a block for G, then the set $\{\Delta^g : g \in G\}$ is a partition of Ω called a **block system**. A converse of this statement is also true. A partition $\mathcal{P}$ of Ω is G**-invariant** if, for every $\Delta \in \mathcal{P}$ and $g \in G$, we have that $\Delta^g \in \mathcal{P}$. If a partition $\mathcal{P}$ is G-invariant, then any member of $\mathcal{P}$ is a block for G. So an imprimitive action of a group can be thought of as an action which leaves invariant a nontrivial partition of the set it is acting on.

A larger class of groups, which includes the primitive ones, is the class of quasiprimitive groups.

Definition 2.5.1. Let G be a permutation group on Ω. We say that G is **quasiprimitive** if all its nontrivial normal subgroups are transitive. In the particular case when Ω is finite, it is equivalent to saying that all the minimal normal subgroups of G are transitive. We say that G is **innately transitive** if it has a transitive minimal normal subgroup. Such a transitive minimal normal subgroup is called a **plinth** of G.

We prove below that every primitive group is quasiprimitive. Furthermore, it follows from the above definition that any finite quasiprimitive group is innately transitive.

Lemma 2.5.2. *Any primitive permutation group is quasiprimitive.*

Proof. Let G be a primitive permutation group, and consider H a nontrivial normal subgroup of G. Since the orbits of H form a block system for G, it follows from the primitivity that H is transitive or H lies in the kernel of the action. Since the kernel is trivial, we obtain that H is transitive. Therefore, we conclude that all nontrivial normal subgroups of G are transitive, which means that G is quasiprimitive. □

Example 2.5.3.

1. Consider $G = A_5$ and let P be a 5-Sylow subgroup of G. Then G acts on $\Omega := [G : P]$ by right coset action. Thus G is quasiprimitive and $\mathrm{soc}(G) = G$. Since P is the stabilizer G_P, and P is contained in a dihedral subgroup of order 10, then P is not a maximal subgroup of G, so G is imprimitive. Therefore, this is an example of an imprimitive quasiprimitive permutation group.

2. Consider $P \cong C_5 \leq A_5$ and let $G = A_5 \times P$ acting on $\Omega := A_5$ such that $x^{(s,p)} := s^{-1}xp$ for all $s, x \in A_5$ and $p \in P$. Then P is an intransitive normal subgroup of G but G contains a transitive minimal normal subgroup isomorphic to A_5. Therefore, G is an example of an innately transitive group which is not quasiprimitive.

It is well known [8, Lemma 5.1] that a permutation group G has at most two transitive minimal normal subgroups, and in case it has two, S and $\overline{S}$, they are isomorphic and centralize each other, then we can write

$$\mathrm{soc}(G) = S \times \overline{S} = S \times C_G(S).$$

So as for the primitive groups (see [25] or [27]), there is an O'Nan-Scott Theorem for finite quasiprimitive groups, dividing them into eight distinct classes according to their socles: HA, HS, HC, As, Tw, SD, CD and PA. The description we give here is the same as the one presented in [2]. For explicit examples, please consult [7, Section 3.3].

The first three types below are necessarily primitive, and they are permutationally isomorphic to primitive subgroups of the holomorphs of a plinth S of G considered as a permutation group on S via the base group action, and they contain the socle of the holomorph. These are:

HA: Certain subgroups of the *holomorph* of an *abelian* group; these have a unique minimal normal subgroup S (namely the base group of the holomorph), and S is both regular and abelian.

HS: Certain subgroups of the *holomorph* of a nonabelian *simple* group; these have precisely two minimal normal subgroups S and $\bar{S}$ (namely the base group of the holomorph and its centralizer), and both S and $\bar{S}$ are regular, nonabelian and simple.

HC: Certain subgroups of the *holomorph* of a *composite* nonabelian characteristically simple group; these have precisely two minimal normal subgroups S and $\bar{S}$ (namely the base group of the holomorph and its centralizer), and both S and $\bar{S}$ are regular, nonabelian but are not simple.

The five remaining types correspond to quasiprimitive permutation groups that may be primitive or imprimitive. We have:

As: An *almost simple* group; such groups have a unique minimal normal subgroup S that is nonabelian and simple. Here S can be regular or not, but if it is then G must be imprimitive.

Tw: A *twisted wreath* product; such a group has a unique minimal normal subgroup S that is nonabelian and regular, but not simple.

Quasiprimitive permutation groups of the three remaining classes have a unique minimal normal subgroup S that is nonabelian, nonregular and nonsimple. These types are distinguished by the nature of a point stabilizer in S which is necessarily nontrivial. For what follows, a **subdirect subgroup** is a subgroup of a direct product that projects surjectively onto each factor.

Sd: A group of *simple diagonal* type; for such a group a point-stabilizer in S is simple and a subdirect subgroup of S.

Cd: A group of *compound diagonal* type; for such a group a point-stabilizer in S is nonsimple and a subdirect subgroup of S.

Pa: A group of *product action* type; for such a group a point-stabilizer in S is not a subdirect subgroup of S and it is nontrivial.

Theorem 2.5.4. *[28, Praeger, 1993] The types of quasiprimitive permutation groups as defined above are disjoint and exhaustive; in other words, the type of any quasiprimitive permutation group is defined above and is unique.*

Figure 2.4 shows that the classes in the theorem are in fact disjoint. In this figure, each "up arrow" represents an affirmative answer, and each "down arrow" represents a negative answer.

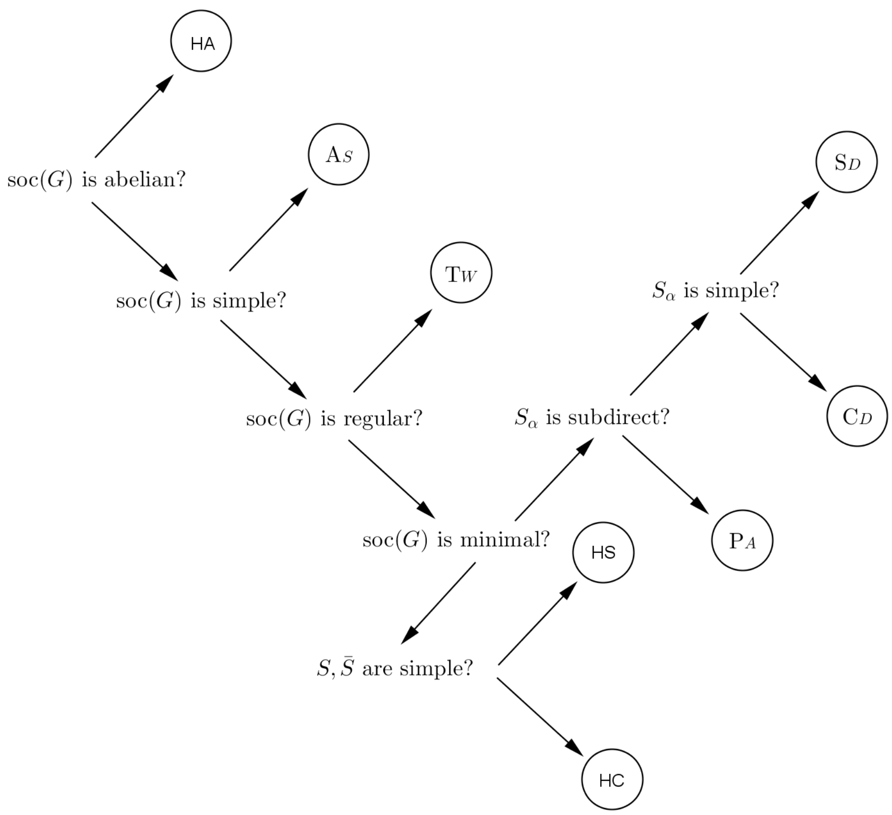

Figure 2.4: O'Nan-Scott classes for quasiprimitive permutation groups

Chapter 3

Tool box

This chapter covers many topics related to permutation groups that will be useful in the next chapters. In particular, we present Scott's Lemma, which characterizes subdirect subgroups of direct products of simple groups. We also give some results about finite simple groups. Finally, the last section is devoted to number theoretic results, such as Legendre's Formula. These results will be applied to embed characteristically simple subgroups in wreath products and to show that the Main Theorem is valid for the O'Nan-Scott classes Sd and Cd.

3.1 Subgroups of direct products

Dealing with direct products, it will be useful to look at their projections on each direct factor. The next definition treats subgroups for which the projections are surjective or injective.

Definition 3.1.1. Let $S = Q_1 \times \cdots \times Q_r$ be a group and consider the projections

$$\begin{array}{cccc} \pi_i & : & S & \to & Q_i \\ & & (q_1, \ldots, q_r) & \mapsto & q_i, \end{array}$$

where $i \in \underline{r}$. Given a subgroup P of S, we say that

(i) P is a **strip** of S if $P \neq 1$ and, for each $i \in \underline{r}$, either the restriction of π_i to P is injective or $P\pi_i = 1$. If it is injective, $P\pi_i \cong P$ for some $i \in \underline{r}$, and we say that P **covers** Q_i. We define the **support** of P as

$$\mathrm{supp}(P) := \{Q_i \colon P \text{ covers } Q_i\}.$$

(ii) A strip P of S is said to be **non-trivial** if $|\operatorname{supp}(P)| > 1$.

(iii) A strip P of S is said to be a **full strip** if $P\pi_i = Q_i$ for all $Q_i \in \operatorname{supp}(P)$.

(iv) P is a **subdirect subgroup** of S if $P\pi_i = Q_i$ for each $i \in \underline{r}$.

(v) P is a **diagonal subgroup** of S if the restriction of π_i to P is injective for each $i \in \underline{r}$.

(vi) P is a **full diagonal subgroup** of S if P is both a subdirect and diagonal subgroup of S.

Example 3.1.2. Let G be a group, $1 \neq H < G$ and set $S = G^3$. Suppose that G_1, G_2, G_3 are the internal direct factors of S, so $S = G_1 \times G_2 \times G_3$, and consider the subgroups

$$X = \{(g, g, 1) \colon g \in G\},$$
$$Y = \{(1, 1, g) \colon g \in G\},$$
$$Z = \{(h, h, h) \colon h \in H\}.$$

We have that X, Y and Z are strips of S, where $\operatorname{supp}(X) = \{G_1, G_2\}$, $\operatorname{supp}(Y) = \{G_3\}$ and $\operatorname{supp}(Z) = \{G_1, G_2, G_3\}$, thus X and Z are non-trivial strips of S. In fact, both X and Y are full strips of S, and Z is a diagonal subgroup of S. We have that XY is a full diagonal subgroup of S. Moreover, Z is a full diagonal subgroup of $H^3 < S$.

We observe that in terms of strips, a diagonal subgroup is a strip with full support. The next result states that every strip is a diagonal subgroup in a possibly smaller group. It also justifies the term *diagonal*.

Lemma 3.1.3. *Let $S = Q_1 \times \cdots \times Q_r$ be a group and let π_i be the projection of S onto Q_i. If P is a strip of S with $\operatorname{supp}(P) = \{Q_1, \ldots, Q_r\}$, then for each $i \in \underline{r}$ there is an isomorphism $\varphi_i \colon P\pi_1 \to P\pi_i$ such that*

$$P = \{(a, a\varphi_2, \ldots, a\varphi_r) \colon a \in P\pi_1\}.$$

Proof. Consider the restrictions of the projections $\overline{\pi}_i \colon P \to P\pi_i$. Since P is a diagonal subgroup of S, then each $\overline{\pi}_i$ is an isomorphism. Take $\varphi_i := \overline{\pi}_1^{-1}\pi_i$ and let $X := \{(a, a\varphi_2, \ldots, a\varphi_r) \colon a \in P\pi_1\}$. Given $p \in P$, let $a := p\pi_1$. Then

$$p = (p\pi_1, p\pi_2, \ldots, p\pi_r) = (p\overline{\pi}_1\varphi_1, p\overline{\pi}_1\varphi_2, \ldots, p\overline{\pi}_1\varphi_r) = (a, a\varphi_2, \ldots, a\varphi_r),$$

which shows that $p \in X$. Conversely, given $x = (a, a\varphi_2, \ldots, a\varphi_r) \in X$, where $a \in P\pi_1$, let $p \in P$ such that $a = p\pi_1$. Then for each $i \in \underline{r}$, $p\pi_i = p\overline{\pi}_1\varphi_i = a\varphi_i$. So $x = (p\pi_1, \ldots, p\pi_r) = p$, which shows that $x \in P$. Therefore $P = X$, as required. □

The first part of the following lemma appears in Scott's paper [36, Lemma p. 328], and it is known as Scott's Lemma. It describes the structure of the subdirect subgroups of a direct product of nonabelian simple groups. The second part is a result that can be found in [18, Proposition 5.2.5(i)], and it characterizes the normal subgroups of a direct product of finite nonabelian simple groups.

Lemma 3.1.4. *Consider $S = Q_1 \times \cdots \times Q_r$, where each Q_i is a nonabelian simple group, and let P be a nontrivial subgroup of S.*

1. *If P is a subdirect subgroup of S, then P is the direct product $\prod P_j$ of full diagonal subgroups of subproducts $\prod_{i\in I_j} Q_i$, where $I_j \subseteq \underline{r}$ and the I_j form a partition of $\underline{r}$.*

2. *If P is a normal subgroup of S, then $P = \prod_{j\in J} Q_j$, where J is a subset of $\underline{r}$.*

In case we have a direct product of nonabelian simple groups, it is possible to determine all its minimal normal subgroups, as well all its maximal normal subgroups.

Corollary 3.1.5. *[12, Exercise 4.3.6] Let $S = Q_1 \times \cdots \times Q_r$ be a direct product of nonabelian simple groups. Then $Q_1, \ldots, Q_r$ are the only minimal normal subgroups of S, and the centralizers $C_S(Q_j) = \prod_{i\neq j} Q_i$ are the only maximal normal subgroups of S.*

Proof. First we will prove that $Q_1, \ldots, Q_r$ are the only minimal normal subgroups of S. Since each Q_i is simple, it is clear that Q_i is a minimal normal subgroup of S. To see that they are unique, let P be a minimal normal subgroup of S. Considering that P is normal and applying Lemma 3.1.4 (item 2) for P, we obtain that $P = \prod_{j\in J} Q_j$, where J is a nonempty subset of $\underline{r}$. But since each Q_j is a normal subgroup of S and P is minimal, it follows that $P = Q_{j_0}$ for some $j_0 \in J$. This shows that $Q_1, \ldots, Q_r$ are the only minimal normal subgroups of S.

Now, to see the second part, first notice that since each $S/C_S(Q_j)$ is simple, then each $C_S(Q_j)$ is a maximal normal subgroup of S. To see that they are unique, let H be a maximal normal subgroup of S. Again, because H is normal, then $H = \prod_{j \in J'} Q_j$, where J' is a nonempty subset of $\underline{r}$. Since H is maximal, we have that S/H is simple, and then it is isomorphic to some Q_{j_1}. This means that $H = \prod_{i \neq j_1} Q_i = C_S(Q_{j_1})$. Therefore, we conclude that the centralizers $C_S(Q_j)$ are the only maximal normal subgroups of S. $\square$

Using the lemma above, we obtain the following criterion to decide if a nonabelian normal subgroup is minimal. An alternative proof can be found in [12, Theorem 4.3A].

Corollary 3.1.6. *Let G be a group and $S = Q_1 \times \cdots \times Q_r$ be a normal subgroup of G, where each Q_i is a nonabelian simple group. Then S is a minimal normal subgroup of G if, and only if, G acts transitively on $\Sigma := \{Q_1, \ldots, Q_r\}$ by conjugation.*

Proof. First we note that G acts on $\Sigma := \{Q_1, \ldots, Q_r\}$ by conjugation. In fact, given $g \in G$ and Q_i, since Q_i is simple, we have that ${Q_i}^g$ is a minimal normal subgroup of $S^g = S$. Then, according to the previous corollary, we have that ${Q_i}^g \in \Sigma$, which means that G acts on Σ.

Suppose that S is a minimal normal subgroup of G and let $\Gamma := {Q_1}^G$ be the orbit of Q_1 under G. We will prove that $\Gamma = \Sigma$. Consider the product H of all the elements Q_i in Γ. As the action of G permutes the elements of Γ among themselves, we have that $H \trianglelefteq G$. Thus, since $H \leq S$ and S is minimal, we conclude that $H = S$. This implies that $\Gamma = \Sigma$, that is, G is transitive on Σ.

On the other hand, suppose that G is transitive on Σ. We will prove that S is minimal. Given a nontrivial $H \leq S$ such that $H \trianglelefteq G$, we apply Lemma 3.1.4 (item 2) to obtain that $H = \prod_{i \in I} Q_i$, where I is a subset of $\underline{r}$. Let $\Gamma = \{Q_i \colon i \in I\}$ be the set of the direct factors of H. The normality of H implies that Γ is an orbit to the action of G on Σ. But since G is transitive on Σ, we must have $\Gamma = \Sigma$, which means that $H = S$. Therefore, S is a minimal normal subgroup of G. $\square$

Given a direct product of two groups, the following lemma characterizes its subgroups which are also themselves direct products.

Lemma 3.1.7. *Let $G = G_1 \times G_2$ be the direct product of the groups G_1 and G_2, and consider the projections $\pi_i \colon G \to G_i$. If $H \leq G$, then the following are equivalent:*

1. $H = H\pi_1 \times H\pi_2$.

2. $H\pi_1 \leq H$.

3. $H\pi_2 \leq H$.

Proof. Clearly (1) implies (2) and (3). We prove that (2) implies (1). Since $H \leq H\pi_1 \times H\pi_2$, we just have to show the inclusion $H\pi_1 \times H\pi_2 \leq H$. So given $(a, b), (c, d) \in H$, we have to prove that

$$((a, b)\pi_1, (c, d)\pi_2) = (a, d) \in H.$$

Since $H\pi_1 \leq H$, then $a, c \in H$. Note that

$$(a, d) = (a, b)(c^{-1}a, 1)(a^{-1}, b^{-1})(c, d),$$

where the right side is in H, so $(a, d) \in H$. Then $H\pi_1 \times H\pi_2 \leq H$. Therefore, $H = H\pi_1 \times H\pi_2$ and we conclude that (2) implies (1). The case (3) implies (1) is analogous. Thus the result is proved. □

Lemma 3.1.8. *Let $G = G_1 \times G_2$ acting on Ω and $\alpha \in \Omega$. If G_1 and G_2 are transitive on Ω, then G_α is a subdirect subgroup of G.*

Proof. Consider the projections $\pi_i \colon G \to G_i$. First we prove that $G_\alpha\pi_1 = G_1$. Given $x \in G_1$, since G_2 is transitive, there is $y \in G_2$ such that $\alpha^x = \alpha^y$. Thus $xy^{-1} \in G_\alpha$ and since x was arbitrary, then $G_\alpha\pi_1 = G_1$. Analogously, we prove that $G_\alpha\pi_2 = G_2$. Then G_α is a subdirect subgroup of G and the result is proved. □

3.2 Minimal normal subgroups and stabilizers

The following result gives us a way to determine when a group action is faithful. It will be useful when dealing with groups having just one minimal normal subgroup.

Lemma 3.2.1. *Let G be a group acting on Ω. Consider $\alpha \in \Omega$ and let S be a minimal normal subgroup of G such that $S_\alpha < S$. If K is the kernel of the action of G, then $K \leq C_G(S)$. In particular, if $C_G(S) = 1$, we have that G is faithful on Ω.*

Proof. Since $K \trianglelefteq G$, we have that $K \cap S \leq S$ and $K \cap S \trianglelefteq G$. As S is a minimal normal subgroup of G, then $K \cap S = S$ or $K \cap S = 1$. Since by hypothesis $K \cap S \leq S_\alpha \neq S$, we conclude that $K \cap S = 1$. Then $K \leq C_G(S)$. In fact, given $k \in K$ and $s \in S$, $k^{-1}l^{-1}kl \in K \cap S = 1$, thus $kl = lk$. Since k and l are arbitrary, then $K \leq C_G(S)$. Therefore, in case $C_G(S) = 1$, the faithfulness is immediate. □

Dealing with minimal normal subgroups and their stabilizers, it is important to understand the relation between the projections of the stabilizer on the direct factors of the minimal normal subgroup.

