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Degree of the Exceptional Component of the Space of Holomorphic Foliations of Degree Two and Codimension One in \mathbb{P}^3

Belo Horizonte - MG May, 2018

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Tese apresentada ao Departamento de Matemática da Universidade Federal de Minas Gerais, como requisito para a obtenção do grau de doutor em Matemática.

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Orientador: Israel Vainsencher

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RESUMO

O propósito deste trabalho é obter o grau da componente excepcional, E(3), do espaço de folheações holomorfas de grau 2 e codimensão um em \mathbb{P}^3 . Trata-se de uma componente de dimensão treze, descrita no célebre trabalho de Alcides Lins-Neto e Dominique Cerveau, [13]. E(3) é o fecho da órbita, sob a ação natural de Aut \mathbb{P}^3 , da folheação definida pela forma diferencial

$$\omega = (3fdg - 2gdf)/x_0$$
, onde $f = x_0^2 x_3 - x_0 x_1 x_2 + \frac{x_1^3}{3}$, $g = x_0 x_2 - \frac{x_1^2}{2}$.

Nossa tarefa inicial é descrever uma caracterização geométrica para o par g, f. Isto nos levará à construção de um espaço de parâmetros como um fibrado explícito sobre a variedade de bandeiras completas de \mathbb{P}^3 . Fazendo uso de ferramentas de teoria da interseção equivariante, podemos calcular o número desejado como uma integral sobre o nosso espaço de parâmetros.

Palavras-chave: Folheações Holomorfas. Componente Excepcional. Grau.

ABSTRACT

The purpose of this work is to obtain the degree of the exceptional component, E(3), of the space of holomorphic foliations of degree two and codimension one in \mathbb{P}^3 . As shown in the celebrated work by Dominique Cerveau and Alcides Lins Neto [13], E(3) is a 13dimensional component. It is the closure of the orbit under the natural action of Aut \mathbb{P}^3 of the foliation defined by the differential form

$$\omega = (3fdg - 2gdf)/x_0$$
, where $f = x_0^2 x_3 - x_0 x_1 x_2 + \frac{x_1^3}{3}$, $g = x_0 x_2 - \frac{x_1^2}{2}$

Our first task is to unravel a geometric characterization of the pair g, f. This leads us to the construction of a parameter space as an explicit fiber bundle over the variety of complete flags. Using tools from equivariant intersection theory, especially Bott's formula, the degree is expressed as an integral over our parameter space.

Keywords: Holomorphic Foliations. Exceptional Component. Degree.

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1 INTRODUCTION

A holomorphic foliation of codimension one and degree d in the complex projective space \mathbb{P}^n is given by a differential 1-form

$$\omega = A_0 dx_0 + \dots + A_n dx_n$$

where A_0, \ldots, A_n are homogeneous polynomials of degree d + 1, satisfying the conditions

- 1. (projectivity) $A_0x_0 + \cdots + A_nx_n = 0$
- 2. (integrability) $\omega \wedge d\omega = 0$.

These conditions define a closed scheme $\overline{\mathcal{F}(d,n)}$ in the projective space of global sections of the twisted cotangent sheaf $\Omega_{\mathbb{P}^n}^1(d+2)$. It is further required that the singular locus of ω (defined by the common zeros of the polynomials A_i , $i = 0, \ldots, n$) be of codimension greater than or equal to two. This corresponds to the inexistence of a common factor of positive degree for the polynomials A_i thus defining a Zariski open subset $\mathcal{F}(d, n)$. Naturally, the closure $\overline{\mathcal{F}(d, n)}$ possesses a decomposition into irreducible components.

In [13], a full description of the components of $\overline{\mathcal{F}(2,n)}$, $n \geq 3$ is given. We learn that there are six components of $\overline{\mathcal{F}(2,3)}$:

- 1. Linear pull-backs $\mathcal{S}(2,3)$: For a degree 2 foliation \mathcal{F} in \mathbb{P}^2 and a linear rational map $\alpha : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, take the pull-back $\mathcal{F}^* = \alpha^*(\mathcal{F})$.
- 2. Rational $\mathcal{R}(2,2)$: Take homogeneous polynomials f, g both of degree 2 and make the 1-form $\omega = f dg g df$.
- 3. Rational $\mathcal{R}(1,3)$: Take homogeneous polynomials f of degree 1 and g of degree 3, and form $\omega = fdg 3gdf$.
- 4. Logarithmic $\mathcal{L}(1, 1, 1, 1)$: Take four linear polynomials $f_1 \cdots f_4$ and complex numbers $\lambda_1, \ldots, \lambda_4$ with $\lambda_1 + \cdots + \lambda_4 = 0$. Then, write

$$\omega = \lambda_1 f_2 f_3 f_4 df_1 + \lambda_2 f_1 f_3 f_4 df_2 + \lambda_3 f_1 f_2 f_4 df_3 + \lambda_4 f_1 f_2 f_3 df_4$$

5. Logarithmic $\mathcal{L}(1,1,2)$: Take f_1, f_2 linear polynomials, f_3 quadratic, and complex numbers $\lambda_1, \ldots, \lambda_3$ with $\lambda_1 + \lambda_2 + 2\lambda_3 = 0$. Then, write

$$\omega = \lambda_1 f_2 f_3 df_1 + \lambda_2 f_1 f_3 df_2 + \lambda_3 f_1 f_2 df_3.$$

6. The exceptional component E(3): This is the orbit closure of the foliation defined by the differential form

$$\omega = \frac{3f_0 dg_0 - 2g_0 df_0}{x_0}, \quad \text{where} \quad f_0 = x_0^2 x_3 - x_0 x_1 x_2 + \frac{x_1^3}{3}, \quad g_0 = x_0 x_2 - \frac{x_1^2}{2}. \quad (1.1)$$

We will study in detail the component E(3) along this work. The main point is to describe the geometry of the family of pairs g, f in the orbit closure.

Two important invariants of a projective scheme are its dimension and degree.

For some components of spaces of foliations the degree has been found in recent works. For instance, in [6] the authors managed to compute the degrees of the rational components of type $\mathcal{R}(n, d, d)$ in \mathbb{P}^n , $n \geq 3, d > 1$:

$$\deg \mathcal{R}(n,d,d) = \frac{1}{N_d - 1} \binom{2N_d - 2}{N_d},$$

where $N_d = \binom{n+d}{d} - 1$. As a particular case, the degree of the rational component $\mathcal{R}(2,2)$ is 1430. Also the degrees of rational components of type $\mathcal{R}(n, d_0, d_1)$ where d_0 divides d_1 were found:

deg
$$\mathcal{R}(n, d_0, d_1) = \binom{N_{d_1} + N_{d_0} - 1}{N_{d_0}} - \frac{d_1}{d_0} \binom{N_{d_1} + N_{d_0} - 1}{N_{d_0} - 1},$$

and so the degree of $\mathcal{R}(1,3)$ is 700.

The degrees of the rational components $\mathcal{R}(n, 2, 3)$ were also computed in [6] for $n \leq 5$ and in [11] this is extended to the degree of rational components $\mathcal{R}(n, 2, 2r+1)$ for general n, r.

The case of linear pullbacks was done by Viviane Ferrer and Israel Vainsencher (to appear). For linear pullbacks of foliations of degree d in \mathbb{P}^2 , the degree of the component $\mathcal{S}(d,3)$ is

deg
$$S(d,3) = \frac{1}{162} \frac{(d+4)!}{(d+1)!} (d^2 + 6d + 11)(d^2 + 2d + 3),$$

so we have deg S(2,3) = 1320.

To the best of my knowledge, for the logarithmic components the degrees have not been determined. In [5] the authors showed that these components are generically reduced.

In this thesis we handle the construction of an adequate parameter space for the exceptional component, enabling us to achieve the calculation of the degree of

$$E(3) \subset \mathbb{P}(H^0(\Omega^1_{\mathbb{P}^3}(4))) = \mathbb{P}^{44}.$$

As customary in enumerative questions, the bulk of the work lies in the description of an appropriate parameter space. The description of the exceptional component E(3) could in principle suggest using Aluffi & Faber's approach to the calculation of degrees of orbits closures, cf. [2]. We took another path, essentially trying and deciphering an algebrogeometric characterization of that pair (cubic,quadric) from the explicit expressions (1.1) given just above. Eventually, this lead to our construction of an explicit smooth, projective variety \mathbb{Y}_4 (cf. Proposition 2, page 32) endowed with a generically injective morphism onto E(3).

Our strategy for the exceptional component starts with a complete flag in \mathbb{P}^3

$$p \text{ (point)} \in \ell \text{ (line) } \subset v \text{ (plane)},$$

over which we describe suitable cubic forms f and quadratic forms g that fulfill some special conditions. The polynomials g, f will give us the differential 1-form

$$\omega = \frac{3fdg - 2gdf}{h}$$

where h is an equation of the plane v of the given flag.

It turns out that ω can be zero for certain values of otherwise acceptable g, f, for instance, $g = x_0^2$, $f = x_0^3$. The main technical difficulty is how to handle the indeterminacy locus of the rational map

$$(g, f) \mapsto \omega$$

leading to a resolution of its indeterminacies. As we will see, this indeterminacy locus is a non-reduced and reducible scheme. The resolution of indeterminacies is done by a careful analysis of the irreducible components of the indeterminacy locus, blown up one at a time,

$$\mathbb{Y}_4 \longrightarrow \mathbb{Y}_3 \longrightarrow \mathbb{Y}_2 \longrightarrow \mathbb{Y}_1 \longrightarrow \mathbb{Y} \dashrightarrow E(3) \subset \mathbb{P}^{44}.$$

Once accomplished the resolution of the map, we obtain a line bundle \mathcal{W} over \mathbb{Y}_4 , pullback of $\mathcal{O}_{\mathbb{P}^{44}}(-1)$, with each fiber spanned by a differential 1-form possibly in the boundary of E(3). Since the dimension of the exceptional component is 13, the required degree is (see Theorem 3)

$$\int -c_1^{13}(\mathcal{W}).$$

This will be done in Chapter 2. In Chapter 3, we proceed to compute this integral. In order to do this, we will apply Bott's formula

$$\int -c_1^{13}(\mathcal{W}) = \sum_F \frac{-c_1^T(\mathcal{W})^{13} \cap [F]_T}{c_{top}^T(\mathcal{N}_F)},$$

where the sum runs through all the fixed components of a suitable action of the torus $T := \mathbb{C}^*$ on \mathbb{Y}_4 , induced from an action on \mathbb{P}^3 . The \mathcal{N}_F appearing in the denominator denotes the normal bundle of a fixed component F in \mathbb{Y}_4 .

Putting together the contributions from a total of 1728 fixed isolated points and 120 fixed lines we find in Theorem 4 the number **168208** as the desired degree.

In the appendix we include the scripts for *Macaulay2* used to study the resolution of the indeterminacies and to compute the sum of all contributions in Bott's formula.

1.1 NOTATION AND CONVENTIONS

Let x_0, x_1, x_2, x_3 denote homogeneous coordinates of \mathbb{P}^3 . Write

$$S_d = \langle x_0^d, x_0^{d-1} x_1, \dots, x_3^d \rangle$$

for the vector space of homogeneous polynomials of degree d. If no confusion arises, we abuse and write S_d for the trivial vector bundle $S_d \times X$ over some base variety X.

The dual projective space $\check{\mathbb{P}}^3 = \mathbb{P}(S_1)$ comes equipped with a tautological line subbundle,

$$\mathcal{O}_{\check{\mathbb{P}}^3}(-1) \hookrightarrow S_1$$

Given a vector bundle \mathcal{V} , we write $\mathbb{P}(\mathcal{V})$ for the projective bundle of rank one subspaces of the fibers of \mathcal{V} . It comes endowed with a tautological rank one subline bundle

$$\mathcal{O}_{\mathcal{V}}(-1) \hookrightarrow \mathcal{V},$$

where we omit the pullback $\mathcal{V}_{|\mathbb{P}(\mathcal{V})}$.

For an action of \mathbb{C}^* in X, we denote $t \circ v$ for the image of the point $(t, v) \in \mathbb{C}^* \times X$ by the action.

For clarity, we will use a left square (\Box) also at the end of examples.

2 DESCRIPTION OF THE EXCEPTIONAL COMPONENT

2.1 ASSOCIATED COMPLETE FLAG

In [13] the authors describe the exceptional component of foliations in \mathbb{P}^3 starting with the two homogeneous polynomials

$$f_0 := x_0^2 x_3 - x_0 x_1 x_2 + \frac{x_1^3}{3}$$
 and $g_0 := x_0 x_2 - \frac{x_1^2}{2}$ (2.1)

from which arises the integrable differential 1-form

$$\omega_0 := \frac{2g_0 df_0 - 3f_0 dg_0}{x_0}$$

Explicitly,

$$\omega_0 = (x_1 x_2^2 - 2x_1^2 x_3 + x_0 x_2 x_3) dx_0 + x_0 (3x_1 x_3 - 2x_2^2) dx_1 + x_0 (x_1 x_2 - 3x_0 x_3) dx_2 + x_0 (2x_0 x_2 - x_1^2) dx_3.$$

The singular locus of such a foliation consists of a union of three curves (cf. appendix 4.1):

- 1. the conic given by the ideal $\langle x_0, x_2^2 2x_1x_3 \rangle$ (it lies in the plane $x_0 = 0$);
- 2. the line ℓ_0 defined by $x_0 = x_1 = 0$;
- 3. the twisted cubic given by $\langle 2x_2^2 3x_1x_3, x_1x_2 3x_0x_3, x_1^2 2x_0x_2 \rangle$.

Notice that these three components meet at the point $p_0 := (0:0:0:1) \in \mathbb{P}^3$. The line ℓ_0 is tangent to the twisted cubic, the plane is the osculating plane at p_0 and the conic is the osculating conic. The component E(3) consists of the orbit of ω_0 under the group Aut \mathbb{P}^3 of automorphism of \mathbb{P}^3 ; its dimension is equal to 13 (see [3]).

Let us examine the geometry of the surface defined by our cubic form $(2.1) f_0$.

We see that it is an irreducible cubic, singular along the line ℓ_0 . Moreover, given a point $p_t = (0:0:t:1) \in \ell_0$, the tangent cone to our cubic surface at the point p_t has equation

$$x_0^2 - tx_0x_1 = x_0(x_0 - tx_1)$$

Thus, the tangent cone is a pair of planes containing the double line ℓ_0 , one of which is fixed and the other varies with the point p_t .

We have also perceived that at the special point $p_0 = (0 : 0 : 0 : 1)$ the tangent cone to the cubic is the double plane $x_0^2 = 0$. Therefore, our cubic f_0 comes endowed with a companion complete flag

$$\varphi_0: \ p_0 = \{x_0 = x_1 = x_2 = 0\} \in \ell_0 = \{x_0 = x_1 = 0\} \subset v_0 = \{x_0 = 0\}.$$
(2.2)

Such a cubic belongs to one of the strata of cubics singular along a line, as described in [1] and revisited in [4].

As for the quadric

$$g_0 = x_0 x_2 - \frac{x_1^2}{2},$$

notice we have a cone containing the line ℓ_0 and vertex p_0 . Moreover, the tangent plane to the cone at the smooth points on ℓ_0 is precisely the same plane v_0 .

Remark 1. The flag can move under the action of an automorphism of \mathbb{P}^3 , but it can be recovered directly from the 1-form ω defining the exceptional foliation, just by looking at the singular locus. For any such ω , the singular locus has three components - a conic living on the new plane, the new line and a twisted cubic that meets the other two components just at the new point; the line is tangent to the twisted cubic, the plane is the osculating plane and the conic is the osculating conic. By the way, the dimension 13 mentioned just above is the dimension of the family of pointed twisted cubics.

These considerations about the cubic/quadric pairs as in (2.1) lead us to consider the construction of a parameter space for the family of exceptional foliations by starting with the variety \mathbb{F} of complete flags on \mathbb{P}^3 ,

$$p(\text{point}) \in \ell(\text{line}) \subset \pi(\text{plane}).$$

Given such a flag we take:

- A cubic surface f with the properties:
- 1. f is singular along ℓ ;
- 2. the tangent cone to f at each point $q \in \ell$ is the union of two planes $\pi \cup \alpha_q$.
- 3. $\alpha_p = \pi$, that is, at the point p the tangent cone is the double plane.
- A quadratic cone g with the properties:
- 1. the line ℓ is contained in the cone g;
- 2. the plane π is tangent to g along the line ℓ ;
- 3. the point p is in the vertex of the cone.

Definition 1. For simplicity we will call here a cubic or a quadric that meet the requirements above as *special*.

An exceptional foliation is then given by the differential form

$$\omega = \frac{3fdg - 2gdf}{h},$$

where h is an equation of the plane π .

At this point, we realize that after the construction of special pairs (f, g), it is still necessary to impose the condition that the differential form 3fdg - 2gdf be divisible by the equation of the plane. We refer to this as the *divisibility condition*. It deserves a closer look. In order to better understand this divisibility condition, let us fix a complete flag as in (2.2).

The conditions about the cubic surface show that its equation can be expressed as

 $f = (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3)x_0^2 + a_4x_0x_1^2 + a_5x_0x_1x_2 + a_6x_1^3,$

while the conditions about the quadric implies

$$g = b_0 x_0^2 + b_1 x_0 x_1 + b_2 x_0 x_2 + b_3 x_1^2.$$

We register for later use the following invariant description.

Lemma 1. The 7 monomials appearing in f $x_0^3, x_0^2 x_1, x_0^2 x_2, x_0^2 x_3, x_0 x_1^2, x_0 x_1 x_2, x_1^3$ are the monomial generators of the subspace of cubics, $x_0^2 \langle x_0, \ldots, x_3 \rangle + x_0 \langle x_0, x_1 \rangle \langle x_0, x_1, x_2 \rangle + \langle x_0, x_1 \rangle^3 \subset S_3.$ Likewise, the 4 monomials $x_0^2, x_0 x_1, x_0 x_2, x_1^2$ in the quadric g generate the subspace $x_0 \langle x_0, x_1, x_2 \rangle + \langle x_0, x_1 \rangle^2 \subset S_2.$

(2.3)

Varying the flag, we obtain equivariant vector subbundles with respective ranks 7 and 4,

$$\mathcal{A} \subset S_3 \text{ and } \mathcal{B} \subset S_2.$$
 (2.4)

Projectively we obtain, at this stage, the variety

$$\mathbb{P}(\mathcal{A}) \times_{\mathbb{F}} \mathbb{P}(\mathcal{B})$$

a $\mathbb{P}^6 \times \mathbb{P}^3$ bundle of pairs (f, g) over a fixed flag.

Continuing the discussion about the divisibility condition, we may write

$$3fdg - 2gdf = x_0w_0 + \left[(3a_6b_1 - 2a_4b_3)x_1^4 + (3a_6b_2 - 2a_5b_3)x_1^3x_2 \right] dx_0,$$

where w_0 is a 1-form. From this it follows that the divisibility condition is given, on the fiber over φ_0 , by the equations

$$\begin{cases} 3a_6b_1 = 2a_4b_3, \\ 3a_6b_2 = 2a_5b_3. \end{cases}$$

This locus consists of two irreducible components inside $\mathbb{P}^6 \times \mathbb{P}^3$, both of codimension two:

$$(\star) \begin{cases} 3a_6b_1 = 2a_4b_3\\ 3a_6b_2 = 2a_5b_3\\ a_4b_2 = a_5b_1 \end{cases}$$
(2.5)
and $(\star\star) \qquad a_6 = b_3 = 0.$

For a pair (g, f) satisfying $(\star\star)$, we actually get

З

$$\begin{cases} f = x_0 \left(a_0 x_0^2 + a_1 x_0 x_1 + a_2 x_0 x_2 + a_3 x_0 x_3 + a_4 x_1^2 + a_5 x_1 x_2 \right), \\ g = x_0 \left(b_0 x_0 + b_1 x_1 + b_2 x_2 \right). \end{cases}$$

It means that a general element in the second component $(\star\star)$ consists of a cubic and a quadric both divisible by the equation of the plane, certainly not interesting for our study of the exceptional component.

Henceforth, we refer to the divisibility condition as the three equations (2.5). Recalling the dimension, 13, of the exceptional component, it is reassuring to find



2.2 CONSTRUCTION OF A PARAMETER SPACE

Let $\mathbb{G} = \mathbb{G}(1,3)$ be the Grassmann variety of lines in \mathbb{P}^3 , with tautological sequence

 $\mathcal{S} \longrightarrow \mathbb{G} \times \mathbb{C}^4 \longrightarrow \mathcal{Q},$

where rank(\mathcal{S})=2. Denote by \mathcal{Q}^{\vee} the dual of \mathcal{Q} . We also have the tautological sequence over \mathbb{P}^3 ,

$$\mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \mathbb{P}^3 \times \mathbb{C}^4 \longrightarrow \mathcal{P}$$

Then,

$$\begin{split} \mathbb{P}(\mathcal{S}) &= \{(p,\ell) \,|\, p \in \ell\} \quad \subset \quad \mathbb{P}^3 \times \mathbb{G} \\ \mathbb{P}(\mathcal{Q}^{\vee}) &= \{(\ell,\pi) \,|\, \ell \subset \pi\} \quad \subset \quad \mathbb{G} \times \check{\mathbb{P}}^3. \end{split}$$

The variety of complete flags is just the fiber product

$$\mathbb{F} = \mathbb{P}(\mathcal{S}) \times_{\mathbb{G}} \mathbb{P}(\mathcal{Q}^{\vee}) = \{ (p, \ell, \pi) | \ p \in \ell \subset \pi \} \subset \mathbb{P}^3 \times \mathbb{G} \times \check{\mathbb{P}}^3.$$

The tautological bundles of these spaces lift to bundles over \mathbb{F} still denoted by the same letters. They fit together into the diagram (pullbacks omitted),

$$\mathcal{O}_{\mathcal{Q}^{\vee}}(-1) = \mathcal{O}_{\check{\mathbb{P}}^3}(-1) \hookrightarrow \mathcal{Q}^{\vee} \hookrightarrow \mathcal{P}^{\vee} \hookrightarrow S_1 := (\mathbb{C}^4)^{\vee}$$
$$\varphi_0 : \langle x_0 \rangle \subset \langle x_0, x_1 \rangle \subset \langle x_0, x_1, x_2 \rangle \subset \langle x_0, \dots, x_3 \rangle$$

where the bottom row indicates the corresponding fibers over our favorite flag (2.2) φ_0 .

Now, recall from (2.4) the rank 7 subbundle $\mathcal{A} \subset S_3$ of special cubics and the rank 4 vector subbundle $\mathcal{B} \subset S_2$ of special quadric cones over the variety \mathbb{F} of complete flags. In view of Lemma 1, we have the surjections

$$(\mathcal{O}_{\mathcal{Q}^{\vee}}(-2) \otimes S_1) \oplus (\mathcal{O}_{\mathcal{Q}^{\vee}}(-1) \otimes \mathcal{Q}^{\vee} \otimes \mathcal{P}^{\vee}) \oplus \operatorname{Sym}_3 \mathcal{Q}^{\vee} \longrightarrow \mathcal{A} \subset S_3$$
$$(\mathcal{O}_{\mathcal{Q}^{\vee}}(-1) \otimes \mathcal{P}^{\vee}) \oplus \operatorname{Sym}_2 \mathcal{Q}^{\vee} \longrightarrow \mathcal{B} \subset S_2.$$

By construction, the vector bundles \mathcal{A}, \mathcal{B} fit into the exact sequences

$$S_{1} \otimes \mathcal{O}_{\check{\mathbb{P}}^{3}}(-2) \longrightarrow \mathcal{A} \longrightarrow \overline{\mathcal{A}} := \mathcal{A} / \left(S_{1} \otimes \mathcal{O}_{\check{\mathbb{P}}^{3}}(-2) \right)$$

$$\mathcal{O}_{\check{\mathbb{P}}^{3}}(-2) \longrightarrow \mathcal{B} \longrightarrow \overline{\mathcal{B}}.$$

$$(2.6)$$

Remark 2. For later use, remember that knowing the fibers of an equivariant, say \mathbb{C}^* -vector bundle over the variety \mathbb{F} of complete flags suffices to get our hands on the \mathbb{C}^* -equivariant Chern classes.

We recall the fiber $\overline{\mathcal{A}}_{\varphi_0}$ is spanned by the classes

$$\overline{x_0x_1^2}, \overline{x_0x_1x_2}, \overline{x_1^3} \mod x_0^2 S_1.$$

We have likewise the subbundle $\mathcal{O}_{\check{\mathbb{P}}^3}(-2) \subset \mathcal{B}$ and the corresponding quotient $\overline{\mathcal{B}}$. The fiber of $\overline{\mathcal{B}}$ over φ_0 has the generators

$$\overline{x_0x_1}, \overline{x_0x_2}, \overline{x_1^2}$$
 mod. $\langle x_0^2 \rangle$.

This shows the following:

Lemma 2. $\overline{\mathcal{A}}$ is isomorphic to $\left(\mathcal{Q}^{\vee}/\mathcal{O}_{\mathbb{P}^3}(-1)\right) \otimes \overline{\mathcal{B}}$.

Proof. Indeed, on the fiber over φ_0 we have

$$(\mathcal{Q}^{\vee}/\mathcal{O}_{\check{\mathbb{P}}^3}(-1))_{\varphi_0} = \langle x_0, x_1 \rangle / \langle x_0 \rangle = \langle \overline{x_1} \rangle$$

hence

$$\left(\left(\mathcal{Q}^{\vee} / \mathcal{O}_{\mathbb{P}^3}(-1) \right) \otimes \overline{\mathcal{B}} \right)_{\varphi_0} = \langle \overline{x_1} \rangle \otimes \langle \overline{x_0 x_1}, \overline{x_0 x_2}, \overline{x_1^2} \rangle = \overline{\mathcal{A}}_{\varphi_0},$$

where the = signs mean isomorphisms of representations of the stabilizer of the flag φ_0 .

We define

$$\mathbb{X} = \mathbb{P}(\mathcal{B}) \tag{2.7}$$

the corresponding projective bundle of special quadrics. The fiber \mathbb{X}_{φ_0} is the \mathbb{P}^3 of special quadrics over the flag $\varphi_0 = (p_0, \ell_0, v_0)$ fixed in (2.2).

Next, we rewrite the divisibility condition (2.5) as the following system of linear equations in the variables $(\underline{a}) = (a_4, a_5, a_6)$ with coefficients $(\underline{b}) = (b_0, b_1, b_2, b_3) \in \mathbb{P}^3$,

$$\begin{bmatrix} 2b_3 & 0 & -3b_1 \\ 0 & -2b_3 & 3b_2 \\ -b_2 & b_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
 (2.8)

The matrix of the coefficients has determinant zero and generic rank two. Computing the 2×2-minors, we find that the rank of the coefficients matrix will be less than two when $b_1 = b_2 = b_3 = 0 \leftrightarrow x_0^2$. Off the point x_0^2 the solution space is spanned by the vector product of two rows,

$$(-3b_1b_2: -3b_2^2: -2b_2b_3) = (3b_1: 3b_2: 2b_3)$$

Our idea now is to describe the locus of special cubic forms with the divisibility condition as a \mathbb{P}^4 -bundle over \mathbb{X} (since it has codimension two in the \mathbb{P}^6 of special cubics). But to make it possible, we need to replace \mathbb{X} by a new space, \mathbb{X}' , for which the rank of the coefficients matrix is two everywhere.

Precisely, we think of the fiberwise solution space to (2.8) as defining a rational map $\psi : \mathbb{X} \dashrightarrow \mathbb{P}(\overline{\mathcal{A}})$, notation as in (2.6), which on the fiber over the standard flag φ_0 (2.2) reads

$$\psi: (\underline{b}) \mapsto (a_4: a_5: a_6) = (3b_1: 3b_2: 2b_3).$$
(2.9)

Look at the closure \mathbb{X}' of the graph of ψ . On the fiber over φ_0 , this is the blowup of $\mathbb{P}^3 = \mathbb{P}(\mathcal{B}_{\varphi_0})$ at the point x_0^2 . As a matter of fact, this turns out to be the restriction to the fiber over φ_0 of the blowup of \mathbb{X} along the section $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^3}(-2)\right) \hookrightarrow \mathbb{P}(\mathcal{B})$ over \mathbb{F} . We have the diagram



where the leftmost inclusion is defined by taking the square of the equation of the plane in the flag. The equality $\mathbb{P}(\overline{\mathcal{A}}) = \mathbb{P}(\overline{\mathcal{B}})$ comes from Lemma 2; under this identification, we have

$$\mathcal{O}_{\overline{\mathcal{A}}}(-1) = \mathcal{O}_{\overline{\mathcal{B}}}(-1) \otimes (\mathcal{Q}^{\vee}/\mathcal{O}_{\mathbb{P}^3}(-1)).$$

Pulling back the tautological line subbundle via ψ' , we get the diagrams

and

where rank $\mathcal{A}' = 5$, rank $\mathcal{B}' = 2$. We have

$$\mathbb{X}' = \mathbb{P}(\mathcal{B}'), \tag{2.12}$$

a \mathbb{P}^1 bundle over the \mathbb{P}^2 bundle $\mathbb{P}(\overline{\mathcal{B}})$ over \mathbb{F} . We define

$$\mathbb{Y} = \mathbb{P}(\mathcal{A}'),\tag{2.13}$$

an equivariant \mathbb{P}^4 -bundle over \mathbb{X}' .

Proposition 1. A general point in \mathbb{Y} corresponds to a pair (g, f) of special quadric, cubic satisfying the divisibility condition.

Proof. The assertion follows from the previous considerations. \Box

Remark. Since all constructions performed so far, as well as in the sequel are equivariant, henceforth we drop the reference to the fiber over our favorite flag φ_0 , and simplify notation writing, until further notice,

$$\mathbb{X}=\mathbb{X}_{arphi_0}, \ \mathcal{A}=\mathcal{A}_{arphi_0},\ldots, \ etc.$$

In view of Lemma 2, the rational map (2.9) can also be written as the rational linear projection map

$$\psi_{\overline{\mathcal{B}}}: \qquad \mathbb{X} \qquad \dashrightarrow \qquad \mathbb{P}(\overline{\mathcal{B}})$$
$$g := b_0 x_0^2 + b_1 x_0 x_1 + b_2 x_0 x_2 + b_3 x_1^2 \qquad \longmapsto \qquad g' := b_1 \overline{x_0 x_1} + b_2 \overline{x_0 x_2} + b_3 \overline{x_1^2}$$

(class mod. x_0^2). Write for short

$$\mathbb{X}' = \{ (g, g') = ((b_0 : b_1 : b_2 : b_3), (u_1 : u_2 : u_3)) \mid b_i u_j = b_j u_i \},$$
(2.14)

where

$$g' := u_1 \overline{x_0 x_1} + u_2 \overline{x_0 x_2} + u_3 \overline{x_1^2}.$$
 (2.15)

Here, the \underline{u} are homogeneous coordinates in \mathbb{P}^2 = fiber of $\mathbb{P}(\overline{\mathcal{B}})$ over φ_0 . The map ψ' is (see (2.9))

$$\psi': \quad \mathbb{X}' \quad \longrightarrow \qquad \mathbb{P}(\overline{\mathcal{A}}) \\ (g,g') \quad \mapsto \quad 3u_1 \overline{x_0 x_1^2} + 3u_2 \overline{x_0 x_1 x_2} + 2u_3 \overline{x_1^3}$$

Notice that the divisibility condition (2.5) has now the expression

$$\begin{cases}
3a_6u_1 = 2a_4u_3 \\
3a_6u_2 = 2a_5u_3 \\
a_4u_2 = a_5u_1
\end{cases}$$
(2.16)

 Set

$$\mathbb{U}_0 := \mathbb{A}^3 \subset \mathbb{X} = \mathbb{P}^3 \ni (1:0:0:0) \leftrightarrow x_0^2,$$

i.e., we put $b_0 = 1$ and b_1, b_2, b_3 are affine coordinates on \mathbb{U}_0 . We write

$$g = x_0^2 + b_1 x_0 x_1 + b_2 x_0 x_2 + b_3 x_1^2 \in \mathbb{U}_0$$

Let $\mathbb{U}'_0 \subset \mathbb{U}_0 \times \mathbb{P}^2$ be defined by $u_1 = 1$. So $b_2 = b_1 u_2, b_3 = b_1 u_3$. Thus the affine coordinates on \mathbb{U}'_0 are b_1, u_2, u_3 . We have

$$g' = x_0 x_1 + u_2 x_0 x_2 + u_3 x_1^2.$$

The rational map (2.9) gives

$$(a_4:a_5:a_6) = (3:3u_2,2u_3)$$
, so $\frac{a_4}{3} = \frac{a_5}{3u_2} = \frac{a_6}{2u_3}$ and $a_5 = u_2a_4, a_6 = \frac{2}{3}u_3a_4$.

Recall $\overline{\mathcal{A}} = \langle \overline{x_0 x_1^2}, \overline{x_0 x_1 x_2}, \overline{x_1^3} \mod x_0^2 S_1 \rangle$. We have the trivialization,

$$\mathcal{O}_{\overline{\mathcal{A}}}(-1)_{|\mathbb{U}_0'} = \langle \overline{x_0 x_1^2} + u_2 \overline{x_0 x_1 x_2} + \frac{2}{3} u_3 \overline{x_1^3} \rangle$$

and

$$\mathcal{A}'_{|\mathbb{U}'_0} = x_0^2 S_1 + \langle x_0 x_1^2 + u_2 x_0 x_1 x_2 + \frac{2}{3} u_3 x_1^3 \rangle.$$

We have $\mathbb{Y}_{|\mathbb{U}'_0} = \mathbb{U}'_0 \times \mathbb{P}^4$. We redefine homogeneous coordinates (a_i) on the \mathbb{P}^4 factor and write

$$\begin{cases} f = (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3)x_0^2 + a_4(x_0x_1^2 + u_2x_0x_1x_2 + \frac{2}{3}u_3x_1^3), \\ g = x_0^2 + b_1(x_0x_1 + u_2x_0x_2 + u_3x_1^2). \end{cases}$$
(2.17)

For example, choose $g = x_0^2$, $g' = x_0x_1 + u_2x_0x_2 + u_3x_1^2 \in \mathbb{X}'$, $(u_2, u_3) \in \mathbb{A}^2$. We may pick $f = x_0x_1^2 + u_2x_0x_1x_2 + \frac{2}{3}u_3x_1^3$ in the fiber of \mathbb{Y} over (g, g') and evaluate

$$\omega = \frac{3fdg - 2gdf}{x_0} = \tag{2.18}$$

$$= (4x_0x_1^2 + 4u_2x_0x_1x_2 + 4u_3x_1^3)dx_0 - (4x_0^2x_1 + 2u_2x_0^2x_2 + 4u_3x_0x_1^2)dx_1 - 2u_2x_0^2x_1dx_2.$$

Thus, ω spans a linear space of dimension one over the chosen points. If this space were well defined for all points of \mathbb{Y} , we would have a line subbundle \mathcal{W} of the trivial bundle of twisted differential forms, pullback of the $\mathcal{O}_{\mathbb{P}^{44}}(-1)$ of the corresponding projective space where E(3) lives. So \mathbb{Y} would be an adequate parameter space for the exceptional component.

However, there are indeterminacies in \mathbb{Y} corresponding to the vanishing of the form w = 3fdg - 2gdf, e.g., when $f = x_0^3$, $g = x_0^2$, just to give an example.

The next step is to solve the indeterminacies of the rational map

$$\mathbb{Y} \quad - \rightarrow \quad \mathbb{P}(H^0(\Omega^1_{\mathbb{P}^3}(4))) \\
(g, g', f) \quad \longmapsto \quad \omega = \frac{3fdg - 2gdf}{x_0}$$
(2.19)

Note that g' is implicit in this formula: its coefficients appear in f, cf. (2.17).

We will see below (cf. $\S2.3$) that the indeterminacy locus of this map is a non-reduced and reducible scheme. But its reduction consists of two beautiful components.

One of them corresponds to the points (g, g', f) parametrized by $(s: t) \in \mathbb{P}^1$, where

$$f = (sx_0 + tx_1)^3$$
, $g = (sx_0 + tx_1)^2$, $g' = x_1(2sx_0 + tx_1)$.

The other component corresponds to a section over the whole \mathbb{P}^2 fiber of the exceptional divisor of X' over $g = x_0^2$, with $f = x_0^3$, *i.e.*, the points of the form (x_0^2, g', x_0^3) , with g' as in (2.15), cf. the diagram below:

This second component is then the \mathbb{P}^2 projectivization of the space

$$\langle \overline{x_0 x_1}, \overline{x_0 x_2}, \overline{x_1^2} \rangle.$$
 (2.21)

The (global) equations of the indeterminacy locus of the map (2.19) are too hard to treat. However, locally they look rather friendly. Let us work over an affine cover of X' to deal with it.

2.3 INDETERMINACIES OF ω

Let's sketch our plan to solve the indeterminacies of the map $(g, g', f) \mapsto \omega$ (2.19).

The links to appendix 4 refer to scripts for the software Macaulay2 that perform the calculations.

Notation as in (2.14), we list the typical neighborhoods to consider.

2.3.1 • $b_3 = 1$

In this case, $u_1 = b_1, u_2 = b_2, u_3 = 1$ and the divisibility condition (2.5), (2.16) is

$$\begin{cases} 3a_6b_1 = 2a_4 \\ 3a_6b_2 = 2a_5 \\ a_4b_2 = a_5b_1 \end{cases}$$

Observe that the third equation is irrelevant, and we can write

$$\begin{cases} f = (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3)x_0^2 + a_6\left(\frac{3}{2}b_1x_0x_1^2 + \frac{3}{2}b_2x_0x_1x_2 + x_1^3\right), \\ g = b_0x_0^2 + b_1x_0x_1 + b_2x_0x_2 + x_1^2. \end{cases}$$
(2.22)

Presently our \mathbb{P}^4 of special cubics has homogeneous coordinates a_0, a_1, a_2, a_3, a_6 . Compute ω as in 4.2.1.1. We find

$$\omega = \left((3a_0b_1 - 2a_1b_0)x_0^2x_1 + \dots + a_3b_2x_0x_2x_3 \right) dx_0 + \dots + 2a_3 \left(b_0x_0^3 + \dots + b_2x_0^2x_2 \right) dx_3.$$

Notice that if we set $a_6 = 0$ the expression for ω acquires the terms

$$[\dots + (a_1b_1 + 6a_0)x_0x_1^2 + 4a_1x_1^3 + 4a_2x_1^2x_2 + 4a_3x_1^2x_3 + \dots]dx_0 + \dots$$

Hence, if this is zero then

$$a_0 = a_1 = a_2 = a_3 = a_6 = 0, (2.23)$$

which is impossible since $(a_0 : a_1 : a_2 : a_3 : a_6) \in \mathbb{P}^4$.

It means that over the neighborhood $b_3 = 1$ we do not have any indeterminacies when $a_6 = 0$. Thus in order to study the indeterminacy locus we can take $a_6 = 1$.