Lemma 3.2.2. *Let G be a permutation group on Ω, let $S = Q_1 \times \cdots \times Q_r$ be a transitive normal subgroup of G, where each Q_i is a nonabelian characteristically simple group, and assume that G acts transitively on $\Sigma := \{Q_1, \ldots, Q_r\}$ by conjugation. Consider the projections $\pi_i \colon S \to Q_i$. For a fixed $\alpha \in \Omega$ define $P := (S_\alpha\pi_1) \times \cdots \times (S_\alpha\pi_r)$. Then G_α normalizes P. In particular, if $P = S_\alpha$, then $S_\alpha\pi_i = (Q_i)_\alpha \neq Q_i$ for every $i \in \underline{r}$.*

Proof. As S is transitive on Ω, we have by Lemma 2.1.1 that $G = G_\alpha S$. Further, since G is transitive on $\Sigma := \{Q_1, \ldots, Q_r\}$ and S acts trivially by conjugation on this set, G_α is transitive on Σ. Given $g \in G_\alpha$ and $i \in \underline{r}$, let $j \in \underline{r}$ such that ${Q_i}^g = Q_j$. Then

$$(S_\alpha\pi_i)^g = ({S_\alpha}^g)\pi_j = S_\alpha\pi_j. \tag{3.1}$$

Therefore, G_α normalizes P, and since G_α is transitive on Σ, we obtain that all the $S_\alpha\pi_i$'s are isomorphic. Still, for all $i \in \underline{r}$, we have $S_\alpha\pi_i = Q_i \cap P$. In particular, if $P = S_\alpha$, then $S_\alpha\pi_i = (Q_i)_\alpha$ for every $i \in \underline{r}$. Thus, if we had $S_\alpha\pi_i = Q_i$ for some i, then we would get $Q_i = (Q_i)_\alpha$ for every i, and so $S = S_\alpha$, which is not possible because S is transitive. Therefore, $S_\alpha\pi_i \neq Q_i$ for every $i \in \underline{r}$. □

The next result, which will be useful in Chapter 5, gives us a special partition for the direct factors of a minimal normal subgroup based on the form of its stabilizer.

Lemma 3.2.3. *Let G be a permutation group on Ω and let $S = Q_1 \times \cdots \times Q_r$ be a transitive minimal normal subgroup of G, where each Q_i is a nonabelian simple group and $r \geq 2$. Suppose that for some $\alpha \in \Omega$, S_α is nonsimple and a subdirect subgroup of S. Then there exists a set $\overline{\Sigma} = \{S_1, \ldots, S_k\}$, where $k \geq 2$, and each S_j is a nonabelian characteristically simple subgroup of S, such that $S = S_1 \times \ldots \times S_k$, G_α acts transitively by conjugation on $\overline{\Sigma}$ and, considering the* *projections* $\overline{\pi}_i\colon S \to S_i$*, we have that* $S_\alpha\overline{\pi}_i$ *is a simple group isomorphic to* Q_1 *and that*

$$S_\alpha = S_\alpha\overline{\pi}_1 \times \cdots \times S_\alpha\overline{\pi}_k. \tag{3.2}$$

Proof. We have by Corollary 3.1.6 that each Q_i is isomorphic to Q_1. Since S_α is a subdirect subgroup of S, it follows from Scott's Lemma (Lemma 3.1.4, item 1) that

$$S_\alpha = D_1 \times \cdots \times D_k, \tag{3.3}$$

where $k \leq r$ and each group D_i is a full diagonal subgroup of a subproduct $S_i := \prod_{Q_j \in P_i} Q_j$, where $\mathcal{P} := \{P_1, \ldots, P_k\}$ is a partition of $\Sigma := \{Q_1, \ldots, Q_r\}$. Hence P_i consists of the Q_j's that compose S_i, and clearly $S = S_1 \times \cdots \times S_k$. Since each D_i is a full diagonal subgroup, each $D_i \cong Q_1$, which means that D_i is a nonabelian simple group. Therefore, since S_α is nonsimple, we must have $k \geq 2$. Besides that, according to Corollary 3.1.5, $D_1, \ldots, D_k$ are the only minimal normal subgroups of S_α.

We know that $S_\alpha \trianglelefteq G_\alpha$. So we assert that the set $\{D_1, \ldots, D_k\}$ is G_α-invariant by conjugation. In fact, given $g \in G_\alpha$ and D_i, since D_i is nonabelian and simple, ${D_i}^g$ is a minimal normal subgroup of S_α. Therefore, ${D_i}^g \in \{D_1, \ldots, D_k\}$. So we conclude that $\{D_1, \ldots, D_k\}$ is G_α-invariant.

For each $i \in \underline{k}$, since D_i is a subdirect subgroup of S_i, we have that

$$D_i\pi_l = \begin{cases} Q_l, & \text{if } Q_l \in P_i, \\ 1, & \text{otherwise.} \end{cases}$$

Since S acts trivially by conjugation on Σ and $G = SG_\alpha$ by Lemma 2.1.1, then G_α acts transitively by conjugation on Σ. We assert that G_α acts by conjugation on $\overline{\Sigma} := \{S_1, \ldots, S_k\}$. In fact, given $g \in G_\alpha$ and S_i, consider $j \in \underline{k}$ such that ${D_i}^g = D_j$. We will prove that ${S_i}^g = S_j$. The idea is to show that these groups have the same direct factors. In fact, given Q_l a direct factor of ${S_i}^g$, there exists $Q_p \in P_i$ such that ${Q_p}^g = Q_l$. Since ${D_i}^g\pi_l = (D_i\pi_p)^g$, we conclude that $D_j\pi_l = {Q_p}^g = Q_l$. But this means that $Q_l \in P_j$, that is, Q_l is a direct factor of S_j. The other inclusion is analogous. So we conclude that $\overline{\Sigma}$ is G_α-invariant. In particular, this means that the partition $\mathcal{P}$ is G_α-invariant, and so it is uniform.

Moreover, we observe that the action of G_α on $\overline{\Sigma}$ is transitive. In fact, given $S_i, S_j \in \overline{\Sigma}$, consider $Q_p \in P_i$ and $Q_l \in P_j$. Since G_α is transitive on Σ, there exists $g \in G_\alpha$ such that ${Q_p}^g = Q_l \in P_j \cap {P_i}^g$. As the sets in $\mathcal{P}$ are disjoint, we conclude that ${P_i}^g = P_j$, which means that ${S_i}^g = S_j$. Therefore, G_α is transitive on $\overline{\Sigma}$.

Now consider the projections $\overline{\pi}_i \colon S \to S_i$. In order to prove the validity of the relation (3.2), we will show that $S_\alpha\overline{\pi}_i = D_i$. According to (3.3), $S_\alpha\overline{\pi}_i \leq D_i$. On the other hand, if $d \in D_i$, then consider $s := (1, \dots, 1, d, 1, \dots, 1)$, where d appears in position i. So $s \in S_\alpha$ and $s\overline{\pi}_i = d$, that is, $d \in S_\alpha\overline{\pi}_i$. Therefore, $S_\alpha\overline{\pi}_i = D_i$. So using (3.3), we can write

$$S_\alpha = S_\alpha\overline{\pi}_1 \times \cdots \times S_\alpha\overline{\pi}_k,$$

which concludes the proof. □

In terms of Section 2.5, the result above says that if G is a permutation group of type CD, then there exists a direct factorization of S such that a stabilizer in G acts transitively on it and a stabilizer in S factorizes accordingly.

3.3 Finite simple groups

The next four results, which will be used in subsequent chapters, rely on the Classification of the Finite Simple Groups. The first one is the well-known Schreier's Conjecture, which can be proved using the classification.

Lemma 3.3.1. *[12, Schreier's Conjecture, p. 133] Let S be a finite simple group. Then the group* Out(S) *of outer automorphisms of S is solvable.*

Lemma 3.3.2. *[17, Theorem 4.9.] Every finite simple group has a cyclic Sylow subgroup.*

Feit-Thompson Theorem states that every finite group of odd order is solvable. As a consequence, it is not difficult to prove that the order of a finite nonabelian simple group is divisible by four.

Lemma 3.3.3. *[14, Theorem 25.2] If Q is a finite nonabelian simple group, then* $4 \mid |Q|$.

The last result is a property of the sporadic Mathieu group M_{11}.

Lemma 3.3.4. *[11, p. 18] The Mathieu group M_{11} has only one conjugacy class of subgroups whose orders are 660.*

3.4 Number theoretic results

This section presents some number theoretic results related to the factorial of an integer. The first result is the well-known Legendre's Formula.

Lemma 3.4.1. *[20, Legendre's Formula, pp.8-10] Let p be a prime and let $n = a_m p^m + a_{m-1} p^{m-1} + \cdots + a_0$ be the p-adic expansion of n. Consider the sum $s_p(n) := a_m + a_{m-1} + \cdots + a_0$ of the digits of n. Then the largest p-power that divides $n!$ is p^l where*

$$l = \frac{n - s_p(n)}{p - 1}.$$

The following useful lemma is a corollary of Legendre's Formula, and it gives some properties about the divisibility of the factorial.

Lemma 3.4.2. *Given a prime number p and a natural number $n \geq 2$, we have that*

1. $p^n \nmid n!$.

2. *If* $p^{n-1} \mid n!$, *then* $p = 2$ *and* n *is a power of 2.*

3. $4^{n-1} \nmid n!$.

Proof. Consider the p-adic expansion of n and let l be the exponent of the largest p-power p^l that divides $n!$. Since $n \geq 2$, we have by Legendre's Formula that $l \leq n - 1$, which proves item (1). If $l = n - 1$, then $s_p(n) = 1$ and $p = 2$, thus n is a power of 2, which proves item (2). In order to see the last assertion, let $p = 2$ and $l = n - 1$. Since $n - 1 < 2n - 2$ for $n \geq 2$, the definition of l gives that $2^{2n-2} = 4^{n-1} \nmid n!$, which proves item (3). □

Chapter 4

Factorizations of groups

In Chapter 1 we emphasized the importance of group factorizations in dealing with the inclusion problem. The first section of this chapter presents some results about factorizations of finite simple groups, and defines *full factorizations* and *strong multiple factorizations*. The former type of factorizations depends on the primes that divide the order of the finite simple group. The results in this chapter will be applied in the last chapter, to analyzing the consequences of the Main Theorem in each O'Nan-Scott class.

4.1 Factorizations of a simple group

Factorizations of groups appear naturally in studying permutation groups. For example, if $G \leq \mathrm{Sym}(\Omega)$ has a transitive subgroup Y and let $\alpha \in \Omega$, then G factorizes through the stabilizer G_α, that is, $G = G_\alpha Y$ (Lemma 2.1.1). However, for our purpose, we need more than the usual notion of factorizations, as described in the definition below.

Definition 4.1.1. Let G be a group and $p(G)$ be the set of primes that divide $|G|$. If A and B are proper subgroups of G such that $G = AB$, then we call this expression and the set $\{A, B\}$ a **factorization** of G. A factorization $Q = AB$ of a finite simple group Q is said to be a **full factorization** if $p(Q) = p(A) = p(B)$. A set $\mathcal{A} := \{A_1, \ldots, A_l\}$ of proper subgroups of Q is said to be a **multiple factorization** if $Q = A_iA_j$ whenever $i \neq j$. Moreover, $\mathcal{A}$ is said to be a **strong multiple factorization** if, in addition $l \geq 3$ and $Q = A_i(A_j \cap A_k)$ whenever $|\{i, j, k\}| = 3$.

Baddeley and Praeger [1] characterized all the full and strong multiple factorizations, as described in the following lemma. We use the notation $G \cdot 2$ to the extension of G by the cyclic group C_2.

Lemma 4.1.2. *([1, Theorems 1.1. and 1.2.])*

1. *Suppose that Q is a finite simple group and $Q = AB$ is a full factorization of Q. Then Q, A and B are as in one of the rows of Table 4.1. Conversely, each row of Table 4.1 is a full factorization of a simple group.*

Q	A	B
A_6	A_5	A_5
M_{12}	M_{11}	M_{11}, $\mathrm{PSL}_2(11)$
$\mathrm{P}\Omega_8^+(q)$ $(q > 2)$	$\Omega_7(q)$	$\Omega_7(q)$
$\mathrm{P}\Omega_8^+(2)$	$\mathrm{Sp}(6, 2)$	A_7, A_8, S_7, S_8, $\mathrm{Sp}(6, 2)$, ${\mathbb{Z}_2}^6 \rtimes A_7$, ${\mathbb{Z}_2}^6 \rtimes A_8$
	A_9	A_8, S_8, $\mathrm{Sp}(6, 2)$, ${\mathbb{Z}_2}^6 \rtimes A_7$, ${\mathbb{Z}_2}^6 \rtimes A_8$
$\mathrm{Sp}(4, q)$ $(q \geq 4$ even$)$	$\mathrm{Sp}(2, q^2) \cdot 2$	$\mathrm{Sp}(2, q^2) \cdot 2$, $\mathrm{Sp}(2, q^2)$

Table 4.1: Full factorizations $Q = AB$

2. *If $\mathcal{A} = \{A_1, \ldots, A_l\}$ is a strong multiple factorization of a simple group Q, then $l = 3$. Further, if $\{A_1, A_2, A_3\}$ is a strong multiple factorization of Q, then Q, A_1, A_2, and A_3 are as in one of the rows of Table 4.2. Conversely, each row of Table 4.2 is a strong multiple factorization of a simple group.*

Q	A_1	A_2	A_3
$\mathrm{Sp}(4a, 2)$ $(a \geq 2)$	$\mathrm{Sp}(2a, 4) \cdot 2$	$O_{4a}^-(2)$	$O_{4a}^+(2)$
$\mathrm{P}\Omega_8^+(3)$	$\Omega_7(3)$	${\mathbb{Z}_3}^6 \rtimes \mathrm{PSL}_4(3)$	$\mathrm{P}\Omega_8^+(2)$
$\mathrm{Sp}(6, 2)$	$G_2(2)$	$O_6^-(2)$	$O_6^+(2)$
	$G_2(2)'$	$O_6^-(2)$	$O_6^+(2)$
	$G_2(2)$	$O_6^-(2)'$	$O_6^+(2)$
	$G_2(2)$	$O_6^-(2)$	$O_6^+(2)'$

Table 4.2: Strong multiple factorizations $Q = A_1(A_2 \cap A_3)$

The following corollary characterizes the factorizations $Q = AB$ of finite simple groups Q, in which both A and B are direct powers of the same finite simple group. The options for such Q, A and B are very restricted.

Corollary 4.1.3. *Let Q be a finite simple group, and suppose that $Q = AB$ is a factorization of Q where $A \cong T^{s_1}$ and $B \cong T^{s_2}$ for some finite nonabelian simple group T and integers s_1, s_2. Then $s_1 = s_2 = 1$ and Q and T are as in Table 4.3.*

Proof. We assert that $Q = AB$ is a full factorization. In fact, given a prime number p, since $|Q| = \frac{|A||B|}{|A \cap B|}$ and $|A|$ and $|B|$ are powers of $|T|$, then if p divides $|Q|$, we have that p divides $|T|$, so p divides $|A|$ and $|B|$. Thus $Q = AB$ is a full factorization, and such factorizations are completely characterized in Lemma 4.1.2. Looking in Appendix A at the orders of the groups in Table 4.1, and using that $\mathrm{Sp}(6,2) \cong \Omega_7(2)$ [37, 3.8.2], we observe that the only options where $|A|$ and $|B|$ are powers of the same finite simple group occur when $A \cong B \cong T$. The options for Q and T are in Table 4.3. □

Q	$A \cong T \cong B$
A_6	A_5
M_{12}	M_{11}
$\mathrm{P}\Omega_8^+(q)$ $(q \geq 2)$	$\Omega_7(q)$

Table 4.3: Factorizations of $Q = AB$, where $|A| = |T|^{s_1}$ and $|B| = |T|^{s_2}$

The next result characterizes the factorizations $S = HD$ of finite nonabelian and nonsimple characteristically simple groups S, in which H is also a nonabelian characteristically simple subgroup and D is a full diagonal subgroup of S. We will see that the options for such S and H are also very restricted.

Corollary 4.1.4. *Let $S = Q_1 \times \cdots \times Q_r$, where $r \geq 2$ and each Q_i is isomorphic to a finite nonabelian simple group Q, and let $H \cong T^k$ be a nonabelian characteristically simple subgroup of S. Consider the projections $\pi_i \colon S \to Q_i$, where $i \in \underline{r}$, and suppose that $1 < H\pi_i < Q_i$ for all $i \in \underline{r}$. If there is a full diagonal subgroup D of S such that $S = HD$, then $r = k = 2$, $T \cong A \cong B$, $H = A \times B$ and $Q = \overline{A}\,\overline{B}$, where $\overline{A} \cong A$, $\overline{B} \cong B$, and Q, A and B are described in Table 4.3.*

Proof. First we assume that $r = 2$, that is, $S = Q_1 \times Q_2$. By Lemma 3.1.3 there exist isomorphisms $\varphi_i \colon Q \to Q_i$ such that

$$D = \{(q\varphi_1, q\varphi_2) \colon q \in Q\} \cong Q.$$

Denote $A := H\pi_1$, $B := H\pi_2$, and consider $\overline{H} := A \times B \geq H$. Since A and B are homomorphic images of H, there are integers s_1 and s_2 such that $A \cong T^{s_1}$ and $B \cong T^{s_2}$. As $S = HD = \overline{H}D$, then given $q \in Q$, there exist $a, b, h \in Q$ such that

$$(q\varphi_1, 1) = (a\varphi_1, b\varphi_2)(h\varphi_1, h\varphi_2).$$

This means that $q = ah$ and $h = b^{-1}$, so $q = ab^{-1} \in (A{\varphi_1}^{-1})(B{\varphi_2}^{-1})$. Denoting $\overline{A} := A{\varphi_1}^{-1} \cong A$ and $\overline{B} := B{\varphi_2}^{-1} \cong B$, we conclude that $Q = \overline{A}\,\overline{B}$. From Lemma 4.1.3, the possibilities for Q, A and B are described in Table 4.3. In this case we see that $A \cong T \cong B$.

Assume now that $r \geq 3$. We will see that this option is not possible.

Let $\overline{H} := A_1 \times \cdots \times A_r \geq H$, where $A_i := H\pi_i$, and write $S = \overline{H}D$. Analogously to the previous case, there exist integers s_i and isomorphisms $\varphi_i \colon Q \to Q_i$ such that $A_i \cong T^{s_i}$ and

$$D = \{(q\varphi_1, q\varphi_2, \ldots, q\varphi_r) \colon q \in Q\} \cong Q.$$

So given $q \in Q$, there exist $a_1, \ldots, a_r, h \in Q$ such that

$$(q\varphi_1, 1, \ldots, 1) = (a_1\varphi_1, \ldots, a_r\varphi_r)(h\varphi_1, \ldots, h\varphi_r).$$

This means that $q = a_1 h$ and $h = {a_i}^{-1}$ for each $i \geq 2$. So $q = a_1{a_i}^{-1}$ for each $i \geq 2$. Denoting $\overline{A}_i := A_i{\varphi_i}^{-1} \leq Q$, then $q \in \overline{A}_1(\overline{A}_2 \cap \ldots \cap \overline{A}_r)$. Since q was arbitrary, we obtain

$$Q = \overline{A}_1(\overline{A}_2 \cap \ldots \cap \overline{A}_r) = \overline{A}_1(\overline{A}_i \cap \overline{A}_j)$$

for all $i \neq j$, where $i, j \geq 2$. But note that there is nothing special about working with the first coordinate. Then we conclude that

$$Q = \overline{A}_i(\overline{A}_j \cap \overline{A}_l)$$

for all distinct $i, j, l \in \underline{r}$. This means that $\{\overline{A}_1, \ldots, \overline{A}_r\}$ is a strong multiple-factorization of Q, and such factorizations are completely characterized in Lemma 4.1.2. Looking at the characterization and Appendix A, we observe that it is not possible to obtain $r \geq 3$ with all $\overline{A}_i$ being a direct power of the same finite simple group. Therefore, the only possible case is $r = 2$.

Thus we have $\overline{H} = A \times B \cong T^2$. Since H is a subdirect subgroup of $\overline{H}$, we have two options, either H is a diagonal group isomorphic to $A \cong T$ or $H = \overline{H}$. The first option is not possible, because otherwise, since $S = HD$ and $A < Q_1$, we would have

$$|Q|^2 = |S| = \frac{|D||H|}{|D \cap H|} = \frac{|Q||A|}{|D \cap H|} < |Q|^2,$$

which is a contradiction. Therefore, $H = \overline{H} = A \times B$ and the result is proved. $\square$

4.2 Factorizations and uniform automorphisms

The existence of factorizations of direct products with strips as factors is related to the existence of *uniform automorphisms*.

Definition 4.2.1. Let G be a group and τ be an automorphism of G. We say that τ is **uniform** if the map $\overline{\tau}\colon g \mapsto g^{-1}(g\tau)$ is surjective.

Since we are dealing with finite groups, we have that a map $G \to G$ is surjective if, and only if, it is injective. So $\tau \in \mathrm{Aut}(G)$ is not uniform if, and only if, $g \mapsto g^{-1}(g\tau)$ is not injective, that is, $g^{-1}(g\tau) = h^{-1}(h\tau)$ for some distinct $g, h \in G$. The last equation says that hg^{-1} is a non-identity fixed point of τ. Then $\tau \in \mathrm{Aut}(G)$ is not uniform if, and only if, τ admits a non-identity fixed point. It is a consequence of the Finite Simple Group Classification that every automorphism of a finite nonabelian simple group Q has non-identity fixed points [15, Theorem 1.48], which means that Q does not have uniform automorphisms. In fact, the following strong result holds.

Lemma 4.2.2. *([19, 9.5.3]) A finite nonsolvable group has no uniform (that is, fixed-point-free) automorphisms.*

The relation between uniform automorphisms and factorizations is given by the result below.

Lemma 4.2.3. *([31, Lemma 2.2]) The following assertions are equivalent for a group G.*

1. *There exist nontrivial full strips X and Y in $G \times G$ such that $G \times G = XY$.*

2. *G admits a uniform automorphism.*

The next lemma generalizes the previous result for the case where G does not admit a uniform automorphism and we want to factorize at least two copies of G.

Lemma 4.2.4. *([31, Theorem 1.2]) Let G be a group that does not admit a uniform automorphism and let X and Y be direct products of pairwise disjoint nontrivial full strips in G^r with $r \geq 2$. Then $XY \neq G^r$.*

Lemma 4.2.6 below will be used in the last chapter, but can be viewed as an example in which the subgroup Y is the product of (nonfull) strips isomorphic to the Mathieu group M_{11} and $XY \neq G^r$ anyway. First we need a lemma.

Lemma 4.2.5. *Let X and Y be proper subgroups of a group S, such that X and Y are in the same conjugacy class. Then $S \neq XY$.*

Proof. Assume that $S = XY$ and let $s = ab \in S$ such that $Y = X^s$, $a \in X$ and $b \in Y$. Then $Y = X^b$ and

$$S = S^b = X^b Y^b = YY = Y,$$

which is an absurd, since by hypothesis Y is proper in S. Therefore, $S \neq XY$, as asserted. □

Lemma 4.2.6. *Let $S = Q^4$ and $A_1, \ldots, A_4 \leq Q$ such that $Q = M_{12}$ and each $A_i \cong M_{11}$. Let $X = \{(p, p, q, q)\colon p, q \in Q\}$ and $Y = \{(a_1, a_2, a_2\psi, a_4)\colon a_i \in A_i\}$ be subgroups of S, where $\psi\colon A_2 \to A_3$ is an isomorphism. Then $S \neq XY$.*

Proof. We have that $X = D_1 \times D_2 \cong Q^2$ is the direct product of two full strips of S and $Y \cong {M_{11}}^3$ is the direct product of three strips of S, where the second strip is a diagonal subgroup of Q^2.

Assume that $Q^4 = XY$ and consider the projections $\pi_{12}\colon S \to Q^2$ and $\pi_{34}\colon S \to Q^2$, where the first one projects onto the first two coordinates and the second one projects onto the last two coordinates. By applying these projections in $Q^4 = XY$, we obtain that

$$Q^2 = D_1(A_1 \times A_2), \tag{4.1}$$

$$Q^2 = D_2(A_3 \times A_4). \tag{4.2}$$

We claim that $Q = A_1A_2 = A_3A_4$. In fact, given $q \in Q$, we have by (4.1) that there exist $p \in Q$ and $a_i \in A_i$ such that $(q, 1) = (p, p)(a_1, a_2)$. Then $p = a_2^{-1}$ and $q = pa_1 = a_2^{-1}a_1 \in A_2A_1$. As q is arbitrary, $Q = A_1A_2$. Analogously, by (4.2), $Q = A_3A_4$. Thus by Appendix A, $|A_1 \cap A_2| = |A_3 \cap A_4| = 660$. On the other hand, denote $C_1 := A_1 \cap A_2$ and $C_2 := (A_3 \cap A_4)\psi^{-1}$. Note that

$$X \cap Y = \{(c, c, c\psi, c\psi)\colon c \in C_1 \cap C_2\} \cong C_1 \cap C_2.$$

Since $Q^4 = XY$,

$$|X \cap Y| = \frac{|X||Y|}{|Q|^4} = \frac{|M_{11}|^3}{|M_{12}|^2} = 55.$$

As $X \cap Y \cong C_1 \cap C_2 \leq A_2$ and

$$|C_1C_2| = \frac{|C_1||C_2|}{|X \cap Y|} = \frac{660^2}{55} = 7920 = |A_2|,$$

then $A_2 = C_1C_2$. By Lemma 3.3.4, M_{11} has just one conjugacy class of subgroups whose orders are 660, and so C_1 and C_2 are conjugate. However, this an absurd by Lemma 4.2.5. Then $Q^4 \neq XY$ and the result is proved. □

4.3 An algorithm

Let $S = Q_1 \times \cdots \times Q_r$ be a direct product of nontrivial groups, and let π_i be the projection of S onto Q_i. If P is a full strip of S with $\operatorname{supp}(P) = \{Q_{i_1}, \ldots, Q_{i_m}\}$, where $i_j \in \underline{r}$ for all $j \in \underline{m}$, then P is a full diagonal subgroup of the direct product of the elements on its support. In this case we will write $P = D(Q_{i_1}, \ldots, Q_{i_m})$. We emphasize that this notation just gives the shape of P, and the precise definition of P depends on the isomorphisms given by Lemma 3.1.3.

According to Example 4.3.1, in case either X or Y contains a trivial full strip of S, then we may have $XY = S$. However, as we can see in Example 4.3.2, this condition does not guarantee that $XY = S$.

Example 4.3.1. Let Q be a finite group, $S = Q^{12}$, and consider

$$\begin{aligned} X =& \{(x,x,x,y,y,y,z,z,z,w,w,w) \colon x,y,z,w \in Q\} \cong Q^4, \\ Y =& \{(a_1,a_2,a_3,a_1,a_5,a_6,a_7,a_5,a_9,a_{10},a_{11},a_9) \colon a_i \in Q\} \cong Q^9, \end{aligned}$$

that are direct products of full strips of S. We want to prove that $XY = S$. Notice that

$$X \cap Y = \{(x,x,x,x,x,x,x,x,x,x,x,x) \colon x \in Q\} \cong Q.$$

Since

$$|XY| = \frac{|X||Y|}{|X \cap Y|} = \frac{|Q|^4|Q|^9}{|Q|} = |Q|^{12} = |S|,$$

we conclude that $XY = S$.

Example 4.3.2. Let $S = Q_1 \times Q_2 \times Q_3 \times Q_4$, where each Q_i is isomorphic to a finite group Q, and consider

$$X = D(Q_1, Q_2) \times D(Q_3, Q_4) \cong Q^2$$

and

$$Y = D(Q_1, Q_2) \times Q_3 \cong Q^2$$

be direct products of full strips of S. We will prove that $XY \neq S$.

Since $D(Q_1, Q_2) \leq X \cap Y$, we have that $|Q| \leq |X \cap Y|$. Then

$$|XY| = \frac{|X||Y|}{|X \cap Y|} \leq \frac{|Q|^2|Q|^2}{|Q|} = |Q|^3 < |S|,$$

and we conclude that $XY \neq S$.

As a corollary of Lemma 4.2.4, we characterize, under certain conditions, the factorizations of finite nonsimple and nonabelian characteristically simple groups. In case the group can be factorized, we give some restrictions to the factorization. Although the proof relies on Lemma 4.2.4, note that its application is purely combinatorial.

Corollary 4.3.3. *Let $S = Q_1 \times \cdots \times Q_r$, where $r \geq 2$ and each Q_i is isomorphic to a finite nonabelian simple group Q, and consider the projections $\pi_i \colon S \to Q_i$. Let X and Y be nontrivial subgroups of S such that $X\pi_i, Y\pi_i \in \{1, Q_i\}$ for all $i \in \underline{r}$. Then*

1. *$X = X_1 \times \cdots \times X_p$ and $Y = Y_1 \times \cdots \times Y_q$, where the sets $\{X_i \colon i \in \underline{p}\}$ and $\{Y_j \colon j \in \underline{q}\}$ consist of pairwise disjoint full strips of S.*
2. *Algorithm 4.1 decides if $XY = S$.*
3. *If $XY = S$, denote $\operatorname{supp}(Y) := \cup_i \operatorname{supp}(Y_i)$. Then*
   (a) *$|\operatorname{supp}(X_i) \cap \operatorname{supp}(Y_j)| \leq 1$ for all $i \in \underline{p}$ and $j \in \underline{q}$.*
   (b) *$|\operatorname{supp}(X_i) \cap \operatorname{supp}(Y)| \geq |\operatorname{supp}(X_i)| - 1$ for all $i \in \underline{p}$.*

Proof. 1. Let $J_X := \{i \in \underline{r} \colon X\pi_i = Q_i\}$ and $J_Y := \{j \in \underline{r} \colon Y\pi_j = Q_j\}$. Then X is subdirect subgroup of $\prod_{i \in J_X} Q_i$ and Y is a subdirect subgroup of $\prod_{j \in J_Y} Q_j$. Then according to Scott's Lemma (Lemma 3.1.4, item 1) we have that $X = X_1 \times \cdots \times X_p$ and $Y = Y_1 \times \cdots \times Y_q$, where the sets $\{X_i \colon i \in \underline{p}\}$ and $\{Y_j \colon j \in \underline{q}\}$ consist of pairwise disjoint full strips of S.

2. To see that Algorithm 4.1 works, we have to verify that in each step, given A, B and C, then $AB = C$ if, and only if $(AB)\pi_J = C\pi_J$. Since π_J is a homomorphism, one direction is clear. Then suppose that $(AB)\pi_J = C\pi_J$, and let $G_1 := \prod_{i \in \overline{I}} Q_i$ and $G_2 := \prod_{j \in J} Q_j$. Thus $C = G_1 \times G_2$ and, by hypothesis, $(AB)\pi_J = G_2$. Consider the projection $\pi_{\overline{I}} \colon S \to G_1$. Then $(AB)\pi_{\overline{I}} = G_1 \leq AB$. So according to Lemma 3.1.7, we have that

$$AB = (AB)\pi_{\overline{I}} \times (AB)\pi_J = G_1 \times G_2 = C.$$

Therefore, $AB = C$ if, and only if $(AB)\pi_J = C\pi_J$. As a consequence, $AB = C$ if, and only if $XY = S$.

We also have to verify that if neither A nor B contains a direct factor Q_j, $j \in J$, then $XY \neq S$. In fact, if that is the case, A and B are direct products of pairwise disjoint nontrivial full strips in S. Then by Lemma 4.2.4, we have that $AB \neq C$. However, from the previous considerations, this implies that $XY \neq S$. Therefore, the algorithm is correct.

3. Assume that $XY = S$.

   (a) By contradiction, suppose that there exist $i_0 \in \underline{p}$ and $j_0 \in \underline{q}$ such that

   $$|\operatorname{supp}(X_{i_0}) \cap \operatorname{supp}(Y_{j_0})| \geq 2.$$

   Reindexing, if necessary, assume that $Q_1, Q_2 \leq \operatorname{supp}(X_{i_0}) \cap \operatorname{supp}(Y_{j_0})$, and consider the projection $\overline{\pi}\colon S \to Q_1 \times Q_2$. As $XY = S$, then

   $$Q_1 \times Q_2 = S\overline{\pi} = (X\overline{\pi})(Y\overline{\pi}) = (X_{i_0}\overline{\pi})(Y_{j_0}\overline{\pi}),$$

   where $X_{i_0}\overline{\pi}$ and $Y_{j_0}\overline{\pi}$ are nontrivial full strips in $Q_1 \times Q_2$. However, from Lemmas 4.2.2 and 4.2.3, this is not possible. Then

   $$|\operatorname{supp}(X_i) \cap \operatorname{supp}(Y_j)| \leq 1$$

   for all $i \in \underline{p}$ and $j \in \underline{q}$, and the item is proved.

   (b) Given X_{i_0}, $i_0 \in \underline{p}$, let $P := \prod_{Q_j \in \operatorname{supp}(X_{i_0})} Q_j$ and consider the projection $\overline{\pi}\colon S \to P$. Since $XY = S$, then

   $$Q^{|\operatorname{supp}(X_{i_0})|} \cong P = S\overline{\pi} = (X\overline{\pi})(Y\overline{\pi}) = (X_{i_0})(Y\overline{\pi}).$$

   As $X_{i_0} \cong Q$, the relation above implies that $Y\overline{\pi} \cong Q^{j_0}$ for some $j_0 \geq |\operatorname{supp}(X_{i_0})| - 1$. Since $i_0 \in \underline{p}$ is arbitrary, then

   $$|\operatorname{supp}(X_i) \cap \operatorname{supp}(Y)| \geq |\operatorname{supp}(X_i)| - 1$$

   for all $i \in \underline{p}$.

□

INPUT: S, X and Y.
OUTPUT: Decides if $XY = S$ or not.
BEGIN: $J := \underline{r}$, $A := X$, $B := Y$ and $C := S$.
IF $X\pi_i \cup Y\pi_i = \{Q_i\}$ for all $i \in \underline{r}$ **THEN**
  IF neither A nor B contains a direct factor Q_j, where $j \in J$, **THEN RETURN FALSE**.
  ELSE $I := \{i \in J \colon Q_i$ is a direct factor of either A or $B\}$.
    IF $I = J$ **THEN RETURN TRUE** .
    ELSE $\overline{I} := I$ and $J := J - I$.
      $\pi_J :=$ the projection map $\pi \colon S \to \prod_{j\in J} Q_j$.
      $A := A\pi_J$, $B := B\pi_J$ and $C := C\pi_J$.
      GOTO first **IF** statement.
ELSE RETURN FALSE.

Algorithm 4.1

To see the algorithm working, consider the following example.

Example 4.3.4. Let S, X and Y as in Example 4.3.1. Suppose that $Q_1, \ldots, Q_{12}$ are the internal direct factors of S. Then we can write

$$\begin{aligned} X =& D(Q_1, Q_2, Q_3) \times D(Q_4, Q_5, Q_6) \times D(Q_7, Q_8, Q_9) \times D(Q_{10}, Q_{11}, Q_{12}), \\ Y =& D(Q_1, Q_4) \times D(Q_5, Q_8) \times D(Q_9, Q_{12}) \times Q_2 \times Q_3 \times Q_6 \times Q_7 \times Q_{10} \times Q_{11}. \end{aligned}$$

Each of the following paragraphs follows the steps of the algorithm.

Let $J := \underline{12}$, $A := X$, $B := Y$ and $C := S$. Since $X\pi_i = Q_i$ for all $i \in \underline{12}$, then $I := \{2, 3, 6, 7, 10, 11\}$. As $I \neq J$, we have that $\overline{I} := \{2, 3, 6, 7, 10, 11\}$. Hence after the first recursive step, we have $J := \{1, 4, 5, 8, 9, 12\}$,

$$\begin{aligned} A :=& Q_1 \times D(Q_4, Q_5) \times D(Q_8, Q_9) \times Q_{12}, \\ B :=& D(Q_1, Q_4) \times D(Q_5, Q_8) \times D(Q_9, Q_{12}), \\ C :=& Q_1 \times Q_4 \times Q_5 \times Q_8 \times Q_9 \times Q_{12}. \end{aligned}$$

Now $I := \{1, 12\}$. Since $I \neq J$, then $\overline{I} := \{1, 12\}$. Thus after the second recursive step, we have $J := \{4, 5, 8, 9\}$,

$$\begin{aligned} A :=& D(Q_4, Q_5) \times D(Q_8, Q_9), \\ B :=& Q_4 \times D(Q_5, Q_8) \times Q_9, \\ C :=& Q_4 \times Q_5 \times Q_8 \times Q_9. \end{aligned}$$

Now $I := \{4, 9\}$, and since $I \neq J$, then $\overline{I} := \{4, 9\}$. In the last recursive step $J := \{5, 8\}$,

$$\begin{aligned} A :=& Q_5 \times Q_8, \\ B :=& D(Q_5, Q_8), \\ C :=& Q_5 \times Q_8. \end{aligned}$$

Finally, $I := \{5, 8\}$ and since $I = J$, then $XY = S$, which is according to Example 4.3.1.

Chapter 5

Wreath products

In this chapter we define wreath products and the product action, and next we give the definition of cartesian decompositions.

We prove that given a nonabelian transitive characteristically simple subgroup of a wreath product in product action, then such subgroup has to be in the base group of the wreath product.

We also state and demonstrate the Embedding Theorem (Theorem 5.4.2). This result says explicitly how to embed a finite quasiprimitive permutation group in a wreath product in product action.

5.1 Wreath products and product action

Let G be a group and let H be a subgroup of S_n. Then considering the direct product G^n of n copies of G, we define the **wreath product** $G \,\mathrm{wr}\, H$ of G and H to be the semidirect product $G^n \rtimes H$, in which the conjugation action of H on G^n is given by

$$(g_1, \ldots, g_n)^h := (g_{1^{h^{-1}}}, \ldots, g_{n^{h^{-1}}}),$$

for all $g_i \in G$ and $h \in H$. We say that G^n is the **base group** of the wreath product. Suppose now that G acts on a finite set Ω. Then there is an important action of $G \,\mathrm{wr}\, H$ on Ω^n, the **product action**, defined by

$$(\alpha_1, \ldots, \alpha_n)^{(g_1,\ldots,g_n)h^{-1}} := ((\alpha_{1^h})^{g_{1^h}}, \ldots, (\alpha_{n^h})^{g_{n^h}}),$$

for all $\alpha_i \in \Omega$, $g_i \in G$ and $h \in H$.

The wreath product $G \,\mathrm{wr}\, H$ acts primitively in product action on Ω^n if, and only if, H is transitive on $\underline{n}$ and G is primitive and not regular on Ω [12, Lemma

2.7A]. Moreover, this result also holds if we replace *primitive* with *innately transitive* [7, Theorem 2.7.4].

5.2 Cartesian decompositions

In this section we introduce the concept of cartesian decompositions and we present its relation to minimal normal subgroups.