Setting $a_6 = 1$ in the expression of ω , and collecting the coefficients in the resulting 1-form, we see that the indeterminacy locus is given by the ideal (cf. 4.2.1.2)

$$J = \langle a_3, 3b_1b_2 - 4a_2, 6a_0 - 6b_0b_1 + a_1b_1, 3b_1^2 - 8a_1 + 12b_0, b_2^2, b_2(a_1 - 3b_0), (a_1 - 3b_0)^2 \rangle.$$

Note that it contains the square $\langle b_2, a_1 - 3b_0 \rangle^2$.

The radical is

$$J_{red} := rad(J) = \langle 8a_0 - b_1^3, a_2, b_2, a_3, 4b_0 - b_1^2, 4a_1 - 3b_1^2 \rangle.$$
(2.24)

It is generated by a regular sequence (the affine coordinates here are $a_0, a_1, a_2, a_3, b_0, b_1, b_2$), therefore we have a locally complete intersection. Plugging (2.24) into (2.22) we find

$$f = \left(\frac{b_1}{2}x_0 + x_1\right)^3, \quad g = \left(\frac{b_1}{2}x_0 + x_1\right)^2.$$
 (2.25)

Definition 2. We call C the curve defined locally by the above ideal J_{red} .

Blowing up \mathbb{Y} along this one dimensional variety C does not solve the indeterminacies yet, but makes the resulting ones very nice.

Indeed, let \mathbb{A}^7 the open dense set defined by $b_3 = a_6 = 1$. The blowup of this \mathbb{A}^7 along C is

$$\mathbb{Y}_1 = \{ ((\underline{a}, \underline{b}), (s_0 : \ldots : s_5)) \mid s_i \cdot e_j = s_j \cdot e_i \} \subset \mathbb{A}^7 \times \mathbb{P}^5,$$
(2.26)

where e_i , $0 \le i \le 5$, are the equations of J_{red} , ordered as displayed in (2.24).

We will write the equations of \mathbb{Y}_1 as

$$\frac{8a_0 - b_1^3}{s_0} = \frac{a_2}{s_1} = \frac{b_2}{s_2} = \frac{a_3}{s_3} = \frac{4b_0 - b_1^2}{s_4} = \frac{4a_1 - 3b_1^2}{s_5} .$$
(2.27)

The exceptional divisor $\mathbb{P}(\mathcal{N}_{C|\mathbb{Y}})$ is a \mathbb{P}^5 -bundle over C, where $\mathcal{N}_{C|\mathbb{Y}}$ denotes the normal bundle of C in \mathbb{Y} .

Choose a_2 for the local equation of the exceptional divisor, so take $s_1 = 1$. The equations (2.27) become (cf. 4.2.1.3),

$$\begin{cases}
8a_0 - b_1^3 = s_0a_2 \\
b_2 = s_2a_2 \\
a_3 = s_3a_2 \\
4b_0 - b_1^2 = s_4a_2 \\
4a_1 - b_1^2 = s_5a_2.
\end{cases}$$

Substituting into the expression of ω , we find (see the expression for wnt_1 in 4.2.1.3), yes, a horrible expression. But is easy to see that the equation a_2 of the exceptional divisor divides this new ω . Then, we can perform this division to obtain the new expression of ω in the present coordinates $b_1, a_2, s_0, s_2, s_3, s_4, s_5$. And now we are able to look at the indeterminacy locus of this new ω , finding the generators (ideal Jn_1 in 4.2.1.3)

$$J'' = \langle 3s_4 - 2s_5, s_3, s_0 - b_1 s_5, 4 - 3b_1 s_2, a_2 \rangle$$
(2.28)

(remember that we are doing this on the neighborhood $s_1 = 1$ of \mathbb{P}^5).

We can perform this analysis on the other 5 standard neighborhoods of \mathbb{P}^5 to see that the ideal of the new indeterminacy locus is, in homogeneous coordinates of \mathbb{P}^5 , given by the dehomogenization of (2.29) w.r.t. the variable s_i , cf. the ideal wn_i in 4.2.1.4

$$J'' = \langle 3s_4 - 2s_5, s_3, s_0 - b_1 s_5, 4s_1 - 3b_1 s_2, \text{exc} \rangle, \qquad (2.29)$$

where the exc in ideal J'' represents the chosen local equation of the exceptional divisor (in (2.28) this is the equation a_2).

This ideal J'' corresponds to a ruled surface with basis C (due to the presence of the exc equation), which is a \mathbb{P}^1 -subbundle of the exceptional divisor: note that the other four equations are homogeneous linear equations in the variables <u>s</u>.

Hence, we have now a reduced and irreducible local complete intersection of dimension two. Since it is the full indeterminacy locus, a blowup along this subscheme will solve the indeterminacies over the neighborhood $b_3 = 1$. Thus, the map to ω becomes a morphism after blowing up this neighborhood. (cf. [9], ex. 7.17.3, page 168).

Definition 3. We call R the ruled surface defined by the ideal J'' (2.29).

Let us register for future use (see Table 3, page 45) the blowup of \mathbb{Y}_1 along R as

$$\{((\underline{b},\underline{s}), (v_0:\ldots:v_4)) \mid v_i \cdot e_j = v_j \cdot e_i\} \subset \mathbb{A}^7 \times \mathbb{P}^4,$$
(2.30)

where e_i , $0 \le i \le 4$, are the equations of J'', ordered as in (2.29),

As in (2.27), the equations of this blowup are

$$\frac{3s_4 - 2s_5}{v_0} = \frac{s_3}{v_1} = \frac{s_0 - b_1 s_5}{v_2} = \frac{4s_1 - 3b_1 s_2}{v_3} = \frac{\text{exc}}{v_4} .$$
(2.31)

2.3.2 • $b_0 = 1$ and $u_1 = 1$.

Keep the notation as in (2.14).

Taking $u_1 = 1$, the interesting equations are

$$\left\{\begin{array}{c}b_2 = b_1 u_2\\b_3 = b_1 u_3\end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{c}3a_6 = 2a_4 u_3\\a_5 = a_4 u_2\end{array}\right\},$$

where the former corresponds to the blowup of our \mathbb{P}^3 of special quadrics at x_0^2 and the latter to divisibility. We have

$$\begin{cases} f = (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3)x_0^2 + a_4(x_0x_1^2 + u_2x_0x_1x_2 + \frac{2}{3}u_3x_1^3) \\ g = x_0^2 + b_1x_0x_1 + b_1u_2x_0x_2 + b_1u_3x_1^2. \end{cases}$$

After computing ω and collecting coefficients, we realize that if $a_0 = 0$ then there is no indeterminacy (similar to (2.23)). Hence, take $a_0 = 1$ (cf. 4.2.2.1).

The indeterminacy locus is again a non-reduced scheme. The reduction now presents a surprise in relation to the previous neighborhood: its ideal can be written as 4.2.2.2

$$J_{red} = rad(J) = \langle 3b_1^2 - 4a_4, a_3, a_2, 2a_1 - 3b_1, b_1u_2, b_1(b_1 - 4u_3) \rangle,$$
(2.32)

so it is reducible. There are two components: (see primaryDecomposition K in 4.2.2.2)

1. A component C, with ideal

$$J_C = \langle b_1 - 4u_3, a_1 - 6u_3, a_2, a_3, a_4 - 12u_3^2, u_2 \rangle$$
(2.33)

which represents (compare with (2.25))

$$f = (x_0 + 2u_3x_1)^3, \quad g = (x_0 + 2u_3x_1)^2,$$

the same curve C as before, viewed in the present neighborhood.

2. A component with ideal

$$J_E = \langle a_1, a_2, a_3, a_4, b_1 \rangle,$$
(2.34)

which is the whole \mathbb{P}^2 -fiber of the exceptional divisor of \mathbb{X}' over $g = x_0^2$, and $f = x_0^3$. See (2.20). Notice that there is precisely one point in the intersection of these two components, since

$$J_C + J_E = \langle a_1, a_2, a_3, a_4, b_1, u_2, u_3 \rangle.$$

This ideal represents the single point $(g, g', f) = (x_0^2, x_0 x_1, x_0^3) \in \mathbb{Y}$.

Definition 4. We call E the component given by the ideal J_E (2.34).

Let's see what happens when we make the blowup of \mathbb{Y} first along C, followed by a blowup on E', the strict transform of E.

2.3.2.1 Blowing up along C

Now, let \mathbb{A}^7 be the affine neighborhood defined by $b_0 = a_0 = u_1 = 1$. The blowup of this \mathbb{A}^7 along C is

$$\mathbb{Y}_1 = \{ ((\underline{a}, \underline{b}), (s_0 : \ldots : s_5)) \mid s_i \cdot e_j = s_j \cdot e_i \} \subset \mathbb{A}^7 \times \mathbb{P}^5,$$

where e_i , $0 \le i \le 5$, are the equations of J_C , ordered as in (2.33).

We will write the equations of \mathbb{Y}_1 as

$$\frac{b_1 - 4u_3}{s_0} = \frac{a_1 - 6u_3}{s_1} = \frac{a_2}{s_2} = \frac{a_3}{s_3} = \frac{a_4 - 12u_3^2}{s_4} = \frac{u_2}{s_5} .$$
(2.35)

There are many choices for the local equation of the first exceptional divisor, as we see in (2.33). Let us pick, say, $\exp_1 = u_2$. Then, we make the following substitutions on the expression of ω , which corresponds to $s_5 = 1$ in (2.35), (cf. 4.2.2.3):

```
\begin{cases} b_1 - 4u_3 &= s_0 u_2 \\ a_1 - 6u_3 &= s_1 u_2 \\ a_2 &= s_2 u_2 \\ a_3 &= s_3 u_2 \\ a_4 - 12u_3^2 &= s_4 u_2 \end{cases}
```

The new 7 affine coordinates on the blowup are

$$s_0, s_1, s_2, s_3, s_4, u_2, u_3.$$
 (2.36)

After these substitutions, one can check that the new expression for ω is divisible by u_2 . Dividing, we obtain the transform of ω over the first blowup (cf. wn_5 in 4.2.2.3).

Definition 5. Denote by $\mathbb{Y}_1 \to \mathbb{Y}$ the blowup of \mathbb{Y} along C.

Recall (2.32) the radical $J_{red} = rad(J)$ represents the union $C \cup E$. After the blowup of \mathbb{Y} along C, calculations as in 4.2.2.3 reveal that the strict transform of J_{red} is given by

$$J'_E = \langle s_3, s_2, 3s_0 - 2s_1, 6u_3 + s_1u_2, 3s_4 + s_1^2u_2 \rangle.$$
(2.37)

It coincides with the strict transform E' of E under the first blowup.

2.3.2.2 Blowup along E'

We take E' as our new blowup center.

Definition 6. Denote by $\mathbb{Y}_2 \to \mathbb{Y}_1$ the blowup of \mathbb{Y}_1 along E'.

After the blowup along C, the reduction of the new indeterminacy locus in \mathbb{Y}_1 consists again of two components: the component E' (2.37), and the ruled surface R (Definition 3), with ideal (cf. 4.2.2.4)

$$\langle u_2, s_3, s_2 - 6u_3, 3s_0 - 2s_1, 3s_1u_3 - s_4 \rangle$$
.

Note the presence of the local equation $\exp_1 = u_2$, representing the base C, and the other four linear equations in the "fiber" variables <u>s</u>.

These two components intersect in codimension one in R, since the sum of their ideals is (cf. 4.2.2.4)

$$\langle u_3, u_2, s_4, s_3, s_2, 3s_0 - 2s_1 \rangle$$
.

Thus, the blowup along E' does not change the structure of R, that is, its strict transform $R' \subset \mathbb{Y}_2$ is a ruled surface, a \mathbb{P}^1 -bundle over the (transform of the) curve C.

The blowup of the \mathbb{A}^7 (2.36) along E' is

$$\mathbb{Y}_2 = \{ ((\underline{u}, \underline{s}), (t_0 : \ldots : t_4)) \mid t_i \cdot e_j = t_j \cdot e_i \} \subset \mathbb{A}^7 \times \mathbb{P}^4,$$

where e_i , $0 \le i \le 4$, are the equations of $J_{E'}$, ordered as in (2.37), and ($\underline{u}, \underline{s}$) as in (2.36).

We will write the equations of $\mathbb{Y}_2 \subset \mathbb{A}^7 \times \mathbb{P}^4$ as

$$\frac{s_3}{t_0} = \frac{s_2}{t_1} = \frac{3s_0 - 2s_1}{t_2} = \frac{6u_3 + s_1u_2}{t_3} = \frac{3s_4 + s_1^2u_2}{t_4} .$$
(2.38)

Choose now the equation $exc_2 = s_2$ in (2.37) as the new local exceptional equation, *i.e.*, take $t_1 = 1$. Equations (2.38) become

$$s_{3} = t_{0}s_{2}$$

$$3s_{0} - 2s_{1} = t_{2}s_{2}$$

$$6u_{3} + s_{1}u_{2} = t_{3}s_{2}$$

$$3s_{4} + s_{1}^{2}u_{2} = t_{4}s_{2}$$

The new 7 affine coordinates on \mathbb{Y}_2 are

$$t_0, t_2, t_3, t_4, u_2, s_1, s_2. (2.39)$$

Again, the expression of the renewed ω , obtained by these substitutions, is divisible by the equation of the exceptional divisor (s_2 in this case).

However, after this division, the map to ω is not solved yet, (cf. 4.2.2.4).

There are now two components in the new indeterminacy locus. One of them is

$$J_{R'} = \langle u_2, t_3 - 1, t_2, t_0, 3s_1 - 2t_4 \rangle.$$
(2.40)

The ideal $J_{R'}$ represents the ruled surface R' which is a \mathbb{P}^1 -subbundle of the (transform of the) exceptional divisor $\mathbb{P}(\mathcal{N}_{C|\mathbb{Y}})$ (the same as (2.29), viewed in other neighborhood).

2.3.2.3 Blowup along R'

Definition 7. Denote by $\mathbb{Y}_3 \to \mathbb{Y}_2$ the blowup of \mathbb{Y}_2 along R'.

This blowup, in the affine chart $s_5 = 1$, $t_1 = 1$, is

$$\mathbb{Y}_3 = \{ ((\underline{u}, \underline{s}, \underline{t}), (v_0 : \ldots : v_4)) \mid v_i \cdot e_j = v_j \cdot e_i \} \subset \mathbb{A}^7 \times \mathbb{P}^4,$$

where e_i , $0 \le i \le 4$, are the equations of $J_{R'}$, ordered as in (2.40), and $(\underline{u}, \underline{s}, \underline{t})$ as in (2.39).

We will write the equations of \mathbb{Y}_3 as

$$\frac{u_2}{v_0} = \frac{t_3 - 1}{v_1} = \frac{t_2}{v_2} = \frac{t_0}{v_3} = \frac{3s_1 - 2t_4}{v_4} .$$
(2.41)

Choose now the equation $exc_3 = u_2$ in (2.40) as the new local exceptional equation, *i.e.*, take $v_0 = 1$. Equations (2.41) become

$$\begin{cases} t_3 - 1 &= v_1 u_2 \\ t_2 &= v_2 u_2 \\ t_0 &= v_3 u_2 \\ 3s_1 - 2t_4 &= v_4 u_2 \end{cases}$$

The new 7 affine coordinates on \mathbb{Y}_3 are

$$v_1, v_2, v_3, v_4, u_2, s_2, t_4. \tag{2.42}$$

In \mathbb{Y}_3 the indeterminacy locus reduces to (Jn_11 in 4.2.2.5)

$$J_L = \langle s_2, v_1, v_2, v_3, v_4, t_4 \rangle.$$
(2.43)

2.3.2.4 Blowup along L

The component represented by the ideal J_L (2.43) is easy to describe. Since the affine variables are as listed in (2.42), the six linear equations represent a line inside the (transform of) the second blowup center E (note the presence of the equation $\exp_2 = s_2$ in J_L), a line parametrized by the variable u_2 .

Definition 8. We call L the component described by the ideal J_L (2.43).

Finally, we solve the indeterminacies by blowing up along the still remaining indeterminacy locus, the component L.

Definition 9. We will denote by $\mathbb{Y}_4 \to \mathbb{Y}_3$ the blowup of \mathbb{Y}_3 along L.

To put coordinates for later use (see Table 8, page 56), set

$$\mathbb{Y}_4 = \{ ((\underline{u}, \underline{s}, \underline{t}, \underline{v}), (z_0 : \ldots : z_5)) \mid z_i \cdot e_j = z_j \cdot e_i \} \subset \mathbb{A}^7 \times \mathbb{P}^5,$$

where e_i , $0 \le i \le 5$, are the equations of J_L , ordered as in (2.43), and $(\underline{u}, \underline{s}, \underline{t}, \underline{v})$ as in (2.42). The equations of $\mathbb{Y}_4 \subset \mathbb{A}^7 \times \mathbb{P}^5$ are

$$\frac{s_2}{z_0} = \frac{v_1}{z_1} = \frac{v_2}{z_2} = \frac{v_3}{z_3} = \frac{v_4}{z_4} = \frac{t_4}{z_5} .$$
 (2.44)

Over \mathbb{Y}_4 the map is solved, at least in the affine chart $[s_5 = t_1 = v_0 = 1]$, (cf. 4.2.2.6).

The calculations in all other standard neighborhoods reveal that this sequence of blowups still work to solve the map. (See 4.2.2.7).

Theorem 1. The four blowups of \mathbb{Y} along C, E, R and L will solve the map $(g, f) \mapsto \omega$ on the neighboorhod $b_0 = 1$, $u_1 = 1$.

There are other four standard neighborhoods of \mathbb{Y} to be checked, corresponding to $[b_2 = 1]$, $[b_1 = 1]$, $[b_0 = 1$ and $u_2 = 1]$, $[b_0 = 1$ and $u_3 = 1]$. The four blowups described will solve the maps over each of these neighborhoods, cf. 4.2.3, 4.2.4, 4.2.5

2.4 SUMMARY

The whole discussion of the indeterminacy loci was made over our fixed flag (2.2). Since the blowup centers as described were actually fibers of bundles over the variety of complete flags \mathbb{F} , we have at the end a bundle which constitutes the desired parameter space:

$$\mathbb{Y}_{4} \xrightarrow{\text{blowup } L} \mathbb{Y}_{3} \xrightarrow{\text{blowup } R} \mathbb{Y}_{2} \xrightarrow{\text{blowup } E} \mathbb{Y}_{1} \xrightarrow{\text{blowup } C} \mathbb{Y}$$

$$\mathbb{Y} \xrightarrow{\mathbb{P}^{4} \text{ bundle}} \mathbb{X}' \xrightarrow{\text{blowup } \mathbb{P}(\mathcal{O}_{\tilde{\mathbb{P}}^{3}}(-2))} \mathbb{X} \xrightarrow{\mathbb{P}^{3} \text{ bundle}} \mathbb{F} = \text{ all flags}$$
(2.45)

This can be summarized as follows:

Theorem 2. Let \mathbb{Y}_4 be the variety obtained by the four blowups as described above. Then \mathbb{Y}_4 is equipped with a morphism onto the exceptional component E(3).

By construction, \mathbb{Y}_4 is equipped with a line bundle \mathcal{W} , pullback of the line bundle $\mathcal{O}_{\mathbb{P}^{44}}(-1)$ over $\mathbb{P}^{44} = \mathbb{P}(H^0(\Omega^1_{\mathbb{P}^3}(4)))$. The fiber of \mathcal{W} over a point in \mathbb{Y}_4 is the rank one space spanned by the computed 1-form ω .

$$\begin{split} \mathcal{W} &= \Phi^*(\mathcal{O}(-1)) & \mathcal{O}(-1) \\ & \downarrow & \downarrow \\ & \mathbb{Y}_4 & \stackrel{\Phi}{\longrightarrow} & \mathbb{P}(H^0(\Omega^1_{\mathbb{P}^3}(4))) & \supset & E(3) = \Phi(\mathbb{Y}_4) \end{split}$$

Proposition 2. The map $\Phi : \mathbb{Y}_4 \longrightarrow E(3)$ is generically injective.

Proof. For a given $\omega \in E(3)$ (off the boundary) we already saw that the flag can be recovered (see Remark 1, page 18). Hence, we can look at a fiber over a fixed flag.

So fix the flag $\varphi_0 = (p_0, \ell_0, v_0)(2.2)$. The fiber of X at this flag is the \mathbb{P}^3 of special quadrics

$$g = b_0 x_0^2 + b_1 x_0 x_1 + b_2 x_0 x_2 + b_3 x_1^2$$

Now, we will pay attention to the dense open set where $b_2 = 1$, so

$$g = b_0 x_0^2 + b_1 x_0 x_1 + x_0 x_2 + b_3 x_1^2.$$
(2.46)

The divisibility condition (2.5) shows that the fiber of \mathbb{Y} over a quadric as in (2.46) is the $\mathbb{P}^4 = \{(a_0 : a_1 : a_2 : a_3 : a_5)\}$ of special cubics

$$f = (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3)x_0^2 + a_5\left(b_1x_0x_1^2 + x_0x_1x_2 + \frac{2}{3}b_3x_1^3\right)$$

Computing $\omega = \frac{3fdg - 2gdf}{x_0}$ one can see that there are no indeterminacies on this neighborhood $[b_2 = 1]$ (cf. 4.2.5). Moreover,

$$2g\frac{\partial f}{\partial x_3} - 3f\frac{\partial g}{\partial x_3} = 2a_3gx_0^2,$$

and the quadric g can be recovered from ω just by looking at the coefficient of dx_3 (at least on the dense open set $a_3 = 1$).

The coefficient of dx_2 in ω is

$$\frac{1}{x_0} \left(2g \frac{\partial f}{\partial x_2} - 3f \frac{\partial g}{\partial x_2} \right) =$$

$$(2a_2b_0 - 3a_0)x_0^3 + (2a_5b_0 + 2a_2b_1 - 3a_1)x_0^2x_1 + (-a_5b_1 + 2a_2b_3)x_0x_1^2 - a_2x_0^2x_2 - a_5x_0x_1x_2 - 3a_3x_0^2x_3.$$

Hence the coefficients a_2 and a_5 can be recovered from $x_0^2 x_2 dx_2$ and $x_0 x_1 x_2 dx_2$, respectively. And since the coefficients b_0 and b_1 are also known one can obtain from ω the coefficients a_0 and a_1 from $x_0^3 dx_2$ and $x_0^2 x_1 dx_2$ respectively.

Theorem 3. The degree of the exceptional component of codimension one and degree two foliations in \mathbb{P}^3 is given by

$$\int_{\mathbb{Y}_4} -c_1^{13}(\mathcal{W}) \cap [\mathbb{Y}_4].$$

Proof. We have dim(\mathbb{Y}_4) = 13. Since the map Φ is generically injective the required degree is (cf. definition of deg_f \tilde{X} in [8], page 83)

$$\int_{\mathbb{Y}_4} c_1(\Phi^*\mathcal{O}(1))^{13} \cap [\mathbb{Y}_4] = \int_{\mathbb{Y}_4} -c_1(\Phi^*\mathcal{O}(-1))^{13} \cap [\mathbb{Y}_4] = \int_{\mathbb{Y}_4} -c_1(\mathcal{W})^{13} \cap [\mathbb{Y}_4].$$
3 CALCULATION OF THE DEGREE

3.1 ACTION OF A TORUS

In order to obtain the value of the integral in Theorem 3, we will apply Bott's formula

$$\int_{\mathbb{Y}_4} -c_1^{13}(\mathcal{W}) \cap [\mathbb{Y}_4] = \sum_F \frac{-c_1^T(\mathcal{W})^{13} \cap [F]_T}{c_{top}^T(\mathcal{N}_{F|\mathbb{Y}_4})},$$
(3.1)

where the sum runs through all fixed components F under a convenient action of the torus \mathbb{C}^* . Fix an action

$$\begin{array}{cccc} \mathbb{C}^* \times \check{\mathbb{P}}^3 & \longrightarrow & \check{\mathbb{P}}^3 \\ (t, x_i) & \mapsto & t^{w_i} x_i \end{array} \tag{3.2}$$

of \mathbb{C}^* in $\check{\mathbb{P}}^3$ with distinct weights $w_i, i \in \{0, 1, 2, 3\}$.

For a good choice of these weights, the only fixed flags are the standard ones (cf. 5.1),

$$p_{ijk} = \{x_i = x_j = x_k = 0\} \in \ell_{ij} = \{x_i = x_j = 0\} \subset v_i = \{x_i = 0\}.$$

Hence, we have 24 fixed flags.

The reader not familiarized with Bott's formula may consult the reference [12], in special the examples with a fixed component of positive dimension. See also the appendix 5.2.

3.2 FIXED POINTS OVER A FLAG

Now, consider our favourite fixed flag $\varphi_0 = (p_0, \ell_0, v_0)$ (2.2).

The induced action on \mathbb{X}' is given by

$$\begin{aligned}
g &= b_0 x_0^2 + b_1 x_0 x_1 + b_2 x_0 x_2 + b_3 x_1^2 \mapsto \\
t \circ g &= t^{2w_0} b_0 x_0^2 + t^{w_0 + w_1} b_1 x_0 x_1 + t^{w_0 + w_2} b_2 x_0 x_2 + t^{2w_1} b_3 x_1^2 ; \\
g' &= u_1 x_0 x_1 + u_2 x_0 x_2 + u_3 x_1^2 \mapsto \\
t \circ g' &= t^{w_0 + w_1} u_1 x_0 x_1 + t^{w_0 + w_2} u_2 x_0 x_2 + t^{2w_1} u_3 x_1^2 .
\end{aligned}$$
(3.3)

We recognize at once the six isolated fixed points

$$\mathbf{x}_1' := (x_0 x_1), \ \mathbf{x}_2' := (x_0 x_2), \ \mathbf{x}_3' := (x_1^2), \ \mathbf{x}_4' := (x_0^2, x_0 x_1), \ \mathbf{x}_5' := (x_0^2, x_0 x_2), \ \mathbf{x}_6' := (x_0^2, x_1^2).$$

The last three points lie in the exceptional \mathbb{P}^2 of the blowup of $\mathbb{P}^3 = \{(b_0 : \cdots : b_3)\}$ at the point $x_0^2 \leftrightarrow (1:0:0:0)$.

Over each of these six points, in the fiber $\mathbb{Y}_{\mathbf{x}'_i}$ of \mathbb{Y} we have 5 other isolated fixed points, since this fiber is a \mathbb{P}^4 (again, for a good choice of the weights, cf. 5.1.2).

For example, take the fixed point $\mathbf{x}'_1 = x_0 x_1$. The fiber $\mathbb{Y}_{\mathbf{x}'_1}$ is the \mathbb{P}^4 of special cubics

$$f = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0^2 x_2 + a_3 x_0^2 x_3 + a_4 x_0 x_1^2,$$

by the divisibility condition (2.5) applied to (2.3). Thus, the five fixed cubics here are

$$\mathbf{y}'_1 := x_0^3, \ \mathbf{y}'_2 := x_0^2 x_1, \ \mathbf{y}'_3 := x_0^2 x_2, \ \mathbf{y}'_4 := x_0^2 x_3, \ \mathbf{y}'_5 := x_0 x_1^2.$$

For the pair $(\mathbf{x}'_1, \mathbf{y}'_3) = (x_0 x_1, x_0^2 x_2) \in \mathbb{Y}$, one can compute

$$\omega = \frac{3\mathbf{x}_1' d\mathbf{y}_3' - 2\mathbf{y}_3' d\mathbf{x}_1'}{x_0} = -x_0 x_1 x_2 dx_0 + 3x_0^2 x_2 dx_1 - 2x_0^2 x_1 dx_2.$$

The fiber of the line bundle \mathcal{W} over the fixed point $(x_0x_1, x_0^2x_2)$ is the \mathbb{C} -linear space spanned by

$$\omega = x_0 x_1 x_2 dx_0 - 3x_0^2 x_2 dx_1 + 2x_0^2 x_1 dx_2.$$

Notice that presently there are many fixed points outside of the indeterminacy locus of the map $(g, f) \mapsto \omega$ (see Table 1 below). Their contribution for Bott's formula can be immediately computed.

We present a table of the 30 fixed points described until here. Computations in 4.2.6.

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fixed point at $\mathbb{X}': \mathbf{x}'$	fixed point at $\mathbb{Y}_{\mathbf{r}} : \mathbf{v}'$	generator of fiber in \mathcal{W}
	$-\mathbf{x}_i \cdot \mathbf{y}_j$	
x_0x_1	$\begin{array}{c} x_0^3 \\ x_0^2 x_1 \\ x_0^2 x_2 \\ x_0^2 x_3 \\ x_0 x_1^2 \end{array}$	$ \begin{array}{ll} \rightsquigarrow & \omega = x_0^2 x_1 dx_0 - x_0^3 dx_1 \\ \rightsquigarrow & \omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1 \\ \rightsquigarrow & \omega = x_0 x_1 x_2 dx_0 - 3 x_0^2 x_2 dx_1 + 2 x_0^2 x_1 dx_2 \\ \rightsquigarrow & \omega = x_0 x_1 x_3 dx_0 - 3 x_0^2 x_3 dx_1 + 2 x_0^2 x_1 dx_3 \\ \rightsquigarrow & \omega = x_1^3 dx_0 - x_0 x_1^2 dx_1 \end{array} $
$x_0 x_2$	$egin{array}{c} x_0^3 \ x_0^2 x_1 \ x_0^2 x_2 \ x_0^2 x_3 \ x_0 x_1 x_2 \end{array}$	$ \begin{array}{ll} \rightsquigarrow & \omega = x_0^2 x_2 dx_0 - x_0^3 dx_2 \\ \rightsquigarrow & \omega = x_0 x_1 x_2 dx_0 + 2x_0^2 x_2 dx_1 - 3x_0^2 x_1 dx_2 \\ \rightsquigarrow & \omega = x_0 x_2^2 dx_0 - x_0^2 x_2 dx_2 \\ \rightsquigarrow & \omega = x_0 x_2 x_3 dx_0 - 3x_0^2 x_3 dx_2 + 2x_0^2 x_2 dx_3 \\ \rightsquigarrow & \omega = -x_1 x_2^2 dx_0 + 2x_0 x_2^2 dx_1 - x_0 x_1 x_2 dx_2 \end{array} $
x_1^2	$egin{array}{c} x_0^3 \ x_0^2 x_1 \ x_0^2 x_2 \ x_0^2 x_3 \ x_1^3 \end{array}$	$ \begin{array}{ll} \rightsquigarrow & \omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1 \\ \rightsquigarrow & \omega = x_1^3 dx_0 - x_0 x_1^2 dx_1 \\ \rightsquigarrow & \omega = 2x_1^2 x_2 dx_0 - 3x_0 x_1 x_2 dx_1 + x_0 x_1^2 dx_2 \\ \rightsquigarrow & \omega = 2x_1^2 x_3 dx_0 - 3x_0 x_1 x_3 dx_1 + x_0 x_1^2 dx_3 \\ \rightsquigarrow & \text{not defined} \end{array} $
$x_0^2, x_0 x_1$	$\begin{array}{c} x_0^3 \\ x_0^2 x_1 \\ x_0^2 x_2 \\ x_0^2 x_3 \\ x_0 x_1^2 \end{array}$	$ \begin{array}{ll} \rightsquigarrow & \text{not defined} \\ \rightsquigarrow & \omega = x_0^2 x_1 dx_0 - x_0^3 dx_1 \\ \rightsquigarrow & \omega = x_0^2 x_2 dx_0 - x_0^3 dx_2 \\ \rightsquigarrow & \omega = x_0^2 x_3 dx_0 - x_0^3 dx_3 \\ \rightsquigarrow & \omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1 \end{array} $
$x_0^2, x_0 x_2$	$\begin{array}{c} x_0^3 \\ x_0^2 x_1 \\ x_0^2 x_2 \\ x_0^2 x_3 \\ x_0 x_1 x_2 \end{array}$	$ \begin{array}{ll} & \rightsquigarrow & \text{not defined} \\ & \rightsquigarrow & \omega = x_0^2 x_1 dx_0 - x_0^3 dx_1 \\ & \rightsquigarrow & \omega = x_0^2 x_2 dx_0 - x_0^3 dx_2 \\ & \rightsquigarrow & \omega = x_0^2 x_3 dx_0 - x_0^3 dx_3 \\ & \rightsquigarrow & \omega = 2x_0 x_1 x_2 dx_0 - x_0^2 x_2 dx_1 - x_0^2 x_1 dx_2 \end{array} $
x_0^2, x_1^2	$egin{array}{c} x_0^3 \ x_0^2 x_1 \ x_0^2 x_2 \ x_0^2 x_3 \ x_1^3 \end{array}$	$ \begin{array}{ll} & \longrightarrow & \text{not defined} \\ & \rightsquigarrow & \omega = x_0^2 x_1 dx_0 - x_0^3 dx_1 \\ & \rightsquigarrow & \omega = x_0^2 x_2 dx_0 - x_0^3 dx_2 \\ & \rightsquigarrow & \omega = x_0^2 x_3 dx_0 - x_0^3 dx_3 \\ & \rightsquigarrow & \omega = x_1^3 dx_0 - x_0 x_1^2 dx_1 \end{array} $

Table 1 – Fixed points over $\mathbb {Y}$

There are 26 fixed points in \mathbb{Y} off the indeterminacy locus. In the following examples we will compute the contribution of the fixed points for Bott's formula (3.1).

Example 1. Let us compute the contribution of the point $(g, f) = (x_0 x_1, x_0^2 x_2)$. We have

$$\omega = x_0 x_1 x_2 dx_0 - 3x_0^2 x_2 dx_1 + 2x_0^2 x_1 dx_2,$$

hence the induced action reads

$$t \circ \omega = t^{w_0} x_0 t^{w_1} x_1 t^{w_2} x_2 d(t^{w_0} x_0) - 3t^{2w_0} x_0^2 t^{w_2} x_2 d(t^{w_1} x_1) + 2t^{2w_0} x_0^2 t^{w_1} x_1 d(t^{w_2} x_2)$$
$$= t^{2w_0 + w_1 + w_2} \cdot (x_0 x_1 x_2 dx_0 - 3x_0^2 x_2 dx_1 + 2x_0^2 x_1 dx_2),$$

and so the equivariant Chern class is

$$c_1^T(\mathcal{W})|_{(\mathbf{x}'_1, \mathbf{y}'_3)} = 2w_0 + w_1 + w_2.$$
 (3.4)

Since this component of fixed points consists of a single point, its normal space coincides with the tangent space of \mathbb{Y} at this point. It decomposes as

$$T_{(x_0x_1,x_0^2x_2)} \mathbb{Y} = T_{(p,\ell,v)} \mathbb{F} \oplus T_{(x_0x_1)} \mathbb{X} \oplus T_{(x_0^2x_2)} \mathbb{Y}_{x_0x_1} =$$

$$= \left(\frac{x_1}{x_0} + \frac{x_2}{x_0} + \frac{x_3}{x_0} + \frac{x_2}{x_1} + \frac{x_3}{x_1} + \frac{x_3}{x_2}\right) + \left(\frac{x_0^2}{x_0x_1} + \frac{x_0x_2}{x_0x_1} + \frac{x_1^2}{x_0x_1}\right) +$$

$$+ \left(\frac{x_0^3}{x_0^2x_2} + \frac{x_0^2x_1}{x_0^2x_2} + \frac{x_0^2x_3}{x_0^2x_2} + \frac{x_0x_1^2}{x_0^2x_2}\right),$$
(3.5)

where this sum represents the decomposition into eigenspaces; the weight of $\frac{x_i}{x_j}$ is $w_i - w_j$. From this, it follows that the contribution of the point $(x_0x_1, x_0^2x_2)$ in Bott's formula is the fraction

$$\frac{-(2w_0+w_1+w_2)^{13}}{(w_0-w_1)^3(w_2-w_0)^2(w_3-w_0)(w_2-w_1)^3(w_3-w_1)(w_3-w_2)^2(2w_1-w_0-w_2)}.$$
 (3.6)

Let us illustrate the general process used to compute the tangent spaces in (3.5). For the fiber $\mathbb{Y}_{x_0x_1}$, since $b_2 = b_3 = 0$ and $b_1 = 1$, the divisibility condition (2.5) reads $a_5 = a_6 = 0$, so this fiber consists of special cubics

$$f = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0^2 x_2 + a_3 x_0^2 x_3 + a_4 x_0 x_1^2$$

To compute the tangent space of this \mathbb{P}^4 -fiber over the fixed point $x_0^2 x_2$ (5.8), we write

$$T_{x_{0}^{2}x_{2}} \mathbb{Y}_{x_{0}x_{1}} = \underbrace{\langle x_{0}^{3}, x_{0}^{2}x_{1}, x_{0}^{2}x_{3}, x_{0}x_{1}^{2} \rangle}_{\text{other generators of the fiber}} \otimes \underbrace{\langle x_{0}^{2}x_{2} \rangle^{\vee}}_{\text{point}} = (3.7)$$

$$= [x_{0}^{3} \otimes (x_{0}^{2}x_{2})^{\vee}] \oplus [(x_{0}^{2}x_{1}) \otimes (x_{0}^{2}x_{2})^{\vee}] \oplus [(x_{0}^{2}x_{3}) \otimes (x_{0}^{2}x_{2})^{\vee}] \oplus [(x_{0}x_{1}^{2}) \otimes (x_{0}^{2}x_{2})^{\vee}] = \underbrace{\operatorname{notation}}_{=} \frac{x_{0}^{3}}{x_{0}^{2}x_{2}} + \frac{x_{0}^{2}x_{1}}{x_{0}^{2}x_{2}} + \frac{x_{0}x_{1}}{x_{0}^{2}x_{2}} + \frac{x_{0}x_{1}}{x_{0}^{2}x_{2}}.$$

=

Example 2. Next, take a fixed point over the exceptional divisor of X'. Say, take the fixed point $(x_0^2, x_0x_2, x_0^2x_3)$. The tangent space of Y at this point decomposes as

$$T_{(x_0^2, x_0 x_2, x_0^2 x_3)} \mathbb{Y} = T_{(p,\ell,v)} \mathbb{F} \oplus T_{(x_0^2, x_0 x_2)} \mathbb{X}' \oplus T_{(x_0^2 x_3)} \mathbb{Y}_{x_0^2, x_0 x_2}$$

and the tangent space of \mathbb{X}' decomposes as

$$T_{x_0^2, x_0 x_2} \mathbb{X}' = \mathcal{L}_{x_0^2, x_0 x_2} \oplus T_{[\mathcal{L}_{x_0^2, x_0 x_2}]} \mathbb{P}(\mathcal{N}_{x_0^2|\mathbb{X}}),$$
(3.8)

where $\mathcal{L}_{x_0^2, x_0 x_2}$ is the line represented by the point $(x_0^2, x_0 x_2)$ in $\mathbb{P}(\mathcal{N}_{x_0^2|\mathbb{X}})$.