Definition 5.2.1. A **cartesian decomposition** of a set Ω is a finite set of partitions of Ω, $\mathcal{E} = \{\Gamma_1, \dots, \Gamma_r\}$, such that $|\Gamma_i| \geq 2$ for all i and

$$|\gamma_1 \cap \cdots \cap \gamma_r| = 1 \text{ for all } \gamma_1 \in \Gamma_1, \dots, \gamma_r \in \Gamma_r.$$

A cartesian decomposition is said to be **trivial** if it contains only one partition, namely the partition into singletons. A cartesian decomposition is said to be **homogeneous** if all the Γ_i have the same cardinality. For $G \leq \mathrm{Sym}(\Omega)$, we say that $\mathcal{E}$ is G**-invariant** if ${\Gamma_i}^g \in \mathcal{E}$ for all $\Gamma_i \in \mathcal{E}$ and $g \in G$. Analogously to sets and partitions, we denote by $G_{(\mathcal{E})}$ the **pointwise stabilizer** of $\mathcal{E}$ in G, that is, the set of elements $g \in G$ such that ${\Gamma_i}^g = \Gamma_i$ for all $\Gamma_i \in \mathcal{E}$.

Example 5.2.2. Let Ω be the set of the vertices of the square, as in Figure 5.1. Then we consider the following partitions of Ω.

$$\Gamma_1 = \{\{(0,0),(1,0)\},\{(0,1),(1,1)\}\},$$
$$\Gamma_2 = \{\{(0,0),(0,1)\},\{(1,0),(1,1)\}\}.$$

We observe that if $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$, then $|\gamma_1 \cap \gamma_2| = 1$. So $\mathcal{E} = \{\Gamma_1, \Gamma_2\}$ is a cartesian decomposition of Ω.

If $\mathcal{E} = \{\Gamma_1, \dots, \Gamma_r\}$ is a cartesian decomposition for a set Ω, given $\alpha \in \Omega$, for each $i \in \underline{r}$ let γ_i be the unique block of Γ_i such that $\alpha \in \gamma_i$. This defines a bijection $\lambda \colon \Omega \to \Gamma_1 \times \ldots \times \Gamma_r$. Thus Ω can be naturally identified with the cartesian product $\Gamma_1 \times \ldots \times \Gamma_r$. Moreover, if G is a group acting on Ω, then by Lemma 2.2.2, G also has an action on $\Gamma_1 \times \ldots \times \Gamma_r$ in such a way that these actions are equivalent.

Definition 5.2.3. Suppose that S is a transitive permutation group on Ω, and that $\mathcal{E} = \{\Gamma_1, \dots, \Gamma_r\}$ is an S-invariant cartesian decomposition of Ω such that

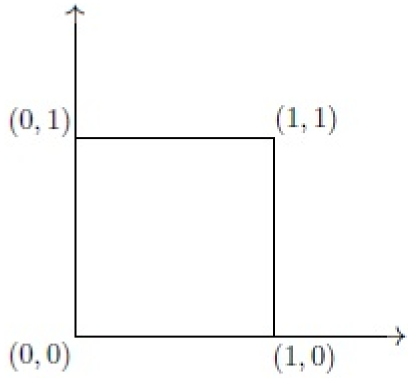

Figure 5.1: Square

$S_{(\mathcal{E})} = S$. For $i \in \underline{r}$, let $\overline{Q_i}$ denote the kernel of the S-action on Γ_i, and set

$$Q_i = \bigcap_{j \neq i} \overline{Q_j}. \tag{5.1}$$

The cartesian decomposition $\mathcal{E}$ is said to be S-**normal** if $S = \prod_{i=1}^{r} Q_i$.

Example 5.2.4. [34, p. 89, Example 4.22] Suppose that G is a permutation group on Ω with a transitive normal subgroup S such that S is the direct product $S = \prod_{i=1}^{r} S_i$, where the S_i form a G-conjugacy class. Suppose that $\alpha \in \Omega$ such that

$$S_\alpha = (S_\alpha \cap S_1) \times \cdots \times (S_\alpha \cap S_r). \tag{5.2}$$

For each $i \in \underline{r}$, let $\overline{S_i} = \prod_{j \neq i} S_j$, and let Γ_i denote the set of $\overline{S_i}$-orbits in Ω. Then we assert that $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_r\}$ is an S-normal cartesian decomposition of Ω.

In fact, S_i and $\overline{S_i}$ are normal subgroups of S such that $S = S_i \times \overline{S_i}$. For each $i \in \underline{r}$, let Γ_i as above. Since S is transitive and $\overline{S_i}$ is normal in S, then Γ_i is a S-invariant partition of Ω. Since S is transitive on Γ_i and $\overline{S_i}$ acts trivially on Γ_i, we find that S_i is transitive on Γ_i. Moreover, G permutes transitively by conjugation the subgroups S_i and $\overline{S_i}$. Therefore, G permutes transitively the partitions Γ_i in such a way that the G-actions on the S_i and on the Γ_i are permutationally isomorphic.

First we want to prove that $\mathcal{E}$ is a cartesian decomposition of Ω. Choose, for all $i \in \underline{r}$, the block $\gamma_i \in \Gamma_i$ such that $\alpha \in \gamma_i$. Then γ_i is an $\overline{S_i}$-orbit stabilized by $(S_i)_\alpha \times \overline{S_i}$. Hence γ_i is also an $((S_i)_\alpha \times \overline{S_i})$-orbit. As by relation 5.2 $S_\alpha \leq (S_i)_\alpha \times \overline{S_i}$, the correspondence between the overgroups of S_α and the S-blocks containing α given in Lemma 2.1.2 implies that $S_{\gamma_i} = (S_i)_\alpha \times \overline{S_i}$. In particular,

$$S_{\gamma_1} \cap \cdots \cap S_{\gamma_r} = (S_1)_\alpha \times \cdots \times (S_r)_\alpha = S_\alpha. \tag{5.3}$$

Let $\gamma = \gamma_1 \cap \cdots \cap \gamma_r$ and note that $\alpha \in \gamma$. Suppose that $\omega \in \gamma$. Then there is $s \in S$ such that $\alpha^s = \omega$, and so $s \in S_{\gamma_1} \cap \cdots \cap S_{\gamma_r}$, so $s \in S_\alpha$, which gives that $\omega = \alpha^s = \alpha$. Thus $|\gamma| = 1$. Suppose now that $\gamma_i' \in \Gamma_i$ for all $i \in \underline{r}$. We have that S_i is transitive on Γ_i while acts trivially on Γ_j for $j \neq i$. Then for all $i \in \underline{r}$, there exists $s_i \in S_i$ such that $(\gamma_i')^{s_i} = \gamma_i$ and $(\gamma_j')^{s_i} = \gamma_j'$ if $j \neq i$. Hence

$$(\gamma_1' \cap \cdots \cap \gamma_r')^{(s_1 \cdots s_r)} = \gamma_1 \cap \cdots \cap \gamma_r = \gamma.$$

Thus $|\gamma_1' \cap \cdots \cap \gamma_r'| = |\gamma| = 1$, which gives that $\mathcal{E}$ is a cartesian decomposition of Ω. As already shown above, $\mathcal{E}$ is G-invariant.

It remains to show that $\mathcal{E}$ is S-normal. Let K_i be the kernel of the S-action on Γ_i. Clearly $\overline{S_i} \leq K_i$. If $\overline{S_i} < K_i$, then there exists a nontrivial $q \in S_i$ such that q acts trivially on Γ_i. On the other hand, q acts trivially on each Γ_j for $j \neq i$. So given $\omega \in \Omega$, consider for each $i \in \underline{r}$, $\gamma_i \in \Gamma_i$ such that $\{\omega\} = \gamma_1 \cap \cdots \cap \gamma_r$. Then $\{\omega^q\} = (\gamma_1 \cap \cdots \cap \gamma_r)^q = \gamma_1 \cap \cdots \cap \gamma_r$, hence q must act trivially on Ω. As G is a permutation group, this is impossible, which gives that $\overline{S_i} = K_i$. According to Definition 5.2.3, we conclude that $\mathcal{E}$ is S-normal. In particular, the argument of this paragraph also shows that $S^{\Gamma_i} = S_i$.

5.3 Characteristically simple groups in wreath products

Recall the definition of wreath product and product action given in Section 5.1.

The following theorem, due to Csaba Schneider, is a key tool to describe CharS-QP inclusions, and it says that a transitive nonabelian characteristically simple subgroup of a wreath product W in product action is in the base group of W.

Theorem 5.3.1. *Let Γ be a finite set such that $|\Gamma| \geq 2$, let $r \geq 2$, and let $W = \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_r$ be considered as a permutation group on $\Omega = \Gamma^r$ in product action. If H is a transitive nonabelian characteristically simple subgroup of W, then H is a subgroup of the base group, that is, $H \leq (\mathrm{Sym}(\Gamma))^r$.*

Before proving the theorem, we need the definition of a *component* of a subgroup of W. We also need a result about how transitivity passes from the group to its components.

Let W be as above and suppose that X is a subgroup of W. For $j \in \underline{r}$, we define the **j-th component** $X^{(j)}$ of X as follows. Suppose that W_j is the stabilizer in W of j under the permutation representation $\pi\colon W \to S_r$. Then

$$W_j = \mathrm{Sym}(\Gamma) \times (\mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_{r-1}), \tag{5.4}$$

where the first factor of the direct product acts on the j-th coordinate, while the second factor acts on the other coordinates. In particular, 'S_{r-1}' is taken to be the stabilizer of j in S_r. We define $X^{(j)}$ as the projection of $X_j = X \cap W_j$ onto the first factor of W_j. We view $X^{(j)}$ as a subgroup of $\mathrm{Sym}(\Gamma)$.

Theorem 5.3.2 (Theorem 1.2, [33])**.** *If X is a transitive subgroup of W, then each component of X is transitive on Γ. Moreover, if X acts transitively on $\underline{r}$, then each component of the intersection $X \cap (\mathrm{Sym}(\Gamma))^r$ is transitive on Γ.*

We turn to the proof of Theorem 5.3.1.

Proof of Theorem 5.3.1. Suppose that $H = T_1 \times \cdots \times T_k = T^k$ for some non-abelian finite simple group T. Suppose, as above, that $\pi : W \to S_r$ is the natural projection. Let B be the base group $(\mathrm{Sym}(\Gamma))^r$ of W. Then $B = \ker \pi$. Assume for contradiction that $H \not\leq B$; that is $H\pi \neq 1$.

First we assume that $H\pi$ is transitive on $\underline{r}$. The case when $H\pi$ is intransitive will be treated afterwards. Set $H_B = H \cap B$. Then H_B is a normal subgroup of H and by Lemma 3.1.4 (item 2) it is of the form T^s, with some s. Further, $H = H_B \times \overline{H}_B$ where similarly $\overline{H}_B = T^{k-s}$, and we have that $\overline{H}_B$ acts transitively and faithfully by π on $\underline{r}$. For $j \in \underline{r}$, consider the component $H_B^{(j)}$ as a permutation group on Γ. By Theorem 5.3.2, $H_B^{(j)}$ is transitive on Γ for all j.

Claim. $H_B^{(j)} \cong H_B$ for all j.

Proof of Claim. Suppose, for $j \in \underline{r}$, that σ_j denotes the projection of W_j onto the first factor of the direct product decomposition in (5.4). Then we have that $H_B^{(j)} \cong H_B/(\ker \sigma_j \cap H_B)$. Let m be an element of $\ker \sigma_1 \cap H_B$. Thus $m = (1, m_2, \ldots, m_r)$ with $m_j \in H_B^{(j)}$. Let $j \in \underline{r}$. Since $\overline{H}_B$ is transitive on $\underline{r}$, there is some element $g = (g_1, \ldots, g_r)h$ of $\overline{H}_B$ such that $1(g\pi) = 1h = j$. Then

$$m^g = (1, m_2, \ldots, m_r)^g = (1, m_2^{g_2}, \ldots, m_r^{g_r})^h,$$

and so the j-th coordinate of m^g is 1. Hence $m^g \in \ker \sigma_j \cap H_B$, and then $(\ker \sigma_1 \cap H_B)^g \leq \ker \sigma_j \cap H_B$. Similarly, the same argument above shows that

$\ker\sigma_j \cap H_B \leq (\ker\sigma_1 \cap H_B)^g$, and so $\ker\sigma_j \cap H_B = (\ker\sigma_1 \cap H_B)^g$. On the other hand, $\ker\sigma_1 \cap H_B$ is a subgroup of H_B and $\overline{H}_B$ centralizes H_B, and so $\ker\sigma_1 \cap H_B = \ker\sigma_j \cap H_B$ for all j. Thus $\ker\sigma_1 \cap H_B$ acts trivially on Ω, and so $\ker\sigma_1 \cap H_B = 1$, which gives $\ker\sigma_j \cap H_B = 1$ for all j. Therefore, $H_B^{(j)} \cong H_B$ for all j. □

Thus the restrictions to H_B of the projection maps σ_j are monomorphisms. Then $\beta_j = \sigma_1^{-1}\sigma_j : H_B^{(1)} \to H_B^{(j)}$ is an isomorphism for all j. As a consequence, every element $m \in H_B$ can be expressed uniquely as $m = (y, y\beta_2, \ldots, y\beta_r)$, for some $y \in H_B^{(1)}$.

Claim. For all $j \in \underline{r}$, there is some element $x_j \in \mathrm{Sym}(\Gamma)$ such that $y\beta_j = y^{x_j}$ for all $y \in H_B^{(1)}$.

Proof of Claim. Suppose that $y \in H_B^{(1)}$. Then $m = (y, y\beta_2, \ldots, y\beta_r) \in H_B$. Let $j \in \underline{r}$ and, using the transitivity of $\overline{H}_B$ in $\underline{r}$, suppose that $g = (g_1, \ldots, g_r)h \in \overline{H}_B$ is such that $1(g\pi) = 1h = j$. Then g centralizes m and hence

$$(y, y\beta_2, \ldots, y\beta_r) = m^g = (y^{g_1}, (y\beta_2)^{g_2}, \ldots, (y\beta_r)^{g_r})^h.$$

Comparing the j-th coordinates in the two sides of the last equation, we find that $y\beta_j = y^{g_1}$. Taking $x_j := g_1$, thus we have $y\beta_j = y^{x_j}$. □

Claim. If Σ is a H_B-orbit in Ω, then $|\Sigma| = |\Gamma|$.

Proof of Claim. Since H is transitive on Ω and $H_B \trianglelefteq H$, all the H_B-orbits have the same size. Hence it suffices to show the claim for just one H_B-orbit. Choose the elements $1, x_2, \ldots, x_r$ as in the previous claim, let $\gamma \in \Gamma$ and consider the element $\omega = (\gamma, \gamma x_2, \ldots, \gamma x_r)$. Suppose that $m \in H_B$. By the previous claim, m has the form $m = (y, y^{x_2}, \ldots, y^{x_r})$ for some $y \in H_B^{(1)}$. Hence $\omega^m = (\gamma y, \gamma y x_2, \ldots, \gamma y x_r)$. Thus m stabilizes ω if, and only if, $y \in \mathrm{Sym}(\Gamma)$ stabilizes γ. Thus $(H_B)_\omega = (H_B^{(1)})_\gamma$. So by applying the Orbit-Stabilizer Theorem twice, and using that $|H_B| = |H_B^{(1)}|$ and that $H_B^{(1)}$ is transitive on Γ, we have

$$|\omega^{H_B}| = \frac{|H_B|}{|(H_B)_\omega|} = \frac{|H_B^{(1)}|}{|(H_B^{(1)})_\gamma|} = |\Gamma|,$$

as desired. □

Claim. The case when $H\pi$ is transitive is impossible.

Proof of Claim. H_B is a normal subgroup of H and every H_B-orbit has size $|\Gamma|$. Hence the number of H_B-orbits is $|\Gamma|^{r-1}$. Since H is transitive on Ω, $\overline{H}_B$ is transitive on the set of H_B-orbits and hence $|\Gamma|^{r-1} \mid |\overline{H}_B|$. Since $\overline{H}_B$ has a faithful action on $\underline{r}$, this leads to $|\Gamma|^{r-1} \mid r!$. Now, since Γ is an orbit for the characteristically simple group $H_B^{(1)}$, we find that $|\Gamma| \geq 5$. Hence $|\Gamma|$ is divisible by p, where p is either an odd prime or $p = 4$, which is a contradiction by Lemma 3.4.2. □

This completes the proof for the case when $H\pi$ is a transitive subgroup of S_r. Let us now turn to the case when $H\pi$ is intransitive. Recall that B is the base group of W. Assuming that $H \not\leq B$, gives that there exists a $H\pi$-orbit Δ in $\underline{r}$ with size at least 2. Set $\overline{\Delta} = \underline{r} \setminus \Delta$ and $r_1 = |\Delta|$. Then, by [33, Proposition 1.4], H can be embedded into the direct product

$$W_1 \times W_2 = \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_{r_1} \times \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_{r-r_1}$$

such that the projection H_1 of H into W_1 acts transitively on $\underline{r_1}$. Now, since H is transitive on Γ^r, H_1 is also transitive on Γ^{r_1}. Further, as H is characteristically simple, so is H_1. Hence using the theorem in the case when $H\pi$ is transitive gives a contradiction. Therefore, $H \leq B$. □

5.4 The Embedding Theorem

We present below the hypotheses under which we will work in this section, as well as the notation to be used.

Hypothesis 5.4.1. *(Embedding Hypothesis) G is a finite quasiprimitive permutation group on Ω and $\alpha \in \Omega$. We assume the following conditions:*

1. *$S = Q_1 \times \cdots \times Q_r$ is a minimal normal subgroup of G, where Q_i is a nonabelian and characteristically simple group and $r \geq 2$.*

2. *G acts transitively on $\Sigma := \{Q_1, \ldots, Q_r\}$ by conjugation. We denote this representation by $\rho\colon G \to \mathrm{Sym}(\Sigma)$.*

3. *Consider the projections $\pi_i\colon S \to Q_i$, where $i \in \underline{r}$. We have that*

$$S_\alpha = (S_\alpha\pi_1) \times \cdots \times (S_\alpha\pi_r). \tag{5.5}$$

In particular, we observe that item 2 above tells us that all the Q_i are isomorphic. Then we can consider $S = Q^r$, where Q is a nonabelian and characteristically simple group. Moreover, since the transitivity of S on Ω allows us to write $G = SG_\alpha$, and that S acts trivially by conjugation on Σ, item 2 says that G_α is transitive on Σ.

We have some work to do, but the purpose of this section is to prove the following Embedding Theorem.

Theorem 5.4.2. *(Embedding Theorem) Assume Hypothesis 5.4.1 as valid. If we consider* $\Gamma := [Q_1\colon (Q_1)_\alpha]$*, then there exists a permutational embedding* $\psi\colon G \hookrightarrow \mathrm{Sym}(\Gamma) \operatorname{wr} S_r$*, where we consider the wreath product as a permutation group in product action on* Γ^r*.*

By *permutational embedding* we mean that G and its image are not just isomorphic as abstract groups, but that in their respective actions, G and $G\psi$ are permutationally isomorphic. This is the reason why we can usually identify Ω with Γ^r, G with $G\psi$, and consider $G \leq \mathrm{Sym}(\Gamma) \operatorname{wr} S_r$.

Although we have required that the group G is finite, the reader will notice that the Embedding Theorem is still true for all innately transitive groups.

According to Theorem 1.2.2, if G has type SD, then G cannot be embedded in a wreath product in product action.

We want to prove the Embedding Theorem. Using the transitivity of S, we have that $G = SG_\alpha$. So consider the set $\Omega' := [S\colon S_\alpha]$. By Corollary 2.3.2, we obtain a transitive action of G on Ω' given by

$$(S_\alpha x)^{sy} := S_\alpha(y^{-1}xsy), \tag{5.6}$$

where $x, s \in S$ and $y \in G_\alpha$. Moreover, the actions of G on Ω and on Ω' are equivalent. Now, since $G \leq \mathrm{Sym}(\Omega)$, the action of G on Ω is faithful. Therefore, as these actions are equivalent, both are faithful, so we have the next result.

Lemma 5.4.3. *The actions of* G *on* Ω *and on* Ω' *are equivalent and faithful.*

We denote $U := (Q_1)_\alpha = S_\alpha\pi_1$. By Lemma 3.2.2 we have $Q_1 \neq U$. So consider $\Gamma := [Q_1\colon U]$. Our goal is to show that the permutation group induced by G on Ω' is permutationally isomorphic to a subgroup of $\mathrm{Sym}(\Gamma) \operatorname{wr} S_r$. In order to do that, we use the transitivity of G_α on Σ to fix, for each $i \in \underline{r}$, an element $t_i \in G_\alpha$ such that

$$Q_i^{\,t_i} = Q_1. \tag{5.7}$$

In particular, we note that ${(Q_i)_\alpha}^{t_i} = U$. Using (5.7), we define a bijection between Ω' and Γ^r given by

$$\begin{array}{cccc} \lambda\colon & \Omega' & \to & \Gamma^r \\ & S_\alpha(q_1, \ldots, q_r) & \mapsto & (Uq_1^{t_1}, \ldots, Uq_r^{t_r}), \end{array}$$

where, for each $i \in \underline{r}$, $q_i \in Q_i$.

Lemma 5.4.4. *The map λ given above defines a bijection between Ω' and Γ^r.*

Proof. To see that λ is well-defined and injective, we consider two elements $S_\alpha(q_1, \ldots, q_r)$ and $S_\alpha(p_1, \ldots, p_r)$ in Ω'. Then $S_\alpha(q_1, \ldots, q_r) = S_\alpha(p_1, \ldots, p_r)$ if, and only if, for all $i \in \underline{r}$, we have $q_i p_i^{-1} \in S_\alpha \pi_i = (Q_i)_\alpha$. But this is equivalent to saying that, for all $i \in \underline{r}$, $(q_i p_i^{-1})^{t_i} = q_i^{t_i}(p_i^{t_i})^{-1} \in {(Q_i)_\alpha}^{t_i} = U$, which means that $(Uq_1^{t_1}, \ldots, Uq_r^{t_r}) = (Up_1^{t_1}, \ldots, Up_r^{t_r})$. So

$$S_\alpha(q_1, \ldots, q_r) = S_\alpha(p_1, \ldots, p_r) \Longleftrightarrow (Uq_1^{t_1}, \ldots, Uq_r^{t_r}) = (Up_1^{t_1}, \ldots, Up_r^{t_r}).$$

This shows that λ is well-defined and injective. Since (5.7) allows us to write every element of Γ^r in the form $(Uq_1^{t_1}, \ldots, Uq_r^{t_r})$, it is also clear that λ is surjective. So we conclude that λ is a bijection. □

We can write the elements of $G = SG_\alpha$ in the form $(s_1, \ldots, s_r)y$, where each $s_i \in Q_i$ and $y \in G_\alpha$. Thus, by Lemma 2.2.2, it follows that G has an action on Γ^r given by

$$(Ux_1, \ldots, Ux_r)^{(s_1,\ldots,s_r)y} := [(Ux_1, \ldots, Ux_r)\lambda^{-1}]^{(s_1,\ldots,s_r)y}\lambda, \tag{5.8}$$

where $x_i \in Q_1$, $s_i \in Q_i$ and $y \in G_\alpha$. Moreover, by the same result, the actions of G on Ω' and on Γ^r are equivalent. We denote this action by $\psi\colon G \to \mathrm{Sym}(\Gamma^r)$.

Lemma 5.4.5. *The actions of G on Ω, Ω' and on Γ^r are equivalent and faithful.*

Proof. Since by Lemma 5.4.3 the actions of G on Ω and on Ω' are equivalent and faithful, and that this last one is equivalent to the action of G on Γ^r, then the actions of G on Ω, Ω' and on Γ^r are all equivalent and faithful. □

Consider the representation $\rho\colon G \to \mathrm{Sym}(\Sigma)$ given in Hypothesis 5.4.1 (item 2). Thus we define $\mu\colon G \to S_r$ to be the induced homomorphism by ρ, that is, each permutation $g\mu$ is given by

$$i^{(g\mu)} = j \Longleftrightarrow {Q_i}^g = Q_j. \tag{5.9}$$

In order to simplify the notation, we just write i^g instead of $i^{(g\mu)}$. The relation above tells us that the actions of G on $\underline{r}$ and on Σ are equivalent.

Lemma 5.4.6. *The representations $\mu\colon G \to S_r$ and $\rho\colon G \to \mathrm{Sym}(\Sigma)$ are equivalent.*

Proof. To see this equivalence, notice that $\varphi\colon \underline{r} \to \Sigma$, defined by $i \mapsto Q_i$, is a bijection such that, given $g \in G$ that satisfies ${Q_i}^g = Q_j$, so by relation (5.9) it follows that

$$(i\varphi)^{g\rho} = {Q_i}^{g\rho} = Q_j = j\varphi = (i^{g\mu})\varphi.$$

Therefore, μ and ρ are equivalent representations. □

We want to understand better how the action of G on Γ^r works. Having in mind relation (5.7) and given $(Ux_1, \ldots, Ux_r) \in \Gamma^r$, we consider elements $q_i \in Q_i$ such that $x_i = q_i^{t_i}$. If $(s_1, \ldots, s_r)y \in G = SG_\alpha$, we have from (5.6) and (5.8) that

$$\begin{aligned}
(Ux_1, \ldots, Ux_r)^{(s_1,\ldots,s_r)y} &= [S_\alpha(q_1, \ldots, q_r)]^{(s_1,\ldots,s_r)y}\lambda \\
&= [S_\alpha(q_1 s_1, \ldots, q_r s_r)]^y \lambda \\
&= [S_\alpha((q_{1^{y^{-1}}} s_{1^{y^{-1}}})^y, \ldots, (q_{r^{y^{-1}}} s_{r^{y^{-1}}})^y)]\lambda \\
&= (U(q_{1^{y^{-1}}} s_{1^{y^{-1}}})^{yt_1}, \ldots, U(q_{r^{y^{-1}}} s_{r^{y^{-1}}})^{yt_r}). \qquad (5.10)
\end{aligned}$$

At this point we have that $G\psi$ is a subgroup of $\mathrm{Sym}(\Gamma^r)$. We want, through the relation above, to see exactly which are the elements of $G\psi$. More specifically, we want to conclude that $G\psi \leq \mathrm{Sym}(\Gamma) \operatorname{wr} S_r$. To do that, using the same notation as above, we define a permutation $\sigma_i \in \mathrm{Sym}(\Gamma)$ for each $i \in \underline{r}$, in a way that $((s_1, \ldots, s_r)y)\psi = (\sigma_1, \ldots, \sigma_r)(g\mu) \in \mathrm{Sym}(\Gamma) \operatorname{wr} S_r$, where this last element acts by product action on Γ^r. Fix $i \in \underline{r}$ and consider

$$\begin{array}{cccc}
\sigma_i\colon & \Gamma & \to & \Gamma \\
 & Uq_i^{t_i} & \mapsto & U(q_i s_i)^{yt_{i^y}},
\end{array}$$

where each $q_i \in Q_i$. We observe that in the definition above i, t_i, s_i and y are fixed. We will prove that σ_i is a permutation.

Lemma 5.4.7. *The map σ_i given above defines a permutation of Γ.*

Proof. To see that σ_i is well-defined and injective, let $Up_i^{t_i}$ and $Uq_i^{t_i}$ be in Γ. We have that $Up_i^{t_i} = Uq_i^{t_i}$ if, and only if, $(q_i p_i^{-1})^{t_i} \in U$, that is, according to equation

(5.7), we have that $q_ip_i^{-1} \in U^{t_i^{-1}} = (Q_i)_\alpha$. Acting now with y, we obtain that this last relation is equivalent to

$$(q_is_is_i^{-1}p_i^{-1})^{yt_{iy}} = (q_is_i)^{yt_{iy}}[(p_is_i)^{yt_{iy}}]^{-1} \in (Q_i)_\alpha^{yt_{iy}} = U,$$

that is,

$$U(q_is_i)^{yt_{iy}} = U(p_is_i)^{yt_{iy}}.$$

Therefore,

$$Uq_i^{t_i} = Up_i^{t_i} \iff U(q_is_i)^{yt_{iy}} = U(p_is_i)^{yt_{iy}}.$$

This means that σ_i is well-defined and injective. To verify the surjectivity, given $Up_i^{t_i} \in \Gamma$, we want to find $Uq_i^{t_i} \in \Gamma$ such that

$$(Uq_i^{t_i})\sigma_i = U(q_is_i)^{yt_{iy}} = Up_i^{t_i}.$$

It is enough to consider $q_i = p_i^{t_it_{iy}^{-1}y^{-1}}s_i^{-1}$. Then σ_i is surjective. So we obtain that σ_i is in fact a permutation of Γ. □

Finally, taking $g = (s_1, \dots, s_r)y \in G$, we note that

$$\begin{aligned}(Uq_1^{t_1}, \dots, Uq_r^{t_r})^{(\sigma_1,\dots,\sigma_r)g\mu} &= (U(q_1s_1)^{yt_{1y}}, \dots, U(q_rs_r)^{yt_{ry}})^{g\mu} \\ &= (U(q_{1y^{-1}}s_{1y^{-1}})^{yt_1}, \dots, U(q_{ry^{-1}}s_{ry^{-1}})^{yt_r}), \qquad (5.11)\end{aligned}$$

which is exactly the expression obtained in (5.10). Therefore,

$$((s_1, \dots, s_r)y)\psi = (\sigma_1, \dots, \sigma_r)(g\mu).$$

So we conclude that $G\psi$ permutes the elements of Γ^r via product action, that is, $G\psi \leq \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_r$. Since Lemma 5.4.5 guarantees that ψ is a permutational embedding, the Embedding Theorem (Theorem 5.4.2) is proved.

5.5 Some consequences

Let $\psi\colon G \to \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_r$ be the injective homomorphism obtained in the Embedding Theorem.

Great, but why, after all, are we so interested in the Embedding Theorem? Well, for two main reasons. It permits us to see that the projection morphism of the wreath product onto S_r composed with ψ is equivalent to the action of

G on the set $\Sigma = \{Q_1, \ldots, Q_r\}$ (corollary below). Besides that, if (H, G) is a CharS-QP inclusion, then using Theorem 5.3.1, we obtain that $H\psi$ is a subgroup of the base group $(\mathrm{Sym}(\Gamma))^r$. This, and some more work, allow us to conclude in next chapter that $H \leq \mathrm{soc}(G)$. Almost there!

Denote $W := \mathrm{Sym}(\Gamma) \,\mathrm{wr}\, S_r$ and $\pi \colon W \to S_r$ the projection of W onto S_r. So $\psi\pi = \mu$. By Lemma 5.4.6, μ is equivalent to ρ, then we obtain that $\psi\pi$ is equivalent to ρ.

Corollary 5.5.1. *We have that $\psi\pi \colon G \to S_r$ and $\rho \colon G \to \mathrm{Sym}(\Sigma)$ are equivalent representations of G.*

Let (H, G) be a CharS-QP inclusion. We apply Theorem 5.3.1 to $G\psi$ and $H\psi$ to obtain that $H\psi\pi = 1$. Therefore, using Corollary 5.5.1, it follows that H normalizes each element of Σ. Then the next result is proved.

Corollary 5.5.2. *Assume Hypothesis 5.4.1 as valid and (H, G) be a* CharS-QP *inclusion. Then H normalizes each element of $\Sigma := \{Q_1, \ldots, Q_r\}$.*

Since the O'Nan-Scott class CD satisfies Hypothesis 5.4.1 by Lemma 3.2.3, groups of this type satisfy the previous corollary. However, we observe that Corollary 5.5.2 does not apply to class SD because, as we already said (Theorem 1.2.2, item 4), this class does not satisfy Hypothesis 5.4.1.

Chapter 6

Characteristically simple subgroups of quasiprimitive permutation groups

Using the classification of quasiprimitive groups developed by Praeger (Section 2.5) and the results of the previous chapter, we prove the main result of this work.

Theorem 6.0.1. *(Main Theorem) Let* (H, G) *be a* CharS-QP *inclusion such that* $\mathrm{soc}(G)$ *is nonabelian. Then* $H \leq \mathrm{soc}(G)$.

The first section states the consequences of the Embedding Theorem to characteristically simple groups, while the last section is devoted to the proof of the Main Theorem.

6.1 That story about characteristically simple groups

Assume Hypothesis 5.4.1 as valid and suppose that each Q_i is isomorphic to a nonabelian simple group Q. Let (H, G) be a CharS-QP inclusion.

Consider the homomorphism $\varsigma\colon G \to \mathrm{Aut}(S)$ where, given $g \in G$, $g\varsigma$ is the conjugation by g. We have that $\ker\varsigma = C_G(S)$. By Corollary 5.5.2,

$$H\varsigma \leq \mathrm{Aut}(Q_1) \times \cdots \times \mathrm{Aut}(Q_r).$$

Moreover, we have that $S\varsigma = \mathrm{Inn}(Q_1)\times\cdots\times\mathrm{Inn}(Q_r)$. Then, by the Isomorphism

Theorem, it follows that

$$\frac{H\varsigma}{H\varsigma \cap S\varsigma} \cong \frac{(H\varsigma)(S\varsigma)}{S\varsigma} \leq \frac{\mathrm{Aut}(Q_1) \times \cdots \times \mathrm{Aut}(Q_r)}{\mathrm{Inn}(Q_1) \times \cdots \times \mathrm{Inn}(Q_r)} \cong (\mathrm{Out}(Q))^r. \tag{6.1}$$

Since Q is simple, we obtain from Schreier's Conjecture (Lemma 3.3.1) that $\mathrm{Out}(Q)$ is soluble. Therefore, as H is nonabelian and characteristically simple, then $H\varsigma$ is nonabelian and characteristically simple, thus nonsoluble, and we must have $H\varsigma \cap S\varsigma = H\varsigma$, that is, $H\varsigma \leq S\varsigma$. So we conclude that $H\varsigma \leq S\varsigma$. Then

$$\frac{H \ker \varsigma}{\ker \varsigma} \cong H\varsigma \leq S\varsigma \cong \frac{S \ker \varsigma}{\ker \varsigma}.$$

Thus, by the Correspondence Theorem, we obtain that $H \ker \varsigma \leq S \ker \varsigma$. Then $H \leq S \ker \varsigma = S \times C_G(S) = \mathrm{soc}(G)$. This proves the following result.

Lemma 6.1.1. *Assume Hypothesis 5.4.1 as valid with Q being a simple group, and let (H, G) be a* CharS-QP *inclusion. Then $H \leq \mathrm{soc}(G)$.*

6.2 Main theorem

We prove in this section the main theorem of this chapter, Theorem 6.0.1. Our strategy is to verify, using the O'Nan-Scott Theorem given in Chapter 2, the assertion for each O'Nan-Scott class. For those groups whose type is HS or As, the result follows from Schreier's Conjecture. Now, for groups of type HC, Tw or Pa, we use the Embedding Theorem. For groups of type Sd, we use Lemma 3.3.2. For groups of type Cd, we use the concept of cartesian decompositions and the case Sd.

Let G be a finite quasiprimitive permutation group on Ω of type HS, HC, As, Tw, Pa, Sd or Cd, and let (H, G) be a CharS-QP inclusion. Assume that S is a nonabelian plinth for G so that $\mathrm{soc}(G) = S \times C_G(S)$. In some cases $C_G(S)$ can be trivial. Moreover, we consider $\pi_i \colon S \to Q_i$ the projections of S on its direct factors, and $\varsigma \colon G \to \mathrm{Aut}(S)$ the representation by conjugation on S, so $\ker \varsigma = C_G(S)$.

G **has type As:** In this case $C_G(S) = 1$, S is a simple group and we have $\mathrm{Inn}(S) \leq G \leq \mathrm{Aut}(S)$. Then ς is injective and $S\varsigma = \mathrm{Inn}(S)$. We observe that

$$\frac{H\varsigma}{H\varsigma \cap S\varsigma} \cong \frac{(H\varsigma)(S\varsigma)}{S\varsigma} \leq \frac{\mathrm{Aut}(S)}{\mathrm{Inn}(S)} = \mathrm{Out}(S),$$

where the isomorphism comes from the Isomorphism Theorem. As S is simple, it follows from the Schreier's Conjecture (Lemma 3.3.1) that $\mathrm{Out}(S)$ is soluble. Since $H\varsigma$ is nonabelian and characteristically simple, thus not soluble, we have that $H\varsigma \cap S\varsigma = H\varsigma$. Therefore $H\varsigma \leq S\varsigma$ and, as ς is injective, we conclude that $H \leq S = \mathrm{soc}(G)$, as desired.

G **has type HS:** In this case S is a simple group that satisfies

$$\mathrm{soc}(G) = S \rtimes \mathrm{Inn}(S) \leq G \leq S \rtimes \mathrm{Aut}(S) = \mathrm{Hol}(S).$$

We assert that $G/\mathrm{soc}(G)$ is embedded in $\mathrm{Out}(S)$. In fact, it follows from Lemma 2.4.1 that

$$\frac{\mathrm{Hol}(S)}{\mathrm{soc}(G)} \cong \mathrm{Out}(S).$$

Since $G \leq \mathrm{Hol}(S)$, then

$$\frac{G}{\mathrm{soc}(G)} \hookrightarrow \mathrm{Out}(S).$$

This proves our assertion. But observe that

$$\frac{H}{H \cap \mathrm{soc}(G)} \cong \frac{H\,\mathrm{soc}(G)}{\mathrm{soc}(G)} \leq \frac{G}{\mathrm{soc}(G)} \hookrightarrow \mathrm{Out}(S),$$

where the isomorphism comes from the Isomorphism Theorem and the embedding comes from our last assertion. Therefore, arguing as we did for the type As, it follows from Schreier's Conjecture that $H \cap \mathrm{soc}(G) = H$. Then we conclude that $H \leq \mathrm{soc}(G)$, as desired.

For the next four O'Nan-Scott types, HC, Tw, Pa and Sd, we will denote $S = Q_1 \times \cdots \times Q_r$, where each Q_i is a nonabelian simple subgroup, and $r \geq 2$, because S is nonsimple. Therefore, using Corollary 3.1.6, we obtain that G acts transitively, by conjugation, on $\Sigma = \{Q_1, \ldots, Q_r\}$. So G satisfies the items 1 and 2 of the Embedding Hypothesis (Hypothesis 5.4.1).

G **has type HC or Tw:** In this case S is regular, so trivially G satisfies item 3 of the Embedding Hypothesis. Thus, by Lemma 6.1.1, we obtain that $H \leq \mathrm{soc}(G)$.

G **has type Pa:** In this case $C_G(S) = 1$ and, for a fixed $\alpha \in \Omega$, S_α is not a subdirect subgroup of S and S is not regular.

In general, the groups of this class do not satisfy item 3 of the Embedding

Hypothesis. In order to fix that, we will define a set $\overline{\Omega}$ on which G acts and, for some $\omega \in \overline{\Omega}$, the following property is satisfied

$$S_\omega = (S_\omega \pi_1) \times \cdots \times (S_\omega \pi_r). \tag{6.2}$$

First we observe that, for a fixed $\alpha \in \Omega$, we have that

$$S_\alpha \leq Q_1 \times \cdots \times Q_r.$$

Therefore,

$$S_\alpha \leq (S_\alpha)\pi_1 \times \cdots \times (S_\alpha)\pi_r. \tag{6.3}$$

Hence we denote $P := (S_\alpha \pi_1) \times \cdots \times (S_\alpha \pi_r) \leq S$ and $\overline{\Omega} := [S \colon P]$. Then S acts transitively by right multiplication on $\overline{\Omega}$. From the transitivity of S on Ω, we have that $G = SG_\alpha$ and, by Lemma 3.2.2, we obtain that G_α normalizes P. Moreover, according to the relation (6.3), $S_\alpha = S \cap G_\alpha \leq P$. Thus we can apply Lemma 2.3.1 to obtain that G has a transitive action on $\overline{\Omega}$ given by

$$(Px)^{sy} := P(y^{-1}xsy), \tag{6.4}$$

where $x, s \in S$ and $y \in G_\alpha$.