Since x_0^2 is a point in X, we have

$$\mathcal{N}_{x_0^2|\mathbb{X}} = T_{x_0^2}\mathbb{X} = \frac{x_0 x_1}{x_0^2} + \frac{x_0 x_2}{x_0^2} + \frac{x_1^2}{x_0^2}$$

and

$$T_{[\mathcal{L}_{x_0^2, x_0 x_2}]} \mathbb{P}(\mathcal{N}_{x_0^2 | \mathbb{X}}) = \left(\frac{x_0 x_1}{x_0^2} \cdot \frac{x_0^2}{x_0 x_2}\right) + \left(\frac{x_1^2}{x_0^2} \cdot \frac{x_0^2}{x_0 x_2}\right) = \frac{x_0 x_1}{x_0 x_2} + \frac{x_1^2}{x_0 x_2}$$

The three eigenspaces in (3.8) are the summands as displayed below:

$$T_{x_0^2, x_0 x_2} \mathbb{X}' = \left(\frac{x_0 x_2}{x_0^2}\right) + \left(\frac{x_0 x_1}{x_0 x_2} + \frac{x_1^2}{x_0 x_2}\right).$$
(3.9)

In a similar way of examples 1 and 2 we can compute the contributions for the 26 "well resolved" points of Table 1.

3.3 DEGREE OF A FIBER

At any point $q \in \mathbb{Y}_4$ lying over $(p, \ell, v) \in \mathbb{F}$, the tangent space of \mathbb{Y}_4 decomposes as

$$T_{(p,\ell,v,q)}\mathbb{Y}_4 = T_{(p,\ell,v)}\mathbb{F} \oplus T_q\mathbb{Y}_{4(p,\ell,v)},$$

where $T_q \mathbb{Y}_{4(p,\ell,v)}$ means the tangent space of the fiber of \mathbb{Y}_4 over the flag $(p,\ell,v) \in \mathbb{F}$. Now, since we have fixed the flag (p_0,ℓ_0,v_0) we will omit the summand $T_{(p,\ell,v)}\mathbb{F}$ in the following sections, and focus on the rank 7 space $T_q \mathbb{Y}_{4(p_0,\ell_0,v_0)} = T_q \mathbb{Y}_4$.

Note that over a fixed flag, the fiber $\mathbb{Y}_{4(p,\ell,v)}$ is a smooth projective variety of dimension 7. This gives a subvariety of exceptional foliations which are obtained from a fixed flag. We can ask about the degree of this 7-dimensional variety, and the answer can be expressed as the integral (in a same way as Theorem 3)

$$\int_{\mathbb{Y}_{4(p,\ell,v)}} -c_1^7(\mathcal{W}) \cap [\mathbb{Y}_{4(p,\ell,v)}].$$

From now on we fix the flag φ_0 in (2.2) to compute the degree of the image of that fiber. This procedure reduces the affine coordinates to 7 instead of 13, simplifying the computational aspects.

3.4 CONTRIBUTIONS OVER THE BLOWUP CENTERS

In \mathbb{Y} there exist 30 fixed points (Table 1); for 26 of them we may compute immediately the contributions as shown in (3.6). There are still four remaining points to consider.

3.4.1 $(g, f) = (x_1^2, x_1^3).$

Notice that this point lives in the neighborhood $a_6 = b_3 = 1$. See 2.3.1.

First, we do the blowup of \mathbb{Y} along the curve C (2.25)

$$f = \left(\frac{b_1}{2}x_0 + x_1\right)^3, g = \left(\frac{b_1}{2}x_0 + x_1\right)^2.$$

It's a local complete intersection. We will look at the local equations, and use a trick as explained below to give adequate weights to the local coefficients of a general quadric and cubic.

From (3.3), the $\mathbb{P}^3 = \{(b_0 : b_1 : b_2 : b_3)\}$ of special quadrics has action

$$[t, (b_0: b_1: b_2: b_3)] \mapsto (t^{2w_0}b_0: t^{w_0+w_1}b_1: t^{w_0+w_2}b_2: t^{2w_1}b_3)$$
(3.10)

The trick is to look at the affine chart $b_3 = 1$,

$$(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2) = \left(\frac{b_0}{b_3}, \frac{b_1}{b_3}, \frac{b_2}{b_3}\right)$$

and set

$$[t, (\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)] \mapsto \left(\frac{t^{2w_0}b_0}{t^{2w_1}b_3}, \frac{t^{w_0+w_1}b_1}{t^{2w_1}b_3}, \frac{t^{w_0+w_2}b_2}{t^{2w_1}b_3}\right)$$

Thus, we gain naturally local weights for coordinates around the fixed point x_1^2 :

$$w_{\mathbf{b}_0} = 2w_0 - 2w_1, \quad w_{\mathbf{b}_1} = w_0 - w_1, \quad w_{\mathbf{b}_2} = w_0 + w_2 - 2w_1$$
(3.11)

These are weights for the *points* of \mathbb{A}^3 . For the respective linear functions the local weights have opposite sign, *i.e.*, the linear function b_1 has local weight $-w_{\mathbf{b}_1} = w_1 - w_0$, and the linear function b_2 has local weight $-w_{\mathbf{b}_2} = 2w_1 - w_0 - w_2$. Then,

$$t \circ (b_1 x_0 x_1^2) = t^{w_1 - w_0} b_1 t^{w_0 + 2w_1} x_0 x_1^2 = t^{3w_1} b_1 x_0 x_1^2 ,$$

$$t \circ (b_2 x_0 x_1 x_2) = t^{2w_1 - w_0 - w_2} b_2 t^{w_0 + w_1 + w_2} x_0 x_1 x_2 = t^{3w_1} b_2 x_0 x_1 x_2$$

The special cubics are (2.22)

$$f = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0^2 x_2 + a_3 x_0^2 x_3 + a_6 \left(\frac{3}{2} b_1 x_0 x_1^2 + \frac{3}{2} b_2 x_0 x_1 x_2 + x_1^3\right).$$

The $\mathbb{P}^4 = \{(a_0, a_1 : a_2 : a_3 : a_6)\}$ of special cubics inherits the induced action

$$[t, (a_0: a_1: a_2: a_3: a_6)] \mapsto (t^{3w_0}a_0: t^{2w_0+w_1}a_1: t^{2w_0+w_2}a_2: t^{2w_0+w_3}a_3: t^{3w_1}a_6)$$
(3.12)

In the affine chart $a_6 = 1$ we write

$$(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \left(\frac{a_0}{a_6}, \frac{a_1}{a_6}, \frac{a_2}{a_6}, \frac{a_3}{a_6}\right)$$

and set

$$[t, (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)] \mapsto \left(\frac{t^{3w_0}a_0}{t^{3w_1}a_6}, \frac{t^{2w_0+w_1}a_1}{t^{3w_1}a_6}, \frac{t^{2w_0+w_2}a_2}{t^{3w_1}a_6}, \frac{t^{2w_0+w_3}a_3}{t^{3w_1}a_6}\right)$$

In this way we can set local weights around the fixed point (x_1^2, x_1^3)

$$\begin{split} w_{\mathbf{a}_0} &= 3w_0 - 3w_1, & w_{\mathbf{b}_0} &= 2w_0 - 2w_1, \\ w_{\mathbf{a}_1} &= 2w_0 - 2w_1, & w_{\mathbf{b}_1} &= w_0 - w_1, \\ w_{\mathbf{a}_2} &= 2w_0 - 3w_1 + w_2, & w_{\mathbf{b}_2} &= w_0 - 2w_1 + w_2 \\ w_{\mathbf{a}_3} &= 2w_0 - 3w_1 + w_3. \end{split}$$

The local equations of the first blowup center are (see ideal J_{red} in (2.24)):

$$\begin{array}{rcl} 8a_0 - b_1^3 & (\text{weight } 3w_1 - 3w_0) \\ a_2 & (\text{weight } 3w_1 - 2w_0 - w_2) \\ b_2 & (\text{weight } 2w_1 - w_0 - w_2) \\ a_3 & (\text{weight } 3w_1 - 2w_0 - w_3) \\ 4b_0 - b_1^2 & (\text{weight } 2w_1 - 2w_0) \\ 4a_1 - 3b_1^2 & (\text{weight } 2w_1 - 2w_0) \end{array}$$

Notice that the above are all weighted homogeneous equations. The first blowup produces as exceptional divisor a \mathbb{P}^5 -bundle over the curve C, to wit, the projectivized normal bundle $\mathbb{P}(\mathcal{N}_{C|\mathbb{Y}})$. Since J_{red}/J_{red}^2 is the conormal sheaf of C, the weights of the normal bundle have opposite sign by duality. Then, over a fixed point in C, we have a (2.26)

$$\mathbb{P}^{5} = \{(s_0 : s_1 : s_2 : s_3 : s_4 : s_5)\}$$

with a natural induced action and weights

$$t \circ (s_0 : s_1 : s_2 : s_3 : s_4 : s_5) = (t^{w_{s_0}} s_0 : t^{w_{s_1}} s_1 : t^{w_{s_2}} s_2 : t^{w_{s_3}} s_3 : t^{w_{s_4}} s_4 : t^{w_{s_5}} s_5)$$
(3.13)

Over the fixed point $q := (x_1^2, x_1^3) \in C$ we have a \mathbb{P}^5 with four isolated fixed points, and since $w_{s_4} = w_{s_5}$, a fixed one dimensional component

$$\ell_1 = \mathbb{P}^1 = \{ (0:0:0:0:s_4:s_5) \} . \tag{3.14}$$

Let us describe the fiber of the normal bundle over q. First of all, we have

$$T_q \mathbb{Y} = T_{(x_1^2)} \mathbb{X} \oplus T_{(x_1^3)} \mathbb{Y}_{x_1^2}$$
(3.15)

At the point $g = x_1^2$ we have $b_1 = b_2 = 0$, so the fiber $\mathbb{Y}_{x_1^2}$ is the \mathbb{P}^4 of special cubics

$$f = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0^2 x_2 + a_3 x_0^2 x_3 + a_6 x_1^3$$

and (3.15) reads

$$T_q \mathbb{Y} = \left(\frac{x_0^2}{x_1^2} + \frac{x_0}{x_1} + \frac{x_0 x_2}{x_1^2}\right) + \left(\frac{x_0^3}{x_1^3} + \frac{x_0^2}{x_1^2} + \frac{x_0^2 x_2}{x_1^3} + \frac{x_0^2 x_3}{x_1^3}\right) \ .$$

The blowup center is the curve C parametrized by $sx_0 + x_1$, so (as in (3.7))

$$T_q C = \frac{x_0}{x_1}.$$

From the short exact sequence

$$T_qC \hookrightarrow T_q \mathbb{Y} \twoheadrightarrow \mathcal{N}_{qC|\mathbb{Y}},$$

one can find

$$\mathcal{N}_{qC|\mathbb{Y}} = \underbrace{\frac{x_0^2}{x_1^2}}_{s_5'} + \underbrace{\frac{x_0x_2}{x_1^2}}_{s_2'} + \underbrace{\frac{x_0^3}{x_1^3}}_{s_0'} + \underbrace{\frac{x_0^2}{x_1^2}}_{s_4'} + \underbrace{\frac{x_0^2x_2}{x_1^3}}_{s_1'} + \underbrace{\frac{x_0^2x_3}{x_1^3}}_{s_3'} + \underbrace{\frac{x_0x_3}{x_1^3}}_{s_3'}$$
(3.16)

Note the presence of the two summands $\frac{x_0^2}{x_1^2}$, representing two (independent) eigenvectors associated to the same weight. A simple comparison of the weights reveals the associations with the homogeneous coordinates s_i ; for example, the fixed point $s'_0 = (1:0:0:0:0:0)$, with weight $w_{s_0} = 3w_0 - 3w_1$ (cf (3.13)) is associated to the eigenvector $\frac{x_0^3}{x_1^3}$, with weight $3w_0 - 3w_1$.

For short, we use the following notation for points of the \mathbb{P}^5 -fiber:

$$s'_{0} = (1:0:0:0:0:0), \quad s'_{1} = (0:1:0:0:0:0)$$
$$s'_{2} = (0:0:1:0:0), \quad s'_{3} = (0:0:0:1:0:0)$$
$$s'_{4} = (0:0:0:0:1:0), \quad s'_{5} = (0:0:0:0:0:1)$$

Example 3. To compute the contribution of the fixed point

$$(q, s'_0) = ((x_1^2, x_1^3), (1:0:0:0:0:0)) \in \mathbb{Y}_1,$$

write

$$T_{(q,s_0')} \mathbb{Y}_1 = T_q C \oplus \mathcal{L}_{x_0^3/x_1^3} \oplus T_{[\mathcal{L}_{x_0^3/x_1^3}]} \mathbb{P}(\mathcal{N}_{qC|\mathbb{Y}}).$$

This leads us to the decomposition

$$T_{(q,s_0')} \mathbb{Y}_1 = \left(\frac{x_0}{x_1}\right) + \left(\frac{x_0^3}{x_1^3}\right) + \left[\left(\frac{x_0^2}{x_1^2} + \frac{x_0^2 x_2}{x_1^3} + \frac{x_0^2 x_3}{x_1^3} + \frac{x_0^2}{x_1^2} + \frac{x_0 x_2}{x_1^2}\right) \cdot \left(\frac{x_0^3}{x_1^3}\right)^*\right] = \\ = \left(\frac{x_0}{x_1}\right) + \left(\frac{x_0^3}{x_1^3}\right) + \left(\frac{x_1}{x_0} + \frac{x_2}{x_0} + \frac{x_3}{x_0} + \frac{x_1}{x_0} + \frac{x_1 x_2}{x_0^2}\right).$$

Thus,

$$c_7^T(T_{(q,s_0')} \mathbb{Y}_1) = 3(w_0 - w_1)^4 (w_2 - w_0)(w_3 - w_0)(w_1 + w_2 - 2w_0)$$
.

Now, we look for the equivariant Chern class $c_1^T(\mathcal{W})$. To obtain the fiber of \mathcal{W} at the point (q, s'_0) perform the blowup, choosing the first equation (exc:= $8a_0 - b_1^3$ (2.24)) to be the equation of the exceptional divisor. Step by step: (cf. 4.2.1.4 and 4.2.1.5)

1. Write

$$\begin{cases} f = a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0^2 x_2 + a_3 x_0^2 x_3 + \frac{3}{2} b_1 x_0 x_1^2 + \frac{3}{2} b_2 x_0 x_1 x_2 + x_1^3 \\ g = b_0 x_0^2 + b_1 x_0 x_1 + b_2 x_0 x_2 + x_1^2; \end{cases}$$

2. Compute

$$\omega = \frac{3fdg - 2gdf}{x_0};$$

3. Do the substitutions into ω ,

$$\begin{array}{rcl}
a_2 &=& s_1(8a_0 - b_1^3) \\
b_2 &=& s_2(8a_0 - b_1^3) \\
a_3 &=& s_3(8a_0 - b_1^3) \\
4b_0 - b_1^2 &=& s_4(8a_0 - b_1^3) \\
4a_1 - b_1^2 &=& s_5(8a_0 - b_1^3)
\end{array}$$

- 4. Divide by the equation $exc=8a_0-b_1^3$;
- 5. Evaluate the new obtained expression at the fixed point, simply by taking the coefficients

$$a_0 = b_1 = 0$$
 (the point $(f = x_1^3, g = x_1^2)$),
 $s_1 = s_2 = s_3 = s_4 = s_5 = 0$ (the point $(1:0:0:0:0:0)$).

After the blowup, we can see that (cf. Fwn_0 in 4.2.1.5)

$$\omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1,$$

so $c_1^T(\mathcal{W})|_{(q,s_0')} = 2w_0 + 2w_1$ (see (3.4)).

However, we will not always find a well defined ω because there are indeterminacies remaining – we don't solve all of them with this single blowup. But in example 3 we were lucky at the chosen point.

Next, we present a table with the four fixed points and the fixed \mathbb{P}^1 : 4.2.1.5

fixed point/ \mathbb{P}^1 ($s_0: s_1: s_2: s_3: s_4: s_5$)	associated eigenvector	generator of fiber in \mathcal{W}
(1:0:0:0:0:0:0)(0:1:0:0:0:0)(0:0:1:0:0:0)(0:0:0:1:0:0)(0:0:0:1:0:0)	$\begin{array}{c} x_0^3/x_1^3 \\ x_0^2 x_2/x_1^3 \\ x_0 x_2/x_1^2 \\ x_0^2 x_3/x_1^3 \end{array}$	$\omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1$ $\omega = 2x_1^2 x_2 dx_0 - 3x_0 x_1 x_2 dx_1 + x_0 x_1^2 dx_2$ not defined $\omega = 2x_1^2 x_3 dx_0 - 3x_0 x_1 x_3 dx_1 + x_0 x_1^2 dx_3$

Table 2 – Fixed points over $f = x_1^3, g = x_1^2$

There are three well resolved points over $q = (x_1^2, x_1^3)$: the points s'_0 , s'_1 and s'_3 . Now we need to study the indeterminacy point (0:0:1:0:0:0) and the fixed line ℓ_1 .

3.4.1.1 Resolution at the point $s'_2 = (0:0:1:0:0:0)$.

The new indeterminacy locus is the ruled surface R, defined by the ideal (see (2.29))

$$J'' = \langle 3s_4 - 2s_5, s_3, s_0 - b_1 s_5, 4s_1 - 3b_1, b_2 \rangle.$$
(3.17)

Over the point $s'_2 = (0:0:1:0:0:0)$ we have a (2.30)

$$\mathbb{P}^4 = \{ (v_0 : v_1 : v_2 : v_3 : v_4) \} = \mathbb{P}(\mathcal{N}_{(q,s_2')R|\mathbb{Y}_1}).$$

The action passes to this \mathbb{P}^4 by looking to the normal space, see (3.22). But we can use our trick to see the induced action, just like it was done in §3.4.1.

Indeed, we are in the neighborhood $s_2 = 1$, and in this affine chart,

$$(\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_3, \mathbf{s}_4, \mathbf{s}_5) = \left(\frac{s_0}{s_2}, \frac{s_1}{s_2}, \frac{s_3}{s_2}, \frac{s_4}{s_2}, \frac{s_5}{s_2}\right)$$

Thus, we have local weights for affine points around $s_2 = 1$, given by (see (3.13))

Remember that the equations have opposite sign weights by duality. The local equations of the blowup center R are

$$\begin{cases} 3\mathbf{s}_4 - 2\mathbf{s}_5 & \text{(weight } w_2 - w_0) \\ \mathbf{s}_3 & \text{(weight } w_1 + w_2 - w_0 - w_3) \\ \mathbf{s}_0 - b_1 \mathbf{s}_5 & \text{(weight } w_1 + w_2 - 2w_0) \\ 4\mathbf{s}_1 - 3b_1 & \text{(weight } w_1 - w_0) \\ b_2 & \text{(weight } 2w_1 - w_0 - w_2) \end{cases}$$

and so relations (2.31) induce

$$w_{v_0} = w_0 - w_2 \qquad w_{v_3} = w_0 - w_1$$

$$w_{v_1} = w_0 + w_3 - w_1 - w_2 \qquad w_{v_4} = w_0 + w_2 - 2w_1$$

$$w_{v_2} = 2w_0 - w_1 - w_2$$

$$t \circ (v_0 : v_1 : v_2 : v_3 : v_4) = (t^{w_{v_0}} v_0 : t^{w_{v_1}} v_1 : t^{w_{v_2}} v_2 : t^{w_{v_3}} v_3 : t^{w_{v_4}} v_4)$$
(3.18)

This \mathbb{P}^4 has five isolated fixed points since the weights are distinct pairwise. (cf. 4.2.1.6).

fixed point $(v_0:v_1:v_2:v_3:v_4)$	associated eigenvector	generator of fiber in \mathcal{W}
(1:0:0:0:0)(0:1:0:0)(0:0:1:0:0)(0:0:1:0)(0:0:0:1:0)(0:0:0:1:0)		$\omega = x_1^3 dx_0 - x_0 x_1^2 dx_1$ $\omega = 2x_1^2 x_3 dx_0 - 3x_0 x_1 x_3 dx_1 + x_0 x_1^2 dx_3$ $\omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1$ $\omega = 2x_1^2 x_2 dx_0 - 3x_0 x_1 x_2 dx_1 + x_0 x_1^2 dx_2$ $\omega = x_0 x_1^2 dx_0 - 3x_0 x_1 x_2 dx_1 + x_0 x_1^2 dx_2$

Table 3 – Fixed points over $(x_1^3, x_1^2), (0:0:1:0:0:0)$

Now we look at the tangent spaces at these points. We are able to write (see (3.16))

$$\mathcal{N}_{qC|\mathbb{Y}} = \frac{x_0^2}{x_1^2} + \frac{x_0 x_2}{x_1^2} + \frac{x_0^3}{x_1^3} + \frac{x_0^2}{x_1^2} + \frac{x_0^2 x_2}{x_1^3} + \frac{x_0^2 x_3}{x_1^3}$$

Remember that the indeterminacy point s'_2 corresponds to $\frac{x_0x_2}{x_1^2}$ (Table 2). The blowup center is the ruled surface R, a \mathbb{P}^1 -bundle over C. The tangent space of this surface R decomposes as

$$T_{(q,s'_2)}R = T_q C \oplus T_{s'_2} \mathbb{P}^1.$$
 (3.19)

The equations of this \mathbb{P}^1 are (3.17)

$$\begin{cases} 3s_4 - 2s_5 = 0\\ s_3 = 0\\ s_0 - b_1 s_5 = 0\\ 4s_1 - 3b_1 = 0, \end{cases}$$

hence this is the \mathbb{P}^1 -fiber of points

$$(b_1s_5:\frac{3}{4}b_1:s_2:0:\frac{2}{3}s_5:s_5).$$

At the fixed point (x_1^3, x_1^2, s_2') we have $b_1 = 0$, so this is the

$$\mathbb{P}^1 = \{ (0:0:s_2:0:\frac{2}{3}s_5:s_5) \},$$
(3.20)

and

$$T_{s_2'} \mathbb{P}^1 = \frac{s_5}{s_2} = \frac{x_0^2}{x_1^2} \cdot \frac{x_1^2}{x_0 x_2} = \frac{x_0}{x_2} .$$
(3.21)

A note for (3.21): at the $\mathbb{P}^1 = \{(s_2 : s_5)\}$ (3.20) we look at the point $s'_2 = (1 : 0)$. The equation of this point in \mathbb{P}^1 is $s_5^{\vee} = 0$, the linear form vanishing at the point. From this, the tangent space of \mathbb{P}^1 at this point is $\frac{s_2^{\vee}}{s_5^{\vee}} = \frac{s_5}{s_2}$. At this moment we "overload" the notation to distinguish between the equation (s_5^{\vee}) and coordinate of the point (s_5) .

Using the expression (3.21) in (3.19) gives

$$T_{(q,s_2')}R = \frac{x_0}{x_1} + \frac{x_0}{x_2}.$$

The short exact sequence

$$T_{(q,s'_2)}R \hookrightarrow T_{(q,s'_2)}\mathbb{Y}_1 \twoheadrightarrow \mathcal{N}_{(q,s'_2)R|\mathbb{Y}_2}$$

allows us to write

$$\mathcal{N}_{(q,s_2')R|\mathbb{Y}_1} = \frac{x_0 x_2}{x_1^2} + \frac{x_0}{x_2} + \frac{x_0^2}{x_1 x_2} + \frac{x_0}{x_1} + \frac{x_0 x_3}{x_1 x_2}.$$
(3.22)

We may conclude from this the existence of the five isolated points associated to these weights, as we have already done in Table 3.

Example 4. For $v'_3 = (0:0:0:1:0)$ in Table 3 one has

$$c_1^T(\mathcal{W})|_{(q,s_2',v_3')} = w_0 + 2w_1 + w_2$$

since $\mathcal{W}|_{(q,s'_2,v'_3)} = \langle 2x_1^2 x_2 dx_0 - 3x_0 x_1 x_2 dx_1 + x_0 x_1^2 dx_2 \rangle.$

The tangent space of \mathbb{Y}_3 at this fixed point is

$$T_{(q,s'_2,v'_3)} \mathbb{Y}_3 = T_{(q,s'_2)} R \oplus \mathcal{L}_{x_0/x_1} \oplus T_{[\mathcal{L}_{x_0/x_1}]} \mathbb{P}(\mathcal{N}_{(q,s'_2)R|\mathbb{Y}_1})$$
$$= \left(\frac{x_0}{x_1} + \frac{x_0}{x_2}\right) + \left(\frac{x_0}{x_1}\right) + \left(\frac{x_2}{x_1} + \frac{x_1}{x_2} + \frac{x_0}{x_2} + \frac{x_3}{x_2}\right).$$

This yields the equivariant class

$$c_7^T(T_{(q,s'_2,v'_3)}\mathbb{Y}_3) = (w_0 - w_1)^2(w_0 - w_2)^2(w_2 - w_1)^2(w_2 - w_3)$$
.

3.4.1.2 Resolution on the fixed \mathbb{P}^1 over (x_1^2, x_1^3) .

Inside the line $\ell_1 = \{(0 : 0 : 0 : 0 : s_4 : s_5)\}$ there's another indeterminacy point $p_{\ell} = (0 : 0 : 0 : 0 : 2 : 3)$: recall the fiber of \mathcal{W} at this line $\ell_1 = \{(0 : 0 : 0 : 0 : 0 : s_4 : s_5)\}$ is

$$\omega = (2s_5 - 3s_4)x_1^3 dx_0 - (2s_5 - 3s_4)x_0x_1^2 dx_1, \qquad (3.23)$$

see Table 2. We may be tempted to divide (3.23) by the common factor $2s_5 - 3s_4$, but we need to pay attention to some things:

• There is a single point in ℓ_1 , the point $p_{\ell_1} = \{(0 : 0 : 0 : 0 : 2 : 3)\}$, in the indeterminacy locus (note that ω (3.23) vanishes at this point). In fact, this point satisfies the equations of our next blowup center R (2.29).

• After the blowup along R, over the point p_{ℓ_1} there is a $\mathbb{P}^4 = \mathbb{P}(\mathcal{N}_{(q,p_{\ell_1})R|\mathbb{Y}_1})$, with five fixed points, for which the contributions to Bott's formula can not be forgotten. The fibers of \mathcal{W} over these points are all distinct, see Table 4.

Then, the contribution of the fixed component ℓ_1 can not be computed yet. We just obtained in (3.16)

$$\mathcal{N}_{qC|\mathbb{Y}} = \frac{x_0^3}{x_1^3} + \frac{x_0^2}{x_1^2} + \frac{x_0^2 x_2}{x_1^3} + \frac{x_0^2 x_3}{x_1^3} + \frac{x_0^2}{x_1^2} + \frac{x_0 x_2}{x_1^2}.$$

Then

$$T_{(q,p_{\ell_1})} \mathbb{Y}_1 = T_q C \oplus \mathcal{L}_{x_0^2/x_1^2} \oplus T_{[\mathcal{L}_{x_0^2/x_1^2}]} \mathbb{P}(\mathcal{N}_{qC|\mathbb{Y}}) =$$
$$= \frac{x_0}{x_1} + \frac{x_0^2}{x_1^2} + \frac{x_2}{x_0} + \frac{x_0}{x_1} + 1 + \frac{x_2}{x_1} + \frac{x_3}{x_1}.$$
(3.24)

Note the summand 1 corresponding to an eigenvector with weight 0. The ruled surface R, with base C has fiber over the point $q = (x_1^2, x_1^3)$

$$\mathbb{P}^1 = \{ (0:0:s_2:0:2s_5:3s_5) \}$$

(compare with (3.20), and remember that p_{ℓ_1} is the point $s_2 = 0$). It follows from this

$$T_{(q,p_{\ell_1})}R = T_qC \oplus T_{p_{\ell_1}}\mathbb{P}^1 = \frac{x_0}{x_1} + \frac{x_2}{x_0}.$$
(3.25)

The exceptional $\mathbb{P}^4 = \mathbb{P}(\mathcal{N}_{(q,p_{\ell_1})R|\mathbb{Y}_1})$, where

$$\mathcal{N}_{(q,p_{\ell_1})R|\mathbb{Y}_1} = \frac{x_0^2}{x_1^2} + \frac{x_0}{x_1} + 1 + \frac{x_2}{x_1} + \frac{x_3}{x_1} \ . \tag{3.26}$$

fixed point $(v_0:v_1:v_2:v_3:v_4)$	associated eigenvector	generator of fiber in \mathcal{W}
(1:0:0:0:0)(0:1:0:0:0)(0:0:1:0:0)(0:0:0:1:0)(0:0:0:1:0)(0:0:0:0:1)	$ \begin{array}{c} 1 \\ x_3/x_1 \\ x_0/x_1 \\ x_2/x_1 \\ x_0^2/x_1^2 \end{array} $	$\begin{split} \omega &= x_1^3 dx_0 - x_0 x_1^2 dx_1 \\ \omega &= 2x_1^2 x_3 dx_0 - 3x_0 x_1 x_3 dx_1 + x_0 x_1^2 dx_3 \\ \omega &= x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1 \\ \omega &= 2x_1^2 x_2 dx_0 - 3x_0 x_1 x_2 dx_1 + x_0 x_1^2 dx_2 \\ \omega &= x_0^2 x_1 dx_0 - x_0^3 dx_1 \end{split}$

Table 4 – Fixed points over $(x_1^3, x_1^2), p_{\ell_1} = (0:0:0:0:1:3/2)$

Notice that we have five fixed points, but the strict transform $\tilde{\ell}_1$ of ℓ_1 passes through one of them (the point $v'_0 := (1 : 0 : 0 : 0 : 0)$, associated to the weight 0). Hence, we gain the contribution of four isolated fixed points. For the point v'_0 , we will compute the contribution of the strict transform $\tilde{\ell}_1$, since this point lies inside a higher dimensional component of the fixed point locus. 4.2.1.7

Example 5. For $v'_2 = (0:0:1:0:0)$ we can compute, using (3.26) and (3.25),

$$T_{(q,p_{\ell_1},v_2')} \mathbb{Y}_3 = T_{(q,p_{\ell_1})} R \oplus \mathcal{L}_{x_0/x_1} \oplus T_{[\mathcal{L}_{x_0/x_1}]} \mathbb{P}(\mathcal{N}_{(q,p_{\ell_1})R|\mathbb{Y}_1}) = \\ = \left(\frac{x_0}{x_1} + \frac{x_2}{x_0}\right) + \left(\frac{x_0}{x_1}\right) + \left(\frac{x_0}{x_1} + \frac{x_1}{x_0} + \frac{x_2}{x_0} + \frac{x_3}{x_0}\right)$$

and

$$c_7^T(T_{(q,p_\ell,t_2')}\mathbb{Y}_3) = (w_0 - w_1)^4(w_2 - w_0)^2(w_0 - w_3).$$

Moreover, Table 4 shows $\mathcal{W}|_{(q,p_{\ell_1},v_2')} = \langle x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1 \rangle$, so

$$c_1^T(\mathcal{W})|_{(q,p_{\ell_1},v_2')} = 2w_0 + 2w_1$$
.

The fixed point $v'_0 = (1:0:0:0:0)$ lies in the strict transform $\tilde{\ell}_1$ of ℓ_1 . The action is trivial on $\ell_1 = \{(0:0:0:0:s_4:s_5)\}$, since $w_{s_4} = w_{s_5}$, and it follows from (3.24) the decomposition of the fiber of normal bundle

$$\mathcal{N}_{(q,p_{\ell_1})\ell_1|\mathbb{Y}_1} = \frac{x_0}{x_1} + \frac{x_0^2}{x_1^2} + \frac{x_0^2}{x_1^2} + \frac{x_2}{x_0} + \frac{x_0}{x_1} + \frac{x_2}{x_1} + \frac{x_3}{x_1} + \frac{x_3}{x_1} + \frac{x_2}{x_1} + \frac{x_2}{x_1} + \frac{x_2}{x_1} + \frac{x_2}{x_1} + \frac{x_2}{x_1} + \frac{x_3}{x_1} + \frac{x_3}{x_1} + \frac{x_3}{x_1} + \frac{x_3}{x_1} + \frac{x_3}{x_1} + \frac{x_2}{x_1} + \frac{x_3}{x_1} + \frac{x$$

The term T_qC corresponds to the pullback of the tangent space of C, restricted to the ℓ_1 that lives in the \mathbb{P}^5 -fiber.

To find the decomposition of $\mathcal{N}_{(q,p_{\ell_1},v'_0)\tilde{\ell}_1|\mathbb{Y}_3}$, remember (3.26) and (3.25) to write

$$T_{(q,p_{\ell_1},v_0')}\mathbb{Y}_3 = T_{(q,p_{\ell_1})}R \oplus \mathcal{L}_1 \oplus T_{[\mathcal{L}_1]}\mathbb{P}(\mathcal{N}_{(q,p_{\ell_1})R|\mathbb{Y}_1}) =$$

$$= \left(\frac{x_0}{x_1} + \frac{x_2}{x_0}\right) + 1 + \left[\left(\frac{x_0^2}{x_1^2} + \frac{x_0}{x_1} + \frac{x_2}{x_1} + \frac{x_3}{x_1}\right) \cdot 1^{\vee}\right]$$
(3.28)

Thus,

$$\mathcal{N}_{(q,p_{\ell_1},v_0')\tilde{\ell}_1|\mathbb{Y}_3} = \frac{x_0}{x_1} + \frac{x_2}{x_0} + \frac{x_0^2}{x_1^2} + \frac{x_0}{x_1} + \frac{x_2}{x_1} + \frac{x_3}{x_1} \ . \tag{3.29}$$

The weights in the sum (3.29) are the same as in (3.27), because the point v'_0 is associated with weight 0. This will always be repeated on the other fixed lines ahead.

Despite this, the global description of normal bundle $\mathcal{N}_{\tilde{\ell}_1|\mathbb{Y}_3}$ do not need to be the same as in (3.27). There are integers d_1, \ldots, d_6 with

$$\mathcal{N}_{(q,p_{\ell_1},v_0')\tilde{\ell}_1|\mathbb{Y}_3} = \frac{x_0}{x_1} + \frac{x_0^2}{x_1^2} + \frac{x_2}{x_0} + \frac{x_0}{x_1} + \frac{x_0}{x_1} + \frac{x_2}{x_1} + \frac{x_3}{x_1} \\ \mathcal{O}_{\tilde{\ell}_1}(d_1) + \mathcal{O}_{\tilde{\ell}_1}(d_2) + \mathcal{O}_{\tilde{\ell}_1}(d_3) + \mathcal{O}_{\tilde{\ell}_1}(d_4) + \mathcal{O}_{\tilde{\ell}_1}(d_5) + \mathcal{O}_{\tilde{\ell}_1}(d_6)$$
(3.30)

Example 6. 4.3.14.1 With (3.30), we can compute the contribution of $\tilde{\ell}_1$ for Bott. The equivariant Chow ring of $\tilde{\ell}_1 = \mathbb{P}^1$ is

$$\mathcal{A}_*^T(\tilde{\ell}_1) = \mathcal{A}_*(\tilde{\ell}_1) \otimes \mathbb{Z}[t],$$

where $\mathcal{A}_*(\tilde{\ell}_1) = \mathbb{Z}[h]/h^2$. The decomposition of $\mathcal{N}_{(q,p_{\ell_1},v'_0)\tilde{\ell}_1|\mathbb{Y}_3}$ gives $c_6^T(\mathcal{N}_{\tilde{\ell}_1|\mathbb{Y}_3}) =$

$$(w_0 - w_1 + d_1h)(2w_0 - 2w_1 + d_2h)(w_2 - w_0 + d_3h)(w_0 - w_1 + d_4h)(w_2 - w_1 + d_5h)(w_3 - w_1 + d_6h)(w_3 - w$$

i.e., $c_6^T(\mathcal{N}_{\tilde{\ell}_1|\mathbb{Y}_3}) = \mathbf{w} + \mathbf{d}h$, where

$$\mathbf{w} = 2(w_0 - w_1)^3(w_2 - w_1)(w_3 - w_1)(w_2 - w_0)$$

$$\mathbf{d} = (w_0 - w_1)^2 [2(w_2 - w_0)(w_2 - w_1)(w_3 - w_1)d_1 + \dots + 2(w_0 - w_1)(w_2 - w_1)(w_2 - w_0)d_6].$$

Along the strict transform $\tilde{\ell}_1$, the line bundle $\mathcal{W}|_{\tilde{\ell}_1}$ is trivial (the fibers are generated by $\omega = x_1^3 dx_0 - x_0 x_1^2 dx_1$ everywhere). Hence $c_1(\mathcal{W}|_{\tilde{\ell}_1}) = 0$, and then $c_1^T(\mathcal{W}|_{\tilde{\ell}_1}) = w_0 + 3w_1$.

The contribution of the fixed $\tilde{\ell}_1$ may be computed by

$$\int_{\tilde{\ell}_1} \frac{-c_1^T (\mathcal{W}|_{\tilde{\ell}_1})^7}{c_6^T (\mathcal{N}_{\tilde{\ell}_1|\mathbb{Y}_3})} = -\int_{\tilde{\ell}_1} \frac{(w_0 + 3w_1)^7}{\mathbf{w} + \mathbf{d}h} = -\int_{\tilde{\ell}_1} \frac{(w_0 + 3w_1)^7 [\mathbf{w} - \mathbf{d}h]}{\mathbf{w}^2}$$

where in the last equality we multiplied by $\mathbf{w} - \mathbf{d}h$ both the numerator and the denominator, and used that $h^2 = 0$ (since $\mathcal{A}_*(\tilde{\ell}_1) = \mathbb{Z}[h]/h^2$).

Taking the coefficient of h we find

$$\int_{\tilde{\ell}_1} \frac{-c_1^T (\mathcal{W}|_{\tilde{\ell}_1})^7}{c_6^T (\mathcal{N}_{\tilde{\ell}_1|\mathbb{Y}_3})} = \frac{d(w_0 + 3w_1)^7}{\mathbf{w}^2} \, .$$

Notice that this contribution of $\tilde{\ell}_1$ is a linear expression on the unknown integers d_1, \ldots, d_6 . Further ahead, we will find some useful relations for $d'_i s$, see Proposition 3.61.

,

3.4.2 $(g, g', f) = (x_0^2, x_0 x_1, x_0^3).$

We will adapt the same ideas of section 3.4.1 to compute the contributions of all fixed components that appear over the point $q := (g, g', f) = (x_0^2, x_0 x_1, x_0^3)$ when solving the indeterminacies as in §2.3.2.

The fixed point lie in the neighborhood $a_0 = b_0 = u_1 = 1$. Take

$$\begin{cases} f = x_0^3 + a_1 x_0^2 x_1 + a_2 x_0^2 x_2 + a_3 x_0^2 x_3 + a_4 (x_0 x_1^2 + u_2 x_0 x_1 x_2 + \frac{2}{3} u_3 x_1^3) \\ g = x_0^2 + b_1 (x_0 x_1 + u_2 x_0 x_2 + u_3 x_1^2). \end{cases}$$

The coefficients inherit the weights

$$w_{\mathbf{a}_{1}} = w_{1} - w_{0}, \qquad w_{\mathbf{b}_{1}} = w_{1} - w_{0},$$

$$w_{\mathbf{a}_{2}} = w_{2} - w_{0}, \qquad w_{\mathbf{u}_{2}} = w_{2} - w_{1},$$

$$w_{\mathbf{a}_{3}} = w_{3} - w_{0}, \qquad w_{\mathbf{u}_{3}} = w_{1} - w_{0},$$

$$w_{\mathbf{a}_{4}} = 2w_{1} - 2w_{0}.$$

(3.31)

Presently the equations of C are (2.33)

$$\begin{cases} b_1 - 4u_3 & \text{(weight } w_0 - w_1\text{)} \\ a_1 - 6u_3 & \text{(weight } w_0 - w_1\text{)} \\ a_2 & \text{(weight } w_0 - w_2\text{)} \\ a_3 & \text{(weight } w_0 - w_3\text{)} \\ a_4 - 12u_3^2 & \text{(weight } 2w_0 - 2w_1\text{)} \\ u_2 & \text{(weight } w_1 - w_2\text{)} \end{cases}$$

(Remember that, for equations, the weights have opposite sign to the dual points). Notice that these are all weighted homogeneous equations. Hence, over the indeterminacy point $q = (x_0^2, x_0 x_1, x_0^3)$ we have a

$$\mathbb{P}^{5} = \{(s_{0}: s_{1}: s_{2}: s_{3}: s_{4}: s_{5})\} = \mathbb{P}(\mathcal{N}_{qC|\mathbb{Y}})$$

with a natural induced action and weights (2.35)

$$\begin{aligned}
 &w_{s_0} &= w_1 - w_0 & w_{s_3} &= w_3 - w_0 \\
 &w_{s_1} &= w_1 - w_0 & w_{s_4} &= 2w_1 - 2w_0 \\
 &w_{s_2} &= w_2 - w_0 & w_{s_5} &= w_2 - w_1.
 \end{aligned}$$
(3.32)

Over this \mathbb{P}^5 we have four isolated fixed points and a fixed line ℓ_2 .

fixed point/ \mathbb{P}^1 ($s_0: s_1: s_2: s_3: s_4: s_5$)	associated eigenvector	generator of fiber in \mathcal{W}
$(s_0:s_1:0:0:0:0) \\ (0:0:1:0:0:0) \\ (0:0:0:1:0:0) \\ (0:0:0:1:0:0) \\ (0:0:0:0:1:0) \\ (0:0:0:0:1:0) \\ (0:0:0:0:0:1) \\ \end{cases}$	$ \begin{array}{c} x_1/x_0 \\ x_2/x_0 \\ x_3/x_0 \\ x_1^2/x_0^2 \\ x_2/x_1 \end{array} $	$\omega = (3s_0 - 2s_1)x_0^2 x_1 dx_0 - (3s_0 - 2s_1)x_0^3 dx_1$ $\omega = x_0^2 x_2 dx_0 - x_0^3 dx_2$ $\omega = x_0^2 x_3 dx_0 - x_0^3 dx_3$ $\omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1$ not defined

Table 5 – Fixed points over $(g, g', f) = (x_0^2, x_0 x_1, x_0^3)$

See 4.2.2.8 for calculations. In this case, there are still two points to solve, to wit,

$$s'_5 = (0:0:0:0:0:1)$$

and the point

$$p_{\ell_2} = (2:3:0:0:0:0) \in \ell_2 = \{(s_0:s_1:0:0:0:0)\}.$$
(3.33)

The second step is to blowup along the transform of E (2.45).

3.4.2.1 Resolution on the point $(x_0^2, x_0x_1, x_0^3, s_5')$

Now, we take the blowup along the strict transform of E and look at the fixed point $s'_5 = (0:0:0:0:0:1) \in \mathbb{Y}_1$.

Remember that to find the equations of the transform, we have an ideal (see (2.32))

$$J_{red} = rad(J) = \langle 3b_1^2 - 4a_4, a_3, a_2, 2a_1 - 3b_1, b_1u_2, b_1(b_1 - 4u_3) \rangle,$$

which represents the two components, the curve C and the surface E. Take the transform of this ideal J_{red} by the blowup along C.

The ideal of the strict transform of E is (2.37)

$$J'_E = \langle s_3, s_2, 3s_0 - 2s_1, 6u_3 + s_1u_2, 3s_4 + s_1^2u_2 \rangle.$$

Over the point $s'_5 = (0:0:0:0:1)$ we have a $\mathbb{P}^4 = \{(t_0: t_1: t_2: t_3: t_4)\} = \mathbb{P}(\mathcal{N}_{(q,s'_5)E|\mathbb{Y}_1}).$

Let's see how the action passes to this \mathbb{P}^4 . We are in the neighborhood $s_5 = 1$, and in this affine chart,

$$(\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4) = \left(\frac{s_0}{s_5}, \frac{s_1}{s_5}, \frac{s_2}{s_5}, \frac{s_3}{s_5}, \frac{s_4}{s_5}\right)$$

Thus, from (3.32) we have local weights for affine coordinates around $s_5 = 1$, given by

Remember that the equations have opposite sign weights by duality. The local equations of the blowup center E are (from (3.31) and (3.34))

$$\begin{cases} \mathbf{s}_{3} & (\text{weight } w_{0} + w_{2} - w_{1} - w_{3}) \\ \mathbf{s}_{2} & (\text{weight } w_{0} - w_{1}) \\ 3\mathbf{s}_{0} - 2\mathbf{s}_{1} & (\text{weight } w_{0} + w_{2} - 2w_{1}) \\ 6u_{3} + \mathbf{s}_{1}u_{2} & (\text{weight } w_{0} - w_{1}) \\ 3\mathbf{s}_{4} + \mathbf{s}_{1}^{2}u_{2} & (\text{weight } 2w_{0} + w_{2} - 3w_{1}) \end{cases}$$

and so relations (2.38) induce

$$w_{t_0} = w_1 + w_3 - w_0 - w_2 \qquad w_{t_3} = w_1 - w_0$$

$$w_{t_1} = w_1 - w_0 \qquad w_{t_4} = 3w_1 - 2w_0 - w_2$$

$$w_{t_2} = 2w_1 - w_0 - w_2$$

$$t \circ (t_0: t_1: t_2: t_3: t_4) = (t^{w_{t_0}} t_0: t^{w_{t_1}} t_1: t^{w_{t_2}} t_2: t^{w_{t_3}} t_3: t^{w_{t_4}} t_4) \qquad (3.35)$$

An important remark is that the point $q = (x_0^2, x_0 x_1, x_0^3)$ lives in the intersection of the two components C and E.

Now we will describe the tangent space of \mathbb{Y}_2 at the fixed points.

The first blowup center C is parametrized by $x_0 + tx_1$, hence

$$T_q C = \frac{x_1}{x_0} . (3.36)$$

The tangent space of \mathbb{Y} at q decomposes as

$$T_{q} \mathbb{Y} = T_{(x_{0}^{2}, x_{0}x_{1})} \mathbb{X}' \oplus T_{x_{0}^{3}} \mathbb{Y}_{(x_{0}^{2}, x_{0}x_{1})},$$

and the fiber $\mathbb{Y}_{(x_0^2, x_0 x_1)}$ is the projectivization of

$$x_0^3 \oplus x_0^2 x_1 \oplus x_0^2 x_2 \oplus x_0^2 x_3 \oplus x_0 x_1^2$$
.

From this results

$$T_q \mathbb{Y} = \left(\frac{x_1}{x_0} + \frac{x_2}{x_1} + \frac{x_1}{x_0}\right) + \left(\frac{x_1}{x_0} + \frac{x_2}{x_0} + \frac{x_3}{x_0} + \frac{x_1^2}{x_0^2}\right)$$

and the the normal bundle of C in \mathbb{Y} at q is

$$\mathcal{N}_{qC|\mathbb{Y}} = \frac{x_2}{x_1} + \frac{x_1}{x_0} + \frac{x_1}{x_0} + \frac{x_2}{x_0} + \frac{x_3}{x_0} + \frac{x_1^2}{x_0^2}$$
(3.37)

(cf. Table 5). On the indeterminacy point $s'_5 = (0:0:0:0:0:1)$, related to $\frac{x_2}{x_1}$ (3.32), we have

$$T_{(q,s'_5)} \mathbb{Y}_1 = T_q C \oplus \mathcal{L}_{\frac{x_2}{x_1}} \oplus T_{[\mathcal{L}_{\frac{x_2}{x_1}}]} \mathbb{P}(\mathcal{N}_{qC|\mathbb{Y}}).$$

This yields

$$T_{(q,s_5')} \mathbb{Y}_1 = \left(\frac{x_1}{x_0}\right) + \left(\frac{x_2}{x_1}\right) + \left(\frac{x_1^2}{x_0 x_2} + \frac{x_1^2}{x_0 x_2} + \frac{x_1}{x_0} + \frac{x_1 x_3}{x_0 x_2} + \frac{x_1^3}{x_0^2 x_2}\right)$$

Now, E' is the transform of E under the first blowup. But $C \cap E = \{q\}$, and so $T_{(q,s'_5)}E'$ is the tangent space of the blowup of \mathbb{P}^2 at one point.

On E, spanned by $\langle x_0 x_1, x_0 x_2, x_1^2 \rangle$ (2.21), we have

$$T_{x_0x_1}E = \frac{x_2}{x_1} + \frac{x_1}{x_0}$$

and

$$T_{(q,s_5')}E' = \mathcal{L}_{\frac{x_2}{x_1}} \oplus T_{[\mathcal{L}_{\frac{x_2}{x_1}}]}\mathbb{P}(\mathcal{N}_{\frac{x_2}{x_1}|E}) = \frac{x_2}{x_1} + \frac{x_1^2}{x_0x_2}$$

The short exact sequence

$$T_{(q,s'_5)}E' \hookrightarrow T_{(q,s'_5)}\mathbb{Y}_1 \twoheadrightarrow \mathcal{N}_{(q,s'_5)E'|\mathbb{Y}_1}$$

shows us that (compare with (3.35))

$$\mathcal{N}_{(q,s_5')E'|\mathbb{Y}_1} = \underbrace{\frac{x_1}{x_0}}_{t_1'} + \underbrace{\frac{x_1^2}{x_0x_2}}_{t_2'} + \underbrace{\frac{x_1}{x_0}}_{t_3'} + \underbrace{\frac{x_1x_3}{x_0x_2}}_{t_0'} + \underbrace{\frac{x_1^3}{x_0^2x_2}}_{t_4'}$$

There are 3 fixed points and a fixed line $\ell_3 := \{(0:t_1:0:t_3:0)\}$. 4.2.2.9

Table 6 – Fixed points over $(q, s'_5) = (x_0^2, x_0 x_1, x_0^3), (0:0:0:0:0:1)$

fixed point/ \mathbb{P}^1 $(t_0: t_1: t_2: t_3: t_4)$	associated eigenvector	generator of fiber in \mathcal{W}
(1:0:0:0:0)(0:0:1:0:0)(0:0:0:0:1)(0:t,:0:t,:0)	$\begin{array}{c} x_1 x_3 / x_0 x_2 \\ x_1^2 / x_0 x_2 \\ x_1^3 / x_0^2 x_2 \\ x_1 / x_0 x_2 \\ x_2 / x_2 \end{array}$	$\omega = x_0^2 x_3 dx_0 - x_0^3 dx_3$ $\omega = x_0^2 x_1 dx_0 - x_0^3 dx_1$ $\omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1$ $\omega = (t_0 - t_0) x_0^2 x_0 dx_0 - (t_0 - t_0) x_0^3 dx_0$

Example 7. For $t'_0 = (1:0:0:0:0)$ we have $\omega = x_0^2 x_3 dx_0 - x_0^3 dx_3$, so

$$c_1^T(\mathcal{W})|_{(q,s'_5,t'_0)} = 3w_0 + w_3.$$

The tangent space at this point, for $c_7^T(T_{(q,s'_5,t'_0)}\mathbb{Y}_2)$, is

$$T_{(q,s'_{5},t'_{0})} \mathbb{Y}_{2} = T_{(q,s'_{5})} E' \oplus \mathcal{L}_{\frac{x_{1}x_{3}}{x_{0}x_{2}}} \oplus T_{[\mathcal{L}_{\frac{x_{1}x_{3}}{x_{0}x_{2}}}]} \mathbb{P}(\mathcal{N}_{(q,s'_{5})E'|\mathbb{Y}_{1}}) =$$

$$= \left(\frac{x_{2}}{x_{1}} + \frac{x_{1}^{2}}{x_{0}x_{2}}\right) + \left(\frac{x_{1}x_{3}}{x_{0}x_{2}}\right) + \left(\frac{x_{2}}{x_{3}} + \frac{x_{1}}{x_{3}} + \frac{x_{2}}{x_{3}} + \frac{x_{1}^{2}}{x_{0}x_{3}}\right).$$
(3.38)

There is still one indeterminacy point $p_{\ell_3} = (0 : 1 : 0 : 1 : 0) \in \ell_3$. The point $q' := (q, s'_5, p_{\ell_3})$ lies in the neighborhood $t_1 = 1$. The ideal of the next blowup center R is (ideal JRn_7 in 4.2.2.9)

$$J_R = \langle u_2, t_3 - 1, t_2, t_0, 3s_1 - 2t_4 \rangle.$$
(3.39)

Let $\mathbb{P}^4 = \{(v_0 : v_1 : v_2 : v_3 : v_4)\} = \mathbb{P}(\mathcal{N}_{q'R|\mathbb{Y}_2})$ the fiber of the exceptional divisor over q'. Here, the fixed points locus consists of three well solved fixed points, the strict transform $\tilde{\ell}_3$ of ℓ_3 and a new indeterminacy point (q', v'_0) 4.2.2.10.

fixed point $(v_0: v_1: v_2: v_3: v_4)$	associated eigenvector	generator of fiber in \mathcal{W}
(1:0:0:0:0)(0:1:0:0:0)(0:0:1:0:0)(0:0:1:0:0)	$\begin{array}{c} x_2/x_1 \\ 1 \\ x_1/x_2 \\ x_2/x_3 \end{array}$	not defined $\omega = x_0^2 x_2 dx_0 - x_0^3 dx_2$ $\omega = x_0^2 x_1 dx_0 - x_0^3 dx_1$ $\omega = x_0^2 x_2 dx_0 - x_0^3 dx_1$
(0:1:0:0:0)(0:0:1:0:0)(0:0:0:1:0)(0:0:0:1:0)(0:0:0:0:1)	$ \begin{array}{c} 1 \\ x_1/x_2 \\ x_3/x_2 \\ x_1^2/x_0x_2 \end{array} $	$\omega = x_0^2 x_2 dx$ $\omega = x_0^2 x_1 dx$ $\omega = x_0^2 x_3 dx$ $\omega = x_0 x_1^2 dx_0$

Table 7 – Fixed points over $q' = [(x_0^2, x_0 x_1, x_0^3), s'_5, p_{\ell_3}]$

Weights to $\mathbb{P}^4 = \mathbb{P}(\mathcal{N}_{q'R|\mathbb{Y}_2})$. In the affine chart $t_1 = 1$,

$$(\mathbf{t}_0, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4) = \left(\frac{t_0}{t_1}, \frac{t_2}{t_1}, \frac{t_3}{t_1}, \frac{t_4}{t_1}\right)$$

Thus, from (3.35) we have local weights for affine points around $t_3 = 1$, given by

$$\begin{aligned}
 &w_{t_0}^L &= w_3 - w_2 & w_{t_3}^L &= 0 \\
 &w_{t_2}^L &= w_1 - w_2 & w_{t_4}^L &= 2w_1 - w_0 - w_2
 \end{aligned}$$
(3.40)

The local equations of the blowup center R (3.39) are, from (3.31), (3.34) and (3.40)

$$\begin{array}{ccc} u_2 & (\text{weight } w_1 - w_2) \\ \mathbf{t}_3 - 1 & (\text{weight } 0) \\ \mathbf{t}_2 & (\text{weight } w_2 - w_1) \\ \mathbf{t}_0 & (\text{weight } w_2 - w_3) \\ 3\mathbf{s}_1 - 2\mathbf{t}_4 & (\text{weight } w_0 + w_2 - 2w_1) \end{array}$$

and so relations (2.38) induce

$$\begin{array}{rclrcl} w_{v_0} &=& w_2 - w_1 & w_{v_3} &=& w_3 - w_2 \\ w_{v_1} &=& 0 & w_{v_4} &=& 2w_1 - w_0 - w_2 \\ w_{v_2} &=& w_1 - w_2 \end{array}$$

$$t \circ (v_0 : v_1 : v_2 : v_3 : v_4) = (t^{w_{v_0}} v_0 : t^{w_{v_1}} v_1 : t^{w_{v_2}} v_2 : t^{w_{v_3}} v_3 : t^{w_{v_4}} v_4)$$
(3.41)

Note that the fixed point $v'_1 = (0 : 1 : 0 : 0 : 0)$ is associated to the weight $w_{v_1} = 0$, and we write 1 as in Table 7 for this eigenvalue. It is quite useful for calculations, like in (3.28).

Let's see the decomposition of $\mathcal{N}_{q'R|\mathbb{Y}_2}$. As done for t'_0 (3.38), for p_{ℓ_3} we have

$$T_{(q,s'_5,p_{\ell_3})} \mathbb{Y}_2 = \left(\frac{x_2}{x_1} + \frac{x_1^2}{x_0 x_2}\right) + \left(\frac{x_1}{x_0}\right) + \left(\frac{x_3}{x_2} + \frac{x_1}{x_2} + \frac{x_1^2}{x_0 x_2} + 1\right).$$
(3.42)

Among the equations of R (3.39) we have $exc_1 = u_2$ (the basis C) and the \mathbb{P}^1

 $\{(t_0:t_1:t_2:t_3:t_4)=(0:1:0:1:3s_1/2)\}.$

The point $p_{\ell_3} = (0:1:0:1:0)$ is the point of equation $s_1 = 0$, and $T_{p_{\ell_3}} \mathbb{P}^1 = \frac{t_3^{\vee}}{s_1^{\vee}} = \frac{\mathbf{s}_1}{\mathbf{t}_3}$.

$$T_{q'}R = T_qC \oplus T_{p_{\ell_3}}\mathbb{P}^1 = \frac{x_1}{x_0}(3.36) + \frac{x_1^2}{x_0x_2}((3.34) + (3.40)),$$
$$\mathcal{N}_{q'R|\mathbb{Y}_2} = \frac{x_2}{x_1} + \frac{x_3}{x_2} + \frac{x_1}{x_2} + \frac{x_1^2}{x_0x_2} + 1.$$

Example 8. 4.3.14.2 The fixed point v'_1 lies in the strict transform $\tilde{\ell}_3$ of ℓ_3 . From (3.42) we have the decomposition of the normal bundle

$$\mathcal{N}_{\ell_3|\mathbb{Y}_2} = \frac{x_2}{x_1} + \frac{x_1^2}{x_0 x_2} + \frac{x_1}{x_0} + \frac{x_1}{x_0} + \frac{x_3}{x_2} + \frac{x_1}{x_2} + \frac{x_1^2}{x_0 x_2} + \frac{x_1^2}{x_0 x$$

The decomposition for the normal bundle of ℓ_3 is, for some integers d_7, \ldots, d_{12} ,

$$\mathcal{N}_{\tilde{\ell}_{3}|\mathbb{Y}_{3}} = \frac{x_{2}}{x_{1}} + \frac{x_{1}^{2}}{x_{0}x_{2}} + \frac{x_{1}}{x_{0}} + \frac{x_{1}}{x_{2}} + \frac{x_{1}}{x_{2}} + \frac{x_{3}}{x_{2}} + \frac{x_{1}^{2}}{x_{0}x_{2}} + \frac{x_{1}^{2$$

The contribution of the fixed line $\tilde{\ell}_3$ may be computed as

$$\int_{\tilde{\ell}_3} \frac{-c_1^T (\mathcal{W}|_{\tilde{\ell}_3})^7}{c_6^T (\mathcal{N}_{\tilde{\ell}_3|\mathbb{Y}_3})} = -\int_{\tilde{\ell}_3} \frac{(3w_0 + w_2)^7}{\mathbf{w} + \mathbf{d}h} = -\int_{\tilde{\ell}_3} \frac{(3w_0 + w_2)^7 [\mathbf{w} - \mathbf{d}h]}{\mathbf{w}^2} ,$$

where

$$\mathbf{w} = -(w_2 - w_1)^2 (2w_1 - w_0 - w_2)^2 (w_1 - w_0)(w_3 - w_2)$$

 $\mathbf{d} = (2w_1 - w_0 - w_2)^2 (w_1 - w_0) (w_1 - w_2) (w_3 - w_2) d_7 + \dots + (w_2 - w_1)^2 (w_0 - w_1) (w_3 - w_2) (2w_1 - w_0 - w_2) d_{12}$ and we multiplied by $\mathbf{w} - \mathbf{d}h$ in both the numerator and the denominator, and used that $h^2 = 0.$

Taking the coefficient of h we find a linear expression on d_7, \ldots, d_{12} ,

$$\int_{\tilde{\ell}_3} \frac{-c_1^T(\mathcal{W}|_{\tilde{\ell}_3})^7}{c_6^T(\mathcal{N}_{\tilde{\ell}_3|\mathbb{Y}_3})} = \frac{\mathbf{d}(3w_0 + w_2)^7}{\mathbf{w}^2} \ .$$

We still have to solve the indeterminacy over the point (q', v'_0) , see Table 7. We will take a look at the neighborhood $v_0 = 1$.

The affine variables now are $(v_1, v_2, v_3, v_4, u_2, t_4, s_2)$. The final blowup center is the one dimensional component L (see page 32) defined by the equations

$$s_2 = v_1 = v_2 = v_3 = v_4 = t_4 = 0$$

The component *L* is parametrized by u_2 , and $T_{(q',v'_0)}L = w_{u_2} = \frac{x_2}{x_1}$ (3.31). Over the point (q', v'_0) is defined a $\mathbb{P}^5 = \mathbb{P}(\mathcal{N}_{(q',v'_0)L|\mathbb{Y}_3}) = \{(z_0 : z_1 : z_2 : z_3 : z_4 : z_5)\}.$ Since we are in the affine chart $v_0 = 1$, we have local weights assigned to coordinates \underline{v} from (3.41), and with (3.34), (3.40) we gain Table 8 (cf. 4.2.2.11).

Table 8 – Fixed	points over	$[(x_0^2, x_0x_1, x_0^2)]$	$(s_{1}), s_{5}',$	(0 :	: 1	:0	:1:	: 0	$), v'_0$	5]
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fixed point $(z_0: z_1: z_2: z_3: z_4: z_5)$	associated eigenvector	generator of fiber in \mathcal{W}
(1:0:0:0:0:0:0)(0:1:0:0:0:0)(0:0:1:0:0:0)(0:0:0:1:0:0)(0:0:0:1:0:0)(0:0:0:0:1:0)	$egin{array}{c} x_1/x_0 \ x_1/x_2 \ x_1^2/x_2^2 \ x_1x_3/x_2^2 \ x_1^3/x_0x_2^2 \end{array}$	$\begin{split} \omega &= x_0 x_2^2 dx_0 - x_0^2 x_2 dx_2 \\ \omega &= x_0^2 x_2 dx_0 - x_0^3 dx_2 \\ \omega &= x_0^2 x_1 dx_0 - x_0^3 dx_1 \\ \omega &= x_0^2 x_3 dx_0 - x_0^3 dx_3 \\ \omega &= x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1 \end{split}$
$\begin{array}{c} (0:0:1:0:0:0)\\ (0:0:0:1:0:0)\\ (0:0:0:0:1:0)\\ (0:0:0:0:1:0)\\ (0:0:0:0:0:1:0) \end{array}$	$\begin{array}{c} x_1^2/x_2^2 \\ x_1x_3/x_2^2 \\ x_1^3/x_0x_2^2 \\ x_1^2/x_0x_2 \end{array}$	$\omega = x_0^2 x_1 dx_0 - x_0^3 dx_1$ $\omega = x_0^2 x_3 dx_0 - x_0^3 dx_3$ $\omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1$ $\omega = 2x_0 x_1 x_2 dx_0 - x_0^2 x_2 dx_1 - x_0^2 x_1 dx$

To get the contributions of these points, write

$$T_{(q',v'_0)} \mathbb{Y}_3 = T_{q'} R \oplus \mathcal{L}_{\frac{x_2}{x_1}} \oplus T_{[\mathcal{L}_{\frac{x_2}{x_1}}]} \mathbb{P}(\mathcal{N}_{(q')R|\mathbb{Y}_2}) =$$
$$= \left(\frac{x_1^2}{x_0 x_2} + \frac{x_1}{x_0}\right) + \left(\frac{x_2}{x_1}\right) + \left(\frac{x_1}{x_2} + \frac{x_1^2}{x_2^2} + \frac{x_1 x_3}{x_2^2} + \frac{x_1^3}{x_0 x_2^2}\right) .$$

The short exact sequence

$$T_{(q',v'_0)}L \hookrightarrow T_{(q,v'_0)}\mathbb{Y}_3 \twoheadrightarrow \mathcal{N}_{(q',v'_0)L|\mathbb{Y}_3}$$

shows

$$\mathcal{N}_{(q',v_0')}L|\mathbb{Y}_3 = \underbrace{\frac{x_1}{x_0}}_{z_0'} + \underbrace{\frac{x_1}{x_2}}_{z_1'} + \underbrace{\frac{x_1^2}{x_2^2}}_{z_2'} + \underbrace{\frac{x_1x_3}{x_2^2}}_{z_3'} + \underbrace{\frac{x_1^3}{x_0x_2^2}}_{z_4'} + \underbrace{\frac{x_1^2}{x_0x_2}}_{z_5'} . \tag{3.43}$$

Example 9. For $z'_4 = (0:0:0:1:0)$ we have $\omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1$, so

$$c_1^T(\mathcal{W})|_{(q',v_0',z_4')} = 2w_0 + 2w_1.$$

The tangent space at this point is

$$T_{(q',v'_0,z'_4)} \mathbb{Y}_4 = T_{(q',v'_0)} L \oplus \mathcal{L}_{\frac{x_1^3}{x_0 x_2^2}} \oplus T_{[\mathcal{L}_{\frac{x_1^3}{x_0 x_2^2}}]} \mathbb{P}(\mathcal{N}_{(q',v'_0)L|\mathbb{Y}_3}) =$$
(3.44)

$$= \left(\frac{x_2}{x_1}\right) + \left(\frac{x_1^3}{x_0 x_2^2}\right) + \left(\frac{x_1}{x_0} + \frac{x_1}{x_2} + \frac{x_1^2}{x_2^2} + \frac{x_1 x_3}{x_2^2} + \frac{x_1^2}{x_0 x_2}\right) \cdot \underbrace{\frac{x_0 x_2^2}{x_1^3}}_{w_{z_4}^{\vee}}$$

$$= \left(\frac{x_2}{x_1}\right) + \left(\frac{x_1^3}{x_0 x_2^2}\right) + \left(\frac{x_2^2}{x_1^2} + \frac{x_0 x_2}{x_1^2} + \frac{x_0}{x_1} + \frac{x_0 x_3}{x_1^2} + \frac{x_2}{x_1}\right) \Rightarrow$$

$$c_7^T (T_{(q', v'_0, z'_4)} \mathbb{Y}_4) = 2(w_2 - w_1)^3 (3w_1 - w_0 - 2w_2)(w_0 + w_2 - 2w_1)(w_0 - w_1)(w_0 + w_3 - 2w_1).$$

3.4.2.2 Resolution on the fixed line over $(x_0^2, x_0 x_1, x_0^3)$

After the blowup of \mathbb{Y} along C we obtain, over the fixed point $q = (x_0^2, x_0 x_1, x_0^3)$, the indeterminacy point s'_5 which was fully solved in the previous section. But there is another indeterminacy point at the fixed line ℓ_2 (3.33, see Table 5).

Let $\mathbb{P}^4 = \{(t_0: t_1: t_2: t_3: t_4)\} = \mathbb{P}(\mathcal{N}_{(q, p_{\ell_2})E'|\mathbb{Y}_2})$ the fiber of the projectivized normal bundle over $p_{\ell_2} = (1: 3/2: 0: 0: 0: 0)$. The ideal of E' is, in affine chart $s_0 = 1$,

$$J_{E'} = \langle s_4 - 3u_3, s_3, s_2, 2s_1 - 3, b_1 \rangle.$$

Note that $s_0 = 1$ represents the choice of $\exp_1 = b_1 - 4u_3$ for the local equation of the first exceptional divisor, see (2.35). The local weights induced by (3.32) are

From equations of $J_{E'}$ and (3.45)+(3.31) we obtain the weights on this \mathbb{P}^4 . 4.2.2.12

fixed point / \mathbb{P}^1 ($t_0: t_1: t_2: t_3: t_4$)	associated eigenvector	generator of fiber in \mathcal{W}
$egin{aligned} (t_0:0:0:0:t_4)\ (0:1:0:0:0)\ (0:0:1:0:0)\ (0:0:0:1:0:0)\ (0:0:0:1:0) \end{aligned}$	$x_1/x_0 \ x_3/x_1 \ x_2/x_1 \ 1$	$\omega = (3t_4 - 8t_0)x_0x_1^2dx_0 - (3t_4 - 8t_0)x_0^2x_1dx_1$ $\omega = x_0^2x_3dx_0 - x_0^3dx_3$ $\omega = x_0^2x_2dx_0 - x_0^3dx_2$ $\omega = x_0^2x_1dx_0 - x_0^3dx_1$

Table 9 – Fixed points over $(x_0^2, x_0 x_1, x_0^3), p_{\ell_2} = (1:3/2:0:0:0:0)$

Oh no! There is another fixed line $\ell_4 = \{(t_0 : 0 : 0 : 0 : t_4)\}$. We also have two fixed points and the strict transform $\tilde{\ell}_2$ of ℓ_2 .

Remember that (3.37)

$$\mathcal{N}_{qC|\mathbb{Y}} = \frac{x_2}{x_1} + \frac{x_1}{x_0} + \frac{x_1}{x_0} + \frac{x_2}{x_0} + \frac{x_3}{x_0} + \frac{x_1^2}{x_0^2}$$

Since p_{ℓ_2} has weight $\frac{x_1}{x_0}$ (Table 5),

$$T_{(q,p_{\ell_2})} \mathbb{Y}_1 = T_q C \oplus \mathcal{L}_{\frac{x_1}{x_0}} \oplus T_{[\mathcal{L}_{\frac{x_1}{x_0}}]} \mathbb{P}(\mathcal{N}_{qC|\mathbb{Y}})$$

$$= \frac{x_1}{x_0} + \frac{x_1}{x_0} + \frac{x_0 x_2}{x_1^2} + \frac{x_2}{x_1} + 1 + \frac{x_3}{x_1} + \frac{x_1}{x_0} .$$
(3.46)

Since E is spanned by $\langle x_0 x_1, x_0 x_2, x_1^2 \rangle$ (2.21) we have

$$T_{x_0 x_1} E = \frac{x_2}{x_1} + \frac{x_1}{x_0}$$

and

$$T_{(q,p_{\ell_2})}E' = \mathcal{L}_{\frac{x_1}{x_0}} \oplus T_{[\mathcal{L}_{\frac{x_1}{x_0}}]}\mathbb{P}(\mathcal{N}_{\frac{x_1}{x_0}|E}) = \frac{x_1}{x_0} + \frac{x_0x_2}{x_1^2} .$$
(3.47)

Hence the fiber $\mathbb{P}^4 = \mathbb{P}(\mathcal{N}_{(q,p_\ell)E'|\mathbb{Y}_1})$ of the exceptional divisor, where (cf. Table 9)

$$\mathcal{N}_{(q,p_{\ell_2})E'|\mathbb{Y}_1} = \frac{x_1}{x_0} + \frac{x_2}{x_1} + 1 + \frac{x_3}{x_1} + \frac{x_1}{x_0} \ . \tag{3.48}$$

This will enable us to find the contributions of the fixed points.

Example 10. 4.3.14.3 For the fixed line ℓ_2 , we have from (3.46)

$$\mathcal{N}_{\ell_2|\mathbb{Y}_1} = \frac{x_1}{x_0} + \frac{x_1}{x_0} + \frac{x_1}{x_0} + \frac{x_0 x_2}{x_1^2} + \frac{x_2}{x_1} + \frac{x_3}{x_1} + \frac{x_1}{x_0} \quad (3.49)$$

The normal bundle of $\tilde{\ell}_2$ is, for some integers d_{13}, \ldots, d_{18} ,

$$\mathcal{N}_{\tilde{\ell}_{2}|\mathbb{Y}_{2}} = \frac{x_{1}}{x_{0}} + \frac{x_{1}}{x_{0}} + \frac{x_{0}x_{2}}{x_{1}^{2}} + \frac{x_{2}}{x_{1}^{2}} + \frac{x_{3}}{x_{1}} + \frac{x_{1}}{x_{0}} + \frac{x_{1$$

 \mathbf{SO}

$$c_{6}^{T}(\mathcal{N}_{\tilde{\ell}_{2}|\mathbb{Y}_{2}}) = [\prod_{i=13,14,18} (w_{1} - w_{0} + d_{i}h)](w_{0} + w_{2} - 2w_{1} + d_{15}h)(w_{2} - w_{1} + d_{16}h)(w_{3} - w_{1} + d_{17}h)$$

(mod h^2). From Table 9, $\mathcal{W}|_{\tilde{p}\in\tilde{\ell}_2} = \langle x_0^2 x_1 dx_0 - x_0^3 dx_1 \rangle$ for all $\tilde{p}\in\tilde{\ell}_2$ (the restriction $\mathcal{W}|_{\tilde{\ell}_2}$ is trivial), and so $c_1^T(\mathcal{W}|_{\tilde{\ell}_2}) = 3w_0 + w_1$. Like Examples 6 and 8, we gain a linear expression on d_{13}, \ldots, d_{18}

$$\int_{\tilde{\ell}_2} \frac{-c_1^T (\mathcal{W}|_{\tilde{\ell}_2})^7}{c_6^T (\mathcal{N}_{\tilde{\ell}_2|\mathbb{Y}_2})} = \frac{\mathbf{d}(3w_0 + w_1)^7}{\mathbf{w}^2}$$

In view of Table 9, the fiber of \mathcal{W} over a point $(t_0:t_4) \in \ell_4$ is spanned by

$$\omega = (3t_4 - 8t_0)x_0x_1^2dx_0 - (3t_4 - 8t_0)x_0^2x_1dx_1.$$

Thus, the fixed line ℓ_4 meets the next blowup center R at the fixed point

$$p_{\ell_4} = (3:0:0:0:8).$$

The ideal of R is locally given by

$$J_R = \langle 3t_4 - 8, t_3, t_1, t_2 - 4s_5, 2s_4 - 9u_3 \rangle.$$
(3.50)

Let $\mathbb{P}^4 = \{(v_0 : v_1 : v_2 : v_3 : v_4)\} = \mathbb{P}(\mathcal{N}_{(q, p_{\ell_2}, p_{\ell_4})R|\mathbb{Y}_2}) \text{ over } p_{\ell_4} = (1 : 0 : 0 : 0 : 8/3).$ After taking the convenient weights we are able to find the Table 10 below (cf. 4.2.2.13).

Table 10 – Fixed points over $(x_0^2, x_0 x_1, x_0^3), p_{\ell_2}, (1:0:0:0:8/3)$

fixed point $(v_0:v_1:v_2:v_3:v_4)$	associated eigenvector	generator of fiber in \mathcal{W}
$egin{array}{llllllllllllllllllllllllllllllllllll$	$ \begin{array}{c} 1 \\ x_0/x_1 \\ x_0x_3/x_1^2 \\ x_0x_2/x_1^2 \\ x_1/x_0 \end{array} $	$\omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1$ $\omega = x_0^2 x_1 dx_0 - x_0^3 dx_1$ $\omega = x_0^2 x_3 dx_0 - x_0^3 dx_3$ $\omega = x_0^2 x_2 dx_0 - x_0^3 dx_2$ $\omega = x_1^3 dx_0 - x_0 x_1^2 dx_1$

For short, write here $q'' = [(x_0^2, x_0 x_1, x_0^3), p_{\ell_2}, p_{\ell_4}]$. From (3.47)+(3.48),

$$T_{q''} \mathbb{Y}_2 = T_{(q,p_{\ell_2})} E' \oplus \mathcal{L}_{x_1/x_0} \oplus T_{[\mathcal{L}_{x_1/x_0}]} \mathbb{P}(\mathcal{N}_{(q,p_{\ell_2})E'|\mathbb{Y}_1})$$

$$= \left(\frac{x_1}{x_0} + \frac{x_0 x_2}{x_1^2}\right) + \left(\frac{x_1}{x_0}\right) + \left(\frac{x_0 x_2}{x_1^2} + \frac{x_0}{x_1} + \frac{x_0 x_3}{x_1^2} + 1\right).$$
(3.51)

The ruled surface R has equations $2s_4 - 9u_3$, representing the (transform of) basis C and the other four linear equations of (3.50) represent the \mathbb{P}^1

$$\{(t_0: t_1: t_2: t_3: t_4) = (1: 0: 4s_5: 0: 8/3)\}.$$

The point $p_{\ell_4} = (1:0:0:0:8/3)$ is the point of equation $s_5 = 0$, thus (3.36) + (3.45)

$$T_{q''}R = T_qC \oplus T_{p_{\ell_4}}\mathbb{P}^1 = \frac{x_1}{x_0} + \frac{t_0^{\vee}}{\mathbf{s}_5^{L^{\vee}}} = \frac{x_1}{x_0} + \frac{\mathbf{s}_5^L}{1} = \frac{x_1}{x_0} + \frac{x_0x_2}{x_1^2} \ .$$

From (3.51) remains

$$\mathcal{N}_{q''R|\mathbb{Y}_2} = \frac{x_1}{x_0} + \frac{x_0x_2}{x_1^2} + \frac{x_0}{x_1} + \frac{x_0x_3}{x_1^2} + 1.$$

Example 11. For $v'_1 = (0:1:0:0:0)$, associated to x_0/x_1 (Table 10),

$$T_{(q'',v_1')} \mathbb{Y}_3 = T_{q''} R \oplus \mathcal{L}_{\frac{x_0}{x_1}} \oplus T_{[\mathcal{L}_{\frac{x_0}{x_1}}]} \mathbb{P}(\mathcal{N}_{q''R|\mathbb{Y}_2}) = \frac{x_1}{x_0} + \frac{x_0 x_2}{x_1^2} + \frac{x_0}{x_1} + \frac{x_1^2}{x_0^2} + \frac{x_2}{x_1} + \frac{x_3}{x_1} + \frac{x_1}{x_0} \Rightarrow$$

$$c_7^T (T_{(q'',v_1')} \mathbb{Y}_3) = -2(w_0 - w_1)^4 (w_0 + w_2 - 2w_1)(w_2 - w_1)(w_3 - w_1).$$
And $\mathcal{W}|_{(q'',v_1')} = \langle x_0^2 x_1 dx_0 - x_0^3 dx_1 \rangle$, so $c_1^T (\mathcal{W}|_{(q'',v_1')}) = 3w_0 + w_1.$

Example 12. 4.3.14.4 For the fixed line ℓ_4 , we have

$$\mathcal{N}_{\ell_4|\mathbb{Y}_2} = \frac{x_1}{x_0} + \frac{x_0 x_2}{x_1^2} + \frac{x_1}{x_0} + \frac{x_1}{x_0} + \frac{x_0 x_2}{x_1^2} + \frac{x_0 x_3}{x_1^2} + \frac{x_0}{x_1} ,$$

$$T_{(q,p_{\ell_2})} E = \mathcal{O}_{\ell_4}^{\oplus 2} - \mathcal{L}_{x_1/x_0} = \mathcal{O}_{\ell_4}(-1) + \frac{\mathcal{N}_{\mathbb{P}^1|\mathbb{P}^4}}{\mathcal{N}_{\mathbb{P}^1|\mathbb{P}^4}} = \mathcal{O}_{\ell_4}(1)^{\oplus 3} ,$$

and for unknown integers d_{19}, \cdots, d_{24} ,

$$\mathcal{N}_{\tilde{\ell}_{4}|\mathbb{Y}_{3}} = \frac{x_{1}}{v_{0}} + \frac{x_{0}x_{2}}{v_{1}^{2}} + \frac{x_{1}}{x_{0}} + \frac{x_{0}x_{2}}{v_{1}^{2}} + \frac{x_{0}x_{3}}{v_{1}^{2}} + \frac{x_{0}}{v_{1}^{2}} + \frac{x_{0}x_{3}}{v_{1}^{2}} + \frac{x_{0}}{v_{1}^{2}} + \frac{x$$

where

$$\mathbf{w} = -(w_1 - w_0)^3(w_0 + w_2 - 2w_1)^2(w_0 + w_3 - 2w_1),$$

$$\mathbf{d} = \left[\prod_{i=19,21} (w_1 - w_0 + d_i h)\right] \left[\prod_{j=20,22} (w_0 + w_2 - 2w_1 + d_j h)\right](w_0 + w_3 - 2w_1 + d_{23}h)(w_0 - w_1 + d_{24}h)$$

(mod h²).