Recall that $S_\alpha < S$ and that $C_G(S) = 1$. Thus we can apply Lemma 3.2.1 to G and $\overline{\Omega}$, to conclude that the action above is faithful. We denote this action by $\eta \colon G \to \mathrm{Sym}(\overline{\Omega})$. This tells us that $G \cong G\eta$ is embedded in $\mathrm{Sym}(\overline{\Omega})$.

Since the items 1 and 2 of the Embedding Hypothesis are properties of G viewed as an abstract group, clearly $G\eta$ satisfies them too. We assert that $G\eta$ satisfies item 3 as well.

Let $\omega := P \in \overline{\Omega}$. Since the action of S on $\overline{\Omega}$ is right multiplication, then $S_\omega = P$. So

$$S_\omega = (S_\alpha \pi_1) \times \cdots \times (S_\alpha \pi_r). \tag{6.5}$$

We observe that the relation above says that, for each $i \in \underline{r}$, we have

$$S_\alpha \pi_i = Q_i \cap S_\omega = (Q_i)_\omega.$$

Therefore, $S_\alpha \pi_i = (Q_i)_\omega$, and we can write

$$S_\omega = (Q_1)_\omega \times \cdots \times (Q_r)_\omega.$$

But this is equivalent to saying that

$$S_\omega = (S_\omega\pi_1) \times \cdots \times (S_\omega\pi_r).$$

This shows that $G\eta$ satisfies item 3 of Embedding Hypothesis, as we wanted.

To show that $H \leq \mathrm{soc}(G)$, first we prove that H is transitive on $\overline{\Omega}$, and after we apply Lemma 6.1.1.

By Lemma 2.3.1 we have that $G_\omega = S_\omega G_\alpha$. So $G_\alpha \leq G_\omega$. Then by Lemma 2.1.2 follows that $\Delta := \alpha^{G_\omega}$ is a block for G. This tells us that the set

$$\Omega' := \{\Delta^g : g \in G\}$$

is a block system for G, where clearly G acts transitively.

Since $G_\alpha \leq G_\omega$, we have $G_\alpha \leq G_\Delta$. We assert that $G_\omega = G_\Delta$. In fact, applying Lemma 2.1.2, we have that $\alpha^{G_\Delta} = \Delta = \alpha^{G_\omega}$. Applying the same lemma again, we obtain that $G_\omega = G_\Delta$. Thus by Lemma 2.2.3 we conclude that the actions of G on $\overline{\Omega}$ and Ω' are equivalent.

Since H is transitive on Ω, we have that H is transitive on Ω'. As the actions of H on Ω' and $\overline{\Omega}$ are equivalent, we conclude that H is transitive on $\overline{\Omega}$.

Applying Corollary 5.5.2 to $H\eta$, we get that $H\eta$ normalizes each element of $\Sigma\eta = \{Q_1\eta, \ldots, Q_r\eta\}$. Therefore, by applying Lemma 6.1.1 to $G\eta$, it follows that $H\eta \leq \mathrm{soc}(G\eta) = S\eta$. However, since η is injective, $H \leq S = \mathrm{soc}(G)$.

G **has type Sᴅ:** In this case $C_G(S) = 1$ and, for a fixed $\alpha \in \Omega$, S_α is a subdirect subgroup of S and is simple. Since S is transitive on Ω and $S_\alpha \cong Q$, then $|\Omega| = |Q|^{r-1}$. For a fixed $j \in \underline{r}$, denote

$$\overline{Q}_j := Q_1 \times \cdots \times Q_{j-1} \times Q_{j+1} \times \cdots \times Q_r.$$

The idea is to decompose $H = H_0 \times H_1$ into two direct factors, where $H_0 \leq S$, and to prove that H_1 has to be trivial.

In this case each $Q_i \cong Q$, for some nonabelian simple group Q, and G can be considered [28, Section 2, case III(a)] as a subgroup of $\overline{G} := (S \cdot \mathrm{Out}(Q)) \rtimes S_r$, where S_r permutes the factors of S naturally and $\mathrm{Out}(Q)$ acts on $S \cong Q^r$ diagonally.

Consider the extension $\overline{Q} := S \cdot \mathrm{Out}(Q)$. We have that $\overline{G}$ permutes the elements in $\Sigma = \{Q_1, \ldots, Q_r\}$ and the kernel of this action is precisely $\overline{Q}$. If we denote by H_0 the kernel of H acting on Σ, we obtain that $H_0 = H \cap \overline{Q}$.

Since H is characteristically simple, we have by Lemma 3.1.4 (item 2) that $H_0 \cong T^{k_0}$ for some integer k_0, and there exists a normal subgroup H_1 of H such that $H = H_0 \times H_1$. It follows from the Isomorphism Theorem and from the definition of $\overline{Q}$ that

$$\frac{H_0}{H_0 \cap S} \cong \frac{H_0 S}{S} \leq \frac{\overline{Q}}{S} \cong \mathrm{Out}(Q),$$

and since $\mathrm{Out}(Q)$ is soluble by Schreier's Conjecture (Lemma 3.3.1), and H_0 is nonabelian and characteristically simple, we conclude that $H_0 = H_0 \cap S$, which means that $H_0 \leq S$.

Observe that if $H_1 = 1$, since $H_0 \leq S$, then $H \leq S$. Our task is precisely to prove that $H_1 = 1$. Then suppose that H_1 is not trivial. Note that

$$H_1 \cap \overline{Q} \leq H \cap \overline{Q} = H_0,$$

so $H_1 \cap \overline{Q} = 1$. This means that H_1 permutes the elements in Σ faithfully, so $|H_1| \mid r!$. In particular, since the size of the smallest nonabelian simple group is 60, we have $r \geq 5$.

We have that $H_0 \neq 1$. In fact, if that is not the case, we have $H = H_1$, so H_1 is transitive on Ω. Then applying the Orbit-Stabilizer Theorem and the transitivity of S, we obtain

$$\frac{|H_1|}{|(H_1)_\alpha|} = |\Omega| = |Q|^{r-1},$$

so $|Q|^{r-1} \mid |H_1|$. Since $|H_1| \mid r!$, we get that $|Q|^{r-1} \mid r!$. Since $4 \mid |Q|$ by Lemma 3.3.3, we have that $4^{r-1} \mid r!$, which is not possible by Lemma 3.4.2. Therefore, $H_0 \neq 1$.

Let's analyze the action of H_0 on Ω. Since $H_0 \trianglelefteq H$ and H is transitive, the orbits of H_0 form a block system for H. This means that H_0 is half transitive, that is, the orbits of H_0 have the same size. The Orbit-Stabilizer Theorem gives that

$$|\alpha^{H_0}| = \frac{|H_0|}{|(H_0)_\alpha|},$$

which means that $|(H_0)_\alpha|$ is independent of α. This enables us to calculate the number of H_0-orbits. In fact, the number of H_0-orbits is equal to

$$\frac{|Q|^{r-1}}{|\alpha^{H_0}|} = \frac{|Q|^{r-1}}{|T|^{k_0}}|(H_0)_\alpha|.$$

Since $H = H_0 \times H_1$ and H is transitive on Ω, we have that H_1 is transitive on the H_0-orbits. Then the Orbit-Stabilizer Theorem gives that the number of H_0-orbits divides $|H_1|$. Therefore, the number of H_0-orbits divides $r!$.

From the Isomorphism Theorem we have that

$$Q_i \geq H_0\pi_i \cong \frac{H_0}{\ker \pi_i \cap H_0} \cong T^{s_i}, \tag{6.6}$$

where $s_i \geq 0$ for all $i \in \underline{r}$. Thus we consider two cases:

Case 1: There exists $i \in \underline{r}$ such that $s_i \geq 2$.

We want to use Lemma 3.4.2 to show that this case is not possible. Since $s_i \geq 2$ for some $i \in \underline{r}$, there is a copy of T^2 in Q. Applying Theorem 3.3.2, let p be a prime such that the Sylow p-subgroups of Q are cyclic. If p divides $|T|$, let P be a Sylow p-subgroup of T and consider P^2, that is a Sylow p-subgroup of T^2. Since P is a p-subgroup of Q, there exists a Sylow p-subgroup of Q containing P, and the same is true for P^2. So P and P^2 are cyclic. However, this is not true, since P cyclic implies that P^2 is not cyclic. Then p does not divide $|T|$. Since the order of a finite nonabelian simple group is even by Lemma 3.4.2, we can assume that p is odd. Recall that the number of H_0-orbits is

$$\frac{|Q|^{r-1}}{|T|^{k_0}}|(H_0)_\alpha|. \tag{6.7}$$

Then p^{r-1} divides the number of H_0-orbits, and so it divides $r!$. However, this contradicts Lemma 3.4.2. Therefore, Case 1 is not possible.

Case 2: $s_i \leq 1$ for all $i \in \underline{r}$.

Since $H_0 \neq 1$, $s_i \geq 1$ for some $i \in \underline{r}$, thus relation 6.6 gives that there is a copy of T in Q. We want to write H_0 as a direct product of diagonal groups and to prove that H_1 can be embedded in the direct product of symmetric groups smaller than S_r.

First we observe that since each $T_i \leq H_0$ is simple, then each $T_i \leq H_0$ is a strip of S. We assert that if $i \neq j$, then $\mathrm{supp}(T_i) \cap \mathrm{supp}(T_j) = \varnothing$. In fact, if there is $k \in \underline{r}$ such that $T_i\pi_k \cong T$ and $T_j\pi_k \cong T$, then $(T_i \times T_j)\pi_k \cong T^2$. However, this is impossible, since hypothesis $s_k \leq 1$ implies that $H_0\pi_k \cong T$. Then the supports of the direct factors of H_0 are disjoint. Then Lemma 3.1.3 permits to write each T_i as a diagonal subgroup of the direct products of the elements on its support. Therefore,

$$H_0 = T_1 \times \cdots \times T_{k_0}, \tag{6.8}$$

where each $T_i \cong T$ is a diagonal subgroup of

$$\prod_{Q_j \in \operatorname{supp}(T_i)} Q_j,$$

in such a way that these subproducts do not have factors in common. As a consequence, we obtain that $k_0 \leq r$. We will treat the case $k_0 = r$ separately. Assume first that $k_0 < r$. Since $|T| \mid |Q|$, we have by 6.7 that $|Q|^{r-k_0-1}$ divides the number of H_0-orbits.

Let d_i be the cardinality of $\operatorname{supp}(T_i)$. Moreover, let m_1 be the number of T_i's for which $d_i \geq 5$, and let m_2 be the number of T_i's whose $d_i < 5$. So $m_1 + m_2 = k_0$. Relabeling if necessary, we can write

$$H_0 = T_1 \times \cdots \times T_{m_1} \times T_{m_1+1} \times \cdots \times T_{m_1+m_2},$$

where the first m_1 factors have length $d_i \geq 5$. Still, denote $m_3 := r - \sum_{i=1}^{k_0} d_i$, that is the number of factors Q_i that do not involve any of the T_i.

Since H_1 centralizes H_0, we have that H_1 centralizes each T_i. As given $h_1 \in H_1$ and $i \in \underline{r}$ we have

$$(\operatorname{supp}(T_i))^{h_1} = \operatorname{supp}(T_i^{h_1}) = \operatorname{supp}(T_i),$$

we conclude that each $\operatorname{supp}(T_i)$ is H_1-invariant.

Observe that if a group has a nonabelian simple group as a direct factor, since the smallest nonabelian simple group is A_5, then its size has to be at least 60. This means that if such a kind of group acts faithfully on a set with less than five elements, then the action is necessarily trivial. Therefore, as the action of H_1 on Σ is faithful, if $d_i < 5$, we obtain that H_1 acts trivially on $\operatorname{supp}(T_i)$.

According to the observation above, we have that H_1 acts trivially on

$$\operatorname{supp}(T_{m_1+1}) \dot{\cup} \cdots \dot{\cup} \operatorname{supp}(T_{k_0}).$$

Since the action of H_1 on Σ is faithful, we obtain that

$$|H_1| \mid (d_1!) \ldots (d_{m_1}!)(m_3!). \tag{6.9}$$

As we assumed $H_1 \neq 1$, we have from the relation above that either $m_1 = 0$ and $m_3 \geq 5$, or $m_1 \neq 0$. In both cases we conclude that

$$d_1 + \cdots + d_{m_1} + m_3 - m_1 - 1 > 0. \tag{6.10}$$

Recall that $|Q|^{r-k_0-1}$ divides $|H_1|$. So relation 6.9 says that

$$|Q|^{r-k_0-1} \mid (d_1!)\ldots(d_{m_1}!)(m_3!). \tag{6.11}$$

On the other hand, we have that $r \geq d_1 + d_2 + \cdots + d_{m_1} + m_2 + m_3$. So

$$|Q|^{d_1+d_2+\cdots+d_{m_1}+m_2+m_3} \mid |Q|^r.$$

Since 6.10 is valid and $k_0 = m_1 + m_2$, then

$$|Q|^{d_1+d_2+\cdots+d_{m_1}+m_2+m_3-m_1-m_2-1} = |Q|^{(d_1-1)+\cdots+(d_{m_1}-1)+m_3-1} \mid |Q|^{r-k_0-1}.$$

Therefore, using equation 6.11, we obtain that

$$|Q|^{(d_1-1)+\cdots+(d_{m_1}-1)+m_3-1} \mid (d_1!)\ldots(d_{m_1}!)(m_3!).$$

Since by Lemma 3.3.3 we have that $4 \mid |Q|$, then the previous line gives that

$$2^{d_1}\cdots 2^{d_{m_1}} 2^{m_3} \mid (d_1!)\ldots(d_{m_1}!)(m_3!),$$

which is a contradiction by Lemma 3.4.2. This implies that $H_1 = 1$, which means that if $k_0 < r$, then $H \leq S$.

Now consider the case where $k_0 = r$. Then relation 6.8 implies that $d_i = 1$ for all $i \in \underline{r}$. Since each $\mathrm{supp}(T_i)$ is H_1-invariant, we have that H_1 acts faithfully and trivially on Σ, thus $H_1 = 1$. Then if $k_0 = r$, we obtain $H = H_0 \leq S$.

Therefore, if (H, G) is a CharS-QP inclusion, where G has type SD, then $H \leq S$.

For the last O'Nan-Scott type CD, we will denote $S = Q_1 \times \cdots \times Q_r$, where each Q_i is a nonabelian simple subgroup and $r \geq 2$, because S is nonsimple.

G **has type CD:** In this case $C_G(S) = 1$ and for a fixed $\alpha \in \Omega$, S_α is a subdirect subgroup of S, but it is not simple. So Lemma 3.2.3 guarantees that there exist two sets $\overline{\Sigma} = \{S_1, \ldots, S_k\}$ and $\{D_1, \ldots, D_k\}$, where $k \geq 2$, each D_i is a full diagonal subgroup of S_i and $S_i = \prod_{Q_j \in \operatorname{supp} D_i} Q_j$, such that $S = S_1 \times \cdots \times S_k$, G_α acts transitively by conjugation on $\overline{\Sigma}$ and, considering the projections $\overline{\pi}_i \colon S \to S_i$, we have that

$$S_\alpha = S_\alpha\overline{\pi}_1 \times \cdots \times S_\alpha\overline{\pi}_k, \tag{6.12}$$

where each $S_\alpha\overline{\pi}_i = D_i$. Moreover, the actions of G_α on $\overline{\Sigma}$ and on $\{D_1, \ldots, D_k\}$ are equivalent.

It means that G satisfies all the items of the Embedding Hypothesis. For each $i \in \underline{k}$, let $\overline{S_i} := \prod_{j \neq i} S_j$. Thus G also satisfies the conditions of Example 5.2.4, and we conclude that $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_k\}$, where each Γ_i is the set of $\overline{S_i}$-orbits, is an S-normal cartesian decomposition of Ω preserved by G. Moreover, $S^{\Gamma_i} = S_i$ and the G-actions on $\overline{\Sigma}$ and on $\mathcal{E}$ are equivalent. That means that the actions of G_α on $\mathcal{E}$ and on $\{D_1, \ldots, D_k\}$ are equivalent.

Let G_{Γ_i} be the stabilizer in G of Γ_i. Now G_{Γ_i} induces a permutation group G^{Γ_i} on Γ_i for each $i \in \underline{k}$. Since $S \trianglelefteq G$, $S^{\Gamma_i} \trianglelefteq G^{\Gamma_i}$. We want to prove that S^{Γ_i} is a minimal normal subgroup of G^{Γ_i}. In order to do that, we want to use Corollary 3.1.6. Let $Q_m, Q_n \leq S_i$. As G_α is transitive on Σ, there exists $g \in G_\alpha$ such that ${Q_m}^g = Q_n$. Since the supports $\operatorname{supp}(D_i)$ are disjoint and ${Q_m}^g = Q_n \leq {S_i}^g \cap S_i$, we have that ${S_i}^g = S_i$. Thus ${\Gamma_i}^g = \Gamma_i$, then $g \in G_{\Gamma_i}$. Therefore, G_{Γ_i} permutes transitively the nonabelian simple factors of $S_i = S^{\Gamma_i}$. So according to Corollary 3.1.6, S^{Γ_i} is a minimal normal subgroup of G^{Γ_i}.

We want to show now that G^{Γ_i} is a permutation group of type SD. For each $i \in \underline{k}$, let γ_i be the part in Γ_i that contains α. Since $D_i \leq S_\alpha$, then by equation 5.3 we have $D_i \leq (S_i)_{\gamma_i}$. Conversely, if $s \in (S_i)_{\gamma_i}$, then s stabilizes γ_j for all $j \in \underline{k}$, and so s stabilizes the intersection $\bigcap_j \gamma_j = \{\alpha\}$. Thus $s \in S_i \cap S_\alpha = D_i$. Therefore, $(S_i)_{\gamma_i} = D_i$, and so each G^{Γ_i} has type SD. In particular, we have $\operatorname{soc}(G^{\Gamma_i}) = S^{\Gamma_i}$.

Since $S^{\Gamma_i} = S_i$, we conclude that

$$\operatorname{soc}(G) = \prod_i S_i = \prod_i \operatorname{soc}(G^{\Gamma_i}). \tag{6.13}$$

As G satisfies Embedding Hypothesis, then by Corollary 5.5.2 we have that H normalizes each element of $\overline{\Sigma}$. Since the G-actions on $\overline{\Sigma}$ and on $\mathcal{E}$ are equivalent, thus H acts trivially on $\mathcal{E}$, that is,

$$H \leq G_{(\mathcal{E})} \leq G^{\Gamma_1} \times \cdots \times G^{\Gamma_k}.$$

Since H is transitive on Ω, then H is transitive on each Γ_i. If we consider the projection map $\sigma_i \colon G^{\Gamma_1} \times \cdots \times G^{\Gamma_k} \to G^{\Gamma_i}$, then each $H\sigma_i$ is transitive on Γ_i. Since G^{Γ_i} has type SD, we can apply the previous case to obtain that $H\sigma_i \leq \mathrm{soc}(G^{\Gamma_i})$ for all $i \in \underline{k}$. Thus by equation (6.13), we have that

$$H \leq \prod_i \mathrm{soc}(G^{\Gamma_i}) = \mathrm{soc}(G) = S.$$

Therefore, if (H, G) is a CharS-QP inclusion, where G has type CD, then $H \leq S$.

This concludes the proof of the main theorem.

Chapter 7

CharS-QP Inclusions

In this chapter we will see how to apply the Main Theorem to describe CharS-QP inclusions by analyzing each O'Nan-Scott class with nonabelian plinth.

Throughout this chapter, let G be a finite quasiprimitive permutation group on Ω of type HS, HC, As, Tw, Pa, Sd or Cd, and let (H, G) be a CharS-QP inclusion. Assume that $S = Q_1 \times \cdots \times Q_r$ is a nonabelian plinth for G, where each $Q_i \cong Q$ for a nonabelian simple group Q, so that $\mathrm{soc}(G) = S \times C_G(S)$. In case $C_G(S)$ is nontrivial, then it is isomorphic to S. Moreover, we consider the projections $\pi_i \colon \mathrm{soc}(G) \to Q_i$ of $\mathrm{soc}(G)$ onto its direct factors. According to Theorem 6.0.1, $H \leq \mathrm{soc}(G)$.