Finally we complete the contributions over the fixed point $(x_0^2, x_0 x_1, x_0^3)$!

3.4.3 $(g,g',f) = (x_0^2, x_0x_2, x_0^3).$

Next, we study the indeterminacy point $q := (x_0^2, x_0 x_2, x_0^3)$.

It lies in the neighborhood $a_0 = b_0 = u_2 = 1$, so take

$$\begin{cases} f = x_0^3 + a_1 x_0^2 x_1 + a_2 x_0^2 x_2 + a_3 x_0^2 x_3 + a_5 (u_1 x_0 x_1^2 + x_0 x_1 x_2 + \frac{2}{3} u_3 x_1^3), \\ g = x_0^2 + b_2 u_1 x_0 x_1 + b_2 x_0 x_2 + b_2 u_3 x_1^2. \end{cases}$$

The local coefficients are equipped with the weights

$$w_{\mathbf{a}_{1}} = w_{1} - w_{0} \qquad w_{\mathbf{b}_{2}} = w_{2} - w_{0}$$

$$w_{\mathbf{a}_{2}} = w_{2} - w_{0} \qquad w_{\mathbf{u}_{1}} = w_{1} - w_{2}$$

$$w_{\mathbf{a}_{3}} = w_{3} - w_{0} \qquad w_{\mathbf{u}_{3}} = 2w_{1} - w_{0} - w_{2}$$

$$w_{\mathbf{a}_{5}} = w_{1} + w_{2} - 2w_{0}.$$
(3.52)

Notice that in this neighborhood there are no points in the first blowup center C. The local equations of E are

$$\begin{cases} a_1 & (\text{degree } w_0 - w_1) \\ a_2 & (\text{degree } w_0 - w_2) \\ a_3 & (\text{degree } w_0 - w_3) \\ a_5 & (\text{degree } 2w_0 - w_1 - w_2) \\ b_2 & (\text{degree } w_0 - w_2) \end{cases}$$

Notice that these are all homogeneous equations. Hence, over the indeterminacy point $(x_0^2, x_0 x_2, x_0^3)$ we have a $\mathbb{P}^4 = \{(t_0 : t_1 : t_2 : t_3 : t_4)\} = \mathbb{P}(\mathcal{N}_{qE|\mathbb{Y}_1})$ with a natural induced action and weights

$$w_{t_0} = w_1 - w_0 \qquad w_{t_3} = -2w_0 + w_1 + w_2$$

$$w_{t_1} = w_2 - w_0 \qquad w_{t_4} = w_2 - w_0$$

$$w_{t_2} = w_3 - w_0$$
(3.53)

The methods to compute the contributions of $c_1^T(\mathcal{W})$ and the tangent space at the fixed points are similar to those explained in the previous section 3.4.2. Now, we have

$$T_q E' = T_q E = \frac{x_1}{x_2} + \frac{x_1^2}{x_0 x_2}$$
(3.54)

(without any changes by the blowup in C).

Over this \mathbb{P}^4 there are three fixed points and a fixed $\ell_5 = \{(0:t_1:0:0:t_4)\}$. 4.2.3.1

fixed point/ \mathbb{P}^1 $(t_0: t_1: t_2: t_3: t_4)$	associated eigenvector	generator of fiber in \mathcal{W}
$egin{array}{llllllllllllllllllllllllllllllllllll$	$ x_1/x_0 x_3/x_0 x_1x_2/x_0^2 x_2/x_0 $	$\begin{aligned} \omega &= x_0^2 x_1 dx_0 - x_0^3 dx_1 \\ \omega &= x_0^2 x_3 dx_0 - x_0^3 dx_3 \\ \omega &= 2x_0 x_1 x_2 dx_0 - x_0^2 x_2 dx_1 - x_0^2 x_1 dx_2 \\ \omega &= (3t_4 - 2t_1) x_0^2 x_2 dx_0 - (3t_4 - 2t_1) x_0^3 dx_2 \end{aligned}$

Table 11 – Fixed points over $(g, g', f) = (x_0^2, x_0 x_2, x_0^3)$

There's one indeterminacy point $p_{\ell_5} := (0:3:0:0:2)$ inside the fixed line ℓ_5 . The remaining indeterminacy locus is

$$L: u_3 = t_3 = t_2 = 2t_1 - 3 = 2t_0 - 3u_1 = b_2 = 0.$$
(3.55)

The affine variables are $b_2, u_1, u_3, t_0, t_1, t_2, t_3$, and L is parametrized by u_1 , so

$$T_{(q,p_{\ell_5})}L = w_{u_1} = \frac{x_1}{x_2}$$
 (3.56)

The blowup of \mathbb{Y}_2 along L gives over $p_{\ell_5} = (0: \frac{3}{2}: 0: 0: 1)$, as fiber of exceptional divisor,

$$\mathbb{P}^5 = \{(z_0: z_1: z_2: z_3: z_4: z_5)\} = \mathbb{P}(\mathcal{N}_{(q, p_{\ell_5})L|\mathbb{Y}_2}).$$

Weights to $\mathbb{P}^5 = \mathbb{P}(\mathcal{N}_{(q,p_{\ell_5})L|\mathbb{Y}_2})$. In the affine chart $t_4 = 1$,

$$(\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) = \left(\frac{t_0}{t_4}, \frac{t_1}{t_4}, \frac{t_2}{t_4}, \frac{t_3}{t_4}\right)$$

Thus, from (3.53) we have local weights for affine points around $t_4 = 1$, given by

$$\begin{aligned}
 &w_{t_0}^L &= w_1 - w_2 & w_{t_2}^L &= w_3 - w_2 \\
 &w_{t_1}^L &= 0 & w_{t_3}^L &= w_1 - w_0
 \end{aligned}$$
(3.57)

The local equations of the blowup center L (3.55) are, from (3.52) and (3.57)

$$\begin{array}{cccc} u_3 & (\text{weight } w_0 - 2w_1 + w_2) \\ \mathbf{t}_3 & (\text{weight } w_0 - w_1) \\ \mathbf{t}_2 & (\text{weight } w_2 - w_3) \\ 2\mathbf{t}_1 - 3 & (\text{weight } 0) \\ 2\mathbf{t}_0 - 3u_1 & (\text{weight } w_2 - w_1) \\ b_2 & (\text{weight } w_0 - w_2) \end{array}$$

thus,

$$\begin{array}{rcrcrcrcrc} w_{z_0} &=& 2w_1 - w_0 - w_2 & w_{z_3} &=& 0 \\ w_{z_1} &=& w_1 - w_0 & w_{z_4} &=& w_1 - w_2 \\ w_{z_2} &=& w_3 - w_2 & w_{z_5} &=& w_2 - w_0 \end{array}$$

$$t \circ (z_0 : z_1 : z_2 : z_3 : z_4 : z_5) = (t^{w_{z_0}} z_0 : t^{w_{z_1}} z_1 : t^{w_{z_2}} z_2 : t^{w_{z_3}} z_3 : t^{w_{z_4}} z_4 : t^{w_{z_5}} z_5)$$
(3.58)

From (3.58) we have the following table 4.2.3.2

Table 12 – Fixed points over $(x_0^2, x_0 x_2, x_0^3), p_{\ell_5}$

$\omega = x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1$ $\omega = 2x_0 x_1 x_2 dx_0 - x_0^2 x_2 dx_1 - x_0^2 x_1 dx_2$ $\omega = x_0^2 x_3 dx_0 - x_0^3 dx_3$ $\omega = x_0^2 x_2 dx_0 - x_0^3 dx_2$ $\omega = x_0^2 x_1 dx_0 - x_0^3 dx_1$

Example 13. 4.3.14.5 On the point $q = (x_0^2, x_0 x_2, x_0^3)$, we have (cf. (3.9))

$$T_q \mathbb{Y} = \frac{x_2}{x_0} + \frac{x_1}{x_2} + \frac{x_1^2}{x_0 x_2} + \frac{x_1}{x_0} + \frac{x_2}{x_0} + \frac{x_3}{x_0} + \frac{x_1 x_2}{x_0^2} .$$

From (3.54), and the summand x_2/x_0 for p_{ℓ_5} (Table 11), one can find

$$\mathcal{N}_{\ell_{5}|\mathbb{Y}_{2}} = \frac{x_{1}}{x_{2}} + \frac{x_{1}^{2}}{x_{0}x_{2}} + \frac{x_{2}}{x_{0}} + \frac{x_{1}}{x_{0}} + \frac{x_{1}}{x_{2}} + \frac{x_{3}}{x_{2}} + \frac{x_{1}}{x_{0}} + \frac{x_{1}}{x_{2}} + \frac{x_{2}}{x_{0}} + \frac{x_{1}}{x_{0}} + \frac{x_{1}}{x_{0$$

For some integers d_{25}, \ldots, d_{30} we have

$$\mathcal{N}_{\tilde{\ell}_5|\mathbb{Y}_4} = \frac{x_1}{x_2} + \frac{x_1^2}{x_0x_2} + \frac{x_2}{x_0} + \frac{x_1}{x_2} + \frac{x_3}{x_2} + \frac{x_1}{x_0} + \frac{x_1}{x_2} + \frac{x_2}{x_0} + \frac{x_1}{x_2} + \frac{x_2}{x_0} + \frac{x_1}{x_0} +$$

Since $c_1^T(\mathcal{W}|_{\ell_5}) = 3w_0 + w_2$ (cf. Table 12), like examples (6), (8), (10), (12) we obtain a linear expression in d_{25}, \ldots, d_{30} ,

$$\int_{\tilde{\ell}_5} \frac{-c_1^T (\mathcal{W}|_{\tilde{\ell}_5})^7}{c_6^T (\mathcal{N}_{\tilde{\ell}_5|\mathbb{Y}_4})} = \frac{\mathbf{d}(3w_0 + w_2)^7}{\mathbf{w}^2} \ .$$

3.4.4
$$(g, g', f) = (x_0^2, x_1^2, x_0^3).$$

At last, let's see how to solve indeterminacies around the point $q := (x_0^2, x_1^2, x_0^3)$.

It lies in the neighborhood $a_0 = b_0 = u_3 = 1$, and then

$$\begin{cases} f = x_0^3 + a_1 x_0^2 x_1 + a_2 x_0^2 x_2 + a_3 x_0^2 x_3 + a_6 (\frac{3}{2} u_1 x_0 x_1^2 + \frac{3}{2} u_2 x_0 x_1 x_2 + x_1^3) \\ g = x_0^2 + b_3 u_1 x_0 x_1 + b_3 u_2 x_0 x_2 + b_3 x_1^2. \end{cases}$$

The weights of the local coefficients are

$$w_{\mathbf{a}_{1}} = w_{1} - w_{0} \qquad w_{\mathbf{b}_{3}} = 2w_{1} - 2w_{0}$$

$$w_{\mathbf{a}_{2}} = w_{2} - w_{0} \qquad w_{\mathbf{u}_{1}} = w_{0} - w_{1}$$

$$w_{\mathbf{a}_{3}} = w_{3} - w_{0} \qquad w_{\mathbf{u}_{2}} = w_{0} - 2w_{1} + w_{2}$$

$$w_{\mathbf{a}_{6}} = 3w_{1} - 3w_{0}$$

$$(3.59)$$

The ideal of the curve C is

$$J_C = \langle u_2, a_3, a_2, 2a_1 - 3b_3u_1, 2a_6 - b_3^2u_1, b_3u_1^2 - 4 \rangle.$$

Note the presence of the equation $b_3u_1^2 = 4$. Since on the fixed point $q = (x_0^2, x_1^2, x_0^3)$ we have $b_3 = u_1 = 0$, it means that the fixed point is outside of this first blowup center.

Next, we look at the second blowup center, the component E with equations

 $\begin{cases} b_3 & (\text{degree } 2w_0 - 2w_1) \\ a_1 & (\text{degree } w_0 - w_1) \\ a_2 & (\text{degree } w_0 - w_2) \\ a_3 & (\text{degree } w_0 - w_3) \\ a_6 & (\text{degree } 3w_0 - 3w_1). \end{cases}$

We have a $\mathbb{P}^4 = \{(t_0: t_1: t_2: t_3: t_4)\} = \mathbb{P}(\mathcal{N}_{qE|\mathbb{Y}})$ over the point $q = (x_0^2, x_1^2, x_0^3)$, with induced action and weights

$$w_{t_0} = 2w_1 - 2w_0 \qquad w_{t_3} = w_3 - w_0$$

$$w_{t_1} = w_1 - w_0 \qquad w_{t_4} = 3w_1 - 3w_0$$

$$w_{t_2} = w_2 - w_0$$

(3.60)

Here, there are five isolated fixed points. The table over this \mathbb{P}^4 is 4.2.4.1

fixed point $(t_0:t_1:t_2:t_3:t_4)$	associated eigenvector	generator of fiber in $\mathcal W$
$\begin{array}{c}(1:0:0:0:0:0)\\(0:1:0:0:0)\\(0:0:1:0:0)\\(0:0:0:1:0)\\(0:0:0:1:0)\\(0:0:0:1:0)\end{array}$	$\begin{array}{c} x_1^2/x_0^2 \\ x_1/x_0 \\ x_2/x_0 \\ x_3/x_0 \\ x_1^3/x_0^3 \end{array}$	$\begin{split} \omega &= x_0 x_1^2 dx_0 - x_0^2 x_1 dx_1 \\ \omega &= x_0^2 x_1 dx_0 - x_0^3 dx_1 \\ \omega &= x_0^2 x_2 dx_0 - x_0^3 dx_2 \\ \omega &= x_0^2 x_3 dx_0 - x_0^3 dx_3 \\ \omega &= x_1^3 dx_0 - x_0 x_1^2 dx_1 \end{split}$

Table 13 – Fixed points over (x_0^2, x_1^2, x_0^3)

This completes the description of the fixed points.

3.5 SUM OF THE CONTRIBUTIONS

The methods to solve the indeterminacies of the map $(g, f) \mapsto \omega$ show us that there are over a fixed flag:

- \cdot 26 fixed points at \mathbb{Y} (see Table 1, page 37);
- · 6 fixed points at \mathbb{Y}_1 (3 on Table 2 and 3 on Table 5);
- \cdot 13 fixed points at \mathbb{Y}_2 (3 on Table 6, 2 on Table 9, 5 on Table 13 and 3 on Table 11);
- · 16 fixed points at \mathbb{Y}_3 (5 on Table 3, 4 on Table 4, 3 on Table 7 and 4 on Table 10);
- 11 fixed points at \mathbb{Y}_4 (6 on Table 8 and 5 on Table 12);
- 5 fixed \mathbb{P}^1 (1 on Tables 4, 7, 9, 10 and 12).

Running over all flags, we obtain a total of $72 \cdot 24 = 1728$ fixed points and $5 \cdot 24 = 120$ fixed lines.

There are so many points! We can do the sum of all contributions using the freeware Macaulay2. The scripts can be found in the appendix 4.3, 4.4.

By examples 6, 8, 10, 12 and 13 this sum is a linear expression in unknowns d_1, \ldots, d_{30} .

Lemma 3. The integers d_1, \ldots, d_{30} satisfy the relations

Proof. Take the particular weights $w_0 = 0$, $w_1 = 1$, $w_2 = 5$, $w_3 = 25$ (cf. 5.2).

The degree of fibers is the same for every base flag that we look. For example, for the flag φ_0 (2.2) the degree is (see somad_(0,0,1) in 4.3.15)

$$-\frac{729}{320}d_1 - \frac{729}{640}d_2 + \frac{729}{1600}d_3 - \frac{729}{320}d_4 + \dots + \frac{3125}{192}d_{30} + \frac{49642909}{3974400}$$
(3.62)

and for the flag φ_1 ,

$$\varphi_1: p_1 = \{x_0 = x_1 = x_3 = 0\} \in \ell_0 = \{x_0 = x_1 = 0\} \subset v_1 = \{x_1 = 0\}$$

the degree is (see somad (1,0,0) in (4.3.15))

$$-\frac{1}{6000}d_1 - \frac{1}{12000}d_2 - \frac{1}{144000}d_3 - \frac{1}{6000}d_4 + \dots - \frac{210827008}{121875}d_{30} + \frac{7491488544437}{5070000000} .$$
(3.63)

Since the degrees are the same, we gain the equation (3.62)=(3.63). And we have 24 fixed flags, so actually we get 23 linear equations on the 30 variables d_i . This system of equations has the 18 independent linear equations (3.61), cf. ideal II in 4.3.16.

Remark 3. The relations 3.61 are enough to compute the Chern classes of the normal bundles of fixed lines, and also for compute the degree of a fiber. To example, for the line $\tilde{\ell}_1$ we have (3.30)

$$\mathcal{N}_{(q,p_{\ell_1},v_0')\tilde{\ell}_1|\mathbb{Y}_3} = \frac{x_0}{x_1} + \frac{x_0^2}{x_1^2} + \frac{x_2}{x_1^2} + \frac{x_2}{x_0} + \frac{x_0}{x_1} + \frac{x_2}{x_1} + \frac{x_3}{x_1},$$

Since $d_3 = 1$, $d_5 = 0$ and $d_6 = 0$, we obtain

$$c_{6}^{T}(\mathcal{N}_{\tilde{\ell}_{1}|\mathbb{Y}_{3}}) = (w_{0} - w_{1} + d_{1}h)(2w_{0} - 2w_{1} + d_{2}h)(w_{0} - w_{1} + d_{4}h)(w_{2} - w_{0} + h)(w_{2} - w_{1})(w_{3} - w_{1})$$

$$c_{6}^{T}(\mathcal{N}_{\tilde{\ell}_{1}|\mathbb{Y}_{3}}) = \mathbf{w} + \mathbf{d}h, \text{ where}$$

$$\mathbf{w} = 2(w_{0} - w_{1})^{3}(w_{2} - w_{1})(w_{3} - w_{1})(w_{2} - w_{0})$$

$$\mathbf{d} = [(w_{0} - w_{1})^{2}(w_{2} - w_{1})(w_{3} - w_{1})] \cdot [\underbrace{(2d_{1} + d_{2} + 2d_{4})}_{=-2}(w_{2} - w_{0}) + 2(w_{0} - w_{1})],$$

$$\mathbf{d} = 2(w_{0} - w_{1})^{2}(w_{2} - w_{1})(w_{3} - w_{1})(2w_{0} - w_{1} - w_{2}).$$

Proposition 3. (Degree of a Fiber) The degree of the 7-dimensional variety of exceptional foliations over a fixed flag is **21**.

Proof. Simply replace the relations (3.61) in (3.62), cf. 4.3.16

When running over all the complete flags of \mathbb{P}^3 we obtain the full exceptional component of foliations. The contributions on Bott's formula are computed replacing the power 7 on the numerators by 13, and multiplying the denominators by $c_6^T(T_{(p,\ell,v)}\mathbb{F})$, where (p,ℓ,v) runs over the fixpoints of the variety \mathbb{F} of complete flags in \mathbb{P}^3 (see (3.1) and (3.5)).

If $\langle x_i, x_j, x_k, x_m \rangle$ is a basis of $(\mathbb{C}^4)^{\vee}$ and the fixed complete flag is

$$p_{ijk} = \{x_i = x_j = x_k = 0\} \in \ell_{ij} = \{x_i = x_j = 0\} \subset v_i = \{x_i = 0\},\$$

then

$$T_{(p_{ijk},\ell_{ij},v_i)}\mathbb{F} = \frac{x_j}{x_i} + \frac{x_k}{x_i} + \frac{x_m}{x_i} + \frac{x_k}{x_j} + \frac{x_m}{x_j} + \frac{x_m}{x_j},$$

and so

$$c_6^T(T_{(p_{ijk},\ell_{ij},v_i)}\mathbb{F}) = (w_j - w_i)(w_k - w_i)(w_m - w_i)(w_k - w_j)(w_m - w_k).$$

Theorem 4. The degree of the exceptional component of foliations of codimension one and degree two in \mathbb{P}^3 is **168208**.

3.6 A GEOMETRIC INTERPRETATION OF THE DEGREE

For a codimension one foliation in \mathbb{P}^n , given by the differential form

$$\omega = \sum_{i=0}^{n} A_i dx_i,$$

the hyperplane defined by the distribution at a point $p\in \mathbb{P}^n$ is

$$H = \sum_{i=0}^{n} A_i(p) x_i = 0.$$

Thus, a tangent direction $v = (v_0 : \ldots : v_n) \in \mathbb{P}(T_p \mathbb{P}^3)$ lies on this hyperplane H if

$$\sum_{i=0}^{n} A_i(p) \cdot v_i = 0.$$

This can be thought of as a linear equation on the coefficients of the A_i . Hence, the point $(p, v) \in \mathbb{P}(T\mathbb{P}^3)$ defines a hyperplane in the projective space of distributions.

$$\omega(p) \cdot v = 0. \tag{3.64}$$

The equation (3.64) shows that the degree of a *m*-dimensional component of the space of codimension one foliations in \mathbb{P}^n can be interpreted as the number of such foliations that are tangent to *m* general directions in \mathbb{P}^n .

In particular, Theorem 4 means that there are 168208 exceptional foliations tangent to 13 general directions in \mathbb{P}^3 .

4 APPENDIX A

Here, we present scripts for the software *Macaulay2* to perform some calculations. You may open a session in http://habanero.math.cornell.edu:3690/ and perform the computations just by cut&paste.

4.1 THE STANDARD EXCEPTIONAL FOLIATION

```
Rx=QQ[x_0..x_3]; R = Rx[dx_0..dx_3,SkewCommutative=>true];
X = vars Rx; dX = vars R;
f=x_0^2*x_3-x_0*x_1*x_2+(1/3)*x_1^3;
g=x_0*x_2-(1/2)*x_1^2;
-- computing df and dg
matrix{{f}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{g}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
-- the explicity 1-form
w = 2*g*df - 3*f*dg; w=w//x_0
-- the singular locus. Taking the polinomials A_i
(m,c)=coefficients w;
c=sub(c,Rx); I=ideal c
-- the components of the singular locus
```

```
primaryDecomposition I
```

4.2 RESOLUTION OF THE MAP

We start by the resolution of the indeterminacies of the map $(g, f) \mapsto \omega$. This was done by looking over standard neighborhoods that cover X'. We present here two neighborhoods, leaving the others to the reader.

4.2.1 Neighborhood $b_3 = a_6 = 1$

4.2.1.1 Computing ω

Ra=QQ[a_0 .. a_6,b_0..b_3,s_0..v_5];

```
Rx = Ra[x_0 \dots x_3]; R = Rx[dx_0 \dots dx_3, SkewCommutative=>true];
X = vars Rx; dX = vars R; X3 = basis(3,Rx); X4 = basis(4,Rx);
f1 = a_0 * x_0^3 + a_1 * x_0^2 * x_1 + a_2 * x_0^2 * x_2 + a_3 * x_0^2 * x_3;
f2 = a_4*x_0*x_1^2 + a_5*x_0*x_1*x_2 + a_6*x_1^3;
f=f1+f2;
g = b_0 * x_0^2 + b_1 * x_0 * x_1 + b_2 * x_0 * x_2 + b_3 * x_1^2;
-- the divisibility condition on this neighborhood
f=sub(f,{a_4=>(3/2)*b_1,a_5=>(3/2)*b_2,a_6=>1});
g=sub(g,b_3=>1);
-- codes to compute df and dg
matrix{{f}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{g}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
w = 2*g*df - 3*f*dg;
-- at this point, one can verify if w is divisible by x_0.
-- for this, calculate w \% x_0 to verify that this remainder is zero.
w=w // x_0;
4.2.1.2 Indeterminacy Locus
-- the following codes will extract the ideal J of the indeterminacy locus
-- execute after loading the codes in previous section
matrix{{w}}; (p,r) = quotientRemainder(oo,dX);
(p,r) = quotientRemainder(sub(transpose p,Rx), X3);
J = ideal p; J = trim J; J=sub(J,Ra);
-- the ideal J_red is the radical of J
-- We mean K=J_red. This is the local ideal of C
K=radical J;
4.2.1.3 Blowup along C
-- the local equations of C are the equations of K
e_0=8*a_0-b_1^3;
e_1=a_2;
e_2=b_2;
```

e_3=a_3;

```
e_4=4*b_0-b_1^2;
e_5=4*a_1-3*b_1^2;
-- the substitution to local equation a_2 are done
-- by list m_1 below
m_11={a_0=>(1/8)*b_1^3+(1/8)*a_2*s_0, b_2=>a_2*s_2, a_3=>a_2*s_3};
m_{12}=\{b_{0}=>(1/4)*b_{1}^{2}+(1/4)*a_{2}*s_{4}, a_{1}=>(3/4)*b_{1}^{2}+(1/4)*a_{2}*s_{5}\};
m_1=flatten(append(m_11,m_12));
wnt_1=sub(w,m_1);
-- to verify that wn_1t is divisible by a_2, compute
-- wnt_1 % a_2, and see if this remainder is zero.
wn_1=wnt_1 // a_2;
-- collect the new indeterminacy locus ideal Jn_1
matrix{{wn_1}}; (p,r) = quotientRemainder(oo,dX);
(p,r) = quotientRemainder(sub(transpose p,Rx), X3);
Jn_1 = ideal p; Jn_1 = trim Jn_1; Jn_1=sub(Jn_1,Ra);
4.2.1.4 Running other neighboorhods
-- now, we will run over all 6 local equations
for i from 0 to 5 do (
exc=e_i;
ll1={a_0=>(1/8)*(s_0*exc + b_1^3), a_2=>s_1*exc, b_2=>s_2*exc};
112={a_3=>s_3*exc, b_0=>(1/4)*(s_4*exc + b_1^2), a_1=>(1/4)*(s_5*exc + 3*b_1^2)};
l=flatten(append(ll1,ll2));
m_i=delete(l_i,l);
Jn_i=sub(J,m_i);
Jn_i=Jn_i : exc;
wn_i=sub(w,m_i);
wn_i=wn_i // exc;
)
-- the Jn_i are the new indeterminacy locus ideals, seen locally.
-- all Jn_i, 0 <= i <= 5, are irreducible and reduced
-- and Jn_3=1 means that the map was solved in this neighborhood
```

```
for i from 0 to 5 do (
print(# primaryDecomposition Jn_i, radical Jn_i == Jn_i);
)
for i from 0 to 5 do (
exc=e_i;
JR_i = ideal(3*s_4-2*s_5, s_3, exc, 3*b_1*s_2-4*s_1, s_0-b_1*s_5);
print(sub(JR_i,s_i=>1)==Jn_i);
)
4.2.1.5 Fibers to Table 2
-- in the previous section, we've computed the expressions
-- wn_i of w after the blowup along C. Now, we look
-- at the fixed points for the referred Table2.
M1={a_0=>0,a_1=>0,a_2=>0,a_3=>0,b_0=>0,b_1=>0,b_2=>0};
M2={s_0=>0,s_1=>0,s_2=>0,s_3=>0};
M3=\{s_4=>0, s_5=>0\};
M12=flatten(append(M1,M2));
-- a little caution now: there's a fixed P1 (s_4:s_5).
-- for this reason we've set separated M3 above.
for i from 0 to 5 do (
Fwn_i=sub(wn_i,M12);
if i<4 then Fwn_i=sub(Fwn_i,M3);</pre>
print(s'_i,Fwn_i);
)
-- look closely at the fibers over the fixed P1!
4.2.1.6 Fibers to Table 3
-- Fwn_2=0 means indeterminacy. We continue with equations of R
e_6=3*s_4-2*s_5;
e_7=s_3;
e_8=s_0-b_1*s_5;
e_9=4*s_1-3*b_1;
e_10=b_2;
for i from 6 to 10 do (
```
```
exc=e_i;
ll1={s_4=>(1/3)*(v_0*exc+2*s_5), s_3=>v_1*exc, s_0=>v_2*exc+b_1*s_5};
ll2={s_1=>(1/4)*(v_3*exc+3*b_1), b_2=>v_4*exc};
l=flatten(append(ll1,ll2));
m_i=delete(l_(i-6),l);
Jn2_i=sub(Jn_2,m_i);
Jn2_i=Jn2_i : exc;
wn2_i=sub(wn_2,m_i);
wn2_i=wn2_i // exc;
)
M4=\{v_0=>0, v_1=>0, v_2=>0, v_3=>0, v_4=>0\};
for i from 6 to 10 do (
Fwn2_i=sub(wn2_i,M12);
Fwn2_i=sub(Fwn2_i,M3);
Fwn2_i=sub(Fwn2_i,M4);
print(v'_(i-6),Fwn2_i);
)
4.2.1.7 Fibers to Table 4
-- the point pl=(0:0:0:0:1:3/2) has s_4=1, s_5=3/2
-- ideal of next blowup center R is Jn_4
Jn_4==ideal(2*s_5-3,s_3,2*s_0-3*b_1,4*s_1-3*b_1*s_2,4*b_0-b_1^2)
e_{6=2*s_{5-3}};
e_7=s_3;
e_8=2*s_0-3*b_1;
e_9=4*s_1-3*b_1*s_2;
e_10=4*b_0-b_1^2;
for i from 6 to 10 do (
exc=e_i;
ll1={s_5=>(1/2)*(v_0*exc + 3), s_3=>v_1*exc, s_0=>(1/2)*(v_2*exc + 3*b_1)};
ll2={s_1=>(1/4)*(v_3*exc+3*b_1*s_2), b_0=>(1/4)*(v_4*exc+b_1^2)};
l=flatten(append(ll1,ll2));
m_i=delete(l_(i-6),1);
Jn4_i=sub(Jn_4,m_i);
Jn4_i=Jn4_i : exc;
```

```
wn4_i=sub(wn_4,m_i);
wn4_i=wn4_i // exc; )
-- all Jn4_i=1, 6<=i<=10.
for i from 6 to 10 do (
Fwn4_i=sub(wn4_i,M12);
Fwn4_i=sub(Fwn4_i,s_5=>3/2);
Fwn4_i=sub(Fwn4_i,M4);
print(v'_(i-6),Fwn4_i);
)
-- Fwn4_i, i=6..10 is the generator of W over (x_1^2,x_1^3,pl,v_(6-i))
-- this completes the resolution over (x_1^2, x_1^3).
4.2.2
       Neighborhood a_0 = b_0 = u_1 = 1
4.2.2.1 Computing \omega
restart
Ra=QQ[a_0..b_6,s_0..v_5,z_0..z_5];
Rx = Ra[x_0 \dots x_3]; R = Rx[dx_0 \dots dx_3, SkewCommutative=>true];
X = vars Rx; dX = vars R; X3 = basis(3,Rx); X4 = basis(4,Rx);
f1 = a_0*x_0^3 + a_1*x_0^2*x_1 + a_2*x_0^2*x_2 + a_3*x_0^2*x_3;
f2 = a_4 x_0 x_1^2 + a_5 x_0 x_1 x_2 + a_6 x_1^3;
f = f1+f2:
g = b_0 * x_0^2 + b_1 * x_0 * x_1 + b_2 * x_0 * x_2 + b_3 * x_1^2;
-- first blowup X' --> X
g=sub(g,{b_0=>1, b_2=>u_2*b_1, b_3=>u_3*b_1});
f=sub(f,{a_5=>a_4*u_2, a_6=>(2/3)*a_4*u_3});
matrix{{f}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{g}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
-- the 1-form w
w = 2*g*df - 3*f*dg; w=w // x_0;
-- indeterminacy locus of (g,f) --> w
```

```
matrix{{w}}; (p,r) = quotientRemainder(oo,dX);
(p,r) = quotientRemainder(sub(transpose p,Rx), X3);
J = ideal p; J = trim J; J=sub(J,Ra);
```

```
-- if a_0=0 there's no indeterminacy
trim(sub(J,a_0=>0))
-- this means that a_0=...=a_4=0, impossible.
```

```
4.2.2.2 Components of reduced indeterminacy locus
```

```
-- next, we can set a_0=1 to play with indeterminacy locus % \left[ \left( {{{\mathbf{x}}_{i}} \right)^{2}} \right]
```

```
w=sub(w,a_0=>1);
J=sub(J,a_0=>1);
K=trim(radical J);
primaryDecomposition K
```

```
The primary decomposition of K reveals two ideals.Namely, the codimension one component is C and thecodimension two component is E.
```

```
-- These two components meet at one point trim(oo_0+oo_1)
```

4.2.2.3 First blowup center: C

```
-- we start with the local ideal of C
-- and choose u_2 for local equation
```

-- the list of substitutions is m_5 below:

```
m_51={b_1=>4*u_3+u_2*s_0, a_1=>6*u_3 + u_2*s_1, a_2=>u_2*s_2};
m_52={a_3=>u_2*s_3, a_4=>12*u_3^2+u_2*s_4};
m_5=flatten(append(m_51,m_52));
```

-- now, plug it into w
wn_5=sub(w,m_5);

```
-- verify if the new expression is divisible by u_2
wn_5 % u_2 == 0
-- since the answer is true, may perform the division
wn_5=wn_5 // u_2;
-- the new indeterminacy locus
Jn_5=sub(J,m_5);
Jn_5=trim(Jn_5 : u_2);
-- and the strict transform of K=J_red
Kn_5=sub(K,m_5);
Kn_5=trim(Kn_5 : u_2);
-- The strict transform of E
JE=ideal(a_1..a_4,b_1);
SJE=sub(JE,m_5):u_2;
SJE==Kn_5
-- this means that the strict transform E' of E
-- coincides with the strict transform of J_red
4.2.2.4 Second blowup center: E
-- After the blowup along C, the ruled surface
-- R appears in the reduction of the
-- indeterminacy locus, rad(Jn_5).
-- This reduction consists of E' and R
primaryDecomposition(radical Jn_5);
trim(oo_0+oo_1)
-- this means that this intersection is a curve
-- Now, we'll perform the blowup along E
-- this is done by changes m_7 below:
m_71={s_3=>t_0*s_2, s_0=>(1/3)*(t_2*s_2+2*s_1)};
m_72={u_3=>(1/6)*(t_3*s_2 - s_1*u_2), s_4=>(1/3)*(t_4*s_2-s_1^2*u_2)};
m_7=flatten(append(m_71,m_72));
```

-- now, put it into wn_5

```
wn_7=sub(wn_5,m_7);
-- verify if the new expression is divisible by s_2
wn_7 % s_2 == 0
-- since the answer is true, perform the division
wn_7=wn_7 // s_2;
-- the new indeterminacy locus
Jn_7=sub(Jn_5,m_7);
Jn_7=trim(Jn_7 : s_2);
-- this indeterminacy locus Jn_7 is reduced
radical Jn_7 == Jn_7
-- but not irreducible
primaryDecomposition Jn_7
4.2.2.5 Third blowup center: R
-- The blowup on v_O=1 is done by changes m_11 below
m_111={t_3=>v_1*u_2+1, t_2=> v_2*u_2, t_0=> v_3*u_2};
m_112={s_1=>(1/3)*(v_4*u_2+2*t_4)};
m_11=flatten(append(m_111,m_112));
-- now, put it into wn_7
wn_11=sub(wn_7,m_11);
-- verify if the new expression is divisible by u_2
wn_11 % u_2 == 0
-- since the answer is true, perform the division
wn_11=wn_11 // u_2;
-- the new indeterminacy locus
Jn_11=sub(Jn_7,m_11);
Jn_11=trim(Jn_11 : u_2);
```

```
-- this indeterminacy locus Jn_11 is reduced and irreducible !!!!
radical Jn_11 == Jn_11
primaryDecomposition Jn_11
Jn_11
4.2.2.6 Fourth blowup center: L
-- we'll run all 6 standard neighborhoods z_i=1, 0<=i<=5
e_16=s_2;
e_17=v_1;
e_18=v_2;
e_19=v_3;
e_20=v_4;
e_21=t_4;
-- in the following codes, the list m_{(16+i)} does the
-- necessary substitutions to the blowup on z_i=1
for i from 16 to 21 do (
exc=e_i;
ll1={s_2=>z_0*exc, v_1=>z_1*exc, v_2=>z_2*exc, v_3=>z_3*exc};
ll2={v_4=>z_4*exc, t_4=>z_5*exc};
l=flatten(append(ll1,ll2));
m_i=delete(l_(i-16),l);
Jn_i=sub(Jn_11,m_i);
Jn_i=Jn_i : exc;
print Jn_i;
)
-- the indeterminacy locus on z_i=1 is given by Jn_(i+16)
-- as they are all equal to 1, the map was solved.
4.2.2.7 Running other neighboorhods
-- equations of C
e_0=b_1-4*u_3;
e_1=a_1-6*u_3;
```

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e_2=a_2;

```
e_3=a_3;
e_4=a_4-12*u_3^2;
e_5=u_2;
-- we'll run all six possible choices of local
-- equation for exceptional divisor.
-- Jn_i is the new indeterminacy locus for s_i=1.
for i from 0 to 5 do (
exc=e_i;
ll1={b_1=>s_0*exc + 4*u_3, a_1=>s_1*exc + 6*u_3, a_2=>s_2*exc};
112={a_3=>s_3*exc, a_4=>s_4*exc + 12*u_3^2, u_2=>s_5*exc};
l=flatten(append(ll1,ll2));
m_i=delete(l_i,l);
Jn_i=sub(J,m_i);
Jn_i=trim(Jn_i : exc);
Kn_i=sub(K,m_i);
Kn_i=trim(Kn_i : exc);
wn_i=sub(w,m_i);
wn_i=wn_i // exc; )
-- wn_i is the expression of w in the new affine coordinates
-- notice that Jn_3=1 means that there's
-- no indeterminacy when s_3=1.
-- the second blowup center is Kn_i. Note Kn_2 = 1.
-- from now on, there are many neighborhoods to consider.
-- we continue by neighborhood s_0=1, leaving the others
-- to the reader since they are similar
-- radical of Jn_0 is composed by two components:
primaryDecomposition Jn_0
-- the components E (represented by Kn_0) and
-- the component R (the other ideal in p.D. Jn_0)
-- we take the ideal of R
JR_0 = (radical Jn_0) : Kn_0;
```

```
-- equations of Kn_0
e_6=s_4-3*u_3;
e_7=s_3;
e_8=s_2;
e_9=2*s_1-3;
e_10=b_1;
for i from 6 to 10 do (
exc=e_i;
ll1={s_4=>t_0*exc + 3*u_3, s_3=>t_1*exc, s_2=>t_2*exc};
ll2={s_1=>(1/2)*(t_3*exc + 3), b_1=>t_4*exc};
l=flatten(append(ll1,ll2));
m_i=delete(l_(i-6),l);
Jn0_i=sub(Jn_0,m_i);
Jn0_i=trim(Jn0_i : exc);
Kn0_i=sub(Kn_0,m_i);
Kn0_i=trim(Kn0_i : exc);
JRn_i=sub(JR_0,m_i);
JRn_i=trim(JRn_i : exc);
wn0_i=sub(wn_0,m_i);
wn0_i=wn0_i // exc;
)
-- the ideal JRn_i represents the transform of R
-- 6 <= i <= 10. The new indeterminacy locus is JnO_i.
-- The magic now is the equality in all neighborhoods
for i from 6 to 10 do print (Jn0_i == JRn_i);
-- Henceforth, the next blowup along R will solve
-- the map in any neighboorhood, without another blowup!
4.2.2.8 Fibers to Table 5
M1={a_1=>0,a_2=>0,a_3=>0,a_4=>0,b_1=>0,u_2=>0,u_3=>0};
M2={s_2=>0,s_3=>0,s_4=>0,s_5=>0};
M3={s_0=>0,s_1=>0};
```

```
M12=flatten(append(M1,M2));
-- a little caution now: there's a fixed P1 (s_0:s_1).
-- it is for this reason the separated M3 above.
for i from 0 to 5 do (
Fwn_i=sub(wn_i,M12);
if i>1 then Fwn_i=sub(Fwn_i,M3);
print(s'_i,Fwn_i);
)
4.2.2.9 Fibers to Table 6
-- equations of E' on s_5=1
e_6=s_3;
e_7=s_2;
e_8=3*s_0-2*s_1;
e_9=6*u_3+s_1*u_2;
e_10=3*s_4+s_1^2*u_2;
-- equations of R
JR_5 = (radical Jn_5) : Kn_5;
for i from 6 to 10 do (
exc=e_i;
ll1={s_3=>t_0*exc, s_2=>t_1*exc, s_0=>(1/3)*(t_2*exc+2*s_1)};
ll2={u_3=>(1/6)*(t_3*exc-s_1*u_2), s_4=>(1/3)*(t_4*exc-s_1^2*u_2)};
l=flatten(append(ll1,ll2));
m_i=delete(l_(i-6),l);
Jn5_i=sub(Jn_5,m_i);
Jn5_i=trim(Jn5_i : exc);
Kn5_i=sub(Kn_5,m_i);
Kn5_i=trim(Kn5_i : exc);
JRn_i=sub(JR_5,m_i);
JRn_i=trim(JRn_i : exc);
wn5_i=sub(wn_5,m_i);
wn5_i=wn5_i // exc;
)
```

```
-- the JRn_i are the ideals of next blowup center R
M41=\{t_0=>0,t_2=>0,t_4=>0\};
M42=\{t_1=>0, t_3=>0\};
for i from 6 to 10 do (
Fwn5_i=sub(wn5_i,M41);
Fwn5_i=sub(Fwn5_i,M12);
Fwn5_i=sub(Fwn5_i,M3);
teste_i=(i-6)*(i-8)*(i-10);
if teste_i==0 then Fwn5_i=sub(Fwn5_i,M42);
print(t'_(i-6),Fwn5_i);
)
4.2.2.10 Fibers to Table 7
-- Equations of R
e_11=u_2;
e_12=t_3-1;
e_13=t_2;
e_14=t_0;
e_15=3*s_1-2*t_4;
for i from 11 to 15 do (
exc=e_i;
ll1={u_2=>v_0*exc, t_3=>v_1*exc+1, t_2=>v_2*exc};
ll2={t_0=>v_3*exc, s_1=>(1/3)*(v_4*exc+2*t_4)};
l=flatten(append(ll1,ll2));
m_i=delete(l_(i-11),l);
Jn57_i=sub(Jn5_7,m_i);
Jn57_i=trim(Jn57_i : exc);
wn57_i=sub(wn5_7,m_i);
wn57_i=wn57_i // exc;
)
M5=apply(5,i->v_i=>0);
-- build list {v_i=>0}
-- remember that t_3=1 on pl_3
```

```
for i from 11 to 15 do (
Fwn57_i=sub(wn57_i,M41);
Fwn57_i=sub(Fwn57_i,M12);
Fwn57_i=sub(Fwn57_i,M3);
Fwn57_i=sub(Fwn57_i,M5);
Fwn57_i=sub(Fwn57_i,t_3=>1);
print(v'_(i-11),Fwn57_i);
)
```

```
4.2.2.11 Fibers to Table 8
-- The blowup on neighborhood s_0=1 was done in
-- "running other neighborhoods".
-- fibers on neighborhood s_0=1 are wn0_i, 6<=i<=10
-- there's a fixed P^1={(t_0:t_4)}
M71={t_1=>0,t_2=>0,t_3=>0};
M72={t_0=>0,t_4=>0};
for i from 6 to 10 do (
Fwn0_i=sub(wn0_i,M12);
Fwn0_i=sub(Fwn0_i,M3);
Fwn0_i=sub(Fwn0_i,M71);
teste_i=(i-7)*(i-8)*(i-9);
if teste_i==0 then Fwn0_i=sub(Fwn0_i,M72);
print(t'_(i-6),Fwn0_i);
)
```

```
-- indeterminacy locus Jn57_11
-- Equations of L
e_16=s_2;
e_17=v_1;
e_18=v_2;
e_19=v_3;
e_20=v_4;
e_21=t_4;
```

```
for i from 16 to 21 do (
exc=e_i;
ll1={s_2=>z_0*exc, v_1=>z_1*exc, v_2=>z_2*exc};
112={v_3=>z_3*exc, v_4=>z_4*exc, t_4=>z_5*exc};
l=flatten(append(ll1,ll2));
m_i=delete(l_(i-16),1);
Jnz_i=sub(Jn57_11,m_i);
Jnz_i=trim(Jnz_i : exc);
wnz_i=sub(wn57_11,m_i);
wnz_i=wnz_i // exc;
)
M6=apply(6,i->z_i=>0);
for i from 16 to 21 do (
Fwnz_i=sub(wnz_i,M41);
Fwnz_i=sub(Fwnz_i,M12);
Fwnz_i=sub(Fwnz_i,M3);
Fwnz_i=sub(Fwnz_i,M5);
Fwnz_i=sub(Fwnz_i,M6);
print(z'_(i-16),Fwnz_i);
)
-- finally done the fixed points over
-- (x_0^2,x_0x_1,x_0^3,s_5')
4.2.2.12 Fibers to Table 9
-- The blowup on neighborhood s_0=1 was done in
-- "running other neighborhoods".
-- fibers on neighborhood s_0=1 are wn0_i, 6<=i<=10
-- there's a fixed P^1=\{(t_0:t_4)\}
M71=\{t_1=>0, t_2=>0, t_3=>0\};
M72=\{t_0=>0, t_4=>0\};
for i from 6 to 10 do (
Fwn0_i=sub(wn0_i,M12);
Fwn0_i=sub(Fwn0_i,M3);
```

```
Fwn0_i=sub(Fwn0_i,M71);
teste_i=(i-7)*(i-8)*(i-9);
if teste_i==0 then Fwn0_i=sub(Fwn0_i,M72);
print(t'_(i-6),Fwn0_i);
)
4.2.2.13 Fibers to Table 10
-- Equations of R
e_11=3*t_4-8;
e_12=t_3;
e_13=t_1;
e_14=t_2-4*s_5;
e_15=2*s_4-9*u_3;
for i from 11 to 15 do (
exc=e_i;
ll1={t_4=>(1/3)*(v_0*exc+8), t_3=>v_1*exc, t_1=>v_2*exc};
ll2={t_2=>v_3*exc+4*s_5, s_4=>(1/2)*(v_4*exc+9*u_3)};
l=flatten(append(ll1,ll2));
m_i=delete(l_(i-11),l);
Jn06_i=sub(Jn0_6,m_i);
Jn06_i=trim(Jn06_i : exc);
wn06_i=sub(wn0_6,m_i);
wn06_i=wn06_i // exc;
)
for i from 11 to 15 do (
Fwn06_i=sub(wn06_i,M12);
Fwn06_i=sub(Fwn06_i,M5);
Fwn06_i=sub(Fwn06_i,t_4=>(8/3));
print(v'_(i-11),Fwn06_i);
)
```

-- Finally solved over (x_0^2,x_0x_1,x_0^3) !!!!

4.2.3 Neighborhood $a_0 = b_0 = u_2 = 1$.

```
restart;
Ra=QQ[a_0 .. a_6,b_0..b_3,s_0..v_5,z_0..z_5];
Rx = Ra[x_0 \dots x_3]; R = Rx[dx_0 \dots dx_3, SkewCommutative=>true];
X = vars Rx; dX = vars R; X3 = basis(3,Rx); X4 = basis(4,Rx);
f1 = a_0 * x_0^3 + a_1 * x_0^2 * x_1 + a_2 * x_0^2 * x_2 + a_3 * x_0^2 * x_3;
f2= a_4*x_0*x_1^2 + a_5*x_0*x_1*x_2 + a_6*x_1^3;
f=f1+f2:
g = b_0 * x_0^2 + b_1 * x_0 * x_1 + b_2 * x_0 * x_2 + b_3 * x_1^2;
g=sub(g,{b_0=>1, b_1=>u_1*b_2, b_3=>u_3*b_2});
f=sub(f,{a_4=>a_5*u_1, a_6=>(2/3)*a_5*u_3});
matrix{{f}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{g}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
w = 2*g*df - 3*f*dg; w=w // x_0;
matrix{{w}}; (p,r) = quotientRemainder(oo,dX);
(p,r) = quotientRemainder(sub(transpose p,Rx), X3);
J = ideal p; J = trim J; J=sub(J,Ra);
-- no indeterminacies when a_0=0
trim(sub(J,a_0=>0))==ideal(a_1,a_2,a_3,a_5)
-- this means (a_0:a_1:a_2:a_3:a_5)=(0:0:0:0:0), impossible.
-- Hence, to deal with indeterminacy locus we can take a_0=1.
w=sub(w,a_0=>1); J=trim(sub(J,a_0=>1));
-- The indeterminacy locus J is non-reduced. But the reduction presents a
-- single component E (this neighborhood is outside
-- of first blowup center C)
K=radical J;
-- equations of blowup center E
e_0=a_1;
```

```
e_1=a_2;
e_2=a_3;
e_3=a_5;
e_4=b_2;
for i from 0 to 4 do (
exc=e_i;
ll1={a_1=>t_0*exc, a_2=>t_1*exc, a_3=>t_2*exc};
112={a_5=>t_3*exc, b_2=>t_4*exc};
l=flatten(append(ll1,ll2));
m_i=delete(l_i,l);
Jn_i=sub(J,m_i);
Jn_i=Jn_i : exc;
wn_i=sub(w,m_i);
wn_i=wn_i // exc; )
-- solved for i=2,3. Continue over t_4=1
-- equations of next blowup center Jn_4
-- the line L
e_5=u_3;
e_6=t_3;
e_7=t_2;
e_8=2*t_1-3;
e_9=2*t_0-3*u_1;
e_10=b_2;
for i from 5 to 10 do (
exc=e_i;
ll1={u_3=>z_0*exc, t_3=>z_1*exc, t_2=>z_2*exc};
ll2={t_1=>(1/2)*(z_3*exc+3), t_0=>(1/2)*(z_4*exc+3*u_1), b_2=>z_5*exc};
l=flatten(append(ll1,ll2));
m_i=delete(1_(i-5),1);
Jn4_i=sub(Jn_4,m_i);
Jn4_i=Jn4_i : exc;
wn4_i=sub(wn_4,m_i);
wn4_i=wn4_i // exc; )
-- solved everywhere, since Jn4_i=1, for 5 <= i <= 10.
```

```
4.2.3.1 Fibers to Table 11
-- the fixed points. Note the P^1=\{(t_1:t_4)\}
M1={a_1=>0,a_2=>0,a_3=>0,a_5=>0,b_2=>0,u_1=>0,u_3=>0};
M2=\{t_0=>0, t_2=>0, t_3=>0\};
M3=\{t_1=>0, t_4=>0\};
M12=flatten(append(M1,M2));
for i from 0 to 4 do (
Fwn_i=sub(wn_i,M12);
teste_i=i*(i-2)*(i-3);
if teste_i==0 then Fwn_i=sub(Fwn_i,M3);
print(t'_i,Fwn_i);
)
4.2.3.2 Fibers to Table 12
-- neighborhood t_4=1. In pl_5 have t_1=3/2.
M4=apply(6,i->z_i=>0);
for i from 5 to 10 do (
Fwn4_i=sub(wn4_i,M12);
Fwn4_i=sub(Fwn4_i,t_1=>(3/2));
Fwn4_i=sub(Fwn4_i,M4);
print(z'_(i-5),Fwn4_i);
)
```

4.2.4 Neighborhood $a_0 = b_0 = u_3 = 1$.

```
Ra=QQ[a_0 .. a_6,b_0..b_3,s_0..v_5,z_0..z_5];
Rx = Ra[x_0 .. x_3]; R = Rx[dx_0 .. dx_3,SkewCommutative=>true];
X = vars Rx; dX = vars R; X3 = basis(3,Rx); X4 = basis(4,Rx);
f1 = a_0*x_0^3 + a_1*x_0^2*x_1 + a_2*x_0^2*x_2 + a_3*x_0^2*x_3;
f2= a_4*x_0*x_1^2 + a_5*x_0*x_1*x_2 + a_6*x_1^3;
f=f1+f2;
g = b_0*x_0^2 + b_1*x_0*x_1 + b_2*x_0*x_2 + b_3*x_1^2;
```

```
g=sub(g,{b_0=>1, b_1=>u_1*b_3, b_2=>u_2*b_3});
f=sub(f,{a_4=>(3/2)*a_6*u_1, a_5=>(3/2)*a_6*u_2});
matrix{{f}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{g}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
w = 2*g*df - 3*f*dg; w=w // x_0;
matrix{{w}}; (p,r) = quotientRemainder(oo,dX);
(p,r) = quotientRemainder(sub(transpose p,Rx), X3);
J = ideal p; J = trim J; J=sub(J,Ra);
-- no indeterminacies when a_0=0
trim(sub(J,a_0=>0))==ideal(a_1,a_2,a_3,a_6)
-- this means (a_0:a_1:a_2:a_3:a_6)=(0:0:0:0:0), impossible.
-- Hence, to deal with indeterminacy locus we can take a_0=1.
w=sub(w,a_0=>1); J=trim(sub(J,a_0=>1));
-- the first blowup center is C, of ideal
MMM=ideal(u_2,a_3,a_2,9*b_3-a_1^2,a_1*u_1-6,a_1^3-27*a_6)
-- the second blowup center is (the transform) JE, the ideal of E
JE= (radical J) : MMM;
-- the p.D. of radical J is composed by these two ideals MMM and JE
primaryDecomposition radical J
-- choose a_1u_1-6 for the exceptional equation
exc = a_1 * u_1 - 6;
m_41={u_2=>s_0*exc, a_3=>s_1*exc, a_2=>s_2*exc};
m_42=\{b_3=>(1/9)*(s_3*exc + a_1^2), a_6=>(1/27)*(s_5*exc + a_1^3)\};
m_4=flatten(append(m_41,m_42));
-- new indeterminacy locus
Jn_4=sub(J,m_4);
Jn_4=Jn_4 : exc;
```

```
-- the second blowup center E
JEn_4=sub(JE,m_4);
JEn_4=JEn_4 : exc;
wn_4=sub(w,m_4);
wn_4 = wn_4 // exc;
-- equations of E'
e_6=s_3;
e_7=a_1;
e_8=s_2;
e_9=s_1;
e_10=s_5;
for i from 6 to 10 do (
exc=e_i;
ll1={s_3=>t_0*exc, a_1=>t_1*exc, s_2=>t_2*exc};
ll2={s_1=>t_3*exc, s_5=>t_4*exc};
l=flatten(append(ll1,ll2));
m_i=delete(1_(i-6),1);
Jn4_i=sub(Jn_4,m_i);
Jn4_i=Jn4_i : exc;
wn4_i=sub(wn_4,m_i);
wn4_i=wn4_i // exc; )
-- After this two blowups, the indeterminacy locus becomes reduced
for i from 6 to 10 do print(radical Jn4_i == Jn4_i);
-- moreover, Jn4_i is irreducible, 6 <= i <= 10
-- note that Jn4_9=1 (the map was solved here), thus
-- # primaryDecomposition Jn4_i will return 0.
-- The ideals Jn4_i, i \not= 9, represents
-- the ruled surface R, and so codim = 5.
```

for i from 6 to 10 do print(# primaryDecomposition Jn4_i, codim Jn4_i);

-- Thus, the next blowup along R will solve the map.

```
-- we leave to the reader as an exercise the verifications over
-- the neighborhood [b_1=1]. Over [b_2=1] there's no indeterminacies.
```

```
4.2.4.1 Fibers to Table 13
```

```
-- since the indeterminacy point q is outside of first
-- blowup center C, we will go directly to equations of E
e_0=b_3;
e_1=a_1;
e_2=a_2;
e_3=a_3;
e_4=a_6;
M1={a_1=>0,a_2=>0,a_3=>0,a_6=>0,b_3=>0,u_1=>0,u_2=>0};
M2=apply(5,i->t_i=>0);
M12=flatten(append(M1,M2));
for i from 0 to 4 do (
exc=e_i;
111={b_3=>t_0*exc, a_1=>t_1*exc, a_2=>t_2*exc};
ll2={a_3=>t_3*exc, a_6=>t_4*exc};
l=flatten(append(ll1,ll2));
m_i=delete(l_i,l);
wn_i=sub(w,m_i);
wn_i=wn_i // exc;
Fwn_i=sub(wn_i,M12);
print(t_i, Fwn_i);
)
```

-- solved in all fixed points.

4.2.5 Neighborhood $b_2 = 1$

```
Ra=QQ[a_0 .. a_6,b_0..b_3,s_0..v_5,z_0..z_5];
Rx = Ra[x_0 .. x_3]; R = Rx[dx_0 .. dx_3,SkewCommutative=>true];
X = vars Rx; dX = vars R; X3 = basis(3,Rx); X4 = basis(4,Rx);
f1 = a_0*x_0^3 + a_1*x_0^2*x_1 + a_2*x_0^2*x_2 + a_3*x_0^2*x_3;
f2 = a_4*x_0*x_1^2 + a_5*x_0*x_1*x_2 + a_6*x_1^3;
```

```
f = f 1 + f 2;
g = b_0 * x_0^2 + b_1 * x_0 * x_1 + b_2 * x_0 * x_2 + b_3 * x_1^2;
-- divisibiliy condition
g=sub(g,b_2=>1);
f=sub(f,{a_4=>a_5*b_1, a_6=>(2/3)*a_5*b_3});
matrix{{f}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{g}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
w = 2*g*df - 3*f*dg; w=w // x_0;
matrix{{w}}; (p,r) = quotientRemainder(oo,dX);
(p,r) = quotientRemainder(sub(transpose p,Rx), X3);
J = ideal p; J = trim J; J=sub(J,Ra);
-- (a_0:a_1:a_2:a_3:a_5)=(0:0:0:0:0) is impossible. This
-- means that there's no ideterminacies when b_2=1.
4.2.6 Fibers to Table 1
restart:
Ra=QQ[a_0 .. a_6,b_0..b_3,s_0..v_5,z_0..z_5];
Rx = Ra[x_0 \dots x_3]; R = Rx[dx_0 \dots dx_3, SkewCommutative=>true];
X = vars Rx; dX = vars R; X3 = basis(3,Rx); X4 = basis(4,Rx);
f1 = a_0*x_0^3 + a_1*x_0^2*x_1 + a_2*x_0^2*x_2 + a_3*x_0^2*x_3;
f2= a_4*x_0*x_1^2 + a_5*x_0*x_1*x_2 + a_6*x_1^3;
f=f1+f2;
g = b_0 * x_0^2 + b_1 * x_0 * x_1 + b_2 * x_0 * x_2 + b_3 * x_1^2;
```

```
Za=apply(7,i->a_i=>0);
```

```
-- g=x_0x_1
```

```
ggg=sub(g,{b_0=>0, b_1=>1, b_2=>0, b_3=>0});
fff=sub(f,{a_5=>0, a_6=>0});
matrix{{fff}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{ggg}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
w = 2*ggg*df - 3*fff*dg; w=w // x_0;
```

```
for i from 0 to 4 do (
ff_i=sub(fff,a_i=>1);
ff_i=sub(ff_i,Za);
ww_i=sub(w,a_i=>1);
ww_i=sub(ww_i,Za);
print(ggg,ff_i,ww_i); )
-- the print corresponds to the three columns of the table
-- g=x_0x_2
ggg=sub(g,{b_0=>0, b_1=>0, b_2=>1, b_3=>0});
fff=sub(f,{a_4=>0, a_6=>0});
matrix{{fff}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{ggg}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
w = 2*ggg*df - 3*fff*dg; w=w // x_0;
for i from 0 to 5 do (
ff_i=sub(fff,a_i=>1);
ff_i=sub(ff_i,Za);
ww_i=sub(w,a_i=>1);
ww_i=sub(ww_i,Za);
if i!=4 then print(ggg,ff_i,ww_i); )
-- g=x_1^2
ggg=sub(g,{b_0=>0, b_1=>0, b_2=>0, b_3=>1});
fff=sub(f,{a_4=>0, a_5=>0});
matrix{{fff}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{ggg}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
w = 2*ggg*df - 3*fff*dg; w=w // x_0;
for i from 0 to 6 do (
ff_i=sub(fff,a_i=>1);
ff_i=sub(ff_i,Za);
ww_i=sub(w,a_i=>1);
ww_i=sub(ww_i,Za);
if i^2-9*i!=-20 then print(ggg,ff_i,ww_i); )
```

```
-- g=x_0^2, g'=x_0x_1
ggg=sub(g,{b_0=>1, b_1=>0, b_2=>0, b_3=>0});
fff=sub(f,{a_5=>0, a_6=>0});
matrix{{fff}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{ggg}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
w = 2*ggg*df - 3*fff*dg; w=w // x_0;
for i from 0 to 4 do (
ff_i=sub(fff,a_i=>1);
ff_i=sub(ff_i,Za);
ww_i=sub(w,a_i=>1);
ww_i=sub(ww_i,Za);
print(ggg,ff_i,ww_i); )
-- g=x_0^2, g'=x_0x_2
ggg=sub(g,{b_0=>1, b_1=>0, b_2=>0, b_3=>0});
fff=sub(f,{a_4=>0, a_6=>0});
matrix{{fff}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{ggg}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
w = 2*ggg*df - 3*fff*dg; w=w // x_0;
for i from 0 to 5 do (
ff_i=sub(fff,a_i=>1);
ff_i=sub(ff_i,Za);
ww_i=sub(w,a_i=>1);
ww_i=sub(ww_i,Za);
if i!=4 then print(ggg,ff_i,ww_i); )
-- g=x_0^2, g'=x_1^2
ggg=sub(g,{b_0=>1, b_1=>0, b_2=>0, b_3=>0});
fff=sub(f,{a_4=>0, a_5=>0});
matrix{{fff}}; diff(X,oo); mdf=sub(oo,R); df = determinant(oo * (transpose dX));
matrix{{ggg}}; diff(X,oo); mdg=sub(oo,R); dg = determinant(oo * (transpose dX));
w = 2*ggg*df - 3*fff*dg; w=w // x_0;
```

for i from 0 to 6 do (

```
ff_i=sub(fff,a_i=>1);
ff_i=sub(ff_i,Za);
ww_i=sub(w,a_i=>1);
ww_i=sub(ww_i,Za);
if i^2-9*i!=-20 then print(ggg,ff_i,ww_i); )
```

4.3 DEGREE OF A FIBER

Now, the script to compute the sum of all contributions. Although similar, first we will present the degree of a fiber for the sake of clarity, and then the degree of the exceptional component.

```
Ra=QQ[d_1..d_30];
R=Ra[x_0..x_3];
B=\{2*x_0, x_0+x_1, x_0+x_2, 2*x_1\};
A={3*x_0,2*x_0+x_1,2*x_0+x_2,2*x_0+x_3};
A0_1=append(A, (x_0+2*x_1));
A0_2=append(A, (x_0+x_1+x_2));
A0_3=append(A,(3*x_1));
-- in A and B we put the weights for
-- cubics and quadrics
4.3.1
       Contributions on Table 1
-- contributions in Y ---> X
-- here, B instead of X and A instead of Y
-- T_(01)B
for k from 1 to 3 do (
B0_k=delete(B_k,B);
TBO_k=1;
for i from 0 to 2 do (
TB_(k,i)=(BO_k)_i-B_k;
TBO_k=TBO_k * TB_(k,i); ); )
-- T_pA_(01)
```

```
for k from 1 to 3 do (
for i from 0 to 4 do (
TA0_{(k,i)=1};
FA0_(k,i)=delete((A0_k)_i,A0_k);
for j from 0 to (\#(FAO_(k,i)) - 1) do (
TA_{(k,i,j)}=(FAO_{(k,i)})_{j} - (AO_{k})_{i};
TAO_(k,i)=TAO_(k,i) * TA_(k,i,j); ); ); )
for k from 1 to 3 do (
for i from 0 to 4 do (
de_(k,i)=TA0_(k,i) * TB0_k;
nu_{(k,i)=B_k+(A0_k)_{i-x_0}};
); )
-- the quotient nu_(k,i)^7 / de_(k,i) represents the contribution of
-- the pair (k,i), 1<=k<=3, 0<=i<=4, (k,i) \not= (3,4), in the order
-- k: (x_0x_1,x_0x_2,x_1^2) = (1,2,3)
-- i: (x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3) = (0, 1, 2, 3)
-- i=4 : x_0x_1^2, or x_0x_1x_2, or x_1^3, depending on k.
-- note that the pair (k,i)=(3,4) does not represent anything
-- since it is an indeterminacy point.
-- contributions in Y ----> X'
B00r=delete(B_0,B);
for k from 1 to 3 do (
B00_k=delete(B_k,B00r);
TB00_k = (B_k - B_0) * ((B00_k)_0 - B_k) * ((B00_k)_1 - B_k);
for i from 0 to 4 do (
de0_(k,i)=(TA0_(k,i)*TB00_k);
nu0_{(k,i)=B_0+(A0_k)_{i-x_0}};
); )
-- the quotient nu0_(k,i)^7 / de0_(k,i)
-- represents the contribution of (x_0^2,k,i), in the order
-- k: (x_0x_1,x_0x_2,x_1^2) = (1,2,3)
-- i: (x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3) = (0, 1, 2, 3)
-- i=4 : x_0x_1^2, ou x_0x_1x_2, ou x_1^3, depending on the k.
-- notice that (x_0^2,k,0) does not represent anything
```

```
-- since they are indeterminacies.
-- 1<=k<=3, 1<=i<=4.</pre>
```

4.3.2 Contributions on Table 2

```
-- contributions over Y_1
-- point (3,4)=x_1^2,x_1^3. Adjusting the tangent space
pesosB={TB_(3,0),TB_(3,1),TB_(3,2)};
pesosA={TA_(3,4,0),TA_(3,4,1),TA_(3,4,2),TA_(3,4,3)};
pesos=flatten(append(pesosA,pesosB));
normal=delete(x_0-x_1,pesos);
-- 3 fixed points. The weights on the table are
curva_1=3*x_0-3*x_1;
curva_2=2*x_0+x_2-3*x_1;
curva_3=2*x_0+x_3-3*x_1;
for k from 1 to 3 do (
Cv_k=delete(curva_k,normal);
TC_{(3,4,k)}=(x_0-x_1)*(curva_k);
for i from 0 to (\#Cv_k - 1) do (
TC1_(k,i)=(Cv_k)_i - curva_k;
TC_(3,4,k)=TC_(3,4,k) * TC1_(k,i); );
de_{(3,4,k)}=TC_{(3,4,k)};
nu_{(3,4,k)=5*x_1 - x_0 + curva_k;})
-- the quotient nu_{(3,4,k)^7} / de_{(3,4,k)}
-- represents the contribution of the three isolated
-- fixed points solved over the pair (x_1^2, x_1^3)
-- 1<=k<=3
```

4.3.3 Contributions on Table 5

```
normalB={(B00_1)_0-B_1 , (B00_1)_1-B_1};
normalA={TA_(1,0,0),TA_(1,0,1),TA_(1,0,2),TA_(1,0,3)};
normal=flatten(append(normalA,normalB));
```

```
-- 3 fixed points. The weights on the table are
-- (curva_4 for future use in the next exceptional)
curva_1=-x_0+x_2;
curva_2=-x_0+x_3;
curva_3=-2*x_0+2*x_1;
curva_4=-x_1+x_2;
for k from 1 to 4 do (
Cv_k=delete(curva_k,normal);
```

TC_(1,0,k)=(x_1-x_0)*(curva_k);
for i from 0 to (#Cv_k - 1) do (
TC1_(k,i)=(Cv_k)_i - curva_k;
TC_(1,0,k)=TC_(1,0,k) * TC1_(k,i););
de_(1,0,k)=(TC_(1,0,k));
nu_(1,0,k)=4*x_0 + curva_k;)

-- the quotient nu_(1,0,k)^7 / de_(1,0,k) -- represents the contribution of three isolated fixed points solved -- over the point (x_0^2,x_0x_1,x_0^3) , 1<=k<=3

4.3.4 Contributions on Table 6

```
TEt=(x_2-x_1)*(2*x_1-x_0-x_2);
normal={TC1_(4,0),TC1_(4,1),TC1_(4,2),TC1_(4,3),x_1-x_0};
P2_1=-x_0+2*x_1-x_2;
P2_2=-x_0+x_1-x_2+x_3;
P2_3=-2*x_0+3*x_1-x_2;
for k from 1 to 3 do (
P2E_k=delete(P2_k,normal);
TE_k=(TEt)*(P2_k);
for i from 0 to (#P2E_k - 1) do (
TP2_(k,i)=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de01A2_k=TE_k;
nu01A2_k=TE_k;
nu01A2_k=4*x_0+x_2-x_1+P2_k; )
-- the quotient nu01A2_k^7 / de01A2_k
```

-- represents contribution of the three isolated fixed points over

-- $(x_0^2, x_0x_1, x_0^3, (x_2/x_1)), 1 \le k \le 3.$

4.3.5 Contributions on Table 11

```
pesosB={x_2-x_0,TB_(2,1),TB_(2,2)};
pesosA={TA_(2,0,0),TA_(2,0,1),TA_(2,0,2),TA_(2,0,3)};
pesos=flatten(append(pesosA,pesosB));
normal=delete(TB_(2,1),pesos);
normal=delete(TB_(2,2),normal);
TEt=(x_1-x_2)*(2*x_1-x_0-x_2);
P2_1=-x_0+x_1;
P2_2=-x_0+x_3;
P2_3=-2*x_0+x_1+x_2;
for k from 1 to 3 do (
P2E_k=delete(P2_k,normal);
TE_k=(TEt)*(P2_k);
for i from 0 to (\#P2E_k - 1) do (
TP2_{k,i}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de02A2_k=TE_k;
nu02A2_k=4*x_0+P2_k; )
-- the quotient nu02A2_k^7 / de02A2_k
-- represents the contribution of the three isolated
-- fixed points over (x_0^2, x_0x_2, x_0^3), 1<=k<=3.
```

4.3.6 Contributions on Table 13

```
pesosB={-TE_(3,0),TE_(3,1),TE_(3,2)};
pesosA={TA_(3,0,0),TA_(3,0,1),TA_(3,0,2),TA_(3,0,3)};
pesos=flatten(append(pesosA,pesosB));
normal=delete(TE_(3,1),pesos);
normal=delete(TE_(3,2),normal);
TEt=(x_0-x_1)*(-2*x_1+x_0+x_2);
for k from 1 to 5 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
TE_k=(TEt)*(P2_k);
for i from 0 to (#P2E_k - 1) do (
TP2_(k,i)=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de03A2_k=TE_k;
nu03A2_k=4*x_0+P2_k; )
```

```
-- the quotient nu03A2_k^7 / de03A2_k
-- represents the contribution of the five isolated
-- fixed points over (x_0^2,x_1^2,x_0^3), 1<=k<=5.</pre>
```

4.3.7 Contributions on Table 3

```
pesosB={TB_(3,0),TB_(3,1),TB_(3,2)};
pesosA={TA_(3,4,0),TA_(3,4,1),TA_(3,4,2),TA_(3,4,3)};
pesos=flatten(append(pesosA,pesosB));
normal=delete(TB_(3,1),pesos);
pesos2=delete(TB_(3,2),normal);
tgreta={};
for i from 0 to 4 do (
nm2_i=pesos2_i-TB_(3,2);
tgreta=append(tgreta,nm2_i); )
tgreta=unique tgreta;
normal2=append(tgreta,TB_(3,2));
TEt=(x_0-x_1)*(x_0-x_2);
for k from 1 to 5 do (
P2_k=normal2_(k-1);
P2E_k=delete(P2_k,normal2);
TE_k=(TEt)*(P2_k);
for i from 0 to (\#P2E_k - 1) do (
TP2_{(k,i)}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
deA3_k=TE_k;
nuA3_k=5*x_1+TB_(3,2)+P2_k-x_0; )
```

```
-- the quotient nuA3_k^7 / deA3_k
-- represents the contribution of the five fixed points over
-- (x_1^2,x_1^2,x_1^3,x_0x_2/x_1^2), 1<=k<=5.</pre>
```

4.3.8 Contribution on Table 4

normal={2*x_0-2*x_1,x_0-x_1,x_2-x_1,x_3-x_1}; TEt=(x_0-x_1)*(x_2-x_0);

```
for k from 1 to 4 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
TE_k=-1*(TEt)*(P2_k)^2;
for i from 0 to (#P2E_k - 1) do (
TP2_(k,i)=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
deA3sp_k=TE_k;
nuA3sp_k=5*x_1+P2_k-x_0+2*x_0-2*x_1; )
```

```
-- the quotient nuA3sp_k^7 / deA3sp_k
-- represents the contribution of the four fixed points over
-- (x_1^2,x_1^2,x_1^3,x_0^2/x_1^2), 1<=k<=4.</pre>
```

4.3.9 Contribution on Table 7

```
normal={x_3-x_2,x_1-x_2,2*x_1-x_0-x_2};
TEt=(2*x_1-x_0-x_2)*(x_1-x_0);
for k from 1 to 3 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
P2E_k=append(P2E_k,x_2-x_1);
TE_k=-1*(TEt)*(P2_k)^2;
for i from 0 to (#P2E_k - 1) do (
TP2_(k,i)=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de01A3sp_k=TE_k;
nu01A3sp_k=4*x_0+x_2-x_1+x_1-x_0+P2_k; )
```

```
-- the quotient nu01A3sp_k^7 / de01A3sp_k
-- represents the contribution of the three fixed points over
-- (x_0^2,x_0x_1,x_0^3,(x_2/x_1),x_1/x_0), 1<=k<=3.</pre>
```

4.3.10 Contribution on Table 8

```
normal11={x_1-x_0,x_1-x_2,2*x_1-2*x_2,x_1+x_3-2*x_2};
normal22={3*x_1-x_0-2*x_2,2*x_1-x_0-x_2};
normal=flatten(append(normal11,normal22));
TEt=(x_2-x_1);
```

for k from 1 to 6 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
TE_k=(TEt)*(P2_k);
for i from 0 to (#P2E_k - 1) do (
TP2_(k,i)=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i););
dev0A4_k=TE_k;
nuv0A4_k=4*x_0+x_2-x_1+x_1-x_0+x_2-x_1+P2_k;)
-- the quotient nuv0A4_k^7 / dev0A4_k

-- represents the contribution of the six fixed points over

-- $(x_0^2, x_0x_1, x_0^3, (x_2/x_1), (x_1/x_0), (x_2/x_1)), 1 \le k \le 6.$

4.3.11 Contribution on Table 9

```
normal={x_3-x_1,x_2-x_1};
TEt=(-2*x_1+x_0+x_2)*(x_1-x_0);
for k from 1 to 2 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
P2E_k=append(P2E_k,x_1-x_0);
P2E_k=append(P2E_k,x_1-x_0);
TE_k=-1*(TEt)*(P2_k)^2;
for i from 0 to (#P2E_k - 1) do (
TP2_(k,i)=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de01p1A2_k=TE_k;
nu01p1A2_k=4*x_0+x_1-x_0+P2_k; )
```

```
-- the quotient nu01plA2_k^7 / de01plA2_k
-- represents the contribution of the two fixed points over
-- (x_0^2,x_0x_1,x_0^3,(x_1/x_0)), 1<=k<=2.</pre>
```

4.3.12 Contribution on Table 10

normal={x_0-x_1,x_0+x_3-2*x_1,x_0+x_2-2*x_1,x_1-x_0}; TEt=(-2*x_1+x_0+x_2)*(x_1-x_0);

```
for k from 1 to 4 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
TE_k=-1*(TEt)*(P2_k)^2;
for i from 0 to (#P2E_k - 1) do (
TP2_(k,i)=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de01qlA3_k=TE_k;
nu01qlA3_k=4*x_0+2*x_1-2*x_0+P2_k; )
```

```
-- the quotient nu01qlA3_k^7 / de01qlA3_k
-- represents the contribution of the four fixed points over
-- (x_0^2,x_0x_1,x_0^3,(x_1/x_0),(x_1/x_0)), 1<=k<=4.</pre>
```