If $C_G(S) = 1$, in order to get some information about H, we will analyze the image of H under the projections $\pi_i \colon S \to Q_i$. In the next lemma we see that this approach restricts the possible O'Nan Scott class of G.

Lemma 7.0.1. *Let (H, G) be a CharS-QP inclusion and let S be a nonabelian plinth of G such that $C_G(S) = 1$. Suppose that for some $i_0 \in \underline{r}$ we have that $H\pi_{i_0} = 1$. Then G has type Sd or Cd.*

Proof. We have that $S = \mathrm{soc}(G)$ and by Theorem 6.0.1, $H \leq S$. Since H is transitive, we can write $S = HS_\alpha$. Therefore, using that π_{i_0} is surjective and that $H\pi_{i_0} = 1$, we have

$$Q_{i_0} = S\pi_{i_0} = (HS_\alpha)\pi_{i_0} = (H\pi_{i_0})(S_\alpha\pi_{i_0}) = S_\alpha\pi_{i_0}.$$

Fixing an arbitrary $j \in \underline{r}$, it follows from the transitivity of G_α on the factors Q_i that there exists $g \in G_\alpha$ such that ${Q_{i_0}}^g = Q_j$. Then

$$Q_j = {Q_{i_0}}^g = (S_\alpha\pi_{i_0})^g = S_\alpha\pi_j.$$

Since j was arbitrary, we conclude that $Q_j = S_\alpha \pi_j$ for all $j \in \underline{r}$. This means that S_α is a subdirect subgroup of S, so it follows from Theorem 2.5.4 that G has type SD or CD, which completes the proof. □

So by applying Scott's Lemma (Lemma 3.1.4), we obtain that if $H\pi_{i_0} = 1$ for some $i_0 \in \underline{r}$, then

$$S_\alpha = D_1 \times \cdots \times D_l,$$

where each D_i is a nonabelian diagonal simple group isomorphic to Q and l divides r.

7.1 G has type As

In this case S is simple and $C_G(S) = 1$, thus $S = \mathrm{soc}(G)$.

Lemma 7.1.1. *Let (H, G) be a* CharS-QP *inclusion where G has type As, H is nonsimple and $S = \mathrm{soc}(G)$. Then $S \cong A_n$ and $G_\alpha \cap S \cong A_{n-1}$, where $n = |G : G_\alpha| = |S : G_\alpha \cap S| \geq 10$.*

Proof. We have that $H \cong T^k$, where $k \geq 2$. Using that H is transitive, it follows from Lemma 2.1.1 that $G = G_\alpha H$, in which $H \nleq G_\alpha \lneq G$. Then we are in the conditions of [2, Theorem 1.4], so

$$S \cong A_n \text{ and } G_\alpha \cap S \cong A_{n-1},$$

where $n = |G : G_\alpha| = |S : G_\alpha \cap S| \geq 10$. □

In fact, this lemma says that either H is a transitive simple subgroup of S, that is also a simple group, or H is a nonsimple transitive characteristically simple subgroup of $S \cong A_n$, in the natural action of A_n on the set $\underline{n}$ for $n \geq 10$.

7.2 G has type Tw

In this case S is regular and $C_G(S) = 1$, thus $S = \mathrm{soc}(G)$.

Lemma 7.2.1. *Let (H, G) be a* CharS-QP *inclusion where G has type Tw and $S = \mathrm{soc}(G)$. Then $H = S$.*

Proof. According to Main Theorem 6.0.1, we have that $H \leq S$. Since S is regular, H is also regular, and we have from the Orbit-Stabilizer Theorem that

$$|S| = |\Omega| = |H|.$$

Then we conclude that $H = S$. □

In particular, the result above says that if (H, G) is a CharS-QP inclusion where G has type Tw, then H is regular.

7.3 G has type SD

In this case $C_G(S) = 1$ and, for a fixed $\alpha \in \Omega$, S_α is a subdirect subgroup of S and is simple.

From Lemma 3.1.3 we have that

$$S_\alpha = D_1 = \{(q\varphi_1, q\varphi_2, \dots, q\varphi_r) \colon q \in Q\} \cong Q,$$

where $\varphi_i \colon Q \to Q_i$ is an isomorphism for each $i \in \underline{r}$. From now on we will use these isomorphisms to write arbitrary elements of S in the form

$$(q_1\varphi_1, q_2\varphi_2, \dots, q_r\varphi_r).$$

Since S is transitive on Ω and $S_\alpha \cong Q$, then $|\Omega| = |Q|^{r-1}$. For a fixed $j \in \underline{r}$, denote

$$\overline{Q}_j := Q_1 \times \cdots \times Q_{j-1} \times Q_{j+1} \times \cdots \times Q_r.$$

Lemma 7.0.1 suggests that to obtain some information about H and G, it is helpful to analize the projections of H on the direct factors of S. So given $i_0 \in \underline{r}$, we have three options: $H\pi_{i_0} = 1$, $H\pi_{i_0} = Q_{i_0}$ or $1 < H\pi_{i_0} < Q_{i_0}$. We obtain the following characterization.

Theorem 7.3.1. *Let G be a finite quasiprimitive permutation group on Ω of type SD and let $S = Q_1 \times \cdots \times Q_r$ be a plinth of G. Consider the projections $\pi_i \colon S \to Q_i$, where $i \in \underline{r}$, and let (H, G) be a* CharS-QP *inclusion in which $H \cong T^k$. Then*

1. *If $H\pi_{i_0} = 1$ for some $i_0 \in \underline{r}$, then $H = \overline{Q}_{i_0}$ (in this case H is regular).*
2. *If $H\pi_{i_0} = Q_{i_0}$ for some $i_0 \in \underline{r}$, then $H = S$ (in this case H is not regular) or $H = \overline{Q}_j$ for some $j \in \underline{r}$ (in this case H is regular).*

3. *If* $1 < H\pi_j < Q_j$ *for all* $j \in \underline{r}$*, then* $r = k = 2$*,* $T \cong A \cong B$*,* $H = A \times B$ *and* $Q = \overline{A}\,\overline{B}$*, where* $\overline{A} \cong A$*,* $\overline{B} \cong B$*, and* Q*,* A *and* B *are described in Table 4.3 (in this case* H *is not regular).*

Proof. **Case 1:** Suppose that $H\pi_{i_0} = 1$ for some $i_0 \in \underline{r}$.

Without loss of generality, assume that $i_0 = 1$ and let

$$\overline{Q}_1 = \{(1, q_2\varphi_2, \ldots, q_r\varphi_r) \colon q_i \in Q\}.$$

Notice that $S = \overline{Q}_1 S_\alpha$. In fact, we have, for all $q_1, \ldots, q_r \in Q$, that

$$(q_1\varphi_1, q_2\varphi_2, \ldots, q_r\varphi_r) = (1, (q_2{q_1}^{-1})\varphi_2, \ldots, (q_r{q_1}^{-1})\varphi_r)(q_1\varphi_1, q_1\varphi_2, \ldots, q_1\varphi_r).$$

Since the first element belongs to $\overline{Q}_1$ and the second one to S_α, we have that $S = \overline{Q}_1 S_\alpha$, which implies that $\overline{Q}_1$ is transitive. Moreover, as $S_\alpha \cap \overline{Q}_1 = 1$, then $\overline{Q}_1$ is a regular group. Since $H \leq \overline{Q}_1$ and H is transitive, we obtain that $\overline{Q}_1 = H$. This proves item 1.

Case 2: Consider now the case in which $H\pi_{i_0} = Q_{i_0}$ for some $i_0 \in \underline{r}$.

In this case Q_{i_0} is a composition factor of H, which means by the Jordan-Hölder Theorem [21, Theorem VII.1.8] that $Q \cong T$, and so $k \leq r$. Since S is transitive on Ω,

$$|\Omega| = |S \colon S_\alpha| = |Q|^{r-1} = |T|^{r-1}.$$

As H is transitive on Ω,

$$|T|^{r-1} = |\Omega| = \frac{|H|}{|H_\alpha|} = \frac{|T|^k}{|H_\alpha|}.$$

Since $k \leq r$, then $k = r - 1$ or $k = r$. If $k = r$, then $H = S$. Otherwise, assume now that $k = r - 1$.

Given $j \in \underline{r}$, since $H\pi_j$ is a homomorphic image of H, by the Isomorphism Theorem

$$H\pi_j \cong T^{s_j} \cong Q^{s_j}.$$

Since $H\pi_j \leq Q_j$, thus $s_j \in \{0, 1\}$. If $s_j = 0$ for some $j \in \underline{r}$, then we are in the previous case ($H = \overline{Q}_j$). So assume that $s_j = 1$ for all $j \in \underline{r}$. We will see that this option is not possible. If it is true, thus H is a subdirect subgroup of S and, by Scott's Lemma, H is a direct product

$$H = R_1 \times \cdots \times R_m,$$

of full diagonal subgroups of subproducts $\prod_{l \in I_j} Q_l$, in which $\{I_j\}$ is a partition for $\underline{r}$. Since each $R_j \cong Q$ and $H \cong Q^{r-1}$, we have that $m = r-1$ and, without loss of generality, we can assume $R_1 \leq Q_1 \times Q_2$ and $R_j \leq Q_{j+1}$ for all $2 \leq j \leq r-1$.

Let $\overline{\pi}\colon S \to Q_1 \times Q_2$ be a projection. Since $\overline{\pi}$ is surjective and $S = HS_\alpha$, then

$$Q^2 \cong Q_1 \times Q_2 = S\overline{\pi} = (H\overline{\pi})(S_\alpha\overline{\pi}).$$

Notice that $H\overline{\pi}$ and $S_\alpha\overline{\pi}$ are non-trivial full strips of $Q_1 \times Q_2$. However, from Lemmas 4.2.2 and 4.2.3, this is not possible. Thus $s_j = 0$ must hold for some $j \in \underline{r}$. Therefore, either $H = S$ or $H = \overline{Q_j}$ for some $j \in \underline{r}$, and item 2 is proved.

Case 3: Consider now the case in which $1 < H\pi_j < Q_j$ for all $j \in \underline{r}$.

We have that S_α is a full diagonal subgroup of S. Since H is transitive, then $S = HS_\alpha$ by Lemma 2.1.1. So we are under the hypotheses of Corollary 4.1.4. Then $r = 2$, $T \cong A \cong B$, $H = A \times B$ and $Q \cong AB$, where Q, A and B are described in Table 4.3. Therefore, item 3 is proved. □

Corollary 7.3.2. *Let G be a finite quasiprimitive permutation group on Ω of type SD and let $S \cong Q^r$ be a plinth of G, where Q is a nonabelian simple group. If (H, G) is a* CharS-QP *inclusion and H is regular, then $H \cong Q^{r-1}$. Otherwise, if H is not regular, then either $H = S$ or $H = A \times B$ and $Q = \overline{A}\,\overline{B}$, where $\overline{A} \cong A$, $\overline{B} \cong B$, and Q, A and B are described in Table 4.3.*

7.4 G has type HS

In this case G is primitive, and $\mathrm{soc}(G) = S \times C_G(S)$, where both S and $C_G(S)$ are simple and regular.

Theorem 7.4.1. *Let G be a finite quasiprimitive permutation group on Ω of type HS and let S be a plinth of G. If (H, G) is a* CharS-QP *inclusion, then one of the following holds.*

1. *Either $H = S$ or $H = C_G(S)$ (in this case H is regular).*

2. *$H = \mathrm{soc}(G)$ (in this case H is not regular).*

3. *$H = A \times B$ and $S \cong AB$, where S, A and B are described in Table 4.3 (in this case H is not regular).*

Proof. According to [8, Lemma 5.1], there is an involution $i \in N_{\mathrm{Sym}(\Omega)}(G)$ that interchanges S and $C_G(S)$. In particular this says that $S \cong C_G(S)$. Consider $\overline{G} := \langle G, i\rangle \leq \mathrm{Sym}(\Omega)$. Since $G \leq \overline{G}$ and G is primitive, thus $\overline{G}$ is also primitive. If $P := \mathrm{soc}(G)$, then $P \trianglelefteq G$ and P is normalized by i, then we have that $P \trianglelefteq \overline{G}$, and by Lemma 3.1.4 (item 2) P is a minimal normal subgroup of $\overline{G}$. By Lemma 3.1.8 P_α is a subdirect subgroup of P. Since $P \cong S^2$, we have by Lemma 3.1.4 (item 1) that P_α is a simple diagonal subgroup of P. Hence by O'Nan-Scott Theorem (Theorem 2.5.4) $P = \mathrm{soc}(\overline{G})$ and $\overline{G}$ has type SD. Since $H \leq G \leq \overline{G}$, then $(H, \overline{G})$ is a CharS-QP inclusion. By Theorem 6.0.1, $H \leq P$. Therefore, applying Theorem 7.3.1 and analyzing the possibilities, the result is proved. □

Corollary 7.4.2. *Let G be a finite quasiprimitive permutation group on Ω of type HS and let S be a plinth of G. If (H, G) is a CharS-QP inclusion and H is regular, then either $H = S$ or $H = C_G(S)$. Otherwise, if H is nonregular, then either $H = \mathrm{soc}(G)$ or $H = A \times B$ and $S \cong AB$, where S, A and B are described in Table 4.3.*

7.5 G has type PA

In this case $C_G(S) = 1$, hence $\mathrm{soc}(G) = S$ and, for a fixed $\alpha \in \Omega$, S_α is not a subdirect subgroup of S and S is not regular.

Theorem 7.5.1. *Let G be a finite quasiprimitive permutation group on Ω of type PA and let $S = Q_1 \times \cdots \times Q_r$ be a plinth of G. Consider the projections $\pi_i \colon S \to Q_i$, where $i \in \underline{r}$, and let (H, G) be a CharS-QP inclusion in which $H \cong T^k$. Then one of the following holds.*

1. *$S \cong (A_n)^r$, $S_\alpha \cong (A_{n-1})^r$ and $|\Omega| = n^r$, where $n \geq 10$.*
2. *T is isomorphic to one of the groups A_6, M_{12}, $\mathrm{PSp}(4, 2^a)$ or $\mathrm{P}\Omega_8^+(q)$.*
3. *$k = r$ and $T_i \leq Q_i$ for all $i \in \underline{r}$.*

Proof. Let $P := S_\alpha\pi_1 \times \cdots \times S_\alpha\pi_r$. According to Lemma 3.2.2, we have that G_α acts transitively on the direct factors of P. As a consequence, since S_α is not a subdirect subgroup of S, then $1 < S_\alpha\pi_i < Q_i$ for all $i \in \underline{r}$.

Since $H \cong T^k$, by Isomorphism Theorem and Lemma 3.1.4 (item 2) we have that $H\pi_i \cong T^{s_i}$ for each $i \in \underline{r}$, where $s_i \geq 0$. However, as $S = S_\alpha H$ (Lemma 2.1.1),

$$Q_i = S\pi_i = (S_\alpha\pi_i)(H\pi_i). \tag{7.1}$$

Since $S_\alpha\pi_i$ is proper in Q_i, then $H\pi_i \neq 1$. Therefore, $s_i \geq 1$ for all $i \in \underline{r}$.

Case 1: Suppose that $s_{i_0} \geq 2$ for some $i_0 \in \underline{r}$.

We have that $H\pi_i \neq Q_i$ for all $i \in \underline{r}$. Otherwise, Q would be a composition factor of H and so $T \cong Q$. Since $s_{i_0} \geq 2$, this is not possible. Therefore, $1 < H\pi_i < Q_i$ for all $i \in \underline{r}$.

Consider the factorization in 7.1. By [2, Theorem 1.4], we have that $Q \cong A_n$ and $S_\alpha\pi_i \cong A_{n-1}$ for all $i \in \underline{r}$, where $n \geq 10$. In particular, $P \cong (A_{n-1})^r$.

Claim. $S_\alpha = P$.

Proof of Claim. Assume that $S_\alpha \neq P$. Since S_α is a subdirect subgroup of P, by Scott's Lemma (Lemma 3.1.4), S_α is the direct product of diagonal groups

$$S_\alpha = D_1 \times \cdots \times D_l$$

for some $l \leq r$. Renumbering if necessary and using that $S_\alpha \neq P$, we can assume that $D_1 \leq Q_1 \times \cdots \times Q_m$, where $2 \leq m \leq r$. Consider the projection $\overline{\pi}\colon S \to Q_1 \times Q_2$. Since $S = S_\alpha H$, then

$$Q_1 \times Q_2 = S\overline{\pi} = (S_\alpha\overline{\pi})(H\overline{\pi}),$$

where $S_\alpha\overline{\pi} = \{(q\varphi_1, q\varphi_2)\colon q \in A_{n-1}\}$ and $\varphi_i\colon A_{n-1} \to S_\alpha\pi_i$ are isomorphisms for $i = 1, 2$. Since $n \geq 10$, and in this case the automorphisms of A_n are conjugations by elements in S_n [37, 2.4.1], we can extend the isomorphisms φ_i to A_n, that is, for $i = 1, 2$ there exist isomorphisms $\overline{\varphi_i}\colon A_n \to Q_i$ such that the restriction of $\overline{\varphi_i}$ to A_{n-1} is equal to φ_i, and so

$$Q_1 \times Q_2 = \overline{D}(H\overline{\pi}) = \overline{D}(H\pi_1 \times H\pi_2),$$

where $\overline{D} = \{(q\overline{\varphi_1}, q\overline{\varphi_2})\colon q \in A_n\}$. By Corollary 4.1.4 we have that the possibilities for Q and H are in Table 4.3. Since we already know that $Q \cong A_n$ with $n \geq 10$, we obtain a contradiction. Therefore $S_\alpha = P$, as desired. □

Applying the Orbit-Stabilizer Theorem, we see that

$$|\Omega| = \frac{|S|}{|S_\alpha|} = n^r.$$

So if $s_i \geq 2$ for some $i \in \underline{r}$, then $S \cong (A_n)^r$, $S_\alpha \cong (A_{n-1})^r$ and $|\Omega| = n^r$, where $n \geq 10$.

Case 2: Suppose that $s_i = 1$ for all $i \in \underline{r}$.

Since T_i is simple, we have that T_i is a strip of S for all $i \in \underline{r}$. Moreover, the supports of each T_i are pairwise disjoint. In fact, if for some l we have $T_i\pi_l \cong T \cong T_j\pi_l$ for distinct $i, j \in \underline{r}$, then

$$T^2 \cong (T_i \times T_j)\pi_l \leq H\pi_l \cong T,$$

that is an absurd. Then the supports are pairwise disjoint and we can write

$$\begin{array}{lcl} T_1 & \leq & Q_1 \times \cdots \times Q_{l_1}, \\ T_2 & \leq & Q_{l_1+1} \times \cdots \times Q_{l_1+l_2}, \\ \vdots & \vdots & \vdots \\ T_k & \leq & Q_{l_1+l_2+\cdots+l_{k-1}+1} \times \cdots \times Q_{l_1+l_2+\cdots+l_k}. \end{array}$$

First suppose that $l_i \geq 2$ for some $i \in \underline{r}$. Renumbering, if necessary, assume that $l_1 \geq 2$.