4.3.13 Contribution on Table 12

```
normal={2*x_1-x_0-x_2,x_1-x_0,x_3-x_2,x_1-x_2,x_2-x_0};
TEt=(x_1-x_2);
for k from 1 to 5 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
TE_k=-1*(TEt)*(P2_k)^2;
for i from 0 to (#P2E_k - 1) do (
TP2_(k,i)=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de02plnA4_k=TE_k;
nu02plnA4_k=4*x_0+x_2-x_0+P2_k; )
```

```
-- the quotient nu02plnA4_k^7 / de02plnA4_k
-- represents the contribution of the five fixed points over
-- (x_0^2,x_0x_2,x_0^3,p_ln=(x_2/x_0)), 1<=k<=5.</pre>
```

4.3.14 Contribution on fixed lines

```
-- function to compute the contribution of a fixed P1
Rh=R[h]/h^2;
contl = (N1,N11,peso) -> (
denl=product(N1+h*N11);
conjl=denl-2*(denl%h);
denl=denl*conj1;
```

```
denl=sub(denl,R);
numl=peso^7;
numl=numl*conjl;
numl=numl//h;
numl=sub(numl,R);
)
```

```
4.3.14.1 \tilde{\ell}_1: Example 6
```

Nl={x_0-x_1,2*x_0-2*x_1,x_2-x_0,x_0-x_1,x_2-x_1,x_3-x_1}; Nll=apply(6,j->d_(j+1)); peso=3*x_1+x_0; contl(Nl,Nll,peso); bp_1=numl; cp_1=denl;

```
-- the quotient bp1 / cp1
-- represents the contribution of the fixed line ~1_1
```

```
4.3.14.2 \tilde{\ell}_3: Example 8
```

```
Nl={x_2-x_1,2*x_1-x_0-x_2,x_1-x_0,x_1-x_2,x_3-x_2,2*x_1-x_0-x_2};
Nll=apply(6,j->d_(j+7));
peso=3*x_0+x_2;
contl(Nl,Nll,peso);
bp_2=numl;
```

```
cp_2=denl;
```

```
-- the quotient bp2 / cp2
-- represents the contribution of the line ~1_3
```

```
4.3.14.3 \tilde{\ell}_2: Example 10
```

```
Nl={x_1-x_0,x_1-x_0,x_0+x_2-2*x_1,x_2-x_1,x_3-x_1,x_1-x_0};
Nll=apply(6,j->d_(j+13));
peso=3*x_0+x_1;
contl(Nl,Nll,peso);
bp_3=numl;
cp_3=denl;
```

-- the quotient bp3 / cp3

-- represents the contribution of the line ~1_2.

```
4.3.14.4 \tilde{\ell}_4: Example 12
```

```
Nl={x_1-x_0,x_0+x_2-2*x_1,x_1-x_0,x_0+x_2-2*x_1,x_0+x_3-2*x_1,x_0-x_1};
Nll=apply(6,j->d_(j+19));
peso=2*x_1+2*x_0;
contl(Nl,Nll,peso);
bp_4=numl;
cp_4=denl;
```

```
-- the quotient bp4 / cp4
-- represents the contribution of the fixed line ~1_4
```

```
4.3.14.5 \tilde{\ell}_5: Example 13
```

```
Nl={x_1-x_2,2*x_1-x_0-x_2,x_2-x_0,x_1-x_2,x_3-x_2,x_1-x_0};
Nll=apply(6,j->d_(j+25));
peso=3*x_0+x_2;
contl(Nl,Nll,peso);
bp_5=numl;
cp_5=denl;
```

```
-- the quotient bp5 / cp5
-- represents the contribution of the fixed line ~1_5
```

4.3.15 Sum of contributions

```
Ry=R[y_0..y_3];
```

```
-- take the weights
-- y_0 => 0
-- y_1 => 1
-- y_2 => 5
-- y_3 => 25
pesos={y_0=>0,y_1=>1,y_2=>5,y_3=>25};
Y=ideal(y_0..y_3);
-- a trick to vary the flag
```

```
for a from 0 to 3 do (
bandeira_a=append({},x_0=>y_a);
Ys_a=sub(Y,Ry/y_a);
Ys_a=trim(sub(Ys_a,Ry));
for b from 0 to 2 do (
bandeira_(a,b)=append(bandeira_a,x_1=>(Ys_a)_(2-b));
Ys_(a,b)=sub(Ys_a,Ry/((Ys_a)_(2-b)));
Ys_(a,b)=trim(sub(Ys_(a,b),Ry));
for c from 0 to 1 do (
bandeira_(a,b,c)=append(bandeira_(a,b),x_2=>(Ys_(a,b))_c);
Ys_(a,b,c)=sub(Ys_(a,b),Ry/(Ys_(a,b))_c);
Ys_(a,b,c)=trim(sub(Ys_(a,b,c),Ry));
bandeira_(a,b,c)=append(bandeira_(a,b,c),x_3=>(Ys_(a,b,c))_0);
); ); )
-- in the ideal II we will store all equalities
-- somad_(a,b,c) = somad_(a',b',c'),
-- where somad_(a,b,c) represents the degree of the
-- fiber over the flag (a,b,c), see bandeira_(a,b,c)
II=ideal();
for a from 0 to 3 do (
for b from 0 to 2 do (
for c from 0 to 1 do (
soma=0;
pntsfixos=0;
P1s=0;
for k from 1 to 3 do (
for i from 0 to 4 do (
nun_(k,i)=sub(nu_(k,i),bandeira_(a,b,c));
nun_(k,i)=sub(nun_(k,i),pesos);
nun_(k,i)=sub(nun_(k,i),QQ);
den_(k,i)=sub(de_(k,i),bandeira_(a,b,c));
den_(k,i)=sub(den_(k,i),pesos);
den_(k,i)=sub(den_(k,i),QQ);
ctr_(k,i)=nun_(k,i)^7 / den_(k,i);
soma=soma+ctr_(k,i);
pntsfixos=pntsfixos + 1; ); );
soma=soma-ctr_(3,4);
```

```
pntsfixos=pntsfixos-1;
for k from 1 to 3 do (
for i from 1 to 4 do (
nuOn_(k,i)=sub(nu0_(k,i),bandeira_(a,b,c));
deOn_(k,i)=sub(deO_(k,i),bandeira_(a,b,c));
nuOn_(k,i)=sub(nuOn_(k,i),pesos);
nuOn_(k,i)=sub(nuOn_(k,i),QQ);
deOn_(k,i)=sub(deOn_(k,i),pesos);
deOn_(k,i)=sub(deOn_(k,i),QQ);
ctr0_(k,i)=nu0n_(k,i)^7 / de0n_(k,i);
soma=soma+ctr0_(k,i);
pntsfixos=pntsfixos + 1;
); );
for k from 1 to 3 do (
nun_(3,4,k)=sub(nu_(3,4,k),bandeira_(a,b,c));
den_(3,4,k)=sub(de_(3,4,k),bandeira_(a,b,c));
nun_{(3,4,k)}=sub(nun_{(3,4,k)},pesos);
nun_{(3,4,k)}=sub(nun_{(3,4,k)},QQ);
den_(3,4,k)=sub(den_(3,4,k),pesos);
den_{(3,4,k)}=sub(den_{(3,4,k)},QQ);
ctr_(3,4,k)=nun_(3,4,k)^7 / den_(3,4,k);
soma=soma+ctr_(3,4,k);
pntsfixos=pntsfixos + 1;
);
for k from 1 to 3 do (
nun_(1,0,k)=sub(nu_(1,0,k),bandeira_(a,b,c));
den_(1,0,k)=sub(de_(1,0,k),bandeira_(a,b,c));
nun_{(1,0,k)=sub(nun_{(1,0,k),pesos)};
nun_(1,0,k)=sub(nun_(1,0,k),QQ);
den_(1,0,k)=sub(den_(1,0,k),pesos);
den_(1,0,k)=sub(den_(1,0,k),QQ);
ctr_(1,0,k)=nun_(1,0,k)^7 / den_(1,0,k);
soma=soma+ctr_(1,0,k);
pntsfixos=pntsfixos + 1;
);
for k from 1 to 3 do (
nun01A2_k=sub(nu01A2_k,bandeira_(a,b,c));
den01A2_k=sub(de01A2_k,bandeira_(a,b,c));
nun01A2_k=sub(nun01A2_k,pesos);
```

```
nun01A2_k=sub(nun01A2_k,QQ);
den01A2_k=sub(den01A2_k,pesos);
den01A2_k=sub(den01A2_k,QQ);
ctr01A2_k=nun01A2_k^7 / den01A2_k;
soma=soma+ctr01A2_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 3 do (
nun02A2_k=sub(nu02A2_k,bandeira_(a,b,c));
den02A2_k=sub(de02A2_k,bandeira_(a,b,c));
nun02A2_k=sub(nun02A2_k,pesos);
nun02A2_k=sub(nun02A2_k,QQ);
den02A2_k=sub(den02A2_k,pesos);
den02A2_k=sub(den02A2_k,QQ);
ctr02A2_k=nun02A2_k^7 / den02A2_k;
soma=soma+ctr02A2_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 5 do (
nun03A2_k=sub(nu03A2_k,bandeira_(a,b,c));
den03A2_k=sub(de03A2_k,bandeira_(a,b,c));
nun03A2_k=sub(nun03A2_k,pesos);
nun03A2_k=sub(nun03A2_k,QQ);
den03A2_k=sub(den03A2_k,pesos);
den03A2_k=sub(den03A2_k,QQ);
ctr03A2_k=nun03A2_k^7 / den03A2_k;
soma=soma+ctr03A2_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 5 do (
nunA3_k=sub(nuA3_k,bandeira_(a,b,c));
denA3_k=sub(deA3_k,bandeira_(a,b,c));
nunA3_k=sub(nunA3_k,pesos);
nunA3_k=sub(nunA3_k,QQ);
denA3_k=sub(denA3_k,pesos);
denA3_k=sub(denA3_k,QQ);
ctrA3_k=nunA3_k^7 / denA3_k;
soma=soma+ctrA3_k;
pntsfixos=pntsfixos + 1;
```
```
);
for k from 1 to 4 do (
nunA3sp_k=sub(nuA3sp_k,bandeira_(a,b,c));
denA3sp_k=sub(deA3sp_k,bandeira_(a,b,c));
nunA3sp_k=sub(nunA3sp_k,pesos);
nunA3sp_k=sub(nunA3sp_k,QQ);
denA3sp_k=sub(denA3sp_k,pesos);
denA3sp_k=sub(denA3sp_k,QQ);
ctrA3sp_k=nunA3sp_k^7 / denA3sp_k;
soma=soma+ctrA3sp_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 3 do (
nun01A3sp_k=sub(nu01A3sp_k,bandeira_(a,b,c));
den01A3sp_k=sub(de01A3sp_k,bandeira_(a,b,c));
nun01A3sp_k=sub(nun01A3sp_k,pesos);
nun01A3sp_k=sub(nun01A3sp_k,QQ);
den01A3sp_k=sub(den01A3sp_k,pesos);
den01A3sp_k=sub(den01A3sp_k,QQ);
ctr01A3sp_k=nun01A3sp_k^7 / den01A3sp_k;
soma=soma+ctr01A3sp_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 6 do (
nunv0A4_k=sub(nuv0A4_k,bandeira_(a,b,c));
denv0A4_k=sub(dev0A4_k,bandeira_(a,b,c));
nunv0A4_k=sub(nunv0A4_k,pesos);
nunv0A4_k=sub(nunv0A4_k,QQ);
denv0A4_k=sub(denv0A4_k,pesos);
denv0A4_k=sub(denv0A4_k,QQ);
ctrv0A4_k=nunv0A4_k^7 / denv0A4_k;
soma=soma+ctrv0A4_k;
pntsfixos=pntsfixos + 1;
):
for k from 1 to 2 do (
nun01plA2_k=sub(nu01plA2_k,bandeira_(a,b,c));
den01plA2_k=sub(de01plA2_k,bandeira_(a,b,c));
nun01plA2_k=sub(nun01plA2_k,pesos);
nun01plA2_k=sub(nun01plA2_k,QQ);
```

```
den01plA2_k=sub(den01plA2_k,pesos);
den01plA2_k=sub(den01plA2_k,QQ);
ctr01plA2_k=nun01plA2_k^7 / den01plA2_k;
soma=soma+ctr01plA2_k;
pntsfixos=pntsfixos + 1;
):
for k from 1 to 4 do (
nun01qlA3_k=sub(nu01qlA3_k,bandeira_(a,b,c));
den01qlA3_k=sub(de01qlA3_k,bandeira_(a,b,c));
nun01qlA3_k=sub(nun01qlA3_k,pesos);
nun01qlA3_k=sub(nun01qlA3_k,QQ);
den01qlA3_k=sub(den01qlA3_k,pesos);
den01qlA3_k=sub(den01qlA3_k,QQ);
ctr01qlA3_k=nun01qlA3_k^7 / den01qlA3_k;
soma=soma+ctr01qlA3_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 5 do (
nun02plnA4_k=sub(nu02plnA4_k,bandeira_(a,b,c));
den02plnA4_k=sub(de02plnA4_k,bandeira_(a,b,c));
nun02plnA4_k=sub(nun02plnA4_k,pesos);
nun02plnA4_k=sub(nun02plnA4_k,QQ);
den02plnA4_k=sub(den02plnA4_k,pesos);
den02plnA4_k=sub(den02plnA4_k,QQ);
ctr02plnA4_k=nun02plnA4_k^7 / den02plnA4_k;
soma=soma+ctr02plnA4_k;
pntsfixos=pntsfixos + 1;
):
for k from 1 to 5 do (
cpn_k=sub(cp_k,bandeira_(a,b,c));
bpn_k=sub(bp_k,bandeira_(a,b,c));
cpn_k=sub(cpn_k,pesos);
cpn_k=sub(cpn_k,QQ);
bpn_k=sub(bpn_k,pesos);
ctrP1_k=bpn_k / cpn_k;
soma=soma + ctrP1_k;
P1s=P1s+1;
);
somad_(a,b,c)=soma;
```

II=II+(somad_(a,b,c)-somad_(0,0,0));
););)

4.3.16 Degree to Proposition 3

```
-- The relations for d's II=trim(sub(II,Ra))
```

-- and the degree of a fiber soma=sub(-somad_(0,0,1),Ra/II)

The minus sign is due to the minus in

$$\int_{\mathbb{Y}_{4(p,\ell,v)}} -c_1^7(\mathcal{W}) \cap [\mathbb{Y}_{4(p,\ell,v)}].$$

The variable soma returns the result, **21**, of Proposition **3**.

4.4 DEGREE OF THE EXCEPTIONAL COMPONENT

The scripts are similar to the presented in the previous section.

```
R=QQ[d_1..d_30];
Rx=R[x_0..x_3];
B={2*x_0,x_0+x_1,x_0+x_2,2*x_1};
A={3*x_0,2*x_0+x_1,2*x_0+x_2,2*x_0+x_3};
A0_1=append(A, (x_0+2*x_1));
A0_2=append(A, (x_0+x_1+x_2));
A0_3=append(A,(3*x_1));
vtgflag={x_1-x_0,x_2-x_0,x_3-x_0,x_2-x_1,x_3-x_1,x_3-x_2};
tgflag=product vtgflag;
-- Y ---> X
for k from 1 to 3 do (
B0_k=delete(B_k,B);
TBO_k=1;
for i from 0 to 2 do (
TB_(k,i)=(BO_k)_i-B_k;
TBO_k=TBO_k * TB_(k,i); ); )
```

```
for k from 1 to 3 do (
for i from 0 to 4 do (
TAO_(k,i)=1;
FAO_(k,i) = delete((AO_k)_i, AO_k);
for j from 0 to (\#(FAO_(k,i)) - 1) do (
TA_{(k,i,j)}=(FAO_{(k,i)})_{j} - (AO_{k})_{i};
TAO_(k,i)=TAO_(k,i) * TA_(k,i,j); ); ); )
for k from 1 to 3 do (
for i from 0 to 4 do (
de_(k,i)=TA0_(k,i) * TB0_k * tgflag;
nu_(k,i)=B_k+(A0_k)_i-x_0; ); )
-- Y ----> X'
B00r=delete(B_0,B);
for k from 1 to 3 do (
B00_k=delete(B_k,B00r);
TB00_k = (B_k - B_0) * ((B00_k)_0 - B_k) * ((B00_k)_1 - B_k);
for i from 0 to 4 do (
de0_(k,i)=(TA0_(k,i)*TB00_k)*(tgflag);
nu0_(k,i)=B_0+(A0_k)_i-x_0;
                                 ); )
-- point (3,4)=x_1^2,x_1^3
pesosB={TB_(3,0),TB_(3,1),TB_(3,2)};
pesosA={TA_(3,4,0),TA_(3,4,1),TA_(3,4,2),TA_(3,4,3)};
pesos=flatten(append(pesosA,pesosB));
normal=delete(x_0-x_1,pesos);
curva_1=3*x_0-3*x_1;
curva_2=2*x_0+x_2-3*x_1;
curva_3=2*x_0+x_3-3*x_1;
for k from 1 to 3 do (
Cv_k=delete(curva_k,normal);
TC_{(3,4,k)}=(x_0-x_1)*(curva_k);
for i from 0 to (\#Cv_k - 1) do (
TC1_(k,i)=(Cv_k)_i - curva_k;
TC_(3,4,k)=TC_(3,4,k) * TC1_(k,i); );
de_(3,4,k)=(TC_(3,4,k))*(tgflag);
nu_{(3,4,k)=5*x_1 - x_0 + curva_k;})
```

```
-- point 0_(1,0)=x_0^2,x_0x_1,x_0^3
normalB={(B00_1)_0-B_1 , (B00_1)_1-B_1};
normalA={TA_(1,0,0),TA_(1,0,1),TA_(1,0,2),TA_(1,0,3)};
normal=flatten(append(normalA,normalB));
curva_1 = -x_0 + x_2;
curva_2 = -x_0 + x_3;
curva_3=-2*x_0+2*x_1;
curva_4 = -x_1 + x_2;
for k from 1 to 4 do (
Cv_k=delete(curva_k,normal);
TC_{(1,0,k)}=(x_1-x_0)*(curva_k);
for i from 0 to (\#Cv_k - 1) do (
TC1_(k,i)=(Cv_k)_i - curva_k;
TC_(1,0,k)=TC_(1,0,k) * TC1_(k,i); );
de_(1,0,k)=(TC_(1,0,k))*(tgflag);
nu_{(1,0,k)=4*x_0 + curva_k;})
-- point 0_{(1,0)}, (x_2/x_1).
TEt=(x_2-x_1)*(2*x_1-x_0-x_2);
normal={TC1_(4,0),TC1_(4,1),TC1_(4,2),TC1_(4,3),x_1-x_0};
P2_1=-x_0+2*x_1-x_2;
P2_2=-x_0+x_1-x_2+x_3;
P2_3=-2*x_0+3*x_1-x_2;
for k from 1 to 3 do (
P2E_k=delete(P2_k,normal);
TE_k=(TEt)*(P2_k);
for i from 0 to (\#P2E_k - 1) do (
TP2_{(k,i)}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de01A2_k=(TE_k)*(tgflag);
nu01A2 k=4*x_0+x_2-x_1+P2k;)
-- point 0_(2,0)=x_0^2,x_0x_2,x_0^3
pesosB={x_2-x_0,TB_(2,1),TB_(2,2)};
pesosA={TA_(2,0,0),TA_(2,0,1),TA_(2,0,2),TA_(2,0,3)};
pesos=flatten(append(pesosA,pesosB));
```

```
normal=delete(TB_(2,1),pesos);
normal=delete(TB_(2,2),normal);
TEt=(x_1-x_2)*(2*x_1-x_0-x_2);
P2_1=-x_0+x_1;
P2_2 = -x_0 + x_3;
P2_3=-2*x_0+x_1+x_2;
for k from 1 to 3 do (
P2E_k=delete(P2_k,normal);
TE_k=(TEt)*(P2_k);
for i from 0 to (\#P2E_k - 1) do (
TP2_{(k,i)}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de02A2_k=(TE_k)*(tgflag);
nu02A2_k=4*x_0+P2_k; )
-- point 0_(3,0)=x_0^2,x_1^2,x_0^3
pesosB={-TB_(3,0),TB_(3,1),TB_(3,2)};
pesosA={TA_(3,0,0),TA_(3,0,1),TA_(3,0,2),TA_(3,0,3)};
pesos=flatten(append(pesosA,pesosB));
normal=delete(TB_(3,1),pesos);
normal=delete(TB_(3,2),normal);
TEt=(x_0-x_1)*(-2*x_1+x_0+x_2);
for k from 1 to 5 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
TE_k=(TEt)*(P2_k);
for i from 0 to (\#P2E_k - 1) do (
TP2_{(k,i)}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de03A2_k=(TE_k)*(tgflag);
nu03A2_k=4*x_0+P2_k; )
-- over (x_1^3,x_1^2, x_0x_2/x_1^2).
```

```
ver (x_1 0, x_1 2, x_0x_2, x_1 2):
pesosB={TB_(3,0),TB_(3,1),TB_(3,2)};
pesosA={TA_(3,4,0),TA_(3,4,1),TA_(3,4,2),TA_(3,4,3)};
pesos=flatten(append(pesosA,pesosB));
normal=delete(TB_(3,1),pesos);
pesos2=delete(TB_(3,2),normal);
tgreta={};
```

```
for i from 0 to 4 do (
nm2_i=pesos2_i-TB_(3,2);
tgreta=append(tgreta,nm2_i); )
tgreta=unique tgreta;
normal2=append(tgreta,TB_(3,2));
TEt=(x_0-x_1)*(x_0-x_2);
for k from 1 to 5 do (
P2_k=normal2_(k-1);
P2E_k=delete(P2_k,normal2);
TE_k=(TEt)*(P2_k);
for i from 0 to (\#P2E_k - 1) do (
TP2_{(k,i)}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
deA3_k=(TE_k)*(tgflag);
nuA3_k=5*x_1+TB_(3,2)+P2_k-x_0; )
   _____
-- ----- contributions of the isolated ------
-- ----- fixed points over fixed P1s ------
-- over (x_1^3,x_1^2, x_0^2/x_1^2).
normal={2*x_0-2*x_1,x_0-x_1,x_2-x_1,x_3-x_1};
TEt=(x_0-x_1)*(x_2-x_0);
for k from 1 to 4 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
TE_k=-1*(TEt)*(P2_k)^2;
for i from 0 to (\#P2E_k - 1) do (
TP2_{(k,i)}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
deA3sp_k=(TE_k)*(tgflag);
nuA3sp_k=5*x_1+P2_k-x_0+2*x_0-2*x_1; )
-- point 0_{(1,0)}, (x_2/x_1), (x_1/x_0).
-- over (x_0^2,x_0x_1,x_0^3,s_5',t_1=t_3)
normal={x_3-x_2,x_1-x_2,2*x_1-x_0-x_2};
```

```
TEt=(2*x_1-x_0-x_2)*(x_1-x_0);
```

```
for k from 1 to 3 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
P2E_k=append(P2E_k,x_2-x_1);
TE_k=-1*(TEt)*(P2_k)^2;
for i from 0 to (\#P2E_k - 1) do (
TP2_{k,i}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de01A3sp_k=(TE_k)*(tgflag);
nu01A3sp_k=4*x_0+x_2-x_1+x_1-x_0+P2_k; )
-- point 0_{(1,0)}, (x_2/x_1), (x_1/x_0), (x_2/x_1).
-- over (x_0^2,x_0x_1,x_0^3,s_5',(t_1=t_3),v_0')
normal={x_1-x_0,x_1-x_2,2*x_1-2*x_2,x_1+x_3-2*x_2,3*x_1-x_0-2*x_2,
2*x_1-x_0-x_2}; TEt=(x_2-x_1);
for k from 1 to 6 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
TE_k=(TEt)*(P2_k);
for i from 0 to (\#P2E_k - 1) do (
TP2_{k,i}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
dev0A4_k=(TE_k)*(tgflag);
nuv0A4_k=4*x_0+x_2-x_1+x_1-x_0+x_2-x_1+P2_k; )
-- point O_{(1,0)}, (x_1/x_0) over (x_0^2, x_0x_1, x_0^3, p_1)
normal={x_3-x_1, x_2-x_1};
TEt=(-2*x_1+x_0+x_2)*(x_1-x_0);
for k from 1 to 2 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
P2E_k=append(P2E_k,x_1-x_0);
P2E_k=append(P2E_k,x_1-x_0);
TE_k=-1*(TEt)*(P2_k)^2;
for i from 0 to (\#P2E_k - 1) do (
TP2_{(k,i)}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de01plA2_k=(TE_k)*(tgflag);
nu01plA2_k=4*x_0+x_1-x_0+P2_k; )
```

```
-- point 0_(1,0),p_l=(x_1/x_0),q_l2=(x_1/x_0)
-- over (x_0^2,x_0x_1,x_0^3,p_1,q_12)
normal={x_0-x_1,x_0+x_3-2*x_1,x_0+x_2-2*x_1,x_1-x_0};
TEt=(-2*x_1+x_0+x_2)*(x_1-x_0);
for k from 1 to 4 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
TE_k=-1*(TEt)*(P2_k)^2;
for i from 0 to (\#P2E_k - 1) do (
TP2_{k,i}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de01qlA3_k=(TE_k)*(tgflag);
nu01qlA3_k=4*x_0+2*x_1-2*x_0+P2_k; )
-- point 0_(2,0),p_l=(x_2/x_0)
-- over (x_0^2,x_0x_2,x_0^3,p_1)
normal={2*x_1-x_0-x_2,x_1-x_0,x_3-x_2,x_1-x_2,x_2-x_0};
TEt=(x_1-x_2);
for k from 1 to 5 do (
P2_k=normal_(k-1);
P2E_k=delete(P2_k,normal);
TE_k=-1*(TEt)*(P2_k)^2;
for i from 0 to (\#P2E_k - 1) do (
TP2_{(k,i)}=(P2E_k)_i - P2_k;
TE_k=TE_k * TP2_(k,i); );
de02plnA4_k=(TE_k)*(tgflag);
nu02plnA4_k=4*x_0+x_2-x_0+P2_k; )
  -------
-- ----- fixed lines ------
-- contributions of fixed P1
Rh=Rx[h]/h^2;
contl = (N1,N11,peso) -> (
denl=product(Nl+h*Nll)*tgflag;
conjl=denl-2*(denl%h);
denl=denl*conjl;
```

```
denl=sub(denl,Rx);
numl=peso^13;
numl=numl*conjl;
numl=numl//h;
numl=sub(numl,Rx);
)
-- fixed P^1 over (x_1^3, x_1^2)
Nl={x_0-x_1,2*x_0-2*x_1,x_2-x_0,x_0-x_1,x_2-x_1,x_3-x_1};
Nll=apply(6, j->d_(j+1));
peso=3*x_1+x_0;
contl(N1,N11,peso);
bp_1=numl;
cp_1=den1;
-- fixed P^1 over (x_0^3,x_0x_1,x_0^2,s_5')
Nl={x_2-x_1,2*x_1-x_0-x_2,x_1-x_0,x_1-x_2,x_3-x_2,2*x_1-x_0-x_2};
Nll=apply(6,j->d_(j+7));
peso=3*x_0+x_2;
contl(N1,N11,peso);
bp_2=numl;
cp_2=den1;
-- fixed P^1 over (x_0^3,x_0x_1,x_0^2)
Nl=\{x_1-x_0, x_1-x_0, x_0+x_2-2*x_1, x_2-x_1, x_3-x_1, x_1-x_0\};
Nll=apply(6, j->d_(j+13));
peso=3*x_0+x_1;
contl(N1,N11,peso);
bp_3=numl;
cp_3=den1;
-- fixed P^1 over (x_0^3,x_0x_1,x_0^2,p_1)
Nl={x_1-x_0,x_0+x_2-2*x_1,x_1-x_0,x_0+x_2-2*x_1,x_0+x_3-2*x_1,x_0-x_1};
Nll=apply(6,j->d_(j+19));
peso=2*x_1+2*x_0;
contl(N1,N11,peso);
bp_4=numl;
cp_4=den1;
```

```
-- fixed P^1 over (x_0^3,x_0x_2,x_0^2)
Nl={x_1-x_2,2*x_1-x_0-x_2,x_2-x_0,x_1-x_2,x_3-x_2,x_1-x_0};
Nll=apply(6,j->d_(j+25));
peso=3*x_0+x_2;
contl(N1,N11,peso);
bp_5=numl;
cp_5=denl;
     -----
-- ----- sum ------
 -- trick to vary the flag
Ry=R[y_0..y_3];
soma=0;
pesos={y_0=>0,y_1=>1,y_2=>5,y_3=>25};
pntsfixos=0;
P1s=0;
Y=ideal(y_0..y_3);
for i from 0 to 3 do (
bandeira_i=append({},x_0=>y_i);
Ys_i=sub(Y,Ry/y_i);
Ys_i=trim(sub(Ys_i,Ry));
for j from 0 to 2 do (
bandeira_(i,j)=append(bandeira_i,x_1=>(Ys_i)_(2-j));
Ys_(i,j)=sub(Ys_i,Ry/((Ys_i)_(2-j)));
Ys_(i,j)=trim(sub(Ys_(i,j),Ry));
for k from 0 to 1 do (
bandeira_(i,j,k)=append(bandeira_(i,j),x_2=>(Ys_(i,j))_k);
Ys_(i,j,k)=sub(Ys_(i,j),Ry/(Ys_(i,j))_k);
Ys_(i,j,k)=trim(sub(Ys_(i,j,k),Ry));
bandeira_(i,j,k)=append(bandeira_(i,j,k),x_3=>(Ys_(i,j,k))_0);
); ); )
```

time

```
for a from 0 to 3 do (
for b from 0 to 2 do (
for c from 0 to 1 do (
for k from 1 to 3 do (
for i from 0 to 4 do (
nun_(k,i)=sub(nu_(k,i),bandeira_(a,b,c));
nun_(k,i)=sub(nun_(k,i),pesos);
nun_{(k,i)=sub(nun_{(k,i),QQ)};
den_(k,i)=sub(de_(k,i),bandeira_(a,b,c));
den_(k,i)=sub(den_(k,i),pesos);
den_(k,i)=sub(den_(k,i),QQ);
ctr_(k,i)=nun_(k,i)^13 / den_(k,i);
soma=soma+ctr_(k,i);
pntsfixos=pntsfixos + 1; ); );
soma=soma-ctr_(3,4);
pntsfixos=pntsfixos-1;
for k from 1 to 3 do (
for i from 1 to 4 do (
nuOn_(k,i)=sub(nu0_(k,i),bandeira_(a,b,c));
deOn_(k,i)=sub(deO_(k,i),bandeira_(a,b,c));
nuOn_(k,i)=sub(nuOn_(k,i),pesos);
nuOn_(k,i)=sub(nuOn_(k,i),QQ);
deOn_(k,i)=sub(deOn_(k,i),pesos);
deOn_(k,i)=sub(deOn_(k,i),QQ);
ctr0_(k,i)=nuOn_(k,i)^13 / deOn_(k,i);
soma=soma+ctr0_(k,i);
pntsfixos=pntsfixos + 1;
); );
for k from 1 to 3 do (
nun_(3,4,k)=sub(nu_(3,4,k),bandeira_(a,b,c));
den_(3,4,k)=sub(de_(3,4,k),bandeira_(a,b,c));
nun_{(3,4,k)}=sub(nun_{(3,4,k)},pesos);
nun_{(3,4,k)}=sub(nun_{(3,4,k)},QQ);
den_(3,4,k)=sub(den_(3,4,k),pesos);
den_(3,4,k)=sub(den_(3,4,k),QQ);
ctr_(3,4,k)=nun_(3,4,k)^13 / den_(3,4,k);
soma=soma+ctr_(3,4,k);
pntsfixos=pntsfixos + 1;
```

```
);
for k from 1 to 3 do (
nun_(1,0,k)=sub(nu_(1,0,k),bandeira_(a,b,c));
den_(1,0,k)=sub(de_(1,0,k),bandeira_(a,b,c));
nun_{(1,0,k)=sub(nun_{(1,0,k),pesos)};
nun_(1,0,k)=sub(nun_(1,0,k),QQ);
den_(1,0,k)=sub(den_(1,0,k),pesos);
den_(1,0,k)=sub(den_(1,0,k),QQ);
ctr_(1,0,k)=nun_(1,0,k)^13 / den_(1,0,k);
soma=soma+ctr_(1,0,k);
pntsfixos=pntsfixos + 1;
);
for k from 1 to 3 do (
nun01A2_k=sub(nu01A2_k,bandeira_(a,b,c));
den01A2_k=sub(de01A2_k,bandeira_(a,b,c));
nun01A2_k=sub(nun01A2_k,pesos);
nun01A2_k=sub(nun01A2_k,QQ);
den01A2_k=sub(den01A2_k,pesos);
den01A2_k=sub(den01A2_k,QQ);
ctr01A2_k=nun01A2_k^13 / den01A2_k;
soma=soma+ctr01A2_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 3 do (
nun02A2_k=sub(nu02A2_k,bandeira_(a,b,c));
den02A2_k=sub(de02A2_k,bandeira_(a,b,c));
nun02A2_k=sub(nun02A2_k,pesos);
nun02A2_k=sub(nun02A2_k,QQ);
den02A2_k=sub(den02A2_k,pesos);
den02A2_k=sub(den02A2_k,QQ);
ctr02A2_k=nun02A2_k^13 / den02A2_k;
soma=soma+ctr02A2_k;
pntsfixos=pntsfixos + 1;
):
for k from 1 to 5 do (
nun03A2_k=sub(nu03A2_k,bandeira_(a,b,c));
den03A2_k=sub(de03A2_k,bandeira_(a,b,c));
nun03A2_k=sub(nun03A2_k,pesos);
nun03A2_k=sub(nun03A2_k,QQ);
```

```
den03A2_k=sub(den03A2_k,pesos);
den03A2_k=sub(den03A2_k,QQ);
ctr03A2_k=nun03A2_k^13 / den03A2_k;
soma=soma+ctr03A2_k;
pntsfixos=pntsfixos + 1;
):
for k from 1 to 5 do (
nunA3_k=sub(nuA3_k,bandeira_(a,b,c));
denA3_k=sub(deA3_k,bandeira_(a,b,c));
nunA3_k=sub(nunA3_k,pesos);
nunA3_k=sub(nunA3_k,QQ);
denA3_k=sub(denA3_k,pesos);
denA3_k=sub(denA3_k,QQ);
ctrA3_k=nunA3_k^13 / denA3_k;
soma=soma+ctrA3_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 4 do (
nunA3sp_k=sub(nuA3sp_k,bandeira_(a,b,c));
denA3sp_k=sub(deA3sp_k,bandeira_(a,b,c));
nunA3sp_k=sub(nunA3sp_k,pesos);
nunA3sp_k=sub(nunA3sp_k,QQ);
denA3sp_k=sub(denA3sp_k,pesos);
denA3sp_k=sub(denA3sp_k,QQ);
ctrA3sp_k=nunA3sp_k^13 / denA3sp_k;
soma=soma+ctrA3sp_k;
pntsfixos=pntsfixos + 1;
):
for k from 1 to 3 do (
nun01A3sp_k=sub(nu01A3sp_k,bandeira_(a,b,c));
den01A3sp_k=sub(de01A3sp_k,bandeira_(a,b,c));
nun01A3sp_k=sub(nun01A3sp_k,pesos);
nun01A3sp_k=sub(nun01A3sp_k,QQ);
den01A3sp_k=sub(den01A3sp_k,pesos);
den01A3sp_k=sub(den01A3sp_k,QQ);
ctr01A3sp_k=nun01A3sp_k^13 / den01A3sp_k;
soma=soma+ctr01A3sp_k;
pntsfixos=pntsfixos + 1;
);
```

```
for k from 1 to 6 do (
nunv0A4_k=sub(nuv0A4_k,bandeira_(a,b,c));
denv0A4_k=sub(dev0A4_k,bandeira_(a,b,c));
nunv0A4_k=sub(nunv0A4_k,pesos);
nunv0A4_k=sub(nunv0A4_k,QQ);
denv0A4_k=sub(denv0A4_k,pesos);
denv0A4_k=sub(denv0A4_k,QQ);
ctrv0A4_k=nunv0A4_k^13 / denv0A4_k;
soma=soma+ctrv0A4_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 2 do (
nun01plA2_k=sub(nu01plA2_k,bandeira_(a,b,c));
den01plA2_k=sub(de01plA2_k,bandeira_(a,b,c));
nun01plA2_k=sub(nun01plA2_k,pesos);
nun01plA2_k=sub(nun01plA2_k,QQ);
den01plA2_k=sub(den01plA2_k,pesos);
den01plA2_k=sub(den01plA2_k,QQ);
ctr01plA2_k=nun01plA2_k^13 / den01plA2_k;
soma=soma+ctr01plA2_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 4 do (
nun01qlA3_k=sub(nu01qlA3_k,bandeira_(a,b,c));
den01qlA3_k=sub(de01qlA3_k,bandeira_(a,b,c));
nun01qlA3_k=sub(nun01qlA3_k,pesos);
nun01qlA3_k=sub(nun01qlA3_k,QQ);
den01qlA3_k=sub(den01qlA3_k,pesos);
den01qlA3_k=sub(den01qlA3_k,QQ);
ctr01qlA3_k=nun01qlA3_k^13 / den01qlA3_k;
soma=soma+ctr01qlA3_k;
pntsfixos=pntsfixos + 1;
):
for k from 1 to 5 do (
nun02plnA4_k=sub(nu02plnA4_k,bandeira_(a,b,c));
den02plnA4_k=sub(de02plnA4_k,bandeira_(a,b,c));
nun02plnA4_k=sub(nun02plnA4_k,pesos);
nun02plnA4_k=sub(nun02plnA4_k,QQ);
den02plnA4_k=sub(den02plnA4_k,pesos);
```

```
den02plnA4_k=sub(den02plnA4_k,QQ);
ctr02plnA4_k=nun02plnA4_k^13 / den02plnA4_k;
soma=soma+ctr02plnA4_k;
pntsfixos=pntsfixos + 1;
);
for k from 1 to 5 do (
cpn_k=sub(cp_k,bandeira_(a,b,c));
bpn_k=sub(bp_k,bandeira_(a,b,c));
cpn_k=sub(cpn_k,pesos);
cpn_k=sub(cpn_k,R);
bpn_k=sub(bpn_k,pesos);
bpn_k=sub(bpn_k,R);
ctrP1_k=bpn_k / cpn_k;
soma=soma + ctrP1_k;
P1s=P1s+1;
);
); ); )
-- soma returns the degree as a linear expression on d's
-- the relations for d's are
II1=ideal(d_29,d_27+2,2*d_25+d_26+2*d_28+2*d_30+1);
II2=ideal(d_23,d_20+d_22,d_19+d_21-d_24+2,d_17,d_16,d_15-1);
II3=ideal(d_13+d_14+d_18+2,d_11,d_9+d_30+1,d_8+d_12);
II4=ideal(2*d_7-2*d_10-d_26-2*d_30+1,d_6,d_5,d_3-1,2*d_1+d_2+2*d_4+2);
II=II1+II2+II3+II4;
-- substitute these relations in soma
-- to obtain the degree of the exceptional component
soma=sub(-soma, R/II)
```

The minus sign is due to the minus in Theorem 3. The variable some returns the result of the main Theorem 4.