Write $l := l_1$ and consider the projection map $\overline{\pi}\colon S \to Q_1 \times \cdots \times Q_l$. As $S = HS_\alpha$, then

$$(H\overline{\pi})(S_\alpha\overline{\pi}) = Q_1 \times \cdots \times Q_l.$$

Write $L := S_\alpha\pi_1 \times \cdots \times S_\alpha\pi_l$. Since $H\overline{\pi} = T_1$ and $S_\alpha\overline{\pi} \leq L$, thus

$$T_1 L = Q_1 \times \cdots \times Q_l.$$

Therefore, T_1 acts transitively, by right multiplication, on the coset

$$[Q_1 \times \cdots \times Q_l \colon L].$$

Since T_1 is simple, this action is faithful. Consider $U := S_\alpha\pi_1$ and fix, for each

$i \in \underline{l}$, $t_i \in G_\alpha$ such that $Q_i^{t_i} = Q_1$. So we define the map

$$\begin{array}{rccc} \lambda\colon & [Q_1 \times \cdots \times Q_l\colon L] & \to & [Q_1\colon U]^l \\ & L(q_1, \ldots, q_l) & \mapsto & (Uq_1{}^{t_1}, \ldots, Uq_l{}^{t_l}). \end{array}$$

Analogously to the proof of Lemma 5.4.4, we obtain that λ is a bijection. So by Lemma 2.2.2, we obtain that the actions of T_1 on $[Q_1 \times \cdots \times Q_l\colon L]$ and $[Q_1\colon U]^l$ are equivalent. If we write $\Gamma := [Q_1\colon U]$, thus T_1 can be considered a transitive subgroup of $\mathrm{Sym}(\Gamma^l)$. We want to prove that $T_1 \leq (\mathrm{Sym}(\Gamma))^l$. To see this, let $t = (x_1, \ldots, x_l) \in T$. Fix $i \in \underline{l}$ and consider

$$\begin{array}{rccc} \sigma_i\colon & \Gamma & \to & \Gamma \\ & Uq_i{}^{t_i} & \mapsto & U(q_i x_i)^{t_i}, \end{array}$$

where each $q_i \in Q_i$. We observe that in the definition above i, t_i and x_i are fixed. Analogously to the proof of Lemma 5.4.7, we obtain that σ_i is a permutation of Γ.

Finally, note that

$$\begin{aligned} (Uq_1^{t_1}, \ldots, Uq_l^{t_l})^t &= (U(q_1x_1)^{t_1}, \ldots, U(q_lx_l)^{t_l}) \\ &= ((Uq_1^{t_1})\sigma_1, \ldots, (Uq_l^{t_l})\sigma_l) \\ &= (Uq_1^{t_1}, \ldots, Uq_l^{t_l})^{(\sigma_1, \ldots, \sigma_l)}. \end{aligned}$$

Therefore, $t = (\sigma_1, \ldots, \sigma_l) \in (\mathrm{Sym}(\Gamma))^l$, and since t was arbitrary, we conclude that $T_1 \leq (\mathrm{Sym}(\Gamma))^l$. According to Theorem 1.2.1, we have that T is isomorphic to one of the groups A_6, M_{12}, $\mathrm{PSp}(4, 2^a)$ or $\mathrm{P}\Omega_8^+(q)$. Further, in this case, $l = 2$.

Now suppose that $l_i = 1$ for all $i \in \underline{r}$. Then $k = r$ and $T_i \leq Q_i$ for all $i \in \underline{r}$. Therefore, the result is proved. □

7.6 G has type C_D

In this case $C_G(S) = 1$ and for a fixed $\alpha \in \Omega$, S_α is a subdirect subgroup of S, but it is not simple.

Theorem 7.6.1. *Let G be a finite quasiprimitive permutation group on Ω of type* C_D *and let $S = Q_1 \times \cdots \times Q_r$ be a plinth of G, where each Q_i is isomorphic to a nonabelian simple group Q. Consider the projections $\pi_i\colon S \to Q_i$, where $i \in \underline{r}$, and let (H, G) be a* CharS-QP *inclusion in which $H = T_1 \times \cdots \times T_k$ and* *each $T_i \cong T$ for a nonabelian simple group T. Then*

1. *If $1 < H\pi_j < Q_j$ for all $j \in \underline{r}$, then $k = r$, $H \cong A^r$ and $Q = \overline{A}\,\overline{B}$, where $\overline{A} \cong A$, $\overline{B} \cong B$, and Q, A and B are described in Table 4.3.*

2. *If $H\pi_{i_0} \in \{1, Q_{i_0}\}$ for some $i_0 \in \underline{r}$, then $S_\alpha = D_1 \times \cdots \times D_l$, where l divides r, and $T \cong Q$, where each D_i and each T_i are full strips of S such that $|\operatorname{supp}(T_i) \cap \operatorname{supp}(D_j)| \leq 1$ and $|\operatorname{supp}(D_j) \cap \operatorname{supp}(H)| = r/l$ or $r/l - 1$, for all $i \in \underline{k}$ and $j \in \underline{l}$. Among the options for H satisfying these properties, Algorithm 4.1 decides those that satisfy $S = S_\alpha H$.*

Proof. Lemma 3.2.3 guarantees that there exist two sets $\overline{\Sigma} := \{S_1, \ldots, S_l\}$ and $\overline{D} := \{D_1, \ldots, D_l\}$, where $l \geq 2$, each D_i is a full diagonal subgroup of S_i and $S_i = \prod_{Q_j \in \operatorname{supp} D_i} Q_j$, such that $S = S_1 \times \cdots \times S_k$ and

$$S_\alpha = D_1 \times \cdots \times D_l \cong Q^l. \tag{7.2}$$

As we saw in the proof of the Main Theorem (Section 6.2, case G has type CD), there is an S-normal cartesian decomposition $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_l\}$ preserved by G such that $G^{\Gamma_i} \leq \operatorname{Sym}(\Gamma_i)$ has type SD with socle S_i, for all $i \in \underline{l}$. Moreover, if we consider the projections $\overline{\pi}_i \colon S \to S_i$, then $(H\overline{\pi}_i, G^{\Gamma_i})$ is a CharS-QP inclusion.

Case 1: Suppose that $1 < H\pi_j < Q_j$ for all $j \in \underline{r}$.

Applying Theorem 7.3.1 (item 3) to the inclusion $(H\overline{\pi}_i, G^{\Gamma_i})$, we obtain that $|\operatorname{supp}(D_i)| = 2$ for all $i \in \underline{l}$, that is, r is even and $S_i \cong Q^2$, $H\pi_i \cong A$ for all $i \in \underline{r}$ and $T \cong A$, which implies that $H \cong A^k$ where $k \leq r$, and $Q = \overline{A}\,\overline{B}$, where $\overline{A} \cong A$, $\overline{B} \cong B$, and Q, A and B are described in Table 4.3.

Renumbering, if necessary, consider

$$S = \underbrace{Q_1 \times Q_2}_{S_1} \times \underbrace{Q_3 \times Q_4}_{S_2} \times \cdots \times \underbrace{Q_{r-1} \times Q_r}_{S_{r/2}}.$$

Claim. $H = H\pi_1 \times \cdots \times H\pi_r$.

Proof of Claim. It is equivalent to prove that $H\pi_i \leq H$ for all $i \in \underline{r}$. So assume the opposite, that is, $H\pi_{i_1} \not\leq H$ for some $i_1 \in \underline{r}$. Then H has a nontrivial strip $X \cong A$ such that $\operatorname{supp}(X) = \{Q_{i_1}, \ldots, Q_{i_m}\}$, where $m \geq 2$.

Notice that the two factors Q_j in each S_i never appear together in $\operatorname{supp}(X)$, that is, $\operatorname{supp}(D_i) \not\subseteq \operatorname{supp}(X)$ for each $i \in \underline{l}$. In fact, if we assume that $Q_1, Q_2 \in$

supp(X) and consider the projection $\pi_1 \colon S \to S_1$, since $S = S_\alpha H$, thus

$$S_1 = S\overline{\pi}_1 = (S_\alpha\overline{\pi}_1)(H\overline{\pi}_1) = (D_1)(X\overline{\pi}_1).$$

However, analyzing the orders,

$$|Q|^2 = \frac{|Q||A|}{|D_1 \cap X\overline{\pi}_1|} < |Q|^2,$$

an absurd. Notice that there is nothing special about Q_1, Q_2. Therefore, $\mathrm{supp}(D_i) \nsubseteq \mathrm{supp}(X)$ for all $i \in \underline{l}$. Again, renumbering if necessary, we may assume that $Q_2, Q_3 \in \mathrm{supp}(X)$. Considering the projection $\overline{\pi} \colon S \to S_1 \times S_2$, we have

$$S_1 \times S_2 = S\overline{\pi} = (S_\alpha\overline{\pi})(H\overline{\pi}) = (D_1 \times D_2)(H\overline{\pi}), \tag{7.3}$$

where $H\overline{\pi}$ is contained in a subgroup $\overline{H}$ of $S_1 \times S_2$ that is isomorphic to A^3. Let $P := (D_1 \times D_2) \cap (\overline{H})$. Then $|Q|^4 = \frac{|Q|^2|A|^3}{|P|}$, thus

$$|P| = \frac{|A|^3}{|Q|^2}. \tag{7.4}$$

For what follows, consult Appendix A if necessary.

Suppose that $Q \cong A_6$ and $A \cong A_5$. Then by (7.4) $|P| = 5/3$, which is an absurd. Therefore, if $Q \cong A_6$, then H does not have strips.

Assume that $Q \cong \mathrm{P}\Omega_8^+(q)$ and $A \cong \Omega_7(q)$. Then by (7.4)

$$|P| = d.q^3.\frac{(q^6 - 1)}{(q^2 + 1)}.$$

We will prove that there exists an odd prime that divides $q^2 + 1$ but does not divide $d.q^3.(q^6 - 1)$. If q is even, then $q^2 + 1$ is odd, then there exists an odd prime p that divides $q^2 + 1$. On the other hand, if q is odd, then $q^2 + 1 \equiv 2 \pmod 4$. Thus $q^2 + 1$ is even but it is not a 2-power, so there exists an odd prime p that divides $q^2 + 1$. Therefore, in any of the cases, there exists an odd prime p that divides $q^2 + 1$. We want to prove that p does not divide $d.q^3.(q^6 - 1)$. Since p is odd, then p does not divide $q^2 - 1$. As $q^6 - 1 = (q^2 - 1).(q^4 + q^2 + 1)$, and p does not divide $q^2 - 1$ but divides $q^2 + 1$, thus p does not divide $q^6 - 1$. Then p is the prime which we are looking for. This means that also $|P|$ is not an integer, which is an absurd. Thus if $Q \cong \mathrm{P}\Omega_8^+(q)$, H does not have strips.

Finally, suppose that $Q \cong M_{12}$ and $A \cong M_{11}$. Then by (7.4), $|P| = 55$.

Denote $A_i := H\pi_i$. By Lemma 3.1.3, there exist isomorphisms $\varphi_2 \colon Q_1 \to Q_2$, $\varphi_4 \colon Q_3 \to Q_4$ and $\psi \colon A_2 \to A_3$ such that

$$\begin{aligned} D_1 \times D_2 =& \{(p, p\varphi_2, q, q\varphi_4) \colon p \in Q_1, q \in Q_3\}, \\ \overline{H} =& \{(a_1, a_2, a_2\psi, a_4) \colon a_i \in A_i\}. \end{aligned}$$

Consider isomorphisms $\eta_i \colon Q \to Q_i$. Then we define the isomorphism

$$\phi \colon S_1 \times S_2 \to Q^4,$$

where $\phi = (\eta_1^{-1}, \varphi_2^{-1}\eta_1^{-1}, \eta_3^{-1}, \varphi_4^{-1}\eta_3^{-1})$. Let $\overline{A}_1 := A_1\eta_1^{-1}$, $\overline{A}_2 := A_2\varphi_2^{-1}\eta_1^{-1}$, $\overline{A}_3 := A_3\eta_3^{-1}$ and $\overline{A}_4 := A_4\varphi_4^{-1}\eta_3^{-1}$. Thus $\overline{A}_i \cong A$ and applying ϕ in (7.3), we obtain $Q^4 = (D_1 \times D_2)\phi(\overline{H}\phi)$, where $\overline{\psi} \colon \overline{A}_2 \to \overline{A}_3$ is an isomorphism and

$$\begin{aligned} (D_1 \times D_2)\phi =& \{(p, p, q, q) \colon p, q \in Q\}, \\ \overline{H}\phi =& \{(\overline{a}_1, \overline{a}_2, \overline{a}_2\overline{\psi}, \overline{a}_4) \colon \overline{a}_i \in \overline{A}_i\}. \end{aligned}$$

However, this an absurd by Lemma 4.2.6. Then if $Q \cong M_{12}$, H does not have strips.

Therefore, $H = H\pi_1 \times \cdots \times H\pi_r$ as asserted. □

It means that if $1 < H\pi_i < Q_i$ for all $i \in \underline{r}$, then $k = r$ and so $H \cong A^r$.

Case 2: Suppose that $H\pi_{i_0} = Q_{i_0}$ for some $i_0 \in \underline{r}$.

In this case Q_{i_0} is a composition factor of H, which means by the Jordan-Hölder Theorem [21, Theorem VII.1.8] that $Q \cong T$, and so $k \leq r$. So each T_i is a full strip of S.

Since $S = S_\alpha H$ by Lemma 2.1.1, we have by Corollary 4.3.3 (item 3) that $|\operatorname{supp}(T_i) \cap \operatorname{supp}(D_j)| \leq 1$ and $|\operatorname{supp}(D_j) \cap \operatorname{supp}(H)| = r/l$ or $r/l - 1$ for all $i \in \underline{k}$ and $j \in \underline{l}$. Among the options for H satisfying these properties, Algorithm 4.1 decides those that satisfy $S = S_\alpha H$.

Case 3: Suppose that $H\pi_{i_0} = 1$ for some $i_0 \in \underline{r}$.

Let $j_0 \in \underline{l}$ such that $Q_{i_0} \leq S_{j_0} = Q_{i_0} \times \cdots Q_{i_{r/l-1}}$. Since $(H\overline{\pi}_{j_0}, G^{\Gamma_{j_0}})$ is a CharS-QP inclusion where $G^{\Gamma_{j_0}}$ has type SD, by applying Theorem 7.3.1 (item 1) we obtain that $H\overline{\pi}_{j_0} = Q_{i_1} \times \cdots Q_{i_{r/l-1}} \cong Q^{r/l-1}$. Then $H\pi_{i_1} = Q_{i_1}$. So this case is a particular case of Case 2. □

7.7 G has type HC

In this case G is primitive, and $\mathrm{soc}(G) = S \times C_G(S)$, where both S and $C_G(S)$ are regular and nonsimple. Since S is isomorphic to $C_G(S)$, if $S \cong Q^r$ then $\mathrm{soc}(G) \cong Q^m$, where $m = 2r$.

Theorem 7.7.1. *Let G be a finite quasiprimitive permutation group on Ω of type HC and let $P := \mathrm{soc}(G) = Q_1 \times \cdots \times Q_m$, where each Q_i is isomorphic to a nonabelian simple group Q. Consider the projections $\pi_i \colon P \to Q_i$, where $i \in \underline{m}$, and let (H, G) be a CharS-QP inclusion in which $H = T_1 \times \cdots \times T_k$ and each $T_i \cong T$ for a nonabelian simple group T. Let $S \cong Q^r$ be a plinth for G and take $\alpha \in \Omega$. Then*

1. *If $1 < H\pi_j < Q_j$ for all $j \in \underline{m}$, then $k = m$, $H \cong A^m$ and $Q = \overline{A}\,\overline{B}$, where $\overline{A} \cong A$, $\overline{B} \cong B$, and Q, A and B are described in Table 4.3.*

2. *If $H\pi_{i_0} \in \{1, Q_{i_0}\}$ for some $i_0 \in \underline{m}$, then $P_\alpha = D_1 \times \cdots \times D_r$, and $T \cong Q$, where each D_i and each T_i are full strips of P in such a way that $|\mathrm{supp}(T_i) \cap \mathrm{supp}(D_j)| \le 1$ and $|\mathrm{supp}(D_j) \cap \mathrm{supp}(H)| = 2$ or 1, for all $i \in \underline{k}$ and $j \in \underline{r}$. Among the options for H satisfying these properties, Algorithm 4.1 decides those that satisfy $P = P_\alpha H$.*

Proof. According to [8, Lemma 5.1], there is an involution $i \in N_{\mathrm{Sym}(\Omega)}(G)$ that interchanges S and $C_G(S)$. In particular this says that $S \cong C_G(S)$. Consider $\overline{G} := \langle G, i\rangle \le \mathrm{Sym}(\Omega)$. Since $G \le \overline{G}$ and G is primitive, thus $\overline{G}$ is also primitive. If $P := \mathrm{soc}(G)$, we have that $P \trianglelefteq \overline{G}$, and by Lemma 3.1.4 (item 2) that P must be a direct product of some of the direct factors of S and $C_G(S)$. Since $\overline{G}$ is transitive on these direct factors, we have by Lemma 3.1.4 (item 2) that P is a minimal normal subgroup of $\overline{G}$. Consider the projections $\pi_1 \colon P \to S$ and $\pi_2 \colon P \to C_S(G)$. By Lemma 3.1.8, $P_\alpha\pi_1 = S$ and $P_\alpha\pi_2 = C_G(S)$. Then P_α is a subdirect subgroup of P. Moreover, $P_\alpha \cap S$ is a subdirect subgroup of S and $P_\alpha \cap C_G(S)$ is a subdirect subgroup of $C_G(S)$. Then P_α is nonsimple. Hence by O'Nan-Scott Theorem (Theorem 2.5.4) $P = \mathrm{soc}(\overline{G})$ and $\overline{G}$ has type CD. Since $H \le G \le \overline{G}$, then $(H, \overline{G})$ is a CharS-QP inclusion. By Theorem 6.0.1, $H \le P$. Therefore, applying Theorem 7.6.1 we obtain that $T \cong Q$ and $P_\alpha = D_1 \times \cdots \times D_l$, where each D_i and each T_i are full strips of P such that $|\mathrm{supp}(T_i) \cap \mathrm{supp}(D_j)| \le 1$ and $|\mathrm{supp}(D_j) \cap \mathrm{supp}(H)| = m/l$ or $m/l - 1$, for all

$i \in \underline{k}$ and $j \in \underline{l}$. However, since S is regular, $|\Omega| = |Q|^r$, and so

$$\frac{|Q|^{2r}}{|Q|^l} = \frac{|P|}{|P_\alpha|} = |\Omega| = |Q|^r.$$

Then $l = r = m/2$, which means that $|\operatorname{supp}(D_i)| = 2$ for all $i \in \underline{r}$. Therefore, the result is proved. □

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Appendix A

Orders of some groups

G	$\|G\|$
A_5	$2^2.3.5$
A_6	$2^3.3^2.5$
A_7	$2^3.3^2.5.7$
A_8	$2^6.3^2.5.7$
A_9	$2^6.3^4.5.7$
S_7	$2^4.3^2.5.7$
S_8	$2^7.3^2.5.7$
M_{11}	$2^4.3^2.5.11$
M_{12}	$2^6.3^3.5.11$
$\mathrm{PSL}_2(11)$	$2^2.3.5.11$
$\mathrm{Sp}(6,2)$	$2^9.3^4.5.7$
${\mathbb{Z}_2}^6 \rtimes A_7$	$2^9.3^2.5.7$
${\mathbb{Z}_2}^6 \rtimes A_8$	$2^{12}.3^2.5.7$
${\mathbb{Z}_3}^6 \rtimes \mathrm{PSL}_4(3)$	$2^7.3^{12}.5.13$
$\Omega_7(3)$	$2^9.3^9.5.7.13$
$G_2(2)$	$2^6.3^3.7$
$G_2(2)'$	$2^5.3^3.7$
$O_6^-(2)$	$2^7.3^2.5.7$
$O_6^+(2)$	$2^7.3^4.5$
$O_{4a}^-(2)$	$2^{4a^2-2a+1}.(2^{2a}+1).\prod_{i=1}^{2a-1}(2^{2i}-1)$
$\Omega_7(q)$	$\frac{1}{d}.q^9.(q^2-1).(q^4-1).(q^6-1)$, where $d = \mathrm{mdc}(2, q-1)$
$\mathrm{P}\Omega_8^+(q)$	$\frac{1}{d^2}.q^{12}.(q^2-1).(q^4-1)^2.(q^6-1)$, where $d = \mathrm{mdc}(2, q-1)$

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