5 APPENDIX B

5.1 INDUCED ACTION

For a given action of \mathbb{C}^* in \mathbb{P}^3 ,

$$\begin{aligned} \xi : & \mathbb{C}^* \times \mathbb{P}^3 & \to \mathbb{P}^3 \\ & [t, (q_0 : q_1 : q_2 : q_3)] & \mapsto (t^{-w_0} q_0 : t^{-w_1} q_1 : t^{-w_2} q_2 : t^{-w_3} q_3) \end{aligned}$$

we have, for each $t \in \mathbb{C}^*$, a linear map $T_t : \mathbb{C}^4 \to \mathbb{C}^4$ defined on canonical basis $\{e_0, e_1, e_2, e_3\}$ by $T_t(e_i) = t^{-w_i} e_i$.

The dual space $(\mathbb{C}^4)^{\vee}$ has dual basis $\{x_0, x_1, x_2, x_3\}$, the linear forms with

$$x_i(e_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.$$

Under T_t we obtain another basis of \mathbb{C}^4 , the basis $\beta = \{t^{-w_0}e_0, t^{-w_1}e_1, t^{-w_2}e_2, t^{-w_3}e_3\}$. The dual basis of β is the basis $\alpha = \{y_0, y_1, y_2, y_3\}$ of $(\mathbb{C}^4)^{\vee}$ given by linear forms with

$$y_i(t^{-w_j}e_j) = \delta_{ij} \therefore y_i(e_j) = t^{w_j}\delta_{ij}.$$

Thus, the dual basis is $\alpha = \{t^{w_0}x_0, t^{w_1}x_1, t^{w_2}x_2, t^{w_3}x_3\}$, and so the action ξ induces an action ξ^{\vee} on $\check{\mathbb{P}}^3$ just by

For a point $(1:q_1:q_2:q_3) \in \mathbb{P}^3$ the action ξ gives

$$t \circ (1:q_1:q_2:q_3) = (t^{-w_0}:t^{-w_1}q_1:t^{-w_2}q_2:t^{-w_3}q_3) = (1:t^{w_0-w_1}q_1:t^{w_0-w_2}q_2:t^{w_0-w_3}q_3).$$

If the weights w_i , $0 \le i \le 3$, are all distinct, then for $t \ne \pm 1$ the point will be changed. Therefore, the only fixed point is the point (1:0:0:0). In a similar way, the other fixed points by the action ξ are (0:1:0:0), (0:0:1:0) and (0:0:0:1).

The same rule holds for planes: for distinct weights w_i the only fixed planes are the standard ones, $x_i = 0, 0 \le i \le 3$, and so there is only four fixed points on $\check{\mathbb{P}}^3$ (Note: the planes $x_i = 0$ are fixed, but not *point-a-point*).

However, if we set $w_0 = w_1$ then all points of the line $(q_0 : q_1 : 0 : 0)$ will be fixed by the action, point-a-point, and so we gain a fixed component in \mathbb{P}^3 of positive dimension.

5.1.1 Action on lines

Let $\mathbb{G} = \mathbb{G}(1,3)$ the grassmannian of lines in \mathbb{P}^3 , and consider the Plücker embedding $\mathbb{G} \xrightarrow{\sim} Q \subset \mathbb{P}^5$, where $Q = \{(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) \in \mathbb{P}^5 | p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0\}$ ([7], example 2.1, pg.68).

The Plücker embedding associates a line

$$\ell = \begin{cases} a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = 0\\ b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 = 0 \end{cases}$$

to the point of Q whose coordinates are the (2×2) -minors of the coefficients matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{bmatrix}.$$

The action (5.1) applied on ℓ gives the line

$$t \circ \ell = \begin{cases} a_0 t^{w_0} x_0 + a_1 t^{w_1} x_1 + a_2 t^{w_2} x_2 + a_3 t^{w_3} x_3 = 0\\ b_0 t^{w_0} x_0 + b_1 t^{w_1} x_1 + b_2 t^{w_2} x_2 + b_3 t^{w_3} x_3 = 0 \end{cases}$$

and the coefficients matrix of $t \circ \ell$ is

$$\begin{bmatrix} t^{w_0}a_0 & t^{w_1}a_1 & t^{w_2}a_2 & t^{w_3}a_3 \\ t^{w_0}b_0 & t^{w_1}b_1 & t^{w_2}b_2 & t^{w_3}b_3 \end{bmatrix}$$

Computing the (2×2) -minors of this matrix, we conclude that the action passes to Q by

$$t \circ \underline{p} = (t^{w_0 + w_1} p_{01} : t^{w_0 + w_2} p_{02} : t^{w_0 + w_3} p_{03} : t^{w_1 + w_2} p_{12} : t^{w_1 + w_3} p_{13} : t^{w_2 + w_3} p_{23}).$$

Thus, for distinct weights, the Plücker relation $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$ guarantees that the only fixed lines by the action are the standard ones $x_i = x_j = 0, i, j \in \{0, 1, 2, 3\}$.

5.1.2 Action on quadrics and cubics

The set of quadric surfaces of \mathbb{P}^3 form a \mathbb{P}^9 , with induced action from ξ^{\vee} (5.1). For $g = b_0 x_0^2 + b_1 x_0 x_1 + \dots + b_9 x_3^2$, the induced action gives

$$t \circ g = t^{2w_0} b_0 x_0^2 + t^{w_0 + w_1} b_1 x_0 x_1 + \dots + t^{2w_3} b_9 x_3^2 .$$

For fixed quadrics to be only the monomials $x_i x_j = 0$, it is not enough that the weights are different. Now, it is also necessary that the sums of weights, taken 2–a–2, to be different.

This is similar to cubics. For $f = a_0 x_0^3 + a_1 x_0^2 x_1 + \cdots + a_{19} x_3^3$, the induced action is

$$t \circ f = t^{3w_0} a_0 x_0^3 + t^{2w_0 + w_1} a_1 x_0^2 x_1 + \dots + t^{3w_3} a_{19} x_3^3.$$

Therefore, to have only isolated fixed cubics we need that the sums of weights, taken 3–a–3, to be different.

As an example, we can take

$$(w_0, w_1, w_2, w_3) = (0, 1, 5, 25) \tag{5.2}$$

and the components of fixed flags, quadrics and cubics are all isolated points.

5.2 THREE PLANES IN \mathbb{P}^3

The intention here is to show with a simpler example some methods used in calculations involving Bott's formula.

A basic example is to find the number of points living in the intersection of three generic planes in \mathbb{P}^3 . The answer to this question is widely known as 1, and can be expressed in terms of the Chow ring of \mathbb{P}^3 as

$$\int_{\mathbb{P}^3} c_1(\mathcal{O}_{\mathbb{P}^3}(1))^3$$

Now allow us to complicate the calculations a little more by computing this number on the blowup of \mathbb{P}^3 along a line, let's say, along the line

$$\ell: x_1 = x_2 = 0.$$

Denote by $\pi: X \to \mathbb{P}^3$ this blowup, and $\mathcal{O}(1)$ the pullback bundle

$$\mathcal{O}(1) = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1)).$$

Now, we want to show how to compute $\int_{X} c_1(\mathcal{O}(1))^3 = 1$ applying Bott's formula,

$$\int_{X} c_1(\mathcal{O}(1))^3 \cap [X] = \sum_{F} \int_{F} \frac{c_1^T(\mathcal{O}(1)|_F)^3 \cap [F]_T}{c_{top}^T(\mathcal{N}_{F|X})}$$
(5.3)

where the sum runs through all fixed components F under an action of the torus \mathbb{C}^* .

To make things more fun, let's set the action

$$\mathbb{C}^* \times \mathbb{P}^3 \longrightarrow \mathbb{P}^3$$

$$(t, (s_0: s_1: s_2: s_3)) \longmapsto (t^{-1}s_0: t^{-1}s_1: t^{-2}s_2: t^{-3}s_3)$$

This action induces an action on the linear forms just by change the signals because of duality, *i.e.*,

Notice that the action on \mathbb{P}^3 gives two isolated fixed points and one fixed line

$$p_2 = (0:0:1:0) \notin \ell;$$

$$p_3 = (0:0:0:1) \in \ell;$$

$$d = (s_0:s_1:0:0)$$

and the fixed component d (the line of equations $x_2 = x_3 = 0$) cuts the blowup center ℓ in one point $p_0 = (1 : 0 : 0 : 0)$. We have to compute the contributions of the components of fixed points in the sum (5.3). In the numerator of (5.3) we have $c_1^T(\mathcal{O}(1)|_F)$. Since $\mathcal{O}(1)$ is a line bundle, the equivariant Chern class of this bundle over a fixed component F is

$$c_1^T(\mathcal{O}(1))|_F = \lambda + c_1(\mathcal{O}(1)|_F),$$
 (5.5)

where λ is the weight of the fiber of the bundle on the fixed component F and $c_1(\mathcal{O}(1)|_F)$ denotes the classical Chern class of $\mathcal{O}(1)$ restricted to F.

For an equivariant rank n bundle E over $F = \mathbb{P}^1$, one can split E as a direct sum of n line bundles $E = \bigoplus_{\lambda} E^{\lambda}$, for which the weight of the fiber of each E^{λ} is λ . Hence (5.5) applies for each E^{λ} ,

$$c_1^T(E^\lambda)|_F = \lambda + c_1(E|_F).$$

It follows that the n-th equivariant Chern class of E can be obtained by the product

$$c_n^T(E)|_F = \prod_{\lambda} c_1^T(E^{\lambda})|_F.$$
(5.6)

• Contribution of p_2

The simplest point to compute the contribution in (5.3) is p_2 . Since it is out of the blowup center, its contribution can be computed already in \mathbb{P}^3 .

The fiber of $\mathcal{O}(1)|_{p_2}$ is generated by x_2 , of weight $\lambda = 2$ (see (5.4)). And dim($\{p_2\}$) = 0, so $c_1(\mathcal{O}(1)|_{p_2}) = 0$. In view of (5.5) we have

$$c_1^T(\mathcal{O}(1)|_{p_2}) = 2.$$
 (5.7)

The denominator is $c_3^T(\mathcal{N}_{p_2|\mathbb{P}^3})$. But $\mathcal{N}_{p_2|\mathbb{P}^3} \simeq T_{p_2}\mathbb{P}^3$ as the normal space of a point in an ambient is equal to the tangent space of this ambient at the point. To compute $c_3^T(T_{p_2}\mathbb{P}^3)$, remember that the tangent bundle to any Grassmannian G = G(k, V) (rank V = r) is (see [7], Theorem 2.4, pg. 73)

$$TG = \operatorname{Hom}_{G}(\mathcal{S}, \mathcal{Q}) = \mathcal{Q} \otimes \mathcal{S}^{\vee},$$
 (5.8)

where

$$\mathcal{S} \hookrightarrow (V = \mathbb{C}^r) \longrightarrow \mathcal{Q}$$

is the tautological sequence over G; the fiber of $\mathcal{Q}^{\vee} \subset V^{\vee}$ over a point in G corresponding to a k-linear subspace of V is the rank r-k space of linear forms spanned by the equations of this space. In particular, for $G = \mathbb{P}^3, k = 1, r = 4$, at the point $p_2 = (0 : 0 : 1 : 0)$, given by the equations $x_0 = x_1 = x_3 = 0$, the fiber of \mathcal{Q} is spanned by the duals of these three equations; and the fiber of \mathcal{S}^{\vee} is spanned by $\overline{x_2} := x_2 \mod \langle x_0, x_1, x_3 \rangle$. Therefore, using (5.8) the tangent space to \mathbb{P}^3 at the point p_2 can be written as

$$T_{p_2}\mathbb{P}^3 = \langle x_0^{\vee}, x_1^{\vee}, x_3^{\vee} \rangle \otimes \langle x_2 \rangle = \langle x_0^{\vee} \otimes x_2, \ x_1^{\vee} \otimes x_2, \ x_3^{\vee} \otimes x_2 \rangle.$$

The action induced on this tangent space is just

$$t \circ (x_0^{\vee} \otimes x_2) = t^{-w_0} x_0^{\vee} \otimes t^{w_2} x_2 = t^{w_2 - w_0} (x_0^{\vee} \otimes x_2)$$

$$t \circ (x_1^{\vee} \otimes x_2) = t^{-w_1} x_1^{\vee} \otimes t^{w_2} x_2 = t^{w_2 - w_1} (x_1^{\vee} \otimes x_2)$$

$$t \circ (x_3^{\vee} \otimes x_2) = t^{-w_3} x_3^{\vee} \otimes t^{w_2} x_2 = t^{w_2 - w_3} (x_3^{\vee} \otimes x_2)$$

where w_i is the weight of x_i , $i \in \{0, 1, 2, 3\}$ (here, $w_0 = w_1 = 1$, $w_2 = 2$ and $w_3 = 3$). We will write

$$T_{p_2}\mathbb{P}^3 = \underbrace{\frac{x_2}{x_0}}_{w_2 - w_0} + \underbrace{\frac{x_2}{x_1}}_{w_2 - w_1} + \underbrace{\frac{x_2}{x_3}}_{w_2 - w_3}$$

In this notation, the $T_{p_2}\mathbb{P}^3$ is split into three line subbundles, and each one is associated to one weight. Using the expression (5.6), we find

$$c_3^T(T_{p_2}\mathbb{P}^3) = (w_2 - w_0)(w_2 - w_1)(w_2 - w_3) = 1 \cdot 1 \cdot (-1) = -1$$

Therefore, the contribution of the point p_2 is

$$\frac{c_1^T(\mathcal{O}(1)|_{p_2})^3}{c_3^T(T_{p_2}\mathbb{P}^3)} = \frac{8}{-1} = -8.$$
(5.9)

• Contribution over p_3

The point $p_3 = (0:0:0:1)$ lives inside the blowup center, the line ℓ .

The exceptional divisor of the blowup X of \mathbb{P}^3 along the line ℓ is a \mathbb{P}^1 -bundle over the line ℓ , the projectivized normal bundle $\mathbb{P}(\mathcal{N}_{\ell|\mathbb{P}^3})$. So, over the point p_3 there is a \mathbb{P}^1 -fiber for which we expect two fixed points.

Since p_3 is the point of equations $x_0 = x_1 = x_2 = 0$, we have

$$T_{p_3}\mathbb{P}^3 = \frac{x_3}{x_0} + \frac{x_3}{x_1} + \frac{x_3}{x_2}.$$
(5.10)

The blowup center is $\ell = \{(x_0 : 0 : 0 : x_3)\}$ (the variables here are only x_0 and x_3). Over ℓ the equation of p_3 is only $x_0 = 0$, and the tangent space to ℓ at the point p_3 is

$$T_{p_3}\ell = \frac{x_3}{x_0}.$$
 (5.11)

Next, beware of the fact that, although the normal bundle $\mathcal{N}_{\ell|\mathbb{P}^3} = \mathcal{O}_{\ell}(1) \oplus \mathcal{O}_{\ell}(1)$, these two summands are not isomorphic as \mathbb{C}^* -modules. The short exact sequence

$$T\ell \longrightarrow T\mathbb{P}^3_{|\ell} \longrightarrow \mathcal{N}_{\ell|\mathbb{P}^3}$$

shows us that

$$(\mathcal{N}_{\ell|\mathbb{P}^3})_{p_3} = \underbrace{\frac{x_3}{x_1}}_{w_3 - w_1 = 2} + \underbrace{\frac{x_3}{x_2}}_{w_3 - w_2 = 1}$$

Thus we have two *isolated* fixed points (since the weights are distinct). We will denote these points by

$$q_1 = \left(p_3, \frac{x_3}{x_1}\right)$$
 and $q_2 = \left(p_3, \frac{x_3}{x_2}\right)$.

The fibers of $\mathcal{O}(1)$ at these two points are the same, $\mathcal{O}_{\mathbb{P}^3}(1)_{p_3} = \langle x_3 \rangle$. Then, $c_1^T(\mathcal{O}(1)|_{q_i}) = 3, \ i = 1, 2$ in view of our choice of weights (5.4).

Now we look at the tangent space to X at these points, recalling $\mathcal{N}_{q_i|X} \simeq T_{q_i}X$.

For an inclusion of a smooth scheme Y into a smooth variety Z, denote by \tilde{Z} the blowup of Z along Y. The exceptional divisor can be identified with $\mathbb{P}(\mathcal{N}_{Y|Z})$, and for a point $(y, s), y \in Y$ and $s \in \mathbb{P}(\mathcal{N}_{yY|Z})$, the tangent space to \tilde{Z} can be split as

$$T_{(y,s)}\tilde{Z} = T_y Y \oplus \mathcal{L}_s \oplus T_{[\mathcal{L}_s]} \mathbb{P}(\mathcal{N}_{yY|Z}), \qquad (5.12)$$

where $\mathcal{L}_s \subset \mathcal{N}_{yY|Z}$ is the line represented by the point $[\mathcal{L}_s] \in \mathbb{P}(T_yZ)$.

For the point q_1 , (5.12) gives

$$T_{q_1}X = T_{p_3}\ell \oplus \mathcal{L}_{\frac{x_3}{x_1}} \oplus T_{[\mathcal{L}_{\frac{x_3}{x_1}}]}\mathbb{P}(\mathcal{N}_{p_3\ell|\mathbb{P}^3}).$$

But $\mathcal{N}_{p_3\ell|\mathbb{P}^3} = \frac{x_3}{x_1} + \frac{x_3}{x_2}$ and so

$$T_{[\mathcal{L}_{\frac{x_3}{x_1}}]}\mathbb{P}(\mathcal{N}_{p_3\ell|\mathbb{P}^3}) = \underbrace{\frac{x_3}{x_2}}_{\mathcal{S}^{\vee}} \cdot \underbrace{\frac{x_1}{x_3}}_{\mathcal{Q}} = \frac{x_1}{x_2}$$

This yields

$$T_{q_1}X = \frac{x_3}{x_0} + \frac{x_3}{x_1} + \frac{x_1}{x_2},$$

and

$$c_3^T(T_{q_1}X) = (w_3 - w_0)(w_3 - w_1)(w_1 - w_2) = 2 \cdot 2 \cdot (-1) = -4.$$

With this, we have the contribution of the fixed point q_1 :

$$\frac{c_1^T(\mathcal{O}(1)|_{q_1})^3}{c_3^T(T_{q_1}X)} = \frac{27}{-4} .$$
(5.13)

In a similar way,

so the contribution of q_2 is

$$T_{q_2}X = \frac{x_3}{x_0} + \frac{x_3}{x_2} + \frac{x_2}{x_1},$$

$$\frac{c_1^T(\mathcal{O}(1)|_{q_2})^3}{c_3^T(T_{q_2}X)} = \frac{27}{2}.$$
 (5.14)

• Contribution over the fixed line d

The fixed component d meets the blowup center ℓ in one point $p_0 = (1 : 0 : 0 : 0)$. Then its contribution to the sum (5.3) cannot be computed yet.

Like (5.10) and (5.11),

$$T_{p_0} \mathbb{P}^3 = \frac{x_0}{x_1} + \frac{x_0}{x_2} + \frac{x_0}{x_3},$$

$$T_{p_0} \ell = \frac{x_0}{x_3}.$$
(5.15)

Hence,

$$\mathcal{N}_{p_0\ell|\mathbb{P}^3} = \underbrace{\frac{x_0}{x_1}}_{w_0 - w_1 = 0} + \underbrace{\frac{x_0}{x_2}}_{w_0 - w_2 = -1} = 1 + \frac{x_0}{x_2}$$

where the notation 1 represents an eigenvector associated to the weight 0 (due to the weights $w_0 = w_1$). Therefore, over the point p_0 there are two isolated fixed points, namely $r_1 = (p_0, 1)$ and $r_2 = \left(p_0, \frac{x_0}{x_2}\right)$. It is over each of these points that we will compute the contributions to the sum (5.3).

The contribution of r_2 follows the same ideas like the points q_1 and q_2 . Write (5.12)

$$T_{r_2}X = \underbrace{\frac{x_0}{x_3}}_{T_{p_0}\ell} + \underbrace{\frac{x_0}{x_2}}_{\mathcal{L}\frac{x_0}{x_2}} + \underbrace{\frac{x_2}{x_0}}_{T_{[\mathcal{L}\frac{x_0}{x_2}]}\mathbb{P}(\mathcal{N}_{p_0\ell|\mathbb{P}^3})}$$

 \mathbf{SO}

$$c_3^T(T_{r_2}X) = (w_0 - w_3)(w_0 - w_2)(w_2 - w_0) = (-2) \cdot (-1) \cdot 1 = 2.$$

The fiber of $\mathcal{O}(1)|_{r_2}$ is spanned by x_0 , so $c_1^T(\mathcal{O}(1)|_{r_2}) = 1$. Then,

$$\frac{c_1^T(\mathcal{O}(1)|_{r_2})^3}{c_3^T(T_{r_2}X)} = \frac{1}{2} .$$
(5.16)

The last point is $r_1 = (p_0, 1)$. But this point lives in the strict transform \tilde{d} of the line d, which constitutes a full component of fixed points and we will use this \tilde{d} to compute the contribution in (5.3).

The bundle $\mathcal{O}(1)|_{\tilde{d}} = \mathcal{O}_{\tilde{d}}(1)$ has weight 1 since both x_0 and x_1 have this weight. But \tilde{d} has dimension one, and $c_1(\mathcal{O}_{\tilde{d}}(1)) = h$. In view of (5.5),

$$c_1^T(\mathcal{O}_{\tilde{d}}(1)) = 1 + h.$$

The denominator in (5.3) is now $c_2^T(\mathcal{N}_{\tilde{d}|X})$. In terms of weights, we can write over the point r_1

$$T_{r_1}X = \frac{x_0}{x_3} + 1 + \frac{x_0}{x_2}$$

From $T_{r_1}\tilde{d} = \frac{x_0}{x_1} = 1$, the normal space can by written as

$$\mathcal{N}_{r_1\tilde{d}|X} = \frac{x_0}{x_3} + \frac{x_0}{x_2}.$$
(5.17)

Notice that (5.17) gives a decomposition of $\mathcal{N}_{r_1\tilde{d}|X}$ in two eigensubspaces, and we can compute $c_2^T(\mathcal{N}_{\tilde{d}|X})$ like (5.6).

Remember that (5.15)

$$T_{p_0}\mathbb{P}^3 = \frac{x_0}{x_1} + \frac{x_0}{x_2} + \frac{x_0}{x_3}$$

and

$$T_{p_0}d = \frac{x_0}{x_1} = 1.$$

Then,

$$\mathcal{N}_{p_0 d|X} = \frac{x_0}{x_3} + \frac{x_0}{x_2}.$$

(Note the same description as (5.17)). But this is the fiber of the normal bundle of a \mathbb{P}^1 in \mathbb{P}^3 , which can be described as $\mathcal{N}_{d|\mathbb{P}^3} = \mathcal{O}_d(1) \oplus \mathcal{O}_d(1)$.

After the blowup, the normal bundle of \tilde{d} in X shall be modified, in accordance with the Proposition 4, page 131. It says that the subbundles related to the blowup center are unchanged, and the other subbundles shall be tensorized with $\mathcal{O}_{\tilde{d}}(-1)$. From $T_{p_0}\ell = \frac{x_0}{x_3}$ it follows that

• the subbundle $\mathcal{O}_d(1)$ related to the weight $\lambda_1 = \frac{x_0}{x_3}$ will not change after the blowup:

$$\mathcal{N}_{\tilde{d}|X}^{\lambda_1} = \mathcal{O}_{\tilde{d}}(1).$$

• the subbundle $\mathcal{O}_d(1)$ related to the weight $\lambda_2 = \frac{x_0}{x_2}$ will change after the blowup:

$$\mathcal{N}_{\tilde{d}|X}^{\lambda_2} = \mathcal{O}_{\tilde{d}}(1) \otimes \mathcal{O}_{\tilde{d}}(-1) = \mathcal{O}_{\tilde{d}}.$$

This will be denoted by

$$\mathcal{N}_{r_1\tilde{d}|X} = \underbrace{\frac{x_0}{x_3}}_{\mathcal{O}_{\tilde{d}}(1)} + \underbrace{\frac{x_0}{x_2}}_{\mathcal{O}_{\tilde{d}}}.$$

Now we are able to use (5.6)

$$c_2^T(\mathcal{N}_{r_1\tilde{d}|X}) = (w_0 - w_3 + h)(w_0 - w_2) = 2 - h,$$

and so

$$\int_{\tilde{d}} \frac{c_1^T (\mathcal{O}_{\tilde{d}}(1))^3}{c_2^T (\mathcal{N}_{\tilde{d}|X})} = \int_{\tilde{d}} \frac{(1+h)^3}{2-h} = \int_{\tilde{d}} \frac{(1+h)^3 (2+h)}{4}$$

where in the last equality we multiplied both numerator and denominator by 2 + h, and used that $h^2 = 0$ (of course, dim $\tilde{d} = 1$).

Expanding the numerator and taking the coefficient of h we obtain the contribution

$$\int_{\tilde{d}} \frac{c_1^T (\mathcal{O}_{\tilde{d}}(1))^3}{c_2^T (\mathcal{N}_{\tilde{d}|X})} = \frac{7}{4}.$$
(5.18)

Putting together the contributions (5.9), (5.13), (5.14), (5.16) and (5.18) the number in (5.3) is

$$-8 - \frac{27}{4} + \frac{27}{2} + \frac{1}{2} + \frac{7}{4} = 1$$

5.3 NORMAL BUNDLE OF STRICT TRANSFORM

Now we will show the following:

Proposition 4. Let $\mathbb{P}^1 = \ell \subset \mathbb{P}^N$ and $Y \subset \mathbb{P}^N$ be a smooth *m*-dimensional subscheme with $\ell \cap Y = \{p\}$, a single point. Let $X = Bl_Y \mathbb{P}^N$, $\tilde{\ell}$ the strict transform of ℓ , and

$$\mathcal{N}_{\ell \mid \mathbb{P}^N} = \mathcal{O}_{\ell}(a_1) \oplus \cdots \oplus \mathcal{O}_{\ell}(a_m) \oplus \mathcal{O}_{\ell}(a_{m+1}) \oplus \cdots \oplus \mathcal{O}_{\ell}(a_{N-1}).$$

If, for any \mathbb{C}^* -action that fix ℓ we have Y invariant and the tangent space to Y at p is isomorphic as \mathbb{C}^* -module with the fibers

$$T_pY \simeq \mathcal{O}_\ell(a_1)|_p \oplus \cdots \oplus \mathcal{O}_\ell(a_m)|_p$$
,

then we have the decomposition for $\mathcal{N}_{\tilde{\ell}|X}$,

$$\mathcal{N}_{\tilde{\ell}|X} = \mathcal{O}_{\tilde{\ell}}(a_1) \oplus \cdots \oplus \mathcal{O}_{\tilde{\ell}}(a_m) \oplus \mathcal{O}_{\tilde{\ell}}(a_{m+1}-1) \oplus \cdots \oplus \mathcal{O}_{\tilde{\ell}}(a_{N-1}-1).$$

Proof. Let $\mathbb{P}^N = \{(x_0 : \ldots : x_N)\}$ and $\ell = \mathbb{P}^1 = \{(x_0 : x_1 : 0 : \ldots : 0)\}, p = (1 : 0 : \ldots : 0).$ We will compute $\int_{\mathbb{P}^N} c_1[\mathcal{O}_{\mathbb{P}^N}(1)]^N$ via Bott in two ways: directly in \mathbb{P}^N and in the blowup along Y.

Fix $v \in \mathbb{Z}$, $v \ge 2$, and put an action on linear forms by

Thus, the weights of x_0 and x_1 are equal to 1 and the weight of x_i , $i \ge 2$, is v + i - 2. The line ℓ is a fixed component, and we can compute the contribution of this line now. We have $c_1^T(\mathcal{O}_{\mathbb{P}^N}(1)) = 1$ (weight of x_0, x_1) + h (= $c_1(\mathcal{O}_\ell(1))$). And from

$$\begin{split} \mathcal{N}_{p\ \ell \mid \mathbb{P}^{N}} &= \frac{x_{0}}{x_{2}} + \frac{x_{0}}{x_{3}} + \dots + \frac{x_{0}}{x_{N}}, \\ \mathcal{O}_{\ell}(a_{(N-1)}) \\ c_{N-1}^{T}(\mathcal{N}_{\ell \mid \mathbb{P}^{N}}) &= [c_{1}(\mathcal{O}_{\ell}(a_{1})) + 1 - v] \cdot [c_{1}(\mathcal{O}_{\ell}(a_{2})) + 1 - v - 1] \cdot \dots \cdot [c_{1}(\mathcal{O}_{\ell}(a_{N-1})) + 1 - v - N + 2] \\ c_{N-1}^{T}(\mathcal{N}_{\ell \mid \mathbb{P}^{N}}) &= [a_{1}h + 1 - v] \cdot [a_{2}h - v] \cdot \dots \cdot [a_{N-1}h - v - N + 3] \\ c_{N-1}^{T}(\mathcal{N}_{\ell \mid \mathbb{P}^{N}}) &= Z + \left(\frac{a_{1}Z}{1 - v} + \frac{a_{2}Z}{-v} + \dots + \frac{a_{N-1}Z}{-v - N + 3}\right)h \\ c_{N-1}^{T}(\mathcal{N}_{\ell \mid \mathbb{P}^{N}}) &= Z \left(1 + \sum_{i=1}^{N-1} \frac{a_{i}}{2 - i - v}h\right), \end{split}$$

where

$$Z = (1-v)(-v)(-v-1)\cdots(-v-N+3) = (-1)^{N-1}\frac{(v+N-3)!}{(v-2)!}$$

The contribution of ℓ is

$$\int_{\ell} \frac{c_1^T (\mathcal{O}_{\ell}(1))^N}{c_{N-1}^T (\mathcal{N}_{\ell|\mathbb{P}^N})} = \int_{\ell} \frac{1+Nh}{Z \cdot (1+\sum \frac{a_i}{2-i-v}h)} = \frac{1}{Z} \int_{\ell} (1+Nh)(1-\sum_{i=1}^{N-1} \frac{a_i}{2-i-v}h)$$
$$\int_{\ell} \frac{c_1^T (\mathcal{O}_{\ell}(1))^N}{c_{N-1}^T (\mathcal{N}_{\ell|\mathbb{P}^N})} = \frac{1}{Z} \left(N - \sum_{i=1}^{N-1} \frac{a_i}{2-i-v}\right) .$$
(5.19)

This contribution is the same if computed on the fixed points that arise over the point p when we do the blowup along Y.

By the hypotheses of tangent space to Y at p, this tangent space is $(1 \le m \le N - 2)$

$$T_p Y = \frac{x_0}{x_2} + \frac{x_0}{x_3} + \dots + \frac{x_0}{x_{m+1}}$$

So, $\mathcal{N}_{p|Y|\mathbb{P}^N} = 1 + \frac{x_0}{x_{m+2}} + \frac{x_0}{x_{m+3}} + \dots + \frac{x_0}{x_N}$. There are N - m - 1 isolated fixed points, corresponding to $\frac{x_0}{x_i}$, and a point $\tilde{p} \in \tilde{\ell}$.

To get the contribution of $q_i := (p, \frac{x_0}{x_i}), m+2 \le i \le N$, note that

$$T_{q_i}X = \underbrace{\frac{x_0}{x_2} + \dots + \frac{x_0}{x_{m+1}}}_{T_pY} + \underbrace{\frac{x_0}{x_i}}_{\mathcal{L}_{x_0/x_i}} + \underbrace{\frac{x_i}{x_0} + \frac{x_i}{x_{m+2}} + \dots + \frac{\overrightarrow{x_i}}{\overrightarrow{x_i}}_{T_{\mathcal{L}_{x_i/x_0}} \mathbb{P}(\mathcal{N}_{p-Y|\mathbb{P}^N})}$$

$$c_N^T(T_{q_i}X) = (1-v)(-v)(-v-1)\cdots(-v-m+2)\cdot(-1)(v+i-3)^2\cdot(i-m-2)(i-m-3)\cdots(i-N)$$
$$c_N^T(T_{q_i}X) = (-1)^{N-1}\frac{(v+m-2)!}{(v-2)!}\cdot(v+i-3)^2\cdot\prod_{\substack{j=m+2\\j\neq i}}^N(j-i)$$

Then, the contribution of the point q_i is

$$\frac{(-1)^{N-1}(v-2)!}{(v+m-2)! \cdot (v+i-3)^2 \cdot \prod_{\substack{j=m+2\\j\neq i}}^{N} (j-i)}$$
(5.20)

Next, we compute the contribution of $\tilde{\ell}$. Write

$$\mathcal{N}_{p\ \tilde{\ell}|X} = \frac{x_0}{x_2} + \frac{x_0}{x_3} + \dots + \frac{x_0}{x_N},$$

$$\mathcal{O}_{\ell}(b_1) = \mathcal{O}_{\ell}(b_2) + \dots + \frac{x_0}{\mathcal{O}_{\ell}(b_{(N-1)})},$$

for unknowns b_i . The computational aspects are the same as for ℓ , just changing \underline{a} by \underline{b} , so the contribution of $\tilde{\ell}$ is

$$\frac{1}{Z} \left(N - \sum_{i=1}^{N-1} \frac{b_i}{2 - i - v} \right) . \tag{5.21}$$

The sum of contributions in (5.20) and (5.21) is equal to (5.19), and so we get the equation

$$\frac{1}{Z}\left(N-\sum_{i=1}^{N-1}\frac{a_i}{2-i-v}\right) = \frac{1}{Z}\left(N-\sum_{i=1}^{N-1}\frac{b_i}{2-i-v}\right) + \sum_{i=m+2}^{N}\frac{(-1)^{N-1}(v-2)!}{(v+m-2)!(v+i-3)^2}\prod_{\substack{j=m+2\\j\neq i}}^{N}(j-i)$$

 \Downarrow multiply everything by Z

$$\sum_{i=1}^{N-1} \frac{a_i}{2-i-v} = \sum_{i=1}^{N-1} \frac{b_i}{2-i-v} - \frac{(v+N-3)!}{(v+m-2)!} \sum_{\substack{i=m+2\\ i=m+2}}^{N} \frac{1}{(v+i-3)^2 \cdot \prod_{\substack{j=m+2\\ j\neq i}}^{N} (j-i)} .$$
 (5.22)

Now, we can verify that the solutions

$$b_1 = a_1, \ b_2 = a_2, \dots, b_m = a_m, \ b_{m+1} = a_{m+1} - 1, \dots, b_{N-1} = a_{N-1} - 1.$$
 (5.23)

holds for any $v \in \mathbb{Z}, v \geq 2$.

Setting the N-1 values v = 2, v = 3, ..., v = N on equation (5.22) we obtain the linear system CB - CA = D, where

$$A = [a_1 \cdots a_{N-1}]^t, \ B = [b_1 \cdots b_{N-1}]^t, \ C = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{N-1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N-1} & \frac{1}{N} & \cdots & \frac{1}{2N-3} \end{bmatrix}_{(N-1)\times(N-1)}$$

Example 14. For N = 4 and m = 1 (*i.e.*, blowup of \mathbb{P}^4 along a smooth curve), equation (5.22) reduces to

$$\frac{a_1}{1-v} + \frac{a_2}{-v} + \frac{a_3}{-1-v} = \frac{b_1}{1-v} + \frac{b_2}{-v} + \frac{b_3}{-1-v} - \frac{2v+1}{v(v+1)}$$

It is a straightforth calculation the verification of the solution, for all $v \in \mathbb{Z}$, $v \ge 2$,

$$b_1 = a_1, \quad b_2 = a_2 - 1, \quad b_3 = a_3 - 1.$$
 (5.24)

Moreover, setting values 2, 3 and 4 to v we obtain the linear system

$$v = 2 \implies 6a_1 + 3a_2 + 2a_3 = 6b_1 + 3b_2 + 2b_3 + 5$$

$$v = 3 \implies 6a_1 + 4a_2 + 3a_3 = 6b_1 + 4b_2 + 3b_3 + 7$$

$$v = 4 \implies 20a_1 + 15a_2 + 12a_3 = 20b_1 + 15b_2 + 12b_3 + 27$$

whose solutions are unique (5.24).

Next, we present scripts for *Macaulay2* to perform the calculations in the above proof. First, set values for N, m as you desire, provided that $2 \le m \le N - 2$.

```
N=4; M=1; -- 1<M<N-1
-- change these values if you want to.
```

```
R=QQ[x_0..x_N,v,a_1..b_N];
Rh=R[h]/h^2;
```

```
-- setting the weights
p01={x_0=>1,x_1=>1};
poth=apply(N-1,i->x_(i+2)=>v+i);
pesos=flatten(append(p01,poth));
```

```
-- tangent and normal spaces to l and Y(=d here)
TPN=apply(N,i->x_0-x_(i+1));
Tl=x_0-x_1; Nl=delete(Tl,TPN);
Td=apply(M,i->x_0-x_(2+i)); Nd=TPN-set(Td);
```

```
-- computing the contribution contl of line l
cn=product(apply(N-1,i->a_(i+1)*h+sub(Nl_i,pesos)));
conj=cn-2*(cn%h); den=cn*conj;
num=((1+h)^N)*conj;
num=num//h;
den=sub(den,R);
num=sub(den,R);
```

```
contl=num/den;
-- after the blowup
-- contributions of points x_i/x_0
for i from 0 to N-M-1 do (
eq=Nd_i;
if eq!=x_0-x_1 then (
Teq=delete(eq,Nd);
Teq=apply(N-M-1,j->Teq_j-eq);
Teq=append(Teq,eq);
Teq=flatten(append(Teq,Td));
Teq=apply(N,j->sub(Teq_j,pesos));
Te_i=product Teq; ) );
-- the contribution contL of strict transform ~1
cnL=product(apply(N-1,i->b_(1+i)*h+sub(Nl_i,pesos)));
conjL=cnL-2*(cnL%h); denL=cnL*conjL;
numL=((1+h)^N)*conjL;
numL=numL//h;
denL=sub(denL,R);
numL=sub(numL,R);
contL=numL/denL;
-- sum of contributions
eq=0;
for i from 0 to N-M-1 do (
if Nd_i!=x_0-x_1 then (
eq=eq+(1/Te_i); ) );
-- matching the contributions before and after the blowup
-- eq represents the obtained equation
eq=eq+contL-contl;
eq=numerator eq;
-- solutions of linear system and the equations like in exemple
I=ideal();
for i from 0 to N-2 do (
eqt_i=sub(eq,v=>i+2);
```

I=I+eqt_i; print(i+2, eqt_i);) trim I

Remark 4. For m = 0, *i.e.*, blowup of X along a point, a similar result of Proposition 4 can be found in [10], Lemma 1.6, page 5.

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